CHAPTER ONE

INTRODUCTION

1.1 COMMUNICATIONS

Communication enters our daily lives in so many different ways that it is easy to overlook the multitude of its facets. The telephones in our homes and offices make it possible for us to communicate with others, no matter how far away. The radio and television sets in our living rooms bring us entertainment from near as well as far-away places. Communication by radio or satellite provides the means for ships on the high seas, aircraft in flight, and rockets and exploratory probes in space to maintain contact with their home bases. Communication keeps the weather forecaster informed of atmospheric conditions that are measured by a multitude of sensors. Communication makes it possible for computers to interact. The list of applications involving the use of communications in one way or another goes on.[1]

In the most fundamental sense, *communication* involves implicitly the transmission of information from one point to another through a succession of processes, as described here:

1. The generation of a thought pattern or image in the mind of an originator.
2. The description of that thought pattern or image, with a certain measure of precision, by a set of aural or visual symbols.
3. The encoding of these symbols in a form that is suitable for transmission over a physical medium (channel) of interest.
4. The transmission of the encoded symbols to the desired destination.
5. The decoding and reproduction of the initial symbols.
6. The re-creation of the original thought pattern or image—with a definable degradation in quality—in the mind of a recipient, with the degradation being caused by imperfections in the system.

The form of communication just described involves a thought pattern or image originating in a human mind. Of course, there are many other forms of communication that do not directly involve the human mind in real time. In space exploration, for example, human decisions may enter only the commands sent to the space probe or to the computer responsible for processing images of far-away planets (e.g., Mars, Jupiter, Saturn) that are sent back by the probe. In computer communications, human decisions enter only in setting up the computer programs or in monitoring the results of computer processing.

Whatever form of communication is used, some basic signal-processing operations are involved in the transmission of information. The next section describes the different types of signals encountered in the study of communication systems. The signal-processing operations of interest are highlighted later in the chapter.

1.2 SIGNALS AND THEIR CLASSIFICATIONS

For our purposes, a *signal* is defined as a single-valued function of time that conveys information. Consequently, for every instant of time there is a unique value of the function. This value may be a real number, in which case we have a *real-valued signal,* or it may be a complex number, in which case we have a *complex-valued signal.* In either case, the independent variable (namely, time) is real-valued.

[1]For an essay on communications, see Berkner (1962).

For a given situation, the most useful method of signal representation hinges on the particular type of signal being considered. Depending on the feature of interest, we may identify four different methods of dividing signals into two classes:

1. PERIODIC SIGNALS, NONPERIODIC SIGNALS

A *periodic signal* $g(t)$ is a function that satisfies the condition

$$g(t) = g(t + T_0) \tag{1.1}$$

for all t, where t denotes time and T_0 is a constant. The smallest value of T_0 that satisfies this condition is called the *period* of $g(t)$. Accordingly, the period T_0 defines the duration of one complete cycle of $g(t)$.

Any signal for which there is no value of T_0 to satisfy the condition of Eq. 1.1 is called a *nonperiodic* or *aperiodic signal.*

2. DETERMINISTIC SIGNALS, RANDOM SIGNALS

A *deterministic signal* is a signal about which there is no uncertainty with respect to its value at any time. Accordingly, we find that deterministic signals may be modeled as completely specified functions of time.

On the other hand, a *random signal* is a signal about which there is uncertainty before its actual occurrence. Such a signal may be viewed as belonging to an ensemble of signals, with each signal in the collection having a different waveform. Moreover, each signal within the ensemble has a certain *probability* of occurrence.

3. ENERGY SIGNALS, POWER SIGNALS

In communication systems, a signal may represent a voltage or a current. Consider a voltage $v(t)$ developed across a resistor R, producing a current $i(t)$. The *instantaneous power* dissipated in this resistor is defined by

$$p = \frac{v^2(t)}{R} \tag{1.2}$$

or, equivalently,

$$p = Ri^2(t) \tag{1.3}$$

In both cases, the instantaneous power p is proportional to the squared amplitude of the signal. Furthermore, for a resistance R of 1 ohm, we see that Eqs. 1.2 and 1.3 take on the same mathematical form. Accordingly, in signal analysis it is customary to work with a 1-ohm resistor, so that, regardless of whether a given signal $g(t)$ represents a voltage or a current, we may express the instantaneous power associated with the signal as

$$p = g^2(t) \tag{1.4}$$

Based on this convention, we define the *total energy* of a signal $g(t)$ as

$$\begin{aligned} E &= \lim_{T\to\infty} \int_{-T}^{T} g^2(t)dt \\ &= \int_{-\infty}^{\infty} g^2(t)dt \end{aligned} \tag{1.5}$$

and its *average power* as

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g^2(t)dt \tag{1.6}$$

We say that the signal $g(t)$ is an *energy signal* if and only if the total energy of the signal satisfies the condition

$$0 < E < \infty$$

We say that the signal $g(t)$ is a *power signal* if and only if the average power of the signal satisfies the condition

$$0 < P < \infty$$

The energy and power classifications of signals are mutually exclusive. In particular, an energy signal has zero average power, whereas a power signal has infinite energy. Also, it is of interest to note that, usually, periodic signals and random signals are power signals, whereas signals that are both deterministic and nonperiodic are energy signals.

4. ANALOG SIGNALS, DIGITAL SIGNALS

An *analog signal* is a signal with an *amplitude* (i.e., value of the signal at some fixed time) that varies continuously for all time; that is, *both amplitude and time are continuous over their respective intervals*. Analog signals arise when a physical waveform such as an acoustic wave or a light wave is converted into an electrical signal. The conversion is effected by means of a *transducer;* examples include the microphone, which converts sound pressure variations into corresponding voltage or current variations, and the photodetector cell, which does the same for light-intensity variations.

On the other hand, a *discrete-time signal* is defined only at discrete instants of time. Thus, in this case, the independent variable takes on only discrete values, which are usually uniformly spaced. Consequently, discrete-time signals are described as sequences of samples that may take on a continuum of values. When each sample of a discrete-time signal is *quantized* (i.e., it is only allowed to take on a finite set of discrete values) and then *coded*, the resulting signal is referred to as a *digital signal*. The output of a digital computer is an example of a digital signal. Naturally, an analog signal may be converted into digital form by *sampling in time, then quantizing and coding*.

1.3 FOURIER ANALYSIS OF SIGNALS AND SYSTEMS

In theory, there are many possible methods for the representation of signals. In practice, however, we find that *Fourier analysis*, involving the resolution of signals into *sinusoidal components*, overshadows all other methods in usefulness. Basically, this is a consequence of the well-known fact that the output of a system to a sine-wave input is another sine wave of the same frequency[2] (but with a different phase and amplitude) under two conditions:

1. The system is *linear* in that it obeys the *principle of superposition*. That is, if $y_1(t)$ and $y_2(t)$ denote the responses of a system to the inputs $x_1(t)$ and $x_2(t)$, respectively, the system is linear if the response to the composite input $a_1x_2(t) + a_2x_2(t)$ is equal to $a_1y_1(t) + a_2y_2(t)$, where a_1 and a_2 are arbitrary constants.
2. The system is *time-invariant*. That is, if $y(t)$ is the response of a system to the input $x(t)$, the system is time-invariant if the response to the time-shifted input $x(t - t_0)$ is equal to $y(t - t_0)$, where t_0 is constant.

Given a linear time-invariant system, the *response* of the system to a single-frequency *excitation* represented by the complex exponential time function $A \exp(j2\pi ft)$ is equal to $AH(f) \exp(j2\pi ft)$, where $H(f)$ is the *transfer function* of the system; the complex exponential $\exp(j2\pi ft)$ contains the cosine function $\cos(2\pi ft)$ as its real part and the sine function $\sin(2\pi ft)$ as its imaginary part. Thus, the response of the system exhibits exactly the same variation with time as the excitation applied to the system. ...s remarkable property of linear time-invariant systems is realized only ...g the complex exponential time function.

...e study of communication systems, we are usually interested in a ... *of frequencies*. For example, although the average voice spectrum extends well beyond 10 kHz, most of the energy is concentrated in the range of 100 to 600 Hz, and a voice signal lying inside the band from 300 to 3400 Hz gives good articulation. Accordingly, we find that telephone

[2] For a historical account of the concept of frequency, see Manley (1982).

circuits that respond well to the band of frequencies from 300 to 3400 Hz give satisfactory commercial telephone service.

To talk meaningfully about the *frequency-domain description* or *spectrum* of a signal, we need to know the *amplitude* and *phase* of each frequency component contained in the signal. We get this information by performing a Fourier analysis on the signal. However, there are several methods of Fourier analysis available for the representation of signals. The particular version that is used in practice depends on the type of signal being considered. For example, if the signal is periodic, then the logical choice is to use the *Fourier series* to represent the signal as a set of harmonically related sine waves. On the other hand, if the signal is an energy signal, then it is customary to use the *Fourier transform* to represent the signal. Irrespective of the type of signal being considered, Fourier methods are invertible. Specifically, if we are given the complete spectrum of a signal, then the original signal (as a function of time) can be reconstructed exactly. The Fourier analysis of signals and systems is considered in Chapters 2 through 4.

1.4 ELEMENTS OF A COMMUNICATION SYSTEM

The purpose of a *communication system* is to transmit information-bearing signals from a *source,* located at one point, to a *user destination,* located at another point some distance away. When the message produced by the source is not electrical in nature, which is often the case, an input transducer is used to convert it into a time-varying electrical signal called the *message signal.* By using another transducer connected to the output end of the system, a "distorted" version of the message is re-created in its original form, so that it is suitable for delivery to the user destination. The distortion mentioned here is due to inherent limitations in the communication system.

Figure 1.1 is a block diagram of a communication system consisting of three basic components: transmitter, channel, and receiver. The *transmitter* has the function of processing the message signal into a form suitable for

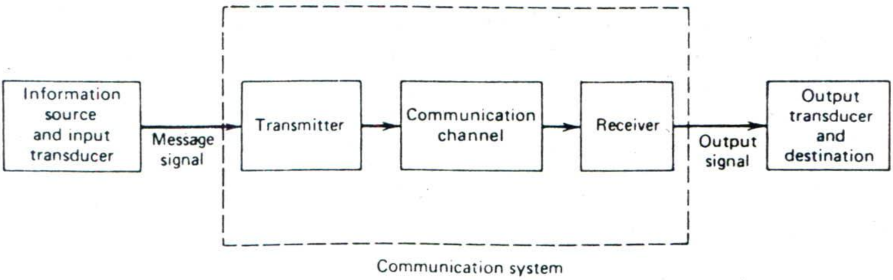


Figure 1.1
Elements of an electrical communication system.

transmission over the channel; such an operation is called *modulation*. The function of the *channel* is to provide a physical connection between the transmitter output and the receiver input. The function of the *receiver* is to process the *received signal* so as to produce an "estimate" of the original message signal; this second operation is called *detection* or *demodulation*.

There are two types of channels, namely, point-to-point channels and broadcast channels. Examples of *point-to-point channels* include wire lines, microwave links, and optical fibers. *Wire lines* operate by guided electromagnetic waves; they are used for local telephone transmission. In *microwave links*, the transmitted signal is radiated as an electromagnetic wave in free space; microwave links are used in long-distance telephone transmission. An *optical fiber* is a low-loss, well-controlled, guided optical medium; optical fibers are used in optical communications.[3] Although these three channels operate differently, they all provide a physical medium for the transmission of signals from one point to another point; hence, the term "point-to-point channels."

Broadcast channels, on the other hand, provide a capability where many receiving stations may be reached simultaneously from a single transmitter. An example of a broadcast channel is a *satellite in geostationary orbit*, which covers about one third of the earth's surface. Thus, three such satellites provide a complete coverage of the earth's surface, except for the polar regions.

1.5 TRANSMISSION OF MESSAGE SIGNALS

To transmit a message (information-bearing) signal over a communication channel, we may use *analog* or *digital* methods. The use of digital methods offers several important operational advantages over analog methods, which include the following:

1. Increased *immunity* to channel noise and external interference.
2. *Flexible operation* of the system.
3. A *common format* for the transmission of different kinds of message signals (e.g., voice signals, video signals, computer data).
4. Improved *security* of communication through the use of encryption.

These advantages are attained, however, at the cost of *increased transmission (channel) bandwidth and increased system complexity*. The first requirement is catered to by the availability of *wideband communication channels* (e.g., optical fibers, satellite channels). The second requirement is taken care of by the use of *very large-scale integration* (VLSI) technology, which offers a cost-effective way of building hardware. Accordingly, there

[3]For a discussion of electronic and photonic (optical) communication systems, see Williams (1987).

is an ever-increasing trend toward the use of digital communications and away from analog communications. This trend is being accelerated by the pervasive influence of digital computers in so many facets of our daily lives. Nevertheless, analog communications remain a force to be reckoned with. Most of the broadcasting systems and a large part of the telephone networks in use today are analog in nature and, moreover, they will remain in service for some time yet. It is therefore important that we understand the operations and requirements of both analog and digital communications.

Notable among the digital methods that may be used for the transmission of message signals over a communication channel is *pulse-code modulation* (PCM). In PCM, the message signal is *sampled, quantized,* and then *encoded.* The sampling operation permits representation of the message signal by a sequence of samples taken at uniformly spaced instants of time. Quantization trims the amplitude of each sample to the nearest value selected from a finite set of representation levels. The combination of sampling and quantization permits the use of a *code* (e.g., binary code) for the transmission of a message signal. Pulse-code modulation and related methods of *analog-to-digital conversion* are covered in Chapter 5.

When digital data are transmitted over a band-limited channel, a form of interference known as *intersymbol interference* may result. The effect of intersymbol interference, if left uncontrolled, is to severely limit the rate at which digital data may be transmitted over the channel. The cure for controlling the effects of intersymbol interference lies in *shaping* the transmitted pulse representing a binary symbol 1 or 0. Intersymbol interference is considered in Chapter 6.

To transmit a message signal (be it in analog or digital form) over a *band-pass* communication channel (e.g., telephone channel, microwave radio link, satellite channel) we need to modify the message signal into a form suitable for *efficient transmission* over the channel. Modification of the message signal is achieved by means of a process known as *modulation.* This process involves varying some parameter of a carrier wave in accordance with the message signal in such a way that the spectrum of the modulated wave matches the assigned channel bandwidth. Correspondingly, the receiver is required to re-create the original message signal from a degraded version of the transmitted signal after propagation through the channel. The re-creation is accomplished by using a process known as *demodulation,* which is the inverse of the modulation process used in the transmitter.

There are other reasons for performing modulation. In particular, the use of modulation permits *multiplexing,* that is, the simultaneous transmission of signals from several message sources over a common channel. Also, modulation may be used to convert the message signal into a form less susceptible to noise and interference.

A carrier wave commonly used to perform modulation is the sinusoidal wave. Such a carrier wave has three independent parameters that can be varied in accordance with the message signal; they are the carrier amplitude, phase, and frequency. The corresponding forms of modulation are known as *amplitude modulation, phase modulation,* and *frequency modulation,* respectively. Amplitude modulation offers simplicity of implementation and a transmission bandwidth requirement equal to twice the message bandwidth; the *message bandwidth* is defined as the extent of significant frequencies contained in the message signal. With special processing, the transmission bandwidth requirement may be reduced to a value equal to the message bandwidth, which is the minimum possible. Phase and frequency modulation, on the other hand, are more complex, requiring transmission bandwidths greater than that of amplitude modulation. In exchange, they offer a superior noise immunity, compared to amplitude modulation. Modulation techniques for analog and digital forms of message signals are studied in Chapter 7.

1.6 *LIMITATIONS AND RESOURCES OF COMMUNICATION SYSTEMS*

Typically, in propagating through a channel, the transmitted signal is distorted because of *nonlinearities* and *imperfections in the frequency response of the channel.* Other sources of degradation are *noise* and *interference* picked up by the signal during the course of transmission through the channel. Noise and distortion constitute two basic *limitations* in the design of communication systems.

There are various sources of noise, internal as well as external to the system. Although noise is random in nature, it may be described in terms of its *statistical properties* such as the *average power* or the spectral distribution of the average power. The mathematical discipline that deals with the statistical characteristics of noise and other random signals is *probability theory.* A discussion of probability theory and the related subject of *random processes* is presented in Chapter 8. Sources of noise and related system calculations are covered in Appendix C.

In any communication system, there are two primary communication resources to be employed, namely, *average transmitted power* and *channel bandwidth.* The average transmitted power is the average power of the transmitted signal. The channel bandwidth defines the range of frequencies that the channel can handle for the transmission of signals with satisfactory fidelity. A general system design objective is to use these two resources as efficiently as possible. In most channels, one resource may be considered more important than the other. Hence, we may also classify communication channels as *power-limited* or *band-limited.* For example, the telephone circuit is a typical band-limited channel, whereas a deep-space communication link or a satellite channel is typically power-limited.

The transmitted power is important because, for a receiver of prescribed *noise figure,* it determines the allowable separation between the transmitter and receiver. Stated in another way, for a receiver of prescribed noise figure and a prescribed distance between it and the transmitter, the available transmitted power determines the *signal-to-noise ratio* at the receiver input. This, in turn, determines the *noise performance* of the receiver. Unless this performance exceeds a certain design level, the transmission of message signals over the channel is not considered to be satisfactory.

The effects of noise in analog communications are evaluated in Chapter 9. This evaluation is traditionally done in terms of signal-to-noise ratios. In the case of digital communications, however, the preferred method of assessing the noise performance of a receiver is in terms of the *average probability of symbol error*. Such an approach leads to considerations of optimum receiver design. In this context, the *matched filter* offers optimum performance for the detection of pulses in an idealized form of receiver (channel) noise known as *additive white Gaussian noise*. As such, the matched-filter receiver or its equivalent, the *correlation receiver*, plays a key role in the design of digital communication systems. The matched filter and related issues are studied in Chapter 10.

Turning next to the other primary communication resource, channel bandwidth, it is important because, for a prescribed band of frequencies characterizing a message signal, the channel bandwidth determines the number of such message signals that can be *multiplexed* over the channel. Stated in another way, for a prescribed number of independent message signals that have to share a common channel, the channel bandwidth determines the band of frequencies that may be allotted to the transmission of each message signal without discernible distortion.

There is another important role for channel bandwidth, which is not that obvious. Specifically, channel bandwidth and transmitted (signal) power are *exchangeable* in that we may trade off one for the other for a prescribed system performance. The choice of one modulation scheme over another or the transmission of a message signal is often dictated by the nature of this trade-off. Indeed, the interplay between channel bandwidth and signal-to-noise ratio, and the limitation that they impose on communication, is highlighted most vividly by Shannon's famous *channel capacity theorem.*[4] Let B denote the channel bandwidth, and SNR denote the received signal-to-noise ratio. The channel capacity theorem states that ideally these two parameters are related by

$$C = B \log_2(1 + \text{SNR}), \text{ bits/s} \tag{1.7}$$

where C is the *channel capacity*, and a *bit* refers to a *bi*nary digi*t*. The channel capacity is defined as the maximum rate at which information may be transmitted without error through the channel; it is measured in *bits per second*. Equation 1.7 clearly shows that for a prescribed channel capacity, we may reduce the required SNR by increasing the channel band-

[4]In 1948, Shannon published a paper that laid the foundations of communication theory (Shannon, 1948). The channel capacity theorem is one of three theorems presented in that classic paper.

width B. Moreover, it provides an idealized framework for comparing the noise performance of one modulation system against another.

Finally, mention should be made of the issue of *system complexity*. We usually find that the efficient exploitation of channel bandwidth or transmitted power or both is achieved at the expense of increased system complexity. We therefore have to keep the issue of system complexity in mind, alongside that of channel bandwidth and transmitted power when considering the various trade-offs involved in the design of communication systems.

CHAPTER TWO

FOURIER ANALYSIS

In this chapter, we begin our study of Fourier analysis. We first review the *Fourier series*, by means of which we are able to represent a periodic signal as an infinite sum of sine-wave components. Next, we develop the *Fourier transform*, which performs a similar role in the analysis of nonperiodic signals. The Fourier transform is more general in application than the Fourier series.[1] The primary motivation for using the Fourier series or the Fourier transform is to obtain the *spectrum* of a

[1]The origin of the theory of Fourier series and Fourier transform is found in J. B. J. Fourier, *The Analytical Theory of Heat* (trans. A. Freeman), Cambridge University Press, London, 1878.

given signal, which describes the frequency content of the signal. In effect, this transformation provides an alternative method of viewing the signal that is often more revealing than the original description of the signal as a function of time.

2.1 FOURIER SERIES

Let $g_p(t)$ denote a periodic signal with period T_0. By using a *Fourier series expansion* of this signal, we are able to resolve the signal into an infinite sum of sine and cosine terms. This expansion may be expressed in the form

$$g_p(t) = a_0 + 2 \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi nt}{T_0}\right) + b_n \sin\left(\frac{2\pi nt}{T_0}\right) \right] \tag{2.1}$$

where the coefficients a_n and b_n represent the unknown amplitudes of the cosine and sine terms, respectively. The quantity n/T_0 represents the nth harmonic of the *fundamental frequency* $f_0 = 1/T_0$. Each of the cosine and sine functions in Eq. 2.1 is called a *basis function*. These basis functions form an *orthogonal set* over the interval T_0 in that they satisfy the following set of relations:

$$\int_{-T_0/2}^{T_0/2} \cos\left(\frac{2\pi mt}{T_0}\right) \cos\left(\frac{2\pi nt}{T_0}\right) dt = \begin{cases} T_0/2, & m = n \\ 0, & m \neq n \end{cases} \tag{2.2}$$

$$\int_{-T_0/2}^{T_0/2} \cos\left(\frac{2\pi mt}{T_0}\right) \sin\left(\frac{2\pi nt}{T_0}\right) dt = 0 \qquad \text{for all } m \text{ and } n \tag{2.3}$$

$$\int_{-T_0/2}^{T_0/2} \sin\left(\frac{2\pi mt}{T_0}\right) \sin\left(\frac{2\pi nt}{T_0}\right) dt = \begin{cases} T_0/2, & m = n \\ 0, & m \neq n \end{cases} \tag{2.4}$$

To determine the coefficient a_0, we integrate both sides of Eq. 2.1 over a complete period. We thus find that a_0 is the *mean value* of the periodic signal $g_p(t)$ over one period, as shown by the *time average*

$$a_0 = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} g_p(t)\, dt \tag{2.5}$$

To determine the coefficient a_n, we multiply both sides of Eq. 2.1 by the cosine function $\cos(2\pi nt/T_0)$ and integrate over the interval $-T_0/2$ to $T_0/2$. Then, using Eqs. 2.2 and 2.3, we find that

$$a_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} g_p(t) \cos\left(\frac{2\pi nt}{T_0}\right) dt, \qquad n = 1, 2, \ldots \tag{2.6}$$

Similarly, we find that

$$b_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} g_p(t) \sin\left(\frac{2\pi nt}{T_0}\right) dt, \qquad n = 1, 2, \ldots \tag{2.7}$$

To apply the Fourier series representation of Eq. 2.1, it is sufficient that inside the interval $-(T_0/2) \leq t \leq (T_0/2)$ the function $g_p(t)$ satisfies the following conditions:

1. The function $g_p(t)$ is single-valued.
2. The function $g_p(t)$ has a finite number of discontinuities.
3. The function $g_p(t)$ has a finite number of maxima and minima.
4. The function $g_p(t)$ is absolutely integrable, that is,

$$\int_{-T_0/2}^{T_0/2} |g_p(t)|\, dt < \infty$$

where $g_p(t)$ is assumed to be complex valued.

These conditions are known as *Dirichlet's conditions*. They are satisfied by the periodic signals usually encountered in communication systems.

COMPLEX EXPONENTIAL FOURIER SERIES

The Fourier series of Eq. 2.1 can be put into a much simpler and more elegant form with the use of complex exponentials. We do this by substituting in Eq. 2.1 the exponential form for the cosine and sine, namely:

$$\cos\left(\frac{2\pi nt}{T_0}\right) = \frac{1}{2}\left[\exp\left(\frac{j2\pi nt}{T_0}\right) + \exp\left(-\frac{j2\pi nt}{T_0}\right)\right]$$

$$\sin\left(\frac{2\pi nt}{T_0}\right) = \frac{1}{2j}\left[\exp\left(\frac{j2\pi nt}{T_0}\right) - \exp\left(-\frac{j2\pi nt}{T_0}\right)\right]$$

We thus obtain

$$g_p(t) = a_0 + \sum_{n=1}^{\infty}\left[(a_n - jb_n)\exp\left(\frac{j2\pi nt}{T_0}\right) + (a_n + jb_n)\exp\left(-\frac{j2\pi nt}{T_0}\right)\right] \tag{2.8}$$

The two product terms inside the square brackets in Eq. 2.8 are the complex conjugate of each other. We may also note the following relation:

$$\sum_{n=1}^{\infty} (a_n + jb_n) \exp\left(-\frac{j2\pi nt}{T_0}\right) = \sum_{n=-\infty}^{-1} (a_n - jb_n) \exp\left(\frac{j2\pi nt}{T_0}\right)$$

Let c_n denote a complex coefficient related to a_n and b_n by

$$c_n = \begin{cases} a_n - jb_n, & n > 0 \\ a_0, & n = 0 \\ a_n + jb_n, & n < 0 \end{cases} \tag{2.9}$$

Accordingly, we may simplify Eq. 2.8 as follows:

$$g_p(t) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{j2\pi nt}{T_0}\right) \tag{2.10}$$

where

$$c_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} g_p(t) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt, \qquad n = 0, \pm 1, \pm 2, \ldots \tag{2.11}$$

The series expansion of Eq. 2.10 is referred to as the *complex exponential Fourier series*. The c_n are called the *complex Fourier coefficients*. Equation 2.11 states that, given a periodic signal $g_p(t)$, we may determine the complete set of complex Fourier coefficients. On the other hand, Eq. 2.10 states that, given this set of values, we may reconstruct the original periodic signal exactly.

According to this representation, a periodic signal contains all frequencies (both positive and negative) that are harmonically related to the fundamental. The presence of negative frequencies is simply a result of the fact that the mathematical model of the signal as described by Eq. 2.10 requires the use of negative frequencies. Indeed, this representation also requires the use of a complex-valued basis function $\exp(j2\pi nt/T_0)$, which has no physical meaning either. The reason for using complex-valued basis functions and negative frequency components is merely to provide a compact mathematical description of a periodic signal, which is well-suited for both theoretical and practical work.

DISCRETE SPECTRUM

The representation of a periodic signal by a Fourier series is equivalent to the resolution of the signal into its various harmonic components. Thus, using the complex exponential Fourier series, we find that a periodic signal $g_p(t)$ with period T_0 has components of frequencies 0, $\pm f_0$, $\pm 2f_0$, $\pm 3f_0$, . . . , and so forth, where $f_0 = 1/T_0$ is the fundamental frequency. That is, while the signal $g_p(t)$ exists in the time domain, we may say that its frequency-domain description consists of components of frequencies, 0, $\pm f_0$, $\pm 2f_0$, . . . , called the *spectrum.*[2] If we specify the periodic signal $g_p(t)$, we can determine its spectrum; conversely, if we specify the spectrum, we can determine the corresponding signal. This means that a periodic signal $g_p(t)$ can be specified in two equivalent ways: (1) the time-domain representation where $g_p(t)$ is defined as a function of time, and (2) the frequency-domain representation where the signal is defined in terms of its spectrum. Although the two descriptions are separate aspects of a given phenomenon, they are not independent of each other, but are related, as Fourier theory shows.

In general, the Fourier coefficient c_n is a complex number; so we may express it in the form

$$c_n = |c_n| \exp[j \arg(c_n)] \tag{2.12}$$

The term $|c_n|$ defines the amplitude of the nth harmonic component of the periodic signal $g_p(t)$, so that a plot of $|c_n|$ versus frequency yields the *discrete amplitude spectrum* of the signal. A plot of $\arg(c_n)$ versus frequency yields the *discrete phase spectrum* of the signal. We refer to the spectrum as a *discrete spectrum* because both the amplitude and phase of c_n have nonzero values only for discrete frequencies that are integer (both positive and negative) multiples of the fundamental frequency.

For a real-valued periodic function $g_p(t)$, we find from the definition of the Fourier coefficient c_n given by Eq. 2.11 that

$$c_{-n} = c_n^* \tag{2.13}$$

where c_n^* is the complex conjugate of c_n. We therefore have

$$|c_{-n}| = |c_n| \tag{2.14}$$

and

$$\arg(c_{-n}) = -\arg(c_n) \tag{2.15}$$

That is, the amplitude spectrum of a real-valued periodic signal is *symmetric* (an even function of n) and the phase spectrum is *asymmetric* (an odd function of n) about the vertical axis passing through the origin.

[2] The term "spectrum" comes from the Latin word for "image." It was originally introduced by Sir Isaac Newton. For a historical account of spectrum analysis, see Gardner (1987).

EXAMPLE 1 PERIODIC PULSE TRAIN

Consider a periodic train of rectangular pulses of duration T and period T_0, as shown in Fig. 2.1. For convenience, the origin has been chosen to coincide with the center of the pulse. This signal may be described analytically over one period, $-(T_0/2) \leq t \leq (T_0/2)$, as follows

$$g_p(t) = \begin{cases} A, & -\dfrac{T}{2} \leq t \leq \dfrac{T}{2} \\ 0, & \text{for the remainder of the period} \end{cases} \tag{2.16}$$

Using Eq. 2.11 to evaluate the complex Fourier coefficient c_n, we get

$$\begin{aligned} c_n &= \frac{1}{T_0} \int_{-T/2}^{T/2} A \exp\left(-\frac{j2\pi nt}{T_0}\right) dt \\ &= \frac{A}{n\pi} \sin\left(\frac{n\pi T}{T_0}\right), \qquad n = 0, \pm 1, \pm 2, \ldots \end{aligned} \tag{2.17}$$

To simplify notation in the foregoing and subsequent results, we will use the *sinc function* defined by

$$\text{sinc}(\lambda) = \frac{\sin(\pi\lambda)}{\pi\lambda} \tag{2.18}$$

where λ is the independent variable. The sinc function plays an important role in communication theory. As shown in Fig. 2.2, it has its maximum value of unity at $\lambda = 0$, and approaches zero as λ approaches infinity, oscillating through positive and negative values. It goes through zero at $\lambda = \pm 1, \pm 2, \ldots$, and so on. Thus, in terms of the sinc function we may rewrite Eq. 2.17 as follows

$$c_n = \frac{TA}{T_0} \text{sinc}\left(\frac{nT}{T_0}\right) \tag{2.19}$$

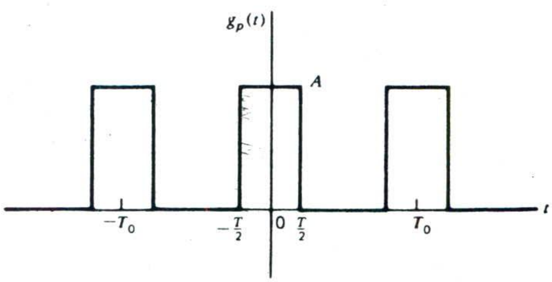


Figure 2.1

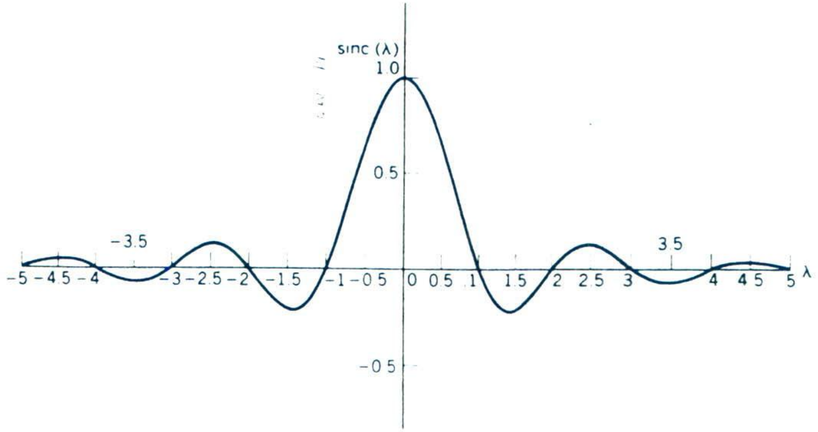


Figure 2.2
The sinc function.

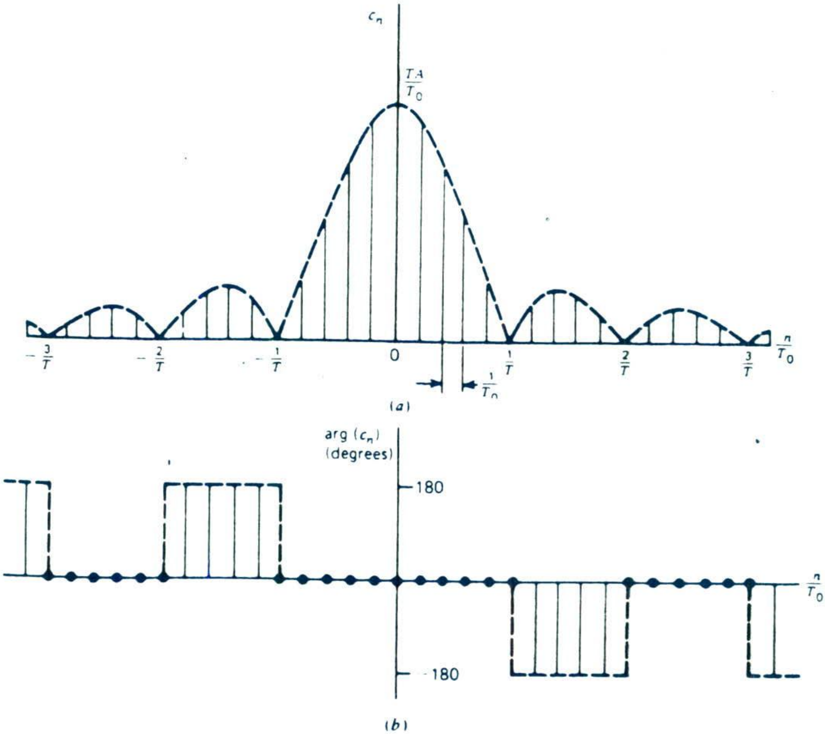


Figure 2.3
Discrete spectrum of periodic train of rectangular pulses for duty cycle $T/T_0 = 0.2$. (a) Amplitude spectrum. (b) Phase spectrum.

where n has discrete values only. In Fig. 2.3 we have plotted the amplitude spectrum $|c_n|$ and phase spectrum $\arg(c_n)$ versus the discrete frequency n/T_0 for a *duty cycle* T/T_0 equal to 0.2. We see that

1. The line spacing in the amplitude spectrum in Fig. 2.3*a* is determined by the period T_0.
2. The envelope of the amplitude spectrum is determined by the pulse amplitude A and duration T.
3. Zero-crossings occur in the envelope of the amplitude spectrum at frequencies that are multiples of $1/T$.
4. The phase spectrum takes on the values 0° and ±180°, depending on the polarity of $\text{sinc}(nT/T_0)$; in Fig. 2.3*b* we have used both 180° and −180° to preserve asymmetry.

EXERCISE 1 **Plot the amplitude spectra of rectangular pulses of unit amplitude and the following two values of duty cycle:**

a. $\dfrac{T}{T_0} = 0.1$

b. $\dfrac{T}{T_0} = 0.4$

2.2 FOURIER TRANSFORM

In the previous sections we used the Fourier series to represent a periodic signal. We now wish to develop a similar representation for a signal $g(t)$ that is nonperiodic, the representation being in terms of exponential time functions. In order to do this, we first construct a periodic function $g_p(t)$ of period T_0 in such a way that $g(t)$ defines one cycle of this periodic function, as illustrated in Fig. 2.4. In the limit we let the period T_0 become infinitely large, so that we may write

$$g(t) = \lim_{T_0 \to \infty} g_p(t) \tag{2.20}$$

Representing the periodic function $g_p(t)$ in terms of the complex exponential form of the Fourier series, we have

$$g_p(t) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{j2\pi nt}{T_0}\right) \tag{2.21}$$

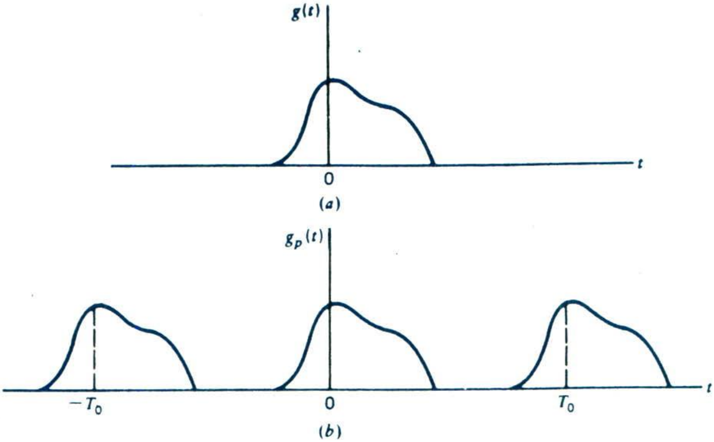


Figure 2.4
The construction of a periodic function from an arbitrarily defined function of time.

where

$$c_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} g_p(t) \exp\left(-\frac{j2\pi nt}{T_0} \right) dt \tag{2.22}$$

Define

$$\Delta f = \frac{1}{T_0}$$

$$f_n = \frac{n}{T_0}$$

and

$$G(f_n) = c_n T_0$$

Thus, making this change of notation in the Fourier series representation of $g_p(t)$, given by Eqs. 2.21 and 2.22, we get the following relations for the interval $-(T_0/2) \leq t \leq (T_0/2)$,

$$g_p(t) = \sum_{n=-\infty}^{\infty} G(f_n) \exp(j2\pi f_n t)\, \Delta f \tag{2.23}$$

where

$$G(f_n) = \int_{-T_0/2}^{T_0/2} g_p(t) \exp(-j2\pi f_n t)\, dt \tag{2.24}$$

Suppose we now let the period T_0 approach infinity or, equivalently, its reciprocal Δf approach zero. Then we find that, in the limit, the discrete frequency f_n approaches the continuous frequency variable f, and the discrete sum in Eq. 2.23 becomes an integral defining the area under a continuous function of frequency f, namely, $G(f)\exp(j2\pi ft)$. Also, as T_0 approaches infinity, the function $g_p(t)$ approaches $g(t)$. Therefore, in the limit, Eqs. 2.23 and 2.24 become, respectively,

$$g(t) = \int_{-\infty}^{\infty} G(f)\exp(j2\pi ft)\,df \tag{2.25}$$

where

$$G(f) = \int_{-\infty}^{\infty} g(t)\exp(-j2\pi ft)\,dt \tag{2.26}$$

We have thus achieved our aim of representing an arbitrarily defined signal $g(t)$ in terms of exponential time functions over the entire time interval from $-\infty$ to ∞. Note that in Eqs. 2.25 and 2.26 we have used a lowercase letter to denote the time function and an uppercase letter to denote the corresponding frequency function.

Equation 2.26 states that, given a time function $g(t)$, we can determine a new function $G(f)$ of the frequency variable f. Equation 2.25 states that, given this new or transformed function $G(f)$, we can recover the original time function $g(t)$. Thus, since from $g(t)$ we can define the function $G(f)$ and from $G(f)$ we can reconstruct $g(t)$, the time function is also specified by $G(f)$. The function $G(f)$ can be thought of as a transformed version of $g(t)$ and is referred to as the *Fourier transform* of $g(t)$. The time function $g(t)$ is similarly referred to as the *inverse Fourier transform* of $G(f)$. The functions $g(t)$ and $G(f)$ are said to constitute a *Fourier transform pair.*

DIRICHLET'S CONDITIONS

For a signal $g(t)$ to be Fourier transformable, it is sufficient that $g(t)$ satisfies *Dirichlet's conditions:*

1. The function $g(t)$ is single-valued, with a finite number of maxima and minima and a finite number of discontinuities in any finite time interval.
2. The function $g(t)$ is absolutely integrable, that is,

$$\int_{-\infty}^{\infty} |g(t)|\,dt < \infty$$

The Dirichlet conditions are not strictly necessary but sufficient for the Fourier transformability of a signal. These conditions include all energy signals, for which we have[3]

$$\int_{-\infty}^{\infty} |g(t)|^2 \, dt < \infty$$

In the two conditions described herein, the signal $g(t)$ is assumed to be complex.

NOTATIONS

The formulas for the Fourier transform and the inverse Fourier transform presented in Eqs. 2.25 and 2.26 are written in terms of *time t* and *frequency f*, with t measured in *seconds* (s) and f measured in *hertz* (Hz). The frequency f is related to the *angular frequency* ω as $\omega = 2\pi f$, which is measured in *radians per second* (rad/s). We may simplify the expressions for the exponents in the integrands of Eqs. 2.25 and 2.26 by using ω instead of f. However, the use of f is preferred over ω for two reasons. First, we have the mathematical *symmetry* of Eqs. 2.25 and 2.26 with respect to each other. Second, the frequency contents of communication signals (i.e., speech and video signals) are usually expressed in hertz.

A convenient *shorthand* notation for the transform relations of Eqs. 2.26 and 2.25 is

$$G(f) = \mathrm{F}[g(t)] \tag{2.27a}$$

$$g(t) = \mathrm{F}^{-1}[G(f)] \tag{2.27b}$$

Another convenient shorthand notation for the Fourier transform pair, represented by $g(t)$ and $G(f)$, is

$$g(t) \rightleftharpoons G(f) \tag{2.28}$$

The shorthand notations described herein are used in the text where appropriate.

SPECTRUM

By using Fourier transformation, an energy signal $g(t)$ is represented by the Fourier transform $G(f)$, which is a function of the frequency variable

[3]If the function $g(t)$ is such that the value of $\int_{-\infty}^{\infty} |g(t)|^2 \, dt$ is defined and finite, then the Fourier transform $G(f)$ of the function $g(t)$ exists and

$$\lim_{A \to \infty} \left[\int_{-\infty}^{\infty} \left| g(t) - \int_{-A}^{A} G(f) \exp(j2\pi ft) \, df \right|^2 dt \right] = 0$$

This result is known as *Plancherel's theorem.*

f. A plot of the Fourier transform $G(f)$ versus the frequency f is called the *spectrum* of the signal $g(t)$. The spectrum is continuous in the sense that it is defined for all frequencies. In general, the Fourier transform $G(f)$ is a complex function of the frequency f. We may therefore express it in the form

$$G(f) = |G(f)| \exp[j\theta(f)] \tag{2.29}$$

where $|G(f)|$ is called the *amplitude spectrum* of $g(t)$, and $\theta(f)$ is called the *phase spectrum* of $g(t)$.

For the special case of a real-valued function $g(t)$, we have

$$G(f) = G^*(-f)$$

Therefore, it follows that if $g(t)$ is a real-valued function of time t, then

$$|G(-f)| = |G(f)| \tag{2.30}$$

and

$$\theta(-f) = -\theta(f) \tag{2.31}$$

Accordingly, we may make the following statements on the spectrum of a *real-valued signal:*

1. The amplitude spectrum of the signal is an even function of the frequency; that is, the amplitude spectrum is *symmetric* about the vertical axis.
2. The phase spectrum of the signal is an odd function of the frequency; that is, the phase spectrum is *antisymmetric* about the vertical axis.

These two statements are often summed up by saying that the spectrum of a real-valued signal exhibits *conjugate symmetry*.

EXAMPLE 2 RECTANGULAR PULSE

Consider a *rectangular pulse* of duration T and amplitude A, as shown in Fig. 2.5. To define this pulse mathematically in a convenient form, we use the following notation

$$\text{rect}(t) = \begin{cases} 1, & -\frac{1}{2} < t < \frac{1}{2} \\ 0, & |t| > \frac{1}{2} \end{cases} \tag{2.32}$$

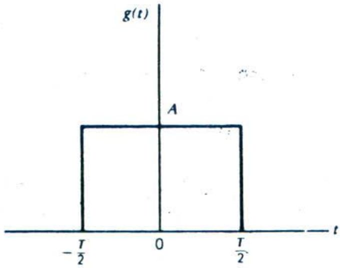


Figure 2.5
Rectangular pulse.

which stands for a *rectangular function* of unit amplitude and unit duration centered at $t = 0$. Then, in terms of this function, we may express the rectangular pulse of Fig. 2.5 simply as follows:

$$g(t) = A \operatorname{rect}\left(\frac{t}{T}\right)$$

The Fourier transform of this rectangular pulse is given by

$$\begin{aligned} G(f) &= \int_{-T/2}^{T/2} A \exp(-j2\pi f t)\, dt \\ &= AT\left[\frac{\sin(\pi f T)}{\pi f T}\right] \\ &= AT \operatorname{sinc}(fT) \end{aligned}$$

We thus have the Fourier transform pair

$$A \operatorname{rect}\left(\frac{t}{T}\right) \rightleftharpoons AT \operatorname{sinc}(fT) \tag{2.33}$$

The amplitude spectrum $|G(f)|$ of the rectangular pulse $g(t)$ is shown plotted in Fig. 2.6*a*. From this spectrum, we may make the following observations:

1. The amplitude spectrum has a *main lobe* of total width $2/T$, centered on the origin.

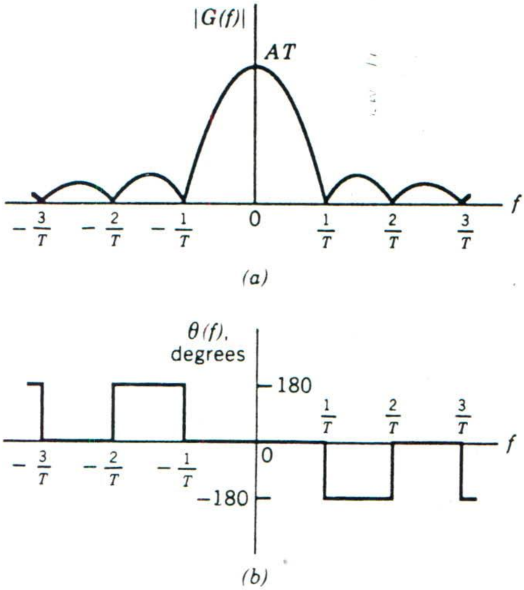


Figure 2.6
Spectrum of rectangular pulse. (a) Amplitude spectrum. (b) Phase spectrum.

2. The *side lobes,* on either side of the main lobe, decrease in amplitude with increasing $|f|$. *Indeed, the amplitudes of the side lobes are bounded by the curve* $1/|f|$.
3. The *zero crossings* of the spectrum occur at $f = \pm 1/T, \pm 2/T, \ldots$

The phase spectrum $\theta(f)$ of the rectangular pulse $g(t)$ is shown plotted in Fig. 2.6*b*. Depending on the sign of the sinc function $\text{sinc}(fT)$, the phase spectrum takes on the values 0° and ±180° in an asymmetric fashion.

EXAMPLE 3 EXPONENTIAL PULSE

A truncated form of decaying *exponential pulse* is shown in Fig. 2.7*a*. We may define this pulse mathematically in a convenient form by using the *unit step function:*

$$u(t) = \begin{cases} 1, & t > 0 \\ \frac{1}{2}, & t = 0 \\ 0, & t < 0 \end{cases} \tag{2.34}$$

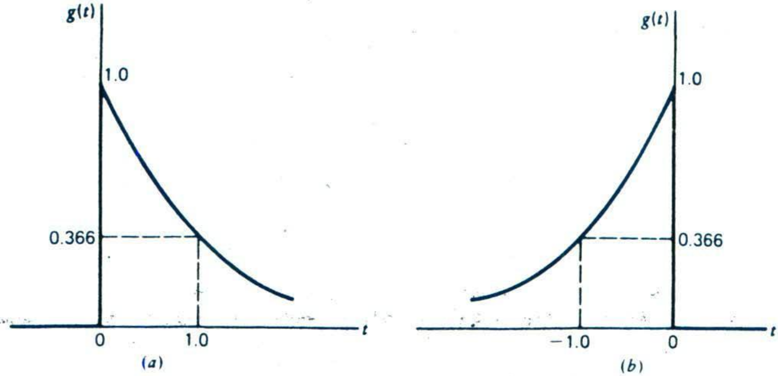


Figure 2.7
(a) Decaying exponential pulse. (b) Rising exponential pulse.

We may then express the exponential pulse of Fig. 2.7*a* as

$$g(t) = \exp(-t)u(t) \tag{2.35}$$

The Fourier transform of this pulse is

$$\begin{aligned} G(f) &= \int_0^\infty \exp(-t)\exp(-j2\pi ft)\,dt \\ &= \int_0^\infty \exp[-t(1+j2\pi f)]\,dt \\ &= \frac{1}{1+j2\pi f} \end{aligned} \tag{2.36}$$

Thus, combining Eqs. 2.35 and 2.36, we obtain the Fourier transform pair:

$$\exp(-t)u(t) \rightleftharpoons \frac{1}{1+j2\pi f} \tag{2.37}$$

Figure 2.8 shows the spectrum of the decaying exponential pulse.

A truncated rising exponential pulse is shown in Fig. 2.7*b*, which is defined by

$$g(t) = \exp(t)u(-t) \tag{2.38}$$

Note that $u(-t)$ is equal to unity for $t < 0$, one-half at $t = 0$, and zero

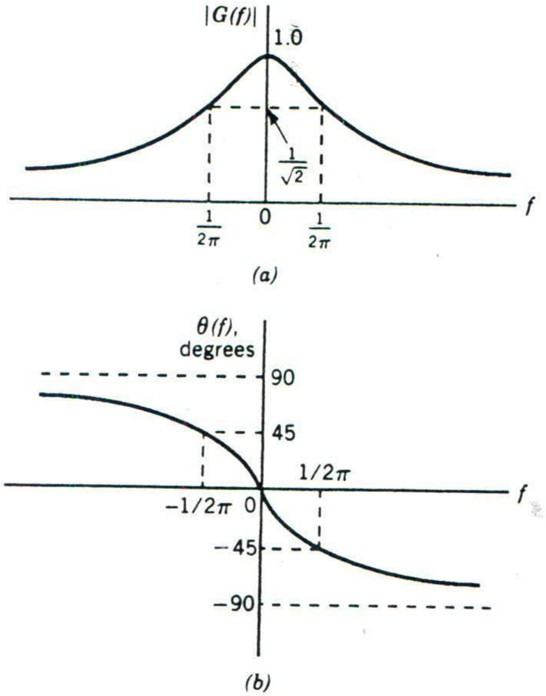


Figure 2.8
Spectrum of decaying exponential pulse. (a) Amplitude spectrum. (b) Phase spectrum.

for $t > 0$. The Fourier transform of this pulse is given by

$$\begin{aligned} G(f) &= \int_{-\infty}^{0} \exp(t) \exp(-j2\pi f t)\, dt \\ &= \int_{-\infty}^{0} \exp[t(1 - j2\pi f)]\, dt \\ &= \frac{1}{1 - j2\pi f} \end{aligned} \tag{2.39}$$

We thus have the Fourier transform pair:

$$\exp(t)u(-t) \rightleftharpoons \frac{1}{1 - j2\pi f} \tag{2.40}$$

Figure 2.9 shows the spectrum of the rising exponential pulse.

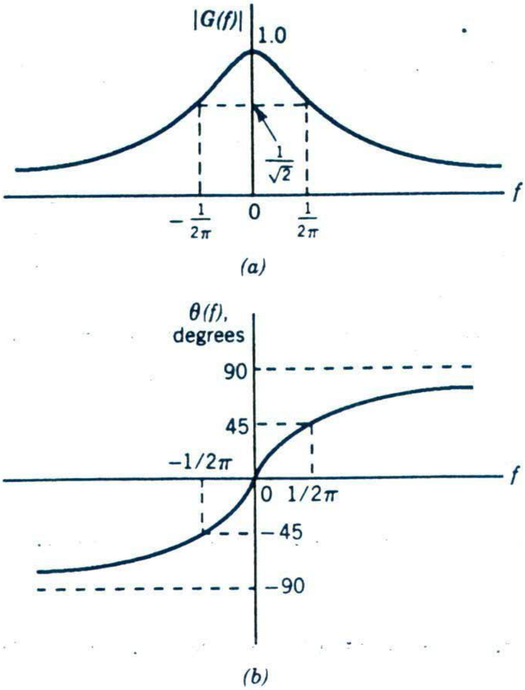


Figure 2.9
Spectrum of rising exponential pulse. (a) Amplitude spectrum. (b) Phase spectrum.

Comparing the spectra of Figs. 2.8 and 2.9, we may make the following two observations:

1. The decaying and rising exponentials of Fig. 2.7 have the same amplitude spectrum.
2. The phase spectrum of the rising exponential is the negative of that of the decaying exponential.

2.3 PROPERTIES OF THE FOURIER TRANSFORM

It is useful to have a feeling for the relationship between a function $g(t)$ and its Fourier transform $G(f)$, and for the effect that various operations on the function $g(t)$ have on the transform $G(f)$. This may be achieved by examining certain properties of the Fourier transform. This section describes 10 of these properties, which will be proved, one by one. These properties are summarized in Table 1 of Appendix D.

PROPERTY 1 LINEARITY (SUPERPOSITION)

Let $g_1(t) \rightleftharpoons G_1(f)$ and $g_2(t) \rightleftharpoons G_2(f)$. Then for all constants a and b, we have

$$ag_1(t) + bg_2(t) \rightleftharpoons aG_1(f) + bG_2(f) \tag{2.41}$$

The proof of this property follows simply from the linearity of the integrals defining $G(f)$ and $g(t)$.

EXAMPLE 4 DOUBLE EXPONENTIAL PULSE

Consider a *double exponential pulse* defined by (see Fig. 2.10)

$$\begin{aligned} g(t) &= \begin{cases} \exp(-t), & t > 0 \\ 1, & t = 0 \\ \exp(t), & t < 0 \end{cases} \\ &= \exp(-|t|) \end{aligned} \tag{2.42}$$

This pulse may be viewed as the sum of a truncated decaying exponential pulse and a truncated rising exponential pulse. Therefore, using the linearity property and the Fourier-transform pairs of Eqs. 2.37 and 2.40, we find that the Fourier transform of the double exponential pulse of Fig. 2.10 is as follows

$$\begin{aligned} G(f) &= \frac{1}{1 + j2\pi f} + \frac{1}{1 - j2\pi f} \\ &= \frac{2}{1 + (2\pi f)^2} \end{aligned}$$

We thus have the Fourier transform pair

$$\exp(-|t|) \rightleftharpoons \frac{2}{1 + (2\pi f)^2} \tag{2.43}$$

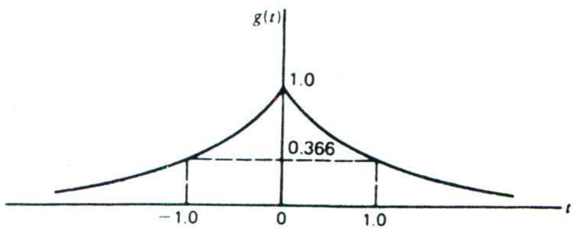


Figure 2.10
Double exponential pulse.

PROPERTY 2 TIME SCALING

Let $g(t) \rightleftharpoons G(f)$. Then,

$$g(at) \rightleftharpoons \frac{1}{|a|} G\left(\frac{f}{a}\right) \tag{2.44}$$

where a is a time-scaling factor that may be positive or negative.

To prove this property, we note that

$$\mathrm{F}[g(at)] = \int_{-\infty}^{\infty} g(at) \exp(-j2\pi ft)\, dt$$

Set $\tau = at$. There are two cases that can arise, depending on whether the scaling factor a is positive or negative. If $a > 0$, we get

$$\begin{aligned}\mathrm{F}[g(at)] &= \frac{1}{a}\int_{-\infty}^{\infty} g(\tau) \exp\left[-j2\pi\left(\frac{f}{a}\right)\tau\right] d\tau \\ &= \frac{1}{a} G\left(\frac{f}{a}\right)\end{aligned}$$

On the other hand, if $a < 0$, the limits of integration are interchanged so that we have the multiplying factor $-(1/a)$ or, equivalently, $1/|a|$. This completes the proof of Eq. 2.44.

Note that the function $g(at)$ represents $g(t)$ compressed in time by a factor a, whereas the function $G(f/a)$ represents $G(f)$ expanded in frequency by the same factor a. Thus the scaling property states that the compression of a function $g(t)$ in the time domain is equivalent to the expansion of its Fourier transform $G(f)$ in the frequency domain by the same factor, and vice versa.

EXAMPLE 5 RECTANGULAR PULSE (CONTINUED)

Example 2 evaluated the Fourier transform of a rectangular pulse; the result of the evaluation is given by the Fourier transform pair of Eq. 2.33. For convenience of presentation, let the rectangular pulse be normalized to have unit amplitude and unit duration. Then, putting $A = 1$ and $T = 1$ in Eq. 2.33, we have

$$\mathrm{rect}(t) \rightleftharpoons \mathrm{sinc}(f)$$

Hence, applying the time-scaling property to this Fourier transform pair, we get

$$\text{rect}(at) \rightleftharpoons \frac{1}{|a|}\,\text{sinc}\left(\frac{f}{a}\right)$$

Figure 2.11 shows the rectangular pulse and its amplitude spectrum for three different values of the time-scaling factor a, namely $a = 1/2, 1, 2$. With $a = 1$ regarded as the frame of reference, we may view the use of $a = 1/2$ as expansion in time, and $a = 2$ as compression in time. These

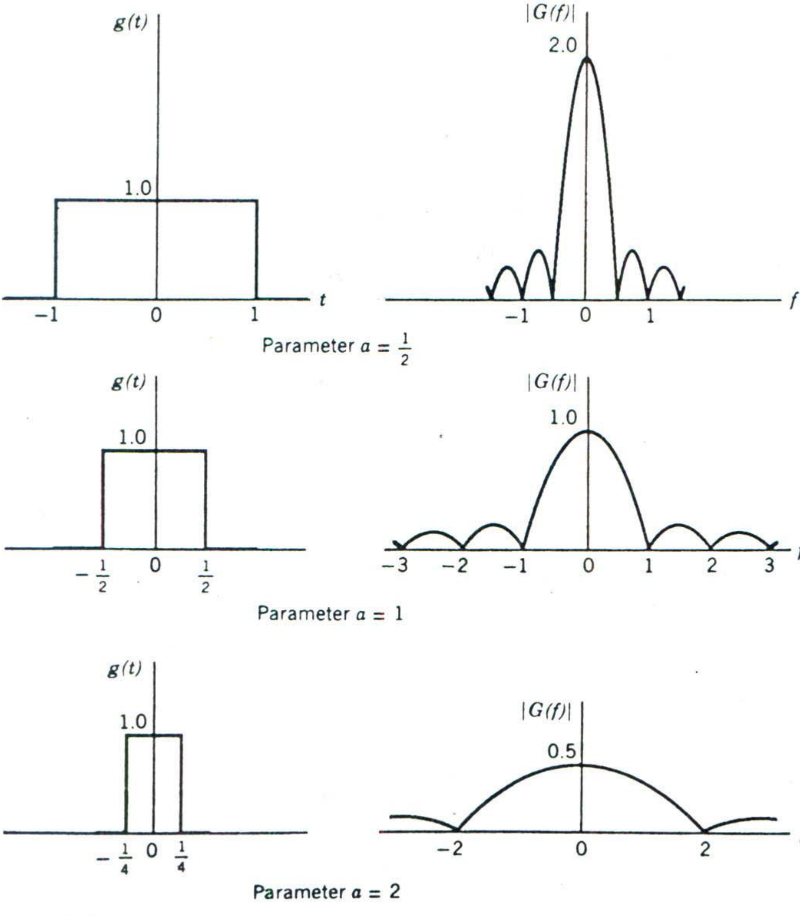


Figure 2.11
The inverse relation between time- and frequency-domain descriptions of rectangular pulse $g(t) = \text{rect}(at)$.

observations are confirmed by the three time-domain descriptions depicted on the left side of Fig. 2.11. The corresponding effects of these time-scale changes on the amplitude spectrum of the rectangular pulse are shown on the right side of Fig. 2.11. The two sets of plots depicted in this figure clearly show that the relationship between the time-domain and frequency-domain descriptions of a signal is an *inverse* one. That is, a narrow pulse (in time) has a wide spectrum (in frequency), and vice versa.

EXERCISE 2 Example 3 showed that the decaying exponential pulse and rising exponential pulse of Fig. 2.7 have the same amplitude spectra but opposite phase spectra. Use the time-scaling property of the Fourier transform to explain this behavior.

PROPERTY 3 DUALITY

If $g(t) \rightleftharpoons G(f)$, then

$$G(t) \rightleftharpoons g(-f) \tag{2.45}$$

This property follows from the relation defining the inverse Fourier transform by writing it in the form

$$g(-t) = \int_{-\infty}^{\infty} G(f) \exp(-j2\pi ft)\, df$$

and then interchanging t and f. Note that $G(t)$ is obtained from $G(f)$ by using t in place of f, and $g(-f)$ is obtained from $g(t)$ by using $-f$ in place of t.

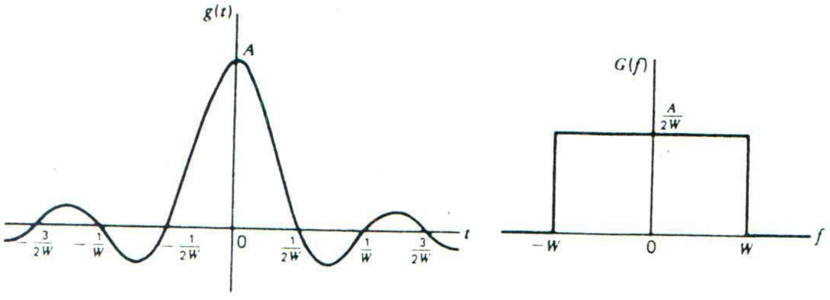
EXAMPLE 6 SINC PULSE

Consider a signal $g(t)$ in the form of a sinc function, as shown by

$$g(t) = A \operatorname{sinc}(2Wt) \tag{2.46}$$

To evaluate the Fourier transform of this function, we apply the duality and time-scaling properties to the Fourier transform pair of Eq. 2.33. Then, recognizing that the rectangular function is an even function, we obtain the following result:

$$A \operatorname{sinc}(2Wt) \rightleftharpoons \frac{A}{2W} \operatorname{rect}\left(\frac{f}{2W}\right) \tag{2.47}$$



(a) (b)

Figure 2.12
(a) Sinc pulse g(t). (b) Fourier transform G(f).

which is illustrated in Fig. 2.12. We thus see that the Fourier transform of a sinc pulse is zero for $|f| > W$. Note also that the sinc pulse itself is only asymptotically limited in time.

EXERCISE 3 Show that the total area under the curve of the sinc function equals one; that is,

$$\int_{-\infty}^{\infty} \operatorname{sinc}(t)\, dt = 1$$

EXERCISE 4 Consider a one-sided frequency function $G(f)$, defined by

$$G(f) = \begin{cases} \exp(-f), & f > 0 \\ \frac{1}{2}, & f = 0 \\ 0, & f < 0 \end{cases}$$

Applying the duality property to the Fourier transform pair of Eq. 2.40, write the inverse Fourier transform of $G(f)$.

PROPERTY 4 TIME SHIFTING

If $g(t) \rightleftharpoons G(f)$, then for a constant time shift t_0,

$$g(t - t_0) \rightleftharpoons G(f) \exp(-j2\pi f t_0). \tag{2.48}$$

To prove this property, we take the Fourier transform of $g(t - t_0)$ and then set $\tau = t - t_0$ to obtain

$$\begin{aligned} \mathrm{F}[g(t - t_0)] &= \exp(-j2\pi f t_0) \int_{-\infty}^{\infty} g(\tau) \exp(-j2\pi f \tau)\, d\tau \\ &= \exp(-j2\pi f t_0) G(f) \end{aligned}$$

The time-shifting property states that if a function $g(t)$ is shifted in the positive direction by an amount t_0, the effect is equivalent to multiplying its Fourier transform $G(f)$ by the factor $\exp(-j2\pi f t_0)$. This means that the amplitude of $G(f)$ is unaffected by the time shift but its phase is changed by the amount $-2\pi f t_0$.

EXAMPLE 7 RECTANGULAR PULSE (CONTINUED)

Consider the rectangular pulse $g_a(t)$ of Fig. 2.13*a*, which starts at time $t = 0$ and terminates at $t = T$. This pulse is defined by

$$g_a(t) = A \operatorname{rect}\left(\frac{t - T/2}{T}\right) \tag{2.49}$$

This pulse is obtained by shifting the rectangular pulse of Fig. 2.5 to the right by $T/2$ seconds. Therefore, applying the time-shifting property to the Fourier transform pair of Eq. 2.33, we find that the Fourier transform $G_a(f)$ of the rectangular pulse $g_a(t)$ defined in Eq. 2.49 is given by

$$G_a(f) = AT \operatorname{sinc}(fT) \exp(-j\pi f T) \tag{2.50}$$

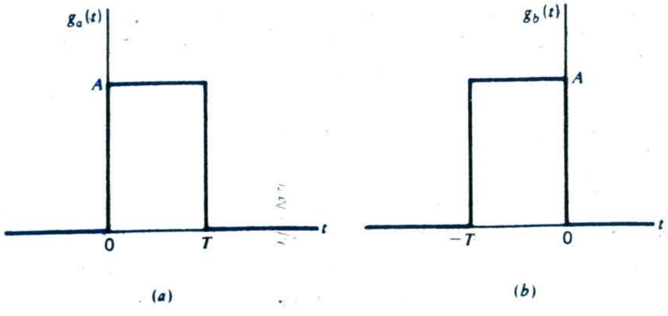


Figure 2.13
Time-shifted versions of a rectangular pulse.

Consider next the rectangular pulse $g_b(t)$ of Fig. 2.13b, which starts at time $t = -T$ and terminates at $t = 0$. This second pulse is defined by

$$g_b(t) = A \operatorname{rect}\left(\frac{t + T/2}{T}\right) \tag{2.51}$$

The pulse $g_b(t)$ is obtained by shifting the rectangular pulse of Fig. 2.5 to the left by $T/2$ seconds. Therefore, applying the time-shifting property to the Fourier transform pair of Eq. 2.33, we find that the Fourier transform $G_b(f)$ of the rectangular pulse $g_b(t)$ defined in Eq. 2.51 is given by

$$G_b(f) = AT \operatorname{sinc}(fT) \exp(j\pi fT) \tag{2.52}$$

PROPERTY 5 FREQUENCY SHIFTING

If $g(t) \rightleftharpoons G(f)$, then for a constant frequency shift f_c,

$$\exp(j2\pi f_c t)\, g(t) \rightleftharpoons G(f - f_c). \tag{2.53}$$

This property follows from the fact that

$$\begin{aligned} F[\exp(j2\pi f_c t) g(t)] &= \int_{-\infty}^{\infty} g(t) \exp[-j2\pi t(f - f_c)]\, dt \\ &= G(f - f_c) \end{aligned}$$

That is, multiplication of a function $g(t)$ by the factor $\exp(j2\pi f_c t)$ is equivalent to shifting its Fourier transform $G(f)$ in the positive direction by the amount f_c. Note the duality between the time-shifting and frequency-shifting operations.

EXAMPLE 8 RADIO FREQUENCY PULSE

Consider the *radio frequency* (*RF*) pulse signal $g(t)$ shown in Fig. 2.14a, which consists of a sinusoidal wave of amplitude A and frequency f_c. The pulse occupies the interval from $t = -T/2$ to $t = T/2$. This signal is referred to as an *RF pulse* when the frequency f_c falls in the radio-frequency band. Such a pulse is commonly used in *radar* for the detection of targets of interest (e.g., aircraft) and for the estimation of useful target parameters (e.g., range).

The signal $g(t)$ of Fig. 2.14a may be expressed mathematically as follows:

$$g(t) = A \operatorname{rect}\left(\frac{t}{T}\right) \cos(2\pi f_c t) \tag{2.54}$$

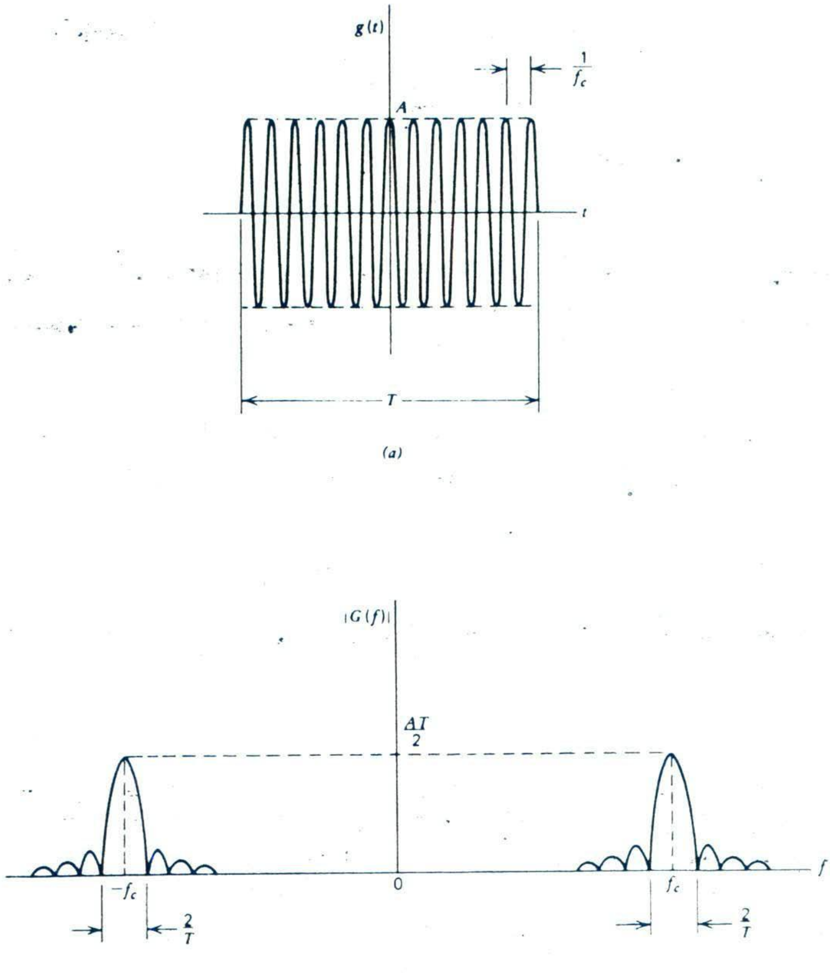


Figure 2.14
(a) RF pulse. (b) Amplitude spectrum.

To find the Fourier transform of this signal, we note that

$$\cos(2\pi f_c t) = \tfrac{1}{2}[\exp(j2\pi f_c t) + \exp(-j2\pi f_c t)]$$

Therefore, applying the frequency-shifting property to the Fourier transform pair of Eq. 2.33, we get the desired result

$$G(f) = \frac{AT}{2}\{\text{sinc}[T(f - f_c)] + \text{sinc}[T(f + f_c)]\} \qquad (2.55)$$

When the number of cycles within the pulse is large, that is, $f_c T \gg 1$, we may use the approximate result

$$G(f) \simeq \begin{cases} \dfrac{AT}{2}\,\mathrm{sinc}[T(f - f_c)], & f > 0 \\ \dfrac{AT}{2}\,\mathrm{sinc}[T(f + f_c)], & f < 0 \end{cases} \tag{2.56}$$

The amplitude spectrum of the RF pulse is shown in Fig. 2.14*b*. This diagram, in relation to Fig. 2.6*a*, clearly illustrates the frequency-shifting property of the Fourier transform.

EXERCISE 5 Consider an exponentially damped sinusoidal wave defined by

$$g(t) = \begin{cases} \exp(-t)\sin(2\pi f_c t), & t > 0 \\ 0, & t \le 0 \end{cases} \tag{2.57}$$

Using the expansion

$$\sin(2\pi f_c t) = \frac{1}{2j}[\exp(j2\pi f_c t) - \exp(-j2\pi f_c t)]$$

and applying the frequency-shifting property to the Fourier transform pair of Eq. 2.37, write the Fourier transform of $g(t)$.

PROPERTY 6 DIFFERENTIATION IN THE TIME DOMAIN

Let $g(t) \rightleftharpoons G(f)$, and assume that the first derivative of $g(t)$ is Fourier transformable. Then

$$\frac{d}{dt}g(t) \rightleftharpoons j2\pi f G(f) \tag{2.58}$$

That is, differentiation of a time function $g(t)$ has the effect of multiplying its Fourier transform $G(f)$ by the factor $j2\pi f$.

This result is obtained simply by taking the first derivative of both sides of the relation defining the inverse Fourier transform of $G(f)$, namely, Eq. 2.25, and then interchanging the operations of integration and differentiation; we are justified to make this interchange because integration and differentiation are both linear operations.

Multiplication of the Fourier transform $G(f)$ by the factor $j2\pi f$ on the right side of Eq. 2.58 implies that differentiation of $g(t)$ with respect to time enhances the high frequency components of the signal $g(t)$.

We may generalize Eq. 2.58 as follows:

$$\frac{d^n}{dt^n} g(t) \rightleftharpoons (j2\pi f)^n G(f) \tag{2.59}$$

EXAMPLE 9 GAUSSIAN PULSE

In this example we wish to use the differentiation property of the Fourier transform to derive the pulse signal $g(t)$ whose Fourier transform $G(f)$ has the same form.

Let $g(t)$ denote the pulse as a function of time, and $G(f)$ its Fourier transform. We note that by differentiating the formula for the Fourier transform $G(f)$ with respect to f, we have

$$-j2\pi t g(t) \rightleftharpoons \frac{d}{df} G(f) \tag{2.60}$$

which expresses the effect of differentiation in the frequency domain. Equation 2.60 is the dual of Eq. 2.58 that describes the time-differentiation property. Dividing both sides of Eq. 2.60 by j, we may also write

$$-2\pi t g(t) \rightleftharpoons \frac{1}{j} \frac{d}{df} G(f) \tag{2.61}$$

Suppose that the pulse-signal $g(t)$ satisfies the first-order differential equation

$$\frac{d}{dt} g(t) = -2\pi t g(t) \tag{2.62}$$

The imposition of this condition on the pulse signal $g(t)$ is equivalent to equating the left-hand members of Eqs. 2.58 and 2.61. Correspondingly, we may equate the right-hand members of Eqs. 2.58 and 2.61, and thus write

$$\frac{1}{j} \frac{d}{df} G(f) = j2\pi f G(f) \tag{2.63}$$

Since $j^2 = -1$, we may rewrite Eq. 2.63 as

$$\frac{d}{df} G(f) = -2\pi f G(f) \tag{2.64}$$

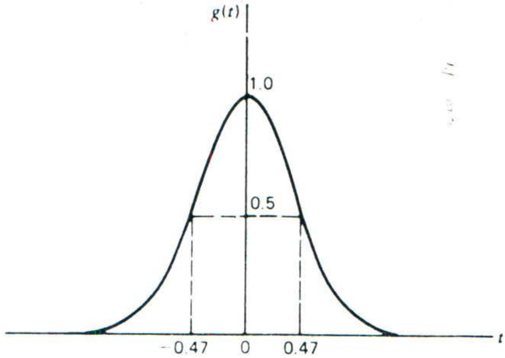


Figure 2.15
Gaussian pulse.

We may now state that if a pulse signal $g(t)$ satisfies the first-order differential equation (2.62), then its Fourier transform $G(f)$ must satisfy the first-order differential equation (2.64). However, these two differential equations have exactly the same mathematical form. Hence, the pulse signal and its transform are the same function. In other words, provided that the pulse signal $g(t)$ satisfies the differential equation (2.62), then $G(f) = g(f)$. Solving Eq. 2.62 for $g(t)$, we obtain

$$g(t) = \exp(-\pi t^2) \tag{2.65}$$

This result is shown plotted in Fig. 2.15.

The pulse defined by Eq. 2.65 is called a *Gaussian pulse,* the name being derived from the similarity of the function to the Gaussian probability density function. We conclude therefore that the Gaussian pulse is its own Fourier transform as shown by

$$\exp(-\pi t^2) \rightleftharpoons \exp(-\pi f^2) \tag{2.66}$$

EXERCISE 6 Show that

$$\int_{-\infty}^{\infty} \exp(-\pi t^2)\, dt = 1 \tag{2.67}$$

Hint: Consider the formula for the Fourier transform of $\exp(-\pi t^2)$ evaluated at $f = 0$.

PROPERTY 7 INTEGRATION IN THE TIME DOMAIN

Let $g(t) \rightleftharpoons G(f)$. Then, provided $G(0) = 0$, we have

$$\int_{-\infty}^{t} g(t)\, dt \rightleftharpoons \frac{1}{j2\pi f} G(f) \qquad (2.68)$$

That is, integration of a time function $g(t)$ has the effect of dividing its Fourier transform $G(f)$ by the factor $j2\pi f$, assuming that $G(0)$ is zero.

To prove this property, we write the Fourier transform of the integrated signal as follows

$$F\left[\int_{-\infty}^{t} g(\tau)\, d\tau\right] = \int_{-\infty}^{\infty} \exp(-j2\pi f t) \int_{-\infty}^{t} g(\tau)\, d\tau\, dt \qquad (2.69)$$

On the right side of this relation, we have a definite integral with respect to the variable t. Clearly, we may view the corresponding integrand as the product of two time functions: the exponential $\exp(-j2\pi f t)$ and the integrated signal $\int_{-\infty}^{t} g(\tau)\, d\tau$. Hence, using the formula for integration by parts and assuming that

$$G(0) = \int_{-\infty}^{\infty} g(\tau)\, d\tau = 0$$

and then simplifying the result, we get the relation of Eq. 2.68. The condition $G(0) = 0$ ensures that $g(\tau)$ integrates out to zero as τ approaches infinity. The more general case, for which $G(0) \neq 0$, is treated later in Section 2.5.

Division of the Fourier transform $G(f)$ by the factor $j2\pi f$ on the right side of Eq. 2.68 implies that integration of $g(t)$ with respect to time suppresses the high-frequency components of $g(t)$. As expected, this effect is the opposite of that produced by differentiation of $g(t)$.

EXAMPLE 10 TRIANGULAR PULSE

Consider the *doublet pulse* $g_1(t)$ shown in Fig. 2.16*a*. By integrating this pulse with respect to time, we obtain the *triangular pulse* $g_2(t)$ shown in Fig. 2.16*b*. The duration of this triangular pulse at the half-amplitude points is the same as the duration of the rectangular pulse of Fig. 2.5. We note that the doublet pulse $g_1(t)$ consists of two rectangular pulses: one of amplitude A, defined for the interval $-T \leq t \leq 0$, and the other of amplitude $-A$, defined for the interval $0 \leq t \leq T$. Therefore, using the results

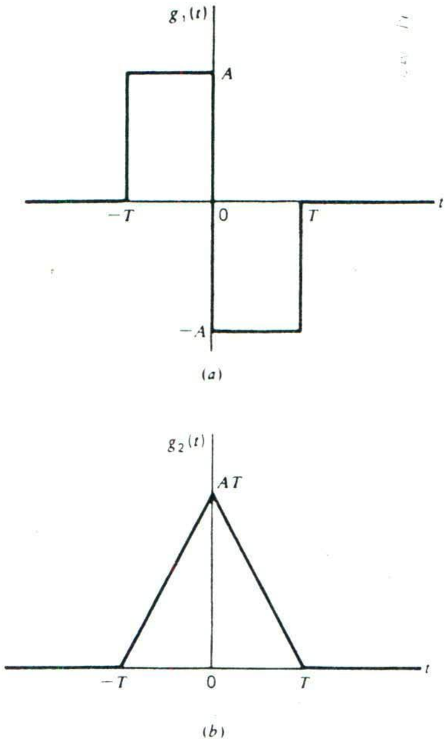


Figure 2.16
(a) Doublet pulse $g_1(t)$. (b) Triangular pulse $g_2(t)$ obtained by integrating $g_1(t)$.

of Example 7, we find that the Fourier transform $G_1(f)$ of the doublet pulse $g_1(t)$ of Fig. 2.16*a* is given by

$$\begin{aligned} G_1(f) &= AT\,\text{sinc}(fT)\,[\exp(j\pi fT) - \exp(-j\pi fT)] \\ &= 2jAT\,\text{sinc}(fT)\sin(\pi fT) \end{aligned} \tag{2.70}$$

We further note that $G_1(0)$ is zero. Hence, using Eqs. 2.68 and 2.70, we find that the Fourier transform $G_2(f)$ of the triangular pulse $g_2(t)$ of Fig. 2.16*b* is given by

$$\begin{aligned} G_2(f) &= \frac{1}{j2\pi f} G_1(f) \\ &= AT\frac{\sin(\pi fT)}{\pi f}\,\text{sinc}(fT) \\ &= AT^2\,\text{sinc}^2(fT) \end{aligned} \tag{2.71}$$

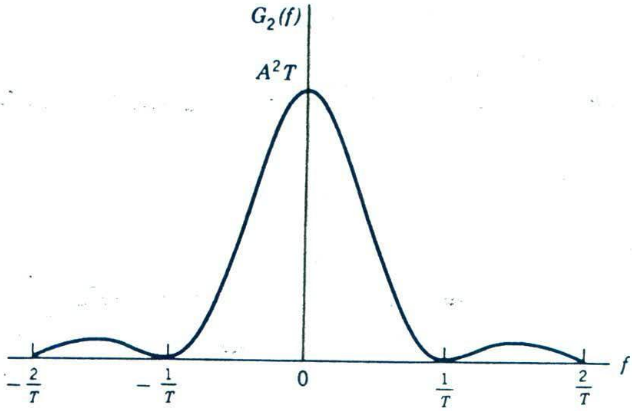


Figure 2.17
Spectrum of triangular pulse.

The Fourier transform $G_2(f)$ is a positive real function of f, which means that the amplitude spectrum of $g_2(t)$ is the same as $G_2(f)$, and its phase spectrum is zero for all f. The Fourier transform $G_2(f)$ is plotted in Fig. 2.17. Note that the spectrum of the triangular pulse is more tightly centered around the origin than the spectrum of the rectangular pulse. Also, the spectrum of the triangular pulse decreases as $1/f^2$, whereas the spectrum of the rectangular pulse is discontinuous and decreases as $1/|f|$.

EXERCISE 7

a. Show that the Fourier transform of a triangular pulse of unit amplitude and unit duration (measured at the half-amplitude points) is equal to $\text{sinc}^2(f)$.

b. Using the result in part a, show that

$$\int_{-\infty}^{\infty} \text{sinc}^2(f)\, df = 1$$

Hint: For part b, consider the formula for the inverse Fourier transform of $\text{sinc}^2(f)$ evaluated at $t = 0$.

PROPERTY 8 CONJUGATE FUNCTIONS

If $g(t) \rightleftharpoons G(f)$, then for a complex-valued time function $g(t)$ we have

$$g^*(t) \rightleftharpoons G^*(-f) \tag{2.72}$$

where the asterisk denotes the complex conjugate operation.

To prove this property, we know from the inverse Fourier transform that

$$g(t) = \int_{-\infty}^{\infty} G(f) \exp(j2\pi ft)\, df$$

Taking the complex conjugates of both sides:

$$g^*(t) = \int_{-\infty}^{\infty} G^*(f) \exp(-j2\pi ft)\, df$$

Next, replacing f with $-f$:

$$\begin{aligned} g^*(t) &= -\int_{\infty}^{-\infty} G^*(-f) \exp(j2\pi ft)\, df \\ &= \int_{-\infty}^{\infty} G^*(-f) \exp(j2\pi ft)\, df \end{aligned}$$

That is, $g^*(t)$ is the inverse Fourier transform of $G^*(-f)$, which is the desired result.

EXAMPLE 11 REAL AND IMAGINARY PARTS OF A TIME FUNCTION

Expressing a complex-valued function $g(t)$ in terms of its real and imaginary parts, we may write

$$g(t) = \mathrm{Re}[g(t)] + j\,\mathrm{Im}[g(t)] \tag{2.73}$$

where Re denotes the "real part of" and Im denotes the "imaginary part of." The complex conjugate of $g(t)$ is

$$g^*(t) = \mathrm{Re}[g(t)] - j\,\mathrm{Im}[g(t)] \tag{2.74}$$

Adding Eqs. 2.73 and 2.74:

$$\mathrm{Re}[g(t)] = \frac{1}{2}[g(t) + g^*(t)] \tag{2.75}$$

and subtracting them:

$$\mathrm{Im}[g(t)] = \frac{1}{2j}[g(t) - g^*(t)] \tag{2.76}$$

Therefore, applying Property 8, we obtain the following two Fourier transform pairs:

$$\mathrm{Re}[g(t)] \rightleftharpoons \frac{1}{2}[G(f) + G^*(-f)] \tag{2.77}$$

$$\mathrm{Im}[g(t)] \rightleftharpoons \frac{1}{2j}[G(f) - G^*(-f)] \tag{2.78}$$

From Eq. 2.78, it is apparent that in the case of a real-valued time function $g(t)$, we have $G(f) = G^*(-f)$; that is, $G(f)$ exhibits *conjugate symmetry*. This result is a restatement of Eqs. 2.30 and 2.31.

EXERCISE 8 Show that for a real-valued signal $g(t)$, Eq. 2.72 may be rewritten in the equivalent form:

$$g(-t) \rightleftharpoons G^*(f)$$

PROPERTY 9 MULTIPLICATION IN THE TIME DOMAIN

Let $g_1(t) \rightleftharpoons G_1(f)$ and $g_2(t) \rightleftharpoons G_2(f)$. Then

$$g_1(t)\, g_2(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(\lambda) G_2(f - \lambda)\, d\lambda \tag{2.79}$$

To prove this property, we first denote the Fourier transform of the product $g_1(t)g_2(t)$ by $G_{12}(f)$, so that we may write

$$g_1(t)\, g_2(t) \rightleftharpoons G_{12}(f)$$

where

$$G_{12}(f) = \int_{-\infty}^{\infty} g_1(t) g_2(t) \exp(-j2\pi f t)\, dt$$

For $g_2(t)$, we next substitute the inverse Fourier transform

$$g_2(t) = \int_{-\infty}^{\infty} G_2(f') \exp(j2\pi f' t)\, df'$$

in the integral defining $G_{12}(f)$ to obtain

$$G_{12}(f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_1(t) G_2(f') \exp[-j2\pi(f - f')t]\, df'\, dt$$

Define $\lambda = f - f'$. Then, interchanging the order of integration, we obtain

$$G_{12}(f) = \int_{-\infty}^{\infty} d\lambda\, G_2(f - \lambda) \int_{-\infty}^{\infty} g_1(t) \exp(-j2\pi\lambda t)\, dt$$

The inner integral is recognized simply as $G_1(\lambda)$, so we may write

$$G_{12}(f) = \int_{-\infty}^{\infty} G_1(\lambda) G_2(f - \lambda)\, d\lambda$$

which is the desired result. This integral is known as the *convolution integral* expressed in the frequency domain, and the function $G_{12}(f)$ is referred to as the *convolution* of $G_1(f)$ and $G_2(f)$. We conclude that *the multiplication of two signals in the time domain is transformed into the convolution of their individual Fourier transforms in the frequency domain*. This property is known as the *multiplication theorem*.

In a discussion of convolution, the following shorthand notation is frequently used:

$$G_{12}(f) = G_1(f) \star G_2(f)$$

where the star $\star$ denotes convolution. Note that convolution is commutative, that is,

$$G_{12}(f) = G_{21}(f)$$

or

$$G_1(f) \star G_2(f) = G_2(f) \star G_1(f)$$

EXAMPLE 12 TRUNCATED SINC PULSE

Consider the truncation of the sinc pulse $\text{sinc}(2Wt)$, so that the resulting signal $g(t)$ is zero outside the interval $-(T/2) \leq t \leq (T/2)$, as shown in Fig. 2.18*a*. This signal may be expressed as the product of a sinc pulse and a rectangular pulse, as shown by

$$g(t) = \text{sinc}(2Wt)\, \text{rect}\left(\frac{t}{T}\right) \tag{2.80}$$

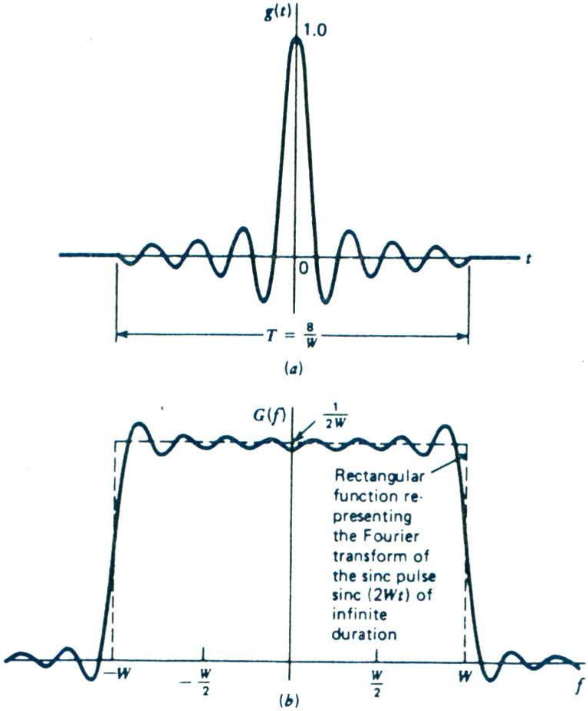


Figure 2.18
The Gibbs phenomenon. (a) A truncated sinc function g(t). (b) Fourier transform G(f).

From Eqs. 2.33 and 2.47, we have

$$F\left[\operatorname{rect}\left(\frac{t}{T}\right)\right] = T\operatorname{sinc}(fT)$$

$$F[\operatorname{sinc}(2Wt)] = \frac{1}{2W}\operatorname{rect}\left(\frac{f}{2W}\right)$$

Therefore, using Eq. 2.79, we find that the Fourier transform of the truncated sinc pulse $g(t)$ is given by

$$\begin{aligned} G(f) &= \frac{T}{2W}\int_{-\infty}^{\infty}\operatorname{rect}\left(\frac{\lambda}{2W}\right)\operatorname{sinc}[(f-\lambda)T]\,d\lambda \\ &= \frac{T}{2W}\int_{-W}^{W}\operatorname{sinc}[(f-\lambda)T]\,d\lambda \\ &= \frac{T}{2W}\int_{-W}^{W}\frac{\sin[\pi(f-\lambda)T]}{\pi(f-\lambda)T}\,d\lambda \end{aligned} \tag{2.81}$$

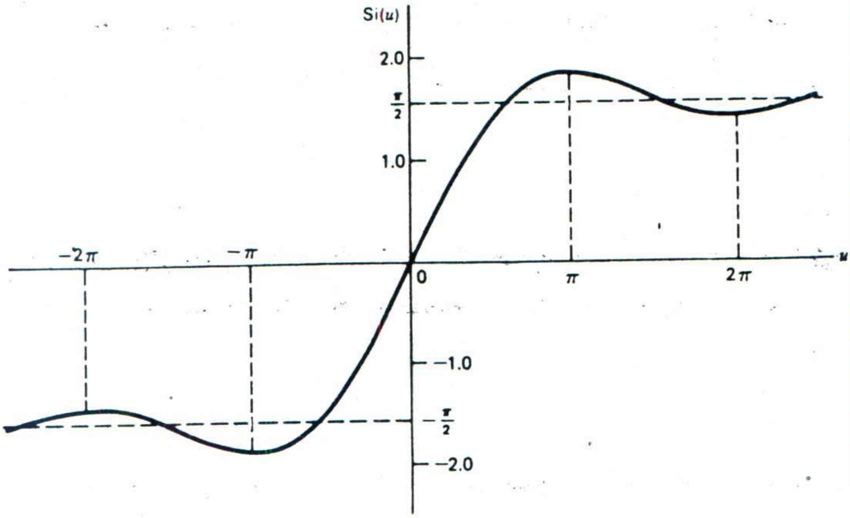


Figure 2.19
The sine integral.

The integral of the function $\sin x/x$ from zero up to some upper limit is called the *sine integral*, which is defined as follows

$$\mathrm{Si}(u) = \int_0^u \frac{\sin x}{x}\, dx \tag{2.82}$$

The sine integral $\mathrm{Si}(u)$ cannot be integrated in closed form in terms of elementary functions, but it can be integrated as a power series.[4] It is plotted in Fig. 2.19. We see that: (1) the sine integral $\mathrm{Si}(u)$ is odd symmetric about $u = 0$; (2) it has its maxima and minima at multiples of π; and (3) it approaches the limiting value $\pi/2$ for large values of u.

Substituting $x = \pi(f - \lambda)T$ in Eq. 2.81, we find that the Fourier transform $G(f)$ of the truncated sinc pulse may be expressed conveniently in terms of the sine integral as follows:

$$G(f) = \frac{1}{2\pi W}[\mathrm{Si}(\pi WT - \pi fT) + \mathrm{Si}(\pi WT + \pi fT)] \tag{2.83}$$

This relation is plotted in Fig. 2.18*b* for the case when $T = 8/W$. We see that $G(f)$ approximates the Fourier transform of a sinc pulse $\mathrm{sinc}(2Wt)$ of infinite duration in an oscillatory fashion, with a maximum deviation of about 9%. Furthermore, for a given value of W, as the pulse duration T

is increased, the ripples in the vicinities of the discontinuities of the rectangular function show a proportionately increased rate of oscillation versus the frequency, f, whereas their amplitudes relative to the magnitude of the discontinuity remain the same. This effect is an example of *Gibbs phenomenon* in Fourier transforms.

EXERCISE 9 Using Eq. 2.79, show that

$$\int_{-\infty}^{\infty} g_1(t)g_2(t)\, dt = \int_{-\infty}^{\infty} G_1(f)G_2(-f)\, df$$

How is the left side of this relation affected by replacing $G_2(-f)$ with $G_2(f)$ in the integral on the right side of the relation?

PROPERTY 10 CONVOLUTION IN THE TIME DOMAIN

Let $g_1(t) \rightleftharpoons G_1(f)$ *and* $g_2(t) \rightleftharpoons G_2(f)$. Then

$$\int_{-\infty}^{\infty} g_1(\tau)g_2(t - \tau)\, d\tau \rightleftharpoons G_1(f)G_2(f) \tag{2.84}$$

This result follows directly by combining Property 3 (duality) and Property 9 (time-domain multiplication). We may thus state that *the convolution of two signals in the time domain is transformed into the multiplication of their individual Fourier transforms in the frequency domain*. This property is known as the *convolution theorem*. Its use permits us to exchange a convolution operation for a transform multiplication, an operation that is ordinarily easier to manipulate.

Using the shorthand notation for convolution, we may rewrite Eq. 2.84 in the form

$$g_1(t) \star g_2(t) \rightleftharpoons G_1(f)G_2(f) \tag{2.85}$$

where the star $\star$ denotes convolution.

EXAMPLE 13 DERIVATIVE OF A CONVOLUTION INTEGRAL

Let $g_{12}(t)$ denote the result of convolving two signals $g_1(t)$ and $g_2(t)$. Then the derivative of $g_{12}(t)$ is equal to the convolution of $g_1(t)$ with the derivative of $g_2(t)$, or vice versa. That is, if

$$g_{12}(t) = g_1(t) \star g_2(t)$$

then

$$\frac{d}{dt} g_{12}(t) = \left[\frac{d}{dt} g_1(t)\right] \star g_2(t)$$

To prove this result, we use the differentiation property (i.e., Eq. 2.58) in conjunction with the convolution property (i.e., Eq. 2.85), obtaining

$$\frac{d}{dt}[g_1(t) \star g_2(t)] \rightleftharpoons j2\pi f[G_1(f)G_2(f)].$$

Associating the factor $j2\pi f$ with $G_1(f)$, we may write

$$\left[\frac{d}{dt} g_1(t)\right] \star g_2(t) \rightleftharpoons [j2\pi f G_1(f)]G_2(f)$$

which yields the desired result:

$$\frac{d}{dt}[g_1(t) \star g_2(t)] = \left[\frac{d}{dt} g_1(t)\right] \star g_2(t) \tag{2.86}$$

Equation 2.86 shows that the derivative of the convolution of two time functions is equal to the convolution of one function with the derivative of the other.

EXERCISE 10 **Using Eq. 2.84, show that**

$$\int_{-\infty}^{\infty} g_1(t)g_2(-t)\,dt = \int_{-\infty}^{\infty} G_1(f)G_2(f)\,df$$

How is the right side of this relation affected by replacing $g_2(-t)$ with $g_2^*(t)$ in the integral on the left side of the relation? How does this result compare with that of Exercise 9?

2.4 INTERPLAY BETWEEN TIME-DOMAIN AND FREQUENCY-DOMAIN DESCRIPTIONS

The properties of the Fourier transform and the various examples used to illustrate them clearly show that the time-domain and frequency-domain descriptions of a signal are *inversely* related. In particular, we may make the following statements:

1. If the time-domain description of a signal is changed, the frequency-domain description of the signal is changed in an *inverse* manner, and

vice versa. This inverse relationship prevents arbitrary specifications of a signal in both domains. In other words, *we may specify an arbitrary function of time or an arbitrary spectrum, but we cannot specify both of them together.*

2. If a signal is strictly limited in frequency, the time-domain description of the signal will trail on indefinitely, even though its amplitude may assume a progressively smaller value. We say a signal is *strictly limited in frequency* or *strictly band-limited* if its Fourier transform is exactly zero outside a finite band of frequencies. The sinc pulse is an example of a strictly band-limited signal, as illustrated in Fig. 2.12. This figure also shows that the sinc pulse is only *asymptotically limited in time*, which confirms the opening statement we made for a strictly band-limited signal. In an inverse manner, if a signal is *strictly limited in time* (i.e., the signal is exactly zero outside a finite time interval), then the spectrum of the signal is infinite in extent, even though the amplitude spectrum may assume a progressively smaller value. This behavior is exemplified by both the rectangular pulse (described in Figs. 2.5 and 2.6) and the triangular pulse (described in Figs. 2.16*b* and 2.17). Accordingly, we may state that *a signal cannot be strictly limited in both time and frequency.*

BANDWIDTH

The *bandwidth* of a signal provides a measure of the *extent of significant spectral content of the signal for positive frequencies*. When the signal is strictly band-limited, the bandwidth is well defined. For example, the sinc pulse described in Fig. 2.12 has a bandwidth equal to W. When, however, the signal is not strictly band-limited, which is generally the case, we encounter difficulty in defining the bandwidth of the signal. The difficulty arises because the meaning of "significant" attached to the spectral content of the signal is mathematically imprecise. Consequently, there is no universally accepted definition of bandwidth.

Nevertheless, there are some commonly used definitions for bandwidth. In this section, we consider two such definitions;[5] the formulation of each definition depends on whether the signal is low-pass or band-pass. A signal is said to be *low-pass* if its significant spectral content is centered around the origin. A signal is said to be *band-pass* if its significant spectral content is centered around $\pm f_c$, where f_c is a nonzero frequency.

When the spectrum of a signal is symmetric with a *main lobe* bounded by well-defined *nulls* (i.e., frequencies at which the spectrum is zero), we may use the main lobe as the basis for defining the bandwidth of the signal. Specifically, if the signal is low-pass, the bandwidth is defined as one half

[5]Another definition for the bandwidth of a signal is presented in Section 4.8.

the total width of the main spectral lobe, since only one half of this lobe lies inside the positive frequency region. For example, a rectangular pulse of duration T seconds has a main spectral lobe of total width $2/T$ hertz centered at the origin, as depicted in Fig. 2.6*a*. Accordingly, we may define the bandwidth of this rectangular pulse as $1/T$ hertz. If, on the other hand, the signal is band pass with main spectral lobes centered around $\pm f_c$, where f_c is large, the bandwidth is defined as the width of the main lobe for positive frequencies. This definition of bandwidth is called the *null-to-null bandwidth*. For example, an RF pulse of duration T seconds and frequency f_c has main spectral lobes of width $2/T$ hertz centered around $\pm f_c$, as depicted in Fig. 2.14*b*. Hence, we may define the null-to-null bandwidth of this RF pulse as $2/T$ hertz.

Another popular definition of bandwidth is the *3-dB bandwidth*.[6] Specifically, if the signal is low-pass, the 3-dB bandwidth is defined as the separation between zero frequency, where the amplitude spectrum attains its peak value, and the positive frequency at which the amplitude spectrum drops to $1/\sqrt{2}$ of its peak value. For example, the decaying exponential and rising exponential pulses defined in Fig. 2.7 have a 3-dB bandwidth of $1/2\pi$ hertz. If, on the other hand, the signal is band pass, centered at $\pm f_c$, the 3-dB bandwidth is defined as the separation (along the positive frequency axis) between the two frequencies at which the amplitude spectrum of the signal drops to $1/\sqrt{2}$ of the peak value at f_c. The 3-dB bandwidth has the advantage in that it can be read directly from a plot of the amplitude spectrum. However, it has the disadvantage in that it may be misleading if the amplitude spectrum has slowly decreasing tails.

EXERCISE 11 Using the idea of a main spectral lobe, what is the bandwidth of a triangular pulse defined in Figs. 2.16*b* and 2.17?

EXERCISE 12 What is the 3-dB bandwidth of the decaying exponential pulse $\exp(-at)$ that is zero for negative time?

TIME-BANDWIDTH PRODUCT

For any family of pulse signals that differ by a time-scaling factor, the product of the signal's duration and its bandwidth is always a constant, as shown by

$$(\text{duration}) \cdot (\text{bandwidth}) = \text{constant}$$

[6] For a discussion of the decibel (dB), see Appendix A.

The product is called the *time–bandwidth product* or *bandwidth–duration product*. The constancy of the time–bandwidth product is another manifestation of the inverse relationship that exists between the time-domain and frequency-domain descriptions of a signal. In particular, if the duration of a pulse signal is decreased by reducing the time scale by a factor a, the frequency scale of the signal's spectrum, and therefore the bandwidth of the signal, is increased by the same factor a, by virtue of Property 2, and the time–bandwidth product of the signal is thereby maintained constant. For example, a rectangular pulse of duration T seconds has a bandwidth (defined on the basis of the positive-frequency part of the main lobe) equal to $1/T$ hertz, making the time–bandwidth product of the pulse equal unity. Whatever definition we use for the bandwidth of a signal, the time–bandwidth product remains constant over certain classes of pulse signals. The choice of a particular definition for bandwidth simply changes the value of the constant.

2.5 DIRAC DELTA FUNCTION

Strictly speaking, the theory of the Fourier transform, as described in Sections 2.2 and 2.3, is applicable only to time functions that satisfy the Dirichlet conditions. Such functions include energy signals. However, it would be highly desirable to extend this theory in two ways:

1. To combine the Fourier series and Fourier transform into a unified theory, so that the Fourier series may be treated as a special case of the Fourier transform.
2. To include power signals in the list of signals to which we may apply the Fourier transform.

It turns out that both these objectives can be met through the "proper use" of the *Dirac delta function* or *unit impulse*.

The Dirac delta function belongs to a special class of functions known as *generalized distributions* that are defined by the use of assignment rules given in Eqs. 2.87 and 2.88. In particular, the Dirac delta function,[7] denoted by $\delta(t)$, is defined as having zero amplitude everywhere except at $t = 0$, where it is infinitely large in such a way that it contains unit area under its curve, as shown by the pair of rules:

$$\delta(t) = 0, \qquad t \neq 0 \tag{2.87}$$

[7]For a detailed treatment of the delta function, see Bracewell (1978) or Lighthill (1959). The notation $\delta(t)$, which was first introduced into quantum mechanics by Dirac, is now in general use; see Dirac (1947).

and

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{2.88}$$

It is important to realize that no function in the ordinary sense can satisfy the two rules of Eqs. 2.87 and 2.88. However, we can imagine a sequence of functions that have progressively taller and thinner peaks at $t = 0$, with the area under the curve remaining equal to unity, whereas the value of the function tends to zero at every point, except at $t = 0$ where it tends to infinity. That is, we may view the delta function as the limiting form of a *unit-area pulse as the pulse duration approaches zero.* It is immaterial what sort of pulse shape is used. For example, we may use a rectangular pulse of unit area, and thus write

$$\delta(t) = \lim_{\tau \to 0} \frac{1}{\tau} \operatorname{rect}\left(\frac{t}{\tau}\right) \tag{2.89}$$

The rectangular pulse is plotted in Fig. 2.20*a* for $\tau = 5, 1, 0.2$. For another example, we may use a Gaussian pulse of unit area and thus write

$$\delta(t) = \lim_{\tau \to 0} \frac{1}{\tau} \exp\left(-\frac{\pi t^2}{\tau^2}\right) \tag{2.90}$$

The Gaussian pulse is plotted in Fig. 2.20*b* for $\tau = 5, 1, 0.2$. From Fig. 2.20, we clearly see that both pulses take on an impulse-like appearance as the parameter τ becomes progressively smaller. Some other examples are considered in Problem 18.

EXERCISE 13 Plot the spectra for the rectangular and Gaussian pulses for the different values of parameter τ given in Fig. 2.20.

PROPERTIES OF THE DELTA FUNCTION

The delta function $\delta(t)$ has several useful properties that are consequences of the two rules defining it, namely, Eqs. 2.87 and 2.88. These properties are discussed here:

1. The delta function is an even function of time; that is,

$$\delta(t) = \delta(-t) \tag{2.91}$$

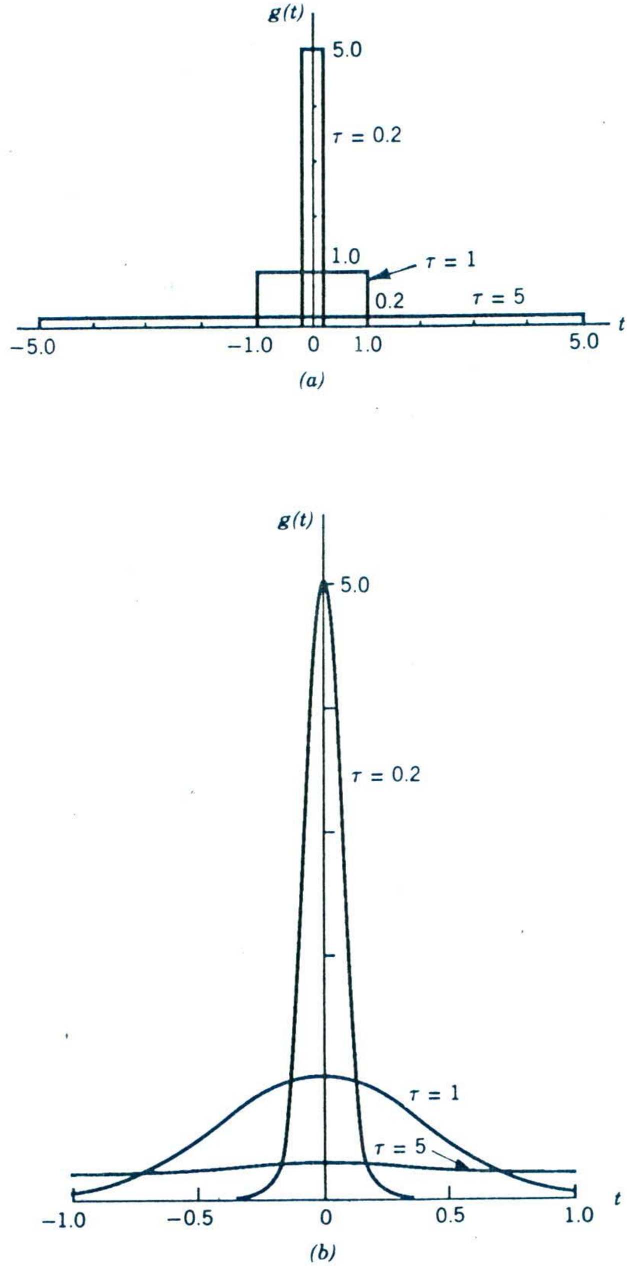


Figure 2.20
(a) Rectangular pulse $g(t) = 1/\tau \, \mathrm{rect}(t/\tau)$ for varying τ. (b) Gaussian pulse $g(t) = 1/\tau \exp(-\pi t^2/\tau^2)$ for varying τ.

2. The integral of the product of $\delta(t)$ and any time function $g(t)$ that is continuous at $t = 0$ is equal to $g(0)$; thus

$$\int_{-\infty}^{\infty} g(t)\,\delta(t)\,dt = g(0) \tag{2.92}$$

We refer to this statement as the *sifting property* of the delta function, since the operation on $g(t)$ indicated on the left side of Eq. 2.92 sifts out a single value of $g(t)$, namely, $g(0)$. Equation 2.92 may also be used as the defining rule for a delta function.

3. The sifting property of the delta function may be generalized by writing

$$\int_{-\infty}^{\infty} g(t)\,\delta(t - t_0)\,dt = g(t_0) \tag{2.93}$$

Since the delta function $\delta(t)$ is an even function of t, we may rewrite Eq. 2.93 in a way emphasizing resemblance to the convolution integral, as follows:

$$\int_{-\infty}^{\infty} g(\tau)\,\delta(t - \tau)\,d\tau = g(t) \tag{2.94}$$

or

$$g(t) \star \delta(t) = g(t) \tag{2.95}$$

That is, the convolution of any function with the delta function leaves that function unchanged. We refer to this statement as the *replication property* of the delta function.

4. The Fourier transform of the delta function is given by

$$\mathrm{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) \exp(-j2\pi ft)\,dt$$

Using the sifting property of the delta function and noting that the exponential function $\exp(-j2\pi ft)$ is equal to unity at $t = 0$, we obtain

$$\mathrm{F}[\delta(t)] = 1$$

We thus have the Fourier transform pair:

$$\delta(t) \rightleftharpoons 1 \tag{2.96}$$

This relation states that the spectrum of the delta function $\delta(t)$ extends uniformly over the entire frequency interval from $-\infty$ to ∞, as shown in Fig. 2.21.

APPLICATIONS OF THE DELTA FUNCTION

dc Signal By applying the duality property to the Fourier transform pair of Eq. 2.96, and noting that the delta function is an even function, we

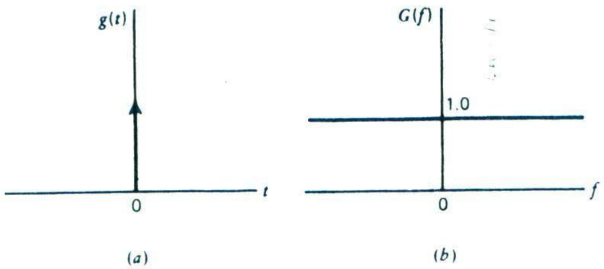


Figure 2.21
(a) Dirac delta function (b) Spectrum.

obtain

$$1 \rightleftharpoons \delta(f) \tag{2.97}$$

Equation 2.97 states that a *dc signal* is transformed in the frequency domain into a delta function $\delta(f)$ occurring at zero frequency, as shown in Fig. 2.22. Of course, this result is intuitively satisfying. From Eq. 2.97 we also deduce the useful relation

$$\int_{-\infty}^{\infty} \exp(-j2\pi ft)\, dt = \delta(f) \tag{2.98}$$

where the integral on the left side is simply the Fourier transform of a function equal to one for all time t.

Complex Exponential Function Next, by applying the frequency-shifting property to Eq. 2.97, we obtain the Fourier transform pair

$$\exp(j2\pi f_c t) \rightleftharpoons \delta(f - f_c) \tag{2.99}$$

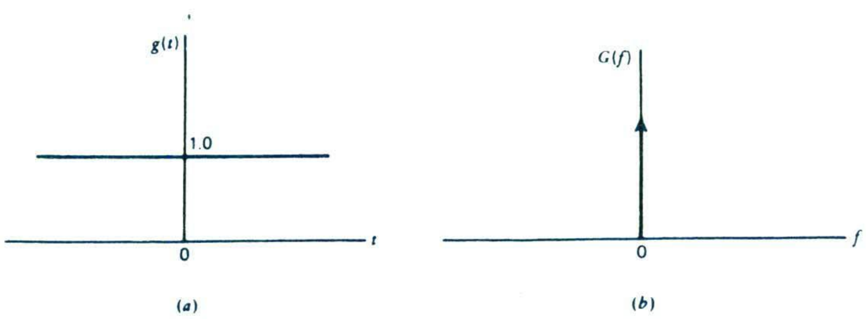


Figure 2.22
(a) dc signal. (b) Spectrum.

for a complex exponential function of frequency f_c. Equation 2.99 states that the complex exponential function $\exp(j2\pi f_c t)$ is transformed in the frequency domain into a delta function $\delta(f - f_c)$ centered at $f = f_c$.

Sinusoidal Functions Consider next the problem of evaluating the Fourier transform of the *cosine function* $\cos(2\pi f_c t)$. We first note that

$$\cos(2\pi f_c t) = \frac{1}{2}[\exp(j2\pi f_c t) + \exp(-j2\pi f_c t)]$$

Therefore, using Eq. 2.99, we find that the cosine function $\cos(2\pi f_c t)$ is represented by the Fourier transform pair

$$\cos(2\pi f_c t) \rightleftharpoons \frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)] \tag{2.100}$$

In other words, the spectrum of the cosine function $\cos(2\pi f_c t)$ consists of a pair of delta functions centered at $f = \pm f_c$, each of which is weighted by the factor ½, as shown in Fig. 2.23.

Similarly, we may show that the *sine function* $\sin(2\pi f_c t)$ is represented by the Fourier transform pair

$$\sin(2\pi f_c t) \rightleftharpoons \frac{1}{2j}[\delta(f - f_c) - \delta(f + f_c)] \tag{2.101}$$

which is illustrated in Fig. 2.24.

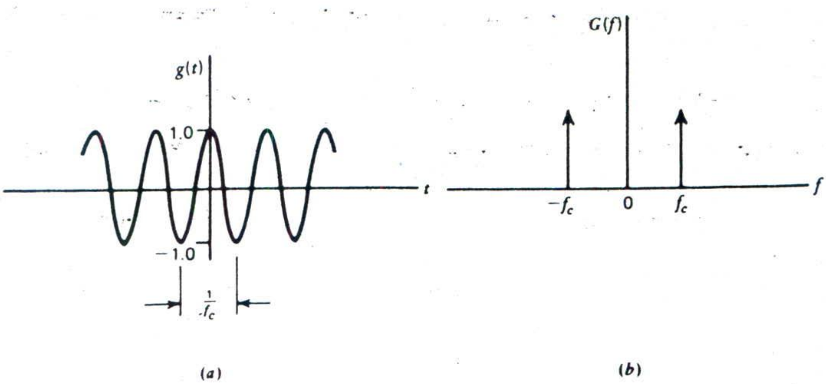


Figure 2.23
(a) Cosine function. (b) Spectrum.

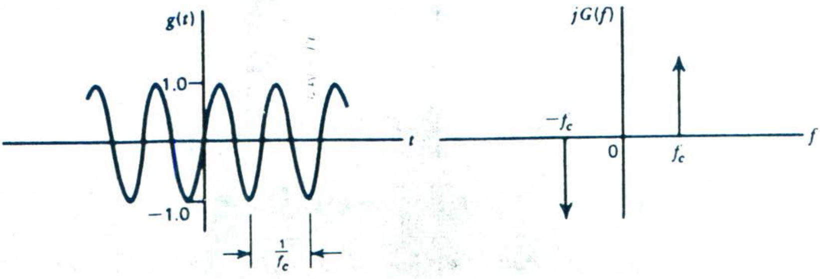


Figure 2.24
(a) Sine function. (b) Spectrum.

Signum Function The *signum function,* denoted by sgn(t), is an odd function of time defined as follows:

$$\text{sgn}(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases} \tag{2.102}$$

The waveform of the signum function is shown in Fig. 2.25*a*. We may view the signum function as the limiting form of a time function that consists of a positive decaying exponential for positive time and a negative rising exponential for negative time. That is, we write

$$\text{sgn}(t) = \lim_{a \to 0} g(a, t) \tag{2.103}$$

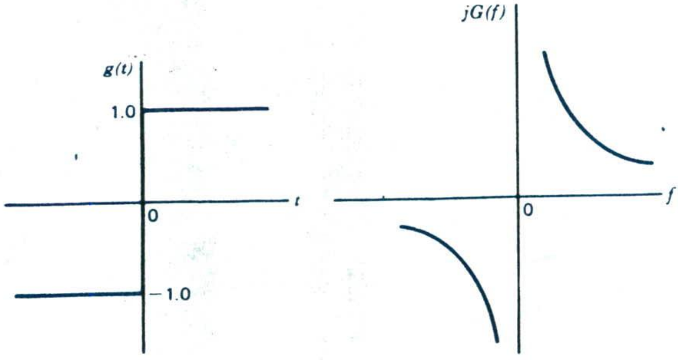


Figure 2.25
(a) Signum function. (b) Spectrum.

where

$$g(a, t) = \begin{cases} \exp(-at), & t > 0 \\ 0, & t = 0 \\ -\exp(at), & t < 0 \end{cases} \quad (2.104)$$

We may also express $g(a, t)$ in the compact form

$$g(a, t) = \exp(-at)u(t) - \exp(at)u(-t) \quad (2.105)$$

where $u(t)$ is the unit step function. The function $g(a, t)$ is plotted in Fig. 2.26 for the parameter $a = 1, 0.5, 0.1$. We clearly see that as the value of parameter a is progressively reduced, the function $g(a, t)$ becomes closer to the signum function in appearance. Applying the time-scaling property to the Fourier transform pairs of Eqs. 2.37 and 2.40, we get

$$\exp(-at)u(t) \rightleftharpoons \frac{1/a}{1 + (j2\pi f/a)}$$

and

$$\exp(at)u(-t) \rightleftharpoons \frac{1/a}{1 - (j2\pi f/a)}$$

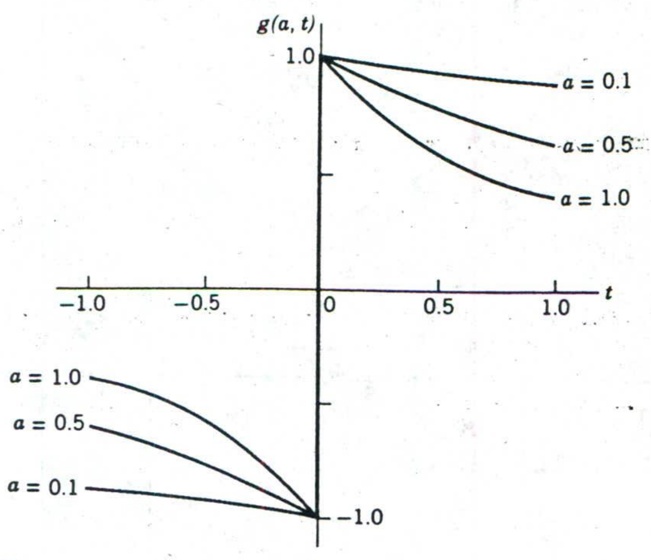


Figure 2.26
The function g(a, t) for varying a.

Subtracting the second Fourier transform pair from the first one, and then using the definition of Eq. 2.105, we get (after combining terms and simplifying):

$$g(a, t) \rightleftharpoons \frac{4\pi f}{j(a^2 + 4\pi^2 f^2)} \tag{2.106}$$

For the limiting condition when the parameter a approaches zero, the function $g(a, t)$ approaches the signum function, in accordance with Eq. 2.103. Therefore, putting $a = 0$ in Eq. 2.106, we obtain the desired Fourier transform pair for the signum function:

$$\text{sgn}(t) \rightleftharpoons \frac{1}{j\pi f} \tag{2.107}$$

The spectrum of the signum function is plotted in Fig. 2.25*b*.

Another useful Fourier transform pair, involving a signum function defined in the frequency domain, is obtained by applying Property 3 (duality) to Eq. 2.107. We thus obtain the following result:

$$\frac{1}{\pi t} \rightleftharpoons j\,\text{sgn}(f) \tag{2.108}$$

where the signum function $\text{sgn}(f)$ is defined by

$$\text{sgn}(f) = \begin{cases} 1, & f > 0 \\ 0, & f = 0 \\ -1, & f < 0 \end{cases}$$

EXERCISE 14 **Plot the spectrum of the function $g(a, t)$ for parameter $a = 1, 0.5, 0.1$, and compare your results with the spectrum of the signum function shown in Fig. 2.25*b*.**

Unit Step Function The *unit step function*, $u(t)$, is defined in Eq. 2.34, reproduced here for convenience:

$$u(t) = \begin{cases} 1, & t > 0 \\ 1/2, & t = 0 \\ 0, & t < 0 \end{cases} \tag{2.109}$$

The waveform of the unit step function is shown in Fig. 2.27*a*. From Eqs. 2.102 and 2.109, or from the corresponding waveforms shown in Figs. 2.25*a*

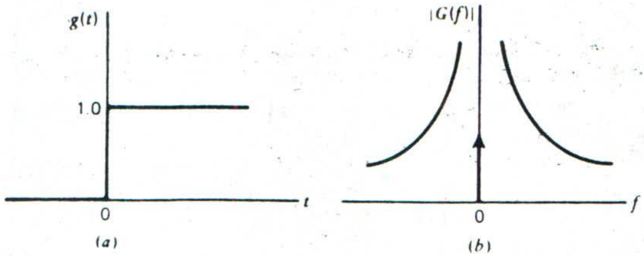


Figure 2.27
(a) Unit step function. (b) Amplitude spectrum.

and 2.27*a*, we see that the unit step function and signum function are related by

$$u(t) = \frac{1}{2}[\text{sgn}(t) + 1] \tag{2.110}$$

Hence, using the linearity property of the Fourier transform and the Fourier transform pairs of Eqs. 2.97, and 2.107, we find that the unit step function is represented by the Fourier transform pair

$$u(t) \rightleftharpoons \frac{1}{j2\pi f} + \frac{1}{2}\delta(f) \tag{2.111}$$

This means that the spectrum of the unit step function contains a delta function weighted by a factor of 1/2 and occurring at zero frequency, as shown in Fig. 2.27*b*.

EXERCISE 15 Using the frequency-shifting property, determine the Fourier transform of the signal

$$g(t) = u(t)\cos(2\pi f_c t)$$

where $u(t)$ is the unit step function.

Integration in the Time Domain (Revisited) The relation of Eq. 2.68 describes the effect of integration on the Fourier transform of a signal $g(t)$, assuming that $G(0)$ is zero. We now consider the more general case, with no such assumption made.

Let

$$y(t) = \int_{-\infty}^{t} g(\tau)\, d\tau \tag{2.112}$$

The integrated signal $y(t)$ can be viewed as the convolution of the original signal $g(t)$ and the unit step function $u(t)$, as shown by

$$y(t) = \int_{-\infty}^{\infty} g(\tau)u(t - \tau)\, d\tau \tag{2.113}$$

where the time-shifted unit step function $u(t - \tau)$ is defined by

$$u(t - \tau) = \begin{cases} 1, & \tau < t \\ \frac{1}{2}, & \tau = t \\ 0, & \tau > t \end{cases} \tag{2.114}$$

Recognizing that convolution in the time domain is transformed into multiplication in the frequency domain, and using the Fourier transform pair of Eq. 2.111 for the unit step function $u(t)$, we find that the Fourier transform of $y(t)$ is

$$Y(f) = G(f)\left[\frac{1}{j2\pi f} + \frac{1}{2}\,\delta(f)\right] \tag{2.115}$$

where $G(f)$ is the Fourier transform of $g(t)$. Since

$$G(f)\,\delta(f) = G(0)\,\delta(f)$$

we may rewrite Eq. 2.115 in the equivalent form:

$$Y(f) = \frac{1}{j2\pi f}\,G(f) + \frac{1}{2}\,G(0)\,\delta(f) \tag{2.116}$$

That is, the effect of integrating the signal $g(t)$ is described by the Fourier transform pair:

$$\int_{-\infty}^{t} g(\tau)\, d\tau \rightleftharpoons \frac{1}{j2\pi f}\,G(f) + \frac{1}{2}\,G(0)\,\delta(f) \tag{2.117}$$

which is the desired result.

This proof is indirect in that it relies on knowledge of the Fourier transform of the unit step function. For a direct proof from first principles, refer to Problem 20.

2.6 FOURIER TRANSFORMS OF PERIODIC SIGNALS

From Section 2.1 we recall that by using the Fourier series, a periodic signal $g_p(t)$ can be represented as a sum of complex exponentials. Also we know that, in a limiting sense, we can define Fourier transforms of complex exponentials. Therefore, it seems reasonable that a periodic signal can be represented in terms of a Fourier transform, provided that this transform is permitted to include delta functions.

Consider a periodic signal $g_p(t)$ of period T_0. We can represent $g_p(t)$ in terms of the complex exponential Fourier series as in Eq. 2.10, which is reproduced here for convenience,

$$g_p(t) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{j2\pi nt}{T_0}\right) \tag{2.118}$$

where c_n is the complex Fourier coefficient defined by

$$c_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g_p(t) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt \tag{2.119}$$

Let $g(t)$ be a pulse-like function, which equals $g_p(t)$ over one period and is zero elsewhere; that is,

$$g(t) = \begin{cases} g_p(t), & -\dfrac{T_0}{2} \leqslant t \leqslant \dfrac{T_0}{2} \\ 0, & \text{elsewhere} \end{cases} \tag{2.120}$$

The periodic signal $g_p(t)$ may be expressed in terms of the function $g(t)$ as an infinite summation, as shown by

$$g_p(t) = \sum_{m=-\infty}^{\infty} g(t - mT_0) \tag{2.121}$$

Based on this representation, we may view $g(t)$ as a *generating function,* which generates the periodic signal $g_p(t)$.

The function $g(t)$ is Fourier transformable. Accordingly, we may rewrite Eq. 2.119 as follows:

$$\begin{aligned} c_n &= \frac{1}{T_0}\int_{-\infty}^{\infty} g(t) \exp\left(-\frac{j2\pi nt}{T_0}\right) dt \\ &= \frac{1}{T_0} G\left(\frac{n}{T_0}\right) \end{aligned} \tag{2.122}$$

where $G(n/T_0)$ is the Fourier transform of $g(t)$, evaluated at the frequency n/T_0. We may thus rewrite Eq. 2.118 as

$$g_p(t) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} G\left(\frac{n}{T_0}\right) \exp\left(\frac{j2\pi nt}{T_0}\right) \tag{2.123}$$

or, equivalently,

$$\sum_{m=-\infty}^{\infty} g(t - mT_0) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} G\left(\frac{n}{T_0}\right) \exp\left(\frac{j2\pi nt}{T_0}\right) \tag{2.124}$$

Equation 2.124 is one form of *Poisson's sum formula.*

Finally, using Eq. 2.99, which defines the Fourier transform of a complex exponential function, and Eq. 2.124, we deduce the following Fourier transform pair for a periodic signal $g_p(t)$ with a generating function $g(t)$ and period T_0:

$$g_p(t) = \sum_{m=-\infty}^{\infty} g(t - mT_0) \rightleftharpoons \frac{1}{T_0} \sum_{n=-\infty}^{\infty} G\left(\frac{n}{T_0}\right) \delta\left(f - \frac{n}{T_0}\right) \tag{2.125}$$

This relation simply states that the Fourier transform of a periodic signal consists of delta functions occurring at integer multiples of the fundamental frequency $1/T_0$, including the origin, and that each delta function is weighted by a factor $G(n/T_0)$. Indeed, this relation merely provides an alternate way of displaying the frequency content of a periodic signal $g_p(t)$.

It is of interest to observe that the function $g(t)$, constituting one period of the periodic signal $g_p(t)$, has a continuous spectrum defined by $G(f)$. On the other hand, the periodic signal $g_p(t)$ itself has a discrete spectrum. We conclude, therefore, that *periodicity in the time domain has the effect of making the spectrum of the signal take on a discrete form, where the separation between adjacent spectral lines equals the reciprocal of the period.*

EXAMPLE 14 IDEAL SAMPLING FUNCTION

An *ideal sampling function,* or *Dirac comb,* consists of an infinite sequence of uniformly spaced delta functions, as shown in Fig. 2.28*a*. We will denote this waveform by

$$\delta_{T_0}(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT_0) \tag{2.126}$$

We observe that the generating function $g(t)$ for the ideal sampling function $\delta_{T_0}(t)$ consists simply of the delta function $\delta(t)$. Therefore, $G(f) = 1$, so that

$$G\left(\frac{n}{T_0}\right) = 1, \qquad \text{for all } n \tag{2.127}$$

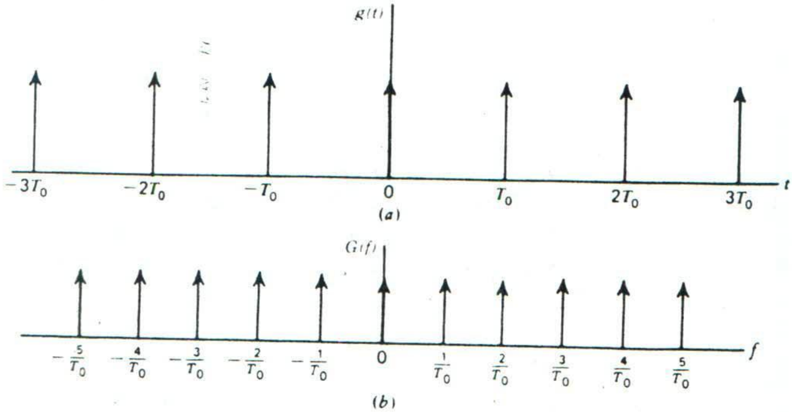


Figure 2.28
(a) Dirac comb. (b) Spectrum.

Thus the use of Eq. 2.125 yields the result

$$\sum_{m=-\infty}^{\infty} \delta(t - mT_0) \rightleftharpoons \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_0}\right) \tag{2.128}$$

Equation 2.128 states that the Fourier transform of a periodic train of delta functions in the time domain consists of another periodic train of delta functions in the frequency domain as in Fig. 2.28*b*. In the special case of the period T_0 equal to 1 second, a periodic train of delta functions is, like a Gaussian pulse, its own Fourier transform.

We also deduce from Poisson's sum formula, Eq. 2.124, the following useful relation

$$\sum_{m=-\infty}^{\infty} \delta(t - mT_0) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \exp\left(\frac{j2\pi nt}{T_0}\right)$$

The dual of this relation is

$$\sum_{m=-\infty}^{\infty} \exp(j2\pi mfT_0) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_0}\right) \tag{2.129}$$

2.7 SAMPLING THEOREM

An operation that is basic to digital signal processing and digital communications is the *sampling process*, whereby an analog signal is converted into a corresponding sequence of samples that are usually spaced uniformly in time. For such a procedure to have practical utility, it is necessary that we choose the sampling rate properly, so that the sequence of samples uniquely defines the original analog signal. This is the essence of the sampling theorem, which is derived in the sequel.

Consider the arbitrary signal $g(t)$ of finite energy, which is specified for all time. A segment of the signal $g(t)$ is shown in Fig. 2.29*a*. Suppose that we sample the signal $g(t)$ instantaneously and at a uniform rate, once every T_s seconds. Consequently, we obtain an infinite sequence of samples spaced T_s seconds apart and denoted by $\{g(nT_s)\}$ where n takes on all possible integer values. We refer to T_s as the *sampling period,* and to its reciprocal $f_s = 1/T_s$ as the *sampling rate*. This ideal form of sampling is called *instantaneous sampling*.

Let $g_\delta(t)$ denote the signal obtained by individually weighting the elements of a periodic sequence of delta functions spaced T_s seconds apart

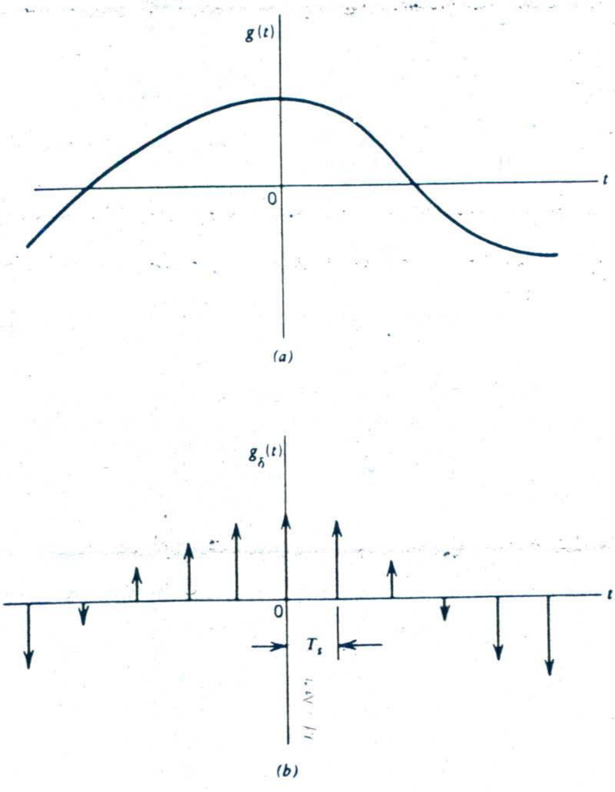


Figure 2.29
The sampling process. (a) Analog signal. (b) Instantaneously sampled version of the signal.

by the sequence of numbers $\{g(nT_s)\}$, as shown by (see Fig. 2.29*b*)

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\,\delta(t - nT_s) \tag{2.130}$$

We refer to $g_\delta(t)$ as the *ideal sampled signal.* The ideal sampled signal $g_\delta(t)$ has a mathematical form similar to that of the Fourier transform of a periodic signal. This is readily established by comparing Eq. 2.130 for $g_\delta(t)$ with the Fourier transform of a periodic signal given in Eq. 2.125. This correspondence suggests that we may determine the Fourier transform of the ideal sampled signal $g_\delta(t)$ by applying the duality property to the Fourier transform of Eq. 2.125. By so doing, and using the fact that a delta function is an even function, we get the desired result:

$$g_\delta(t) \rightleftharpoons f_s \sum_{m=-\infty}^{\infty} G(f - mf_s) \tag{2.131}$$

where $G(f)$ is the Fourier transform of the original signal $g(t)$, and f_s is the sampling rate. Equation 2.131 states that *the process of uniformly sampling a continuous-time signal of finite energy results in a periodic spectrum with a period equal to the sampling rate.*

Another useful expression for the Fourier transform of the ideal sampled signal $g_\delta(t)$ may be obtained by taking the Fourier transform of both sides of Eq. 2.130 and noting that the Fourier transform of the delta function $\delta(t - nT_s)$ is equal to $\exp(-j2\pi nfT_s)$. Let $G_\delta(f)$ denote the Fourier transform of $g_\delta(t)$. We may therefore write

$$G_\delta(f) = \sum_{n=-\infty}^{\infty} g(nT_s)\exp(-j2\pi nfT_s) \tag{2.132}$$

This relation is called the *discrete-time Fourier transform.* It may be viewed as a complex Fourier series representation of the periodic frequency function $G_\delta(f)$, with the sequence of samples $\{g(nT_s)\}$ defining the coefficients of the expansion.

The relations, as derived here, apply to any continuous-time signal $g(t)$ of finite energy and infinite duration. Suppose, however, that the signal is strictly band-limited, with no frequency components higher than W hertz. That is, the Fourier transform $G(f)$ of the signal $g(t)$ has the property that $G(f)$ is zero for $|f| \geq W$, as illustrated in Fig. 2.30*a*; the shape of the spectrum shown in this figure is intended for the purpose of illustration only. Suppose also that we choose the sampling period $T_s = 1/2W$. Then the corresponding spectrum $G_\delta(f)$ of the sampled signal $g_\delta(t)$ is as shown in Fig. 2.30*b*. Putting $T_s = 1/2W$ in Eq. 2.132 yields

$$G_\delta(f) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right)\exp\left(-\frac{j\pi nf}{W}\right) \tag{2.133}$$

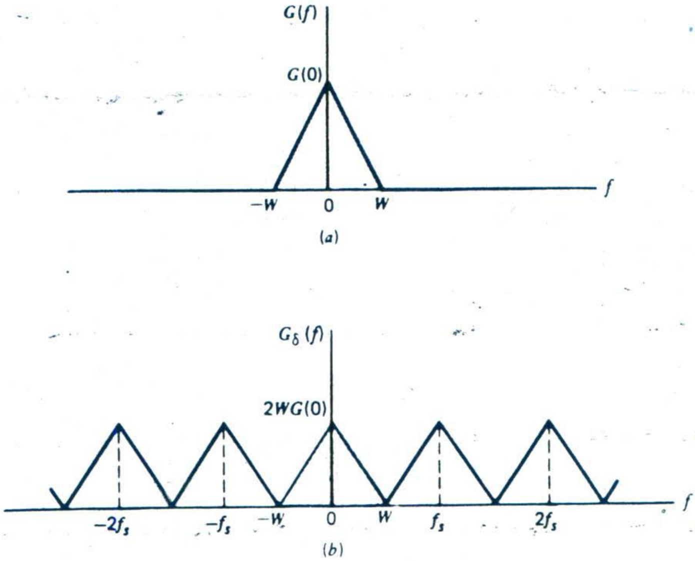


Figure 2.30
(a) Spectrum of a strictly band-limited signal g(t). (b) Spectrum of sampled version of g(t) for a sampling period $T_s = 1/2W$.

From Eq. 2.131, we have

$$G_\delta(f) = f_s G(f) + f_s \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} G(f - mf_s) \tag{2.134}$$

Hence, under the following two conditions:

1. $G(f) = 0$ for $|f| \geq W$
2. $f_s = 2W$

we find from Eq. 2.134 that

$$G(f) = \frac{1}{2W} G_\delta(f), \qquad -W < f < W \tag{2.135}$$

Substituting Eq. 2.133 in Eq. 2.135, we may also write

$$G(f) = \frac{1}{2W} \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \exp\left(-\frac{j\pi nf}{W}\right), \qquad -W < f < W \tag{2.136}$$

Therefore, if the sample values $g(n/2W)$ of a signal $g(t)$ are specified for all time, then the Fourier transform $G(f)$ of the signal is uniquely determined by using the discrete-time Fourier transform of Eq. 2.136. Because $g(t)$ is related to $G(f)$ by the inverse Fourier transform, it follows that the signal $g(t)$ is itself uniquely determined by the sample values $g(n/2W)$ for $-\infty < n < \infty$. In other words, the sequence $\{g(n/2W)\}$ has all the information contained in $g(t)$.

Consider next the problem of reconstructing the signal $g(t)$ from the sequence of sample values $\{g(n/2W)\}$. Substituting Eq. 2.136 in the formula for the inverse Fourier transform defining $g(t)$ in terms of $G(f)$, we get

$$\begin{aligned} g(t) &= \int_{-\infty}^{\infty} G(f) \exp(j2\pi ft)\, df \\ &= \int_{-W}^{W} \frac{1}{2W} \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \exp\left(-\frac{j\pi nf}{W}\right) \exp(j2\pi ft)\, df \end{aligned}$$

Interchanging the order of summation and integration:

$$g(t) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \frac{1}{2W} \int_{-W}^{W} \exp\left[j2\pi f\left(t - \frac{n}{2W}\right)\right] df \qquad (2.137)$$

The integral term in Eq. 2.137 may be readily evaluated yielding

$$\begin{aligned} g(t) &= \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \frac{\sin(2\pi Wt - n\pi)}{(2\pi Wt - n\pi)} \\ &= \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \text{sinc}(2Wt - n), \qquad -\infty < t < \infty \qquad (2.138) \end{aligned}$$

Equation 2.138 provides an *interpolation formula* for reconstructing the original signal $g(t)$ from the sequence of sample values $\{g(n/2W)\}$, with the sinc function $\text{sinc}(2Wt)$ playing the role of an *interpolation function*. Each sample is multiplied by a delayed version of the interpolation function, and all the resulting waveforms are added to obtain $g(t)$.

We may now state the *sampling theorem*[8] for band-limited signals of finite energy in two equivalent parts:

1. *A band-limited signal of finite energy, which has no frequency components higher than W hertz, is completely described by specifying the values of the signal at instants of time separated by 1/2W seconds.*

[8]The sampling theorem was introduced to communication theory by Shannon (1949). It is for this reason that the theorem is sometimes referred to in the literature as the "Shannon sampling theorem." However, the interest of communication engineers in the sampling theorem may be traced back to Nyquist (1928). Indeed, the sampling theorem was known to mathematicians at least since 1915. For historical notes on the sampling theorem, see the review paper by Jerri (1977).

2. *A band-limited signal of finite energy, which has no frequency components higher than W hertz, may be completely recovered from a knowledge of its samples taken at the rate of 2W samples per second.*

The sampling rate of $2W$ samples per second, for a signal bandwidth of W hertz, is called the *Nyquist rate;* its reciprocal $1/2W$ (measured in seconds) is called the *Nyquist interval.* The sampling theorem serves as the basis for the interchangeability of analog signals and digital sequences, which is so valuable in digital signal processing and digital communications.

The derivation of the sampling theorem, as described herein, is based on the assumption that the signal $g(t)$ is strictly band-limited. In practice, however, an information-bearing signal is not strictly band-limited. Hence, distortion may result from the application of the sampling theorem to such a signal. (More will be said on this issue in Chapter 5.)

EXERCISE 16 Apply the duality property to the Fourier transform pair of Eq 2.125 and thereby derive Eq. 2.131 for the ideal sampled signal $g_\delta(t)$.

2.8 NUMERICAL COMPUTATION OF THE FOURIER TRANSFORM

This section briefly describes a procedure for the computation of the Fourier transform, which is particularly well suited for use on a digital computer. We assume that the given signal $g(t)$ is of finite duration. The procedure involves first, the *uniform sampling* of $g(t)$ to obtain a finite sequence of samples denoted by $g(0), g(T_s), g(2T_s), \ldots, g(NT_s - T_s)$, where T_s is the *sampling period* and N is the number of samples. For a correct representation of the signal, the *sampling rate* $1/T_s$ must be equal to or greater than twice the highest frequency component of the signal. For the purpose of our present discussion, it is adequate to assume that this requirement has been satisfied. It is possible, of course, that the signal initially may be in the form of a sequence of samples. In any event, for this sequence of samples, we may define a *discrete Fourier transform* denoted by $\{G(kF_s)\}$, which consists of another sequence of N samples separated in frequency by F_s hertz, as shown by

$$G(kF_s) = T_s \sum_{n=0}^{N-1} g(nT_s) \exp\left(-j\frac{2\pi}{N}kn\right), \qquad k = 0, 1, 2, \ldots, N-1 \tag{2.139}$$

Equation 2.139 is precisely the formula that would be obtained by using the trapezoidal rule for approximating the integral that defines the Fourier transform of the given signal $g(t)$. The difference between the actual Fourier transform and the sequence $\{G(kF_s)\}$ obtained from Eq. 2.139 gives the integration error evaluated at $f = kF_s$. The parameters T_s and F_s are related by

$$T_s F_s = \frac{1}{N} \tag{2.140}$$

To derive the inverse relationship expressing the sequence $\{g(nT_s)\}$ in terms of the discrete spectrum $\{G(kF_s)\}$, we multiply both sides of Eq. 2.139 by $\exp(j2\pi km/N)$ and sum over k, obtaining

$$\sum_{k=0}^{N-1} G(kF_s) \exp\left(j\frac{2\pi}{N}km\right) = T_s \sum_{k=0}^{N-1}\sum_{n=0}^{N-1} g(nT_s) \exp\left[j\frac{2\pi}{N}k(m-n)\right] \tag{2.141}$$

Interchanging the order of summation on the right side of Eq. 2.141, and using the fact that

$$\sum_{k=0}^{N-1} \exp\left[j\frac{2\pi}{N}k(m-n)\right] = \begin{cases} N, & m = n \\ 0, & \text{otherwise} \end{cases} \tag{2.142}$$

we get

$$\sum_{k=0}^{N-1} G(kF_s) \exp\left(j\frac{2\pi}{N}km\right) = NT_s g(mT_s) \tag{2.143}$$

Next, substituting the index n for m and rearranging the terms in Eq. 2.143, we get the desired relation

$$g(nT_s) = F_s \sum_{k=0}^{N-1} G(kF_s) \exp\left(j\frac{2\pi}{N}kn\right), \qquad n = 0, 1, \ldots, N-1 \tag{2.144}$$

which defines the *inverse discrete Fourier transform*. Here again, it is of interest to note that Eq. 2.144 is precisely the formula that would be obtained by using the trapezoidal rule for approximating the integral that defines the inverse Fourier transform.

The discrete Fourier transform, as defined in Eq. 2.139, has properties that are analogous to those of the continuous Fourier transform.

An important feature of the discrete Fourier transform is that the signal $\{g(nT_s)\}$ and its spectrum $\{G(kF_s)\}$ are both in discrete form. Furthermore, they are both periodic, with the period of either one consisting of a finite number of samples N. That is,

$$g(nT_s) = g(nT_s + NT_s) \tag{2.145}$$

and

$$G(kF_s) = G(kF_s + NF_s) \tag{2.146}$$

We thus find that the numerical computation of the discrete Fourier transform is well suited for a digital computer or special-purpose digital processor. Indeed, it is this feature that makes the discrete Fourier transform so eminently useful in practice for spectral analysis and for the simulation of filters on digital computers. This is all the more so by virtue of the availability of an algorithm known as the *fast Fourier transform algorithm* (FFT), which provides a highly efficient procedure for computing the discrete Fourier transform of a finite-duration sequence. This algorithm takes advantage of the fact that the calculation of the coefficients of the discrete Fourier transform may be carried out in an iterative manner, thereby resulting in a considerable saving of computation time.[9] To compute the discrete Fourier transform of a sequence of N samples using the FFT algorithm, we require, in general, $N \log_2 N$ complex additions and $(N/2) \log_2 N$ complex multiplications. On the other hand, by using Eq. 2.139 to compute the discrete Fourier transform directly, we find that for each of the N output samples, we require $(N - 1)$ complex additions and N complex multiplications, so that the direct computation of the discrete Fourier transform requires a total of $N(N - 1)$ complex additions and N^2 complex multiplications. Accordingly, by using the FFT algorithm, the number of arithmetic operations is reduced by a factor of $N/\log_2 N$, which represents a considerable saving in computation effort for large N. For example, with $N = 1024$, we reduce the computation effort by about two orders of magnitude. Indeed, it is this kind of improvement that also makes it possible to use special-purpose digital processors for the hardware implementation of the FFT algorithm.

2.9 *RELATIONSHIP BETWEEN THE FOURIER AND LAPLACE TRANSFORMS*

The Fourier transform (as we have described it) is fully adequate for handling the frequency-domain description of signals encountered in the study of communication theory. Nevertheless, it can be helpful to briefly examine the relation between it and the Laplace transform, which is commonly used in transient analysis.

Consider the special case of a *causal signal* $g(t)$, defined as a signal that is zero for negative time. In other words, the signed $g(t)$ starts at or after $t = 0$. In such a case, the formula for the Fourier transform of $g(t)$ takes

[9] For a description of the FFT algorithm and its applications, see Roberts and Mullis (1987) or Oppenheim and Schafer (1975).

the form

$$G(f) = \int_0^\infty g(t) \exp(-j2\pi ft)\, dt \tag{2.147}$$

This integral bears a close resemblance to the *one-sided Laplace transform* of $g(t)$, as shown by

$$\tilde{G}(s) = \int_0^\infty g(t) \exp(-st)\, dt \tag{2.148}$$

which implies that $g(t) = 0$ for $t < 0$. The quantity

$$s = \sigma + j\omega \tag{2.149}$$

is a *complex variable* whose real and imaginary parts are σ and ω, respectively. Comparing Eqs. 2.147 and 2.148, we see that the Fourier transform $G(f)$ may be obtained from the Laplace transform $\tilde{G}(s)$ by putting

$$s = j\omega = j2\pi f$$

This is the link that connects the Fourier and Laplace transforms.

As mentioned previously, the Fourier transform is adequate for most purposes in communication theory. As such, we will use it exclusively in the rest of the book.

PROBLEMS

The problems are divided into sections that correspond to the major sections in the Chapter. For example, the problems in Section P2.1 pertain to Section 2.1. *This practice is followed in subsequent chapters.*

P2.1 Fourier Series

Problem 1 A signal that is sometimes used in communication systems is a *raised cosine pulse*. Figure P2.1 shows a signal $g_p(t)$ that is a periodic

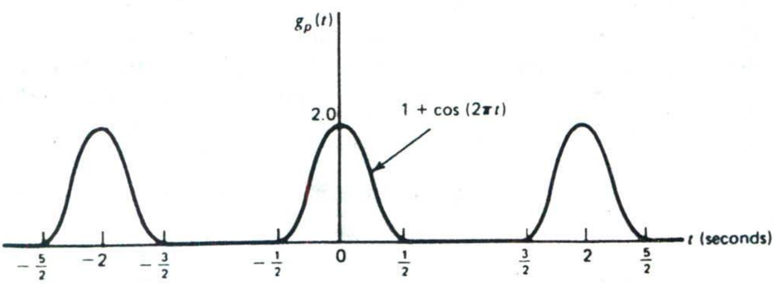


Figure P2.1

sequence of these pulses with equal spacing between them. Show that the first three terms in the Fourier series expansion of $g_p(t)$ are as follows:

$$g_p(t) = \tfrac{1}{2} + \frac{8}{3\pi}\cos(\pi t) + \tfrac{1}{2}\cos(2\pi t) + \cdots$$

Problem 2 Evaluate the amplitude spectrum of the periodic pulsed RF waveform shown in Fig. P2.2, assuming that $f_c T_0 \gg 1$.

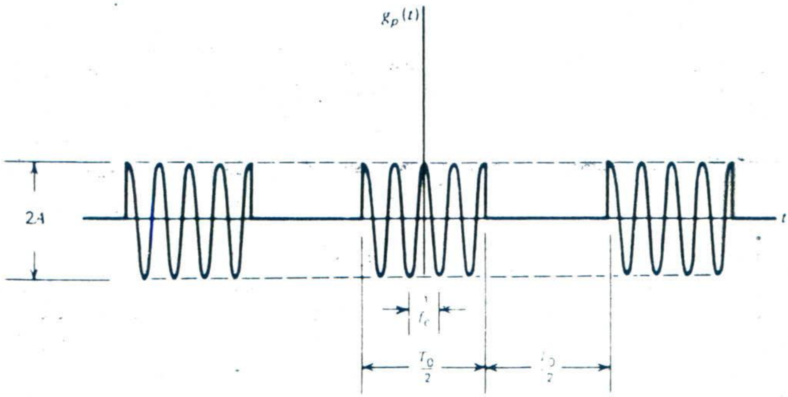


Figure P2.2

Problem 3 Prove the following properties of the Fourier series:

(a) If the periodic function $g_p(t)$ is even, that is,

$$g_p(-t) = g_p(t)$$

then the Fourier coefficients, the c_n, are purely real and even, that is, $c_{-n} = c_n$.

(b) If $g_p(t)$ is odd, that is,

$$g_p(-t) = -g_p(t)$$

then the c_n are purely imaginary and an odd function of n.

(c) If $g_p(t)$ has half-wave symmetry, that is,

$$g_p\left(t \pm \frac{1}{2}T_0\right) = -g_p(t)$$

where T_0 is the period of $g_p(t)$, then the Fourier series of such a signal consists of only odd-order terms.

P2.2 Fourier Transform

Problem 4 Determine the Fourier transform of the signal $g(t)$ consisting of three rectangular pulses, as shown in Fig. P2.3. Sketch the amplitude spectrum of this signal for the case when $T \ll T_0$.

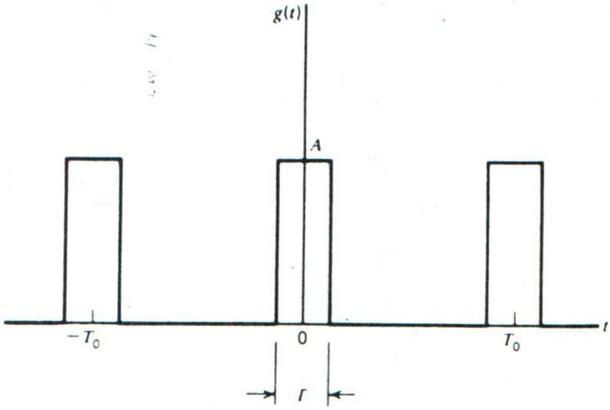


Figure P2.3

Hint: Consider a rectangular pulse of amplitude A and duration T, and use the linearity and time-shifting properties of the Fourier transform.

Problem 5 Determine the inverse Fourier transform of the frequency function $G(f)$ defined by the amplitude and phase spectra shown in Fig. P2.4.

Problem 6 Show that the spectrum of a real symmetric signal is either (a) purely real and even, or (b) purely imaginary and odd.

P2.3 Properties of the Fourier Transform

Problem 7 Let

$$g_1(t) = x\left(\frac{t}{5}\right)$$

$$g_2(t) = x(5t)$$

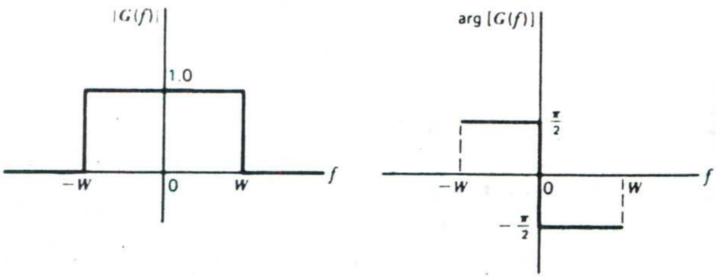


Figure P2.4

(a) Determine the Fourier transforms $G_1(f)$ and $G_2(f)$ in terms of the Fourier transform $X(f)$.
(b) Which of the two time functions, $g_1(t)$ and $g_2(t)$, corresponds to time compression, and which one to time expansion?
(c) Let

$$y(t) = a\, g_1(t)$$

Find the value of scaling factor a required to make $Y(0) = X(0)$, where $Y(f)$ is the Fourier transform of $y(t)$. Repeat your calculation for $g_2(t)$ in place of $g_1(t)$.

Problem 8

(a) Find the Fourier transform of the half-cosine pulse shown in Fig. P2.5*a*.
(b) Apply the time-shifting property to the result obtained in part (a) to evaluate the spectrum of the half-sine pulse shown in Fig. P2.5*b*.
(c) What is the spectrum of a half-sine pulse having a duration equal to aT?
(d) What is the spectrum of the negative half-sine pulse shown in Fig. P2.5*c*?
(e) Find the spectrum of the single sine pulse shown in Fig. P2.5*d*.

Problem 9 Any function $g(t)$ can be split unambiguously into an *even part* and an *odd part*, as shown by

$$g(t) = g_e(t) + g_o(t)$$

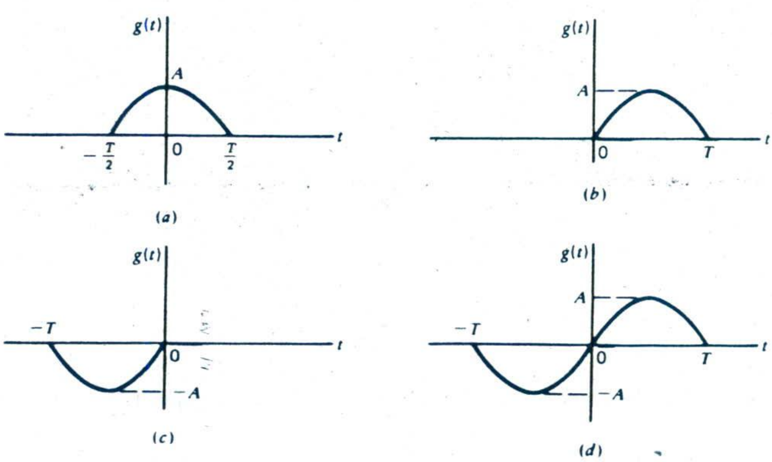


Figure P2.5

The even part is defined by

$$g_e(t) = \tfrac{1}{2}[g(t) + g(-t)]$$

and the odd part is defined by

$$g_o(t) = \tfrac{1}{2}[g(t) - g(-t)]$$

(a) Evaluate the even and odd parts of a rectangular pulse defined by

$$g(t) = A \operatorname{rect}\left(\frac{t}{T} - \frac{1}{2}\right)$$

(b) What are the Fourier transforms of these two parts of the pulse?

Problem 10 Assume the availability of a device that is capable of computing the Fourier transform of an energy signal $g(t)$ used as input. Explain the modifications that will have to be made to the input and output signals of such a device so that it may also be used to compute the inverse Fourier transform of the quantity $G(f)$, where $g(t) \rightleftharpoons G(f)$.

Problem 11 The Fourier transform of a signal $g(t)$ is denoted by $G(f)$. Prove the following properties of the Fourier transform:

(a) The total area under the curve of $g(t)$ is given by

$$\int_{-\infty}^{\infty} g(t)\, dt = G(0)$$

where $G(0)$ is the zero-frequency value of $G(f)$.
(b) The total area under the curve of $G(f)$ is given by

$$\int_{-\infty}^{\infty} G(f)\, df = g(0)$$

where $g(0)$ is the value of $g(t)$ at time $t = 0$
(c) If a real signal $g(t)$ is an even function of time t, the Fourier transform $G(f)$ is real. If a real signal $g(t)$ is an odd function of time t, the Fourier transform $G(f)$ is imaginary.

Problem 12 You are given the Fourier transform pair

$$\exp(-\pi t^2) \rightleftharpoons \exp(-\pi f^2)$$

for a standard Gaussian pulse. Using the time-scaling property, show that

$$\frac{1}{\sqrt{2\pi}\,\tau} \exp\left(-\frac{\pi t^2}{2\tau^2}\right) \rightleftharpoons \exp(-2\pi^2 f^2 \tau^2)$$

Problem 13 Prove the following properties of the convolution process:

(a) The *commutative* property:

$$g_1(t) \star g_2(t) = g_2(t) \star g_1(t)$$

(b) The *associative* property:

$$g_1(t) \star [g_2(t) \star g_3(t)] = [g_1(t) \star g_2(t)] \star g_3(t)$$

(c) The *distributive* property:

$$g_1(t) \star [g_2(t) + g_3(t)] = g_1(t) \star g_2(t) + g_1(t) \star g_3(t)$$

P2.4 Interplay Between Time-Domain and Frequency-Domain Descriptions

Problem 14 Consider a triangular pulse of height A and base $2T$. The duration of the pulse is measured at half-amplitude points. The bandwidth of the pulse is defined as one-half the main lobe of the pulse's spectrum. Show that the time–bandwidth product of the pulse equals unity.

Problem 15 Consider the sinc pulse

$$g(t) = A \operatorname{sinc}(2Wt)$$

The duration of the pulse is defined as the duration of the main lobe of the pulse. Hence, show that the time–bandwidth product of the sinc pulse equals unity.

Problem 16 Consider the Gaussian pulse

$$g(t) = \frac{1}{\sqrt{2\pi}\tau} \exp\left(-\frac{\pi t^2}{2\tau^2}\right)$$

The parameter τ provides one possible measure for the duration of the pulse. Defining the bandwidth of the pulse in a similar manner, show that the time–bandwidth product is 1/4.
Hint: Evaluate the Fourier transform of $g(t)$.

P2.5 Dirac Delta Function

Problem 17 Show that the effect of scaling the argument of the delta function by a constant a is described by

$$\delta(at) = \frac{1}{|a|}\,\delta(t)$$

Problem 18 The delta function may be considered as the limiting form of an ordinary function. Some useful representations are

$$\delta(\tau) = \lim_{\tau \to 0} \frac{1}{2\tau} \exp\left(-\frac{|t|}{\tau}\right)$$

$$= \lim_{\tau \to 0} \frac{\tau}{(t^2 + \tau^2)}$$

$$= \lim_{\tau \to 0} \frac{\sin(t/\tau)}{\pi t}$$

For each representation, plot the time function and its Fourier transform for different values of parameter τ. Hence, demonstrate that each time function approaches the delta function in the limit.

Problem 19 Determine the Fourier transform of the signal

$$g(t) = \cos^2(2\pi f_c t)$$

Problem 20 Let

$$g(t) \rightleftharpoons G(f)$$

and assume that $G(0)$ is nonzero. Starting with the Fourier transform of a signal, evaluate the Fourier transform of the integrated signal

$$\int_{-\infty}^{t} g(\tau)\, d\tau.$$

Hints:

(a) Use the formula for integration by parts.
(b) Use the limiting forms

$$\lim_{t \to 0} \frac{\sin(2\pi f t)}{\pi f} = \delta(f)$$

$$\lim_{t \to 0} \frac{\cos(2\pi f t)}{\pi f} = 0$$

P2.6 Fourier Transforms of Periodic Signals

Problem 21 Consider again the periodic signal $g_p(t)$ defined in Problem 1, which has a period of 2 seconds. The generating function of the signal is defined by

$$g(t) = \begin{cases} 1 + \cos(2\pi t), & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0, & \text{for remainder of the period} \end{cases}$$

(a) Determine the Fourier transform of the generating function $g(t)$.
(b) Hence, using the formula of Eq. 2.125, determine the Fourier transform of the periodic signal $g_p(t)$. Compare your result with that of Problem 1.

P2.7 Sampling Theorem

Problem 22 Specify the Nyquist rate and the Nyquist interval for each of the following energy signals:

(a) $g(t) = \text{sinc}(200t)$
(b) $g(t) = \text{sinc}^2(200t)$
(c) $g(t) = \text{sinc}(200t) + \text{sinc}^2(200t)$

FILTERING AND SIGNAL DISTORTION

In Chapter 2 we used Fourier methods to study *spectral* properties of various kinds of *signals* and relationships between spectra and time-domain characteristics of the signals. We also studied the effects that various time-domain operations on a signal have on the spectrum of the signal. In this chapter we study *filtering* characteristics of *systems*. The system may be a linear time-invariant filter or communication channel. We also consider the linear and nonlinear forms of *signal distortion*, which result from transmission through linear and nonlinear systems respectively. We begin the study by considering the time response of a linear time-invariant system.

3.1 TIME RESPONSE

A *system* refers to any physical device that produces an output signal in response to an input signal. It is customary to refer to the input signal as the *excitation* and to the output signal as the *response*. A system is said to be *linear* if the *principle of superposition* holds; that is, *the response of a linear system to a number of excitations applied simultaneously is equal to the sum of the responses of the system when the excitations are applied individually*. The system is said to be *time-invariant* if a time shift in the excitation applied to the system produces the same time shift in the response of the system. In this section, we study the *time response* of *linear time-invariant systems,* with particular reference to filters and channels. A *filter* refers to a frequency-selective device that is used to limit the spectrum of a signal to some band of frequencies. A *channel* refers to a physical medium that connects the transmitter of a communication system to the receiver. The operation of limiting the spectrum of a signal to some band of frequencies (by passing the signal through a filter or channel) is called *filtering*. In the time domain, a linear system is described in terms of its *impulse response, which is defined as the response of the system (with zero initial conditions) to a unit impulse or delta function $\delta(t)$ applied to the input of the system*. If the system is *time-invariant,* then the shape of the impulse response is the same no matter when the unit impulse is applied to the system. Thus, assuming that the unit impulse or delta function is applied at time $t = 0$, we may denote the impulse response of a linear time-invariant system by $h(t)$. Let this system be subjected to an arbitrary excitation $x(t)$, as in Fig. 3.1*a*. To determine the response $y(t)$ of the system, we begin by first approximating $x(t)$ by a staircase function composed of narrow rectangular pulses, each of duration $\Delta\tau$, as shown in Fig. 3.1*b*. Clearly the approximation becomes better for smaller $\Delta\tau$. As $\Delta\tau$ approaches zero, each pulse approaches, in the limit, a delta function weighted by a factor equal to the height of the pulse times $\Delta\tau$. Consider a typical pulse, shown shaded in Fig. 3.1*b*, which occurs at $t = \tau$. This pulse has an area equal to $x(\tau)\,\Delta\tau$. By definition, the response of the system to a unit impulse or delta function $\delta(t)$, occurring at $t = 0$, is $h(t)$. It follows, therefore, that the response of the system to a delta function, weighted by the factor $x(\tau)\,\Delta\tau$ and occurring at $t = \tau$, must be $x(\tau)h(t - \tau)\,\Delta\tau$. To find the total response $y(t)$ at some time t, we apply the principle of superposition. Thus, summing the various infinitesimal responses due to the various input pulses, we obtain in the limit, as $\Delta\tau$ approaches zero,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)\,d\tau \tag{3.1}$$

This relation is called the ***convolution integral***. Note that for the response $y(t)$ to have the same dimension as the excitation $x(t)$, *the impulse response $h(t)$ must have a dimension that is the inverse of time*.

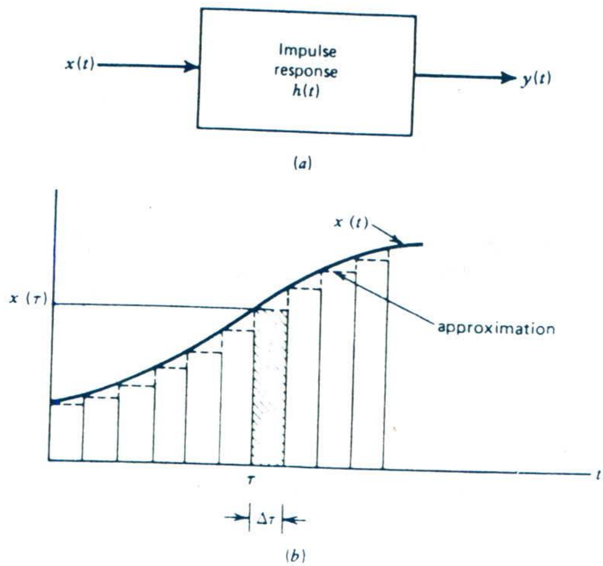


Figure 3.1
(a) Linear system. (b) Approximation of input x(t).

In Eq. 3.1, three different time scales are involved: *excitation time* τ, *response time* t, and *system-memory time* $t - \tau$. This relation is the basis of time-domain analysis of linear time-invariant systems. It states that the present value of the response of a linear time-invariant system is a weighted integral over the past history of the input signal, weighted according to the impulse response of the system. Thus the impulse response acts as a *memory function* for the system.

In Eq. 3.1, the excitation $x(t)$ is convolved with the impulse response $h(t)$ to produce the response $y(t)$. Since convolution is commutative, it follows that we may also write

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)\, d\tau \qquad (3.2)$$

where $h(t)$ is convolved with $x(t)$.

Using the shorthand notation for convolution, we may rewrite Eq. 3.1 simply as

$$y(t) = x(t) \star h(t) \qquad (3.3)$$

where $\star$ denotes convolution. Similarly, we may rewrite Eq. 3.2 as

$$y(t) = h(t) \star x(t) \qquad (3.4)$$

Equations 3.3 and 3.4 highlight the commutative nature of convolution or linear filtering.

EXAMPLE 1 GRAPHICAL INTERPRETATION OF CONVOLUTION

We may develop further insight into convolution by presenting a graphical interpretation of the convolution integral, which is defined in mathematical terms in Eq. 3.1 or 3.2. We will do so in this example by considering Eq. 3.1 first and then 3.2. The example is simple and yet illustrative of the various steps involved in evaluating the convolution integral. Specifically, we consider a linear time-invariant system with an impulse response that is a decaying exponential function and that is driven by a unit step function.

Parts *a* and *b* of Fig. 3.2 depict the impulse response $h(\tau)$ and excitation

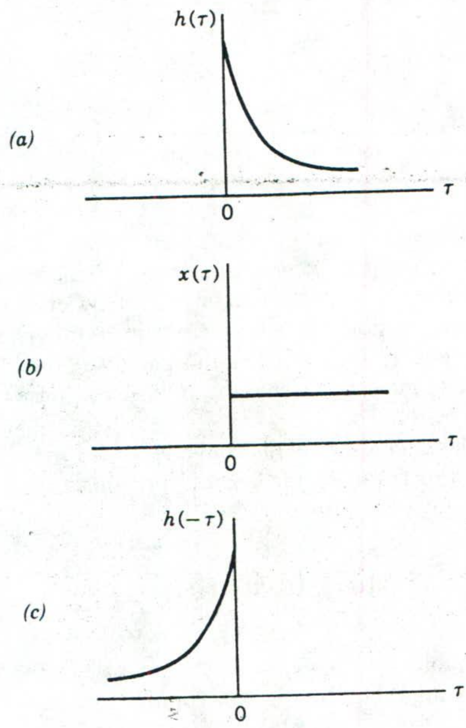


Figure 3.2
The steps involved in computing one form of the convolution integral. (a) Impulse response. (b) Excitation. (c) Image of the impulse response. (d) Time-shifted image of the impulse response. (e) Evaluation of the response.

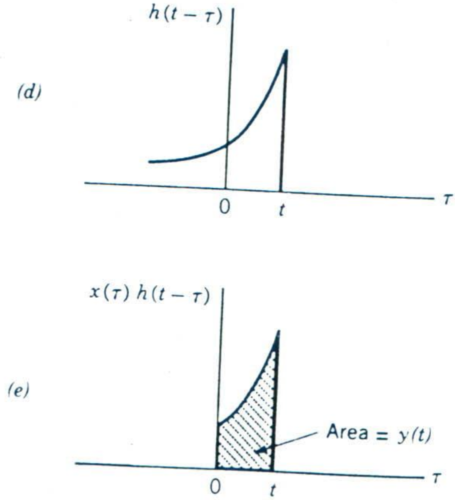


Figure 3.2 *(continued)*

$x(\tau)$, respectively. For reasons that will become apparent presently, the time variable in both cases is shown as τ. In accordance with Eq. 3.1, the integral consists of the product $x(\tau)h(t - \tau)$. We already have $x(\tau)$. To obtain $h(t - \tau)$, we proceed in two steps. First, we formulate $h(-\tau)$, which is the mirror image of $h(\tau)$ with respect to the vertical axis, as shown in Fig. 3.2*c*. Then, we shift $h(-\tau)$ to the right by an amount equal to the specified time t to obtain $h(t - \tau)$; this second step is shown in Fig. 3.2*d*. Next, we multiply $x(\tau)$ by $h(t - \tau)$, as in Fig. 3.2*e*, and thereby obtain the desired integrand $x(\tau)h(t - \tau)$ for the specified value of time t. Finally, we calculate the total area under $x(\tau)h(t - \tau)$, which is shown shaded in Fig. 3.2*e*. This area equals the value of the system response $y(t)$ at time t.

For the graphical interpretation of Eq. 3.2 we may proceed in a similar way, as illustrated in Fig. 3.3. In this second case, the integrand equals $h(\tau)x(t - \tau)$. The first multiplying factor $h(\tau)$ is already available, as in Fig. 3.3*a*. The second multiplying factor $x(t - \tau)$ is obtained by forming the image $x(-\tau)$ of the specified excitation $x(\tau)$, and then shifting the image $x(-\tau)$ to the right by an amount equal to the specified time t. The functions $x(\tau)$, $x(-\tau)$, and $x(t - \tau)$ are depicted in Figs. 3.3*b*, *c*, and *d*, respectively. The resulting product $h(\tau)x(t - \tau)$ is shown in Fig. 3.3*e*. Comparing Figs. 3.2*e* and 3.3*e*, we see that the products $x(\tau)h(t - \tau)$ and $h(\tau)x(t - \tau)$ are reversed with respect to each other. Naturally, they both have the same total area under their individual curves, which confirms the commutative property of convolution.

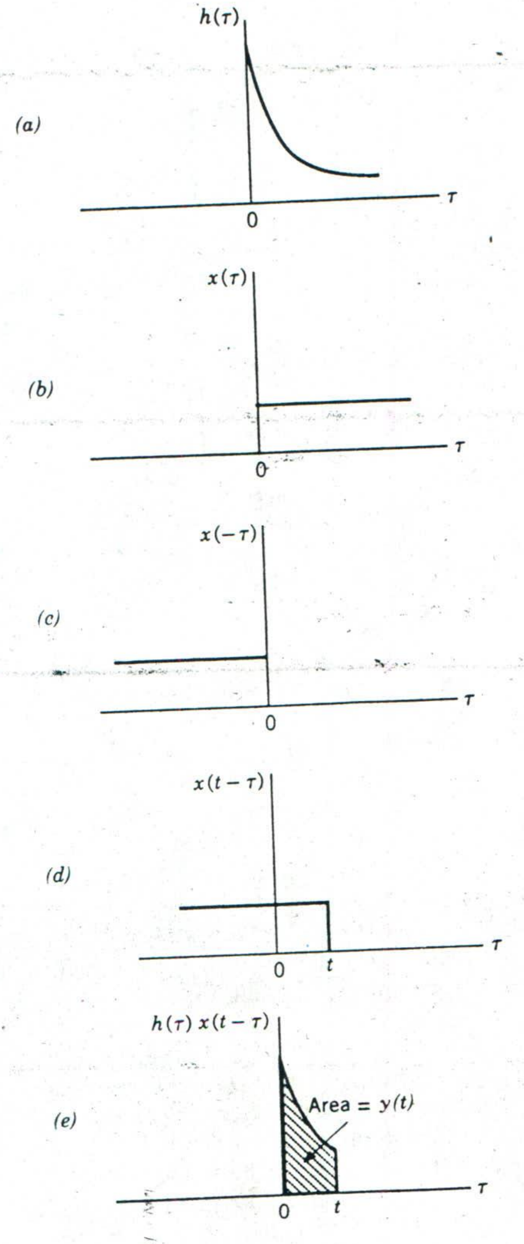


Figure 3.3
The steps involved in computing the second form of the convolution integral. (a) Impulse response. (b) Excitation. (c) Image of the excitation. (d) Time-shifted image of the excitation. (e) Evaluation of the response y(t).

EXAMPLE 2 TAPPED-DELAY-LINE FILTER

Consider a linear time-invariant filter with impulse response $h(t)$. We assume that

1. The impulse response $h(t) = 0$ for $t < 0$.
2. The impulse response of the filter is of finite duration, so that we may write $h(t) = 0$ for $t \geq T_f$.

Then we may express the filter output $y(t)$ produced in response to the input $x(t)$ as follows:

$$y(t) = \int_0^{T_f} h(\tau)x(t - \tau)\, d\tau \tag{3.5}$$

Let the input $x(t)$, impulse response $h(t)$, and output $y(t)$ be *uniformly sampled* at the rate $1/\Delta\tau$ samples per second, so that we may put

$$t = n\, \Delta\tau \tag{3.6}$$

and

$$\tau = k\, \Delta\tau \tag{3.7}$$

where k and n are integers, and $\Delta\tau$ is the *sampling period*. We assume that $\Delta\tau$ is small enough for the product $h(\tau)x(t - \tau)$ to remain essentially constant for $k\, \Delta\tau \leq \tau \leq (k + 1)\, \Delta\tau$ for all values of k and t of interest. Then we can approximate Eq. 3.5 by the *convolution sum:*

$$y(n\, \Delta\tau) = \sum_{k=0}^{N-1} h(k\, \Delta\tau)x(n\, \Delta\tau - k\, \Delta\tau)\, \Delta\tau \tag{3.8}$$

where $N\, \Delta\tau = T_f$. Defining

$$w_k = h(k\, \Delta\tau)\, \Delta\tau$$

we may rewrite Eq. 3.8 as

$$y(n\, \Delta\tau) = \sum_{k=0}^{N-1} w_k x(n\, \Delta\tau - k\, \Delta\tau) \tag{3.9}$$

Equation 3.9 is realized using the circuit shown in Fig. 3.4, which consists of a set of *delay elements* (each producing a delay of $\Delta\tau$ seconds), a set of

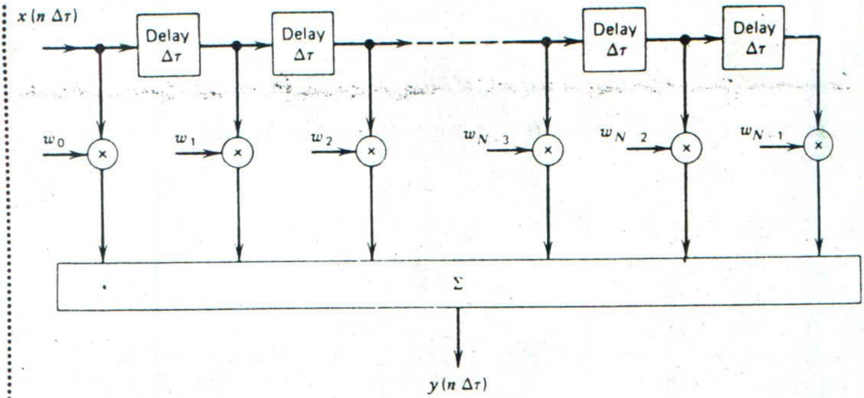


Figure 3.4
Tapped-delay-line filter.

multipliers connected to the *delay-line taps,* a corresponding set of *weights* applied to the multipliers, and a *summer* for adding the multiplier outputs. This circuit is known as a *tapped-delay-line filter* or *transversal filter*. Note that in Fig. 3.4 the tap-spacing or basic increment of delay is equal to the sampling period of the input sequence $\{x(n\ \Delta\tau)\}$.

When a tapped-delay-line filter is implemented using digital hardware, it is commonly referred to as a *finite-duration impulse response* (FIR) *digital filter*. The required delay is provided by means of a *shift register*, with the basic increment of delay, $\Delta\tau$, equal to the clock period. An important feature of a digital filter is that it is programmable, thereby offering a high degree of flexibility in design.[1]

CAUSALITY AND STABILITY

A system is said to be *causal* if it does not respond before the excitation is applied. For a linear time-invariant system to be causal, it is clear that the impulse response $h(t)$ must vanish for negative time. That is, the necessary and sufficient condition for causality is

$$h(t) = 0, \qquad t < 0 \tag{3.10}$$

Clearly, for a system operating in *real time* to be physically realizable, it must be causal. However, there are many applications in which the signal

[1]For a detailed treatment of the theory and design of digital filters, see Roberts and Mullis (1987) or Oppenheim and Schaffer (1975).

to be processed is available in stored form; in these situations the system can be noncausal and yet physically realizable.

The system is said to be *stable* if the output signal is bounded for all bounded input signals. Let the input signal $x(t)$ be bounded, as shown by

$$|x(t)| \leq M, \qquad -\infty < t < \infty \tag{3.11}$$

where M is a positive real finite number. Using Eqs. 3.2 and 3.11, we may write

$$|y(t)| \leq \int_{-\infty}^{\infty} |h(\tau)|\,|x(t-\tau)|\,d\tau = M \int_{-\infty}^{\infty} |h(\tau)|\,d\tau$$

It follows therefore that for a linear time-invariant system to be stable, the impulse response $h(t)$ must be absolutely integrable. That is, the necessary and sufficient condition for stability is

$$\int_{-\infty}^{\infty} |h(t)|\,dt < \infty \tag{3.12}$$

EXERCISE 1 The impulse response of a linear time-invariant system is defined by

$$h(t) = \exp(at)\,u(-t)$$

where $u(-t)$ is the time-reversed version of the unit step function $u(t)$. Is this system casual? Is it stable? Give reasons for your answers.

3.2 *FREQUENCY RESPONSE*

Consider a linear time-invariant system of impulse response $h(t)$ driven by a complex exponential input of unit amplitude and frequency f, that is,

$$x(t) = \exp(j2\pi ft) \tag{3.13}$$

Using Eq. 3.2, the response of the system is obtained as

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)\exp[j2\pi f(t-\tau)]\,d\tau \\ &= \exp(j2\pi ft)\int_{-\infty}^{\infty} h(\tau)\exp(-j2\pi f\tau)\,d\tau \end{aligned} \tag{3.14}$$

Define

$$H(f) = \int_{-\infty}^{\infty} h(\tau) \exp(-j2\pi f\tau)\, d\tau \tag{3.15}$$

Then we may rewrite Eq. 3.14 in the form

$$y(t) = H(f) \exp(j2\pi ft) \tag{3.16}$$

The response of a linear time-invariant system to a complex exponential function of frequency f is, therefore, the same complex exponential function multiplied by a constant coefficient $H(f)$. The quantity $H(f)$ is called the *transfer function* of the system. The transfer function $H(f)$ and impulse response $h(t)$ form a Fourier transform pair, as shown by the pair of relations:

$$H(f) = \int_{-\infty}^{\infty} h(t) \exp(-j2\pi ft)\, dt \tag{3.17}$$

and

$$h(t) = \int_{-\infty}^{\infty} H(f) \exp(j2\pi ft)\, df \tag{3.18}$$

An alternative definition of the transfer function may be deduced by dividing Eq. 3.16 by 3.13 to obtain

$$H(f) = \left. \frac{y(t)}{x(t)} \right|_{x(t)=\exp(j2\pi ft)} \tag{3.19}$$

Consider next an arbitrary signal $x(t)$ applied to the system. The signal $x(t)$ may be expressed in terms of its Fourier transform as

$$x(t) = \int_{-\infty}^{\infty} X(f) \exp(j2\pi ft)\, df \tag{3.20}$$

or, equivalently, in the limiting form

$$x(t) = \lim_{\substack{\Delta f \to 0 \\ f = k\Delta f}} \sum_{k=-\infty}^{\infty} X(f) \exp(j2\pi ft)\, \Delta f \tag{3.21}$$

That is, the input signal $x(t)$ may be viewed as a superposition of complex exponentials of incremental amplitude. Because the system is linear, the response to this superposition of complex exponential inputs is

$$
\begin{aligned}
y(t) &= \lim_{\substack{\Delta f \to 0 \\ f = k\Delta f}} \sum_{k=-\infty}^{\infty} H(f)X(f) \exp(j2\pi f t)\, \Delta f \\
&= \int_{-\infty}^{\infty} H(f)X(f) \exp(j2\pi f t)\, df
\end{aligned} \tag{3.22}
$$

The Fourier transform of the output is therefore

$$
Y(f) = H(f)X(f) \tag{3.23}
$$

A linear time-invariant system may thus be described simply in the frequency domain by noting that the Fourier transform of the output is equal to the product of the transfer function of the system and the Fourier transform of the input.

The result of Eq. 3.23 may, of course, be deduced directly by recognizing that the response $y(t)$ of a linear time-invariant system of impulse response $h(t)$ to an arbitrary input $x(t)$ is obtained by convolving $x(t)$ with $h(t)$, or vice versa, and by the fact that the convolution of a pair of time functions is transformed into the multiplication of their Fourier transforms. The foregoing derivation is presented primarily to develop an understanding of why the Fourier representation of a time function as a superposition of complex exponentials is so useful in analyzing the behavior of linear time-invariant systems.

AMPLITUDE RESPONSE AND PHASE RESPONSE

The transfer function $H(f)$ is a characteristic property of a linear time-invariant system. It is, in general, a complex quantity, so that we may express it in the form

$$
H(f) = |H(f)| \exp[j\beta(f)] \tag{3.24}
$$

where $|H(f)|$ is called the *amplitude response,* and $\beta(f)$ is called the *phase response.* The phase response is related to the transfer function $H(f)$ by

$$
\beta(f) = \arg[H(f)] \tag{3.25}
$$

In the case of a linear system with a real-valued impulse response $h(t)$, the transfer function $H(f)$ exhibits *conjugate symmetry,* which means that

$$
|H(f)| = |H(-f)| \tag{3.26}
$$

and

$$\beta(f) = -\beta(-f) \tag{3.27}$$

That is, the amplitude response $|H(f)|$ is an even function of frequency, whereas the phase response $\beta(f)$ is an odd function of frequency. Plots of the amplitude response $|H(f)|$ and the phase response $\beta(f)$ versus frequency f represent the frequency-domain description of the system. Hence, we may also refer to $H(f)$ as the *frequency response* of the system.

In some applications it is preferable to work with the logarithm of $H(f)$ rather than with $H(f)$ itself. Define

$$\ln H(f) = \alpha(f) + j\beta(f) \tag{3.28}$$

where

$$\alpha(f) = \ln|H(f)| \tag{3.29}$$

The function $\alpha(f)$ is called the *gain* of the system. It is measured in *nepers*, whereas $\beta(f)$ is measured in *radians*. Equation 3.28 indicates that the gain $\alpha(f)$ and phase response $\beta(f)$ are the real and imaginary parts of the logarithm of the transfer function $H(f)$, respectively. The squared amplitude response $|H(f)|^2$ is identified with power. Accordingly, we may also apply the *decibel* (dB) measure to the gain by writing

$$\alpha'(f) = 20 \log_{10}|H(f)| \tag{3.30}$$

The two gain functions $\alpha(f)$ and $\alpha'(f)$ are related by

$$\alpha'(f) = 8.69\alpha(f) \tag{3.31}$$

That is, 1 neper is equal to 8.69 dB.

EXAMPLE 3

Consider a linear time-invariant device with a transfer function defined by

$$\begin{aligned} H(f) &= \begin{cases} -j, & f > 0 \\ 0, & f = 0 \\ j, & f < 0 \end{cases} \\ &= -j\,\mathrm{sgn}(f) \end{aligned} \tag{3.32}$$

where $\mathrm{sgn}(f)$ is the signum function.

The amplitude response and phase response of the device are shown in

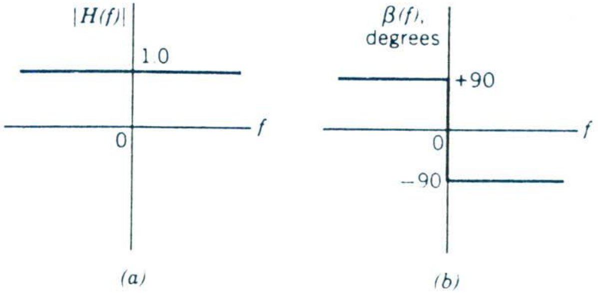


Figure 3.5
Characteristics of a Hilbert transformer. (a) Amplitude response. (b) Phase response.

Fig. 3.5*a* and *b*, respectively. That is, the device produces a phase shift of $-90°$ for all positive frequencies and a phase shift of $+90°$ for all negative frequencies. The amplitudes of all frequency components of the input signal are unaffected by transmission through the device. Such an ideal device is called a *Hilbert transformer*.

Figure 3.6 shows a black-box representation of the Hilbert transformer with a Fourier transformable signal $x(t)$ acting as input, and the resulting output[2] denoted by $\hat{x}(t)$. We wish to determine the output $\hat{x}(t)$, given the input $x(t)$. To do so, we first determine the impulse response of the device. Specifically, we use the Fourier transform pair of Eq. 2.107 to express the impulse response of the Hilbert transformer as

$$h(t) = \frac{1}{\pi t} \tag{3.33}$$

Hence, the convolution of this impulse response with a signal $x(t)$ applied to the input of the Hilbert transformer yields the resulting output $\hat{x}(t)$ as

$$\begin{aligned} \hat{x}(t) &= x(t) \star \left(\frac{1}{\pi t}\right) \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(\tau)}{t - \tau}\, d\tau \end{aligned} \tag{3.34}$$

According to this formula, $\hat{x}(t)$ is the *Hilbert transform* of $x(t)$.

[2]When dealing with Hilbert transformation, it is customary to denote the output by placing a circumflex (or "hat") over the symbol for the input; this explains the reason for using $\hat{x}(t)$ rather than $y(t)$ to denote the output.

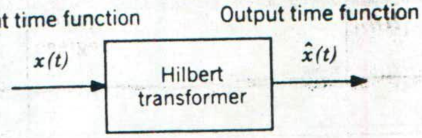


Figure 3.6
Black-box representation emphasizing that both the input and output of a Hilbert transformer are time functions.

EXERCISE 2 The *inverse Hilbert transform,* defining $x(t)$ in terms of $\hat{x}(t)$, is described by

$$x(t) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\hat{x}(\tau)}{t-\tau}\, d\tau \tag{3.35}$$

Starting with the transfer function of Eq. 3.32, derive the formula of Eq. 3.35.

SYSTEM BANDWIDTH

To specify the degree of dispersion of the amplitude response or gain of a system, we use a parameter called the *system bandwidth*. A common definition of system bandwidth is the *3-dB bandwidth,* the exact formulation of which depends on the type of system being considered. In the case of a *low-pass system,* the 3-dB bandwidth is defined as the difference between zero frequency, at which the amplitude response attains its peak value $|H(0)|$, and the frequency at which the amplitude response drops to a value equal to $|H(0)|/\sqrt{2}$, as illustrated in Fig. 3.7*a*. In the case of a *band-pass system,* the 3-dB bandwidth is defined as the difference between the frequencies at which the amplitude response drops to a value equal to $1/\sqrt{2}$ times the peak value $|H(f_c)|$ at the midband frequency f_c, as illustrated in Fig. 3.7*b*. Note that in both cases, the system bandwidth is defined for *positive* frequencies. Note also that an amplitude response value equal to $1/\sqrt{2}$ times the peak value of the amplitude response is equivalent to a drop in the gain of 3-dB below its peak value; hence, the name "3-dB bandwidth."

3.3 LINEAR DISTORTION AND EQUALIZATION

Two basic forms of *signal distortion* result from the transmission of a signal through a physical system: *linear distortion* and *nonlinear distortion*. In the context of telecommunications, the system of interest is comprised of all the components that constitute the path from the source of information to the desired destination. When the system is viewed as being linear and

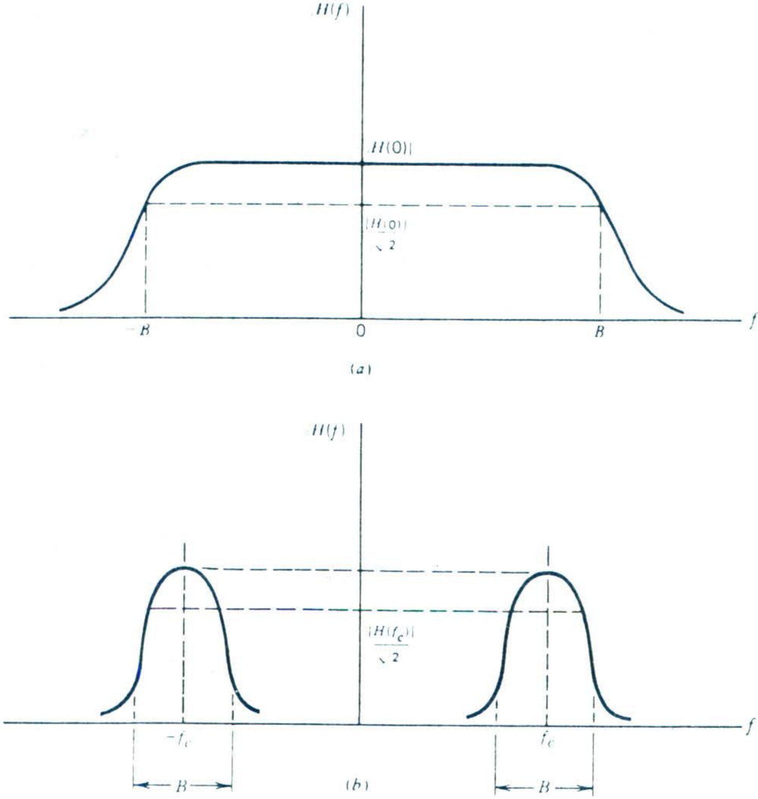


Figure 3.7
The definition of system bandwidth. (a) Low-pass system. (b) Band-pass system.

time invariant, linear distortion arises owing to *imperfections in the frequency response* of the system. On the other hand, nonlinear distortion arises owing to the presence of *nonlinearities* in the makeup of the system. In this section, we discuss the linear distortion problem; nonlinear distortion is considered in Section 3.7. We begin the discussion by formulating the conditions for distortionless transmission of a signal through a linear time-invariant system.

CONDITIONS FOR DISTORTIONLESS TRANSMISSION

By *distortionless transmission* we mean that the output signal of a system is an exact replica of the input signal, except for a possible change of amplitude and a constant time delay. We may therefore say that a signal $x(t)$ is transmitted through the system without distortion if the output signal

$y(t)$ is defined by

$$y(t) = Kx(t - t_0) \tag{3.36}$$

where the constant K accounts for the change in amplitude and the constant t_0 accounts for the delay in transmission.

Let $X(f)$ and $Y(f)$ denote the Fourier transforms of $x(t)$ and $y(t)$, respectively. Then, applying the Fourier transform to Eq. 3.36 and using the time-shifting property of the Fourier transform, we get

$$Y(f) = KX(f)\exp(-j2\pi f t_0) \tag{3.37}$$

The transfer function of a distortionless system is therefore

$$\begin{aligned} H(f) &= \frac{Y(f)}{X(f)} \\ &= K\exp(-j2\pi f t_0) \end{aligned} \tag{3.38}$$

Correspondingly, the impulse response of the system is given by

$$h(t) = K\delta(t - t_0) \tag{3.39}$$

where $\delta(t - t_0)$ is a Dirac delta function shifted by t_0 seconds.

Equation 3.38 indicates that in order to achieve distortionless transmission through a system, the transfer function of the system must satisfy two conditions:

1. The amplitude response $|H(f)|$ is constant for all frequencies, as shown by

$$|H(f)| = K \tag{3.40}$$

2. The phase $\beta(f)$ is linear with frequency, passing through zero as shown by

$$\beta(f) = -2\pi f t_0 \tag{3.41}$$

These two conditions are illustrated in parts *a* and *b* of Fig. 3.8, respectively.

EXERCISE 3 **Using the impulse response of Eq. 3.39 in the convolution integral, show that the input–output relation of a distortionless system is as defined in Eq. 3.36.**

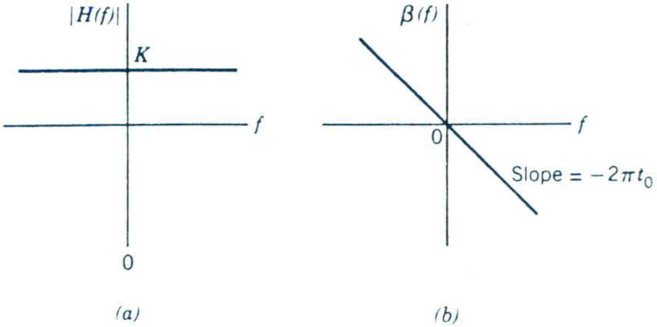


Figure 3.8
Frequency response for distortionless transmission. (a) Amplitude response. (b) Phase response.

EXERCISE 4 Show that the condition of Eq. 3.41 on the phase response $\beta(f)$ for distortionless transmission may be modified by adding a constant equal to a positive or negative integer multiple of 180°. How can such a modification arise in practice?

AMPLITUDE DISTORTION AND PHASE DISTORTION

In practice, the conditions for distortionless transmission, as just described, can only be satisfied approximately. That is to say, there is always a certain amount of linear distortion present in the output signal. In particular, we may distinguish two components of signal distortion produced by transmission through a linear time-invariant system:

1. When the amplitude response $|H(f)|$ of the system is not constant with frequency inside the frequency band of interest, the frequency components of the input signal are transmitted with different amounts of gain or attenuation. This effect is called *amplitude distortion*. The most common form of amplitude distortion is excess gain or attenuation at one or both ends of the frequency band of interest.
2. The second form of distortion arises when the phase response $\beta(f)$ of the system is not linear with frequency. Then if the input signal is divided into a set of components, each one of which occupies a narrow band of frequencies, we find that each of them is subject to a different delay in passing through the system, with the result that the output signal has a different waveform from the input. This form of distortion is called *phase* or *delay distortion*. We will have more to say on this issue in Section 3.6.

You should carefully note the distinction between a *constant delay* and a *constant phase shift*. These two conditions have different implications. Constant delay is a requirement for distortionless transmission. Constant phase shift, on the other hand, causes signal distortion.

EQUALIZATION

To compensate for linear distortion, we may use a network known as an *equalizer* connected in cascade with the system in question. The equalizer is designed in such a way that, *inside the frequency band of interest*, the overall amplitude and phase responses of this cascade connection approximate the conditions for distortionless transmission to within prescribed limits.

Consider, for example, a communication channel with transfer function $H_c(f)$. Let an equalizer of transfer function $H_{eq}(f)$ be connected in cascade with the channel, as in Fig. 3.9. The overall transfer function of this combination is equal to $H_c(f)H_{eq}(f)$. For overall transmission through the cascade connection of Fig. 3.9 to be distortionless, we require that (see Eq. 3.38)

$$H_c(f)H_{eq}(f) = K \exp(-j2\pi f t_0) \tag{3.42}$$

where K is a scaling factor and t_0 is a constant time delay. Ideally, therefore, the transfer function of the equalizer is *inversely related* to that of the channel, as shown by

$$H_{eq}(f) = \frac{K \exp(-j2\pi f t_0)}{H_c(f)} \tag{3.43}$$

In practice, the equalizer is designed such that its transfer function approximates the ideal value of Eq. 3.43 closely enough for the linear distortion to be reduced to a satisfactory level.

A network structure that is well-suited for the design of equalizers is the *tapped-delay-line filter*, depicted in Fig. 3.4. From the time-shifting property of the Fourier transform, we know that when a signal is shifted

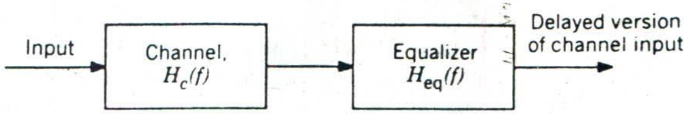


Figure 3.9
Block diagram of equalization.

in time by $\Delta\tau$ seconds, its Fourier transform is multiplied by the complex exponential $\exp(-j2\pi f\,\Delta\tau)$. Accordingly, the transfer function of this tapped-delay-line filter, used as an equalizer, is given by

$$H_{\text{eq}}(f) = \sum_{k=0}^{N-1} w_k \exp(-j2\pi k f\,\Delta\tau) \tag{3.44}$$

For convenience of analysis, let the number of taps be *odd*, as shown by

$$N = 2M + 1 \tag{3.45}$$

where M is an integer. Also, setting

$$k = m + M \qquad \begin{aligned} k &= 0, \ldots, N-1 \\ m &= -M, \ldots, -1, 0, 1, \ldots, M \end{aligned} \tag{3.46}$$

and

$$w_k = c_m \tag{3.47}$$

we may rewrite Eq. 3.44 as

$$H_{\text{eq}}(f) = \left[\sum_{m=-M}^{M} c_m \exp(-j2\pi m f\,\Delta\tau)\right]\exp(-j2\pi M f\,\Delta\tau) \tag{3.48}$$

The expression inside the square brackets on the right side of Eq. 3.48 represents the discrete-time Fourier transform of the sequence of tap coefficients $c_{-M}, \ldots, c_{-1}, c_0, c_1, \ldots, c_M$, with a tap spacing (sampling interval) of $\Delta\tau$ seconds. This discrete-time Fourier transform may be viewed as a truncated version of the complex Fourier series with a frequency periodicity of $1/\Delta\tau$ hertz; note that in this interpretation, the usual roles of time and frequency in the complex Fourier series are interchanged.

We may now describe a procedure for designing the equalizer. Specifically, given a channel of transfer function $H_c(f)$ to be equalized over the interval $-B \leq f \leq B$, we first approximate the reciprocal transfer function $1/H_c(f)$ by a complex Fourier series with periodicity $(1/\Delta\tau) = B$. Typically, $H_c(f)$ is specified numerically in terms of its amplitude and phase components, in which case numerical integration is used to compute the complex Fourier coefficients. The total number of significant terms, $2M + 1$, is chosen to be just big enough to produce a satisfactory approximation to the prescribed $H_c(f)$. The tap coefficients of the equalizer, namely, $c_{-M}, \ldots, c_{-1}, c_0, c_1, \ldots, c_M$ are then matched to the complex Fourier coefficients.

EXERCISE 5 Write the formula for evaluating the coefficients of the complex Fourier series used to approximate $1/H_c(f)$ with periodicity $(1/\Delta\tau) = B$.

3.4 IDEAL LOW-PASS FILTERS

As previously mentioned, a *filter* is a frequency-selective device that is used to limit the spectrum of a signal to some specified band of frequencies. Its frequency response is characterized by a *passband* and a *stopband*, which are separated by a *guardband*. The frequencies inside the passband are transmitted with little or no distortion, whereas those in the stopband are rejected. The filter may be of the *low-pass*, *high-pass*, *band-pass*, or *band-stop* type, depending on whether it transmits low, high, intermediate, or all but intermediate frequencies, respectively.

In this section we study the time response of the *ideal low-pass filter*, which transmits, without any distortion all frequencies inside the passband and completely rejects all frequencies inside the stopband, as illustrated in Fig. 3.10. Note that the conditions for distortionless transmission need only be satisfied inside the pass band of the filter. The transfer function of the ideal low-pass filter so illustrated is defined by

$$H(f) = \begin{cases} \exp(-j2\pi f t_0), & -B \leq f \leq B \\ 0, & |f| > B \end{cases} \tag{3.49}$$

where, for convenience, we have set $K = 1$. The parameter B defines the bandwidth of the filter. For a finite t_0, the ideal low-pass filter is noncasual,

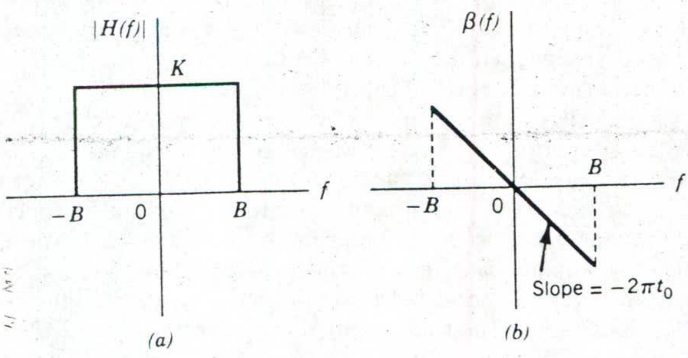


Figure 3.10
Frequency response of ideal low-pass filter. (a) Amplitude response. (b) Phase response.

which may be confirmed by examining the impulse response $h(t)$. Specifically, by evaluating the inverse Fourier transform of the transfer function of Eq. 3.49, we get

$$h(t) = \int_{-B}^{B} \exp[j2\pi f(t - t_0)]\, df \tag{3.50}$$

where the limits of integration have been reduced to the frequency band inside which $H(f)$ does not vanish. Equation 3.50 is readily integrated, yielding

$$\begin{aligned} h(t) &= \frac{\sin[2\pi B(t - t_0)]}{\pi(t - t_0)} \\ &= 2B \operatorname{sinc}[2B(t - t_0)] \end{aligned} \tag{3.51}$$

This impulse response has a peak amplitude of $2B$ centered on time t_0, as shown in Fig. 3.11. The duration of the main lobe of the impulse response is $1/B$, and the build-up time from the zero at the beginning of the main lobe to the peak value is $1/2B$. We see from Fig. 3.11 that, for any finite value of t_0, there is some response from the filter before the time $t = 0$ at which the unit impulse is applied to the input, confirming that the ideal low-pass filter is noncausal. However, despite its noncausality, the ideal low-pass filter serves as a useful standard against which the response of causal filters may be measured.

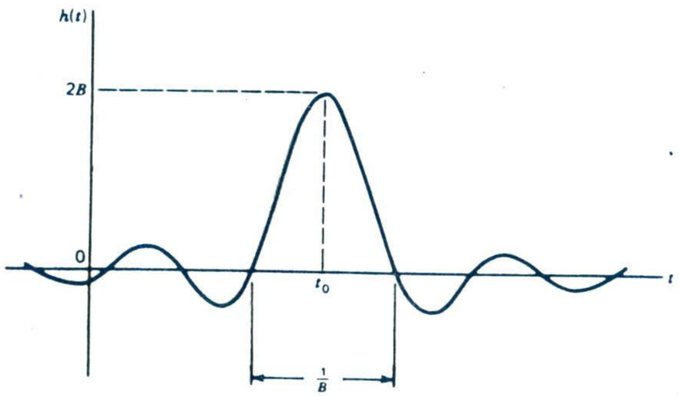


Figure 3.11
Impulse response of ideal low-pass filter.

EXAMPLE 4 PULSE RESPONSE OF IDEAL LOW-PASS FILTER

Consider a rectangular pulse $x(t)$ of unit amplitude and duration T, which is applied to an ideal low-pass filter of bandwidth B. The problem is to determine the response $y(t)$ of the filter.

The impulse response $h(t)$ of the filter is defined by Eq. 3.51. Its response is therefore given by the convolution integral

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \\ &= 2B\int_{-T/2}^{T/2} \frac{\sin[2\pi B(t-t_0-\tau)]}{2\pi B(t-t_0-\tau)}\,d\tau \end{aligned} \tag{3.52}$$

Define

$$\lambda = 2\pi B(t - t_0 - \tau)$$

Then, changing the integration variable from τ to λ, we may rewrite Eq. 3.52 as

$$\begin{aligned} y(t) &= \frac{1}{\pi}\int_{2\pi B(t-t_0-T/2)}^{2\pi B(t-t_0+T/2)} \frac{\sin\lambda}{\lambda}\,d\lambda \\ &= \frac{1}{\pi}\left[\int_0^{2\pi B(t-t_0+T/2)} \frac{\sin\lambda}{\lambda}\,d\lambda - \int_0^{2\pi B(t-t_0-T/2)} \frac{\sin\lambda}{\lambda}\,d\lambda\right] \\ &= \frac{1}{\pi}\{\mathrm{Si}[2\pi B(t-t_0+T/2)] - \mathrm{Si}[2\pi B(t-t_0-T/2)]\} \end{aligned} \tag{3.53}$$

where the *sine integral* is defined by

$$\mathrm{Si}(u) = \int_0^u \frac{\sin\lambda}{\lambda}\,d\lambda \tag{3.54}$$

Figure 3.12 plots the response $y(t)$ for three different values of the filter bandwidth B, assuming that t_0 is zero. We see that, in each case, the output is symmetric about $t = 0$. We further observe that the shape of the output is markedly dependent on the filter bandwidth B. In particular, we note:

1. When B is large compared with $1/T$, as in Fig. 3.12a, the output has approximately the same duration as the input. However, it differs from the input in two major respects. First, the output, unlike the input, has nonzero rise and fall times that are inversely proportional to the filter bandwidth. Second, the output exhibits *ringing* at both the leading and trailing edges.

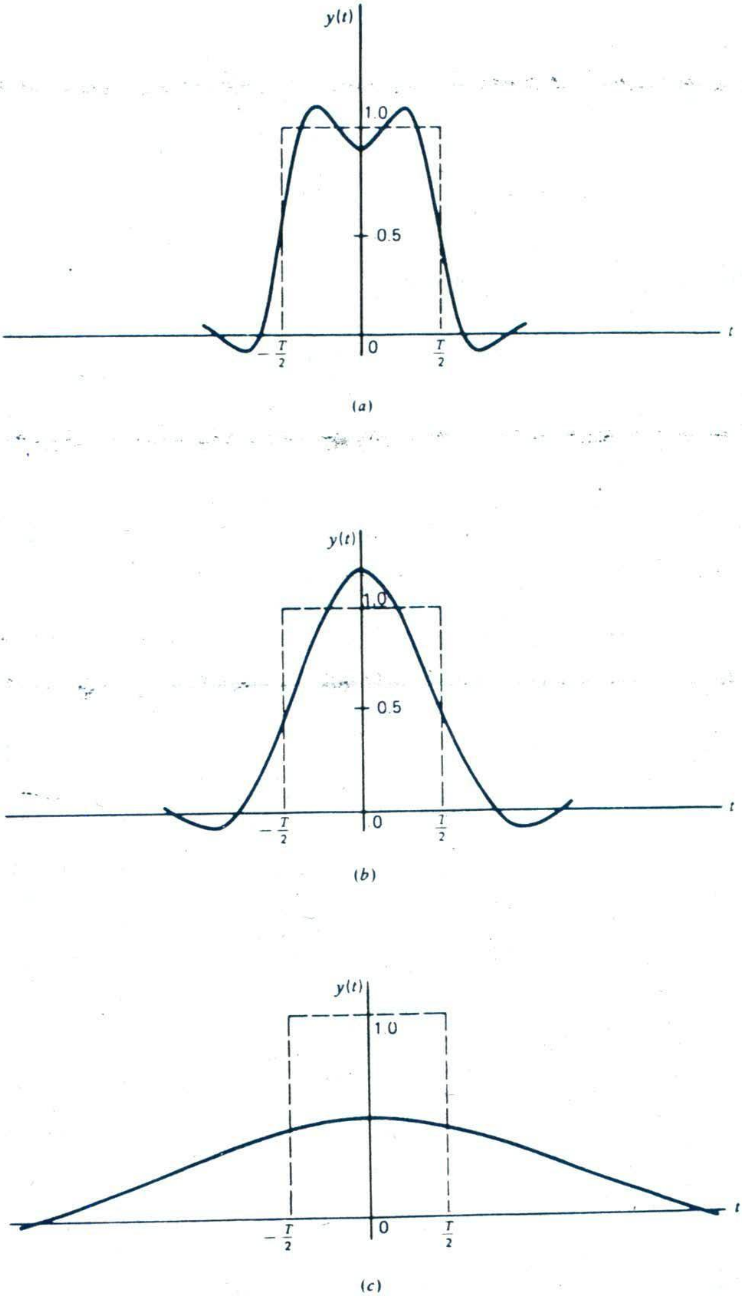


Figure 3.12
Pulse response of ideal low-pass filter for varying filter bandwidth. (a) $B = 2/T$. (b) $B = 1/T$. (c) $B = 1/4T$. The dashed rectangles represent the input signal.

2. When $B = 1/T$, as in Fig. 3.12*b*, the output is recognizable as a pulse; however, the rise and fall times of the output are significant compared with the input pulse duration T.
3. When the filter bandwidth B is small compared with $1/T$, the output is a grossly distorted version of the input, as in Fig. 3.12*c*.

EXERCISE 6 How large would you have to make the delay t_0 for the ideal low-pass filter to be causal?

3.5 BAND-PASS TRANSMISSION

A problem often encountered in the study of communication systems is that of analyzing the transmission of a signal through a band-pass system. Typically, the incoming signal and the system of interest are both *narrow-band* with a common midband frequency. We say that a band-pass signal is narrow-band if the bandwidth of the signal is small compared to its midband frequency. A similar definition holds for a band-pass system. A precise statement about how small the bandwidth must be in order for the signal to be considered narrow-band is not necessary for our present discussion. Obviously, we may analyze the *band-pass transmission problem* directly by using the convolution integral of Eq. 3.1 or its Fourier-transformed version given in Eq. 3.23. However, a more efficient approach is to replace the problem with an *equivalent low-pass transmission model,* the development of which proceeds in two stages. First, a complex low-pass representation is devised for the incoming band-pass signal. Next, a similar representation is devised for the band-pass system. In the sequel, these two representations are considered in turn.

COMPLEX LOW-PASS REPRESENTATION OF NARROW-BAND SIGNALS

Consider a narrow-band signal $x(t)$ with Fourier transform $X(f)$. The amplitude spectrum $|X(f)|$ of the signal is depicted in Fig. 3.13*a*. The *pre-envelope* of the signal $x(t)$ is defined by

$$x_+(t) = x(t) + j\hat{x}(t) \tag{3.55}$$

where $\hat{x}(t)$ is the Hilbert transform of the signal $x(t)$. The pre-envelope $x_+(t)$ is a complex-valued function of time with the original signal $x(t)$ as the real part and the Hilbert transform $\hat{x}(t)$ as the imaginary part. Let $X_+(f)$ denote the Fourier transform of the pre-envelope $x_+(t)$. We may thus write, in the frequency domain,

$$X_+(f) = X(f) + j\hat{X}(f) \tag{3.56}$$

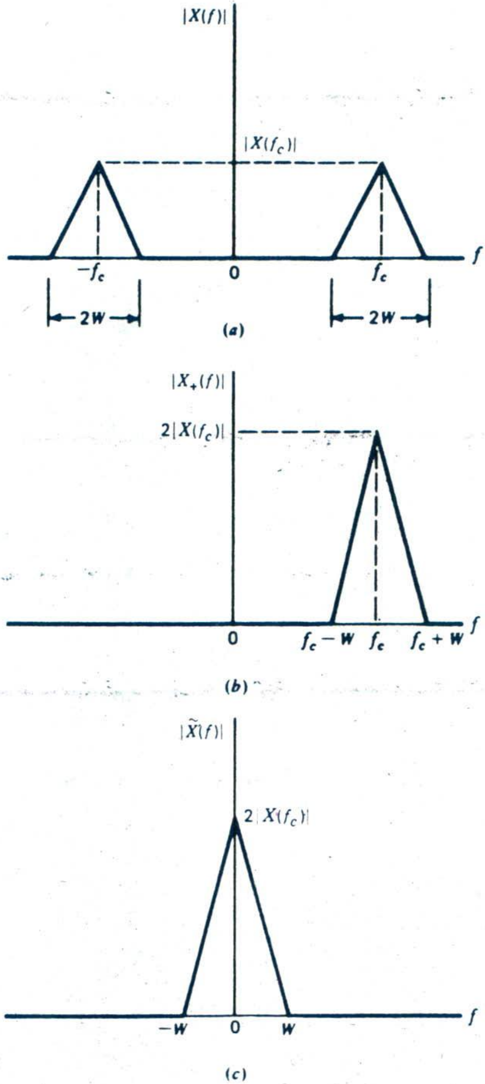


Figure 3.13
(a) Amplitude spectrum of band-pass signal $x(t)$. (b) Amplitude spectrum of pre-envelope $x_+(t)$. (c) Amplitude spectrum of complex envelope $\tilde{x}(t)$.

where $\hat{X}(f)$ is the Fourier transform of $\hat{x}(t)$. From Example 3, we deduce that $\hat{X}(f)$ equals the product $-j\,\text{sgn}\,(f)X(f)$, where $\text{sgn}(f)$ is the signum function. Accordingly, we may rewrite Eq. 3.56 as

$$
\begin{aligned}
X_+(f) &= X(f) + j[-j\,\text{sgn}(f)]X(f) \\
&= X(f) + \text{sgn}(f)X(f)
\end{aligned}
$$

Moreover, using the definition of the signum function, we get

$$X_+(f) = \begin{cases} 2X(f), & f > 0 \\ X(0), & f = 0 \\ 0, & f < 0 \end{cases} \tag{3.57}$$

where $X(0)$ is the zero-frequency value of $X(f)$. Equation 3.57 states that the pre-envelope of a Fourier transformable signal has no frequency content for negative frequencies, as illustrated in Fig. 3.13*b*.

The frequency-shifting property of the Fourier transform suggests that we may express the pre-envelope $x_+(t)$ in the form

$$x_+(t) = \tilde{x}(t) \exp(j2\pi f_c t) \tag{3.58}$$

where $\tilde{x}(t)$ is a complex-valued low-pass signal. The amplitude spectrum of $\tilde{x}(t)$ is illustrated in Fig. 3.13*c*.

Given the narrow-band signal $x(t)$, we may determine the *complex envelope* $\tilde{x}(t)$ by first using Eq. 3.55 to find the pre-envelope $x_+(t)$, and then solving Eq. 3.58 for $\tilde{x}(t)$ in terms of $x_+(t)$. Alternatively, we may determine $\tilde{x}(t)$ by using a frequency-domain approach based on $X(f)$, the Fourier transform of $x(t)$. Specifically, we retain the positive-frequency half of $X(f)$ centered on f_c, shift it to the left by f_c, and then scale it by a factor of two. The spectrum so obtained is the Fourier transform of the complex envelope $\tilde{x}(t)$. The rationale for this second method of determining $\tilde{x}(t)$ follows from the spectra depicted in Fig. 3.13. The second method is usually the preferred method, because it bypasses the need to know the Hilbert transform $\hat{x}(t)$.

The complex envelope $\tilde{x}(t)$ provides the basis for the complex low-pass representation of the narrow-band signal $x(t)$. Indeed, in accordance with Eqs. 3.55 and 3.58, the real part of the product $\tilde{x}(t) \exp(j2\pi f_c t)$ is equal to $x(t)$, as shown by

$$x(t) = \text{Re}[\tilde{x}(t) \exp(j2\pi f_c t)] \tag{3.59}$$

where Re[•] denotes the "real part of" the quantity enclosed in the square brackets. Using the *Euler identity*

$$\exp(j2\pi f_c t) = \cos(2\pi f_c t) + j \sin(2\pi f_c t)$$

and the definition for the complex envelope $\tilde{x}(t)$, we readily find from Eq. 3.59 that $x(t)$ may be expressed as[3]

$$x(t) = x_I(t) \cos(2\pi f_c t) - x_Q(t) \sin(2\pi f_c t) \tag{3.60}$$

[3]Equation 3.60 follows directly from the following rule. Let *a*, *b*, and *c* denote three complex numbers related to one another as

$$c = ab$$

This is the *canonical representation* for a narrow-band signal in terms of the *in-phase component* $x_I(t)$ and *quadrature component* $x_Q(t)$ of the complex envelope associated with the signal. Indeed, it is a representation basic to all linear modulation schemes; more will be said on this issue in Chapter 7.

The complex envelope $\tilde{x}(t)$ is defined in terms of the in-phase component $x_I(t)$ and the quadrature component $x_Q(t)$ as follows:

$$\tilde{x}(t) = x_I(t) + j\, x_Q(t) \tag{3.61}$$

In other words, $x_I(t)$ is the real part of $\tilde{x}(t)$, and $x_Q(t)$ is its imaginary part.

EXERCISE 7 Consider a narrow-band signal $x(t)$ with Fourier transform $X(f)$. Show that the value of $X_+(f)$, the Fourier transform of the pre-envelope of $x(t)$, at frequency $f = 0$ is $X(0)$.

EXERCISE 8 Let $x(t) = m(t) \cos(2\pi f_c t)$, where $m(t)$ is an information-bearing signal. What are the in-phase and quadrature components of $x(t)$? What is the complex envelope of $x(t)$?

COMPLEX LOW-PASS REPRESENTATION OF NARROW-BAND SYSTEM

Consider next a narrow-band system defined by the impulse response $h(t)$ or, equivalently, the transfer function $H(f)$. To develop a complex low-pass representation for this system, we may perform time-domain operations on $h(t)$ or frequency-domain operations on $H(f)$. From the previous discussion of narrow-band signals, we expect the second approach to be the preferred one, as it is computationally less intensive. Accordingly, from analogy with the complex low-pass representation of a narrow-band signal, we may develop the desired complex low-pass representation of the narrow-band system by retaining the positive-frequency half of the transfer function $H(f)$ centered on f_c, and shifting it to the left by f_c. Let $\tilde{H}(f)$ denote the transfer function of the complex low-pass system so defined. Figure 3.14 illustrates the relationship between $H(f)$ and $\tilde{H}(f)$, shown in parts *a* and *b* of the figure, respectively. Note, however, that in going from $H(f)$ to $\tilde{H}(f)$, we have purposely avoided amplitude scaling (see Exercise 9). Note also that for the frequency-domain transformation depicted in Fig. 3.14 to hold, the midband frequency f_c must be larger than half the bandwidth of the narrow-band system.

Then, the real part of c is given by

$\text{Re}[c] = \text{Re}[a]\,\text{Re}[b] - \text{Im}[a]\,\text{Im}[b]$

where Re[•] denotes the "real part of" and Im[•] denotes the "imaginary part of" the respective quantities enclosed in the square brackets.

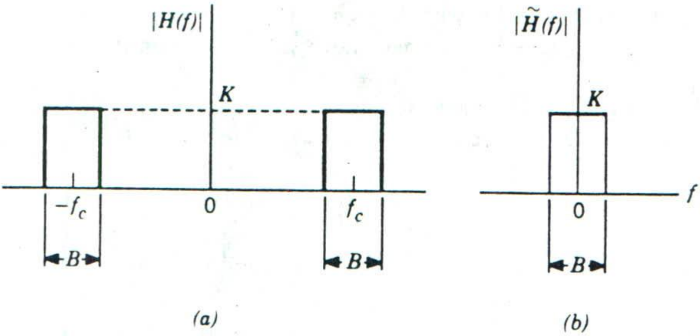


Figure 3.14
(a) Amplitude response of narrow-band system. (b) Amplitude response of complex low-pass system.

EQUIVALENT LOW-PASS TRANSMISSION MODEL

We are now equipped with the tools we need to formulate the equivalent low-pass transmission model for solving the band-pass transmission problem. Specifically, the analysis of a narrow-band system with transfer function $H(f)$ driven by a narrow-band signal with Fourier transform $X(f)$, as depicted in Fig. 3.15*a*, is replaced by an equivalent but simpler analysis of a complex low-pass system with transfer function $\tilde{H}(f)$ driven by a complex low-pass input with Fourier transform $\tilde{X}(f)$, as depicted in Fig. 3.15*b*. This *band-pass to low-pass transformation* completely retains the essence of the filtering process.

According to Fig. 3.15*a*, the Fourier transform of the output of the narrow-band system is given by

$$Y(f) = H(f)X(f)$$

The narrow-band output $y(t)$ itself is given by the inverse Fourier transform of $Y(f)$.

According to Fig. 3.15*b*, the Fourier transform of the output of the complex low-pass system is given by

$$\tilde{Y}(f) = \tilde{H}(f)\tilde{X}(f) \tag{3.62}$$

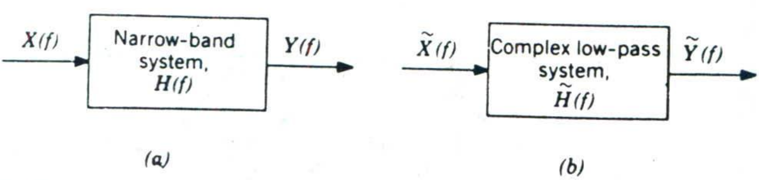


Figure 3.15
Transformation of narrow-band to complex low-pass system.

The complex low-pass output $\tilde{y}(t)$ itself is given by the inverse Fourier transform of $\tilde{Y}(f)$. Having determined $\tilde{y}(t)$, we may find the desired narrow-band output $y(t)$ simply by using the relation:

$$y(t) = \mathrm{Re}[\tilde{y}(t)\exp(j2\pi f_c t)] \tag{3.63}$$

The low-pass transmission model of Fig. 3.15*b* is said to be the *baseband equivalent* of the narrow-band system in Fig. 3.15*a*. The equivalence is in the sense that the model of Fig. 3.15*b* completely *preserves the information content of the incoming narrow-band signal $x(t)$ and also that of the outgoing narrow-band signal $y(t)$*. In general, the term "baseband" is used to designate the band of frequencies representing a signal of interest as delivered by a source of information. In the context of our present situation, the term baseband refers to both input and output.

EXERCISE 9 Evaluate $y(0)$ using the two models of Fig. 3.15. Hence, justify the need for scaling the spectrum of the complex low-pass input $\tilde{x}(t)$ by a factor of two, as depicted in Fig. 3.13*c*.

EXAMPLE 5 RESPONSE OF AN IDEAL BAND-PASS FILTER TO A PULSED RF WAVE

Consider an ideal band-pass filter of midband frequency f_c, and bandwidth B as in Fig. 3.16*a*, with $f_c > B/2$. Note that the conditions for distortionless transmission need only be satisfied for the pass band of the filter. Note also that the phase response of the filter is zero at the mid-band frequence f_c. We wish to determine the response of this filter to an RF pulse of duration T and frequency f_c defined by (see Fig. 3.17*a*)

$$x(t) = A\,\mathrm{rect}\left(\frac{t}{T}\right)\cos(2\pi f_c t)$$

where $f_c T \gg 1$.

Retaining the positive-frequency half of the transfer function $H(f)$, defined in Fig. 3.16*a*, and then shifting it to the origin, we find that the transfer function $\tilde{H}(f)$ of the low-pass equivalent filter is given by [see Fig. 3.16*b*]

$$\tilde{H}(f) = \begin{cases} \exp(-j2\pi f t_0), & -B/2 < f < B/2 \\ 0, & |f| > B/2 \end{cases} \tag{3.64}$$

The complex impulse response in this example has only a real component, as shown by

$$\tilde{h}(t) = B\,\mathrm{sinc}[B(t - t_0)] \tag{3.65}$$

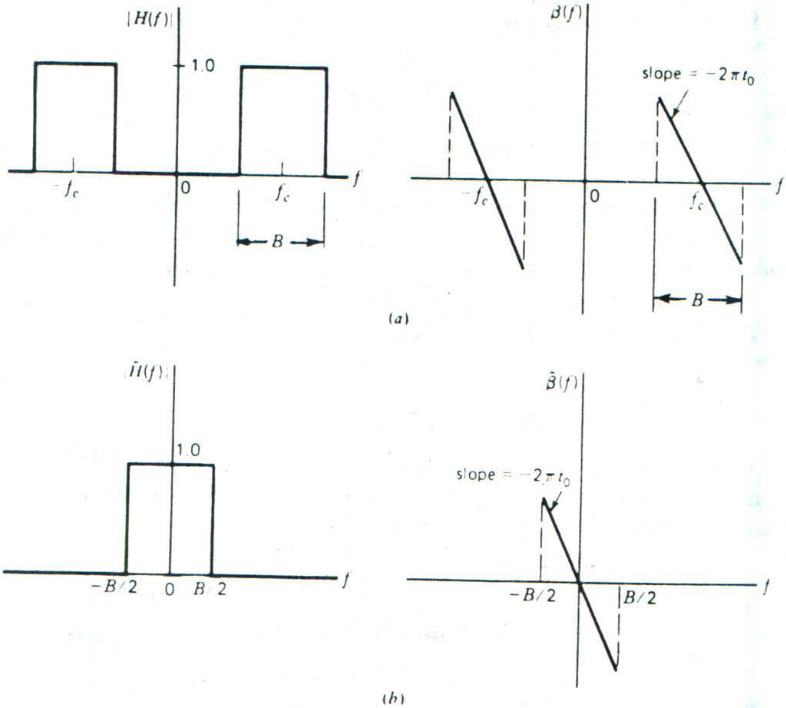


Figure 3.16
(a) Amplitude response $|H(f)|$ and phase response $\beta(f)$ of an ideal band-pass filter. (b) Corresponding components of complex transfer function $\tilde{H}(f)$.

From Example 3 we recall that the complex envelope $\tilde{x}(t)$ of the input RF pulse also has only a real component, as shown by (see Fig. 3.17*b*):

$$\tilde{x}(t) = A \operatorname{rect}\left(\frac{t}{T}\right) \tag{3.66}$$

The complex envelope $\tilde{y}(t)$ of the filter output is obtained by convolving the $\tilde{h}(t)$ of Eq. 3.65 with the $\tilde{x}(t)$ of Eq. 3.66. This convolution is exactly the same as the low-pass filtering operation that we studied in Example 3. Thus, using Eq. 3.53 we may write

$$\tilde{y}(t) = \frac{A}{\pi}\left\{\operatorname{Si}\left[\pi B\left(t + \frac{T}{2} - t_0\right)\right] - \operatorname{Si}\left[\pi B\left(t - \frac{T}{2} - t_0\right)\right]\right\} \tag{3.67}$$

As expected, the complex envelope $\tilde{y}(t)$ of the output has only a real component. Accordingly, from Eqs. 3.63 and 3.67, the output is obtained

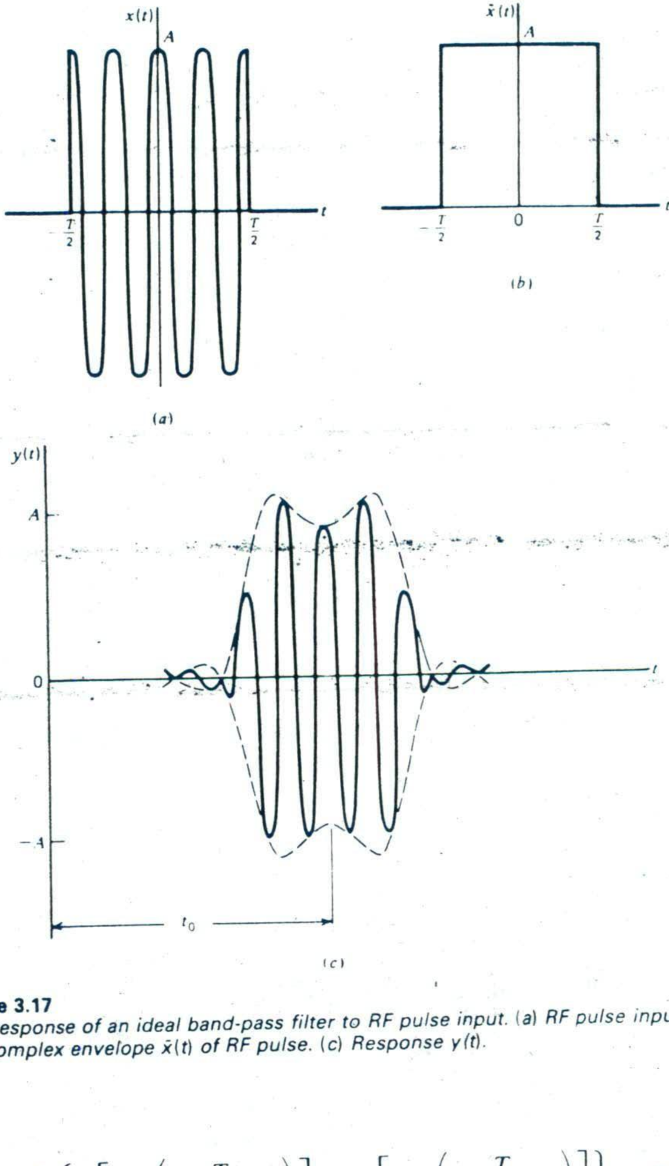


Figure 3.17
The response of an ideal band-pass filter to RF pulse input. (a) RF pulse input x(t). (b) Complex envelope $\tilde{x}(t)$ of RF pulse. (c) Response y(t).

as

$$y(t) = \frac{A}{\pi}\left\{\mathrm{Si}\left[\pi B\left(t + \frac{T}{2} - t_0\right)\right] - \mathrm{Si}\left[\pi B\left(t - \frac{T}{2} - t_0\right)\right]\right\}\cos(2\pi f_c t) \tag{3.68}$$

which is the desired result. Equation 3.68 is shown sketched in Fig. 3.17c for the case when the band-pass filter bandwidth $B = 1/T$.

3.6 PHASE DELAY AND GROUP DELAY

Suppose a steady sinusoidal signal at frequency f_c is transmitted through a *dispersive* channel that has a total phase-shift of $\beta(f_c)$ radians at that frequency. By using two phasors to represent the input signal and the received signal, we see that the received signal phasor lags the input signal phasor by $\beta(f_c)$ radians. The time taken for the received signal phasor to sweep out this phase lag is simply equal to $\beta(f_c)/2\pi f_c$ seconds. This time is called the *phase delay* of the channel.

It is important, however, to realize that the phase delay is not necessarily the true signal delay. This follows from the fact that a steady sinusoidal signal does not carry information. In actual fact, as we will see in subsequent chapters, information can be transmitted only by applying some appropriate change to the sinusoidal wave. Suppose then a slowly varying signal is multiplied by a sinusoidal wave, so that the resulting modulated wave consists of a narrow group of frequencies. When this modulated wave is transmitted through the channel, we find that there is a delay between the envelope of the input signal and that of the received signal. This delay is called the *envelope* or *group delay* of the channel and represents the true signal delay.

Assume that the dispersive channel is described by the transfer function

$$H(f) = K \exp[j\beta(f)] \tag{3.69}$$

where the amplitude K is a constant and the phase $\beta(f)$ is a nonlinear function of frequency. The input signal $x(t)$ consists of a narrow-band signal defined by

$$x(t) = x_c(t) \cos(2\pi f_c t) \tag{3.70}$$

where $x_c(t)$ is a low-pass function with its spectrum limited to the frequency interval $|f| \leq W$. We assume that $f_c \gg W$. By expanding the phase $\beta(f)$ in a *Taylor series*[4] about the point $f = f_c$, and retaining only the first two terms, we may approximate $\beta(f)$ as

$$\beta(f) \simeq \beta(f_c) + (f - f_c) \left.\frac{\partial \beta(f)}{\partial f}\right|_{f=f_c} \tag{3.71}$$

Define

$$\tau_p = -\frac{\beta(f_c)}{2\pi f_c} \tag{3.72}$$

[4]For a general definition of the Taylor series, see Appendix D, Table 4.

and

$$\tau_g = -\frac{1}{2\pi} \frac{\partial \beta(f)}{\partial f}\bigg|_{f=f_c} \tag{3.73}$$

Then we may rewrite Eq. 3.71 in the form

$$\beta(f) \simeq -2\pi f_c \tau_p - 2\pi(f - f_c)\tau_g \tag{3.74}$$

Correspondingly, the transfer function of the channel takes the form

$$H(f) \simeq K \exp[-j2\pi f_c \tau_p - j2\pi(f - f_c)\tau_g] \tag{3.75}$$

Following the procedure described in Section 3.5, we may replace the channel described by $H(f)$ by an equivalent low-pass filter with complex transfer function

$$\tilde{H}(f) \simeq K \exp(-j2\pi f_c \tau_p - j2\pi f \tau_g) \tag{3.76}$$

Similarly, we may replace the input narrow-band signal $x(t)$ by its low-pass complex envelope $\tilde{x}(t)$, which is

$$\tilde{x}(t) = x_c(t) \tag{3.77}$$

The Fourier transform of $\tilde{x}(t)$ is simply

$$\tilde{X}(f) = X_c(f) \tag{3.78}$$

where $X_c(f)$ is the Fourier transform of $x_c(t)$. Therefore, the Fourier transform of the complex envelope of the received signal is given by

$$\begin{aligned} \tilde{Y}(f) &= \tilde{H}(f)\tilde{X}(f) \\ &\simeq K \exp(-j2\pi f_c \tau_p) \exp(-j2\pi f \tau_g) X_c(f) \end{aligned} \tag{3.79}$$

We note that the multiplying factor $K \exp(-j2\pi f_c \tau_p)$ is a constant. We also note, from the time-shifting property of the Fourier transform, that the term $\exp(-j2\pi f \tau_g) X_c(f)$ represents the Fourier transform of the delayed signal $x_c(t - \tau_g)$. Accordingly, the complex envelope of the received signal equals

$$\tilde{y}(t) \simeq K \exp(-j2\pi f_c \tau_p) x_c(t - \tau_g) \tag{3.80}$$

Finally, we find that the received signal is itself given by

$$\begin{aligned} y(t) &= \text{Re}[\tilde{y}(t) \exp(j2\pi f_c t)] \\ &= K x_c(t - \tau_g) \cos[2\pi f_c(t - \tau_p)] \end{aligned} \tag{3.81}$$

Equation 3.81 shows that, as a result of transmission through the channel, two delay effects occur:

1. The sinusoidal carrier wave $\cos(2\pi f_c t)$ is delayed by τ_p seconds; hence τ_p represents the phase delay. Sometimes, τ_p is also referred to as the *carrier delay*.
2. The envelope $x_c(t)$ is delayed by τ_g seconds; hence, τ_g represents the envelope or group delay. Note that τ_g is related to the slope of the phase $\beta(f)$, measured at $f = f_c$, as in Eq. 3.73.

Note also that when the phase response $\beta(f)$ is linear with frequency, and $\beta(0) = 0$, the phase delay and group delay assume a common value.

EXERCISE 10 **Explain why a linear time-invariant system with a phase response equal to a constant suffers from phase distortion.**

3.7 NONLINEAR DISTORTION

Up to this point in our study of signal transmission through a system, we have assumed linearity. In practice, however, we find that the system connecting a source of information to its destination inevitably exhibits some form of *nonlinear* behavior. This occurs whenever the output is increased beyond a limit prescribed by the power that the system is capable of supplying. In such a situation, we say that the system is *overloaded*. When the system is overloaded, a change in the input signal does not produce a corresponding change in the output signal.

Figure 3.18 shows a typical input–output relation, called the *transfer characteristic*, that may give rise to nonlinear distortion. For the purpose of our discussion here, we assume that the system is *memoryless* in the sense that the output $y(t)$ depends only on the input $x(t)$ at time t. We may consider the transfer characteristic of Fig. 3.18 to be composed of the following parts:

1. A reasonably *linear region* centered at the origin, where a change in the input produces a proportional change in the output.
2. Two *saturation regions*, where the output is not affected by the input.
3. Two "knees" that join the linear region to the saturation regions. The useful amplitude range of operation of the system is defined by points P and Q that lie somewhere on the knees of the curve. Their precise locations are determined by the extent of nonlinear distortion that is considered to be tolerable. We may thus view P and Q as *overload points*.

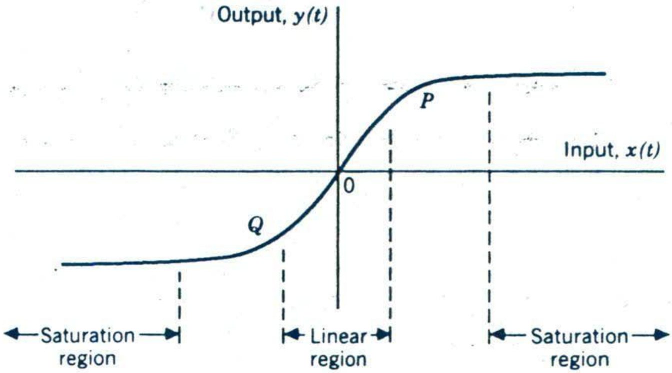


Figure 3.18
Transfer characteristic.

To evaluate the nonlinear distortion, the common procedure is to express the transfer characteristic mathematically by writing the output $y(t)$ as a power series of the input $x(t)$:

$$y(t) = a_1 x(t) + a_2 x^2(t) + a_3 x^3(t) + \cdots \tag{3.82}$$

The first term, $a_1 x(t)$, represents the linear response of the system. The second term, $a_2 x^2(t)$, accounts for a *lack of symmetry* that may exist between the positive and negative parts of the transfer characteristic. (This term would be zero for the symmetric curve shown in Fig. 3.18.) The third term, $a_3 x^3(t)$, provides a first approximation to the flattening of the transfer characteristic due to overloading. Higher order terms on the right side of Eq. 3.82 are usually neglected when operation of the system is bounded by the overload points (P and Q in Fig. 3.18).

Let $X(f)$ denote the Fourier transform of the input $x(t)$. Then, the Fourier transform of the output $y(t)$ is

$$\begin{aligned} Y(f) = {} & a_1 X(f) + a_2 X(f) \star X(f) \\ & + a_3 X(f) \star X(f) \star X(f) + \cdots \end{aligned} \tag{3.83}$$

where $\star$ denotes convolution. Thus, $X(f) \star X(f)$ denotes the convolution of $X(f)$ with itself, and so on. Let $x(t)$ be band-limited in W, such that $X(f) = 0$ for $|f| \geq W$. Then, $x^2(t)$ is band-limited in $2W$, such that $X(f) \star X(f) = 0$ for $|f| \geq 2W$. Similarly, $x^3(t)$ is band-limited in $3W$, such that $X(f) \star X(f) \star X(f) = 0$ for $|f| \geq 3W$, and so on. We may therefore make two observations:

1. The output of a nonlinear system contains new frequency components for $f > W$, which are not present in the input.
2. The presence of nonlinearities (second order, third order, etc.) in the

transfer characteristic produces undesirable frequency components for $|f| \leq W$.

The first set of components can be suppressed by appropriate filtering. On the other hand, the second set of components lying inside the frequency band of interest cannot be removed, thereby giving rise to *nonlinear distortion*.

Two examples to illustrate the analysis of nonlinear distortion follow. In both cases, the problem is simple enough to be handled without having to resort to the use of Fourier transformation.

EXAMPLE 6 HARMONIC DISTORTION

Let the input consist of a single sinusoidal wave:

$$x(t) = A\cos(2\pi f t) \tag{3.84}$$

We assume that only second- and third-order nonlinearities in the transfer characteristic of Fig. 3.18 are of concern, so that fourth- and higher-order terms in Eq. 3.82 may be ignored. Then, substitution of Eq. 3.84 in 3.82 yields the output

$$y(t) = \frac{1}{2}a_2A^2 + \left(a_1A + \frac{3}{4}a_3A^3\right)\cos(2\pi f t) + \frac{1}{2}a_2A^2\cos(4\pi f t) + \frac{1}{4}a_3A^3\cos(6\pi f t) \tag{3.85}$$

Since we are concerned primarily with distortion (i.e., changes in the shape of the waveform), we may ignore the dc component, $\frac{1}{2}a_2A^2$. The components of interest in the output waveform are therefore as follows, with their respective amplitudes shown:

Fundamental:	$a_1A + \frac{3}{4}a_3A^3$
Second harmonic:	$\frac{1}{2}a_2A^2$
Third harmonic:	$\frac{1}{4}a_3A^3$

Accordingly, we define the *second-harmonic distortion*, D_2, as the ratio of the amplitude of the second-harmonic component in the output to that of the fundamental:

$$D_2 = \frac{\frac{1}{2}a_2A}{a_1 + \frac{3}{4}a_3A^2} \tag{3.86}$$

Similarly, we define the *third-harmonic distortion,* D_3, as the ratio of the amplitude of the third-harmonic component in the output to that of the fundamental:

$$D_3 = \frac{\frac{1}{4} a_3 A^2}{a_1 + \frac{3}{4} a_3 A^2} \tag{3.87}$$

The harmonic distortion factors D_2 and D_3 are usually expressed as percentages.

EXAMPLE 7 INTERMODULATION DISTORTION

Let the input $x(t)$ consist of the sum of two sinusoidal waves:

$$x(t) = A_1 \cos(2\pi f_1 t) + A_2 \cos(2\pi f_2 t) \tag{3.88}$$

Here again we assume that fourth- and higher-order terms in Eq. 3.82 may be ignored. Then, substituting this expression for $x(t)$ in Eq. 3.82, we find that the effects produced by the second- and third-order nonlinearities in the transfer characteristic are:

1. The second-order term, $a_2 x^2(t)$, produces a dc and a second-harmonic component corresponding to the single-frequency input, as expected. In addition, however, it produces new components at $f_1 + f_2$ and $f_1 - f_2$ that are the *sum* and *difference frequencies,* respectively. Such components are referred to as *second-order intermodulation products.*

TABLE 3.1

Type of Intermodulation Product	Frequency	Amplitude
Second-order	$f_1 + f_2$	$a_2 A_1 A_2$
	$f_1 - f_2$	$a_2 A_1 A_2$
Third-order	$2f_1 + f_2$	$\frac{3}{4} a_3 A_1^2 A_2$
	$2f_1 - f_2$	$\frac{3}{4} a_3 A_1^2 A_2$
	$2f_2 + f_1$	$\frac{3}{4} a_3 A_1 A_2^2$
	$2f_2 - f_1$	$\frac{3}{4} a_3 A_1 A_2^2$

2. The third-order term, $a_3x^3(t)$, produces the expected fundamental and third-harmonic components. In addition, it gives rise to intermodulation products of its own at the frequencies $2f_1 \pm f_2$ and $2f_2 \pm f_1$, which are referred to as *third-order* intermodulation products.

Table 3.1 presents a summary of the frequencies and amplitudes of the various second- and third-order intermodulation products.

PROBLEMS

P3.1 Time Response

Problem 1 The excitation applied to a linear time-invariant system with impulse response $h(t)$ consists of two delta functions, as shown by

$$x(t) = \delta(t + t_0) + \delta(t - t_0)$$

where t_0 is a constant time shift. Find the response of the system.

Problem 2 The impulse response of a linear time-invariant system is defined by

$$h(t) = \exp(-at)u(t)$$

where $u(t)$ is the unit step function. Determine the response of the system produced by an excitation consisting of the unit step function $u(t)$.

Problem 3 A periodic signal $x_p(t)$ of period T_0 is applied to a linear time-invariant system of impulse response $h(t)$. Use the complex Fourier series representation of $x_p(t)$ and the convolution integral to evaluate the response of the system.

Problem 4 The impulse response of a linear time-invariant system is defined by the Gaussian function:

$$h(t) = \exp[-\pi(t - t_0)^2]$$

where t_0 is a constant.

(a) Is this system causal?
(b) Is it stable?

Give reasons for your answers.

P3.2 Frequency Response

Problem 5 Continuing with the linear time-invariant system described in Problem 2, do the following:

(a) Determine the transfer function of the system.

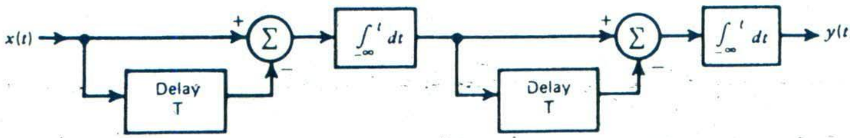


Figure P3.1

(b) Plot the amplitude response and phase response of the system.
(c) Find the 3-dB bandwidth of the system.

Problem 6 Find the transfer function of the linear time-invariant system with its impulse response defined in Problem 4. Hence, plot the amplitude response and phase response of the system. Indicate the 3-dB bandwidth of the system on the plot of the amplitude response.

Problem 7 Evaluate the transfer function of a linear system represented by the block diagram shown in Fig. P3.1.

Problem 8

(a) Determine the overall amplitude response of the cascade connection shown in Fig. P3.2 consisting of N identical stages, each with a time constant RC equal to τ_0.
(b) Show that as N approaches infinity, the amplitude response of the cascade connection approaches the Gaussian function $\exp(-\frac{1}{2}f^2T^2)$, where for each value of N, the time constant τ_0 is selected so that

$$\tau_0^2 = \frac{T^2}{4\pi^2 N}$$

P3.3 Linear Distortion and Equalization

Problem 9

(a) Consider a signal $x(t)$ with Fourier transform $X(f)$ limited to the band $-B \leqslant f \leqslant B$. This signal is applied to a linear time-invariant system with an amplitude response $|H(f)|$ and linear phase, as in Fig. P3.3*a*. Determine the resulting output of the system.

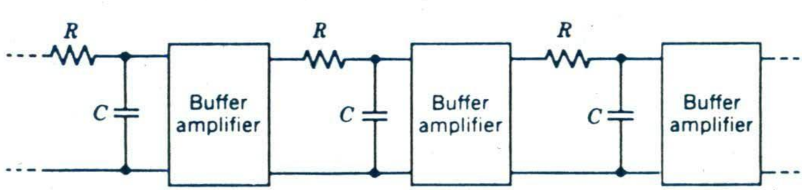


Figure P3.2

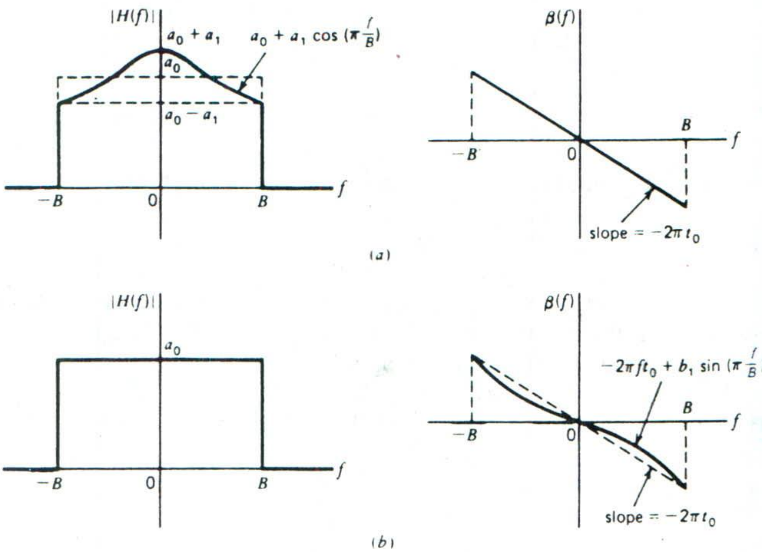


Figure P3.3

(b) Suppose that the system has a constant amplitude response but nonlinear phase, as in Fig. P3.3*b*. Determine the resulting output. Assume that the constant b_1 is small enough to justify using the approximation:

$$\exp\left[jb_1 \sin\left(\frac{\pi f}{B}\right)\right] \simeq 1 + jb_1 \sin\left(\frac{\pi f}{B}\right)$$

Problem 10 Figure P3.4 shows an idealized model of a radio channel. It consists of two paths. One path introduces a propagation delay t_0. The

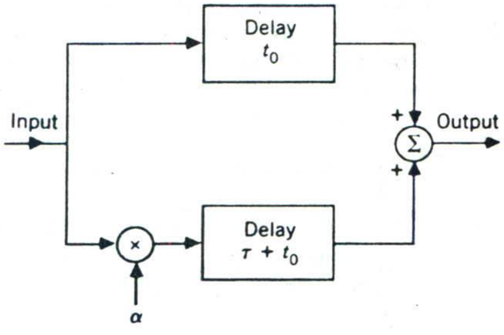


Figure P3.4

other path introduces an additional delay τ and an attenuation represented by the scaling factor α that is less than one. A channel so characterized is referred to as a *multipath channel*. To correct for the multipath distortion, a three-tap equalizer is connected in cascade with the channel. Given that $\alpha^2 \ll 1$, calculate the three tap coefficients of the tapped-delay-line equalizer.
Hint: Use the binomial expansion:

$$\frac{1}{1 + \alpha \exp(-j2\pi f\tau)} = 1 - \alpha \exp(-j2\pi f\tau) + \alpha^2 \exp(-j4\tau f\tau) - \cdots$$

P3.4 Ideal Low-Pass Filters

Problem 11 The transfer function of an ideal low-pass filter is defined by

$$H(f) = \begin{cases} K \exp(-j2\pi f t_0), & |f| < 1 \\ 0, & |f| > 1 \end{cases}$$

where t_0 is a constant. Find the impulse response of the system.

Problem 12 An ideal low-pass filter has zero time delay and bandwidth B. It is driven by a rectangular pulse of unit amplitude and duration T equal to $1/B$ and centered at $t = 0$.

(a) Show that the filter output at $t = 0$ is given by

$$y(0) = \frac{2}{\pi} \mathrm{Si}(\pi)$$

where $\mathrm{Si}(\pi)$ is the value of the sine integral for an argument equal to π.
(b) Show that the filter output at $t = T/2$ is given by

$$y\left(\frac{T}{2}\right) = \frac{1}{\pi} \mathrm{Si}(2\pi)$$

where $\mathrm{Si}(2\pi)$ is the value of the sine integral for an argument of 2π.
(c) Calculate these two values of the filter output and check them against the corresponding pulse response shown in Fig. 3.10*b*.

Note that $\mathrm{Si}(\pi) = 1.85$ and $\mathrm{Si}(2\pi) = 1.42$.

Problem 13 Suppose that, for a given signal $x(t)$, the integrated value of the signal over an interval T is required, as shown by

$$y(t) = \int_{t-T}^{t} x(\tau)\, d\tau$$

(a) Show that $y(t)$ can be obtained by transmitting the input signal $x(t)$ through a filter with its transfer function given by

$$H(f) = T \operatorname{sinc}(fT) \exp(-j\pi f T)$$

(b) An adequate approximation to this transfer function is obtained by using a low-pass filter with a bandwidth equal to $1/T$, passband amplitude response T, and delay $T/2$. Assuming this low-pass filter to be ideal, determine the filter output at time $t = T$ due to a unit step function applied to the filter and compare the result with the corresponding output of the ideal integrator.

Note that $\text{Si}(\pi) = 1.85$ and $\text{Si}(\infty) = \pi/2$.

P3.5 Band-Pass Transmission

Problem 14 An ideal band-pass filter has zero time delay and bandwidth B. An *RF* pulse of unit amplitude, duration $T = 1/2B$, and frequency f_c is applied to the filter; the pulse is centered at $t = 0$. Show that the filter output is given by

$$y(t) = \frac{1}{\pi} [\text{Si}(2\pi Bt + \pi) - \text{Si}(2\pi Bt - \pi)] \cos(2\pi f_c t)$$

where Si(•) is the sine integral. Sketch the waveform of $y(t)$.

Problem 15 Consider an ideal band-pass filter with center frequency f_c and bandwidth B, as defined in Fig. P3.5. The carrier wave $A \cos(2\pi f_0 t)$ is suddenly applied to this filter at time $t = 0$. Assuming that $|f_c - f_0|$ is large compared to the bandwidth B, determine the response of the filter.

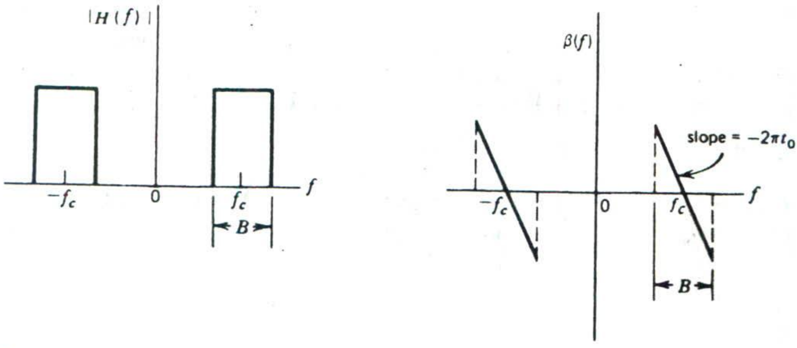


Figure P3.5

P3.6 Phase Delay and Group Delay

Problem 16 The impulse response of a linear time-invariant system is defined by

$$h(t) = \begin{cases} \exp(-t), & t > 0 \\ \frac{1}{2}, & t = 0 \\ 0, & t < 0 \end{cases}$$

(a) Determine the phase delay $\tau_p(f)$ and group delay $\tau_g(f)$ of the system.
(b) Plot both delays versus frequency f, and comment on your results.

P3.7 Nonlinear Distortion

Problem 17 Verify the frequencies and amplitudes of the intermodulation products listed in Table 3.1 for an input consisting of the sum of two sinusoidal waves of frequencies f_1 and f_2 and amplitudes A_1 and A_2.