Solution Hints to the Exercises

from

A Concise Introduction to Mathematical Logic

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Hints to the exercises can as a rule easily be supplemented to a complete solution. Exercises that are essential for the text are solved completely. The reader may mail improved solutions to the author whose website is `www.math.fu-berlin.de/~raut`.

Section 1.1

1. (a): x_k is fictional in f iff $a_k = 0$. (b): Because of the uniqueness, 2^{n+1} (= number of subsets of $\{0, \ldots, n\}$) is the number sought for. (c): induction on formulas in $\neg, +$ and $p_1, \ldots, p_n$.

2. Consider on $\mathcal{F}$ the property $\mathcal{E}\varphi$: 'φ is prime or there are $\alpha, \beta \in \mathcal{F}$ with $\varphi = \neg\alpha$ or $\varphi = (\alpha \circ \beta)$ where $\circ = \wedge$ or $\circ = \vee$.' Formula induction shows $\mathcal{E}\varphi$ for all $\varphi \in \mathcal{F}$.

3. Verify by induction on φ the property $\mathcal{E}\varphi$: 'no proper initial segment of φ is a formula nor can φ be a proper initial segment of a formula'. Induction step: Case $\varphi = \neg\alpha$. Then a proper initial segment of $\neg\alpha$ either equals $\neg$ (hence is not a formula), or has the form $\neg\xi$ where ξ is a proper initial segment of α. Thus $\xi \notin \mathcal{F}$ by the induction hypotheses, hence also $\neg\xi \notin \mathcal{F}$ (since a formula starting with $\neg$ must have the form $\neg\beta$ for some formula β by Exercise 2). Case $\varphi = (\alpha \circ \beta)$. Let ξ be a proper initial segment of φ or conversely. Assume that ξ is a formula so that $\xi = (\alpha' \circ' \beta')$, some $\alpha', \beta' \in \mathcal{F}$ (Exercise 2). Then $\alpha \neq \alpha'$, for otherwise necessarily $\xi = \varphi$. Hence α' is a proper initial segment of α or conversely, a contradiction to the induction hypothesis $\mathcal{E}\alpha$.

4. Assume that $(\alpha \circ \beta) = (\alpha' \circ' \beta')$, hence $\alpha \circ \beta = \alpha' \circ' \beta'$. If $\alpha \neq \alpha'$ then α is a proper initial segment of α' or conversely. This is impossible by Exercise 3. Consequently $\alpha = \alpha'$, hence $\circ = \circ'$ and $\beta = \beta'$.

Section 1.2

1. $w((p \to q_1) \wedge (\neg p \to q_2)) = 0$ *iff* $wp = 1, wq_1 = 0$ or $wp = 0, wq_2 = 0$, and the same condition holds for $w(p \wedge q_1 \vee \neg p \wedge q_2) = 0$. In a similar way the second equivalence is treated.

2. $\neg p \equiv p + 1,\ 1 \equiv p + \neg p,\ p \leftrightarrow q \equiv p + \neg q,\ p + q \equiv p \leftrightarrow \neg q \equiv \neg(p \leftrightarrow q)$.

3. Induction on the $\alpha \in \mathcal{F}_n\{0, 1, \wedge, \vee\}$ (= set of formulas in $0, 1, \wedge, \vee$ and $p_1, \ldots p_n$). If $f, g \in \boldsymbol{B}_n$ are monotonic then so is $\vec{a} \mapsto f\vec{a} \circ g\vec{a}$, where $\circ$ is $\wedge$ or $\vee$. For simplicity, treat first the case $n = 1$. Converse: Induction on the arity n. Clear for $n = 0$, with the formulas 0 and 1 representing the two constants. With $f \in \boldsymbol{B}_{n+1}$ also $f_k : \vec{x} \mapsto f(\vec{x}, k)$ is monotonic ($k = 0, 1$). Let $\alpha_k \in \mathcal{F}_n\{0, 1, \wedge, \vee\}$ represent f_k (induction hypothesis). Then $\alpha_0 \vee (\alpha_1 \wedge p_{n+1})$ represents f. Note that $w\alpha_0 \leqslant w\alpha_1$ for all w.

4. By Exercise 3, a not representable $f \in \boldsymbol{B}_{n+1}$ is not monotonic in the last argument, say. Then $f(\vec{a},1) = 0$ and $f(\vec{a},0) = 1$ for some $\vec{a} \in \{0,1\}^n$, hence $g\colon x \mapsto f(\vec{a},x)$ is negation. This proves the claim.

Section 1.3

1. (a): MP easily yields $p \to q \to r,\ p \to q,\ p \vDash r$. Apply (D) three times.

2. The deduction theorem yields $\vDash (\alpha \to \beta) \to (\gamma \to \alpha) \to (\gamma \to \beta)$.

3. Assume that $w \vDash X, \alpha \vee \beta$. Then clearly $w \vDash X, \alpha$ or $w \vDash X, \beta$.

5. Let $X \vdash \alpha \notin X$ Then $X, \alpha \vdash \beta$ for each β. Thus, $X \vdash \beta$ by (T).

Section 1.4

1. $X \cup \{\neg\alpha \mid \alpha \in Y\} \vdash \bot \Rightarrow X \cup \{\neg\alpha_0, \ldots, \neg\alpha_n\} \vdash \bot$, some $\alpha_0, \ldots, \alpha_n \in Y$. Hence $X \vdash (\bigwedge_{i \leqslant n} \neg\alpha_i) \to \bot$, or equivalently, $X \vdash \bigvee_{i \leqslant n} a_i$. This all is easily verified if $\vdash$ is replaced by $\vDash$.

2. Supplement Lemma 4.4 by proving $X \vdash \alpha \vee \beta \ \Leftrightarrow\ X \vdash \alpha$ or $X \vdash \beta$.

3. Choose X, φ such that $X \nvdash \varphi$ and $X \vdash' \varphi$. Let $Y \supseteq X \cup \{\neg\varphi\}$ be maximally consistent in $\vdash$. Define σ by $p^\sigma = \top$ for $p \in Y$ and $p^\sigma = \neg\top$ otherwise. Induction on α yields with the aid of $(\wedge)$ and $(\neg)$ page 28

$$(*) \qquad \alpha \in Y \Rightarrow\ \vdash \alpha^\sigma \quad ; \quad \alpha \notin Y \Rightarrow\ \vdash \neg\alpha^\sigma.$$

In proving $(*)$, $\vdash \top$, $\vdash \alpha \ \Rightarrow \vdash \neg\neg\alpha$, $\neg\alpha \vdash \neg(\alpha \wedge \beta)$, and $\neg\beta \vdash \neg(\alpha \wedge \beta)$ are needed which easily follow from the $\neg$–rules. By $(*)$, $\vdash \neg\varphi^\sigma$, hence $\vdash' \neg\varphi^\sigma$. Clearly $\vdash Y^\sigma$ (i.e., $\vdash \alpha^\sigma$ for all $\alpha \in Y$), and so $\vdash' Y^\sigma$. But $Y^\sigma \vdash' \varphi^\sigma$ (substitution invariance). Thus, $\vdash' \varphi^\sigma$. Therefore $\vdash' \alpha$ for all α by $(\neg 1)$, so that $\vdash$ is maximal by definition.

4. There is a smallest consequence relation with the properties $(\wedge 1)$–$(\neg 2)$, namely the calculus $\vdash$ of this section. Since $\vdash\ \subseteq\ \vDash$ and $\vdash$ is already maximal according to Exercise 3, $\vdash$ and $\vDash$ must coincide.

Section 1.5

1. For finite M easily shown by induction on the number of elements of M. Note that M has a maximal element. General case: Add to the formulas in Example 1 the set of formulas $\{p_{ab} \mid a \leqslant_0 b\}$.

2. $\Rightarrow$: Assume $M, N \notin F$. Then $\setminus(M \cup N) = \setminus M \cap \setminus N \in F$, because $\neg M, \neg N \in F$. Therefore $M \cup N \notin F$. $\Leftarrow$: $M \in F$ implies $M \cup N \in F$ by condition (b). For proving ($\neg$) from ($\cap$) observe that $M \cup \setminus M \in F$.

3. $\Rightarrow$: Let U be trivial, i.e., $E \in U$ for some finite $E \subseteq I$. Induction on the number of elements in E and Exercise 2 easily show that $\{i_0\} \in U$ for some $i_0 \in E$. The converse is obvious.

Section 1.6

1. First verify the deduction theorem, which holds for each calculus with MP as the only rule and A1, A2 among the axioms; cf. Lemma 6.3. X is consistent iff $X \nvdash \bot$, for $X \vdash \bot \Rightarrow X \vdash (\alpha \to \bot) \to \bot = \neg\neg\alpha$ by A1, hence $X \vdash \alpha$ by A3. Now prove $X \vdash \alpha \to \beta$ *iff* $X \vdash \alpha \Rightarrow X \vdash \beta$, provided X is maximally consistent. This allows one to proceed along the lines of Lemma 4.5 and Theorem 4.6.

2. Apply Zorn's lemma to $H := \{Y \supseteq X \mid Y \nvdash \alpha\}$. Note that if $K \subseteq H$ is a chain then $\bigcup K \in H$ due to the finitarity of $\vdash$.

3. (a): Such a set X satisfies $(*): X \vdash \varphi \to \alpha$ for all α. For otherwise $X, \varphi \to \alpha \vdash \varphi$, hence $X \vdash (\varphi \to \alpha) \to \varphi$, and so $X \vdash \varphi$ by Peirce's axiom. Suppose $\alpha \notin X$. Then $X, \alpha \vdash \varphi, \varphi \to \beta$ by $(*)$, and so $X, \alpha \vdash \beta$. (b): With (a) easily follows $X \vdash \alpha \to \beta$ *iff* $X \vdash \alpha \Rightarrow X \vdash \beta$ as in Exercise 1. Proceed with an adaptation of Lemma 4.5.

4. Crucial for completeness is the proof of (m): $\alpha \vdash \beta \Rightarrow \alpha\gamma \vdash \beta\gamma$ by induction on the rules of $\vdash$. (m) implies (M): $X, \alpha \vdash \beta \Rightarrow X, \alpha\gamma \vdash \beta\gamma$, proving first that a calculus $\vdash$ based solely on unary rules satisfies $X \vdash \beta \Rightarrow \alpha \vdash \beta$ for some $\alpha \in X$. E.g., $\alpha \vdash \alpha\beta \Rightarrow \alpha\gamma \vdash \gamma\alpha \vdash \gamma\alpha\beta \vdash \alpha\beta\gamma$. Although $\alpha(\beta\gamma) \vdash (\alpha\beta)\gamma$ and conversely, it is still tricky to show that $\alpha(\beta\gamma)\delta \vdash (\alpha\beta)\gamma\delta$. (M) implies $X, \alpha \vdash \gamma$ & $X, \beta \vdash \gamma \Rightarrow X, \alpha\beta \vdash \gamma$, because $X, \alpha \vdash \gamma \Rightarrow X, \alpha\beta \vdash \gamma\beta \vdash \beta\gamma$ and $X, \beta\gamma \vdash \gamma\gamma \vdash \gamma$, therefore $X, \alpha\beta \vdash \gamma$. From this it follows [$\vee$]: $X \vdash \alpha\beta \Leftrightarrow X \vdash \alpha$ or $X \vdash \beta$, provided X is φ-maximal, for note that

$$X \nvdash \alpha \ \& \ X \nvdash \beta \Rightarrow X, \alpha \vdash \varphi \ \& \ X, \beta \vdash \varphi \Rightarrow X, \alpha\beta \vdash \varphi \Rightarrow X \nvdash \alpha\beta.$$

Having [$\vee$] one may proceed with a slight modification of Lemma 4.5.

Section 2.1

1. There are 10 essentially binary Boolean functions f. The corresponding algebras $(\{0,1\}, f)$ split into 5 pairs of isomorphic ones. For example, $(\{0,1\}, \land) \simeq (\{1,0\}, \lor)$.

2. $\Leftarrow$: Choose $c = a$ in $a \approx b \;\&\; a \approx c \Rightarrow b \approx c$ to get $a \approx b \Rightarrow b \approx a$.

3. For simplicity, treat first the case $n = 2$ using transitivity.

5. For simplicity, let the signature contain only the symbols r, f, both unary. Then $ra \Rightarrow ra_j \Rightarrow rha$ and $hfa = h(fa_i)_{i \in I} = fa_j = fha$.

Section 2.2

1. Trivial if t is a prime term. A terminal segment of $f\vec{t}$ either equals $f\vec{t}$ or has the form $t'_k t_{k+1} \cdots t_n$ for some $k \leqslant n$ ($t'_k t_{k+1} \cdots t_n$ means t'_n in case $k = n$), where t'_k a terminal segment of t_k. By the induction hypotheses, t'_k is a term concatenation, hence so is $t'_k t_{k+1} \cdots t_n$.

2. It suffices to prove (a') $t\xi = t'\xi' \Rightarrow t = t'$, for all $t, t' \in \mathcal{T}$, all $\xi, \xi' \in \mathcal{S}_{\mathcal{L}}$ by induction on t. This is obvious for prime t. Let $t = ft_1 \cdots t_n$ and $t\xi = t'\xi'$ with $t' = f't'_1 \cdots t'_m$. Then clearly $f = f'$ and $m = n$, hence $t_1 \cdots t_n\xi = t'_1 \cdots t'_n\xi'$. Thus $t_1 = t'_1$ and $t_2 \cdots t_n\xi = t'_2 \cdots t'_n\xi'$ by the induction hypothesis for t_1. Similarly, $t_2 = t'_2 \ldots, t_n = t'_n$ and also $\xi = \xi'$. This proves (a').

3. (a): Similar to Exercise 3 in **1.1**. (b) follows readily from (a). (c): If $\neg\xi \in \mathcal{L}$ then by (b), $\neg\xi = \neg\alpha$ for some $\alpha \in \mathcal{L}$. Hence $\xi = \alpha$. Similarly, $\alpha, \alpha \land \xi \in \mathcal{L} \Rightarrow \alpha \land \xi = \beta \land \gamma$, some $\beta, \gamma \in \mathcal{L}$, hence $\alpha = \beta$ and $\xi = \gamma$.

5. Can completely be reduced to Corollary 1.2.2 by some bijection from X onto a set V of propositional variables.

Section 2.3

1. If $\mathcal{M} \vDash X$ and $x \notin \mathit{free}\, X$ then $\mathcal{M}^a_x \vDash X$ for each a (Theorem 2.3.1).

2. $\forall x(\alpha \to \beta), \forall x\alpha \vDash \alpha \to \beta, \alpha \vDash \beta$ and Exercise 1.

3. The Theorems 3.1 and 3.5 yield $\mathcal{A} \vDash \alpha\,[a] \Leftrightarrow \mathcal{A}' \vDash \alpha\,[a] \Leftrightarrow \mathcal{A}' \vDash \alpha_x(\boldsymbol{a})$.

4. (a): $\exists_n \land \exists_m \equiv \exists_m$ for $n \leqslant m$, and $\exists_n \land \neg\exists_m \equiv \exists_0$ ($\equiv \bot$) for $n \geqslant m$. (b): Exercise 5 in **2.2**, and $\exists_n \land \neg\exists_m \equiv \bigvee_{n \leqslant k < m} \exists_{=k}$ for $n < m$.

Section 2.4

1. $\alpha \equiv \beta \;\Rightarrow\; \vDash \forall\vec{x}\,(\alpha \leftrightarrow \beta) \;\Rightarrow\; \vDash (\alpha \leftrightarrow \beta)\,\frac{\vec{t}}{\vec{x}}\;\big(= \alpha\,\frac{\vec{t}}{\vec{x}} \leftrightarrow \beta\,\frac{\vec{t}}{\vec{x}}\,\big)$.

3. W.l.o.g. $\alpha \equiv \forall\vec{y}\alpha'(\vec{x},\vec{y})$ and $\beta \equiv \forall\vec{z}\beta'(\vec{x},\vec{z})$ with disjoint tuples $\vec{x}$, $\vec{y}$, $\vec{z}$.

4. Simultaneous induction on φ and $\neg\varphi$. Clear if φ is prime. If the claim holds for α, β then also for $(\alpha \wedge \beta)$ and $\neg(\alpha \wedge \beta)$ $(\equiv \neg\alpha \vee \neg\beta)$. The step for $\vee$ is similar. Step for $\neg$: Simply observe that $\neg\neg\alpha \equiv \alpha$.

5. $\exists x(Px \to \forall yPy) \equiv \forall xPx \to \forall yPy$ according to (10) in **2.4**.

Section 2.5

1. Proof very similar to that of Exercise 6 in **2.4**

2. $\Rightarrow$: $S \vDash \alpha\,\frac{t}{x} \;\to \beta \Leftrightarrow S, \alpha\,\frac{t}{x} \vDash \beta$ and (e) page 79. $\Leftarrow$: (9) in **2.4**.

3. $\beta \in T + \alpha \Leftrightarrow T, \alpha \vDash \beta \Leftrightarrow T \vDash \alpha \to \beta$ by the deduction theorem.

Section 2.6

1. The "if" part follows as Theorem 6.1 because $y = f\vec{t} \equiv_{T_f} \delta_f(\vec{t}, y))$. The "only if" part: $y = f\vec{t} \equiv_{T_f} \delta_f(\vec{t}, y)$ and $T_f \vDash \forall\vec{x}\exists!y\,y = f\vec{x}$. Hence also $T_f \vDash \forall\vec{x}\exists!y\,\delta(\vec{x}, y)$.

2. $\mathcal{N} \vDash x = 0 \leftrightarrow \forall y\,x \neq \mathtt{S}y$. Careful calculation confirms the definition $x + y = z \leftrightarrow x = y = z = 0 \vee z \neq 0 \wedge \mathtt{S}(x \cdot z) \cdot \mathtt{S}(y \cdot z) = \mathtt{S}(z^2 \cdot \mathtt{S}(x \cdot y))$. Therein z^2 denotes the term $z \cdot z$.

3. Let $xy = xz = e$ ($\circ$ not written). Choose some y' with $yy' = e$. Then $yx = (yx)(yy') = y(xy)y' = yey' = e$ and so $ex = (xy)x = x(yx) = xe = x$ for all x. In other words, e is a left and right unit element. We hence obtain $y = ye = y(xz) = (yx)z = ez = z$. For the additional claim derive the axioms of $T_G^{=}$ from those of T_G and conversely.

4. If $<$ were definable then $<$ would be invariant under automorphisms of $(\mathbb{Z}, 0, +)$. This is not the case for the automorphism $n \mapsto -n$. This approach to the problem is also called Padoa's method.

Section 3.1

1. Let $X \vdash \alpha\,\frac{t}{x}$. Then $X, \forall x \neg\alpha \vdash \alpha\,\frac{t}{x}, \neg\alpha\,\frac{t}{x}$. Hence $X, \forall x \neg\alpha \vdash \exists x\alpha$. Also $X, \neg\forall x\neg\alpha \vdash \exists x\alpha$ $(= \neg\forall x\neg\alpha)$. Thus $X \vdash \exists x\alpha$ according to $(\neg 2)$.

2. Let $\alpha' := \alpha\,\frac{y}{x}$, $u \notin \mathrm{var}\,\alpha$, $u \neq y$. Then $\forall x\alpha \vdash \alpha'\,\frac{u}{y}$ $(= \alpha\,\frac{u}{x})$ by $(\forall 1)$. Hence $\forall x\alpha \vdash \forall y\alpha'$ by $(\forall 2)$, with $X = \{\forall x\alpha\}$, α' for α, and y for x.

3. $\forall y(\alpha\frac{y}{x}) \vdash \forall x\alpha \vdash \forall z(\alpha\frac{z}{x})$ according to Exercise 2.

4. $\Rightarrow$: $X \nvdash \varphi \Rightarrow X, \varphi \vdash \bot \Rightarrow X \vdash \neg\varphi$. $\Leftarrow$: $X \nvdash \alpha \Rightarrow X \vdash \neg\alpha \Rightarrow X, \alpha \vdash \bot$.

Section 3.2

1. First prove $(*)$ $\mathfrak{T} \vDash \forall\vec{x}\varphi$ iff $\mathfrak{T} \vDash \varphi\,\frac{\vec{t}}{\vec{x}}$ for all $\vec{t} \in \mathcal{T}_0^n$ $(\varphi \in \mathcal{L}$ open); use Theorem 2.3.5. Next prove $\binom{*}{*}$ $X \vdash \alpha \Leftrightarrow \mathfrak{T} \vDash \alpha$ $(\alpha \in \mathcal{L}^0$ open) by induction on $\wedge, \neg$; observe that $\mathcal{L}$ is $=$-free. Let $X \vdash \forall\vec{x}\varphi$ (φ open) and $\vec{t} \in \mathcal{T}_0^n$. Then also $X \vdash \alpha := \varphi\,\frac{\vec{t}}{\vec{x}}$, hence $\mathfrak{T} \vDash \alpha$ by $\binom{*}{*}$. Thus, $\mathfrak{T} \vDash \forall x\varphi$ by $(*)$, and so $\mathfrak{T} \vDash U$.

2. $K \vdash \alpha \;\Rightarrow\; T \vdash \alpha$ for some $T \in K$ (finiteness theorem)

4. (i)$\Rightarrow$(ii): (12) in **2.4**. Observe also $(x = t \to \alpha)\,\frac{t}{x} \equiv \alpha\,\frac{t}{x}$.

Section 3.3

1. Prove $\vdash_{\mathsf{PA}} \forall z(x+y)+z = x+(y+z)$ by induction on z. Obvious for $z = 0$. The induction step follows easily from $\vdash_{\mathsf{PA}} x + \mathtt{S}y = \mathtt{S}(x+y)$. Most proofs of the arithmetical laws in PA need much patience.

2. $z + x = x \to z = 0$ (induction on x) readily yields $x \leqslant y \leqslant x \to x = y$.

3. Informally: $x < y \;\Rightarrow\; \exists z\,\mathtt{S}z + x = y \;\Rightarrow\; \exists z\, z + \mathtt{S}x = y \;\Rightarrow\; \mathtt{S}x \leqslant y$. The converse $\mathtt{S}x \leqslant y \to x < y$ follows from $\vdash_{\mathsf{PA}} x < \mathtt{S}x$. The induction hypothesis of $x \leqslant y \vee y \leqslant x$ may be written as $x < y \vee y \leqslant x$. If $x < y$ then $\mathtt{S}x \leqslant y$, hence $\mathtt{S}x \leqslant y \vee y \leqslant \mathtt{S}x$ (induction claim). We get the same in the case $y \leqslant x$, since then $y \leqslant \mathtt{S}x$ ($\leqslant$ is transitive).

4. (a): Put $\varphi := (\forall y{<}x)\alpha\,\frac{y}{x}$. It suffices to prove (i) $\forall x(\varphi \to \alpha) \vdash_{\mathsf{PA}} \varphi\,\frac{0}{x}$ (which is trivial) and (ii) $\forall x(\varphi \to \alpha) \vdash_{\mathsf{PA}} \varphi \to \varphi\,\frac{\mathtt{S}x}{x}$ since by IS then $\forall x(\varphi \to \alpha) \vdash_{\mathsf{PA}} \forall x\varphi \vdash_{\mathsf{PA}} \forall x\alpha$. Now, $\varphi, \varphi \to \alpha \vdash_{\mathsf{PA}} \varphi \wedge \alpha \equiv_{\mathsf{PA}} \varphi\frac{\mathtt{S}x}{x}$, hence $\forall x(\varphi \to \alpha) \vdash_{\mathsf{PA}} \varphi \to \varphi\frac{\mathtt{S}x}{x}$ which confirms (ii). (b): Follows from

(a) by contraposition. (c): For $\varphi := (\forall x{<}v)\exists y\gamma \to \exists z(\forall x{<}v)(\exists y{<}z)\gamma$ holds $\vdash_{\mathsf{PA}} \varphi\,\frac{0}{v}$, and $\varphi \vdash_{\mathsf{PA}} \varphi\,\frac{\mathtt{S}v}{v}$. This yields the claim by IS.

Section 3.4

1. $T \cup \{\boldsymbol{v}_i \neq \boldsymbol{v}_j \mid i \neq j\}$ is satisfiable because each finite subset is.

2. $Th\mathcal{A} \cup \{\boldsymbol{v}_{n+1} < \boldsymbol{v}_n \mid n \in \mathbb{N}\}$ has a model with a descending ω-chain.

3. If $\alpha \notin T$ then T has a completion T' with $\neg\alpha \in T'$, hence $\alpha \notin T'$.

4. Consider the identical operator on the universe V and restrict it to a given set u in AS.

5. Informally: Suppose $\mathsf{fin}(a)$, $\varphi_x(\emptyset)$, and $\forall u \forall e(\varphi_x(u) \to \varphi_x(u \cup \{e\}))$. Then holds also $\emptyset \in s \wedge (\forall u \in s)(a \setminus u \neq \emptyset \to (\exists e \in a \setminus u) u \cup \{e\} \in s)$ for the set $s := \{u \in \mathfrak{P}a \mid \varphi_x(u)\}$. Hence $a \in s$, i.e. $\varphi_x(a)$.

Section 3.5

2. Let $T + \{\alpha_i \mid i \in \mathbb{N}\}$ be an infinite extension of T. We may assume $\bigwedge_{i \leqslant n} \alpha_i \nvdash_T \alpha_{n+1}$. Hence, $T + \bigwedge_{i \leqslant n} \alpha_i \wedge \neg\alpha_{n+1}$ is consistent. Let T_n be a completion of $T + \bigwedge_{i \leqslant n} \alpha_i \wedge \neg\alpha_{n+1}$. Then $T_n \neq T_m$. Thus, a theory with finitely many completions cannot have an infinite extension and, in particular, no infinite completion.

3. Let $T_0, \ldots, T_n$ be the completions of T. According to Exercise 3 in **3.4**, $\alpha \in T$ *iff* $\alpha \in T_i$ for all $i \leqslant n$. Thus, T is decidable provided each T_i is, and this follows from Theorem **5.2**, for each T_i is a finite extension of T according to Exercise 2, hence is axiomatizable as well.

4. Starting with a effective enumeration $(\alpha_n)_{n \in \mathbb{N}}$ of $\mathcal{L}^0$, a Lindenbaum completion of T as constructed in **1.4** is effectively enumerable.

5. According to Exercise 3 in **3.4**, there is a bijection between the set of consistent extensions of T (including T) and the set of nonempty subsets of the collection $\{T_1, \ldots, T_n\}$ of all completions of T.

Section 3.6

1. $x = y \nvDash \forall x\, x = y$. The same holds for $\vdash$, since $\vdash\ \subseteq \vDash$.

2. (a): Let $(\varphi_n)_{n\in\mathbb{N}}$ and $(\mathcal{A}_n)_{n\in\mathbb{N}}$ be effective enumerations of all sentences and of all finite T-models (up to isomorphism). In step n write down all φ_i for $i \leqslant n$ with $\mathcal{A}_n \nvDash \varphi_i$. (b): Let $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ be effective enumerations of sentences provable or refutable in T, respectively. Each $\alpha \in \mathcal{L}^0$ occurs in one of these sequences. In the first case is $\alpha \in T$.

3. Condition (ii) from Exercise 2 is then granted because the validity of only finitely many axioms has to be tested in a finite structure.

Section 3.7

1. For **H**: Let $\mathcal{B} = h\mathcal{A}$ be a homomorphic image of $\mathcal{A}$, $w\colon Var \to A$, and define $hw\colon Var \to B$ by $x^{hw} := hx^w$. Then $ht^{\mathcal{A},w} = t^{\mathcal{B},hw}$ for all terms t and there is some $w\colon Var \to A$ with $hw = w'$ for any given $w'\colon Var \to B$. For **S**: (3) in **2.3** page 66. For **P**: Set $\mathcal{B} = \prod_{i\in I}\mathcal{A}_i$. Then $t^{\mathcal{B},w} = (t^{\mathcal{A}_i,w_i})_{i\in I}$ with $x^w = (x^{w_i})_{i\in I}$.

Section 3.8

1. (a): Let α_{unc} in $\mathcal{L}_{II}$ formalize the sentence ‘there is a continuous order’. α_{unc} has no countable model. In $\mathcal{L}^1_Q$ one may take $\mho x\, x = x$ for α_{unc}. (b): $X = \{i \neq j \mid i, j \in I,\ i \neq j\} \cup \{\neg\mho x\, x = x\}$ has no model if I is uncountable, although each finite subset of X has a model.

2. Define $\mathbb{R}$ as a continuously ordered set with a countable dense subset.

4. Let x be a variable not in $\mathcal{P}, \mathcal{Q}$. A possible definition is provided by

$$x := 0\,;\ \texttt{WHILE}\ \alpha \vee x = 0\ \texttt{DO}\ \mathcal{P}\,;\ x := \texttt{S0}\ \texttt{END}.$$

Section 4.1

1. Note that $\bar{t}$ in case $k = 0$ is defined for ground terms only.

2. The most important case is $k = 0$. It deals with ground terms only.

Section 4.2

1. First prove (a) $(\forall i{\in}I)\mathcal{A}_i \vDash \pi\,[w_i] \Leftrightarrow \mathcal{B} \vDash \pi\,[w]$ $(x^w = (x^{w_i})_{i\in I})$, π prime and $\mathcal{B} = \prod_{i\in I}\mathcal{A}_i$. Then prove (b) $(\forall i{\in}I)\mathcal{A}_i \vDash \alpha\,[w_i] \Rightarrow \mathcal{B} \vDash \alpha\,[w]$ by induction over basic Horn formulas α as in Theorem 2.1. (b) yields the induction steps over $\wedge, \forall, \exists$. Observe $t^{\mathcal{B},w} = (t^{\mathcal{A}_i,w_i})_{i\in I}$. For a universal Horn theory apply (ii)⇒(i) of Theorem 2.3.2.

2. A set of positive Horn formulas has the trivial (one-element) model.

Section 4.4

1. With $w_1 \vDash p_1, p_3, \neg p_2$ and $w_2 \vDash p_2, p_3, \neg p_1$ we have $w_1, w_2 \vDash \mathcal{P}$. Since $w \vDash \mathcal{P}$ implies $w \vDash p_3$ and either $w \vDash p_1$ or $w \vDash p_2$, there is no valuation $w \leqslant w_1, w_2$ such that $w \vDash \mathcal{P}$.

2. For arbitrary $w \vDash \mathcal{P}$, $w \vDash p_{m,n,m+n}$ follows inductively on n. Hence $w_s \leqslant w_{\mathcal{P}}$, and consequently $w_s = w_{\mathcal{P}}$.

3. (a): Theorem **4.2**. (b): $w_{\mathcal{P}} \nvDash p_{n,m,k}$ if $k \neq n{+}m$, so $\mathcal{P}, \neg p_{n,m,k} \nvdash^{HR} \square$.

Section 4.5

2. ⇒: $x_i \in \mathrm{var}\, t_j \Rightarrow x_j^\sigma = t_j \neq t_j^\sigma = x_j^{\sigma^2}$, hence $\sigma \neq \sigma^2$. ⇐: $t_i^\sigma = t_i$ since necessarily $x^\sigma = x$ for all $x \in \mathrm{var}\, t_i$.

3. Let ω be a unifier of $K_0 \cup K_1$. Then $K_0^\omega = K_1^\omega$ is a singleton. Put $x^{\omega'} = x^{\rho\omega}$ for $x \in \mathrm{var}\, K_0^\rho$ and $x^{\omega'} = x^\omega$ else. Then $K_0^{\rho\omega'} = K_0^{\rho^2\omega} = K_0^\omega$ since $\rho^2 = \iota$, and $K_1^{\omega'} = K_1^\omega$. Thus, $K_0^\rho \cup K_1$ is unified by ω'. The converse need not hold. Let r_2 be a binary relation symbol, f a unary operation symbol, and 0 a constant. $K_0 = \{r_2 fvfx\}$ and $K_1 = \{r_2 f0v\}$ are not unifiable, but K_0^ρ and K_1 are, with $\rho = \binom{v}{u}$. Indeed, for $\omega = \frac{0}{u}\,\frac{fx}{v}$ we get $K_0^{\rho\omega} = \{r_2 fufx\}^\omega = \{r_2 f0fx\} = K_1^\omega$.

Section 4.6

1. Join $\mathcal{P}_g$ and $\mathcal{P}_h$ and add to the resulting program the rules $r_f(\vec{x}, 0, u) \coloneq r_g(\vec{x}, u)$ and $r_f(\vec{x}, Sy, u) \coloneq r_f(\vec{x}, y, v), r_h(\vec{x}, y, v, u)$.

2. Add to the programs the rule $r_f\vec{x}u \coloneq r_{g_1}\vec{x}y_1, \dots, r_{g_m}\vec{x}y_m, r_h\vec{y}u$.

Section 5.1

1. Let $\alpha = \alpha(\vec{x})$, $\vec{a} \in A^n$, and $\mathcal{A} \vDash \alpha(\vec{a})$. Then $\mathcal{C} \vDash \alpha(\vec{a})$ as well, and since $\mathcal{B} \preccurlyeq \mathcal{C}$, also $\mathcal{B} \vDash \alpha(\vec{a})$.

3. Prove first the following simple **lemma**: Let $0 < b < c < 1$. Then there is a strictly monotonic bijection $f\colon [0,1] \to [0,1]$ (an automorphism of the closed interval $[0,1]$) such that $fb = c$. W.l.o.g. $a_1 < \cdots < a_n$, $n \geqslant 2$, and $b \in [a_1, a_n]$ irrational. Let $a_k < b < a_{k+1}$. W.l.o.g. we may assume $a_k = 0$ and $a_{k+1} = 1$. Choose some $c \in \mathbb{Q}$ with $b < c < 1$ and an automorphism $f\colon [0,1] \to [0,1]$ with $fb = c$ according to the above lemma. f can be extended in a trivial way to an automorphism of the whole of $(\mathbb{R}, <)$ by setting $fx = x$ outside $[0,1]$.

4. W.l.o.g. $A \cap B = \emptyset$. It suffices to show that $D_{el}\mathcal{A} \cup D_{el}\mathcal{B}$ is consistent. Assume the contrary. Then there is some conjunction $\gamma(\vec{b})$ of members of $D_{el}\mathcal{B}$ and some $\vec{b} \in B^n$ such that $D_{el}\mathcal{A}, \gamma(\vec{b}) \vdash \bot$. Thus, $D_{el}\mathcal{A} \vdash \neg\gamma(\vec{b})$. Since $A \cap B = \emptyset$, the $b_1, \ldots, b_n$ do not occur in A, hence constant quantification yields $D_{el}\mathcal{A} \vdash \forall\vec{x}\neg\gamma$ and so $\mathcal{A} \vDash \forall\vec{x}\neg\gamma$. But clearly $\mathcal{B} \vDash \exists\vec{x}\gamma$. a contradiction to $\mathcal{A} \equiv \mathcal{B}$.

5. (a): $\{t^{\mathcal{A}} \mid t \in \mathcal{T}_G\}$ is closed with respect to all $f^{\mathcal{A}}$. It is the smallest such set and hence exhausts A. (b): By (a), we may choose to each $a \in A \setminus G$ some $t_a \in \mathcal{T}_G$ such that $D\mathcal{A} \vdash a = t_a$. Thus, $T + D\mathcal{A}$ can be regarded as a definitorial and hence conservative extension of $T + D_G\mathcal{A}$, so that $D\mathcal{A} \vdash_T \alpha \Leftrightarrow D\mathcal{A}^E \vdash_T \alpha$ for all sentences $\alpha \in \mathcal{L}G$.

Section 5.2

2. $T_{\mathrm{suc}} \vdash \mathrm{IS}$ because $(\mathbb{N}, 0, \mathtt{S}) \vDash \mathrm{IS}$ and T_{suc} is complete. To prove the "no circle" scheme which is equivalent to $(*)$ $\forall x\, \mathtt{S}^n x \neq x$ $(n \geqslant 1)$, we start from $(\#)$ $\mathtt{S}^{n+1}x = \mathtt{S}^n(\mathtt{S}x)$ for every n. $(\#)$ is easily verified by metainduction on n, while the induction schema IS is needed in order to prove $(*)$ by induction on x. Clearly, $\mathtt{S}^n 0 \neq 0$ by the axiom $\forall x 0 \neq \mathtt{S}x$. From the induction hypothesis $\mathtt{S}^n x \neq x$ we get the induction claim $\mathtt{S}^n(\mathtt{S}x) = \mathtt{S}(\mathtt{S}^n x) \neq \mathtt{S}x$ by applying $(\#)$ and the second axiom of T_{suc}.

3. Let $a \in G \vDash T$ and $\frac{a}{n}$ the element with $n\,\frac{a}{n} = a$, and $\frac{m}{n}\colon a \mapsto m\,\frac{a}{n}$ for $\frac{m}{n} \in \mathbb{Q}$. Then G becomes the vector group of a $\mathbb{Q}$-vector space. This group is easily shown to be $\aleph_1$-categorical.

4. Each consistent $T' \supseteq T$ is the intersection of its completions in $\mathcal{L}$.

5. Each $\mathcal{A} \vDash T$ has a countable elementary substructure (Theorem 1.5).

Section 5.3

1. For SO_{00}: In the first round player II may play arbitrarily, then according to the winning strategies for models of SO_{01} or SO_{10} in the decomposed segments.

2. If player I starts with $a \in A$ and to the right and the left of a remain at least 2^{k-1} elements, player II should choose correspondingly. Otherwise he should answer with the elements of the same distance from the left or right edge element, respectively.

3. $\mathsf{SO}_{11} \subseteq \mathsf{FO}$ is obvious. $\mathsf{FO} \subseteq \mathsf{SO}_{11}$: If $\mathcal{A} \vDash \mathsf{SO}_{11}$ then for each $k > 0$ there is some finite $\mathcal{B} \vDash \mathsf{SO}_{11}$ such that $\mathcal{A} \sim_k \mathcal{B}$.

4. Prove first that $\mathsf{SO}_{11} \cup \{\exists_i \mid i > 0\}$ is complete. Then apply Theorem 2.3.

Section 5.4

1. Let $h\colon \mathcal{A} \to \mathcal{B}$ be a homomorphism, $\mathcal{M} = (\mathcal{A}, w)$, $\mathcal{M}' = (\mathcal{B}, w')$ with $x^{w'} = hx^w$. Verify $\mathcal{M} \vDash \varphi\,[\vec{a}] \Rightarrow \mathcal{M}' \vDash \varphi\,[h\vec{a}]$ by induction on φ.

2. Let $\mathcal{A} = (A, <)$ be ordered. Replacing each $a \in A$ by a copy of $(\mathbb{Z}, <)$ or of $(\mathbb{Q}, <)$ results in a discrete or a dense order $\mathcal{B} \supseteq \mathcal{A}$, respectively.

3. Let $\mathcal{A}_0 \vDash T_0$. Choose $\mathcal{A}_1$ with $\mathcal{A}_0 \subseteq \mathcal{A}_1 \vDash T_1$, $\mathcal{A}_2$ with $\mathcal{A}_1 \subseteq \mathcal{A}_2 \vDash T_0$ etc. This results in a chain $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots$ such that $\mathcal{A}_{2i} \vDash T_0$ and $\mathcal{A}_{2i+1} \vDash T_1$. Then $\mathcal{A}^* := \bigcup_{i\in\mathbb{N}} \mathcal{A}_{2i} = \bigcup_{i\in\mathbb{N}} \mathcal{A}_{2i+1} \vDash T_0, T_1$ and hence $\mathcal{A}^* \vDash T := T_0 + T_1$. This shows that T is consistent and model compatible with T_0 (hence likewise with T_1). Clearly, T is an $\forall\exists$-theory and therefore also inductive.

4. The union S of a chain of inductive theories model compatible with T has again these properties. By Zorn's lemma there exists a maximal, hence in view of Exercise 3 a largest theory of this kind.

Section 5.5

1. Let $(i,j) \neq (0,0)$. Then DO_{ij} has models $\mathcal{A} \subseteq \mathcal{B}$ with $\mathcal{A} \not\preccurlyeq \mathcal{B}$. To show that DO_{00} is the model completion of DO note first that $T := \mathsf{DO}_{00} + \mathcal{D}\mathcal{A}$ is model complete for each $\mathcal{A} \vDash \mathsf{DO}$. Moreover, T is complete since T has a prime model: For instance, let $\mathcal{A} \vDash \mathsf{DO}_{10}$. Then the ordered sum $\mathbb{Q} + \mathcal{A}$ (i.e., $(\forall x{\in}\mathbb{Q})(\forall y{\in}A)x < y$) is a prime model of T.

2. (a) Lindström's criterion. T is $\aleph_1$-categorical because a T-model can be understood as a $\mathbb{Q}$-vector space. (b) Each T_0-model G is embeddable in a T-model H. One gains such H by defining a suitable equivalence relation on the set of all pairs $\frac{a}{n}$ with $a \in G$ and $n \in \mathbb{Z}\setminus\{0\}$.

3. Uniqueness follows similarly to uniqueness of the model completion. If $\mathcal{A} \vDash T^*$ and $\mathcal{A} \subseteq \mathcal{B} \vDash T$ then $\mathcal{B} \subseteq \mathcal{C} \vDash T^*$ for some $\mathcal{C}$, hence $\mathcal{A} \preccurlyeq \mathcal{C}$ in view of $\mathcal{A} \subseteq \mathcal{C}$, and therefore $\mathcal{A} \subseteq_{ec} \mathcal{B}$ according to Lemma 4.8.

4. The algebraic closure $\overline{\mathcal{F}_p}$ of the prime field $\mathcal{F}_p$ is equal to $\bigcup_{n\geqslant 1} \mathcal{F}_{p^n}$, where $\mathcal{F}_{p^n}$ is the finite field of p^n elements. Thus, an $\forall\exists$-sentence valid in all finite fields is valid in all a.c. fields of prime characteristics and hence in all a.c. fields (proof indirectly with (1) in **3.3**).

Section 5.6

1. Let $\mathcal{A}, \mathcal{B} \vDash \mathsf{ZG}$, $\mathcal{A} \subseteq \mathcal{B}$. Then also $\mathcal{A}' \subseteq \mathcal{B}'$ for the ZGE-expansions $\mathcal{A}'$ and $\mathcal{B}'$ of $\mathcal{A}$ and $\mathcal{B}$, respectively, because $m\,|$ has in ZG both an $\forall$- and an $\exists$-Definition. Thus $\mathcal{A}' \preccurlyeq \mathcal{B}'$ and hence $\mathcal{A} \preccurlyeq \mathcal{B}$.

2. Similiar to quantifier elimination in ZGE but somewhat more simple.

3. Inductively over quantifier-free $\varphi = \varphi(x)$ follows: either $\varphi^{\mathcal{A}}$ or $(\neg\varphi)^{\mathcal{A}}$ is finite for each $\mathcal{A} \vDash \mathsf{RCF}^{\circ}$. This is not the case for $\alpha(x)$.

4. CS holds in the real closed field $\mathbb{R}$, hence in each $\mathcal{A} \in \mathsf{RCF}$. The proofs from CS of $(\forall x{\geqslant}0)\exists y\, x = y \cdot y$, and that each polynomial of odd degree has a zero must be carried out without a theory of continuous functions, which is very instructive.

Section 5.7

1. If F is trivial then there is some $i_0 \in I$ with $i_0 \in J$ for each $J \in F$ by Exercise 3 in **1.5**. Then $a \approx_F b \Leftrightarrow i_0 \in I_{a=b} \Leftrightarrow a_{i_0} = b_{i_0}$, for all $a, b \in \prod_{i\in I} A_i$. This implies $\prod_{i\in I}^F \mathcal{A}_i \simeq \mathcal{A}_{i_0}$.

2. $x \mapsto x^I/F$ $(x \in A)$ is an embedding (to be checked in detail) and moreover an elementary embedding.

3. Let $X \vDash_{\boldsymbol{K}} \varphi$ and I, J_α F defined as in the proof of Theorem **7.3** and assume that for each $i \in I$ there is some $\mathcal{A}_i \in \boldsymbol{K}$ and $w_i \colon Var \to A_i$ such that $w_i\alpha \in D^{\mathcal{A}_i}$ for all $\alpha \in i$ but $w_i\varphi \notin D^{\mathcal{A}_i}$. Put $\mathcal{C} := \prod_{i\in I}^F \mathcal{A}_i$ $(\in \boldsymbol{K})$ and $w = (w_i)_{i\in I}$. Then $wX \subseteq D^{\mathcal{C}}$ and $w\varphi \notin D^{\mathcal{C}}$, hence $X \nvDash_{\mathcal{C}} \varphi$, a contradiction to $X \vDash_{\boldsymbol{K}} \varphi$.

4. W.l.o.g. $\mathcal{A} = \mathit{2}$ and $\mathit{2} \subseteq \mathcal{B} \subseteq \mathit{2}^I$ for some set I by Stone's representation theorem. $\mathit{2} \vDash \alpha \Rightarrow \mathit{2}^I \vDash \alpha \Rightarrow \mathcal{B} \vDash \alpha$ according to Theorem 7.5.

Section 6.1

1. $b \in ran\, f \;\Leftrightarrow\; (\exists a{\leqslant}b) fa = b$ (this predicate is p.r. iff f is p.r.).

2. *Injectivity*: Let $\wp(a, b) = \wp(c, d)$. In order to prove $a = c$ and $b = d$ assume first that $a + b < c + d$. This leads to a contradiction since $\wp(a, b) < \wp(a, b) + b + 1 = \mathsf{t}_{a+b} + a + b + 1 = \mathsf{t}_{a+b+1} \leqslant \mathsf{t}_{c+d} \leqslant \wp(c, d)$. Thus $a + b = c + d$. But then $a = \wp(a, b) - \mathsf{t}_{a+b} = \wp(c, d) - \mathsf{t}_{c+d} = c$, hence also $b = d$. *Surjectivity*: Since $\wp(0, 0) = 0 \in ran\, \wp$ it suffices to prove $\wp(a, b) + 1 \in ran\, \wp$, for all a, b. Clear for $b = 0$ because $\wp(a, 0) + 1 = \mathsf{t}_a + a + 1 = \mathsf{t}_{a+1} = \wp(0, a + 1)$. In case $b \neq 0$ is $\wp(a, b) + 1 = \mathsf{t}_{a+b} + a + 1 = \mathsf{t}_{a+1+b-1} + a + 1 = \wp(a + 1, b - 1)$. This proof also confirms the correctness of the diagram for $\wp$, that is, the arrows truly reflect the successively growing values of $\wp$.

3. $\varkappa_1 n = (\mu k \leqslant n)[(\exists m \leqslant n)\wp(k, m) = n]$.

4. $\mathrm{lcm}\{f\nu \mid \nu{\leqslant}n\} = \mu k \leqslant \prod_{\nu\leqslant n} f\nu[k \neq 0 \;\&\; (\forall \nu \leqslant n) f\nu \mid k]$.

5. $\Rightarrow$: Let R be recursive, $M = \{a \in \mathbb{N} \mid \exists bRab\}$, and $c \in M$ fixed. Put $fn = k$ in case $(\exists m{\leqslant}n)\, n = \wp(m, k) \;\&\; Rkm$, and $fn = c$ otherwise.

Section 6.2

1. Let $\alpha_0, \alpha_1, \ldots$ be a recursive enumeration of X, $\beta_n = \underbrace{\alpha_n \wedge \ldots \wedge \alpha_n}_{n}$. By Exercise 1 in **6.1**, $\{\beta_n \mid n \in \mathbb{N}\}$ is recursive and axiomatizes T as well.

2. Follow the proof of the unique term reconstruction property.

3. Similar to Exercise 2 with the unique formula reconstruction.

4. (a): A proof $\Phi = (\varphi_0, \ldots, \varphi_n)$ of $\varphi = \varphi_n$ from an axiom system X in $T + \alpha$ can easily and in a p.r. manner be converted into a somewhat longer proof Φ' of $\alpha \to \varphi$ in T, following the case distinction in Lemma 1.6.3: $\varphi_i \in \Phi$ should in case $\varphi_i = \alpha$ be replaced by a proof of $\alpha \to \alpha$ in T, and in case $\varphi_i \in X \cup \Lambda$ by a proof of $\varphi_i \to \alpha \to \varphi_i$ in T followed by φ_i and $\alpha \to \varphi_i$. If $\varphi_k \in \Phi$ results from $\varphi_i \in \Phi$ and $\varphi_j = \varphi_i \to \varphi_k \in \Phi$ by applying MP, then the axiom $(\alpha \to \varphi_i \to \varphi_k) \to (\alpha \to \varphi_i) \to \alpha \to \varphi_k$, followed by $(\alpha \to \varphi_i) \to \alpha \to \varphi_k$ and $\alpha \to \varphi_k$ should replace φ_k. One may also proceed inductively on the length of Φ in constructing Φ'.

Section 6.3

1. $\exists x \exists y \alpha \equiv_{\mathcal{N}} \exists z (\exists x \leqslant z)(\exists y \leqslant z)(z = \wp(x, y) \wedge \alpha)$ where $z \notin \mathit{var}\, \alpha$. Similarly for $\forall x \forall y \alpha$. Note also that $\exists x \exists y \alpha \equiv_{\mathcal{N}} \exists z (\exists x \leqslant z)(\exists y \leqslant z)\alpha$. In all these equivalences, $\equiv_{\mathcal{N}}$ can be replaced by $\equiv_{\mathsf{PA}}$.

2. $(\forall z < y) \exists x \alpha \equiv_{\mathsf{PA}} \exists u (\forall z < y)(\exists x < u)\alpha$. Contraposition and renaming of α readily yields $(\exists z < y) \forall x \alpha \equiv_{\mathsf{PA}} \forall u (\exists z < y)(\forall x < u)\alpha$.

3. Prove $\mathrm{R}^{=}$ by case distinction.

4. Prove by induction on φ that both φ and $\neg\varphi$ satisfy the claim.

Section 6.4

1. (a): $p \nmid a \;\Rightarrow\; a \perp p \;\Rightarrow\; \exists xy\, xa + 1 = yp$ (Euclid's lemma)
   $\Rightarrow\; \exists xy\, b = ypb - xab \;\Rightarrow\; p \mid b$.

   (b): Let $m := \mathrm{lcm}\{a_\nu \mid \nu \leqslant n\}$, so that $m = a_\nu c_\nu$ for suitable c_ν. Assume that $(\forall \nu \leqslant n) p \nmid a_\nu$. Then $(\forall \nu \leqslant n) p \mid c_\nu$ by (a). Thus $m = pm'$ and $c_\nu = pc'_\nu$ for suitable m', c'_ν. This leads to contradiction to the definition of m. (c) easily follows from (b).

2. $\exists u[\texttt{beta}\,u0\underline{2} \wedge (\forall v{<}x)(\exists w, w' {\leqslant} y)(\texttt{beta}\,uvw \wedge \texttt{beta}\,uSvw' \wedge w < w'$
$\wedge\ \mathsf{prim}\,w \wedge \mathsf{prim}\,w' \wedge (\forall z{<}w')(\mathsf{prim}\,z \to z{\leqslant}w) \wedge \texttt{beta}\,uxy)]$.

3. (a): Prove this first for x instead of $\vec{x}$. (b): It suffices to show that $\mathrm{sb}_x(\dot{\varphi}, x) = \dot{\varphi}$ for $x \notin \mathit{free}\,\varphi$. ($\mathrm{sb}_x((\forall x\alpha)^{\cdot}, x) = (\forall x\alpha)^{\cdot}$ for closed α).

Section 6.5

2. (ii)⇒(i): If T is complete and $T'{+}T$ is consistent then $T' \subseteq T$ provided T and T' belong to the same language.

4. Trivial if $T + \Delta$ is inconsistent. Otherwise let $\varkappa$ be the conjunction of all sentences $\forall\vec{x}\,\exists! y\alpha(\vec{x}, y)$, α running through all defining formulas for operations from Δ. If T is decidable than so is $T + \varkappa$. Moreover $\vdash_{T+\Delta} \alpha \Leftrightarrow \vdash_{T+\varkappa} \alpha^{rd}$.

5. Set $fa = (\dot{\Phi})_{last}$ if there is a proof Φ in Q with $a = \dot{\Phi}$, and $fa = 0$ otherwise. $\mathit{ran}\,f = \{0\} \cup \{\dot{\varphi} \mid \vdash_{\mathsf{Q}} \varphi\}$ is not recursive, since otherwise $\dot{\mathsf{Q}}$ would be recursive which is not the case.

Section 6.6

1. Let $T \supseteq T_1$ be consistent. $S = \{\alpha \in \mathcal{L}_0 \mid \alpha^{\mathsf{P}} \in T^{\Delta} + \mathit{CA}\}$ is a theory, see the proof of Theorem 6.2. S extends T_0 consistently, hence is undecidable. The same then holds for $T^{\Delta} + \mathit{CA}$, hence for T^{Δ} (since CA is finite), and therefore also for T.

2. Identify $\texttt{P}$ with ω and define for arbitrary $n, m, k \in \omega$

$$n + m = k \leftrightarrow \exists ab(a \sim n \wedge b \sim m \wedge a \cap b = \emptyset \wedge k \sim a \cup b).$$

   For an explicit definition of multiplication on ω the cross product has to be used. These definitions reflect the naive set-theoretic standard definitions of addition and multiplication in $\mathbb{N}$.

Section 6.7

2. Δ_0 is r.e. but not Δ_1 (Remark 2 in **6.4**). $\dot{\mathsf{Q}}$ is Σ_1 but not Δ_1.

3. T is ω-inconsistent *iff* $(\exists\varphi{\in}\mathcal{L}^1)(\forall n\; \mathit{bwb}_T\, \neg\varphi(\underline{n})\ \&\ \mathit{bwb}_T\, \exists x\varphi)$.

Section 7.1

1. Prove $\vdash_{\mathsf{PA}} \exists r\delta_{\mathrm{rem}}(a,b,r)$ for $b \neq 0$ by induction on a.

2. (a): Follow the proof of Euclid's lemma in **6.4**. (b): Use <-induction. (c): Let $p \mid ab$. $p \nmid a \Rightarrow \exists x,y\, xa{+}1 = yp \Rightarrow \exists x,y\, xab{+}b = ybp \Rightarrow p \mid b$.

3. Similar to part (c) of Exercise 1 in **6.4**.

4. Existence: <-induction. Uniqueness: Prove first $p \nmid q^k$ (p,q prime) by induction on k, applying Exercise 2(c).

5. (a): $\Box_{T+\alpha}\varphi \vdash_T \Box_T(\alpha \to \varphi)$ formalizes part (b) of Exercise 4 in **6.2**.

Section 7.3

1. $\vdash_T \Box\alpha \to \alpha \;\Rightarrow\; \vdash_{T'} \neg\Box\alpha \;\Rightarrow\; \vdash_{T'} \mathtt{Con}_{T'}$, since $\mathtt{Con}_{T'} \equiv_T \neg\Box\alpha$ by (5). Thus, T' is inconsistent by (1), hence $\vdash_T \alpha$.

3. Clear if $n = 0$. Let $T^n = T + \neg\Box^n\bot$ and $\mathtt{Con}_{T^n} \equiv_T \neg\Box^{n+1}\bot$ (the induction hypothesis). Now, $\Box^n\bot \vdash_T \Box^{n+1}\bot$ by $D3$. Hence, we obtain $T^{n+1} = (T{+}\neg\Box^n\bot){+}\neg\Box^{n+1}\bot = T{+}\neg\Box^{n+1}\bot$. Further, by (5) page 281, $\mathtt{Con}_{T^{n+1}} \equiv_T \neg\Box\neg(\neg\Box^{n+1}\bot) \equiv_T \neg\Box^{n+2}\bot$.

4. For arithmetical sentences α the statement 'If α is provable in PA then α is true in $\mathcal{N}$' is provable in ZFC. Formalized: $\vdash_{\mathsf{ZFC}} \Box_{\mathsf{PA}}\alpha \to \alpha$.

Section 7.4

1. $\Box p \to \Box\Box p$ is responsible for transitivity, Löb's formula for irreflexivity.

2. $\vdash_{\mathsf{G}} p \to \Box p \to p \;\Rightarrow \vdash_{\mathsf{G}} \Box(p \to \Box p \to p) \;\Rightarrow \vdash_{\mathsf{G}} \Box p \to \Box(\Box p \to p)$.

Section 7.5

1. Prove first $(*)\ \vdash_{\mathsf{G}_n} H \Leftrightarrow\ \vdash_{\mathsf{G}} \Box^n\bot \to H$ for all $H \in \mathcal{F}_\Box$. The direction $\Rightarrow$ in $(*)$ follows by induction on $\vdash_{\mathsf{G}_n} H$. Then continue as follows:

$$\begin{array}{lll}
\vdash_{\mathsf{G}_n} H & \Leftrightarrow \vdash_{\mathsf{G}} \Box^n\bot \to H & (\text{by}(*)) \\
& \Leftrightarrow \vdash_{\mathsf{PA}} (\Box^n\bot \to H)^\imath \text{ for all } \imath & (\text{Theorem 5.2}) \\
& \Leftrightarrow \vdash_{\mathsf{PA}} \Box^n\bot \to H^\imath \text{ for all } \imath & (\text{property of } \imath) \\
& \Leftrightarrow \vdash_{\mathsf{PA}_n} H^\imath \text{ for all } \imath & (\mathsf{PA}_n = \mathsf{PA} + \Box^n\bot).
\end{array}$$

2. The first claim follows immediately from Exercise 3 in **7.3**. For determining the provability logic of $\mathsf{PA}^n_\bot$, use (6) in **7.3** and Theorem 5.3.

4. Prove that $\nvdash_{\mathsf{GS}} \neg[\neg\Box(p \to q) \wedge \neg\Box(p \to \neg q) \wedge \neg\Box(q \to p) \wedge \neg\Box(q \to \neg p)]$ and observe Theorem 5.4.

Section 7.7

1. We show there is some $\pi\colon g \to n$ with $P < Q \Leftrightarrow \pi P < \pi Q$ for $n := \mathrm{lh}\, g$ (the length of a longest path in g). Trivial for $\mathrm{lh}\, g = 0$, with $\pi P = 0$ for all $P \in g$. Let $\mathrm{lh}\, g = n+1$ and $g' := g \setminus \max g$ where $\max g$ denotes the set of all maximal points in g. Then $\mathrm{lh}\, g' = n$ and g' has property (**p**) as well as is readily checked. Hence g' is a preference order with a mapping $\pi'\colon g' \to n$ by the induction hypothesis. Extend π' to $\pi\colon g \to n+1$ by putting $\pi P = n$ for all $P \in \max g$. Obviously, $P < Q \Rightarrow \pi P < \pi Q$. For proving the converse let $\pi P < \pi Q$ with $Q \in \max g$. Then certainly $P' \in \max g$ for some $P' > P$. Hence, by (**p**), either $P < Q$ or $Q < P'$. The latter is impossible since $Q \in \max g$. Thus $P < Q$.

2. If (**i**) is falsified in g (that is, if $\Diamond(\Box p \wedge \Diamond\neg q) \wedge \Diamond(\Box q \wedge \neg p)$ is satisfiable in some point $O \in g$) then g contains the diagram from page 296 as a subdiagram, with no arrow from P to Q and from Q to P'. It easily follows that the finite poset g cannot be a preference order.

3. It is a matter of routine to check that $\Box(\Box p \wedge p \to q) \vee \Box(\Box q \to p)$ is satisfied in an ordered G-frame. For the converse assume that g is initial but not (totally) ordered. Then g contains the "fork" from page 298 as a subframe, in which the Gj-axiom can easily be refuted.

4. Soundness of the G-axioms and rules is shown as the soundness part of Theorem 7.3 which was given in the text. Soundness of the Gj-axiom follows by contraposition. Assume that there are cardinals κ, λ such that $V_\kappa \vDash \Box\alpha \wedge \alpha \wedge \neg\beta$, and $V_\lambda \vDash \Box\beta \wedge \neg\alpha$. Then each of the assumptions $\kappa < \lambda$, $\kappa > \lambda$, or $\kappa = \lambda$ yields a contradiction.