PART I

DIFFERENTIAL EQUATIONS

1

Introduction

1·1. Definitions.

Differential Equation : Equations such as

(i) $\frac{dy}{dx}=\frac{\sqrt{(1-x^2)}}{\sqrt{(2-y^2)}}$,

(ii) $\left(\frac{dy}{dx}\right)^2+2y^2=4\left(\frac{dy}{dx}\right)+4x$,

(iii) $\frac{d^3y}{dx^3}+7\frac{d^2y}{dx^2}+8\frac{dy}{dx}-9y=\log x$,

(iv) $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=kz$,

(v) $\frac{\partial^2 z}{\partial x^2}+2\frac{\partial^2 z}{\partial x\,\partial y}+\frac{\partial^2 z}{\partial y^2}=0$,

which involve differential coefficients, are called the *differential equations.*

Ordinary Differential Equations. Equations like (i), (ii), (iii) which involve a single independent variable are called *ordinary differential equations.*

Partial Differential Equations. Equations like (iv) and (v) which involve partial differential coefficients with respect to more than one independent variable are called *partial differential equations.*

Order and Degree of Differential Equations. An equation like (iii) which involves a third order differential coefficient but none of higher order is said to be of third order. Equation (v) is also of second order.

The degree of a differential equation is the power (or degree) of the highest differential coefficient when the equation has been made rational. Thus equations (i), (iii), (iv) and (v) are all of first degree and equation (ii) is of second degree.

General Solution. The relation containing n arbitrary constants which satisfies an ordinary differential equation of nth order is called its *complete primitive* or *general solution.*

It can be shown that by eliminating n arbitrary constants from an equation in x, y, we get a differential equation of nth order. Such a process is called *formation of differential equations.*

Particular Solution. A particular solution of differential equaation is one obtained from the primitive by assigning definite values to the arbitrary constants.

Geometrically, the primitive is the equation of a family of curves satisfying the differential equation and a particular solution is the equation of some one of this family of curves.

We now give examples on the formation of differential equations by eliminating the arbitrary constants.

Ex. 1. *Eliminate the constants from* $y=ax+bx^2$. [Nag. 61 (S)]

Solution. We have $\frac{dy}{dx}=a+2bx$, $\frac{d^2y}{dx^2}=2b$.

Now $a=\frac{dy}{dx}-2bx=\frac{dy}{dx}-x\frac{d^2y}{dx^2}$, $b=\frac{1}{2}\frac{d^2y}{dx^2}$.

Putting these values of the constants in given equation, we get

$$y=x\left(\frac{dy}{dx}-x\frac{d^2y}{dx^2}\right)+\frac{1}{2}x^2\frac{d^2y}{dx^2}$$

or $$x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0,$$

which is a differential equation of second order, obtained from $y=ax+bx^2$ after eliminating the arbitrary constants a and b.

Ex. 2. *Eliminate the constant* a *from*

$$\sqrt{(1-x^2)}+\sqrt{(1-y^2)}=a\,(x-y). \quad ...(1)$$

Solution. Differentiating the given equation, we get

$$\frac{-x}{\sqrt{(1-x^2)}}-\frac{y}{\sqrt{(1-y^2)}}\frac{dy}{dx}=a\left(1-\frac{dy}{dx}\right). \quad ...(2)$$

Dividing (1) and (2) to eliminate a, we get

$$-\frac{\sqrt{(1-x^2)}+\sqrt{(1-y^2)}}{x/\sqrt{(1-x^2)}+y/\sqrt{(1-y^2)}\,(dy/dx)}=\frac{x-y}{(1-dy/dx)}$$

This can be simplified to give $\frac{dy}{dx}=\frac{\sqrt{(1-y^2)}}{\sqrt{(1-x^2)}}$.

We find here that after eliminating one arbitrary constant a, we get a differential equation of first order.

Ex. 3. *Form the differential equation of which*

$$c\,(y+c)^2=x^3, \quad ...(1)$$

is the complete integral.

Solution. Differentiating the given equation, we get

$$2c\,(y+c)\frac{dy}{dx}=3x^2. \quad ...(2)$$

Dividing (1) by (2),

$$\frac{y+c}{2\,dy/dx}=\frac{x}{3}, \text{ i.e., } 3\,(y+c)=2x\frac{dy}{dx}$$

or $$c=\tfrac{1}{3}\left(2x\frac{dy}{dx}-3y\right).$$

Putting this value of c in (2), we get

$$\tfrac{2}{3}\left(2x\frac{dy}{dx}-3y\right).\ \tfrac{2}{3}x\frac{dy}{dx}\frac{dy}{dx}=3x^2,$$

$$i.e.,\ 8x\left(\frac{dy}{dx}\right)^3-12y\left(\frac{dy}{dx}\right)^2=27x,$$

a differential equation of first order and third degree.

Ex. 4. *Form the differential equation corresponding to the family of curves $y=c(x-c)^2$ where c is an arbitrary constant.*

[Karnatak 1960]

Solution. Here $\dfrac{dy}{dx}=2c(x-c)$.

Dividing the given equation by (1),

$$\frac{y}{dy/dx}=\frac{x-c}{2}\quad\text{or}\quad c=x-\frac{2y}{p},\qquad\text{where } p=\frac{dy}{dx}.$$

Putting this value of c in (1), we get

$$p=2\left(x-\frac{2y}{p}\right)\left(\frac{2y}{p}\right),\quad i.e.,\ p^3=4y(px-2y),$$

which is the required differential equation.

Ex. 5. *Find the differential equation of all circles passing through the origin and having their centres on the x-axis.*

[Nag. T.D.C. 1961]

Solution. Equations of circles passing through the origin and having their centres on the x-axis is

$$x^2+y^2+2gx=0,$$

where g is an arbitrary constant.

Differentiating, $x+y\dfrac{dy}{dx}+g=0$, i.e., $g=-\left(x+y\dfrac{dy}{dx}\right)$.

Putting this value of g in the equation of circles, we get

$$x^2+y^2-2x\left(x+y\frac{dy}{dx}\right)=0,\quad i.e.,\ y^2=x^2+2xy\frac{dy}{dx}$$

which is the required differential equation.

Ex 5. *Find the differential equation of the family of parabolas with foci at the origin and axis along the x-axis.*

Solution. Let the directrix be $x=-2a$ and latus rectum be $4a$. Then equation of the parabola is

(distance from focus=distance from directrix),

$$x^2+y^2=(2a+x)^2\quad\text{or}\quad y^2=4a(a+x).\qquad\ldots(1)$$

Differentiating, $y\left(\dfrac{dy}{dx}\right)=2a$, or $a=\tfrac{1}{2}y\dfrac{dy}{dx}$.

Putting this value of a in (1), the differential equation is

$$y^2=2y\frac{dy}{dx}\left(\tfrac{1}{2}y\frac{dy}{dx}+x\right)\quad\text{or}\quad y\left(\frac{dy}{dx}\right)^2+2x\left(\frac{dy}{dx}\right)-y=0.$$

Ex. 7. *Form the differential equation that represents all parabolas each of which has a latus rectum 4a and whose axes are parallel to x-axis.*

Solution. Equation of the family of such parabolas is

$$(y-k)^2=4a\,(x-h), \quad \text{...(1)}$$

where h and k are arbitrary constant.

Differentiating, $(y-k)\dfrac{dy}{dx}=2a.$ ...(2)

Differentiating again, $(y-k)\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^2=0.$ ...(3)

Putting value of $y-k$ from (2) in (3), we get

$$2a\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3=0,$$

which is the required differential equation.

Ex. 9. *Form the differential equation of all parabolas whose axes are parallel to the axis of y.*

Solution. Such parabolas are given by

$$(x-h)^2=4a\,(y-k),$$

where h, k, a are three arbitrary constants.

Differentiating, $(x-h)\quad 2a\dfrac{dy}{dx}.$

Differentiating again, $1=2a\dfrac{d^2y}{dx^2}$ *i.e.,* $\dfrac{d^2y}{dx^2}=\dfrac{1}{2a}.$

Differentiating once again, $\dfrac{d^3y}{dx^3}=0.$

This is the required differential equation.

Ex. 9. *Form differential equation of all conics whose axes coincide with the axes of co-ordinates.* **[Delhi Hons. 1958]**

Solution. Such conics are given by

$$ax^2+by^2=1, \quad \text{...(1)}$$

where a and b are two arbitrary constants.

Differentiating, $\dfrac{dy}{dx}=-\dfrac{ax}{by},\ \dfrac{d^2y}{dx^2}=-\dfrac{a}{b}\left(\dfrac{1}{y}-\dfrac{x}{y^2}\dfrac{dy}{dx}\right),$

i.e. $\dfrac{d^2y}{dx^2}=\dfrac{y}{x}\dfrac{dy}{dx}\left(\dfrac{1}{y}-\dfrac{x}{y^2}\dfrac{dy}{dx}\right)$ as $-\dfrac{a}{b}=\dfrac{y}{x}\dfrac{dy}{dx}$

or $xy\dfrac{d^2y}{dx^2}+x\left(\dfrac{dy}{dx}\right)^2=y\dfrac{dy}{dx},$

which is the required differential equation.

Ex. 10. *Form the differential equation in the following cases:*

(i) [illegible] (*parameter*). **Ans.** $y=\sqrt{(1-x^2)}\dfrac{dy}{dx}$

(ii) $y=ae^{2x}+be^{-3x}+ce^{x}$ (a, b, c *parameters*).

Ans. $\dfrac{d^3y}{dx^3}-7\dfrac{dy}{dy}+6y=0.$

(iii) $ay^2=(x-c)^3$ (c *parameter*). Ans. $8a\left(\dfrac{dy}{dx}\right)^3=27y.$

(iv) $y=cx+c-c^3$ (c *parameter*).

Ans. $y=x\dfrac{dy}{dx}+\dfrac{dy}{dx}-\left(\dfrac{dy}{dx}\right)^3.$

(v) $e^{2y}+2cxe^{y}+c^2=0$ (c *parameter*).

Ans. $(1-x^2)\left(\dfrac{dy}{dx}\right)^2+1=0.$

(vi) $y=a\cos(mx+b)$ (a, b *parameters*).

Ans. $\dfrac{d^2y}{dx^2}+m^2y=0.$

(vii) $xy=ae^{x}+be^{-x}$ (a, b *parameters*).

Ans. $x\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}-xy=0.$

(viii) $xy=Ae^{x}+Be^{-x}+x^2$. [Osmania 60]

Ans. $x\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}=(xy-x^2)+2$

Ex. 11. *Find the differential equation of all circles of radius a.* [Delhi Hons. 66 ; Poona 63]

Hint. Equation of the circle is $(x-h)^2+(y-k)^2=a^2$. Eliminate h and k to get the diff. equation in the usual way.

Ex. 11. (b) *Find the differential equation of all circles which have their centres on x-axis and have a given radius.* [Marathwada 60]

Hint. Equation of the circle is $(x-h)^2+y^2=a^2$, where h is the parameter.

Ex. 12. *Define (i) General solution, (ii) Particular solution of a differential equation and obtain the differential equation of the family of curves $y=e^{x}(A\cos x+B\sin x)$.* [Poona 64]

Hint. Curve is $y=e^{x}(A\cos x+B\sin x)$.

$$\therefore\quad \frac{dy}{dx}=e^{x}(A\cos x+B\sin x)+e^{x}(-A\sin x+B\cos x)$$

$$=y+e^{x}(-A\sin x+B\cos x).$$

Also
$$\frac{d^2y}{dx^2}=\frac{dy}{dx}+e^{x}(-A\sin x+B\cos x)-e^{x}(A\cos x+B\sin x)$$

$$=\frac{dy}{dx}+\left(\frac{dy}{dx}-y\right)-y.$$

Thus
$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=0$$

is the required differential equation.

2

Equations of First Order and First Degree

2·1. Differential equation of the first order and first degree. A differential equation of the type

$$M+N\frac{dy}{dx}=0,$$

where M and N are functions of x and y or constants, is called *a differential equation of the first order and first degree.*

We give below some methods of solving such equations.

2·2. Solution of the differential equation when variables are separable.

If an equation can be written in such a way that dx and all the terms containing x are on one side and dy and all the terms containing y on the other side, then this is an equation in which variables are separable. Such equations can therefore be written as $f_1(x)\,dx=f_2(y)\,dy$ and can be solved by integrating directly and adding a constant on either side.

Ex. 1. *Solve* $\dfrac{dy}{dx}=\dfrac{1+y^2}{1+x^2}.$

Solution Separating the variables the equation becomes

$$\frac{dy}{1+y^2}=\frac{dx}{1+x^2}$$

Integrating, we get $\tan^{-1} y=\tan^{-1} x+A$

or $\tan^{-1} y-\tan^{-1} x=A$ *i.e.*, $\tan^{-1}\dfrac{y-x}{1+xy}=A=\tan^{-1} C$ (say).

$\therefore \quad y-x=C(1+xy)$

which is the solution.

Ex. 2. *Solve* $\dfrac{dy}{dx}=e^{x-y}+x^2e^{-y}.$

[Gorakhpur 59 ; Andhra 60 ; Sagar 54]

Solution. The given equation can be written as

$$e^y\,dy=(e^x+x^2)\,dx.$$

Integrating, $\quad e^y=e^x+\frac{1}{3}x^3+C.$

Ex. 3. *Solve* $\sec^2 x \tan y\,dx+\sec^2 y \tan x\,dy=0$

[Nagpur T [illegible] Delhi 51]

Solution. Separating the variables, we get

$$\frac{\sec^2 x}{\tan x}dx+\frac{\sec^2 y}{\tan y}dy=0.$$

Integrating, $\log \tan x+\log \tan y=A$

or $\tan x \tan y=e^A=C.$

Ex. 4. *Solve* $(y-px)\,x=y.$ [Saugar 62]

Solution. Equation is $px^2=y\,(x-1)$, *i.e.*, $\dfrac{dy}{dx}=\dfrac{y\,(x-1)}{x^2}$,

i.e., $$\frac{dy}{y}=\frac{x-1}{x^2}dx=\left(\frac{1}{x}-\frac{1}{x^2}\right)dx.$$

Integrating, $\log y=\log x+\dfrac{1}{x}+\log A$ or $\dfrac{y}{x}=Ae^{1/x}$.

Ex. 5. *Solve* $y-x\dfrac{dy}{dx}=a\left(y^2+\dfrac{dy}{dx}\right).$ [Saugar 63]

Solution. The equation can be written as

$$\frac{dx}{x+a}=\frac{dy}{y\,(1-ay)}=\left(\frac{1}{y}+\frac{a}{1-ay}\right)dy.$$

Integrating, $x+a=C\dfrac{y}{1-ay}$.

Ex. 6. *Solve*

(i) $(3+2\sin x+\cos x)\,dy=(1+2\sin y+\cos y)\,dx.$

(ii) $(e^y+1)\cos x\,dx+e^y\sin x\,dx=0.$ [Poona 64]

2·3. Equations reducible to the form in which variables are separable.

Equations of the form

$$\frac{dy}{dx}=f(ax+cy+c)$$

can be reduced to an equation in which variables can be separated What is required is that we put

$$ax+by+c=v,$$

so that $a+b\dfrac{dy}{dx}=\dfrac{dv}{dx}$, *i.e.*, $\dfrac{dy}{dx}=\dfrac{1}{b}\left[\dfrac{dv}{dx}-a\right)$.

Then the equation becomes

$$\frac{1}{b}\left(\frac{dv}{dx}-a\right)=f(v) \text{ or } \frac{dv}{dx}=a+bf(v),$$

in which variables are separable.

Ex. 1. *Solve* $\dfrac{dy}{dx}=(4x+y+1)^2.$

[Raj. 61 ; Agra 54 ; Gujrat 65, 58]

Solution. Put $4x+y+1=v$, so that $4+\dfrac{dy}{dx}=\dfrac{dv}{dx}$.

The equation then reduces to

$$\frac{dv}{dx}-4=v^2 \quad \text{or} \quad \frac{dv}{dx}=v^2+4.$$

The variables are now separable and we can write $\dfrac{dv}{v^2+4}=dx.$

Integrating $\frac{1}{2}\tan^{-1}\left(\dfrac{v}{2}\right)=x+C$

or $\quad \frac{1}{2}\tan^{-1}\left(\dfrac{4x+y+1}{2}\right)=x+C$ is the solution.

Ex. 2. *Solve* $\dfrac{dy}{dx}=\sin(x+y)+\cos(x+y).$ [Agra B.Sc. 67]

Solution. Put $x+y=v$, $\quad 1+\dfrac{dy}{dx}=\dfrac{dv}{dx}.$

$\therefore$ equation is $\dfrac{dv}{dx}-1=\sin v+\cos v$ or $\dfrac{dv}{dx}=1+\sin v+\cos v$

or $\quad dx=\dfrac{dv}{1+\sin v+\cos v}=\dfrac{dv}{2\cos^2\frac{1}{2}v+2\sin\frac{1}{2}v\cos\frac{1}{2}v}$

or $\quad \dfrac{dv}{2\cos^2\frac{1}{2}v\,(1+\tan\frac{1}{2}v)}=dx$ or $\dfrac{\frac{1}{2}\sec^2\frac{1}{2}v\,dv}{1+\tan\frac{1}{2}v}=dx.$

Integrating, $\log(1+\tan\frac{1}{2}v)=x+C$, where $v=x+y$.

$\therefore \quad \log[1+\tan\frac{1}{2}(x+y)]=x+C$ is the required solution.

Ex. 3. *Solve* $(x-y)^2\dfrac{dy}{dx}=a^2.$

[Calcutta Hons. 63; Bihar 61; Vikram 65]

Solution. Put $x-y=v$, so that $1-\dfrac{dy}{dx}=\dfrac{dv}{dx}.$

$\therefore$ equation is $v^2\left[1-\dfrac{dv}{dx}\right]=a^2$ or $\dfrac{dv}{dx}=\dfrac{v^2-a^2}{v^2}$

or $\quad dx=\dfrac{v^2}{v^2-a^2}\,dv=\left(1+\dfrac{a^2}{v^2-a^2}\right)dv.$

Integrating, $x+C=v+a^2\,\dfrac{1}{2a}\log\dfrac{v-a}{v+a}$

or $\quad x+C=(x-y)+\frac{1}{2}a\log\dfrac{x-y-a}{x-y+a}$ is the solution.

Ex. 4. *Solve* $(x+y)^2\dfrac{dy}{dx}=a^2$

[Poona 64; Raj. 63; Delhi Hons. 60; Alld 60]

Solution. Put $x+y=v$, so that $1+\dfrac{dy}{dx}=\dfrac{dv}{dx}.$

$\therefore \quad v^2\left(\dfrac{dv}{dx}-1\right)=a^2,\ \dfrac{dv}{dx}=1+\dfrac{a^2}{v^2}=\dfrac{a^2+v^2}{v^2}$

$$\therefore \quad dx=\frac{v^2}{a^2+v^2}\,dv=\left(1-\frac{a^2}{a^2+v^2}\right)dv.$$

Integrating, $x+C=v-a\tan^{-1}\dfrac{v}{a}$

or $x+C=(x+y)-a\tan^{-1}\dfrac{x+y}{a}$

or $y=C+a\tan^{-1}\dfrac{x+y}{a}$ is the solution.

***Ex. 5.** *Solve* $\dfrac{x\,dx+y\,dy}{x\,dy-y\,dx}=\sqrt{\left(\dfrac{a^2-x^2-y^2}{x^2+y^2}\right)}.$

[Delhi Hons. 62; Agra B.Sc. 55]

Solution. Here we change to polar co-ordinates by putting

$x=r\cos\theta,\ y=r\sin\theta,\ x^2+y^2=r^2,\ x\,dx+y\,dy=r\,dr.$

$\dfrac{y}{x}=\tan\theta,\quad \therefore\ \dfrac{x\,dy-y\,dx}{x^2}=\sec^2\theta\,d\theta$ or $x\,dy-y\,dx=r^2\,d\theta.$

$\therefore$ the equation becomes $\dfrac{1}{r}\dfrac{dr}{d\theta}=\sqrt{\left(\dfrac{a^2-r^2}{r^2}\right)}$

Separating the variables, $\dfrac{dr}{\sqrt{(a^2-r^2)}}=d\theta.$

Integrating, $\sin^{-1}(r/a)=\theta+C$ or $r=a\sin(\theta+C).$

i.e., $\sqrt{(x^2+y^2)}=a\sin[\tan^{-1}(y/x)+C].$

Ex. 6. *Solve* $x\dfrac{dy}{dx}-y=x\sqrt{(x^2+y^2)}.$ [Bombay 61; Agra 56]

Solution. The equation can be put as

$x\,dy-y\,dx=x\sqrt{(x^2+y^2)}\,dx$ or $\dfrac{x\,dy-y\,dx}{x^2}=\sec^2\theta\,d\theta.$

Changing to polars as above, the equation becomes

$x^2\sec^2\theta\,d\theta=xr\,dx$

or $x\sec^2\theta\,d\theta=r\,dx$ or $r\cos\theta\sec^2\theta\,d\theta=r\,dx$

or $\sec\theta\,d\theta=dx$, variables separated.

Integrating, $\log(\sec\theta+\tan\theta)=x+\log C.$

$\therefore\ \sec\theta+\tan\theta=ce^x$ or $\sqrt{(1+y^2/x^2)}+y/x=ce^x.$

Ex. 7. *Solve* $\left(\dfrac{x+y-a}{x+y-b}\right)\dfrac{dy}{dx}=\left(\dfrac{x+y+a}{x+y+b}\right)$

[Delhi Hons. 63; Nagpur 55]

Solution. Put $x+y=v$, so that $1+\dfrac{dy}{dx}=\dfrac{dv}{dx}$

i.e., $\dfrac{dv}{dx}=1+\left(\dfrac{v+a}{v+b}\right)\left(\dfrac{v-b}{v-a}\right)=\dfrac{2(v^2-ab)}{v^2+(b-a)v-ab}$

or $2\,dx=\left(1+\dfrac{b-a}{2}\cdot\dfrac{2v}{v^2-ab}\right)dv.$

Integrating, $2x+C=v+\frac{b-a}{2}\log(v^2-ab)$

or $2x+C=x+y+\frac{1}{2}(b-a)\log[(x+y)^2-ab]$ etc.

Ex. 8. $\frac{dy}{dx}=(x+y)^2$. **[Gauhati 62; Delhi 62; Raj. 62]**

Hint. Put $x+y=v$ etc.

2·4. Homogeneous Differential Equations. [Poona 61 (S)]

An equation of the form $\frac{dy}{dx}=\frac{f_1(x,y)}{f_2(x,y)}$ in which $f_1(x,y)$ and $f_2(x,y)$ are homogeneous functions* of x and y of the same degree can be reduced to an equation in which variables are separable by putting $y=vx$, $\frac{dy}{dx}=v+x\frac{dv}{dx}$.

The following few examples will illustrate the method.

Ex. 1. *Solve* $(x^2+y^2)\,dx+2xy\,dy=0$.

Solution. We have $\frac{dy}{dx}=-\frac{x^2+y^2}{2xy}$ (homogeneous).

Putting $y=vx$, $\frac{dy}{dx}=v+x\frac{dv}{dx}$, the equation becomes

$$v+x\frac{dv}{dx}=-\frac{x^2+x^2v^2}{2x.vx}=-\frac{1+v^2}{2v}$$

or $$x\frac{dv}{dx}=-\frac{1+v^2}{2v}-v=-\frac{1+3v^2}{2v} \text{ (variable separable).}$$

$$\therefore \quad \frac{dx}{x}=-\frac{2v}{1+3v^2}\,dv.$$

Integrating, $\log x+\frac{1}{3}\log(1+3v^2)=\log C$

or $x(1+3v^2)^{1/3}=C$ or $x(1+3y^2/x^2)^{1/3}=C$.

Ex. 2. *Solve* $x^2y\,dx-(x^3+y^3)\,dy=0$. [Agra B.Sc. 54]

Solution. We have $\frac{dy}{dx}=\frac{x^2y}{x^3+y^3}$ (homogeneous).

Putting $y=vx$, $\frac{dy}{dx}=v+x\frac{dv}{dx}$, the equation becomes

$$v+x\frac{dv}{dx}=\frac{v}{1+v^3} \quad\text{or}\quad x\frac{dv}{dx}=\frac{v}{1+v^3}-v=-\frac{v^4}{1+v^3}$$

or $$\frac{dx}{x}=-\frac{1+v^3}{v^4}\,dv=-\left[\frac{1}{v^4}+\frac{1}{v}\right]dv.$$

Integrating, $\log x=\frac{1}{3v^3}-\log v+C$; $\log vx=\frac{1}{3v^3}+C$

*A function $f(x, y)$ is called homogeneous of degree n, if $f(tx, ty)=t^n f(x, y)$.

or $\log y = \dfrac{x^3}{3y^3} + C$ as $v = \dfrac{y}{x}$.

Ex. 3. *Solve* $\dfrac{dy}{dx} = \dfrac{y^3 + 3x^2y}{x^3 + 3xy^2}$. **[Lucknow Pass 60]**

Solution. Putting $y = vx$, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$, we get

$$x\frac{dv}{dx} = \frac{dy}{dx} - v = \frac{v^3 + 3v}{1 + 3v^2} - v = \frac{2v(1 - v^2)}{1 + 3v^2}$$

or $\dfrac{2\,dx}{x} = \dfrac{1 + 3v^2}{2v(1 - v^2)}\,dv = \left(\dfrac{1}{v} - \dfrac{2}{1+v} + \dfrac{2}{1-v}\right)dv.$

Integrating,

$$2\log x = \log v - 2\log(1-v) - 2\log(1+v) + \log C$$

or $x^2(1-v)^2(1+v)^2 = Cv$. Put $v = y/x$ etc.

Ex. 4. *Solve* $y^2 + x^2\dfrac{dy}{dx} = xy\dfrac{dy}{dx}$.

[Delhi Hons. 66; Cal. Hons. 61, 56; Osmania 60; Gujrat 61]

Solution. The equation is $\dfrac{dy}{dx} = \dfrac{y^2}{xy - x^2}$ [homogeneous].

Putting $y = vx$, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$, we get

$$v + x\frac{dv}{dx} = \frac{v^2}{v-1} \quad \text{or} \quad x\frac{dv}{dx} = \frac{v^2}{v-1} - v$$

or $x\dfrac{dv}{dx} = \dfrac{v}{v-1}$ or $\dfrac{dx}{x} = \dfrac{v-1}{v}\,dv$

or $\dfrac{dx}{x} = \left(1 - \dfrac{1}{v}\right)dv.$

Integrating, $\log x = v - \log v + \log c$

or $\log xv = v + \log c$ or $xv = ce^v$

or $y = ce^{y/x}$ as $y = vx$.

Ex. 5. *Solve* $(x^2 + y^2)\,dy = xy\,dx$. **[Nagpur T.D.C. 1961]**

Hint. Homogeneous. Put $y = vx$. **Ans** $y = Ce^{x^2/2y^2}$.

Ex. 6. *Solve the following homogeneous equations :*

(i) $y(y^2 - 2x^2)\,dx + x(2y^2 - x^2)\,dy = 0.$

[Karnatak B.Sc (Sub.) 1960]

(ii) $\dfrac{1}{2x}\dfrac{dy}{dx} + \dfrac{x+y}{x^2+y^2} = 0.$ **[Lucknow Pass 1955]**

(iii) $\dfrac{dy}{dx} + \dfrac{y(x+y)}{x^2} = 0.$ **Ans.** $x^2y = c^2(y + 2x)$

[Poona 1964; Nag. 58; Kerala 61; Vikram 61]

(iv) [illegible] dx [illegible] $x^2\,dy - y^3\,dy$. **Ans.** $\log$ [illegible] $\dfrac{\ }{3y}$ [illegible] C.

(v) $(x^2-y^2)\frac{dy}{dx}=xy$.

(vi) $(x+y)^2=xy\,\frac{dy}{dx}$.

[Poona 1964]

(vii) $x\frac{dy}{dx}-y=\sqrt{(x^2+y^2)}$. [Sagar 1963; Cal. Hons. 62; Raj. 56]

(Cf. Ex. 6 P. 10) Ans. $x^2+y^2=(Cx^2-y)^2$.

Ex. 7. $\left(x\cos\frac{y}{x}+y\sin\frac{y}{x}\right)y=\left(y\sin\frac{y}{x}-x\cos\frac{y}{x}\right)x\frac{dy}{dx}$

[Cal. Hons 1962]

or $x\cos\frac{y}{x}(y\,dx+x\,dy)=y\sin\frac{y}{x}(x\,dy-y\,dx)$.

[Raj. 1959; Cal. Hons. 61, 55; Delhi 68, 61]

Solution. The equation is $\frac{dy}{dx}=\frac{y\,(y\sin y/x+x\cos y/x)}{x\,(y\sin y/x-x\cos y/x)}$.

Putting $y=vx$, $\frac{dv}{dx}=\frac{dy}{dx}-v=\frac{2v\cos v}{v\sin v-\cos v}$

or $\left(\tan v-\frac{1}{v}\right)dv=2\frac{dx}{x}$, *i.e.*, $\log\frac{\sec v}{v}=\log C+2\log x$

or $\sec(y/x)=Cxy$ is the solution.

Ex. 8. *Solve* $\left(x\sin\frac{y}{x}\right)\frac{dy}{dx}=\left(y\sin\frac{y}{x}-x\right)$ [Delhi Pass 67]

Solution. Equation is $\frac{dy}{dx}=\frac{y}{x}-\operatorname{cosec}\frac{y}{x}$.

Putting $y=vx$, $\frac{dy}{dx}=x\frac{dv}{dx}+v$.

Equation reduces to $\sin v\,dv=-\frac{dx}{x}$.

Integrating, $-\cos v=-\log Cx$

or $\cos\frac{y}{x}=\log Cx$ is the solution.

Ex. 9. *Solve* $(x^2+2xy-y^2)\,dx+(y^2+2xy-x^2)\,dy=0$.

[Gujrat B.Sc. (Prin.) 1961]

Solution. $\frac{dy}{dx}=-\frac{x^2+2xy-y^2}{y^2+2xy-x^2}$. Put $y=vx$.

$\therefore\quad v+x\frac{dv}{dx}=-\frac{1+2v-v^2}{v^2+2v-1}$.

$x\frac{dv}{dx}=-\frac{1+2v-v^2}{v^2+2v-1}-v=-\frac{v^3+v^2+v+1}{v^2+2v-1}$.

$\therefore\quad \frac{dx}{x}=-\frac{v^2+2v-1}{v^3+v^2+v+1}\,dv=-\frac{v^2+2v-1}{(v+1)(v^2+1)}\,dv$

$$=\left(\frac{1}{v+1}-\frac{2v}{v^2+1}\right)dv.$$

Integrating, $\log x=\log(v+1)-\log(v^2+1)+\log C$

or $\dfrac{x}{v^2+1}=C(v+1)$ or $\dfrac{x}{y^2/x^2+1}=C\left(\dfrac{y}{x}+1\right)$.

Ex. 10. *Solve* $2y^3\,dx+(x^2-3y^2)\,x\,dy=0.$

[Bombay B.Sc. (Sub.) 1962]

Solution. Proceed yourself.

2·5. Equation Reducible to Homogeneous Form.

An equation of the type $\dfrac{dy}{dx}=\dfrac{ax+by+c}{a'x+b'y+c'}$, when $\dfrac{a}{a'}\neq\dfrac{b}{b'}$ can be reduced to homogeneous form as follows :

Put $x=X+h,\ y=Y+k$; then $\dfrac{dy}{dx}=\dfrac{dY}{dX}$, where $X,\ Y$ are new variables and h,k are arbitrary constants. The equation now becomes

$$\frac{dY}{dX}=\frac{aX+bY+(ah+bk+c)}{a'X+b'Y+(a'h+b'k+c')}.$$

We choose the constants h and k in such a way that

$$ah+bk+c=0,\quad a'h+b'k+c'=0.$$

With this substitution the differential equation reduces to $\dfrac{dY}{dX}=\dfrac{aX+bY}{a'X+b'Y}$ which is a homogeneous equation in X, Y and can be solved by putting $Y=vX$ as earlier.

Special Case. When $\dfrac{a}{a'}=\dfrac{b}{b'}=\dfrac{1}{m}$ (say), then the differential equation can be written as

$$\frac{dy}{dx}=\frac{ax+by+c}{m\,(ax+by)+c'}.$$

Put $ax+by=v$, so that $a+b\,\dfrac{dy}{dx}=\dfrac{dv}{dx}$.

(1) then becomes $\dfrac{1}{b}\left(\dfrac{dv}{dx}-a\right)=\dfrac{v+c}{mv+c'}$ in which variables can be separated.

Ex. 1. *Solve* $\dfrac{dy}{dx}=\dfrac{x+2y-3}{2x+y-3}$.

[Vikram 60]

Solution. Put $x=X+h,\ y=Y+k$, where h,k are some constants; then $\dfrac{dy}{dx}=\dfrac{dY}{dX}$. The given equation then becomes

$$\frac{dY}{dX}=\frac{X+2Y+(h+2k-3)}{2X+Y+(2h+k-3)}.$$

Now choose h, k such that $h+2k-3=0$ and $2h+k-3=0$. Solving these we get $h=1$, $k=1$.

$$\therefore \quad \frac{dY}{dX}=\frac{X+2Y}{2X+Y} \text{ homogeneous in } X \text{ and } Y.$$

Put $Y=vX$, so that $\dfrac{dY}{dX}=v+X\dfrac{dv}{dX}$.

$$\therefore \quad v+X\frac{dv}{dX}=\frac{X+2vX}{2X+vX}=\frac{1+2v}{2+v}, \quad i.e., \quad X\frac{dv}{dX}=\frac{1+2v}{2+v}-v$$

or $\quad \dfrac{dX}{X}=\dfrac{2+v}{1-v^2}\,dv=\left(\dfrac{1}{1-v^2}+\dfrac{v}{1-v^2}\right)dv.$

Integrating, $\log X=2.\frac{1}{2}\log\dfrac{1+v}{1-v}-\frac{1}{2}\log(1-v^2)+\log C$

or $\quad X=C\,\dfrac{1+v}{1-v}\cdot\dfrac{1}{\sqrt{(1-v^2)}}=\dfrac{C\sqrt{(1+v)}}{(1-v)^{3/2}}$

or $$X^2(1-v)^3=C^2(1+v)$$

or $$X^2\left(1-\frac{Y}{X}\right)^3=C^2\left(1+\frac{Y}{X}\right) \text{ as } v=\frac{Y}{X}$$

or $$(X-Y)^3=C^2(X+Y) \text{ but } x=X+1,\ y=Y+1.$$

$\therefore \quad (x-y)^3=C^2(x+y-2)$ is the required solution.

Ex. 2. *Solve* $(3x-7y-3)\dfrac{dy}{dx}=3y-7x+7.$ **[Raj. M.Sc. 61]**

Solution. $\dfrac{dy}{dx}=\dfrac{3y-7x+7}{3x-7y-3}.$

Put $x=X+h$, $y=Y+k$, where h, k are some constants. Then $\dfrac{dy}{dx}=\dfrac{dY}{dX}$. And the given equation becomes

$$\frac{dY}{dX}=\frac{3Y-7X+(3k-7h+7)}{3X-7Y+(3h-7k-3)}.$$

Choose h, k such that $3h-7k-3=0$ and $3k-7h+7=0$, which give $h=1$, $k=0$.

$$\therefore \quad \frac{dY}{dX}=\frac{3Y-7X}{3X-7Y} \text{ [homogeneous].}$$

Put $Y=vX$, $\dfrac{dY}{dX}=v+X\dfrac{dv}{dX}$.

$$\therefore \quad v+X\frac{dv}{dX}=\frac{3vX-7X}{3X-7vX}=\frac{3v-7}{3-7v}$$

or $$X\frac{dv}{dX}=\frac{3v-7}{3-7v}-v=\frac{7(v^2-1)}{3-7v}$$

or $$\frac{7\,dX}{X}=\frac{3-7v}{(v^2-1)}\,dv=-\left(\frac{2}{v-1}+\frac{5}{v+1}\right)dv.$$

Integrating, $7 \log X = -2 \log (v-1) - 5 \log (v+1) + \log C$

or $\quad X^7 (v-1)^2 (v+1)^5 = C$

or $\quad X^7 \left(\dfrac{Y}{X} - 1\right)^2 \left(\dfrac{Y}{X} + 1\right)^5 = C \quad$ as $\quad Y = vX$

or $\quad (Y-X)^2 (Y+X)^5 = C$

or $\quad (y-x+1)^2 (y+x-1)^5 = C \quad$ as $\quad x = X+1, \; y = Y+0.$

Ex. 3. *Solve* $\quad (2x+y+3) \dfrac{dy}{dx} = x+2y+3.$

[Karnatak B.Sc. (Princ.) 60]

Solution. $\quad \dfrac{dy}{dx} = \dfrac{x+2y+3}{2x+y+3}.$

Put $x = X+h, \; y = Y+k$, where h, k are constants.

$$dx = dX, \quad dy = dY; \quad \therefore \quad \frac{dy}{dx} = \frac{dY}{dX}.$$

$$\therefore \quad \frac{dY}{dX} = \frac{X+2Y+(h+2k+3)}{2X+Y+(2h+k+3)}.$$

Choose h, k such that $h+2k+3=0, \; 2h+k+3=0$. Solving these, we get $h=-1, \; k=-1$.

$$\therefore \quad \frac{dY}{dX} = \frac{X+2Y}{2X+Y}. \quad \text{Put } Y = vX, \; \frac{dY}{dX} = v + X\frac{dv}{dX}.$$

$$\therefore \quad v + X\frac{dv}{dX} = \frac{X+2vX}{2X+vX} \quad \text{or} \quad X\frac{dv}{dX} = \frac{1+2v}{2+v} - v$$

or $\quad \dfrac{dX}{X} = \dfrac{2+v}{1-v^2}\, dv = \left(\dfrac{\frac{3}{2}}{1-v} + \dfrac{\frac{1}{2}}{1+v}\right) dv.$

Integrating, $2 \log X = -3 \log (1-v) + \log (1+v) + \log C$

or $\quad X^2 \dfrac{(1-v)^3}{1+v} = C$, or $\quad X^2 \dfrac{(1-Y/X)^3}{(1+Y/X)} = C$

or $\quad (X-Y)^3 = C\,(X+Y)$; where $x = X-1, \; y = Y-1$

or $\quad (X-y)^3 = C\,(x+y-2)$ is the solution.

Ex. 4 *Solve* $(2x-2y+5) \dfrac{dy}{dx} = x-y+3.$

[Sagar 63; Agra B.Sc. 61, 52]

Solution. The equation is $\dfrac{dy}{dx} = \dfrac{x-y+3}{2(x-y)+5}.$

Put $x-y=v$, so that $1 - \dfrac{dy}{dx} = \dfrac{dv}{dx}$ or $\dfrac{dy}{dx} = 1 - \dfrac{dv}{dx}.$

$\therefore$ The equation becomes

$$1 - \frac{dv}{dx} = \frac{v+3}{2v+5} \quad \text{or} \quad \frac{dv}{dx} = 1 - \frac{v+3}{2v+5} = \frac{v+2}{2v+5}$$

or $\quad dx = \dfrac{2v+5}{v+2}\, dv = \left(2 + \dfrac{1}{v+2}\right) dv$, separating the variables.

Integrating, $x=2v+\log(v+2)+C$,

$$x=2(x-y)+\log(x-y+2)+C \text{ as } v=x-y$$

or $\quad 2y-x=\log(x-y+2)+C$ is the required solution.

Ex. 5. *Solve* $\dfrac{dy}{dx}=\dfrac{6x-4y+3}{3x-2y+1}$

[Poona 64; Karnatak B.Sc. (Princ.) 61]

Solution. Put $3x-2y=v$, *i.e.*, $3-2\dfrac{dy}{dx}=\dfrac{dv}{dx}$

$$\therefore \quad \frac{dv}{dx}=3-2\,\frac{2v+3}{v+1}=-\frac{v+3}{v+1}.$$

$$\therefore \quad dx=-\frac{v+1}{v+3}\,dv=-\left(1-\frac{2}{v+3}\right)dv.$$

Integrating, $x=-v+2\log(v+3)+C$

or $\quad x=(2y-3x)+2\log(3x-2y+3)+C$

or $\quad 2x-y=\log(3x-2y+3)+\frac{1}{2}C$ is the solution.

Ex. 6. *Solve* $(6x-4y+1)\dfrac{dy}{dx}=(3x-2y+1)$.

[Karnatak B.Sc. (Sub.) 61]

Solution. $\dfrac{dy}{dx}=\dfrac{3x-2y+1}{2(3x-2y)+1}$. Put $3x-2y=v$.

$$\therefore \quad \frac{dv}{dx}=3-2\,\frac{dy}{dx}=3-2\,\frac{v+1}{2v+1}=\frac{4v+1}{2v+1}$$

or $\quad dx=\dfrac{2v+1}{4v+1}\,dv$ or $2\,dx=\left(1+\dfrac{1}{4v+1}\right)dv$ etc.

Ex. 7. *Solve the following equations :*

(i) $(2x+y+1)\,dx+(4x+2y-1)\,dy=0$. **[Gujrat B.Sc. (Princ.) 61]**

(ii) $\dfrac{dy}{dx}=\dfrac{6x-2y-7}{3x-y+4}$. **[Luck. Pass 56]**

(iii) $(2x-5y+3)\,dx-(2x+4y-6)\,dy=0$. **[Delhi Hons. 61]**

(iv) $\dfrac{dy}{dx}=\dfrac{y-x+1}{y-x-5}$ **[Poona 62; Nag. 62]**

(v) $\dfrac{dy}{dx}=\dfrac{3x-4y-2}{3x-4y-3}$. **[Cal. Hons 63]**

(vi) $(3y+2x+4)\,dx-(4x+6y+5)\,dy=0$. **[Karnatak 63]**

(vii) $(2x-5y+3)\,dx-(2x+4y-6)\,dy=0$. **[Delhi Hons. 65]**

(viii) $(x-y-2)\,dx+(x-2y-3)\,dy=0$. **[All. 66]**

(ix) $(4x+2y+1)\,dy=(2x+y+3)\,dx$. **[Delhi Pass 67]**

Hint. In (*i*) put $2x+y=v$, in (*ii*) put $3x-y=v$ and (*iii*) can be reduced to homogeneous form as usual. In (*ix*) putting $v=2x+y$, variables can be separated

Ex. 8. *Solve* $2y\dfrac{dy}{dx}=\dfrac{x+y^2}{x+4y^2}$ [Bombay B.Sc. 61]

Solution. Put $y^2=v$, $2y\dfrac{dy}{dx}=\dfrac{dv}{dx}$.

$\therefore\quad \dfrac{dv}{dx}=\dfrac{x+v}{x+4v}$ [homogeneous]. Now put $v=xz$ etc.

2·6. A particular case

A differential equation of the form

$$\frac{dy}{dx}=\frac{ax+by+c}{-bx+hy+k},$$

in which coefficient of y in the numerator is equal to the coefficient of x in the denominator with sign changed, can be integrated as follows :

The equation (1) can be written as

$$-b\,(x\,dy+y\,dx)+(hy+k)\,dy-(ax+c)\,dx=0.$$

Integrating, we get $-bxy+(\tfrac{1}{2}hy^2+ky)-(\tfrac{1}{2}ax^2+cx)=A.$

Ex. 1. *Solve* $\dfrac{dy}{dx}+\dfrac{ax+hy+g}{hx+by+f}=0.$

[Raj. B.Sc. 66; Agra B.Sc. 57; Delhi B.A. 57; Raj. M.Sc. 62]

Solution. The equation can be written as

$$(hx+by+f)\,dy+(ax+hy+g)\,dx=0$$

or $\quad h\,(x\,dy+y\,dx)+(by+f)\,dy+(ax+g)\,dx=0.$

Integrating, $\quad hxy+\tfrac{1}{2}by^2+fy+\tfrac{1}{2}ax^2+gx=A$

or $\quad ax^2+2hxy+by^2+2fy+2gx+c=0$, writing $c=-2A$.

Ex. 2. *Solve* $\dfrac{dy}{dx}=\dfrac{2x-y+1}{x+2y-3}$ [Agra B Sc. 59; Nag. 53 (S)]

Solution. Here coefficient of y in numerator is equal to coefficient of x in the denominator with sign changed. Hence write it as

$$(x+2y-3)\,dy-(2x-y+1)\,dx=0$$

or $\quad (x\,dy+y\,dx)+(2y-3)\,dy-(2x+1)\,dx=0.$

Integrating, $\quad xy+y^2-3y-x^2-x=C.$

Ex. 3. *Solve* $(2x-y+1)\,dx+(2y-x-1)\,dy=0.$

[Bombay B Sc. (Sub.) 61; Poona 61]

Solution. The equation is of above type. Hence after regrouping, we have

$$(2x+1)\,dx+(2y-1)\,dy-(y\,dx+x\,dy)=0.$$

Integrating, $\quad (x^2+x)+(y^2-y)-xy=C,$

which is the solution.

Ex. 4. *Solve* $\dfrac{dy}{dx}+\dfrac{2x+3y+1}{3x+4y-1}=0.$ [Delhi Hons. 60]

Solution. The equation is of the above type and can be written as $(3x+4y-1)\,dy+(2x+3y+1)\,dx=0,$

i.e., $3\,(x\,dy+y\,dx)+(4y-1)\,dy+(2x+1)\,dx=0.$

Integrating, $3xy+2y^2-y+x^2+x=C$ is the solution.

2.7. Linear Differential Equations

[Poona 63, 61; Nagpur 62, 61; Guj. 61]

A differential equation of the form

$$\frac{dy}{dx}+Py=Q,$$

where P, Q are functions of x or constants, is called the *linear differential equation of the first order.*

To solve this equation, multiply both the sides by $e^{\int P\,dx}$

Then it becomes $e^{\int P\,dx}\dfrac{dy}{dx}+Py\,e^{\int P\,dx}=Qe^{\int P\,dx}.$

or $$\frac{d}{dx}\left[ye^{\int P\,dx}\right]=Q\,e^{\int P\,dx}.$$

Integrating both the sides, w.r.t. x, we get

$$ye^{\int P\,dx}=\int\left[Qe^{\int P\,dx}\right]dx+C,$$

which is the required solution.

Integrating factor (I.F.). It will be noticed that for solving (1), we multiplied it by a factor $e^{\int P\,dx}$ and the equation became readily (directly) integrable. Such a factor is called the integrating factor.

Note. Sometimes a differential equation takes linear form if we *regard x as dependent variable and y as independent variable.* The equation can then be put as $\dfrac{dx}{dy}+Px=Q$, where P, Q are functions of y or constants.

The integrating factor in this case is $e^{\int P\,dy}$ and solution is

$$xe^{\int P\,dy}=\int\left[Qe^{\int P\,dy}\right]dy+C.$$

(See Ex. 1 to 4 pages 21 and 22).

Ex. 1. *Solve* $(1-x^2)\dfrac{dy}{dx}-xy=1.$

[Delhi 68 : Nag. 61]

Solution. The equation can be written as

$$\frac{dy}{dx}-\frac{x}{1-x^2}\,y=\frac{1}{1-x^2}$$

This is now expressed in the linear form

$$P=-\frac{x}{1-x^2}, \quad \text{I.F.}=e^{\int P\,dx}=e^{\int\frac{-x}{1-x^2}dx}=e^{\frac{1}{2}\log(1-x^2)}$$
$$=\sqrt{(1-x^2)}.$$

Hence the solution is

$$y.\sqrt{(1-x^2)}=\int\frac{1}{1-x^2}\sqrt{(1-x^2)}\,dx+C.$$

Ex. 2. (a) *Solve* $x\dfrac{dy}{dx}+2y=x^2\log x.$ **[Lucknow 52]**

Solution. The equation is $\dfrac{dy}{dx}+\dfrac{2}{x}y=x\log x.$

$$\text{I.F.}=e^{\int(2/x)\,dx}=e^{2\log x}=x^2.$$

Hence the solution is

$$y.x^2=C+\int x^2.x\log x\,dx=C+\int x^3\log x\,dx$$
$$=C+\log x.\frac{x^4}{4}-\int\frac{1}{x}.\frac{x^4}{4}dx$$
$$=C+\tfrac{1}{4}x^4\log x-\tfrac{1}{16}x^4$$

or $\quad y=Cx^{-2}+\tfrac{1}{4}x^2(\log x-\tfrac{1}{4}).$

Ex. 2. (b) *Solve* $x\dfrac{dy}{dx}+2y=x^4.$

[Bombay B.Sc. 61]

Solution. Equation is $\dfrac{dy}{dx}+\dfrac{2}{x}y=x^3.$ I.F. $=x^2$ as above.

Solution is $\quad y.x^2=C+\int x^3.x^2\,dx=C+\tfrac{1}{6}x^6.$

Ex. 3. *Solve* $(x^3-x)\dfrac{dy}{dx}-(3x^2-1)\,y=x^5-2x^3+x.$

[Gujrat B.Sc. (Sub.) 1961]

Solution. The equation is

$$\frac{dy}{dx}-\frac{3x^2-1}{x^3-x}\,y=(x^2-1).$$

$$\text{I.F.}=e^{-\int\frac{(3x^2-1)}{x^3-x}}=e^{-\log(x^3-x)}=\frac{1}{x^3-x}.$$

$\therefore$ Solution is $y.\dfrac{1}{x^3-x}=C+\displaystyle\int\frac{x^2-1}{x^3-1}dx$

$$=C+\int\frac{1}{x}\,dx=C+\log x.$$

Ex 4. *Solve* $xp+y=ax^2+bx+c,\ p=\dfrac{dy}{dx}.$

[Delhi Hons. 1957]

Solution. The equation can be written as

$$\frac{dy}{dx}+\frac{1}{x}y=ax+b+\frac{c}{x} \text{ [linear].}$$

$$\text{I.F.}=e^{\int \frac{1}{x}dx}=e^{\log x}=x.$$

$$\therefore \quad y.x=C+\int\left(ax+b+\frac{c}{x}\right)x\,dx=C+\int(ax^2+6x+c)\,dx$$

$$=C+\tfrac{1}{3}ax^3+\tfrac{1}{2}bx^2+cx.$$

Ex. 5. *If* $\frac{dy}{dx}+2y\tan x=\sin x$ *and if* $y=0$ *when* $x=\frac{1}{3}\pi$, *express y in terms of x.* **[Poona 1964 ; Nagpur 61]**

Solution. The equation is linear.

$$\text{I.F.}=e^{\int 2\tan x\,dx}=e^{-2\log\cos x}=\sec^2 x.$$

Hence general solution is

$$y.\sec^2 x=C+\int\sin x\sec^2 x\,dx=C+\int\sec x\tan x\,dx$$

or $\quad y\sec^2 x=C+\sec x.$

When $y=0$, $x=\frac{1}{3}\pi$, $\therefore\ 0=C+\sec\frac{1}{3}\pi$ or $C+2=0$, $C=-2$.

Hence solution is $\quad y\sec^2 x=\sec x-2,$

$$y=\cos x-2\cos^2 x.$$

Ex. 6. *Solve* $x(x-1)\frac{dy}{dx}-y=x^2(x-1)^2.$ **[Luck. Pass 1958]**

Solution. Equation is $\frac{dy}{dx}-\frac{1}{x(x-1)}y=x(x-1).$

$$\text{I.F.}=e^{-\int\frac{1}{x(x-1)}dx}=e^{\int\left(\frac{1}{x}-\frac{1}{x-1}\right)dx}=\frac{x}{x-1}.$$

Hence $y.\frac{x}{x-1}=C+\int x(x-1).\frac{x}{x-1}\,dx=C+\int x^2\,dx$

or $$y\frac{x}{x-1}=C+\tfrac{1}{3}x^3.$$

Ex. 7. *Solve* $(1+x)\frac{dy}{dx}+3y=\frac{1+x+x^2}{(1+x)}.$

[Lucknow Pass 1957]

Solution. Equation is $\frac{dy}{dx}+\frac{3}{1+x}y=\frac{1+x+x^2}{(1+x)^4}.$

$$\text{I.F.}=e^{\int\frac{3}{1+x}dx}=e^{3\log(1+x)}=(1+x)^3$$

$$\therefore\quad y(1+x)^3=C+\int\frac{(1+x+x^2)}{(1+x)^4}(1+x)^3\,dx$$

$$=C+\int\frac{1+x+x^2}{1+x}=C+\int\left(\frac{1}{1+x}+x\right)dx$$
$$=C+\log(1+x)+\tfrac{1}{2}x^2.$$

Ex. 8. *Solve* $x\dfrac{dy}{dx}+2y=\dfrac{dy}{dx}+4.$ **[Nagpur T.D.C. 1961 (S)]**

Solution. The equation can be written as

$$(x-1)\frac{dy}{dx}+2y=4 \quad \text{or} \quad \frac{dy}{dx}+\frac{2}{x-1}y=\frac{4}{x-1}.$$

Linear, $\text{I.F.}=e^{\int\frac{2}{x-1}dx}=e^{2\log(x-1)}=(x-1)^2.$

$$y(x-1)^2=\int\frac{4}{x-1}(x-1)^2\,dx+C$$
$y(x-1)^2=2(x-1)^2+C$, which is the solution.

Ex. 9. *Solve* $x\dfrac{dy}{dx}-2y=x^2+\sin\dfrac{1}{x^2}.$

[Bombay B.A. (Sub.) 1958]

Solution. The equation is $\dfrac{dy}{dx}-\dfrac{2}{x}y=x+\dfrac{1}{x}\sin\dfrac{1}{x^2}.$

$$\text{I.F.}=e^{-\int\frac{2}{x}dx}=e^{-2\log x}=\frac{1}{x^2}.$$

$$\therefore\quad y.\frac{1}{x^2}=C+\int x\frac{1}{x^2}dx+\int\frac{1}{x^3}\sin\frac{1}{x^2}dx.$$

$=C+\log x-\frac{1}{2}\int\sin t\,dt$, where $\dfrac{1}{x^2}=t,\ \dfrac{-2}{x^3}dx=dt$

$$=C+\log x+\tfrac{1}{2}\cos t$$
$$=C+\log x+\tfrac{1}{2}\cos\frac{1}{x^2}.$$

Ex. 10. *Solve* $\dfrac{dy}{dx}-2y\cos x=-2\sin 2x.$

[Vikram 65; Gujrat B.Sc. (Sub.) 61]

Solution. $\text{I.F.}=e^{-2\int\cos x\,dx}=e^{-2\sin x}.$

$\therefore$ Solution is

$$ye^{-2\sin x}=C-2\int\sin 2x e^{-2\sin x}\,dx$$
$=C-4\int\sin x\cos x e^{-2\sin x}\,dx$; put $-2\sin x=t$
$$=C-\int te^t\,dt=C-e^t(t-1).$$

$\therefore\quad y=Ce^{2\sin x}+(2\sin x+1)$ is the solution.

Equations which become linear when x is treated as dependent variable.

Ex. 1. *Solve* $y\log y\,dx+(x-\log y)\,dy=0.$

[Poona T.D.C. 61(S)]

Solution. Write the equation as

$$\frac{dx}{dy}+\frac{1}{y\log y}x=\frac{1}{y}.$$

$$\text{I.F.}=e^{\int\frac{1}{y}\frac{1}{\log y}dy}=e^{\log(\log y)}=\log y.$$

$$\therefore\quad x\log y=C+\int\frac{1}{y}\log y\,dy$$

$$=C+\tfrac{1}{2}(\log y)^2 \text{ is the solution.}$$

Ex 2. *Solve* $dx+x\,dy=e^{-y}\log y\,dy.$ **[Poona 61]**

Solution. The equation can be written as

$$\frac{dx}{dy}+x=e^{-y}\log y,\ \text{I.F.}=e^{y}.$$

$$\therefore\quad xe^{y}=C+\int e^{-y}\log y.e^{y}\,dy$$

$$=C+\int\log y\,dy=C+\log y.y-\int y.\frac{1}{y}dy$$

$$=C+y\log y-y.$$

Ex. 3. *Solve* $(1+y^2)\,dx+(x-\tan^{-1}y)\,dy=0.$ **[Gujrat 65; Delhi Hons. 65 ; Pb. 62; Cal. Hons. 62; Agra 67, 58]**

Solution. The equation can be written as

$$\frac{dx}{dy}+\frac{1}{1+y^2}x=\frac{\tan^{-1}y}{1+y^2}.$$

$$\text{I.F.}=e^{\int\frac{1}{1+y^2}dy}=e^{\tan^{-1}y}.$$

$$\therefore\quad xe^{\tan^{-1}y}=\int\frac{\tan^{-1}y}{1+y^2}e^{\tan^{-1}y}dy+C$$

$$=\int te^{t}\,dt+C \text{ where } t=\tan^{-1}y$$

$$=e^{t}(t-1)+C=e^{\tan^{-1}y}(\tan^{-1}y-1)+C.$$

Hence $x=(\tan^{-1}y-1)+Ce^{-\tan^{-1}y}$ is the solution.

Ex. 4. *Solve* $(x+2y^3)\dfrac{dy}{dx}=y.$

[Agra B.Sc. 1956 ; Raj B.Sc. 56]

Hint. The equation can be written as

$$\frac{dx}{dy}=x+2y^3 \text{ [linear].}$$

Ans. $x=y^3+Cy.$

2·8. Equations reducible to linear form

***I. Bernoulli Equation*.** $\dfrac{dy}{dx}+Py=Qy^n,$

*Known after James Bernoulli. The method of solution was discovered by Leibnitz.

where P and Q are functions of x or constants.

[Nag. T.D.C. 1961; Poona T.D.C. 61 ; Gujrat B.Sc. (Prin.) 58; Poona B.A. (Gen.) 60]

Dividing both the sides by y^n we have

$$y^{-n}\frac{dy}{dx}+Py^{-n+1}=Q. \quad ...(1)$$

Now put $y^{-n+1}=v$ so that $(1-n)\,y^{-n}\frac{dy}{dx}=\frac{dv}{dx}$.

Then (1) becomes $\frac{1}{1-n}\frac{dv}{dx}+Pv=Q$

or $\frac{dv}{dx}+P\,(1-n)\,v=(1-n)\,Q$

which is a linear equation in v and x.

II. Equation $f'(y)\frac{dy}{dx}+Pf(y)=Q$,

where P and Q are functions of x or constants.

Put $f(y)=v$ so that $f'(y)\frac{dy}{dx}=\frac{dv}{dx}$.

$\therefore$ equation becomes $\frac{dv}{dx}+Pv=Q$,

which is a linear equation in v and x.

Note. In each of these equations, single out Q (function of on the right) and then make suitable substitution to reduce the equation in linear form.

Ex. 1. *Solve* $\frac{dy}{dx}=x^3y^3-xy$.

[Karnatak B.Sc. (Prin.) 1960, 62; Agra 61; Bihar 62; Gujrat B.Sc. (Sub.) 61]

Solution. The equation is $\frac{dy}{dx}+xy=x^3y^3$.

Dividing by y^3; $\frac{1}{y^3}\frac{dy}{dx}+x.\,\frac{1}{y^2}=x^3$.

Put $\frac{1}{y^2}=v$, so that $-\frac{2}{y^3}\frac{dy}{dx}=\frac{dv}{dx}$, *i.e.*, $\frac{1}{y^3}\frac{dy}{dx}=-\frac{1}{2}\frac{dv}{dx}$.

$\therefore$ equation becomes $-\frac{1}{2}\frac{dv}{dx}+xv=x^3$

or $\frac{dv}{dx}-2x.v=-2x^3$.

Linear, I. F. $=e^{\int-2x\,dx}=e^{-x^2}$.

Hence $ve^{-x^2}=\int-2x^3e^{-x^2}\,dx+C$

$=\int x^2\,(-2x)\,e^{-x^2}\,dx+C$

$$= \int -te^t \, dt + C \text{ where } t = -x^2$$
$$= -te^t + e^t + C = -e^{-x^2}(x^2 - 1) + C$$

Hence $\quad v = 1 - x^2 + Ce^{x^2}$ or $\dfrac{1}{y^2} = 1 - x^2 + Ce^{x^2}$.

Ex. 2. *Solve* $\dfrac{dy}{dx} + xy = xy^2$. **[Karnatak 1960]**

Solution. Dividing by y^2, $y^{-2}\dfrac{dy}{dx} + xy^{-1} = x$.

Put $y^{-1} = v$, so that $-y^2\dfrac{dy}{dx} = \dfrac{dv}{dx}$.

$\therefore$ equation is $\dfrac{dv}{dx} - xv = -x$.

I. F. $= e^{\int -x\,dx} = e^{-\frac{1}{2}x^2}$.

$$\therefore \quad ve^{-\frac{1}{2}x^2} = C - \int xe^{-\frac{1}{2}x^2}$$
$$= C + \int e^t \, dt, \text{ where } -\tfrac{1}{2}x^2 = t, \; -x\,dx = dt$$

or $\quad y^{-1}e^{-\frac{1}{2}x^2} = C + e^t = C + e^{-\frac{1}{2}x^2}$

or $\quad y^{-1} = Ce^{\frac{1}{2}x^2} + 1$ is the solution.

Ex 3. *Solve* $\dfrac{dy}{dx} + \dfrac{2}{x}y = \dfrac{y^3}{x^3}$. **[Nag. 1958]**

Solution. Dividing by y^3, $y^{-3}\dfrac{dy}{dx} + \dfrac{2}{x}y^{-2} = \dfrac{1}{x^3}$.

Put $y^{-2} = v$, so that $-2y^{-3}\dfrac{dy}{dx} = \dfrac{dv}{dx}$.

$\therefore$ equation becomes $-\frac{1}{2}\dfrac{dv}{dx} + \dfrac{2}{x}v = \dfrac{1}{x^3}$

or $\dfrac{dv}{dx} - \dfrac{4}{x}v = -\dfrac{2}{x^3}$.

I. F. $= e^{\int(-4/x)\,dx)} = e^{-4\log x} = \dfrac{1}{x^4}$.

$$\therefore \quad v\frac{1}{x^4} = \int -\frac{2}{x^3}\cdot\frac{1}{x^4}\,dx + C \quad C + \frac{1}{3x^6}$$

or $\dfrac{1}{y^2}\cdot\dfrac{1}{x^4} = \dfrac{1}{3x^6} + C$ is the solution.

***Ex. 4.** *Solve* $\dfrac{dy}{dx}(x^2y^3 + xy) = 1$.

[Sagar 1962; Raj. 63; Cal. Hons. 62; Luck. 63]

Solution. The equation can be written as

$$\frac{dx}{dy} - xy = x^2y^3.$$

Dividing by x^2, $x^{-2}\dfrac{dx}{dy}-\dfrac{1}{x}y=y^3$.

Put $-\dfrac{1}{x}=v$; $\therefore$ $x^{-2}\dfrac{dx}{dy}=\dfrac{dv}{dy}$.

$\therefore$ equation becomes $\dfrac{dv}{dy}+vy=y^3$

Linear in v and y. I. F. $=e^{\int y\,dy}=e^{\frac{1}{2}y^2}$.

$\therefore$ $ve^{\frac{1}{2}y^2}=\int y^3e^{\frac{1}{2}y^2}\,dy+C$, $\frac{1}{2}y^2=t$, $y\,dy=dt$

$$=2\int te^t\,dt+C=2e^t(t-1)+C$$

or $-\dfrac{1}{x}e^{\frac{1}{2}y^2}=2e^{\frac{1}{2}y^2}(\frac{1}{2}y^2-1)+C$

or $\dfrac{1}{x}=(2-y^2)-Ce^{-\frac{1}{2}y^2}$ is the solution.

Ex. 5. *Solve* $\dfrac{dy}{dx}=1-x(y-x)-x^3(y-x)^2$. **[Karnatak 1961]**

Solution. Put $y-x=v$, $\dfrac{dy}{dx}-1=\dfrac{dv}{dx}$.

$\therefore$ equation is $\dfrac{dv}{dx}+1=1-xv-x^3v^2$

or $\dfrac{dv}{dx}+xv=-x^3v^2$ or $v^{-3}\dfrac{dv}{dx}+xv^{-2}=-x^3$.

Put $v^{-2}=u$, $-2v^{-3}\dfrac{dv}{dx}=\dfrac{du}{dx}$.

$\therefore$ the equation is $-\dfrac{1}{2}\dfrac{du}{dx}+xu=-x^3$

or $\dfrac{du}{dx}-2xu=2x^3$.

Linear in u and x. I. F. $=e^{\int -2x\,dx}=e^{-x^2}$.

$\therefore$ $ue^{-x^2}=\int 2x^3e^{-x^2}+C$ $\quad -x^2=t$, $-2x\,dx=dt$

$$=\int te^t\,dt+C=e^t(t-1)+C$$

or $v^{-2}e^{-x^2}=e^{-x^2}(-x^2-1)+C$

or $(y-x)^{-2}=Ce^{x^2}-(1+x^2)$ is the solution.

Ex. 6. *Solve* $2\dfrac{dy}{dx}-\dfrac{y}{x}=\dfrac{y^2}{x^2}$.

[Nagpur 1961; Nagpur 51; Delhi Pass 57]

Solution. The equation is $2y^{-2}\dfrac{dy}{dx}-\dfrac{1}{x}y^{-1}=\dfrac{1}{x^2}$.

Put $-y^{-1}=v$, $\therefore \quad y^{-2}\dfrac{dy}{dx}=\dfrac{dv}{dx}$.

$\therefore \quad 2\dfrac{dv}{dx}+\dfrac{1}{x}v=\dfrac{1}{x^2}$ or $\dfrac{dv}{dx}+\dfrac{1}{2x}v=\dfrac{1}{2x^2}$

Linear. I. F $=e^{\int \frac{1}{2}\cdot\frac{1}{x}dx}=e^{\frac{1}{2}\log x}=\sqrt{x}$.

$\therefore \quad v\sqrt{x}=C+\int \dfrac{1}{2x^2}\sqrt{x}\,dx=C+\dfrac{1}{2}\int x^{-3/2}\,dx$.

$\therefore \quad -y^{-1}\sqrt{x}=C-x^{-1/2}$

or $\dfrac{1}{y}=-Cx^{-1/2}+x^{-1}$, is the solution.

Ex. 7. *Solve* $x\dfrac{dy}{dx}+y=y^2 \log x$. [Luck. 1956]

Solution. Dividing by xy^2, $y^{-2}\dfrac{dy}{dx}+\dfrac{1}{x}y^{-1}=\dfrac{1}{x}\log x$.

Put $-y^{-1}=v$, $y^{-2}\dfrac{dy}{dx}=\dfrac{dv}{dx}$.

$\therefore \quad \dfrac{dv}{dx}-\dfrac{1}{x}\cdot v=\dfrac{1}{x}\log x$. I. F. $=e^{\int -\frac{1}{x}dx}=\dfrac{1}{x}$.

Hence solution is $v\cdot\dfrac{1}{x}=C+\int \dfrac{1}{x}\log x\,dx$

or $-\dfrac{1}{y}\cdot\dfrac{1}{x}=C+\log x\left(-\dfrac{1}{x}\right)-\int \dfrac{1}{x}\cdot\left(-\dfrac{1}{x}\right)dx$

integrating by parts

or $-\dfrac{1}{xy}=C-\dfrac{1}{x}\log x-\dfrac{1}{x}$

or $\dfrac{1}{y}=(1+\log x)-Cx$ is the solution.

Ex 8. *Solve* $(x-y^2)\,dx+2xy\,dy=0$. [Poona 1961]

Solution. The equation is

$$2y\frac{dy}{dx}-\frac{1}{x}y^2=-1.$$

Put $y^2=v$, $2y\dfrac{dy}{dx}=\dfrac{dv}{dx}$, $\therefore$ equation is $\dfrac{dv}{dx}-\dfrac{1}{x}v=-1$.

$$\text{I. F.}=e^{\int -\frac{1}{x}dx}=e^{-\log x}=\frac{1}{x}.$$

$\therefore \quad v\cdot\dfrac{1}{x}=C-\int \dfrac{1}{x}\,dx$ or $\dfrac{y^2}{x}=C-\log x$.

Ex. 9. (a) *Solve* $(x^2+3x+2)\frac{dy}{dx}+(2x+1)\,y=(xy+2y)^2.$

[Bombay B.Sc. (Prin.) 1961]

Solution. The equation can be written as

$$(x+2)(x+1)\frac{dy}{dx}+(2x+1)\,y=y^2(x+2)^2$$

or $$y^{-2}\frac{dy}{dx}+\frac{2x+1}{(x+2)(x+1)}y^{-1}=\frac{x+2}{x+1}$$

[dividing by $y^2(x+2)(x+1)$].

Put $-y^{-1}=v,\ y^{-2}\frac{dy}{dx}=\frac{dv}{dx}.$

$\therefore$ equation is $\frac{dv}{dx}-\frac{2x+1}{(x+2)(x+1)}v=\frac{x+2}{x+1}.$

This is a linear equation.

$$\text{I. F.}=e^{-\int\frac{2x+1}{(x+2)(x+1)}dx}=e^{\int\left(\frac{3}{x+2}-\frac{1}{x+1}\right)dx}$$

$$=e^{\log(x+1)-3\log(x+2)}=\frac{x+1}{(x+2)^3}.$$

Hence $$v.\frac{x+1}{(x+2)^3}=C+\int\frac{x+2}{x+1}\cdot\frac{x+1}{(x+2)^3}dx$$

or $$-\frac{1}{y}\frac{x+1}{(x+2)^3}=C+\int\frac{1}{(x+2)^2}dx=C-\frac{1}{(x+2)}.$$

Ex. 10. $\frac{dy}{dx}-2y\tan x=y^2\tan^2 x.$ **[Bombay 61]**

Solution. Dividing by y^2, the equation becomes

$$y^{-2}\frac{dy}{dx}-2y^{-1}\tan x=\tan^2 x.$$

Put $-y^{-1}=v.\,y^{-2}\frac{dy}{dx}=\frac{dv}{dx}.$

$\therefore$ equation is $\frac{dv}{dx}+2\tan x.v=\tan^2 x$, I.F.$=\sec^2 x$.

$\therefore$ $v\sec^2 x=C+\int\tan^2 x.\sec^2 x\,dx=C+\frac{1}{3}\tan^3 x$

or $-\frac{1}{y}\sec^2 x=C+\frac{1}{3}\tan^3 x$ is the solution.

Ex. 11. *Solve* $xy-\frac{dy}{dx}=y^3e^{-x^2}.$

[Bombay 58; Poona B.A. 60]

Solution. Write the equation as

$$y^{-3}\frac{dy}{dx}-xy^{-2}=-e^{-x^2}.$$

Put $-\frac{1}{y} = v, y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$.

$\therefore$ equation is $\frac{dv}{dx} - v = x$, linear, I. F. $= e^{-x}$.

$\therefore \quad ve^{-x} = \int xe^{-x}\,dx + C = -xe^{-x} - e^{-x} + C$

or $\quad -\frac{1}{y} = -(x+1) + Ce^{x}$ or $\frac{1}{y} = (x+1) - Ce^{x}$.

It passes through (0, 1), *i.e.* when $x = 0, y = 1$, $\quad \therefore 1 = 1 - C$ or $C = 0$.

Therefore the curve is given by $\frac{1}{y} = (x+1)$ or $1 = y(x+1)$.

Ex. 15. *Solve,* $\sec^2 y\frac{dy}{dx} + (\tan y)\,2x = x^3$

[Agra 63, 61; Gujrat 51; Delhi Hons. 64]

Solution. Putting $\tan y = v \therefore \sec^2 y\frac{dy}{dx} = \frac{dv}{dx}$, The equation becomes

$$\frac{dv}{dx} + 2xv = x^3$$

Linear. I. F. $= e^{\int 2x\,dx} = e^{x^2}$

$\therefore \quad ve^{x^2} = C + \int x^3 e^{x^2}\,dx$, $x^2 = t$, $\therefore 2x\,dx = dt$

$\therefore \quad x\,dx = \frac{1}{2}dt = C + \frac{1}{2}\int te^{t}\,dt = C + \frac{1}{2}e^{t}(t-1)$

$\therefore \quad \tan y\, e^{x^2} = C + \frac{1}{2}e^{x^2}(\tan y - 1)$ [which is wrong]

The correct solution is $\tan y\, e^{x^2} = e + \frac{1}{2}e^{x^2}(x^2 - 1)$

Ex. 16. *Solve* $\frac{dy}{dx} + y\cos x = y^n \sin 2x$. **[Delhi Hons. 58]**

Solution. Dividing by y^n, we get $y^{-n}\frac{dy}{dx} + y^{-n+1}\cos x = \sin 2x$.

Putting $y^{-n+1} = v, (1-n)y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$, the equation becomes

$$\frac{dv}{dx} + v(1-n)\cos x = (1-n)\sin 2x.$$

I.F. $= e^{\int (1-n)\cos x\,dx} = e^{(1-n)\sin x}$.

$v.\, e^{(1-n)\sin x} = C + \int e^{(1-n)\sin x} \cdot 2\sin x\cos x\,dx$.

Now put $(1-n)\sin x = t$ and integrate.

Ex. 17. *Solve the following linear eautions :*

(i) $(1 + x^2)\frac{dy}{dx} + y = e^{\tan^{-1}x}$,

I. F. $= e^{\tan^{-1}x}$. **Ans.** $y = \frac{1}{2}e^{\tan^{-1}x} + Ce^{-\tan^{-1}x}$.

(ii) $x\cos x\frac{dy}{dx} + y(x\sin x + \cos x) = 1$

$\therefore \quad \frac{dy}{dx} + (\tan x + \frac{1}{x}) y = \frac{1}{x \cos x}$. [Linear]

$P = \tan x + \frac{1}{x}$.

I. F. $= e^{\int (\tan x + \frac{1}{x}) dx} = e^{-\text{Log} \cos x + \log x} = e^{\log x \sec x}$

$= \log x \sec x$ which is wrong solution.

The correct solution is, I. F. $= x \sec x$.

(iii) $\sin 2x \frac{dy}{dx} - y = \tan x$.

$P = -\operatorname{cosec} 2x$. I.F. $= (\tan x)^{-1/2}$ Ans. $y = \tan x + C\sqrt{(\tan x)}$.

(iv) $x(x^2+1)\frac{dy}{dx} = x(1-x^2) + x^3 \log x$. [Agra 59]

$P = \frac{x^2-1}{x(x^2+1)}$. I. F. $= \frac{x^2+1}{x}$. Ans. $y \cdot \frac{x^2+1}{x} + \frac{1}{2}x^2 \log x - \frac{1}{4}x^4 + C$.

(v) $\sqrt{a^2+x^2}\frac{dy}{dx} + y = \sqrt{a^2+x^2} - x$

$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{a^2+x^2}} y = 1 - \frac{x}{\sqrt{a^2+x^2}}$ which is Linear form.

I. F. $= \frac{1}{\sqrt{a^2+x^2}}$, I. F. $= e^{\int \frac{1}{\sqrt{a^2+x^2}} dx}$

$= e^{\log[x + \sqrt{x^2+a^2}]} = \frac{x+\sqrt{a^2+x^2}}{a}$ which is wrong

The correct solun. is I. F. $= x + \sqrt{a^2+x^2}$

(vi) $x \log x \frac{dy}{dx} + y = 2 \log x$ [Punjab 56, 54]

I. F. $= \log x$. Ans. $y \log x = (\log x)^2 + C$.

(vii) $\frac{dy}{dx} + \frac{y}{(1-x)\sqrt{x}} = 1 - \sqrt{x}$. [Raj. B.sc. 59]

I. F. $= \frac{1+\sqrt{x}}{1-\sqrt{x}}$ Ans. $y \cdot \frac{1+\sqrt{x}}{1-\sqrt{x}} = x + \frac{2}{3}x^{3/2} + C$.

(viii) $\frac{dy}{dx} + y \tan x = \sec x$. [Raj. B. Sc. 56]

I. F. $= e^{\log \sec x} = \sec x$. Ans. $y \sec x = \tan x + C$.

(ix) $\cos^2 x \frac{dy}{dx} + y \tan x = \sin x \cos x - \cos^2 x$. [Punjab 56]

I. F. $= e^{\tan x}$. Ans. $y\, e^{\sin x} = e^{\tan x}(\tan x - 1) + C$.

(x) $x(1-x^2)\,dy + (2x^2 y - y - ax^3)\,dx = 0$. [Nag. 62; Agra 60; Cal. Hons. 61]

Equation is $\frac{dy}{dx} + y\frac{2x^2-1}{x(1-x^2)} = \frac{ax^2}{1-x^2}$. I. F. $= \frac{1}{x\sqrt{(1-x^2)}}$

(xi) $(1-x^2)\,dy + y\,dx = a\,dx$. [Cal. Hons. 62]

Equation is $\frac{dy}{dx} + \frac{x}{1-x^2} y = \frac{a}{1-x^2}$. I. F. $= \frac{1}{\sqrt{(1-x^2)}}$.

(xii) $\frac{dy}{dx}+\frac{4x}{x^2+1}y=\frac{1}{(x^2+1)^3}$. [Poona 64]

I.F. $=(x^2+1)^2$. **Ans.** $y(x^2+1)^2=C+\tan^{-1}x$.

(xiii) $(x+a)\frac{dy}{dx}-3y=(x+a)^5$. [Poona 68]

Equation is $\frac{dy}{dx}-\frac{3}{x+a}y=(x+a)^4$.

I. F. $=\frac{1}{(x+a)^3}$. **Ans.** $y(x+a)^{-3}=C+\frac{1}{2}(x+a)^2$

(xiv) $x\frac{dy}{dx}+e^{-x}+x^2+y=0$. [Nagpur 1963]

Equation is $\frac{dy}{dx}+\frac{1}{x}y=-\left(x+\frac{1}{x}e^{-x}\right)$, I.F. $=x$.

Ans. $yx=C-\int(x^2+e^{-x})\,dx=C-\frac{1}{3}x^3+e^{-x}$.

(xv) $\frac{dy}{dx}+\frac{y}{(1-x^2)^{3/2}}=\frac{x+\sqrt{(1-x^2)}}{(1-x^2)^2}$. [Poona 1964]

We have $\int\frac{1}{(1-x^2)^{3/2}}\,dx=\int\frac{\cos\theta\,d\theta}{\cos^3\theta}$, putting $x=\sin\theta$

$$=\int\sec^2\theta\,d\theta=\tan\theta=\frac{x}{\sqrt{(1-x^2)}}.$$

I. F. $=e^{x/\sqrt{(1-x^2)}}$.

Solution is $ye^{x/\sqrt{(1-x^2)}}=C+\int\frac{x+\sqrt{(1-x^2)}}{(1-x^2)^2}e^{x/\sqrt{(1-x^2)}}\,dx$

$$=C+\int\left[\frac{x}{\sqrt{(1-x^2)}}+1\right]e^{x/\sqrt{(1-x^2)}}\frac{dx}{(1-x^2)^{3/2}}$$

$=C+\int(t+1)e^t\,dt$, where $t=\frac{1}{\sqrt{(1-x^2)}}$

$=C+te^t$ etc.

(xvi) $(x+a)\frac{dy}{dx}=xy-y^2$. [Allahabad 1965]

Ex. 18. *Show that the following equations can be reduced to linear form and solve them:*

(i) $\frac{dy}{dx}+2xy+xy^4=0$. [Poona 1963]

Dividing by y^4, $y^{-4}\frac{dy}{dx}+2xy^{-3}=-x$. Put $y^{-3}=v$.

(ii) $x\,dy=y(1+xy)\,dx$. [Delhi Pass 1967]

Equation is $\frac{dy}{dx}=\frac{y}{x}+y^2$.

Divide by y^2. I. F. $=1/x^2$.

(iii) $\frac{dy}{dx}+\frac{1}{x}=\frac{e^y}{x^2}$. [Sagar 1963; Gorakhpur 59]

Divide by e^y, and then put $e^{-y}=v$. **Ans.** $2x=e^y(2Cx^2+1)$

(iv) $y(2xy+e^x)\,dx-e^x\,dy=0$. **[Raj. 1954, 51]**

Divide by $e^x y^2$. **Ans.** $y(x^2+C)+e^x=0$.

(v) $2xy\,dy-(x^2+y^2-1)\,dx=0$. **[Vikram 1959]**

(vi) $\frac{dy}{dx}+(2x\tan^{-1}y-x^3)(1+y^2)=0$.

Divide by $1+y^2$, put $\tan^{-1}y=v$ etc.

Ans. $2\tan^{-1}y=x^2-1+2Ce^{-x^2}$

(vii) $(y\log x-1)\,y\,dx=x\,dy$. **[Cal. Hons. 1961; Vikram 60]**

Divide by xy^2 etc. **Ans.** $1/y+1+\log x=Cx$.

(viii) $y+2\frac{dy}{dx}=y^3(x-1)$. **[Gujrat 1958]**

Divide by y^3 and put $y^{-2}=v$ etc.

(ix) $\cos x\frac{dy}{dx}+y\sin x+2y^3=0$. **[Nag. 1962]**

Dividing by $y^3\cos x$, we get

$$y^{-3}\frac{dy}{dx}+y^{-2}\tan x=-2\sec x.\ \text{Now put } y^{-2}=v.$$

(x) $\frac{dy}{dx}+\frac{xy}{1-x^2}=x\sqrt{y}$. **[Cal. Hons. 1957; Vikram 63]**

Hint. Divide by $\sqrt{y}$ and put $\sqrt{y}=v$ etc.

Ans. $2\sqrt{y}=-\frac{2}{3}(1-x^2)+C(1-x^2)^{1/4}$

(xi) $3e^x\tan y+(1-e^x)\sec^2 y\frac{dy}{dx}=0$. **[Alld. 1965]**

Put $\tan y=v$. **Ans.** $\tan y=C+3\log(1-e^x)$.

Problems of curves leading to the differential equations of the first order and first degree.

Ex. 1. *Find the equations of the curves for which the cartesian subtangent is constant.* **[Nagpur 1956 (S)]**

Solution. Cartesian subtangent is given by $y\frac{dx}{dy}$.

$$\therefore\ y\frac{dx}{dy}=a, \text{ where } a \text{ is a constant}$$

or $\frac{dy}{y}=\frac{1}{a}dx$, separating the variables.

Integrating, $\log y=\frac{1}{a}x+C=\log e^{(x/a+C)}$

or $y=e^{x/a+C}=Ae^{x/a}$

is the curve.

Ex. 2. *Find the equations of the curve for which the cartesian subnormal is constant.* **[Delhi 1950]**

Solution. Cartesian subnormal is given by $y\dfrac{dy}{dx}$.

$$\therefore \quad y\frac{dy}{dx}=a, \text{ where } a \text{ is constant}$$

or $\quad y\,dy=a\,dx.$

Integrating. $\frac{1}{2}y^2=ax+C$ or $y^2=2ax+A$ is the curve, where $A=2C$ is an arbitrary constant.

Ex. 3. *Find the equation of the curve for which the polar subtangent is constant.*

Solution. Polar subtangent is given by $r^2\dfrac{d\theta}{dr}$.

$$\therefore \quad r^2\frac{d\theta}{dr}=a, \quad \text{where } a \text{ is a constant}$$

or $\quad \dfrac{a}{r^2}\,dr=d\theta.$

Integrating, $\quad -\dfrac{a}{r}=\theta+C$ or $r(\theta+C)+a=0$ is the curve.

Ex. 4. *Find the equation of the curve for which polar subnormal is constant.* [Delhi Hons. 1963]

Solution. Polar subnormal is given by $\dfrac{dr}{d\theta}$

$$\therefore \quad \frac{dr}{d\theta}=a \quad \text{or} \quad dr=a\,d\theta.$$

or $\quad r=a\theta+C$ is the curve.

Ex. 5 *Find the curve for which the tangent at each point makes a constant angle α with the radius vector.*

Solution. Let ϕ denote the angle between the radius vector and tangent, then

$$\tan\phi=r\frac{d\theta}{dr}. \quad \text{But } \phi=\alpha \text{ (const.)}$$

So $\quad \tan\alpha=r\dfrac{d\theta}{dr} \quad$ or $\quad \dfrac{dr}{r}=\cot\alpha\,d\theta.$

Integrating, $\log r=\theta\cot\alpha+\log C$

or $\quad \log\dfrac{r}{C}=\theta\cot\alpha \quad$ or $\quad r=Ce^{\theta\cot\alpha}.$

Ex. 6. *Show that all curves for which the square of the normal is equal to the square of the radius vector are either circles or rectangular hyperbolas.*

Solution. At any point $P(x, y)$ of a curve

length of the normal $=y\sqrt{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]}$

Length of the radius vector $=\sqrt{(x^2+y^2)}$.

As given $\quad y^2\left[1+\left(\frac{dy}{dx}\right)^2\right]=x^2+y^2$

or $\quad y^2\left(\frac{dy}{dx}\right)^2=x^2, \quad$ so that $y\frac{dy}{dx}=\pm x.$

When $y\frac{dy}{dx}=+x$, we get

$x\,dx=y\,dy$, *i.e.*, $\frac{1}{2}x^2=\frac{1}{2}y^2+C$ (integrating)

or $\quad x^2-y^2=2C$ which is rectangular hyperbola.

Again when $y\frac{dy}{dx}=-x$, we have

$y\,dy+x\,dx=0.$

Integrating, $y^2+x^2=C$, which is a circle.

Ex. 7. *Find the curve for which the sum of the reciprocals of the radius vector and the polar subtangent is constant.* **[Agra 1956]**

Solution. We know that polar subtangent$=r^2\frac{d\theta}{dr}$.

$\therefore$ as given $\frac{1}{r}+\frac{1}{r^2}\frac{dr}{d\theta}=k$ (const.)

Let $\frac{1}{r}=v$, so that $-\frac{1}{r^2}\frac{dr}{d\theta}=\frac{dv}{d\theta}$.

$\therefore$ (1) becomes $v-\frac{dv}{d\theta}=k$ or $\frac{dv}{d\theta}-v=-k$,

which is linear in v. $\therefore$ I. F.$=e^{\int -1\,d\theta}=e^{-\theta}$.

$\therefore$ The solution is $ve^{-\theta}=\int -ke^{-\theta}\,d\theta+C$

or $\frac{1}{r}e^{-\theta}=ke^{-\theta}+C$ as $v=\frac{1}{r}$

or $\frac{1}{r}=k+Ce^{\theta}$ is the curve.

Ex. 8. *Find the equation of the curve in which the angle between the radius vector and the tangent is one half of the vectorial angle.* **[Agra B.Sc. 1957]**

Solution. If ϕ is the angle between radius vector and tangent,

$\tan\phi=r\frac{d\theta}{dr}$.

As given, $\phi=\frac{1}{2}\theta$ or $\tan\phi=\tan\frac{1}{2}\theta$

or $r\frac{d\theta}{dr}=\tan\frac{1}{2}\theta$.

Separating the variables, $\frac{dr}{r}=\cot\frac{1}{2}\theta\,d\theta$.

Integrating, $\log r=2\log\sin\frac{1}{2}\theta+\log C$

or $\frac{r}{C}=\sin^2\frac{1}{2}\theta=\frac{1}{2}(1-\cos\theta)$

or $r=a(1-\cos\theta)$ where $a=\frac{1}{2}C$. The curve is a cardioid.

Ex. 9. *Find the equation of the curve in which the angle between the radius vector and tangent is supplementary of half the vectorial angle.* **[Agra B.Sc. 58]**

Solution. Here $\phi=\pi-\frac{1}{2}\theta$ or $\tan\phi=\tan(\pi-\frac{1}{2}\theta)$.

$$r\frac{d\theta}{dr}=-\tan\tfrac{1}{2}\theta$$

Separating the variables, we get

$$\frac{dr}{r}+\cot\tfrac{1}{2}\theta\,d\theta=0.$$

Integrating, $\log r+2\log\sin\frac{1}{2}\theta=\log C$ or $r\sin^2\frac{1}{2}\theta=C$

or $\frac{1}{2}r(1-\cos\theta)=C$ or $\dfrac{2C}{r}=1-\cos\theta$.

The curve is a parabola.

Ex. 10. *Show that if y_1 and y_2 be solutions of the equation*

$$\frac{dy}{dx}+Py=Q,$$

where P and Q are functions of x alone, and $y_2=y_1 z$, then

$$z=1+ae^{\int -Q/y_1\,dx},$$

where a is an arbitrary constant. **[Sagar 62]**

Solution. $y_2=y_1 z$. $\therefore$ $\dfrac{dy_2}{dx}=z\dfrac{dy_1}{dx}+y_1\dfrac{dz}{dx}$.

As y_2 is a solution of the given equation, $\dfrac{dy_2}{dx}+Py_2=Q$.

Substituting in this value of $\dfrac{dy_2}{dx}$ and y_2, we get

$$z\frac{dy_1}{dx}+y_1\frac{dz}{dx}+P\,y_1 z=Q$$

or $$z\left(\frac{dy_1}{dx}+Py_1\right)+y_1\frac{dz}{dx}=Q$$

or $$zQ+y_1\frac{dz}{dx}=Q \quad \text{as} \quad \frac{dy_1}{dx}+Py_1=Q$$

or $$\frac{dz}{z-1}=-\frac{Q}{y_1}\,dx.$$

Integrating, $\log(z-1)=C+\displaystyle\int -\frac{Q}{y_1}\,dx$

or $z=1+e^{\int -Q/y_1\,dx}$. This proves the result.

3

Equations of First Order and First Degree

Exact Differential Equations and Reduction to Exact Equations

3·1. Exact Differential Equations. **[Bombay 61 ; Karnatak 60]**

Study the following two differential eqnations :

1. $x\,dy+y\,dx=0$. Solution is $xy=C$.
2. $\sin x \cos y\,dy+\cos x \sin y\,dx=0$.

Solution is $\sin x \sin y=C$.

We see that these differential equations can be obtained by directly differentiating their solutions. Differential equations of this type are called exact equations and bear the following property :

An exact differential equation can always be obtained from its primitive directly by differentiation, without any subsequent multiplication, elimination etc.

***3·2. Necessarry and Sufficient Condition**

To find the necessary and sufficient condition for a differential equation of first degree being exact.

[Poona 63, 61 ; Delhi Hons. 57, 55 ; Nag. 63 ; Gujrat 59; Bombay 61]

Let the equation be $M+N\dfrac{dy}{dx}=0$. ...(1)

Let $u=C$ be its primitive. ...(2)

If (1) is exact, it can be obtained by directly differentiating its primitive.

Differentiating (2), we have $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}\cdot\dfrac{dy}{dx}=0.$ ...(3)

Comparing (1) and (3) we get $M=\dfrac{\partial u}{\partial x}$ and $N=\dfrac{\partial u}{\partial y}$, so that

$$\frac{\partial M}{\partial y}=\frac{\partial^2 u}{\partial y\,\partial x},\quad \frac{\partial N}{\partial x}=\frac{\partial^2 u}{\partial x\,\partial y}.$$

Hence the condition is $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$.

That the condition is necessary has been proved. Now we prove that it is sufficient also, *i.e.* if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then we show that $M+N\frac{dy}{dx}=0$ or $M\,dx+N dy=0$ is an exact equation.

Let $\int M\,dx=U$, then $\frac{\partial U}{\partial x}=M$, so that

$$\frac{\partial^2 U}{\partial y\,\partial x}=\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \text{ as } \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x},$$

i.e. $\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y}\right)$

Integrating, $N=\frac{\partial U}{\partial y}+f(y)$, where $f(y)$ is a function of y free from x.

$$\therefore \quad M+N\frac{dy}{dx}=\frac{\partial U}{\partial x}+\left[\frac{\partial U}{\partial y}+f(y)\right]\frac{dy}{dx}$$
$$=\frac{d}{dx}\left[U+\int f(y)\,\frac{dy}{dx}\,dx\right]$$
$$=\frac{d}{dx}\,[U+F(y)].$$

This shows that $M+N\frac{dy}{dx}=0$ is an exact equation.

3·3. Working Rule (Remember it).

If the equation $M\,dx+N\,dy=0$ satisfies the condition

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x},$$

then it is exact. To integrate it,

(*i*) integrate M with regard to x regarding y as constant;

(*ii*) find out those terms in N which are free from x and integrate them with regard to y;

(*iii*) add the two expressions so obtained and equate the sum to an arbitrary constant.

This gives the general solution of the given exact equation.

Ex. 1. $(y^4+4x^3y+3x)\,dx+(x^4+4xy^3+y+1)\,dy=0$

[Karnatak 60]

Solution Here $M=y^4+4x^3y+3x$ and $N=x^4+4xy^3+y+1$.

$$\frac{\partial M}{\partial y}=4y^3+4x^3 \text{ and } \frac{\partial N}{\partial x}=4x^3+4y^3.$$

Since these are equal, the equation is exact.

To find solution of the differential equation, integrating M *i.e.* y^4+4x^3y+3x w.r.t. x, keeping y as constant, we get $y^4x+x^4y+\frac{3}{2}x^2$.

In x^4+4xy^3+y+1, terms free from x are $y+1$ whose integral with respect to y is $\frac{1}{2}y^2+y$.

Therefore the general solution is

$$y^4x+x^4y+\tfrac{2}{3}x^3+\tfrac{1}{2}y^2+y=C.$$

Ex. 2. *Solve* $x(x^2+y^2-a^2)\,dx+y(x^2-y^2-b^2)\,dy=0.$ **[Nag. 63; Poona 61]**

Solution. Here $M=x^3+xy^2-a^2x$, $N=yx^2-y^3-b^2y$.

$$\frac{\partial M}{\partial y}=2xy \text{ and } \frac{\partial N}{\partial x}=2xy.$$

Since these are equal, the equation is exact,

Integrating M w.r.t. x keeping y as constant, we get

$$\tfrac{1}{4}x^4+\tfrac{1}{2}x^2y^2-\tfrac{1}{2}a^2x^2.$$

In N, terms free from x are $-y^3-b^2y$ whose integral is

$$-\tfrac{1}{4}y^4-\tfrac{1}{2}b^2y^2.$$

Hence the general solution is

$$\tfrac{1}{4}x^4+\tfrac{1}{2}x^2y^2-\tfrac{1}{2}a^2x^2-\tfrac{1}{4}y^4-\tfrac{1}{2}b^2y^2=\text{const.}$$

or $\quad x^4-y^4+2x^2y^2-2a^2x^2-2b^2y^2=C.$

Ex. 3. *Solve* $(x^2-2xy+3y^2)\,dx+(4y^3+6xy-x^2)\,dy=0.$ **[Delhi Hons. 55]**

Solution. Here $\dfrac{\partial M}{\partial y}=-2x+6y$, $\dfrac{\partial N}{\partial x}=6y-2x.$

Since these are equal the equation is exact.

Integrating M, *i.e.* $x^2-2xy+3y^2$ w.r.t. x keeping y as constant, we get $\quad \tfrac{1}{3}x^3-x^2y+3y^2x$

In N, term free from x is $+4y^3$ whose integral is y^4.

Hence the solution is $\tfrac{1}{3}x^3-x^2y+3y^2x+y^4=C.$

Ex. 4. *Solve* $(x-2e^y)\,dy+(y+x\sin x)\,dx=0.$ **[Gujrat 61]**

Solution. Here $M=y+x\sin x$, $N=x-2e^y$.

$\therefore \quad \dfrac{\partial M}{\partial y}=1, \dfrac{\partial N}{\partial x}=1$; therefore equation is exact.

Integrating $y+x\sin x$ with respect to x keeping y as constant, we get $\quad xy+\int x\sin x\,dx=xy-x\cos x+\sin x.$

In N, term free from x is $-2e^y$ whose integral with respect to y is $-2e^y$.

Hence the complete solution is

$$xy-x\cos x+\sin x-2e^y=C.$$

***Ex. 5. (a)** *Solve* $x\,dx+y\,dy=\dfrac{a^2(x\,dy-y\,dx)}{x^2+y^2}.$ **[Delhi Hons. 62**

Solution. The equation can be put as

$$\left(x+\frac{a^2y}{x^2+y^2}\right)dx+\left(y-\frac{a^2x}{x^2+y^2}\right)dy=0.$$

Here $M=x+\dfrac{a^2y}{x^2+y^2}$ and $N=y-\dfrac{a^2x}{x^2+y^2}$.

$\therefore \quad \dfrac{\partial M}{\partial y}=\dfrac{(x^2+y^2)\,a^2-a^2y.2y}{(x^2+y^2)^2}=\dfrac{a^2(x^2-y^2)}{(x^2+y^2)^2}$

and $\dfrac{\partial N}{\partial x}=\dfrac{-a^2(x^2+y^2)+2a^2x^2}{(x^2+y^2)^2}=\dfrac{a^2(x^2-y^2)}{(x^2+y^2)^2}$.

Since $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$, the equation is exact.

Integrating M w.r.t. x regarding y as constant, we get

$$\tfrac{1}{2}x^2+a^2y\frac{1}{y}\tan^{-1}\frac{x}{y} \quad \text{or} \quad \tfrac{1}{2}x^2+a^2\tan^{-1}\frac{x}{y}.$$

In N, term free from x is y whose integral is $\tfrac{1}{2}y^2$.

Hence the solution is $\tfrac{1}{2}x^2+a^2\tan^{-1}\dfrac{x}{y}+\tfrac{1}{2}y^2=\text{const.}$

or $\quad x^2+y^2+2a^2\tan^{-1}\dfrac{x}{y}=C.$

Ex. 5. (b) *Solve* $x\,dx+y\,dy+\dfrac{x\,dy-y\,dx}{x^2+y^2}=0.$

The equation is exact; proceed as in the above example.

***Ex. 6.** *Solve* $(1+e^{x/y})\,dx+e^{x/y}(1-x/y)\,dy=0.$

[Karnatak 61; Bombay 50; Gujrat 59; Poona 61]

Solution. Here $M=1+e^{x/y}$ and $N=e^{x/y}(1-x/y)$

$$\frac{\partial M}{\partial y}=e^{x/y}\left(-\frac{x}{y^2}\right)$$

and $\dfrac{\partial N}{\partial x}=e^{x/y}\dfrac{1}{y}.\left(1-\dfrac{x}{y}\right)+e^{x/y}\left(-\dfrac{1}{y}\right)=e^{x/y}\left(-\dfrac{x}{y^2}\right).$

Since $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$, the equation is exact.

Now integrating $1+e^{x/y}$ with respect to x keeping y as constant, we get $\quad x+\dfrac{e^{x/y}}{1/y}$ *i.e.*, $x+ye^{x/y}$

In N *i.e.*, in $e^{x/y}(1-x/y)$ there is no term free from x.

Hence the required solution is $x+ye^{x/y}=C.$

Ex. 7. $[\cos x \tan y+\cos(x+y)]\,dx$

$$+[\sin x \sec^2 y+\cos(x+y)]\,dy=0.$$

[Bombay 61; Gujrat 61]

Solution. Here $M=\cos x\tan y+\cos(x+y)$,

and $\quad N=\sin x\sec^2 y+\cos(x+y)$.

Now $\quad \dfrac{\partial M}{\partial y}=\cos x\sec^2 y-\sin(x+y)$,

$$\frac{\partial N}{\partial x}=\cos x \sec^2 y-\sin (x-y).$$

Since these are equal, the equation is exact.

Now integrating M, *i.e.* $\cos x \tan y-\cos (x+y)$ with respect to x keeping y as constant, we get

$$\sin x \tan y+\sin (x+y)$$

In N, there is no term free from x.

Hence the general solution is

$$\sin x \tan y+\sin (x+y)=C.$$

Ex. 8. $(\cos x \tan y-\sin x \sec y)\, dx$
$+(\sin x \sec^2 y+\cos x \tan^2 y \operatorname{cosec} y)\, dy=0.$

[Bombay B. A. (Sub.) 58]

Solution. We have $M=\cos x \tan y-\sin x \sec y$,

and $N=\sin x \sec^2 y+\cos x \tan^2 y \operatorname{cosec} y$.

$$\therefore \quad \frac{\partial M}{\partial y}=\cos x \sec^2 y-\sin x \sec y \tan y$$

$$\frac{\partial N}{\partial x}=\cos x \sec^2 y-\sin x \tan y \sec y.$$

as $\tan^2 y \operatorname{cosec} y=\tan y \sec y$.

Since $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$, the equation is exact.

Integrating M *i.e.* $\cos x \tan y-\sin x \sec y$ with regard to x keeping y as constant we get

$$\sin x \tan y+\cos x \sec y.$$

In N there is no term free from x.

Hence the general solution is

$$\sin x \tan y+\cos x \sec y=C.$$

Ex. 9. *Solve* $(\sin x \cos y+e^{2x})\, dx$
$+(\cos x \sin y+\tan y)\, dy=0.$

[Poona 59]

Solution. Here $\dfrac{\partial M}{\partial y}=-\sin x \sin y$, $\dfrac{\partial N}{\partial x}=-\sin x \sin y$.

Since these are equal, the equation is exact.

Integrating M *i.e.*, $\sin x \cos y+e^{2x}$ w.r.t. x, keeping y as constant, we get $-\cos x \cos y+\frac{1}{2}e^{2x}$.

Also in N the term free from x is $\tan y$ whose integral w.r.t. y is $\log \sec y$.

Hence the solution is

$$-\cos x \cos y+\tfrac{1}{2}e^{2x}+\log \sec y=C.$$

Ex. 10. *Solve the following equations (which are exact)* :

(*i*) $(2x^3+3y)\, dx+(3x+y-1)\, dy=0.$ **[Poona 93]**

Ans. $\frac{1}{2}x^4+3yx+\frac{1}{2}y^2-y=C$

(ii) $(x^2-4xy-2y^2)\,dx+(y^2-4xy-2x^2)\,dy.$

Ans. $x^3+y^3-6xy\,(x+y)=C.$

(iii) $\cos x\,(\cos x-\sin a \sin y)\,dx$

$+\cos y\,(\cos y-\sin a \sin x)\,dy=0.$

Ans. $2(x+y)\sin 2x+\sin 2y-4 \sin a \sin x \sin y=C.$

(iv) $(2xy+y-\tan y)\,dx+(x^2-x\tan^2 y+\sec^2 y)\,dy=0.$ [Poona 1964]

Ans. $x^2y+xy-x\tan y+\tan y=C.$

(v) $(2x^2y+4x^3-12xy^2+3y^2-xe^y+e^{2x})\,dy$

$+(12x^2y+2yx^2+4x^3-4y^3+2ye^{2x}-e^y)\,dx=0.$ [Poona 64]

Ans. $4x^3y+x^2y^2+x^4-4y^3x+ye^{2x}-xe^y+y^3=C.$

3·4. Integrating factors.

If an equation becomes exact after it has been multiplied by a function of x and y, then such a function is called an integrating factor [Karnatak 61]

3·5. Number of integrating factors.

To show that there is an infinite number of integrating factors for an equation.

$M\,dx+N\,dy=0.$ [Karnatak 61]

To prove this let μ be an integrating factor: then

$\mu\,(M\,dx+N\,dy)=du.$

Integrating, $u=c$ is a solution.

Now multiplying both the sides by $f(u)$, a function of u, we get $\mu\, f(u)\,[M\,dx+N\,dy]=f(u)\,du.$

Expression on the right is directly integrable and therefore so must be the left hand side.

Hence $\mu f(u)$ is also an integrating factor. Since $f(u)$ is an arbitrary function of u, the number of integrating factors is infinite.

3·6. Integrating factor by inspection.

Sometimes an integrating factor can be found by inspection. For this the reader should study the following results :—

Group of terms	I F.	Exact Differential
$x\,dy-y\,dx$	$\frac{1}{x^2}$	$\frac{x\,dy-y\,dx}{x^2}=d\left(\frac{y}{x}\right)$
$x\,dy-y\,dx$	$\frac{1}{y^2}$	$\frac{y\,dx-x\,dy}{-y^2}=d\left(-\frac{x}{y}\right)$
$x\,dy-y\,dx$	$\frac{1}{xy}$	$\frac{dy}{y}-\frac{dx}{x}=d\left(\log\frac{y}{x}\right)$
$x\,dy-y\,dx$	$\frac{1}{x^2+y^2}$	$\frac{x\,dy-y\,dx}{x^2+y^2}=\dfrac{\frac{x\,dy-y\,dx}{x^2}}{1+\left(\frac{y}{x}\right)^2}=d\left[\tan^{-1}\frac{y}{x}\right]$

Groups of terms	I.F.	Exact Differential
$x\,dy+y\,dx$	$\dfrac{1}{(xy)^n}$	$\dfrac{x\,dy+y\,dx}{xy}=d\,[\log(xy)]$ for $n=1$
$x\,dx+y\,dy$	$\dfrac{1}{(x^2+y^2)^n}$	$\dfrac{x\,dx+y\,dy}{(x^2+y^2)^n}=d\left[-\dfrac{1}{2\,(n-1)\,(x^2+y^2)^{n-1}}\right]$ or $=\dfrac{x\,dx+y\,dy}{x^2+y^2}=d\,[\frac{1}{2}\log(x^2+y^2)]$ if $n=1$.

Ex. 1. *Solve* $(x+y^2)\,dy+(y-x^2)\,dx=0$. **[Nagpur 61]**

Solution. The equation can be written as

$$x\,dy+y\,dx+y^2\,dy-x^2\,dx=0.$$

or $$d\,(xy)+y^2\,dy-x^2 dx=0.$$

Integrating, $xy+\frac{1}{3}y^3-\frac{1}{3}x^3=A$ or $y^3-x^3+3xy=c$.

Ex. 2. *Solve* $y\,dx-x\,dy+3x^2y^2e^{x^3}\,dx=0$. **[Nagpur 61]**

Solution. The equation can be written as

$$\frac{y\,dx-x\,dy}{y^2}+3x^2e^{x^3}\,dx=0,$$

$$d\left(\frac{x}{y}\right)+e^{x^3}\,d(x^3)=0.$$

Integrating, $\dfrac{x}{y}+e^{x^3}=c$.

Ex. 3. *Solve* $x\,dy-y\,dx-x\,(x^2-y^2)^{1/2}\,dx=0$. **[Delhi Hons. 61]**

Solution. The equation can be written as

$$\frac{x\,dy-y\,dx}{(x^2-y^2)^{1/2}}-x\,dx=0$$

i.e. $$\frac{\dfrac{x\,dy-y\,dx}{x^2}}{\left[1-\left(\dfrac{y}{x}\right)^2\right]^{1/2}}=dx,\text{ put }\frac{y}{x}=t,\text{ then }\frac{x\,dy-y\,dx}{x^2}=dt$$

i.e., $$\frac{dt}{\sqrt{(1-t^2)}}=dx\text{ or }x+c=\sin^{-1}t=\sin^{-1}\left(\frac{y}{x}\right).$$

Ex. 4. *Solve* $a\,(x\,dy+2y\,dx)=xy\,dy$.

Solution. The equation can be written as

$$(a-y)\,x\,dy+2ay\;dx=0\quad\text{or}\quad\frac{a-y}{y}\,dy+\frac{2a}{x}\,dx=0.$$

Integrating, $\quad a\log y-y+2a\log x=C_1$

or $\log yx^2 = \frac{y}{a} + \log C$ or $yx^2 = Ce^{y/a}$,

Ex. 5. *Solve* $y\,dx - x\,dy + \log x\,dx = 0$.

Solution. The equation is $x\frac{dy}{dx} - y = \log x$

or $\frac{dy}{dx} - \frac{1}{x}y = \frac{\log x}{x}$. Linear, I.F. $= e^{-\int \frac{1}{x}dx} = \frac{1}{x}$.

$\therefore\ y.\frac{1}{x} = \int \frac{1}{x^2}\log x\,dx - C$

$= -\frac{1}{x}(1 + \log x) - C$

or $y + \log x + Cx + 1 = 0$ is the solution.

Ex. 6. *Solve* $(1+xy)\,y\,dx + (1-xy)\,x\,dy = 0$. **[Bihar 62]**

Solution. Write the equation as

$y\,dx + x\,dy + xy\,(y\,dx - x\,dy) = 0$

or $d(xy) + xy\,(y\,dx - x\,dy) = 0$.

We readily find that $\frac{1}{x^2y^2}$ is the I.F. So the equation becomes

$$\frac{d(xy)}{x^2y^2} + \frac{y\,dx - x\,dy}{xy} = 0 \text{ or } \frac{d(xy)}{(xy)^2} + \left(\frac{dx}{x} - \frac{dy}{y}\right) = 0.$$

Integrating, $-\frac{1}{xy} + \log x - \log y = C_1$ or $x = Cye^{1/xy}$.

Ex. 7. *Solve* $(x^4e^x - 2mxy^2)\,dx + 2mx^2y\,dy = 0$.

Solution. Equation is $2y\frac{dy}{dx} - \frac{2y^2}{x} + \frac{x^2e^x}{m} = 0$.

Putting $y^2 = z$, the equation becomes $\frac{dz}{dx} - \frac{2}{x}z + \frac{x^2e^x}{m} = 0$.

I.F. $= e^{-\int \frac{2}{x}dx} = \frac{1}{x^2}$, etc.

Ex. 8. *Solve* $y\,(2xy + e^x)\,dx - e^x\,dy = 0$. **[Vikram 61]**

Solution. The equation is $e^x\frac{dy}{dx} = 2xy^2 + ye^x$

or $-y^{-2}\frac{dy}{dx} + y^{-1} = -2xe^{-x}$. Put $y^{-1} = v$, $-y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$.

$\therefore$ the equation is $\frac{dv}{dx} + e = -2xe^{-x}$. I.F. $= e^x$ etc.

Solution is $y^{-1}e^x = -x^2 + C$.

3·7. Rules for finding the integrating factor.

Rule. I. If $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=f(x)$, a function of x only, then $e^{\int f(x)\,dx}$ is an integrating factor. **[Delhi Hons. 64]**

Rule II. If $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{M}=g(y)$ is a function of y alone, then $e^{\int -g(y)\,dy}$ is an integrating factor.

We give below some examples to illustrate these rules.

Ex. 1. *Solve* $(x^2+y^2+x)\,dx+xy\,dy=0$.

Solution $M=x^2+y^2+x,\ N=xy$.

$\dfrac{\partial M}{\partial y}=2y,\ \dfrac{\partial N}{\partial x}=y$, equation is not exact.

However, $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=\dfrac{2y-y}{xy}=\dfrac{1}{x}$, a function of x alone.

Hence I.F. $=e^{\int \frac{1}{x}dx}=e^{\log x}=x$.

Multiplying by I.F., the equation becomes

$(x^3+xy^2+x^2)\,dx+x^2y\,dy=0$, exact now (check up).

Integrating, $x^3+xy^2+x^2$ with regard to x, keeping y as constant, we get $\frac{1}{4}x^4+\frac{1}{2}x^2y^2+\frac{1}{3}x^3$

and in x^2y^2 there is no term free from x. Therefore the solution is

$\frac{1}{4}x^4+\frac{1}{2}x^2y^2+\frac{1}{3}x^3=C'$ or $3x^4+4x^3+6x^2y^2=C$.

Ex. 2. *Solve* $(x^2+y^2+1)\,dx-2xy\,dy=0$.

Solution. $\dfrac{\partial M}{\partial y}=2y,\ \dfrac{\partial N}{\partial x}=-2y$, not exact.

However, $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=\dfrac{2y+2y}{-2xy}=-\dfrac{2}{x}$ function of x alone.

$\therefore$ I.F. $=e^{-\int \frac{2}{x}dx}=e^{-2\log x}=\dfrac{2}{x^2}$.

Multiplying by $\dfrac{1}{x^2}$ the equation becomes

$\left(1+\dfrac{y^2}{x^2}+\dfrac{1}{x^2}\right)dx-\dfrac{2y}{x}\,dy=0$, exact now.

Integrating, $1+\dfrac{y^2}{x^2}+\dfrac{1}{x_2}$ with regard to x keeping y as constant.

we get $x-\dfrac{y^2}{x}-\dfrac{1}{x}$

and in $-\frac{2y}{x}$ there in no term free from x.

Hence the solution is

$$x-\frac{y^2}{x}-\frac{1}{x}=C \text{ or } x^2-y^2=Cx+1.$$

Ex. 3. *Solve* $(x^2+y^2)\,dx-2xy\,dy=0.$

Solution Just as in the above example, I.F.$=\frac{1}{x^2}$.

Hence multiplying by $\frac{1}{x^2}$ the equation becomes

$$\left(1+\frac{y^2}{x^2}\right)dx-\frac{2y}{x}dy=0, \text{ exact.}$$

$\therefore$ Solution is $x-\frac{y^2}{x}=c$ or $x^2-y^2=cx$.

Ex. 4. *Solve* $(x^2+y^2+2x)\,dx+2y\,dy=0.$

[Vikram 1959 ; Alld. 59]

Solution. $\frac{\partial M}{\partial y}=2y,\ \frac{\partial N}{\partial x}=0$, not exact.

$$\text{However,}\quad \frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{2y}{2y}=1.$$

$$\therefore \quad \text{I.F.}=e^x.$$

Multiplying by e^x, the equation becomes

$e^x(x^2+y^2+2x)\,dx+2ye^x\,dy=0$, now exact.

This can be written as

$(x^2+2x)\,e^x\,dx+(y^2e^x\,dx+e^x.2y\,dy)=0$

or $d(x^2e^x)+d(y^2e^x)=0.$

$\therefore$ Integrating, $x^2e^x+y^2e^x=C$ or $(x^2+y^2)\,e^x=C.$

Aliter. The equation can also be written as

$$2y\frac{dy}{dx}+y^2=-(x^2+2x).$$

Putting $y^2=v$, $\frac{dv}{dx}+v=-(x^2+2x)$. Linear, I.F.$=e^x$ etc

***Ex. 5.** *Solve* $(\frac{1}{3}y+y^3+\frac{1}{2}x^2)\,dx+\frac{1}{4}(x+xy^2)\,dy=0.$

[Delhi Hons. 1965 ; Agra M.Sc. 63 : Banaras 56]

Solution. $\frac{\partial M}{\partial y}=1+y^2,\ \frac{\partial N}{\partial x}=\frac{1}{4}(1+y^2)$, not exact.

$$\text{However,}\quad \frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{(1+y^2)-\frac{1}{4}(1+y^2)}{\frac{1}{4}x(1+y^2)}=\frac{3}{x},$$

a function of x alone.

$\therefore$ I.F. $= e^{\int \frac{3}{x} dx} = e^{3 \log x} = x^3$.

Multiplying by x^3, the equation becomes

$(x^3y + \frac{1}{3}x^3y^3 + \frac{1}{2}x^5)\, dx + \frac{1}{4}\,(x^4 + x^4y^2)\, dy = 0$, exact now.

Integrating $x^3y + \frac{1}{3}x^3y^3 + \frac{1}{2}x^5$ with respect to x keeping y as constant, we get $\frac{1}{4}x^4y + \frac{1}{12}x^4y^3 + \frac{1}{12}x^6$.

In $\frac{1}{4}\,(x^4 + x^4y^2)$ there is no term free from x.

$\therefore$ the solution is $\frac{1}{4}x^4y + \frac{1}{12}x^4y^3 + \frac{1}{12}x^6 =$ constant

or $$3x^4y + y^3x^4 + x^6 = C.$$

Ex. 6. *Is the differential equation* $(x^3 - 2y^2)\, dx + 2xy\, dy = 0$ *exact ? Solve the equation.* **[Cal. Hons. 1963]**

Solution. The equation is not exact ; however we have

$$\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{-4y - 2y}{2xy} = -\frac{3}{x}; \quad \therefore \text{ I.F.} = e^{\int -3\, dx/x} = \frac{1}{x^3}.$$

Proceed as above.

Ex. 7. $(2x^3y^2 + 4x^2y + 2xy^2 + xy^4 + 2y)\, dx$
$+ 2\,(y^3 - x^2y + x)\, dy = 0.$

Solution. Equation is not exact.

$$\frac{1}{N}\left\{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right\} = 2x. \quad \text{I.F.} = e^{\int 2x\, dx} = e^{x^2}.$$

The solution is $(2x^2y^2 + 4xy + y^4)\, e^{x^2} = C$.

Ex. 8. *Solve* $(y^4 + 2y)\, dx + (xy^3 + 2y^4 - 4x)\, dy = 0.$

[Cal. Hons. 1962, 61]

Solution. $\frac{\partial M}{\partial y} = 4y^3 + 2$, $\frac{\partial N}{\partial x} = y^3 - 4$, not exact.

However, $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{4y^3 + 2 - (y^3 - 4)}{y^4 + 2y} = \frac{3}{y} \text{ a function of } y \text{ alone.}$$

$$\therefore \text{ I.F.} = e^{-\int \frac{3}{y} dy} = e^{-3 \log y} = \frac{1}{y^3}.$$

Multiplying by $1/y^3$, the equation becomes

$$\left(y + \frac{2}{y^2}\right) dx + \left(x + 2y - \frac{4x}{y^3}\right) dy = 0, \text{ exact now.}$$

Integrating $y + \frac{2}{y^2}$ w.r.t. x keeping y as constant, we have

$$yx + \frac{2}{y^2}\, x.$$

In $x + 2y - \frac{4x}{y^3}$, the term free from x is $2y$. So integrating $2y$ w.r.t. y, we get y^2.

Therefore the solution is $yx+\frac{2}{y^2}x+y^2=C$.

Ex. 9. *Solve* $(3x^2y^4+2xy)\,dx+(2x^3y^3-x^2)\,dy=0$. **[Cal. Hons. 54, 53]**

Solution. Here $\frac{\partial M}{\partial y}=12x^2y^3+2x,\ \frac{\partial N}{\partial x}=6x^2y^3-2x$.

Now $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{M}=\dfrac{6x^2y^3+4x}{y(3x^2y^3+2x)}=\dfrac{2}{y}$ function of y alone.

$\therefore$ I. F. $=e^{\int-(2/y)\,dy}=e^{-2\log y}=\frac{1}{y^2}$.

Multiplying by $\frac{1}{y^2}$, the equation becomes

$$\left(3x^2y^2+\frac{2x}{y}\right)dx+\left(2x^3y-\frac{x^2}{y^2}\right)dy=0\text{, exact now.}$$

Integrating $3x^2y^2+\frac{2x}{y}$ w.r.t. x keeping y as constant, we get

$$x^3y^2+\frac{x^2}{y}.$$

In $2x^3y-\frac{x^2}{y^2}$, there is no term free from x.

Hence the solution is $x^3y^2+\frac{x^2}{y}=C$

or $$x^3y^3+x^2=Cy.$$

Ex. 10. $(2xy^4e^y+2xy^3+y)\,dx+(x^2y^4e^y-x^2y^2-3x)\,dy=0$.

Solution. We have $\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=-\frac{4}{y}$. $\therefore$ I.F. $=\frac{1}{y^4}$.

Solution is $$x^2e^y+\frac{x^2}{y}+\frac{x}{y^3}=C.$$

3·8. Rule III.

If $M\,dx+N\,dy=0$ is homogeneous and $Mx+Ny\neq 0$,

then $\frac{1}{Mx+Ny}$ is an integrating factor.

Rule IV. **[Delhi Hons. 61]**

If the equation can be written in the form

$$yf(xy)\,dx+xg(xy)\,dy=0,\ f(xy)\neq g(xy),$$

then $\frac{1}{xy[f(xy)-g(xy)]}=\frac{1}{Mx-Ny}$ is an integrating factor.

Ex. 1. *Solve* $x^2y\,dx-(x^3+y^3)\,dy=0$.

Solution. The equation is homogeneous and

$\therefore Mx+Ny\neq 0.$

Hence $\dfrac{1}{Mx+Ny}=\dfrac{1}{x^3y-(x^3+y^3)y}=-\dfrac{1}{y^4}$ is integrating factor.

Multiplying by $-\dfrac{1}{y^4}$ the equation becomes

$$-\frac{x^2}{y^3}\,dx+\frac{x^3+y^3}{y^4}\,dy=0, \text{ exact now.}$$

Integrating $-\dfrac{x^2}{y^3}$ with respect to x treating y as constant,

we get $-\dfrac{x^3}{3y^3}$.

In $\dfrac{x^3+y^3}{y^4}$ term free from x is $\dfrac{y^3}{y^4}$, *i.e.*, $\dfrac{1}{y}$. Integrating it w.r.t. y, we get $\log y$.

Hence the solution is $-\dfrac{x^3}{3y^3}+\log y=\text{const.}=\log C$

or $\log y=\log C+\dfrac{x^3}{3y^3}$ or $y=Ce^{x^3/3y^3}$.

Ex. 2. *Solve* $(x^4+y^4)\,dx-xy^3\,dy=0$.

Solution. Equation is homogeneous.

$$\therefore \text{ I.F.}=\frac{1}{Mx+Ny}=\frac{1}{x(x^4+y^4)-xy^4}=\frac{1}{x^5}.$$

Multiplying by $\dfrac{1}{x^5}$, the equation becomes

$$\left(\frac{1}{x}+\frac{y^4}{x^5}\right)dx-\frac{y^3}{x^4}\,dy=0, \text{ exact now.}$$

Integrating $\dfrac{1}{x}+\dfrac{y^4}{x^5}$ with respect to x keeping y constant, we get

$$\log x-\frac{y^4}{4x^4}.$$

Also $-\dfrac{y^3}{x^4}$ is not free from x.

Hence the complete solution is $\log x-\dfrac{y^4}{4x^4}=C'$.

$$y^4=4x^4\log x+Cx^4,$$

Ex. 3. *Solve* $y^2\,dx+(x^2-xy-y^2)\,dy=0$.

Solution. The equation is homogeneous.

$$\text{I. F.}=\frac{1}{Mx+Ny}=\frac{1}{y\,(x^2-y^2)}$$

Multiplying by $\dfrac{1}{y\,(x^2-y^2)}$, the equation becomes

Now $\int \frac{x^2y^2+2}{3x^3y^3}dx=\int\left(\frac{1}{3x}+\frac{2}{3x^3y^2}\right)dx=\frac{1}{3}\log x-\frac{1}{3x^2y^2}$

treating y as constant.

In coefficient of dy term free from x is $-\frac{2}{3y}$, whose integral w.r.t. y is $-\frac{2}{3}\log y$. Hence the solution is

$$\frac{1}{3}\log x-\frac{1}{3x^2y^2}-\frac{2}{3}\log y=\log C, \text{ or } x=Cy^2e^{1/x^2y^2}.$$

Ex. 6. *Solve* $y(2xy+1)\,dx+x\,(1+2xy-x^3y^3)dy=0$.

Solution. $\text{I.F.}=\frac{1}{Mx-Ny}=\frac{1}{x^4y^4}$.

The equation after multiplying I.F. becomes

$$\left(\frac{2}{x^3y^2}+\frac{1}{x^4y^3}\right)dx+\left(\frac{1}{x^3y^4}+\frac{2}{x^2y}-\frac{1}{y}\right)dy=0.$$

Solution is $-\frac{1}{x^2y^2}-\frac{1}{3x^3y^3}-\log y=C.$

Ex. 7 *Solve*

$(xy\sin xy+\cos xy)\,y\,dx+(xy\sin xy-\cos xy)\,x\,dy=0.$

[Calcutta Hons. 59]

Solution. Equation is of the form

$yf(xy)\,dx+xg(xy)\,dy=0.$

$\therefore\quad \text{I.F.}=\frac{1}{Mx+Ny}=\frac{1}{2xy\cos xy}.$

Multiplying by I.F., the equation becomes

$$\frac{1}{2}\left(y\tan xy+\frac{1}{x}\right)dx+\frac{1}{2}\left(x\tan xy-\frac{1}{y}\right)dy=0 \text{ (exact now).}$$

Integrating $\frac{1}{2}\left(y\tan xy+\frac{1}{x}\right)$ with respect to x, treating y as constant, we get $\frac{1}{2}[-\log\cos xy+\log x]$.

In coefficients of dy the term free from x is $-\frac{1}{2}\frac{1}{y}$ whose integral is $-\frac{1}{2}\log y$.

Hence the solution is

$\frac{1}{2}[-\log\cos xy+\log x-\log y]=\text{constant}$

or $x=cy\cos(xy)$.

Ex. 8. *Solve* $(y^4-2x^3y)\,dx+(x^4-2xy^3)\,dy=0.$ [Bombay 58]

Solution. The equation is homogeneous.

$$\text{I.F.}=\frac{1}{Mx+Ny}=\frac{1}{-xy(x^3+y^3)}.$$

Now proceed yourself.

Ex. 9. *Solve*

$$(x^2y^2+xy+1)\, y\, dx+(x^2y^2-xy+1)\, x\, dy=0. \qquad \text{[Allahabad 66]}$$

Ex. 10. *Solve* $\quad y^2+\left(x^2-\dfrac{1}{y}\right)\dfrac{dy}{dx}=0.$ **[Delhi Pass 67]**

Solution. Equation is $y^3\, dx+(x^2y-1)\, dy=0.$

$$\text{I.F.}=\frac{1}{Mx+Ny}=\frac{1}{y^3x+x^2y^2-y}.$$

Now integrate after multiplying by I. F.

3·9. Rule V. Let the equation be of the form

$$x^ay^b\,(my\, dx+nx\, dy)+x^cy^d\,(\mu y\, dx+\nu x\, dy),$$

where $a, b, c, d, m, n, \mu, \nu$ are all constants. Then it has an integrating factor $x^\alpha y^\beta$, where α, β are so chosen that after multiplying by $x^\alpha y^\beta$ the equation becomes exact.

Following few examples will illustrate the procedure.

Ex. 1. *Solve* $(y^3-3yx^2)dx+(2xy^2-x^3)\, dy=0.$

Solution. The above equation can be written as

$$y^2\,(y\, dx+2x dy)-x^2\,(2y\, dx+x\, dy)=0.$$

Now let $x^\alpha y^\beta$ be an integrating factor of the equation.

Multiplying by $x^\alpha y^\beta$, the equation becomes

$$(y^{3+\beta}x^\alpha-2y^{\beta+1}x^{2+\alpha})dx+(2x^{1+\alpha}y^{2+\beta}-x^{\alpha+3}y^\beta)\, dy=0.$$

In this (exact) equation,

$$M=y^{3+\beta}x^\alpha-2y^{\beta+1}x^{2+\alpha},\; N=2x^{1+\alpha}y^{2+\beta}-x^{\alpha+3}y^\beta.$$

Hence α and β are such that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x},$$

i.e., $\quad (3+\beta)\, y^{2+\beta}x^\alpha-2\,(\beta+1)\, y^\beta x^{2+\alpha}$

$$=2\,(1+\alpha)\, x^\alpha y^{2+\beta}-(\alpha+3)\, x^{\alpha+2}y^\beta,$$

so that $\qquad 3+\beta=2\,(1+\alpha)$ and $2\,(\beta+1)=\alpha+3.$

Solving these $\alpha=1$ and $\beta=1$. Hence xy is an integrating factor.

Now multiplying by xy, the equation becomes

$$(xy^4-2y^2x^3)\, dx+(2x^2y^3-x^4y)\, dy=0 \text{ (exact).}$$

Integrating $xy^4-2y^2x^3$ with regard to x keeping y constant we have $\frac{1}{2}x^2y^4-\frac{1}{2}y^2x^4$.

In coefficient of dy there is no term free from x.

Hence the solution is $\frac{1}{2}x^2y^4-\frac{1}{2}y^2x^4=$ constant,

i.e., $\quad x^2y^2\,(y^2-x^2)=C.$

Ex. 2. *Prove that* x^hy^k *is an integrating factor of*

$$(py\, dx+qx\, dy)+x^my^n\,(ry\, dx+sx\, dy)=0,$$

if $\quad \dfrac{h+1}{p}=\dfrac{k+1}{q}$ *and* $\dfrac{h+m+1}{r}=\dfrac{k+n+1}{s}.$ **[Delhi Hons. 59]**

Just the article.

Ex. 3. *Solve* $(20x^2+8xy+4y^2+3y^3)\, y\, dx$

$$+4(x^2+xy+y^2+y^3)\, x\, dy=0. \qquad \text{[Raj. M.Sc. 62]}$$

Solution. Let $x^\alpha y^\beta$ be an integrating factor of the equation. Multiplying by $x^\alpha y^\beta$, we get

$$(20x^{\alpha+2}y^{\beta+1}+8x^{\alpha+1}y^{\beta+2}+4x^{\alpha}y^{\beta+3}+3x^{\alpha}y^{\beta+4})\,dx$$
$$+4\,(x^{\alpha+3}y^{\beta}+x^{\alpha+2}y^{\beta+1}+x^{\alpha+1}y^{\beta+2}+x^{\alpha+1}y^{\beta+3})\,dy=0.$$

This is exact for values of α and β for which $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$,

or $$20\,(\beta+1)\,x^{\alpha+2}y^{\beta}+8\,(\beta+2)\,x^{\alpha+1}y^{\beta+1}+4x^{\alpha}(\beta+3)\,y^{\beta+2}+3x^{\alpha}\,(\beta+4)\,y^{\beta+3}$$
$$=4\,(\alpha+3)\,x^{\alpha+2}y^{\beta}+4\,(\alpha+2)\,x^{\alpha+1}y^{\beta+1}+4\,(\alpha+1)\,x^{\alpha}y^{\beta+2}+4\,(\alpha+1)\,x^{\alpha}y^{\beta+3}.$$

$\therefore\quad 20\,(\beta+1)=4\,(\alpha+3),\ 8\,(\beta+2)=4\,(\alpha+2),$
$4\,(\beta+3)=4\,(\alpha+1),\ 3\,(\beta+4)=4\,(\alpha+1)$

These equations are all satisfied for $\beta=0,\ \alpha=2$.

Hence the integrating factor $=x^2$.

Now multiplying by x^2, the equation becomes

$$(20x^4+8x^3y+4x^2y^2+3x^2y^3)\,y\,dx+4\,(x^4+x^3y+x^2y^2+x^2y^3)\,x\,dy=0.$$

This is an exact equation.

Integrating $(20x^4+8x^3y+4x^2y^2+3x^2y^3)\,y$, with respect to x treating y as constant, we get

$$(4x^5+2x^4y+\tfrac{4}{3}x^3y^2+x^3y^3)\,y.$$

In N there is no term free from x.

Hence the solution is

$$4x^5+2x^4y+\tfrac{4}{3}x^3y^2+x^3y^3=c/y.$$

Ex. 4. *Solve* $(8y\,dx+8x\,dy)+x^2y^3\,(4y\,dx+5x\,dy)=0.$

Solution. $x^\alpha y^\beta$ be an integrating factor. Multiplying by $x^\alpha y^\beta$ and applying the condition $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$, we get

$\alpha=1,\ \beta=1.$

$\therefore\quad xy$ is an integrating factor.

The equation on multiplying by xy becomes

$$(8xy^2+4x^3y^5)\,dx+(8x^2y+5x^4y^4)\,dy=0.$$

The solution is $\quad 4x^2y^2+x^4y^5=C$

Ex. 5. *Solve* $x\,(4y\,dx+2x\,dy)+y^3\,(3y\,dx+5x\,dy)=0.$ **[Delhi 68]**

Solution. If $x^\alpha y^\beta$ be an I.F., then $\alpha=2,\ \beta=1$.

The equation after multiplying by x^2y becomes

$$(4x^3y^2+3x^2y^5)\,dx+(2x^4y+5x^3y^4)\,dy=0$$

whose solution is $\qquad x^4y^2+x^3y^5=C.$

Aliter. The equation can be written as

$$(4xy+3y^4)\,dx+(2x^2+5xy^3)\,dy=0.$$

Now $\qquad Mx-Ny=2xy\,[x-y^3].$

Thus an integrating factor is

$$\frac{1}{Mx-Ny}=\frac{1}{2xy\,[x-y^3]}$$

Multiplying by I.F., the equation after simplification becomes

$$\left(4+\frac{3y^3}{x}\right)dx+\left(\frac{2x}{y}+5y^2\right)dy=0$$

which is an exact equation and its solution is

$$4x+3y^3\log x+\tfrac{5}{3}y^3=C.$$

Ex. 6. *Solve* $x^3y^2(2y\,dx+x\,dy)-(5y\,dx+7x\,dy)=0.$

[Delhi Hons. 61]

Solution. Multiplying by $x^{\alpha}y^{\beta}$ and then applying the condition of exactness, we get $\alpha=-\frac{8}{3}$, $\beta=-\frac{10}{3}$

$\therefore$ I.F.$=x^{-8/3}y^{-10/3}$. The equation then becomes

$$(2x^{1/3}y^{2/3}-5x^{-8/3}y^{-7/3})\,dx+(x^{4/3}y^{-1/3}-7x^{-5/3}y^{-10/3})\,dy=0.$$

Solution is

$$\tfrac{3}{2}x^{4/3}y^{2/3}+3x^{-5/3}y^{-7/3}=C_1 \quad\text{or}\quad x^3y^3+2=Cx^{5/3}y^{7/3}.$$

Ex. 7. *Solve* $(y^2+2x^2y)\,dx+(2x^3-xy)\,dy=0.$

Solution. Multiplying by $x^{\alpha}y^{\beta}$ and then applying the conditions of exactness, we get $\alpha=-\frac{5}{2}$, $\beta=-\frac{1}{2}$.

Solution is $\quad 5\sqrt{(xy)}-x^{-3/2}y^{3/2}=C.$

Ex. 8. *Solve* $(2x^2y-3y^4)\,dx+(3x^3+2xy^3)\,dy=0.$

Solution. I.F.$=x^{-49/13}y^{-28/13}$.

Solution is $\quad 5x^{-36/13}y^{24/13}-12x^{-10/13}y^{-15/13}=C.$

Ex. 9. *Given that for some constant* α, $(x+y)^{\alpha}$ *is an integrating factor of*

$$(4x^2+2xy+6y)\,dx+(2x^2+9y+3x)\,dy=0,$$

find α *and solve the differential equation.* **[Karnatak 61]**

Multiply by $(x+y)^{\alpha}$ and apply the condition of exactness to find the value of α. Then solve the resulting exact differential equation.

Ex. 10. *Solve* $3y\,dx-2x\,dy+x^2y^{-1}(10y\,dx-6x\,dy)=0.$

[Delhi Hons. 59]

Find the integrating factor as usual.

Ex. 11. *Prove that* $P(x,y)\,dx+Q(x,y)\,dy=0$ *will have an integrating factor of the form* $\phi(x+y)$ *if*

$$\frac{1}{P-Q}\left(\frac{\partial P}{\partial x}-\frac{\partial Q}{\partial x}\right)$$

is a function of $x+y$. **[Cal. Hons. 62]**

Just the article.

Ex. 12. *Prove that* $\dfrac{1}{(x+y+1)^4}$ *is an integrating factor of*

$$(2xy-y^2-y)\,dx+(2xy-x^2-x)\,dy=0$$

and hence integrate the equation. **[Cal. Hons. 62]**

Multiply by the integrating factor and show that the equation becomes exact.

4

Trajectories

4·1. Trajectories. A curve which cuts every member of a given family of curves at a constant angle α is called an α *trajectory*.

If $\alpha=90°$, then it is called an *orthogonal trajectory* of the family or curves.

4·2. Equation of the trajectories. If a family of curves be given by the differential equation $f(x, y, p)=0$, then its α-trajectory is given by

$$f\left(x, y, \frac{p-\tan\alpha}{1+p\tan\alpha}\right)=0.$$

Orthogonal trajectories. If $\alpha=90°$, then

$$\frac{p-\tan\alpha}{1+p\tan\alpha}=\frac{p\cot\alpha-1}{\cot\alpha+p}=\frac{-1}{p} \text{ as } \cot 90°=0.$$

Hence corresponding to the family of curves whose differential equation is $f(x, y, p)=0$ the differential equation of the orthogonal trajectories is $f(x, y, -1/p)=0$ which on integration gives family of trajectories orthogonal to the given family of curves.

Polar Coordinates. If family of curves by given by

$$f\left(r, \theta, \frac{dr}{d\theta}\right)=0,$$

then their orthogonal trajectory is given by

$$f\left(r, \theta, -r^2\frac{d\theta}{dr}\right)=0.$$

Ex. 1. *Find orthogonal trajectories of hyperbolas* $xy=c^2$.

Solution. Family of hyperbolas is $xy=c^2$.

Differentiating $y+x\dfrac{dy}{dx}=0.$

Replacing $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$, the differential equation of orthogonal trajectories is

$$y-x\frac{dx}{dy}=0 \quad\text{or}\quad x\,dx-y\,dy=0.$$

Integrating, $x^2-y^2=c.$

This gives family of orthogonal trajectories of the hyperbolas $xy=c^2$.

Ex. 2. *Show that the system of confocal conics*

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1,$$

is self-orthogonal. **(Alld. 1965; Delhi Hons. 60, 57; Patna Hons. 58)**

Solution. Differentiating the curve w.r.t. x, we get

$$\frac{2x}{a^2+\lambda}+\frac{2y}{b^2+\lambda}p=0\ ;\quad \therefore\quad \lambda=-\frac{b^2x+a^2yp}{x+yp}.$$

$$\therefore\ a^2+\lambda=\frac{(a^2-b^2)\,x}{x+yp},\ b^2+\lambda=-\frac{(a^2-b^2)\,yp}{x+yp}.$$

Hence the differential equation of the given conics is

$$\frac{x^2\,(x+yp)}{(a^2-b^2)\,x}-\frac{y^2\,(x+yp)}{(a^2-b^2)\,yp}=1$$

or $\quad (x+yp)\,(x-y/p)=a^2-b^2.$ ...(1)

Now replacing p by $-1/p$, the differential equation of the orthogonal trajectories is

$$(x-y/p)\,(x+yp)=a^2-b^2,$$

which is just the same as (1). Thus the system of confocal conics is self-orthogonal.

Ex. 3. *Find the orthogonal trajectories of*

$x^2/a^2+y^2/(a^2+\lambda)=1.$ **(Cal. Hons. 1950; Patna Hons. 54)**

Proceed yourself.

Ex. 4. *Show that the system of confocal and coaxial parabolas* $y^2=4a\,(x+a)$ *is self-orthogonal.* **(Delhi Hons, 1959)**

Solution. Parabolas are given by

$y^2=4ax+4a^2.$ ...(1)

Differentiating w.r.t. we get

$$2yp=4a\quad\text{or}\quad a=\frac{yp}{2}.$$

Putting this value of a in the equation of parabolas, the differential equation of the family of given parabolas is

$y^2=2ypx+y^2p^2.$ ...(2)

Now replacing p oy $-1/p$, the differential equation of the orthogonal trajectories is

$$y^2=-2yx/p+y^2/p^2$$

or $\quad y^2p^2+2xyp=y^2,$

which is just the same as (2). Hence the parabolas (1) are self-orthogonal.

Ex. 5. (a) *Find the orthogonal trajectories of the family of coaxial circles* $x^2+y^2+2gx+c=0$, *where* g *is a parameter and* c *a constant.* **(Delhi Hons. 1966; Nag. 61)**

(b) *Find the orthogonal trajectories of the family of circles* $x^2+y^2+2fy+1=0$, f *being the parameter.* **(Delhi Pass 1967)**

Solution. (a) Differentiating, $x+yp+g=0$.

Putting $g=-(x+yp)$, the differential equation of the family, of of coaxial circles is

$$x^2+y^2-2x(x+yp)+c=0.$$

or $$y^2-x^2-2xyp+c=0.$$

Putting $-1/p$ for p, the differential equation of orthogonal trajectories is

$$y^2-x^2+2xy\frac{dx}{dy}+c=0$$

or $$2xy\frac{dx}{dy}-x^2=-c-y^2. \text{ Put } x^2=t,\ 2x.\frac{dx}{dy}=\frac{dt}{dy}.$$

$$\therefore\ y.\frac{dt}{dy}-t=-c-y^2$$

or $$\frac{dt}{dy}-\frac{1}{y}t=-\frac{c}{y}-y. \quad \text{I.F.}=\frac{1}{y}.$$

$$\therefore\ t\frac{1}{y}=\int-\left(\frac{c}{y}+y\right)\frac{1}{y}\,dy=\frac{c}{y}-y-f\ \text{(const.)}$$

or $$x^2+y^2+fy-c=0.$$

(b) Proceed as in part (a).

Ex. 6. *Determine the 45° trajectories of the family of concentric circles* $x^2+y^2=c^2$. **[Delhi Hons. 1961]**

Solution. Differentiating $x^2+y^2=c^2$,
the differential equation of the family of circles is

$$x+yp=0. \qquad \text{...(1)}$$

Now to find differential equation of the 45° trajectories, we shall replace p by

$$\frac{p-\tan 45^\circ}{1+p\tan 45^\circ} \text{ i.e., by } \frac{p-1}{1+p}$$

Hence the diff. equation of the 45° trajectories is

$$x+y.\frac{p-1}{1+p}=0 \text{ or } (x+y)\,dy+(x-y)\,dx=0.$$

This is a homogeneous equation. Putting $y=vx$, we get

$$(x+vx)\left[v+x\frac{dv}{dx}\right]+x-vx=0$$

i.e., $$x\frac{dv}{dx}+\frac{v^2+1}{v+1}=0$$

or $$\frac{dx}{x}+\frac{v+1}{v^2+1}\,dv=0$$

or $$\frac{dx}{x}+\left(\frac{1}{2}\frac{2v}{v^2+1}+\frac{1}{v^2+1}\right)dv=0.$$

Integrating, $\log x+\frac{1}{2}\log(v^2+1)+\tan^{-1}v=\log C$

or $\qquad \log \{x^2(1+v^2)\}=\log C-2\tan^{-1} v$

i.e., $\qquad x^2+y^2=Ce^{-2\tan^{-1}(y/x)}$ as $y=vx$.

Ex. 7. *Find the equation of a set of curves each member of which cuts every member of the family $xy=$ const. at the angle $\frac{1}{4}\pi$.* **[Delhi Hons. 56]**

Solution. Diff. equation of $xy=$ const is $y+xp=0$.

Replacing p by $\dfrac{p-\tan 45°}{1+p\tan 45°}=\dfrac{p-1}{1+p}$, the differential equation of $\frac{1}{4}\pi$-trajectories is

$$y+x\frac{p-1}{1+p}=0, \quad \text{i.e.,} \quad (x+y)\,dy=(x-y)\,dx.$$

Homogeneous. Putting $y=vx$, we get

$$(x+vx)\left(v+x\frac{dv}{dx}\right)=x-xv \quad \text{or} \quad x.\frac{dv}{dx}=\frac{1-2v-v^2}{1+v}$$

or $$\frac{dx}{x}=\frac{1+v}{1-2v-v^2}\,dv=-\frac{1}{2}\,\frac{-2-2v}{(1-2v-v^2)}\,dv.$$

Integrating, $\log x+\frac{1}{2}\log(1-2v-v^2)=\log C.$

Ex. 8. *Find the orthogonal trajectories of the cardioid $r=a(1-\cos\theta)$, where a is the parameter.* **[Delhi Pass 1968; Saugar 62: Delhi Hons. 64, 62, 58; Bihar Hons. 62, 5?]**

Solution. The cardioid is $r=a(1-\cos\theta)$.

$$\therefore \quad \frac{dr}{d\theta}=a\sin\theta, \text{ i.e., } a=\frac{1}{\sin\theta}.\frac{dr}{d\theta}$$

Hence differential equation of the family of cardioids is

$$r=\frac{1}{\sin\theta}\frac{dr}{d\theta}(1-\cos\theta), \quad \text{i.e.,} \quad \frac{1}{r}\frac{dr}{d\theta}=\frac{\sin\theta}{1-\cos\theta}$$

Now replacing $\dfrac{dr}{d\theta}$ by $-r^2\dfrac{d\theta}{dr}$, the differential equation of the orthogonal trajectories is $-r\dfrac{d\theta}{dr}=\dfrac{\sin\theta}{1-\cos\theta}$,

$$\text{i.e.,} \quad -\frac{dr}{r}=\frac{1-\cos\theta}{\sin\theta}\,d\theta=\frac{2\sin^2\frac{1}{2}\theta\,d\theta}{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}=\tan\tfrac{1}{2}\theta\,d\theta.$$

Integrating, $\log r=2\log\cos\frac{1}{2}\theta+\log 2C.$

$$\therefore \quad r=2C\cos^2\tfrac{1}{2}\theta \quad \text{or} \quad r=C(1+\cos\theta).$$

Ex. 9. *Find orthogonal trajectories of $r=a(1+\cos\theta)$.* **[Patna Hons. 1957; Bihar Hons. 56]**

Proceed as in above example.

Ex. 10. *Find orthogonal trajectories of the series of logarithmic $r=a^\theta$, where a varies.*

Solution. We have $\dfrac{1}{r}\dfrac{dr}{d\theta}=\log a;\ \therefore\ \dfrac{1}{r}\dfrac{dr}{d\theta}=\dfrac{\log r}{\theta}.$

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$, the differential equation of orthogonal trajectory is

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right)=\frac{\log r}{\theta} \quad \text{or} \quad \frac{\log r}{r}\,dr=-\theta\,d\theta,$$

i.e.,
$$\tfrac{1}{2}(\log r)^2=-\frac{\theta^2}{2}+C.$$

Ex. 11. *Find orthogonal trajectories of $r^n \sin n\theta=a^n$.*

[Osmania 56]

Solution. Differentiating logarithmically, we have

$$\frac{n}{r}\frac{dr}{d\theta}+\frac{n\cos n\theta}{\sin n\theta}=0, \text{ i.e., } r\frac{d\theta}{dr}=-\tan n\theta.$$

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{r}$, the differential equation of orthogonal trajectories is

$$-\frac{1}{r}\frac{dr}{d\theta}=-\tan n\theta \quad \text{or} \quad \frac{dr}{r}=\tan n\theta\,d\theta.$$

Integrating, $\log r=\frac{1}{n}\log\sec n\theta+\log c$,

i.e., $r^n=c^n\sec n\theta$ or $r^n\cos n\theta=c^n$.

Ex. 12. *Find orthogonal trajectories of $r^n\cos n\theta=c^n$.*

Proceed as above **Ans.** $r^n\sin n\theta=c^n$

Ex. 13. *Find orthogonal trajectories of the following curves :*

[Poona 64 ; Delhi 59]

(*i*) $ay^2=x^3$ (*semi-cubical parabola*).

Hint. Differential equation of curves is $3y=2px$.

Differential equation of orthogonal trajectories is

$$2x\,dx+3y\,dy=0; \quad \therefore \quad x^2+\tfrac{3}{2}y^2=c^2.$$ **Ans.**

(*ii*) $x^{2/3}+y^{2/3}=$ (*hypo-cycloids*).

Hint. Differential equation of orthogonal trajectories is

$$x^{-1/3}\,dy=y^{-1/3}\,dx \quad \text{or} \quad y^{1/3}\,dy=x^{1/3}\,dx.$$

Ans $x^{4/3}-y^{4/3}=c^{4/3}$.

(*iii*) $x^2+y^2+c^2=1+2cxy$. **[Patna Hons. 52]**

Hint. Differential equation of curves is $\frac{dy}{dx}=\pm\frac{\sqrt{(1-y^2)}}{\sqrt{(1-x^2)}}$.

Differential equation of orthogonal trajectories is

$$\sqrt{(1-x^2)}\,dx\pm\sqrt{(1-y^2)}\,dy=0.$$

(*iv*) $x^2+y^2-ay=0$, **[Allahabad 60]**

Ans. $x^2+y^2+bx=0$.

(*v*) $x^2+y^2=2cy$. **[Bihar Hons. 55]**

Hint. Differential equation of curves is $(x^2-y^2)\,p=2xy$.

Put $-1/p$ for p and then $y=vx$ is the resulting homogeneous equation.

Ans. $(x^2+y^2+bx)=0$.

(*vi*) $a^{n-1}\,y=x^n$. **[Bihar Hons. 53]**

Hint. Differential equation of curves is $xp=ny$.

Differential equation of orthogonal trajectories is

$$x\,dx+ny\,dy=0.$$

Ans. $x^2+ny^2=c$.

(*vii*) $y=ax^2$. **[Karnatak 62 ; Vikram 61 ; Delhi Hons. 53]**

Ans. $x^2+2y^2=c^2$ as in (*vi*).

Ex. 14. *A family of parabolas has a common focus and common axis. Find the orthogonal family.*

[Cal. Hons. 54 ; Delhi Hons. 47 ; Patna Hon 53]

Solution. The parabolas are given by $\dfrac{2a}{r}=1+\cos\theta$.

Their differential equation is $r\dfrac{d\theta}{dr}=\cot\dfrac{\theta}{2}$.

Differential equation of orthogonal trajectories is

$$r\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)=\cot\frac{\theta}{2}$$

or $-\dfrac{dr}{r}=\cot\tfrac{1}{2}\theta\,d\theta$.

Ans. $r=\dfrac{2c}{(1-\cos\theta)}$.

Ex. 15. *Find the orthogonal trajectories of the family of the system of co-axial circles represented by*

$x^2+y^2=2gx$. **[Poona 62 ; Karnatak 63]**

Solution Differentiating $2x+2y\dfrac{dy}{dx}=2g$,

i.e., $g=x+yp$.

Therefore differential equation of system of coaxial circles is

$$x^2+y^2=2x\,(x+yp).$$

Putting $-1/p$ for p the differential equation of the orthogonal trajectories is

$$x^2+y^2=2x\,(x-y/p) \quad i.e., \ (x^2-y^2)\,p=2xy.$$

Now solve it as a homogeneous equation by putting $y=vx$.

Ex. 16. *Find the orthogonal trajectories of*

$x^2-cx+4y=0$. **[Lucknow 51]**

Ans. $x^2+4y=cx$.

5

Linear Differential Equations with Constant Coefficients

5·1. Linear Differential Equation

A differential equation of the form

$$\frac{d^n y}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots+P_n y=X$$

where $P_1, P_2 \ldots, P_n$, and X are functions of x or constants, is called a *linear differential equation of n^{th} order.*

And if $P_1, P_2, \ldots, P_n$ are all constants (not functions of x) and X is some function of x, then the equation is a *linear differential equation with constant coefficients.*

5.2. The Operator D.

It is usual to write

$$D \text{ for } \frac{d}{dx},\ D^2 \text{ for } \frac{d^2}{dx^2},\ldots, D^n \text{ for } \frac{d^n}{dx^n}.$$

And in terms of the operator D the differential equation (1) can be written as $[D^n+P_1D^{n-1}+P_2D^{n-2}+\ldots+P_n]\ y=X.$

Note. It can be proved that D can be treated as an algebraic quantity in several respects.

5.3. A Theorem.

If $y=y_1, y=y_2, \ldots, y=y_n$ are linearly independent solutions of

$$(D^n+a_1D^{n-1}+a_2D^{n-2}+\ldots+a^n)\ y=0, \qquad \ldots(1)$$

then $y=C_1y_1+C_2y_2+\ldots+C_ny_n$ is the general or complete solution of the differential equation, where $C_1, C_2, \ldots, C_n$ are n arbitrary constants.

Let us denote the given equation (1) by $f(D)\ y=0$,

where $\quad f(D)=D^n+a_1D^{n-1}+a_2D^{n-2}+\ldots+a_n.$

Since $y=y_1, y=y_2, \ldots, y=y_n$ are solutions of the equation,

$$\therefore\ f(D)\ y_1=0, f(D)\ y_2=0, \ldots, f(D)\ y_n=0. \qquad \ldots(2)$$

Now putting $y=C_1y_1+C_2y_2+\ldots+C_ny_n$ in (1), we have

$$D^n(C_1y_1+\ldots+C_ny_n)+a_1D^{n-1}(C_1y_1+C_2y_2+\ldots+C_ny_n)$$
$$+\ldots+a_n(C_1y_1+C_2y_2+\ldots+C_ny_n)=0$$

or $\quad C_1\ (D^ny_1+a_1D^{n-1}y_1+\ldots+a_n)+C_2\ (D^ny_2+a_1D^{n-1}y_2+\ldots+a_n)$
$$+\ldots+C_n(D^ny_n+a_1D^{n-1}y_n+\ldots+a_n)=0$$

or $\quad C_1f(D)\ y_1+C_2f(D)\ y_2+\ldots+C_nf(D)\ y_n=0$

or $\quad C_1.0+C_2.0+\ldots+C_n.0=0$ by (2).

Since (1) is satisfied by $y=C_1y_1+C_2y_2+\dots+C_ny_n$, it is a solution of (1). Also since it contains n arbitrary constants, it is the general or complete solution of the equation.

5·4. Auxiliary Equation. Consider the differential equation

$$(D^n+a_1D^{n-1}+a_2D^{n-2}+\dots+a_n)\,y=0 \qquad \dots(1)$$

where $a_1, a_2, \dots, a_n$ are all constants.

Let $y=e^x$ be a solution of this equation. Then putting

$$y=e^{mx},\ Dy=me^{mx},\ D^2y=m^2e^{mx},\ \dots,\ D^ny=m^ne^{mx},$$

the equation becomes

$$(m^n+a_1m^{n-1}+a_2m^{n-2}+\dots+a_n)\,e^{mx}=0.$$

Hence e^{mx} will be a solution of (1) if m is a root of the algebraic equation

$$\mathbf{m^n+a_1m^{n-1}+a_2m^{n-2}+\dots+a_n=0} \qquad \dots(2)$$

This equation in m is called the *Auxiliary equation.*

Note It is observed that the auxiliary equation $f(m)=0$ gives the same values of m as the equation $f(D)=0$ gives of D.

Hence $f(D)=0$, *i.e.*, $D^n+a_1D^{n-2}+\dots+a_n=0$ can in general be regarded as the auxiliary equation.

Therefore in practice we do not replace D by m to form the auxiliary equation. The equation in D may be regarded as auxiliary equation.

5·5. Solution of equation (1) of the above article.

[Gujrat B.Sc. (Prin.) 58 ; Gujrat B.Sc. (Subsi.) 65]

Case I. When all the roots of auxiliary equation are real and different.

If $m_1, m_2, \dots, m_n$ be the n different roots of (2), then $y=e^{m_1x}, y=e^{m_2x}, \dots, y=e^{m_nx}$ are all independent solutions of (1). Therefore the general solution of (1) is

$$y=C_1e^{m_1x}+C_2e^{m_2x}+C_3e^{m_3x}+\dots+C_ne^{m_nx}.$$

Ex. 1. *Solve* $\dfrac{d^3y}{dx^3}-13\dfrac{dy}{dx}-12y=0.$

Solution Equation is $(D^3-13D-12)\,y=0.$

The auxiliary equation is $D^3-13D-12=0$,

i.e., $(D+1)(D+3)(D-4)=0$, $D=-1, -3, 4$

Hence the complete solution is

$$y=C_1e^{-x}+C_2e^{-3x}+C_3e^{4x}.$$

Ex. 2. *Solve* $(D^3+6D^2+11D+6)\,y=0.$ **[Delhi Pass 67]**

Solution. A.E. is $(D+1)(D+2)(D+3)=0$, $D=-1, -2, -3.$

The complete solution is

$$y=c_1e^{-x}+c_2e^{-2x}+c_3e^{-3x}.$$

5·6. Case II. Auxiliary equation having equal roots.

[Gujrat B. Sc (Princ.) 59; Poona T.D.C. 61 (S)]

We have shown in case I § 5·5, that when $m_1, m_2, \ldots, m_n$ are all different, the general solution is

$$y=C_1e^{m_1x}+C_2e^{m_2x}+\ldots+C_ne^{m_nx}.$$

But if $m_1=m_2$ (two roots equal) then this becomes

$$y=(C_1+C_2)\,e^{m_1x}+C_3e^{m_3x}+\ldots+C_ne^{m_nx},$$

which clearly contains only $n-1$ arbitrary constants (since C_1+C_2 is equivalent to only one arbitrary constant)

Therefore this is no longer a general solution.

Consider an equation $(D-m_1)^2\,y=0,$ ...(1)
a differential equation of second order having both the roots equal.

Put $(D-m_1)\,y=v$; then (1) becomes

$$(D-m_1)\,v=0 \quad \text{or} \quad \frac{dv}{dx}=m_1v,$$

Separating the variables, $\dfrac{dv}{v}=m_1\,dx.$

Integrating, $\log v=\log C+m_1x$ or $v=Ce^{m_1x}$

or $(D-m_1)\,y=Ce^{m_1x}$ as $v=(D-m_1)\,y$.

or $\dfrac{dy}{dx}-m_1y=Ce^{m_1x}$

which is a linear equation of the first order, its I.F. $=e^{-mx}$

$$\therefore\quad ye^{-m_1x}=\int Ce^{m_1x}\,e^{-m_1x}\,dx+C_2$$

or $y=(Cx+C_2)\,e^{m_1x}.$

Therefore the most general solution of

$$(D^n+a_1D^{n-1}+a_2D^{n-2}+\ldots+a_n)\,y=0,$$

when two roots of A.E. are equal, is

$$y=(C_1+C_2x)\,e^{m_1x}+C_3e^{m_3x}\ldots+C_ne^{m_nx}.$$

Cor. In case three roots are equal, *i.e.*, $m_1=m_2=m_3$, the general solution is

$$y=(C_1+C_2x+C_3x^2)\,e^{m_1x}+C_4e^{m_4x}+\ldots+C_ne^{m_nx}.$$

Ex. 1. *Solve* $\dfrac{d^4y}{dx^4}-\dfrac{d^3y}{dx^3}-9\dfrac{d^2y}{dx^2}-11\dfrac{dy}{dx}-4y=0.$

Solution. A.E. is $D^4-D^3-9D^2-11D-4=0,$

i.e., $(D+1)^3\,(D-4)=0,\ D=-1,\,-1,\,-1,\,4.$

Hence the general solution is

$$y=(C_1+C_2x+C_3x^2)\,e^{-x}+C_4e^{4x}.$$

Ex. 2. *Solve* $(D^3-2D^2-4D+8)\,y=0.$ **(Delhi Pass 1968)**

Solution. Auxiliary equation is

$$D^3-2D^2-4D+8=0 \quad \text{or} \quad (D+2)\,(D-2)^2=0,$$
$$D=-2,\,2,\,2.$$
$$\therefore\quad y=(C_1+C_2x)\,e^{2x}+C_3e^{-2x}.$$

5·7 Case III. Auxiliary equation having imaginary roots.

Let $\alpha \pm i\beta$ be the imaginary roots of an equation of second order (since imaginary roots occur in pairs).

Then its general solution is

$$y = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x}$$
$$= e^{\alpha x}[C_1 e^{i\beta x} + C_2 e^{-i\beta x}]$$
$$= e^{\alpha x}[C_1(\cos \beta x + i \sin \beta x) + C_2(\cos \beta x - i \sin \beta x)]$$
$$= e^{\alpha x}[(C_1+C_2)\cos \beta x + (C_1 - C_2) i \sin \beta x]$$
$$= e^{\alpha x}[A \cos \beta x + B \sin \beta x].$$

Note. The above result after suitably adjusting constants may also be written as

$$y = e^{\alpha x}.A \cos(\beta x + B) \quad \text{or} \quad y = e^{\alpha x}.A \sin(\beta x + B).$$

Imaginary roots repeated. If auxiliary equation has two equal pairs of imaginary roots, *i.e.*, if $\alpha + i\beta$ and $\alpha - i\beta$ occur twice, then general solution is obtained as

$$y = e^{\alpha x}[(C_1 + C_2 x)\cos \beta x + (C_3 + C_4 x)\sin \beta x].$$

Cor. If a pair of roots of the auxiliary equation occur in the form of quadratic surd $\alpha \pm \sqrt{\beta}$, where β is +ive, then the corresponding term in the solution may be written as

$$e^{\alpha x}[C_1 \cosh x\sqrt{\beta} + C_2 \sinh x\sqrt{\beta}]$$

or $C_1 e^{\alpha x} \cosh (x\sqrt{\beta} + C_2)$ or $C_1 e^{\alpha x} \sinh (x\sqrt{\beta} + C_2)$.

Ex. 1. *Solve* $(D^4 + 5D^2 + 6)y = 0.$ **(Karnatak M. A. 61)**

Solution Auxiliary equation is $(D^4 + 5D^2 + 6) = 0$,

i.e., $(D^2+3)(D^2+2) = 0 \quad \therefore \quad D = \pm\sqrt{3}i, \pm\sqrt{2}i.$

Hence the complete solution is

$$y = C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x + C_3 \cos \sqrt{2}x + C_4 \sin \sqrt{2}x.$$

Ex. 2. *Solve* $(D^4 - D^3 - D + 1)x = 0$ **(Gujrat 58)**

Solution. Auxiliary equation is $D^4 - D^3 - D + 1 = 0$

or $(D^3-1)(D-1) = 0$ or $(D-1)^2(D^2+D+1) = 0$

or $D = 1, 1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i.$

Hence the complete solution is

$$y = (C_1 + C_2 x)e^x + e^{-x/2}\left[C_3 \cos \frac{\sqrt{3}}{2}x + C_4 \sin \frac{\sqrt{3}}{2}x\right].$$

Ex. 3. *Solve the differential equation*

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0,$$

a, b being constants. **(Delhi Hons. 66)**

Solution. Proceed yourself.

5·8. Synopsis of the forms of solutions

To solve an equation of the from

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n)y = 0;$$

1. Find the roots of the auxiliary equation, *viz.* $D^n+a_1D^{n-1}+a_2D^{n-2}+\dots+a_n=4$.
2. Put the *General Solution* as follows :

Roots of Auxi. Equation	Complete Solution
Case I	
All roots $m_1, m_2, m_3, \dots m_n$ real and different.	$y=C_1e^{m_1x}+C_2e^{m_2x}+\dots+C_ne^{m_nx}$
Case II	
$m_1=m_2$ but other roots real and different.	$y=(C_1+C_2x)\,e^{m_1x}+C_3e^{m_3x}+\dots+C_ne^{m_nx}$
Case III (Imag. Roots)	
1. $\alpha\pm i\beta$, a pair of imaginary roots.	Corresponding part of the general solution is $e^{\alpha x}(C_1\cos\beta x+C_2\sin\beta x)$ or $C_1e^{\alpha x}\cos(\beta x+C_2)$ or $C_1e^{\alpha x}\sin(\beta x+C_2)$.
2. $(\alpha\pm i\beta)$, $(\alpha\pm i\beta)$ repeated twice.	Corresponding part of general solution is $y=e^{\alpha x}[(C_1+C_2x)\cos\beta x+(C_3+C_4x)\sin\beta x]$.

Ex. 1. *Solve* $\dfrac{d^4y}{dx^4}-a^4y=0$.

Solution. The auxiliary equation is $(D^4-a^4)=0$

or $\qquad (D^2-a^2)(D^2+a^2)=0,\ D=\pm a,\ \pm ai.$

$\therefore$ solution is $y=C_1e^{ax}+C_2e^{-ax}+(C_3\cos ax+C_4\sin ax)$.

Ex. 2 *Solve* $\dfrac{d^4y}{dx^4}+m^4y=0$. [Agra B. Sc, 55]

Solution. Auxiliary equation is $D^4+m^4=0$

or $\qquad (D^2+m^2)^2-2m^2D^2=0$

or $\qquad (D^2-\sqrt{2}mD+m^2)(D^2+\sqrt{2}mD+m^2)=0.$

When $D^2-\sqrt{2}mD+m^2=0$, $D=\dfrac{m\pm mi}{\sqrt{2}}$.

When $D^2+\sqrt{2}mD+m^2=0$, $D=\dfrac{-m\pm mi}{\sqrt{2}}$, *i.e.*, roots of auxiliary equation are $\frac{m}{\sqrt{2}} \pm \frac{m}{\sqrt{2}} i, \; -\frac{m}{\sqrt{2}} \pm \frac{m}{\sqrt{2}} i.$

Hence the general solution is

$$y = e^{(m/\sqrt{2}) x} C_1 \cos\left(\frac{m}{\sqrt{2}} x + C_2\right) + e^{(-m/\sqrt{2}) x} C_3 \cos\left(\frac{m}{\sqrt{2}} x + C_4\right).$$

5·9. General solution of $(D^n + a_1 D^{n-1} + \ldots a_n)\, y = X.$ ...(1)

[Bombay 61 : Gujrat 52]

To show that if $y = Y$ is a complete solution of

$$(D^n + a_1 D^{n-1} + \ldots + a_n)\, y = 0 \qquad \ldots(2)$$

and $y = u$ is a particular solution of (1); then $y = Y + u$ is a general solution of (1). **[Nagpur B.Sc. 55 (S)]**

Since $y = Y$ is a solution of (2), we have

$$(D^n + a_1 D^{n-1} + \ldots + a_n)(Y) = 0. \qquad \ldots(3)$$

Also since $y = u$ is a solution of (1), we have

$$(D^n + a_1 D^{n-1} - \ldots + a_n)\, u = X. \qquad \ldots(4)$$

Adding (3) and (4), we have

$$(D^n + a_1 D^{n-1} + \ldots + a_n)(Y + u) = X.$$

This shows that $y = Y + u$ is a solution of (1). Now Y being a general solution of (2) contains n arbitrary constants and as such $Y + u$ also contains n arbitrary constants. Therefore $y = Y + u$ is a general solution of (1).

Note 1. In the general solution $y = Y + u$ of the equation (1), Y is called the **Complementary Function** (C.F.) and u is called the **Particular Integral** (P. I.) and thus

The General Solution = C.F. + P.I.

2. The solution Y of (2) can be determined by the methods discussed above. The problem is now to find the particular integral u of (1). We give below certain methods of finding u.

Ex. *Define the Complementary Function and Particular Integral for the linear differential equation with constant coefficients $f(D)\, y = X$.* **[Karnatak 62]**

5·10. Meaning of the symbol $\frac{1}{f(D)}$.

Def. $\frac{1}{f(D)} X$ *is that function of x, free from arbitrary constants, which when operated by $f(D)$ gives X.*

Thus $f(D) . \frac{1}{f(D)} X = X.$

Therefore $f(D)$ and $\frac{1}{f(D)}$ are inverse operators (*i.e.* they cancel each other's effect on the function on which they operate.)

Thus the symbol $\frac{1}{D}$ stands for integration.

5.11. $\mathbf{\frac{1}{f(D)} X}$ **is the particular integral of** $\mathbf{f(D)\ y = X}$**.**

Clearly $\frac{1}{f(D)} X$ will be solution of (1) if it satisfies (1).

So putting $\frac{1}{f(D)} X$ for y in (1), we get

$$f(D) \frac{1}{f(D)} X = X \text{ i.e., } X = X, \text{ which is true.}$$

It means that $\frac{1}{f(D)} X$ is a particular solution of (1).

Therefore to find the particular solution of $f(D)\ y = X$, we should find the value of $\frac{1}{f(D)} X$.

Note. We know that in solving $f(D)\ y = 0$, $f(D) = 0$ forms the auxiliary equation, which can be resolved into linear factors (real or imaginary). Therefore $\frac{1}{f(D)}$ can be resolved into partial fractions. The partial fractions will be of the form $\frac{1}{D - \alpha}$ where α is real or imaginary.

5·12. To show that $\mathbf{\frac{1}{D - \alpha} X = e^{\alpha x} . \frac{1}{D} (e^{-\alpha x} X)}$**.**

Suppose $y = \frac{1}{D - \alpha} X$; then $(D - \alpha)\ y = X$,

or $$\frac{dy}{dx} - \alpha y = X; \text{ this is linear in } y, \text{ as } D \equiv \frac{d}{dx}.$$

$\therefore$ Integrating factor $= e^{\int P\, dx} = e^{\int -\alpha\, dx} = e^{-\alpha x}$

and the solution is $ye^{-\alpha x} = \int e^{-\alpha x} X\, dx$.

(constant is not added as it is the particular solution)

or $$y = e^{\alpha x} \int e^{-\alpha x} . X\, dx$$

$$= e^{\alpha x} \frac{1}{D} (Xe^{-\alpha x}) \text{ as } \frac{1}{D} \equiv \text{integration.}$$

5·13. Working rule for finding the Particular integral of f(D) y = X.

Let $f(D) = (D - \alpha_1)(D - \alpha_2) \ldots (D - \alpha_n)$.

Then resolving into partial fraction, we get

$$\frac{1}{f(D)}=\frac{A_1}{D-\alpha_1}+\frac{A_2}{D-\alpha_2}+\dots+\frac{A_n}{D-\alpha_n} \text{ say.}$$

Now particular integral

$$=\frac{1}{f(D)}X=\left\{\frac{A_1}{D-\alpha_1}+\frac{A_2}{D-\alpha_2}+\dots+\frac{A_n}{D-\alpha_n}\right\}X$$

$$=A_1\frac{1}{D-\alpha_1}X+A_2\frac{1}{D-\alpha_2}X+\dots+A_n\frac{1}{D-\alpha_n}X$$

$$=A_1e^{\alpha_1 x}\int e^{-\alpha_1 x}X\,dx+A_2e^{\alpha_2 x}\int e^{-\alpha_2 x}X\,dx+\dots$$

$$+A_ne^{\alpha_n x}\int e^{-\alpha_n x}X\,dx.$$

which can in general be evaluated and thus the particular integral can be found.

Particular Integral in some special cases.

5·14. Particular Integral when $X=e^{\alpha x}$

[Nagpur 61 ; Poona 61 ; Karnatak 61 ; Gujrat 59 ; Bombay 61]

By successive differentiation, we find that

$$e^{\alpha x}=e^{\alpha x}, \quad \dots(1)$$

$$De^{\alpha x}=ae^{\alpha x} \quad \dots(2)$$

$$D^2e^{ax}=a^2e^{ax}, \quad \dots(3)$$

$$\dots\dots\dots\dots \quad \dots$$

$$D^ne^{ax}=a^ne^{ax}. \quad \dots(n)$$

If $f(D)=(D^n+a_1D^{n-1}+a_2D^{n-2}+\dots+a_{n-1}D+a_n)$, then multiplying (1), (2), (3)......(n) by a_n, a_{n-1}......, 1 respectively and adding, we obtain

$$f(D)\,e^{ax}=f(a)\,e^{ax}$$

Now operating on both the sides by $\frac{1}{f(D)}$,

$$\frac{1}{f(D)}f(D)\,e^{ax}=\frac{1}{f(D)}f(a)\,e^{ax}$$

or $\quad e^{ax}=f(a)\frac{1}{f(D)}e^{ax}$ or $\frac{1}{f(a)}e^{ax}=\frac{1}{f(D)}e^{ax}$,

dividing by $f(a)\neq 0$

Therefore $\dfrac{1}{f(D)}e^{ax}=\dfrac{1}{f(a)}e^{ax}$, provided that $f(a)\neq 0$.

Ex. 1. *Solve* $\dfrac{d^2y}{dx^2}-2k\dfrac{dy}{dx}+k^2y=0.$ **[Nagpur 1957]**

Solution. Auxiliary equation is $D^2-2kD+k^2=0$,

i.e., $\quad (D-k)^2=0$ or $D=k, k.$

$\therefore$ C.F.$=(C_1+C_2x)\,e^{kx}.$

$$\text{P. I.} = \frac{1}{D^2-2kD+k^2}e^x = \frac{1}{1-2k+k^2}e^x = \frac{1}{(1-k)^2}e^x.$$

Hence the general solution is

$$y=(C_1+C_2x)\,e^{kx}+\frac{1}{(1-k)^2}\,e^x,\ k\neq 1.$$

5·15. To show that $\frac{1}{f(D^2)}$ sin ax $=\frac{1}{f(-a^2)}$ sin ax, except when $f(-a^2)=0$. **[Poona 1964 ; Delhi 55]**

By successive differentiation, we get

$$\sin ax=\sin ax, \quad \text{...(1)}$$
$$D\sin ax=a\cos ax,$$
$$D^2\sin ax=-a^2\sin ax, \quad \text{...(2)}$$
$$D^3\sin ax=-a^3\cos ax,$$
$$D^4\sin ax=a^4\sin ax$$

or $(D^2)^2\sin ax=(-a^2)^2\sin ax,$ ...(3)

... ...

Similarly $(D^2)^n\sin ax=(-a^2)^n\sin ax.$

Thus $f(D^2)\sin ax=f(-a^2)\sin ax.$

Operating by $\frac{1}{f(D^2)}$ on both the sides, we get

$$\frac{1}{f(D^2)}f(D^2)\sin ax=\frac{1}{f(D^2)}f(-a^2)\sin ax$$

i.e.. $\sin ax=f(-a^2).\frac{1}{f(D^2)}\sin ax.$

Dividing by $f(-a^2)$, we get

$$\frac{1}{f(D^2)}\sin ax=\frac{1}{f(-a)^2}\sin ax,\ \text{ if } f(-a^2)\neq 0.$$

Similarly $\frac{1}{f(D^2)}\cos ax=\frac{1}{f(-a^2)}\cos ax.$

Important. It follows from the result above that we put $-a^2$ in place of D^2. **We cannot put anything in place of D.**

Thus for D^2 put $-a^2$.

for $D^3=D^2.D$ put $-a^2D$.

for $D^4=D^2.D^2$ put $-a^2(-a^2)$, *i.e.*, a^4 etc.

Thus ultimately $f(D)$ becomes linear in D say of the form $(D+\alpha)$. Then we proceed as follows :

$$\frac{1}{D+\alpha}\sin ax=\frac{(D-\alpha)}{(D+\alpha)(D-\alpha)}\sin ax$$
$$=\frac{(D-\alpha)}{D^2-\alpha^2}\sin ax=\frac{D-\alpha}{-a^2-\alpha^2}\sin ax$$

putting $-a^2$ for D^2 in the denominator

$$=\frac{1}{-a^2-\alpha^2}\left(\frac{d}{dx}\sin ax-\alpha\sin ax\right) \text{ as } D=\frac{d}{dx}$$

$$=\frac{1}{-a^2-\alpha^2}(a\cos ax-\alpha\sin ax),$$

And thus the particular integral in case of $\sin ax$ and $\cos ax$ can be completely evaluated.

Ex. 1. *Solve* $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=\sin 2x$.

[Calcutta Hons 1962 ; Karnatak 60 ; Saugar 59 ; Raj. 59 ; Gujrat 61]

Solution. A.E. is $D^2+D+1=0$, $D=-\frac{1}{2}\pm\frac{1}{2}\sqrt{3}i$.

$\therefore$ C.F.$=e^{-(1/2)x}\,C_1\cos\{\frac{1}{2}\sqrt{3}x+C_2\}$.

$$\text{P.I.}=\frac{1}{D^2+D+1}\sin 2x=\frac{1}{-4+D+1}\sin 2x$$

$$=\frac{1}{D-3}\sin 2x=\frac{D+3}{D^2-9}\sin 2x$$

$$=-\tfrac{1}{13}(2\cos 2x+3\sin 2x).$$

Hence the complete solution is

$$y=e^{-x/2}\,C_1\cos(\tfrac{1}{2}\sqrt{3}x+C_2)-\tfrac{1}{13}(2\cos 2x+3\sin 2x).$$

Ex. 2. *Solve* $(D^2+1)^2\,y=\cos 3x$. **[Bombay 1958]**

Solution. A.E. is $(D^2+1)^2=0$, $D=\pm i, \pm i$.

$\therefore$ C.F.$=(C_1+C_2x)\cos x+(C_3+C_4x)\sin x$.

$$\text{P.I.}=\frac{\cos 3x}{(D^2+1)^2}=\frac{\cos 3x}{(-9+1)^2}=\tfrac{1}{64}\cos 3x.$$

Hence the complete solution is

$$y=(C_1+C_2x)\cos x+(C_3+C_4x)\sin x+\tfrac{1}{64}\cos 3x.$$

Ex. 3. *Solve* $(D^3+D^2+D+1)\,y=\sin 2x$. **[Poona 1963]**

Solution. A.E. is $(D^2+1)(D+1)=0$, $D=-1, \pm i$.

C.F.$=C_1e^{-x}+C_2\cos(x+C_3)$ or $C_1e^{-x}+C_2\cos x+C_3\sin x$.

$$\text{P. I.}=\frac{1}{(D^2+1)(D+1)}\sin 2x=\frac{1}{(-4+1)(D+1)}\sin 2x$$

$$=-\tfrac{1}{3}\,\frac{D-1}{D^2-1}\sin 2x=-\tfrac{1}{3}\,\frac{D-1}{-4-1}\sin 2x$$

$$=\tfrac{1}{15}[2\cos 2x-\sin 2x].$$

Complete solution is $y=$C.F.$+$P.I.

Ex. 4. *Prove that the solution of the differential equation* $\frac{d^2y}{dx^2}+4y=\sin ax$ *when* $a\neq 2$, *under the conditions* $y=0$ *and* $\frac{dy}{dx}=0$ *when* $x=0$ *is* $y=\frac{2\sin ax-a\sin 2x}{2(4-a^2)}$. **[Nagpur 1961]**

Solution. A.E. is $D^2+4=0$, $D=\pm 2i$.

C.F.$=C_1 \sin (2x+C_2)$.

P. I.$=\dfrac{1}{D^2+4} \sin ax=\dfrac{\sin ax}{4-a^2}$.

$\therefore$ The general solution is

$$y=C_1 \sin (2x+C_2)+\frac{\sin ax}{4-a^2}. \qquad \ldots(1)$$

so that $\dfrac{dy}{dx}=2C_1 \cos (2x+C_2)+\dfrac{a \cos ax}{4-a^2}$. ...(2)

But $y=0$ when $x=0$,

$\therefore$ (1) gives $\quad 0=C_1 \sin C_2$. ...(3)

Again $\dfrac{dy}{dx}=0$ when $x=0$.

$\therefore$ (2) gives $\quad 0=2C_1 \cos C_2+\dfrac{a}{4-a^2}$. ...(4)

From (3), $C_1=0$ or $C_2=0$ but if $C_1=0$, (4) does not hold.

Hence $C_2=0$ and then from (4), $C_1=-\dfrac{a}{2(4-a^2)}$.

Putting these values of C_1 and C_2 in (1), the required solution is

$$y=-\frac{a \sin 2x}{2(4-a^2)}+\frac{\sin ax}{4-a^2}=\frac{2 \sin ax-a \sin 2x}{2(4-a^2)}.$$

This proves the result.

5·16. Exceptional case of $\dfrac{1}{f(D)} e^{ax}$ when $f(a)=0$.

[Poona 61 ; Bombay 61]

We have from 5·14, $\dfrac{1}{f(D)} e^{ax}=\dfrac{e^{ax}}{f(a)}$ if $f(a)\neq 0$.

But if $f(a)=0$, this becomes infinite and our method fails.

Now $f(a)=0$ means that $(D-a)$ is a factor of $f(D)$.

Therefore let $\quad f(D)=(D-a)\,\phi(D)$,

such that $\quad \phi(a)\neq 0$. ...(1)

$$\therefore \quad \frac{1}{f(D)} e^{ax}=\frac{1}{(D-a)\,\phi(D)} e^{ax}$$

$$=\frac{1}{D-a}\cdot\frac{1}{\phi(a)} e^{ax} \text{ as } \phi(a)\neq 0$$

$$=\frac{1}{\phi(a)}\frac{1}{D-a} e^{ax}=\frac{1}{\phi(a)}\cdot e^{ax}\int e^{-ax} e^{ax}\,dx \text{ [§ 5·12]}$$

$$=\frac{1}{\phi(a)} e^{ax}\int dx=\frac{xe^{ax}}{\phi(a)}. \qquad \ldots(2)$$

Now differentiating both the sides of (1) w. r. t. D,

$f'(D)=(D-a)\,\phi'(D)+\phi(D)$.

Putting $D=a$, $\quad f'(a)=0+\phi(a)$.

It means $\phi(a)=f'(a)$.

Hence (2) becomes

$$\frac{1}{f(D)}e^{ax}=\frac{xe^{ax}}{f'(a)} \quad \text{or} \quad x.\frac{1}{f'(D)}e^{ax}.$$

Again if $f'(a)=0$ and $f''(a)\neq 0$ then $D-a$ is a factor repeated twice ; and applying the above result once again, we get

$$\frac{1}{f(D)}e^{ax}=x^2\frac{1}{f''(D)}e^{ax} \text{ and so on.}$$

***5·17. Exceptional case of $\frac{1}{f(D^2)}$ sin ax when $f(-a^2)=0$.**

[Delhi Hons. 65, 64]

From § 5·15 P. 68, $\frac{1}{f(D^2)}\sin ax=\frac{1}{f(-a^2)}\sin ax, f(-a^2)\neq 0$.

But if $f(-a^2)=0$, it becomes infinite and our method fails.

Now $f(-a^2)=0$ means that D^2+a^2 is a factor of $f(D^2)$.

Let $f(D^2)=(D^2+a^2)\,\phi(D^2)$, such that $\phi(-a^2)\neq 0$.

Now $\frac{1}{f(D^2)}(\cos ax+i\sin ax)=\frac{1}{f(D^2)}e^{aix}$

$$=x\frac{1}{f'(D^2)}e^{aix}$$

where dashes denote differentiation w.r.t. D

$$=x\frac{1}{f'(D^2)}(\cos ax+i\sin ax).$$

Equating real and imaginary parts, we have

$$\frac{1}{f(D^2)}\cos ax=x\frac{1}{f'(D^2)}\cos ax$$

and $$\frac{1}{f(D^2)}\sin ax=x\frac{1}{f'(D^2)}\sin ax.$$

In case $f'(-a^2)=0$ and $f''(-a^2)\neq 0$, D^2+a^2 is a twice repeated factor of $f(D^2)$. Applying the above result once again, we get

$$\frac{1}{f(D^2)}\sin ax=x^2\frac{1}{f''(D^2)}\sin ax$$

and $$\frac{1}{f(D^2)}\cos ax=x^2\frac{1}{f''(D^2)}\cos ax$$

Ex. 1. *Solve* $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=e^x$ [Karnatak 60]

Solution. Auxiliary equation is

$D^2-3D+2=0$ *i.e.*, $(D-2)(D-1)=0$.

$\therefore$ C.F. $=C_1e^x+C_2e^{2x}$.

P.I. $=\frac{e^x}{D^2-3D+2}$ (case of failure)

$$=x\frac{e^x}{2D-3}$$ multiplying by x and differentiating the deno. w.r.t. D.

$$=x\frac{e^x}{2.1-3}=-xe^x.$$

Hence the complete solution is $y=C_1e^x+C_2e^{2x}-xe^x$.

Ex. 2. *Solve* $(D^2+4D+3)\,y=e^{-3x}$. [Gujrat 61]

Solution. Auxiliary equation is

$D^2+4D+3=0$, $(D+3)(D+1)=0$.

$\therefore$ C. F. $=C_1e^{-x}+C_2e^{-3x}$.

$$\text{P. I.}=\frac{e^{-3x}}{D^2+4D+3},$$ case of failure

$$=x\frac{e^{-3x}}{2D+4}$$ multiplying by x and differentiating the denominator w.r.t. D

$$=x\frac{e^{-3x}}{2(-3)+4}=-\tfrac{1}{2}xe^{-3x}.$$

Hence the general solution is

$y=C_1e^{-x}+C_2e^{-3x}-\tfrac{1}{2}xe^{-3x}$.

Ex. 3. *Solve* $\frac{d^3y}{dx^3}+3\frac{d^2y}{dx^2}+3\frac{dy}{dx}+y=e^{-x}$. [Gujrat 61]

Solution. Auxiliary equation is

$D^3+3D^2+3D+1=0$, $(D+1)^3=0$, $D=-1,-1,-1$.

$\therefore$ C. F. $=(C_1+C_2x+C_3x^2)\,e^{-x}$.

$$\text{P. I.}=\frac{e^{-x}}{(D+1)^3},$$ (case of failure)

$$=x\frac{e^{-x}}{3(D+1)^2}$$ multiplying by x and differentiating the denominator w.r.t. D (this is again a case of failure)

$$=x^2\frac{e^{-x}}{6(D+1)}$$ multiplying again by x and differentiating the denominator w.r.t. D (again case of failure)

$$=x^3\frac{e^{-x}}{6}$$ multiplying by x again and differentiating the denominator w.r.t. D.

Hence the complete solution is

$y=(C_1+C_2x+C_3x^2)\,e^{-x}+\tfrac{1}{6}x^3e^{-x}$.

Ex. 4. *Solve* $2\frac{d^3y}{dx^3}-3\frac{d^2y}{dx^2}+y=e^x+1$. [Poona 61]

Solution. Auxiliary equation is $2D^3-3D^2+1=0$

or $(D-1)(D-1)(2D+1)=0$, $D=1,1,-\tfrac{1}{2}$

C. F. $=(C_1+C_2x)\,e^x+C_3e^{-x/2}$

$$\text{P. I.}=\frac{e^x}{2D^3-3D^2+1}+\frac{1}{2D^3-3D^2+1} \quad \text{first term case of failure}$$

$$=x\frac{e^x}{6D^2-6D}+\frac{e^{0x}}{0-3.0+1} \quad \text{differentiating the denominator of the first and multiplying it by } x \text{ (again case of failure)}$$

$$=x^2\frac{e^x}{12D-6}+1 \quad \text{again differentiating the denominator and multiplying by } x.$$

$$=\tfrac{1}{6}x^2e^x+1.$$

Hence the complete solution is

$$y=(C_1+C_2x)\,e^x+C_3e^{-x/2}+\tfrac{1}{6}x^2e^x+1.$$

Ex. 5. $(D^3-2D^2-5D+6)\,y=e^{3x}$. **[Poona 63]**

Solution. A. E. is $(D-3)\,(D^2+D-2)=0$.

i.e., $(D-3)\,(D+2)\,(D-1)=0$.

$\therefore$ C. F. $=C_1e^x+C_2e^{-2x}+C_3e^{3x}$.

$$\text{P. I.}=\frac{1}{(D-3)\,(D+2)\,(D-1)}\,e^{3x}=\frac{1}{(D-3)\,(3+2)\,(3-1)}\,e^{3x}$$

$$=\frac{1}{10\,(D-3)}\,e^{3x}$$

$=x.\frac{1}{10}e^{3x}$ as it is a case of failure.

$\therefore$ The complete solution is $y=$ C. F. $+$ P. I.

Ex. 6. (a) *Solve* $\dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}+4y=e^{2x}+e^{-2x}$. **[Karnatak 60]**

Solution. A. E. is $(D^2+4D+4)=0$, *i.e.*, $(D+2)^2=0$.

$\therefore$ C. F. $=(C_1+C_2x)\,e^{-2x}$.

$$\text{P. I.}=\frac{e^{2x}}{(D+2)^2}+\frac{e^{-2x}}{(D+2)^2} \quad \text{second is a case of failure}$$

$$=\frac{e^{2x}}{(2+2)^2}+\frac{x^2e^{-2x}}{2} \quad \text{differentiating denominator of the second twice w.r.t. } D \text{ and multiplying by } x^2$$

$$=\frac{e^{2x}}{16}+\tfrac{1}{2}x^2e^{-2x}.$$

Hence the complete solution is $y=$ C. F. $+$ P. I.

Ex. 6. (b) *Solve* $\dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}+4y=2\sinh 2x$. **[Delhi Hons. 62]**

Hint. $2\sinh 2x=e^{2x}-e^{-2x}$. Now proceed as in Ex. 6 (a).

Ex. 6. (c) *Solve the following* :

$$\frac{d^2x}{dt^2}=x+e^t+e^{-t}.$$

[Delhi Pass 68]

Solution. Auxiliary equation is

$D^2-1=0$, *i.e.*, $D=\pm 1$ $\qquad (D\equiv d/dt)$.

$\therefore$ C.F. $= C_1 e^{-t} + C_2 e^t$

$$\text{P.I.} = \frac{e^t + e^{-t}}{(D^2-1)} = \frac{e^t}{D^2-1} + \frac{e^{-t}}{D^2-1} \text{ exceptional cases}$$

$$= t\,\frac{e^t}{2D} + t\,\frac{e^{-t}}{2D}$$

$$= \tfrac{1}{2} t e^t - \tfrac{1}{2} t e^{-t}.$$

$\therefore\ x = C_1 e^{-t} + C_2 e^t + \frac{1}{2} t\,(e^t - e^{-t})$ is the general solution.

Ex. 7. *Solve* $(D^2 + a^2)\,y = \sin ax$. [Poona 62; Saugar 63]

Solution. A. E. is $D^2 + a^2 = 0$, $D = \pm ai$.

$$\therefore\ \text{C. F.} = C_1 \cos(ax + C_2).$$

$$\text{P. I.} = \frac{\sin ax}{D^2 + a^2} \text{ case of failure}$$

$$= x\,\frac{\sin ax}{2D} \text{ multiplying by } x \text{ and differentiating the denominator w.r.t. } D$$

$$= -\frac{x}{2a}\cos ax.$$

Hence $y = C_1 \cos(ax + C_2) - \frac{x}{2a}\cos ax$ is the complete solutions.

Ex. 8. *Solve* $\frac{d^6y}{dx^6} + y = \sin \frac{3}{2}x \sin \frac{1}{2}x$. [Raj. 61]

Solution. Auxiliary equation is

$$D^6 + 1 = 0 \quad \text{or} \quad (D^2+1)(D^4 - D^2 + 1) = 0$$

or $$(D^2+1)\,[(D^2+1)^2 - 3D^2] = 0$$

or $$(D^2+1)(D^2 - \sqrt{3}D + 1)(D^2 + \sqrt{3}D + 1) = 0.$$

When $D^2 + 1 = 0$, $D = \pm i$.

When $D^2 - \sqrt{3}D + 1 = 0$, $D = \frac{\sqrt{3} \pm i}{2}$.

When $D^2 + \sqrt{3}D + 1 = 0$, $D = \frac{\sqrt{3} \pm i}{2}$.

Hence C. F. $= C_1 \cos(x + C_2) + C_3 e^{\frac{1}{2}\sqrt{3}x} \cos(\frac{1}{2}x + C_4)$
$+ C_5 e^{-\frac{1}{2}\sqrt{3}x} \cos(\frac{1}{2}x + C_6)$.

Now $\sin \frac{3}{2}x \sin \frac{1}{2}x = \frac{1}{2}(\cos x - \cos 2x)$.

$$\therefore\ \text{P. I.} = \tfrac{1}{2}\cdot\frac{\cos x}{D^6+1} - \tfrac{1}{2}\cdot\frac{\cos 2x}{D^6+1} \text{ (first term case of failure)}$$

$$= \tfrac{1}{2}x\cdot\frac{\cos x}{6D^5} - \tfrac{1}{2}\cdot\frac{\cos 2x}{(-4)^3+1}$$

$$= \tfrac{1}{2}x\cdot\frac{\cos x}{6(-1)^2 D} + \frac{1}{126}\cos 2x$$

$$= \frac{1}{12} x \sin x + \frac{1}{126}\cos 2x \text{ as } \frac{1}{D} \text{ means integration.}$$

Hence the complete solution is $y = \text{C.F.} + \text{P.I.}$

Ex. 9. *Solve* $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} - 2y = e^x + \cos x.$

[Agra 60; Punjab M.A. 57; Vikram 62; Poona 64; Bombay 58]

Solution. Auxiliary equation is $D^3 - 3D^2 + 4D - 2 = 0$,

i.e., $(D-1)(D^2 - 2D + 2) = 0$, *i.e.*, $(D-1)[(D-1)^2 + 1] = 0$

or $(D=1)\ 1 \pm i$; $\therefore$ C. F. $= C_1 e^x + e^x (C_2 \cos x + C_3 \sin x)$

$$\text{P. I.} = \frac{1}{D^3 - 3D^2 + 4D - 2} e^x + \frac{1}{D^3 - 3D^2 + 4D - 2} \cos x$$

first term is case of failure

$$= x \frac{1}{3D^2 - 6D + 4} e^x + \frac{1}{(-1)D - 3(-1) + 4D - 2} \cos x$$

$$= xe^x + \frac{1}{3D+1} \cos x = xe^x + \frac{3D-1}{9D^2 - 1} \cos x$$

$$= xe^x + \tfrac{1}{10}(3 \sin x + \cos x).$$

Hence the complete solution is $y = \text{C. F.} + \text{P. I.}$

Ex. 10. *Solve* $(D^3 - 5D^2 + 7D - 2)\, y = e^{2x} \cosh x.$

[Delhi Hons. 55]

Solution. A. E. is $D^3 - 5D^2 + 7D - 3 = 0.$

$(D-1)(D^2 - 4D + 3) = 0$ or $(D-1)(D-3)(D-1) = 0.$

$\therefore$ C.F. $= (C_1 + C_2 x) e^x + C_3 e^{3x}.$

$$\text{P.I.} = \frac{e^{2x} \cosh x}{(D-1)^2 (D-3)} = \frac{e^{2x} \cdot \frac{1}{2}(e^x + e^{-x})}{(D-1)^2 (D-3)}$$

$$= \tfrac{1}{2} \frac{e^{3x}}{(D^3 - 5D^2 + 7D - 3)} + \tfrac{1}{2} \frac{e^x}{D^3 - 5D^2 + 7D - 3}$$

both cases of failure

$$= \tfrac{1}{2} x \frac{e^{3x}}{3D^2 - 10D + 7} + \tfrac{1}{2} x \frac{e^x}{3D^2 - 10D + 7}$$

$$= \tfrac{1}{2} x \frac{e^{3x}}{3.3^2 - 10.3 + 7} + \tfrac{1}{2} x^2 \frac{e^x}{6D - 10}$$

$$= \tfrac{1}{8} xe^{3x} - \tfrac{1}{8} x^2 e^x.$$

Hence the complete solution is $y = \text{C.F.} + \text{P.I.}$

Ex. 11. *Solve* $[D^4 + (m^2 + n^2) D^2 + m^2 n^2]\, y$

$= \cos \frac{1}{2}(m+n)\, x \cos \frac{1}{2}(m-n)\, x.$ **[Delhi Hons. 53]**

Solution. A.E. is $(D^2 + m^2)(D^2 + n^2) = 0$, $D = \pm mi, \pm ni.$

C.F. $= C_1 \cos (mx + C_2) + C_3 \cos (nx + C_4).$

$$\text{P. I.} = \frac{\cos \frac{1}{2}(m+n)\, x \cos \frac{1}{2}(m-n)\, x}{D^4 + (m^2 + n^2) D^2 + m^2 n^2}$$

$$= \tfrac{1}{2} \frac{\cos mx + \cos nx}{D^4 + (m^2 + n^2) D^2 + m^2 n^2} \quad \text{cases of failure}$$

$$= \tfrac{1}{2} x \frac{\cos mx + \cos nx}{4D^3 + 2(m^2 + n^2) D}$$

$$=\frac{1}{2}x\,\frac{1}{2D}\left[\frac{\cos mx}{-2m^2+(m^2+n^2)}+\frac{\cos nx}{+2n^2+(m^2-n^2)}\right]$$

$$=\frac{x}{4\,(m^2-n^2)}\left[-\frac{\cos mx}{D}+\frac{\cos nx}{D}\right]$$

$$=\frac{x}{4\,(m^2-n^2)}\left[-\frac{1}{m}\sin mx+\frac{1}{n}\sin nx\right].$$

The complete solution is $y=$C.F.$+$P.I.

5·18. $\frac{1}{f(D)}x^m$**, where m is a positive integer.** **[Gujrat 59]**

Consider $\frac{1}{D-\alpha}x^m$

$$=-\frac{1}{\alpha\,(1-D/\alpha)}x^m=-\frac{1}{\alpha}\left(1-\frac{D}{\alpha}\right)^{-1}x^m$$

$$=-\frac{1}{\alpha}\left(1+\frac{D}{\alpha}+\frac{D^2}{\alpha^2}+\ldots+\frac{D^m}{\alpha^m}+\ldots\right)x^m$$

$$=-\frac{1}{\alpha}\left(x^m+\frac{mx^{m-1}}{\alpha}+\frac{m\,(m-1)\,x^{m-2}}{\alpha^2}+\ldots+\frac{m\,!}{\alpha^m}\right).$$

Therefore to evaluate $\frac{1}{f(D)}x^m$ expand $[f(D)]^{-1}$ in ascending powers of D, retaining terms as far D^m and operate each term on x^m.

We need not retain terms containing D^{m+1}, D^{m+2} etc. as $D^{m+1}x=0$, $D^{m+2}x^m=0$ etc.

Ex. *Solve* $(D^3+2D^2+D)\,y=e^{2x}+x^2+x.$ **[Poona 64]**

Solution. A.E. is $D\,(D+1)^2=0$, *i.e.*, $D=0,\,-1,\,-1.$

$\therefore$ C.F.$=C_1+(C_2+C_3x)\,e^{-x}.$

$$\text{P. I.}=\frac{e^{2x}}{D\,(D+1)^2}+\frac{1}{D\,(1+D)^2}\,(x^2+x)$$

$$=\frac{e^{2x}}{2\,(2+1)^2}+\frac{1}{D}\,(1+D)^{-2}\,(x^2+x)$$

$$=\frac{e^{2x}}{18}+\frac{1}{D}\,[1-2D+3D^2\ldots]\,(x^2+x)$$

$$=\frac{e^{2x}}{18}+\frac{1}{D}\,[x^2+x-4x-2+6]$$

$$=\frac{e^{2x}}{18}+\frac{x^3}{3}-\frac{3x^2}{2}+4x.$$

The complete solution is $y=$C.F.$+$P.I.

***5·19. To show that** $\frac{1}{f(D)}(e^{ax}V)=e^{ax}\frac{1}{f(D+a)}V$**,**
where V is function of x.

[Delhi Hons. 62, 55; Karnatak 61; Bombay 58]

We have on successive differentiation (by parts),

$D(e^{ax}V)=e^{ax}DV+ae^{ax}=e^{ax}(D+a)V,$

$D^2(e^{ax}V)=e^{ax}D^2V+ae^{ax}V+a^2e^{ax}+ae^{ax}DV$

$=e^{ax}(D^2+2aD+a^2)V=e^{ax}(D+a)^2V.$

Similarly, $D^3(e^{ax}V)=e^{ax}(D+a)^3V$

and $D^n(e^{ax}V)=e^{ax}(D+a)^nV.$

Therefore $f(D)(e^{ax}V)=e^{ax}f(D+a)V.$ [Poona 62]

Taking the inverse operators, we have

$$\frac{1}{f(D)}(e^{ax}V)=e^{ax}\frac{1}{f(D+a)}V.$$

Thus we find that operator $\frac{1}{f(D)}$ on $(e^{ax}V)$ is equivalent to $\frac{1}{f(D+a)}$ on V taking e^{ax} outside.

Therefore in practice take out e^{ax} and put $(D+a)$ in place of D and then find $\frac{1}{f(D+a)}V$ as usual.

Ex. 1. *Solve* $\frac{d^2y}{dx^2}-9y=6e^{3x}+xe^{3x}.$ [Bombay 61]

Solution. Auxiliary equation is $D^2-9=0$, $D=\pm3$.

C.F. $=C_1e^{3x}+C_2e^{-3x}$.

$$\text{P. I.}=\frac{1}{D^2-9}e^{3x}(6+x)=e^{3x}\frac{1}{(D+3)^2-9}(6+x)$$

$$=e^{3x}\frac{1}{D^2+6D}(6+x)=e^{3x}\frac{1}{6D}(1+\tfrac{1}{6}D)^{-1}(6+x)$$

$$=e^{3x}\frac{1}{6D}(1-\tfrac{1}{6}D-\ldots)(6+x)$$

$$=e^{3x}\frac{1}{6D}(6+x-\tfrac{1}{6})=\tfrac{1}{36}e^{3x}(35x+3x^2).$$

Hence the complete solution is

$$y=C_1e^{3x}+C_2e^{-3x}+\tfrac{1}{36}e^{3x}(35x+3x^2).$$

Ex. 2. *Solve* $\frac{d^3y}{dx^3}-3\frac{d^2y}{dx^2}+3\frac{dy}{dx}-y=xe^x+e^x.$

[Agra 61 ; Bombay 58]

Solution. A.E. is $D^3-3D^2+3D-1=0$,

i.e., $(D-1)^3=0$ or $D=1, 1, 1.$

$\therefore$ C.F. $=(C_1+C_2x+C_3x^2)e^x$.

$$\text{P. I.}=\frac{1}{(D-1)^3}e^x(x+1)$$

$$=e^x\frac{1}{(D+1-1)^3}(x+1)=e^x\frac{1}{D^3}(x+1)$$

$$=e^x\frac{1}{D^2}\frac{(x+1)^2}{2}=e^x.\frac{1}{D}\frac{(x+1)^3}{6}$$

$$=e^x \cdot \frac{(x+1)^4}{24}.$$

Hence the general solution is $y=$C.F.+P.I.

Ex. 3. *Solve* $(D^3-7D-6)\,y=e^{2x}\cdot x^2.$

[Bombay B. Sc. 61]

Solution. A.E. is $D^3-7D-6=0$;

i.e., $(D+1)(D^2-D-6)=0$ or $(D+1)(D-3)(D+2)=0.$

$\therefore$ C.F.$=C_1e^{-x}+C_2e^{3x}+C_3e^{-2x}.$

$$\text{P. I.}=\frac{e^{2x}.x^2}{D^3-7D-6}=e^{2x}\frac{1}{(D+2)^3-7(D+2)-6}x^2$$

$$=e^{2x}\frac{1}{D^3+6D^2+5D-12}x^2$$

$$=-\frac{e^{2x}}{12}\left(1-\frac{5}{12}D-\frac{1}{2}D^2-\frac{1}{12}D^3\right)^{-1}x^2$$

$$=-\frac{e^{2x}}{12}\left(1+\frac{5}{12}D+\frac{1}{2}D^2+\frac{25}{12^2}D^2\right)x^2$$

$$=-\frac{e^{2x}}{12}\left(x^2+\frac{5}{6}x+\frac{97}{72}\right)\text{ etc.}$$

Ex. 4. *Solve* $\dfrac{d^3y}{dx^3}-2\dfrac{dy}{dx}+4y=e^x\cos x.$

[Delhi Hons. 54; Karnatak 61]

Solution. Auxiliary equation is $D^3-2D+4=0$,

i.e., $(D+2)(D^2-2D+2)=0$

or $(D+2)[(D-1)^2+1]=0.$

$D=-2,\ 1\pm i,$ C.F.$=C_1e^{-2x}+C_2e^x\cos(x+C_3).$

$$\text{P.I.}=\frac{1}{D^3-2D+4}e^x\cos x$$

$$=e^x\frac{1}{(D+1)^3-2(D+1)+4}\cos x$$

$$=e^x\frac{1}{D^3+3D^2+D+3}\cos x \qquad \text{(case of failure)}$$

$$=e^x.x\frac{1}{3D^2+6D+1}\cos x$$

$$=xe^x\frac{1}{-3+6D+1}\cos x$$

$$=xe^x\frac{1}{6D-2}\cos x$$

$$=\tfrac{1}{2}xe^x\frac{3D+1}{9D^2-1}\cos x$$

$$=-\tfrac{1}{20}xe^x(-3\sin x+\cos x)$$

$$=\tfrac{1}{20}xe^x(3\sin x-\cos x).$$

Hence the complete solution is $y=$ C.F. + P.I

Ex. 5. *Solve* $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+4y=e^x\cos x.$ **[Agra 59]**

Solution. A.E. is $D^2-2D+4=0$

or $[(D-1)^2+3]=0$ or $D=1\pm\sqrt{3}i$

$\therefore$ C.F. $=e^x\,[C_1\cos\sqrt{3}x+C_2\sin\sqrt{3}x]$

$$\text{P.I.}=\frac{1}{D^2-2D+4}e^x\cos x$$

$$=e^x\frac{1}{(D+1)^2-2(D+1)+4}\cos x$$

$$=e^x\frac{1}{D^2+3}\cos x=e^x\frac{1}{-1+3}\cos x$$

$$=\tfrac{1}{2}e^x\cos x,\text{ etc.}$$

Ex 6. (a) *Solve* $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=e^x\sin x.$ **[Osmania 62]**

Solution. A. E. is $D^2-2D+2=0$,

i.e., $(D-1)^2+1=0$ or $D=1\pm i.$

C. F. $=e^x\,[C_1\cos x+C_2\sin x]$.

$$\text{P. I.}=\frac{e^x\sin x}{D^2-2D+2}=e^x.\frac{1}{(D+1)^2-2(D+1)+1}\sin x$$

$$=e^x\frac{1}{D^2+1}\sin x \qquad \text{(case of failure)}$$

$$=e^x.x\frac{1}{2D}\sin x=-\frac{x}{2}e^x\cos x,\text{ etc.}$$

Ex. 6. (b) *Solve* $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+2y=xe^{-x}.$ **[Poona 61]**

Solution. A. E. is $D^2+2D+2=0$, $(D+1)^2+1=0.$

$D=-1\pm i$; $\therefore$ C. F. $=C_1e^{-x}\cos(x+C_2)$

$$\text{P. I.}=\frac{1}{D^2+2D+2}xe^{-x}$$

$$=e^{-x}\frac{1}{(D-1)^2+2(D-1)+2}x$$

$$=e^{-x}\frac{1}{D^2+1}x=e^{-x}(1-D^2\ldots)x$$

$$=xe^{-x},\text{ etc.}$$

5 20. $\frac{1}{f(D)}(xV)$, **where V is any function of x.**

[Poona 60; Karnatak 60, 61, 62]

We have

$D^n(x)V=xD^nV+nD^{n-1}V$ by Leibnitz's rule

$$=xD^nV+\frac{d}{dD}(D^n)V \quad \text{as} \quad \frac{d}{dD}D^n=nD^{n-1}.$$

$$\therefore \quad f(D)(xV)=xf(D)V+f'(D)V.$$

Taking the inverse operator, we get

$$\frac{1}{f(D)}(xV)=x\frac{1}{f(D)}V+\left[\frac{d}{dD}\frac{1}{f(D)}\right]V.$$

or $$\frac{1}{f(D)}(xV)=x\frac{1}{f(D)}V-\frac{f'(D)}{[f(D)]^2}V.$$

5·21. $\frac{1}{f(D)}(x^mV)$, **where V is some function of x.**

Now V can have the following different forms :

1. *V has the form x^m*, then x^mV becomes x^{m+n} which can be evaluated by the method of § 5·18 P. 76.

2. *V has the form e^{ax}*, then x^mV becomes x^me^{ax} which can be evaluated by the method of § 5·19 P. 76.

3. *V has the form $\cos ax$ or $\sin ax$*, then x^mV becomes $x^m\cos ax$ or $x^m\sin ax$, *i.e.*, it is real or imaginary part of x^me^{aix}, which can be easily evaluated.

Ex. 1. *Solve* $\frac{d^4y}{dx^4}-y=x\sin x.$

[Mysore 68; Agra 66; Lucknow Pass 57]

Solution. Auxiliary equation is

$$D^4-1=0,\ (D^2+1)(D^2-1)=0,\ D=\pm i,\ \pm 1.$$

Hence C.F.$=C_1\cos x+C_2\sin x+C_3e^x+C_4e^{-x}$.

$$\text{P. I.}=\frac{1}{D^4-1}x\sin x.$$

$$=\text{Imaginary Part of }\frac{1}{D^4-1}xe^{ix}$$

$$=\text{I. P. of } e^{ix}\frac{1}{(D+i)^4-1}x$$

$$=\text{I. P. of } e^{ix}\frac{1}{D^4+4iD^3-6D^2-4iD}x$$

$$=\text{I. P. of } -e^{ix}\frac{1}{4iD}[1-\tfrac{3}{2}iD-D^2+\tfrac{1}{4}iD^3]^{-1}x \quad \left(\because \frac{1}{i}=-i\right)$$

$$=\text{I. P. of } -e^{ix}\frac{1}{4iD}[1+\tfrac{3}{2}iD]x$$

$$=\text{I. P. of } -e^{ix}\frac{1}{4iD}[x+\tfrac{3}{2}i]$$

$$=\text{I. P. of }\frac{i}{4}(\cos x+i\sin x)[\tfrac{1}{2}x^2+\tfrac{3}{2}ix]$$

$$=\tfrac{1}{8}x^2\cos x-\tfrac{3}{8}x\sin x.$$

Hence the complete solution is

$$y = C_1 \cos x + C_2 \sin x + C_3 e^x + C_4 e^{-x} + \tfrac{1}{8} x^2 \cos x - \tfrac{3}{8} x \sin x.$$

Ex. 2. *Solve* $\dfrac{d^2y}{dx^2} + 4y = x \sin x.$ **[Delhi Hons. 63]**

Solution. A. E. is $D^2 + 4 = 0$, $D = \pm 2i$.

C. F. $= C_1 \cos 2x + C_2 \sin 2x$.

$$\text{P. I.} = \frac{1}{D^2+4}\, x \sin x$$

$$= \text{I. P. of } \frac{1}{D^2+4}\, x e^{ix}$$

$$= \text{I. P. of } e^{ix} \frac{1}{(D+i)^2+4}\, x$$

$$= \text{I. P. of } e^{ix} \frac{1}{D^2+2Di+3}\, x$$

$$= \text{I. P. of } \tfrac{1}{3} e^{ix} \left(1 + \tfrac{2}{3} Di + \tfrac{1}{3} D^2\right)^{-1} x$$

$$= \text{I. P. of } \tfrac{1}{3} e^{ix} \left(1 - \tfrac{2}{3} Di\right) x$$

$$= \text{I. P. of } \tfrac{1}{3} (\cos x + i \sin x)\left(x - \tfrac{2}{3} i\right)$$

$$= \tfrac{1}{9} (3x \sin x - 2 \cos x).$$

The complete solution is $y = \text{C.F.} + \text{P.I.}$

Ex. 3. (a) *Solve* $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 3x^2 e^{2x} \sin 2x.$

[Lucknow Pass 58]

Solution. A. E. is $D^2 - 4D + 4 = 0$, $(D-2)^2 = 0$.

$\therefore$ C.F. $= (C_1 + C_2 x)\, e^{2x}$.

$$\text{P. I.} = \frac{3x^2 e^{2x} \sin 2x}{(D-2)^2} = 3e^{2x} \cdot \frac{1}{(D+2-2)^2}\, x^2 \sin 2x$$

$$= 3e^{2x} \frac{1}{D^2}\, x^2 \sin 2x$$

$$= \text{I. P. of } 3e^{2x} \frac{1}{D^2}\, x^2 e^{2ix}$$

$$= \text{I. P. of } 3e^{2x} . e^{2ix} \frac{1}{(D+2i)^2}\, x^2$$

$$= \text{I. P. of } 3e^{2x}\, e^{2ix} \frac{1}{D^2 + 4iD - 4}\, x^2$$

$$= \text{I. P. of } -3e^{2x} e^{2ix} . \tfrac{1}{4} \left(1 - iD - \tfrac{1}{4} D^2\right)^{-1} x^2$$

$$= \text{I. P. of } -3e^{2x} e^{2ix} . \tfrac{1}{4} \left(1 + iD + \tfrac{1}{4} D^2 - D^2\right) x^2$$

$$= \text{I. P. of } -3e^{2x} e^{2ix} . \tfrac{1}{4} \left(x^2 + 2ix - \tfrac{3}{2}\right)$$

$$= \text{I. P. of } -\tfrac{3}{4} e^{2x} (\cos 2x + i \sin 2x)\left(x^2 + 2ix - \tfrac{3}{2}\right)$$

$$= -\tfrac{3}{8} e^{x} \left[(2x^2 - 3) \sin 2x + 4x \cos 2x\right].$$

The complete solution is $y = \text{C. F.} + \text{P. I.}$

Proceed as above.

Ex. 3. (b) *Solve* $(D^2+1)\,y=8x^2e^{2x}\sin 2x$. **[Allahabad 66]**

***Ex. 4.** *Solve* $(D^4+2D^2+1)\,y=x^2\cos x$.

[Delhi Hons. 64, 62; Lucknow Pass 56; Benaras B.Sc. 59; Ujjain 61, 59; Karnatak 62; Nagpur B Sc. 55]

Solution. The auxiliary equation is

$$(D^4+2D^2+1)=0 \quad i.e., \quad (D^2+1)^2=0,$$

$$D=\pm i,\ \pm i.$$

$$\therefore \quad \text{C. F.}=(C_1+C_2x)\cos x+(C_3+C_4x)\sin x.$$

$$\text{P. I.}=\frac{1}{D^4+2D^2+1}\,x^2\cos x=\text{real part of }\frac{1}{(D^2+1)^2}\,x^2e^{ix}.$$

But $\dfrac{1}{(D^2+1)^2}\,x^2e^{ix}=e^{ix}\,\dfrac{1}{[(D+i)^2+1]^2}\,x^2$

$$=e^{ix}\,\frac{1}{(D^2+2iD)^2}\,x^2=e^{ix}\,\frac{1}{-4D^2\,(1-\frac{1}{2}iD)^2}\,x^2$$

$$=-\tfrac{1}{4}e^{ix}\,\frac{1}{D^2}\,(1-\tfrac{1}{2}iD)^{-2}\,x^2$$

$$=-\tfrac{1}{4}e^{ix}\,\frac{1}{D^2}\,(1+iD-\tfrac{3}{4}D^2-\tfrac{1}{2}iD^3+\tfrac{5}{16}D^4+\ldots)\,x^2$$

$$=-\tfrac{1}{4}e^{ix}\,\frac{1}{D^2}\,[x^2+2ix-\tfrac{3}{2}]$$

$$=-\tfrac{1}{4}e^{ix}\left[\frac{x^4}{12}+i\,\frac{x^3}{3}-\tfrac{3}{4}x^2\right]$$

$$=-\tfrac{1}{4}\,(\cos x+i\sin x)\left[\frac{x^4}{12}-\frac{3}{4}\,x^2+i\,\frac{x^3}{3}\right]$$

$\therefore$ P.I. = real part of the above

$$=-\tfrac{1}{4}\left[\left(\frac{x^4}{12}-\tfrac{3}{4}x^2\right)\cos x-\tfrac{1}{3}x^3\sin x\right]$$

The complete solution is $y=\text{C. F.}+\text{P. I.}$

Ex. 5. (a) *Solve* $\dfrac{d^2y}{dx^2}+a^2y=x\cos ax$. **[Nagpur 57]**

Solution. A. E. is $D^2+a^2=0$, $D=\pm ai$.

$$\text{C.F.}=C_1\cos ax+C_2\sin ax.$$

$$\text{P. I.}=\frac{x\cos ax}{D^2+a^2}$$

$$=\text{Real part of }\frac{xe^{aix}}{D^2+a^2}$$

$$=\text{R. P. of }e^{aix}\,\frac{1}{(D+ai)^2+a^2}\,x$$

$$=\text{R. P. of }e^{aix}\,\frac{1}{D^2+2aiD}\,x$$

$$=\text{R. P. of } e^{aix}\frac{i}{2aiD}\left[1-\frac{1}{2a}iD\right]^{-1}x$$

$$=\text{R. P. of } e^{aix}\frac{1}{2ai\,D}\left[x+\frac{1}{2a}i\right]$$

$$=\text{R. P. of } -\frac{i}{2a}(\cos ax+i\sin ax)\left[\frac{x^2}{2}+\frac{1}{2a}ix\right]$$

$$=\frac{1}{4a^2}[ax^2\sin ax+x\cos ax]$$

The complete solution is $y=$C. F.+P. I.

Ex. 5. (b) *Solve* $(D^2+4)\,y=x\cos 2x.$ **[Karnatak 61]**

Hint. Put $a=2$ in the above example.

Ex. 5. (c) *Solve* $\dfrac{d^2y}{dx^2}+y=x\cos x.$ **[Poona 61 (S)]**

Hint. Put $a=1$ in the above example.

Ex. 6. (a) *Solve* $\dfrac{d^2y}{dx^2}-y=x^2\cos x.$ **[Karnatak 60, 61]**

Solution. Auxiliary equation is $D^2-1=0$ *i.e.* $D=\pm 1$.

$\therefore$ C.F.$=C_1e^x+C_2e^{-x}$.

$$\text{P. I.}=\frac{1}{D^2-1}x^2\cos x$$

$$=\text{Real part of }\frac{1}{D^2-1}x^2e^{ix}$$

$$=\text{R.P. of } e^{ix}\frac{1}{(D+i)^2-1}x^2$$

$$=\text{R.P. of } e^{ix}\frac{1}{D^2+2Di-2}x^2$$

$$=\text{R.P. of } -e^{ix}.\tfrac{1}{2}(1-iD-\tfrac{1}{2}D^2)^{-1}x^2.$$

$$=\text{R.P. of } -\tfrac{1}{2}e^{ix}(1+iD+\tfrac{1}{2}D^2-D^2)\,x^2$$

$$=\text{R.P. of } -\tfrac{1}{2}(\cos x+i\sin x)(x^2+2ix-1)$$

$$=-\tfrac{1}{2}[(x^2-1)\cos x-2x\sin x].$$

Hence the complete solution is $y=$C.F.+P.I.

Ex. 6. (b) $(D^4-1)\,y=x^2\sin x.$ **[Delhi 72]**

Proceed as above.

Ex. 7. *Solve* $\dfrac{d^4y}{dx^4}+2\cdot\dfrac{d^2y}{dx^2}+y=x^2\cos^2 x.$ **[Gujrat B.Sc. 62]**

Solution. A.E. is

$D^4+2D^2+1=0,\ (D^2+1)^2=0,\ m=\pm i,\ \pm i$

$\therefore$ C.F.$=(C_1+C_2x)\cos x+(C_3+C_4x)\sin x.$

Now $x^2\cos^2 x=\frac{1}{2}x^2(1+\cos 2x)=\frac{1}{2}x^2+\frac{1}{2}x^2\cos 2x.$

P.I. corresponding to $\frac{1}{2}x^2=(1+D^2)^{-2}\,\frac{1}{2}x^2$

$=(1-2D^2\ldots)\ \frac{1}{2}x^2=\frac{1}{2}x^2-2.$

P.I. corresponding to $\frac{1}{2}x^2 \cos 2x$

$$=\text{R.P. of } \frac{1}{(1+D^2)^2}\cdot \tfrac{1}{2}x^2e^{2ix}$$

$$=\text{R.P. of } \tfrac{1}{2}e^{2ix} \frac{1}{[1+(D+2i)^2]^2}\ x^2$$

$$=\text{R.P. of } \tfrac{1}{2}e^{2ix} \frac{1}{(D^2+4iD-3)^2}\ x^2$$

$$=\text{R.P. of } \tfrac{1}{18}e^{2ix}\ (1-\tfrac{4}{3}iD-\tfrac{1}{3}D^2)^{-2}\ x^2$$

$$=\text{R.P. of } \tfrac{1}{18}e^{2ix}\ (1+\tfrac{8}{3}iD+\tfrac{2}{3}D^2-\tfrac{16}{3}D^2\ldots)\ x^2$$

$$=\text{R.P. of } \tfrac{1}{18}\ (\cos 2x+i \sin 2x)\ [x^2+\tfrac{16}{3}ix-\tfrac{28}{3}]$$

$$=\tfrac{1}{54}\ [(3x^2-28) \cos 2x-16x \sin 2x].$$

The general solution is $y=\text{C.F.}+\text{P.I.}$

Miscellaneous Examples

Ex. 1. *Solve* $(D^2-5D+6)\,y=4e^x+5.$ **[Nagpur 61]**

Solution. A.E. is $(D-3)\,(D-2)=0.$

$\text{C.F.}=C_1e^{3x}+C_2e^{3x}.$

$$\text{P. I.}=\frac{4e^x+5e^{0x}}{D^2-5D+6}=\frac{4e^x}{1-5+6}+\tfrac{5}{6}=2e^x+\tfrac{5}{6}.$$

Hence $y=C_1e^{2x}+C_2e^{3x}+2e^x+\frac{5}{6}.$

Ex. 2. *Solve* $\dfrac{d^3y}{dx^3}-6\dfrac{d^2y}{dx^2}+11\dfrac{dy}{dx}-6y=e^{2x}.$ **[Delhi Hons. 62]**

Solution. A.E. is $D^3-6D^2+11D-6=0$

or $(D-1)\,(D^2-5D+6)=0$ or $(D-1)\,(D-2)\,(D-3)=0.$

$\therefore$ $\text{C.F.}=C_1e^x+C_2e^{2x}+C_3e^{3x}.$

$$\text{P. I.}=\frac{e^{2x}}{D^3-6D^2+11D-6}\ \text{case of failure}$$

$$=x\frac{e^{2x}}{3D^2-12D+11}=x\frac{e^{2x}}{12-24+11}=-xe^{2x}.$$

Hence $y=C_1e^x+C_2e^{2x}+C_3e^{3x}-xe^{2x}.$

Ex. 3. *Solve* $\dfrac{d^3y}{dx^3}-3\dfrac{dy}{dx}+2y=e^{-2x}.$ **[Poona 61]**

Solution. A.E. is $(D-1)\,(D^2+D-2)=0$

or $(D-1)\,(D+2)(D-1)=0.$

$\text{C.F.}=(C_1+C_2x)\,e^x+C_3e^{-2x}.$

$$\text{P. I.}=\frac{e^{-2x}}{D^3-3D+2}\ \text{(case of failure)}$$

$$=x\frac{e^{-2x}}{3D^2-3}=\tfrac{1}{9}xe^{-2x}.$$

Hence $y=(C_1+C_2x)\,e^x+C_3e^{-2x}+\frac{1}{9}xe^{-2x}.$

Ex. 4. *Solve* $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=e^{2x}+\sin 2x.$ **[Luck. Pass 57]**

Solution. The A.E. is $D^2-4D+4=0$

or $(D-2)^2=0$ or $D=2, 2.$

$\therefore$ C.F. $=(C_1+C_2x)\,e^{2x}.$

P. I. $=\frac{e^{2x}}{D^2-4D+4}+\frac{\sin 2x}{D^2-4D+4}$ first term, case of failure

$=x\frac{e^{2x}}{2D-4}+\frac{\sin 2x}{-4-4D+4}$ first term, again case of failure

$=x^2\frac{e^{2x}}{2}-\frac{1}{4}\cdot\frac{1}{D}\sin 2x$

$=\frac{1}{2}x^2e^{2x}+\frac{1}{8}\cos 2x.$

Hence $y=(C_1+C_2x)\,e^{2x}+\frac{1}{2}x^2e^{2x}+\frac{1}{8}\cos 2x.$

***Ex. 5.** *Solve* $(D^3+1)\,y=e^{2x}\sin x+e^{x/2}\sin(\frac{1}{2}\sqrt{3}x).$

[Lucknow Pass 60]

Solution. A.E. is $D^3+1=0$, $(D+1)(D^2-D+1)=0.$

$D=-1, \frac{1\pm\sqrt{3}i}{2}.$ C.F. $=C_1e^{-x}+C_2e^{x/2}\cos(\frac{1}{2}\sqrt{3}x+C_3).$

P.I. corresponding to $e^{2x}\sin x$

$=\frac{e^{2x}\sin x}{D^3+1}=e^{2x}.\frac{1}{(D+2)^3+1}\sin x$

$=e^{2x}\frac{1}{D^3+6D^2+12D+9}\sin x=e^{2x}.\frac{1}{-D-6+12D+9}\sin x$

$=e^{2x}\frac{1}{11D+3}\sin x=e^{2x}.\frac{11D-3}{-11^2-3^2}\sin x$

$=-\frac{1}{130}e^{2x}(11\cos x-3\sin x).$

P.I. corresponding to $e^{x/2}\sin(\frac{1}{2}\sqrt{3}x)$

$=e^{x/2}\frac{1}{(D+\frac{1}{2})^3+1}\sin\left(\frac{\sqrt{3}}{2}x\right)$

$=e^{x/2}\frac{1}{D^3+\frac{3}{2}D^2+\frac{3}{4}D+\frac{9}{8}}\sin\frac{\sqrt{3}}{2}x$ (case of failure)

$=e^{x/2}x.\frac{1}{3D^2+3D+\frac{3}{4}}\sin\frac{\sqrt{3}}{2}x$ differentiating the denominator w.r.t. D and multiplying by x

$=e^{x/2}x.\frac{1}{-3\times\frac{3}{4}+3D+\frac{3}{4}}\sin\frac{\sqrt{3}}{2}x$

$=e^{x/2}.\frac{x}{3}.\frac{1}{D-\frac{1}{2}}\sin\frac{\sqrt{3}}{2}x$

$=e^{x/2}.\frac{x}{3}.\frac{1}{-\frac{3}{4}-\frac{1}{4}}\left(\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}x+\frac{1}{2}\sin\frac{\sqrt{3}}{2}x\right)$

$$=-\frac{x}{6}e^{x/2}\left(\sqrt{3}\cos\frac{\sqrt{3}}{2}x+\sin\frac{\sqrt{3}}{2}x\right)$$

Thus the P.I. $=-\frac{1}{130}e^{2x}(11\cos x-3\sin x)$

$$-\frac{x}{6}e^{x/2}\left(\sqrt{3}\cos\frac{\sqrt{3}}{2}x+\sin\frac{\sqrt{3}}{2}x\right)$$

The complete solution is $y=$C.F.+P.I.

Ex. 6. *Solve* $(D^4+D^2+1)y=e^{-x/2}\cos\left(x\frac{\sqrt{3}}{2}\right)$

[Luck. Pass 59; Punjab M.A. 56; Raj. M.A. 51]

Solution. The auxiliary equation is $(D^4+D^2+1)=0$

or $[(D^2+1)^2-D^2]=0$ or $(D^2-D+1)(D^2+D+1)=0.$

When $D^2-D+1=0$, $D=\frac{1\pm\sqrt{3}i}{2}$,

and when $D^2+D+1=0$, $D=\frac{-1+3i}{2}$.

$\therefore$ C.F. $=C_1e^{x/2}\cos(\frac{1}{2}\sqrt{3}x+C_2)+C_3e^{-x/2}\cos(\frac{1}{2}\sqrt{3}x+C_4)$

P. I. $=\frac{1}{D^4+D^2+1}e^{-x/2}\cos(\frac{1}{2}\sqrt{3}x)$

$$=e^{-x/2}\frac{1}{(D-\frac{1}{2})^4+(D-\frac{1}{2})^2+1}\cos(\tfrac{1}{2}\sqrt{3}x)$$

$$=e^{-x/2}\frac{1}{D^4-2D^3+\frac{5}{2}D^2-\frac{3}{2}D+\frac{21}{16}}\cos(\tfrac{1}{2}\sqrt{3})\text{ case of failure}$$

$$=e^{-x/2}x.\frac{1}{4D^3-6D^2+5D-\frac{3}{2}}\cos(\tfrac{1}{2}\sqrt{3}x)$$

multiplying by x and differentiating the denominator w.r.t. D

$$=e^{-x/2}x.\frac{1}{4D(-\frac{3}{4})-6(-\frac{3}{4})+5D-\frac{3}{2}}\cos(\tfrac{1}{2}\sqrt{3}x)$$

$$=e^{-x/2}x.\frac{1}{2D+3}\cos(\tfrac{1}{2}\sqrt{3}x)=e^{-x/2}x.\frac{2D-3}{4D^2-9}\cos(\tfrac{1}{2}\sqrt{3}x)$$

$$=e^{-x/2}x.\frac{1}{4(-\frac{3}{4})-9}\left[-2.\frac{\sqrt{3}}{2}\sin(\tfrac{1}{2}\sqrt{3}x)-3\cos(\tfrac{1}{2}\sqrt{3}x)\right]$$

$$=-\frac{e^{-x/2}x}{12}[-\sqrt{3}\sin(\tfrac{1}{2}\sqrt{3}x)-3\cos(\tfrac{1}{2}\sqrt{3}x)]$$

$$=\tfrac{1}{4}e^{-x/2}x.\left[\frac{1}{\sqrt{3}}\sin(\tfrac{1}{2}\sqrt{3}x)+\cos(\tfrac{1}{2}\sqrt{3}x)\right]$$

The complete solution is $y=$C.F.+P.I.

Ex. 7. *Solve* $(D^4+2D^3-3D^2)y=x^2+3e^{2x}+4\sin x.$

[Delhi Hons. 62, 66]

Solution. The auxiliary equation is $(D^4+2D^3-3D^2)=0$

or $\quad D^2(D^2+2D-3)=0$ or $D^2(D+3)(D-1)=0.$

$\therefore \quad D=0, 0, -3, 1.$

Hence $\text{C.F.}=(C_1+C_2x)+C_3e^x+C_4e^{-3x}.$

$$\text{P. I.}=\frac{1}{D^2(D^2+2D-3)}(x^2+3e^{2x}+4\sin x)$$

$$=\frac{3e^{2x}}{20}+\frac{4\sin x}{-2(D-2)}-\frac{1}{3D^2}(1-\tfrac{2}{3}D-\tfrac{1}{3}D^2)^{-1}x^2$$

$$=\tfrac{3}{20}e^{2x}-\frac{2(D-2)}{D^2-4}\sin x-\frac{1}{3D^2}(1+\tfrac{2}{3}D+\tfrac{1}{3}D^2+\tfrac{4}{9}D^2+\ldots)x^2$$

$$=\tfrac{3}{20}e^{2x}+\tfrac{2}{5}(\cos x+2\sin x)-\frac{1}{3D^2}(x^2+\tfrac{4}{3}x+\tfrac{14}{9})$$

$$=\tfrac{3}{20}e^{2x}+\tfrac{2}{5}(\cos x+2\sin x)-(\tfrac{1}{36}x^4+\tfrac{4}{27}x^3+\tfrac{7}{27}x^2).$$

Therefore the complete solution is $y=\text{C. F.}+\text{P. I.}$

Ex. 8. *Solve* $\dfrac{d^2y}{dx^2}-6\dfrac{dy}{dx}+13y=8e^{3x}\sin 2x.$ **[Agra 1954]**

Solution. Auxiliary equation is $D^2-6D+13=0.$

$$D=\frac{6\pm\sqrt{(36-52)}}{2}=3\pm 2i.$$

$\therefore \quad \text{C. F.}=e^{3x}[C_1\cos 2x+C_2\sin 2x].$

$$\text{P. I.}=\frac{1}{(D^2-6D+13)}8e^{3x}\sin 2x$$

$$=8e^{3x}\frac{1}{(D+3)^2-6(D+3)+13}\sin 2x$$

$$=8e^{3x}\frac{1}{D^2+4}\sin 2x \text{ (case of failure)}$$

$$=8e^{3x}.\frac{1}{2D}\sin 2x$$

$$=-2xe^{3x}\cos 2x.$$

Hence the complete solution is

$$y=e^{2x}(C_1\cos 2x+C_2\sin 2x)-2e^{3x}\cos 2x.$$

Ex. 9. (a) *If* $(D+b)^4 y=\cos ax$, *show that the complete solution is* $y=(C_1+C_2x+C_3x^2+C_4x^3)e^{-bx}+\dfrac{\cos[bx-4\tan^{-1}(a/b)]}{(a^2+b^2)^2}.$

[Bombay 1961]

(b) *Solve* $(D^4+2D^3-3D^2)y=x^2+3e^x+4\sin x.$ **[Delhi 1972]**

Ex. 10. *Solve* $\dfrac{d^3y}{dx^3}-2\dfrac{d^2y}{dx^2}-19\dfrac{dy}{dx}+20y=xe^x+2e^{-4x}\sin x.$

[Delhi Hons. 1956]

Solution. Auxiliary equation is $(D^3-2D^2-19D+20=0$

or $(D-1)(D^2-D-20)=0$ or $(D-1)(D-5)(D+4)=0,$

$\therefore \quad D=1, 5, -4,$ $\text{C. F.}=C_1e^x+C_2e^{5x}+C_3e^{-4x}.$

P. I. corresponding to xe^x

$$=\frac{1}{(D-1)(D-5)(D+4)}\,xe^x$$

$$=e^x\,\frac{1}{[(D+1)-1]\,[(D+1)-5]\,[(D+1)+4]}\,x$$

$$=e^x\,\frac{1}{D(D-4)(D+5)}\,x=e^x\,\frac{1}{D}\,\frac{1}{D^2+D-20}\,x$$

$$=-e^x\,\frac{1}{20}\,\frac{1}{D}\,(1-\tfrac{1}{20}D-\tfrac{1}{20}D^2)^{-1}\,x$$

$$=-e^x\,\frac{1}{20}\,\frac{1}{D}\,(1+\tfrac{1}{20}D)\,x=-e^x\,\frac{1}{20}\,\frac{1}{D}\,(x+\tfrac{1}{20})$$

$$=-\tfrac{1}{20}e^x\left(\frac{x^2}{2}+\tfrac{1}{20}x\right).$$

P. I. corresponding to $2e^{-4x}\sin x$

$$=\text{Imaginary part of }\frac{2e^{-4x}e^{ix}}{D^3-2D^2-19D+20}$$

$$=\text{Imaginary part of }\frac{2e^{x(i-4)}}{(i-4)^3-2(i-4)^2-19(i-4)+20}$$

$$=\text{Imaginary part of }\frac{e^{x(i-4)}}{1+22i}$$

$$=\text{Imaginary part of }\frac{e^{-4x}(\cos x+i\sin x)}{1^2+22^2}\,(1-22i)$$

$$=\frac{e^{-4x}}{485}\,(\sin x-22\cos x).$$

Hence P.I. $=-\tfrac{1}{20}e^x(\tfrac{1}{2}x^2+\tfrac{1}{20}x)=\tfrac{1}{485}e^{-4x}(\sin x-22\cos x)$.

$\therefore$ Complete solution is $y=$ C.F.$+$P.I.

Ex. 11. *Show that* $\dfrac{1}{f(D)}[e^{ax}\cos bx]=e^{ax}\dfrac{1}{f(D+a)}\cos bx.$

[Bombay 1961]

Just the article 5·19 P. 76. $V=\cos bx$.

Ex. 12. *Solve* $(D-1)^2(D^2+1)^2\,y=\sin^2\tfrac{1}{2}x+e^x+x.$

[Indore 1963; Punjab 65; Raj 61]

Solution. A.E. is $(D-1)^2(D^2+1)^2=0$, $D=1, 1, \pm i, \pm i$.

$\therefore$ C.F.$=(C_1+C_2x)e^x+(C_3+C_4x)\cos x+(C_5+C_6x)\sin x$.

Now $\sin^2\tfrac{1}{2}x+e^x+x=\tfrac{1}{2}(1-\cos x)+e^x+x$

$$=(\tfrac{1}{2}+x)-\tfrac{1}{2}\cos x+e^x.$$

P.I. corresponding to $(\tfrac{1}{2}+x)=\tfrac{5}{2}+x$ (evaluate it).

P.I. corresponding to $e^x=\tfrac{1}{8}x^2e^x$.

P.I. corresponding to $(-\tfrac{1}{2}\cos x)=-\tfrac{1}{32}x^2\sin x$.

$\therefore$ Total P.I. $=\tfrac{5}{2}+x+\tfrac{1}{8}x^2e^x-\tfrac{1}{32}x^2\sin x$.

The complete solution is $y=$ C.F.$+$P.I.

Ex. 13. *Solve* $(D^4-D+1)\, y=e^x+\cos(\tfrac{1}{2}\sqrt{3}x)+x.$

[Lucknow 1954]

Proceed as in the above example.

Ex. 14. *Solve* $(D^2-1)\, y=x\sin x+(1+x^2)\, e^x.$

[Poona 1964; Rajasthan 61; Lucknow 52]

Solution. A.E. is $D^2-1=0$, $D=\pm 1$, C.F. $=C_1e^x+C_2e^{-x}$.

P.I. corresponding to $x\sin x=\dfrac{x\sin x}{D^2-1}$

$$=x\left[\frac{1}{D^2-1}\right]\sin x-\frac{2D}{(D^2-1)^2}\sin x \qquad \text{[see § 5·20 P. 79]}$$

$$=x\,\frac{1}{-1-1}\sin x-\frac{2\cos x}{(-1-1)^2}=-\tfrac{1}{2}(x\sin x+\cos x).$$

P.I. corresponding to $(1+x^2)\, e^x=\dfrac{(1+x^2)\, e^x}{D^2-1}$

$$=e^x\,\frac{1}{(D+1)^2-1}(1+x^2)=e^x\,\frac{1}{D^2+2D}(1+x^2)$$

$$=e^x\,\frac{1}{2D}(1+\tfrac{1}{2}D)^{-1}(1+x^2)$$

$$=e^x\,\frac{1}{2D}(1-\tfrac{1}{2}D+\tfrac{1}{4}D^2\ldots)(1+x^2)$$

$$=e^x\,\frac{1}{2D}(1+x^2-x+\tfrac{1}{2})=e^x(\tfrac{3}{4}x-\tfrac{1}{4}x^2+\tfrac{1}{6}x^3)$$

$$=\tfrac{1}{12}e^x(9x-3x^2+2x^3).$$

The complete solution is $y=$ C.F. $+$ P.I.

Ex. 15. *Solve* $\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}+2y=4x-20\cos 2x$ [Poona 1959]

Solution. A.E. is $D^2+3D+2=0$, $(D+2)(D+1)=0$.

C.F. $=C_1e^{-x}+C_2e^{-2x}$.

P.I. corresponding to $-20\cos 2x$

$$=-\frac{20\cos 2x}{D^2+3D+2}=\frac{-20\cos 2x}{-4+3D+2}=\frac{20\cos 2x}{3D-2}$$

$$=-\frac{20(3D+2)\cos 2x}{9D^2-4}=-\frac{20(-6\sin 2x+2\cos 2x)}{-36-4}$$

$$=(\cos 2x-3\sin 2x).$$

P.I. corresponding to $4x$

$$=\tfrac{1}{2}(1+\tfrac{3}{2}D+\tfrac{1}{2}D^2)^{-1}\, 4x$$

$$=\tfrac{1}{2}(1-\tfrac{3}{2}D)\, 4x=(2x-3).$$

Hence the general solution is

$$y=C_1e^{-x}+C_2e^{-2x}+(\cos 2x-3\sin 2x)+(2x-3).$$

Ex. 16. *Solve* $(D^2y+3Dy+2y)=x^2\cos x.$ [Poona B.A 60]

Solution. C.F. $=C_1e^{-x}+C_2e^{-2x}$ as above.

$$\text{P.I.}=\frac{1}{D^2+3D+2}x^2\cos x \text{ etc.}$$

Now proceed as in Ex. 4 P. 82.

Ex. 17. *Solve* $(D^3-D^2+3D+5)\,y=x^2+e^x\cos 2x.$

[Raj. B.Sc. 60]

Solution. Auxiliary equation is $D^3-D^2+3D+5=0$

or $(D+1)(D^2-2D+5)=0$ or $(D+1)[(D-1)^2+4]=0.$

$$\therefore\quad D=-1,\quad 1\pm 2i.$$

$$\therefore\quad \text{C.F.}=C_1e^{-x}+e^x(C_2\cos 2x+C_3\sin 2x).$$

P.I. corresponding to $x^2=\frac{1}{125}(25x^2-30x+28).$

P.I. corresponding to $e^x\cos 2x=\frac{1}{16}xe^x(\sin 2x-\cos 2x).$

Ex. 18. *Solve* $\frac{d^4y}{dx^4}-2\frac{d^3y}{dx^3}+2\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=\cos x+\cosh x.$

[Bombay 61]

Solution. A.E. is $D^4-2D^3+2D^2-2D+1=0,$

i.e., $(D^2+1)-2D(D^2+1)=0$ or $(D^2+1)(D^2-2D+1)=0$

or $(D^2+1)(D-1)^2=0,\ D=\pm i,\ 1,\ 1.$

$$\therefore\quad \text{C.F.}=C_1\cos(x+C_2)+(C_3+C_4x)\,e^x$$

P.I. corresponding to $\cos x$

$$=\frac{\cos x}{D^4-2D^3+2D^2-2D+1}\ \text{(case of failure)}$$

$$=x.\frac{\cos x}{4D^3-6D^2+4D-2}=x.\frac{\cos x}{-4D+6D+4-2}$$

$$=\frac{x\cos x}{4}$$

P.I. corresponding to $\cosh x$, *i.e.* $\cos(ix)$

$$=\frac{\cos ix}{D^4-2D^3+2D^2-2D+1}=\frac{\cos ix}{1-2D+2-2D-1}$$

putting $D^2=-i^2=1$

$$=\frac{1}{4}\frac{\cos ix}{1-D}=\frac{1}{4}\frac{(1+D)\cos ix}{1-D^2}\ \text{(case of failure)}$$

$$=\frac{1}{4}x\frac{\cos ix-i\sin ix}{-2D}$$

$$=-\frac{1}{8}x\left(\frac{1}{i}\sin ix+\cos ix\right)$$

$$=\tfrac{1}{8}x\,(i\sin ix-\cos ix)=\tfrac{1}{8}x\,(\sinh x-\cosh x).$$

Hence etc.

Ex. 19. *Solve* $(D^5-D^4+2D^3-2D^2+D-1)\,y=\cos x.$

[Rajasthan 60]

Solution. Auxiliary equation is

$(D-1)(D^2+1)^2=0\quad D=1,\ \pm i,\ \pm i.$

$\therefore \quad$ C.F. $= C_1e^x + (C_2 + C_3x) \cos x + (C_4 + C_5x) \sin x$,

P.I. $= \dfrac{1}{D^5 - D^4 + 2D^3 - 2D^2 + D - 1} \cos x$ (case of failure)

$= x \dfrac{1}{5D^4 - 4D^3 + 6D^2 - 4D + 4} \cos x$

differentiating denominator w.r.t. D and multiplying by x (again a case of failure)

$= x^2 \dfrac{1}{20D^3 - 12D^2 + 12D - 4} \cos x$

$= x^2 \dfrac{1}{-20D + 12 + 12D - 4} \cos x = \dfrac{1}{8} x^2 \dfrac{1}{1 - D} \cos x$

putting $D^2 = -1$

$= \dfrac{1}{8} x^2 \dfrac{1 + D}{1 - D^2} \cos x = \dfrac{1}{16} x^2 (\cos x - \sin x)$.

Hence $y =$ C.F. + P.I. is complete solution.

Ex. 20. $\dfrac{d^4y}{dx^4} + \dfrac{d^2y}{dx^2} + y = ax^2 + be^{-x} \sin 2x$.

[Allahabad 1966; Delhi Hons. 60, 53]

Solution. A.E. is $D^4 + D^2 + 1 = 0$

or $\quad (D^2 - D + 1)(D^2 + D + 1) = 0$

or $\quad D = \dfrac{1 \pm \sqrt{3}i}{2}, \; D = \dfrac{-1 \pm \sqrt{3}i}{2}$.

C.F. $= C_1e^{x/2} \cos \left(\dfrac{\sqrt{3}}{2} x + C_2 \right) + C_3e^{-x/2} \cos \left(\dfrac{\sqrt{3}}{2} x + C_4 \right)$

Now P.I. corresponding to ax^2

$= \dfrac{1}{D^4 + D^2 + 1} ax^2 = (1 + D^2 + D^4)^{-1} (ax^2)$

$= (1 - D^2 \ldots) ax^2 = (ax^2 - 2a)$

and P.I. corresponding to $be^{-x} \sin 2x$

$= \dfrac{1}{D^4 + D^2 + 1} be^{-x} \sin 2x$

$=$ imaginary part of $\dfrac{1}{D^4 + D^2 + 1} be^{-x} . e^{2ix}$

$=$ I.P. of $\dfrac{1}{D^4 + D^2 + 1} be^{-x(1-2i)}$

$=$ I.P. of $b \dfrac{e^{-x(1-2i)}}{(1 - 2i)^4 + (1 - 2i)^2 + 1}$

$=$ I.P. of $\dfrac{be^{-x(1-2i)}}{-9 + 20i}$

$=$ I.P. of $-\dfrac{be^{-x(1-2i)}(9 + 20i)}{20^2 + 9^2}$

$$=\text{I. P. of } -\frac{be^{-x}(\cos 2x+i\sin 2x)}{481}(9+20i)$$

$$=-\frac{be^{-x}}{481}(9\sin 2x+20\cos 2x).$$

$$\therefore \quad \text{P.I.}=ax^2-2a-\frac{be^{-x}}{481}(9\sin 2x+20\cos 2x).$$

Hence the complete solution is $y=\text{C.F.}+\text{P.I.}$

***Ex 21.** *Solve* $\dfrac{d^2y}{dx^2}+a^2y=\sec ax.$

[Vikram 1964; Luck. 56; Osmania 62; Calcutta Hons. 62]

Solution. A.E. is $D^2+a^2=0$, *i e.*, $D=\pm ai$

$\therefore$ C.F. $=(C_1\cos ax+C_2\sin ax)$.

Now

$$\text{P.I.}=\frac{1}{(D^2+a^2)}\sec ax=\frac{1}{(D+ai)(D-ai)}\sec ax$$

$$=\frac{1}{2ai}\left[\frac{1}{D-ai}-\frac{1}{D+ai}\right]e^{aix}.\frac{e^{-aix}}{\cos ax}$$

$$=\frac{e^{aix}}{2ai}\left[\frac{1}{D+ai-ai}\right]\frac{e^{-aix}}{\cos ax}-\frac{e^{-aix}}{2ai}\frac{1}{D-ai+ai}\frac{e^{aix}}{\cos ax}$$

$$=\frac{e^{aix}}{2ai}\frac{1}{D}\frac{e^{-aix}}{\cos ax}-\frac{e^{-aix}}{2ai}\frac{1}{D}\frac{e^{aix}}{\cos ax}$$

$$=\frac{e^{aix}}{2ai}\int\frac{\cos ax-i\sin ax}{\cos ax}dx-\frac{e^{-aix}}{2ai}\int\frac{\cos ax+i\sin ax}{\cos ax}dx$$

as $\dfrac{1}{D}$ stands for integration

$$=\frac{e^{aix}}{2ai}\int(1-i\tan ax)\,dx-\frac{e^{-aix}}{2ai}\int(1+i\tan ax)\,dx$$

$$=\frac{e^{aix}}{2ai}\left[x+\frac{i}{a}\log\cos ax\right]-\frac{e^{-aix}}{2ai}\left[x-\frac{i}{a}\log\cos ax\right]$$

$$=\frac{1}{2ai}[x(e^{aix}-e^{-aix})]+\frac{1}{2a^2}\log\cos ax.(e^{aix}+e^{-aix})$$

$$=\frac{x}{a}\sin ax+\frac{1}{a^2}\log(\cos ax)\cos ax.$$

Hence the complete solution is

$$y=(C_1\cos ax+C_2\sin ax)+\frac{x}{a}\sin ax+\frac{1}{a^2}\log(\cos ax)\cos ax.$$

Ex. 22. *Solve* $\dfrac{d^2x}{dt^2}+2b\dfrac{dx}{dt}+k^2x=0$, $k>b>0$ *having been given that when* $t=0$, $x=0$, $\dfrac{dx}{dt}=u_0$.

[Saugar 63]

Solution. Simple.

Ex. 23. *Solve* $(D^2-9D+18)\ y=e^{e^{-3x}}$.

Solution. C.F. $=C_1e^{3x}+C_2xe^{6x}$.

$$\text{P.I.}=\frac{1}{(D-6)(D-3)}\ e^{e^{-3x}}$$

$$=\frac{1}{D-6}\left(e^{3x}\int e^{e^{-3x}}\cdot e^{-3x}\ dx\right)$$

$$=e^{6x}\int e^{-3x}\left(-\tfrac{1}{3}e^{e^{-3x}}\right)dx$$

$$=\tfrac{1}{3}e^{6x}\int e^{e^{-3x}}\ (-e^{-3x})\ dx$$

$$=\tfrac{1}{9}\ e^{e^{-3x}}\cdot e^{6x}.$$

The general solution is $y=$C.F.$+$P.I.

Ex. 24. *What do you understand by linearly independent solutions ?*

Show that e^{ax}, xe^{ax} *and* x^2e^{ax} *are solutions of*

$$\frac{d^3y}{dx^3}-3a\ \frac{d^2y}{dx^2}+3a^2\ \frac{dy}{dx}-a^3y=0.$$

Hence find the general solution of this equation.

[Cal. Hons. 63]

Hint. See 5·6 Case II p. 61.

Ex. 25. *Solve*

(i) $(D^2-5D+6)\ y=e^{2x}.\ x^2$. [Poona 62]

Ans. $y=C_1e^{2x}+C_2e^{3x}-\frac{1}{4}e^{2x}\ [x^4+4x^3+12x^2+24x]$

(ii) $(D^3-3D^2-6D+8)\ y=x$. [Poona 64]

Ans. $y=C_1e^x+C_2e^{4x}+C_3e^{-2x}+\frac{1}{8}\ (x+\frac{3}{4})$

(iii) $(D^4-1)\ y=e^x\cos x$. [Vikram 63]

Ans. $y=C_1e^x+C_2e^{-x}+C_3\cos(x+C_4)-\frac{1}{5}e^x\cos x$.

(iv) $(D^2-4D+3)\ y=3e^x\cos 2x$. [Poona 63]

Ans $y=C_1e^x+C_2e^{3x}+\frac{3}{8}e^x\ (\sin 2x-\cos 2x)$.

(v) $(D^4-2D^2+1)\ y=e^x+\sin 2x$. [Karnatak 63]

Ans. $y=(C_1+C_2x)\ e^x+(C_3+C_4x)\ e^{-x}+\frac{1}{25}\sin 2x+\frac{1}{8}x^2e^x$.

(vi) $(D^2-4D-4)\ y=8\ (x^2+e^{2x}+\sin 2x$. [Nagpur 63]

Ans. $y=(C_1+C_2x)\ e^{2x}-\cos 2x+4x^2e^{2x}+2\ (x^2+2x+\frac{3}{2})$.

(vii) $(D^2+2D+1)\ y=x\ \text{cosec}\ x$. [Nagpur 62]

Solution. A.E. is $(D+1)^2=0$.

$$\text{P.I.}=\frac{1}{(D+1)^2}\ [x\ \text{cosec}\ x]$$

$$=\frac{1}{(D+1)^2}\text{cosec}\ x-\frac{2}{(D+1)^3}\ (\text{cosec}\ x)$$ by 5·20 p. 79.

Now evaluate it as in Ex. 21 p. 92

(viii) $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=x^2e^{3x}$. [Delhi Hons. 65]

(ix) $(D^2+4D-12)\,y=(x-1)\,e^{2x}$. [Alld. 65]

(x) $(D^5-D)\,y=12e^x+8\sin x-2x$. [Alld. 65]

(xi) $(D^4+D^3+D^2-D-2)\,y=x^2+e^x$. [Vikram 65]

(xii) $(D^4-2D^3+D^2)\,y=x^3$. [Agra 67]

(xiii) $(D^2-2D+1)\,y=x^2e^x$. [Agra 67, 75]

6

Homogeneous Linear Equations

6·1. Homogeneous Linear Equations.

An equation of the form

$$x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + P_n y = X, \qquad \ldots(1)$$

where $P_1, P_2, \ldots, P_n$ are constants and X is a function of x, is called the *Homogeneous Linear Equation.*

Important Substitution. If we put

$$x = e^z \quad \text{or} \quad z = \log x,$$

the equation (1) is transformed into an equation with constant coefficients changing the independent variable from x to z.

Thus if $x = e^z$ or $z = \log x$, $\dfrac{dz}{dx} = \dfrac{1}{x}$. $\qquad \ldots(2)$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz} \quad \text{or} \quad x \frac{dy}{dz} = \frac{dy}{dz}. \qquad \ldots(3)$$

$$\text{Again } \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = \frac{x \cdot \frac{d^2y}{dz^2} \cdot \frac{dz}{dx} - \frac{dy}{dz}}{x^2} = \frac{x \cdot \frac{d^2y}{dz^2} \cdot \frac{1}{x} - \frac{dy}{dz}}{x^2}$$

or
$$x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dx^2} - \frac{dy}{dz}.$$

Also
$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left[\frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)\right]$$

$$= \frac{x^2\left(\frac{d^3y}{dz^3}\frac{dz}{dx} - \frac{d^2y}{dz^2}\frac{dz}{dx}\right) - 2x\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)}{x^4} \text{ but } \frac{dz}{dx} = \frac{1}{x}.$$

or
$$x^3 \frac{d^3y}{dx^3} = \frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}. \qquad \ldots(5)$$

Thus if we put $x\dfrac{d}{dx} = \dfrac{d}{dz} = D$, (3), (4), (5) etc. can be put as

$$x \frac{dy}{dx} = Dy,$$

$$x^2 \frac{d^2y}{dx^2} = D(D-1)\,y,$$

$$x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)\,y.$$

and
$$x^n \frac{d^ny}{d^nx} = D(D-1)(D-2)\ldots(D-n+1)\,y.$$

[Bombay 61, Poona 60]

Making this substitution, the equation (1) becomes

$$[\{D(D-1)\ldots(D-n+1)\}+P_1\{D(D-1)\ldots(D-n+2)\}$$
$$+\ldots P_{n-1}D+P_n]\,y=Z$$

or $f(D)\,y=Z$,

where Z is the function of z into which X is changed.

This is now a linear differential equation with constant coefficients and can be solved by the methods of previous chapter.

Ex. 1. *Solve* $x^2 \frac{d^2y}{dx^2}+y=3x^2$.

Solution. Putting $x=e^z$ and $D\equiv\frac{d}{dz}$, the equation becomes

$D(D-1)\,y+y=3e^{2z}$ or $(D^2-D+1)=3e^{2z}$.

The auxiliary equation is $D^2-D+1=0$.

$$D=\frac{1\pm\sqrt{(1-4)}}{2}=\frac{1\pm\sqrt{3}i}{2}.$$

$\therefore$ C.F. $=C_1e^{z/2}\cos(\frac{1}{2}\sqrt{3}z+C_2)$.

Also P.I. $=\frac{3e^{2z}}{D^2-D+1}=\frac{3e^{2z}}{2^2-2+1}=e^{2z}$.

Therefore the solution is

$y=C_1e^{z/2}\cos(\frac{1}{2}\sqrt{3}z+C_2)+e^{2z}$

or $y=C_1x^{1/2}\cos(\frac{1}{2}\sqrt{3}\log x+C_2)+x^2$ as $e^z=x$ or $z=\log x$.

Ex. 2. *Solve* $x^2 \frac{d^2y}{dx^2}-2y=x^2+\frac{1}{x}$. [Sagar 62; Raj 51]

Solution. Putting $x=e^z$, $D\equiv\frac{d}{dz}$, the equation becomes

$[D(D-1)-2]\,y=e^{2z}+e^{-z}$

A.E. is $D^2-D-2=0$, $(D-2)(D+1)=0$, $D=2, -1$.

$\therefore$ C.F. $=C_1e^{2z}+C_2e^{-z}=C_1x^2+C_2\frac{1}{x}$ as $e^z=x$

P.I. $=\frac{1}{(D-2)(D+1)}[e^{2z}+e^{-z}]$ (case of failure for both)

$=z\frac{1}{2D-1}[e^{2z}+e^{-z}]$ multiplying by z and differentiaiing the the denominator w.r.t. D

$$=z\frac{e^{2z}}{2.2-1}+\frac{e^{-z}}{2(-1)-1}=z\left[\frac{e^{2z}}{3}-\frac{e^{-z}}{3}\right]$$

$=\frac{1}{3}(\log x)\left(x^2-\frac{1}{x}\right)$ as $x=e^z$ or $\log x=z$.

Hence the complete solution is

$$y=C_1x^2+C_2\frac{1}{x}+\frac{1}{3}\log x\left(x^2-\frac{1}{x}\right).$$

Ex. 3. *Solve* $x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-4y=x^4.$

[Vikram 1964: Agra 58, 48, 76; Allahabad 55; Raj. 58]

Solution. Putting $x=e^z$ and $D\equiv\frac{d}{dz}$, the equation becomes

$$[D(D-1)-2D-4]\,y=e^{4z} \quad\text{or}\quad (D^2-3D-4)\,y=e^{4z}.$$

Auxiliary equation is $D^2-3D-4=0$ or $(D-4)(D+1)=0$.

$\therefore$ C.F. $=C_1e^{4z}+C_2e^{-z}=C_1x^4+C_2/x$ as $e^z=x$.

And P. I. $=\frac{e^{4z}}{D^2-3D-4}=z\,\frac{e^{4z}}{2D-3}$ differentiating denominator w.r.t. D and multiplying by z

$$=z\,\frac{e^{4z}}{2.4-3}=\frac{ze^{4z}}{5}=\tfrac{1}{5}(\log x)\,x^4 \text{ as } x=e^z.$$

Therefore the required equation is

$$y=C_1x^4+C_2/x+\tfrac{1}{5}x^4\log x.$$

Ex. 4. (a) *Solve* $x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+4y=2x^2.$ **[Delhi 1963, 59]**

Solution. Putting $x=e^z$ and $D\equiv\frac{d}{dz}$, the equation becomes

$$[D(D-1)-3D+4]\,y=2e^{2z} \quad\text{or}\quad (D^2-4D+4)\,y=2e^{2z}.$$

A.E. is $(D^2-4D+4)=0$ or $(D-2)^2=0$, $D=2, 2$.

$\therefore$ C.F. $=(C_1+C_2z)\,e^{2z}=(C_1+C_2\log x)\,x^2$ as $x=e^z$.

Again P.I. $=\frac{1}{(D-2)^2}\,2e^{2z}$ (case of failure)

$=z\,\frac{1}{2(D-2)}\,2e^{2z}$ differentiating denominator w.r.t. D and multiplying by z (case of failure again)

$=z^2.\frac{1}{2}2e^{2z}$ differentiating denominator again w.r.t. D and multiplying by z

$=z^2e^{2z}=(\log x)^2\,x^2.$

Hence the complete solution is

$$y=(C_1+C_2\log x)\,x^2+(\log x)^2\,x^2$$

***Ex. 4. (b)** *Solve* $(x^2D^2-3xD+4)\,y=x^m.$

[Gujrat 1959 ; Bombay 58]

Solution. As in the above example,

C.P.$=(C_1+C_2 \log x)\, x^2$.

$$\text{P.I.}=\frac{1}{(D'-2)^2}\, e^{mz}=\frac{1}{(m-2)^2} e^{mz}, \text{ where } D'\equiv\frac{d}{dz}, m\neq 2$$

$$=\frac{1}{(m-2)^2}\, x^m \text{ as } e^z=x.$$

Hence $y=(C_1+C_2 \log x)\, x^2+\dfrac{1}{(m-2)^2}\, x^m$.

Ex. 4. (c) *Solve* $(x^2D^2-3xD+4)\, y=(x-1)^2$.

Solution. C.F.$=(C_1+C_2 \log x)\, x^2$.

$$\text{P.I.}=\frac{1}{(D'-2)^2}\,(e^z-1)^2=\frac{1}{(D'-2)^2}\,[e^{2z}-2e^z+1]$$

$$=z^2\frac{e^{2z}}{2}-2e^z+\tfrac{1}{4}^* \text{ (first term was a case of failure)}$$

$$=\tfrac{1}{2}x^2 (\log x)^2-2x+\tfrac{1}{4}.$$

Hence $y=(C_1+C_2 \log x)\, x^2+\tfrac{1}{2}x^2 (\log x)^2-2x+\tfrac{1}{4}$.

Ex. 5. *Solve* $x^2\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}-20y=(x+1)^2$.

[Delhi Hon's. 1959]

Solution. Putting $x=e^z$, $D\equiv\dfrac{d}{dz}$, the equation becomes

$$[D(D-1)+2D-20]\, y=(e^z+1)^2=e^{2z}+2e^z+1.$$

A.E. is $D^2+D-20=0$, $(D+5)(D-4)=0$.

C.F.$=C_1e^{4z}+C_2e^{-5z}=C_1x^4+C_2x^{-5}$ as $e^z=x$,

$$\text{P.I.}=\frac{1}{D^2+D-20}\,(e^{2z}+2e^z+1)$$

$$=\frac{e^{2z}}{4+2-20}+\frac{2e^z}{1+1-20}+\frac{1}{0+0-20}$$

$$=-\tfrac{1}{14}e^{2z}-\tfrac{1}{9}e^z-\tfrac{1}{20}=-\tfrac{1}{14}x^2-\tfrac{1}{9}x-\tfrac{1}{20}.$$

Hence the complete solution is

$$y=C_1x^4+C_2x^{-5}-\tfrac{1}{14}x^2-\tfrac{1}{9}x-\tfrac{1}{20}.$$

Ex. 6. (a) *Solve* $x^3\dfrac{d^3y}{dx^3}-x^2\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}-2y=x^3+3x$.

[Karnatak 1963; Raj' 55]

Solution. Putting $x=e^z$ and $D\equiv\dfrac{d}{dz}$, the equation becomes

$$[D(D-1)(D-2)-D(D-1)+2D-2]\, y=e^{3z}+3e^z$$

or $[D^3-4D^2+5D-2]\, y=e^{3z}+3e^z$

A.E. is $D^3-4D^2+5D-2=0$, *i.e.* $(D-1)^2(D-2)=0$

or $D=1, 1, 2$.

$\therefore$ C.F.$=(C_1+C_2z)\, e^z+C_3e^{2z}$

$=(C_1+C_2 \log x)\, x+C_3x^2$ as $x=e^z$, *i.e.* $z=\log x$.

---

* In case of constant 1, we may write it as $e^{0.z}$. Thus

$$\frac{1}{(D'-2)^2}=\frac{e^{0.z}}{(D'-2)^2}=\frac{e^{0.z}}{(0-2)^2}=\tfrac{1}{4}.$$

Also P.I. $= \dfrac{e^{3z}}{(D-1)^2 (D-2)} + 3\dfrac{e^z}{(D-1)^2 (D-2)}$

$= \dfrac{e^{3z}}{(3-1)^2 \ (3-2)} + 3z^2 \dfrac{e^z}{6D-8}$ multiplying the second term by z^2 and differentiating the denominator twice w.r.t. D

$= \frac{1}{4}e^{3z} - \frac{3}{2}z^2 e^z = \frac{1}{4}x^3 - \frac{3}{2} (\log x)^2 x.$

Therefore the general solution is

$$y = (C_1 + C_2 \log x)\, x + C_3 x^2 + \tfrac{1}{4}x^3 - \tfrac{3}{2} (\log x)^2 x.$$

Ex 6. (b) $x^3 \dfrac{d^3y}{dx^3} - x^2 \dfrac{d^2y}{dx^2} + 2x \dfrac{dy}{dx} - 2y = x^2.$ **[Karnatak 63]**

As above.

Ex. 7 *Solve* $x^4 \dfrac{d^3y}{dx^3} + 2x^3 \dfrac{d^2y}{dx^2} - x^2 \dfrac{dy}{dx} + xy = 1.$

[Agra 77, 72, 55; Bombay 61]

Solution. Dividing by x, the equation can be written as

$$x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = \frac{1}{x}.$$

Now putting $x = e^z$ and $D \equiv d/dz$, this becomes

$$[D (D-1)(D-2) + 2D (D-1) - D + 1]\, y = e^{-z}.$$

The A.E. is $D^3 - D^2 - D + 1 = 0$ or $(D-1)^2 (D+1) = 0.$

$\therefore$ C.F. $= (C_1 + C_2 z)\, e^z + C_3 e^{-z}$

$= (C_1 + C_2 \log x)\, x + C_3 x^{-1}$ as $x = e^z$

and P.I. $= \dfrac{e^{-z}}{(D-1)^2 (D+1)}$ (case of failure)

$= z \dfrac{e^{-z}}{3D^2 - 2D - 1}$ multiplying by z and differentiating the denominator w.r.t. D

$= z \dfrac{e^{-z}}{3(-1)^2 - 2(-1) - 1} = \frac{1}{4} z e^{-z} = \frac{1}{4} (\log x) . \dfrac{1}{x}.$

Hence the complete solution is

$$y = (C_1 + C_2 \log x)\, x + C_3 x^{-1} + \tfrac{1}{4} (\log x) . 1/x.$$

Ex. 8. *Solve* $x^3 \dfrac{d^3y}{dx^3} + 6x^2 \dfrac{d^2y}{dx^2} + 8x \dfrac{dy}{dx} + 2y = x^2 + 3x - 4.$

[Nagpur 63

Solution. Putting $x = e^z$ and $D \equiv d/dz$, the equation is

$$[D (D-1)(D-2) + 6D (D-1) + 8D + 2]\, y = e^{2z} + 3e^z - 4.$$

The A.E. is $(D^3 + 3D^2 + 4D + 2) = 0$

or $(D+1)(D^2 + 2D + 2) = 0$

or $D = -1, \dfrac{-2 \pm \sqrt{(4-8)}}{2}$, *i.e.* $D = -1, -1 \pm i$

$\therefore$ C.F. $=C_1e^{-z}+C_2e^{-z}\cos(z+C_3)$.

$$\text{P.I.}=\frac{e^{2z}+3e^z-4}{D^3+3D^2+4D+2}$$

$$=\frac{e^{2z}}{2^3+3.2^2+4.2+2}+\frac{3e^z}{1^3+3.1^2+4.1+2}-\frac{4}{0+0+0+2}$$

$$=\frac{e^{2z}}{30}+\frac{3e^z}{10}-2=\frac{x^2}{30}+\frac{3}{10}x-2.$$

$\therefore$ **the complete solution is**

$$y=C_1x^{-1}+C_2x^{-1}\cos(\log x+C_3)+\tfrac{1}{30}x^2+\tfrac{3}{10}x-2.$$

***Ex. 9.** *Solve* $x^2\dfrac{d^3y}{dx^3}+2x^2\dfrac{d^2y}{dx^2}+2y=10\left(x+\dfrac{1}{x}\right)$.

[Agra 78, 69, 67, 52; Pb. 62; Delhi Hons. 60, 58]

Solution. Putting $x=e^z$, $D\equiv d/dz$, the equation becomes

$$[D(D-1)(D-2)+2D(D-1)+2]y=10(e^z+e^{-z}).$$

The A.E. is $D^3-D^2+2=0$, *i.e.* $(D^2-2D+2)=0$

or $D=-1,\ \dfrac{2\pm\sqrt{(4-8)}}{2}$ *i.e.* $D=-1,\ 1\pm i$

$$\therefore\ \text{C.F.}=C_1e^{-z}+C_2e^z\cos(z+C_3)$$
$$=C_1x^{-1}+C_2x\cos(\log x+C_3)$$

$$\text{P.I.}=\frac{10e^z}{(D+1)(D^2-2D+2)}+\frac{10e^{-z}}{(D+1)(D^2-2D+2)}$$

second term a case of failure

$$=\frac{10e^z}{(1+1)(1^2-2.1+2)}+z\,\frac{10e^{-z}}{3D^2-2D}$$

multiplying second term by z and differentiating its denominator w.r.t. D

$$=5e^z+z.\frac{10e^{-z}}{3.(-1)^2-2(-1)}=5e^z+z.2e^{-z}$$

$$=\left(5x+2\log x\,\frac{1}{x}\right)\text{ as } x=e^z,\ z=\log x.$$

Hence the complete solution is

$$y=C_1x^{-1}+C_2x\cos(\log x+C_3)+5x+2\log x.(1/x).$$

Ex. 10. *Solve* $x^2\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}-3y=x^2\log x$. **[Agra 73, 68; Raj. 61]**

Solution. Putting $x=e^z$, $D\equiv d/dz$, the equation becomes

$$[D(D-1)-D-3]y=ze^{2z}.$$

Auxiliary equation is $(D^2-2D-3)=0$, $(D-3)(D+1)=0$.

C.F. $=C_1e^{3z}+C_2e^{-z}=C_1x^3+C_2x^{-1}$

$$\text{P.I.}=\frac{ze^{2z}}{D^2-2D-3}=e^{2z}\frac{1}{(D+2)^2-2(D+2)-3}z$$ (§ 5·19 P. 76)

$$=e^{2z}\frac{1}{D^2+2D-3}z=-\frac{e^{2z}}{3}[1-\tfrac{2}{3}D-\tfrac{1}{3}D^2]^{-1}z$$
$$=-\tfrac{1}{3}e^{2z}(1+\tfrac{2}{3}D+\dots)z=-\tfrac{1}{3}e^{2z}(z+\tfrac{2}{3})$$
$$=-\tfrac{1}{3}x^2(\log x+\tfrac{2}{3}) \text{ as } e^z=x.$$

Hence the complete solution is

$$y=C_1x^3+C_2x^{-1}-\tfrac{1}{3}x^2(\log x+\tfrac{2}{3}).$$

***Ex. 11. (a)** $(x^2D^2+3xD+1)\,y=\dfrac{1}{(1-x)^2}$.

[Agra 70, 66, 57; Raj. 52]

Solution. Putting $x=e^z$ and $D'\equiv d/dz$, the equation is

$$[D'(D'-1)+3D'+1]\,y=\frac{1}{(1-e^z)^2}.$$

The A.E. is $D'^2+2D'+1=0$. *i.e.* $(D'+1)^2=0$.

$$\therefore \quad \text{C.F.}=(C_1+C_2z)\,e^{-z}=(C_1+C_2\log x)\frac{1}{x}.$$

$$\text{P.I.}=\frac{1}{(D'+1)^2}\frac{1}{(1-e^z)^2}=\frac{1}{(D'+1)(D'+1)}\cdot\frac{1}{(1-e^z)^2}.$$

Let $\dfrac{1}{(D'+1)}\cdot\dfrac{1}{(1-e^z)^2}=u$, *i.e.* $(D'+1)\,u=\dfrac{1}{(1-e^z)^2}$

or $\dfrac{du}{dz}+u=\dfrac{1}{(1-e^z)^2}$, linear equation, I.F.$=e^z$.

$$\therefore \quad ue^z=\int\frac{e^z}{(1-e^z)^2}\,dz=\frac{1}{(1-e^z)}$$

or $u=\dfrac{e^{-z}}{(1-e^z)}$.

$$\therefore \quad \text{P.I.}=\frac{1}{(D'+1)}\cdot\frac{1}{D'+1}\cdot\frac{1}{(1-e^z)^2}=\frac{1}{D'+1}\cdot u$$
$$=\frac{1}{(D'+1)}\frac{e^{-z}}{(1-e^z)}=v \text{ (say)}.$$

Then $(D'+1)\,v=\dfrac{e^{-z}}{(1-e^z)}$ or $\dfrac{dv}{dz}+v=\dfrac{e^{-z}}{(1-e^z)}$.

This is a linear equation. I.F.$=e^z$.

$$\therefore \quad ve^z=\int\frac{e^z}{(1-e^z)}\cdot e^{-z}\,dz=\int\frac{1}{(1-e^z)}\,dz$$
$$=\int\frac{dx}{x(1-x)} \qquad e^z=x,\ dz=\frac{1}{x}dx$$
$$=\int\left(\frac{1}{x}+\frac{1}{1-x}\right)dx=\log x-\log(1-x)=\log\frac{x}{1+x}$$

$$\therefore \quad \text{P.I.}=v=\frac{1}{e^z}\left[\log\frac{x}{1-x}\right]=\frac{1}{x}\left[\log\frac{x}{1-x}\right].$$

The complete solution therefore is

$$y=(C_1+C_2 \log x).\frac{1}{x}+\frac{1}{x}\log\frac{x}{1-x}.$$

Ex. 11. (b) *Solve* $x^2\dfrac{d^2y}{dx^2}+3x\dfrac{dy}{dx}+y=0.$ **[Karnatak 61]**

C.F. of above example is the answer.

***Ex. 12** *Solve* $x^2\dfrac{d^2y}{dx^2}+4x\dfrac{dy}{dx}+2y=x+\sin x.$

Solution. Putting $x=e^z$, $D\equiv d/dz$, the equation becomes

$$[D(D-1)+4D+2]\, y=e^z+\sin e^z.$$

The A.E. is $D^2+3D+2=0$, *i.e.* $(D+2)(D+1)=0.$

$$\text{C.F.}=C_1e^{-z}+C_2e^{-2z}=C_1x^{-1}+C_2x^{-2}.$$

$$\text{P.I.}=\frac{e^z}{(D+2)(D+1)}+\frac{\sin e^z}{(D+2)(D+1)}$$

$$=\tfrac{1}{6}e^z+\frac{1}{D+2}\cdot\frac{1}{D+1}\sin e^z.$$

Now let $\dfrac{1}{D+1}\sin e^z=u$, *i.e.* $(D+1)\,u=\sin e^z$

or $\dfrac{du}{dz}+u=\sin e^z$, Linear, I.F.$=e^z$.

$$\therefore\quad ue^z=\int e^z\sin e^z\,dz$$

$$=\int \sin x\,dx, \text{ as } x=e^z,\ dx=e^z\,dz$$

$$=-\cos x=-\cos e^z;$$

or $\dfrac{1}{D+2}\cdot\dfrac{1}{D+1}\sin e^z=\dfrac{1}{D+2}\,u=\dfrac{1}{D+2}(-e^{-z}\cos e^z)=v$, say.

$\therefore\quad (D+2)\,v=-e^{-z}\cos e^z$ or $\dfrac{dv}{dz}+2v=-e^{-z}\cos e^z$

which is a linear equation with I.F.$=e^{2z}$.

$$\therefore\quad ve^{2z}=-\int e^{-z}\cos e^z\, e^{2z}\,dz=-\int e^z\cos e^z\,dz$$

$$=-\int \cos x\,dx \quad\text{as}\quad x=e^z$$

$$=-\sin x,\quad \therefore\quad v=-\frac{1}{e^{2z}}.\sin x=-\frac{1}{x^2}\sin x.$$

$$\therefore\quad \text{P.I.}=\tfrac{1}{6}e^z+v=\tfrac{1}{6}x-\frac{1}{x^2}\sin x.$$

Therefore the complete solution is

$$y=C_1x^{-1}+C_2x^{-2}+\tfrac{1}{6}x-\frac{1}{x^2}\sin x.$$

Ex. 13. *Solve* $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = 2 \log x.$ **[Agra 64]**

Solution. Putting $x=e^z$, $D \equiv d/dz$, the equation becomes

$$[D(D-1)-D+1]\,y=2z, \text{ i.e. } [D^2-2D+1]\,y=2z.$$

A.E. is $(D-1)^2=0$, $D=1, 1$.

C.F. $=(c_1+c_2z)\,e^z=(c_1+c_2 \log x)\,x$ as $x=e^z$.

Now P.I. $=\dfrac{1}{(1-D)^2}\,2z=(1-D)^{-2}\,2z$

$$=(1+2D+\ldots)\,2z=2z+4=2\log x+4.$$

Hence complete solution is $y=$ C.F. $+$ P.I.

Ex. 14. *Solve* $\dfrac{d^2y}{dx^2}+\dfrac{1}{x}\dfrac{dy}{dx}=\dfrac{12 \log x}{x^2}$.

[Gujrat B.Sc. (Prin.) 68]

Solution. The equation is $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}=12 \log x.$

Putting $x=e^z$, $D \equiv d/dz$, the equation becomes

$$[D(D-1)+D]\,y=12z.$$

A.E. is $D^2=0$, $D=0, 0$.

$\therefore$ C.F. $=C_1+C_2z=C_1+C_2 \log x$

P.I. $=\dfrac{12z}{D^2}=2z^3=2(\log x)^3.$

Hence $y=C_1+C_2\log x+2(\log x)^3$ is the complete solution.

Ex. 15 (a) $(x^4D^4+6x^3D^3+9x^2D^2+3xD+1)\,y=(1+\log x)^2$.

[Karnatak 60]

Solution. Putting $x=e^z$, $D'\equiv\dfrac{d}{dz}$, the equation becomes

$$[D'(D'-1)(D'-2)(D'-3)+6D'(D'-1)(D'-2)+9D'(D'-1)+3D'+1]\,y=(1+z)^2$$

or $(D'^4+2D'^2+1)\,y=(1+z)^2$.

A.E. is $D'^4+2D'^2+1=0$, i.e. $(D'^2+1)^2=0$, $D'=\pm i, \pm i$.

$\therefore$ C.F. $=(C_1+C_2z)\cos z+(C_3+C_4z)\sin z$

$$=(C_1+C_2\log x)\cos(\log x)+(C_3+C_4\log x)\sin(\log x)$$

P.I. $=\dfrac{1}{D'^4+2D'^2+1}(1+z)^2$

$$=[1+(2D'^2+D'^4)]^{-1}(1+z)^2$$

$$=(1-2D'^2-D'^4)(1+z)^2=(1+z)^2-2.2$$

$$=z^2+2z-3=(\log x)^2+2\log x-3.$$

Hence the complete solution is

$$y=(C_1+C_2\log x)\cos\log x+(C_3+C_4\log x)\sin\log x+(\log x)^2+2\log x-3.$$

Ex. 15. (b) $x^4\frac{d^4y}{dx^4}+6x^3\frac{d^3y}{dx^3}+9x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y=0.$

[Karnatak (Sub.) 60]

The C.F. of the above example is the answer here.

***Ex. 16.** *Solve* $x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=e^x.$

[Delhi Hons. 66 ; Vikram 63 ; Agra 46 ; Karnatak 62, 60 ; Sagar 64 ; Marathwada 64]

Solution. Putting $x=e^z$, $D\equiv\frac{d}{dz}$, the equation becomes

$$[D(D-1)+4D-2]\ y=e^{e^z}.$$

A.E. is $D^2+3D+2=0$, $\therefore$ $(D+2)(D+1)=0$.

$\therefore$ C.F. $=C_1e^{-2z}+C_2e^{-z}=C_1x^{-2}+C_2x^{-1}$.

$$\text{P.I.}=\frac{1}{(D+2)(D+1)}e^{e^z}=\left(\frac{1}{D+1}-\frac{1}{D+2}\right)e^{e^z}. \quad \ldots(1)$$

Now let $\frac{1}{D+1}e^{e^z}=u$, *i.e.* $(D+1)u=e^{e^z}$,

or $\frac{du}{dz}+u=e^{e^z}$, linear, I.F. $=e^z$.

$$\therefore\quad ue^z=\int e^z.e^{e^z}\,dz=\int e^x\,dx \text{ as } e^z=x$$

$$=e^x \text{ or } u=\frac{1}{x}e^x \text{ as } e^z=x.$$

Also let $\frac{1}{D+2}e^{e^z}=v$, $(D+2)v=e^{e^z}$

or $\frac{dv}{dz}+2v=e^{e^z}$, linear equation, I.F. $=e^{2z}$.

$$\therefore\quad ve^{2z}=\int e^{2z}.e^{e^z}\,dz=\int e^z\ e^{e^z}.e^z.dz$$

$$=\int xe^x\,dx=e^x(x-1)$$

or $v=\frac{1}{e^{2z}}[e^x(x-1)]=\frac{e^x}{x^2}(x-1)=\frac{e^x}{x}-\frac{e^x}{x^2}.$

$\therefore$ (1) gives P.I. $=u-v=\frac{1}{x}e^x-\left(\frac{e^x}{x}-\frac{e^x}{x^2}\right)=\frac{e^x}{x^2}.$

Hence the complete solution is

$$y=C_1x^{-2}+C_2x^{-1}+\frac{e^x}{x^2}.$$

Ex. 17. *Solve* $x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+2y=x\log x.$ **[Luck. 48]**

Solution. Putting $x=e^z$, $D\equiv\frac{d}{dz}$, the equation becomes

$$[D(D-1)-D+2]\,y=e^z.z.$$

The A.E. is $D^2-2D+2=0$, $D=1\pm i$.

$\therefore$ C.F.$=e^zC_1\cos(z+C_2)=xC_1\cos(\log x+C_2)$.

$$\text{P.I.}=\frac{e^z z}{(D^2-2D+2)}=e^z\frac{1}{(D+1)^2-2(D+1)+2}z$$

$$=e^z\frac{1}{D^2+1}z=e^z(1+D^2)^{-1}z$$

$$=e^z(1-D^2-\ldots)z=ze^z=x\log x.$$

Hence the complete solution is

$$y=xC_1\cos(\log x+C_2)+x\log x.$$

Ex. 18. *Solve* $x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=x\log x.$

Solution. Putting $x=e^z$, $D\equiv\frac{d}{dz}$, the equation becomes

$$[D(D-1)(D-2)+3D(D-1)+D+1]\,y=e^z.z$$

or $(D^3+1)\,y=e^z.z.$

A.E. is $(D^3+1)=0$, *i.e.* $(D+1)(D^2-D+1)=0$.

$$D=-1,\ \frac{1\pm\sqrt{3}i}{2}.$$

$\therefore$ C.F.$=C_1e^{-z}+C_2e^{(1/2)z}\cos(\frac{1}{2}\sqrt{3}z+C_3)$

$=C_1x^{-1}+C_2\sqrt{x}\cos(\frac{1}{2}\sqrt{3}\log x+C_3)$.

$$\text{P.I.}=\frac{1}{D^3+1}ze^z=e^z\frac{1}{(D+1)^3+1}.z=e^z.\frac{2}{2+3D+3D^2+D^3}z$$

$$=\frac{e^z}{2}(1+\tfrac{3}{2}D+\tfrac{3}{2}D^2+\tfrac{1}{2}D^3)^{-1}z=\tfrac{1}{2}e^z(1-\tfrac{3}{2}D\ldots)z$$

$$=\tfrac{1}{2}e^z(z-\tfrac{3}{2})=\tfrac{1}{2}x(\log x-\tfrac{3}{2}).$$

Therefore the complete solution is

$$y=C_1x^{-1}+C_2\sqrt{x}\cos(\tfrac{1}{2}\sqrt{3}\log x+C_3)+\tfrac{1}{2}x(\log x-\tfrac{3}{2}).$$

Ex. 19. *Solve* $x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=\log x.$ [Nagpur 71]

Solution. Putting $x=e^z$, $D\equiv\frac{d}{dz}$, we get

$$[D(D-1)+2D]\,y=z.$$

A.E. is $D^2+D=0$; $\therefore$ $D(D+1)=0$, $D=0,-1$.

$\therefore$ C.F.$=C_1+C_2e^{-z}=C_1+C_2x^{-1}$.

$$\text{P.I.}=\frac{1}{(D^2+D)}z=\frac{1}{D}(1+D)^{-1}z=\frac{1}{D}[1-D-\ldots]z$$

$$=\frac{1}{D}(z-1)=\frac{z^2}{2}-z=\frac{(\log x)^2}{2}-\log x.$$

Therefore the complete solution is

$$y=C_1+C_2x^{-1}+\frac{(\log x)^2}{2}-\log x.$$

Ex. 20. *Solve* $x^4\frac{d^4y}{dx^4}+2x^3\frac{d^3y}{dx^3}+x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+1=x+\log x.$

[Mysore 49]

Solution. Putting $x=e^z$, $D\equiv\frac{d}{dz}$, the equation becomes

$$[D(D-1)(D-2)(D-3)+2D(D-1)(D-2)$$
$$+D(D-1)-D+1]\,y=e^z+z$$

or $(D-1)^4\,y=e^z+z.$

A.E is $(D-1)^4=0$, $D=1, 1, 1, 1.$

$\therefore$ C.F.$=(C_1+C_2z+C_3z^2+C_4z^3)\,e^z.$

P.I.$=\frac{e^z}{(D-1)^4}+\frac{1}{(D-1)^4}z.$ (first term case of failure)

$=z^4\frac{1}{4.3.2.1}e^z+(1-D)^{-4}z$

$=\frac{z^4}{4!}e^z+(1+4D+\ldots)\,z$

$=\frac{z^4}{4!}e^z+z+4=\frac{(\log x)^4}{4!}x+\log x+4.$

{ multiplying by z^4 and differentiating the denominator of first term four times

Therefore the complete solution is

$$y=[C_1+C_2\log x+C_3(\log x)^2+C_4(\log x)^3]\,x$$
$$+\frac{(\log x)^4}{4!}x+\log x+4.$$

Ex. 21. *Solve* $[x^2D^2-(2m-1)\,xD+(m^2+n^2)]\,y=n^2x^m\log x.$

Solution. Putting $x=e^z$, $D'\equiv\frac{d}{dz}$, the equation becomes

$$[D'(D'-1)-(2m-1)D'+(m^2+n^2)]\,y=n^2e^{mz}.z.$$

A.E. is $D'^2-2mD'+m^2+n^2=2$, $(D'-m)^2+n^2=0$, $D'=m\pm in.$

C.F.$=e^{mz}C_1\cos(nz+C_2)=x^mC_1\cos(n\log x+C_2).$

P.I.$=\frac{1}{(D'-m)^2+n^2}n^2ze^{mz}$

$=e^{mz}\frac{1}{(D'+m-m)^2+n^2}n^2z=n^2e^{mz}\frac{1}{D'^2+n^2}z$

$=n^2e^{mz}.\frac{1}{n^2}\left(1+\frac{D'^2}{n^2}\right)^{-1}z=e^{mz}\left(1-\frac{D'^2}{n^2}\cdots\right)z$

$=e^{mz}.z=x^m\log x.$

$\therefore$ The complete solution is

$$y=x^mC_1\cos(n\log x+C_2)+x^m\log x.$$

***Ex. 22.** *Solve* $x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+y=\frac{\log x\sin(\log x)+1}{x}.$

[Agra 62; Delhi Hons 57; Karnatak 61]

Solution. Putting $x=e^z$, $D\equiv\frac{d}{dz}$, the equation becomes

$$[D(D-1)-3D+1]\,y=\frac{z\sin z+1}{e^z}$$

or $(D^2-4D+1)\,y=(z\sin z)\,e^{-z}+e^{-z}$.

A.E. is $D^2-4D+1=0$. $D=2\pm\sqrt{3}$.

$$\text{C.F.}=C_1e^{(2+\sqrt{3})z}+C_2e^{(2-\sqrt{3})z}=e^{2z}\,(C_1e^{\sqrt{3}z}+C_2e^{-\sqrt{3}z}.$$
$$=x^2\,(C_1x^{\sqrt{3}}+C_2x^{-\sqrt{3}}).$$

P. I. corresponding to e^{-z}

$$=\frac{1}{D^2-4D+1}\,e^{-z}=\frac{1}{(-1)^2-0\,(-1)+1}\,e^{-z}$$
$$=\frac{e^{-z}}{6}=\frac{x^{-1}}{6}.$$

P. I. corresponding to $e^{-z}z\sin z$

$$=\frac{1}{(D^2-4D+1)}\,e^{-z}\,z\sin z$$
$$=e^{-z}\,\frac{1}{(D-1)^2-4\,(D-1)+1}\,z\sin z$$
$$=e^{-z}\,\frac{1}{D^2-6D+6}\,z\sin z$$
$$=\text{Imaginary part of } e^{-z}\,\frac{1}{D^2-6D+6}\,ze^{iz}$$
$$=\text{Imaginary part of } e^{-z}\,e^{iz}\,\frac{1}{(D+i)^2-6\,(D+i)+6}z$$
$$=\text{Imaginary part of } e^{-z}.e^{iz}\,\frac{1}{D^2+D(2i-6)+(5-6i)}\,z$$
$$=\text{Imaginary part of } e^{-z(1-i)}\,\frac{1}{5-6i}\left[z-\frac{D\,(2i-6)}{5-6i}+\ldots\right]z$$
$$=\text{Imaginary part of } e^{-z\,(1-i)}\,\frac{1}{5-6i}\left[z-\frac{2i-6}{5-6i}\right]$$
$$=\text{Imaginary part of } e^{-z}\,(\cos z+i\sin z)\,\frac{5+6i}{5^2+6^2}$$
$$\times\left[z-\frac{(2i-6)\,(5+6i)}{5^2+6^2}\right]$$
$$=e^{-z}\left[\frac{6}{61}\,z\cos z+\frac{5}{61}\,z\sin z+\frac{130}{61^2}\cos z-\frac{156}{61^2}\sin z\right.$$
$$\left.+\frac{252}{61^2}\cos z+\frac{210}{61^2}\sin z\right]$$
$$=\frac{e^{-z}z}{61}\,(6\cos z+5\sin z)+\frac{2e^{-z}}{61^2}[191\cos z+27\sin z]$$

$$=\frac{x^{-1}\log x}{61}[6\cos(\log x)+5\sin(\log x)]$$

$$+\frac{2x^{-1}}{3721}[191\cos(\log x)+27\sin(\log x)].$$

Hence the complete solution is

$$y=x^2(C_1x^{\sqrt{3}}+C_2x^{-\sqrt{3}})+\frac{x^{-1}}{6}+\frac{x^{-1}\log x}{11}(6\cos\log x$$

$$+5\sin\log x)+\frac{2x^{-1}}{3721}(191\cos\log x+27\sin\log x).$$

Ex. 23. *Solve* $(x^2D^2-xD+4)\,y=\cos(\log x)+x\sin(\log x)$. **[Vikram 62]**

Solution. Putting $x=e^z$, $D'\equiv\frac{d}{dz}$, the equation becomes

$$[D'(D'-1)-D'+4]\,y=\cos z+e^z\sin z.$$

A.E. is $D'^2-2D'+4=0$, $D'=1\pm\sqrt{3}i$.

$\therefore$ C.F. $=e^z[C_1\cos\sqrt{3}z+C_2\sin\sqrt{3}z]$
$=x[C_1\cos(\sqrt{3}\log x)+C_2\sin(\sqrt{3}\log x)]$.

$$\text{P. I.}=\frac{\cos z}{D'^2-2D'+4}+\frac{e^z\sin z}{D'^2-2D'+4}$$

$$=\frac{\cos z}{-1^2-2D'+4}+e^z\frac{1}{(D'+1)^2-2(D'+1)+4}\cdot\sin z$$

$$=\frac{3+2D'}{9-4D'^2}\cos z+e^z\frac{1}{D'^2+3}\sin z$$

$$=\frac{3\cos z-2\sin z}{9+4}+e^z\frac{1}{-1^2+3}\sin z$$

$$=\tfrac{1}{13}[3\cos(\log x)-2\sin(\log x)]+\tfrac{1}{2}x\sin(\log x).$$

Hence the complete solution is $y=\text{C.F.}+\text{P.I.}$

Ex. 24. *Solve* $x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+4y=x$ *given that* $y=0$ *when* $x=1$ and $y=e^2$ *when* $x=e$. **[Poona 64]**

Solution. Putting $x=e^z$, $D\equiv\frac{d}{dz}$, the equation becomes

$$[D(D-1)-3D+4]\,y=e^z.$$

A.E. is $D^2-4D+4=0$, *i.e.* $D=2, 2$.

C.F. $=(C_1+C_2z)\,e^{2z}$.

$$\text{P.I.}=\frac{e^z}{(D-2)}=e^z.$$

$\therefore$ Complete solution is $y=(C_1+C_2z)\,e^{2z}+e^z$.

But when $x=1$, *i.e.* $z=0$, $y=0$.

And when $x=e$, *i.e.* $z=1$, $y=e^2$.

$\therefore$ $0=C_1+1$, *i.e.* $C_1=-1$

and $e^2=(C_1+C_2)\,e^2+e$, *i.e.* $2e^2-e=C_2e^2$ or $C_2=2-e^{-1}$

Hence the solution is

$$y=[-1+(2-e^{-1})\log x]\,x^2+x \quad \text{as} \quad e^z=x.$$

Ex. 25. (a) $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-4y=x^2.$ **[Delhi 1967 ; Alld. 65]**

(b) $(x^2D^2-3xD+5)\,y=x^2\sin(\log x)$ **[Delhi 1972]**

6·2. Equations reducible to homogeneous form.

Consider an equation of the type

$$(a+bx)^n\frac{d^ny}{dx^n}+P_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\ldots+P_{n-1}(a+bx)\frac{dy}{dx}+P_ny=X(x),$$

where $P_1, P_2, \ldots, P_n$ are constants

If we put $a+bx=u$, then

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=b\frac{dy}{du},$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(b\frac{dy}{du}\right)=b^2\frac{d^2y}{du^2}\text{ etc.}$$

... ...

and $\dfrac{d^ny}{dx^n}=b^n\dfrac{d^ny}{du^n}.$

Thus the equation, after dividing by b^n, becomes

$$u^n\frac{d^ny}{du^n}+\frac{P_1}{b}\cdot u^{n-1}\frac{d^{n-1}y}{du^{n-1}}+\ldots+\frac{P_{n-1}}{b^{n-1}}\frac{dy}{du}+\frac{P_n}{b^n}y=\frac{1}{b^n}X\left(\frac{u-a}{b}\right),$$

which is a standard homogeneous equation.

Now putting $u=a+bx=e^z$ and $D\equiv\dfrac{d}{dz}$, the equation becomes

$$D(D-1)\ldots(D-n+1)\,y+\frac{P_1}{b}D(D-1)\ldots(D-n+2)\,y+\ldots+\frac{P_{n-1}}{b^{n-1}}Dy+\frac{P_n}{b^n}y=\frac{1}{b^n}X\left(\frac{e^z-a}{b}\right).$$

This is a linear equation with constant coefficients and can be solved by appropriate method.

Ex. 1. *Solve* $(x+a)^2\dfrac{d^2y}{dx^2}-4(x+a)\dfrac{dy}{dx}+6y=x.$

[Poona 1964 ; Agra 74, 62, 56]

Solution. Putting $x+a=e^z$, $D\equiv\dfrac{d}{dz}$, the equation becomes

$$[D(D-1)-4D+6]\,y=(e^z-a).$$

A.E is $D^2-5D+6=0.\ (D-3)(D-2)=0$

$\therefore$ C.F. $=C_1e^{2z}+C_2e^{3z}=C_1(x+a)^2+C_2(x+a)^3.$

$$\text{P.I}=\frac{1}{D^2-5D+6}(e^z-a)=\frac{1}{1-5+6}e^z-\frac{a}{6}$$

$$=\frac{e^z}{2}-\frac{a}{6}=\frac{(x+a)}{2}-\frac{a}{6}.$$

Therefore the complete solution is

$$y=C_1(x+a)^2+C_2(x+a)^3+\frac{x+a}{2}-\frac{a}{6}.$$

Ex. 2. *Solve* $(1+x)^2\dfrac{d^2y}{dx^2}+(1+x)\dfrac{dy}{dx}+y=4\cos\log(1+x).$

[Delhi 1968; Agra 71, 50]

Solution. Putting $1+x=e^z$, $D\equiv\dfrac{d}{dz}$, the equation becomes

$$[D(D-1)+D+1]\,y=4\cos z.$$

A.E. is $D^2+1=0$, $D=\pm i$.

C.F.$=C_1\cos(z+C_2)=C_1\cos[\log(1+x)+C_2]$.

P.I.$=\dfrac{1}{D^2+1}.4\cos z$ (case of failure)

$=4z.\dfrac{1}{2D}\cos z$ multiplying by z and differentiating the denominator w.r.t. D

$=2z\sin z=2\log(1+x)\sin\log(1+x)$.

Therefore the complete solution is

$$y=C_1\cos[\log(1+x)+C_2]+2\log(1+x)\sin\log(1+x).$$

Ex. 3. *Solve*

$$[(3x+2)^2D^2+3(3x+2)D-36]\,y=3x^2+4x+1.$$

[Delhi Hons. 1972, 70, 61]

Solution. Putting $3x+2=e^z$, $D\equiv\dfrac{d}{dz}$, the equation becomes

$$[3^2D'(D'-1)+3.3D'-36]\,y=3\left(\frac{e^z-2}{3}\right)^2+4\left(\frac{e^z-2}{3}\right)+1.$$

A.E. is $9(D'^2-4)=0$, $D'=\pm 2$.

$\therefore$ C.F.$=C_1e^{2z}+C_2e^{-2z}=C_1(3x+2)^2+C_2(3x-2)^{-2}$.

$$\text{P.I.}=\frac{1}{9(D'^2-4)}\left(\frac{e^{2z}-1}{3}\right)=\frac{1}{27}\left[\frac{e^{2z}}{D'^2-4}-\frac{1}{D'^2-4}\right]$$

$=\dfrac{1}{27}\left[z\dfrac{e^{2z}}{2D'}+\dfrac{1}{4}\right]$ since first term is case of failure

$=\frac{1}{27}\left[\frac{1}{4}ze^{2z}+\frac{1}{4}\right]=\frac{1}{108}[ze^{2z}+1]$

$=\frac{1}{108}[(3x+2)^2\log(3x+2)+1]$ as $3x+2=e^z$.

Hence the complete solution is

$$y=C_1(3x+2)^2+C_2(3x+2)^{-2}+\tfrac{1}{108}[(3x+2)^2\log(3x+2)+1].$$

Ex. 4. $(3x+2)^2\dfrac{d^2y}{dx^2}+5(3x+2)\dfrac{dy}{dx}-3y=x^2+x+1.$

Solution. Putting $3x+2=e^z$, $D\equiv\dfrac{d}{dz}$, the equation becomes

$$[3^2.D(D-1)+5.3D-3]\,y=\left(\frac{e^z-2}{3}\right)^2+\left(\frac{e^z-2}{3}\right)+1.$$

A.E. is $3(3D^2+2D-1)=0$ or $(3D-1)(D+1)=0$.

C.F. $=C_1e^{z/3}+C_2e^{-z}=C_1(3x+2)^{1/3}+C_2(3x+2)^{-1}$.

$$\text{P.I.}=\frac{1}{3(3D^2+2D-1)}\left(\frac{e^{2z}-e^z+7}{9}\right)$$

$$=\frac{e^{2z}}{27(3.2^2+2.2-1)}-\frac{e^z}{27.(3.1^2+2.1-1)}+\frac{7}{27(0+0-1)}$$

Hence the complete solution is

$$y=C_1(3x+2)^{1/3}+C_2(3x+2)^{-1}+\frac{(3x+2)^2}{405}-\frac{(3x+2)}{108}-\frac{7}{27}.$$

Ex. 5. *Solve* $(1+2x)^2\frac{d^2y}{dx^2}-6(1+2x)\frac{dy}{dx}+16y=8(1+2x)^2$.

Solution. Putting $1+2x=e^z$, $D\equiv\frac{d}{dz}$, the equation becomes

$$[2^2D(D-1)-6.2D+16]\,y=8e^{2z}$$

or $4[D^2-4D+4]\,y=8e^{2z}$ or $(D-2)^2=2e^{2z}$.

C.F. $=(C_1+C_2z)e^{2z}=[C_1+C_2\log(1+2x)](1+2x)^2$

P.I. $=\frac{2e^{2z}}{(D-2)^2}$ (case of failure)

$=z.\frac{2e^{2z}}{2.(D-2)}$ multiplying by z and differentiating the denominator w.r.t. D

$=z.\frac{2e^{2z}}{2}$ again multiplying by z and differentiating the denominator w.r.t. D

$=z^2e^{2z}=[\log(1+2x)]^2(1+2x)^2$.

Therefore the complete solution is

$$y=[C_1+C_2\log(1+2x)](1+2x)^2+[(\log(1+2x)]^2(1+2x)^2.$$

Ex. 6. *Solve* $(x+1)^2\frac{d^2y}{dx^2}+(x+1)\frac{dy}{dx}=(2x+3)(2x+4)$.

[Agra 1970; Nagpur 63]

Solution. Putting $x+1=e^z$, $D\equiv\frac{d}{dz}$, the equation becomes

$$[D(D-1)+D]\,y=(2e^z+1)(2e^z+2).$$

A.E. $D^2=0$, $D=0, 0$

∴ C.F. $=(C_1+C_2z)e^{0z}=C_1+C_2z=C_1+C_2\log(1+x)$.

P.I. $=\frac{1}{D^2}(4e^{2z}+6e^z+2)$

$=\frac{4e^{2z}}{2^2}+\frac{6e^z}{1^2}+z^2$ $\left(\text{as }\frac{1}{D^2}\text{ means integration twice}\right)$

$=e^{2z}+6e^z+z^2=(x+1)^2+6(x+1)+[\log(x+2)]^2$

$=x^2+8x+[\log(x+1)]^2$ leaving the constant term 7, which can be considered to be included in part C_1 of the C.F.

Hence the complete solution is

$$y=C_1+C_2 \log (1+x)+x^2+8x+[\log (x+1)]^2.$$

Ex. 7. *Solve* $16 (x+1)^4 \frac{d^4y}{dx^4}+96 (x+1)^3 \frac{d^3y}{dx^3}$

$$+104 (x+1)^2 \frac{d^2y}{dx^2}+8 (x+1) \frac{dy}{dx}+y=x^2+4x+3.$$

Solution. Putting $(x+1)=e^z$, $D\equiv d/dz$, the equation becomes

$$[16D (D-1) (D-2) (D-3)+96D (D-1) (D-2)+104D (D-1)$$
$$+8D+1] y=e^{2z}+2e^z, \text{ as } x^2+4x+3=(x+1) (x+3),$$

i.e. $(16D^4-8D^2+1) y=e^{2z}+2e^z$.

A.E. is $16D^4-8D^2+1=0$, $(4D^2-1)^2=0$.

$D=\pm\frac{1}{2}, \pm\frac{1}{2}$ repeated twice.

$\therefore$ C.F. $=(C_1+C_2z) e^{z/2}+(C_3+C_4z) e^{-z/2}$

$$=[C_1+C_2 \log (1+x)] (x+1)^{1/2}$$
$$+[C_3+C_4 \log (1+x)] (x+1)^{-1/2},$$

$$\text{P.I.}=\frac{e^{2z}+2e^z}{(4D^2-1)^2}=\frac{e^{2z}}{(4.2^2-1)^2}+\frac{2e^z}{(4.1^2-1)^2}$$

$$=\frac{e^{2z}}{225}+\frac{2e^z}{9}=\frac{(x+1)^2}{225}+\frac{2 (x+1)}{9}.$$

Thus the complete solution is

y=C.F.+P.I.

Ex. 8. *Solve* $(5+2x)^2 \frac{d^2y}{dx^2}-6 (5+2x) \frac{dy}{dx}+8y=0.$

[Saugar 1963; Marathwada 64]

Solution. Putting $5+2x=e^z$, $D\equiv d/dz$, the equation becomes

$$[2^2D (D-1) - 6.2D+8] y=0,$$

i.e. $(D^2-4D+2) y=0$, $D=2\pm\sqrt{2}$.

Therefore the solution is $y=e^{2z}C_1 \cos (\sqrt{2}z+C_2)$,

i.e. $y=(5+2x)^2 C_1 \cos [\sqrt{2} \log (5+2x)+C_2]$.

Ex. 9. *Solve* $(2x-1)^3 \frac{d^3y}{dx^3}+(2x-1) \frac{dy}{dx}-2y=0.$

Solution Putting $(2x-1)=e^z$, $D\equiv d/dz$, the equation is

$$[2^3 D (D-1) (D-2)+2D-2] y=0,$$

i.e. $(4D^3-12D^2+9D-1) y=0$.

A.E. is $(4D^3-12D^2+9D-1=0$, $(D-1) (4D^2-8D+1)=0$,

i.e. $D=1, \frac{8\pm\sqrt{(64-16)}}{8}$, $D=1, 1\pm\frac{\sqrt{3}}{2}$.

$\therefore$ solution is $y=C_1e^z+C_2e^z \cos \left[\frac{\sqrt{3}}{2} z+C_3\right]$

i.e. $y=C_1 (2x-1)+C_2 (2x-1) \cos\left[\frac{\sqrt{3}}{2} \log (2x-1)+C_3\right]$.

Miscellaneous Examples

Ex. 1. *Putting $y=z^2$, reduce the equation*

$$2x^2y\frac{d^2y}{dx^2}+4y^2=x^2\left(\frac{dy}{dx}\right)^2+2xy\frac{dy}{dx}$$

to homogeneous form and hence solve it.

Solution. We have $y=z^2$; $\therefore$ $\frac{dy}{dx}=2z\frac{dz}{dx}$

and $\frac{d^2y}{dx^2}=2\left(\frac{dz}{dx}\right)^2+2z\frac{d^2z}{dx^2}$.

Putting these values in the given equation, we get

$$2x^2.z^2\left[2\left(\frac{dz}{dx}\right)^2+2z\frac{d^2z}{dx^2}\right]+4z^4=x^2.4z^2\left(\frac{dz}{dx}\right)^2+2xz^2.2z\frac{dz}{dx},$$

i.e. $x^2\frac{d^2z}{dx^2}-x\frac{dz}{dx}+z=0$, homogeneous

Now to solve it, put $x=e^u$, $D\equiv\frac{d}{du}$; then the equation becomes

$$(D(D-1)-D+1]z=0$$

or $(D^2-2D+1)z=0$ or $(D-1)^2z=0$.

$\therefore$ $z=(C_1+C_2u)e^u=(C_1+C_2\log x)x$.

$\therefore$ $y=z^2=x^2(C_1+C_2\log x)^2$.

Ex. 2. *Solve the following equations :*

(i) $(x^2D^2+2xD-2)y=0$, [Nagpur 61]

Ans. $y=C_1x^{-1}+C_2x^2$.

(ii) $(x^2D^2+xD-4)y=x^2$. [Karnatak 59]

Ans. $y=C_1x^2+C_2x^{-2}+\frac{1}{4}x^2\log x$.

(iii) $(x^3D^3+6x^2D^2+8xD-8)y=x^2$, [Bombay 58]

Putting $x=e^z$, $(D'^3+3D'^2+4D'-8)y=e^{2z}$, $D'=1$, $-2\pm 2i$ etc.

(iv) $(x^4D^3+2x^3D^2-x^2D+x)y=1$. [Bombay 61]

$(D'-1)^2(D'+1)y=e^{-z}$ etc.

(v) $(x^2D^2+xD+1)y=\log x\sin(\log x)$.

Equation is $(D'^2+1)y=z\sin z$. Now refer Ex. 2 P. 81.

(vi) $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-9x=0$. [Nagpur 63]

Ans. $y=C_1x^3+C_2x^{-3}$.

(vii) $(x^3D^3+3x^2D^2+xD)y=24x^2$. [Poona 62]

Equation is $D'^3y=24e^{2z}$.

Ans. $y=C_1+C_2z+C_3z^2+\frac{24e^{2z}}{2^3}$, where $z=\log x$.

(viii) $(x^2D^2+5xD+4)y=x^4$. [Cal. Hons. 62, 61. 58]

Equation is $(D'+2)^2=e^{4z}$. **Ans.** $y=(C_1+C_2\log x)x^{-2}+\frac{1}{36}x^4$.

(ix) $(x^2D^2-3xD+5)y=x^2\sin(\log x)$. [Osmania 60 ; Karnatak 64]

Equation is $(D'^2-4D'+5)\, y=e^{2z} \sin z$ etc.

Ex. 3. *If* $D=x\dfrac{d}{dx}$, *prove*

(i) $\dfrac{1}{(D-m)^r\,\phi(D)}\, x^m=\dfrac{x^m\,(\log x)^r}{r\,!\,\phi(m)}$,

(ii) $f\left(x\dfrac{d}{dx}\right) x^m \log x=x^m\,[f(m)\log x+f'(m)]$.

[Bombay 64]

Solution. As $D=x\dfrac{d}{dx}=\dfrac{d}{dz}$ $\therefore$ $x=e^z$.

(i) $\because$ $\dfrac{1}{(D-m)^r\,\phi(D)}\, x^m=\dfrac{1}{(D-m)^r\,\phi(D)}\, e^{mz}$

case of failure, multiplying by z and differentiating the deno. w.r.t. D

$=z\,\dfrac{1}{r\,(D-m)^{r-1}\,\phi(D)}\, e^{mz}$ case of failure again

$=z^2\,\dfrac{1}{r\,(r-1)\,(D-m)^{r-2}\,\phi(D)}$

case of failure again, so differentiating $(r-2)$ times and multiplying by z^{r-2}

$=z^r\,\dfrac{1}{r\,!\,\phi(D)}\, e^{mz}=z^r\,\dfrac{1}{r\,!}\,\dfrac{e^{mz}}{\phi(m)}$

$=\dfrac{(\log x)^r}{r\,!\,\phi(m)}\, x^m$ as $x=e^z$, $z=\log x$.

(ii) Now $f\left(x\dfrac{d}{dx}\right) x^m \log x=f(D)\, e^{mz}.z$

$=[zf(D)\, e^{mz}+f'(D)\, e^{mz}]$ [see § 5·20 P. 79]

$=[zf(m)e^{mz}+f'(m)e^{mz}]$

$=e^{mz}\,[zf(m)+f'(m)]$

$=x^m\,[\log x f(m)+f'(m)]$

This proves the result.

Ex. 4. *Show that the equation* $x^2\dfrac{d^2y}{dx^2}+Px\,\dfrac{dy}{dx}+Q=0$ *can be reduced by substitution to* $\dfrac{d^2y}{dx^2}+(P-1)\dfrac{dy}{dt}+Qy=0$. [Cal. Hons. 63]

Hint. Refer § 6 1 P. 99.

$x^2\,\dfrac{d^2y}{dx^2}=D\,(D-1)\, y$, where $D\equiv\dfrac{d}{dt}$, $x\,\dfrac{dy}{dx}=Dy$ etc.

7

Equations of the First Order But not of the First Degree

7·1. Definition

The differential equations of first order do not contain differential coefficient higher then $\frac{dy}{dx}$. In this chapter we shall consider differential equations which involve powers of $\frac{dy}{dx}$. It is usual to denote $\frac{dy}{dx}$ by p. Thus an equation

$$p^n+P_1p^{n-1}+P^2p^{n-2}+\ldots+P_n=0,$$

where $P_1, P_2, \ldots\ldots, P_n$ are functions of x and y, is the equation of first order and nth degree.

7·2. Types of Equations

It may be possible to solve such equations by one or more of the *four methods* given below. *In each case the problem is reduced to that of solving one or more equations of first order and first degree.*

7·3. Equations solvable for p.

Suppose the equation

$$p^n+P_1p^{n-1}+P_2p^{n-2}+\ldots+P_n=0$$

can be put in the form

$$[p-F_1(x, y) \; [p-F_2(x, y)]\ldots[p-F_n(x, y)]=0.$$

Then equating to zero each factor of the above form, we get n equations of first order and first degree, namely

$$\frac{dy}{dx}=F_1(x, y), \frac{dy}{dx}=F_2(x, y), \ldots, \frac{dy}{dx}=F_n(x, y)$$

If solutions of the above n componet equations are given by

$$f_1(x, y, c_1)=0, f_2(x, y, c_2)=0, \ldots. f_n(x, y, c_n)=0.$$

then the relation

$$f_1(x, y, c_1) f_2(x, y, c_2)\ldots f_n(x, y, c_n)=0,$$

is the most general solution of the equation (I).

There is no loss of generality* if we take

*The general solution of a differeniiai equatlons of the nrst order should contain one arbitrary constant.

$c_1=c_2=\ldots=c_n=c$ (say).

Therefore the general solution of the equation is put as

$f_1(x, y, c)\, f_2(x, y, c) \ldots f_n(x, y, c)=0,$

Ex. 1. *Solve*

$$p^4-(x+2y+1)\,p^3+(x+2y+2xy)\,p^2-2xyp=0.$$

Solution. On factorization the given equation becomes

$$p\,(p-1)\,(p-x)\,(p-2y)=0.$$

The component equations of first order and first degree are

$p=0, p=1, p=x, p=2y.$

or $\dfrac{dy}{dx}=0, \dfrac{dy}{dx}=1, \dfrac{dy}{dx}=x, \dfrac{dy}{dx}=2y.$

Solutions of these component equations are respectively

$y-c=0, y-x-c=0, 2y-x^2-c=0, y-ce^{2x}=0.$

and therefore the most general solution of the given equation is

$$(y-c)\,(y-x-c)\,(2y-x^2-c)\,(y-ce^{2x})=0.$$

Ex. 2. *Solve* $p^3+2xp^2-y^2p^2-2xy^2p=0.$

Solution. The equation can be written as

$p\,(p^2+2xp-y^2p-2xy^2)=0$ or $p\,(p+2x)\,(p-y^2)=0.$

The component equations are

$$\frac{dy}{dx}=0, \frac{dy}{dx}+2x=0, \frac{dy}{dx}-y^2=0.$$

Solutions of these component equations are

$y-c=3, y+x^2-c=0, xy+yc+1=0.$

Therefore the most general solution is

$$(y-c)\,(y+x^2-c)\,(xy+yc+1)=0.$$

Ex. 3. *Solve* $xy\,(p^2+1)=(x^2+y^2)\,p.$ [Poona 61]

Solution. The equations can be written as

$xyp^2-(x^2+y^2)\,p+xy=0,$

i.e. $(yp-x)\,(xp-y)=0.$

Thus the component equations are

$$y\frac{dy}{dx}-x=0, \quad x\frac{dy}{dx}-y=0,$$

i.e. $y\,dy-x\,dx=0, \dfrac{dy}{y}-\dfrac{dx}{x}=0,$

whose solutions are $y^2-x^2=c, y/x=c$

Hence the general solution is

$$(y^2-x^2-c)\,(y-cx)=0.$$

Ex. 4. *Solve* $x^2\left(\dfrac{dy}{dx}\right)^2+xy\dfrac{dy}{dx}-6y^2=0.$ [Banaras 51]

Solution. Writting p for dy/dx, the equation becomes

$x^2p^2+pxy-6y^2=0,$ *i.e.* $(px+3y)\,(px-2y)=0.$

The component equations are

$$x\frac{dy}{dx}+3y=0 \quad \text{and} \quad x\frac{dy}{dx}-2y=0$$

or $$\frac{dy}{y}+3\frac{dx}{x}=0 \quad \text{and} \quad \frac{dy}{y}-\frac{2\,dx}{x}=0.$$

Integrating these $yx^3=c$ and $y/x^2=c$.
Hence the solution is $(yx^3-c)(y/x^2-c)=0$.

***Ex. 5.** *Solve* $x^2\left(\frac{dy}{dx}\right)^2-2xy\frac{dy}{dx}+2y^2-x^2=0.$

[Saugar 62 ; Cal. Hons. 61 ; Gorakhpur 59]

Solution. Writing p for $\frac{dy}{dx}$, the equation becomes

$$x^2p^2-2xyp+2y^2-x^2=0.$$

Solving for p,

$$p=\frac{2xy\pm\sqrt{[4x^2y^2-4x^2(2y^2-x^2)]}}{2x^2}$$

$$=\frac{y\pm\sqrt{(x^2-y^2)}}{x}.$$

The component equations are $\frac{dy}{dx}=\frac{y\pm\sqrt{(x^2-y^2)}}{x}$.

These are homogeneous ; $\therefore$ put $p=vx$, so that

$$v+x\frac{dv}{dx}=\frac{v\pm\sqrt{(1-v^2)}}{1}, \quad i.e., \quad x\frac{dv}{dx}=\pm\sqrt{(1-v^2)}$$

or $$\frac{dv}{\sqrt{(1-v^2)}}=\frac{dx}{x} \quad \text{and} \quad \frac{dv}{\sqrt{(1-v^2)}}=-\frac{dx}{x}.$$

Integrating, $\sin^{-1}v=\log cx$ and $\sin^{-1}v=-\log cx$,
i.e., $\sin^{-1}(y/x)=\pm\log cx$ as $v=y/x$,
which form the required solution.

Ex. 6. *Solve* $x^2\left(\frac{dy}{dx}\right)^2+3xy\frac{dy}{dx}+2y^2=0.$ **[Nagpur 61]**

Solution. The equation can be written as

$$x^2p^2+3xyp+2y^2=0, \quad i.e. \quad (xp+y)(xp+2y)=0.$$

The component equations are

$$x\frac{dy}{dx}+y=0 \text{ and } x\frac{dy}{dx}+2y=0,$$

or $$\frac{dy}{y}+\frac{dx}{x}=0 \quad \text{and} \quad \frac{dy}{y}+\frac{2\,dx}{x}=0.$$

Integrating, $xy=c$ and $yx^2=c$.
Hence the solution is $(xy-c)(yx^2-c)=0$.

Ex. 7. (a) *Solve* $yp^2+(x-y)p-x=0.$

[Delhi Hons. 54 ; Alld. 51]

Solution. We have $(p-1)(yp+x)=0$.

i.e. $\frac{dy}{dx}=1$ and $y\frac{dy}{dx}+x=0$ or $y\,dy+x\,dx=0$

Integrating, $y=x+c$ and $x^2+y^2=c$.

The solution is $(y-x-c)(x^2+y^2-c)=0$.

Ex. 7. (b) *Solve* $xp^2+(y-x)p-y=0$. **[Jabalpur 62]**

Ans. $(y-x+c)(xy+c)=0$.

Ex. 8. *Solve* $p^2-p(x^2+xy+y^2)+xy(x+y)=0$.

Solution. After factorizing, the equation can be written as

$$(p-x)[p^2+px-y(x+y)]=0$$

or $(p-x)(p-y)[p+(x+y)]=0$.

The component equations are

$$\frac{dy}{dx}=x,\ \frac{dy}{dx}=y,\ \frac{dy}{dx}+x+y=0.$$

Solution of $\frac{dy}{dx}=x$ is $y=\frac{x^2}{2}+\text{const.}$, *i.e.* $2y-x^2=c$,

Solution of $\frac{dy}{dx}=y$ is $\log y=x+\text{const.}$, *i.e.* $y=ce^x$,

and solution of $\frac{dy}{dx}+y=-x$ (linear equation) is

$$ye^x=c+\int -xe^x\,dx,$$

i.e. $ye^x=c-(x-1)e^x$

or $y+x-1-ce^{-x}=0$.

Therefore the complete solution is

$$(2y-x^2-c)(y-ce^x)(y+x-1-ce^{-x})=0.$$

Ex. 9. (a) *Solve* $p^3(x+2y)+3p^2(x+y)+(y+2x)p=0$.

Solution. On factorizing, the equation is

$$p(p+1)(px+2py+2x+y)=0.$$

The component equations are

$$\frac{dy}{dx}=0,\ \frac{dy}{dx}+1=0,\ \frac{dy}{dx}(x+2y)+2x+y=0.$$

Solution of $\frac{dy}{dx}=0$ is $y=c$.

Solution of $\frac{dy}{dx}+1=0$, is $y+x=c$.

Solution of $\frac{dy}{dx}(x+2y)+2x+y=0$,

i.e. $(x\,dy+y\,dx)+2y\,dy+2x\,dx=0$ is $xy+y^2+x^2=c$.

Therefore the complete solution of the given equation is

$$(y-c)(y+x-c)(xy+y^2+x^2-c)=0.$$

Ex. 9. (b) *Solve* $p^2+px+py+xy=0.$ **[Cal. Hons. 63]**

Hint. Equation is $(p+x)(p+y)=0$ etc.

Ex. 10. $p^3-(x^2+xy+y^2)p^2+(x^3y+x^2y^2+xy^3)p-x^3y^3=0.$

Solution. The equation on factorization is

$$(p-x^2)(p-y^2)(y-xy)=0.$$

The component equations are

$$\frac{dy}{dx}=x^2, \frac{dy}{dx}=y^2, \frac{dy}{dx}=xy\left(\equiv\frac{dy}{y}=x\,dx\right).$$

Solutions of these equations are respectively

$$3y-x^3=c,\ xy+cy+1=0,\ y=ce^{\frac{1}{2}x^2}.$$

Therefore the complete solution is

$$(3y-x^3-c)(xy+cy+1)(y-ce^{\frac{1}{2}x^2})=0.$$

Ex. 11. *Solve* $xyp^2+(x^2+xy+y^2)p+x^2+xy=0.$

Solution. The equation can be written as

$$(xp+x+y)(yp+x)=0.$$

The component equations are

$$x\frac{dy}{dx}+x+y=0 \quad\text{and}\quad y\frac{dy}{dx}+x=0$$

or $x\,dy+y\,dx+x\,dx=0$ and $y\,dy+x\,dx=0$.

Their solutions clearly are

$$xy+\frac{x^2}{2}-c=0 \quad\text{and}\quad \frac{y^2}{2}+\frac{x^2}{2}-c=0.$$

Therefore the most general solution is

$$\left(xy+\frac{x^2}{2}-c\right)\left(\frac{y^2}{2}+\frac{x^2}{2}-c\right)=0.$$

Ex. 12. *Solve* $(x^2+x)p^2+(x^2+x-2xy-y)p+y^2-xy=0.$

Solution. After factorizing, the equation becomes

$$[xp+x-y][(x+1)p-y]=0$$

The component equations are

$$x\frac{dy}{dx}+x-y=0 \quad\text{and}\quad (x+1)\frac{dy}{dx}-y=0$$

or $\quad x\,dy-y\,dx+x\,dx=0 \quad$ and $\quad \dfrac{dx}{x+1}-\dfrac{dy}{y}=0$

or $\quad \dfrac{x\,dy-y\,dx}{x^2}+\dfrac{1}{x}\,dx=0 \quad$ and $\quad \dfrac{dx}{x+1}-\dfrac{dy}{y}=0.$

Integrating these, we get

$$\frac{y}{x}+\log x+\log c=0,\ y-c(x+1)=0$$

or $\quad y+x\log(xc)=0$ and $y-c(x+1)=0.$

Therefore the most general solution is

$$[y+x \log (xc)]\,[y-c\,(x+1)]=0.$$

***Ex. 13.** *Solve* $\left(1-y^2+\frac{y^2}{x^2}\right)p^2-2\frac{y}{x}p+\frac{y^2}{x^2}=0.$

[Raj. 57, 53; Agra 62; Patna Hons. 59]

Solution. The given equation is

$$p^2-p^2y^2+\frac{y^4}{x^2}p^2-2\frac{y}{x}p+\frac{y^2}{x^2}=0$$

or $\left(p^2-2\frac{y}{x}p+\frac{y^2}{x^2}\right)=p^2y^2\left(1-\frac{y^2}{x^2}\right).$

or $\left(p-\frac{y}{x}\right)^2=p^2y^2\left(1-\frac{y^2}{x^2}\right)$

or $(px-y)=\pm py\,(x^2-y^2)^{1/2}$ or $p\,[x\pm y\sqrt{(x^2-y^2)}]-y=0.$

Thus the component equations are

$$\frac{dy}{dx}[x\pm y\sqrt{(x^2-y^2)}]-y=0 \quad\text{or}\quad \frac{dx}{dy}=\frac{x\pm y\sqrt{(x^2-y^2)}}{y}.$$

To solve it put $x=vy$, $\therefore\ \frac{dx}{dy}=v+y\frac{dv}{dy};$

$\therefore$ component equations become $v+y\frac{dv}{dy}=v\pm\sqrt{(v^2-1)}$

or $\frac{dv}{dy}=\pm\sqrt{(v^2-1)}$ or $\frac{dv}{\sqrt{(v^2-1)}}=\pm dy.$

Integrating, $\log\,[v+\sqrt{(v^2-1)}]=\pm y+c$

or $\log\frac{x+\sqrt{(x^2-y^2)}}{y}=\pm y+c.$

Ex. 14. *Solve* $p^2+2py\cot x=y^2.$ **[Banaras 59; Raj. 58]**

Solution. The equation can be written as

$$(p+y\cot x)^2=y^2\,(1+\cot^2 x)$$

or $p+y\cot x=\pm y\,\text{cosec}\,x.$

The component equations are

$$\frac{dy}{dx}=y\,(-\cot x+\text{cosec}\,x) \quad\text{and}\quad \frac{dy}{dx}=y\,(-\cot x-\text{cosec}\,x)$$

or $\frac{dy}{y}=(-\cot x+\text{cosec}\,x)\,dx,\ \frac{dy}{y}=(-\cot x-\text{cosec}\,x)\,dx.$

Integrating the first of these, we get

$$\log y=-\log\sin x+\log\tan\frac{x}{2}+\log c=\log\frac{c\tan\frac{1}{2}x}{x\sin x}$$

or $y=c\,\frac{\tan\frac{1}{2}x}{2\sin\frac{1}{2}x\cos\frac{1}{2}x}=\frac{c}{2\cos^2\frac{1}{2}x}=\frac{c}{1+\cos x}.$

Solution of first component equation is

$y(1+\cos x)=c.$

Similarly solution of the other equation is $y(1-\cos x)=c$.

Thus the complete solution of the given equation is

$$[y(1+\cos x)-c]\,[y(1-\cos x)-c]=0.$$

Ex. 15. *If the curve whose differential equation is $p^2+2py\cot x=y^2$ passes through the point $(\frac{1}{2}\pi, 1)$, show that the equation of the curve is given by*

$$(2y-\sec^2\tfrac{1}{2}x)(2y-\operatorname{cosec}^2\tfrac{1}{2}x)=0.$$ [Bombay 61]

Solution. Proceeding as in the above example the general solution of the equation is

$$[y(1+\cos x)-c]\,[y(1-\cos x)-c]=0.$$

It passes through $(\frac{1}{2}\pi, 1)$; hence

$$[1(1+\cos\tfrac{1}{2}\pi)-c]\,[1(1-\cos\tfrac{1}{2}\pi)-c]=0,$$

i.e. $(1-c)(1-c)=0,\quad 1-c=0 \quad \text{or} \quad c=1,$

$\therefore$ The required curve through $(\frac{1}{2}\pi, 1)$ is

$$[y(1+\cos x)-1]\,[y(1-\cos x)-1]=0.$$

$\therefore\ (2y\cos^2\tfrac{1}{2}x-1)(2y\sin^2\tfrac{1}{2}x-1)=0.$

$\therefore\ \cos^2\tfrac{1}{2}x\sin^2\tfrac{1}{2}x\,(2y-\sec^2\tfrac{1}{2}x)(2y-\operatorname{cosec}^2\tfrac{1}{2}x)=0$

or $(2y-\sec^2\tfrac{1}{2}x)(2y-\operatorname{cosec}^2\tfrac{1}{2}x)=0.$

Ex. 16. *Solve* $4y^2p^2+2pxy(3x+1)+3x^3=0.$

Solution. The equation can be written as

$$4y^2p^2+6px^2y+2pxy+3x^3=0,$$

i.e. $2yp(2yp+3x^2)+x(2yp+3x^2)=0$

or $(2yp+3x^2)(2yp+x)=0.$

The component equations are

$$2y\frac{dy}{dx}+3x^2=0 \quad \text{and} \quad 2y\frac{dy}{dx}+x=0.$$

Solutions of these equations are

$$y^2+x^3=c,\ y^2+\frac{x^2}{2}=c.$$

Therefore the complete solution is

$$(y^2+x^3-c)(y^2+\tfrac{1}{2}x^2-c)=0.$$

Ex. 17. *Solve* $p^2-2p\cosh x+1=0.$

Solution. The equation can be written as

$$p^2-p(e^x+e^{-x})+1=0 \quad \text{or} \quad (p-e^x)(p-e^{-x})=0.$$

The component equations are

$$\frac{dy}{dx}=e^x \quad \text{and} \quad \frac{dy}{dx}=e^{-x}.$$

Their solutions are $y=e^x+c,\ y=-e^{-x}+c.$

Therefore the comp[illegible] [illegible]tion is

$$(y-e^x+c)(y+e$$ [illegible]

Ex. 18. *Solve*

(i) $p^2-5p+6=0$. [Delhi 1959]

Ans. $(y-2x-c)(y-3x-c)=0$

(ii) $p^2-ax^3=0$. **Ans.** $25(y+c)^2-4ax^5=0$

(iii) $p^3=ax^4$. **Ans.** $27ax^7=343(y+c)^3$

(vi) $p^2-7p+12=0$. **Ans.** $y=4x+c, y=3x+c$

(v) $p^2-9p+18=0$. **Ans.** $(y-6x+c)(y^2-3x-c)=0$

(vi) $xyp^2+p(3x^2-2y^2)-6xy=0$.

Ans. $(y-cx^2)(y^2+3x^2-c)=0$

(vii) $xy^2(p^2+2)=2py^3+x^3$. [Nagpur 1958]

Ans. $(x^2-y^2+c)(x^2-y^2+cx^4)=0$

(viii) $p^2+x^3y-x^3p-yp=0$. **Ans.** $(y-ce^x)(4y-x^4+c)=0$

(ix) $3p^2y^2-2xyp+4y^2-x^2=0$.

Hint. Put $x^2-3y^2=v^2$. **Ans.** $x^2-3y^2=(c\pm 2x)^2$

7·4. Equations solvable for y*. [Karnatak 1961]

If the equation is solvable for y, we can express y explicitly in terms of x and p. Thus the equations of this type can be put as

$$y=f(x, p). \qquad ...(1)$$

Now differentiating with respect to x, we get

$$\frac{dy}{dx}=p=F\left(x, p, \frac{dp}{dx}\right), \qquad ...(2)$$

which is now an equation in two variables x and p.

Suppose the solution of (2) is

$$\phi(x, p, c)=0. \qquad ...(3)$$

Then eliminating p from (1) and (3), we get the required solution.

If p cannot be easily eliminated, then express values of x and y in terms of the parameter p in the form

$$x=\phi_1(p, c), y=\phi_2(p, c).$$

These two relations together give the complete solution of the given equation.

7·5. Lagrange's Equations

To solve the equation

$$\mathbf{y=x\phi(p)+f(p)},$$

[Bombay 1961, 58 (S); Poona 58]

Differentiating with regard to x, we get

$$p=\phi(p)+\{x\phi'(p)+f'(p)\}\frac{dp}{dx}$$

or $$p-\phi(p)=[x\phi'(p)+f'(p)]\frac{dp}{dx}$$

or $$\frac{dx}{dp}-x\frac{\phi'(p)}{p-\phi(p)}=\frac{f'(p)}{p-\phi(p)}.$$

*This will be possible only when the equation is of first degree in y.

This is linear equation in x and p and can be solved in the usual way.

Note. In case $\phi(p)=p$, the above method fails since $p-\phi(p)=0$ and we do not get a linear equation in x and p. In this case the equation is of Clairaut's form and we solve it as in § 7·7 P. 130.

Ex. 1. *Solve* $y=2px+p^4x^2$. ...(1)

Solution. Differentiating with respect to x,

$$p=2p+2x\frac{dp}{dx}+2p^4x+4p^3x^2\frac{dp}{dx}$$

or $\left(p+2x\dfrac{dp}{dx}\right)(1+2p^3x)=0.$

We discard the factor $1+2p^3x=0$.

The factor $p+2x\dfrac{dp}{dx}=0$ gives $\dfrac{2\,dp}{p}+\dfrac{dx}{x}=0$.

Integrating, $p^2x=c$. ...(2)

From (2), $p^2=c/x$. Putting this value in (1),

$y=2px+c^2$ or $y-c^2=2px$.

Squaring, $(y-c^2)^2=4p^2x^2=4\dfrac{c}{x}\cdot x^2$

or $(y-c^2)^2=4cx$ is the complete solution.

Note. From (2), $x=c/p^2$.

$\therefore$ (1) gives $y=2p.\dfrac{c}{p^2}+p^4\dfrac{c^2}{p^4}=\dfrac{2c}{p}+c^2$.

Thus $x=\dfrac{c}{p^2}$, $y=\dfrac{2c}{p}+c^2$ also together constitute the complete solution of (1).

Ex. 2. *Solve* $y=2px-p^2$. [Bombay 1961]

Solution. The equation is solved for y. Differentiating with respect to x,

$$p=2p+2x\frac{dp}{dx}-2p\frac{dp}{dx} \quad \text{or} \quad p\frac{dx}{dp}+2x-2y=0$$

or $\dfrac{dx}{dp}+\dfrac{2}{p}x=2$, linear, I.F. $=e^{\int \frac{2}{p}dp}=p^2$.

$$\therefore \quad xp^2=c+\int 2p^2\,dp=c+\tfrac{2}{3}p^3$$

or $x=cp^{-2}+\frac{2}{3}p$. ...(1)

Also putting this value of x in given equation,

$$y=2p\,(cp^{-2}+\tfrac{2}{3}p)-p^2$$
$$=2cp^{-1}+\tfrac{1}{3}p^2. \quad ...(2)$$

(1) and (2) together constitute general solution of the given equation.

***Ex. 3.** *Solve* $y=-px+x^4p^2$. **[Calcutta 59, 54 ; Gujrat 61 ; Poona 65, 60 ; Delhi Hons. 59 ; Raj. 56]**

Solution. Differentiating with respect to x,

$$p=-p-x\frac{dp}{dx}+4x^3p^2+2x^4p\,\frac{dp}{dx},$$

i.e. $$2p+x\frac{dp}{dx}-2px^3\left(2p+x\frac{dp}{dx}\right)=0$$

or $$\left(2p+x\frac{dp}{dx}\right)(1-2px^3)=0.$$

Rejecting the factor $1-2px^3$, we get

$$x\frac{dp}{dx}+2p=0 \quad \text{or} \quad \frac{dp}{p}+\frac{2\,dx}{x}=0.$$

Integrating, $p=\dfrac{c}{x^2}$.

Putting this value of p in (1), we get

$$y=-(c/x^2)\,x+x^4\,(c^2/x^4) \quad \text{or} \quad y=-c/x+c^2,$$

which is the required solution of the equation.

Ex. 4. *Solve* $x-yp=ap^2$. **[Karnatak 63]**

Solution. Solving for y, $y=x/p-ap$.

Differentiating, $p=\dfrac{1}{p}-\dfrac{x}{p^2}\dfrac{dp}{dx}-a\dfrac{dp}{dx}$,

i.e. $$\frac{dp}{dx}(ap^2+x)=p\,(1-p^2).$$

This can be put as $$\frac{dx}{dp}-x.\frac{1}{p\,(1-p^2)}=\frac{ap}{1+p^2}, \quad \ldots(1)$$

which is a linear equation in x and p.

Integrating factor $=e^{\int \frac{dp}{p\,(1-p^2)}}$.

Now $$\int\frac{dp}{p\,(1-p^2)}$$

$$=\int\frac{dp}{p\,(1-p)\,(1+p)}=\int\left\{\frac{1}{p}+\frac{1}{2\,(1-p)}-\frac{1}{2\,(1+p)}\right\}dp$$

$$=\log p-\tfrac{1}{2}\log(1-p)-\tfrac{1}{2}\log(1+p)=\log\frac{p}{\sqrt{(1-p^2)}}.$$

$\therefore$ Integrating factor $=e^{-\log\frac{p}{\sqrt{(1-p^2)}}}=\dfrac{\sqrt{(1-p^2)}}{p}$.

Solution of (1) is

$$\frac{x\sqrt{(1-p^2)}}{p}=c+\int\frac{ap}{1-p^2}\cdot\frac{\sqrt{(1-p^2)}}{p}dp$$

or $$x=\frac{p}{\sqrt{(1-p^2)}}(c+a\sin^{-1}p).$$

Putting this value of x in the given equation, we get

$$y=\frac{1}{\sqrt{(1-p^2)}}(c+a\sin^{-1}p)-ap. \quad ...(3)$$

(2) and (3) together constitute solution of the given equation.

Ex. 5. *Solve* $p^2-py+x=0$.

Solution. Solving for y, $y=p+x/p$.

Differentiating, $p=\dfrac{dp}{dx}+\dfrac{1}{p}-\dfrac{x}{p^2}\dfrac{dp}{dx}$

or $\left(p-\dfrac{1}{p}\right)\dfrac{dx}{dp}+\dfrac{x}{p^2}=1$ or $\dfrac{dx}{dp}+\dfrac{1}{p(p^2-1)}x=\dfrac{p}{p^2-1}$,

which is a linear equation in x and p.

Now proceed as in the above example or put $a=-1$ in the above example.

Ex. 6. *Solve* $y=3x+\log p$. [Nagpur 61]

Solution. The equation is solved for y. Differentiating w.r.t. 'x', we get $p=3+\dfrac{1}{p}\dfrac{dp}{dx}$ or $p(p-3)=\dfrac{dp}{dx}$

or $$dx=\frac{dp}{p(p-3)}=\tfrac{1}{3}\left[\frac{1}{(p-3)}-\frac{1}{p}\right]dp.$$

Integrating, $x=\tfrac{1}{3}\log\dfrac{p-3}{p}+\log C_1$

or $\dfrac{p-3}{p}=ce^{3x}$ or $p=\dfrac{3}{(1-ce^{3x})}$.

Putting this value of p in the given equation, the solution is

$$y=3x+\log\frac{3}{(1-ce^{3x})}.$$

Ex. 7. (a) *Solve* $y-2px=f(xp^2)$. [Allahabad 59]

Solution. Solving for y, $y=2px+f(xp^2)$.

Differentiating w.r.t. 'x', we get

$$p=2p+2x\frac{dp}{dx}+f'(xp^2)\left[p^2+x.2p\frac{dp}{dx}\right].$$

or $$\left(p+2x\frac{dp}{dx}\right)[1+pf'(xp^2)]=0,$$

so that $p+2x\dfrac{dp}{dx}=0$, *i.e.* $\dfrac{2dp}{p}+\dfrac{dx}{x}=0$.

Integrating, $2\log p+\log x=\log c$, *i.e.* $p^2x=c$.

Putting $p=\sqrt{c}/\sqrt{x}$ in the given equation,

$$y=2\sqrt{(cx)}+f(c),$$

which is the required solution.

Ex. 7. (b) *Solve* $y=2px-xp^2$. **[Rajasthan 60]**

Solution. This is a particular case of the above example.

Ex. 8. *Solve* $\left(\frac{dy}{dx}\right)^3+m\left(\frac{dy}{dx}\right)^2=a(y+mx)$. **[Karnatak 61]**

Solution. Solving for y, the equation is

$$ay=-amx+mp^2+p^3. \quad ...(1)$$

Differentiating w.r.t. 'x' we get

$$ap=-am+2mp\frac{dp}{dx}+3p^2\frac{dp}{dx}$$

or $\frac{dp}{dx}=\frac{a(p+m)}{2mp+3p^2}$

or $a\,dx=\frac{2mp+3p^2}{p+m}dp=\left(3p-m+\frac{m^2}{p+m}\right)dp.$

Integrating, $ax=c+\frac{3}{2}p^2-mp+m^2\log(p+m)$, ...(2)

so that from (1),

$$ay=-m[c+\tfrac{3}{2}p^2+mp+m^2\log(p+m)]+mp^2+p^3. \quad ...(3)$$

(2) and (3) together constitute solution of the equation.

Ex. 9. *Solve* $y=x+a\tan^{-1}p$.

Solution. The equation is solved for y.

Differentiating, $p=1+\frac{a}{1+p^2}\frac{dp}{dx}$ or $a\frac{dp}{dx}=(p-1)(1+p^2)$.

or $\frac{a\,dp}{(p-1)(p^2+1)}=dx$, i.e. $\frac{a}{2}\left[\frac{1}{p-1}-\frac{p+1}{p^2+1}\right]dp=dx.$

Integrating, $\frac{a}{2}[\log(p-1)-\frac{1}{2}\log(p^2+1)-\tan^{-1}p]=x+c.$

This relation together with the given equation constitutes the solution of the equation.

Ex. 10. *Solve* $xp^2-2yp+ax=0$.

Solution. Solving for y, $y=\frac{1}{2}\frac{ax}{p}+\frac{1}{2}xp.$

Differentiating, $p=\frac{1}{2}\left(\frac{a}{p}-\frac{ax}{p^2}\frac{dp}{dx}\right)+\frac{1}{2}\left(p+x\frac{dp}{dx}\right)$

i.e. $x\frac{dp}{dx}\left(1-\frac{a}{p^2}\right)=\left(p-\frac{a}{p}\right)$ or $\frac{(p^2-a)}{p^2}\left[x\frac{dp}{dx}-p\right]=0$

i.e. $x\frac{dp}{dx}=p$ or $\frac{dp}{p}=\frac{dx}{x}$.

Integrating, $p=cx$.

Putting this value of p in the given equation, we have

$$c^2x^3-2ycx+ax=0 \quad i.e., \quad 2y=cx^2+a/c,$$

which is the required solution.

Ex 11. *Solve :*

(*i*) $4y=x^2+p^2$. **Ans.** $\log(p-x)=\dfrac{x}{p-x}+c$.

(*ii*) $y=(1+p)x+p^2$. **Ans.** $x=2(1-p)+ce^{-p}$.

(*iii*) $4p^3+3px=y$. **Ans.** $x=-\frac{12}{7}p^2+\dfrac{c}{3}p^{-3/2}$.

(*iv*) $y=\dfrac{1}{\sqrt{(1+p^2)}}+b$. **Ans.** $(x+c)^2+(y-b)^2=1$.

7·6. Equations solvable for x

If x can be expressed explicitly in terms of y and p, then the equation is said to be solvable for x. Such an equation can be put in the form

$$x=f(y,p). \qquad \text{...(1)}$$

Differentiate it with respect to y to obtain

$$\frac{dx}{dy}=\frac{1}{p}=F\left(y,p,\frac{dp}{dy}\right)$$

which can be solved as an equation in y and p.

Suppose the solution is $\phi(y,p,c)=0$. ...(2)

Then eliminating p from (1) and (2), we get the primitive of the equation.

If elimination is not possible then values of x and y expressed in terms of parameter p together constitute the solution of the equation.

Ex. 1. *Solve* $y=3px+6p^2y^2$.

Solution. Solving for x, $3x=\dfrac{y}{p}-6py^2$.

Differentiating w.r.t. y, $\dfrac{3}{p}=\dfrac{1}{p}-\dfrac{y}{p^2}\dfrac{dp}{dy}-6y^2\dfrac{dp}{dy}-12py$

i.e. $(1+6p^2y)\left(2p+y\dfrac{dp}{dy}\right)=0.$

Neglecting the first factor, we get $2p+y\dfrac{dp}{dy}=0$.

Integrating it, $py^2=c$, *i.e.* $p=c/y^2$.

Putting this value of p in the given equation,

$$y=3x\frac{c}{y^2}+6y^2\cdot\frac{c^2}{y^4}, \quad i.e. \quad y^3=3cx+6c^2$$

which is the required solution.

***Ex. 2.** *Solve* $y=2px+y^2p^3$. [Rajasthan 1960, 65; Saugar 63; Delhi 63, 61; Patna Hons. 60, 51; Bihar Hons. 56; Gujrat 61]

Solution. Solving for x, $2x=\dfrac{y}{p}-y^2p^2$.

Differentiating w.r.t. y, $\dfrac{2}{p}=\dfrac{1}{p}-\dfrac{y}{p^2}\dfrac{dp}{dy}-2y^2p\dfrac{dp}{dy}-2p^2y$

or $\dfrac{1}{p}+2p^2y+\dfrac{y}{p}\dfrac{dp}{dy}\left(\dfrac{1}{p}+2p^2y\right)=0$

or $\left(\dfrac{1}{p}+2p^2y\right)\left(1+\dfrac{y}{p}\dfrac{dp}{dy}\right)=0.$

Neglecting the first factor, we get $1+\dfrac{y}{p}\cdot\dfrac{dp}{dy}=0$

i.e. $\dfrac{dp}{p}+\dfrac{dy}{y}=0$, *i.e.* $py=c$, on integration.

Now putting $p=c/y$ in the given equation.

$$y=2\frac{c}{y}.x+y^2.\frac{c^3}{y^3} \quad\text{or}\quad y^2=2cx+c^3$$

which is the required solution.

Ex. 3. *Solve* $p=\tan\left(x-\dfrac{p}{1+p^2}\right)$.

Solution. When solved for x, the equation becomes

$$x=\tan^{-1}p+\frac{p}{1+p^2}, \qquad \text{...(1)}$$

Differentiating w.r.t. y, $\dfrac{1}{p}=\dfrac{1}{(1+p^2)}\dfrac{dp}{dy}+\dfrac{(1+p^2)-2p^2}{(1+p^2)^2}\dfrac{dp}{dy}$

or $dy=\dfrac{2p\,dp}{(1+p^2)^2}$.

Integrating, $y=c-\dfrac{1}{(1+p^2)}$. ...(2)

Equations (1) and (2) together constitute the solution.

Ex. 4. *Solve* $x=y+p^2$.

Solution. The equation is solved for x; differentiating w.r.t. y,

$\dfrac{1}{p}=1+2p\dfrac{dp}{dy}$, *i.e.* $\dfrac{dp}{dy}=\dfrac{1+p}{2p^2}$

or $\dfrac{2p^2}{1-p}dp=dy$ or $-2\left(p+1+\dfrac{1}{p-1}\right)dp=dy$.

Integrating, $c-2\left[\dfrac{p^2}{2}+p+\log(p-1)\right]=y$

or $y=c-[p^2+2p+2\log(p-1)]$. ...(1)

Putting this value of y in given equation,

$x=c-[2p+2\log(p-1)]$. ...(2)

(1) and (2) together constitute the solution.

Ex. 5. *Solve* $y^2\log y=xpy+p^2$. [Allahabad 1959]

Solution. When solved for x, we get

$$x=\frac{y\log y}{p}-\frac{p}{y}.$$

Differentiating w.r.t. 'y', we get

$$\frac{dx}{dy}=\frac{1}{p}=(1+\log y)\frac{1}{p}-\frac{1}{p^2}\frac{dp}{dy}.y\ \log y-\frac{1}{y}\frac{dp}{dy}+\frac{p}{y^2}$$

or $\left(1+\frac{y^2}{p^2}\log y\right)\left(\frac{p}{y^2}-\frac{1}{y}\frac{dp}{dy}\right)=0.$

Neglecting the first factor, $\frac{dp}{dy}=\frac{p}{y}$ or $\frac{dp}{p}=\frac{dy}{y}.$

Integrating, $\log p=\log y+\log c$ *i.e.* $p=cy.$

Putting this value of p in given equation,

$$y^2\log y=xy.cy+c^2y^2 \quad \text{or} \quad \log y=cx+c^2.$$

Ex. 6. *Solve* $p^3-4xyp+8y^2=0.$ **[Agra 1955; Raj. 57]**

Solution. When solved for x, we get

$$x=\frac{2y}{p}+\frac{p^2}{4y}$$

Differentiating w.r.t. y,

$$\frac{dx}{dy}=\frac{1}{p}=\frac{2}{p}-\frac{2y}{p^2}\frac{dp}{dy}+\frac{p}{2y}\frac{dp}{dy}-\frac{p^2}{4y^2}$$

or $\left(\frac{2y}{p^2}\frac{dp}{dy}-\frac{1}{p}\right)\left(1-\frac{p^3}{4y^2}\right)=0$

i.e. $\frac{2y}{p^2}\frac{dp}{dy}-\frac{1}{p}=0$ or $\frac{2\,dp}{p}-\frac{dy}{y}=0.$

Integrating, $p^2=cy.$

Putting this value of p^2 in given equation,

$$(cy-4xy)\,p=-8y^2 \quad \text{or} \quad 64y^2=(c-4x)^2\,cy$$

or $64y=c\,(c-4x)^2$ which is the required solution.

Ex. 7. *Solve* $y=2px+p^2y.$

Solution. Solving for x, we get $2x=-py+\frac{y}{p}.$

Differentiating w.r.t. y, we get

$$\frac{2}{p}=-p-y\frac{dp}{dy}+\frac{1}{p}-\frac{y}{p^2}\frac{dp}{dy}$$

i.e. $\frac{1}{p}+p=-y\left(1+\frac{1}{p^2}\right)$

or $\left(y\frac{dp}{dy}+p\right)\left(1+\frac{1}{p^2}\right)=0$

or $y\frac{dp}{dy}+p=0$ or $\frac{dp}{p}+\frac{dy}{y}=0.$

Integrating, $\log p+\log y=\log c$ or $py=c,\ p=c/y.$

Putting this value of p in the given equation, the solution is

$$y=\frac{2c}{y}x+\frac{c^2}{y^2}y$$

or $y^2=2cy+c^2$.

Ex. 8. *Solve* $yp^2-2xp+y=0$.

Solution. Solving for x, $2x=yx+y/p$.

Differentiating w.r.t. y,

$$\frac{2}{p}=p+y\frac{dp}{dy}+\frac{1}{p}-\frac{y}{p^2}\frac{dp}{dy}$$

or $\frac{1}{p}-p=y\frac{dp}{dy}\left(1+\frac{1}{p^2}\right)$.

or $\left(y\frac{dp}{dy}+p\right)\left(1-\frac{1}{p^2}\right)=0$ or $y\frac{dp}{dy}+p=0$

i.e. $\frac{dp}{p}+\frac{dy}{y}=0$ or $py=c$, $p=c/y$.

Putting this value of p in the given equation, the solution is

$$y.\frac{c^2}{y^2}-2x.\frac{c}{y}+y=0 \quad i.e, \quad y^2=2cx-c^2.$$

Ex. 9. *Solve* (*i*) $x=y+a\log p$.

(*ii*) $y=yp^2+2px$. **[Delhi 62 ; Lucknow 63]**

Ans. $y^2=2cx+c^2$.

(*iii*) $ayp^2+(2x-b)p-y=0$. **[Calcutta Hons. 58; Andhra 50]**

Differentiating w.r.t. y, we get $py=c$.

Ans. $ac^2+(2x-b)c-y^2=0$.

(*iv*) $x^2+p^2x=yp$. **Ans.** $x=[\frac{1}{3}p^2+c\sqrt{p}]$,

$$y=\frac{(c\sqrt{p}-\frac{1}{3}p^2)^2}{p}+p(c\sqrt{p}-\frac{1}{3}p^2).$$

*7·7. Clairaut's Equation $y=px+f(p)$. ...(1)

[Calcutta Hons. 61 ; Gujarat (Prin.) 61 ; Bombay (Sub.) 61]

The differential equation of the form (1) is known as Clairaut's equation.

To solve $y=px+f(p)$.

Differentiating it w r.t. x, we get

$$p=p+[x+f'(p)]\frac{dp}{dx}, \ i.e. \ [x+f'(p)]\frac{dp}{dx}=0.$$

Neglecting $x+f'(p)=0$, we get $\frac{dp}{dx}=0$.

Integrating it, we get $p=c$.

Putting $p=c$ in (1), the required solution is

$$y=cx+f(c).$$

Thus to find the solutions of Clairaut's equation put c for p in the equation.

Note. If we eliminate p between $x+f'(p)=0$ and the given equation, we get an equation involving no constant ; this is called the singular solution of the equation and will be discussed in the next chapter.

7·8. Equations Reducible to Clairaut's Form

It is sometimes possible to reducc a given equation in Clairaut's form with the help of suitable substitutions. The following two substitutions may be noted in this connection :

1. *Equation* $y^2=pxy+f\left(p\dfrac{y}{x}\right)$.

Putting $y^2=Y$, and $x^2=X$, *i.e.* $\dfrac{y}{x}\dfrac{dy}{dx}=\dfrac{dY}{dX}$, the equation

becomes $y^2=p\dfrac{y}{x}.x^2+f\left(p\dfrac{y}{x}\right)$ or $Y=\dfrac{dY}{dX}Y+f\left(\dfrac{dY}{dX}\right)$

which is of Clairaut's form.

2. *Equation* $e^{my}(c-mp)=f(pe^{my-ex})$.

This can be reduced to Clairaut's form by the substitutions
$e^{my}=Y$ and $e^{cx}=X$.

Ex. 1. *Solve* $px-y+p^3=\dfrac{m^3}{p^3}$. [Bombay 61]

Solution. The equation is $y=px+p^3-\dfrac{m^3}{p^3}$.

This is of Clairaut's form. Hence putting c for p, the solution is $y=cx+c^3-\dfrac{m^3}{c^3}$.

Ex. 2. *Solve* $y=px+p-p^2$. [Bombay 61 ; Calcutta 63]

Solution. The equation is of Clairaut's form.

Hence putting c for p, the general solution is

$$y=cx+c-c^2$$

Ex. 3 *Solve* $(y-px)(p-1)=p$. [Poona 64 ; Nag. T.D.C 61]

Solution. The equation can be writ'en as

$$y-px=\frac{p}{p-1} \quad \text{or} \quad y=px+\frac{p}{p-1}$$

which is of Clairaut's form. Hence putting c for p, the solution is

$$y=cx+\frac{c}{c-1}.$$

Ex. 4. *Solve* $\sin px \cos y=\cos px \sin y+p$. [Agra 1950, 77]

Solution. The equation can be written as

$\sin(px-y)=p$ or $y=px-\sin^{-1}p$.

This is of Clairaut's form.

Hence putting c for p, the solution is $y=px-\sin^{-1}c$.

***Ex. 5.** *Solve* $p=\tan(px-y)$. [Poona 1961]

Solution. The equation can be written as

$$\tan^{-1} p = px - y \quad \text{or} \quad y = px - \tan^{-1} p$$

which is of Clairaut's form. Hence putting c for p, the solution is

$$y = cx - \tan^{-1} c.$$

Ex. 6. *Solve* $(y-px)^2 = 1+p^2$.

Solution. Here we have $y = px \pm \sqrt{(1+p^2)}$.

Both the factors are of the Clairaut's form ; their solutions are

$$y = cx \pm \sqrt{(1+c^2)}.$$

Therefore the primitive is

$$[y - cx - \sqrt{(1+c^2)}]\,[y - cx + \sqrt{(1+c^2)}] = 0$$

or $(y-cx)^2 = 1+c^2$.

Ex. 7. *Solve* $p^2 x\,(x-2) + p\,(2y - 2xy - x + 2) + y^2 + y = 0$.

Solution. The equation may be written as

$$(y - px + 2p)\,(y - px + 1) = 0.$$

Each factor is of Clairaut's form. Hence putting c for p in each factor, the solution is

$$(y - cx + 2c)\,(y - cx + 1) = 0.$$

Ex. 8. *Solve* $\left(\dfrac{dy}{dx}\right)^2 (x^2 - a^2) - 2\left(\dfrac{dy}{dx}\right) xy + y^2 - b^2 = 0.$ [Gauhati 1962]

Solution. We have $p^2x^2 - 2pxy + y^2 = a^2p^2 + b^2$

or $(y-px)^2 = a^2p^2 + b^2$

i.e. $y = px \pm \sqrt{(a^2p^2 + b^2)}$.

Both these are in Clairaut's form. Hence the solution is

$$y = cx \pm \sqrt{(a^2c^2 + b^2)}.$$

Ex. 9. *Solve*

(*i*) $y = px + p^2$. [Delhi 1951]

Ans. $y = cx + c^2$.

(*ii*) $xp^2 - yp + 2 = 0$. [Rajasthan 1952]

Equation is $y = px + \dfrac{2}{p}$ Ans. $y = cx + \dfrac{2}{c}$.

(*iii*) $p = \log\,(px - y)$. [Karnatak 1964]

Equation is $y = px - e^p$. Ans. $y = cx - e^c$.

(*iv*) $y^2 + x^2\left(\dfrac{dy}{dx}\right)^2 - 2xy\,\dfrac{dy}{dx} = 4\left(\dfrac{dx}{dy}\right)^2$. [Gauhati 1964]

Equation is $(y-px)^2 = \dfrac{4}{p^2}$ or $y = px \pm \dfrac{2}{p}$. Ans. $(y-cx)^2 = \dfrac{4}{c^2}$

(*v*) $(x-a)\,p^2 + (x-y)\,p - y = 0$. [Bihar 1954 ; Patna 53]

Equation is $y = px - \dfrac{ap}{p+1}$. Ans. $y = cx - \dfrac{ac}{c+1}$.

(*vi*) $y = px + \sqrt{(a^2p^2 + b^2)}$. [Delhi Hons. 1967]

Examples Reducible to Clairaut's Form. Sometimes by suitably changing the variables the differential equation can be reduced to Clairaut's form and then its solution can be easily found.

Ex. 10. *Solve* $x^2(y-px)=yp^2$.

[Saugar 1966 ; Delhi 50 ; Nagpur 57 ; Patna Hon's. 53]

Solution. Put $x^2=u,\ 2x\,dx=du$ and $y^2=v,\ 2y\,dy=dv$ $\Big\}\ \dfrac{y}{x}\dfrac{dy}{dx}=\dfrac{dv}{du}$

or $\dfrac{y}{x}p=\dfrac{dv}{du}$ or $p=\dfrac{x}{y}\dfrac{dv}{du}$.

Thus the given equation becomes

$$x^2\left(y-\frac{x^2}{y}\frac{dv}{du}\right)=y.\frac{x^2}{y^2}\left(\frac{dv}{du}\right)^2$$

or $\left(y^2-x^2\dfrac{dv}{du}\right)=\left(\dfrac{dv}{du}\right)^2$ or $v=u\dfrac{dv}{du}+\left(\dfrac{dv}{du}\right)^2$

which is an equation in Clairaut's form. The solution is

$v=cu+c^2$ or $y^2=cx^2+c^2$.

Ex. 11. *Reduce the equation* $y^2(y-xp)=x^4p^2$ *where* $p\equiv\dfrac{dy}{dx}$, *to Clairaut's form by the substitution* $x=1/X,\ y=1/Y$ *and hence solve the equatian.*

[Patna Hons. 1945]

Solution When $x=1/X,\ y=1/Y$, we have

$$dx=-\frac{1}{X^2}\,dX,\ dy=-\frac{1}{Y^2}\,dY \text{ so that } p=\frac{dy}{dx}=\frac{X^2}{Y^2}\frac{dY}{dX}.$$

Putting this value of p in given equation, we get

$$\frac{1}{Y^2}\left(\frac{1}{Y}-\frac{1}{X}.\frac{X^2}{Y^2}\frac{dY}{dX}\right)=\frac{1}{X^4}.\frac{X^4}{Y^4}\left(\frac{dY}{dX}\right)^2$$

or $\left(Y-X\dfrac{dY}{dX}\right)=\left(\dfrac{dY}{dX}\right)^2$ or $Y=X\dfrac{dY}{dX}+\left(\dfrac{dY}{dX}\right)^2$

which is of Clairaut's form. The solution is

$Y=Xc+c^2$ or $1/y=c/x+c^2$ as $x=1/X$ and $y=1/Y$.

***Ex. 12.** *Solve* $(y+xp)^2=x^2p$ *(put* $xy=v$*).* [Poona 61]

Solution. Putting $xy=v$,

$$x\frac{dy}{dx}+y=\frac{dv}{dx},\quad i.e.\ xp+y=P \text{ where } \frac{dv}{dx}=P.$$

$\therefore$ Equation becomes $p^2=x(P-y)=xP-v$,

or $$v=xP-P^2$$

which is of Clairaut's form. Hence solution is

$v=xc-c^2$ or $xy=cx-c^2$.

Ex. 13. *Solve* $e^{3x}(p-1)+p^3e^{2y}=0$.

[Alld. 60; Bombay 61 ; Gujrat 61; Raj. 59]

Solution Put $e^x=u,\ e^y=v$, so that

$$\frac{dv}{du}=\frac{e^y}{e^x}\frac{dy}{dx}=\frac{v\,dy}{u\,dx}.$$

$\therefore \quad p=\dfrac{dy}{dx}=\dfrac{u}{v}P$ where $P=\dfrac{dv}{du}$

and then the given equation becomes

$$u^3\left(\frac{u}{v}P-1\right)+\frac{u^3}{v^3}P^3v^2=0,$$

i.e. $uP-v+P^3=0$

or $v=uP+P^3$, Clairaut's form.

Hence solution is

$v=uc+c^3$ or $e^y=ce^x+c^3$.

Ex. 13. *Solve* $(px-y)(py+x)=h^2p$, *where* $x\equiv dy/dx$ *using the transformation* $x^2=u$, $z^2=v$.

[Gorakhpur 59 ; Bihar Hons. 55 ; Delhi Hons. 59; Allahabad 59 ; Saugar 59]

Solution. When $x^2=u$, $y^2=v$,

$$p=\frac{dy}{dx}=\frac{x}{y}\frac{dv}{du}=\frac{x}{y}P \text{ (say).}$$

Then the given equation becomes

$$\left(\frac{x}{y}Px-y\right)\left(\frac{x}{y}Py+x\right)=h^2.\frac{x}{y}P \quad \text{or} \quad (Px^2-y^2)(P+1)=h^2P$$

or $(Pu-v)(P+1)=h^2P$ or $v=Pu-\dfrac{h^2P}{P+1}$

which is of Clairaut's form. The solution is

$$v=cu-\frac{h^2c}{c+1} \quad \text{or} \quad y^2=cx^2-\frac{ch^2}{c+1}.$$

Ex. 15. *Reduce the differential equation* $(px-y)(x-yp)=2p$ *to Clairaut's form by the substitution* $x^2=u$, $y^2=v$ *and find its complete primitive.* **[Bihar 61; Calcutta 54, Agra 71, 54; Raj. 49]**

Solution. When $x^2=u$, $y^2=v$

$$p=\frac{dy}{dx}=\frac{x}{y}\frac{dv}{du}=\frac{x}{y}P \text{ (say);}$$

then the equation becomes

$$\left(\frac{x^2}{y}P-y\right)(x-xP)=2\frac{x}{y}P \quad \text{or} \quad (x^2P-y^2)(1-P)=2P$$

or $(uP-v)(1-P)=2P$ or $v=Pu-\dfrac{2P}{1-P}$

which is of Clairaut's form. The solution is

$$v=cu-\frac{2c}{1-c} \quad \text{or} \quad y^2=cx^2-\frac{2c}{1-c}$$

or $c^2x^2-c(x^2-y^2-2)+y^2=0$.

Ex. 16. *Solve* $x^2p^2+yp(2x+y)+y^2=0$; *use the substitution* $y=u$, $xy=v$. **[Bombay 58 (S)]**

Solution. We have $y=u,\ xy=v$.

Now $\dfrac{du}{dx}=\dfrac{dy}{dx}=p$ and $\dfrac{dv}{dx}=x\dfrac{dy}{dx}+y=px+y$.

$$\therefore\quad P=\frac{dv}{du}=\frac{dv/dy}{du/dx}=\frac{xp+y}{p}$$

so that $p=\dfrac{y}{P-x}$.

Putting this value of p in the given equation, we have

$$\frac{x^2y^2}{(P-x)^2}+\frac{y^2}{(P-x)}(2x+y)+y^2=0$$

i.e. $x^2+(P-x)(2x+y)+(P-x)^2=0,$

i.e. $Py-xy+P^2=0$

i.e. $xy=Py+P^2$ or $v=Pu+P^2$

which is of Clairaut's form. Hence the solution is

$v=cu+c^2$ or $xy=cy+c^2$.

Ex. 17. *Solve* $y\left(\dfrac{dy}{dx}\right)^2+x^3\dfrac{dy}{dx}-x^2y=0$. *By putting* $x^2=u$, $y^2=v$. **[Poona 59 (S)]**

Solution. We have $P=\dfrac{dv}{du}=\dfrac{2y}{2x}\dfrac{dy}{dx}=\dfrac{y}{x}\dfrac{dy}{dx}=\dfrac{y}{x}p$

so that $p=\dfrac{x}{y}P$.

Putting this value of p, the given equation becomes

$$y\frac{x^2}{y^2}P^2+x^3\frac{x}{y}P-x^2y=0$$

i.e. $P^2+x^2P-y^2=0$, *i.e.* $P^2+uP-v=0$

or $v=uP+P^2$, Clairaut's form.

Solution is $v=cu+c^2$ or $y^2=cx^2+c^2$.

Ex. 18. *Reduce the equation*

$$axyp^2+(x^2-ay^2-b)\,p-xy=0$$

to Clairaut's form and hence solve the equation.

[Allahabad 1960 ; Raj. 52, Patna Hons. 46]

Solution. Let us put $x^2=u,\ ay^2+b=v$

or $\dfrac{dv}{du}=\dfrac{dv/dx}{du/dx}=\dfrac{2ayp}{2x}=a\dfrac{y}{x}p$

$\therefore\quad p=\dfrac{x}{ay}P$ where $P=\dfrac{dv}{du}$.

Putting this value of p in the given equation, it becomes

$$axy.\ \frac{x^2}{a^2y^2}P^2+(x^2-ay^2-b)\,\frac{x}{ay}P-xy=0,$$

$$uP^2+(u-v)\,P-(v-b)=0$$

or $Pu\,(P+1)-v\,(P+1)=-b$ or $Pu-v=-b/(P+1)$

or $\quad v=uP+b/(P+1)$, Clairaut's form.

$\therefore$ solution is $v=uc+b/(c+1)$.

or $ay^2+b=cx^2+b/(c+1)$.

Ex. 19. *Solve* $y=2px+yp^2$. **[Patna Hons. 1950]**

Solution. Put $y^2=v$, so that $2yp=\dfrac{dv}{dx}=P$ (say).

The equation becomes $y^2=2pxy+y^2p^2$

or $v=Px+\frac{1}{4}P^2$, Clairaut's form.

Therefore the solution is $v=cx+\frac{1}{4}c^2$ or $y^2=cx+\frac{1}{4}c^2$.

Ex. 20. *Reduce the following equations to Clairaut's form by suitable substitutions :*

(i) $(xp-y)(xp-2y)+x^3=0$. **[Patna Hons. 1958]**

Put $\dfrac{y}{x}=v, \qquad P=\dfrac{dv}{dx}=\dfrac{xp-y}{x^2}$.

The equation becomes $v=Px+\dfrac{1}{P}$ etc.

(ii) $x^2p^2+yp(2x+y)+y^2=0$. *Put* $y=u$, $xy=v$.

[Bombay 1961, Bihar 62, 59, Patna Hons. 57, Luck. 62]

Ans. $xy=cy+c^2$.

(iii) $3y^2p^2-2xyp+4y^2-x^2=0$. *Put* $x^2-3y^2=v^2$.

Ans. $x^2-y^2-4cx+3c^2=0$.

(iv) $ayp^2+(2x-b)p-y=0$. *Put* $y^2=v$, $2x-b=u$.

$v=uP+aP^2$. Ans. $y^2=c(2x-b)+ac^2$.

(v) $(xp-y)^2=(x^2-y^2)[\sin^{-1}(y/x)]^2$. **[Delhi Hons. 1956]**

(vi) $(x^2+y^2)(1+p^2)=(y-px)^2(1+x^2)$. **[Raj. 1955]**

Ans. $[c^2(x^2+y^2)-(y^2-1)]^2=4c^2(x^2+y^2)$.

Ex. 21. *Solve*

$(x^2+y^2)(1+p)^2-2(x+y)(1+p)(x+yp)+(x+yp)^2=0$.

[Raj. 1954 ; Delhi Hons. 63]

Solution. Let us put $x+y=u$, $x^2+y^2=v$, so that

$$P=\frac{dv}{du}=\frac{2x+2yp}{1+p}=\frac{2(x+yp)}{1+p}.$$

Now dividing the given equation by $(1+p)^2$, we get

$$(x^2+y^2)-2(x+y).\frac{x+yp}{1+p}+\frac{(x+yp)^2}{(1+p)^2}=0.$$

i.e. $v-uP+\frac{1}{4}P^2=0$

or $v=uP-\frac{1}{4}P^2$, Clairaut's form.

Hence primitive is $v=cu-\frac{1}{4}c^2$

or $x^2+y^2=c(x+y)-\frac{1}{4}c^2$

or $x^2+y^2+2a(x+y)+a^2=0$, where $2a=-c$.

Miscellaneous Examples*

Ex. 1. *Solve* $(x-a)\,p^2+(x-y)\,y-p=0$.

[Bihar Hons. 1954; Patna Hons. 53, 50]

Solution. The equation can be written as

$$y=px-\frac{ap^2}{p+1}, \qquad \text{Clairaut's form.}$$

$\therefore$ General solution is $y=cx-\dfrac{ac^2}{c+1}$

or $(x-a)\,c^2+(x-y)\,c-y=0$.

Ex. 2. *Solve* $y+x^2=(p+x)^2$.

Solution The equation is $y=2px+p^2$ etc.

Ex 3. *Solve* $x^2-p^2-2xyp+y^2=x^2\,(x^2+y^2)$.

Solution. The equation is

$(xp-y)^2=x^2\,(x^2+y^2)$ or $xp-y=\pm x\sqrt{(x^2+y^2)}$.

Putting $y=vx,\ \dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$, the equation is

$$x\left(v+x\frac{dv}{dx}\right)-vx=\pm x\sqrt{(x^2+v^2x^2)} \quad \text{or} \quad x\frac{dv}{dx}=\pm x\sqrt{(1+v^2)}$$

or $\pm dx=\dfrac{dv}{\sqrt{(1+v^2)}}$ or $c\pm x=\sinh^{-1} v$

or $v=\sinh\,(c\pm x)$ or $y=a\sinh\,(c\pm x)$.

***Ex. 4.** *Solve* $(py+nx)^2=(y^2+nx^2)\,(1+p^2)$.

[Delhi Hons. 58, Gujrat 59]

Solution. On simplification the equation becomes

$nx^2p^2-2pnxy+y^2+nx^2-n^2x^2=0$.

Now put $y=vx$, so that $p=\dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$.

$\therefore$ Equation becomes

$nx^2p^2-2pnx^2v+v^2x^2+nx^2-n^2x^2=0$.

Cancelling x^2, $np^2-2pnv+v^2+n-n^2=0$

or $(p-v)^2=(v-1)+\dfrac{v^2\,(n-1)}{n}$

or $\left(x\dfrac{dv}{dx}\right)^2=\dfrac{n-1}{n}\,(v^2+n)$ as $p-v=x\dfrac{dv}{dx}$

or $\dfrac{dv}{\sqrt{(v^2+n)}}=\pm\sqrt{\left(\dfrac{n-1}{n}\right)}\dfrac{dx}{x}$.

Integrating,

$$\log\,[v+\sqrt{(v^2+n)}]=\pm\sqrt{\left(\frac{n-1}{n}\right)}\log x+\log c$$

*For many more examples that can be reduced to Clairaut's form, see next chapter on 'Singular Solutions'.

or $\quad v+\sqrt{(v^2+n)}=cx^{\pm\sqrt{\left(\frac{n-1}{n}\right)}}$

or $\quad y+\sqrt{(y^2+nx^2)}=cx^{1\pm\left(\frac{n-1}{n}\right)}$ as $y=vx$.

Ex. 5. *Solve* $y=px+\sqrt{(1+p^2)}\,\phi(x^2+y^2)$.

Solution. Let us put $x=r\cos\theta$, $y=r\sin\theta$.

Then $dx=\cos\theta\,dr-r\sin\theta\,d\theta$, $dy=\sin\theta\,dr+r\cos\theta\,d\theta$, so that

$$p=\frac{dy}{dx}=\frac{\sin\theta\,(dr/d\theta)+r\cos\theta}{\cos\theta\,(dr/d\theta)-r\sin\theta}.$$

$$\therefore\quad p^2+1=\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]\Big/\left[\cos\theta\left(\frac{dr}{d\theta}\right)-r\sin\theta\right]^2.$$

Again $px-y=\dfrac{r^2}{\cos\theta\,(dr/d\theta)-r\sin\theta}$ on simplification.

Therefore the equation becomes

$$\frac{-r^2}{\sqrt{[r^2+(dr/d\theta)^2]}}=\phi(r^2)$$

or $\quad \left(\dfrac{dr}{d\theta}\right)^2=\dfrac{r^4}{[\phi(r^2)]^2}-r^2 \quad$ or $\quad \dfrac{dr}{d\theta}=\dfrac{\sqrt{[r^2-\{\phi(r^2)\}^2]}}{\phi(r^2)}$.

Integrating, $\quad \theta=c+\displaystyle\int\frac{\phi(r^2)}{\sqrt{[r^2-\{\phi(r^2)\}^2]}}\,dr.$

Ex. 6. *Solve* $(xp-y)^2=a\,(1+p^2)\,(x^2+y^2)^{3/2}$.

Solution. Proceeding as above, the equation becomes

$$\frac{dr}{d\theta}=\sqrt{\left(2r.\frac{1}{2a}-r^2\right)}$$

or $\quad \theta=c+\displaystyle\int\frac{dr}{\sqrt{[2r\;(1/2a)-r^2]}}=c+\text{vers}^{-1}\left(\frac{r}{\frac{1}{2}a}\right)$

as $\displaystyle\int\frac{dx}{\sqrt{(2ax-x^2)}}=\text{vers}^{-1}\frac{x}{a}$

or $\quad \tan^{-1}(y/x)=c+\text{vers}^{-1}\,[2a\sqrt{(x^2+y^2)}]$

as $\theta=\tan^{-1}y/x$, $x=\sqrt{(x^2+y^2)}$.

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*A differential equation which does not belong to any of the forms discussed so far may sometimes be solved by suitable transformation. Ex. 6. is solved here by changing to polars.

8

Singular Solutions

8·1. Singular Solutions

[Delhi Hons. 1963; Poona 64; Pb. 59; Bombay 58]

Sometimes a particular solution satisfies a differential equation but this solution cannot be obtained for any particular value of the arbitrary constant in the general solution. This is called *singular solution, i.e.* to say, a solution which is not contained in the general solution of the equation, is called a singular solution.

Illustration. Consider a differential equation

$$y=px+\frac{a}{p} \qquad \text{...(1)}$$

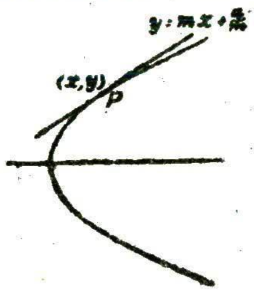
This is of Clairaut's form. Hence its solution is

$$y=mx+\frac{a}{m}, \qquad \text{...(2)}$$

where m is any arbitrary constant.

Giving different values of m we obtain different solutions, all of which satisfy (1) and they all touch a parabola

$$y^2=4ax. \qquad \text{...(3)}$$



Now consider a point P (x, y) on the parabola, the tangent at which is

$$y=mx+\frac{a}{m}.$$

At point P the tangent and the parabola have the same direction. Therefore at P, (2) and (3) both have same $\frac{dy}{dx}$ and x, y.

And since P is any point on the parabola, the equation of the parabola, *i.e.* $y^2=4ax$ must be a solution of the differential equation (1). It is evident that this solution is not contained in (2).

Therefore $y^2=4ax$, the envelope, is a singular solution of equation (1).

8·2. Discriminant

Of a quadratic equation

$$ax^2+bx+c=0$$

the discriminant is b^2-4ac.

If $b^2-4ac=0$, then the equation has equal roots. But if equation is of higher degree than two, then *the condition of two equal roots is obtained by eliminating x between $f(x)=0$ and $f'(x)=0$.*

8·3. p-discriminant and c-discriminant. [Delhi Hons. 1963]

Let $f(x, y, p)=0$

be the differential equation whose solution is

$$\phi(x, y, c)=0.$$

Then p-discriminant is obtained by eliminating p between

$$f(x, y, p)=0 \text{ and } \frac{\partial f}{\partial p}=0. \quad ...(1)$$

Also c-discriminant is obtained by eliminating c between

$$\phi(x, y, c)=0 \text{ and } \frac{\partial \phi}{\partial c}=0.$$

8·4. Important

If $E(x, y)=0$ is a singular solution of the differential equation $f(x, y, p)=0$ whose primitive is $\phi(x, y, c)=0$ then $E(x, y)$ is a factor of both the discriminants.

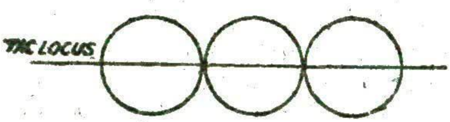
However, each discriminant may have other factors which correspond to other loci associated with the primitive. Generally equations of these loci do not satisfy the differential equation; therefore they are sometimes called *extraneous loci.*

Ex. 4. *Explain what is meant by Clairaut's form of a differential equation of the first order, and show that its complete solution is a family of curves and their envelope.* [Poona 1960]

8·5. Types of Extraneous Loci [Pb. 1956]

Tac-locus. [Pb. 1961; Delhi Hons. 62]

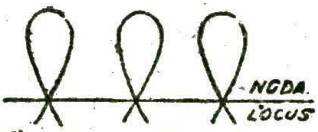
The vanishing of p-discriminant simply gives locus of the points for which two values of p become equal. In case two particular curves touch each other, the two values of p at the point of contact become equal. The point of contact is by no means a point on the envelope. If $T(x, y)=0$ is a locus of such points (pts. of contact), then $T(x, y)$ is a factor of p discriminant. In



It will be noted that a singular solution does not contain arbitrary constant. For working rule read § 8·6 and then read solved examples.

general, $T(x, y)$ is not a factor of c-discriminant and it does not satisfy the differential equation.

Nodal locus. **[Delhi Hons. 63]**



Let one of the curves of the family have a node at P. So at P there is a double point with distinct tangents. Thus at P two values of p satisfy and there can be no more than $n-1$ (if n values of p) distinct curves through P. Therefore c-discriminant vanishes at P. If there is a locus of such points it is called *nodal locus* $N(x, y)=0$. Clearly $N(x, y)$ is a factor of c-discriminant. In general, $N(x, y)$ is not a factor of the p-discriminant and it does not satisfy the differential equation.

Cusp locus. **[Punjab 61; Delhi 62]**

Let one of the curves of the family have a cusp at P. So at P there is a double point with coincident angents. So at P two values of p are equal and the p discriminant at P vanishes. Also as in the case of a node there can be no more than $n-1$ curves through P and therefore the c-discriminant also vanishes at P. The locus, if there is any, of all such points is called a *cusp*

locus $C(x, y)=0$. Clea ly $C(x, y)$ is a factor of both p and c-discriminants and it in general does not satisfy the differential equation.

Note. If the curves of the family $\phi(x, y, c)=0$ are straight lines, then there may not be a Tac locus, a Nodal locus or a Cusp locus.

8·6. General Procedure. **[Bombay 58; Karnatak 62]**

To find singular solutions of a differential equation

$f(x, y, p)=0.$

1. Find its primitive $\phi(x, y, c)=0$.
2. Find p-discriminant.
3. Find c-discriminant.

Now **p-discriminant** equated to zero may include as a factor :

1. **Envelope,** *i.e.* singular solution **once** (E).
2. **Cusp locus once** (C).
3. **Tac locus, twice** (T^2),

i.e. **p-discriminant**$\equiv ET^2C$ (*Remember*)

and **c-discriminant** equated to zero may include as a factor :

1. **Envelope** or Singular solution **once** (E),
2. **Cusp locus thrice** (C^3)

3. **Nodal locus twice (N^2).**

i.e. **c-discriminant$\equiv$EN^{2}C^3** (*Remember*).

Ex. 1. *Solve and find complete primitive and singular solution of the equation* $3y=2px-\dfrac{2p^2}{x}$. **[Gujrat 61]**

Solution. Differentiating the given equation w.r.t. x, we get

$$3p=2p+2x\frac{dp}{dx}+\frac{2p^2}{x^2}-\frac{4p}{x}\frac{dp}{dx},$$

i.e. $$\left(2x\frac{dp}{dx}-p\right)\left(1-\frac{2p}{x^2}\right)=0,$$

i.e $$2x\frac{dp}{dx}-p=0 \quad \text{or} \quad 2\frac{dp}{p}=\frac{dx}{x}$$

Integrating, $p^2=cx$.

Putting the value of p^2 in given equation, we get

$$3y=2px-2c \quad \text{or} \quad (3y+2c)^2=4p^2x^2$$

or $$(3y+2c)^2=4cx^3, \qquad \text{...(1)}$$

which is the complete primitive.

Now the given differential equation can be written as

$$2p^2-2x^2p+3xy=0 \qquad \text{...(2)}$$

and (1) is $4c^2+4c\,(3y-x^3)+9y^2=0.$ ...(3)

From (3) c-discriminant (EN^2C^3) is

$$16\,(x^3-3y)^2-144y^2=0, \quad i.e. \quad x^6-6xy^3=0$$

or $$x^3\,(x^3-6y)=0. \qquad \text{...(4)}$$

Also from (2) p-discriminant (ET^2C) is

$$x\,(x^3-6y)=0.$$

We find that factor x^3-6y occurs only once in both p and c-discriminants. Hence $x^3-6y=0$ is the singular solution.

Also x which occurs once in p-discriminant and thrice in c-discriminant is the cusp locus.

Ex. 2. *Reduce the equation* $x^2p^2+py\,(2x+y)+y^2=0$, *where* $p=dy/dx$ *to Clairaut's form by putting* $u=y$ *and* $v=xy$ *and find its complete primitive and also its singular solution.*

[Karnatak 1964; Agra 62, 78; Raj. 64, 58; Bombay 61]

Solution. We have $u=y$ and $v=xy$,

so that $$\frac{du}{dx}=\frac{dy}{dx} \quad \text{and} \quad \frac{dv}{dx}=x\frac{dy}{dx}+y$$

Now $$\frac{dv}{du}=\frac{dv/dx}{du/dx}=\frac{x\dfrac{dy}{dx}+y}{dy/dx}=\frac{xp+y}{p},$$

so that $p\dfrac{dv}{du}=xp+y$, *i.e.* $p\left(\dfrac{dv}{du}-x\right)=y$

or $p=\dfrac{y}{P-x}$, where $P=\dfrac{dv}{du}$.

Putting this value of p in $x^2p^2+py\,(2x+y)+y^2=0$, we get

$$\frac{x^2y^2}{(P-x)^2}+\frac{y}{P-x}\,y\,(2x+y)+y^2=0$$

or $y^2\,[x^2+(2x+y)\,(P-x)+(P-x)^2]=0$

or $yP-xy+P^2=0$

or $xy=yP+P^2$, *i.e.* $v=uP+P^2$,

which is of Clairaut's form.

Hence replacing P by c, the general solution is

$$v=uc+c^2 \quad \text{or} \quad xy=yc+c^2.$$

Now c-discriminant (EN^2C^3) is $y^2+4xy=0$,

i.e. $y\,(y+4x)=0$. ...(1)

And from the equation $x^2p^2+py\,(2x+y)+y^2=0$.

p-discriminant (ET^2C) is $y^2\,(2x+y)^2-4x^2y^2=0$

or $y^3.(y+4x)=0$

or $y^2.y\,(y+4x)=0$ (ET^2C). ...(2)

Now $y\,(y+4x)$ occurs both in c-and p-discriminants and both $y=0$ and $y+4x=0$ satisfy the given differential equation. Therefore $y=0$ and $y+4x=0$ are both singular solutions.

Ex. 3. *Obtain the primitive and singular solution (if it exists) of the equation* $xp^2-2yp+4x=0$. **[Raj. 1961 ; Gujrat 61]**

Solution. Equation is $xp^2-2yp+4x=0$.

To find its primitive, write the equation as

$$y=\tfrac{1}{2}xp-\frac{2x}{p} \quad \text{(solved for } y\text{)}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=p=\tfrac{1}{2}p+\tfrac{1}{2}x\frac{dp}{dx}-\frac{2}{p}+\frac{2x}{p^2}\frac{dp}{dx}$$

or $\left(\dfrac{x\,dp}{dx}-p\right)\left(\tfrac{1}{2}+\dfrac{2}{p^2}\right)=0.$

Factor $\dfrac{x\,dp}{dx}-p=0$ gives $\dfrac{dx}{x}=\dfrac{dp}{p}$,

i.e. $p=cx$.

Putting this value of p in (1), we get

$$c^2x^3-2ycx+4x=0$$

or $c^2x^2-2yc+4=0$. ...(2)

This is complete primitive of (1).

From (1) p-discriminant (EN^2C^3) is

$$y^2-4x^2=0$$

and from (2) c-discriminant (ET^2C) is

$$y^2-4x^2=0.$$

Since $y^2-4x^2=0$ is non-repeated common factor in p and c-discriminants and it satisfies the differential equation, the singular solution is

$$y^2-4x^2=0, \text{ i.e. } y=\pm 2x.$$

Ex. 4. *Obtain the complete primitive and singular solution of the equation $4xp^2=(3x-a)^2$, explaining the geometrical significance of the irrelevant factors that present themselves.*

[Agra 1976, 54, 52 ; Vikram 62]

Solution. The equation is $4xp^2=(3x-a)^2$. ...(1)

From this $p=\dfrac{dy}{dx}=\pm\frac{1}{2}\dfrac{3x-a}{\sqrt{x}}=\pm\left(\frac{3}{2}\sqrt{x}-\frac{1}{2}\dfrac{a}{\sqrt{x}}\right)$

or $dy=\pm\left(\frac{3}{2}\sqrt{x}-\frac{1}{2}\dfrac{a}{\sqrt{x}}\right)dx.$

Integrating, $y=c\pm(x^{3/2}-a\sqrt{x})$

or $(y-c)^2=x(x-a)^2$

which is the complete primitive of the equation (1).

Write it as $c^2-2cy+y^2-x(x-a)^2=0.$ ...(2)

Now from (1), p-discriminant (ET^2C) is

$4x(3x-a)^2=0$, i.e. $x(3x-a)^2=0$ ...(3)

and from (2) c-discriminant (EN^2C^3) is

$y^2-[y^2-x(x-a)^2]=0$, i.e. $x(x-a)^2=0.$ ...(4)

Non-repeated factors common in (3) and (4) give singular solution. Thus $x=0$ is singular solution.

The factor $3x-a=0$ which occurs twice in p-discriminant and is not in c-discriminant is Tac-locus.

Similarly $x-a=0$ which occurs squared only in c-discriminant gives nodal locus.

Ex. 5. *Solve and examine for singular solution the equation $xp^2-(x-a)^2=0$.* **[Vikram 1963 ; Delhi Hons. 58]**

Solution. Proceed as above. Complete primitive is

$$(y-c)^2=\tfrac{4}{9}x(x-3a)^2.$$

p-discriminant is $x(x-a)^2=0$ (ET^2C).

c-discriminant is $\frac{4}{9}x(x-3a)^2=0$ (EN^2C^3).

Thus $x=0$ is envelope or the singular solution,

$x-a=0$ is Tac-locus

and $x-3a=0$ is Nodal locus.

Ex. 6. *Find the general and singular solutions of*

$$y^2-2pxy+p^2(x^2-1)=m^2.$$

[Raj. 62 ; Pb. 56 ; Luck. Pass 59]

Solution. Equation is

$p^2(x^2-1)-2pxy+y^2-m^2=0.$ ...(1)

This can be written as $(px-y)^2=p^2+m^2$

or $px-y=\pm\sqrt{(p^2+m^2)}$

or $y=px\pm\sqrt{(p^2+m^2)}$, (Clairaut's form)

Hence general solution is

$$y=cx\pm\sqrt{(c^2+m^2)} \quad \text{or} \quad (y-cx)^2=c^2+m^2$$

or $c^2(x^2-1)-2xyc+y^2-m^2=0$. ...(2)

From (1) and (2) both p-and c-discriminants* are

$$x^2y^2-(x^2-1)(y^2-m^2)=0, \quad i.e. \quad y^2+m^2x^2=m^2,$$

which is therefore the singular solution.

Ex. 7. *Find the general and singular solution of the differential equation $(xp-y)^2=p^2-1$ where p has the usual meaning.* **[Raj. 60]**

Solution. The equation is

$$p^2(x^2-1)-2xyp+(y^2+1)=0 \quad ...(1)$$

or $xp-y=\pm\sqrt{(p^2-1)}$

$y=xp\pm\sqrt{(p^2-1)}$, Clairaut's form.

The general solution is

$$y=cx\pm\sqrt{(c^2-1)} \quad \text{or} \quad (y-cx)^2=c^2-1$$

or $c^2(x^2-1)-2xyc+y^2+1=0$. ...(2)

Obviously from (1) and (2), we have the same p-and c-discriminants, namely

$$x^2y^2-(x^2-1)(y^2+1)=0$$

or $y^2-x^2+1=0$ or $x^2-y^2=1$ (rect. hyperbola)

which are therefore singular solutions of the differential equation.

Ex. 8. *Find the camplete primitive and the singular soiution of the differential equation*

$$\sin\left(x\frac{dy}{dx}\right)\cos y=\cos\left(x\frac{dy}{dx}\right)\sin y+\frac{dy}{dx}.$$

[Agra 69 ; Raj. 53]

Solution. The equation can be written as

$$\sin(xp)\cos y-\cos(xp)\sin y=p,\ p=\frac{dy}{dx}$$

or $\sin(xp-y)=p$ or $xp-y=\sin^{-1}p$

or $y=xp-\sin^{-1}p$ Clairaut's foım.

$\therefore$ Solution is $y=cx-\sin^{-1}c$. ...(1)

Here p-and c-discriminants shall be just the same, so we find any one of them (say c-discriminant) which will be obtained by eliminating c between (1) and its differential w.r.t. c.

Differentiating (1) w.r.t. c, we get $0=x-\dfrac{1}{\sqrt{(1-c^2)}}$

or $x^2=\dfrac{1}{1-c^2}$ or $c^2=\dfrac{x^2-1}{x^2}$ or $c=\dfrac{\sqrt{(x^2-1)}}{x}$

*In the case of Clairaut's equation p-and c-discriminants are always identical.

Putting this value of c in (1), the c-discriminant is

$$=\sqrt{(x^2-1)}=\sin^{-1}\frac{\sqrt{(x^2-1)}}{x},$$

which is the singular solution.

Ex. 9. *Solve the differential equation*

$$(px^2+y^2)(px+y)=(p+1)^2$$

by reducing it to Clairaut's form and find its singular solution if it exists. [Raj. 65, 59, 55]

Solution. Let us put $x+y=u$ and $xy=v$, ...(1)

so that $1+\frac{dy}{dx}=\frac{du}{dx}$ and $x\frac{dy}{dx}+y=\frac{dv}{dx}$.

$$\therefore \quad P=\frac{dv}{du}=\frac{dv/dx}{du/dx}=\frac{xp+y}{1+p}. \quad ...(2)$$

The given equation can be written as

$$[(px+y)(x+y)-xy(p+1)](px+y)=(p+1)^2$$

Dividing by $(p+1)^2$, we get

$$\left[\frac{px+y}{p+1}(x+y)-xy\right]\frac{px+y}{p+1}=1$$

or $[Pu-v]P=1$ from (1) and (2)

or $v=Pu-\frac{1}{p}$ Clairaut's form.

$\therefore$ Its primitive is $v=cu-\frac{1}{c}$ or $c^2u-cv-1=0$

or $c^2(x+y)-cxy-1=0$ ...(3)

The c-discriminant (EN^2C^3) is

$$x^2y^2+4(x+y)=0. \quad ...(4)$$

Again equation is $p^2(x^3-1)+p(xy^2+x^2y-2)+y^3-1=0$.

$\therefore$ p-discriminant (ET^2C) is

$$(xy^2+x^2y-2)^2-4(x^3-1)(y^3-1)=0$$

or $4(x-y)^2[x^2y^2+4(x+y)]=0$. ...(5)

Therefore from (4) and (5), $x^2y^2+4(x+y)=0$, which occurs once both in p-and c-discriminants is singular solution.

Also $x-y=0$, which occurs in p-discriminant only, is the tac-locus.

Ex. 10. *Find the complete primitive and singular solution of*

$$(a^2-x^2)\left(\frac{dy}{dx}\right)^2+2xy\frac{dy}{dx}+(b^2-y^2)=0.$$

[Gujrat 1958; Delhi Hons. 56]

Solution. We have $p^2x^2-2pxy+y^2=a^2p^2+b^2$ ...(1)

or $y=px\pm\sqrt{(a^2p^2+b^2)}$. Clairaut's form

Hence the complete primitive is

$$y=cx\pm\sqrt{(a^2c^2+b^2)}.$$

The p- (or c-) discriminant is

$$4x^2y^2-4(a^2-x^2)(b^2-y^2)=0$$

or $x^2y^2-(a^2b^2-b^2x^2-a^2y^2+x^2y^2)=0$

or $b^2x^2+a^2y^2=a^2b^2$

or $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$

which is the singular solution.

Ex. 11. *Obtain the general and singular solution of $y=px+\sqrt{(b^2+a^2p^2)}$ and interpret the result geometrically.* **[Nag. 1962; Gujrat 58]**

Solution. Equation is in Clairaut's form.

∴ General solution is in Clairaut's form

As in the above example singular solution is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \qquad \text{..(2)}$$

The general solution (1) represents system of lines which all envelope the ellipse (2).

Ex. 12. *Find the singular solution of*

$y=px+\sqrt{(1-p^2)}-p\cos^{-1}p.$ **[Gujrat 1958]**

Proceed as in the above example.

***Ex. 13.** *Reduce the equation $xyp^2-(x^2+y^2-1)p+xy=0$ to Clairaut's form by substituting $x^2=u$ and $y^2=v$. Hence show that the equation represents a family of conics touching the four sides of a square.* **[Punjab 1966; Agra 58; Delhi Hons. 57; Bombay 61; Bihar Hons. 56; Patna Hons. 54]**

Solution. The equation is

$$xyp^2-(x^2+y^2-1)p+xy=0. \qquad \text{...(1)}$$

Since $x^2=u$ and $y^2=v$.

$$2x=\frac{du}{dx} \text{ and } 2y\frac{dy}{dx}=\frac{dv}{dx}, \text{ i.e. } 2yp=\frac{dv}{dx}.$$

$$\therefore\ P=\frac{dv}{du}=\frac{dv/dx}{du/dx}=\frac{2yp}{2x}=\frac{yp}{x} \quad \text{or} \quad p=\frac{xP}{y}$$

Putting this value of p in (1), the equation becomes

$$xy\frac{x^2P^2}{x^2}-(x^2+y^2-1)\frac{xP}{y}+xy=0$$

i.e. $x^2P^2-(x^2+y^2-1)P+y^2=0$

or $uP^2-(u+v-1)P+v=0$

or $u(P^2-P)-v(P-1)+P=0$

or $v=uP+\frac{P}{P-1}$, Clairaut's form.

$$\therefore\ \text{Solution is } v=uc+\frac{c}{c-1} \quad \text{or} \quad y^2=cx^2+\frac{c}{c-1}$$

or $c^2x^2-(x^2+y^2-1)\,c+y^2=0$. ...(2)

From (1) p-discriminant (ET^2C) is

$$(x^2+y^2-1)^2-4x^2y^2=0$$

or $(x^2+y^2-2xy-1)\,(x^2+y^2+2xy-1)=0$

or $[(x-y)^2-1]\,[(x+y)^2-1]=0$

or $(x-y-1)\,(x-y+1)\,(x+y-1)\,(x+y+1)=0$. ...(3)

Again from (2), c-discriminant (EN^2C^3) is

$$(x^2+y^2-1)^2-4x^2y^2=0. \quad ...(4)$$

Since (3) and (4) are the same, the singular solutions, *i.e.* envelope of the family of conics given by (2) is

$$x-y-1=0,\ x-y+1=0,\ x+y-1=0,\ x+y+1=0.$$

These four lines clearly form a square. Hence differential equation (1) represents conics (2) touching the four sides of a square.

Ex. 14. *Reduce the equation $xp^2-2py+x+2y=0$ to Clairaut's form, by putting $x^2=u$ and $y-x=v$. Hence obtain and interpret the primitive and singular solution of the equation.* [Agra 1959]

Solution. We have $x^2=u,\ y-x=v$,

$$2x=\frac{du}{dx} \text{ and } \frac{dy}{dx}-1=\frac{dv}{dx}.$$

$$\therefore\ P=\frac{dv}{du}=\frac{dv/dx}{du/dx}=\frac{p-1}{2x}$$

or $p=1+2xP$.

Putting this value of p in given equation, we get

$$x(1+2xP)^2-2y\,(1+2xP)+x+2y=0$$

i.e. $4x^3P^2+4x^2P+2x-4xyP=0$

or $4x^2P^2+4P\,(x-y)+2=0$ or $4uP^2-4vP+2=0$

or $v=uP+\dfrac{1}{2P}$, Clairaut's form.

Hence replacing P by c, the solution is

$$v=uc+\frac{1}{2c} \text{ or } y-x=x^2c+\frac{1}{2c}$$

or $2c^2x^2-2c\,(y-x)+1=0$. ...(1)

Now p-discriminant (ET^2C) from equation (1) is

$$y^2-x\,(x+2y)=0, \quad i.e. \quad y^2-x^2-2xy=0$$

or $(y-x)^2-2x^2=0$, ...(2)

and c-discriminant from (1) (EN^2C^2) is

$$(y-x)^2-2x^2=0. \quad ...(3)$$

Since $(y-x)^2-2x^2=0$, *i.e.* $y-x=\pm\sqrt{2}x$ occur only once both in p-and c-discriminants, these represent the singular solution.

Therefore the general solution (1) which represents a system of parabolas touches a pair of lines

$$y-x=\pm\sqrt{2}x.$$

Ex. 15. *Reduce the differential equation* $(px-y)(x-py)=2p$ *to Clairaut's form by the substitution* $x^2=u$ *and* $y^2=v$ *and find its complete primitive and its singular solution, if any.*

[Delhi Hons. 65, Sagar 63, Agra 66, 54; Raj. 49]

Solution. When $x^2=u$, $y^2=v$, $p=\dfrac{dy}{dx}=\dfrac{x}{v}\dfrac{dv}{du}=\dfrac{x}{y}P$ (say), the equation becomes

$$\left(\frac{x^2}{y}P-y\right)(x-xP)=\frac{x}{y}P \quad \text{or} \quad (x^2P-y^2)(1-P)=2P$$

or $(uP-v)(1-P)=2P$ or $v=Pu-\dfrac{2P}{1-P}$. Clairaut's form.

$\therefore$ General solution is $v=cu-\dfrac{2c}{1-c}$

or $y^2=cx^2-\dfrac{2c}{1-c}$ or $c^2x^2-c(x^2+y^2-2)+y^2=0$. ...(1)

The given equation is $p^2xy-p(x^2+y^2-2)+xy=0$. ...(2)

Now p-discriminant (ET^2C) is

$(x^2+y^2-2)^2-4x^2y^2=0$

or $[x^2+y^2-2-2xy][x^2+y^2-2+2xy]=0$

$[(x-y)^2-2][(x+y)^2-2]=0$

or $(x-y+\sqrt{2})(x-y-\sqrt{2})(x+y+\sqrt{2})(x+y-\sqrt{2})=0$. ...(4)

The c-discriminant (EN^2C^3) from (1) is

$(x^2+y^2-2)^2-4x^2y^2=0$. ...(5)

which is same as p-discriminant.

Hence the common factors of p-and c-discriminant occuring once in them, *i.e.*

$(x-y+\sqrt{2})(x-y-\sqrt{2})(x+y+\sqrt{2})(x+y-\sqrt{2})=0$

give singular solutions.

Ex. 16. *Solve and test for singular solution.*

$p^3-4pxy+8y^2=0$. **[Agra 55 ; Raj. 67]**

Solution. Let us put $y=z^2$, $\dfrac{dy}{dx}=2z\dfrac{dz}{dx}$,

i.e. $p=2zP$, where $P=\dfrac{dz}{dx}$.

Then equation becomes $(2zP)^3-4(2zP)xy+8y^2=0$

or $8z^3P^3-8xz^3P+8z^4=0$ as $y=z^2$.

or $z=xP-P^3$, Clairaut's form.

Hence solution is $z=c_1x-c_1^3$ or $y^{1/2}=c_1x-c_1^3$

or $y=c_1^2(x-c_1^2)^2$ or $y=c(x-c)^2$, $c_1^2=c$. ...(1)

Now the equation is $p^3-4xpy+8y^2=0$. ...(2)

This is cubic in p, hence p-discriminant will be obtained by eliminating p from (2) and differential of (2) w.r.t. p.

i.e. $3p^2-4xy=0$. ...(3)

Eliminating p between (2) and (3), the p-discriminant (ET^2C) is $y(27y-4x^3)=0$. ...(4)

Again differentiating (1) w.r t. c, we get

$(x-c)^2-2c\,(x-c)=0$ or $(x-c)\,(x-3c)=0$;

$x=c$ or $x=3c$.

Putting $c=x$ in (1), we get $y=0$.

Putting $c=\frac{1}{3}x$ in (1), we get $27y-4x^3=0$.

Therefore c-discriminant (EN^2C^3) is

$y\,(27y-4x^3)=0$. ...(5)

Once occuring factors both in p-and c-discriminant give singular solutions. Hence

$y=0$ and $27y-4x^3=0$ are singular solutions.

Hence when $c=0, y=0$ from (1) also, hence $y=0$ is particular integral of the equation.

Ex. 17. *Find the singular solution of*

$$x^2\left(y-x\frac{dy}{dx}\right)=y\left(\frac{dy}{dx}\right). \qquad ...(1)$$

[Poona 60]

Solution. Let us put $x^2=u$, $y^2=v$,

so that $2x=\frac{du}{dx}$, $2y\frac{dy}{dx}=\frac{dv}{dx}$

or $P=\frac{dv}{du}=\frac{2yp}{2x}=\frac{y}{x}p$ or $p=\frac{x}{y}P$.

Putting this value of p in the given equation, it becomes

$$x^2\left(y-\frac{x^2}{y}P\right)=y.\frac{x^2}{y^2}P^2 \text{ or } (y^2-x^2P)=P^2,$$

i.e. $v-uP=P^2$ or $v=uP+P^2$,

which is of Clairaut's form. Hence complete primitive is

$v=uc+c^2$. ...(2)

From (1) and (2), p- and c-discriminants are

$x^6+4x^2y^2=0$ or $x^2\,(x^4+4y^2)=0$, $u^2+4v=0$ or $x^4+4y^2=0$.

$\therefore$ $x^4+4y^2=0$ which is the singular solution.

Ex. 18. *Find the singular solution of*

$$y^2\left(y-x\frac{dy}{dx}\right)=x^4\left(\frac{dy}{dx}\right)^2.$$

[Agra 1970]

Solution. Put $u=\frac{1}{x}$ and $v=\frac{1}{y}$.

This gives $P=\frac{dv}{du}=\frac{x^2}{y^2}\frac{dy}{dx}$. ...(1)

Now dividing the given equation by y^4, we get

$$\frac{1}{y}-\frac{1}{x}\cdot\frac{x^2}{y^2}\cdot\frac{dy}{dx}=\left(\frac{x^2}{y^2}\frac{dy}{dx}\right)^2$$

or $v=uP+P^2$ from (1),

which is of Clairaut's form; therefore its solution is

$v=uc+c^2$ or $(1/y)=(1/x)\,c^2+c^2$

or $c^2xy+cy-x=0.$

$\therefore$ c-discriminant (EN^2C^3) is

$y\,(y^2+4x^2)=0.$

Also given equation is

$p^2x^4+pxy^2-y^3=0.$

$\therefore$ p-discriminant (ET^2C) is

$x^2y^3\,(y+4x^2)=0.$

Clearly $y+4x^2=0$ is the singular solution.

***Ex. 19.** *Obtain the singular solution of the equation*

$p^2y^2\cos^2\alpha-2pxy\sin^2\alpha+y^2-x^2\sin^2\alpha=0,$

directly from the equation and also from its complete primitive explaining the geometrical significance of the irrelevent factors that present themselves. [Agra 1967, 63; Raj. 54; Delhi Hons. 62

Solution. The equation is

$p^2y^2\cos^2\alpha-2pxy\sin^2\alpha+y^2-x^2\sin^2\alpha=0,$ ...(1

which is quadratic in p; therefore p-discriminant gives

$4x^2y^2\sin^4\alpha-4y^2\cos^2\alpha\,(y^2-x^2\sin^2\alpha)=0$

or $y^2\,[x^2\sin^2\alpha\,(\sin^2\alpha+\cos^2\alpha)-y^2\cos^2\alpha]=0$

or $y^2\cos^2\alpha\,(x^2\tan^2\alpha-y^2)=0$ (ET^2C). ...(2

Again solving (1) for p, we get

$py=x\tan^2\alpha\pm\sec\alpha\pm\sec\alpha\sqrt{(x^2\tan^2\alpha-y^2)}$

or $\pm\dfrac{y\,dy-x\tan^2\alpha\,dx}{\sqrt{(x^2\tan^2\alpha-y^2)}}=\sec\alpha\,dx$

or $\pm(x^2\tan^2\alpha-y^2)^{1/2}=c-x\sec\alpha$

or $x^2\tan^2\alpha-y^2=c^2-2cx\sec\alpha+x^2\sec^2\alpha$

or $c^2-2cx\sec\alpha+x^2+y^2=0,$

which clearly represents a family of circles for all values of c.

Now c-discriminant is

$x^2\sec^2\alpha-(x^2+y^2)=0$

or $x^2\tan^2\alpha-y^2=0$ (EN^2C^3). ...(3)

Clearly once occuring factors in (2) and (3) are the singular solutions. Therefore,

$y=\pm x\tan\alpha,$

which occur once in (2) and (3) both, give singular solutions.

The linear factor occuring twice in p-discriminant gives tac-locus. Hence $y=0$, which occurs twice in p-discriminant, is a tac-locus.

Thus the equation (1) represents a family of circles (2), whose envelope is given by

$y = \pm x \tan \alpha$.

Ex. 20. *Find the differential equation of the family of curves $x^2+y^2-2cx+c^2 \cos^2 \alpha=0$ where c is an arbitrary parameter and α is a given arc between 0 and $\frac{1}{2}\pi$.*

Determine the singulār solution of this differential equation.

[Punjab 1959]

Solution. Equation is

$x^2+y^2-2cx-c^2 \cos^2 \alpha=0$. ...(1)

Differentiating, $2x+2yp-2c=0$ or $c=(x+yp)$.

Putting this value of c in (1), the differential equation of family of curv's is

$x^2+y^2-2(x+yp)x+(x+yp)^2 \cos^2 \alpha=0$.

i.e. $p^2y^2 \cos^2 \alpha-2xyp(1-\cos^2 \alpha)+y^2-x^2(1-\cos^2 \alpha)=0$,

i.e. $p^2y^2 \cos^2 \alpha-2xyp \sin^2 \alpha+y^2-x^2 \sin^2 \alpha=0$

which is same as equation in above Ex. 19 P. 151.

Ex. 21. (a) *Find the differential equation of the family of circles $x^2+y^2+2cx+2c^2-1=0$ (c arbitrary constant). Determine the singular solution of this differential equation.*

Solution. Eliminating c as above, the differential equation of the family of circle is

$$2y^2p^2+2xyp+x^2+y^2-1=0.$$

c-discriminant is $x^2-2(x^2+y^2-1)=0$

i.e. $x^2+2y^2-2=0$ (EN^2C^3).

p discriminant is $x^2y^2-2y^2(x^2+y^2-1)=0$

i.e. $y^2(x^2+2y^2-2)=0$ (ET^2C).

Thus $x^2+2y^2-2=0$ (ellipse) is envelope of all such circles and $y=0$ which occurs squared only in p-discriminant is its tac-locus.

Ex. 21 (b) *Find singular solution of the differential equation which rcpresents the circles, $x^2+y^2+2cxy+c^2-1=0$.*

[Poona 1962]

Proceed as above. Differential equation is

$p^2(1-x^2)-(1-y^2)=0$.

S.S. is $x=\pm 1$, $y=\pm 1$.

Ex. 22. *Obtain the primitive and the singular solutions of the equation* $p^2(1-x^2)=1-y^2$.

Specify the nature of the geometrical loci which are no singular solutions but which may be obtained along with singular solutions.

Solution. We have $p^2(1-x^2)-(1-y^2)=0$. ...(1)

p-discriminant (ET^2C) is

$0+4(1-x^2)(1-y^2)=0$ (ET^2C)

i.e. $(1-x)(1+x)(1-y)(1+y)=0$. ...(2)

Again to solve (1), we have

$$p=\frac{dy}{dx}=\pm\frac{\sqrt{(1-y^2)}}{\sqrt{(1-x^2)}} \quad \text{or} \quad \frac{dy}{\sqrt{(1-y^2)}}\pm\frac{dx}{\sqrt{(1-x^2)}}=0.$$

Integrating, $\sin^{-1} y\pm\sin^{-1} x=c_1$

or $\sin^{-1}[y\sqrt{(1-x^2)}\pm x\sqrt{(1-y^2)}]=c_1$

or $y\sqrt{(1-x^2)}\pm x\sqrt{(1-y^2)}=\sin c_1=c$

or $y\sqrt{(1-x^2)}=c\mp x\sqrt{(1-y^2)}$.

Squaring, $y^2(1-x^2)=c^2\mp 2cx\sqrt{(1-y^2)}+x^2(1-y^2)$

or $c^2\mp 2cx\sqrt{(1-y^2)}+x^2-y^2=0$.

c-discriminant (EN^2C^3) is

$$x^2(1-y^2)-(x^2-y^2)=0$$

or $$y^2(1-x^2)=0$$

or $$y^2(1-x)(1+x)=0. \qquad \ldots(3)$$

The common factors occuring once in (2) and (3) represent singular solutions. Hence

$1-x=0$ and $1+x=0$, *i.e.* $x=\pm 1$ are the singular solutions.

Again $y=0$ which occurs twice in c-discriminant only and does not satisfy the differential equation represents nodal locus.

***Ex. 23.** *Obtain the primitive and singular solutions of the following equation :*

$$4p^2x(x-a)(x-b)=\{3x^2-2x(a+b)+ab\}^2.$$

Specify the nature of the loci which are not solutions but which are obtained with the singular solutions. **[Agra 1960, 51]**

Solution. The equation is

$$4p^2x(x-a)(x-b)=\{3x^2-2x(a+b)+ab\}^2. \qquad \ldots(1)$$

The p-discriminant is (ET^2C)

$$[3x^2-2x(a+b)+ab]^2\, x(x-a)(x-b)=0. \qquad \ldots(2)$$

Again to solve (1), we have

$$p=\frac{dy}{dx}=\pm\frac{[3x^2-2x(a+b)+ab]}{2\sqrt{\{x(x-a)(x-b)\}}}$$

or $$\frac{dy}{dx}=\pm\frac{3x^2-2x(a+b)+ab}{2\sqrt{\{x^3-x^2(a+b)+abx\}}}$$

Integrating,

$$y=\pm\sqrt{\{x^3-x^2(a+b)+abx\}}+c$$

(numerator being differential of the expression under radical sign.)

or $(y-c)^2=x^3-x^2(a+b)+abx$,

which is the complete primitive of the equation. This can be written as

$$c^2-2cy+y^2-x(x-a)(x-b)=0.$$

$\therefore$ c-discriminant is (ENC^3)

$$4y^2-4\{y^2-x(x-a)(x-b)\}=0$$

or $$x(x-a)(x-b)=0. \qquad \ldots(3)$$

The non-repeated factors common in (2) and (3) give singular solutions. Hence $x(x-a)(x-b)=0$, which occurs once in (2) and (3) both, gives singular solutions.

Again $3x^2-2x(a+b)+ab=0$ which occurs twice only in p-discriminant represents tac-locus.

Ex. 24. *Investigate fully for singular solution, explaining the geometrical significance of irrelevant factors that present themselves in* $4x(x-1)(x-2)\left(\frac{dy}{dx}\right)^2=(3x^2-6x+2)^2$.

[Agra 1956; Raj. 51; Delhi Hons. 53]

Solution. This is exactly Ex. 23 when $a=1$, $b=2$.

Hence proceeding as in that example,

$x(x-1)(x-2)=0$ is singular solution

and $3x^2-6x+2=0$ is tac-locus.

Ex. 25. *Transform the equation*

$$(2x^2+1)\left(\frac{dy}{dx}\right)^2+(x^2+2xy+y^2+2)\frac{dy}{dx}+2y^2+1=0$$

to Clairaut's forms by the substitution $x+y=u$, $xy-1=v$ *and interpret it. Find its singular solution also.* **[Karnatak 61]**

Solution. $x+y=u$, $xy-1=v$.

$$\frac{du}{dx}=1+p,\ \frac{dv}{dx}=xp+y.$$

$$\therefore\ P=\frac{dv}{du}=\frac{y+xp}{p+1}.$$

Now write the given equations as

$$p^2x^2+(p^2x^2+2pxy+y^2)+(x^2+y^2)p+y^2-(p^2+2p+1)=0,$$

i.e. $px^2(p+1)+y^2(p+1)+(px+y)^2+(p+1)^2=0$

or $(p+1)(px+y)(x+y)-(xy-1)(p+1)^2+(px+y)^2=0.$

Now dividing by $(p+1)^2$, it becomes

$$xy-1=\frac{px+y}{p+1}(x+y)+\left(\frac{px+y}{p+1}\right)^2$$

or $v=uP+P^2$, Clairaut's form.

Hence its solution is $v=uc+c^2$.

From p-and c-discriminants the singular solution is

$(x+y)^2+4(xy-1)=0.$

Ex. 26. *Interpret geometrically the factors in the p-and the c-discriminants of the equation*

$8p^3x=y(12p^2-9)$, *where* $p=dy/dx$. **[Punjab 56]**

Solution. Put $3y^2=v^3$, so that $6yp=3v^2\frac{dv}{dx}$.

$$\therefore\ p=\tfrac{1}{2}\frac{v^2}{y}P,\text{ where } P=\frac{dv}{dx}.$$

Putting this value of p, the equation becomes

$$x\frac{v^6}{y^3}P^3=y\left(3\frac{v^4}{y^2}P^2-9\right) \quad \text{or} \quad xv^6P^3=y^2(3v^4P^2-9y^2)$$

or $xv^6P^3=(v^7P^2-v^6)$ or $xP^3=vP^2-1$

or $v=xP+\frac{1}{P^2}$ Clairaut's form.

Solution is $v=xa+\frac{1}{a^2}$, where a is a constant

or $v^3=\left(ax+\frac{1}{a^2}\right)$ or $3y^2=a^3\left(x+\frac{1}{a^3}\right)^3$

or $3cy^2=(x+c)^3$, where $c=\frac{1}{a^2}$. ...(1)

This is the complete solution.

Now the equation is $8p^3x-12p^2y+9y=0$. ...(2)

To find p discriminant, we differentiate (2) w.r.t. p which gives $24p^2x-24py=0$, *i.e.*, $p(px-y)=0$.

When $p=0$, (2) gives $y=0$,

When $px-y=0$, $p=y/x$, (2) gives $9x^2y-4y^3=0$.

Therefore, p-discriminant, (ET^2C) is

$y(9x^2y-4y^3)=0$ or $y^2(9x^2-4y^2)=0$. ...(2)

Again to find c-discriminant, differentiating (1) w.r.t. c, we get

$3y^2=3(x+c)^2$, *i.e.* $(x+c)=\pm y$.

When $(x+c)=-y$, (1) gives $3(-y-x)y^2=-y^3$,

i.e. $y^2(2y+3x)=0$,

and when $(x+c)=y$, (1) gives $3(y-x)y^2=y$,

i.e $y^2(3x-2y)=0$.

Therefore, c-discriminant (EN^2C^3) is

$y^4(2y+3x)(3x-2y)=0$, *i.e.* $y^4(9x^2-4y^2)=0$. ...(4)

Now write

p-discriminant as $y.y(9x^2-4y^2)=0$,

c-discriminant as $y^3.y(9x^2-4y^2)=0$.

$y(9x^2-4y^2)=0$, which occurs both in p-and c-discriminants, is a singular solution. So geometrically interpreting $y=0$, $3x=\pm 2y$ represent envelopes of the family of curves in (1).

Again y occurs cubed in c-and once in p-discriminant. Hence $y=0$ is cusp locus also.

Ex. 27. *Solve the differential equation* $x^2(y-px)=yp^2$ *and find its singular solution.* **[Delhi Hons. 60]**

Solution. Ref. Ex. 10, P. 153. The complete primitive is

$y^2=x^2c+c^2$. ...(1)

Equation is $yp^2+x^3p-x^2y=0$ ...(2)

From (2), p-discriminant (ET^2C) is

$x^6+4x^2y^2=0$, *i.e.* $x^2(x^4+4y^2)=0$. ...(3)

Also from (1) *c*-discriminant (EN^2C^3) is

$x^4+4y^2=0$. ...(4)

Hence $x^4+4y^2=0$, which occurs only once both in *p*-and *c*-discriminant, is the singular solution. Also $x=0$, which occurs twice in *p*-discriminant only, gives a tac-locus

Ex. 28 *Verify that $y=cx+c^2$ and $x^2+4y=0$ are both solutions of the differential equation*

$$\left(\frac{dy}{dx}\right)^2+x\left(\frac{dy}{dx}\right)-y=0.$$ **[Poona 1961]**

Solution. The differential equation can be written as

$y=px+p^2$. ...(1)

This is Clairaut's form. Hence putting *c* for *p*, the general solution of the equation is

$y=cx+c^2$. ...(2)

To investigate the singular solution, which is not included in the general solution, from (1) *p*-discriminant (ET^2C) gives

$x^2+4y=0$.

Also from (2) *c*-discriminant (EN^2C^3) gives

$x^2+4y=0$.

Now since x^2+4y occurs only once both in *p*-and *c*-discriminants, $x^2+4y=0$ gives the singular solution of the equation.

Ex. 29. *Solve and examine for singular solution of the equation*

$$\left(1+\frac{dy}{dx}\right)=\frac{27}{8a}(x+y),\left(1-\frac{dy}{dx}\right)^3.$$ **[Delhi Hons. 1959]**

Solution. Let us put $x+y=u$, $x-y=v$.

$$\therefore\ \frac{dv}{du}=P=\frac{1-p}{1+p}, \text{ where } p=\frac{dy}{dx}.$$

The given differential equation can be put as

$$(1+p)^3=\frac{27}{8a}(x+y)(1-p)^3 \quad\text{or}\quad 1=\frac{27}{8a}(x+y)\left(\frac{1-p}{1+p}\right)^3,$$

which becomes $1=\dfrac{27}{8a}uP^3$ or $P=\dfrac{dv}{du}=\frac{2}{3}a^{1/3}u^{-1/3}$.

Integrating $v+c=a^{1/3}u^{2/3}$,

so that $(v+c)^3=au^2$ or $(x-y+c)^3=a(x+y)^2$. ...(1)

This is the general solution.

Differentiating (1) w.r.t. *c*, we get

$3(x-y+c)^2=0$, *i.e.* $x-y+c=0$. ...(2)

Eliminating *c* from (1) and (2), the *c*-discriminant is

$x+y=0$.

Similarly the *p*-discriminant is $x+y=0$.

Therefore $x+y=0$ is the singular solution.

Ex. 30. *Solve* $p^2 (2-3y)^2=4 (1-y)$.

[Delhi Hons. 1964 ; Punjab 61]

Solution. The equation can be written as

$$\frac{dx}{dy}=\pm\frac{2-3y}{2\sqrt{(1-y)}}=\pm\frac{3-3y-1}{2\sqrt{(1-y)}}$$

or $$dx=\pm\left[\frac{3}{2}\sqrt{(1-y)}-\frac{1}{2}\frac{1}{\sqrt{(1-y)}}\right]dy.$$

Integrating, $x=c\pm[-(1-y)^{3/2}+(1-y)^{1/2}]$

or $x-c=\pm(1-y)^{1/2}[1-(1-y)]$

or $(x-c)^2=(1-y)\,y^2$ or $c^2-2cx+x^2-y^2(1-y)=0.$ ...(1)

From the given equation p-discriminant (ET^2C) is

$(2-3y)^2(1-y)=0$ ...(2)

and from (1) c-discriminant (EN^2C^3) is

$x^2-[x^2-y^2(1-y)]=0$ or $y^2(1-y)=0.$ ...(3)

The common non-repeated factor in (2) and (3), *i.e.* $1-y=0$ is singular solution.

$2-3y=0$ which occurs twice in p-discriminant only represents tac-locus.

And $y=0$ which occurs twice in c-discriminant only represents nodal locus.

Ex. 31. *Find singular solution* $p^2+y^2=1$ *and interpret the result geometrically.*

[Gujrat 1959]

Solutiun. The equation is $p=\dfrac{dy}{dx}=\sqrt{(1-y^2)}$

i.e. $\dfrac{dy}{\sqrt{(1-y^2)}}=dx$, *i.e.* $\sin^{-1}y=x+e$ or $y=\sin(x+c)$. ...(1)

This is complete primitive.

Differentiating (1) w.r.t. c, we get

$0=\cos(x+c).$ ...(2)

Squaring and adding (1) and (2) to eliminate c from them, the c-discriminant is $y^2=1.$ ...(3)

Also from the given differential equation, the p-discriminant is $y^2-1=0.$

Therefore $y^2-1=0$, *i.e.* $y=\pm1$ gives envelope (Singular solution) of the sine curves given by (1).

Ex. 33. *Find the complete primitive and singular solution of* $3p^2e^y-px+1=0.$

[Poona 1959 (S)]

Solution. Solving for x, $x=3pe^y+1/p$.

Differentiating w.r.t. y, $\dfrac{1}{p}=3pe^y+3e^y\dfrac{dp}{dy}-\dfrac{1}{p^2}\dfrac{dp}{dy}$

or $\left(\dfrac{dp}{dy}-p\right)\left(3e^y-\dfrac{1}{p^2}\right)-0$, $\dfrac{dp}{dy}-p=0$, $p=ce^y$.

Putting this value of p in the equation, the complete primitive is

$3c^2e^{3y}-ce^yx+1=0.$ ...(1)

Now from the given differential equation p-discriminant is (ET^2C), *i.e.* $x^2-12e^y=0.$

From (1), c-discriminant is $e^{2y}(x^2-12e^y)=0$ (EN^2C^3).

$\therefore$ $x^2-12e^y=0$, which occurs only once in p-and c-discrimnants, is singular solution.

Ex. 33. *Obtain the complete primtive (C.P.) and singular solutions (S.S.) of the following equations :*

(i) $\left(\dfrac{dy}{dx}\right)^4=4y\left(x\dfrac{dy}{dx}-2y\right)^2.$

[Karnatak 63 ; Calcutta Hons. 62]

S.S. is $x^4-16y=0$, $y=0$.

Hint. Put $y=Y^2$ to change into Clairaut's form.

(ii) $yy^2-3xp+y=0.$ C.P. $y^2-2cx+c^2=0$, S.S. $y^2=x^2$.

(iii) $3xp^2-6xp+x-2y=0.$

C.P. $x^2+c(x-3y)+c^2=0$, S.S. $(3y+x)(y-x)=0$.

(iv) $p^2+2px^3-4x^2y=0,$

C P. $y-cx^2-c^2=0$, S.S. $x^4+4y=0$, Tac-locus $x=0$.

(v) $y=px+p^2.$ C.P. $y=cx+c^2$, S.S. $x^2+4y=0$.

(vi) $y=px+p^3.$ C.P. $y=cx+c^3$, S.S. $27y^2+4x^3=0$.

(vii) $y=px+\cos p.$

C.P. $y=cx+\cos c$, S.S. $(y-x\sin^{-1}x)^2=1-x^2$.

(viii) $p=\log(px-y).$ C.P. $y=cx-e^c$, S.S. $y=x(\log x-1)$.

(ix) $p^2+2xp=3x^2.$ No. S.S., $x=0$ is Tac-locus.

(x) $y^2(y-xp)=x^4p^2.$ C.P. $y=cy+xyc^2$, S.S. $y+4x^2=0$.

Hint. Put $x=\dfrac{1}{u}$ (See Ex. 18 above).

(xi) $y^2(1+4p^2)-2pxy-1=0.$ [Agra 72]

C.P. $y^2+4c^2=1+2cx$, S.S. $x^2-4y^2+4=0$, Tac-locus $y=0$.

(xii) $y^2p^2+y^2=r^2.$

C.P. $(x+c)^2+y^2=r^2$, S.S. $y=\pm r$, Tac-locus $y=0$.

(xiii) $x^3p^2+x^2yp+a^3=0.$

[Delhi Hons. 63 ; Poona 64 ; Patna Hons. 60]

Hint. Put $u=\dfrac{1}{x}$ $v=y$, Clairaut's form $v+Px=a^3P^2$.

C.P. $1=cxy-a^3c^2x$. S.S. $x(xy^2-4a^3)=0$. Tac-locus $x=0$.

(xiv) $y=xp+a\sqrt{(1+p^2)}.$ S.S. $x^2+y^2=a^2$.

(xv) $xp^2-yp-y=0.$ [Delhi Hons. 54]

S.S. $y^2+4xy=0$.

(xvi) $x^2p^2-3xyp+2y^2+x^3=0.$ S.S. $x^2(y-4x^3)=0$.

(xvii) $dy\sqrt{x}=dx\sqrt{y}.$ C.P. $(x+y-c)^2=4xy$, S.S. $xy=0$.

(xviii) $4p^2=9x.$ C.P. $(y+c)^2=x^3$, No. S.S. $x=0$ is cusp locus.

(xix) $8y=px(12-9p^2)$. [Delhi Hons. 55]

(xx) $\cos^2 yp^2+\sin x\cos x\cos yp-\sin y\cos^2 x=0$. [Delhi Hons. 61]

Ex. 34. *Solve* $\left(\frac{dy}{dx}\right)^2+(y-a)(y-b)=0,\ a<b.$

Find the singular solution if any. [Cal. Hons. 62]

Ex. 35. *From the differential equation corresponding to the family of curves $y=c(x-c)^2$, where c is an arbitrary constant.*

Show that the resulting differential equation reduces to Clairaut's type by the substitution $y=v^2$. Hence solve the equation and find the singular solution. [Karnatak 60]

Solution As found in Ex. 4 page 3 the differential equation of curves is $p^3=4y(px-2y)$. ...(1)

Now put $y=v^2,\ \frac{dy}{dx}=2v\frac{dv}{dx}=2vP$ (say).

Hence the equation reduces to

$8v^3P^3=4v^2(2vPx-2v^2)$ or $P^3=Px-v$

or $v=Px-P^3$, Clairaut's form.

Hence solution is $v=kx-k^3$ or $\sqrt{y}=kx-k^3$

or $y=k^2(x-k^2)^2$ or $y=c(x-c)^2$, where $c=k^2$. ...(2)

Now differentiating it w.r.t. c, $(x-c)^2-2c(x-c)=0$,

i.e. $(x-c)(x-c-2c)=0$, *i.e.* $x=c,\ x=3c$.

When $x=c$, we have from (2), $y=0$.

When $c=\frac{1}{3}x$, we have $y=\frac{1}{3}x(x-\frac{1}{3}x)^2$, $y=\frac{4}{27}x^3$.

Hence c-discriminant is $y(y-\frac{4}{27}x^3)=0$ ...(3)

Again differentiating (1) w.r.t. p, $3p^2=4xy$.

Putting this value in (1). $\frac{4}{3}xyp=xyp-8y^2$

or $y^2=\frac{1}{3}yxp$, *i.e.* $y^4=\frac{1}{9}x^2y^2.\frac{4}{3}xy$

or $y^3(y-\frac{4}{27}x^3)=0$ or $y^2y(y-\frac{4}{27}x^3)=0$. ...(4)

This is p-discriminant (ET^2C).

From p and c-discriminants, $y(y-\frac{4}{27}x^3)=0$ is the singular solution $y=0$, which occurs squared only in p-discriminant gives Tac-locus also.

Ex. 36. *olve the differential equation $(8p^3-27)x-12p^2y$ and investigate whether a singular solution exists.*

[Agra 65 ; Punjab 62 ; Delhi Hons. 57 ; Karnarak 60, 62]

Solution. Equation is $(8p^3-27)x=27p^2y$ ...(1)

or $y=\frac{2}{3}px-\frac{9x}{4p^2}$.

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=p=\frac{2}{3}p=\frac{2}{3}x\frac{dp}{dx}-\frac{9}{4p^2}+\frac{9}{2}\frac{x}{p^3}\frac{dp}{dx}$$

or $$\left(x\frac{dp}{dx}-\frac{p}{2}\right)\left(\frac{2}{3}+\frac{9x}{2p^3}\right)=0.$$

Now $x\dfrac{dp}{dx}-\dfrac{p}{2}=0$ gives $2\dfrac{dp}{p}=\dfrac{dx}{x}$.

Integrating, $p^2=kx$.

Putting this value in (1), we get

$(8kxp-27)=x=12kxy$,

i.e. $(8kx)^2\,kx=(12ky+27)^2$ and $x=0$

or $64k^3x^3=144k^2\left(y-\dfrac{27}{12k}\right)^2$ or $x^3-\dfrac{9}{4k}\left(y+\dfrac{9}{4k}\right)^2=0$

or $x^3-c\,(y+c)^2=0$, where $c=\dfrac{9}{4k}$.

Hence the complete primitive is

$x\,[x^3-c\,(y+c)^2]=0$.

Now proceeding as usual, $4y^3+27x^3=0$ is the singular solution.

$x=0$ is a part of general solution. It is cusp-locus for one part of the general solution and the envelope-locus for the other part.