PART II

DIFFERENTIAL EQUATIONS

1

Exact Differential Equations and Equations of Particular Forms

1·1. Introduction.

In this chapter differential equations of order higher than the first and with variable coefficients will be considered. There is, as a matter of fact, no general method to solve these differential equations. However, we consider below some particular types of xthese equations.

1·2. Dependent variable absent.

If an equation does not contain y directly, it can be written in the form

$$f\left(\frac{d^n y}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \ldots, \frac{dy}{dx}, x\right)=0. \quad \ldots(1)$$

Let us put $\dfrac{dy}{dx}=p, \dfrac{d^2y}{dx^2}=\dfrac{dp}{dx}, \ldots$, etc.

Then the equation (1) becomes

$$f\left(\frac{d^{n-1}p}{dx^{n-1}}, \frac{d^{n-2}p}{dx^{n-2}}, \ldots, p, x\right)=0,$$

which is reduced by order one and may be solved for p to give

$$p=F(x), \quad i.e. \quad \frac{dy}{dx}=F(x),$$

so that $y=\int F(x)\, dx+c$ is the required solution.

The following few examples will illustrate the method.

Note. If the equation is of the form

$$f\left(\frac{d^n y}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \ldots, \frac{d^k y}{dx^k}, x\right)=0,$$

in which lowest derivative is of order k, then substitution made is

$$\frac{d^k y}{dx^k}=q, \quad \frac{d^{k-1}y}{dx^{k+1}}=\frac{dq}{dx}, \ldots \text{ etc.}$$

This would reduce the order of the equation by k.

Ex. 1. *Solve* $2\dfrac{d^2y}{dx^2}-\left(\dfrac{dy}{dx}\right)^2+4=0.$

Solution. The equation is free from y.

Putting $\frac{dy}{dx}=p$, $\frac{d^2y}{dx^2}=\frac{dp}{dx}$, the equation becomes

$$2\frac{dp}{dx}-p^2+4=0 \text{ or } \frac{2\,dp}{p^2-4}=dx$$

separating the variables

or $\frac{1}{2}\left(\frac{1}{p-2}-\frac{p}{p+2}\right)dp=dx.$

Integrating, $\frac{1}{2}\log\frac{p-2}{p+2}=x+\log c_1$

or $\frac{p-2}{p+2}=c_1e^{2x}$, $(p-2)=(p+2)\,c_1e^{2x}$.

This gives $p=\frac{dy}{dx}=\frac{2(1+c_1e^{2x})}{1-c_1e^{2x}}=2\left(1+\frac{2c_1e^{2x}}{1-c_1e^{2x}}\right)$.

Integrating again $y=2x-2\log(1-c_1e^{2x})+c_2$ which is the complete solution.

Ex. 2. *Solve* $\frac{d^2y}{dx^2}+\frac{dy}{dx}+\left(\frac{dy}{dx}\right)^3=0.$ **[Agra 55]**

Solution. The equation does contain y directly.

$\therefore$ Putting $\frac{dy}{dx}=p$, $\frac{d^2y}{dx^2}=\frac{dp}{dx}$, the equation becomes

$$\frac{dp}{dx}+p+p^3=0 \text{ or } \frac{dp}{p(1+p^2)}=-dx$$

or $$\left(\frac{1}{p}-\frac{p}{1+p^2}\right)dp=-dx.$$

Integrating, $\log\frac{p}{\sqrt{(1+p^2)}}=-x+\log c_1$ or $\frac{p^2}{1+p^2}=c_1^2e^{-2x}$

or $p^2=(1+p^2)\,c_1^2e^{-2x}$ or $p=\frac{c_1e^{-x}}{\sqrt{(1-c_1^2e^{-2x})}}$.

Integrating again, $y=-\sin^{-1}(c_1e^{-x})+c_2$.

Ex 3. *Solve* $(1+x^2)\frac{d^2y}{dx^2}+1+\left(\frac{dy}{dx}\right)^2=0.$ **[Agra 57]**

Solution. The equation does not contain y directly.

Putting $\frac{dy}{dx}=p$, $\frac{d^2y}{dx^2}=\frac{dp}{dx}$, the equation becomes

$(1+x^2)\frac{dp}{dx}+1+p^2=0$ (variables separable)

i.e. $\frac{dp}{1+p^2}+\frac{dx}{1+x^2}=0$; $\therefore$ $\tan^{-1}p+\tan^{-1}x=\tan^{-1}c_1$

or $\tan^{-1}\frac{p+x}{1-px}=\tan^{-1}c_1$ or $\frac{p+x}{1-px}=c_1$

i.e. $(p+x)=c_1(1-px)$ or $p=\dfrac{dy}{dx}=\dfrac{c_1-x}{1+c_1x}=\dfrac{1}{c_2}\left[\dfrac{1+c_1^2}{1+c_1x}-1\right].$

Integrating, $y=\dfrac{1+c_1^2}{c_1^2}\log(1+c_1x)-\dfrac{1}{c_1}x+c_2$ which is the general solution.

Ex. 4. *Solve* $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}=2.$ **[Agra 52]**

Solution. The equation does not contain y directly.

Thus putting $\dfrac{dy}{dx}=p,\ \dfrac{d^2y}{dx^2}=\dfrac{dp}{dx}$, the equation becomes

$$(1-x^2)\frac{dp}{dx}-xp=2 \quad\text{or}\quad \frac{dp}{dx}-\frac{x}{1-x^2}p=\frac{2}{1-x^2}.$$

This is a linear equation in p and x.

$$\text{I.F.}=e^{-\int\frac{1}{1-x^2}}=e^{\frac{1}{2}\log(1-x^2)}=\sqrt{(1-x^2)}$$

$$\therefore\quad p\sqrt{(1-x^2)}=c_1+\int\frac{2}{1-x^2}\sqrt{(1-x^2)}\,dx$$

$$=c_1+2\sin^{-1}x.$$

$$\therefore\quad p=\frac{dy}{dx}=\frac{c_1}{\sqrt{(1-x^2)}}+2\sin^{-1}x.\frac{1}{\sqrt{(1-x^2)}}.$$

Integrating, $y=c_1\sin^{-1}x+(\sin^{-1}x)^2+c_2$
which is the required solution.

Ex. 5. *Solve* $x\dfrac{d^2y}{dx^2}+x\left(\dfrac{dy}{dx}\right)^2-\dfrac{dy}{dx}=0.$

Solution. The equation does not contain y directly. So putting $\dfrac{dy}{dx}=p,\ \dfrac{d^2y}{dx^2}=\dfrac{dp}{dx}$, the equation becomes.

$$x\frac{dp}{dx}+xp^2-p=0 \quad\text{or}\quad \frac{1}{p^2}\frac{dp}{dx}-\frac{1}{x}\frac{1}{p}=-1.$$

Putting $\dfrac{1}{p}=u,\ -\dfrac{1}{p^2}\dfrac{dp}{dx}=\dfrac{du}{dx}$, this becomes

$$\frac{du}{dx}+\frac{1}{x}u=1.\quad \text{Liner equation,}\quad \text{I.F.}=e^{\int\frac{1}{x}dx}=x.$$

$$\therefore\quad ux=c_1+\int x\,dx \quad\text{or}\quad ux=c_1+\tfrac{1}{2}x^2$$

or $$\frac{1}{p}x=c_1+\tfrac{1}{2}x^2 \quad\text{or}\quad p=\frac{dy}{dx}=\frac{2x}{x^2+2c_1}.$$

Integrating, $y=\log(x^2+2c_1)+c_2$
which is the required solution.

Ex. 6. *Solve* $2x\dfrac{d^3y}{dx^3}\cdot\dfrac{d^2y}{dx^2}=\left(\dfrac{d^2y}{dx^2}\right)^2-a^2.$

Solution. The equation does not contain y directly.

Also lowest differential is $\frac{d^2y}{dx^2}$. So put $\frac{d^2y}{dx^2}=q,\ \frac{d^3y}{dx^3}=\frac{dq}{dx}$.

Then the equation becomes

$$2x.\frac{dq}{dx}.q=q^2-a^2 \quad \text{or} \quad 2q\frac{dq}{dx}-\frac{q^2}{x}=-\frac{a^2}{x}.$$

Putting $q^2=u,\ 2q\frac{dq}{dx}=\frac{du}{dx}$, the equation becomes

$$\frac{du}{dx}-\frac{1}{x}u=-\frac{a^2}{x}. \quad \text{Linear equation, I F.}=e^{\int -\frac{1}{x}dx}=\frac{1}{x}.$$

$$\therefore \quad u.\frac{1}{x}=c_1-\int\frac{a^2}{x^2}\,dx=c_1+\frac{a^2}{x}$$

or $u=q^2=c_1x+a^2$, *i.e.* $q=\frac{d^2y}{dx^2}=(c_1x+a^2)^{1/2}$.

Integrating, $\frac{dy}{dx}=\frac{(c_1x+a^2)^{3/2}}{\frac{3}{2}.c_1}+c_2$.

Integrating again, $y=\frac{(c_1x+a^2)^{5/2}}{\frac{3}{2}.\frac{5}{2}c_1^2}+c_2x+c_3$,

which is required general solution.

Ex. 7. *Solve* $\left(\frac{d^3y}{dx^3}\right)^2+x\frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}=0.$

Solution. The equation is free from y and the lowest differential coefficient is $\frac{d^2y}{dx^2}$. So putting $\frac{d^2y}{dx^2}=q, \frac{d^3y}{dx^3}=\frac{dq}{dx}$ the equation becomes

$$\left(\frac{dq}{dx}\right)^2-x\frac{dq}{dx}-q=0 \quad \text{or} \quad q=x\frac{dq}{dx}+\left(\frac{dq}{dx}\right)^2$$

which is of Clairaut's form. Hence putting c_1 for dq/dx,

$$q=xc_1+c_1^2 \quad \text{or} \quad \frac{d^2y}{dx^2}=c_1x+c_1^2.$$

Integrating, $\frac{dy}{dx}=\frac{1}{2}c_1x^2+c_1^2x+c_2$.

Integrating again, $y=\frac{1}{6}c_1x^3+\frac{1}{2}c_1^2x^2+c_2x+c_3$,

which is the required solution.

Ex. 8. *Solve* $\frac{dy}{dx}-x\frac{d^2y}{dx^2}=f\left(\frac{d^2y}{dx^2}\right)$.

Solution. Putting $\frac{dy}{dx}=p,\ \frac{d^2y}{dx^2}=\frac{dp}{dx}$, the equation becomes

$$p-x\frac{dp}{dx}=f\left(\frac{dp}{dx}\right) \quad \text{or} \quad p=x\frac{dp}{dx}+f\left(\frac{dp}{dx}\right)$$

which is of Clairaut's form. Hence putting c for dp/dx,

$$p=xc+f(c) \text{ or } \frac{dy}{dx}=xc+f(c).$$

Integrating, $y=\frac{1}{2}cx^2+f(c)\,x+c'$.

Ex. 9. *Solve* $\frac{d^4y}{dx^4}-\cot x\,\frac{d^3y}{dx^3}=0$.

Solution. Equation is free from y and lowest differential coefficient is $\frac{d^3y}{dx^3}$; hence putting $\frac{d^3y}{dx^3}=q, \frac{d^4y}{dx^4}=\frac{dq}{dx}$, the equation becomes $\frac{dq}{dx}-\cot x\cdot q=0$ or $\frac{dq}{q}-\cot x\,dx=0$.

Integrating $\log q-\log \sin x=\log c_1$ or $q=c_1 \sin x$

i e. $\frac{d^3y}{dx^3}=c_1 \sin x$. Integrating, $\frac{d^2y}{dx^2}=-c_1 \cos x+c_2$.

Integrating again, $\frac{dy}{dx}=-c_1 \sin x+c_2x+c_3$.

and then $y=c_1 \cos x+\frac{1}{2}c_2x^2+c_3x+c_4$.

Ex. 10. *Solve* $\frac{d^4y}{dx^4}\cdot\frac{d^3y}{dx^3}=1$.

Solution. Equation is free from y. Putting $\frac{d^3y}{dx^3}=q$, the equation becomes

$$\frac{dq}{dx}\cdot q=1 \text{ or } q\,dq=dx \text{ or } q^2=2x+c_1$$

or $q=\frac{d^3y}{dx^3}=\pm(2x+c_1)^{1/2}$, $\frac{d^2y}{dx^2}=\pm\frac{1}{3}(2x+c_1)^{3/2}+c_2$,

$$\frac{dy}{dx}=\pm\frac{1}{15}(2x+c_1)^{5/2}+c_2x+c^3.$$

$$y=\pm\tfrac{1}{105}(2x+c_1)^{7/2}+\tfrac{1}{2}c_2x^2+c_3x+c_4.$$

Ex. 11. *Solve* $\frac{d^2y}{dx^2}=\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}$

Solution. Putting $\frac{dy}{dx}=p, \frac{d^2y}{dx^2}=\frac{dp}{dx}$, the equation becomes

$$\frac{dp}{dx}=\sqrt{(1+p^2)} \text{ or } \frac{dp}{\sqrt{(1+p^2)}}=dx.$$

Integrating, $\sinh^{-1} p=p+c_1$ or $p=\frac{dy}{dx}=\sinh(x+c_1)$.

Integrating again, $y=c_2+\cosh(x+c_1)$.

Ex. 12. *Solve* $\frac{d^2y}{dx^2}-\frac{a^2}{x(a^2-x^2)}\cdot\frac{dy}{dx}=\frac{x^2}{a(a^2-x^2)}$.

Solution. Putting $\frac{dy}{dx}=p$, $\frac{d^2y}{dx^2}=\frac{dp}{dx}$, the equation becomes

$$\frac{dp}{dx}-\frac{a^2}{a\,(a^2-x^2)}\,p=\frac{x^2}{a\,(a^2-x^2)},\text{ linear equation.}$$

$$\text{I. F.}=e^{-\int\frac{a^2}{x\,(a^2-x^2)}dx}=e^{-\int\left(\frac{1}{x}+\frac{\frac{1}{2}}{a-x}-\frac{1}{a+x}\right)dx}$$

$$=e^{-\log x+\frac{1}{2}\log(a-x)+\frac{1}{2}\log(a+x)}=\frac{\sqrt{(a^2-x^2)}}{x}$$

$$\therefore\ p\,\frac{\sqrt{(a^2-x^2)}}{x}=c_1+\int\frac{x}{a\,(a^2-x^2)^{1/2}}dx$$

$$=c_1-\frac{1}{a}\,(a^2-x^2)^{1/2}$$

or $$p=\frac{dy}{dx}=\frac{c_1x}{\sqrt{(a^2-x^2)}}-\frac{1}{a}\,x.$$

Integrating, $y=-c_1\,(a^2-x^2)^{1/2}-\frac{1}{2a}\,x^2+c_1$,

which is the required solution.

Ex. 13. *Solve* $\frac{d^2y}{dx^2}=a^2+k^2\left(\frac{dy}{dx}\right)^2.$

Solution. The equation does not contain y directly. Therefore putting $\frac{dy}{dx}=p, \frac{d^2y}{dx^2}=\frac{dp}{dx}$, the equation becomes

$$\frac{dp}{dx}=a^2+k^2p^2\quad\text{or}\quad\frac{dp}{a^2+k^2p^2}=dx.$$

Integrating $$\frac{1}{ak}\tan^{-1}\left(\frac{pk}{a}\right)=x+c_1$$

or $pk=a\tan\{ak\,(x+c_1)\}$ or $p=\frac{dy}{dx}=\frac{a}{k}\tan\{ak\,(x+c_1)\}$,

Integrating, $$y=\frac{1}{k^2}\log\sec\{ak\,(x+c_1)\}+c_2.$$

1·3. Equations in which x is absent

An equation which is free from x can be put in the form

$$f\left(\frac{d^ny}{dx^n},\frac{d^{n-1}y}{dx^{n-1}},\ldots,\frac{dy}{dx},y\right)=0.$$

If we put $\frac{dy}{dx}=p$, then $\frac{d^2y}{dx^2}=\frac{dp}{dx}=\frac{dp}{dy}\cdot\frac{dy}{dx}=p\,\frac{dp}{dy}$

and $$\frac{d^3y}{dx^3}=\frac{d}{dx}\left(p\,\frac{dp}{dy}\right)=\frac{d}{dy}\left(p\,\frac{dp}{dx}\right)\frac{dy}{dx}$$

$$=\left[p\,\frac{d^2p}{dy^2}+\left(\frac{dp}{dy}\right)^2\right]p=p^2\,\frac{d^2p}{dy^2}+p\left(\frac{dp}{dy}\right)^2,\text{ etc.}$$

The equation thus takes the form

$$f\left(\frac{d^{n-1}p}{dy^{n-1}}, \ldots, p, y\right)^3=0,$$

which is reduced by one order and may possibly be integrated for p and then for y.

The following few examples will make the procedure clear.

Ex. 1. *Solve* $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+\left(\dfrac{dy}{dx}\right)^3=0.$ **[Agra 1955]**

Solution. The equation does not contain x directly; so putting

$$\frac{dy}{dx}=p, \frac{d^2y}{dx^2}=\frac{d}{dx}(p)=\frac{d}{dy}(p)\cdot\frac{dy}{dx}=p\frac{dp}{dy},$$

the equation becomes

$$p\frac{dp}{dy}+p+p^3=0 \quad \text{or} \quad \frac{dp}{1+p^2}+dy=0.$$

Integrating, $\tan^{-1}p=c_1-y$ or $p=\dfrac{dy}{dx}=\tan(c_1-y)$

or $\cot(c_1-y)\,dy=dx$ (separating the variables).

Integrating, $-\log\sin(c_1-y)+\log c_2=x.$

or $\log\dfrac{\sin(c_1-y)}{c_2}=-x$ or $\sin(c_1-y)=c_2e^{-x}$

or $c_1-y=\sin^{-1}(c_2e^{-x})$ or $y=c_1-\sin^{-1}(c_2e^{-x}).$

Ex. 2. *Solve* $y\dfrac{d^2y}{dx^2}-\left(\dfrac{dy}{dx}\right)^2=y^2\log y.$ **[Delhi Hons. 1958]**

Solution. The equation does not contain x directly; hence putting $\dfrac{dy}{dx}=p, \dfrac{d^2y}{dx^2}=\dfrac{dp}{dy}\cdot\dfrac{dy}{dx}=p\dfrac{dp}{dy}$, the equation becomes

$$yp\frac{dp}{dy}-p^2=y^2\log y \quad \text{or} \quad p\frac{dp}{dy}-\frac{1}{y}p^2=y\log y.$$

Putting $p^2=v, 2p\dfrac{dp}{dy}=\dfrac{dv}{dy}$; the above equation becomes

$$\frac{dv}{dy}-\frac{2}{y}v=2y\log y.$$

Linear equation, $\text{I.F.}=e^{-\int\frac{2}{y}dy}=\dfrac{1}{y^2}.$

$$\therefore\ v\frac{1}{y^2}=c_1+\int 2y\log y\cdot\frac{1}{y^2}\,dy=c_1+\int\frac{2}{y}\log y\,dy$$

or $p^2\cdot\dfrac{1}{y^2}=c_1+(\log y)^2$ or $p=\dfrac{dy}{dx}=\pm y\,[c_1+(\log y)^2]^{1/2}$

or $\dfrac{dy}{y\sqrt{[c_1+(\log y)^2]}}=dx\ \left[\text{put } \log y=u, \dfrac{1}{y}dy=du\right].$

$$\therefore \quad \frac{du}{\sqrt{(c_1+u^2)}}=dx \quad \text{or} \quad \log[u+\sqrt{(c_1+u^2)}]=\log c_2+x$$

or $u+\sqrt{(c_1+u^2)}=c_2e^x$ or $\log y+\sqrt{[c_1+(\log y)^2]}=c_2e^x$

or $c_1+(\log y)^2=(c_2e^x-\log y)^2$

or $c_1=c^2e^{2x}-2c_2e^x \log y$

or $\log y=k_1e^x+k_2e^{-x}$ is the solution.

***Ex. 3** *Solve* $\left(\frac{dy}{dx}\right)^2-y\frac{d^2y}{dx^2}=n\left\{\left(\frac{dy}{dx}\right)^2+a^2\left(\frac{d^2y}{dx^2}\right)\right\}^{1/2}$.

[Agra 1951]

Solution. The equation does not contain x directly. So putting $\frac{dy}{dx}=p=, \frac{d^2y}{dx^2}=\frac{dp}{dx}=\frac{dp}{dy}\frac{dy}{dx}=p\frac{dp}{dy}$, the equation becomes

$$p^2-py\frac{dp}{dy}=n\left[p^2+a^2p^2\left(\frac{dp}{dy}\right)^2\right]^{1/2}$$

or $\quad p=y\frac{dp}{dy}+n\left[1+a^2\left(\frac{dp}{dy}\right)^2\right]^{1/2}$,

which is of Clairaut's form. Putting c for dp/dy,

$$p=cy+n[1+a^2c^2]^{1/2} \quad \text{or} \quad \frac{dy}{dx}=cy+n(1+a^2c^2)^{1/2}$$

or $\quad \frac{dy}{cy+n(1+a^2c^2)^{1/2}}=dx.$

Now integrating $\frac{1}{c}\log\; cy+n(1+a^2c^2)^{1/2}]=x+c'$

or $\quad cy+n(1+a^2c^2)^{1/2}=c_2e^{cx}$.

***Ex. 4.** *Solve* $y(1-\log y)\frac{d^2y}{dx^2}+(1+\log y)\left(\frac{dy}{dx}\right)^2=0$.

[Delhi Hons. 1969 ; Bombay 61 ; Agra 58; Raj. 63, 55 ; Gujrat 61]

Solution. The equation does not coniain x directly. So putting $\frac{dy}{dx}=p, \frac{d^2y}{ax^2}=p\frac{dp}{dy}$, the equation becomes

$$y(1-\log y)p\frac{dp}{dy}+(1+\log y)p^2=0$$

or $\quad \frac{dp}{p}+\frac{1+\log y}{y(1-\log y)}dy=0$ (variable separated)

Now put $\log y=t$ so that $\frac{1}{y}dy=dt$.

$$\therefore \quad \frac{dp}{p}+\frac{1+t}{1-t}dt=0 \quad \text{or} \quad \frac{dp}{p}=\left(1+\frac{2}{t-1}\right)dt.$$

Integrating, $\log p=t+2\log(t-1)+\text{const.}$

or $p=c_1e^t(t-1)^2$ or $\frac{dy}{dx}=c_1y(\log y-1)^2$ as $y=e^t$

or $\frac{dy}{y(\log y-1)^2}=c_1\,dx$. Integrating $-\frac{1}{(\log y-1)}=c_1x+c_2$

or $1-\log y=\frac{1}{c_1x+c_2}$ is the required solution.

Ex. 5. (a) *Solve* $y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=y^2$.

Solution. Equation is free from x. Putting $\frac{dy}{dx}=p$ and $\frac{d^2y}{dx^2}=p\frac{dp}{dy}$, the equation becomes $yp\frac{dp}{dy}+p^2=y^2$

or $p\frac{dp}{dy}+\frac{1}{y}p^2=y$.

Put $p^2=u$, $2p\frac{dp}{dy}=\frac{du}{dy}$; then the equation becomes

$\frac{du}{dy}+\frac{2}{y}u=2y$, Linear, I.F. $=y^2$.

$\therefore\ uy^2=c_1'+\int 2y.y^2\,dy$, *i.e.* $uy^2=c_1'+\frac{1}{2}y^4$

or $2p^2y^2=y^4+c_1^2$ or $\sqrt{2}p=\sqrt{2}\frac{dy}{dx}=\frac{\sqrt{(y^4+c_1^2)}}{y}$

Integrating, $\sqrt{2}\sinh^{-1}\frac{y^2}{c_1}=2x+\sqrt{2}c_2$

or $y^2=c_1\sinh(\sqrt{2}x+c_2)$ is the solution.

Ex. 5. (b) *Solve* $y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=1$. [Karnatak 60]

Hint. Just like above Ex.

Ex. 6. *Solve* $y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=\frac{dy}{dx}$.

Solution. The equation is free from x; hence putting $\frac{dy}{dx}=p$, $\frac{d^2y}{dx^2}=p\frac{dp}{dy}$, the equation becomes

$$yp\frac{dp}{dy}+p^2=p \quad\text{or}\quad \frac{dp}{dy}+\frac{1}{y}p=\frac{1}{y}.$$

This is a linear equation, its I.F. $=e^{\int\frac{1}{y}dy}=y$.

$\therefore\ p.y=c_1+\int\frac{1}{y}.y\,dy=c_1+y$ or $p=\frac{dy}{dx}=\frac{c_1+y}{y}$

or $\frac{y}{c_1+y}dy=dx$ or $\left(1-\frac{c_1}{c_1+y}\right)dy=dx$,

Integrating, $y-c_1\log(c_1+y)=x+c_2$

or $y-x-c_2=c_1\log(c_1+y)$ is the solution.

Ex. 7. *Solve* $(1-y^2)\dfrac{d^2y}{dx^2}=2\dfrac{dy}{dx}\left(1+y\dfrac{dy}{dx}\right)$.

Solution. Equation is free from x; hence putting

$$\frac{dy}{dx}=p,\ \frac{d^2y}{dx^2}=p\frac{dp}{dy},$$

the equation becomes

$$(1-y^2)p\frac{dp}{dy}=2p\,(1+yp)\quad\text{or}\quad\frac{dp}{dy}-\frac{2yp}{1-y^2}=\frac{2}{1-y^2}$$

which is a linear equation, I.F.$=e^{-\int\frac{2y}{1-y^2}dy}=(1-y^2)$.

$$\therefore\quad p.(1-y^2)=c_1+\int\frac{2}{1-y^2}(1-y^2)\,dy=c_1+2y,$$

so that $\quad p=\dfrac{dy}{dx}=\dfrac{c_1+2y}{1-y^2}\quad$ or $\quad\dfrac{1-y^2}{c_1+2y}\,dy=dx$

or $\quad\left(-\frac{1}{2}y+\frac{1}{4}c_1-\dfrac{\frac{1}{4}c_1^2}{c_1+2y}\right)dy=dx.$

Integrating, $\quad -\frac{1}{4}y^2+\frac{1}{4}c_1y-\frac{1}{8}c_1^2\log(c_1+2y)=x+c.$

$$-2y^2+2c_1y-c_1^2\log(c_1+2y)=8x+c_2.$$

Ex. 8. *Solve* $\dfrac{d^2y}{dx^2}+a\left(\dfrac{dy}{dx}\right)^2=0.$

Solution. The equation does not contain x directly.

$\therefore$ putting $\dfrac{dy}{dx}=p,\ \dfrac{d^2y}{dx^2}=p\dfrac{dp}{dy}$, the equation becomes

$$p\frac{dp}{dy}-ap^2=0\quad\text{or}\quad\frac{dp}{p}-a\,dy=0$$

Integrating, $\log p-ay=\log c_1$ or $p=c_1e^{ay}$,

i.e. $\dfrac{dy}{dx}=c_1e^{ay}$ or $e^{-ay}\,dy=c_1\,dx.$

Integrating, $-\dfrac{e^{-ay}}{a}=c_1x+c_2$ or $e^{-ay}=c_1'x+k'$

which is the required solution.

Ex. 9. *Solve* $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+4\left(\dfrac{dy}{dx}\right)^2=0.$

Solution. The equation is free from x; so putting $\dfrac{dy}{dx}=p$, $\dfrac{d^2y}{dx^2}=p\dfrac{dp}{dy}$, the equation becomes

$$p\frac{dp}{dy}+2p+4p^2=0\quad\text{or}\quad\frac{dp}{(1+2p^2)}=-2dy.$$

Integrating, $\dfrac{1}{\sqrt{2}}\tan^{-1}\sqrt{2}p=-2y+k$

or $\tan^{-1}\sqrt{2p}=c_1-2\sqrt{2y}$ or $\sqrt{2}\dfrac{dy}{dx}=\tan(c_1-2\sqrt{2y})$

or $\sqrt{2}\cot(c_1-2\sqrt{2y})\,dy=dx$.

Integrating, $\log\sin(c_1-2\sqrt{2y})=-2x+c_2'$

or $\sin(c_1-2\sqrt{2y})=e^{c_2'}.e^{-2x}=c_2e^{-2x}$,

which is the required solution.

1·4. Exact Equation.

The differential equation

$$f\left(\frac{d^ny}{dx^n},\frac{d^{n-1}y}{dx^{n-1}},\ldots,\frac{dy}{dx},y,x\right)=Q(x),$$

is called an exact differential equation if it can be obtained by differentiating once and without any further process from an equation of the next lower order

$$f\left(\frac{d^{n-1}y}{dx^{n-1}},\frac{d^{n-2}y}{dx^{n-1}},\ldots,\frac{dy}{dx},y,x\right)=\int Q(x)\,dx+c.$$

For example,

$$3y^2\frac{d^3y}{dx^3}+14y\frac{dy}{dx}\frac{d^2y}{dx^2}+4\left(\frac{dy}{dx}\right)^2+12\frac{dy}{dx}\frac{d^2y}{dx^2}=2x.$$

is an exact equation since it may be obtained merely by differentiating once the equation

$$3y^2\frac{d^2y}{dx^2}+4y\left(\frac{dy}{dx}\right)^2+6\left(\frac{dy}{dx}\right)^2=x^2+c.$$

*** 1·5. Condition of exactness for a linear equation of order n.**

[Nagpur 1962 ; Bombay 61 ; Delhi Hons. 62, 61, 60]

Let the linear differential equation of order n be

$$P_0\frac{d^ny}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+\ldots+P_ny=Q(x), \qquad \ldots(1)$$

where $P_0, P_1,\ldots, P_n, Q$ are all functions of x.

Since $P_1\dfrac{d^ny}{dx^n}$ is obtained by differentiating once $P_0\dfrac{d^{n-1}y}{dx^{n-1}}$ hence let the equation (1) be obtained by differentiating once the equation

$$P_0\frac{d^{n-1}y}{dx^{n-2}}+Q_1\frac{d^{n-2}y}{dx^{n-2}}+\ldots+Q_{n-1}y=\int Q(x)\,dx+c. \qquad \ldots(2)$$

Differentiating (2) once w.r.t. x, we obtain

$$P_0\frac{d^ny}{dx^n}+(P_0'+Q_1)\frac{d^{n-1}y}{dx^{n-1}}+(Q_1'+Q_2)\frac{d^{n-2}y}{dx^{n-2}}+(Q_2'+Q_3)\frac{d^{n-3}y}{dx^{n-3}}$$
$$+\ldots+(Q'_{n-2}+Q_{n-1})\frac{dy}{dx}+Q'_{n-1}y=Q(x)$$

(1) and (3) are just the same equations; hence equating coefficients of various terms, we get

$$P_0=P_0,\ P_1=P_0'+Q_1,\ P_1=Q_1'+Q_2,\ P_3=Q_2'+Q_3$$
$$\ldots P_{n-1}=Q'_{n-2}+Q_{n-1} \text{ and } P_n=Q'_{n-1}$$

These relations give

$$Q_1=P_1-P'_0,$$
$$Q_2=P_2-P'_1+P''_0,$$
$$Q_{n-1}=P_{n-1}-P'_{n-2}+P''_{n-3}-\ldots+(-1)^{n-1}P_0^{(n-1)}.$$

These give coefficients of various terms in (2).

Also from the relation $P_n=Q'_{n-1}$, we get

$$P_n=P'_{n-1}-P''_{n-2}+P'''_{n-3}-\ldots+(-1)^{n-1}P_0^{(n)}$$

or $P_n-P'_{n-1}+P''_{n-2}-P'''_{n-3}-\ldots+(-1)^n P_0^{(n)}=0.$ ...(4)

(4) is the condition of exactness for the equation (1).

Thus if condition (4) is satisfied for (1), the first integral of (1) is (2) which takes the form

$$P_0\frac{d^{n-1}y}{dx^{n-1}}+(P_1-P_0')\frac{d^{n-2}y}{dx^{n-2}}+(P_2-P_1'+P_0'')\frac{d^{n-3}y}{dx^{n-3}}+\ldots$$
$$+\{P_{n-1}-P'_{n-2}+P''_{n-3}+\ldots+(-1)^{n-1}P_0^{n-1}\}y$$
$$=\int Q(x)\,dx+c.$$

Note. How to write condition (4) :

(1) Write P_n, P_{n-1}, P_{n-2} ... the various coefficients starting from the highest.

(2) Put as many dashes as the number which is subtracted from n in the suffix. Thus P_{n-2} whose suffix is 2 lower than n should be written with two dashes and P_0 whose suffix is n lower than n should be written with n dashes.

(3) Put +xe and —ve before these coefficients alternately.

Ex. 1. *Solve* $(1+x+x^2)\dfrac{d^3y}{dx^3}+(3+6x)\dfrac{d^2y}{dx^2}+6\dfrac{dy}{dx}=0.$

[Agra 1972. 56, 56 ; Raj. 64, 50]

Solution. The equation is of order 3. Here

$$P_0=1+x+x^2,\ P_1=3+6x,\ P_2=6,\ P_3=0.$$

Therefore the equation will be exact if

$$P_3-P_2'+P_1''-P'''_0=0$$

i.e. $0-0+0-0=0$, which is true.

Hence the equation is exact.

The first integral of the given equation is

$$P_0\frac{d^2y}{dx^2}+(P_1-P_0')\frac{dy}{dx}+(P_2-P_1'+P_0'')\,y=c_1$$

or $(1+x+x^2)\dfrac{d^2y}{dx^2}+\{3+6x-(1+2x)\}\dfrac{dy}{dx}+(6-6+2)\,y=c_1$ ...(1)

Now considering (1) as an equation to be tested for exactness, we have for (1), $P_0=1+x+x^2$, $P_1=2(1+2x)$, $P_2=2$

and here $P_2-P_1'+P_0''=2-4+2=0$

i.e. for (1) also the condition of exactness is satisfied ; therefore the first integral of (1) is

$$P_0 \frac{dy}{dx}+(P_1-P'_0)\, y=\int c_1\, dx+c_2$$

or $$(1+x+x^2)\frac{dy}{dx}[2+4x-(1+2x)]\, y=c_1x+c_2$$

or $$(1+x+x^2)\frac{dy}{dx}+(1+2x)\, y=c_1x+c_2. \qquad \ldots(2)$$

Here again $P_0=1+x+x^2$, $P_1=(1+2x)$

and $P_1-P_0'=(1+2x)-(1+2x)=0.$

Therefore (2) is also exact. Its solution is

$$P_0\, y=\int(c_1x+c_2)\, dx+c_3$$

i.e. $$(1+x+x^2)\, y=\tfrac{1}{2}c_1x^2+c_2x+c_3,$$

which is the required solution of the given equation.

Ex. 2. *Solve* $x\dfrac{d^3y}{dx^3}+(x^2+x+3)\dfrac{d^2y}{dx^2}+(4x+2)\dfrac{dy}{dx}+2y=0.$

[Delhi Hons. 1970, 61]

Solution. The equation is of order three. Also here

$P_0=x$, $P_1=x^2+x+3$, $P_2=4x+2$, $P_3=2$.

The condition of exactness is

$P_3-P_2'+P''-P_0'''=0,$

i.e. $2-4+2-0=0$, which is satisfied.

Hence the equation is exact. The first integral is

$$P_0\frac{d^2y}{dx^2}+(P_1-P_0')\frac{dy}{dx}+(P_2-P_1'+P_0'')\, y=\text{const.}$$

or $$x\frac{d^2y}{dx^2}+(x^2+x+2)\frac{dy}{dx}+(2x+1)\, y=c_1. \qquad \ldots(1)$$

In (1), $P_0=x$, $P=x^2+x+2$, $P_2=2x+1$.

The condition of exactness, *i.e.* $P_2-P_1'+P_0''=0$ is satisfied for (1).

$\therefore$ the integral of (1) is

$$P_0\frac{dy}{dx}+(P_1-P_0')\, y=\int c_1\, dx+c_2$$

i.e. $$x\frac{dy}{dx}+(x^2+x+1)\, y=c_1x+c_2. \qquad \ldots(2)$$

This is not exact and can be written as

$$\frac{dy}{dx}+\left\{x+1+\frac{1}{x}\right\}y=c_1+\frac{c_2}{x}$$

Linear equation, I.F. $=e^{\int\left(x+1+\frac{1}{x}\right)dx}=xe^{\frac{1}{2}x(x+2)}.$

Hence the complete solution is

$$y.xe^{\frac{1}{2}x(x+2)}=\int\left\{c_1+\frac{c_2}{x}\right\}xe^{\frac{1}{2}x(x+2)}dx+c_3.$$

Ex. 3. *Solve* $x\dfrac{d^3y}{dx^3}+(x^2-3)\dfrac{d^2y}{dx^2}+4x\dfrac{dy}{dx}+2y=0.$

[Bombay 1958; Allahabad 52]

Solution. The equation is of third order. Here $P_0=x$, $P_1=x^2-3$, $P_2=4x$, $P_3=2$, and condition of exactness,

$P_3-P_2'+P_1''-P_0'''=2-4+2=0$ is satisfied.

Hence the eqation is exact. The first integral is

$$P_0\frac{d^2y}{dx^2}+(P_1-P_0')\frac{dy}{dx}+(P_2-P_1'+P_0'')\,y=c_1$$

or $$x\frac{d^2x}{dx^2}+(x^2-4)\frac{dy}{dx}+2xy=c_1. \qquad ...(1)$$

For (1) $P_0=x$, $P_1=x^2-4$, $P_2=2x$ and $P_2-P_1'+P_0''=0$ is satisfied for (1). Hence (1) is also exact. The intergral of (1) is

$$P_0\frac{dy}{dx}+(P_1-P_0')\,y=\int c_1\,dx+c_2$$

or $x\dfrac{dy}{dx}+(x^2-5)\,y=c_1x+c_2$ or $\dfrac{dy}{dx}+\left\{x-\dfrac{5}{x}\right\}y=c_1+\dfrac{c_2}{x}.$

This is a linear equation.

$$\text{I.F.}=e^{\int\left(x-\frac{5}{x}\right)dx}=e^{\frac{1}{2}x^2-5\log x}=e^{\frac{1}{2}x^2}x^{-5}.$$

Hence the solution is

$$y\frac{e^{\frac{1}{2}x^2}}{x^5}=c_1\int\frac{e^{\frac{1}{2}x^2}}{x^5}dx+c_2\int\frac{e^{\frac{1}{2}x^2}}{x^6}dx+c_3.$$

***Ex. 4.** *Solve* $x^2\dfrac{d^2y}{dx^2}+3x\dfrac{dy}{dx}+y=\dfrac{2}{(1-x)^2},$

[Agra 1966, 57, 55, 51; Raj. 52]

Solution. The equation is of second order.

Here $P_0=x^2$, $P_1=3x$, $P_2=1$.

The condition of exactness, *i.e.*, $P_2-P_1'+P_0''=0$ gives $1-3+2=0$ which is satisfied.

Hence the equation is exact. The first integral is

$$P_0\frac{dy}{dx}+(P_1-P_0')y=\int\frac{1}{(1-x)^2}dx+c_1,$$

i.e., $x^2\dfrac{dy}{dx}+xy=\dfrac{1}{1-x}+c_1$ or $\dfrac{dy}{dx}+\dfrac{1}{x}y=\dfrac{1}{x^2(1-x)}+\dfrac{c_1}{x^2},$

* The equation can be solved by the method of homogeneous equations. See Ex. 11 (a) page 100 Part I.

which is linear. Its I.F. $=e^{\int \frac{1}{x}\,dx}=x$.

Hence $y.x=\int\left\{\frac{1}{x^2(1-x)}+\frac{c_1}{x^2}\right\}x\,dx+c_2$

$$=\int\left(\frac{1}{x}+\frac{1}{1-x}+\frac{c}{x^2}\right)x\,dx+c_2$$

or $$yx=\log\frac{x}{1-x}+c_1\log x+c_2,$$

which is the required solution.

Ex. 5. *Solve* $(1+x^2)\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y=0.$ **[Agra 69, 53]**

Solution. The equation is of second order. Here

$P_0=1+x^2,\ P_1=3x,\ P_2=1.$

The condition of exactness $P_2-P_1'+P_0''=0$,

i.e. $1-3+2=0$ is satisfied.

Hence the equation is exact. The first integral of the given equation is $P_0\frac{dy}{dx}+(P_1-P_0')\,y+c_1$,

i.e. $(1+x^2)\frac{dy}{dx}+xy=c_1$ or $\frac{dy}{dx}+\frac{x}{1+x^2}y=\frac{c_1}{1+x^2}$.

Linear equation, I.F. $=e^{\int\frac{x}{1+x^2}dx}=e^{\frac{1}{2}\log(1+x^2)}=\sqrt{(1+x^2)}$.

$\therefore$ Solution is $y.\sqrt{(1+x^2)}=c_2+\int\frac{c_1}{(1+x^2)}\sqrt{(1+x^2)}\,dx$

$$=c_2+c_1\log\left[x+\sqrt{(1+x^2)}\right].$$

Ex. 6. *Solve* $(x^3-x)\frac{d^3y}{dx^3}+(8x^2-3)\frac{d^2y}{dx^2}+14x\frac{dy}{dx}+4y=\frac{2}{x^3}.$

[Agra 46 ; Raj. 59]

Solution. Equation is of third order. Here

$P_0=x^3-x,\ P_1=8x^2-3,\ P_2=14x,\ P_3=4.$

The condition of exactness $P_3-P_2'+P_1''-P_0'''=0$,

i.e. $4-14+16-6=0$ is satisfied

$\therefore$ The equation is exact. The first integral is

$$P_0\frac{d^2y}{dx^2}+(P_1-P_0')\frac{dy}{dx}+(P_2-P_1'+P_0'')\,y=\int\frac{2}{x^3}\,dx+c_1$$

os $(x^3-x)\frac{d^2y}{dx^2}+(5x^2-2)\frac{dy}{dx}+4xy=-\frac{1}{x^2}+c_1.$ ...(1)

For this equation $P_0=x^3-x,\ P_1=5x^2-2,\ P_2=4x.$

The condition of exactness $P_2-P_1'+P_0''=0$ is satisfied again.

$\therefore$ The integral of (1) is

$$P_0\frac{dy}{dx}+(P_1-P_0')\,y=\int\left(-\frac{1}{x^2}+c_1\right)dx+c_2$$

or $(x^3-x)\dfrac{dy}{dx}+(2x^2-1)\,y=\dfrac{1}{x}+c_1x+c_2$

or $\dfrac{dy}{dx}+\dfrac{2x^2-1}{x\,(x^2-1)}=\dfrac{1}{x^2\,(x^2-1)}+\dfrac{c_1}{x^2-1}+\dfrac{c_2}{x\,(x^2-1)}$.

Linear equation, I.F. $=e^{\int \frac{2x^2-1}{x(x^2-1)}dx}=e^{\int \frac{2x^2-1}{x^2(x^2-1)}x\,dx}$

$=e^{\frac{1}{2}\int\left(\frac{1}{t}+\frac{1}{t-1}\right)dt}=e^{\log t\,(t-1)}$ where $t=x^2$

$=x\sqrt{(x^2-1)}$.

Hence the solution is

$$yx\sqrt{(x^2-1)}=\int\left[\frac{1}{x\sqrt{(x^2-1)}}+\frac{c_1x}{\sqrt{(x^2-1)}}+\frac{c_2}{\sqrt{x^2-1)}}\right]dx+c_3$$
$$=\sec^{-1}x+c_1\sqrt{(x^2-1)}+c_2\log[x+\sqrt{(x^2-1)}]+c_3.$$

Ex. 7. *Solve* $\sin x\dfrac{d^2y}{dx^2}-\cos x\dfrac{dy}{dx}+2y\sin x=0$.

[Agra 78, 68, 64: Delhi Hons. 57]

Solution. Equation is of second order. Here

$P_0=\sin x,\ P_1=-\cos x,\ P_2=2\sin x.$

Condition of exactness $P_2-P_1'+P_0''=0$.

i.e. $2\sin x-\sin x-\sin x=0$ is satisfied.

Hence the equation is exact. The first integral is

$P_0\dfrac{dy}{dx}+(P_1-P_0')\,y=c_1$ or $\sin x\dfrac{dy}{dx}-(2\cos x)\,y=c_1$

or $\dfrac{dy}{dx}-(2\cos x)\,y=c_1\operatorname{cosec} x$, linear (not exact).

I.F. $=e^{-\int 2\cot x\,dx}=e^{-2\log\sin x}=\dfrac{1}{\sin^2 x}=\operatorname{cosec}^2 x$.

Hence the solution is

$$y\operatorname{cosec}^2 x=c_1\int \operatorname{cosec} x.\operatorname{cosec}^2 x\,dx^*+c_2$$
$$=-\tfrac{1}{2}c_1\operatorname{cosec} x\cot x+\tfrac{1}{2}c_1\log\tan\tfrac{1}{2}x+c_2$$

or $y=-\frac{1}{2}c_1\cos x+\frac{1}{2}c_1\sin^2 x\log\tan\frac{1}{2}x+c_2\sin^2 x$.

***Ex. 8.** *Solve* $\dfrac{d^3y}{dx^3}+\cos x\dfrac{d^2y}{dx^2}-2\sin x\dfrac{dy}{dx}-y\cos x=\sin 2x$.

[Agra 57; Allahabad 53; Pb. 62; Delhi Hons. 59]

Solution. $P_0=1,\ P_1=\cos x,\ P_2=-2\sin x,\ P_3=-\cos x$.

Condition of exactness $P_3-P_2'+P_1''-P_0'''=0$,

i.e. $-\cos x+2\cos x-\cos x-0=0$ is satisfied.

$^*\displaystyle\int \operatorname{cosec}^n x\,dx=-\frac{\cot x\operatorname{cosec}^{n-2}x}{n-1}+\frac{n-2}{n-1}\int \operatorname{cosec}^{n-2}x\,dx.$

Therefore the equation is exact. The first integral is

$$P_0 \frac{d^2y}{dx^2}+(P_1-P_0')\frac{dy}{dx}+(P_2-P_1'+P_0'')\,y=\int \sin 2x\,dx+c_1$$

or $$\frac{d^2y}{dx^2}+\cos x\frac{dy}{dx}-\sin x\,.\,y=-\tfrac{1}{2}\cos 2x+c_1,$$

which is again exact. Hence its solution is

$$\frac{dy}{dx}+\cos x\,.\,y=-\tfrac{1}{4}\sin 2x+c_1x+c_2,$$

which is linear, I.F $=e^{\sin x}$. Hence the solution is

$$ye^{\sin x}=-\tfrac{1}{4}\int \sin 2x\,.\,e^{\sin x}\,dx+\int (c_1x+c_2)\,e^{\sin x}\,dx+c_3.$$

Now $$\int \sin 2x\, e^{\sin x}\,dx=2\int \sin x\cos x\, e^{\sin x}\,dx$$

$$=2\int te^t\,dt=2e^t(t-1),\quad \text{where } t=\sin x$$

$$=2e^{\sin x}(\sin x-1).$$

Hence the solution is

$$ye^{\sin x}=-\tfrac{1}{2}e^{\sin x}(\sin x-1)+\int (c_1x+c_2)\,e^{\sin x}\,dx+c_3.$$

Ex. 9. *Solve* $(ax-bx^2)\dfrac{d^2y}{dx^2}+2a\dfrac{dy}{dx}+2by=x.$ **[Raj. 1954]**

Solution. Here $P_0=ax-bx^2$, $P_1=2a$, $P_2=2b$ and condition of exactness, $P_2-P_1'+P_0''=0$ is satisfied.

Hence the equation is exact. First integral is

$$P_0\frac{dy}{dx}+(P_1-P_0')\,y=\int x\,dx+c_1$$

or $$(ax-bx^2)\frac{dy}{dx}+(a+2bx)\,y=\tfrac{1}{2}x^2+c_1$$

or $$\frac{dy}{dx}+\frac{a+2bx}{x(a-bx)}\,y=\frac{\tfrac{1}{2}x}{(a-bx)}+\frac{c_1}{x(a-bx)},\ \text{linear equation}$$

$$\text{I.F.}=e^{\int \frac{a+2bx}{x(a-bx)}dx}=e^{\int\left(\frac{1}{x}+\frac{3b}{a-bx}\right)dx}$$

$$=e^{[\log x-3\log(a-bx)]}=\frac{x}{(a-bx)^3}.$$

Hence the solution is

$$y.\frac{x}{(a-bx)^3}=\tfrac{1}{2}\int\frac{x^2}{(a-bx)^4}\,dx+\int\frac{c_1}{(a-bx)^4}\,dx+c_2$$

$$=\int\frac{\frac{a^2}{b^2}\sin^4\theta\,\frac{2a}{b}\sin\theta\cos\theta\,d\theta}{2a^4\cos^8\theta}+\frac{c_1}{3b(a-bx)^3}+c_2$$

putting $bx=a\sin^2\theta$ in first integral

$$\left.\begin{aligned}&=\frac{1}{ab^3}\int \tan^3\theta \sec^2\theta \, d\theta+\frac{c_1}{3b(a-bx)^3}+c_2\\&=\frac{1}{6a^2b^3}\tan^6\theta+\frac{c_1}{3b(a-bx)^3}+c_2\\&=\frac{1}{6ab^3}\left(\frac{bx}{a-bx}\right)^3+\frac{c_1}{3b(a-bx)^3}+c_2\end{aligned}\right\}\quad \begin{aligned}&\sin^2\theta=\frac{bx}{a},\\&\tan^2\theta=\frac{\sin^2\theta}{\cos^2\theta}\\&=\frac{bx/a}{(1-bx/a)}\\&=\frac{bx}{a-bx}\end{aligned}$$

or $yx=\frac{x^3}{6a}+\frac{c_1}{3b}+c_2(a-bx)^3$ is the solution.

Ex. 10. $(x^3-x)\frac{d^2y}{dx^2}-2(x-1)\frac{dy}{dx}-4y=0.$

Solution. Exact equation. Proceed as usual.

$y=c_1(4x^3-2x^2-\frac{2}{3}x-\frac{1}{3})+x^3(x-1)[c_2-4c_1\log x-\log(x-1)]$,

Ex. 11, *Solve* $\frac{d^2y}{dx^2}+2\sin x\frac{dy}{dx}+2y\cos x=0.$

Solution. $P_0=1$, $P_1=2\sin x$, $P_2=2\cos x$,

$P_2-P_1'+P_0''=2\cos x-2\cos x+0=0.$

Therefore the equation is exact. The first integral is

$P_0\frac{dy}{dx}+(P_1-P'_0)y=c_1$, *i.e.* $\frac{dy}{dx}+2\sin x.y=c_1$,

which is linear. I.F.$=e^{\int 2\sin x\,dx}=e^{-2\cos x}$.

$\therefore$ Solution is $ye^{-2\cos x}=\int c_1e^{-2\cos x}\,dx+c_2$

or $y=e^{2\cos x}\int c_1e^{-2\cos x}\,dx+c_2e^{2\cos x}$.

Ex. 12. *Solve* $x\frac{d^2y}{dx^2}+(1-x)\frac{dy}{dx}-y=e^x.$

$P_2-P_1'+P_0''=-1-(-1)+0=0$, therefore the equation is exact.

First differential is $P_0\frac{dy}{dx}+(P_1-P_0')y=\int e^x\,dx+c_1$

or $x\frac{dy}{dx}-xy=e^x+c_1$ or $\frac{dy}{dx}-y=\frac{e^x}{x}+\frac{c_1}{x}$,

which is a linear equation. I.F.$=e^{-\int dx}=e^{-x}$.

Hence solution is $ye^{-x}=\int\frac{e^x}{x}.e^{-x}\,dx+\int\frac{c_1}{x}e^{-x}\,dx+c_2$

or $y=e^x\log x+c_1e^x\int\frac{e^{-x}}{x}\,dx+c_2e^x$.

Ex. 13. $(2x^2+3x)\frac{d^2y}{dx^2}+(6x+3)\frac{dy}{dx}+2=(x+1)e^x.$

[Vikram 62]

Solution. $P_0=2x^2+3x$, $P_1=6x+3$, $P_2=2$
and $P_2-P_1'+P_0''=2-6+4=0$; $\because$ the equation is exact.

First integral is $(2x^2+3x)\dfrac{dy}{dx}+2xy=\int(x+1)\,e^x+c_1$

or $$\frac{dy}{dx}+\frac{2}{(3+2x)}y=\frac{e^x}{3+2x}+\frac{c_1}{x(3+2x)}.$$

Linear equation, I.F. $=e^{\int\frac{2}{3+2x}dx}=e^{\log(3+2x)}=3+2x.$

$\therefore$ Solution is $y\,(3+2x)=\int e^x\,dx+\int\frac{1}{x}\,dx+c_2$

$$=e^x+c_1\log x=c_2.$$

Ex. 14. *Solve* $\dfrac{d^2y}{dx^2}+2e^x\dfrac{dy}{dx}+2e^xy=x^2.$

Solution. $P_0=1$, $P_1=2e^x$, $P_2=2e^x$. $P_2-P_1'+P_0''=0$ is satisfied. Therefore the equation is exact.

First integral is $P_0\dfrac{dy}{dx}+(P_1-P_0')\,y=\int x^2\,dx+c_1$

cr $\dfrac{dy}{dx}+2e^xy=\frac{1}{3}x^3+c_1$. Linear, I.F. $=e^{\int 2e^x\,dx}=e$

$\therefore$ Solution is $ye^{2e^x}=\int\frac{1}{3}x^3e^{2e^x}\,dx+\int c_1e^{2e^x}\,dx+c_2.$

Ex. 15. *Solve* $(x^2-x)\dfrac{d^2y}{dx^2}+2\,(2x+1)\dfrac{dy}{dx}+2y=0.$

Solution. $P_0=x^2-x$. $P_1=2\,(2x+1)$. $P_2=2$.
$P_2-P_1'+P_0''=2-4+2=0$ is satisfied.
The equation is exact. First integral is

$$(x^2-x)\frac{dy}{dx}+(2x+3)\,y=c_1$$

or $\dfrac{dy}{dx}+\dfrac{2x+3}{x\,(x-1)}y=\dfrac{c_1}{x\,(x-1)}$, Linear equation.

$$\text{I.F.}=e^{\int\frac{2x+3}{x(x-1)}dx}=e^{\int\left(\frac{-3}{x}+\frac{5}{x-1}\right)dx}=\frac{(x-1)^5}{x^3}$$

Hence the solution is

$$y\,\frac{(x-1)^5}{x^3}=c_2+\int\frac{c_1}{x\,(x-1)}\,\frac{(x-1)^5}{x^3}\,dx$$

or $$y\,\frac{(x-1)^5}{x^3}=c_2+c_1\int\frac{(x-1)^4}{x^4}\,dx$$

$$=c_2+c_1\int\left(1-\frac{4}{x}+\frac{6}{x^2}+\frac{4}{x^3}+\frac{1}{x^4}\right)dx$$

$$=c_2+c_1\left(x-4\log x-\frac{6}{x}+\frac{2}{x^2}-\frac{1}{3x^4}\right)$$

or $y(x-1)^5=c_2x^3+c_1(x^4-4x^3\log x-6x^2+2x-\frac{1}{3})$.

Ex. 16. *Solve* $\dfrac{d^2y}{dx^2}+\dfrac{dx}{dy}=e^x$.

Solution. $P_0=1$, $P_1=1$, $P_2=0$, $P_2-P_1'+P_0''=0$ is satisfied. Therefore the equation is exact. First integral is

$$\frac{dy}{dx}+y=e^x+c_1. \quad \text{Linear, I.F.}=e^x.$$

Hence the solution is $ye^x=\int(e^x+c_1)\,e^x\,dx+c_2$

$$=\tfrac{1}{2}e^{2x}+c_1e^x+c_2,$$

$$y=\tfrac{1}{2}e^x+c_1+c_2e^{-x}.$$

Ex. 17. *Solve* $(x^3+4x)\,y'''+(9x^2-12)\,y''+18xy'+6y=0$.

[Delhi Hons. 56]

Solution. Third order equation. $P_0=x^3-4x$, $P_1=9x^2-12$, $P_2=18x$, $P_3=6$ and the condition of exactness $P_3-P_2'+P_1''-P_0'''=6-18+18-6=0$ is satisfied.

Hence the equation is exact. The first integral is

$$P_0\frac{d^2y}{dx^2}+(P_1-P_0')\frac{dy}{dx}+(P_2-P_1'+P_0'')\,y=c_1$$

or $$(x^3-4x)\frac{d^2y}{dx^2}+(6x^2-8)\frac{dy}{dx}+6xy=c_1.$$

which again satisfies the condition of exactness.

Hence next integral is

$$(x^3-4x)\frac{dy}{dx}+(6x-12x)\,y=c_1x+c_2,$$

i.e. $$\frac{dy}{dx}-\frac{6}{x^2-4}=\frac{c_1x+c_2}{x^3-4x}, \text{ linear.}$$

$$\text{I.F.}=e^{-\int\frac{6}{x^2-4}dx}=e^{-\int\frac{6}{(x-2)(x+2)}dx}=e^{-\int\left(\frac{3/2}{x-2}-\frac{3/2}{x+2}\right)dx}$$

$$=\left(\frac{x+2}{x-2}\right)^{3/2}.$$

Hence $$y\left(\frac{x+2}{x-2}\right)^{3/2}=c_3+\int\frac{(c_1x+c_2)}{x^3-4x}\left(\frac{x+2}{x-2}\right)^{3/2}dx=\text{etc.}$$

Ex. 18. *Find the first integral of*

$$x^5\frac{d^6y}{dx^6}+x^4\frac{d^5y}{dx^5}+x\frac{dy}{dx}+y=\log x.$$

Solution. Exact. First integral is

$$x^5y_5-4x^4y_4+16x^3y_3-48x^2y_1+96xy_2-96y=x\log x-x+c.$$

1·6. Integrating factor.

It may be noticed that sometimes an equation becomes exact after it has been multiplied by a suitable factor called the integrating factor.

In case coefficients $P_0, P_1, \ldots, P_n$ etc. are of the type $(A_0x^s+A_1x^t+\ldots)$ etc. the integrating factor is of the form x^m. To determine the integrating factor in such case we multiply the equation by x^m and apply the condition of exactness. This condition will be satisfied for a particular value of m and corresponding to this value of m, x^m is an integrating factor.

Again if coefficients $P_0, P_1, \ldots, P_n$ etc. are trigonometrical functions, integrating factor is also a trigonometrical function, which can be determined by trial and error method.

The following examples will make the procedure clear.

***Ex. 1.** *Solve* $\sqrt{x}\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}+3y=x.$

[Agra 1965 ; Raj. 51; Delhi Hons. 62; Bombay 61]

Solution. The equation in its present form does not satisfy condition of exactness. Let it become exact after multiplying by x^m. So multiplying by x^m it becomes

$$x^{m+\frac{1}{2}}\frac{d^2y}{dx^2}+2x^{m+1}\frac{dy}{dx}+3x^my=x^{m+1}$$

for which $P_0=x^{m+\frac{1}{2}}$, $P_1=2x^{m+1}$, $P_2=3x^m$

and condition of exactness is $P_2-P_1'+P_0''=0$,

i.e. $3x^m-2(m+1)x^m+(m+\frac{1}{2})(m-\frac{1}{2})x^{m-3/2}=0$

or $(1-2m)[x^m-\frac{1}{4}(2m+1)x^{m-3/2}]=0$

which is clearly satisfied by $m=\frac{1}{2}$ (from factor $1-2m=0$).

Hence the I F.$=x^{1/2}=\sqrt{x}$. Therefore multiplying by $\sqrt{x}$, the exact equation is $x\dfrac{d^2y}{dx^2}+2x^{3/2}\dfrac{dy}{dx}+3x^{1/2}y=x^{3/2}$.

Its first integral is $P_0\dfrac{dy}{dx}+(P_1-P_0')\,y=\displaystyle\int x^{3/2}\,dx+c_1$

or $\quad x\dfrac{dy}{dx}+(2x^{3/2}-1)\,y=\frac{2}{5}x^{5/2}+c_1$

or $\quad \dfrac{dy}{dx}+\left(2x^{1/2}-\dfrac{1}{x}\right)y=\frac{2}{5}x^{3/2}+\dfrac{c_1}{x}$,

which is linear, I.F.$=e^{\int(2x^{1/2}-1/x)\,dx}=\dfrac{1}{x}e^{\frac{4}{3}x^{3/2}}$.

Hence the solution is

$$y.\frac{1}{x}e^{\frac{4}{3}x^{3/2}}=\frac{2}{5}\int x^{3/2}\frac{e^{\frac{4}{3}x^{3/2}}}{x}\,dx+c_1\int\frac{e^{\frac{4}{3}x^{3/2}}}{x^2}\,dx+c_2$$

$$=e^{\frac{4}{3}x^{3/2}}+c_1\int\frac{e^{\frac{4}{3}x^{3/2}}}{x^2}\,dx+c_2.$$

Ex. 2. *Solve* $x^4\dfrac{d^2y}{dx^2}+x^2(x-1)\dfrac{dy}{dx}+xy=x^3-4.$

Solution The equation in its present form is not exact as it does not satisfy condition of exactness ; suppose it becomes exact after it has been multiplied by x^m. So multiplying by x^m, it becomes

$$x^{m+4}\frac{d^2y}{dx^2}+x^{m+2}(x-1)\frac{dy}{dx}+x^{m+1}y=x^m(x^3-4).$$

The condition of exactness, *i.e* $P_2-P_1'+P_0''=0$ gives

$$x^{m+1}-[(m+3)x^{m+2}-(m+2)x^{m+1}]+(m+4)(m+3)x^{m+2}=0$$

or $x^{m+2}(m+3)^2+x^{m+1}(m+3)=0$

or $(m+3)[(m+3)x^{m+2}-x^{m-1}]=0,$

which is satisfied when $m+3=0$, or $m=-3$.

Hence x^{-3} is an integrating factor.

Multiplying by x^{-3} the equation becomes

$$x\frac{d^2y}{dx^2}+\left(1-\frac{1}{x}\right)\frac{dy}{dx}+\frac{1}{x^2}y=1-\frac{4}{x^4}\text{ (now exact).}$$

First integral is $P_0\dfrac{dy}{dx}+(P_1-P_0')y=\displaystyle\int\left(1-\frac{4}{x^3}\right)dx+c$

i.e. $x\dfrac{dy}{dx}-\dfrac{1}{x}y=x+\dfrac{2}{x^2}+c_1$

or $\dfrac{dy}{dx}-\dfrac{1}{x^2}y=1+\dfrac{2}{x^3}+\dfrac{c_1}{x}$. Linear, I.F. $=e^{-\int 1/x^2\,dx}=e^{1/x}$.

Hence the solution is

$$ye^{1/x}=\int\left(1+\frac{2}{x^3}+\frac{c_1}{x}\right)e^{1/x}\,dx+c_2$$

$$=\int\left(1+\frac{c_1}{x}\right)e^{1/x}\,dx-2\int\frac{1}{x}e^{1/x}\left(-\frac{1}{x^2}\,dx\right)+c_2$$

$$=\int\left(1+\frac{c_1}{x}\right)e^{1/x}\,dx-2e^{1/x}\left(\frac{1}{x}-1\right)+c_2.$$

Ex. 3. *Solve* $2x^2(x+1)\dfrac{d^2y}{dx^2}+x(x+3)\dfrac{dy}{dx}-3y=x^2.$

Solution. Equation is not exact, Let x^m be its integrating factor ; then multiplying by x^m, it becomes

$$2x^{m+2}(x+1)\frac{d^2y}{dx^2}+x^{m+1}(7x+3\frac{dy}{dx}-3x^my=x^{m+2}.$$

Now $P_0=2x^{m+2}(x+1)$, $P_1=x^{m+1}(7x+3)$, $P_2=-3x^m$.

Condition of exactness, *i e.* $P_2-P_1'+P_0''=0$ gives

$$-3x^m-[7(m+2)x^{m+1}+3(m+1)x^m]$$
$$+2\{(m+3)(m+2)x^{m-1}+(m+2)(m+1)x^m\}=0,$$

i.e. $(m+2)(2m-1)(x^{m+1}+x^m)=0.$

which gives $m+2=0$ or $2m-1=0$, *i.e.* $m=-2,\ \frac{1}{2}$.

Thus when $m=-2$, the integrating factor is x^{-2} and multiplying by x^{-2} the equation (exact now) is

$$2(x+1)\frac{d^2y}{dx^2}+\left(7+\frac{3}{x}\right)\frac{dy}{dx}-\frac{3}{x^2}y=1.$$

The first integral is $2(x+1)\dfrac{dy}{dx}+\left(5+\dfrac{3}{x}\right)y=x+c_1$, ...(1)

which is linear and may be integrated further.

Again when $m=\frac{1}{2}$, the I.F.$=x^{1/2}$ and multiplying by $x^{1/2}$, the original equation becomes (exact now)

$$2x^{5/2}(x+1)\frac{d^2y}{dx^2}+x^{3/2}(7x+3)\frac{dy}{dx}-3x^{1/2}y=x^{5/2}$$

whose first integral is

$$2x^{5/2}(x+1)\frac{dy}{dx}-2x^{3/2}y=\tfrac{2}{7}x^{7/2}+c_2,$$

which is again linear and may be integrated further. ...(2)

However, the solution may also be obtained by eliminating $\dfrac{dy}{dx}$ between (1) and (2). For this, multiply (1) by $x^{5/2}$ and subtract from (2). The solution so obtained is

$$5(x+1)=\tfrac{5}{7}x^3+c_1x-c_2x^{-5/2}.$$

Ex. 4. *Solve* $2x^2\dfrac{d^2y}{dx^2}+15x\dfrac{dy}{dx}-7y=3x^2.$

Solution. Proceeding as in the above Ex., $m=-\frac{3}{2}$, 6.

Ans. $y=\dfrac{x^2}{9}+\dfrac{c_1}{x^7}+c_2\sqrt{x}.$

Ex. 5. *Solve* $x^5\dfrac{d^2y}{dx^2}+3x^3\dfrac{dy}{dx}+(3-6x)x^2y=x^4+2x-5.$

[Agra 70 ; Delhi Hons. 60]

Solution. Here the equation is not exact as the coefficients of the equation do not satisfy the condition of exactness.

So suppose the equation becomes exact after it has been multiplied by x^m

Multiplying by x^m, the equation becomes

$$x^{m+5}\frac{d^2y}{dx^2}+3x^{m+3}\frac{dy}{dx}+(3-6x)x^{m+2}y=x^m(x^4+2x-5)$$

For this $P_0=x^{m+5}$, $P_1=3x^{m+3}$, $P_2=(3-6x)x^{m+2}$.

Since the equation is exact, $P_2-P_1'+P_0''=0$,

i.e. $(3-6x)x^{m+2}-3(m+3)x^{m+2}+(m+5)(m+4)x^{m+3}=0,$

i.e. $(m+2)(m+7)x^{m+3}-3(m+2)x^{m+2}=0.$

This is satisfied clearly when $m=-2$. Therefore x^{-2} is an integrating factor.

On multiplying by x^{-2}, the equation becomes

$$x^3\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+(3-6x)y=x^2+\frac{2}{x}-\frac{5}{x^2},$$

which is an exact equation now.

Its first integral is

$$P_0\frac{dy}{dx}+(P_1-P_0')\,y=\int\left(x^2+\frac{2}{x}-\frac{5}{x^2}\right)dx+c_1,$$

i.e. $$x^3\frac{dy}{dx}+3x(1-x)\,y=\frac{x^3}{3}+2\log x+\frac{5}{x}+c_1,$$

which is a linear equation I.F.$=e^{\int\frac{3(1-x)}{x^3}dx}=e^{-\frac{3}{x}-3\log x}$

$$=e^{\frac{-3}{x}}\cdot\frac{1}{x^3}.$$

Hence the solution is

$$y\,.\,e^{-3/x}\cdot\frac{1}{x^3}=c_2+\int\left(\frac{1}{3}+\frac{2}{x^3}\log x+\frac{5}{x^4}+\frac{c_1}{x^3}\right)e^{-3/x}\cdot\frac{1}{x^3}\cdot dx.$$

Ex. 6. *Find first integral of*

$$x^2\frac{d^3y}{dx^3}+4x\frac{d^2y}{dx^2}+(x^2+2)\frac{dy}{dx}+3xy=2.$$

Solution. Proceed as in above. I.F.$=x$.

First integral is $x^3\dfrac{d^2y}{dx^2}+x^2\dfrac{dy}{dx}+x^3y=x^2+c.$

***Ex. 7.** *Solve* $\sin^2 x\dfrac{d^2y}{dx^2}=2y.$ **[Agta 62, 56; Raj. 52]**

Solution. The equation can be written as

$$\frac{d^2y}{dx^2}-2\operatorname{cosec}^2 x\,.\,y=0.$$

Multiplying by $\cot x$, $\cot x\dfrac{d^2y}{dx^2}-2\cot x\operatorname{cosec}^2 x\,.\,y=0$ in which $P_0=\cot x$, $P_1=0\left(\text{coeff. of }\dfrac{dy}{dx}\right)$, $P_2=-2\cot x\operatorname{cosec}^2 x$.

$\therefore\quad P_2-P_1'+P_0''=-2\cot x\operatorname{cosec}^2 x-0+2\operatorname{cosec}^2 x\cot x=0.$

Thus the equation is exact now.

Its first integral being $P_0\dfrac{dy}{dx}+(P_1-P_0')\,y=c_1$

or $\quad\cot x\dfrac{dy}{dx}+\operatorname{cosec}^2 x\,.\,y=c_1\quad$ or $\quad\dfrac{dy}{dx}+\dfrac{\operatorname{cosec}^2 x}{\cot x}y=c_1\tan x.$

Linear equation, I.F.$=e^{-\log\cot x}=e^{\log\tan x}=\tan x.$

Hence the solution is

$$y\tan x=\int c_1\tan^2 x\,dx+c_2=\int c_1(\sec^2 x-1)\,dx+c_2$$
$$=c_1(\tan x-x)+c_2.$$

***Ex. 8.** *Solve* $\dfrac{d^2y}{dx^2}+2\tan x\dfrac{dy}{dx}+3y=\tan^2 x\sec x.$

Solution. The equation is not exact. However, if we multiply by $\cos x$, it becomes

$$\cos x \frac{d^2y}{dx^2}+2 \sin x \frac{dy}{dx}+3 \cos x . y=\tan^2 x, \qquad ...(1)$$

for which $P_2-P_1'+P_0''=3 \cos x-2 \cos x-\cos x=0$,

i.e. condition of exactness is satisfied. Therefore by inspection $\cos x$ is an integrating factor.

First integral of (1) is

$$P_0 \frac{dy}{dx}+(P_1-P_0') \, y=\int \tan^2 x \, dx+c_1$$

i.e. $$\cos x \frac{dy}{dx}+3 \sin x . y=\int (\sec^2 x-1) \, dx+c_1$$

$$=\tan x-x+c_1$$

or $$\frac{dy}{dx}+3 \tan x . y=\sec x \tan x-x \sec x+c_1 \sec x.$$

Linear, I.F. $=e^{\int 3 \tan x \, dx}=e^{3 \log \sec x}=\sec^3 x.$

Hence the solution is

$$y \sec^3 x \quad c_2+\int (\sec x \tan x-x \sec x+c_1 \sec x) \sec^3 x \, dx.$$

Now $$\int \sec^4 x \tan x \, dx=\int (1+\tan^2 x) \tan x . \sec^2 x \, dx$$

$$=\tfrac{1}{2} \tan^2 x+\tfrac{1}{4} \tan^4 x.$$

$$\int x \sec^4 x \, dx$$

$$=\int x \, (1+\tan^2 x) \sec^2 x \, dx$$

$$=x \, (\tan x+\tfrac{1}{3} \tan^3 x)-\int (\tan x+\tfrac{1}{3} \tan^3 x) \, dx$$

$$=x(\tan x+\tfrac{1}{3} \tan^3 x)-\int \tan x \, dx$$

$$-\tfrac{1}{3}\int (\tan x \sec^2 x-\tan x) \, dx$$

$$=x \, (\tan x+\tfrac{1}{3} \tan^3 x)-\tfrac{2}{3} \int \tan x \, dx-\tfrac{1}{3} \int \tan x \sec^2 x \, dx$$

$$=x \, (\tan x+\tfrac{1}{3} \tan^3 x)-\tfrac{2}{3} \log \sec x-\tfrac{1}{3} \cdot \tfrac{1}{2} \tan^2 x$$

and $$\int \sec^4 x \, dx=\int (1+\tan^2 x) \sec^2 x \, dx=\tan x+\tfrac{1}{3} \tan^3 x.$$

Hence the complete solution is

$$y \sec^3 x=c_2+\tfrac{1}{2} \tan^2 x+\tfrac{1}{4} \tan^4 x-x \, (\tan x+\tfrac{1}{3} \tan^3 x)$$
$$+\tfrac{2}{3} \log \sec x+\tfrac{1}{6} \tan^2 x+c_1 \, (\tan x+\tfrac{1}{3} \tan^3 x).$$

Ex. 9. *Solve* $\dfrac{d^2y}{dx^2}-\cot x \dfrac{dy}{dx}+2y=\cos x.$

Solution. Multiplying by $\sin x$, the equation becomes

$$\sin x \frac{d^2y}{dx^2} - \cos x \frac{dy}{dx} + 2y \sin x = \sin x \cos x.$$

Here $P_0 = \sin x$, $P_1 = -\cos x$, $P_2 = 2 \sin x$.
Also $P_2 - P_1' + P_0'' = 2 \sin x - \sin x - \sin x = 0$.
Therefore the equation is exact. First integral is

$$P_0 \frac{dy}{dx} + (P_1 + P_0')\, y = \int \sin x \cos x \, dx + c_1$$

i.e. $\sin x \dfrac{dy}{dx} - 2 \cos x . y = \frac{1}{2} \sin^2 x + c\}$

o $\dfrac{dy}{dx} - \dfrac{2 \cos x}{\sin x}\, y = \frac{1}{2} \sin x + \dfrac{c_1}{\sin x}$.

Linear. I.F. $= e^{-2 \log \sin x} = \dfrac{1}{\sin^2 x} = \operatorname{cosec}^2 x$.

The solution is

$$y \operatorname{cosec}^2 x = c_2 + \int \left(\tfrac{1}{2} \sin x + \frac{c_1}{\sin x} \right) \operatorname{cosec}^2 x \, dx$$

$$= c_2 + \int \left(\tfrac{1}{2} \operatorname{cosec} x + c_1 \operatorname{cosec}^3 x \right) dx$$

$$= c_2 + \tfrac{1}{2} \log \tan \tfrac{1}{2}x - c_1 \left(\tfrac{1}{2} \operatorname{cosec} x \cot x + \tfrac{1}{2} \log \tan \tfrac{1}{2}x \right).$$

1·7. Non-linear Equations

Exactness. So far we have been discussing exactness of linear equations. The equations which are not linear may also be exact, in such a but thereis no simple test for their exactness. We group terms way that they become perfect differential and their integrals may be written direetly. Much depends on success of trial for such arragements.

The method will be fully illustrated in thc following examples.

Ex. 1. *Solve* $2y \dfrac{d^3y}{dx^3} + 2\left(y + 3 \dfrac{dy}{dx}\right) \dfrac{d^2y}{dx^2} + 2\left(\dfrac{dy}{dx}\right)^2 = 2.$

Solution. The given equation may be written as

$$2y \frac{d^3y}{dx^3} + 2y \frac{d^2y}{dx^2} + 6 \frac{dy}{dx} \frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 2 \qquad ...(1)$$

The first term may be obtained by differentiating the term $2y \dfrac{d^2y}{dx^2}$.

But $$\frac{d}{dx}\left(2y \frac{d^2y}{dx^2}\right) = 2y \frac{d^3y}{dx^3} + 2 \frac{dy}{dx} \frac{d^2y}{dx^2}. \qquad ...(2)$$

So leaving apart from (1) the terms on the right of (2) we are left with $2y \dfrac{d^2y}{dx^2} + 4 \dfrac{dy}{dx} \dfrac{d^2y}{dx^2} + 2\left(\dfrac{dy}{dx}\right)^2$

and the first term of it is obtained by differentiating $2y \dfrac{dy}{dx}$.

But $$\frac{d}{dx}\left(2y \frac{dy}{dx}\right) = 2y \frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 \qquad ...(3)$$

The remaining term $4\dfrac{dy}{dx}\dfrac{d^2y}{dx^2}=\dfrac{d}{dx}2\left(\dfrac{dy}{dx}\right)^2$. ...(4)

Thus combining terms on the right of (2), (3) and (4), we get left hand side of (1). Thus (1) can be written as

$$\frac{d}{dx}\left(2y\frac{d^2y}{dx^2}\right)+\frac{d}{dx}\left(2y\frac{dy}{dx}\right)+\frac{d}{dx}2\left(\frac{dy}{dx}\right)^2=2.$$

Integrating, $2y\dfrac{d^2y}{dx^2}+2y\dfrac{dy}{dx}+2\left(\dfrac{dy}{dx}\right)^2=2x+c_1$. ...(5)

In (5) the first term is obtained by the differentiation of $2y\dfrac{dy}{dx}$ but $\dfrac{d}{dx}\left(2y\dfrac{dy}{dx}\right)=2y\dfrac{d^2y}{dx^2}+2\left(\dfrac{dy}{dx}\right)^2$

The remaining term $2y\dfrac{dy}{dx}=\dfrac{d}{dx}(y^2)$.

Therefore (5) may be written as

$$\frac{d}{dx}\left(2y\frac{dy}{dx}\right)+\frac{d}{dx}(y^2)=2x+c_1.$$

Integrating, $2y\dfrac{dy}{dx}+y^2=x^2+c_1x+c_2$. ...(6)

Now putting $y^2=u$, $2y\dfrac{dy}{dx}=\dfrac{du}{dx}$.

$\therefore$ the equation (6) becomes $\dfrac{du}{dx}+u=x^2+c_1x+c_2$

This is linear equation, I.F. $=e^x$.

$\therefore$ The solution is $ue^x=c_3+\int(x^2+c_1x+c_2)\,e^x\,dx$

or $y^2e^x-c_3+e^x(x^2-2x+2)+c_1e^x(x-1)+c_2e^x$

or $y^2=x^2+k_1+k_2x+k_3e^{-x}$ is required solution.

Note. The following scheme may be noted :—

$$\begin{array}{lll}
 & 2yy'''+2yy''+6y'y''+2(y')^2=2 & \\
\dfrac{d}{dx}(2yy'')=2yy''' & \qquad +2y'y'' & \\
\hline
 & 2yy''+4y'y''+2(y')^2 & \\
\dfrac{d}{dx}(2yy')= & 2yy'' \qquad +2(y')^2 & \\
\hline
 & 4y'y'' & \\
\dfrac{d}{dx}(2y'^2)= & 4y'y'' & \\
\hline
 & \times &
\end{array}$$

Therefore, the equation is

$$\frac{d}{dx}(2yy'')+\frac{d}{dx}(2yy')+\frac{d}{dx}(2y'^2)=2.$$

Integrating, $2yy''+2yy'+2y'^2=2x+c_1$ ...(A)

For $\qquad 2yy''+2yy'+2y'^2=2x+c_1$

$$\begin{array}{ll}
\dfrac{d}{dx}(yy'')= & 2yy'' \qquad +2(y')^2 \\
\hline
 & \qquad 2yy' \\
\dfrac{d}{dx}(y^2) & \qquad 2yy' \\
\hline
 & \qquad \times
\end{array}$$

$\therefore$ (A) can be written as $\dfrac{d}{dx}(2yy')+\dfrac{d}{dx}(y^2)=2x+c_1$.

Integrating, $\quad 2yy'+y^2=x^2+c_1x+c_2$

which is just (6) and may be integrated as above.

***Ex. 2.** *Solve* $x^2y\dfrac{d^2y}{dx^2}+\left(x\dfrac{dy}{dx}-y\right)^2-3y^2=0.$

[Agra 71, 67, 63, 58; Raj. 65, 63, 58]

Solution. The equation may be written as

$$x^2yy''+x^2(y')^2-2xyy'-2y^2=0,$$

$$\begin{array}{lll}
\dfrac{d}{dx}(x^2yy')= & x^2yy''+x^2y'^2 & +2xyy' \\
\hline
 & & -4xyy'-2y^2 \\
\dfrac{d}{dx}(-2xy^2)= & & -4xyy'-2y^2 \\
\hline
 & & \qquad \times
\end{array}$$

Thus the equation may be written as

$$\frac{d}{dx}(x^2yy')+\frac{d}{dx}(-2xy^2)=0.$$

Integrating, $\quad x^2yy'-2xy^2=c_1$

or $\quad y\dfrac{dy}{dx}-\dfrac{2}{y}y^2=\dfrac{c_1}{x^2}.$

Put $y^2=u$, $2y\dfrac{dy}{dx}=\dfrac{du}{dx}$; the equation thus becomes

$$\frac{du}{dx}-\frac{4}{x}u=\frac{2c_1}{x^2}, \text{ linear, I.F.}=e^{-\int\left(\frac{4}{x}\right)dx}=\frac{1}{x^4}.$$

Hence the solution is $u.\dfrac{1}{x^4}=c_2+\displaystyle\int\frac{2c_1}{x^2}\cdot\frac{1}{x^4}\,dx$

or $\quad y^2\dfrac{1}{x^4}=c_2-\dfrac{2c_1}{5}\dfrac{1}{x^5}$

or $\quad xy^2=c_2x^5-\dfrac{2c_1}{5}$ or $xy^2=k_1x^5+k_2$.

Ex. 3. *Solve* $2y\dfrac{d^3y}{dx^3}+6\dfrac{d^2y}{dx^2}\cdot\dfrac{dy}{dx}=-\dfrac{1}{x^2}.$

Solution. The equation may be written as

$$2yy'''+6y''y'=-(1/x^2). \quad ...(1)$$

$$\frac{d}{dx}(2yy'')=2yy'''+2y''y'$$

$$\overline{\quad 4y''y'}$$

$$\frac{d}{dx}(2y'^2)= \quad 4y''y'$$

$$\overline{\quad \times \quad}$$

Thus equation is $\dfrac{d}{dx}(2yy'')+\dfrac{d}{dx}(2y'^2)=\dfrac{1}{x^2}$

Integrating, $2yy''+2y'^2=\dfrac{1}{x}+c_1$ ...(2)

Now $\dfrac{d}{dx}(2yy')= \quad 2yy''+2y'^2$

$$\overline{\quad \times \quad}$$

$\because$ (2) may be written as $\dfrac{d}{dx}(2yy')=\dfrac{1}{x}+c_1$.

Integrating, $\quad 2yy'=\log x+c_1x+c_2$

or $\quad 2y\dfrac{dy}{dx}=\log x+c_1x+c_2$.

Integrating it, $y^2=\displaystyle\int \log x\, dx+\tfrac{1}{2}c_1x^2+c_2x+c_3$.

Now $\displaystyle\int \log x\, dx=\int 1.\log x\, dx=\log x.x-\int x.\frac{1}{x}\,dx$

$$=x\log x-x.$$

$\therefore \quad y^2=x\log x+\tfrac{1}{2}c_1x^2+(c_2-1)x+c_3$.

or $\quad y^2=x\log x+k_1x^2+k_2x+k_3$ is the solution.

Ex. 4. *Show that* $x^2\dfrac{d^3y}{dx^3}+x\dfrac{d^2y}{dx^2}+(2xy-1)\dfrac{dy}{dx}+y^2=0$ *is exact and first integral is* $x^2\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+xy^2=c$.

Solution. The given equation may be written as

$$x^2y'''+xy''+2xyy'-y'+y^2=0$$

$$\frac{d}{dx}(x^2y'')= x^2y'''+2xy''$$

$$-xy''+2xyy'-y'+y^2$$

$$\frac{d}{dx}(-xy')= \quad -xy'' \quad -y'$$

$$2xyy' \quad +y^2$$

$$\frac{d}{dx}(xy^2)= \quad 2xyy' \quad +y^2$$

$$\overline{\quad \times \quad}$$

Thus the given equation is

$$\frac{d}{dx}(x^2y'')+\frac{d}{dx}(-xy')+\frac{d}{dx}(xy^2)=0.$$

Integrating directly (form shows that equation is exact), the first integral is $x^2y''-xy'+xy^2+c$.

This proves the result.

Ex. 5. *Show that the equation*

$$\left(y^2+2x^2\frac{dy}{dx}\right)\frac{d^2y}{dx^2}+2(x+y)\left(\frac{dy}{dx}\right)^2+x\frac{dy}{dx}+y=0$$

Solution The equation may be written as

$$y^2y''+2x^2y'y''+2x(y')^2+2y(y')^2+xy'+y=0$$

$$\begin{array}{lrr}
\frac{d}{dx}(y^2y')= & y^2y'' & +2y(y')^2 \\
\hline
 & 2x^2y'y''+2x(y')^2 & +xy'+y \\
\frac{d}{dx}(x^2y'^2)= & 2x^2y'y''+2x(y')^2 & \\
\hline
 & & xy'+y \\
\frac{d}{dx}(xy)= & & xy'+y \\
\hline
 & & \times
\end{array}$$

Therefore the equation may be written as

$$\frac{d}{dx}(y^2y')+\frac{d}{dx}(x^2y'^2)+\frac{d}{dx}(xy)=0 \text{ (exact form).}$$

Integrating, $y^2y'+x^2y'^2+xy=c$ or $y^2\frac{dy}{dx}+x^2\left(\frac{dy}{dx}\right)^2+xy=c$.

Ex. 6. *Solve* $(2y+x)\frac{d^2y}{dx^2}+2\frac{dy}{dx}\left(1+\frac{dy}{dx}\right)=0.$

Solution. The equation may be written as

$$2yy''+xy''+2y'+2y'^2=0. \qquad \ldots(1)$$

$$\begin{array}{lrr}
\frac{d}{dx}(2yy')= & 2yy'' & +2y'^2 \\
\hline
 & xy''+2y' & \\
\frac{d}{dx}(xy')= & xy''+y' & \\
\hline
 & y' & \\
\frac{dy}{dx}= & y' & \\
\hline
 & \times &
\end{array}$$

The equation becomes $\frac{d}{dx}(2yy')+\frac{d}{dx}(xy')+\frac{dy}{dx}=0.$

Integrating, $\quad 2yy'+xy'+y=c_1. \qquad \ldots(2)$

$$\frac{d}{dx}(y^2)=2yy'$$

$$\overline{\qquad xy'+y}$$

$$\frac{d}{dx}(xy)=\qquad xy'+y$$

$$\overline{\qquad \times}$$

$\therefore$ (2) can be written as $\dfrac{d}{dx}(y^2)+\dfrac{d}{dx}(xy)=c_1$.

Integrating, $y^2+xy=c_1x+c_2$ is the complete solution.

Ex. 7. *Solve* $\quad xy\dfrac{d^2y}{dx^2}+x\left(\dfrac{dy}{dx}\right)^2+y\dfrac{dy}{dx}=0.$

Solution. The equation may be written as

$$xyy''+xy'^2+yy'=0. \qquad \text{...(1)}$$

$$\frac{d}{dx}(xyy')=xyy''+xy'^2+yy'$$

$$\overline{\qquad \times}$$

$\therefore$ the given equation is $\dfrac{d}{dx}(xyy')=0$.

Integrating, $xy\dfrac{dy}{dx}=c_1$ or $y\,dy=c_1\dfrac{dx}{x}$.

Integrating, $\frac{1}{2}y^2=c_1\log x+c_2$ or $y^2=k_1\log x+k_2$.

Ex. 8. *Show that the equation*

$$y+3x\frac{dy}{dx}+2y\left(\frac{dy}{dx}\right)^2+\left(x^2+2y^2\frac{dy}{dx}\right)\frac{d^2y}{dx^2}=0$$

is exact and find its first integral. **[Delhi Hons. 63 ; Pb. 60]**

Solution. The equation may be written as

$$x^2y''+2y^2y'y''+2yy'^2+3xy'+y=0$$

$$\frac{d}{dx}(x^2y')=x^2y'' \qquad\qquad +2xy'$$

$$\overline{\qquad 2y^2y'y''+2yy'^2+xy'+y}$$

$$\frac{d}{dx}(y^2y'^2)=\qquad 2y^2y'y''+2yy'^2$$

$$\overline{\qquad xy'+y}$$

$$\frac{d}{dx}(xy)=\qquad xy'+y$$

$$\overline{\qquad \times}$$

The equation may be written as

$$\frac{d}{dx}(x^2y')+\frac{d}{dx}(y^2y'^2)+\frac{d}{dx}(xy)=0 \text{ (exact form).}$$

Integrating, $x^2\dfrac{dy}{dx}+y^2\left(\dfrac{dy}{dx}\right)^2+xy=c_1$ is the first integral.

Ex. 9. *Solve* $\cos y \dfrac{d^2y}{dx^2} - \sin y \left(\dfrac{dy}{dx}\right)^2 + \cos y \dfrac{dy}{dx} = x+1.$

Solution. The equation may be written as

$$\cos y . y'' - \sin y . y'^2 + \cos y . y' = x+1,$$

$$\frac{d}{dx}(\cos y . y') = \cos y . y'' - \sin y . y'^2$$

$$\qquad\qquad\qquad\qquad\qquad \cos y . y'$$

$$\frac{d}{dx}(\sin y) = \qquad\qquad \cos y\ y'$$

$$\qquad\qquad\qquad\qquad\qquad \times$$

Hence the equation is $\dfrac{d}{dx}(\cos y . y') + \dfrac{d}{dx}(\sin y) = x+1.$

Integrating, $\cos y \dfrac{dy}{dx} + \sin y = \dfrac{(x+1)^2}{2} + c_1$

Putting $\sin y = u$, $\cos y \dfrac{dy}{dx} = \dfrac{du}{dx}$, the equation becomes

$$\frac{du}{dx} + u = \frac{(x+1)^2}{2} + c_1, \text{ Linear ; I.F.} = e^x.$$

$$\therefore\ ue^x = \int \tfrac{1}{2}(x+1)^2 e^x\, dx + \int c_1 e^x\, dx + c_2$$

$$= e^x \left[\frac{(x+1)^2}{2} - (x+1) + 1 + c_1\right] + c_2$$

or $\quad \sin y = \dfrac{(x+1)^2}{2} - x + c_1 + c_2 e^{-x}$

or $\quad 2 \sin y = x^2 + k_1 + k_2 e^{-x}$ is the complete solution.

Ex. 10. *Solve* $2x^2 \cos y \dfrac{d^2y}{dx^2} - 2x^2 \sin y \left(\dfrac{dy}{dx}\right)^2$

$$+ x \cos y \frac{dy}{dx} - \sin y = \log x.$$

[Raj. 56

Solution. The equation may be written as

$$2x^2 \cos y . y'' - 2x^2 \sin y . y'^2 + x \cos y . y' - \sin y = \log x \qquad \ldots(1)$$

$$\frac{d}{dx}(2x^2 \cos y . y') = 2x^2 \cos y . y'' - 2x^2 \sin y . y'^2 + 4x \cos y . y'$$

$$\qquad\qquad\qquad\qquad\qquad -3x \cos y\ y' - \sin y$$

$$\frac{d}{dx}(-3x \sin y) = \qquad -3x \cos y . y' - 3 \sin y$$

$$\qquad\qquad\qquad\qquad\qquad 2 \sin y$$

Equation is not exact.

So dividing by x^2, the equation becomes

$$\cos y . y'' - 2 \sin y . y'^2 + \frac{1}{x} \cos y . y' - \frac{1}{x^2} \sin y = \frac{1}{x^2} \log x. \qquad \ldots(2)$$

$$\frac{d}{dx}(2\cos y.y') = 2\cos y.y'' - 2\sin y.y'^2$$

$$\frac{1}{x}\cos y.y' - \frac{1}{x^2}\sin y$$

$$\frac{d}{dx}\left(\frac{\sin y}{x}\right) = \qquad \frac{1}{x}\cos y.y' - \frac{1}{x^2}\sin y$$

$$\times$$

Therefore (2) is exact and can be written as

$$\frac{d}{dx}(2\cos y.y') + \frac{d}{dx}\left(\frac{\sin y}{x}\right) = \frac{1}{x^2}\log x.$$

Integrating, $2\cos y\dfrac{dy}{dx} + \dfrac{\sin y}{x} = -\dfrac{1}{x}(\log x+1) + c_1.$

Putting $\sin y = u$, $\cos y\dfrac{dy}{dx} = \dfrac{du}{dx}$,

$$\frac{du}{dx} + \frac{1}{2x}u = -\frac{1}{2x}(\log x+1) + \tfrac{1}{2}c_1.$$

Linear, I.F. $= \sqrt{x}$. Hence the solution is

$$u\sqrt{x} = \int\left[-\frac{1}{2\sqrt{x}}(\log x+1) + \tfrac{1}{2}c_1\sqrt{x}\right]dx + c_2$$

$$= \int -\tfrac{1}{2}(z+1)\,e^{z/2}\,dz + \frac{c_1x^{3/2}}{3} + c_2, \text{ where } x = e^z$$

$$= -e^{z/2}[(z+1)-2] + \frac{c_1x^{3/2}}{3} + c_2$$

or $\sin y.\sqrt{x} = -\sqrt{x}[\log x - 1] + \dfrac{c_1x^{3/2}}{3} + c_2$

or $\sin y = -\log x + 1 + \dfrac{c_1x}{3} + \dfrac{c_2}{\sqrt{x}}.$

Ex. 11. *Solve* $x^2y\dfrac{d^2y}{dx^2} + \left(x\dfrac{dy}{dx} - y\right)^2 = 0.$ **[Raj. 53]**

Solution. The equation may be written as

$$x^2yy'' + x^2y'^2 - 2xyy' + y^2 = 0. \qquad ...(1)$$

$$\frac{d}{dx}(x^2yy') = x^2yy'' + x^2y'^2 + 2xyy'$$

$$-4xyy' + y^2$$

$$\frac{d}{dx}(-2xy^2) = \qquad -4xyy' + 2y^2$$

$$3y^2$$

Therefore the equation in its present form is not exact. Now dividing by x^2, it becomes

$$yy'' + y'^2 - \frac{2yy'}{x} + \frac{y^2}{x^2} = 0. \qquad (2)$$

$$\frac{d}{dx}(yy')=yy''+y'^2$$

$$\begin{array}{rr} & \overline{-\dfrac{2yy'}{x}+\dfrac{y^2}{x^2}} \\ \dfrac{d}{dx}\left(-\dfrac{y^2}{x}\right)= & -\dfrac{2yy'}{x}+\dfrac{y^2}{x^2} \\ & \overline{\qquad\times\qquad} \end{array}$$

Therefore (2) is exact. It can be written as

$$\frac{d}{dx}(yy')+\frac{d}{dx}\left(-\frac{y^2}{x}\right)=0.$$

Integrating, $y\dfrac{dy}{dx}-\dfrac{y^2}{x}=c_1$. Put $y^2=u$, $2y\dfrac{dy}{dx}=\dfrac{du}{dx}$

i.e. $\dfrac{du}{dx}-\dfrac{2}{x}u=2c_1$. Linear; I.F. $=e^{-\int\frac{2}{x}dx}=\dfrac{1}{x^2}$.

$$\therefore \qquad u.\frac{1}{x^2}=\int 2c_1\frac{1}{x^2}dx+c_2$$

or $\qquad y^2.\dfrac{1}{x^2}=-\dfrac{2c_1}{x}+c_2$ or $y^2=x(c_2x-2c_1)$.

Ex. 12. *Solve* $x^3\dfrac{d^2y}{dx^2}+(4x^2-3x)\dfrac{dy}{dx}+(2x-3)y=0$ *without using the condition of exactness.*

Solution. The condition of exactness, *i.e.* $P_2-P_1'+P_0''=0$ is satisfied but we would solve the equation without using this condition.

The equation may be written as

$$x^3y''+4x^2y'-3xy'+2xy-3y=0$$

$$\begin{array}{lll} \dfrac{d}{dx}(x^3y')=x^3y''+3x^2y' & & \\ & \overline{x^2y'-3xy'+2xy-3y} & \\ \dfrac{d}{dx}(x^2y)= & x^2y' \qquad +2xy & \\ & \overline{-3xy' \qquad -3y} & \\ \dfrac{d}{dx}(-3xy)= & -3xy' \qquad -3y & \\ & \overline{\qquad\qquad\times\qquad\qquad} & \end{array}$$

Hence the equation can be written as

$$\frac{d}{dx}(x^3y')+\frac{d}{dx}(x^2y)+\frac{d}{dx}(-3xy)=0.$$

Integrating, $x^3\dfrac{dy}{dx}+x^2y-3xy=c_1$

or $\quad \dfrac{dy}{dx}+\left(\dfrac{1}{x}-\dfrac{3}{x^2}\right) y=\dfrac{c_1}{x^3}.$ Linear, I.F.$=xe^{3/x}$.

$$\therefore \quad yxe^{3/x} \int \frac{c_1}{x^3} . xe^{3/x}\, dx+c_2=-\tfrac{1}{3} c_1 e^{3/x}+c_2$$

or $\quad xy=-\frac{1}{3}c_1+c_2e^{-3/x}$ which is the required solution.

Note. It is always possible to apply the above method of trial to all linear equations which satisfy the condition of exactness of § 1·8 p. 15.

Ex. 13. *Solve*

$$2 \sin x \frac{d^2y}{dx^2}+2 \cos x \frac{dy}{dx}+2 \sin x \frac{dy}{dx}+2y \cos x=\cos x.$$

Solution. The equation may be put as

$$2 \sin x . y''+2 \cos x . y'+2 \sin x . y'+2y \cos x=\cos x$$

$$\frac{d}{dx}(2 \sin x . y')=2 \sin x . y''+2 \cos x . y'$$

$$2 \sin x . y'+2y \cos x$$

$$\frac{d}{dx}(2 \sin x . y)= \qquad 2 \sin x . y'+2y \cos x$$

$$\times$$

Thus the equation can be written as

$$\frac{d}{dx}(2 \sin x . y')+\frac{d}{dx}(2 \sin x . y)=\cos x.$$

Integrating, $\quad 2 \sin x \dfrac{dy}{dx}+2 \sin x . y=\sin x+c_1$

or $\quad \dfrac{dy}{dx}+y=\frac{1}{2}+c_1 \operatorname{cosec} x.$ Linear, I F.$=e^x$.

$$\therefore \quad ye^x=c_2+\int (\tfrac{1}{2}+c_1 \operatorname{cosec} x)\, e^x\, dx.$$

Ex. 14. *Solve* $x \dfrac{d^3y}{dx^3}-x \dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}=0.$

Solution. The equation is free from y. So putting $\dfrac{dy}{dx}=p$, $\dfrac{d^2y}{dx^2}=\dfrac{dp}{dx}$, the equation becomes

$$x \frac{d^2p}{dx^2}-x \frac{dp}{dx} \quad p=0.$$

This satisfies condition of exactness, *i.e.* $P_2-P_1'+P_0''=0$.

Hence the first integral is $P_0 \dfrac{dp}{dx}+(P_1-P'_0)\, p=c$

or $\quad x\dfrac{dp}{dx}+(-x-1)\, p=c \quad$ or $\quad \dfrac{dp}{dx}-\left(1+\dfrac{1}{x}\right) p=\dfrac{c}{x}.$

Linear equation, I.F. $=e^{-\int\left(1+\frac{1}{x}\right)dx}=\frac{1}{x}e^{-x}$.

Hence $p\cdot\frac{1}{x}\,e^{-x}=\int\frac{c}{x}\cdot\frac{e^{-x}}{x}dx+c'$.

or $p=\frac{dy}{dx}=c'xe^{x}+c.xe^{x}\int\frac{1}{x^2}\,e^{-x}\,dx$

which on integration further gives the solution.

Ex. 15. *Find the first intergal of*

$$\frac{dy}{dx}\frac{d^2y}{dx^2}-x^2y\ \frac{dy}{dx}-xy^2=0.$$

Solution. The equation is $\frac{1}{2}\,\frac{d}{dx}\left(\frac{dy}{dx}\right)^2-\frac{1}{2}\,\frac{d}{dx}\,(x^2y^2)=0.$

Integrating, $\left(\frac{dy}{dx}\right)^2=c_1+x^2y^2$. This is first integral.

1·8. Equation of the form $\frac{d^ny}{dx^n}=f(x)$

The equation can be integrated successively to give the required solution.

1·9. Equations of the form $\frac{d^2y}{dx^2}=f(y)$

To integrate such equations, the equation is multiplied by $2\frac{dy}{dx}$. The equation thus becomes

$$2\ \frac{dy}{dx}\frac{d^2y}{dx^2}=2f(y)\ \frac{dy}{dx}$$

which on integration gives

$$\left(\frac{dy}{dx}\right)^2=2\int f(y)\frac{dy}{dx}\,dx+c_1$$

i.e. $$\left(\frac{dy}{dx}\right)^2=2\int f(y)\ dy+c_1$$

which can be integrated further.

Ex. 1. *Solve* $\frac{d^ny}{dx^n}=x^m$.

Solution. Integrating the equation directly,

$$\frac{d^{n-1}y}{dx^{n-1}}=\frac{x^{m+1}}{m+1}+k_1.$$

ating again, $\frac{d^{n-1}y}{dx^{n-2}}=\frac{x^{m+2}}{(m+1)\,(m+2)}+k_1x+k_2.$

... ...

$$y=\frac{x^{m+n}}{(m+1)(m+2)\dots(m+n)}+c_1x^{n-1}+c_2x^{n-2}+\dots+c_n$$
$$=\frac{m!\,x^{m+n}}{(m+n)!}+(c_1x^{n-1}+c_2x^{n-2}+\dots+c_n)$$

where constants are suitably adjusted.

Ex 2. *Solve* $\frac{d^4y}{dx^4}=x+e^{-x}-\cos x.$

Solution. Integrating the equation once,

$$\frac{d^3y}{dx^3}=\frac{x^2}{2}-e^{-x}-\sin x+c_1.$$

Integrating again, $\frac{d^2y}{dx^2}=\frac{x^3}{6}+e^{-x}+\cos x+c_1x+c_2.$

Again integrating, $\frac{dy}{dx}=\frac{x^4}{24}-e^{-x}+\sin x+c_1\frac{x^2}{2}+c_2x+c_3.$

Integrating once again, the solution is

$$y=\frac{x^5}{120}+e^{-x}-\cos x+c_1\frac{x^3}{6}+c_2\frac{x^2}{2}+c_3x+c_4.$$

Ex. 3. *Solve* $\frac{d^2y}{dx^2}=x^2\sin x.$

Solution. Integrating, $\frac{dy}{dx}=x^2(-\cos x)+\int 2x\cos x\,dx+c_1$

or $\frac{dy}{dx}=-x^2\cos x+2x\sin x+2\cos x+c_1.$

Integrating again, the solution is

$$y=\int(-x^2\cos x+2x\sin x+2\cos x)\,dx+c_1x+c_2$$
$$=-x^2\sin x-4x\cos x+6\sin x+c_1x+c_2.$$

Ex. 4. *Solve* $\frac{d^3y}{dx^3}=\sin^2 x=\frac{1}{2}(1-\cos 2x).$

Solution Integrating, $\frac{d^2y}{dx^2}=\frac{1}{2}x-\frac{1}{4}\sin 2x+c_1.$

$$\therefore\quad \frac{dy}{dx}=\tfrac{1}{4}x^2+\tfrac{1}{8}\cos 2x+c_1x+c_2$$

and finally $y=\frac{1}{12}x^3+\frac{1}{16}\sin 2x+\frac{1}{2}c_1x^2+c_2x+c_3.$

Ex. 5. *Solve* $\frac{d^3y}{dx^3}=\log x.$

Solution. Integrating successively, $\frac{d^2y}{dx^2}=\int\log x\,dx+k_1$

or $\frac{d^2y}{dx^2}=x\log x-x+k_1$ etc.

Finally, $36y=6x^3\log x-11x^3+c_1x^2+c_2x+c_3.$

Ex. 6. *Solve* $\dfrac{d^3y}{dx^3}=xe^x$.

Solution. Integrating successively, $\dfrac{d^2y}{dx^2}=e^x(x-1)+c_1$,

$$\frac{dy}{dx}=e^x(x-2)+c_1x+c_2$$

and $\quad y=e^x(x-3)+\frac{1}{2}c_1x^2+c_2x+c_3$.

Ex. 7. *Solve* $\dfrac{d^2y}{dx^2}=\sec^2 y \tan y$. $\quad\left(\textit{Type } \dfrac{d^2y}{dx^2}=f(y)\right)$

[Delhi Hons. 63]

Solution. Multiplying by $2\dfrac{dy}{dx}$ and integrating,

$$\left(\frac{dy}{dx}\right)^2=\int 2\sec^2 y \tan y \frac{dy}{dx}\,dx+c_1$$

$$=\tan^2 y+c_1=\frac{\sin^2 y+c_1\cos^2 y}{\cos^2 y}$$

or $$\frac{\cos y\,dy}{\sqrt{\{c_1-(c_1-1)\sin^2 y\}}}=dx$$

Integrating, $$\frac{1}{\sqrt{(c_1-1)}}\sin^{-1}\left[\sqrt{\left(\frac{c_1-1}{c_1}\right)}\sin y\right]=x+c_2,$$

$$\sin^{-1}\left\{\sqrt{\left(\frac{k_1}{k_1+1}\right)}\sin y\right\}=\sqrt{k_1}x+k_2.$$

if $\quad c_1-1=k_1,\ c_2\sqrt{k_1}=k_2$

or $\quad \sin(\sqrt{k_1}x+k_2)=\sqrt{\left(\dfrac{k_1}{k_1+1}\right)}\sin y.$

Ex. 8. *Solve* $\sin^3 y\,\dfrac{d^2y}{dx^2}=\cos y$ or $\dfrac{d^2y}{dx^2}=\operatorname{cosec}^2 y \cot y$.

Solution. Multiplying by $2\dfrac{dy}{dx}$ and integrating,

$$\left(\frac{dy}{dx}\right)^2=c_1-\cot^2 y=\frac{c_1\sin^2 y-\cos^2 y}{\sin^2 y}$$

or $$\frac{\sin y\,dy}{\sqrt{\{c_2-(1+c_1)\cos^2 y\}}}=dx.$$

Integrating, $$-\frac{1}{\sqrt{(1+c_1)}}\sin^{-1}\left\{\sqrt{\left(\frac{1+c_1}{c_1}\right)}\cos y\right\}=x+c_2$$

or $\quad \sin(\sqrt{k_1}x+k_2)+\sqrt{\left(\dfrac{k_1}{k_1-1}\right)}\cos y=0,$

where $1+c_1=k_1,\ \sqrt{k_1}c_2=k_2$.

Ex. 9. $\dfrac{d^2y}{dx^2}=\dfrac{1}{\sqrt{(ay)}}$.

[Cal. Hons. 62]

Solution. Multiplying by $2\dfrac{dy}{dx}$ and integrating, we get

$$\left(\frac{dy}{dx}\right)^2=\frac{4}{\sqrt{a}}y^{1/2}+c_1 \text{ etc.}$$

Ex. 10. $\dfrac{d^2y}{dx^2}+\dfrac{a^2}{y^2}=0.$ **[Karnatak B. Sc. (Sub.) 60]**

Solution. Multiplying by $2\dfrac{dy}{dx}$ and integrating, we get

$$\left(\frac{dy}{dx}\right)^2-\frac{2a^2}{y}=c, \text{ i.e. } \frac{dy}{dx}=\sqrt{\left(\frac{2a^2+cy}{y}\right)} \text{ etc.}$$

Ex. 11. $y^3.\dfrac{d^2y}{dx^2}=\mu$, *$\mu$ being a constant.*

[Gujrat B. Sc. (Prin.) 51]

Solution. $\dfrac{d^2y}{dx^2}=\dfrac{\mu}{y^3}.$

Multiplying by $2\dfrac{dy}{dx}$ and integrating, we get

$$\left(\frac{dy}{dx}\right)^2=-\frac{\mu}{y^2}+a^2$$

$$\frac{dy}{dx}=\frac{1}{y}\sqrt{\{(a^2y^2-\mu)\}} \text{ etc.}$$

Ex. 12. *In the case of a stretched elastic string, which has one end fixed and a particle of mass m attached to the other end, the equation of motion is* $m\dfrac{d^2s}{dt^2}=-\dfrac{mg}{e}(s-l)$, *where l is the natural length of the string and e its elongation due to a weight mg. Find s and v, determining the constants, so that $s=s_0$ at the time $t=0$ and $v=0$ when $t=0$.* **[Delhi Hon's 51, 58]**

Solution. The equation is

$$\frac{d^2s}{dt^2}=-\frac{g}{e}(s-l) \qquad \left[\text{form } \frac{d^2y}{dx^2}=f(y)\right].$$

Multiplying by $2\dfrac{ds}{dt}$ and integrating, we get

$$\left(\frac{ds}{dt}\right)^2=-\frac{g}{e}(s-l)^2+c.$$

But when $s=s_0$, $t=0$, $v=\dfrac{ds}{dt}=0.$

$$\therefore \quad 0=-\frac{g}{e}(s-l)^2+c \quad \text{or} \quad c=\frac{g}{e}(s_0-l)^2$$

$$\therefore \quad \left(\frac{ds}{dt}\right)^2=\frac{g}{e}[(s_0-l)^2-(s-l)^2].$$

$$\therefore \quad \frac{ds}{dt}=-\sqrt{\left(\frac{g}{e}\right)}\sqrt{[(s_0-l)^2-(s-l)^2]}$$

or $\sqrt{\left(\frac{g}{e}\right)}\,dt=-\frac{ds}{\sqrt{[(s_0-l)^2-(s-l)^2]}}$.

Integrating, $\sqrt{\left(\frac{g}{e}\right)}\,t=\cos^{-1}\frac{s-l}{s_0-l}+c.$

When $s=0$, $s=s_0$; $\therefore\ 0=\cos^{-1}1+c$ or $c=0$.

Hence $\sqrt{\left(\frac{g}{e}\right)}\,t=\cos^{-1}\frac{s-l}{s_0-l}$

or $s=l+(s_0-l)\cos\left[\sqrt{\left(\frac{g}{e}\right)}\,t\right]$.

Ex. 13 *A particle whose mass m is acted upon by a force* $m\mu\left(x+\frac{a^4}{x^3}\right)$ *towards the origin ; if it starts from rest at a distance a, show that it will arrive at the origin in time* $\frac{1}{4}\frac{\pi}{\sqrt{\mu}}$. **[Raj. 61]**

Solution The differential equation of the motion is

$$m\frac{d^2x}{dt^2}=-m\mu\left(x+\frac{a^4}{x^3}\right),\ i.e.\ \frac{d^2x}{dt^2}=-\mu\left(x+\frac{a^4}{x^3}\right)$$

Multiplying by $2\frac{dx}{dt}$ and integrating,

$$\left(\frac{dx}{dt}\right)^2=-\mu\left(x^2-\frac{a^4}{x^2}\right)+c.$$

When $x=a$, $\frac{dx}{dt}=0$; $\therefore\ c=0$.

$$\therefore\ \left(\frac{dx}{dt}\right)^2=\frac{\mu}{x^2}(x^4-a^4)\quad\text{or}\quad\frac{dx}{dt}=-\frac{\sqrt{\mu}}{x}\sqrt{(x^4-a^4)}$$

or $-\frac{x\,dx}{\sqrt{(x^4-a^4)}}=+\sqrt{\mu}\,dt.$ Putting $x^2=z$, $2x\,dx=dz$,

$$-\frac{dz}{\sqrt{(z^2-a^4)}}=+2\sqrt{\mu}\,dt.$$

Integrating, $\cos^{-1}\frac{z}{a^2}=2\sqrt{\mu}\,t+c.$

When $t=0$, $z=x^2=a^2$: $\therefore\ c=0$

or $\cos^{-1}\frac{z}{a^2}=2\sqrt{\mu}\,t.$

If t is the time of reaching the origin where $z=0$, then

$$t=\frac{1}{2\sqrt{\mu}}\cos^{-1}0=\frac{1}{2\sqrt{\mu}}\cdot\tfrac{1}{2}\pi=\frac{\pi}{4\sqrt{\mu}}.$$

Ex 14. *Determine the curve whose radius of curvature varies as the cube of the length of the normal intercepted between the curve and the x-axis.* **[Delhi Hons. 1957]**

Solution. $\rho=\frac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2}$, Normal$=y\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}$

$\rho=\lambda$ (Normal)^3 given,

i.e. $$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}=\lambda y^3\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2},$$

i.e. $$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}\left[1-\lambda y^3\left(\frac{d^2y}{dx^2}\right)\right]=0.$$

If $1+\left(\frac{dy}{dx}\right)^2=0$, then $\frac{dy}{dx}$ is imaginary.

Hence the differential equation of the curve is given by

$$1-\lambda y^3\frac{d^2y}{dx^2}=0 \quad \text{or} \quad \frac{d^2y}{dx^2}=-\frac{\mu}{y^3}, \text{ where } \mu=-\frac{1}{\lambda}.$$

Multiplying by $2\frac{dy}{dx}$ and integrating, we get

$$\left(\frac{dy}{dx}\right)=+\frac{\mu}{y^2}+a,\ \frac{dy}{dx}=\frac{\sqrt{(\mu+ay^2)}}{y}$$

i.e. $$\frac{y\,dy}{\sqrt{(\mu+ay)}}=dx$$

or $$\sqrt{(\mu+ay^2)}=ax+b.$$

So the curve is $\mu+ay^2=(ax+b)^2$,

1·10. Equation in which y occurs in only two derivatives of the form $\frac{d^ny}{dx^n}$ and $\frac{d^{n-2}y}{dx^{n-2}}$.

Such equations can be written as

$$f\left(\frac{d^ny}{dx^n},\ \frac{d^{n-2}y}{dx^{n-2}},\ x\right)=0.$$

Put $\frac{d^{n-2}y}{dx^{n-2}}=q$, so that $\frac{d^ny}{dx^n}=\frac{d^2q}{dx^2}$.

Hence the equation can be written as

$$f\left(\frac{d^2y}{dx^2},\ q,\ x\right)=0.$$

This can be solved for q giving

$$q=\frac{d^{n-2}q}{dx^{n-2}}=\phi(x).$$

It can then be solved in a suitable way.

Ex. 1. *Solve* $\frac{d^4y}{dx^4}-a^2\frac{d^2y}{dx^2}=0.$

Solution. Putting $\frac{d^2y}{dx^2}=q$, $\frac{d^4y}{dx^4}=\frac{d^2q}{dx^2}$, the equation becomes

$$\frac{d^2q}{dx^2}-a^2q=0 \quad \text{or} \quad (D^2-a^2)\,q=0.$$

A.E. is $D^2-a^2=0$. $D=\pm a$.

$\therefore$ Solution is $q=c_1e^{ax}+c_2e^{-ax}$

or $\dfrac{d^2y}{dx^2}=c_1e^{ax}+c_2e^{-ax}$.

Integrating twice successively, we get

$$y=\frac{c_1}{a^2}e^{ax}+\frac{c_2}{a^2}e^{-ax}+c_3\frac{x^2}{2}+c_4.$$

1 11. Equations in which order of differentiation differs by one

Such an equation can be written as

$$f\left(\frac{d^ny}{dx^n},\frac{d^{n-1}y}{dx^{n-1}},x\right)=0.$$

Put $\dfrac{d^{n-1}y}{dx^{n-1}}=q$, so that $\dfrac{d^ny}{dx^n}=\dfrac{dq}{dx}$.

The equation thus becomes $f\left(\dfrac{dq}{dx},q,x\right)$

which can be solved for q, giving

$$q=\frac{d^{n-1}y}{dx^{n-1}}=\theta(x)$$

which after being integrated successively $n-1$ times gives the solution.

Ex 1. *Solve* $\dfrac{d^3y}{dx^3}\dfrac{d^2y}{dx^2}=2$.

Solution. Putting $\dfrac{d^2y}{dx^2}=q$, $\dfrac{d^3y}{dx^3}=\dfrac{dq}{dx}$, the equation becomes

$$\frac{dq}{dx}q=2 \quad \text{or} \quad q\,dq=2\,dx.$$

Integrating, $q^2=4x+c_1$ or $q=\sqrt{(4x+c_1)}$

or $\dfrac{d^2y}{dx^2}=\sqrt{(4x+c_1)}$.

Integrating, $\dfrac{dy}{dx}=\dfrac{(4x+c_1)^{3/2}}{4.\frac{3}{2}}+c_2$.

Integrating again, $y=\dfrac{(4x+c_1)^{5/2}}{4.4.\frac{3}{2}.\frac{5}{2}}+c_2x+c_3$

or $y=\frac{1}{60}(4x+c_1)^{5/2}+c_2x+c_3$.

2

Linear Equations of Second Degree

2·1. Linear Equation of Second Degree

An equation of the type

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=X,$$

where P, Q, X are functions of x alone, is called the linear equation of second degree.

If the coefficients P and Q are constants, the equation can be solved by the methods of chapter V Pt. I, otherwise there is no general method for solving such equations. We give below certain procedures which at times yield a solution.

***2 2. Complete solution of $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=X$, when one integral of the complementary function is known.**

To solve completely the above equation when one integral belonging to the complementary function is known.

[Agra 67, 62 ; Punjab 60 ; Bombay 58, 61 ; Gauhati 67, 64 ; Poona 61 ; Karnatak 61 ; Gujrat B. Sc 65, 61 ; Cal. Hons. 62 ; Nag. 63]

The equation is $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=X.$

The equation is $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=X.$

Let $y=y_1$ be a known part of the complementary function, *i.e.* it is a solution of

$$\frac{d^2y}{dx^2}+p\frac{dy}{dx}+Qy=0; \quad \therefore \quad \frac{d^2y_1}{dx^2}+P\frac{dy_1}{dx}+Qy_1=0. \qquad \text{...(2)}$$

Now putting $y=vy_1$, we get $\frac{dy}{dx}=v\frac{dy_1}{dx}+y_1\frac{dv}{dx}$

and $$\frac{d^2y}{dx^2}=v\frac{d^2y_1}{dx^2}+2\frac{dy_1}{dx}\frac{dv}{dx}+y_1\frac{d^2v}{dx^2}.$$

With this substitution the equation (1) becomes

$$\left(v\frac{d^2y_1}{dx^2}+2\frac{dy_1}{dx}\cdot\frac{dv}{dx}+y_1\frac{d^2v}{dx^2}\right)+P\left(v\frac{dy_1}{dx}+y_1\frac{dv}{dx}\right)+Q.vy_1=X$$

or $$v\left[\frac{d^2y_1}{dx^2}+P\frac{dy_1}{dx}+Qy_1\right]+y_1\left[\frac{d^2v}{dx^2}+P\frac{dv}{dx}\right]+2\frac{dy_1}{dx}\frac{dy}{dx}=X.$$

But by (2), $\dfrac{d^2y_1}{dx^2}+P\dfrac{dy_1}{dx}+Qy_1=0.$

Therefore the equation after dividing by y_1 becomes

$$\frac{d^2v}{dx^2}+\left[P+\frac{2}{y_1}\frac{dy_1}{dx}\right]\frac{dv}{dx}=\frac{X}{y_1}. \qquad \ldots(3)$$

Now putting $\dfrac{dv}{dx}=p$, it becomes

$$\frac{dp}{dx}+\left[P+\frac{2}{y_1}\frac{dy_1}{dx}\right]p=\frac{X}{y_1}, \qquad \ldots(4)$$

which is a linear differential equation of order 1 in p and x.

Its $\quad$ $\text{I.F.}=e^{\int\left(P+\frac{2}{y_1}\frac{dy_1}{dx}\right)dx}=e^{\int\left(P\,dx+\frac{2}{y_1}dy_1\right)}$

$$=y_1^2e^{\int P\,dx}.$$

Hence solution of (4) is

$$py_1^2e^{\int P\,dx}=\int\left[\frac{X}{y_1}y_1^2e^{\int P\,dx}\right]dx+c, \qquad \ldots(5)$$

which gives $p=\dfrac{dv}{dx}$ and on direct integration gives the solution of the equation (3).

After integrating $\dfrac{dv}{dx}$, we get value of v containing two constants.

Having found v, we find y with the help of the relation $y=vy_1$

This is the complete primitive of (1) since it contains two arbitrary constants,

Cor. *If* $y=y_1(x)$ *and* $y=y_2(x)$ *are two solutions of the equation* $\dfrac{d^2y}{dx^2}+P(x)\dfrac{dy}{dx}+Q(x)\,y=0$, *where* $P(x)$ *and* $Q(x)$ *are continuous functions of* x, *prove that*

$$y_1\frac{dy_2}{dx}-y_2\frac{dy_1}{dx}=ce^{-\int P\,dx}$$

[Calcutta Hons. 62]

When $X=0$ in above article, we have from (5)

$$py_1^2e^{\int P\,dx}=c \quad \text{or} \quad py_1^2=ce^{-\int P\,dx}$$

i.e. $\quad \dfrac{dv}{dx}y_1^2=ce^{-\int P\,dx}$

or $\quad y_1^2\dfrac{d\,(y_2/y_1)}{dx}=ce^{-\int P\,dx}$ $\qquad$ as $y_2=vy_1$

or $\quad y_1\dfrac{dy_2}{dx}-y_2\dfrac{dy_1}{dx}=ce^{-\int P\,dx}$

2·3. Search for a particular integral of

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0$$

The article discussed above gives us a method of finding the complete solution if integral belonging to the C.F. is known. Therefore the main problem is that of finding an integral of the given differential equation. If this is given, then the question can be solved straight away. If however a solution is not known, one of the following rules may help us in determining an integral of the complementary function.

Rule I. $y=e^{mx}$ **is to be a solution if** $m^2+Pm+Q=0$.

$y=e^{mx}$, then $\frac{dy}{dx}=me^{mx}, \frac{d^2y}{dx^2}=m^2e^{mx}$.

Hence if $y=e^{mx}$ is a solution then

$m^2e^{mx}+Pme^{mx}+Qe^{mx}=0$

or $m^2+Pm+Q=0$.

Deductions (i) Thus $y=e^x$ will be solution if

$1+P+Q=0$ (sum of the coefficients is zero).

[Karnatak 61; Bombay 61]

(ii) $y=e^{-x}$ will be a solution of

$1-P+Q=0$.

Rull II. $y=x^m$ **to be a solution.**

If $y=x^m$, then $\frac{dy}{dx}=mx^{m-1}, \frac{d^2y}{dx^2}=m\,(m-1)\,x^{m-2}$.

Hence if $y=x^m$ is a solution, then

$m\,(m-1)\,x^{m-2}+Pmx^{m-1}+Qx^m=0$

or $m\,(m-1)+Pmx+Qx^2=0$.

Deductions. (*i*) $y=x$ will be solution if $P+Qx=0$.

[Karnatak 61; Bombay 61]

(*ii*) $y=x^2$ will be solution if (taking $m=2$),

$2+2Px+Qx^2=0$.

2.4. Summary. Thus for

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0$$

(*i*) $y=e^x$ is a particular integral	if	$1+P+Q=0$.
(*ii*) $y=e^{-x}$,, ,, ,,	if	$1-P+Q=0$.
(*iii*) $y=e^{mx}$,, ,, ,,	if	$m^2+mP+Q=0$.
(*iv*) $y=x$,, ,, ,,	if	$P+Qx=0$,
(*v*) $y=x^2$,, ,, ,,	if	$2+2Px+Qx^2=0$,
(*vi*) $y=x^m$,, ,, ,,	if	$m\,(m-1)+Pmx+Qx^2=0$.

2.5. Procedure (*Important*). Adopt the following steps in solving problems of this type :

(i) Put the equation in the standard form

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=X$$

in which coefficient of $\frac{d^2y}{dx^2}$ is unity.

(ii) Test for a particular solution of the C.E. and remember that if

$$\left.\begin{array}{ll} 1+P+Q=0, & y=e^x \text{ is a solution} \\ P+xQ=0, & y=x \text{ is a solution.} \end{array}\right\} \text{etc.}$$

(see summary § 2·4)

(iii) Put $y=vy_1$ and simplify; the reduced equation will be

$$\frac{d^2v}{dx^2}+\left(P+\frac{2}{y_1}\frac{dy_1}{dx}\right)\frac{dv}{dx}=\frac{X}{y_1}.$$ [equation (3) P. 46].

(iv) Put $\frac{dv}{dx}=p$ and solve the resulting linear equation in p and x.

The following examples will fully illustrate the method.

Ex. 1. *Solve* $\frac{d^2y}{dx^2}-\frac{3}{x}\frac{dy}{dx}+\frac{3}{x^2}y=2x-1.$

Solution. Here $P=-\frac{3}{x}$, $Q=\frac{3}{x^2}$. Since $P+Qx=0$, therefore $y=x$ is a part of C.F.

Putting $y=vx$, $\frac{dy}{dx}=x\frac{dv}{dx}+v$ and $\frac{d^2y}{dx^2}=x\frac{d^2v}{dx^2}+2\frac{dv}{dx}$, the equation becomes

$$x\frac{d^2v}{dx^2}+2\frac{dv}{dx}-\frac{3}{x}\left(x\frac{dv}{dx}+v\right)+\frac{3}{x^2}vx=2x-1,$$

i.e. $x\frac{d^2v}{dy^2}-\frac{dv}{dx}=2x-1$ or $\frac{d^2v}{dx^2}-\frac{1}{x}\frac{dv}{dx}=2-\frac{1}{x}.$

Now putting $\frac{dv}{dx}=p$, $\frac{d^2v}{dx^2}=\frac{dp}{dx}$, the equation becomes

$$\frac{dp}{dx}-\frac{1}{x}p=2-\frac{1}{x}.$$

This is linear in p and x.

Its I.F. $=e^{\int\left(-\frac{1}{x}\right)dx}=\frac{1}{x}.$

Therefore the solution of this equation is

$$p\frac{1}{x}=k_1+\int\left(2-\frac{1}{x}\right)\cdot\frac{1}{x}\,dx=k_1+2\log x+\frac{1}{x}$$

or $p=\frac{dv}{dx}=2x\log x+1+k_1x.$

Integrating, $v=\int(2x\log x+1+kx)\,dx$

$$=x^2\log x+x+c_1x^2+c_2.$$

The complete solution is $y=vx$

or $y=x^3 \log x+x^2+c_1x^3+c_2x$.

*Ex. 2. *Solve* $\dfrac{d^2y}{dx^2}-x^2\dfrac{dy}{dx}+xy=x$.

[Karnatak 69; Sagar 52; Agra 49; Delhi Hons. 59]

Solution. Here $P=-x^2, Q=x$, since $P+Qx=0$, therefore $y=x$ is a part of C.F.

Putting $y=vx, \dfrac{dy}{dx}=x\dfrac{dv}{dx}+v, \dfrac{d^2y}{dx^2}=x\dfrac{d^2v}{dx^2}+2\dfrac{dv}{dx}$, the equation becomes $\left(x\dfrac{d^2v}{dx^2}+2\dfrac{dv}{dx}\right)-x^2\left(x\dfrac{dv}{dx}+v\right)+x.vx=x$

or $x\dfrac{d^2v}{dx^2}+(2-x^3)\dfrac{dv}{dx}=x$. Now putting $\dfrac{dv}{dx}=p$,

$$x\frac{dp}{dx}+(2-x^3)\,p=x \quad \text{or} \quad \frac{dp}{dx}+\left(\frac{2}{x}-x^2\right)p=1, \qquad \ldots(1)$$

This is linear. Its I.F. $=e^{\int\left(\frac{2}{x}-x^2\right)dx}=x^2e^{-\frac{1}{3}x^3}$

Hence solution of (1) is

$$p.x^2e^{-\frac{1}{3}x^3}=\int x^2e^{-\frac{1}{3}x^3}\,dx+c_1$$

$$=-e^{-\frac{1}{3}x^3}+c_1,$$

the integration on the right has been done by taking $\frac{1}{3}x^3=t$, $x^2\,dx=dt$.

This gives $\quad p=\dfrac{dv}{dx}=-\dfrac{1}{x^2}+\dfrac{c_1e^{\frac{1}{3}x^3}}{x^2}$.

$$\therefore \quad v=\frac{1}{x}+\int c_1e^{\frac{1}{3}x^3}.x^{-2}\,dx+c_2.$$

Hence $y=vx=1+xc_1\displaystyle\int e^{\frac{1}{3}x^3}.x^{-2}\,dx+c_2x$

is the complete solution.

Ex. 3. *Solve* $x^2\dfrac{d^2y}{dx^2}-2x(1+x)\dfrac{dy}{dx}+2(1+x)y=x^3$.

[Agra 67, 66; Delhi 63; Raj. 64; Karnatak 63, 61; Bombay 58]

Solution. Dividing by x^2, the equation in standard form is

$$\frac{d^2y}{dx^2}-\frac{2(1+x)}{x}\frac{dy}{dx}+\frac{2(1+x)}{x^2}y=x,$$

so that $P=-\dfrac{2(1+x)}{x}, Q=\dfrac{2(1+x)}{x^2}$. Since $P+Qx=0$, therefore

$y=x$ is a part of C.F. Putting $y=vx$, the equation becomes

$$\left(x\frac{d^2v}{dx^2}+2\frac{dv}{dx}\right)-\frac{2(1+x)}{x}\left(x\frac{dv}{dx}+v\right)+\frac{2(1+x)}{x^2}vx=x$$

or $x\dfrac{d^2v}{dx^2}-2x\dfrac{dv}{dx}=x.$ Putting $\dfrac{dv}{dx}=p$ this becomes

$$x\frac{dp}{dx}-2xp=x \quad \text{or} \quad \frac{dp}{dx}-2p=1.$$

This is a linear equation. Its I.F. $=e^{-2x}$.

$$\therefore \quad pe^{-2x}=c_1+\int e^{-2x}\,dx=c_1-\tfrac{1}{2}e^{-2x}$$

or $$p=\frac{dv}{dx}=c_1e^{2x}-\tfrac{1}{2}.$$

Integrating again, $v=\dfrac{c_1}{2}e^{2x}-\tfrac{1}{2}x+c_2$.

Hence $y=vx=\dfrac{c_1x.e^{2x}}{2}-\tfrac{1}{2}x^2+c_2x$ is the complete solution.

Ex. 4. (a) $(1-x^2)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=x(1-x^2)^{3/2}$.

[Raj. 1961; Delhi Hons. 69, 57]

Solution. The equation in the standard form is

$$\frac{d^2x}{dx^2}+\frac{x}{1-x^2}\frac{dy}{dx}-\frac{1}{1-x^2}y=x(1-x^2)^{1/2},$$

$$P=\frac{x}{1-x^2}, Q=-\frac{1}{1-x^2}, \ P+Qx=0; \quad \therefore \quad y=x \text{ is part of C.F.}$$

Putting $y=vx$, the equation becomes

$$\left(x\frac{d^2v}{dx^2}+2\frac{dv}{dx}\right)+\frac{x}{1-x^2}\left(x\frac{dv}{dx}+v\right)-\frac{1}{1-x^2}vx=x(1-x^2)^{1/2}$$

or $$\frac{d^2v}{dx^2}+\left(\frac{2}{x}+\frac{x}{1-x^2}\right)\frac{dv}{dx}=(1-x^2)^{1/2}$$

or $$\frac{dp}{dx}+\left(\frac{2}{x}+\frac{x}{1-x^2}\right)p=(1-x^2)^{1/2}, \text{ where } \frac{dv}{dx}=p.$$

Linear equation, I.F. $=e^{\int\left(\frac{2}{x}+\frac{x}{1-x^2}\right)dx}=\dfrac{x^2}{\sqrt{(1-x^2)}}$.

Hence $$p.\frac{x^2}{\sqrt{(1-x^2)}}=c_1+\int(1-x^2)^{1/2}.\frac{x^2}{\sqrt{(1-x^2)}}dx$$
$$=c_1+\tfrac{1}{3}x^3$$

or $$p=\frac{dv}{dx}=c_1\frac{\sqrt{(1-x^2)}}{x^2}+\tfrac{1}{3}x\sqrt{(1-x^2)}.$$

Integrating,

$$v=c_1\int\frac{\sqrt{(1-x^2)}}{x^2}dx+\tfrac{1}{3}\int x\sqrt{(1-x^2)}\,dx+c_2$$

$$=c_1\left[\sqrt{(1-x^2)}\left(-\frac{1}{x}\right)-\int \tfrac{1}{2}(1-x^2)^{-1/2}(-2x)\left(-\frac{1}{x}\right)dx\right]$$

$$-\tfrac{1}{3}\cdot\tfrac{1}{2}\cdot\frac{(1-x^2)^{3/2}}{\frac{3}{2}}+c,$$ integrating first integral by parts

$$=-c_1\frac{\sqrt{(1-x^2)}}{x}-c_1\int\frac{1}{\sqrt{(1-x^2)}}dx-\tfrac{1}{9}(1-x^2)^{3/2}+c_2$$

$$=-c_1\frac{\sqrt{(1-x^2)}}{x}-c_1\sin^{-1}x-\tfrac{1}{9}(1-x^2)^{3/2}+c_2.$$

Hence complete solution is $y=vx$,

i.e. $y=-c_1\sqrt{(1-x^2)}-c_1 x\sin^{-1}x-\frac{1}{9}x(1-x^2)^{3/2}+c_2x.$

Ex. 4. (b) *Verify that one solution of the equation*

$$(1-x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0$$

is $y=x$, and find another solution valid in the interval $-1<x<1$.

[Punjab 1967]

Proceed as above. The solution is valid only in $-1<x<1$.

Ex. 5. *Solve* $x^2\dfrac{d^2y}{dx^2}-(x^2+2x)\dfrac{dy}{dx}+(x+2)y=x^3e^x.$

[Agra 1971, 68, 58; Gujrat 61]

Solution. Dividing by x^2, the equation in standard form is

$$\frac{d^2y}{dx^2}-\left(1+\frac{2}{x}\right)\frac{dy}{dx}+\left(\frac{1}{x}+\frac{2}{x^2}\right)y=xe^x.$$

Here $P=-\left(1+\dfrac{2}{x}\right)$, $Q=\dfrac{1}{x}+\dfrac{2}{x^2}$ and $P+xQ=0$.

Therefore $y=x$ is a part of C F. Putting $y=vx$

$$\frac{dy}{dx}=x\frac{dv}{dx}+v,\ \frac{d^2y}{dx^2}=x\frac{d^2v}{dx^2}+2\frac{dv}{dx},$$

the equation becomes

$$\left(x\frac{d^2v}{dx^2}+2\frac{dv}{dx}\right)-\left(1+\frac{2}{x}\right)\left(x\frac{dv}{dx}+v\right)+\left(\frac{1}{x}+\frac{2}{x^2}\right)vx=xe^x$$

or $$x\frac{d^2v}{dx^2}+\left(2-\frac{x^2+2x}{x}\right)\frac{dv}{dx}=xe^x$$

or $\dfrac{d^2v}{dx^2}-\dfrac{dv}{dx}=e^x.$ Putting $\dfrac{dv}{dx}=p$, the equation becomes

$\dfrac{dp}{dx}-p=e^x.$ This is a linear equation, its I.F. $=e^{-x}$.

Hence $pe^{-x}=\displaystyle\int e^x.e^{-x}\,dx+c_1=x+c_1,$

i.e. $p=\dfrac{dv}{dx}=xe^x+c_1e^x$, *i.e.* $v=e^x(x-1)+c_1e^x+c_2$

The complete solution is $y=vx$

or $y=e^x(x^2-x)+c_1xe^x+c_2x.$

Ex. 6. *Solve* $x\dfrac{dy}{dx}-y=(x-1)\left(\dfrac{d^2y}{dx^2}-x+1\right).$

[Punjab 61 ; Agra 51]

Solution. The equation, in the standard form, is

$$\frac{d^2y}{dx^2}-\frac{x}{x-1}\frac{dy}{dx}+\frac{1}{x-1}y=x-1.$$

$$P=-\frac{x}{x-1},\ Q=\frac{1}{x-1},\ P+xQ=0,\quad \therefore\quad y=x \text{ is a part of C.F.}$$

Putting $y=vx$, the equation becomes

$$\left(x\frac{d^2v}{dx^2}+2\frac{dv}{dx}\right)-\frac{x}{x-1}\left(x\frac{dv}{dx}+v\right)+\frac{1}{x-1}vx=x-1$$

or $\dfrac{d^2v}{dx^2}+\left(\dfrac{2}{x}-\dfrac{x}{x-1}\right)\dfrac{dv}{dx}=\dfrac{x-1}{x}$ Put $\dfrac{dv}{dx}=p.$

$\therefore\quad \dfrac{dp}{dx}+\left(\dfrac{2}{x}-\dfrac{x}{x-1}\right)p=\dfrac{x-1}{x}.$ This is linear.

Its I.F. $=e^{2\log x-x-\log(x-1)}=\dfrac{x^2}{x-1}e^{-x}.$

$$\therefore\ p\frac{x^2}{x-1}e^{-x}=\int xe^{-x}\,dx+c_1=-e^{-x}(x+1)+c_1$$

or $p=\dfrac{dv}{dx}=-\dfrac{x^2-1}{x^2}+c_1\left(\dfrac{x-1}{x^2}\right)e^x=-1+\dfrac{1}{x^2}+c_1\left(\dfrac{1}{x}-\dfrac{1}{x^2}\right)e^x.$

Integrating, $v=-x-\dfrac{1}{x}+c_1\dfrac{e^x}{x}+c_2.$

The complete solution is $y=vx$

or $y=-x^2-1+c_1e^x+c_2x=c_1e^x+c_2x-(1+x^2).$

Ex. 7. *Solve* $x\dfrac{d^2y}{dx^2}-(2x-1)\dfrac{dy}{dx}+(x-1)y=0.$

[Agra 55; Alld. 58 ; Delhi Hons. 60, 58]

Solution. The equation in standard form is

$$\frac{d^2y}{dx^2}-\left(2-\frac{1}{x}\right)\frac{dy}{dx}+\left(1-\frac{1}{x}\right)y=0.$$

Here $P=-2+\dfrac{1}{x},\ Q=1-\dfrac{1}{x}$ and $1+P+Q=0;$

therefore $y=e^x$ is a solution. Putting $y=ve^x$,

$$\frac{dy}{dx}=e^x\frac{dv}{dx}+ve^x,\ \frac{d^2y}{dx^2}=e^x\frac{d^2v}{dx^2}+2e^x\frac{dv}{dx}+ve^x$$

Hence the equation becomes

$$e^x\left(\frac{d^2v}{dx^2}+2\frac{dv}{dx}+v\right)-\left(2-\frac{1}{x}\right)e^x\left(\frac{dv}{dx}+v\right)+\left(1-\frac{1}{x}\right)ve^x=0$$

or $\dfrac{d^2v}{dx^2}+\left[2-\left(2-\dfrac{1}{x}\right)\right]\dfrac{dv}{dx}=0$ or $\dfrac{d^2v}{dx^2}+\dfrac{1}{x}\dfrac{dv}{dx}=0.$

Putting $\dfrac{dv}{dx}=p$, $\dfrac{dp}{dx}+\dfrac{1}{x}\,p=0$ or $\dfrac{dp}{p}+\dfrac{dx}{x}=0.$

Integrating, $\log p+\log x=\log c_1$ or $px=c_1$.

$\therefore\ p=\dfrac{dv}{dx}=\dfrac{c_1}{x}.\quad \therefore\ v=c_1\log x+c_2.$

The complete solution is $y=ve^x$

or $y=e^x(c_1\log x+c_2)$.

Ex. 8. *Solve* $\dfrac{d^2y}{dx^2}-\cot x\,\dfrac{dy}{dx}-(1-\cot x)\,y=e^x\sin x.$

[Karnatak 64; Raj. 57]

Solution. Here $P=-\cot x$, $Q=-(1-\cot x)$.

$1+P+Q=0.$

Therefore, $y=e^x$ is a part of C.F. Putting $y=ve^x$,

$\dfrac{dy}{dx}=e^x\left(\dfrac{dv}{dx}+v\right),\ \dfrac{d^2y}{dx^2}=e^x\left(\dfrac{d^2v}{dx^2}+2\dfrac{dv}{dx}+v\right).$

The equation becomes

$$e^x\left[\left(\frac{d^2v}{dx^2}+2\frac{dv}{dx}+v\right)-\cot x\left(\frac{dv}{dx}+v\right)-(1-\cot x)\,v\right]=e^x\sin x$$

or $\dfrac{d^2v}{dx^2}+(2-\cot x)\dfrac{dv}{dx}=\sin x.$ Put $\dfrac{dv}{dx}=p$

then $\dfrac{dp}{dx}+(2-\cot x)\,p=\sin x$, a linear equation.

Its I.F. $=e^{\int(2-\cot x)\,dx}=e^{2x-\log\sin x}=\dfrac{e^{2x}}{\sin x}.$

$\therefore\ p.\dfrac{e^{2x}}{\sin x}=\displaystyle\int\sin x\,\frac{e^{2x}}{\sin x}\,dx+c_1=\tfrac{1}{2}e^{2x}+c_1$

or $p=\dfrac{dv}{dx}=\tfrac{1}{2}\sin x+c_1\sin x\,e^{-2x}.$

Integrating, $v=-\tfrac{1}{2}\cos x-\dfrac{c_1}{2^2-1}\,e^{-2x}(\cos x+2\sin x)^*+c_2.$

$\therefore$ The complete solution is $y=ve^x$,

or $y=-\tfrac{1}{2}e^x\cos x-\dfrac{c_1}{5}\,e^{-x}(\cos x+2\sin x)+c_2e^x.$

Ex. 9. $x\dfrac{d^2y}{dx^2}-2(x+1)\dfrac{dy}{dx}+(x+2)\,y=(x-2)\,e^x.$

Solution. Equation in standard form is

$^*\displaystyle\int e^{ax}\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)$

$$\frac{d^2y}{dx^2}-2\left(1+\frac{1}{x}\right)\frac{dy}{dx}+\left(1+\frac{2}{x}\right)y=\frac{x-2}{x}e^x.$$

$$P=-2\left(1+\frac{1}{x}\right),\quad Q=1+\frac{2}{x},\ 1+P+Q=0.$$

$\therefore\quad y=e^x$ is a part of C.F. Putting $y=ve^x$, the equation becomes

$$e^x\left[\left(\frac{d^2v}{dx^2}+2\frac{dv}{dx}+2\right)-2\left(1+\frac{1}{x}\right)\left(\frac{dv}{dx}+v\right)+\left(1+\frac{2}{x}\right)v\right]=\frac{x-2}{x}e^x$$

or $\quad \dfrac{d^2v}{dx^2}-\dfrac{2dv}{xdx}=\dfrac{x-2}{x}=1-\dfrac{2}{x}.$

Putting $\dfrac{dv}{dx}=p$, the equation becomes $\dfrac{dp}{dx}-\dfrac{2}{x}p=1-\dfrac{2}{x}.$

Linear, I.F. $=e^{\int -\frac{2}{x}dx}=\dfrac{1}{x^2}$

$$\therefore\quad p\cdot\frac{1}{x^2}=\int\left(\frac{1}{x^2}-\frac{2}{x^3}\right)dx+c_1=-\frac{1}{x}+\frac{1}{x^2}+c_1$$

or $\quad p=\dfrac{dv}{dx}=-x+1+c_1x^2.$

Integrating, $v=-\frac{1}{2}x^2+x+c_1\dfrac{x^3}{3}+c_2.$

The complete solution is $y=ve^x$,

i.e. $\quad v=-\frac{1}{2}x^2e^x+xe^x+\frac{1}{3}c_1x^3e^x+c_2e^x.$

Ex. 10. *Solve* $x\dfrac{d^2y}{dx^2}+(1-x)\dfrac{dy}{dx}-y=e^x.$

[Agra 1950 ; Lucknow 51]

Solution. The equation in the standard form is

$$\frac{d^2y}{dx^2}+\left(\frac{1-x}{x}\right)\frac{dy}{dx}-\frac{1}{x}y-\frac{e^x}{x}.$$

$P=\dfrac{1-x}{x},\ Q=-\dfrac{1}{x},\ 1+P+Q=0.$ Hence $y=e^x$ is a part of C.F.

Putting $y=ve^x$, the equation becomes

$$e^x\left[\left(\frac{d^2v}{dx^2}+2\frac{dv}{dx}+v\right)+\frac{1-x}{x}\left(\frac{dv}{dx}+v\right)-\frac{1}{x}v\right]=\frac{e^x}{x}$$

or $\quad \dfrac{d^2v}{dx^2}+\dfrac{1+x}{x}\dfrac{dv}{dx}=\dfrac{1}{x}.$ Put $\dfrac{dv}{dx}=p$;

then $\dfrac{dp}{dx}+\dfrac{1+x}{x}p=\dfrac{1}{x}$ Linear, I.F. $=e^{\int\left(1+\frac{1}{x}\right)dx}$

$$\therefore\quad pxe^x=\int e^x\,dx+c_1=e^x+c_1$$

or $\quad p=\dfrac{dv}{dx}=\dfrac{1}{x}+\dfrac{c_1}{x}e^{-x}.$

Integrating, $v=\log x+c_1\int x^{-1}e^{-x}\,dx+c_2.$

The complete solution is $y=ve^x$,

i.e. $\quad y=e^x\log x+c_1e^x\int x^{-1}e^{-x}\,dx+c_2e^x.$

Ex. 11. *Solve* $(3-x)\dfrac{d^2y}{dx^2}-(9-4x)\dfrac{dy}{dx}+(6-3x)\,y=0.$

[Marathwada 1964 : Alld 53 ; Karnatak 60, 63]

Solution. The equation in standard form is

$$\frac{d^2y}{dx^2}-\frac{9-4x}{3-x}\frac{dy}{dx}+\frac{6-3x}{3-x}\,y=0.$$

$$P=-\frac{9-4x}{3-x},\ Q=\frac{6-3x}{3-x},\ 1+P+Q=0^*.$$

Hence $y=e^x$ is a part of C.F.

Putting $y=ve^x$, the equation becomes

$$e^x\left[\left(\frac{d^2v}{dx^2}+2\frac{dv}{dx}+v\right)-\frac{9-3x}{3-x}\left(\frac{dv}{dx}+v\right)+\frac{6-3x}{3-x}\,v\right]=0$$

or $\quad \dfrac{dp}{dx}-\dfrac{3-2x}{3-x}\dfrac{dv}{dx}=0$; putting $\dfrac{dv}{dx}=p$,

$$\frac{dp}{dx}-\frac{3-2x}{3-x}\,p=0 \quad\text{or}\quad \frac{dp}{p}-\left(2+\frac{3}{x-3}\right)dx=0.$$

Integrating, $\log p-2x-3\log(x-3)=\log c_1$,

$$p=\frac{dv}{dx}=c_1(x-3)^3e^{2x}.$$

Integrating, $v=A+Be^{2x}(4x^3-42x^2+150x-183)$,

The complete solution is $y=ve^x$

or $\quad y=Ae^x+Be^{3x}(4x^3-42x^2+150x-183).$

***Ex. 12.** *Solve* $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=0$, *given that* $x+\dfrac{1}{x}$ *is one integral.*

(Rajasthan 59)

Solution. The equation is $\dfrac{d^2y}{dx^2}+\dfrac{1}{x}\dfrac{dy}{dx}-\dfrac{y}{x}=0.$

Putting $y=v\left(x+\dfrac{1}{x}\right)$, the equation becomes

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*It is not always necessary to put the equation in standard form to find a part of C.F.

Sum of the coefficients in given form $=3-x-(9-4x)+6-3x=0$. This also shows that $y=e^x$ is a solution.

$$\frac{d^2v}{dx^2}+\left(P+\frac{2}{y_1}\frac{dy_1}{dx}\right)\frac{dv}{dx}=0, \text{ where } y_1=x+\frac{1}{x}$$

or
$$\frac{d^2v}{dx^2}+\left[\frac{1}{x}+\frac{2\left(1-\frac{1}{x^2}\right)}{x+\frac{1}{x}}\right]\frac{dv}{dx}=0$$

or $\dfrac{d^2v}{dx^2}+\dfrac{3x^2-1}{3(x^2+1)}\dfrac{dv}{dx}=0,\ \dfrac{dp}{dx}+\dfrac{3x^2-1}{x(x^2+1)}p=0,\ p=\dfrac{dv}{dx}$

or $\dfrac{dp}{p}+\dfrac{3x^2-1}{x(x^2+1)}dx=0$ or $\dfrac{dp}{p}+\left(-\dfrac{1}{x}+\dfrac{4x}{x^2+1}\right)dx=0.$

Integrating, $\log p-\log x+2\log(x^2+1)=\log c_1$

or $p=\dfrac{dv}{dx}=\dfrac{c_1x}{(x^2+1)^2},\ v=-\dfrac{c_1}{2(x^2+1)}+c_2.$

The complete solution is $y=v\left(x+\dfrac{1}{x}\right)=v\dfrac{(x^2+1)}{x}$

$$=\frac{A'}{x}+c_2\left(x+\frac{1}{x}\right)=\frac{A}{x}+Bx.$$

Aliter. The equation is homogeneous with variable coefficients. So putting $x=e^z$, $D\equiv d/dz$, the equation becomes

$$[D(D-1)+D-1]\,y=0 \quad\text{or}\quad (D-1)(D+1)\,y=0,$$

so that
$$y=c_1e^z+c_2e^{-z}=c_1x+\frac{c_2}{x}.$$

Ex. 13. $(x+1)\dfrac{d^2y}{dx^2}-2(x+3)\dfrac{dy}{dx}+(x+5)\,y=e^x.$

[Nagpur 53]

Solution. The sum of the coefficients is zero. Hence $y=e^x$ is a part of C. F. The equation in standard form is

$$\frac{d^2y}{dx^2}-\frac{2(x+3)}{x+1}\frac{dy}{dx}+\frac{x+5}{x+1}y=\frac{e^x}{x+1}.$$

Putting $y=ve^x$, the equation becomes

$$\frac{d^2v}{dx^2}+\left(P+\frac{2}{y_1}\frac{dy_1}{dx}\right)\frac{dv}{dx}=\frac{X}{y_1} \qquad \text{where } y_1=e^x$$

or
$$\frac{d^2v}{dx^2}+\left[-\frac{2(x+3)}{x+1}+\frac{2}{e^x}e^x\right]\frac{dv}{dx}=\frac{e^x/(x+1)}{e^x}$$

or $\dfrac{d^2v}{dx^2}-\dfrac{4}{x+1}\dfrac{dv}{dx}=\dfrac{1}{x+1}$; next put $\dfrac{dv}{dx}=p,$

$$\frac{dp}{dx}-\frac{4}{x+1}p=\frac{1}{x+1},\ \text{I.F.}=e^{\int -\frac{4}{x+1}dx}=\frac{1}{(x+1)^4}.$$

$$\therefore \quad p\frac{1}{(x+1)^4}=c_1+\int\frac{1}{(x+1)^5}dx=c_1-\frac{1}{4(x+1)^4}$$

or
$$p=\frac{dv}{dx}=c_1(x+1)^4-\tfrac{1}{4}.$$

Integrating, $v=\dfrac{c_1(x+1)^5}{5}-\tfrac{1}{4}x+c_2.$

Hence $y=ve^x=\frac{1}{5}c_1e^x(x+1)^5-\frac{1}{4}ye^x+c_2e^x$ is the complete solution.

Ex. 14. *Solve* $\dfrac{d^2y}{dx^2}-(1+x)\dfrac{dy}{dx}xy=x.$

[Gujrat 65; Bombay 61; Karnatak 61]

Solution. Here $1+P+Q=0$. Hence $y=e^x$ is a part of C F.

The equation after putting $y=ve^x$ becomes

$$\frac{d^2v}{dx^2}+\left(P+\frac{2}{y_1}\frac{dy_1}{dx}\right)\frac{dv}{dx}=\frac{x}{y_1} \quad \text{where } y_1=e^x$$

or
$$\frac{d^2v}{dx^2}+\left[-(1+x)+\frac{2}{e^x}.e^x\right]\frac{dv}{dx}=\frac{x}{e^x}$$

or
$$\frac{d^2v}{dx^2}+(1-x)\frac{dv}{dx}=xe^{-x} \text{ or } \frac{dp}{dx}+(1-x)p=xe^{-x},\ p=\frac{dv}{dx}$$

Linear; I.F. $=e^{\int(1-x)\,dx}=e^{x-\frac{1}{2}x^2}$

$$\therefore \quad pe^{x-\frac{1}{2}x^2}=\int xe^{-\frac{1}{2}x^2}\,dx+c_1=-e^{-\frac{1}{2}x^2}+c_2$$

putting $\frac{1}{2}x^2=t$, $x\,dx=dt$

i.e. $p=\dfrac{dv}{dx}=c^{-x}+c_1\,e^{-x+\frac{1}{2}x^2}.$

Integrating, $v=-e^{-x}+c_1\int e^{-x+\frac{1}{2}x^2}\,dx+c_2.$

Hence $y=ve^x=-1+c_1e^x\int e^{-x+\frac{1}{2}x^2}\,dx+c_2e^x$

is the complete solution.

Ex. 15. *Solve* $\dfrac{d^2y}{dx^2}+(1-\cot x)\dfrac{dy}{dx}-y\cot x=\sin^2 x.$

Solution. Here $P=1-\cot x$, $Q=-\cot x$;

$\therefore \quad 1-P+Q=0.$

Therefore $y=e^{-x}$ is a part of C F. Putting $y=ve^{-x}$, the equation becomes $\dfrac{d^2v}{dx^2}+\left(P+\dfrac{2}{y_1}\dfrac{dy_1}{dx}\right)\dfrac{dv}{dx}=\dfrac{\sin^2 x}{y_1}$ $\quad y_1=e^{-x}$

or
$$\frac{d^2v}{dx^2}+\left[(1-\cot x)+\frac{2}{e^{-x}}(e^{-x})\right]\frac{dv}{dx}=\frac{\sin^2 x}{x}$$

or $\dfrac{d^2v}{dx^2}-(1+\cot x)\dfrac{dv}{dx}=e^x\sin^2 x$; putting $\dfrac{dv}{dx}=p$.

$\dfrac{dp}{dx}-(1-\cot x)\,p=e^x\sin^2 x$, I.F. $=e^{-\int(1+\cot x)dx}=e^{-x}\dfrac{1}{\sin x}$

$$\therefore\quad pe^{-x}\cdot\frac{1}{\sin x}=c_1+\int e^x\sin^2 x e^{-x}\cdot\frac{1}{\sin x}\,dx$$

$$=c_1+\int\sin x\,dx=c_1-\cos x.$$

$$\therefore\quad p=\frac{dv}{dx}=c_1e^x\sin x-e^x\sin x\cos x=c_1e^x\sin x-\tfrac{1}{2}e^x\sin 2x.$$

Integrating,

$$v=c_2+c_1e^x(\sin x-\cos x)-\tfrac{1}{2}\cdot\frac{e^x}{2^2+1}(\sin 2x-2\cos 2x).$$

The complete solution is $y=ve^{-x}$

or $y=c_2e^{-x}+c_1(\sin x-\cos x)-\frac{1}{10}(\sin 2x-2\cos 2x)$.

Ex. 16. $(x+2)\dfrac{d^2y}{dx^2}-(2x+5)\dfrac{dy}{dx}+2y=(x+1)\,e^x$.

[Delhi Hons. 64: Agra 52; Raj. 59; Pb. 61; Meerut 76]

Solution. The equation in standard form is

$$\frac{d^2y}{dx^2}-\frac{2x+5}{x+2}\frac{dy}{dx}+\frac{2}{x+2}\,y=\frac{(x+1)}{x+2}\,e^x$$

Here $m^2+mP+Q=m^2-\dfrac{2x+5}{x+2}\,m+\dfrac{2}{x+2}$

$$=\frac{m-2}{x+2}\,[mx+(2m-1)]=0 \text{ gives } m=2.$$

Therefore by § 2·3 P 48, $y_1=e^{2x}$ is a part of C.F.

Putting $y=ve^{2x}$, is the equation reduces to

$$\frac{d^2v}{dx^2}+\left(P+\frac{2}{y_1}\frac{dy_1}{dx}\right)\frac{dv}{dx}=\frac{X}{y_1},\ y_1=e^{2x}\left(\text{Put } p=\frac{dv}{dx}\right).$$

or $\dfrac{dp}{dx}+\dfrac{2x+3}{x+2}\,p=\dfrac{x+1}{x+2}\,e^{-x}$. Linear, I.F. $=\dfrac{e^{2x}}{x+2}$.

$$\therefore\quad p\frac{e^{2x}}{x+2}=\int\frac{x+1}{x+2}\,e^{-x}\cdot\frac{e^{2x}}{x+2}\,dx+c_1$$

$$=\int e^x\left[\frac{1}{x+2}-\frac{1}{(x+2)^2}\right]dx+c_1$$

$$=\frac{1}{x+2}\,e^x+\int e^x\cdot\frac{1}{(x+2)^2}\,dx-\int e^x\frac{1}{(x+2)^2}\,dx+c_1$$

integrating first integral by parts.

$$=\frac{e^x}{x+2}+c_1$$

or $p=\frac{dv}{dx}=e^{-x}+c_1e^{-2x}(x+2)$.

$\therefore \quad v=-e^{-x}-\frac{1}{4}c_1e^{-2x}(2x+5)+c_2$.

Hence $y=e^{2x}=e^{2x}[-e^{-x}-\frac{1}{4}c_1e^{-2x}(2x+5)+c_2]$

$=-e^{x}+\frac{1}{4}c_1(2x+5)+c_2e^{2x}$

is the complete solution.

Ex. 17. *Solve*

$$(x\sin x+\cos x)\frac{d^2y}{dx^2}-x\cos x\frac{dy}{dx}+y\cos x=0$$

of which $y=x$ is a solution.

[Agra 70; Delhi Hons. 53; Punjab 62]

Solution. Here $P=\frac{-x\cos x}{x\sin x+\cos x}$, $Q=\frac{\cos x}{x\sin x+\cos x}$.

Clearly $P+Qx=0$, so that $y=x$ is a solution, putting $y=vx$, the equation becomes

$$\frac{d^2v}{dx^2}+\left[P+\frac{2}{y_1}\frac{dy_1}{dx}\right]\frac{dv}{dx}=0,\text{ where } y_1=x$$

i.e. $$\frac{d^2v}{dx^2}+\left[-\frac{x\cos x}{x\sin x+\cos x}+\frac{2}{x}\right]\frac{dv}{dx}=0$$

or $$\frac{dp}{dx}+\left[\frac{-x\cos x}{x\sin x+\cos x}+\frac{2}{x}\right]p=0,\quad \text{where } p=\frac{dv}{dx}$$

or $$\frac{dp}{p}+\left[\frac{-x\cos x}{x\sin x+\cos x}+\frac{2}{x}\right]dx=0$$

Integrating, $\log p-\log(x\sin x+\cos x)+2\log x=\log c_1$

i.e. $$\frac{px^2}{x\sin x+\cos x}=c_1 \text{ or } p=\frac{dv}{dx}=c_1\left[\frac{\sin x}{x}+\frac{\cos x}{x^2}\right]$$

$$\therefore \quad v=c_1\int\left(\frac{\sin x}{x}+\frac{\cos x}{x^2}\right)dx+c_2$$

$$=c_2\left[-\frac{1}{x}\cos x-\int-\frac{1}{x^2}(-\cos x)\,dx+\int\frac{\cos x}{x^2}dx\right]+c_2$$

integrating first integral by parts.

$$=-\frac{c_1}{x}\cos x+c_2$$

$\therefore \quad y=vx=-c_1\cos x+c_2x$ is the solution.

Ex. 18. $(x\sin x+\cos x)\frac{d^2y}{dx^2}-x\cos x\frac{dy}{dx}+y\cos x$

$=\sin x\,(x\sin x+\cos x)^2$

[Delhi Hons. 70; Bombay 61]

Solution. Here $P+xQ=0$.

Hence $y=x$ is a part of C.F.

Putting $y=vx$, the equation becomes

$$\frac{d^2v}{dx^2}+\left[P+\frac{2}{y_1}\frac{dy_1}{dx}\right]\frac{dv}{dx}=\frac{X}{y_1},\text{ where } y_1=x$$

i.e. $\frac{dp}{dx}+\left[-\frac{x\cos x}{x\sin x+\cos x}+\frac{2}{x}\right]p=\frac{\sin x\,(x\sin x+\cos x)}{x}.$

This is linear, I.F. $=\frac{x^2}{x\sin x+\cos x}$.

$$\therefore \quad p\cdot\frac{x^2}{x\sin x+\cos x}=\int x\sin x\,dx+c_1.$$

$$=(-x\cos x+\sin x)+c_1.$$

$$\therefore \quad p=\frac{dv}{dx}=\frac{(x\sin x+\cos x)\,[(-x\cos x+\sin x)+c_1]}{x^2}$$

$$=\left[-\sin x\cos x-\frac{\cos^2 x-\sin x^2}{x}-\frac{\sin x\cos x}{x^2}\right]$$
$$+c_1\left[\frac{\sin x}{x}+\frac{\cos x}{x^2}\right]$$

Integrating,

$$v=c_2+\int\left[-\tfrac{1}{2}\sin 2x-\frac{\cos 2x}{x}+\tfrac{1}{2}\,\frac{\cos 2x}{x^2}+c_1\left(\frac{\sin x}{x}+\frac{\cos x}{x^2}\right)\right]dx$$

$$=c_2+\tfrac{1}{4}\cos 2x-\tfrac{1}{2}\,\frac{\sin 2x}{x}-c_1\,\frac{\cos x}{x}.$$

$$\therefore \quad y=vx=c_2x+\tfrac{1}{4}x\cos 2x-\tfrac{1}{2}x\sin 2x-c_1\cos x.$$

Ex. 19. $x\,\frac{d^2y}{dx^2}\,(x\cos x-2\sin x)+(x^2+2)\frac{dy}{dx}\sin x$
$-2y\,(x\sin x+\cos x)=0$

given that $y=x^2$ is a solution.

Solution.

$$P=\frac{x^2+2}{x\,(x\cos x-2\sin x)},\quad Q=-\frac{2\,(x\sin x+\cos x)}{x\,(x\cos x-2\sin x)}.$$

Putting $y=vy_1=vx^2$ the equation becomes

$$\frac{d^2v}{dx^2}+\left[P+\frac{2}{y_1}\frac{dy_1}{dx}\right]\frac{dv}{dx}=0$$

i.e. $\frac{dp}{dx}+\frac{(x^2+2)\sin x}{x\,(x\cos x-2\sin x)}+\frac{2}{x^2}\cdot dx\,\Big]\,p=0$, where $p=\frac{dv}{dx}$

or $\frac{dp}{p}+\left[\frac{(x^2+2)\sin x}{x^2\cos x-2x\sin x)}+\frac{4}{x}\right]dx=0.$

Integrating, $\log p-\log(x^2\cos x-2x\sin x)+4\log x=\log c_1$

or $p=\frac{dv}{dx}=c_1\,\frac{(x^2\cos x-2x\sin x)}{x^4}=c_1\left(\frac{\cos x}{x^2}-\frac{2\sin x}{x^3}\right).$

Integrating, $v=c_1\,\frac{\sin x}{x^2}+c_2.$

Hence the complete solution is $y=vy_1$ etc.

***Ex. 20.** *Solve* $\sin^2 x\,\frac{d^2y}{dx^2}=2y$, *given that $y=\cot x$ is a solution.*

[Agra 62, 56 ; Raj. 52]

Solution. We have $\frac{d^2y}{dx^2}-2y \operatorname{cosec}^2 x=0$, $P=0$.

Putting $y=vy=v \cot x$, the equation becomes

$$\frac{d^2v}{dx^2}+\left[P+\frac{2}{y_1}\frac{dy_1}{dx}\right]\frac{dv}{dx}=0, \text{ where } y_1=\cot x$$

i.e. $$\frac{dp}{dx}-\frac{2}{\sin x \cos x}p=0, \text{ where } p=\frac{dv}{dx}$$

or $$\frac{dp}{p}=4 \operatorname{cosec} 2x\, dx.$$

Integrating, $\log p=\log \tan x+\log c_1$

or $$p=\frac{dv}{dx}=c_1 \tan^2 x$$

or $$\frac{dv}{dx}=c_1 (\sec^2 x-1); \quad \therefore \quad v=c_1 \tan x-c_1x+c_2.$$

Hence the complete solution is $y=vy_1$

i.e. $y=\cot x\,(c_1 \tan x-c_1x+c_2)=c_1-c_1x \cot x+c_2 \cot x.$

Ex. 21. *Solve* $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-a^2y=0$, *given that* $y=ce^{a \sin^{-1}x}$ *is integral.*

Solution. $$\frac{d^2y}{dx^2}-\frac{x}{1-x^2}\frac{dy}{dx}-\frac{a^2}{1-x^2}y=0.$$

Putting $y=vy_1=vce^{a \sin^{-1}x}$, the equation becomes

$$\frac{dv^2}{dx^2}+\left(\frac{2a}{\sqrt{(1-x^2)}}-\frac{x}{1-x^2}\right)\frac{dv}{dx}=0$$

or $$\frac{dp}{dx}+\left(\frac{2a}{\sqrt{(1-x^2)}}-\frac{x}{1-x^2}\right)p=0, \text{ where } p=\frac{dv}{dx}$$

or $$\frac{dp}{p}+\left(\frac{2a}{\sqrt{(1-x^2)}}-\frac{x}{1-x^2}\right)dx=0$$

Integrating, $\log p+2a \sin^{-1} x+\frac{1}{2} \log (1-x^2)=\log c_1$

or $p\sqrt{(1-x^2)}=c_1e^{-2a \sin^{-1}x}$ or $p=\frac{dv}{dx}=c_1\frac{e^{-2a \sin^{-1}x}}{\sqrt{(1-x^2)}}$.

$$\therefore \quad v=-\frac{c_1}{2a}e^{-2a \sin^{-1}x}+c_2.$$

Hence the complete solution is $y=vy_1$, etc.

Ex. 22. *Solve* $x^2\frac{d^2y}{dx^2}+\frac{dy}{dx}-9y=0$, *given that* $y=x^3$ *is a solution.*

Proceeding as above, $y=c_1x^3+c_2x^{-3}$ is a solution.

Ex. 23. *Given that the equation*

$$x(1-x)\frac{d^2y}{dx^2}+(\tfrac{3}{2}-2x)\frac{dy}{dx}-\tfrac{1}{4}y=0.$$

has a particular integral of the form x^n, prove that $n=-\frac{1}{2}$, and that the primitive of the equation is $y=x^{-1/2}[A+B\sin^{-1}(x^{1/2})]$, where A and B are arbitrary constants. **[Poona M. A 1961]**

Solution Let $y=x^n$, than $\frac{dy}{dx}=nx^{n-1}, \frac{d^2y}{dx^2}=n(n-1)x^{n-2}$.

Putting these values in the given differential equation,

$$x(1-x)n(n-1)x^{n-2}+(\tfrac{3}{2}-2x)nx^{n-1}-\tfrac{1}{4}x^n=0$$

i.e. $x^n[-n(n-1)-2n-\frac{1}{4}]+x^{n-1}[n(n-1)-\frac{3}{2}n]=0$

$y=x^n$ will be a solution if

$n(n-1)+2n+\frac{1}{4}=0$, i.e. $4n^2+4n+1=0$ i.e. $(2n+1)^2=0$

i.e. $n=-\frac{1}{2}$

and $n(n-1)-\frac{3}{2}n=0$ which is also satisfied by $n=-\frac{1}{2}$.

Hence $y=x^n=x^{-1/2}$ is a solution of the given differential equation,

Now put $y=vx^{-1/2}$ and proceed as above to find the complete primitive.

Ex. 24. *Solve* $(x^2+x)y'''-(x^2+3x+1)y''$

$$+\left(x+y+\frac{2}{x}\right)y'-\left(1+\frac{4}{x}+\frac{2}{x^2}\right)y=3x^2(x+1)^2$$

of which $y=x$ is a particular integral. **[Delhi Hons. 1961]**

Solution. Putting $y=vx$, the equation reduces to

$$(x^2+x)v'''-(x^2-2)v''-(x+2)v'=3x(x+1)^2.$$

Now putting $v'=p$, it becomes

$$(x^2+x)p''-(x^2-2)p'-(x+2)p=3x(x+1)^2.$$

Sum of the coefficients is zero ; $\therefore$ $p=e^x$ is a part of C.F.

Putting $p=e^x\omega$, $p'=e^x(\omega'+\omega)$, $p''=e^x(\omega''+2\omega'+\omega)$.

the equation becomes

$$(x^2+x)\omega''+(x^2+x+2)\omega'=3xe^{-x}(x+1)^2$$

Putting $\omega'=q$, this becomes

$$(x^2+x)q'+(x^2+2x+2)q=3xe^{-x}(x+1)^2$$

or $\frac{dq}{dx}+\left(1+\frac{2}{x}-\frac{1}{1+x}\right)q=3e^{-x}(x+1)$. I.F.$=\frac{x^2e^x}{x+1}$.

$\therefore$ $q\frac{x^2e^x}{x+1}=x^3+k_1$ or $\frac{d\omega}{dx}=q=x(x+1)e^{-x}+k_1\frac{x+1}{x^2}e^{-x}$

or $\frac{p}{e^x}=\omega=-x^2e^{-x}-3xe^{-x}-3e^{-x}+c_1\frac{e^{-x}}{x}+c_2$

or $p=\frac{dv}{dx}=-x^2-3x-3+\frac{c_1}{x}+c_2e^x$

or $v=-\frac{1}{3}x^3-\frac{3}{2}x^2-3x+c_1\log x+c_2e^x+c_3$.

The complete solution is $y=vx$

i.e. $y=-\frac{1}{3}x^4-\frac{3}{2}x^3-3x^2+c_1x\log x+c_2xe^x+c_3x$.

***2 6. Removal of the First Derivative.** (*Reduction to Normal form*). **[Gujrat 1961 ; Agra 53 ; Nagpur 62, 61 ; Poona 59, 60; Jiwaji 66 ; Delhi 64, 56; Nag. 61 ; Karnatak 61]**

When we fail to obtain a part of C.F. we cannot apply § 2·2 p. 45. In such cases the equation may be solved after removing the first derivative. This is done as follows:

Put $y=vy_1$, which gives $\dfrac{dy}{dx}=v\dfrac{dy_1}{dx}+y_1\dfrac{dv}{dx}$

and $\dfrac{d^2y}{dx^2}=v\dfrac{d^2y_1}{dx^2}+2\dfrac{dv}{dx}\cdot\dfrac{dy}{dx}+y_1\dfrac{d^2v}{dx^2}$

The equation $\dfrac{d^2y}{dx^2}+P\dfrac{dy}{dx}+Qy=X$ then becomes

$$\left[v\frac{d^2y_1}{dx^2}+2\frac{dv}{dx}\frac{dy_1}{dx}+y_1\frac{d^2y}{dx^2}\right]+P\left[v\frac{dy_1}{dx}+y_1\frac{dv}{dx}\right]+Qvy_1=X$$

or $$y_1\frac{d^2v}{dx^2}+y_1\left[P+\frac{2}{y_1}\frac{dy_1}{dx}\right]\frac{dv}{dx}+P\left[\frac{d^2y_1}{dx^2}+P\frac{dy_1}{dx}+Qy_1\right]^*=X.$$

Now, let us choose y_1 such that first derivative is removed

i.e. $P+\dfrac{2}{y_1}\dfrac{dy_1}{dx}=0$ or $\dfrac{dy_1}{y_1}=-\tfrac{1}{2}P\,dx$

i.e. $\log y_1=-\tfrac{1}{2}\int P\,dx$ or $y_1=e^{-\frac{1}{2}\int P dx}$. ...(1)

Thus the above equation becomes

$$\frac{d^2v}{dx^2}+\frac{v}{y_1}\left[\frac{d^2y_1}{dx^2}+P\frac{dy_1}{dx}+Qy_1\right]=\frac{X}{y_1}. \qquad ...(2)$$

From (1), $\dfrac{dy_1}{dx}=e^{-\frac{1}{2}\int P\,dx)}(-\tfrac{1}{2}P)=-\tfrac{1}{2}Py_1$,

$$\frac{d^2y_1}{dx^2}=-\tfrac{1}{2}P\frac{dy_1}{dx}-\tfrac{1}{2}\frac{dP}{dx}y_1=-\tfrac{1}{2}P(-\tfrac{1}{2}Py_1)-\tfrac{1}{2}\frac{dP}{dx}y_1$$

$$=\tfrac{1}{4}P^2y_1-\tfrac{1}{2}y_1\frac{dP}{dx}$$

Putting these values in (2), it becomes

$$\frac{d^2v}{dx^2}+v\left[\tfrac{1}{4}P^2-\tfrac{1}{2}\frac{dP}{dx}-P.\tfrac{1}{2}P+Q\right]=\frac{X}{e^{-\frac{1}{2}\int Pdx}}$$

or $$\frac{d^2y}{dx^2}+v\left[Q-\tfrac{1}{4}P^2-\tfrac{1}{2}\frac{dP}{dx}\right]=Xe^{\int\frac{1}{2}Pdx}$$

or $$\frac{d^2v}{dx^2}-Q_1v=X_1 \qquad ...(3)$$

where $Q_1=Q-\tfrac{1}{4}P^2-\tfrac{1}{2}\dfrac{dP}{dx}$ and $X_1=Xe^{\int\frac{1}{2}Pdx}$

The reduced equation (3) (which is called of normal form) can be easily integrated.

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*Here unlike § 2·2 p. 45 the exprsss on in the last bracket is not zero since here y_1 is not a part of C.F.

Note. Students should remember the coefficients Q_1, X_1 of (3) and should apply in the examples directly. The equation should, however, first be reduced to standard form to get values P, Q and X etc.

Ex. 1. *Solve* $\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}+(x^2+2)\,y=e^{\frac{1}{2}(x^2+2x)}$.

Solution. Here $P=-2x$, $Q=x^2+2$, $X=e^{\frac{1}{2}(x^2+2x)}$.

To remove the first derivative, we choose

$$y_1=e^{-\int\frac{1}{2}P\,dx}=e^{\frac{1}{2}x^2},$$

Now putting $y=vy_1$ the above equation becomes

$$\frac{d^2v}{dx^2}+Q_1v=X_1,$$

where $Q_1=Q-\frac{1}{4}P^2-\frac{1}{2}\dfrac{dP}{dx}=3$, $X_1=\dfrac{X_1}{y_1}=\dfrac{e^{\frac{1}{2}(x^2+2x)}}{e^{\frac{1}{2}x^2}}=e^x$.

Hence after removing the first term, the equation reduces to

$$\frac{d^2v}{dx^2}+3v=e^x \quad\text{or}\quad (D^2+3)\,v=e^x.$$

Its A.E. is $D^2+3=0$, $D=\pm 3i$, C.F.$=c_1\cos(\sqrt{3}x+c_2)$,

$$\text{P.I.}=\frac{e^x}{D^2+3}=\frac{e^x}{4}.$$

$$\therefore\qquad v=e_1\cos(\sqrt{3}x+c_3)+\frac{e^x}{4}.$$

The complete solution is $y=vy_1$

i.e. $y=e^{\frac{1}{2}x^2}\,c_1\cos(\sqrt{3}x+c_2)+\frac{1}{4}e^x.\,e^{\frac{1}{2}x^2}$

$\quad=c_1e^{\frac{1}{2}x^2}\,c_1\cos(\sqrt{3}x+c_2)+\frac{1}{4}e^{\frac{1}{2}(x^2+2x)}$.

Ex 2. *Solve* $\dfrac{d^2y}{d^2x}-4x\dfrac{dy}{dx}+(4x^2-3)\,y=e^{x^2}$.

[Delhi Hons. 72; Gujrat 65; Nagpur 63, 61]

Solution. Here $P=-4x$, $Q=4x^2-3$; $X=e^{x^2}$.

To remove the first derivative choose

$$y_1=e^{-\frac{1}{2}\int P\,dx}=e^{\int 2x\,dx}=e^{x^2}.$$

Putting $y=vy_1$ the equation after removing the first derivative is $\dfrac{d^2v}{dx^2}+Q_1v=X_1$

where $\quad Q_1=Q-\frac{1}{4}P^2-\frac{1}{2}\dfrac{dP}{dx}=(4x^2-3)-4x^2+2=-1,$

$$X_1=\frac{X}{y_1}=\frac{e^{x^2}}{e^{x^2}}=1,$$

i.e. $\dfrac{d^2v}{dx^2}-v=1$ or $(D^2-1)\,v=1$.

A.E. is $D^2-1=0$, $\qquad D=\pm 1$, C.F.$=c_1e^x+c_2e^{-x}$

P. I.$=\frac{1}{D^2-1}=-1$. Hence $v=c_1e^x+c_2e^{-x}-1$.

The complete solution is $y=vy_1$

or $y=e^{x^2}(c_1e^x+c_2e^{-x}-1)$.

Ex. 3. *Solve* $\frac{d^2y}{dx^2}-2\tan x\frac{dy}{dx}+5y=\sec x\, e^x$. **[Agra 51]**

Solution. To remove first derivative, we choose

$$y'=e^{-\frac{1}{2}\int P\,dx}=e^{\int \tan x\,dx}=e^{\log \sec x}=\sec x.$$

Now, putting $y=-vy_1$, the above equation becomes

$$\frac{d^2v}{dx^2}+Q_1v=X_1,$$

where $Q_1=Q-\frac{1}{4}P^2-\frac{1}{2}\frac{dP}{dx}=5-\tan^2 x+\sec^2 x=6.$

$X_1=X/y_1=\sec x\,e^x/\sec x=e^x$.

Hence after removing first term, the equation becomes

$$\frac{d^2v}{dx^2}+6v=e^x.$$

A.E. is $D^2+6=0$, $D=\pm\sqrt{6}i$, C.F.$=c_1\cos(\sqrt{6}x+c_2)$

P.I.$=e^x/(D^2+6)=e^x/7$.

Hence $=c_1\cos(\sqrt{6}x+c_2)+\frac{1}{7}e^x$.

The complete solution is, $y=vy_1$

i.e. $y=c_1\sec x\cos(\sqrt{6}x+c_2)+\frac{1}{7}\sec x\,e^x$.

Ex. 4. *Solve* $\frac{d^2y}{dx^2}-2\tan x\frac{dy}{dx}+5y=0$.

[Karnatak 62; Agra 58, 76; Delhi Hons. 68, 63; Nagpur 61 (S)]

Solution. Just the above example with $X=0$. Here removing the first term, we get $(D^2+6)v=0$.

Hence $v=c_1\cos(\sqrt{6}x+c_2)$

or $y=vy_1=c_1\cos(\sqrt{6}x+c_2)\sec x$.

Ex. 5. *Solve* $\frac{d}{dx}\left(\cos^2 x\frac{dy}{dx}\right)+\cos^2 x.y=0$. **[Delhi Hons. 56]**

Solution. The equation is

$$\cos^2 x\frac{d^2y}{dx^2}-2\sin x\cos x\frac{dy}{dx}+\cos^2 x.y=0,$$

i.e. $\frac{d^2y}{dx^2}-2\tan x\frac{dy}{dx}+y=0$.

Choose $y_1=e^{-\frac{1}{2}\int P\,dx}=\sec x$,

$$Q_1=Q-\frac{1}{4}P^2-\frac{1}{2}\frac{dP}{dx}=1-\tan^2 x+\sec^2 x=2.$$

Normal form is $\dfrac{d^2v}{dx^2}+2v=0$, $(D^2+2)\,v=0$,

$v=c_1\cos(\sqrt{2}x+c_2)$.

Therefore $y=vy_1=c_1\cos(\sqrt{2}x+c_2)\sec x$ is the solution.

***Ex. 6.** *Solve* $\dfrac{d^2y}{dx^2}-4x\dfrac{dy}{dx}+(4x^2-1)\,y=-3e^{x^2}\sin 2x.$

[Nag 62; Agra 77, 72, 55, 50; Luck. 56, 51]

Solution. Here $P=-4x$, $Q=(4x^2-1)$, $X=-3e^{x^2}\sin 2x$.

To remove the first derivative, choose

$y_1=e^{-\frac{1}{2}\int P\,dx}=e^{\int 2x\,dx}=e^{x^2}$

And putting $y=vy_1$, the reduced equation becomes

$\dfrac{d^2v}{dx^2}+Q_1v=X_1,$

where $Q_1=Q-\frac{1}{4}P^2-\frac{1}{2}\dfrac{dP}{dx}=4x^2-1-\frac{1}{4}(16x^2)-\frac{1}{2}(-2x)=1$

$X_1=\dfrac{X}{y_1}=-\dfrac{3e^{x^2}\sin 2x}{e^{x^2}}=-3\sin 2x.$

Therefore, the equation reduces to

$\dfrac{d^2v}{dx^2}+v=-3\sin 2x$ or $(D^2+1)\,v=-3\sin 2x.$

A.E. is $D^2+1=0$, $D=\pm i$, C.F.$=c_1\cos(x+c_2)$,

P. I.$=\dfrac{-3\sin 2x}{D^2+1}=\dfrac{-3\sin 2x}{-4+1}=\sin 2x.$

Hence $v=c_1\cos(x+c_2)+\sin 2x$.

Therefore the complete solution is $y=vy_1$

or $y=e^{x^2}c_1\cos(x+c_2)+e^{x^2}\sin 2x$.

Ex. 7. *Solve* $\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}+(x^2+5)\,y=xe^{-\frac{1}{2}x^2}$. [Luck. 54, 49]

Solution. Here $P=2x$, $Q=x^2+5$, $X=xe^{-\frac{1}{2}x^2}$.

To remove the first derivative, choose $y_1=e^{-\frac{1}{2}\int P\,dx}=e^{-\frac{1}{2}x^2}$.

If $Q_1=Q-\frac{1}{4}P^2-\frac{1}{2}dP/dx=(x^2+5)-x^2-1=4$, $X_1=X/y_1=x$.

The transformation $y=vy_1$ gives $\dfrac{d^2v}{dx^2}+Q_1v=X_1$

i.e. $\dfrac{d^2v}{dx^2}+4v=x$ or $(D^2+4)\,v=x$.

A.E. is $D^2+4=0$, $D=\pm 2i$, C.F.$=c_1\cos(2x+c_2)$.

P. I.$=\dfrac{x}{D^2+4}=\frac{1}{4}\left(1-\dfrac{D^2}{4}+\dots\right)x=\frac{1}{4}x.$

$\therefore\ v=c_1\cos(2x+c_2+\frac{1}{4}x.$

The complete solution is $y=vy_1$, *i.e.*

$y=e^{-\frac{1}{2}x^2}\,[c_1 \cos (2x+c_2)+\frac{1}{4}x]$.

Ex. 8. *Solve* $\dfrac{d^2y}{dx^2}+4x\dfrac{dy}{dx}+4x^2y=0$ *by removing the first derivative.* **[Karnatak 61; Meerut 71; Agra 53]**

Solution. Here $P=4x$, $Q=4x^2$, $X=0$.

To remove the first derivative, choose $y_1=e^{-\frac{1}{2}\int P\,dx}=e^{-x^2}$.

Let $Q_1=Q-\frac{1}{4}P^2-\frac{1}{2}dP/dx=4x^2-4x^2-2=-2$, $X_1=X/y_1=0$.

The transformation $y=vy_1$ reduces the given equation to

$$\frac{d^2v}{dx^2}+Q_1v=X_1 \quad\text{or}\quad \frac{d^2v}{dx^2}-2v=0 \quad\text{or}\quad (D^2-2)\,v=0.$$

$\therefore \quad v=c_1e^{\sqrt{2}x}+c_2e^{-\sqrt{2}x}$.

Therefore the complete solution is $y=vy_1$, *i.e.*

$y=e^{-x^2}\,(c_1e^{\sqrt{2}x}+c_2e^{-\sqrt{2}x})$.

Ex. 9. *Solve* $\dfrac{d^2y}{dx^2}-4x\dfrac{dy}{dx}+4x^2y=e^{x^2}$ **[Indore 66]**

Solution. $P=-4x$, $Q=4x^2$, $X=e^{x^2}$, $y_1=e^{-\frac{1}{2}\int P\,dx}=e^{x^2}$,

$Q_1=2$, $X_1=1$.

$\therefore$ The equation after removing the first derivative becomes

$$\frac{d^2v}{dx^2}+2v=1 \quad\text{or}\quad (D^2+2)\,v=1.$$

C.F. $=c_1 \cos (\sqrt{2}x+c_2)$.

P.I. $=\dfrac{1}{D^2+2}=\frac{1}{2}$.

$\therefore \quad v=c_1 \cos (\sqrt{2}x+c_2)+\frac{1}{2}$.

The complete solution is

$y=vy_1=e^{x^2}\,c_1 \cos (\sqrt{2}x+c_2)+\frac{1}{2}e^{x^2}$,

Ex. 10. *Solve* $\dfrac{d^2y}{dx^2}-2bx\dfrac{dy}{dx}+b^2x^2y=0.$ **[Agra 56; Gujrat 61]**

Solution. Here $P=-2bx$, $Q=b^2x^2$, $X=0$.

To remove the first derivative, choose

$y=e^{-\frac{1}{2}\int P\,dx}=e^{\int bx\,dx}=e^{\frac{1}{2}bx^2}$.

If $Q_1=Q-\frac{1}{4}P^2-\frac{1}{2}dP/dx=b^2x^2-b^2x^2+b=b$, $X_1=X/y_1=0$,
the transformation $y=vy_1$ reduces the given equation to

$$\frac{d^2v}{dx^2}+Q_1v=X_1 \quad\text{or}\quad \frac{d^2v}{dx^2}+bv=0 \quad\text{or}\quad (D^2+b)\,v=0.$$

$\therefore \quad v=c_1 \cos (\sqrt{b}x+c_2)$ as $D=\pm\sqrt{b}i$.

Therefore the complete solution is $y=vy_1$

i.e. $\quad y=e^{\frac{1}{2}bx^2}\,c_1 \cos (\sqrt{b}x+c_2)$.

Ex. 11. *Solve* $\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}+(x^2-8)\ y=x^2e^{-\frac{1}{2}x^2}$. **[Gujrat 61]**

Solution. Choose $y_1=e^{-\int\frac{1}{2}P\,dx}=e^{-\frac{1}{2}x^2}$.

$Q_1=Q-\frac{1}{4}P^2-\frac{1}{2}\,dP/dx=(x^2-8)-x^2-1=-9,\ X_1=X/y_1=x^2$.

Hence equation in normal form is

$$\frac{d^2v}{dx^2}-9v=x^2,\ i.e.\ (D^2-9)\ v=x^2.$$

$$\text{P. I.}=\frac{x^2}{D^2-9}=-\tfrac{1}{9}\left(1-\tfrac{1}{9}D^2\right)^{-1}x^2=-\tfrac{1}{9}\left(1+\tfrac{1}{9}D^2+\ldots\right)x^2$$
$$=-\tfrac{1}{9}\left(x^2+\tfrac{2}{9}\right).$$

Hence $v=c_1e^{3x}+c_2e^{-3x}-\frac{1}{9}x^2-\frac{2}{81}$.

Complete solution is $y=vy_1$, *i e* $y=e^{-\frac{1}{2}x^2}\ v$.

***Ex. 12.** *Solve* $x^2\dfrac{d^2y}{dx^2}-2\ (x^2+x)\dfrac{dy}{dx}+(x^2+2x+2)\ y=0.$

[Agra 57; Nag. 61; Poona (Gen.) 60; Delhi Hons. 62; Meerut 73]

Solution. Dividing by x^2, the equation in standard form is

$$\frac{d^2y}{dx^2}-2\left(1+\frac{1}{x}\right)\frac{dy}{dx}+\left(1+\frac{2}{x}+\frac{x}{x^2}\right)y=0.$$

Here $P=-2\left(1+\dfrac{1}{x}\right),\ Q=\left(1+\dfrac{2}{x}+\dfrac{2}{x^2}\right),\ X=0.$

To remove first derivative, choose

$$y_1=e^{-\frac{1}{2}\int P\ dx}=e^{\int\left(1+\frac{1}{x}\right)dx}=e^{x+\log\ x}=xe^x.$$

Putting $y=vy_1$ the transformed equation becomes

$$\frac{d^2v}{dx^2}+Q_1v=0 \text{ as } X=0,$$

where $\quad Q_1=Q-\frac{1}{4}P^2-\frac{1}{2}\dfrac{dP}{dx}=1+\dfrac{2}{x}+\dfrac{2}{x^2}-\left(1+\dfrac{1}{x}\right)^2-\dfrac{1}{x^2}=0.$

$\therefore$ Reduced equation is $\dfrac{d^2v}{dx^2}=0.$

Integrating, $\dfrac{dv}{dx}=c_1$, and then $v=c_1x+c_2$.

The complete solution is $y=vy_1=xe^x\ (c_1x+c_2)$.

Ex. 13. *Solve* $x\dfrac{d}{dx}\left(x\dfrac{dy}{dx}-y\right)-2x\dfrac{dy}{dx}+2y+x^2y=0.$ **[Raj. 57]**

Solution. The equation after simplification becomes

$$x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+(x^2+2)\ y=0$$

or in standard form, $\dfrac{d^2y}{dx^2}-\dfrac{2}{x}\dfrac{dy}{dx}+\left(1+\dfrac{2}{x^2}\right)y=0.$

To remove the first derivative choose $y_1 = e^{-\frac{1}{2}\int -\frac{2}{x}dx} = x$.
Then transformation $y = vy_1$ reduces the given equation to

$$\frac{d^2v}{dx^2} + Q_1 v = X_1,$$

where $Q_1 = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} = \left(1 + \frac{2}{x^2}\right) - \frac{1}{x^2} - \frac{1}{x^2} = 1$, $X_1 = \frac{X}{y_1} = 0$.

Thus the reduced equation is

$$\frac{d^2v}{dx^2} + v = 0 \quad \text{or} \quad (D^2+1)\, v = 0.$$

$\therefore \quad v = c_1 \cos(x + c_2)$ as $D = \pm i$.

Hence the complete solution is $y = vy_1$

or $\quad y = xc_1 \cos(x + c_2)$.

Ex. 14. *Solve* $\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \left(n^2 + \frac{2}{x^2}\right) y = 0.$ **[Karnatak 1963]**

Hint. Proceed as in the above example, $y_1 = x$.

Reduced equation is $\frac{d^2v}{dx^2} + n^2 v = 0$.

Complete solution is $y = xc_1 \cos(nx + c_2)$.

Ex. 15. *Solve* $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} - n^2 y = 0.$ **[Lucknow 1951 ; Agra 49]**

Solution. To remove first derivative choose $y_1 = e^{-\frac{1}{2}\int P\,dx}$,

i.e. $\quad y_1 = e^{-\frac{1}{2}\int \frac{2}{x}dx} = e^{-\int \frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$.

$$Q_1 = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} = -n^2 - \frac{1}{x^2} + \frac{1}{x^2} = -n^2,\ X_1 = 0.$$

Hence the reduced equation is $\frac{d^2v}{dx^2} - n^2 v = 0$,

$(D^2 - n^2)\, v = 0$, $D = \pm n$

$\therefore \quad v = c_1 e^{nx} + c_2 e^{-nx}$.

The complete solution is $y = vy_1 = (c_1 e^{nx} + c_2 e^{-nx})/x$.

Ex. 16. *Solve* $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + n^2 y = 0.$ **[Karnatak 1960]**

Hint. As in the above example, choose $y_1 = 1/x$.

$\therefore \ Q_1 = n^2$, $X_1 = 0$.

Reduced equation is $\frac{d^2v}{dx^2} + n^2 v = 0$. $D = \pm ni$.

$v = c_1 \cos(nx + c_2)$.

Complete solution is $y = vy_1 = c_1 \cos(nx + c_2)/x$.

Ex. 17. *Solve* $\dfrac{d^2y}{dx^2}+\dfrac{2}{x}\dfrac{dy}{dx}+y=\dfrac{\sin 2x}{x}.$ **[Delhi Hons. 1959]**

Just like above example.

Ex. 18. *Solve* $\dfrac{d^2y}{dx^2}+\dfrac{1}{x^{1/3}}\dfrac{dy}{dx}+\left(\dfrac{1}{4x^{2/3}}-\dfrac{1}{6x^{4/3}}-\dfrac{6}{x^2}\right)y=0.$

[Rajasthan 1951 ; Punjab 60]

Solution. Here $P=\dfrac{1}{x^{1/3}},\ Q=\dfrac{1}{4x^{2/3}}-\dfrac{1}{6x^{4/3}}-\dfrac{6}{x^2},\ X=0.$

To remove the first derivative, choose

$$y_1=e^{-\frac{1}{2}\int P\,dx}=e^{-\frac{1}{2}\int\frac{1}{x^{1/3}}dx}=e^{-\frac{3}{4}x^{2/3}}.$$

Putting $y=vy_1$ the given equation becomes

$$\frac{d^2v}{dx^2}+Q_1v=0 \text{ as } X=0,$$

where $Q_1=Q-\frac{1}{4}P^2-\frac{1}{2}\dfrac{dP}{dx}$

$$=\left(\frac{1}{4x^{2/3}}-\frac{1}{6x^{4/3}}-\frac{6}{x^2}\right)-\frac{1}{4}\cdot\frac{1}{x^{2/3}}-\frac{1}{2}\left(-\frac{1}{3x^{4/3}}\right)=-\frac{6}{x^2}.$$

$\therefore$ The equation becomes

$$\frac{d^2v}{dx^2}-\frac{6}{x^2}v=0 \quad\text{or}\quad x^2\frac{d^2v}{dx^2}-6v=0$$

This is a homogeneous equation. To solve it put $x=e^z$, $D\equiv d/dz$, then the equation becomes

$$D(D-1)\,v-6v=0 \qquad\text{or}\qquad [D^2-D-6]\,v=0.$$

A.E. is $D^2-D-6=0$ or $(D+2)(D-3)=0,\ D=-2,\ 3.$

$\therefore$ Solution is $v=c_1e^{-2z}=c_2e^{3z}=c_1x^{-2}+c_2x^3.$

Hence the complete solution is $y=vy_1$, *i.e.*

$$y=(c_1x^{-2}+c_2x^3)\,e^{-\frac{4}{3}x^{2/3}}.$$

Ex. 19. *Solve* $\dfrac{d^2y}{dx^2}-2\tan x\dfrac{dy}{dx}-(a^2+1)\,y=0.$

Solution. Here $P=-2\tan x,\ Q=-(a^2+1),\ X=1$

To remove first derivative, we choose

$$y_1=e^{-\frac{1}{2}\int P\,dx}=e^{\int\tan x\,dx}=e^{\log\sec x}=\sec x.$$

Putting $y=vy_1$, the equation becomes

$$\frac{d^2v}{dx^2}+Q_1v=0 \text{ as } X=0,$$

where $Q_1=Q-\frac{1}{4}P^2-\frac{1}{2}\dfrac{dP}{dx}=-(a^2+1)-\tan^2 x+\sec^2 x$

$$=-a^2-(1+\tan^2 x)+\sec^2 x=-a^2.$$

Hence the reduced equation is $\dfrac{d^2v}{dx^2}-a^2v=0$

or $(D^2-a^2)\,v=0$ or $(D-a)\,(D+a)\,v=0$.

$\therefore\quad v=c_1e^{ax}+c_2e^{-ax}$.

The complete solution is $y=vy_1$,

i.e. $y=(c_1e^{ax}+c_2e^{-ax})\sec x$.

Ex. 20. *Solve* $\dfrac{d^2y}{dx^2}+2n\cot nx\,\dfrac{dy}{dx}+(a^2-n^2)\,y=0$.

Solution. $P=2n\cot nx$, $Q=a^2-n^2$, $X=0$.

To remove first derivative choose

$$y_1=e^{-\frac{1}{2}\int P\,dx}=e^{-\int n\cot nx\,dx}=e^{-\log\sin nx}=1/\sin nx.$$

Putting $y=vy_1$ the reduced equation is

$$\frac{d^2v}{dx^2}+Q_1v=0,$$

where $Q_1=Q-\frac{1}{4}P^2-\frac{1}{2}\dfrac{dP}{dx}=(a^2-n^2)-n^2\cot^2 nx+n^2\operatorname{cosec}^2 nx$

$$=a^2-n^2\,(1+\cot^2 nx)+n^2\operatorname{cosec}^2 nx=a^2$$

Hence the reduced equation becomes

$$\frac{d^2v}{dx^2}+a^2v=0\quad\text{or}\quad (D^2+a^2)\,v=0,\ D=\pm ai.$$

$\therefore\quad v=c_1\cos(ax+c_2)$.

The complete solution is $y=vy_1$,

i.e. $y=c_1\cos(ax+c_2)\sin nx$

Ex. 21. *Solve* $\dfrac{d^2y}{dx^2}-\dfrac{1}{x^{1/2}}\dfrac{dy}{dx}+\dfrac{y}{4x^2}(-8+x^{1/2}+x)=0$.

[Punjab 1960; Delhi Hons. 53]

Solution. Choose y_1 such that $y_1=e^{-\frac{1}{2}\int P\,dx}$, i.e.

$$y_1=e^{\frac{1}{2}\int x^{-1/2}\,dx}=e^{x/2}.$$

$$Q_1=Q-\tfrac{1}{4}P^2-\tfrac{1}{2}\frac{dP}{dx}=\frac{1}{4x^2}(-8+x^{1/2}+x)-\frac{1}{4x}+\frac{1}{4x^{3/2}}$$

$$=-\frac{2}{x^2},\ X_1=0.$$

Hence after removing the first derivative, the equation is

$$\frac{d^2v}{dx^2}-\frac{2}{x^2}v=0,\quad\text{or}\quad x^2\frac{d^2v}{dx^2}-2v=0.$$

Homogeneous. Put $x=e^z$, $D\equiv d/dz$; then equation becomes

$[D\,(D-1)-2]\,v=0$,

i.e. $[D^2-D-2]\,v=0$, $(D-2)\,(D+1)\,v=0$.

$\therefore\quad v=c_1e^{2z}+c_2e^{-z}=(c_1x^2+c_2x^{-1})$ as $x=e^z$.

Hence complete solution is $y=vy_1$,

i.e. $y=(c_1x^2+c_2x^{-1})\,e^{x/2}$ as $y_1=e^{x/2}$.

Ex. 22. *Solve* $4x^2 \dfrac{d^2y}{dx^2}+4x^5 \dfrac{dy}{dx}+(x^8+6x^4+4)\, y=0.$

[Vikram 1962]

Solution. $P=x^3$, $Q=\frac{1}{4}\,(x^6+6x^2+4/x^2)$.

Choose y_1 such that $y=e^{-\frac{1}{2}\int x^3\, dx}=e^{-\frac{1}{8}x^4}$.

$$Q_1=Q-\tfrac{1}{4}P^2-\tfrac{1}{2}\frac{dP}{dx}=\tfrac{1}{4}\,(x^6+6x^2+4/x^2)-\tfrac{1}{4}x^6-\tfrac{1}{2}.3x^2$$
$$=1/x^2.$$

Hence the equation after removing first derivative is

$$\frac{d^2y}{dx^2}+\frac{1}{x^2}\,v=0 \quad \text{or} \quad x^2\,\frac{d^2v}{dx^2}+v=0.$$

Homogeneous; put $x=e^z$; equation is $[D\,(D-1)+1]\,v=0$.

A.E. is $(D^2-D+1)=0$, $D=\frac{1}{2}\pm\frac{1}{2}\sqrt{3}i$.

Hence $v=c_1e^{z/2}\cos\,(\frac{1}{2}\sqrt{3}z+c_2)$.

The complete solution is $y=vy_1$,

i.e. $y=c_1\sqrt{x}\,\cos\,(\frac{1}{2}\sqrt{3}\,\log x+c_2)\times e^{-\frac{1}{8}x^4}$ as $x=e^z$.

Ex. 23. *Solve* $x^2\,\dfrac{d^2y}{dx^2}-2x\,(3x-2)\,\dfrac{dy}{dx}+3x\,(3x-4)\,y=e^{3x}.$

[Poona 1959]

Solution. The equation is

$$\frac{d^2y}{dx^2}-2\left(3-\frac{2}{x}\right)\frac{dy}{dx}+\left(9-\frac{12}{x}\right)y=\frac{e^{3x}}{x^2}.$$

Choose $y_1=e^{-\frac{1}{2}\int P\, dx}=e^{(3x-2\log x)}=e^{3x}.\ 1/x^2$,

$$Q_1=Q-\tfrac{1}{4}P^2-\tfrac{1}{2}\frac{dP}{dx}=-\frac{2}{x^2},\ X_1=\frac{X}{y_1}=\frac{e^{3x}/x^2}{e^{3x}/x^2}=1.$$

Hence after removing first derivative, the equation becomes

$$\frac{d^2v}{dx^2}-\frac{2}{x^2}v=1 \quad \text{or} \quad x^2\,\frac{d^2v}{dx^2}-2v=x^2.$$

Homogeneous; put $x=e^z$, $D\equiv d/dz$.

$\therefore\ [D\,(D-1)-2]\,v=e^{2z}$, A.E. $(D-2)\,(D+1)=0$.

C.F. $=c_1e^{2z}+c_2e^{-z}=c_1x^2+c_2x^{-1}$.

$$\text{P.I.}=\frac{e^{2z}}{D^2-D-1}=\frac{ze^{2z}}{2D-1},\ \text{case of failure}.$$

$$\frac{ze^{2z}}{4-1}=\tfrac{1}{3}ze^{2z}=\tfrac{1}{3}x^2\log x,\ \text{as } x=e^z.$$

Hence $v=c_1x^2+c_2x^{-1}+\frac{1}{3}x^2\log x$.

Therefore the complete solution is $y=vy_1$,

i.e. $y=\dfrac{1}{x^2}\,e^{3x}\cdot\,(c_1x^2+c_2x^{-1}+\frac{1}{3}x^2\log x)$.

Ex. 24. *Solve* $\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+(x^2+1)\,y=x\,(x^2+3).$

Solution. Choose $y_1=e^{-\frac{1}{2}\int P\,dx}=e^{-\frac{1}{2}x^2}.$

$$Q_1=Q-\tfrac{1}{4}P^2-\tfrac{1}{2}\frac{dX}{dx}=0,\ X_1=\frac{X}{y_1}=x\ (x^2+3)\ e^{\frac{1}{2}x^2}.$$

Normal form equation is $\frac{d^2v}{dx^2}=x\ (x^2+3)\ e^{\frac{1}{2}x^2}.$

Integrating $\frac{dv}{dx}=c_1+\int x\,(x^2+3)\ e^{\frac{1}{2}x^2}\,dx$, put $\frac{1}{2}x^2=t$

$$=c_1+\int e^t\,(2t+3)\,dt=e^t\,(2t+1)$$

$$=c_1+e^{\frac{1}{2}x^2}(x^2+1).$$

Integrating again $v=c_1x+c_2+xe^{\frac{1}{2}x^2}.$
The complete solution is

$$y=vy_1=e^{-\frac{1}{2}x^2}\ (c_1x+c_2+xe^{\frac{1}{2}x^2}).$$

Ex. 25 (a) *Solve* $\frac{d^2y}{dx^2}-\frac{2}{x}\left(\frac{dy}{dx}\right)+\left(a^2+\frac{2}{x^2}\right)y=0.$

[Calcutta Hons. 62]

Solution Choose $y_1=e^{-\frac{1}{2}\int P\,dx}=e^{\int\frac{1}{x}dx}=x.$

$$Q_1=Q-\tfrac{1}{4}P^2-\tfrac{1}{2}\frac{dP}{dx}=\left(a^2+\frac{2}{x^2}\right)-\frac{1}{x^2}-\frac{1}{x^2}=a^2.$$

Hence after removing first derivative, the equation is

$$\frac{d^2v}{dx^2}+a^2v=0\quad\text{or}\quad(D^2+a^2)\ v=0,$$

$$v=c_1\cos\,(ax+c_2).$$

The complete solution is $y=vy_1$,

i.e. $y=xc_1\cos\,(ax+c_2)$

Ex. 25. (b) *Solve* $\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}+\left(1+\frac{2}{x^2}\right)\ y=xe^x.$

[Guru Nanak 73]

Ex. 26. *Solve* $x^2\frac{d^2y}{dx^2}+(x-4x^2)\frac{dy}{dx}+(1-2x+4x^2)\,y=0.$

Solution. Choose $y_1=e^{-\int\frac{1}{2}\left(\frac{x-4x^2}{x^2}\right)dx}=x^{-1/2}.e^{2x}.$

$$Q_1=Q-\tfrac{1}{4}P^2-\tfrac{1}{2}\frac{dP}{dx}=\frac{5}{4x^2}.$$

Hence normal form is $\frac{d^2v}{dx^2}+\frac{5}{4x^2}\ v=0.$

Putting $x=e^z$, $[4D(D-1)+5]\,v=0$, where $D\equiv d/dz$.
A.E. is $4D^2-4D+5=0$, $D=\frac{1}{2}\pm i$,

$v=e^{\frac{1}{2}z}\, c_1 \cos(z+c_2)$ etc.

Ex. 27. *Solve* $\left(\dfrac{d^2y}{dx^2}+y\right)\cot x+2\left(\dfrac{dy}{dx}+y\tan x\right)=\sec x$.

Solution Equation is

$$\frac{d^2y}{dx^2}+2\tan x\,\frac{dy}{dx}+y\,(1+2\tan^2 x)=\sec x\tan x.$$

Choose $y_1=e^{-\int \frac{1}{2} Pdx}=\cos x$, $Q_1=0$, $X_1=X/y_1=\sec^2 x\tan x$.

Normal form is $\dfrac{d^2v}{dx^2}=\sec^2 x\tan x$.

Integrating $\dfrac{dv}{dx}=\frac{1}{2}\tan^2 x+c_1=\frac{1}{2}(\sec^2 x-1)+c_1$.

Integrating again,

$v=\frac{1}{2}(\tan x-x)+c_1x+c_2$.

Complete solution is $y=vy_1=$ etc.

Ex. 28. *Solve* $x^2(\log x)^2\dfrac{d^2y}{dx^2}-2x\log x\dfrac{dy}{dx}$

$$+[2+\log x-2(\log x)^2]\,y=x^2(\log x)^3$$

Solution. $P=-\dfrac{2}{x\log x}$, $Q=\dfrac{2+\log x-2(\log x)^2}{x^2(\log x)^2}$, $X=\log x$.

Choose $y_1=e^{-\frac{1}{2}\int Pdx}=e^{\log(\log x)}=\log x$.

$$Q_1=Q-\tfrac{1}{4}P^2-\tfrac{1}{2}\frac{dP}{dx}=-\frac{2}{x^2},\quad X_1=\frac{X}{y_1}=1.$$

$\therefore$ After removing first derivative, the equation becomes

$$\frac{d^2v}{dx^2}-\frac{2}{x^2}v=1 \quad\text{or}\quad x^2\frac{d^2v}{dx^2}-2v=x^2.$$

This is differential equation of Ex. 23 P. 72.

2·7. Method of changing the independent variable.

[Karnatak 61; Gujrat 58; Delhi Hons. 64; Meerut 76; Sagar 64; Poona 64, 62]

Sometimes by the change of independent variable, the equation may become easily integrable

Let the equation of second order be

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=X. \qquad \text{...(1)}$$

If we change independent variable from x to z with the help of the relation $z=f(x)$, then

$$\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx},\quad \frac{d^2y}{dx^2}=\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2+\frac{dy}{dz}\cdot\frac{d^2z}{dx^2}.$$

Putting these values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2+\frac{dy}{dz}\cdot\frac{d^2z}{dx^2}+P\left[\frac{dy}{dz}\cdot\frac{dz}{dx}\right]+Qy=X$$

or $$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2+\frac{dy}{dz}\left(\frac{d^2z}{dx^2}+P\frac{dz}{dx}\right)+Qy=X$$

or $$\frac{d^2y}{dz^2}=\frac{\frac{d^2z}{dx^2}+P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}\frac{dy}{dz}+\frac{Qy}{\left(\frac{dz}{dx}\right)^2}=\frac{X}{\left(\frac{dz}{dx}\right)^2}$$

or $$\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1y=X_1,$$

where $\mathbf{P_1}=\left(\frac{d^2z}{dx^2}+P\frac{dz}{dx}\right)\bigg/\left(\frac{dz}{dx}\right)^2$, $Q_1=Q\bigg/\left(\frac{dz}{dx}\right)^2$ and $X_1=X\bigg/\left(\frac{dz}{dx}\right)^2$

Here P_1, Q_1, X_1 are functions of x and can be expressed as functions of z with the help of the relation $z=f(x)$.

How to choose z ? After obtaining equation (2) we would like to choose z in such a way that (2) can be easily integrated.

Case I. $P_1=0$. [Bombay 61]

If we choose z such that

$$P_1=0, \text{ i.e. } \frac{d^2z}{dx^2}+P\frac{dz}{dx}=0, \text{ i.e. } \frac{d}{dx}\left(\frac{dz}{dx}\right)+P\frac{dz}{dx}=0$$

then $\frac{dz}{dx}=e^{-\int P\,dx}$ or $z=\int\left[e^{-\int P\,dx}\right]dx.$

The equation (2) consequently becomes $\frac{d^2y}{dz^2}+Q_1y=X_1.$

This equation can be solved, if

(*i*) Q_1 is a constant (then it being a linear equation with constant coefficients),

or (*ii*) Q_1 is of the form k/z^2 (then it being a linear homogeneous equation with variable coefficients).

Case II. $Q_1=a^2$. We can choose z such that $Q_1=a^2$,

i.e. $Q\bigg/\left(\frac{dz}{dx}\right)^2=\pm a^2$ or $a\frac{dz}{dx}=\sqrt{(\pm Q)}$,*

or $az=\int\sqrt{(\pm Q)}\,dx.$

With this choice the equation (`) becomes

$$\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+a^2y=X_1.$$

* +ive or −ive sign is taken to form the expression under the radical sign +ive.

This can be readily integrated if P_1 comes out to be a constant in which case this equation is a linear equation with constant coefficients.

Note (1). The student should remember the values of P_1, Q_1 and R_1 for immediate use.

Note (2). Only two choices, $P_1=0$ or $Q_1=a^2$, should be made. Sometimes it is possible to make both the choices to get the solution of the given equation.

The following few examples will fully illustrate the method.

Ex. 1. *Solve* $\dfrac{d^2y}{dx^2}-(1+4e^x)\dfrac{dy}{dx}+3e^{2x}\,y=e^{2(x+e^x)}$. [Meerut 75]

Solution. Here $P=-(1+4e^x)$, $Q=3e^{2x}$. Changing the independent variable x to z, the equation becomes

$$\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1y=X_1,$$

where $P_1=\left(\dfrac{d^2z}{dx^2}+P\dfrac{dz}{dx}\right)\Big/\left(\dfrac{dz}{dx}\right)^2$, $Q_1=Q\Big/\left(\dfrac{dz}{dx}\right)^2$, $X_1=X\Big/\left(\dfrac{dz}{dx}\right)^2$.

Let us choose z such that $Q_1=a^2+3$ (say); then

$$Q=3\left(\frac{dz}{dx}\right)^2 \quad\text{or}\quad \frac{dz}{dx}=e^x \text{ and } z=e^x.$$

Putting $z=e^x$, $P_1=[e^x-(1+4e^x)\,e^x]/(e^x)^2=-4$

and $X_1=\dfrac{X}{(dz/dx)^2}=\dfrac{e^{2(x+e^x)}}{e^{2x}}=e^{2e^x}=e^{2z}$.

Hence the equation (when $z=e^x$) reduces to

$$\frac{d^2y}{dz^2}-4\frac{dy}{dz}+3y=e^{2z}$$

or $(D^2-4D+3)\,y=e^{2z}$, $D\equiv d/dz$.

A.E. is $D^2-4D+3=0$, *i.e.* $(D-3)(D-1)=0$,

C.F. is $y=c_1e^z+c_2e^{3z}$.

$$\text{P.I.}=\frac{e^{2z}}{D^2-4D+3}=\frac{e^{2z}}{2^2-4.2+3}=\frac{e^{2z}}{-1}.$$

Hence $y=c_1e^z+c_2e^{3z}-e^{2z}$.

Putting $z=e^x$, $y=c_1e^{e^x}+c_2e^{3e^x}-e^{2e^x}$ is the required solution.

Note. We have taken $a^2=3$ for convenience only; any other positive value of a^2 would lead to the same result.

Ex. 2. *Solve* $(1+x^2)^2\dfrac{d^2y}{dx^2}+2x(1+x^2)\dfrac{dy}{dx}+4y=0$.

[Agra 1961, 73]

Solution. The equation in the standard form is

$$\frac{d^2y}{dx^2}+\frac{2x}{1+x^2}\frac{dy}{dx}+\frac{4}{(1+x^2)^2}y=0.$$

Here $P=\frac{2x}{1+x^2}$, $Q=\frac{4}{(1+x^2)^2}$, $X=0$.

Changing the independent variable from x to z, the equation becomes

$$\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1y=0$$

where $P_1=\left(\frac{d^2z}{dx^2}+P\frac{dz}{dx}\right)\Big/\left(\frac{dz}{dx}\right)^2, Q_1=Q\Big/\left(\frac{dz}{dx}\right)^2.$

Let us choose z, such that $Q_1=\text{const.}=1$ (say).

$$\therefore \quad \left(\frac{dz}{dx}\right)^2=Q=\frac{4}{(1+x^2)^2}, \quad \frac{dz}{dx}=\frac{2}{(1+x^2)}, \quad z=2\tan^{-1}x.$$

Then $P_1=\dfrac{-\frac{4}{(1+x^2)^2}+\frac{2x}{1+x^2}\times\frac{2}{1+x^2}}{\{2/(1+x^2)\}^2}=0.$

Hence the r duced equation is

$$\frac{d^2y}{dz^2}+y=0 \quad \text{or} \quad (D^2+1)\,y=0,\ D=\pm i.$$

$$\begin{aligned}\therefore\quad y&=c_1\cos z+c_2\sin z\\ &=c_1\cos(2\tan^{-1}x)\\ &\quad+c_2\sin(2\tan^{-1}x)\\ &=c_1\frac{1-x^2}{1+x^2}+c_2\frac{2x}{1+x^2}\end{aligned}$$

$$\left\{\begin{aligned}&\text{Let } \tan^{-1}x=\theta, \text{ i.e. } x=\tan\theta,\\ &\cos(2\tan^{-1}x)=\cos 2\theta\\ &=\frac{1-\tan^2\theta}{1+\tan^2\theta}=\frac{1-x^2}{1+x^2}\text{etc.}\end{aligned}\right.$$

or $y\,(1+x^2)=c_1(1-x^2)+2c_2x$ is required solution.

Aliter. If we choose z such that $P_1=0$, *i.e.*

$$\frac{d^2z}{dx^2}+P\frac{dz}{dx}=0, \text{ i.e. } \frac{dp}{dx}+\frac{2x}{1+x^2}p=0, \text{ where } p=\frac{dz}{dx},$$

i.e. $\frac{dp}{p}+\frac{2x}{1+x^2}dx=0$ or $\log p=-\log(1+x^2)$.

$$\therefore\quad P=\frac{dz}{dx}=\frac{1}{1+x^2}=\tan^{-1}x.$$

Then $Q_1=Q\Big/\left(\frac{dz}{dx}\right)^2=\frac{4}{(1+x^2)^2}\Big/\frac{1}{(1+x^2)^2}=4$, a constant.

Hence the transformed equation is

$$\frac{d^2y}{dx^2}+4y=0 \quad \text{or} \quad (D^2+4)\,y=0.$$

$$\begin{aligned}\therefore\quad y&=c_1\cos 2z+c_2\sin 2z\\ &=c_1\cos(2\tan^{-1}x)+c_2\sin(2\tan^{-1}x)\end{aligned}$$

which is same as obtained when we supposed $Q_1=\text{constant}$.

Note. Almost all the examples can be solved by taking $Q_1=\text{constant}$ and $P_1=0$ like the above examples. We shall solve one more example by both the considerations. Students should adopt one method for practice. The case $Q_1=\text{constant}$ is usually easier.

***Ex. 3.** *Solve* $\cos x\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}\sin x-2y\cos^2 x=2\cos^5 x.$

[Delhi Hons. 1967 ; Raj. 64 ; Agra 77, 72, 60 ; Gujrat 58 ; Gauhati Hons. 64]

Solution. The equation in standard form is

$$\frac{d^2y}{dx^2}+\tan x\frac{dy}{dx}-2\cos^2 x\,.\,y=2\cos^4 x.$$

Changing the independent variables from x to z, the equation becomes $\dfrac{d^2y}{dz^2}+P_1\dfrac{dy}{dz}+Q_1y=X_1$,

where $P_1=\left[\dfrac{d^2z}{dx^2}+P\dfrac{dz}{dx}\right]\Big/\left(\dfrac{dz}{dx}\right)^2$, $Q_1=\dfrac{Q}{(dz/dx)^2}$, $X_1=\dfrac{X}{(dz/dx)^2}$

Let us choose z such that $Q_1=\text{constant}=-2$ (say)*

i.e. $2(dz/dx)^2=-Q=-2\cos^2 x$, *i.e.* $dz/dx=\cos x$.

Integrating, $z=\sin x$.

Now $P_1=\dfrac{-\sin x+\tan x\cos x}{\cos^2 x}=0$

and $X_1=2\cos^4 x/\cos^2 x=\cos^2 x$.

Hence the reduced equation is

$$\frac{d^2y}{dx^2}-2y=2\cos^2 x,\ (D^2-2)\,y=2\,(1-z^2).$$

A.E. is $D^2-2=0$, $D=\pm\sqrt{2}$, C.F.$=c_1e^{\sqrt{2}z}+c_2e^{-\sqrt{2}z}$.

$$\text{P.I.}=\frac{2\,(1-z^2)}{D^2-2}=-\left(1-\frac{D^2}{2}\right)^{-1}(1-z^2)$$

$$=-\left[1+\frac{D^2}{2}-\ldots\ldots\right](1-z^2)=-(1-z^2)+1=z^2.$$

(Using Binomial Theorem)

Hence $y=c_1e^{\sqrt{2}z}+c_2e^{-\sqrt{2}z}+z^2$

$=c_1e^{\sqrt{2}\sin x}+c_3e^{-\sqrt{2}\sin x}+\sin^2 x$, as $z=\sin x$.

Aliter. We may choose z such that $P_1=$,

i.e. $\dfrac{d^2z}{dx^2}+P\dfrac{dz}{dx}=0$, $\dfrac{dp}{dx}+p\tan x=0$, $p=\dfrac{dz}{dx}$;

then $\dfrac{dp}{p}=-\tan x\,dx$, *i.e.* $\log p=\log\cos x$,

or $p=\dfrac{dz}{dx}=\cos x$; $z=\sin x$

and now proceed as in the above case to get the same solution.

Ex. 4. *Solve* $\dfrac{d^2y}{dx^2}-\cot x\dfrac{dy}{dx}-\sin^2 x\,.\,y=\cos x-\cos^3 x.$

[Rajasthan 54 ; Karnatak 62]

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*A negative value of Q_1 is taken to make $(dz/dx)^2$ +ve.

Solution Here $P=-\cot x$, $Q=-\sin^2 x$,
$X=\cos x-\cos^3 x$

Changing the independent variable from x to z, the equation becomes $\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1y=X_1$,

where $P_1=\left(\frac{d^2z}{dx^2}+P\frac{dz}{dx}\right)\Big/\left(\frac{dz}{dx}\right)^2$, $Q_1=Q\Big/\left(\frac{dz}{dx}\right)^2$, $X_1=X\Big/\left(\frac{dz}{dx}\right)^2$

Let us choose z such that $Q_1=\pm a^2=-1$ (say) ; then

$$Q=-\left(\frac{dz}{dx}\right)^2, \text{ i.e } \left(\frac{dz}{dx}\right)^2=-Q=\sin^2 x$$

or $\frac{dz}{dx}=\sin x$, $z=-\cos x$.

Then $P_1=(\cos x-\cot x.\ \sin x)/\sin^2 x=0$,

and $X_1=(\cos x-\cos^3 x)/\sin^2 x=\cos x\,(1-\cos^2 x)/\sin^2 x$
$=\cos x=-z$.

Hence the reduced equation is

$$\frac{d^2y}{dz^2}-y=-z \text{ or } (D^2-1)\,y=-z,\ D\equiv\frac{d}{dz}$$

A.E is $D^2-1=0$, $D=\pm 1$, C.F.$=c_1e^z+c_2e^{-z}$.

$$\text{P.I.}=\frac{-z}{D^2-1}=z.$$

Hence $y=c_1e^z+c_1e^{-z}+z$
$=c_1e^{-\cos x}+c_2e^{\cos x}-\cos x$ as $z=-\cos x$.

Ex. 5. *Solve* $\frac{d^2y}{dx^2}+(3\sin x-\cot x)\frac{dy}{dx}+2y\sin^2 x$
$=e^{\cos x}\sin^2 x$. **Raj. 57 ; Bombay 61]**

Solution. Here
$P=(3\sin x-\cot x)$, $Q=2\sin^2 x$, $X=e^{-\cos x}\sin^2 x$.

Changing independent variable from x to z, the equation becomes $\frac{d^2y}{dx^2}+P_1\frac{dy}{dz}+Q_1y=X_1$;

choose z such that $Q_1=Q/(dz/dx)^2=a^2=2$ (say)

i.e. $dz/dx=\sin x$, $z=-\cos x$,

$$P_1=\left(\frac{d^2z}{dx^2}+P\frac{dz}{dx}\right)\Big/\left(\frac{dz}{dx}\right)^2=\frac{\cos x+(3\sin x-\cot x)\sin x}{\sin^2 x}=3,$$

$X_1=X/(dz/dx)^2=e^{-\cos x}\sin 2x/\sin 2x=e^{-\cos x}=e^z$.

Hence the transformed equation is

$$\frac{d^2y}{dz^2}+3\frac{dy}{dz}+2y=e^z, \text{ i.e. } (D^2+3D+2)\,y=e^z.$$

A.E. is $(D^2+3D+2)=0$; $(D+2)(D+1)=0$, $D=-2,-1$.

C.F.$=c_1e^{-z}+c_2e^{-2z}$, P.I.$=e^z/(D^2+3D+2)=\frac{1}{6}e^z$.

Hence $y=c_1e^{-z}+c_2e^{-2z}+\frac{1}{6}e^{z}$.
$=c_1e^{\cos x}+c_2e^{2\cos x}+\frac{1}{6}e^{-\cos x}$

is the complete solution.

*Ex. 6. *Solve* $\frac{d^2y}{dx^2}+\cot x\frac{dy}{dx}+4y \operatorname{cosec}^2 x=0$.

[Vikram 63 ; Agra 74, 63, 55 ; Raj. 56 ; Delhi 72, 68, 59 ; Karnatak 63]

or $\sin^2 x\frac{d^2y}{dx^2}+\sin x\cos x\frac{dy}{dx}+4y=0$. [Rajasthan 63]

Solution. Here $P=\cot x$, $Q=4 \operatorname{cosec}^2 x$, $X=0$.

Changing the independent variable from x to z, the equation because $\frac{d^2y}{dz^2}+P_1\frac{dy}{dx}+Q_1y=0$, as $X=0$.

where $P_1=\left(\frac{d^2z}{dx^2}+P\frac{dz}{dx}\right)\Big/\left(\frac{dz}{dx}\right)^2$ and $Q_1=Q\Big/\left(\frac{dz}{dx}\right)$.

Let us choose z such $Q_1=a^2=1$ (say) ; then

$\left(\frac{dz}{dx}\right)^2=Q=4 \operatorname{cosec}^2 x$, $\frac{dz}{dx}=2 \operatorname{cosec} x$,

$z=2 \log \tan \frac{1}{2}x$.

Then $P_1=\frac{(-2 \operatorname{cosec} x \cot x+2 \cot x . \operatorname{cosec} x)}{(4 \operatorname{cosec}^2 x)}=0$.

Hence the transf rmed equation $d^2y/dz^2+y=0$

or $(D^2+1)y=0$, $D=\pm i$, $y=c_1 \cos(z+c_2)$

or $y=c_1 \cos(2 \log \tan \frac{1}{2}x+c_2)$

is the required solution.

*Ex. 7. *Salve* $\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}+\frac{a^2}{x^4}y=0$.

[Agra 56 ; Karnatak 61 ; Poona 60]

Solution. Hete $P=\frac{2}{x}$, $Q=\frac{a^2}{x^4}$, $X=0$

Changing the independent variable from x to z, the equation becomes $\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1y=0$,

where $P_1=\left(\frac{d^2z}{dx^2}+P\frac{dz}{dx}\right)\Big/\left(\frac{dz}{dx}\right)^2$, $Q_1=\frac{Q}{(dz/dx)^2}$.

Now choose z such that $Q_1=\text{const.}=a^2$ (say),

i.e. $a^2\left(\frac{dz}{dx}\right)^2=Q=\frac{a^2}{x^4}$, *i.e.* $\frac{dz}{dx}=\frac{1}{x^2}$, $z=-\frac{1}{x}$

Then $P_1=\frac{-\frac{2}{x^3}+\frac{2}{x}\cdot\frac{1}{x^2}}{(1/x^2)^2}=0$.

$\therefore$ The transformed equation is $d^2y/dx^2+a^2y=0$ or $(D^2+a^2)\,y=0$. A.E. is $(D^2+a^2)=0$, $D=\pm ai$.

$\therefore$ $y=c_1 \cos(az+c_2)=c_1 \cos(c_2-a/x)$ as $z=-1/x$.

Ex. 8. *Solve* $(x^3-x)\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+n^2x^3y=0.$ **[Agra 1960]**

Solution. The equation in the standard form is

$$\frac{d^2y}{dx^2}+\frac{1}{x\,(x^2-1)}\frac{dy}{dx}+\frac{n^2x^2}{x^2-1}y=0.$$

Here $P=\dfrac{1}{x\,(x^2-1)}$, $Q=\dfrac{n^2x^2}{x^2-1}$

Changing the independent variable from x to z, the equation becomes $\dfrac{d^2y}{dz^2}+P_1\dfrac{dy}{dz}+Q_1y=0$,

where $P_1=\left(\dfrac{d^2z}{dx^2}+P\right)\Big/\left(\dfrac{dz}{dx}\right)^2$, $Q_1=\dfrac{Q}{(dz/dx)^2}$.

Choose z such that $Q_1=$const.$=n^2$ (say), then

$$n^2\left(\frac{dz}{dx}\right)^2=Q=\frac{n^2x^2}{x^2-1} \text{ or } \frac{dz}{dx}=\frac{x}{\sqrt{(x^2-1)}}.$$

$\therefore$ $z=\sqrt{(x^2-1)}$.

Also $P_1=\left[\dfrac{x^2-1-x^2}{(x^2-1)^{3/2}}+\dfrac{1}{x\,(x^2-1)}\cdot\dfrac{x}{\sqrt{(x^2-1)}}\right]\Big/\left(\dfrac{dz}{dx}\right)^2=0.$

Hence the reduced equation is $d^2y/dz^2+n^2y=0$.

$\therefore$ $y=c_1\cos(nz+c_2)$

or $y=c_1 \cos[n\sqrt{(x^2-1)}+c_2]$.

Ex. 9. *Solve* $(a^2-x^2)\dfrac{d^2y}{dx^2}-\dfrac{a^2}{x}\dfrac{dy}{dx}+\dfrac{x^2}{a}y=0.$ **[Agra 57]**

Solution. The equation in the standard form is

$$\frac{d^2y}{dx^2}-\frac{a^2}{x\,(a^2-x^2)}\frac{dy}{dx}+\frac{x^2}{a\,(a^2-x^2)}y=0,$$

$$P=-\frac{a^2}{x\,(a^2-x^2)},\ Q=\frac{x^2}{a\,(a^2-x^2)}.$$

Changing the independent variable from x to z, the equation becomes $\dfrac{d^2y}{dz^2}+P_1\dfrac{dy}{dz}+Q_1y=0$,

where $P_1=\left(\dfrac{d^2z}{dx^2}+P\dfrac{dz}{dx}\right)\Big/\left(\dfrac{dz}{dx}\right)^2$, $Q_1=\dfrac{Q}{(dz/dx)^2}$.

Let us choose z, such that $Q_1=$const.$=1/a$ (say) : then

$$\frac{1}{a}\left(\frac{dz}{dx}\right)^2=Q=\frac{x^2}{a\,(a^2-x^2)}, \frac{dz}{dx}=\pm\frac{x}{\sqrt{(a^2-x^2)}}.$$

Then $P_1=\left[\left\{\frac{1}{\sqrt{(a^2-x^2)}}-\frac{x^2}{(a^2-x^2)^{3/2}}\right\}\right.$

$$\left.-\frac{a^2}{x\,(a^2-x^2)}\frac{x}{\sqrt{(a^2-x^2)}}\right]\Big/\left(\frac{dz}{dx}\right)^2=0.$$

Hence the transformed equation is

$$\frac{d^2y}{dz^2}+\frac{1}{a}y=0,\left(D^2+\frac{1}{a}\right)y=0,\ D=\pm\frac{1}{\sqrt{a}}i$$

$$y=c_1\cos\left(\frac{z}{\sqrt{a}}+c_2\right)=c_1\cos\left\{c_2\pm\sqrt{\left(\frac{a^2-x^2}{a}\right)}\right\}$$

***Ex. 10.** *Solve* $x^6\frac{d^2y}{dx^2}+3x^5\frac{dy}{dx}+a^2y=\frac{1}{x^2}$

[Vikram 64 ; Agra 53 ; Delhi Hons. 60, 57]

Solution. Equation is $\frac{d^2y}{dx^2}+\frac{3}{x}\frac{dy}{dx}+\frac{a^2}{x^6}y=\frac{1}{x^8}$.

$$P=\frac{3}{x},\ Q=\frac{a^2}{x^6},\ Y=\frac{1}{x^8}.$$

Changing the independent variable from x to z the equation is

$$\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1y=X_1,$$

where $P_1=\left(\frac{d^2z}{dx^2}+P\frac{dz}{dx}\right)\Big/\left(\frac{dz}{dx}\right)^2,\ Q_1=\frac{Q}{(dz/dx)^2},\ X_1=\frac{X}{(dz/dx)^2}.$

Let us choose z such that $Q_1=\text{constant}=a^2$ (say) ; then

$$a^2\left(\frac{dz}{dx}\right)^2=Q=\frac{a^2}{x^6},\ i.e.\ \frac{dz}{dx}=+\frac{1}{x^3},\ z=-\frac{1}{2x^2}.$$

$$P_1=\left[\left(-\frac{3}{x^4}\right)+\frac{3}{x}\cdot\frac{1}{x^3}\right]\Big/\left(\frac{dz}{dx}\right)^2=0,\ X_1=\frac{1/x^8}{1/x^6}=\frac{1}{x^2}=-2z.$$

Hence the transformed equation is

$d^2y/dz^2+a^2y=-2z$ or $(D^2+a^2)\,y=-2z$.

C.F.$=c_1\cos(az+c_2)$, P.I.$=-2z/(D^2+a^2)=-2z/a^2$.

$\therefore\ y=c_1\cos(az+c_2)-2z/a^2$

$$=c_1\cos\left(c_2-\frac{a}{2x^2}\right)+\frac{1}{a^2x^2}.$$

Ex. 11. *Solve* $\frac{d^2y}{dx^2}+(\tan x-3\cos x)\frac{dy}{dx}+2y\cos^2 x+\cos^4 x.$

[Rajasthan 57]

Solution. With usual notations putting

$$Q_1=2,\ 2\left(\frac{dz}{dx}\right)^2=2\cos^2 x,\ z=\sin x,$$

$$P_1=\frac{-\sin x+(\tan x-3\cos x)\cos x}{\cos^2 x}=-3,$$

and $\quad X_1=\dfrac{\cos^4 x}{\cos^2 x}=\cos^2 x=(1-z^2).$

Hence after changing variable from x to z, the equation becomes $\dfrac{d^2y}{dz^2}-3\dfrac{dy}{dz}+2y=(1-z^2).$

A.E. is $(D^2-3D+2)=0$, *i.e.* $(D-2)(D-1)=0.$

C.F.$=c_1e^{2z}+c_2e^z.$

$$\text{P.I.}=\frac{1}{(D^2-3D+2)}(1-z^2)=\tfrac{1}{2}(1-\tfrac{3}{2}D+\tfrac{1}{2}D^2)^{-1}(1-z^2)$$
$$=\tfrac{1}{2}(1+\tfrac{3}{2}D-\tfrac{1}{2}D^2+\tfrac{9}{4}D^2+\ldots)(1-z^2)$$
$$=\tfrac{1}{2}(1-z^2)-\tfrac{3}{2}.\ z-\tfrac{7}{4}=-\tfrac{1}{2}z^2-\tfrac{3}{2}z-\tfrac{5}{4}.$$
$$\therefore\ y=c_1e^{2z}+c_2e^z-\tfrac{1}{2}z^2-\tfrac{3}{2}z-\tfrac{5}{4}$$
$$=c_1e^{2\sin x}+c_1e^{\sin x}-\tfrac{1}{2}\sin^2 x-\tfrac{3}{2}\sin x-\tfrac{5}{4}.$$

Ex. 12. *Solve the equation* $\dfrac{d^2y}{dx^2}+\tan x\dfrac{dy}{dx}+y\cos^2 x=0,$ *by putting* $z=\sin x$. **[Calcutta Hons. 63]**

Solution. Putting $z=\sin x$,

$$Q_1=\frac{Q}{(dz/dx)^2}=\frac{\cos^2 x}{\cos^2 x}=1,$$
$$P_1=\left(\frac{d^2z}{dx^2}+P\frac{dz}{dx}\right)\Big/\left(\frac{dz}{dx}\right)^2=0,\ X_1=0.$$

$\therefore$ The transformed equation is $d^2y/dz^2+y=0.$

The solution is $\quad y=c_1\cos(z+c_2)$

$\qquad\qquad\qquad\qquad =c_1\cos(\sin x+c_2).$

Ex. 13. *Solve* $\dfrac{d^2y}{dx^2}-\cot x\dfrac{dy}{dx}-y\sin^2 x=0.$ **[Raj. 55]**

Solution. Putting $Q_1=-1, \left(\dfrac{dz}{dx}\right)^2=\sin^2 x,\ z=-\cos x.$

$$P_1=\frac{\cos x-\cot x\sin x}{\sin^2 x}=0,\ X_1=0.$$

Hence after changing variables from x to z, the equation becomes $(d^2y/dz^2)-y=0$, $(D^2-1)\,y=0,$

$$y=c_1e^z+c_2e^{-z}$$
$$=c_1e^{-\cos x}+c_2e^{\cos x}$$

Ex. 14. *Solve* $\dfrac{d^2y}{dx^2}-\dfrac{1}{x}\dfrac{dy}{dx}+4x^2y=x^4.$

[Vikram-62 ; Raj. 49 ; Delhi Hons. 58 ; Poona 62]

or $\ x\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}+4x^3y=x^5.$ **[Poona 64]**

Solution. Putting $Q_1=1, \left(\dfrac{dz}{dx}\right)^2=4x^2,\ z=x^2.$

$P_1=0,\ X_1=\dfrac{x^4}{4x^2}=\frac{1}{4}x^2=\frac{1}{4}z.$

The transformed equation is $\dfrac{d^2y}{dx^2}+y=\frac{1}{4}z.$

$y=c_1\cos(z+c_2)+\frac{1}{4}z$
$\quad=c_1\cos(x^2+c_2)+\frac{1}{4}x^2.$

Ex. 15. *Solve* $x\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}-4x^3y=8x^3\sin x^2.$ **[Raj. 55]**

Solution. Standard form is $\dfrac{d^2y}{dx^2}-\dfrac{1}{x}\dfrac{dy}{dx}-4x^2y=8x^2\sin x^2.$

Putting $Q_1=-1,\ \left(\dfrac{dz}{dx}\right)^2=4x^2\ z=x^3.$

$$P_1=\frac{2-\frac{1}{x}.2x}{4x^2}=0,\ X_1=\frac{8x^2\sin x^2}{4x^2}=2\sin x^2=2\sin z.$$

The equation in z is $(d^2y/dz^2)-y=2\sin z,$

$(D^2-1)\ y=2\sin z,\ y=c_1e^z+c_2e^{-z}-\sin z$

or $y=c_1e^{x^2}+c_2e^{-x^2}-\sin x^2.$

Ex. 16. *Solve* $x^4\dfrac{d^2y}{dx^2}+2x^3\dfrac{dy}{dx}+n^2y=0.$ **[Agra 52]**

Solution. Standard form $\dfrac{d^2y}{dx^2}+\dfrac{2}{x}\dfrac{dy}{dx}+\dfrac{n^2}{x^4}y=0.$

Putting $Q_1=\dfrac{Q}{(dz/dx)^2}=n^2,\ \dfrac{dz}{dx}=\dfrac{1}{x^2},\ z=-\dfrac{1}{x}$

$$P_1=\frac{-\frac{2}{x^3}+\frac{2}{x}\cdot\frac{1}{x^2}}{(dz/dx)^2}=0,\ X_1=0.$$

Equation in z is $(d^2y/dz^2)+n^2y=0,\ y=c_1\cos(nz+c_2)$

or $y=c_1\cos\left(c_2-\dfrac{n}{x}\right)$

Ex. 17. *Solve* $\dfrac{d^2y}{dx^2}+\left(1-\dfrac{1}{x}\right)\dfrac{dy}{dx}+4x^2e^{-2x}\ y=4\ (1+x)\ e^{-x}.$

Solution. Putting $Q_1=4,\ z=-e^{-x}\ (x+1),\ P_1=0,$

equation in z is $\dfrac{d^2y}{dz^2}+4y=-4z.$

C.F.$=c_1\cos(2z+c_2)=c_1\cos[c_2-2e^{-x}(x+1)].$

Ex. 18. *Solve* $x\dfrac{d^2y}{dx^2}+(4x^2-1)\dfrac{dy}{dx}+4x^3y=2x^3.$

Solution. St. form is $\dfrac{d^2y}{dx^2}+\left(4x-\dfrac{1}{x}\right)\dfrac{dy}{dx}+4x^2y=2x^2.$

Putting $Q_1=\dfrac{4x^2}{(dz/dx)^2}=1,\ \dfrac{dz}{dx}=2x,\ z=x^2,$

$$P_1=\frac{2+(4x-1/x)\,2x}{4x^2}=2,\ X_1=\frac{2x^2}{4x^2}=\tfrac{1}{2}.$$

Equation in z is $\dfrac{d^2y}{dz^2}+2\dfrac{dy}{dz}+y=\tfrac{1}{2},\ (D+1)^2\,y=\tfrac{1}{2}.$

$$y=(c_1+c_2z)\,e^{-z}+\tfrac{1}{2}=(c_1+c_2x^2)\,e^{-x^2}+\tfrac{1}{2}.$$

***2·8. Method of Variation is Parameters**

[Meerut 70 ; Raj. 66 ; Nag. 63 ; Delhi Hons. 66, 63]

We shall now explain a somewhat artificial but elegant method for finding the complete primitive of a linear equation *whose complementary function is known.*

The equation is $\dfrac{d^2y}{dx^2}+P\dfrac{dy}{dx}+Qy=X.$ ...(1)

Let C.F. be $y=c_1u+c_2v,$ ...(2)

where c_1 and c_2 are constant ; then u and v are two integral solutions of $\dfrac{d^2y}{dx^2}+P\dfrac{dy}{dx}+Qy=0,$ ...(3)

i.e. $u_2+Pu_1+Qu=0$ and $v_2+Pv_1+Qv=0$*. ...(4)

Clearly (2) will not be solution of (1) as there X is not zero.

Now let us replace the constants c_1 and c_2 by A and B (parameters) which are functions of x and let

$y=Au+Bv$ ...(5)

be a solution of (1).

Differentiating (5), $\dfrac{dy}{dx}=A_1u+B_1v+Au_1+Bv_1.$

Let A and B satisfy the additional equation

$\mathbf{A_1u+A_1v=0,}$ ...(6)

Then $\dfrac{dy}{dx}=Au_1+Bv_1$

and $\dfrac{d^2y}{dx^2}=Au_2+Bv_2+A_1u_1+B_1v_1.$

Substituting these values in (1), we get

$$[Au_2+Bv_2+A_1u_1+B_1v_1]+P\,[Au_1+Bv_1]+Q\,(Au+Bv)=X$$

or $\quad A\,[u_2+Pu_1+Qu]+B\,[v_2+Pv_1+Qv]+(A_1u_1+B_1v_1)=X.$

The expressions within the first two brackets are zero by (4).

$\therefore\quad \mathbf{A_1u_1+B_1v_1=X}.$ ...(7

Equations (6) and (7) can be solved for A_1, B_1 which on int

*Here suffixes denote differentiation with respect to x ; thus $u_1=\dfrac{du}{dx},\ A_1=\dfrac{dA}{dx}$, etc.

gration give A and B. Then the complete solution of (1) is written as $y=Au+Bv$.

***Working Rule.**

Given the equation $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=X$,

find the solution of $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0$,

i.e. complementary function of given equation.

Take the C.E. as $y=Au+Av$, ...(1)

treating A and B as functions of x.

To determine A and B, use the following two equations :

(*i*) $y=Au+Bv$ is a solution of given equation which gives a condition $A_1u_1+B_1v_1=X$.

(*ii*) $A_1u+B_1v=0$.

From these two relations between A_1, B_1, find A_1 and B_1 and integrate to get A and B.

The complete solution is then $y=Au+Bv$.

***Ex. 1 (a)** *Apply the method of variation of parameters to solve* $\frac{d^2y}{dx^2}+n^2y=\sec nx$.

[Meerut 70; Sagar 63 ; Agra 69, 64, Raj. 65, 52, Vikram 64; Nagpur 61; Banaras 59]

Solution. The equation is $\frac{d^2y}{dx^2}+n^2y=\sec nx$. ...(1)

C.F. or solution of $\frac{d^2y}{dx^2}+n^2y=0$ is

$y=c_1 \cos nx+c_2 \sin nx$

where c_1 and c_2 are constants.

Let us suppose that

$y=A \cos nx+B \sin nx$ ...(2)

be a solution of (1) where A and B are functions of x.

From (2),

$$\frac{dy}{dx}=-An \sin nx+Bn \cos nx+A_1 \cos nx+B_1 \sin nx.$$

Choose A and B such that

$A_1 \cos nx+B_1 \sin nx=0$; ...(3)

then $\frac{dy}{dx}=-An \sin nx+Bn \cos nx$.

$$\therefore \quad \frac{d^2y}{dx^2}=-n^2 (A \cos nx+B \sin nx)-A_1n \sin nx+B_1n \cos nx$$

$$=-n^2y-A_1n \sin nx+B_1n \cos nx.$$

Putting values of $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$ etc. in (1), we get

$$[-n^2y-A_1n\sin nx+B_1n\cos nx]+n^2y=\sec nx$$

or $\quad -A_1n\sin nx+B_1n\cos nx=\sec nx.$ (4)

Multiplying (3) by $n\sin nx$, (4) by $\cos nx$ and adding, we get

$B_1n=1$ or $B_1=1/n.$

Then $A_1=-{}^1/n\tan nx$;

so $\quad A=\int A_1\,dx=\dfrac{1}{n^2}\log\cos nx+c_1$

and $B_1=\int B_1\,dx=\dfrac{x}{n}+c_2.$

Therefore the complete solution is

$$y=A\cos nx+B\sin nx$$

$$=\left(\frac{1}{n^2}\log\cos nx+c_1\right)\cos nx+\left(\frac{x}{n}+c_2\right)\sin nx$$

$$=c_1\cos nx+c_2\sin nx+\frac{\cos nx}{n^2}\log\cos nx+\frac{x}{n}\sin nx.$$

Ex. 1. (b) *Using the method of variation of parameters solve the differential equation* $\dfrac{d^2y}{dx^2}+9y=\sec 3x.$ **[Nagpur 61]**

Put $n=3$ in the above example.

Ex. 2 *Apply the method of variation of parameters to solve* $\dfrac{d^2y}{dx^2}+4y=4\tan 2x.$ **[Sagar 64: Agra 71, 54]**

Solution. First let us find out C.F. *i.e.* solution of

$\dfrac{d^2y}{dx^2}+4y=0,\quad$ *i.e.* $\quad (D^2+4)\,y=0.$

$D=\pm 2i,\quad \therefore\quad y=c_1\cos 2x+c_2\sin 2x.$

So let $y=A\cos 2x+B\sin 2x$ be a solution of the given equation, where A and B are functions of x.

Then $\dfrac{dy}{dx}=-2A\sin 2x+2B\cos 2x+A_1\cos 2x+B_1\sin 2x.$

Choose A, B such that $A_1\cos 2x+B_1\sin 2x=0.$ ...(1)

$\therefore\quad dy/dx=-2A\sin 2x+2B\cos 2x$

and $d^2y/dx^2=-4\,(A\cos 2x+B\sin 2x)-2A_1\sin 2x+2B_1\cos 2x$

$$=-4y-2A_1\sin 2x+2B_1\cos 2x.$$

Putting values of $\dfrac{d^2y}{dx^2},\dfrac{dy}{dx}$ in given equation, it becomes

$-4y-4A_1\sin 2x+2B_1\cos 2x=4\tan 2x$

Equation (3) and (4) can be directly written from § 2·8 which are

$A_1u+B_1v=0$

and $\quad A_1u_1+B_1v_1=X$

where $u=\cos nx$, $v=\sin nx$, $X=\sec nx$.

i.e. $-2A_1 \sin 2x + 2B_1 \cos 2x = 4 \tan 2x.$
or $\quad A_1 \sin 2x - B_1 \cos 2x = -2 \tan 2x.$ ...(2)

Solving (1) and (2) for A_1 and B_1, we get

$$A_1 = -\frac{2 \sin^2 2x}{\cos 2x} = -\frac{2(1-\cos^2 2x)}{\cos 2x} = 2 \cos 2x - 2 \sec 2x.$$

$\therefore \quad A = \sin 2x - \log(\sec 2x + \tan 2x) + c_1$
and $\quad B_1 = 2 \sin 2x; \quad \therefore \quad B = -\cos 2x + c_2.$

Hence the complete solution is

$$y = A \cos 2x + B \sin 2x$$
$$= c_1 \cos 2x + c_2 \sin 2x - \log(\sec 2x + \tan 2x).\cos 2x.$$

***Ex. 3 (a).** *Apply the method of variation of parameters to solve*

$$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$$ **[Nagpur 63; Agra 50; Vikram 62]**

Solution. The C. F., *i.e.* solution of $d^2y/dx^2 - y = 0$ is

$$y = c_1 e^x + c_2 e^{-x}.$$

Now let $\quad y = Ae^x + Be^{-x},$

where A and B are functions of x, be a solution of the given equation.

Then $\quad dy/dx = Ae^x - Be^{-x} + A_1 e^x + B_1 e^{-x}.$

Choose A, B, such that $A_1 e^x + B_1 e^{-x} = 0.$ ...(1)

Then $\quad dy/dx = Ae^x - Be^{-x}$

and $\quad d^2y/dx^2 = Ae^x + Be^{-x} + A_1 e^x - B_1 e^{-x}$
$\quad = y + A_1 e^x - B_1 e^{-x}.$

Putting these values in given equation, we get

$(y + A_1 e^x - B_1 e^{-x}) - y = 2/(1+e^x)$

or $\quad A_1 e^x - B_1 e^{-x} = 1/(1+e^x).$ ...(2)

From (1), $Be^{-x} = -Ae^x$; $\therefore$ (2) gives

$$2A_1 e^x = \frac{2}{1+e^x} \quad \text{or} \quad A_1 = \frac{e^{-x}}{1+e^x}$$

Also $\quad B_1 = -\frac{e^x}{1+e^x}$, so that $B = -\int \frac{e^x}{1+e^x} = -\log(1+e^x) + c_2$

and
$$A = \int A_1 \, dx = \int \frac{e^{-x}}{1+e^x} = \int \frac{1}{z^2(1+z)} \, dz, \text{ where } z = e^x$$
$$= \int \left(\frac{1}{z^2} - \frac{1}{z} + \frac{1}{1+z} \right) dz = -\frac{1}{z} - \log z + \log(1+z) + c_1$$
$$= \log \frac{1+z}{z} - \frac{1}{z} + c_1 = \log \frac{1+e^x}{e^x} - e^{-x} + c_1.$$

Hence the complete solution is

$$y = Ae^x + Be^{-x}$$
$$= e^x \log \frac{1+e^x}{e^x} - 1 + c\, e^x - e^{-x} \log(1+e^x) + c_2 e^{-x}.$$

Ex. 3. (b) *Apply the method of variation of parameters to solve*

$$x^2 \frac{d^2y}{dx^2} - y = 2e^x.$$ [Nagpur 61 (S)]

Divide by x^2 and then proceed as in the above example with $X=2e^x/x^2$.

Ex. 4. *Apply the method of variation of parameters to solve*

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=x^2e^x.$$

[Meerut 70 ; Raj. 64 ; Agra 66, 60. 53]

Solution. We find the C.F. *i.e.*, solution of

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0 \text{ (Homogeneous equation).}$$

Put $x=e^z$, $D\equiv d/dz$, then it becomes

$$[D(D-1)+D-1]\,y=0 \quad \text{or} \quad (D^2-1)\,y=0.$$

$$\therefore \quad y=c_1e^z+c_2e^{-z}=c_1x+c_2(1/x) \text{ as } e^z=x.$$

Let us suppose that $y=Ax+B/x$

be a solution of the given equation, where A, B (parameters) are the functions of x.

$$\therefore \quad \frac{dy}{dx}=A-\frac{B}{x^2}+A_1x+\frac{B_1}{x}=0.$$

Choose A and B, such that $A_1x+\dfrac{B_1}{x}=0$ ...(1)

Then $\dfrac{dy}{dx}=A-\dfrac{B}{x^2}$

and $\dfrac{d^2y}{dx^2}=A_1+\dfrac{2B}{x^3}-\dfrac{B_1}{x^2}$.

Putting these values of $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}$ in given equation, we get

$$\left[\text{the equation in standard form is } \frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}-\frac{y}{x^2}=e^x\right]$$

$$\left[A_1+\frac{2B}{x^3}-\frac{B_1}{x^2}\right]+\frac{1}{x}\left[A-\frac{B}{x^2}\right]-\frac{1}{x^2}\left[Ax+\frac{B}{x}\right]=e^x$$

or $\quad A_1-\dfrac{B_1}{x^2}=e^x$ ...(2)

Solving (1) and (2) for A_1 and B_1, we have

$$A_1=\tfrac{1}{2}e^x \text{ and } B_1=-\tfrac{1}{2}x^2e^x.$$

Thus $\quad A=\int A_1\,dx=\tfrac{1}{2}e^x+c_1$

and $\quad B=\int B_1\,dx=-\tfrac{1}{2}\int(x^2e^x)\,dx=-\tfrac{1}{2}e^x\,[x^2-2x+2]+c_2.$

Hence the complete solution is $y=Ax+B/x$

or $\quad y=(\tfrac{1}{2}e^x+c_1)\,x+[-\tfrac{1}{2}e^x(x^2-2x+2)+c_2]/x$

$$=c_1x+\frac{c_2}{x}+e^x-\frac{e^x}{x}.$$

Ex. 5. (a) *Apply the method of variation of parameters to solve the equation* $(1-x)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=(1-x)^2$.

[Delhi Hons. 66 ; Agra 55 ; Raj. 66]

Solution The equation in standard form is

$$\frac{d^2y}{dx^2}+\frac{x}{1-x}\frac{dy}{dx}-\frac{1}{1-x}y=1-x. \quad ...(1)$$

Here $P+xQ=\frac{x}{1-x}-x.\frac{1}{1-x}=0.$

Hence $y=x$ is a part of C.F.

Now we find C.F., *i.e.* solution of

$$\frac{d^2y}{dx^2}+\frac{x}{1-x}\frac{dy}{dx}-\frac{1}{1-x}y=0.$$

Putting y vx, the reduced equation is

$$\frac{d^2v}{dx^2}+\left(P+\frac{2}{y_1}\frac{dy_1}{dx}\right)\frac{dv}{dx}=0, \text{ where } y_1=x$$

$$\frac{d^2v}{dx^2}+\left(\frac{x}{1-x}+\frac{2}{x}\right)\frac{dv}{dx}=0$$

or $$\frac{dp}{dx}+\left(-1-\frac{1}{x-1}+\frac{2}{x}\right)p=0, \text{ where } p=\frac{dv}{dx}.$$

Integrating, $\log p-x-\log(x-1)+2\log x=\log c_1$

i.e. $p=\frac{dv}{dx}=\frac{c_1(x-1)e^x}{x^2}=c_1\left[\frac{1}{x}-\frac{1}{x^2}\right]e^x$

$\therefore$ $v=c_1e^x/x+c_2$, $y=vx=c_1e^x+c_2x$.

Let us now suppose that $y=Ae^x+Bx$ be a solution of (1), where A and B are functions of x. Then proceeding as usual, A, B must satisfy the conditions

$$A_1u+B_1v=0, \text{ i.e., } A_1e^x+B_1x=0$$

and $$A_1u_1+B_1v_1=X, \text{ i.e., } A_1e^x+B_1.1=1-x.$$

Solving these for A_1 and B_1, we get $B_2=1$, $A_1=-xe^{-x}$.

$$\therefore \quad B=\int B_1\,dx=x+c_1,\ A=\int A_1\,dx=e^{-x}(x+1)+c_2.$$

Hence the complete solution is $y=Ae^x+Bx$

or $y=[e^{-x}(x+1)+c_2]e^x+(x+c_1)x$

$=c_1x+c_2e^x+(1+x+x^2).$

Ex 5. (b). *Apply the method of variation of parameters to solve*

$$(x-1)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=(x-1)^2.$$ **[Agra 70]**

Solution. This is just the above example.

Ex. 6. *Apply the method of variation of parameters to solve the equation* $x\frac{dy}{dx}-y=(x-1)\left(\frac{d^2y}{dx^2}-x+1\right)$.

[Sagar 62 ; Raj. 63 ; Agra 72, 51 ; Delhi Hons. 63]

Solution. The equation is

$$(x-1)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=(x-1)^2.$$

Now proceed as in the above example.

$$y=c_1e^x+c_2x+(x^2+x+1).$$

Ex. 7. *Solve by the method of variation of parameters*

$$x^2\frac{d^2y}{dx^2}-2x(1+x)\frac{dy}{dx}+2(1+x)y=x^3.$$

[Delhi Hons. 65, 63 ; Agra 67, 63 ; Raj. 64, 55]

Solution The equation in standard form is

$$\frac{d^2y}{dx^2}-\frac{2(1+x)}{x}\frac{dy}{dx}+\frac{2(1+x)}{x^2}y=x. \quad ...(1)$$

Here $P+Qx=-\frac{2(2+x)}{x}+\frac{2(1+x)}{x}=0.$

Hence $y=x$ is part of C.F.

So we find C.F., *i.e.* solution of

$$\frac{d^2y}{dx^2}-\frac{2(1+x)}{x}\frac{dy}{dx}+\frac{2(1+x)}{x^2}y=0.$$

Put $y=vx$; then this equation becomes

$$\frac{d^2v}{dx^2}+\left(P+\frac{2}{y_1}\frac{dy}{dx}\right)\frac{dv}{dx}=0, \text{ where } y_1=x$$

or $\frac{d^2v}{dx^2}-2\frac{dv}{dx}=0$ or $(D^2-2D)v=0, D=0, 2.$

$\therefore \quad v=c_1+c_2e^{2x}.$

$\therefore \quad$ C.F. is $y=vx=c_1x+c_2xe^{2x}$

Now let $y=Ax+Bxe^{2x}$ be a solution of (1), where A and B are functions of x. Then proceeding as usual, A and B satisfy the conditions

$A_1u+B_1v=0,$ *i.e.* $A_1x+B_1xe^{2x}=0.$

and $A_1u_1+B_1v_1=X$ *i.e.* $A_1+B_1e^{2x}(2x+1)=x.$

Solving these for A_1 and B_1, we have

$B_1=\frac{1}{2}e^{-2x}$ and $A=-\frac{1}{2}$

i.e. $B=\frac{1}{4}e^{-2x}+c_1$ and $A=-\frac{1}{2}x+c_2$

Therefore the complete solution is $y=Ax+Bxe^{2x}$.

i.e. $y=(-\frac{1}{2}x+c_2)x+xe^{2x}(-\frac{1}{4}e^{-2x}+c_1)$

$=-\frac{1}{4}x-\frac{1}{2}x^2+c_2x+c_1xe^{2x}.$

Ex. 8. *Apply the method of variation of parameters to solve*

$$(x+2)\frac{d^2y}{dx^2}-(2x+5)\frac{dy}{dx}+2y=(x+1)e^x.$$

[Jiwaji 66 ; Vikram 64 ; Agra 59 : Raj. 59]

Example 7 has been solved otherwise also. See Ex. 3 P. 49.

Solution. The equation in the standard form is

$$\frac{d^2y}{dx^2}-\frac{2x+5}{x+2}\frac{dy}{dx}+\frac{2}{x+2}y=\frac{x+1}{x+2}e^x. \quad \text{...(1)}$$

We find C.F., *i.e.* solution of

$$\frac{d^2y}{dx^2}-\frac{2x+5}{x+2}\frac{dy}{dx}+\frac{2}{x+2}y=0. \quad \text{...(2)}$$

We note that e^{mx} will be solution if

$$m^2+P.m+Q=0, \quad i.e. \quad m^2-\frac{2x+5}{x+5}m+\frac{2}{x+2}=0,$$

i.e. $(m-2)\,[mx+(2m-1)]=0$, *i.e.* $m=2$.

Hence $y_1=e^{2x}$ is a solution (2)

Putting $y=vy_1=ve^{2x}$ the equation (2) becomes

$$\frac{dp}{dx}+\frac{2x+3}{x+2}p=0 \text{ (See Ex. 16 P. 58) when } \frac{dv}{dx}=p$$

or $\dfrac{dp}{p}+\left(2-\dfrac{1}{x+2}\right)dx=0$

or $\log p-2x+\log(x+2)=\log c_1$ or $p=c_1(x+2)\,e^{-2x}$

or $dv/dx=c_1(x+2)\,e^{-2x}$.

$$v=\int c_1(x+2)\,e^{-2x}+c_2=-c_1.\tfrac{1}{4}e^{-2x}(2x+5)+c_2,$$

$\therefore$ the solution of (2) is $y=ve^{2x}$.

i.e. $\qquad y=-\tfrac{1}{4}c_1(2x+5)+c_2e^{2x}$.

Let us take $\quad y=A(2x+5)+Be^{2x}$. ...(3)

to be solution of (1) where A and B (parameters) are functions of x.

Then A and B satisfy the conditions

$$A_1u+B_1v=0, \quad i.e. \quad A_1(2x+5)+B_1e^{2x}=0,$$

and $\quad A_1u_1+B_1v_1=X$, *i.e.*, $A_12+2B_1e^{2x}=\dfrac{x+1}{x+2}e^x$.

Solving these, $A_1=-\dfrac{e^x}{4}\dfrac{x+1}{(x+2)^2}=-\dfrac{e^x}{4}\left[\dfrac{1}{x+2}-\dfrac{1}{(x+2)^2}\right]$.

Integrating, $A=\dfrac{-e^x}{4(x+2)}+c_1$

and $\quad B_1=\dfrac{(2x+5)(x+1)e^{-x}}{4(x+2)^2}=\dfrac{e^{-x}}{4}\left[2-\dfrac{1}{x+2}-\dfrac{1}{(x+2)^2}\right]$.

$\therefore \quad B=\dfrac{e^{-x}}{4}\left(\dfrac{1}{x+2}-2\right)+c_2$.

Therefore the complete solution is $y=A(2x+5)+Be^{2x}$.

or $\quad y=\left[-\dfrac{e^x}{4(x+2)}+c_1\right](2x+5)+\dfrac{e^{-x}}{4}\left[\left(\dfrac{1}{x+2}-2\right)+c_2\right]e^{2x}$

$\qquad =c_1(2x+5)+c_2e^{2x}-e^x$.

Ex. 9. *By the method of variation of parameters, solve the equation* $\dfrac{d^2y}{dx^2}+(1-\cot x)\dfrac{dy}{dx}-y\cot x=\sin^2 x$. **[Rajasthan 58, 56]**

Solution. Here $1-P+Q=0$; $\therefore$ $y=e^{-x}$ is a solution of A.E.

Putting $y=ve^{-x}$, the equation which gives C.F. ($X=0$) reduces

to $\dfrac{d^2v}{dx^2}+\left[P+\dfrac{2}{y_1}\dfrac{dy_1}{dx}\right]\dfrac{dv}{dx}=0$, where $y_1=e^{-x}$

or $\dfrac{d^2v}{dx^2}+\left[1-\cot x-\dfrac{2}{e^{-x}}e^{-x}\right]\dfrac{dv}{dx}=0$. Put $\dfrac{dv}{dx}=p$

$\therefore$ $dp/dx=(1+\cot x)\,p$ or $dp/p=(1+\cot x)\,dx$.

$\therefore$ $\log p-\log c_1+x+\log\sin x=\log(c_1 \sin xe^x)$

or $p=dv/dx=c_1 \sin xe^x$.

Integrating $v=\frac{1}{2}c_1e^x(\sin x-\cos x)+c_2$.

$\therefore$ $y=vy_1=\frac{1}{2}c_1(\sin x-\cos x)+c_2e^x$

which is the C.F. of the given equation

Now let $y=A(\sin x-\cos x)+Be^{-x}$

be a solution of the given equation, where A and B are functions of x.

Then A and B satisfy the relations

$A_1(\sin x-\cos x)+B_1e^{-x}=0$

and $A_1(\cos x+\sin x)-B_1e^{-x}=\sin^2 x$.

Solving these, $A_1=\frac{1}{2}\sin x$, *i.e.* $A=-\frac{1}{2}\cos x+c_1$

and $B_1=\frac{1}{2}e^x(\sin x\cos x-\sin^2 x)=\frac{1}{4}e^x[\sin 2x-(1-\cos 2x)]$.

$$\therefore\quad B=\frac{e^x}{4(-5)}(2\cos 2x-\sin 2x)-\frac{e^x}{4}$$
$$+\frac{e^x}{4(-5)}(-2\sin 2x-\cos 2x)+c_2.$$

Hence the solution is

$$y=(-\tfrac{1}{2}\cos x+c_1)(\cos x-\sin x)+e^{-x}\left[\frac{e^x}{4}\left(\frac{-\cos 2x}{5}-\frac{3\sin 2x}{-5}\right)-\frac{e^x}{4}+c_2\right]$$

$=c_1(\cos x+\sin x)-\frac{1}{10}(\sin 2x-2\cos 2x)+c_2e^{-x}$.

Ex. 10. *Solve* $d^2y/dx^2+y=\operatorname{cosec} x$.

[Indore 66 ; Nagpur 63 ; Agra 65, 49]

Solution. C.F. is $y=c_1\cos x+c_2\sin x$.

Let $y=A\cos x+B\sin x$ be solution of the given equation, where A and B are functions of x. Then proceeding as in Ex. 9 above

$A_1\cos x+B_1\sin x=0$,

and $-A_1\sin x+B_1\cos x=\operatorname{cosec} x$,

so that $A_1=-1$, $B_1=\cot x$.

$\therefore$ $A=-x+c_1$, $B=\log\sin x+c_2$.

Hence the complete solution is

$y=(-x+c_1)\cos x+(\log\sin x+c_2)\sin x$

$=c_1\cos x+c_2\sin x-x\cos x+\sin x\log\sin x$.

Ex. 11. *Solve* $d^2y/dx^2+y=x$.

Solution. As above let the solution be $y=A\cos x+B\sin x$, A and B here satisfy the conditions

$$A_1\cos x+B_1\sin x=0,$$
$$-A_1\sin x+B_1\cos x=x.$$

These give $A_1=-x\sin x$, $B_1=x\cos x$.

$$A=-\int x\sin x\,dx=-[-x\cos x+\int\cos x\,dx]$$
$$=x\cos x-\sin x+c_1,$$
$$B=\int x\cos x\,dx=x\sin x+\cos x+c_2.$$

Hence the complete solution is

$$y=(x\cos x-\sin x+c_1)\cos x+(x\sin x+\cos x+c_2)\sin x$$
$$=x+c_1\cos x+c_2\sin x.$$

Ex. 12. *By the method of variation of parameters, solve the equation* $(1-x^2)\dfrac{d^2y}{dx^2}-4x\dfrac{dy}{dx}-(1+x^2)\,y=x.$

[Sagar 66 : Vikram 63 ; Raj. 53]

Solution. The equation in standard form is

$$\frac{d^2y}{dx^2}-\frac{4x}{1-x^2}\frac{dy}{dx}-\frac{(1+x^2)}{1-x^2}\,y=\frac{x}{1-x^2}. \qquad ...(1)$$

To remove the first derivative, we choose (see § 2·6 P. 62)

$$y_1=e^{-\frac{1}{2}\int P\,dx}=e^{\frac{1}{2}\int\frac{4x}{1-x^2}dx}=e^{-\log(1-x^2)}=\frac{1}{1-x^2}.$$

Putting $y=vy_1$, the equation becomes

$$\frac{d^2v}{dx^2}+Q_1v=X_1.$$

$$Q_1=Q-\tfrac{1}{4}P^2-\tfrac{1}{2}\frac{dP}{dx}=-\frac{1+x^2}{1-x^2}-\frac{4x^2}{(1-x^2)^2}+\tfrac{4}{2}\left[\frac{1}{1-x^2}+\frac{2x^2}{(1-x^2)^2}\right]$$
$$=1,\ X_1=\frac{X}{y_1}=x.$$

The equation reduces to $d^2v/dx^2+v=x$. ...(2)

C.F. of (2) is $v=A\cos x+B\sin x$. ...(3)

Consider A and B as functions of x and let (3) be a solution of (2); then A and B satisfy the relations

$$A_1\cos x+B_1\sin x=0,$$
$$-A_1\sin x+B_1\cos x=x.$$

These give $A_1=-x\sin x$, $B_1=x\cos x$,

$$A=-x\cos x-\sin x+c_1,\ B=x\sin x+\cos x+c_2.$$

Complete solution of (2) is

$$v=A\cos x+B\sin x$$
$$=c_1\cos x+c_2\sin x+x.$$

And the complete solution (1) is $y=vy_1$,

i.e. $$y=[c_1\cos x+c_2\sin x+x]\,\frac{1}{1-x^2}.$$

Ex. 13 *By the method of variation of parameters, solve the equation* $(1-x^2)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=x(1-x^2)^{3/2}$. [Raj. 54]

Solution. The equation can be written as

$$\frac{d^2y}{dx^2}+\frac{x}{1-x^2}\frac{dy}{dx}-\frac{1}{1-x^2}y=x(1-x^2)^{1/2}.$$

Here $P+Qx=0$; $\therefore$ $y=x$ is part of C.F

To find C.F., *i.e.* solution of $\dfrac{d^2y}{dx^2}+\dfrac{x}{1-x^2}\dfrac{dv}{dx}-\dfrac{1}{1-x^2}y=0$,

put $y=vx$; the equation then becomes

$$\frac{d^2v}{dx^2}+\left[P+\frac{2}{y_1}\frac{dy_1}{dx}\right]\frac{dv}{dx}=0,\ y_1=x$$

or $\dfrac{d^2v}{dx^2}+\left(\dfrac{x}{1-x^2}+\dfrac{2}{x}\right)\dfrac{dv}{dx}=0$ or $\dfrac{dp}{dx}+\left(\dfrac{x}{1-x^2}+\dfrac{2}{x}\right)p=0,\ p=\dfrac{dv}{dx}$

$$\frac{dp}{p}+\left(\frac{2}{x}+\frac{x}{1-x^2}\right)dx=0.$$

Integrating, $\log p+2\log x-\frac{1}{2}\log(1-x^2)=\log c_1$

or $p=\dfrac{dv}{dx}=\dfrac{c_1\sqrt{(1-x^2)}}{x^2}$.

$$\therefore\quad v=c_1\left[\sqrt{(1-x^2)}\left(\frac{1}{x}\right)\right.$$

$$\left.-\int\tfrac{1}{2}(1-x^2)^{-1/2}(-2x).\left(-\frac{1}{x}\right)dx\right]+c_2$$

$$=-c_1\left[\frac{\sqrt{(1-x^2)}}{x}+\sin^{-1}x+c^2\right]$$

Hence C.F. is $y=vx=-c_1[\sqrt{(1-x^2)}+x\sin^{-1}x]+c_2x$.

Let $y=A[\sqrt{(1-x^2)}+x\sin^{-1}x]+Bx$

be a solution of the given equation, where A and B are functions of x. Then A and B satisfy the relations

$$A_1[\sqrt{(1-x^2)}+x\sin^{-1}x]+B_1x=0 \quad ...(1)$$

and $A_1\left[\dfrac{-x}{\sqrt{(1-x^2)}}+\sin^{-1}x+\dfrac{x}{\sqrt{(1-x^2)}}\right]+B_1.1=x(1-x^2)^{1/2}$

or $A_1[\sin^{-1}x]+B_1=x(1-x^2)^{1/2}$.(2)

Multiplying (2) by x and subtracting from (1), we get

$$A_1\sqrt{(1-x^2)}=-x^2(1-x^2)^{1/2} \text{ or } A_1=-x^2,\ A=-\tfrac{1}{3}x^3+c_1,$$

$$B_1=-\frac{A_1}{x}[\sqrt{(1-x^2)}+x\sin^{-1}x]=x\sqrt{(1-x^2)}+x^2\sin^{-1}x,$$

$$B=\int[x\sqrt{(1-x^2)}+x^2\sin^{-1}x]\,dx+c_2$$

$$=\int[x\sqrt{(1-x^2)}\,dx+\sin^{-1}x.\tfrac{1}{3}x^3-\int\tfrac{1}{3}x^3\frac{1}{\sqrt{(1-x^2)}}dx+c_2$$

For an altrenative solution, see Ex. 4 P. 50.

$$=\int x\sqrt{(1-x^2)}\,dx+\tfrac{1}{3}x^3\sin^{-1}x+\tfrac{1}{3}\int\frac{x\,(1-x^2-1)}{\sqrt{(1-x^2)}}\,dx+c_2$$
$$=\frac{4}{3}\int x\sqrt{(1-x^2)}\,dx-\tfrac{1}{3}\int\frac{x}{\sqrt{(1-x^2)}}\,dx+\tfrac{1}{3}x^3\sin^{-1}x+c_2$$
$$=-\frac{2}{3}\,\frac{(1-x^2)^{3/2}}{\frac{3}{2}}+\tfrac{1}{3}\,(1-x^2)^{1/2}+\tfrac{1}{3}x^3\sin^{-1}x+c_2.$$

Hence the complete solution is

$$y=A\,[\sqrt{(1-x^2)}+x\sin^{-1}x]+Bx$$
$$=(-\tfrac{1}{3}x^3+c_1)\,[\sqrt{(1-x^2)}+x\sin^{-1}x]$$
$$+[-\tfrac{4}{9}(1-x^2)^{3/2}+\tfrac{1}{3}(1-x^2)^{1/2}+\tfrac{1}{3}x^3\sin^{-1}x+c_2]\,x$$
$$=c_1\,[\sqrt{(1-x^2)}+x\sin^{-1}x]+c_2x+\tfrac{1}{3}x\,(1-x^2)\sqrt{(1-x^2)}$$
$$-\tfrac{4}{9}x\,(1-x^2)^{3/2}$$
$$=c_1\,[\sqrt{(1-x^2)}+x\sin^{-1}x]+c_2x-\tfrac{1}{9}x\,(1-x^2)^{3/2}.$$

Ex. 14. *Apply the method of variation of parameters to solve the equation*

$$\frac{d^2y}{dx^2}+(\tan x-3\cos x)\,\frac{dy}{dx}+2y\cos^2 x=\cos^4 x. \qquad \text{[Raj. 57]}$$

Solution. As found in Ex. 11 P. 82, the C.F. is

$$y=c_1e^{2\sin x}+c_2e^{\sin x}$$

Let $\qquad y=Ae^{2\sin x}+Be^{\sin x}, \qquad \ldots(1)$

be the solution of the equation, where A, B (parameters) are functions of x.

Then A and B satisfy the following conditions :

$$A_1e^{2\sin x}+B_1e^{\sin x}=0,\ i.e.\ A_1e^{\sin x}=-B_1$$

and $\qquad 2A_1e^{2\sin x}.\cos x+B_1e^{\sin x}.\cos x=\cos^4 x,$

i.e. $\qquad 2A_1e^{2\sin x}+B_1e^{\sin x}=\cos^3 x,\ i.e.\ A_1e^{2\sin x}=\cos^3 x,$

$$A_1=\cos^3 x\cdot e^{-2\sin x},\ B_1=-A_1e^{\sin x}=-\cos^3 x\,.e^{-\sin x}.$$

$$\therefore\quad A=\int\cos^3 x\,.e^{-2\sin x}\,dx=\int(1-\sin^2 x)\,e^{-2\sin x}\cos x\,dx$$
$$=-\int(1-t^2)\,e^{2t}\,dt,\quad \text{where } -\sin x=t,\ -\cos x\,dx=dt$$
$$=-e^{2t}\,[\tfrac{1}{2}-\tfrac{1}{2}t^2+\tfrac{1}{2}t-\tfrac{1}{4}]$$
$$=-e^{-2\sin x}\,[\tfrac{1}{2}-\tfrac{1}{2}\sin^2 x-\tfrac{1}{2}\sin x-\tfrac{1}{4}]+c_1.$$

$$B=-\int\cos^3 x e^{-\sin x}\,dx=-\int(1-\sin^2 x)\cos x\,.e^{-\sin x}\,dx$$
$$=\int(1-z^2)\,e^z\,dz,\ \text{where } z=-\sin x,\ dz=-\cos x\,dx$$
$$=e^z\,[1-z^2+2z-2]=e^z\,[2z-z^2-1]$$
$$=e^{-\sin x}\,[-2\sin x-\sin^2 x-2]+c_2.$$

Putting these values of A and B in (1), we have

$$y=c_1e^{2\sin x}+\tfrac{1}{2}\sin^2 x+\tfrac{1}{2}\sin x-\tfrac{1}{4}+c_2e^{\sin x}-2\sin x-\sin^2 x-1$$
$$=c_1e^{2\sin x}+c_2e^{\sin x}-\tfrac{1}{2}\sin^2 x-\tfrac{3}{2}\sin x-\tfrac{5}{4},$$

which is the complete solution.

2·9. Method of operational factors.

Let the equation of second order be

$$P_0 \frac{d^2y}{dx^2}+P_1 \frac{dy}{dx}+P_2 y=X. \qquad ...(1)$$

Write $\frac{d}{dx}\equiv D$; then (1) is

$$P_0D^2y+P_1Dy+P_2y=X \quad \text{or} \quad f(D)\,y=X.$$

If $f(D)$ can be resolved into two factors $F_1(D)$ and $F_2(D)$ such that when $F_1(D)$ operates on y and $F_2(D)$ operates upon the result $[F_1(D)\,y]$, then the equation may be written as

$$f(D)\,y=F_2(D)\,\{F_1(D)\,y\}$$

or $f(D)\,y=F_2(D)\,F_1(D)\,y.$

[Here we cannot write $f(D)\,y=F_1(D)\,F_2(D)\,y$].

The order in which the factors are written must be verified so as to form the same equation.

The following few examples will fully illustrate the method :

***Ex. 1.** *Solve* $x\frac{d^2y}{dx^2}+(1-x)\frac{dy}{dx}-y=e^x.$

[Agra 50 ; Luck. 51]

Solution. Writing D for d/dx, the equation is

$$[xD^2+(1-x)\,D-1]\,y=e^x$$

or $(xD+1)\,(D-1)\,y=e^x. \qquad ...(1)$

[This cannot be written as $(D-1)\,(xD+1)\,y=e^x$ as this does not give equation on expansion.]

Now let $(D-1)\,y=v$; then $(xD+1)\,v=e^x$

or $v=cx^{-1}+e^xx^{-1}.$

Then equation (1) becomes $(D-1)\,y=cx^{-1}+e^xx^{-1}$

or $y=c_1e^x+ce^x\int e^{-x}x^{-1}\,dx+e^x+\log x.$

Ex. 2. *Solve* $[(x+3)\,D^2-(2x+7)\,D+2]\,y=(x-3)^2\,e^x$

Solution The equation may be written as

$$[(x+3)\,D-1]\,(D-2)\,y=(x+3)^2\,e^x.$$

Put $(D-2)\,y=v$; $\therefore$ $[(x+3)\,D-1]\,v=(x+3)^2\,e^x$

or $\left(D-\frac{1}{x+3}\right)v=(x+3)\,e^x$; I.F. $=e^{\int -\frac{1}{x+}\,dx}=\frac{1}{x+3}.$

$$\therefore \quad v.\frac{1}{x+3}=\int e^x+k=e^x+k$$

or $(D-2)\,y=v=(x+3)\,e^x+k\,(x+3).$

Linear, I.F. $=e^{-2x}$, thus the solution is

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*For an alternative solution, we have $1+P+Q=0$ $\therefore$ $y=e^x$ is a part of C.F. (See. Ex. 10 p. 54).

$$ye^{-2x}=\int [(x+3)\,e^{-x}+k\,(x+3)]\,e^{-2x}\,dx$$
$$=-xe^{-x}-4e^{-x}+k\,[-\tfrac{1}{2}xe^{-2x}-\tfrac{7}{4}e^{-2x}]+c_2$$
or $$y=-xe^{x}-4e^{x}-c_1\,(2x+7)+c_2e^{2x}.$$

Ex. 3. *Solve* $[xD^2+(1-x)\,D-2\,(1+x)]\,y=e^{-x}\,(1-6x)$.

Solution. The equation may be written as
$$[xD+(1+x)]\,[D-2]\,y=e^{-x}\,(1-6x).$$
Putting $(D-2)\,y=v$, $v=(1-3x)\,e^{-x}+ke^{-x}x^{-1}$.
$$\therefore\quad (D-2)\,y=(1-3x)\,e^{-x}+ke^{-x}x^{-1}$$
or $$y=xe^{-x}+c_1e^{2x}\int\frac{e^{-3x}}{x}\,dx+c_2e^{2x}$$

Ex. 4. *Solve* $[xD^2-(x+2)\,D+2]\,y=x^3$.

Solution. The equation is $(xD-2)\,(D-1)\,y=x^3$.
If $(D-1)\,y=v$, $v=x^3+kx^2$
or $(D-1)\,y=x^3+kx^2$.
$$\therefore\quad y=-x^3+c_1\,(x^2+2x+2)+c_2e^{x}.$$

Ex. 5. *Solve* $3x^2\dfrac{d^2y}{dx^2}+(2-6x^2)\dfrac{dy}{dx}-4y=0$.

Solution. The equation is $(3x^2D+2)\,(D-2)\,y=0$.

Proceeding as usual, $y=c_1e^{2x}\displaystyle\int e^{(3/2x)-2x}\,dx$.

Ex. 6. *Solve* $3x^2\dfrac{d^2y}{dx^2}+(2+6x-6x^2)\dfrac{dy}{dx}-4y=0$.

Solution. The equation is $(D-2)\,(3x^2D+2)\,y=0$.
If $(3x^2D+2)\,y=v$, $v=k\,.\,e^{2x}$.
$$\therefore\quad (3x^2D+2)\,y=ke^{2x},$$
$$y=c_1e^{2/3x}+c_2e^{2/3x}\int\frac{1}{3x^2}\,e^{(2x-2/3x)}\,dx.$$

3

Simultaneous Differential Equations

3·1. So far we have discussed differential equations involving two variables, *viz* one dependent (usually y) and the other independent (usually x). In the present chapter, we shall discuss differential equations in which there is one **independent variable and two or more than two dependent variables.** To completely solve such equations we shall require as many simultaneous equations as is the number of dependent variables.

3·2. Methods of solving simultaneous linear differential equations with constant coefficients.

Let x and y be the two dependent variables and t the independent variable. Thus, in simultaneous equations there occur differential coefficients with regard to t,

First Method. Use of operator D. Write D for d/dt and put equations in the form

$$f_1(D)\,x+f_2(D)\,y=T_1, \qquad \ldots(1)$$

$$\phi_1(D)\,x+\phi_2(D)\,y=T_2 \qquad \ldots(2)$$

where T_1 and T_2 are functions of independent variables t. Now **to eliminate y,** operate (1) by $\phi_2(D)$ and (2) by $f_2(D)$; then these equations become

$$f_1(D)\,\phi_2(D)\,x+f_2(D)\,\phi_2(D)\,y=\phi_2(D)\,T_1$$

and
$$\phi_1(D)\,f_2(D)\,x+f_2(D)\,\phi_2(D)\,y=f_2(D)\,T_2$$

Subtracting,

$$[f_1(D)\,\phi_2(D)-\phi_1(D)\,f_2(D)]\,x=\phi_2(D)\,T_1-f_2(D)\,T_2$$

i.e.
$$F_1(D)\,x=T,$$

which is linear equation in x and t and can be solved to give x.

Putting this value of x in (1) or (2), we get value of y.

Note. We can also eliminate x to get a linear equation in y and t which when solved gives y. And x can be obtained from (1) and (2) after putting the value of y there.

Second Method. Method of Differentiation. Sometimes x or y can be conveniently eliminated if we differentiate (1) or (2) or both. The resulting equations after eliminating one dependent variable (x or y) are solved to give the value of another dependent variable. And then the value of the other variable can be found.

3·3. (Important) Number of Arbitrary Constants

The number of arbitrary constants in the general solutions of (1) and (2) of § 3·2 is equal to the degree of D in

$$\Delta=\begin{vmatrix} f_1(D) & f_2(D) \\ \phi_1(D) & \phi_2(D) \end{vmatrix} \quad \text{if } \Delta\neq 0.$$

In case $\Delta=0$, the system is dependent; such cases will not be considered.

Ex. 1. *Solve* $\dfrac{dx}{dt}+2x-3y=t,\ \dfrac{dy}{dt}-3x+2y=e^{2t}.$

[Karnatak B.Sc. (Snb.) 61]

Solution. Writing D for d/dt, the equations are

$$(D+2)\,x-3y=t. \qquad \text{...(1)}$$

$$-3x+(D+2)\,y=e^{2t}. \qquad \text{...(2)}$$

Multiplying (1) by $(D+2)$, (2) by 3 and adding, we get

$$[(D+2)^2-9]\,x=(D+2)\,t+3e^{2t}$$

or $$(D^2+4D-5)\,x=2t+1+3e^{2t}.$$

A.E. is $(D+5)(D-1)=0,\ D=1,\ -5.$

$$\therefore\ \text{C.F.}=c_1e^t+c_2e^{-5t}.$$

$$\text{P.I.}=\frac{2t+1}{D^2+4D-5}+\frac{3e^{2t}}{D^2+4D-5}$$

$$=-\tfrac{1}{5}\,(1-\tfrac{4}{5}D-\tfrac{1}{5}D^2)^{-1}\,(2t+1)+\frac{3e^{2t}}{4+8-5}$$

$$=-\tfrac{1}{5}\,(1+\tfrac{4}{5}D+\dots)\,(2t+1)+\tfrac{3}{7}e^{2t}$$

$$=-\tfrac{1}{5}\,(2t+1+\tfrac{8}{5})+\tfrac{3}{7}e^{2t}=\tfrac{3}{7}e^{2t}-\tfrac{2}{5}t-\tfrac{13}{25}.$$

$$\therefore\ x=c_1e^t+c_2e^{-5t}+\tfrac{3}{7}e^{2t}-\tfrac{2}{5}t-\tfrac{13}{25}.$$

This gives $\dfrac{dx}{dt}=c_1e^t-5c_2e^{-5t}+\tfrac{6}{7}e^{2t}-\tfrac{2}{5}.$

Now from (1), $3y=\dfrac{dx}{dt}+2x-t$

$$=3c_1e^t-3c_2e^{-5t}+\tfrac{12}{7}e^{2t}-\tfrac{9}{5}t=\tfrac{36}{25}.$$

$$\therefore\ y=c_1e^t-c_2e^{-5t}+\tfrac{4}{7}e^{2t}-\tfrac{3}{5}t-\tfrac{12}{25}.$$

Ex. 2. *Solve* $\dfrac{dx}{dt}+5x+y=e^t,\ \dfrac{dy}{dt}-x+3y=e^{2t}.$

[Delhi Hons. 68]

Solution. The equations are

$$(D+5)\,x+y=e^t,\ -x+(D+3)\,y=e^{2t}.$$

Multiplying first by $(D+3)$ and subtracting second from it, we get

$$(D+5)(D+3)\,x+x=(D+3)\,e^t-e^{2t}$$

or $$(D^2+8D+15)\,x=4e^t-e^{2t}.$$

From this A.E is $(D+5)(D+3)=0.$

$$\therefore\ x=c_1e^{-5t}+c_2e^{-3t}+\frac{4e^t-e^{2t}}{D^2+8D+15}$$

$$=c_1e^{-5t}+c_2e^{-3t}+\tfrac{1}{6}e^t-\tfrac{1}{35}e^{2t}.$$

Also $y=e^t-(D+5)\,x$ etc.

Ex 3. (a) *Solve the simultaneous equations*

$$\frac{dx}{dt}-7x+y=0\ ;\ \frac{dy}{dt}-2x-5y=0.$$

[Delhi Hons. 72, 70, 67 ; Karnatak 62 ; Agra 71, 58 ; Bombay 65]

Solution. Writing D for d/dt, the equations are

$$(D-7)\,x+y=0 \qquad ...(1)$$

$$-2x+(D-5)\,y=0. \qquad ...(2)$$

Eliminating y from these, we get

$(D-7)\,(D-5)\,x+2x=0$, *i.e.* $(D^2-12D+37)\,x=0$

Auxiliary equation is $D^2-12D+37=0$, $D=6\pm i$.

$\therefore\quad x=e^{6t}\,(c_2 \cos t+c_2 \sin t)$,

so that $\dfrac{dx}{dt}=6e^{6t}\,(c_1 \cos t+c_2 \sin t)+e^{6t}\,(-c_1 \sin t+c_2 \cos t)$

$=6x+e^{6t}\,(-c_1 \sin t+c_2 \cos t)$.

From (1), $y=7x-\dfrac{dx}{dt}$

$=7x-[6x+e^{6t}\,\{-c_1 \sin t+c_2 \cos t\}]$

$=x-e^{6t}\,(-c_1 \sin t+c_2 \cos t)$

$=e^{6t}\,[(c_1 \cos t+c_2 \sin t)-(-c_1 \sin t+c_2 \cos t)]$

$=e^{6t}\,[(c_1-c_2) \cos t+(c_2+c_1) \sin t]$.

Hence the solution is

$x=e^{6t}\,[c_1 \cos t+c_2 \sin t]$,

$y=e^{6t}\,[(c_1-c_2) \cos t+(c_2+c_1) \sin t]$.

Ex. 3. (b) *Solve* $\dfrac{dx}{dt}+7x-y=0,\ \dfrac{dy}{dt}+2x+5y=0.$ **[Raj. 61]**

Hint. Proceed as above.

***Ex. 4.** *Solve the simultaneous equations*

$$\frac{d^2x}{dt^2}-3x-4y=0,\ \frac{d^2y}{dt^2}+x+y=0.$$

[Delhi Hons. 68 ; Agra 69, 62, 56]

Solution. Writing D for (d/dt), the equations are

$$(D^2-3)\,x-4y=0 \qquad ...(1)$$

and $$x+(D^2+1)\,y=0. \qquad ...(2)$$

Eliminating y, we get $(D^2+1)\,(D^2-3)\,x+4x=0$

or $(D^4-2D^2+1)\,x=0$ or $(D^2-1)^2\,x=0$,

$D=-1, -1, +1, +1$,

$$\therefore\quad x=(c_1+c_2t)\,e^{-t}+(c_3+c_4t)\,e^t. \qquad ..(3)$$

Now $Dx=(dx/dt)=-(c_1+c_2t)\,e^{-t}+c_2e^{-t}+(c_2+c_4t)\,e^t+c_4e^t$

and $D^2x=(c_1+c_2t)\,e^{-t}-2c_2e^{-t}+(c_3+c_4t)\,e^t+2c_4e^t$.

From (1), $y=\frac{1}{2}(D^2-3)\,x=\frac{1}{2}(D^2x-3x)$

$$=\tfrac{1}{2}\,[(c_1+c_2t)\,e^{-t}+(c_3+c_4t)\,e^t-2c_2e^{-t}+2c_4e^t-3\,(c_1+c_2t)\,e^{-t}-3\,(c_3+c_4t)\,e^t]$$

$$=\tfrac{1}{2}\,[e^t\,(c_4-c_3-c_4t)-e^{-t}\,(c_1+c_2t+c_2)] \qquad \ldots(4)$$

Hence the solution is

$$x=(c_1+c_2t)\,e^{-t}+(c_3+c_4t)\,e^t,$$
$$y=\tfrac{1}{2}\,[e^t\,(c_4-c_3-c_4t)-e^{-t}\,(c_1+c_2t+c_2)].$$

Ex. 5. *Solve* $\dfrac{dy}{dx}=2z-y,\ \dfrac{dz}{dx}=4z-2y.$ **[Bombay 61]**

Sulution. Equations are

$$(D+1)\,y+3z=0,\ (D-4)\,z+2y=0.$$

Eliminating y, $[(D+1)\,(D-4)-6]\,z=0.$

Auxiliary equation is

$$D^2-3D-10=0,\ (D-5)\,(D+2)=0.$$

Thus $\quad z=c_1e^{5x}+c_2e^{-2x},\ \dfrac{dz}{dx}=5c_1e^{5x}-2c_2e^{-2x}.$

From second equation, $2y=4z-(dz/dx)$,

i.e. $\quad 2y=4\,(c_1e^{5x}+c_2e^{-2x})-(5c_1e^{5x}-2c_2e^{-2x})$,

or $\quad y=-\frac{1}{2}c_1e^{5x}+3c_2e^{-2x}.$

Ex. 6 *Solve* $\dfrac{dy}{dx}=y,\ \dfrac{dz}{dx}=2y+z.$ **[Bombay 61 (New Course)]**

Solution. First equation is $(D-1)\,y=0$, giving $y=c_1e^x$.

$\therefore$ 2nd equation gives $dz/dx-z=2c_1e^x$, linear, I.F.$=e^{-x}$.

$$\therefore\quad ze^{-x}=c_2+2c_1\int e^xe^{-x}\,dx=c_2+2c_1x.$$

Hence $\quad y=c_1e^x,\ z=c_2e^x+2c_1xe^x.$

Ex. 7. *Solve* $\dfrac{dx}{dt}=3x+2y,\ \dfrac{dy}{dt}+5x+3y=0$ **[Karnatak 61]**

Solution. Equations are

$$(D-3)\,x-2y=0,\ 5x+(D+2)\,y=3.$$

Eliminating y, $[D-3)\,(D+3)+10]\,x=0.\quad (D^2+1)\,x=0.$

$\therefore\quad x=c_1\cos t+c_2\sin t,\ \dfrac{dx}{dt}=-c_1\sin t+c_2\cos t.$

$\therefore\quad 2y=\dfrac{dx}{dt}-3x=(-c_1\sin t+c_2\cos t)-3\,(c_1\cos t+c_2\sin t)$

or $\quad y=\frac{1}{2}\,(c_2-3c_1)\cos t-\frac{1}{2}\,(c_1+3c_2)\sin t.$

Ex 8. *Solve* $\dfrac{dx}{dt}=3x-y,\ \dfrac{dy}{dt}=x+y$ **[Poona 61]**

Solution. Equations are

$$(D-3)\,x+y=0,\ x-(D-1)\,y=0.$$

Eliminating y, $[(D-3)(D-1)+1]\, x=0$

or $(D^2-4D+4)\, x=0$, $(D-2)^2\, x=0$.

$\therefore\ x=(c_1+c_2t)\, e^{2t},\ \dfrac{dx}{dt}=e^{2t}\,(2c_1+2c_2t+c_2).$

Now $y=3x-\dfrac{dx}{dt}=(c_1+c_2t-c_2)\, e^{2t}.$

Ex. 9. *Solve* $\dfrac{dx}{dt}+\dfrac{dy}{dt}+2x+y=0,\ \dfrac{dy}{dt}+5x+3y=0.$

[Delhi Hons. 69 ; Gujrat 61]

Solution. Equations are

$(D+2)\, x+(D+1)\, y=0,\ (D+3)\, y+5x=0.$

Eliminating y, $[(D+3)(D+2)-5(D+1)]\, x=0.$

Auxiliary equation is $(D^2+1)=0$, $D=\pm i$.

$\therefore\ x=c_1 \cos t+c_2 \sin t,\ \dfrac{dx}{dt}=-c_1 \sin t+c_2 \cos t.$

Now subtracting first equation from second, $2y=\dfrac{dx}{dt}+2x$,

i.e. $y=\frac{1}{2}\,[(-c_1 \sin t+c_2 \cos t)+2\,(c_1 \cos t+c_2 \sin t)]$

$=\frac{1}{2}\,(c_2+2c_1) \cot t+\frac{1}{2}\,(2c_2-c_1) \sin i.$

Ex. 10. *Solve* $\dfrac{d^2x}{dt^2}-3x-y=e^t,\ \dfrac{dy}{dt}-2x=0:$ **[Poona 59]**

Solution. $(D^2-3)\, x-y=e^t,\ Dy-2x=0.$

Eliminating x, we get $D\,(D^2-3)\, y-2y=2e^t.$

A.E. is $(D^3-3D-2)=0$, *i.e.* $(D+1)^2\,(D-2)=0.$

$\therefore\ y=(c_1+c_2t)\, e^{-t}+c_3e^{2t}+\dfrac{2e^t}{D^3-3D-2}$

$=(c_1+c_2t)\, e^{-t}+c_3e^{2t}=\frac{1}{2}e^t.$

and now $x=\frac{1}{2}\,\dfrac{dy}{dt}$ etc.

***Ex. 11. (a)** *Solve the simultaneous equations*

$\dfrac{dx}{dt}+4x+3y=t,\ \dfrac{dy}{dt}+2x+5y=e^t.$

[Agra 68, 54 ; Bombay 58]

Solution. Writing D for d/dt, the equations are

$(D+4)\, x+3y=t$ ...(1)

and $2x+(D+5)\, y=e^t.$ ...(2)

Multiplying (1) by $(D+5)$, (2) by 3 and then subtracting (2) from (1), we get

$(D+5)(D+4)\, x-6x=(D+5)\, t-3e^t$

or $(D^2+9D+14)\, x=1+5t-3e^t.$

Auxiliary equation is

$D^2+9D+14=0$, i.e $(D+7)(D+2)=0$.

$\therefore$ C.F.$=c_1e^{-7t}+c_2e^{-2t}$.

$$\text{P.I.}=\frac{1+5t-3e^t}{D^2+9D+14}$$

$$=\tfrac{1}{14}\left[1+\tfrac{9}{14}D+\tfrac{1}{14}D^2\right]^{-1}(1+5t)-\frac{3e^t}{1+9+14}$$

$$=\tfrac{1}{14}\left[1-\tfrac{9}{14}D\ldots\right](1+5t)-\frac{3e^t}{24}$$

$$=\tfrac{1}{14}\left[1+5t-\tfrac{45}{14}\right]-\tfrac{1}{8}e^t=\tfrac{5}{14}t-\tfrac{31}{196}-\tfrac{1}{8}e^t.$$

Hence $x=c_2e^{-7t}+c_2e^{-2t}+\frac{5}{14}t-\frac{31}{196}-\frac{1}{8}e^t$.

Now from (1),

$$y=\tfrac{1}{3}\left[t-(D+4)x\right]=\tfrac{1}{3}\left[t-\frac{dx}{dt}-4x\right]$$

$$=\tfrac{1}{3}\left[t+7c_1e^{-7t}+2c_2e^{-2t}-\tfrac{5}{14}+\tfrac{1}{8}e^t\right.$$
$$\left.-4\left(c_1e^{-7t}+c_2e^{-2t}+\tfrac{5}{14}t-\tfrac{31}{196}-\tfrac{1}{8}e^t\right)\right]$$

or $y=\frac{1}{3}\left[3c_1e^{-7t}-2c_2e^{-2t}-\frac{3}{7}t+\frac{5}{8}e^t+\frac{27}{98}\right]$.

Ex. 11. (b) *Solve* $\dfrac{dx}{dt}+4x+3y=t^2,\ \dfrac{dy}{dt}+2x+5y=e^{2t}$.

[Karnatak 63]

Proceed as in the above example

Ex. 11. (c) *Solve the simultaneous equations*

$$\left.\begin{aligned}(5D+4)\,y-(2D+1)\,z&=e^{-x}\\(D+8)\,y-3z&=5e^x\end{aligned}\right\}\ D=\frac{d}{dx}.$$

[Meerut 77, 70]

Solution. Multiplying first by 3 and second by $(2D+1)$ and subtracting, we get

$$[3(5D+4)-(D+8)(2D+1)]\,y=3e^{-x}-(2D+1)\,5e^x.$$

$D=1$, -2 etc.

***Ex. 12.** *Solve the equations*

$$\frac{dx}{dt}=-\omega y \text{ and } \frac{dy}{dt}=\omega x.$$

[Agra 57, 52]

Solution. Differentiating the first equation w.r.t. t, we have

$$\frac{d^2x}{dt^2}=-\omega\frac{dy}{dt}=-\omega(\omega x).$$

$\therefore$ $(D^2+\omega^2)\,x=0$. $D=\pm\omega i$.

$\therefore$ $x=c_1\cos\omega t+c_2\sin\omega t$

Now $y=-\dfrac{1}{\omega}\left(\dfrac{dx}{dt}\right)=-\dfrac{1}{\omega}\left[-c_1\omega\sin\omega t+c_2\omega\cos\omega t\right]$

$=c_1\sin\omega t-c_2\cos\omega t$.

A Deduction. *Show that point (x, y) lies on a circle.*

We have $x=c_1\cos\omega t+c_2\sin\omega t$

and $y=c_1 \sin \omega t-c_2 \cos \omega t$.

Squaring and adding, $x^2+y^2=c_1^2+c_2^2$, a circle.

Hence point (x, y) lies on a circle.

Ex. 13. *Solve the equations*

$$\frac{d^2x}{dt^2}-\frac{dy}{dt}=2x+2t, \frac{dx}{dt}+4\frac{dy}{dt}=3y. \quad \text{[Raj. 56]}$$

Solution. The equations are

$$(D^2-2)\, x-Dy=2t, \quad \ldots(1)$$

$$(Dx+(4D-3)\, y=0 \quad \ldots(2)$$

Multiplying (1) by $(4D-3)$, (2) by D and adding, we get

$$[(D^2-2)(4D-3)+D^2]\, x=(4D-3)\, 2t$$

or $(4D^3-2D^2-8D+6)\, x=8-6t$

or $(2D^3-D^2-4D+3)\, x=4-3t$.

Auxiliary equation is

$$2D^3-D^2-4D+3=0, (D-1)(2D^2+D-3)=0$$

or $(D-1)(2D+3)(D-1)=3, D=1, 1, -\frac{3}{2}$.

$\therefore$ C F.$+(c_1+c_2t)\, e^t-c_3e^{-3t/2}$.

$$\text{P.I.}=\tfrac{1}{3}\,(1-\tfrac{4}{3}D-\tfrac{1}{3}D^2+\tfrac{2}{3}D^3)^{-1}(4-3t)$$

$$=\tfrac{1}{3}\,(1+\tfrac{4}{3}D\ldots)(4-3t)=\tfrac{1}{3}\,(4-3t-4)$$

$$=-t.$$

Hence, $x=(c_1+c_2t)\, e^t-c_2e^{-3t/2}-t$.

$\therefore$ $(dx/dt)=(c_1+c_2t+c_2)\, e^t+\frac{3}{2}c_3e^{-3t/2}-1$.

$\therefore$ (2) gives

$$4\,(dy/dx)-3y=-\{(c_1+c_2t+c_2)\, e^t+\tfrac{3}{2}c_3e^{-3t/2}-1\}.$$

Linear, I.F.$=e^{-3t/4}$.

$$\therefore\ ye^{-3t/4}=c_4-\tfrac{1}{4}\int e^{-3t/4}\{(c_1+c_2t+c_2)\, e^t+\tfrac{3}{2}c_3e^{-3t/2}-1\}\, dt.$$

Ex. 14. *Solve the simultaneous equations*

$$\frac{d^2x}{dt^2}+m^2y=0, \frac{d^2y}{dt^2}-m^2x=0 \quad \text{[Poona 64; Agra 53]}$$

Solution. Writing D for d/dt, the equations are

$$D^2x+m^2y=0, D^2y-m^2x=0.$$

Eliminating y from these, we get

$$(D^4+m^4)\, x=0.$$

Auxiliary equation is $D^4+m^4=0, (D^2+m^2)^2-2m^2D^2=0$

or $(D^2-\sqrt{2}mD+m^2)(D^2+\sqrt{2}mD+m^2)=0$

These give $D=\dfrac{m\pm im}{\sqrt{2}}, \dfrac{-m\pm im}{\sqrt{2}}$.

Hence

$$x=c_1e^{(m/\sqrt{2})\,t}\cos\left(\frac{m}{\sqrt{2}}\,t+c_2\right)+c_3e^{-(m/\sqrt{2})\,t}\cos\left(\frac{m}{\sqrt{2}}\,t+c_4\right)$$

Now

$$\frac{dx}{dt}=c_1e^{(m/\sqrt{2})\,t}\,\frac{m}{\sqrt{2}}\left[\cos\left(\frac{m}{\sqrt{2}}t+c_2\right)-\sin\left(\frac{m}{\sqrt{2}}t+c_2\right)\right]$$
$$-c_3e^{-(m/\sqrt{2})\,t}\,\frac{m}{\sqrt{2}}\left[\cos\left(\frac{m}{\sqrt{2}}t+c_4\right)+\sin\left(\frac{m}{\sqrt{2}}t+c_4\right)\right]$$

and $$\frac{d^2x}{dt^2}=c_1e^{(m/\sqrt{2})\,t}\,\frac{m^2}{2}\left[\cos\left(\frac{m}{2}\,t+c_2\right)-2\sin\left(\frac{m}{\sqrt{2}}t+c_2\right)\right.$$
$$\left.-\cos\left(\frac{m}{\sqrt{2}}\,t+c_2\right)\right]$$
$$+c_3e^{(m/\sqrt{2})\,t}\,\frac{m^2}{2}\left[\cos\left(\frac{m}{\sqrt{2}}t+c_4\right)+2\sin\left(\frac{m}{\sqrt{2}}t+c_4\right)\right.$$
$$\left.-\cos\left(\frac{m}{\sqrt{2}}t+c_4\right)\right]$$
$$=m^2\left[c_3e^{-(m/\sqrt{2})\,t}\sin\left(\frac{m}{\sqrt{2}}t+c_4\right)-c_1e^{(m/\sqrt{2})\,t}\sin\left(\frac{m}{\sqrt{2}}\,t+c_2\right)\right].$$

From given equation, $y=-\dfrac{1}{m^2}\dfrac{d^2x}{dt^2}$.

$$\therefore\quad y=c_1e^{(m/\sqrt{2})\,t}\sin\left(\frac{m}{\sqrt{2}}\,t+c_2\right)-c_3e^{-(m/\sqrt{2})\,t}\sin\left(\frac{m}{\sqrt{2}}t+c_4\right)].$$

Ex. 15. *Solve* $\dfrac{dx}{dt}+\dfrac{2}{t}(x-y)=1.$ ...(1)

$\dfrac{dy}{dt}+\dfrac{1}{t}(x+5y)=t.$ **[Raj. 64; Agra 63]**

Solution. (1) is $t\,(dx/dt)+2x-2y=t$.

Differentiating it w.r.t. t, we get

$$t\,\frac{d^2x}{dt^2}+\frac{dx}{dt}+2\,\frac{dx}{dt}-2\,\frac{dy}{dt}=1.$$

Now putting in it value of $\dfrac{dy}{dt}$ from (2),

$$t\,\frac{d^2x}{dt^2}+3\,\frac{dx}{dt}-2\left[t-\frac{x+5y}{t}\right]=1$$

Now putting in it value of $\dfrac{dy}{dt}$ from (2)

$$t\,\frac{d^2x}{dt^2}+3\,\frac{dx}{dt}-2\left[t-\frac{x+5y}{t}\right]=1$$

or $$t^2\frac{d^2x}{dt^2}+3t\,\frac{dx}{dt}-2t^2+2x+10y=t.$$

Putting value of y from (1), we get

$$t^2\frac{d^2x}{dt^2}+3t\,\frac{dx}{dt}-2t^2+2x+5\left[t\,\frac{dx}{dt}+2x-t\right]=t$$

or $\quad t^2 \dfrac{d^2x}{dt^2}+8t\dfrac{dx}{dt}+12x=2t^2+6t.$

This is homogeneous, so putting $t=e^z$, $D=t\,(d/dt)$, we get

$$[D\,(D-1)+8D+12]\;x=2e^{2z}+6e^z.$$

A.E. is $D^2+7D+12]=0$, $\quad$ *i.e.* $\quad (D+3)\,(D+4)=0.$

$\therefore \quad$ C.F. $=c_1e^{-3z}+c_2e^{-4z}=\dfrac{c_1}{t^3}+\dfrac{c_2}{t^4}.$

$$\text{P.I.}=\frac{2e^{2z}+6e^z}{D^2+7D+12}=\frac{2e^{2z}}{4+14+12}+\frac{6e^z}{1+7+12}$$

$$=\frac{e^{2z}}{15}+\frac{3e^z}{10}=\frac{t^2}{15}+\frac{3t}{10} \text{ as } e^z=t.$$

Hence $\quad x=\dfrac{c_1}{t^3}+\dfrac{c_2}{t^4}+\dfrac{t^2}{15}+\dfrac{3t}{10},$

so that $\quad \dfrac{dx}{dt}=-\dfrac{3c_1}{t^4}-\dfrac{4c_2}{t^5}+\dfrac{2t}{15}+\dfrac{3}{10}.$

From (1),

$$2y=t\,\frac{dx}{dt}+2x-t$$

$$=-\frac{3c_1}{t^3}-\frac{4c_2}{t^4}+\frac{2t^2}{15}+\frac{3t}{10}+2\left(\frac{c_1}{t^3}+\frac{c_2}{t^4}+\frac{t^2}{15}+\frac{3t}{10}\right)-t$$

$$=-\frac{c_1}{t^3}-\frac{2c_2}{t^4}+\frac{4t^2}{15}-\frac{1}{10}\,t.$$

$\therefore \quad y=-\tfrac{1}{2}c_1t^{-3}-c_2t^{-4}+\tfrac{2}{15}t^2-\tfrac{1}{20}t.$

Ex. 16. (a) *Solve* $\dfrac{dx}{dt}+\dfrac{dy}{dt}-2y=2\cos t-7\sin t,$

$\dfrac{dx}{dt}-\dfrac{dy}{dt}+2x=4\cos t-3\sin t.$ **[Raj. 60]**

Solution. The equations can be written as

$Dx+(D-2)\,y=2\cos t-7\sin t,$ ...(1)

$(D+2)\,x-4y=4\cos t-3\sin t$ ...(2)

Multiplying (1) by D, (2) by $(D-2)$ and adding, we get

$$[D^2+(D+2)\,(D-2)]\;x$$

$$=D\,[2\cos t-7\sin t]+(D-2)\,[4\cos t-3\sin t]$$

or $\quad (2D^2-4)\,x=-18\cos t \quad$ or $\quad (D^2-2)\,x=-9\cos t.$

C.F. $=e^{\sqrt{2}t}+c_2e^{-\sqrt{2}t}.$

$$\text{P.I.}=\frac{-9\cos t}{D^2-2}=\frac{-9\cos t}{-1-2}=3\cos t.$$

$\therefore \quad x=c_1e^{\sqrt{2}t}+c_2e^{-\sqrt{2}t}+3\cos t.$

$dx/dt=\sqrt{2}c_1e^{\sqrt{2}t}-2c_2e^{-\sqrt{2}t}-3\sin t.$

From (2), $\dfrac{dy}{dt}=\dfrac{dx}{dt}+2x-4\cos t+3\sin t$

$$=(2+\sqrt{2})\,c_1e^{\sqrt{2}t}+(2-\sqrt{2})\,c_2e^{-\sqrt{2}t}+2\cos t.$$

From (1), $2y=\dfrac{dx}{dt}+\dfrac{dy}{dt}-2\cos t-7\sin t$

$$=(2+2\sqrt{2})\,c_1e^{\sqrt{2}t}+(2-2\sqrt{2})\,c_2e^{-\sqrt{2}t}-10\sin t$$

or $$y=(1+\sqrt{2})\,c_1e^{\sqrt{2}t}+(1-\sqrt{2})\,c_2e^{-\sqrt{2}t}-5\sin t.$$

Ex. 16 (b) *Solve* $\dfrac{dx}{dt}+2\dfrac{dy}{dt}-2x+2y=3e^t.$ ...(1)

$$3\frac{dx}{dt}+\frac{dy}{dt}+2x+y=4e^{2t}$$ ...(2)

[Agra 64; Poona 64]

Solution. Multiplying (2) by 2 and subtracting from it (1), we get $5\,(dx/dt)+6x=8e^{2t}-3e^t$

or $dx/dt+\frac{6}{5}x=\frac{8}{5}e^{2t}-\frac{3}{5}e^t$. I.F.$={}^{6t/5}$.

$$\therefore\quad xe^{6t/5}=\int\left(\tfrac{8}{5}e^{2t}-\tfrac{3}{5}e^t\right)e^{6t/5}\,dt+c_1$$

or $x=\frac{1}{2}e^{2t}+c_1e^{-6t/5}-\frac{3}{11}e^t$

and then $y=c_2e^{-t}-8c_1e^{-6t/5}+\frac{15}{22}e^t.$

Ex. 16. (c) *Find a general solution of the system of the equations*

$(D-1)\,x+y=t^2,\ (D+1)\,x-Dy=t,$

where $D\equiv d/dt.$ **[Rajasthan 67]**

Proceed as above.

Ex. 17. *Solve* $4\dfrac{dx}{dt}+9\dfrac{dy}{dt}+2x+31y=e^t,$

$$3\frac{dx}{dt}+7\frac{dy}{dt}+x+24y=3.$$

Solution. The equations are

$(4D+2)\,x+(9D+31)\,y=e^t,$

$(3D+1)\,x+(7D+24)\,y=3.$

Eliminating x, we get

$(D^2+8D+17)\,y=6-4e^t,\ D=-4\pm i,$

$y=e^{-4t}\,[c_1\sin t+c_2\cos t]-\frac{2}{13}e^t+\frac{6}{17}.$

Again eliminating dx/dt from the given equations, we get

$$\frac{dy}{dt}+3y-2x=12-3e^t,$$

i.e $2x=\dfrac{dy}{dt}+3y-12+3e^t.$

$$x=\tfrac{1}{2}\,[-(c_1+c_2)\sin t+(c_1-c_2)\cos t]\,e^{-4t}+\tfrac{37}{26}e^t-\tfrac{93}{17}.$$

Ex. 18 *Solve* $t\,dx=(t-2x)\,dt,$ ...(1)

$t\,dy=(tx+ty+2x-t)\,dt.$ ...(2)

[Agra 72, 70 ; Bombay 61]

Solution. From (1), $\dfrac{dx}{dt}+\dfrac{2}{t}x=1$, I.F.$=t^2$,

$$\therefore \quad xt^2=\int t^2\,dt+c=\tfrac{1}{3}t^3+c$$

or $\qquad x=\tfrac{1}{3}t+ct^{-2}.$ ...(3)

Now (2) can be written as

$t\,dy=t\,(x+y)\,dt-(t-2x)\,dt,$

i.e. $t\,dy=t\,(x+y)\,dt-t\,dx$ from (1)

i.e. $(dx+dy)=(x+y)\,dt$

or $\dfrac{dx+dy}{x+y}=dt;\qquad \therefore \quad \log(x+y)=t+\log c_1.$

$\therefore \quad x+y=c_1e^t.$ ...(4)

(3) and (4) from the solu'ion of equations.

Ex. 19. *Solve* $2\dfrac{d^2y}{dx^2}-\dfrac{dz}{dx}-4y=2x,$

$2\dfrac{dy}{dx}+4\dfrac{dz}{dx}-3z=0.$ **[Raj. 55, 52]**

Solution. Putting D for d/dx, the equation is

$(2D^2-4)\,y-Dz=2x,$ ...(1)

$2Dy+(4D-3)\,z=0.$ ...(2)

Multiplying (2) by D, (1) by $(4D-3)$ and adding, we get

$[(2D^2-4)(4D-3)+2D^2]\,y=(4D-3)\,2x,$

i.e. $(8D^3-4D^2-16D+12)\,y=8-6x$

or $(2D^3-D^2-4D+3)\,y=\tfrac{1}{2}(4-3x).$

A.E. is $2D^3-D^2-4D+3=0$, $(D-1)(2D^2+D-3)=0,$

i.e. $(D-1)(2D+3)(D-1)=0$, $D=1,1,-\tfrac{3}{2}.$

$\therefore \quad$ C.F.$=(c_1+c_2x)\,e^x+c_3e^{-(3/2)x}.$

$$\text{P.I.}=\frac{\tfrac{1}{2}(4-3x)}{2D^3-D^2-4D+3}=\tfrac{1}{6}\left(1-\tfrac{4}{3}D-\tfrac{1}{3}D^2+\tfrac{2}{3}D^3\right)^{-1}\left(\tfrac{1}{4}-3x\right)$$

$$=\tfrac{1}{6}\left(1+\tfrac{4}{3}D+\ldots\right)(4-3x)$$

$$=\tfrac{1}{6}(4-3x-4)=-x/2.$$

Hence $y=(c_1+c_2x)\,e^x+c_3e^{-3x/2}-\tfrac{1}{2}x,$

$$\frac{dy}{dx}=e^x\,(c_1+c_2x+c_2)-\tfrac{3}{2}c_3e^{-3x/2}-\tfrac{1}{2},$$

$$\frac{d^2y}{dx^2}=e^x\,[c_1+c_2x+2c_2]+\tfrac{9}{4}c_3\,e^{-3x/2}.$$

Now from (1), $\frac{dz}{dx}=2\frac{d^2y}{dx^2}-4y-2x$

$$=e^x\,[-2c_1-2c_2x+4c_2]+\tfrac{3}{2}c_3e^{-3x/2}$$

Now putting values of $\frac{dy}{dx}$ and $\frac{dz}{dx}$ in (2),

$$3z=2\,\frac{dy}{dx}+4\,\frac{dz}{dx}$$

$$=e^x\,[-6c_1+6c_2x+18c_2]-c_3e^{-3x/2}-1.$$

$\therefore\ z=-2e^x\,(c_1+c_2x-3c_2)-\frac{1}{3}c_3e^{-3x/2}-\frac{1}{3}.$

Ex. 20. *Solve the simultaneous equations*

$$\frac{d^2x}{dt^2}+4x+y=te^t,\ \frac{d^2y}{dt^2}+y-2x=\sin^2 t.$$ [Pb. 58]

Solution. The equation can be written as

$(D^2+4)\,x+y=te^t,$ ...(1)

$-2x+(D^2+1)\,y=\sin^2 t.$ ...(2)

Multiplying (1) by (D^2+1) and subtracting (2) from it, we get

$$(D^2+1)\,(D^2+4)\,x+2x=(D^2+1)\,te^t-\sin^2 t$$

or $(D^4+5D^2+6)\,x=2e^t\,(t+1)-\sin^2 t.$

A.E. is $D^4+5D^2+6=0,\ (D^2+3)\,(D^2+2)=0,$

i.e. $D=\pm\sqrt{3}i,\ \pm\sqrt{2}i.$

$\therefore$ C.F.$=c_1\cos(\sqrt{3}t+c_2)+c_3\cos(\sqrt{2}t+c_4).$

$$\text{P.I.}=\frac{2e^t\,(t+1)}{D^4+5D^2+6}-\frac{\frac{1}{2}\,(1-\cos 2t)}{D^4+5D^2+6}$$

$$=2e^t\,\frac{1}{(D+1)^4+5\,(D+1)^2+6}\,(t+1)-\tfrac{1}{12}+\frac{\frac{1}{2}\cos 2t}{D^4+5D^2+6}$$

$$=2e^t\,\frac{1}{D^4+4D^3+11D^2+14D+12}\,(t+1)-\tfrac{1}{12}+\tfrac{1}{2}\,\frac{\cos 2t}{16-20+6}$$

$$=e^t/6\,(1+\tfrac{7}{6}D+\tfrac{11}{12}D^2+\ldots)^{-1}\,(t+1)-\tfrac{1}{12}+\tfrac{1}{4}\cos 2t$$

$$=\tfrac{1}{6}e^t\,(1-\tfrac{7}{6}D+\ldots)\,(t+1)-\tfrac{1}{12}+\tfrac{1}{4}\cos 2t$$

$$=\tfrac{1}{6}e^t\,(t+1-\tfrac{7}{6})-\tfrac{1}{12}+\tfrac{1}{4}\cos 2t$$

$$=\tfrac{1}{6}e^t\,(t-\tfrac{1}{6})-\tfrac{1}{12}+\tfrac{1}{4}\cos 2t.$$

$$\therefore\ x=c_1\cos(\sqrt{3}t+c_2)+c_3\cos(\sqrt{2}t+c_4)+\tfrac{1}{6}e^t\,(t-\tfrac{1}{6})$$
$$-\tfrac{1}{12}+\tfrac{1}{4}\cos 2t.$$

$$dx/dt=-\sqrt{3}c_1\sin(\sqrt{3}t+c_2)-\sqrt{2}c_3\sin(\sqrt{2}t+c_4)$$
$$+\tfrac{1}{6}e^t\,(t-\tfrac{1}{6}+1)-\tfrac{1}{2}\sin 2t,$$

$$d^2x/dt^2=-3c_1\cos(\sqrt{3}t-c_2)-2c_3\cos(\sqrt{2}t+c_4)$$
$$+\tfrac{1}{6}e^t\,(t-\tfrac{11}{6})-\cos 2t,$$

From (1), $y=te^t-d^2x/dt^2-4x$ etc.

Ex. 21. *Solve* $d^2x/dt^2+4x+y=te^{3t}$,

$d^2y/dt^2+y-2x=\cos^2 t.$

Solution. Eliminating y from these, we get

$(D^4+5C^2+6)\ x=10te^{3t}+6e^{3t}-\cos^2 t.$

C.F.$=(c_1 \cos \sqrt{3}t+c_2 \sin \sqrt{3}t)+(c_3 \cos \sqrt{2}t+c_4 \sin \sqrt{2}t).$

$$\text{P.I.}=\frac{10te^{3t}+6e^{3t}}{D^4+5D^2+6}-\frac{\frac{1}{2}(1+\cos 2t)}{D^4+5D^2+6}$$

$$=-\tfrac{49}{1452}e^{3t}+\tfrac{5}{66}te^{3t}-\tfrac{1}{12}-\tfrac{1}{4}\cos 2t.$$

$$\therefore\quad x=(c_1 \cos \sqrt{3}t+c_2 \sin \sqrt{3}t)+(c_2 \cos \sqrt{2}t+c_4 \sin \sqrt{2}t)$$
$$-\tfrac{42}{1452}e^{3t}+\tfrac{5}{66}e^{3t}-\tfrac{1}{12}-\tfrac{1}{4}\cos 2t.$$

Now from the first equation,

$$y=te^{3t}-4x-d^2x/dt^2$$
$$=-3c_1 \cos \sqrt{3}t-c_2 \sin \sqrt{3}t-2c_3 \cos \sqrt{2}t$$
$$-2c_4 \sin \sqrt{2}t+\tfrac{1}{66}te^{3t}+\tfrac{1}{3}-\tfrac{23}{1452}e^{3t}.$$

Ex. 22. *Solve* $4\dfrac{dx}{dt}+9\dfrac{dy}{dt}+44x+49y=t,$

$$3\frac{dx}{dt}+7\frac{dy}{dt}+34x+38y=e^t$$

Solution. The equations can be written as

$$(4D+44)\ x+(9D+49)\ y=t,$$
$$(3D+34)\ x+(7D+38)\ y=e^t.$$

Eliminating y as usual, we get on simplification

$$(D^2+7D+6)\ x=7-58e^t+38t,\ D=-1,\ -6.$$

$$\therefore\quad x=c_1e^{-t}+c_2e^{-6t}+\tfrac{19}{3}t-\tfrac{58}{9}-\tfrac{29}{7}e^t,$$
$$y=c_1e^{-t}+4c_2e^{-6t}+\tfrac{17}{3}t+\tfrac{55}{9}+\tfrac{24}{7}e^t.$$

Ex. 23. *Solve* $\dfrac{dx}{dt}=ny-mz,\ \dfrac{dy}{dt}=lz-nx,\ \dfrac{dz}{dt}=mx-ly.$

[Raj 62, 53]

Solution. Multiplying these by x, y, z respectively and adding, we get

$$x\frac{dx}{dt}+y\frac{dy}{dt}+z\frac{dz}{dt}=0.$$

Integrating, $\quad x^2+y^2+z^2=c_1.$ ...(1)

Next multiplying by l, m, n respectively and adding, we get

$$l\frac{dx}{dt}+m\frac{dy}{dt}+n\frac{dz}{dt}=0.$$

Integrating, $\quad lx+my+nz=c_2.$ ...(2)

Again multiplying by lx, my, nz respectively, we get

$$lx\frac{dx}{dt}+my\frac{dy}{dt}+nz\frac{dz}{dt}=0.$$

Integrating, $\quad lx^2+my^2+nz^2=0.$ ...(3)

(1), (2) and (3) from the complete solution.

Ex. 24. *Solve the simultaneous equations*

$$\frac{dx}{dt}=ax+by,\ \frac{dy}{dt}=a'x+b'y.$$

[Agra 61]

Solution. The given equations are

$$(D-a)\,x-by=0,\ (D-b')\,y-a'x=0.$$

Eliminating y, we get

$$(D-a)\,(D-b')\,x-a'\,bx=0.$$

A E. is $\quad D^2-(a+b')\,D+(ab'-a'b)=0.$

If m_1 and m_2 are its two roots, then

$$x=c_1e^{m_1t}+c_2e^{m_2t}$$

so that $\quad dx/dt=c_1m_1e^{m_1t}+c_2m_2e^{m_2t}$

$$\therefore\quad y=\frac{1}{b}\left[\frac{dx}{dt}-ax\right]$$

$$=[(m_1-a)\,c_1e^{m_1t}+(m_2-a)\,c_2e^{m_2t}]/b$$

Ex. 25. *Solve* $\dfrac{dx}{dt}+\dfrac{dy}{dt}+2x+y=0,\ \dfrac{dy}{dt}+5x+3y=0.$

[**Hint.** Eliminate y. $(D^2+1)\,x=0,\ x=c_1\cos t+c_2\sin t,$
$y=-\frac{1}{2}\,(c_1+2c_2)\sin t+\frac{1}{2}\,(c_2-3c_1)\cos t.$

Ex. 26. *Solve* $2\dfrac{dx}{dt}-\dfrac{dy}{dt}+2x+y=11t,$

$$2\frac{dx}{dt}+3\frac{dy}{dt}+5x-3y=2.$$

[Nagpur 53]

Hint. Eliminating y, $(8D+11)\ x=33t+2,$
$x=Ae^{-11t/8}+3t-2,\ y=\frac{6}{16}Ae^{-11t/8}+5t+3.$

Ex. 27. *Solve* $t\dfrac{dx}{dt}+y=0,\ t\dfrac{dy}{dt}+x=0.$ [Luck. 54]

Ans. $x=c_1t+c_2t^{-1},\ y=-c_1t+c_2t^{-1}.$

Ex. 28. *Solve* $t^2\dfrac{d^2x}{dt^2}+t\dfrac{dx}{dt}+2y=0,\ t^2\dfrac{d^2y}{dt^2}+t\dfrac{dy}{dt}-2x=0.$

[Meerut 68 ; Poona 60]

Ans. $x=At\cos(\log t-\alpha)+Bt^{-1}\cos(\log t-\beta)$
$y=At\sin(\log t-\alpha)-Bt^{-1}\sin(\log t-\beta).$

Ex. 29 *Solve* $\dfrac{d^2x}{dt^2}-2\dfrac{dy}{dt}-x=e^t\cos t,\ \dfrac{d^2y}{dt^2}+2\dfrac{dx}{dt}-y=e^t\sin t.$

Hint. Eliminate y.

$x=(c_1+c_2t)\cos t+(c_3+c_4t)\sin t+\frac{1}{25}e^t\,(4\sin t-3\cos t),$
$y=-(c_1+c_3t)\sin t+(c_2+c_4t)\cot t-\frac{1}{25}e^t\,(3\sin t+4\cos t).$

Ex. 30. *Solve*

(*i*) $dx/dt+x-y=e^t,\ dy/dt+y-x=0.$ [Mysore 70]

(*ii*) $dx/dt+x=y+e^t,\ dx/dt+y=x+e^t.$ [Mysore 68]

Proceed yourself.

Exercise

Solve the following simultaneous differential equations :

1. $\dfrac{dx}{dt}+2x+3y=0,\ \dfrac{dy}{dt}+3x+2y=2e^{2t}.$ [Poona 63]

Ans. $x=c_1e^t+c_2e^{-5t}-\frac{6}{7}e^{2t},\ y=-c_1e^t+c_2e^{-5t}+\frac{8}{7}e^{2t}$.

2. $5\dfrac{dy}{dx}-2\dfrac{dz}{dx}+4y-z=e^{-x},\ \dfrac{dy}{dx}+8y-3z=5e^{-x}$.

[Calcutta Hons. 63; Poona 62]

Hint. Eliminating z.

$(D^2+D-2)\,y=-4e^{-x};\quad \therefore\quad y=2e^{-x}+A_1e^x+B_1e^{-2x}$

then $(-2D^2-2D+4)\,z=12e^{-x},\ z=3e^{-x}+A_2e^{-x}+B_2e^{-2x}$

where $A_2=3A_1$ and $B_2=2B_1$.

3. $\dfrac{dx}{dt}+\dfrac{dy}{dt}-y=2t+1,\ 2\dfrac{dx}{dt}+2\dfrac{dy}{dt}+y=t$ **[Poona 63]**

Ans. $x=-t-\frac{8}{3},\ y=\frac{1}{2}t^2+\frac{4}{3}t+c_1$.

4. $\dfrac{dy}{dx}+y=z+e^x,\ \dfrac{dz}{dx}+z=y+e^x$. **[Nagpur 63]**

Ans. $y=e^x+A+Be^{-2x},\ z=e^x+A-Be^{-2x}$.

5. $\dfrac{d^2x}{dt^2}+x-2\dfrac{dy}{dt}=2t,\ 2\dfrac{dx}{dt}-x+\dfrac{dy}{dt}-2y=7$. **[Sagar 63]**

6. $2\dfrac{dx}{dt}+\dfrac{dy}{dt}-4x-y=e^t,\ \dfrac{dx}{dt}+3x+y=0$. **[Poona 63]**

Ans. $x=Ae^t+Be^{-t}-\frac{1}{2}e^t,\ y=-(D+3)\,x$.

7 $\dfrac{d^4y}{dx^4}-a\dfrac{d^3y}{dx^3}+b\dfrac{d^2z}{dx^2}+cy=0$.

$\dfrac{d^4z}{dx^4}+a\dfrac{d^3y}{dx^3}+b\dfrac{d^2z}{dx^2}+cz=0$. **[Sagar 60]**

8. $(D+1)\,x+(D-1)\,y=e^t$

$(D^2+D+1)\,x+(D^2-D+1)\,y=t^2$. **[Mysore 69]**

3·4. Simultaneous equations of the form

$$\mathbf{P_1\,dx+Q_1\,dy+R_1\,dz=0,} \qquad ...(1)$$

$$\mathbf{P_2\,dx+Q_2\,dy+R_2\,dz=0,} \qquad ...(2)$$

where $P_1, P_2, \ldots$ are all functions of x, y, z.

Solving these equations simultaneously, we get

$$\frac{dx}{Q_1R_2-Q_2R_1}=\frac{dy}{R_1P_2-R_2P_1}=\frac{dz}{P_1Q_2-P_2Q_1},$$

which is of the form

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}. \qquad ...(3)$$

Thus simultaneous equations (1) and (2) can always be put in the form (3).

3·5. Method of Solving $\dfrac{dx}{P}=\dfrac{dy}{Q}=\dfrac{dz}{R}$.

First method. We have

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{l\,dx+m\,dy+n\,dz}{lP+mQ+nR}.$$

If l, m, n are such that $lP+mQ+nR=0$,

then we get $\qquad l\,dx+m\,dy+n\,dz=0.$

If it is an exact differential equation du (say), then $u=a$ is one part of the complete solution.

Similarly, if we can choose l', m', n' such that

$$l'\,P+m'\,Q+n'\,R=0,$$

we get $\quad l'\,dx+m'\,dy+n'\,dz=0.$

This gives another equation on integration.

The two equations so obtained form the complete solution.

Second Method. The equations are $\dfrac{dx}{P}=\dfrac{dy}{Q}=\dfrac{dz}{R}$.

Take any two members $\dfrac{dx}{P}=\dfrac{dy}{Q}$ (say) and integrate this equation to obtain an integral.

Next choose other two members $\dfrac{dx}{P}=\dfrac{dz}{R}$ (say).

This on integration gives an another integral.

The two integrals so obtained form the complete solution.

Note. Sometimes one solution can be used to simplify the other differential equation in the integrable form.

3·6. Geometrical Interpretation.

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}.$$

[Bombay 61]

From solid geometry, we know that direction cosines of the tangent to a curve are proportional to dx, dy, dz. Hence the above differential equations represent curves, the direction ratios of the tangent at (x, y, z) being proportional to P, Q and R. If $u=a$ and $v=b$ are two simultaneous solutions of the above equation, then the curves are obtained by intersection of the surfaces $u=a$, $v=b$. Since a and b both can have any values in infinite number of ways, the curves are doubly infinite in number.

Ex. 1. *Solve the simultaneous equations*

$$\frac{a\,dx}{(b-c)\,yz}=\frac{b\,dy}{(c-a)\,zx}=\frac{c\,dz}{(a-b)\,xy}.$$

[Agra 66; Raj. 51; Nag. 61]

Solution. Choosing ax, by, cz as multiplier, we get

$$\frac{a\,dx}{(b-c)\,yz}=\ldots=\frac{a^2x\,dx+b^2y\,dy+c^2z\,dz}{xyz\,\Sigma a\,(b-c)=0}$$

$$\therefore\quad a^2x\,dx+b^2y\,dy+c^2z\,dz=0.$$

Integrating, $\quad a^2x^2+b^2y^2+c^2z^2=c_1. \qquad \ldots(1)$

Again choosing x, y, z as multipliers, we get

$$\frac{a\,dx}{(b-c)\,yz}=\ldots=\frac{ax\,dx+by\,dy+cz\,dz}{xyz\,\Sigma\,(b-c)=0}.$$

Hence $ax\,dx+by\,dy+cz\,dz=0$.

Integrating, $ax^2+by^2+cz^2=c_2$, ...(2)

(1) and (2) together form the complete solution of the given simultaneous equations.

Ex. 2. *Solve* $\dfrac{dx}{mz-ny}=\dfrac{dy}{nx-lz}=\dfrac{dz}{ly-mx}$

[Vikram 63; Raj. 58]

Solution. We have

$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx}$$

$$=\frac{l\,dx+m\,dy+n\,dz}{l(mz-ny)+m(nx-lz)+n(ly-mx)}$$

$$=\frac{x\,dx+y\,dy+z\,dz}{x\,(mz-ny)+y\,(nx-lz)+z\,(ly-mx)}$$

or $\Sigma l\,(mz-ny)=0$ and $\Sigma x\,(mz-ny)=0$.

We get $l\,dx+m\,dy+n\,dz=0$ and $x\,dx+y\,dy+z\,dz=0$.

Integrating these, we get

$lx+my+nz=c_1$ and $x^2+y^2+z^2=c_2$.

These give the required solution.

Ex. 3 *Solve* $\dfrac{dx}{y-z}=\dfrac{dy}{z-x}=\dfrac{dz}{x-y}.$

Ans. $x+y+z=c_1$, $x^2+y^2+z^2=c_2$.

Ex. 4. *Solve* $\dfrac{dx}{x\,(y-z)}=\dfrac{dy}{y\,(z-x)}=\dfrac{dz}{z\,(x-y)}$

Solution. We have $dx+dy+dz=0$ and $\dfrac{dx}{x}+\dfrac{dy}{y}+\dfrac{dz}{z}=0$.

$\therefore$ $x+y+z=c_1$, $xyz=c_2$ form solution.

Ex. 5. *Solve* $\dfrac{l\,dx}{mn\,(y-z)}=\dfrac{m\,dy}{nl\,(z-x)}=\dfrac{n\,dz}{lm\,(x-y)}.$

Solution. We have $l^2\,dx+m^2\,dy+n^2\,dz=0$

and $l^2x\,dx+m^2y\,dy+n^2z\,dz=0$.

$\therefore$ $l^2x+m^2y+n^2z=c_1$, $l^2x^2+m^2y^2+n^2z^2=c_2$.

Ex. 6. *Integrate the equation*

$$\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy}.$$

[Raj. 56]

Solution. We have

$$\frac{dx-dy}{(x-y)\,(x+y+z)}=\frac{dy-dz}{(y-z)\,(x+y+z)}=\frac{dz-dx}{(z-x)\,(x+y+z)}$$

i.e. $\dfrac{dx-dy}{x-y}=\dfrac{dy-dz}{y-z}=\dfrac{dz-dx}{z-x}$.

The first two give $\log(x-y)=\log(y-z)+\log c$

i.e. $\dfrac{x-y}{y-z}=c_1$.

Similarly from the last two, $\dfrac{y-z}{z-x}=c$.

These form the complete solution.

Ex. 7. *Solve* $\dfrac{dx}{x+y-xy^2}=\dfrac{dy}{xy^2-x-y}=\dfrac{dz}{z(y^2-x^2)}$. **[Raj. 57]**

Solution. $\dfrac{dx}{x+y-xy^2}=\dfrac{dy}{xy^2-x-y}=\dfrac{dz}{z(y^2-x^2)}$

$$=\frac{dx+dy}{0}=\frac{zx\,dx+zy\,dy+dz}{zx\,[(x+y-xy^2)+yz\,(xy^2-x-y\,+z\,(y^2-x^2)]}$$

$$=\frac{zx\,dx+yz\,dy+dz}{z\,[x^2-y^2+y^2-x^2]}.$$

Thus $dx+dy=0$ and $zx\,dx+yz\,dy+dz=0$

or $dx+dy=0$ and $x\,dx+y\,dy+\dfrac{1}{z}\,dz=0$.

Integrating, $x+y=c_1$

and $x^2+y^2+2\log z=c_2$.

These form the solution.

Ex. 8. $\dfrac{dx}{x(2y^4-z^4)}=\dfrac{dy}{y(z^4-2x^4)}=\dfrac{dz}{z(x^4-y^4)}$.

Ans. $xyz^2=c_1,\ x^4+y^4+z^4=c_2$.

Ex. 9. *Solve* $\dfrac{dx}{xy}=\dfrac{dy}{y^2}=\dfrac{dz}{zxy-2x^2}$. **[Agra 59]**

Solution. From $\dfrac{dx}{xy}=\dfrac{dy}{y^2}$, we get $\dfrac{dx}{x}=\dfrac{dy}{y}$.

Integrating, $\log x=\log y+\log c$, i.e. $x=cy$.

Now taking $\dfrac{dy}{y^2}=\dfrac{dz}{zxy-2x^2}$, we get

$\dfrac{dy}{y^2}=\dfrac{dz}{zcy^2-2c^2y^2}$, i.e. $dy=\dfrac{dz}{z(c-2c^2)}$.

Integrating. $(c-2c^2)\,y=\log z+c_2$

i.e. $x-2x\left(\dfrac{x}{y}\right)=\log z+c_2$ or $x-\dfrac{2x^2}{y}=\log z-c_2$.

These form the complete solution.

Ex. 10. *Solve* $\dfrac{dx}{x^2}=\dfrac{dy}{y^2}=\dfrac{dz}{nxy}$.

Solution From $\frac{dx}{x^2}=\frac{dy}{y^2}$, we get $\frac{1}{x}=\frac{1}{y}+c_1$

i.e. $y-x=c_1xy$. ...(1)

Again from given equations

$$\frac{dx/x-dy/y+c_1\ dz/n}{x-y+c_1\ (xy)}=0;\ \therefore\ \frac{dx}{x}-\frac{dy}{y}+c_1\ \frac{dz}{n}=0.$$

Integrating, $\log x-\log y+\frac{c_1}{n}\ z=c_2$.

or $z=\frac{n}{c_1}\ \log\left(\frac{y}{x}\right)+c_2=\frac{nxy}{y-x}\ \log\left(\frac{y}{x}\right)+c_2.$ ...(2)

(1) and (2) together form the solution.

Ex. 11. *Solve* $\frac{dx}{y^2}=\frac{dy}{x^2}=\frac{dz}{x^2y^2z^2}$.

Solution. $\frac{dx}{y^2}=\frac{dy}{x^2}$, give $x^2\ dx=y^2\ dy$, $\therefore$ $x^3-y^3=c_1$.

Again $\frac{x^2\ dx+y^2\ dy-2\ dz/z^2}{x^2y^2+x^2y^2-2x^2y^2=0}$.

Hence $x^2\ dx+y^2\ dy-\frac{2}{z^2}\ dz=0.$

Integrating, $\frac{1}{3}x^3+\frac{1}{3}y^3+\frac{2}{z}=c_2$ or $x^3+y^3+\frac{6}{z}=c_2$.

Ex. 12. *Solve the simultaneous equations*

$$\frac{dx}{x^2-y^2-z^2}=\frac{dy}{2xy}=\frac{dz}{2xz}.$$ [Mysore 70]

Solution. From $\frac{dy}{2xy}=\frac{dz}{2xz}$, $\frac{dy}{y}=\frac{dz}{z}$.

Integrating, $y=cz$. ...(1)

Now using x, y, z as multipliers, we get

$$\frac{dx}{x^2-y^2-z^2}=\frac{dy}{2xy}=\frac{dz}{2xz}=\frac{x\ dx+y\ dy+z\ dz}{x\ (x^2+y^2+z^2)};$$

so when $\frac{dz}{2xz}=\frac{x\ dx+y\ dy+z\ dz}{x\ (x^2+y^2+z^2)}$,

we have $\frac{dz}{z}=\frac{2x\ dx+2y\ dy+2z\ dz}{x^2+y^2+z^2}$.

Integrating, $\log c_2+\log z=\log\ (x^2+y^2+z^2)$

or $x^2+y^2+z^2=c_2z$. ...(2)

(1) and (2) together form the complete solution.

Ex. 13. *Solve* $\frac{dx}{1+y}=\frac{dy}{1+x}=\frac{dz}{z}$.

Solution. Here $\frac{dz}{z}=\frac{dx+dy}{2+x+y}=\frac{dx-dy}{y-x}$.

$$\left.\begin{aligned} \therefore \quad \log z &= \log (2+x+y)+\log c_1 \\ \text{and} \quad \log z &= -\log (x-y)+\log c_2 \end{aligned}\right\} \quad \textbf{Ans.}$$

Ex. 14. *Solve* $\dfrac{dx}{y^3x-2x^4}=\dfrac{dy}{2y^4-x^3y}=\dfrac{dz}{9z\,(x^3-y^3)}$. **[Raj. 55]**

Solution. The equations can be written as

$$\frac{dx/x}{y^3-2x^3}=\frac{dy/y}{2y^3-x^3}=\frac{dz/3z}{3\,(x^3-y^3)}$$
$$=\frac{dx/x+dy/y+dz/3z}{0}.$$

Hence $\dfrac{dx}{x}+\dfrac{dy}{y}+\frac{1}{3}\dfrac{dz}{z}=0.$

Integrating, $\log x+\log y+\frac{1}{3}\log z=\log c_1$

i.e. $xyz^{1/3}=c_1$. ...(1)

Also from the first two terms, we have

$$(2y^4-x^3y)\,dx-(y^3x-2x^4)\,dy=0$$

or $y^3\,(2y\,dx-x\,dy)-x^3\,(y\,dx-2x\,dy)=0.$

*By trial the I.F. $=\dfrac{1}{x^3y^3}$.

Hence multiplying by $\dfrac{1}{x^3y^3}$, equation becomes

$$\left(\frac{2y}{x^3}-\frac{1}{y^2}\right)dx-\left(\frac{1}{x^2}-\frac{2x}{y^3}\right)dy=0. \text{ exact now.}$$

Integrating $\dfrac{2y}{x^3}-\dfrac{1}{y^2}$ with respect to x treating y as constant, we get $-\dfrac{y}{x^2}+\dfrac{x}{y^2}$.

In coefficients of dy there is no term free from x.

Hence the solution is $\dfrac{y}{x^2}-\dfrac{x}{y^2}=c_2$. ...(2)

(1) and (2) together form the complete solution of the given simultaneous equations.

Ex. 15. *Solve* $\dfrac{dx}{y+z}=\dfrac{dy}{z+x}=\dfrac{dz}{x+y}$. **[Gujrat 61]**

Solution. We have

$$\frac{dx-dy}{y-x}=\frac{dy-dz}{z-y}=\frac{dx+dy+dz}{2\,(x+y+z)}.$$

From first two members, we have

$\log (y-x)=\log (z-y)+\log c_1$ or $(y-x)=c_1\,(z-y)$. ...(1)

*For a rule of finding I. F. in such a case see § 3·9 page 52 of Part I of this ...

Again last two members give

$$-\log (y-z)=\tfrac{1}{2} \log (x+y+z)-\tfrac{1}{2} \log c_2$$

or $(y-z)^2 (x+y+z)=c_2$. ...(2)

(1) and (2) together give the complete integral.

Ex. 16. *Solve* $\dfrac{dx}{x (y^2-z^2)}=\dfrac{dy}{y (z^2-x^2)}=\dfrac{dz}{z (x^2-y^2)}$.

[Raj. 54; Karnatak 60]

Solution. We have

$$\frac{dx}{x (y^2-z^2)}=\frac{dy}{y (z^2-x^2)}=\frac{dz}{z (x^2-y^2)}$$

$$=\frac{x\, dx+y\, dy+z\, dz}{\Sigma x^2 (y^2-z^2)=0}=\frac{dx/x+dy/y+dz/z}{\Sigma (y^2-z^2)=0}.$$

Hence $x\, dx+y\, dy+z\, dz=0$ and $\dfrac{dx}{x}+\dfrac{dy}{y}+\dfrac{dz}{z}=0$.

Integrating these, we get

$$x^2+y^2+z^2=c_1, \log xyz=\log c_2, \text{ i.e. } xyz=c_2.$$

These constitute the complete integral.

Ex. 17. *Solve* $\dfrac{dx}{z^2-2yz-y^2}=\dfrac{dy}{y+z}=\dfrac{dz}{y-z}$. **[Vikram 62]**

Solution. Using $1, y, z$ as multipliers, we get

$$\frac{dx+y\, dy+z\, dz}{z^2-2yz-y^2+y(y+z)+z(y-z)}=0,$$

i.e. $dx+y\, dy+z\, dy=0$.

Integrating, $2x+y^2+z^2=c_1$.

Also $\dfrac{dy}{y+z}=\dfrac{dz}{y-z}$ can be written as

$$y\, dy-(z\, dy+y\, dz)-z\, dz=0$$

Integrating, $y^2-2yz-z^2=c_2$. ...(2)

(1) and (2) constitute the complete integral.

Ex. 18. *Solve* $\dfrac{dx}{x\ y^2-z^2)}=\dfrac{dy}{y (z^2+x^2)}=\dfrac{dz}{-z (x^2+y^2)}$.

Hint. Multipliers are $x, -y, -z$ and $\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$.

Integrals are $x^2-y^2-z^2=c_1, xyz=c_2$.

Ex. 19. *Solve* $\dfrac{dx}{y-zx}=\dfrac{dy}{x+yz}=\dfrac{dz}{x^2+y^2}$. **[Karnatak 61]**

Solution. Using $y, x, -1$ as multipliers, we get

$$y\, dx+x\, dy-dz=0, d\,(yx)-dz=0.$$

Integrating, $xy-z=c_1$. ...(1)

Again using $x, -y, z$ as multipliers, we get

$$x\, dx-y\, dy+z\, dz=0.$$

Integrating, $x^2-y^2+z^2=c_2$. ...(2)

(1) and (2) together form the complete integral.

Ex. 20. *Solve* $\dfrac{dx}{z(x+y)}=\dfrac{dy}{z(x-y)}=\dfrac{dz}{x^2+y^2}$.

[Agra 72; Poona 60; Delhi Hons. 72]

Solution. Using $x, -y, -z$ as multipliers, we get

$x\,dx-y\,dy-z\,dz=0$, $\therefore\ x^2-y^2-z^2=c_1$. ...(1)

Again using $y, x, -z$ as multipliers, we get

$y\,dx+x\,dy-z\,dz=0$, *i.e.* $xy-\frac{1}{2}z^2=c_2$. ...(2)

(1) and (2) form the complete integral.

Note. We can also use the first two members of the given equation to give one integral.

Ex. 21. *Solve* $\dfrac{dx}{x^2+y^2+yz}=\dfrac{dy}{x^2+y^2-yz}=\dfrac{dz}{z(x+y)}$.

[Bombay 61]

Solution. We have

$$\frac{dx-dy}{z(x+y)}=\frac{dz}{z(x+y)},\ \text{i.e. } dx-dy-dz=0.$$

$\therefore\ x-y-z=c_1$. ...(1)

Again $\dfrac{x\,dx+y\,dy}{x^3+y^3+xy\,(x+y)}=\dfrac{dz}{z(x+y)}$,

i.e. $\dfrac{x\,dx+y\,dy}{x^2-xy+y^2+xy}=\dfrac{dz}{z}$ or $\dfrac{x\,dx+y\,dy}{x^2+y^2}=\dfrac{dz}{z}$.

Integrating,

$\log(x^2+y^2)=2\log z+\log c_2$ or $\dfrac{x^2+y^2}{z^2}=c_2$ ...(2)

(1) and (2) together form the complete integral.

Ex. 22. *Solve* $\dfrac{dx}{x^2-y^2-yz}=\dfrac{dy}{x^2-y^2-zx}=\dfrac{dz}{z(x-y)}$.

[Bombay 61]

Solution. As above $x-y-z=c_1$ is an integral.

Again $\dfrac{x\,dx-y\,dy}{x^2-y^2}=\dfrac{dz}{z}$, *i.e.* $\dfrac{x^2-y^2}{z^2}=c_2$.

Ex. 23. *Solve* $\dfrac{dx}{1}=\dfrac{dy}{3}=\dfrac{dz}{5z+\tan(y-3x)}$.

[Bombay 61]

Solution. When $\dfrac{dx}{1}=\dfrac{dy}{3}$, *i.e.* $dy-3\,dx=0$, we have

$y-3x=c$. ...(1)

Now $\dfrac{dx}{1}=\dfrac{dz}{5z+\tan(y-3x)}$ becomes

$\dfrac{5\,dx}{1}=\dfrac{5\,dz}{5z+\tan c}$ as $y-3x=c$.

Integrating, $\log(5z+\tan c)=\log c_2+5x$,

i.e. $5z+\tan c=c_2e^{5x}$, *i.e.* $5z+\tan(y-5x)=c_2e^{5x}$. ...(2)

(1) and (2) form the complete integral.

Ex. 24. $\dfrac{dx}{1}=\dfrac{dy}{-2}=\dfrac{dz}{3x^2\sin(y+2x)}$.

Ans. $2x+y=c_1$, $x^3\sin(y+2x)-z=c_2$.

Ex. 25. *Solve* $\dfrac{dx}{y}=\dfrac{dy}{x}=\dfrac{dz}{xyz^2(x^2-y^2)}$. [Bombay 61]

Solution. $\dfrac{dx}{y}=\dfrac{dy}{x}$ gives $x\,dx-y\,dy=0$, *i.e.* $x^2-y^2=c_1$. ...(1)

Now $\dfrac{dx}{y}=\dfrac{dz}{xyz^2(x^2-y^2)}$ gives $\dfrac{dx}{1}=\dfrac{dz}{xz^2c_1}$

or $c_1x\,dx=\dfrac{dz}{z^2}$ or $c_1x^2=-\dfrac{2}{z}+c_2$

or $(x^2-y^2)x^2=-\dfrac{2}{z}+c_2$. ...(2)

(1) and (2) form the complete integral.

Ex. 26. $\dfrac{dx}{y}=\dfrac{dy}{-z}=\dfrac{dz}{z^2+(y+x)^2}$.

Ans. $x+y=c_1$, $\log[z^2+(y+x)^2]-2x=c_2$.

Ex. 27. $\dfrac{dx}{xz(z^2+xy)}=\dfrac{dy}{-yz(z^2+xy)}=\dfrac{dz}{x^4}$

Ans. $xy=c_1$, $(z^2+xy)^2-x^4=c_2$.

Ex. 28. $\dfrac{dx}{y}=\dfrac{dy}{-x}=\dfrac{dz}{2x-3y}$. [Mysore 69]

4

Total Differential Equations

4·1. Total Differential Equation

An equation of the form

$$P\,dx+Q\,dy+R\,dz=0,$$

where P, Q, R are functions of x, y, z is called the *total differential equation.*

Thus equations

$$(3x^2y^2-e^xz)\,dx+(2x^3y+\sin z)\,dy+(y\cos z-e^x)\,dz=0, \quad ...(1)$$

$$(3xz+2y)\,dx+x\,dy+x^2\,dz=0, \quad ...(2)$$

$$dx+dy+x\,dz=0, \quad ...(3)$$

are all total differential equations.

Integrable equations. It can be seen that (1) is exact differential of $\quad x^3y^2-e^xz+y\sin z=c.$

Thus (1) is an exact equation.

On the other hand (2) is not exact; but on multiplying by x (integrating factor) it gives

$$(3x^2z+2xy)\,dx+x^2\,dy+x^3\,dz=0,$$

which is exact whose integral is $x^3z+x^2y=c$.

Thus (1) and (2) are integrable equations.*

Note. It can be seen that (3) is not integrable and its primitive cannot be found.

*4·2. Condition of Integrability.

[Agra 63, 51; Delhi 67; Gujrat 61, 59, 58; Bombay 58, 61; Meerut 70; Raj. 60, 58; Poona 62; Sagar 63; Karnatak 62, 61]

The equation is $\quad P\,dx+Q\,dy+R\,dz=0. \quad ...(1)$

Let $u=a$ be a solution of (1) or of $\mu\,(P\,dx+Q\,dy+R\,dz)$, where μ is a function of x, y, z. In general, let

$$du=\mu\,(P\,dx+Q\,dy+R\,dz).$$

But $\quad du=\dfrac{\partial u}{\partial x}dx+\dfrac{\partial u}{\partial y}dy+\dfrac{\partial u}{\partial z}dz,$

so that $\quad \dfrac{\partial u}{\partial x}=\mu P,\ \dfrac{\partial u}{\partial y}=\mu Q,\ \dfrac{\partial u}{\partial z}=\mu R.$

---

*It can be verified that equations (1) and (2) both satisfy the condition of integrability given in § .·2 below.

Also since $\frac{\partial^2 \mu}{\partial x\, \partial y}=\frac{\partial^2 \mu}{\partial y\, \partial x}$,

$$\frac{\partial}{\partial x}(\mu Q)=\frac{\partial}{\partial y}(\mu P)$$

or $$\mu\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=Q\frac{\partial u}{\partial x}-P\frac{\partial u}{\partial z}. \quad ...(2)$$

Similarly, since $\frac{\partial^2 u}{\partial x\, \partial z}=\frac{\partial^2 u}{\partial z\, \partial x}$ and $\frac{\partial^2 u}{\partial y\, \partial z}+\frac{\partial^2 u}{\partial z\, \partial y}$, we get

$$\mu\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial y}\right)=P\frac{\partial \mu}{\partial z}-R\frac{\partial \mu}{\partial x} \quad ...(3)$$

and $$\mu\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)=R\frac{\partial \mu}{\partial y}-Q\frac{\partial \mu}{\partial z}. \quad ...(4)$$

Multiplying (2), (3), (4) by R, Q, P respectively and adding (to eliminate μ), we get

$$\mathbf{P}\left(\frac{\partial \mathbf{Q}}{\partial \mathbf{z}}-\frac{\partial \mathbf{R}}{\partial \mathbf{y}}\right)+\mathbf{Q}\left(\frac{\partial \mathbf{R}}{\partial \mathbf{x}}-\frac{\partial \mathbf{P}}{\partial \mathbf{z}}\right)+\mathbf{R}\left(\frac{\partial \mathbf{P}}{\partial \mathbf{y}}-\frac{\partial \mathbf{Q}}{\partial \mathbf{x}}\right)=0. \quad ...(5)$$

This is the required condition of integrability.

Note. We can show that if (5) is satisfied, the equation (1) has an integral. Thus the condition (1) is necessary and sufficient for the integrability of (1).

4·3. Exact Equation. *Condition of exactness.*

In case $\quad P\,dx+Q\,dy+R\,dz \quad ...(1)$

is exact differential of $\quad u=a$,

then $\quad du=P\,dx+Q\,dy+R\,dz.$

Also $du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz$,

so that $P=\frac{\partial u}{\partial x}$, $Q=\frac{\partial u}{\partial y}$, $R=\frac{\partial u}{\partial z}$.

But $\frac{\partial^2 u}{\partial x\, \partial y}=\frac{\partial^2 u}{\partial y\, \partial x}$.

$$\therefore \quad \frac{\partial \mathbf{P}}{\partial \mathbf{y}}=\frac{\partial \mathbf{Q}}{\partial \mathbf{x}}$$

Similarly $\frac{\partial \mathbf{Q}}{\partial \mathbf{z}}=\frac{\partial \mathbf{R}}{\partial \mathbf{y}}$ and $\frac{\partial \mathbf{R}}{\partial \mathbf{x}}=\frac{\partial \mathbf{P}}{\partial \mathbf{z}}$.

These are the conditions of exactness.

Note. These can be obtained by putting $\mu=1$ in above article.

***4·4. Method of solving**

$\mathbf{P\,dx+Q\,dy+R\,dz=0}$. [Saugar 62]

First check up that the condition of integrability,

i.e. $$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0, \quad ...(2)$$

is satisfied.

Case I. Exact Equation. If the equation is exact, *i.e.* if

$$\frac{\partial Q}{\partial z}=\frac{\partial R}{\partial y},\ \frac{\partial R}{\partial x}=\frac{\partial P}{\partial z},\ \frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x},$$

i.e. coefficients of P, Q, R in (2) are zero, then the equation can be integrated after properly regrouping the terms.

Case II. Homogeneous Equation. If P, Q, R are homogeneous functions of x, y, z, then one variable say z can be separated from the other by the substitution $x=zu$, $y=zv$, so that $dx=z\,du+u\,dz$ and $dy=z\,dv+v\,dz$.

Case III. One variable constant. If any two terms say $P\,dx+Q\,dy=0$ can be readily solved, then we take the third variable $z=$constant, so that $dz=0$.

Thus let solution of $P\,dx+Q\,dy=0$ be ...(3)

$$u=\phi(z),$$

where $\phi(z)$ is function of z only and is as such constant with respect to x and y.

Then to completely find the solution, differentiate (3) with respect to x, y and z and compare this with the given equation (1). We thus get the value of $d\phi/dz$ which on integration gives value of ϕ and the complete solution (3).

Case IV. Method of auxiliary equations. If none of the above methods is found convenient, then comparing (1) and (2), we get simultaneous equations

$$\frac{dx}{\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}}=\frac{dy}{\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}}=\frac{dz}{\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}},$$

which are called auxiliary equations and can be solved like simultaneous equations. If $u=a$, $v=b$ be the two integrals of auxiliary equations, then by comparing $A\,du+B\,dv=0$ with (1), we get the values of A and B and then the complete primitive.*

We shall illustrate these methods by some of the solved examples.

Ex. 1. *Solve* $(x-y)\,dx-x\,dy+z\,dz=0$. ...(1)

Solution. We have $P=x-y$, $Q=-x$, $R=z$.

The condition of integrability is

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial x}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0$$

or $\quad (x-y)\,[0-0]+(-x)\,[0-0]+z\,(-1+1)=0.$

The condition is satisfied Hence the equation is integrable. Hence we see that in the above condition coefficients of P, Q, R are separately zero. Hence the equation is exact.

*Fourth method of auxiliary equations will fail in case the equation is exact.

Second method cannot be applied in case P, Q, R are not homogeneous.

I. Method. Exact Equation.

The equation can be written as

$$x\,dx-(y\,dx+x\,dy)+z\,dz=0.$$

Integrating, $\frac{1}{2}x^2-xy+\frac{1}{2}z^2=\text{constant}$

or $x^2-2xy+z^2=c$ is the solution.

II. Method. Homogeneous Equations. [Vikram 69

Here we find that P, Q, R are homogeneous functions. Hence putting $x=uz, y=vz$, we get

$$dx=u\,dz+z\,du,\ dy=v\,dz+z\,dv$$

and then the equation becomes

$$(uz-vz)(u\,dz+z\,du)-uz\,(v\,dz+z\,dv)+z\,dz=0$$

or $$z\,(u-v)\,du-zu\,dv+[(u-v)\,u-uv+1]\,dz=0$$

or $$\frac{(u-v)\,du-u\,dv}{[u^2-2uv+1]}+\frac{dz}{z}=0.$$

Integrating, $\log(u^2-2uv+1)+2\log z=\log c$

or $$z^2\,(u^2-2uv+1)=c$$

or $$u^2z^2-2uz.\,vz+z^2=c$$

or $$x^2-2xy+z^2=c \qquad \text{as } uz=x,\ vz=y.$$

III Method. Regarding one variable constant.

If we put $z=\text{constant}$, then $dz=0$ and then the equation becomes $(x-y)\,dx-x\,dy=0$, *i.e.*, $x\,dx-(y\,dx+x\,dy)=0$.

Integrating $\frac{1}{2}x^2-xy=\phi(z)$. (2)

Differentiating it, we get $(x-y)\,dx-x\,dy-\phi'(z)\,dz=0$.

Comparing this with the given equation, we get

$z=-\phi'(z)$, so that $\phi(z)=-\frac{1}{2}z^2+c$.

Putting this value of $\phi(z)$ in (2), the solution is

$$\tfrac{1}{2}x^2-xy+\tfrac{1}{2}z^2=k$$

or $x^2-2xy+z^2=2k=c_3$ is the solution

IV Method. Method of auxiliary equations.

Since the equation is exact, the auxiliary equations are

$$\frac{dx}{0}=\frac{dy}{0}=\frac{dz}{0},$$

which fails to give any solution. [See footnote on P. 124]

4·5. Example which can be solved by inspection

Now we solve certain examples which can be solved as exact equations or which can be solved by inspection. All these examples can be solved by other methods also, but it is easiest if it is possible to solve a particular equation by inspection.

Ex. 1. *Solve* $2yz\,dx-3zx\,dy-4xy\,dz=0$. **[Bombay 61]**

Solution. Here $P=2yz$, $Q=-3zx$, $R=-4xy$.

$\therefore$ Condition of exactness, *i.e.*

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial z}-\frac{\partial Q}{\partial x}\right)=0$$

gives $2yz(-3x+4x)-3zx(-4y+2y)-4xy(2z+3z)=0$

i.e. $$2xyz+18xyz-20xyz=0.$$

Thus the condition of exactness is satisfied.

Dividing by xyz, the equation becomes

$$\frac{2dx}{x}-\frac{3dy}{y}-\frac{4dz}{z}=0.$$

Integrating, $2\log x-3\log y-4\log z=\log c.$

i.e. $\frac{x^2}{y^3z^4}=c$ is the solution.

Note. All the equations, that we will be solving, satisfy the condition of integrability. Therefore, we shall not show in every example that this condition is satisfied. Students should verify this for themselves and in examination they should inevitably show it.

Ex. 2. *Solve* $(y+z)\,dx+dy+dz=0.$ **[Agra 54]**

Solution. Write the equation as

$$dx+\frac{dy+dz}{y+z}=0.$$

Integrating, $x+\log(y+z)=\log c$

or $y+z=ce^{-x}$ is the solution.

Ex. 3. *Show that*

$$(yz+2x)\,dx+(zx-2z)\,dy+(xy-2y)\,dz=0$$

is integrable and solve the equation. **[Poona 60]**

Solution. Show for yourself that the condition of integrability is satisfied for equation.

To solve it, write it as

$$(yz\,dx+zx\,dy+xy\,dz)+2x\,dx-2(z\,dy+y\,dz)=0.$$

Integrating, $xyz+x^2-2yz=c.$

Ex. 4. *Solve*

$$(yz+2x)\,dx+(zx+2y)\,dy+(xy+2z)\,dz=0.$$

[Delhi Hons. 68]

Solution. The equation can be written as

$$(yz\,dx+zx\,dy+xy\,dz)+2(x\,dx+y\,dy+z\,dz)=0$$

or $d(xyz)+d(x^2+y^2+z^2)=0.$

Integrating, the solution is

$$xyz+(x^2+y^2+z^2)=c.$$

Ex. 5. *Solve*

$$yz(1+x)\,dx+zx(1+y)\,dy+xy(1+z)\,dz=0$$

[Gujrat 61]

or $(yz+xyz)\,dx+(zx+xyz)\,dy+(xy+xyz)\,dz=0.$

[Agra 69, 64, 58; Raj. 55]

Solution. Dividing by xyz, the equation becomes

$$\left(\frac{1}{x}+1\right) dx+\left(\frac{1}{y}+1\right) dy+\left(\frac{1}{z}+1\right) dz=0.$$

Integrating, $\log xyz+x+y+z=c$ is the solution.

Ex. 6. *Solve* $(a-z)(y\,dx+x\,dy)+xy\,dz=0.$

[Raj 61; Poona 69; Bombay 61]

Solution. We can write the equation as

$$\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{a-z}=0.$$

Integrating, $\log x+\log y-\log (a-z)=\log c$

i.e. $xy=c(a-z).$

Ex. 7. *Solve* $yz \log z\,dz-zx \log z\,dy+xy\,dy=0.$

[Meerut 68 ; Agra 68]

Solution. Dividing by $xyz \log z$, the equation becomes

$$\frac{1}{x} dx-\frac{1}{y} dy+\frac{1}{z \log z} dz=0.$$

Integrating, $\log x-\log y+\log (\log z)=\log c,$

i.e. $x \log z=cy.$

Ex. 8. *Solve* $(y+z)\,dx+(z+x)\,dx+(x+y)\,dz=0.$

[Raj. 59; Karnatak 61, 62]

Solution. Write the equation as

$$(y\,dx+x\,dy)+(z\,dx+x\,dz)+(z\,dy+y\,dz)=0,$$

i.e. $d(xy)+d(zx)+d(yz)=0.$

Integrating, $xy+yz+zx=c$ is the solution.

Ex. 9. *Solve* $(y^2+z^2-x^2)\,dx-2xy\,dy-2xz\,dz=0.$

[Delhi Hons. 68; Karnatak 61]

Solution. The equation can be written as

$$(x^2+y^2+z^2)\,dx=2x\,(x\,dx+y\,dy+z\,dz),$$

i e. $$\frac{2(x\,dx+y\,dy+z\,dz)}{x^2+y^2+z^2}=\frac{dx}{x}.$$

Integrating, $\log (x^2+y^2+z^2)=\log x+\log c.$

$\therefore$ $x^2+y^2+z^2=cx$ is the solution.

Ex. 10. *Solve* $x\,dx+y\,dy-\sqrt{(a^2-x^2-y^2)}\,dz=0.$

[Gujrat 58]

Solution. The equations can be written as

$$\frac{x\,dx+y\,dy}{\sqrt{(a^2-x^2-y^2)}}=dz,$$

i.e., $$\frac{d(x^2+y^2)}{\sqrt{\{a^2-(x^2+y^2)\}}}=2\,dz.$$

Integrating, $\sin^{-1} \dfrac{x^2+y^2}{a}=2z+c,$

or $x^2+y^2=a \sin (2z+c)$ is the solution.

Ex. 11. *Solve* $z\,dz+(x-a)\,dx=\sqrt{\{h^2-z^2-(x-a)^2\}}\,dy.$ **[Agra 47]**

Solution The equation can be written as

$$\frac{z\,dz+(x-a)\,dx}{\sqrt{[h^2-\{z^2+(x-a)^2\}]}}=dy$$

or $$\tfrac{1}{2}\cdot\frac{d\,[z^2+(x-a)^2]}{\sqrt{[h^2-\{z^2+(x-a)^2\}]}}=dy.$$

Integrating, $\sqrt{[h^2-z^2-(x-a)^2}=-y+c.$

$\therefore\quad h^2-z^2-(x-a)^2=(y-a)^2$ is the solution.

Ex. 12. *Integrate* $(2x+y^2+2xz)\,dx+2xy\,dy+x^2\,dz=du.$ **[Agra 55]**

Solution Write the equation as

$$2x\,dx+(y^2\,dx+2xy\,dy)+(x^2\,dz+2xz\,dx)=du,$$

i.e. $2x\,dx+d\,(y^2x)+d\,(x^2z)=du$

Integrating, $x^2+y^2x+x^2z=u+c$ is the solution.

Ex. 13. *Solve*

$$yz^2\,(x^2-yz)\,dx+zx^2\,(y^2-xz)\,dy+xy^2\,(z^2-xy)\,dz=0.$$

Solution. Dividing by $x^2y^2z^2$, the equation becomes

$$\left(\frac{1}{y}-\frac{z}{x^2}\right)dx+\left(\frac{1}{z}-\frac{x}{y^2}\right)dy+\left(\frac{1}{x}-\frac{y}{z^2}\right)dz=0$$

or $$\frac{y\,dx-x\,dy}{y^2}+\frac{y\,dz-z\,dy}{z^2}+\frac{z\,dx-x\,dz}{x^2}=0,$$

i.e. $$d\left[\frac{x}{y}\right]+d\left[\frac{y}{z}\right]+d\left[\frac{z}{x}\right]=0.$$

Integrating, $\dfrac{x}{y}+\dfrac{y}{z}+\dfrac{z}{x}=c$ is the solution.

Ex. 14. *Solve* $(x^2y-y^3-y^2z)\,dx+(xy^2-x^2z-x^3)\,dy$
$+(xy^2+x^2y)\,dz=0.$

[Agra 63, 57; Raj. 54, 52; Sagar 63]

Solution. Dividing by x^2y^2, the equation becomes

$$\left(\frac{1}{y}-\frac{y}{x^2}-\frac{z}{x^2}\right)dx+\left(\frac{1}{x}-\frac{z}{y^2}-\frac{x}{y^2}\right)dy+\left(\frac{1}{x}+\frac{1}{y}\right)dz=0.$$

i.e. $$\frac{y\,dx-x\,dy}{y^2}+\frac{x\,dy-y\,dx}{x^2}+\frac{x\,dz-z\,dx}{x^2}+\frac{y\,dz-z\,dy}{y^2}=0.$$

Integrating, $\dfrac{x}{y}+\dfrac{y}{x}+\dfrac{z}{x}+\dfrac{z}{y}=c$

i.e. $x^2+y^2+z\,(x+y)=cxy.$

Note. For an alternate solution of this example as a homogeneous equation, see Ex. 2 P. 134.

Ex 15. *Solve* $\dfrac{yz}{x^2+y^2}\,dx-\dfrac{xz}{x^2+y^2}\,dy-\tan^{-1}\dfrac{y}{x}\,dz=0.$

Solution. Write the equation as

$$\frac{y\,dx-x\,dy}{(x^2+y^2)\tan^{-1}(y/x)}=\frac{dz}{z}.$$

Hence the equation becomes $-\dfrac{du}{u}=\dfrac{dz}{z}.$

Integrating, $\quad -\log u=\log z+\log c$

or $\dfrac{1}{u}=cz$ or $u=\dfrac{1}{cz}$ *i.e.* $\tan^{-1}\left(\dfrac{y}{x}\right)=\dfrac{1}{cz}$ or $\dfrac{y}{x}=\tan\left(\dfrac{1}{cz}\right).$

4·6. Examples by taking one variable constant.

If by taking a variable constant (*i.e.* by omitting any one of the terms) the equation can be easily integrated, then this method is preferred.

It will not be difficult to decide as to which variable should be taken constant, since equation free from it should be integrable. The following examples will make the method clear.

Ex 1. *Solve* $3x^2\,dx+3y^2\,dy-(x^3+y^3+e^{2z})\,dz=0.$

[Agra 70 ; Karnatak 61]

Solution. Taking $z=$const.*, $dz=0$, the equation becomes

$$3x^2\,dx+3y^2\,dy=0.$$

Integrating, $\quad x^3+y^3=\phi(z),$ ...(1)

where $\phi(z)$ is a function of z (regarded const.).

Now differentiating (1), we get

$$3x^2\,dx+3y^2\,dy-\phi'(z)\,dz=0. \quad ...(2)$$

Comparing (2) with the given equation, we get

$$\phi'(z)=x^3+y^3+e^{2z}$$
$$=\phi(z)+e^{2z}$$

i.e. $\dfrac{d\phi}{dz}-\phi=e^{2z}$, linear, I.F.$=e^{-z}$.

$\therefore\quad \phi e^{-z}=c+\int e^{2z}\,.\,e^{-z}\,dz=c+e^z.$

Hence $\phi=ce^z+e^{2z}$

$\therefore\quad$ (1) becomes $\quad x^3+y^3=ce^z+e^{2z}.$

which is the required solution.

Note. This example has been solved by another method also. See Ex. 3 P. 134.

Ex. 2. $(2xz-yz)\,dx+(2yz-zx)\,dy-(x^2-xz+y^2)\,dz=0.$

[Agta 59, 53]

Solution. Taking $z=$const. †, $dz=0$.

The equation becomes

$$(2xz-yz)\,dx+(2yz-zx)\,dy=0$$

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*We cannot take $x=$const. or $y=$const. because in that case the remaining equations are not integrable. This suggests that only z be taken constant.

†Here also if we take x or y constant, the resulting equations cannot be integrated. Hence z is taken constant.

i.e. $(2x-y)\,dx+(2y-x)\,dy=0$

or $2x\,dx-(y\,dx+x\,dy)+2y\,dy=0.$

Integrating, $x^2-xy+y^2=\phi(z),$ ...(1)

where $\phi(z)$ is a function of z, which is regarded constant.

Now differentiating (1), we get

$(2x-y)\,dx+(2y-x)\,dy-\phi'(z)\,dz=0.$

Multiplying it by z, it becomes

$(2xz-yz)\,dx+(2yz-zx)\,dy-z\phi'(z)\,dz=0.$ ...(2)

Comparing it with the given equation, we get

$z\phi'(z)=x^2-xy+y^2=\phi(z).$

$\therefore\quad z\dfrac{d\phi}{dz}=\phi$ or $\dfrac{d\phi}{\phi}=\dfrac{dz}{z}.$

$\therefore\quad \phi=cz.$

Putting this value in (1), the solution is

$x^2-xy+y^2=cz.$

Ex. 3. *Verify that the condition of integrability is satisfied by the following equation and solve it.*

$(2x^2+2xy+2xz^2+1)\,dx+dy+2dz=0.$

[Delhi Hons. 67; Bombay 62; Raj. 57; Nagpur 61; Punjab 67; Agra 71, 65; Sagar 62; Vikram 64]

Solution. Verify for yourself that the condition of integrability is satisfied.

Taking x=constant, $dx=0$, the equation becomes

$dy+2z\,dz=0.$ Integrating, $y+z^2=\phi(x),$ ...(1)

where $\phi(x)$ is a function of x, which is taken constant.

Differentiating it, we get

$dy+2z\,dz-\phi'(x)\,dx=0.$

Comparing it with the given equati n, we get

$-\phi'(x)=2x^2+2xy+2xz^2+1$

$=2x^2+2x(y+z^2)+1=2x^2+2x\phi(x)+1=0$

or $\dfrac{d\phi}{dx}+2x\phi+2x^2+1=0.$ Linear, I.F.$=e^{x^2}$.

$$\therefore\quad \phi e^{x^2}=c-\int(2x^2+1)\,e^{x^2}\,dx=c-\int 2x^2e^{x^2}\,dx-\int e^{x^2}\,dx$$

$$=c-\int x.e^{x^2}\,2x\,dx-\int e^{x^2}\,dx$$

$$=c-xe^{x^2}+\int e^{x^2}\,dx-\int e^{x^2}\,dx=c-xe^{x^2}$$

$\therefore\quad \phi=ce^{-x^2}-x.$

Putting this value in (1), the general solution is

$(y+z^2)=ce^{-x^2}-x$ or $(x+y+z^2)\,e^{x^2}=c.$

Ex. 4. *Solve*

$$(e^x y+e^z)\,dx+(e^y z+e^x)\,dy+(e^y-e^x y-e^y z)\,dz=0.$$

Solution. Taking $z=$ const., $dz=0$.

The equation becomes

$$(e^x y\,dx+e^x\,dy)+e^y z\,dy+e^z\,dx=0.$$

Integrating, $\quad e^x y+e^y z+e^z x=\phi(z)$. ...(1)

Differentiating it with respect to all variables, we get

$$(e^x y+e^z)\,dx+(e^y z+e^x)\,dy+(e^y+e^z x)\,dz-\phi'(z)\,dz=0.$$

Comparing this with the given equation, we get

$$e^y+e^z x-\phi'(z)=e^y-e^x y-e^y z$$

or $\quad \phi'(z)=e^x y+e^y z+e^z x=\phi(z).$

$$\therefore \quad \frac{d\phi}{dz}=\phi, \quad i.e. \quad \frac{d\phi}{\phi}=dz \quad \text{or} \quad \phi=ce^z.$$

Putting this value in (1), the general integral is

$$e^x y+e^y z+e^z x=ce^z.$$

Ex. 5. *Solve* $\quad (\cos x+e^x y)\,dx+(e^x+e^y z)\,dy+e^y\,dz=0.$

Solution. Take $x=$ const., $dx=0$.

$\therefore \quad e^x\,dy+e^y z\,dy+e^y\,dz=0$ gives $e^x y+e^y z=\phi(x)$ etc.

Ans. $\quad e^x y+e^y z+\sin x=c.$

Ex. 6. *Solve* $\quad x\,dy-y\,dx-2x^2 z\,dz=0.$ [Raj. 60]

Solution. Taking $z=$ const., $dz=0$.

$$\therefore \quad x\,dy-y\,dx=z, \quad i.e. \quad \frac{dx}{x}=\frac{dy}{y}.$$

Integrating, $\quad \log x-\log y=\phi(z)$. ...(1)

$$\text{Differentiating,} \quad \frac{1}{x}\,dx-\frac{1}{y}\,dy=\phi'(z)\,dz,$$

i.e. $\quad y\,dx-x\,dy=xy\,\phi'(z)\,dz.$

Comparing it with given equation, we get

$$2x^2 z=xy\,\phi'(z), \quad i.e. \quad 2\frac{x}{y}z=\phi'(z)$$

$$i.e. \quad 2ze^{\phi}=\frac{d\phi}{dz} \quad \text{or} \quad 2z\,dz=e^{-\phi}\,d\phi$$

$$\text{or} \quad z^2=c-e^{\phi} \quad \text{or} \quad e^{-\phi}=c-z^2; \quad \therefore \quad \phi=-\log(c-z^2).$$

Putting this in (1), the complete solution is

$$\log x-\log y=-\log(c-z^2) \quad \text{or} \quad y=x(c-z^2).$$

Ex. 7. *Solve* $\quad 3y\,dx=3x\,dy+y^2\,dz.$ [illegible]

Solution. Taking $z=$ const., $dz=0$

$$\therefore \quad y\,dx-x\,dy=0 \quad \text{or} \quad \frac{dx}{x}-\frac{dy}{y}=0.$$

$$\text{Integrating,} \quad \log\left(\frac{x}{y}\right)=\phi(z) \quad \text{or} \quad \frac{x}{y}=e^{\phi(z)}. \qquad ...(1)$$

Differentiating again $\frac{1}{x}\,dx-\frac{1}{y}\,dy=\phi'(z)\,dz$

or $\quad y\,dx-x\,dy=xy\,\phi'(z)\,dz.$

The given equation is

$$y\,dx-x\,dy=\frac{y^2}{z}\,dz.$$

Comparing, $xy\,\phi'(z)=\frac{y^2}{z}$.

or $\quad z\phi'(z)=\frac{y}{x}=e^{-\phi(z)}$

or $\quad z\frac{d\phi}{dz}=e^{-\phi}$ where $\phi\equiv\phi(z)$

or $\quad e^{\phi}\,d\phi=dz/z)$

i.e. $\quad e^{\phi}=\log Az$, where A is a const.

Putting this in (1), the complete solution is

$(x/y)=\log Az \quad$ or $\quad x=y\log Az.$

Ex. 8. *Solve*

$$(z+z^2)\cos x\,dx-(z+z^2)\,dy+(1-z^2)(y-\sin x)\,dz=0.$$

Solution. Taking $z=$const., $dz=0$

$\therefore$ the equation becomes

$$\cos x\,dx-dy=0 \quad\text{or}\quad \sin x-y=\phi(z). \qquad ...(1)$$

Differentiating, $\quad \cos x\,dx-dy=\phi'(z)\,dz$

or $\quad (z+z^2)\cos x\,dx-(z+z^2)\,dy=(z+z^2)\,\phi'(z)\,dz.$

Comparing it with given equation, we get

$$(1-z^2)(y-\sin x)=-(z+z^2)\,\phi'(z)$$

i.e. $\qquad -(1-z^2)\,\phi(z)=-z\,(1+z)\,\phi'(z).$

$$\therefore\quad \frac{d\phi}{\phi}=\frac{1-z}{z}\,dz=\left(\frac{1}{z}-1\right)dz$$

Integrating, $\log\phi-\log z=-z+\log c$, $\phi=cze^{-z}$.

Hence from (1), the general solution is

$\sin x-y=cze^{-z} \quad$ or $\quad y=\sin x-cze^{-z}.$

Ex. 9. *Show that the differential equation*

$$z\,(1-z^2)\,dz+z\,dy=(x+y+z^{2x})\,dz$$

satisfies the condition of integrability, and hence solve it.

[Delhi Hons. III 72]

Ex. 10. $(2x^3-z)\,z\,dx+2x^2yz\,dy+x\,(z+x)\,dz=0.$

[Guru Nanak 73]

4·7 Examples on Homogeneous Equations and by the Method of Auxiliary Equations.

We now give some examples which can be solved as homogeneous equations or by forming auxiliary equations.

The method of forming auxiliary equation is sometimes convenient.

***Ex. 1.** *Solve* $(y^2+yz)\,dx=(xz+z^2)\,dy+(y^2-xy)\,dz=0.$

[Delhi Hons 70; Agra 72, 60; Vikram 62; Karnatak 60; Sagar 65; Bombay 58; Mysore 70; Raj 56, 53, 51]

Solution. The equation satisfies the condition of integrability.

First Method. The equation is *homogeneous*; hence put $x=zu,\ y=zv,$ (Refer § 4·4 Case II P. 123)

$$dx=z\,du+u\,dz,\quad dy=z\,dv+v\,dz.$$

Hence the equation becomes

$$z^2(v^2+v)\,[z\,du+u\,dz]+z^2(u+1)\,(z\,dv+v\,dz)+z^2(v^2-uv)\,dz=0$$

or $\quad z(v^2+v)\,du+z(u+1)\,dv+[u(v^2+v)+v(u+1)+v^2-uv]\,dz=0$

or $\quad \dfrac{v^2+v}{uv^2+uv+v+v^2}\,du+\dfrac{u+1}{uv^2+uv+v+v^2}\,dv+\dfrac{dz}{z}=0$

or $\quad \dfrac{du}{u+1}+\dfrac{dv}{v^2+v}-\dfrac{dz}{z}=0$ as $uv^2+uv+v+v^2=(u+1)\,(u^2+v)$

or $\quad \dfrac{du}{u+1}+\left(\dfrac{1}{v}-\dfrac{1}{v+1}\right)dv+\dfrac{dz}{z}=0.$

Integrating, $\log\dfrac{(u+1)\,vz}{v+1}=\log c$, *i.e.* $\dfrac{(u+1)\,vz}{v+1}=c,$

i.e. $\quad \dfrac{(x/z+1)\,(y/z)\,z}{(y/z+1)}=c$ as $x=zu,\ y=zv$

or $\quad (x+z)\,y=c\,(y+z).$

Second Method. The auxiliary equations are

$$\frac{dx}{\dfrac{\partial Q}{\partial z}-\dfrac{\partial R}{\partial y}}=\frac{dy}{\dfrac{\partial R}{\partial x}-\dfrac{\partial P}{\partial z}}=\frac{dz}{\dfrac{\partial P}{\partial y}-\dfrac{\partial Q}{\partial x}}$$ (Refer Case IV P. 124)

i.e. $\quad \dfrac{dx}{2(x-y+z)}=\dfrac{dy}{-2y}=\dfrac{dz}{2y}$ or $\dfrac{dx}{x-y+z}=\dfrac{dy}{-y}=\dfrac{dz}{y}.$

Last two members give $dy+dz=0$, *i.e.* $y+z=u$ (say).

Also $\quad \dfrac{dx+dz}{x+z}=\dfrac{dy}{-y}$.

Integrating, $\log(x+z)+\log y=\log c,$

i.e. $\quad y\,(x+z)=v$ (say).

Then $\quad A\,dv=B\,dv=0$ ...(2)

gives $\quad A\,(dy+dz)+B\,(y\,dx+x\,dy+y\,dz+z\,dy)=0,$

i.e. $\quad By\,dx+[A+B\,(x+z)]\,dy+(A+By)\,dz=0.$

Comparing this with the given equation, we get

$By=y^2+yz,\quad$ *i.e.*, $\quad B=y+z=u.$

$A+B\,(x+z)=xz+z^2=z\,(x+z)\ ;\quad \therefore\ A=(x+z)\,(z-B)$

or $\quad A=(x+z)\,(z-y-z)=-y\,(x+z)=-v$

Hence (2) is $-v\,du+u\,dv=0$, *i.e.* $\dfrac{du}{u}-\dfrac{dv}{v}=0.$

Integrating, $\log(u/v)=\log c$, *i.e.* $u/v=c$

or $\dfrac{y+z}{y(x+z)}=c$ or $y(x+z)=\dfrac{1}{c}(y+z)$.

which is the required solution.

Note. We can do this example by taking any one of the variables constant also.

***Ex. 2.** *Solve the differential equation*

$$(x^2y-y^3-y^2z)\,dx+(xy^2-x^2z-x^3)\,dy+(xy^2+x^2y)\,dz=0.$$

[Agra 63, 57; Raj. 52]

Solution. The equation satisfies the condition of integrability (verify yourself) and is homogeneous.

Put $x=uz,\ y=vz,$

$$dx=u\,dz+z\,du,\ dy=v\,dz+z\,dv.$$

Then the equation becomes

$$(u^2vz^2-v^3z^3-v^2z^3)(u\,dz+z\,du)+(uv^2z^2+u^2z^3-u^2z^3)$$
$$\times(v\,dz+z\,dv)+(uv^2z^3+u^2vz^3)\,dz=0$$

i.e. $v(u^2-v^2-v)\,du+u(v^2-u^2-u)\,dv=0$

or $(u^2-v^2)(v\,du-u\,dv)-v^2\,du-u^2\,dv=0.$

Dividing by u^2v^2, it becomes

$$\left(\frac{1}{v^2}-\frac{1}{u^2}\right)(v\,du-u\,dv)-\frac{1}{u^2}\,du-\frac{1}{v^2}\,dv=0$$

i.e. $$\frac{v\,du-u\,dv}{v^2}+\frac{u\,dv-v\,du}{u^2}-\frac{1}{u^2}\,du-\frac{1}{v^2}\,dv=0.$$

Integrating, $\dfrac{u}{v}+\dfrac{v}{u}+\dfrac{1}{u}+\dfrac{1}{v}=c$

i.e. $\dfrac{x}{y}+\dfrac{y}{x}+\dfrac{z}{x}+\dfrac{z}{y}=c$ as $u=\dfrac{x}{z}$ and $v=\dfrac{y}{z}$.

Ex. 3. *Solve* $3x^2\,dx+3y^2\,dy-(x^3+y^3+e^{2z})\,dz=0.$

[Karnatak 61]

Solution. The equation satisfies the condition of integrability. Here auxiliary equations are

$$\frac{dx}{\dfrac{\partial Q}{\partial z}+\dfrac{\partial R}{\partial y}}=\frac{dy}{\dfrac{\partial R}{\partial y}-\dfrac{\partial P}{\partial z}}=\frac{dz}{\dfrac{\partial P}{\partial y}-\dfrac{\partial Q}{\partial x}}$$

i.e. $$\frac{dx}{3y^2}=\frac{dy}{-3x^2}=\frac{dz}{0+0}$$

i.e. $$\frac{dx}{y^2}=\frac{dy}{-x^2}=\frac{dz}{0}.$$

First two give $x^2\,dx+y^2\,dy=0$, *i.e.* $x^3+y^3=u$ (say).

Last gives $dz=0$, *i.e.* $z=v$ (say).

So $\quad A\,du+B\,dv=0$ ...(1)

gives $\quad A\,(3x^2\,dx+3y^2\,dy)+B\,dz=0$

or $\quad 3Ax^2\,dx+3Ay^2\,dy+B\,dz=0.$

Comparing this with given equation, we get

$$3x^2=3Ax^2,\ B=-(x^3+y^3+e^{2z}).$$

$\therefore\quad A=1,\ B=-(u+e^{2v}).$

Hence (1) becomes $du-(u+e^{2v})\,dv=0.$

i.e. $\quad \dfrac{du}{dv}-u=e^{2v}.$ Linear, I.F.$=e^{-v}$.

$$\therefore\quad ue^{-v}=c_1+\int e^{2v}.e^{-v}\,dv=c_1+e^v$$

or $\quad u=c_1e^v+e^{2v}$

or $\quad x^3+y^3=c_1e^z+e^{2z}$ is the solution.

Note. This example has been solved by taking $z=$const. also. See Ex. 1 P. 129.

Ex. 4. *Show that the condition of integrability is satisfied by the equation $z\,(z-y)\,dx+(z+x)\,z\,dy+x\,(x+y)\,dz=0$ and solve it.* **[Bombay 61]**

Solution. We have

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)$$

$=z\,(z-y)\,[2z+x-x]+z\,(z+x)\,[2x+2y-2z]+x\,(x+y)\,[-z-z]$
$=0.$

Hence the condition of integrability is satisfied.

Now the auxiliary equations are

$$\frac{dx}{2z}=\frac{dy}{2x+2y-2z}=\frac{dz}{-2z},\ \text{i.e.}\ \frac{dx}{z}=\frac{dy}{x+y-z}=\frac{dz}{-z}.$$

First and third give $dx+dz=0$, *i.e.* $x+z=u$ (say) ...(1)

Also $\dfrac{dx+dy}{x+y}=\dfrac{dz}{-z}$, *i.e.* $\log(x+y)+\log z=\log v.$

i.e. $\quad (x+y)\,z=v.$ ...(2)

Now $\quad A\,du+B\,dv=0$ ...(3)

gives $\quad A\,(dx+dz)+B\,[z\,dx+z\,dy+(x+y)\,dz]=0$

i.e. $\quad (A+Bz)\,dx+Bz\,dy+(A+x+y)\,dz=0.$

Comparing it with the given equation, we get

$$A+Bz=z\,(z-y),\ Bz=z\,(z+x),\ A+x+y=x\,(x+y).$$

$\therefore\quad B=(z+x)=u,\ A=z\,(z-y)-Bz=-z\,(x+y)=-v,$

$\therefore\quad$ (3) becomes $\quad -v\,du+u\,dv=0,$

Integrating, $\quad v/u=c \quad$ or $\quad v=cu$

i.e. $\quad (x+y)\,z=c\,(x+z)$ is the solution.

Ex. 5. *Solve* $\ xz^2\,dx-z\,dy+2y\,dz=0.$ **[Gujrat 61]**

Solution. The condition of integrability is satisfied, since

$$xz^3(-1-2)-z(0-3xz^2)+2y(0-0)=0.$$

The auxiliary equations are

$$\frac{dx}{-3}=\frac{dy}{-3xz^2}=\frac{dz}{0}, \quad i.e. \quad \frac{dx}{1}=\frac{dy}{xz^2}=\frac{dz}{0}.$$

From $dz=0$, $\quad z=u$ (say).

From $xz^2\,dx=dy$, $\quad x^2z^2-2y=v.$

Hence $A\,du+B\,dv=0$ becomes ...(1)

$$A\,du+B(2xz^2\,dx+2x^2z\,dz-2\,dy)=0$$

i.e. $\quad 2Bxz^2\,dx-2B\,dy+(A+B.2x^2z)\,dz=0$

Comparing this with the given equation, we get

$$2Bxz^2=xz^3, \quad -2B=-z, \quad A+2x^2zB=2y.$$

$\therefore \quad B=\frac{1}{2}z=\frac{1}{2}u, \quad A=2y-x^2z^2=-v.$

Hence (1) becomes

$$-v\,du+\tfrac{1}{2}u\,dv=0, \quad i.e. \quad v=cu^2$$

or $\quad x^2z^2-2y=cz^2$ is the solution.

Ex. 6. *Solve* $\quad (y^2+yz+z^2)\,dx+(z^2+xz+x^2)\,dy$

$$+(x^2+xy+y^2)\,dz=0.$$

[Delhi Hons. 69; Karnatak 62; Agra 61, 52; Rajputana 63, 58, 54]

Solution. *Method of auxiliary equations.* The auxiliary equations are

$$\frac{dx}{\dfrac{\partial Q}{\partial z}-\dfrac{\partial R}{\partial y}}=\frac{dy}{\dfrac{\partial R}{\partial y}-\dfrac{\partial P}{\partial z}}=\frac{dz}{\dfrac{\partial P}{\partial y}-\dfrac{\partial Q}{\partial x}},$$

i.e. $$\frac{dx}{(2z+x)-(x+2y)}=\frac{dy}{(2x+y)-(y+2x)}=\frac{dz}{(2y+z)-(z+2x)}$$

i.e. $$\frac{dx}{y-z}=\frac{dy}{z-x}=\frac{dz}{x-y}.$$

This gives $\quad dx+dy+dz=0.$

Integrating, $\quad x+y+z=u$ (say). ...(1)

Also $\quad (y+z)\,dx+(z+x)\,dy+(x+y)\,dz=0$

i.e. $\quad (x\,dy+y\,dx)+(y\,dz+z\,dy)+(z\,dx+x\,dz)=0.$

Integrating, $\quad xy+yz+zx=v.$ ...(2)

Then $\quad A\,du+B\,dv=0$ gives

$$[A+B(y+z)]\,dx+[A+B(z+x)]\,dy+[A+B(x+y)]\,dz=0.$$

Comparing this with given equation, we get

$$A+B(y+z)=y^2+yz+z^2.$$

$$A+B(z+x)=z^2+xz+x^2,$$

$$A+B(x+y)=x^2+xy+y^2.$$

Subtracting first two of these,

$$B(y-x)=y^2-x^2+z(y-x);$$

$\therefore \quad B=x+y+z=u.$

Then $\quad A=-(xy+yz+zx)=-v.$

Integrating, $u/v=$constant or $x+y+z=c\,(xy+yz+zx)$ is the solution.

Aliter. *Homogeneous equation.* This is clearly a homogeneous equation.

So putting $x=uz$, $y=vz$, $dx=u\,dz+z\,du$ $dy=v\,dz+z\,dv$, the given equation becomes,

$$z^2(v^2+v+1)(u\,dz+z\,du)+z^2(u^2+u+1)(v\,dz+z\,dv)$$
$$+z^2(u^2+uv+v^2)\,dz=0,$$

i.e. $$\frac{(v^2+v+1)\,du+(u^2+u+1)\,dv}{(u+v+1)(uv+v+u)}+\frac{dz}{z}=0.$$

This can be written as

$$\frac{d[(u+v+1)(uv+u+v)]-2[(u+v+uv)](du+dv)}{(u+v+1)(uv+v+u)}+\frac{dz}{z}=0.$$

or $$\frac{d[(u+v+1)(uv+u+v)]}{(u+v+1)(uv+v+u)}-\frac{2(du+dv)}{u+v+1}+\frac{dz}{z}=0.$$

Integrating, $$\frac{u+v+uv}{u+v+1}\,z=c$$

or $$\frac{xz+yz+xy}{z^2}\cdot\frac{z}{x+y+z}\times z=c, \text{ as } u=x/z,\ v=y/z$$

or $xy+yz+zx=c\,(x+y+z)$ is the solution.

Ex. 7. *Solve equation in Ex. 2 P. 129 by the method of homogeneous equation.*

Proceed yourself.

4·8. An integrating factor of homogeneous equation

If $Px+Qy+Rz\neq 0$, then

$$\frac{1}{Px+Qy+Rz}$$

is an integrating factor of the homogeneous equation.

Ex. 1. *Solve Ex. 6 P. 136.*

Solution. Here $Px+Qy+Rz=(x+y+z)(xy+yz+zx)$.

Hence multiplying by I.F. $=\dfrac{1}{(x+y+z)(xy+yz+zx)}$,

the equation becomes

$$\frac{(y^2+yz+z^2)\,dx+(\quad)\,dy+(\quad)\,dz}{(x+y+z)(xy+yz+zx)}=0.$$

$$\frac{dL}{L}-\frac{2(dx+dy+dz)}{x+y+z}=0,$$

where $L=(x+y+z)(xy+yz+zx)$.

Integrating, $\quad L=c\,(x+y+z)^2$

or $\quad (yz+zx+xy)=c\,(x+y+z).$

Ex. 2. $z^2\,dx+(z^2-2yz)\,dy+(2y^2-yz-xz)\,dz=0.$ [Agra 40]

Solution. Here $Px+Qy+Rz=0$; hence there cannot be an integrating factor of the equation. So proceed as usual by putting $x=uz$, $y=vz$.

Aliter. Taking $z=$ constant, $dz=0$, the equation becomes $z^2\,dx+(z^2-2yz)\,dy=0$.

Integrating, $\quad z^2x+(z^2y-y^2z)=\phi(z).$ ...(1)

Differentiating,

$$z^2\,dx+(z^2-2yz)\,dy+(2zx+2zy-y^2)\,dy=\phi'(z).$$

Comparing, $2y^2-yz-xz=2zx+2zy-y^2-\phi'$

or $\quad \dfrac{d\phi}{dz}=3\,(-y^2+yz+zx)=\dfrac{3}{z}\,\phi.$

$$\therefore \quad \frac{d\phi}{\phi}=\frac{3\,dz}{z} \quad \text{or} \quad \phi=cz^3.$$

Hence from (1), the solution is

$$z^2x+(z^2y-y^2z)=cz^3,$$

i.e. $\quad zx+yz-y^2=cz^2.$

4·9. Geometrical interpretation of $P\,dx+Q\,dy+R\,dz=0$.

Let (x_1, y_1, z_1) be a general point in space for which all $P_1=P\,(x_1, y_1, z_1)$, $Q_1=Q\,(x_1, y_1, z_1)$, $R_1=R\,(x_1, y_1, z_1)$ are not zero.

If P, Q, R are single-valued, then set P_1, Q_1, R_1 may be considered numbers of a unique line through the point. Hence the given differential equation may be considered to define at each point (x_1, y_1, z_1), a line $\dfrac{x-x_1}{P_1}=\dfrac{y-y_1}{Q_1}=\dfrac{z-z_1}{R_1}$ and a plane $P_1\,(x-x_1)+Q_1\,(y-y_1)+R_1\,(z-z_1)=0$, which is normal to the above line.

If $f(x, y, z)=c$ is the solution of the differential equation then it represents a family of surfaces such that through a general point (x_1, y_1, z_1) of space there passes a single surface S_1 of the family, the tangent plane to this surface at (x_1, y_1, z_1), being

$$(x-x_1)\,\frac{\partial f}{\partial x_1}+(y-y_1)\,\frac{\partial f}{\partial y_1}+(z-z_1)\,\frac{\partial f}{\partial z_1}=0$$

and normal being $\dfrac{x-x_1}{\partial f/\partial x_1}=\dfrac{y-y_1}{\partial f/\partial y_1}=\dfrac{z-z_1}{\partial f/\partial z_1}$.

*4·10. Orthogonality of integral surfaces of

$$\mathbf{P\,dx+Q\,dy+R\,dz=0} \qquad \text{...(1)}$$

and $\quad \dfrac{\mathbf{dx}}{\mathbf{P}}=\dfrac{\mathbf{dy}}{\mathbf{Q}}=\dfrac{\mathbf{dz}}{\mathbf{R}}$ ...(2)

[Karnatak 63 ; Gujrat 59]

This follows from the fact that at any point (x_1, y_1, z_1) the direction P_1, Q_1, R_1 is

(*i*) normal to the integral surface (1) through the point (x_1, y_1, z_1) and

(*ii*) the direction of the integral curve of (2) through the point (x_1, y_1, z_1).

Hence these cut orthogonally.

Note. (2) is also cut orthogonally by

$$\frac{dx}{\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}}=\frac{dy}{\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}}=\frac{dz}{\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}} \qquad ...(3)$$

since the condition of their orthogonality is

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0, \qquad ...(4)$$

which is the condition of integrability of (1), see § 4·2 P. 12)

Geometrical interpretation of the condition of integrability of

$$\mathbf{P\,dx+Q\,dy+R\,dz=0} \qquad ...(1)$$

[Meerut 68]

As outlined earlier the geometrical interpretation of the condition of integrability of (1) is that the two surfaces given by

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

and $$\frac{dx}{\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)}=\frac{dy}{\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)}=\frac{dz}{\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)}$$

cut orthogonally.

Ex. 1. *Verify that (2) and (3) cut orthogonally, when*

$P=ny-mz,\ Q=lz-nx,\ R=mx-ly.$ [Gujrat 59]

Hint : Show that (4) is satisfied. Hence the result.

4·11. Non-integrable Single Equation.

If equation $\qquad P\,dx+Q\,dy+R\,dz=0, \qquad ...(1)$

does not satisfy the condition of integrability of § 4·2, then (1) in general cannot be integrated.

However, if an arbitrary relation

$$f(x, y, z)=c \qquad ...(2)$$

is given in x, y, z, then solution of (1) can be determined subject to the relation (2) as follows :

From (2), we get

$$\frac{\partial f}{\partial x}\,dx+\frac{\partial f}{\partial y}\,dy+\frac{\partial f}{\partial z}\,dz=0. \qquad ...(3)$$

When form of the relation (2) is known, then one variable and its differential can be determined in terms of other variables and their differentials. Therefore from (1), (2) and (3) one variable and its differential can be eliminated. If this variable is z, then we get a differential equation of the form

$$P_1\,dx+Q_1\,dy=0, \qquad \ldots(4)$$

where P_1 and Q_1 are functions of x and y only. The forms of P_1 and Q_1 depend upon (2) containing c of (2).

Solution of (4) may now be determined. This solution together with (2) constitutes a solution. For different values of f, different solutions can be obtained.

Ex 1. *Solve* $y\,dx+x\,dy-(x+y+2z)\,dz=0$, *when*

(*i*) $z=a$,

(*ii*) $x+y+2z=0$,

Solution. The equation does not satisfy the condition of integrability; so it cannot be integrated in general.

(*i*) However if $z=a$, $dz=0$, and then the equation becomes

$$y\,dx+x\,dy=0, \quad i.e., \quad xy=c.$$

Hence $xy=c$, $z=a$ constitute the solution of the differential equation, when $z=a$.

(*ii*) When $x+y+2z=0$, the solution is given by

$$xy=c,\ x+y+2z=0.$$

Ex. 2. *Find the system of curves satisfying the differential equation* $x\,dx+y\,dy+c\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}\,dz=0$ $\ldots(1)$

which lie on the surface $x^2/a^2+y^2/b^2+z^2/c^2=1$. $\ldots(2)$

Solution. From (2), $1-\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{z^2}{c^2}$.

$\therefore$ (1) becomes $x\,dx+y\,dy+z\,dz=0$.

Integrating, $x^2+y^2+z^2=c^2$. $\ldots(3)$

Therefore the curves are given by the intersection of (2) and (3).

Ex. 3. *Find the most general solution of the equation*

$$y\,dx+(z-y)\,dy+x\,dz=0, \qquad \ldots(1)$$

which is consistent with the relation

$$2x-y-z=1. \qquad \ldots(2)$$

Solution. From (2), $z=2x-y-1$,

$$dz=2\,dx-dy.$$

Putting values of z and dz in (1), we get

$$y\,dx+(2x+2y-1)\,dy+x\,(2dx-dy)=0$$

or $\quad (y+2x)\,dx+(x-2y-1)\,dy=0$

or $\quad d\,(xy+x^2-y^2-y)=0.$

Integrating, $xy+x^2-y^2-y=c$

is the solution of (1) consistent with (2).

5

Integration in Series

5·1. Introdnction

In this chapter we shall consider the solutions of the linear differential equations of second order, expressed in the form of series.

A linear differential equation of second order in the standard form is put as

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)\,y=0 \qquad \ldots(1)$$

where $P(x)$ and $Q(x)$ are functions of x.

A point $x=a$ may have special character with respect to the differential equation (1). To determine the nature of a point a with respect to (1), it is convenient to shift origin to the point a and suitably change the differential equation also, the form of the differential equation remains essentially unchanged by such a shift of origin. And now the nature of origin is determined with respect to the transformed equation. Therefore without loss of generality, we can restrict our study to nature of origin with respect to (1).

Now $x=0$ or origin is called an **ordinary point** of the differential equation if $P(x)$ and $Q(x)$ do not become infinite in a neighbcurhood of origin and these can be expanded in the form of power series given by

$$P(x)=\sum_{n=0}^{\infty} p_n x^n$$

and $$Q(x)=\sum_{n=0}^{\infty} q_n x^n.$$

If origin is not an ordinary point of the differential equation, then it is called a **singular point** of the differential equation. There are two types of singular points :

(*i*) regular singular points,

and (*ii*) irregular singular points.

These are defined as follows :

If $x\,P(x)$ and $x^2\,Q(x)$ can be expanded as power series in x in a neighbourhood of origin, then origin is a **regular singularity**.

A singularity which is not regular is called an **irregular singularity**.

The nature of the series solution in a neighbourhood of irregular singularity is quite complex and will not be considered here. We shall find the series solutions when

(*i*) origin is an ordinary point;

(*ii*) when origin is a regular singularity.

5·2. Solution near an ordinary point

We begin by looking for a formal solution of

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)\,y=0 \qquad ...(1)$$

in the form of the series

$$y=\sum_{n=0}^{\infty} c_n x^n. \qquad ...(2)$$

If $P(x)$ and $Q(x)$ are not polynomials of x, then certainly these can be expressed as

$$P(x)=\sum_{n=0}^{\infty} p_n x^n \text{ and } Q(x)=\sum_{n=0}^{\infty} q_n x^n. \qquad ...(3)$$

From (2),

$$\frac{dy}{dx}=\Sigma\, nc_n x^{n-1} \text{ and } \frac{d^2y}{dx^2}=\Sigma n\,(n-1)\,c_n x^{n-2} \qquad ...(4)$$

Putting values from (2), (3), (4) in (1), we get

$$\Sigma n\,(n-1)\,c_n x^{n-2}+(\Sigma p_n x^n)\,(\Sigma nc_n x^{n-1})+(\Sigma q_n x^n)\,(\Sigma c_n x^n)=0.$$

Now equating to zero coefficients of various powers of x we get the values of various coefficients of (2).

This solution will in general consist of two arbitrary constants and is therefore the general solution of (1). Following example would make the procedure clear.

Ex. 1. *Solve* $\dfrac{d^2y}{dx^2}-x^2\dfrac{dy}{dx}-y=0$ *in powers of* x.

Solution. Here $x=0$ is an ordinary point. Therefore assume that the series solution be

$$y=c_0=c_1x+c_2x^2+c_3x^3+\ldots+c_nx^n+\ldots$$

so that $\dfrac{dy}{dx}=c_1+2c_2x+3c_3x^2+\ldots+nc_nx^{n-1}+\ldots$

and $\dfrac{d^2y}{dx^2}=2c_2++6c_3x+\ldots+n\,(n-1)\,c_nx^{n-2}+\ldots$

Putting these values in equation, we get

$$(2c_2-c_0)-(6c_3-c_1)\,x+(12c_4-c_1-c_2)\,x^2+(20c_5-2c_2-c_3)\,x^3+\ldots$$
$$+[(n+2)\,(n+1)\,c_{n+2}-(n-1)\,c_{n-1}-c_n]\,x^n+\ldots=0.$$

Equating to zero the coefficients of various powers of x,

$$2c_2-c_0=0 \quad i.e. \quad c_2=\tfrac{1}{2}c_0,$$
$$6c_3-c_1=0 \quad i.e. \quad c_3=\tfrac{1}{6}c_1,$$
$$12c_4-c_1-c_2=0 \quad i.e. \quad c_4=\tfrac{1}{24}c_0+\tfrac{1}{12}c_1,\ldots$$

$$(n+2)(n+1)c_{n+2}-(n-1)c_{n-1}-c_n=0$$

i.e. $$c_{n+2}=\frac{(n-1)c_{n-1}}{(n+1)(n+2)}+\frac{c_n}{(n+1)(n+2)},\ n \geqslant 1,$$

Hence the solution in the form of the series is

$$z=c_0+c_1x+\tfrac{1}{2}c_0x^2+\tfrac{1}{6}c_1x^3+(\tfrac{1}{24}c_0+\tfrac{1}{12}c_1)x^4\ldots$$

or $$y=c_0(1+\tfrac{1}{2}x^2+\tfrac{1}{24}x^4+\ldots)+c_1(x+\tfrac{1}{6}x^3+\tfrac{1}{12}x^4+\ldots).$$

Ex. 2. *Solve* $\dfrac{d^2y}{dx^2}-2x^2\dfrac{dy}{dx}+4xy=x^2+2x+2$ *in powers of* x.

Solution. Let the solution in series be

$$y=c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+c_5x^5+\ldots+c_nx^n+\ldots$$

$$\therefore\quad \frac{d^2y}{dx^2}-2x^2\frac{dy}{dx}+4xy-x^2-2x-2=0 \text{ gives}$$

$$(2c_2-2)+(6c_3+4c_0-2)x+(12c_4+2c_1-1)x^2-20c_5x^3+\ldots$$
$$+[(n+2)(n+1)c_{n+2}-2(n-1)c_{n-1}+4c_{n-1}]x^n+\ldots$$

Equating to zero the coefficients of various powers of x,

$2c_2-2=0$ *i.e.* $c_2=1$

$6c_3+4c_0-2=0$ *i.e.* $c_3=\frac{1}{3}-\frac{2}{3}c_0,\ c_4=\frac{1}{12}-\frac{1}{6}c_1,\ c_5=0,$

$$c_{n+3}=\frac{2(n-3)}{(n+1)(n+2)}c_{n-1},\ n \geqslant 3.$$

Therefore the complete solution is

$$y=c_0(1-\tfrac{2}{3}x^3-\tfrac{2}{45}x^6\ldots)+c_1(x+\tfrac{1}{6}x^4-\tfrac{1}{63}x^7-\ldots)$$
$$+x^2+\tfrac{1}{3}x^3+\tfrac{1}{12}x^4+\tfrac{1}{45}x^6+\ldots$$

5·3. When x=0 is a singular (regular) point

In this case we shall assume a trial solution,

$$y=x^k(c_0+c_2x+c_2x^2+\ldots)=x^k\sum_{n=0}^{\infty}c_nx^n$$

where all c's are constants,

Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ and put their values in the given differential equation.

The index k will be determined by the quadratic equation which will be obtained by equating to zero the coefficient of the lowest power of x.

This equation in k is called the **Indicial equation.**

The values of $c_1, c_2, c_3, \ldots$ etc. are all determined in terms of c_0 by equating to zero coefficients of other various powers of x.

Now there arise following cases depending upon the nature of the roots of Indicial equation :

(*i*) The roots of Indicial equation *unequal and not differing by an integer.*

(*ii*) The roots of indicial equation *equal.*

(*iii*) The roots of indicial equation *unequel and differing by an integer.*

We shall discuss these cases one by one by taking examples of each case. A general theory is developed later in § 5·9 p. 152.

5·4. Case I. Roots of the indicial equation unequal and not differing by an integer.

Let α and β be the roots of the indicial equation. If α and β do not differ by integer, then in general two independent solutions are obtained by putting $k=\alpha$ and β, in the series. Let u and v be these two solutions : then the general solution is $y=cu+c'v$ where c and c' are arbitrary constants.

Ex. 1. *Solve completely in series the equation*

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)\,y=0$$

taking $2n$ as non-integral. **[Vikram 1964]**

Solution Let $y=x^k\,[c_0+c_1x+c_2x^2+c_3x^3+\ldots]$

so that $\dfrac{dy}{dx}=x^{k-1}\,[c_0k+c_1\,(k+1)\,x+c_2\,(k+2)\,x^2+\ldots]$

and $\dfrac{d^2y}{dx^2}=x^{k-2}\,[c_0k\,(k-1)+c_1\,(k+1)\,kx+c_2\,(k+2)\,(k+1)\,x^2+\ldots]$

Putting these values, in the given equation, we get

$$c_0\,[k\,(k-1)+k-n^2]\,x^k+c_1\,[(k+1)^2+n^2]\,x^{k+1}$$
$$+[c_2\,\{(k+2)^2-n^2\}+c_0]\,x^{k+2}+\ldots+\ldots=0.$$

Equating to zero coefficients of various powers of x, we get

$k^2-n^2=0,\quad i.e.\quad k=+n,\,-n,$

difference of the roots$=2n=$not integral as given.

Also $c_1\,[(k+1)^2-n^2]=0,\quad i.e.\quad c_1=0.$

$$c_2=\frac{1}{n^2-(k+2)^2}\,c_0,\; c_3=\frac{1}{n^2-(k+3)^2}\,c_1=0.$$

$\therefore\quad c_1=c_3=c_5=c_7=\ldots=0,$

$$c_4=\frac{1}{n^2-(k+4)^2}\,c_2=\frac{1}{n^2-(k+4)^2}\,\frac{1}{n^2-(k+2)^2}\,c_0 \text{ etc.}$$

Hence

$$z=c_0x^k\left[1+\frac{1}{n^2-(k+2)^2}\,x^2+\frac{1}{n^2-(k+4)^2}\,\frac{1}{n^2-(k+2)^2}\,x^4+\ldots\right].$$

Putting $k=n$ and $-n$ and taking $c_0=c$ and c' the two independent solutions are

$$cx^n\left[1-\frac{1}{4\,(n+1)}\,x^2+\frac{1}{4.8\,(n+2)\,(n+1)}\,x^4-\ldots\right]=cu \text{ (say)}$$

and $$c'x^{-n}\left[1-\frac{1}{4\,(-n+1)}\,x^2+\frac{1}{4.8\,(-n+2)\,(-n+1)}\,x^4-\ldots\right]$$
$$=c'v \text{ (say)}.$$

Hence $y=cu+c'v$ is the complete solution.

Note. If $c=\dfrac{1}{2^n\Gamma(n+1)}$, then cu is called the Bessel's function of order n and is denoted by $J_n(x)$ (see chapter VIII).

Ex. 2. *Solve completely in series the equation*

$$(2x+x^3)\frac{d^2y}{dx^2}-\frac{dy}{dx}-6xy=0.$$

Solution. Let $y=x^k(c_0+c_1x+c_2x^2+\ldots)$.

Putting the values of y, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in the equation and equating to zero the coefficients of various powers of x, we get :

Indicial equation $k(2k-3)=0$, *i.e.* $k=0, \frac{3}{2}$

$c_1\{2(k+1)-(k+1)\}=0$, *i.e.* $c_1=0$,

$c_2\{2(k+2)(k+1)-(k+2)\}-c_0\{k(k-1)-6\}=0$,

i.e. $c_2(2k+1)+c_0(k-3)=0$.

Similarly $c_3(2k+5)+c_1(k-2)=0$,

$c_4(2k+5)+c_2(k-1)=0$ and so on.

Clearly $c_1=c_3=c_5=\ldots=0$

and $c_2=-\dfrac{k-3}{2k+1}c_0$, $c_4=\dfrac{k-1}{2k+5}\,\dfrac{k-3}{2k+1}c_0$ etc.

Hence $z=c_0x^k\left[1-\dfrac{k-3}{2k+1}x^2+\dfrac{(k-1)(k-3)}{(2k+1)(2k+5)}x^4-\ldots\right]$.

Putting $k=0$ and $\frac{3}{2}$ and putting $c_0=c$ and c', the two independent solutions are

$$c\,[1+3x^2+\tfrac{3}{5}x^4-\tfrac{1}{15}x^6+\ldots]=cv \text{ (say)}$$

and $c'x^{3/2}\left[1+\frac{3}{8}x^2-\dfrac{1.3}{8.16}x^4+\dfrac{1.3.5}{8.16.24}x^6-\ldots\right]=c'v$ (say),

then the complete solution is $y=cu+c'v$.

Ex. 3. *Solve in series the equations*

(i) $9x(1-x)\dfrac{d^2y}{dx^2}-12\dfrac{dy}{dx}+4y=0.$

(ii) $2x(1-x)\dfrac{d^2y}{dx^2}+(1-x)\dfrac{dy}{dx}+3y=0.$

(iii) $4x\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+y=0.$

(iv) $2x\dfrac{d^2y}{dx^2}+(x+1)\dfrac{dy}{dx}+3y=0.$

(v) $2x^2\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+(x^2+1)y=0.$

(vi) $3x\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+x^2y=0.$

5·5. Case II. Roots of the Indicial equation, equal.

We shall later illustrate the following principle in this regard :

If $k=\alpha$ be the repeated (equal) root of the indicial equation, then obtain the solution in terms of k and call it y; the two independent solutions are obtained by putting $k=\alpha$ in y and in $\partial y/\partial k$.

It will be seen that the second solution always consists of the product or a numerical multiple of the first solution and log x plus a series (c.f. § 5·9 case II p. 154).

Ex. 1. *Obtain a general solution in ser es of powers x of the equation* $x\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+xy=0.$ **[Raj. 1964]**

Solution. Let the solution in series be

$$y=x^k\,[c_0+c_1x+c_2x^2+c_3x^3+\ldots].$$

Putting values of y, $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$ in the given differential equation, we get

$$c_0\,[k\,(k-1)+k]\,x^{k-1}+c_1[\,k\,(k+1)+(k+1)]\,x^k$$
$$+[c_2\{(k+2)\,(k+1)+(k+2)\}+c_0]\,x^{k-1}$$
$$+[c_3\{(k+3)\,(k+2)+(k+3)\}+c_1]\,x^{k+2}+\ldots=0.$$

The incidial equation is $k\,(k-1)+k=0$ as $c_0\neq 0$,
i.e. $k^2=0$.

Other coefficients are given by $c_1\,(k+1)^2=0$, *i.e.* $c_1=0$,

$$c_2\,(k+2)^2+c_0=0,\quad i.e.\quad c_2=-\frac{1}{(k+2)^2}\,c_0,$$
$$c_3\,(k+3)^2+c_1=0,\quad i.e.\quad c_3=0 \text{ as } c_1=0,$$
$$c_4\,(k+4)^2+c_2=0,\quad i.e.\quad c_4=\frac{1}{(k+4)^2\,(k+2)}\,c_0,$$

......and so on.

Hence one solution for $k=0$ may be obtained by putting $k=0$ in

$$y=c_0x^k\left[1-\frac{1}{(k+2)^2}\,x^2+\frac{1}{(k+4)^2\,(k+2)^2}\,x^4\ldots\right]. \qquad \ldots(1)$$

*Differentiating it w.r.t. k without putting $k=0$,, we get

* Because if we put the series in y in the left hand side of the given differential equation, we get the single term $c_0k^2x^{k-1}$. As this involves the square of k, its partial differential coefficient with respect to k, *i.e.* $2c_0kx^{k-1}+c_0k^2x^{k-1}\log x^{k-1}\log x$ will also vanish at $k=0$.

In other words

$$\frac{\partial}{\partial k}\left[x\frac{d^2}{dx^2}+\frac{d}{dx}+x\right]y=2c_0kx^{k-1}+c_0k^2x^{k-1}\log x$$

or
$$\left[x\frac{d^2}{dx^2}+\frac{d}{dx}+x\right]\frac{\partial y}{\partial k}=2c_0kx^{k-1}+c_0k^2x^{k-1}\log x$$

as operators are commutative.

Therefore $\dfrac{\partial y}{\partial k}$ is a second solution of the differential equation, if k is put equal to zero after differentiation. (Ref. Case II p. 151).

$$\frac{\partial y}{\partial k}=y\log x+c_0x^k\left[\frac{2}{(k+2)^3}x^2\right.$$
$$\left.-\left\{\frac{2}{(k+2)^3(k+4)^2}+\frac{2}{(k+2)^2(k+4)^3}\right\}x^4\ldots\right]. \quad \ldots(2)$$

Now putting $k=0$ and $c_0=c$ and c' in (1) and (2) respectively, the two independent solutions are

$$y=c\left[1-\frac{1}{2^2}x^2+\frac{1}{2^2.4^4}x^4-\frac{1}{2^2.4^2.6^2}x^6+\ldots\right]=cu \text{ (say)},$$

$$\frac{\partial y}{\partial k}=c'u\log x+c'\left[\frac{1}{2}x^2-\frac{1}{2^2.4^2}(1+\tfrac{1}{2})x^4\right.$$
$$\left.+\frac{1}{2^2.4^2.6^2}(1+\tfrac{1}{2}+\tfrac{1}{3})x^6-\ldots\right]$$

$=c'v$ say.

And then the complete solution is

$y=cu+c'v.$

Ex. 2 *Integrate in series the equation*

$$(x-x^2)\frac{d^2y}{dx^2}+(1-5x)\frac{dy}{dx}-4y=0.$$ [Roorkee 70]

Solution Let a solution in series be

$y=x^k\,(c_0+c_1x+c_2x^2+c_3x^3+\ldots).$

Putting values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the equation and collecting coefficients of various powers of x, and equating to zero these coefficients, we get

$c_0\{k(k-1)+k\}=3,$ *i.e.*, $k^2=0$

$c_1\{(k-1)k+k+1\}-c_0\{k(k-1)+5k+4\}=0,$

i.e $c_1(k+1)^2-c_0(k+2)^2=0,$ *i.e.* $c_1=\left(\frac{k+2}{k+1}\right)^2 c_0.$

Similarly $c_2=\left(\frac{k+3}{k+1}\right)^2 c_0,\ c_3=\left(\frac{k+4}{k+1}\right)^2 c_0,\ldots$

Hence

$$y=c_0x^k\left[1+\left(\frac{k+2}{k+1}\right)^2x+\left(\frac{k+3}{k+1}\right)^2x^2+\left(\frac{k+4}{k+1}\right)^2x^3+\ldots\right] \quad \ldots(1)$$

is a solution if $k=0$.

Differentiating it w.r.t. k, without putting $k=0$, we get

$$\frac{\partial y}{\partial x}=y\log x+c_0x^k\left[2\left(\frac{k+2}{k+1}\right)\frac{-1}{(k+1)^2}x\right.$$
$$\left.+2\left(\frac{k+3}{k+1}\right)\frac{-2}{(k+1)^3}x^2+2\left(\frac{k+4}{k+1}\right)\frac{-3}{(k+1)^2}x^3+\ldots\right] \quad \ldots(2)$$

Putting $k=0$ and $c_0=c$ and c' in (1) and (2), the two independent solutions of the equations are

$y=c\,[1+2^2x+3^2x^2+4^2x^3+\ldots]=cu$, say,

$\partial y/\partial k=c'u\log x-2c'\,[1.2x+2.3x^2+3.4x^3+\ldots]=c'v$ say.

Hence the complete primitive is

$y=cu+c'v.$

Ex. 3. *Solve completely in series* $x\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+x^2y=0.$

Solution. Let $y=x^k\,[c_0+c_1x+c_2x^2+\ldots]$.

Putting for y, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in the given equation, we get

$$k^2c_0x^{k-1}+(k+1)^2\,c_1x^k+(k+2)^2\,c_2x^{k+1}+[(k+3)^2\,c_3+c_0]\,x^{k+2}$$
$$+\ldots+[(k+n)^2\,c_n+c_{n-3}]\,x^{k+n-1}+\ldots=0.$$

Equating to zero coefficients of various powers of x, we get

$c_0k^2=0$, i.e. $k^2=0$.

$(k+1)^2\,c_1=0$, i.e. $c_1=0$, $(k+2)^2\,c_2=0$, i.e. $c_2=0$.

$$c_3=-\frac{1}{(k+3)^2}\,c_0,\quad c_4=-\frac{1}{(k+4)^2}\,c_1=0,\quad c_5=0.$$

$$c_6=-\frac{1}{(k+6)^2}\,c_3=\frac{x}{(k+3)^2\,(k+6)^2}\,c_0,\quad c_7=c_8=0.$$

Hence $$y=c_0x^k\left[1-\frac{1}{(k+3)^2}x^3+\frac{1}{(k+3)^2\,(k+6)^2}x^6\right.$$
$$\left.-\frac{1}{(k+3)^2\,(k+6)^2\,(k+9)^2}\,x^9+\ldots\right] \quad \ldots(1)$$

$$\frac{\partial y}{\partial k}=y\log x+2c_0x^k\left[\frac{1}{(k+3)^3}\,x^3-\left(\frac{1}{(k+3)^3\,(k+6)^2}\right.\right.$$
$$\left.\left.+\frac{1}{(k+3)^2\,(k+6)^3}\right)x^6+\ldots\right]. \quad \ldots(2)$$

Putting $k=0$ and $c_0=0$ and c' in (1) and (2), the two solutions are

$$y=c\left[1-\frac{1}{3^2}x^3+\frac{1}{3^4\,(2\,!)^2}x^6-\frac{1}{3^6\,(3\,!)^2}x^9+\ldots\right]=cu\text{ (say)}.$$

$$\frac{\partial y}{\partial x}=c'u\log x+c'\left[\frac{1}{3^3}x^3-\frac{1}{3^5\,(2\,!)^2}\,(1+\tfrac{1}{2})\,x^6\right.$$
$$\left.+\frac{1}{3^7\,(3\,!)^2}\,(1+\tfrac{1}{2}+\tfrac{1}{3})\,x^9+\ldots\right]=c'v\text{ (say)}.$$

The complete solution is $y=cu+c'v$.

Ex. 4. *Solve completely in series :*

(i) $(x-x^2)\dfrac{d^2y}{dx^2}+(1-x)\dfrac{dy}{dx}-y=0.$ [Meerut 68]

(ii) $x\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}-y=0.$

(iii) $x\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+2y=0.$

(iv) $4(x^4-x^2)\frac{d^2y}{dx^2}+8x^3\frac{dy}{dx}-y=0.$

(v) $\frac{d^2y}{dx^2}+(x-1)\frac{dy}{dx}+y=0$ *in powers of* $x-2$. [Agra 72]

Proceed as in the above solved examples.

5·6. A. Case III. Roots of the indicial equatoin differing by an integer.

This case can further be sub-divided into two sub-cases.

1. If one of the roots makes z infinite.
2. If one of the roots makes z indeterminate.

We shall discuss these two possibilities separately.

5·7. Case III. (a) *The indicial equation has two roots α and β ($\alpha>\beta$) differing by an integer and some of the coefficients of x become infinite for $k=\beta$.*

In this case put $c(k-\beta)$ for c_0.

This would lead to two independent solutions for $k=\beta$, namely the modified y and $\partial y/\partial k$ as in case II.

We thus find the three solutions :

(*i*) the solution by putting $k=\alpha$ in y,

(*ii*) the solution by putting $k=\beta$ in modified y,

(*iii*) putting $k=\beta$ in the differential of modified z namely $\partial y/\partial k$.

But only two of these are independent as solution (*i*) is a numerical multiple of (*ii*).

The following examples shall fully illustrate the method.

Ex. 1. *Obtain a general solution in series of powers of x of the equation (Bessel's equation of order one)*

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-1)y=0.$$

[Rajasthan 63]

Solution. Let $y=x^k[c_0+c_1x+c_2x^2+\ldots]$.

Then putting values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given equation and then equating to zero coefficients of various powers of x, we get

$c_0[k(k+1)+k-1]=0$, i.e. $k^2-1=0$,

i.e. $k=1,-1$, ...(1)

$c_1\{(k+1)^2-1\}=0$, i.e. $c_1=0$, ...(2)

$c_2\{(k+2)^2-1\}+c_0=0\ldots.$

$c_n\{(k+n)^2-1\}+c_{n-2}=0.$

This gives

$$y=c_0x^k\left[1-\frac{1}{(k+1)(k+3)}x^2+\frac{1}{(k+1)(k+3)^2(k+5)}x^4-\frac{1}{(k+1)(k+3)^2(k+5)^2(k+7)}x^6+\ldots\right]. \quad \ldots(1)$$

where k may have values 1 or -1.

But if we take $k=-1$ in the above series, the coefficients become infinite because of the factor $(k+1)$ in the denominator.

So we put $c\,(k+1)$ for c_0 in (1).

Thus
$$y=cx^k\left\{(k+1)-\frac{1}{(k+3)}x^2+\frac{1}{(k+3)^2(k+5)}x^4-\frac{1}{(k+3)^2(k+5)^2(k+7)}x^6+\dots\right\}, \quad \dots(2)$$

$$\frac{\partial y}{\partial k}=y\log x+cx^k\left[1+\frac{1}{(k+3)^2}x^2-\left\{\frac{2}{(k+3)^3(k+5)}+\frac{1}{(k+3)^2(k+5)^2}\right\}x^4+\dots\right]. \quad \dots(3)$$

Putting $k=-1$ in (2) and (3), we get

$$cx^{-1}\left[-\tfrac{1}{2}x^2+\frac{1}{2^2.4}x^4+\frac{1}{2^2.4^2.6}x^6\dots\right]=cu \text{ (say)},$$

and
$$cu\log x+cx^{-1}\left[1+\frac{1}{2^2}x^2-\frac{1}{2^2.4}\left(\tfrac{2}{2}+\tfrac{1}{4}\right)x^4-\frac{1}{2^2.4^2.6}\left(\tfrac{2}{2}+\tfrac{2}{4}+\tfrac{1}{6}\right)x^2+\dots\right]=c'v \text{ (say)}.$$

Hence the general solution is

$$y=cu+c'v.$$

5·8. Case III B. Roots of Indicial Equation differing by an integer making a coefficient of x indeterminate.

Let α and β be the two roots of the indicial equation ($\alpha>\beta$), differing by an integer. If one of the coefficients of y becomes indeterminate when $k=\beta$, the complete primitive is given by putting $k=\beta$ in y which then contains two arbitrary constants. The result on putting $k=\alpha$ in y simply gives a numerical multiple of the series contained in the first solution.

The following examples will make the procedure clear.

Ex. 1. *Integrate in series Legendre's differential equation,*

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0,$$

when n is a positive integer. [Agra 70; Vikram 63]

Solution. Let $y=x^k(c_0+c_1x+c_2x^2+\dots)$ be a solution.

Then putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given differential equation, we get

$$(1-x^2)x^{k-2}[k(k-1)c_0+(k+1)kc_1x+(k+2)(k+1)c_2x^2+(k+3)(k+2)c_3x^3+\dots]$$
$$-2x.x^{k-1}[kc_0+(k+1)c_1x+(k+2)c_2x^2+\dots]+n(n+1)x^k(c_0+c_1x+c_2x^2+\dots)=0.$$

Equating the coefficients of $x^{k-2}, x^{k-1}, x^k, x^{k+1}$, ...etc. to zero, we get

$$\left.\begin{array}{l} c_0 k\,(k-1)=0 \text{ (Indicial equation),} \\ \text{which gives } k=0 \text{ or } 1 \text{ as } c_0 \neq 0. \end{array}\right\} \qquad \ldots(1)$$

$c_1 k\,(k+1)=0$ which when $k=0$, gives indeterminate value of c_1. However, $c_1=0$ when $k=1$.

$$c_2\,(k+2)\,(k+1)-c_0\,[k\,(k+1)-n\,(n-1)]=0, \qquad \ldots(2)$$

$$c_3\,(k+3)\,(k+2)-c_1\,[(k+1)\,(k+2)-n.(n+1)=0, \qquad \ldots(3)$$

...and so on.

Thus the solution containing two arbitrary constants c_0 and c_1 will be obtained by taking $k=0$.

When $k=0$, $c_2=-\dfrac{n\,(n+1)}{2\,!}\,c_0$

$$c_3=\frac{n\,(n+1)-2}{3\,!}\,c_1=-\frac{(n+2)\,(n-1)}{3\,!}\,c_1.$$

Similarly $c_4=-\dfrac{n\,(n-2)\,(n+1)\,(n+3)}{4\,!}\,c_0$...etc.

Therefore the solution is

$$[y]_{k=0}=c_0+c_1x-\frac{n\,(n+1)}{2\,!}\,c_0x^2-\frac{(n+2)\,(n-1)}{3\,!}\,c_1x^3$$

$$+\frac{n\,(n-2)\,(n+1)(n+3)}{4\,!}\,c_0x^4-\ldots$$

$$=c_0\left[1-\frac{n\,(n+1)}{2\,!}\,x^2+\frac{n\,(n-2)\,(n+1)\,(n+3)}{4\,!}\,x^4-\ldots\right]$$

$$+c_1\left[x-\frac{(n-1)\,(n+2)}{3\,!}\,x^3-\frac{(n-1)\,(n-3)\,(n+2)\,(n+4)}{5\,!}\,x^5-\ldots\right]$$

Ex. 2. *Integrate in series* $(1-x^2)\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}+2y=0.$

Hint. Put $n=1$ in the above series.

Ex. 3. *Integrate in series the differential equation*

$$(1-x^2)\frac{d^2y}{dx^2}+2x\,\frac{dy}{dx}+y=0.$$

Solution. Let $y=x^k\,(c_0+c_1x+c_2x^2+\ldots)$ be a solution of the differential equation.

Then proceeding in the usual way, we get

$k\,(k-1)=0$ as $c_0\neq 0$, $k=0, 1$

$c_1\,(k+1)\,k=0,$

If we put $k=1$ in (1), we get

$$\tfrac{1}{2}c_0x\left[2-\frac{1}{4}\,x^2+\frac{1}{4^2.6}\,x^4-\frac{1}{4^2.6^2.8}\,x^6+\ldots\right]=\tfrac{1}{2}c_0w \text{ (say)}$$

then obviously $\frac{1}{2}w=4u$ if $c_0=c$, which makes c_1 indeterminate when $k=0$,

$c_2(k+2)(k+1)-c_0\{k(k-1)-2k-1\}=0,$

$c_3(k+3)(k+2)-c_1\{k(k+1)-2(k+1)-1\}=0,$

...and so on.

The solution containing two arbitrary constants c_0 and c_1 by taking $k=0$ is given by

$$[y]_{k=0}=c_0\left[1-\tfrac{1}{2}x^2+\tfrac{1}{8}x^4+\tfrac{1}{80}x^6-\ldots\right]+c_1\left[x-\tfrac{1}{2}x^3+\tfrac{1}{40}x^5+\ldots\right].$$

Ex. 4. *Integrate in series the equations*

(i) $\dfrac{d^2y}{dx^2}+x^2y=0.$

(ii) $(2+x^2)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+(1+x)y=0.$

Ans. (*i*) $y=c_0\left[2-\dfrac{1}{3.4}x^4+\dfrac{1}{3.4.7.8}x^8-\ldots\right]$

$+c_1\left[x-\dfrac{1}{4.5}x^5+\dfrac{1}{4.5.8.9}x^9-\ldots\right.$

(*ii*) $y=c_0\left[1-\tfrac{1}{4}x^2-\tfrac{1}{12}x^3+\tfrac{5}{96}x^3\ldots\right]+c_1\left[x-\tfrac{1}{6}x^3-\tfrac{1}{24}x^4+\ldots\right].$

5·9. General theory of the series solution near a regular singular point : (Forbenius Method).

We have already defined above the regular singular point of differential equation. We now come to a general theory of solution in series.

Equivalently a point $x=a$ is called a regular singular point of a linear differential equation of second order, if the differential equation when written in the form

$$L(y)\equiv(x-a)^2\frac{d^2y}{dx^2}+(x-a)P(x)\frac{dy}{dx}+Q(x)y=0, \qquad \ldots(1)$$

is such that $P(x)$ and $Q(x)$ have Taylor's expansions in a neighbourhood of $x=a$, *i.e.*,

$$P(x)=\sum_{n=0}^{\infty}p_n(x-a)^n,\ Q(x)=\sum_{n=0}^{\infty}q_n(x-a)^n.$$

If $|P(x)|\leqslant M$ and $|Q(x)|\leqslant M$ in a neighbourhood of a given by $|x-a|<r$, then

$$|p_n|\leqslant\frac{M}{r^n}\text{ and }|q_n|\leqslant\frac{M}{r^n}\ (n=0,1,2,\ldots)$$

We shall derive the series solution of (1) near the regular singular point $x=a$ and would discuss its convergence.

Assume that a series solution of (1) be

$$y(x,\rho)=(x-a)^\rho\sum_{n=0}^{\infty}c_n(x-a)^n, \qquad \ldots(2)$$

where $c_0\neq0$ is arbitrary.

On substituting (2) in (1), we get

$$L(y)=\sum_{n=0}^{\infty}[(\rho+n)(\rho+n-1)+p_0(\rho+n)+q_0]\,c_n$$

$$+\sum_{k=0}^{n-1}\{p_{n-k}(\rho+k)+q_{n-k}\}\,c_k].\,(x-a)^{\rho+n}.$$

Putting $[(\rho+n)(\rho+n-1)+p_0(\rho+n)+q_0]\,c_n$

$$=-\sum_{k=0}^{n-1}\{p_{n-k}(\rho+k)+q_{n-k}\}\,c_k. \qquad ...(3)$$

$$L(y(x,\rho))=c_0\{\rho(\rho-1)+p_0\rho+q_0\}(x-q)^\rho.$$

The indicial equation on equating co-efficient of $(x-a)^\rho$ to zero is ($\because c_0\neq 0$),

$$\rho(\rho-1)+p_0\rho+q_0=0.$$

If ρ_1, ρ_2 be the roots of this equation, then

$$\rho_1+\rho_2=1-p_0$$

and $L(y(x,\rho))=c_0(\rho-\rho_1)(\rho-\rho_2)(x-a)^\rho.$

Case I. $\rho_1-\rho_2=\lambda$, where λ is not an integer (positive, negative or zero).

Since $L(y(x,\rho_1))=0$; $L(y(x,\rho_2))=0$, therefore

$$y(x,\rho_1)=\sum_{n=0}^{\infty}c_n(x-a)^{\rho_1+n} \qquad ...(5)$$

and $$y(x,\rho_2)=\sum_{n=1}^{\infty}c_n(x-a)^{\rho_2+n} \qquad ...(6)$$

are two linearly independent solutions of (1), provided the series in (5) and (6) are convergent in some neighbourhood of the point $x=a$.

Now $(\rho+n)(\rho+n-1)+p_0(\rho+n)+q_0$

$$=\rho(\rho-1)+p_0\rho+q_0+n^2+n(p_0+2\rho-1)$$

$$=(\rho-\rho_1)(\rho-\rho_2)+n^2+n(2\rho-\rho_1-\rho_2)$$

$$=\begin{cases}n^2+n\lambda & \text{for } \rho=\rho_1\\ n^2-n\lambda & \text{for } \rho=\rho_2\end{cases}$$

Thus, we have from (3)

$$n(n+\lambda)\,c_n=-\sum_{k=0}^{n-1}\{p_{n-k}(\rho_1+k)+q_{n-k}\}\,c_k$$

and $$n(n-\lambda)\,c_n=-\sum_{k=0}^{n-1}\{p_{n-k}(\rho_2+k)+q_{n-k}\}\,c_k. \qquad ...(7)$$

Let when $\rho=\rho_1$, $b_n=|c_n|$, for $0\leqslant n<|\lambda|$.

$$n(n+\lambda)\, b_n = M \sum_{k=0}^{n-1} \frac{|\rho_1|+k+1}{r^{n-k}}\, b_k \text{ for } n > |\lambda|.$$

It can be easily seen that when $\rho=\rho_1$, $b_n \geqslant |c_n|$ for all values of n, This series $\sum_{k=0}^{\infty} b_n (x-a)^n$ dominates the series

$$\sum_{n=0}^{\infty} c_n (x-a)^n.$$

Again for sufficiently large values of n,

$$n(n+\lambda)\, b_n - \frac{(n-1)(n-1+\lambda)}{r}\, b_{n-1} = M \frac{|\rho_1|+n}{r}\, b_{n-1}.$$

whence we obtain

$$\frac{b_n}{b_{n-1}} = \frac{(n-1)(n-1+\lambda)}{n(n+\lambda) r} = \frac{|\rho_1|+n}{n(n+\lambda) r}$$

and hence

$$\lim_{n\to\infty} \frac{b_n}{b_{n-1}} = \frac{1}{r}.$$

Therefore the series $\sum_{n=0}^{\infty} b_n (x-a)^n$ is convergent in $|x-a| < r$, and hence the series $\sum_{n=0}^{\infty} c_n (x-a)^n$ is also convergent in $|x-a| < r$, when $\rho=\rho_1$.

On the same lines, we can show that

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

is convergent in $|x-a| < r$, when $\rho=\rho_2$.

Case II. *When the two roots of indicial equation are equal.*

Let $\rho_1=\rho_2=\sigma$ (say).

In this case

$$L(y(x, \rho)) = c_0 (\rho-\sigma)^2 (x-a)^\rho.$$

The recurrence relation (3) [compare from (7)] becomes

$$n^2 c_n = -\sum_{k=0}^{\infty} \{p_{n-k} (\sigma+k)+q_{n-k}\}\, c_k.$$

It can now be shown by the method similar to the one used in Case I, that the series, $\sum_{n=0}^{\infty} c_n (x-a)^n$ is convergent in $|x-a| < r$ where $\rho_1=\sigma=\rho_2$.

Also $\because$ $L(y(x, \sigma))=0$,

$$\therefore \quad y(x, \sigma) = \sum_{n=0}^{\infty} (x-a)^{n+\sigma}$$

is one solution of the differential equation (1).

Further,

$$\frac{\partial}{\partial\rho} L(y(x, \rho)) = 2c_0 (\rho-\sigma)(x-a)^\rho + c_0 (\rho-\sigma)^2 (x-a)^\rho \log(x-a).$$

$$\therefore \quad L\left(\frac{\partial}{\partial\rho} y(x, \rho)\right) = \frac{\partial}{\partial\rho} L(y(x, \rho)) = 0 \text{ when } \rho=\sigma.$$

Hence $\left[\frac{\partial}{\partial\rho} y(x, \rho)\right]_{\rho=\sigma}$ also satisfies the differential equation (1) and is therefore a solution.

Now $\frac{\partial}{\partial\rho} y(x, \rho) = \sum_{n=0}^{\infty} c_n' (x-a)^{\rho+n} + \sum_{n=0}^{\infty} c_n (x-a)^{\rho+n} \log(x-a).$

The coefficients c_n are to be considered as functions of ρ.

Hence

$$\left[\frac{\partial}{\partial\rho} y(x, \rho)\right]_{\rho=\sigma} = \log(x-a)\, y(x, \sigma) + \sum_{n=0}^{\infty} c_n' (x-a)^{\sigma+n}$$
$$= \log(x-a)\, y(x, \sigma) + \phi(x, \sigma),$$

where $\phi(x, \sigma) = \sum_{n=0}^{\infty} c_n' (x-a)^{\sigma+n}$ can be shown to be convergent in $|x-a| < r$.

Thus in this case the two linearly independent solutions of the differential equations are

$$y(x, \sigma) \quad \text{and} \quad y(x, \sigma)\log(x-a) + \phi(x, \sigma).$$

Case III. *When ρ_1 and ρ_2 differ by an integer.*

Let $\rho_2 - \rho_1 = m$ (a positive integer).

Let $f(\rho+n) = (n+\rho)(n+\rho-1) + p_0(\rho+n) + q_0,$

whence $f(\rho) = \rho(\rho-1) + p_0\rho + q_0$
$= (\rho-\rho_1)(\rho-\rho_2).$

Hence $f(\rho_1)=0,\ f(\rho_2)=f(\rho_1+m)=0.$

The recurrence relation (3) gives

$$f(\rho_1+n)\, c_n(\rho_1) = -\sum_{k=0}^{n-1} \{p_{n-k}(\rho_1+k) + q_{n-k}\}\, c_k(\rho_1)$$

and $f(\rho_2+n)\, c_n(\rho_2) = -\sum_{k=0}^{n-1} \{p_{n-k}(\rho_2+k) + q_{n-k}\}\, c_k(\rho_2),$

where $n=1, 2,$ and $c_n(\rho_1)$ and $c_n(\rho_2)$ are values of c_n when $\rho=\rho_1$ and ρ_2 respectively.

The coefficients $c_n(\rho_2)$ can be easily determined, but the coefficients $c_m(\rho_1), c_{m+1}(\rho_1), \ldots$ all become infinite because $f(\rho_1+m)=0$. To overcome this difficulty we replace c_0 by $K_0(\rho-\rho_2)$, where $K_0 (\neq 0)$ is a constant. Then

$$L(y(x, \rho)) = c_0(\rho-\rho_1)(\rho-\rho_2)(x-a)^\rho$$
$$= K_0(\rho-\rho_1)(\rho-\rho_2)^2(x-a)^\rho.$$

Thereby we obtain

$$L(y(x, \rho_2))=1 \quad \text{and} \quad L\left[\frac{\partial}{\partial \rho} y(x, \rho)\right]_{\rho=\rho_2}=0.$$

$$\therefore \quad y(x, \rho_2)=\sum_{n=0}^{\infty} c_n(\rho_2)(x-a)^{\rho_2+n}$$

and $\left[\dfrac{\partial}{\partial \rho} y(x, \rho)\right]_{\rho=\rho_2}=y(x, \rho_2)\log(x-a)+\psi(x, \rho_2),$

where $\psi(x, \rho)=\sum\limits_{n=0}^{\infty} c_n'(\rho_2)(x-a)^{\rho_2+n}$

are two linearly independent solutions of (1).

The two series $\sum\limits_{n=0}^{\infty} c_n(\rho_2)(x-a)^n$

and $\sum\limits_{n=0}^{\infty} c^{n\prime}(\rho_2)(x-a)^n$

can again be shown to be convergent in $|x-a|<r$.

6
Numerical Solutions

6·1. Introduction

So far we obtained analytical expressions for the solutions of a differential equation. Sometimes analytical solutions cannot be evaluated and sometimes these are not required. In the present chapter we shall consider the problem of obtaining numerical values for the solutions of first order differential equations.

6·2 Picard's method of successive approximations.

Given a differential equation

$$\frac{dy}{dx}=f(x, y) \qquad \ldots(1)$$

and the initial condition $y(x_0)=y_0$, $\qquad \ldots(2)$
to determine values of y *for values of* x *other than* x_0.

From (1), we have

$$\Big[\, y \,\Big]_{y_0}^{y}=\int_{x_0}^{x} f(x, y)\, dx$$

or $$y-y_0=\int_{x_0}^{x} f(x, y)dx. \qquad \ldots(3)$$

Now the integral on the right hand side can be evaluated if we only know the expression of y in terms of x. This is not known. Therefore we cannot proceed. What we know is the value of y at x_0 As an approximation, we replace y by y_0 in the integral on the right and call the value of y on left as first approximation y, so that

$$y_1=y_0+\int_{x_0}^{x} f(x, y_0)\, dx. \qquad \ldots(4)$$

Now that this better approximation of y is obtained (y_1 is better approximation than y_0 for value of y at any point x), we replace y by y_1 in right hand side of (3), and get the second approximation y_2, given by

$$y_2=y_0=\int_{x_0}^{x} f(x, y_1)\, dx. \qquad \ldots(5)$$

Continuing this procedure, successive approximations

$y_0, y_1, y_2, y_3\ldots\ldots$

can be obtained, each giving a better approximation than the preceding one.

Ex. 1. *Apply Picard's method upto third approximation to solve*

$$\frac{dy}{dx}=x+y^2$$

where $y=0$, *when* $x=0$. [Agra 1969]

Solution. We have $f(x, y)=x+y^2$, $y_0=0$, $x_0=0$.

The first approximation y_1 is given by

$$y_1=y_0+\int_0^x f(x, y_0)\, dx \qquad \text{where } f(x, y)=x+0^2=x$$

$$=0+\int_0^x x\, dx=\tfrac{1}{2}x^2.$$

Now the second approximation y_2 is given by

$$y_2=y_0+\int_0^x f(x, y_1)\, dx, \qquad f(x, y_1)=x+y_1^2$$

$$=0+\int_0^x [x+(\tfrac{1}{2}x^2)^2]\, dx$$

$$=\tfrac{1}{2}x^2+\tfrac{1}{20}x^5$$

The third approximation y_3 is given by

$$y_3=y_0+\int_0^x f(x, y_2)\, dx, \qquad f(x, y_2)=x+y_2^2$$

$$=0+\int_0^x \left[x+(\tfrac{1}{2}x^2+\tfrac{1}{20}x^5)^2\right] dx$$

$$=\int_0^x [x+\tfrac{1}{4}x^4+\tfrac{1}{20}x^7+\tfrac{1}{400}x^{10}]\, dx$$

$$=\tfrac{1}{2}x^2+\tfrac{1}{20}x^5+\tfrac{1}{160}x^8+\tfrac{1}{4400}x^{11}.$$

Ex. 2. *Apply Picard's method upto third approximation to solve*

$$\frac{dy}{dx}=2y-2x^2-3$$

given that $y=2$, *when* $x=0$. [Agra 1970]

Solution. We have

$y_0=2$, $x_0=0$, $\qquad f(x, y)=2y-2x^2-3.$

Now the first approximation, y_1 is given by

$$y_1=y_0+\int_0^x f(x, y_0)\, dx, f(x, y_0)=2\cdot 2-2x^2-3$$

$$=2+\int_0^x (4-2x^2-3)\, dx=2+\int_0^x (1-2x^2)\, dx$$

$$=2+\left[x-\tfrac{2}{3}x^3\right]_0^x=2+x-\tfrac{2}{3}x^3.$$

Now for second approximation, y_2, we have

$$y_2=y_0+\int_0^x f(x, y_1)\, dx, f(x, y_1)=2y_1-2x^2-3$$

$$=2+\int_0^x \left[2\left(2+x-\tfrac{2}{3}x^3\right)-2x^2-3\right] dx$$

$$=2+\int_0^x \left[1+2x-2x^2-\tfrac{4}{3}x^3\right] dx$$

$$=2+x+x^2-\tfrac{2}{3}x^3-\tfrac{1}{3}x^4.$$

The third approximation y_3, is given by

$$y_3=y_0+\int_0^x f(x, y_2)\, dx, f(x, y_2)=2y_2-2x^2-3$$

$$=2+\int_0^x \left\{2\left(2+x+x^2-\tfrac{2}{3}x^3-\tfrac{1}{3}x^4\right)-2x^2-3\right\} dx$$

$$=2+\int_0^x \left\{1+2x-\tfrac{4}{3}x^3-\tfrac{2}{3}x^4\right\} dx$$

$$=2+x+x^2-\tfrac{1}{3}x^4-\tfrac{2}{15}x^5.$$

Ex. 3. *Apply Picard's method upto third approximation to solve*

$$\frac{dy}{dx}=3e^x+2y$$

where $y=0$, *when* $x=0$.

Solution. We have $y_0=0$, $x_0=0$, $f(x, y)=3e^x+2y$.

The first approximation, y_1 is given by

$$y_1=y_0+\int_0^x f(x, y_0)\, dx, \qquad f(x, y_0)=3e^x$$

$$=0+\int_0^x 3e^x\, dx=3\,(e^x-1).$$

Again the second approximation, y_2, is given by

$$y_2=y_0+\int_0^x f(x, y_1)\, dx, \qquad f(x, y_1)=3e^x+2y_1$$

$$=0+\int_0^x [3e^x+2.3\,(e^x-1)]\, dx=\int_0^x (9e^x-6)\, dx$$

$$=9e^x-6x-9.$$

Now the third approximation, y_3 is given by

$$y_3=y_0+\int_0^x f(x, y_2), \qquad f(x, y_2)=3c^x+2y_2$$

$$=0+\int_0^x [3e^x+2\,(9e^x-6x-9)]\, dx$$

$$=\int_0^x (21e^x-12x-18)\, dx$$

$$=21e^x-6x^2-18x-21.$$

Ex. 4. *Apply Picard's method to solve*

$$\frac{dy}{dx}=e^x+y^2, \qquad y(0)=0.$$

Solution. $y_1=0+\int_0^x e^x\, dx=(e^x-1),$

$$y_2=0+\int_0^x [e^x+(e^x-1)^2]\,dx=\tfrac{1}{2}e^{2x}-e^x+x+\tfrac{1}{2},$$

$$y_3=0+\int_0^x [e^x+(\tfrac{1}{2}e^{2x}-e^x+x+\tfrac{1}{2})^2]\,dx$$

$$=\tfrac{1}{16}e^{4x}-\tfrac{1}{3}e^{3x}+\tfrac{1}{2}xe^{2x}+\tfrac{1}{2}e^{2x}-2xe^x+2e^x$$
$$+\tfrac{1}{3}x^3+\tfrac{1}{2}x^2+\tfrac{1}{4}x-\tfrac{107}{48}.$$

Ex. 5. *Apply Picard's method to solve :*

$$\frac{dy}{dx}=2-\frac{y}{x},$$
$$y(1)=2.$$

[Agra 71]

Solution. $y_1=2+\int_1^x \left(2-\frac{2}{x}\right)dx$ as $f(x, y_0)=2-\frac{2}{x}$

$$=2+\Big[2x-2\log x\Big]_1^x=2x-2\log x,$$

$$y_2=2+\int_1^x \left[2-\frac{2x-2\log x}{x}\right]dx=2+\int_1^x \frac{2}{x}\log x\,dx$$
$$=2+(\log x)^2,$$

$$y_3=2+\int_1^x \left[2-\frac{2+(\log x)^2}{x}\right]dx$$
$$=2x-2\log x-\tfrac{1}{3}(\log x)^3.$$

Ex 6. *Solve the differential equation $dy/dx+x-y$ with the initial conditions $y=1$ when $x=0$; and show that the sequence of approximations given by Picard's method tend to the exact solution as a limit.*

[Agra 72]

Ex. 7. *Solve the following equations by Picard's method :*

(*i*) $\dfrac{dy}{dx}=x+y,\ y(0)=1$

(*ii*) $\dfrac{dy}{dx}=x^2+y^2,\ y(0)=1.$

Proceed as above.

Ans. (*i*) $y_1=1+x+\tfrac{1}{2}x^2,\ y_2=1+x+x^2+\tfrac{1}{6}x^3,$

$y_3=1+x+x^2+\tfrac{1}{3}x^3+\tfrac{1}{24}x^4,$

(*ii*) $y_1=1+x+\tfrac{1}{3}x^3,\ y_2=1+x+x^2+\tfrac{2}{3}x^3+\tfrac{1}{6}x^4$

$+\tfrac{1}{15}x^5+\tfrac{1}{63}x^7.$

Higher approximation can be similarly determined.

Ex. 8. *Find the third approximation of the solution of the equation* $\dfrac{dy}{dz}=z,\ \dfrac{dz}{dx}=x^3(y+z)$

by Picard's method, where $y=1$, $z=\frac{1}{2}$ when $x=0$. **[Meerut 1970]**

Solution. Let $f_1(x, y, z)=z,\ f_2(x, y, z)=x^3(y+z).$

Now first approximations are given by

$$y_1=y_0+\int_0^x f_1(x, y_0, z_0)\,dx=y_0+\int_0^x z_0\,dx$$

$$=1+\int_0^x \tfrac{1}{2}dx=1+\tfrac{1}{2}x$$

$$z_1=z_0+\int_0^x f_2(x, y_0, z_0)\,dx=z_0+\int_0^x x^3(y_0+z_0)\,dx$$

$$=\tfrac{1}{2}+\int_0^x x^3(1+\tfrac{1}{2})\,dx=\tfrac{1}{2}+\tfrac{3}{8}x^4.$$

And the second approximations are

$$y_2=y_0+\int_0^x f_1(x, y_1, z_1)=y_0+\int_0^x z_1\,dx$$

$$=1+\int_0^x (\tfrac{1}{2}+\tfrac{3}{8}x^4)\,dx=1+\tfrac{1}{2}x+\tfrac{3}{40}x^5$$

$$z_2=z_0+\int_0^x f_2(x, y_1, z_1)=z_0+\int_0^x x^3(y_1+z_1)\,dx$$

$$=\tfrac{1}{2}+\int_0^x x^3(1+\tfrac{1}{2}x+\tfrac{1}{2}+\tfrac{3}{8}x^4)\,dx$$

$$=\tfrac{1}{2}+\int_0^x (\tfrac{3}{2}x^3+\tfrac{1}{2}x^4+\tfrac{3}{8}x^7)\,dx$$

$$=\tfrac{1}{2}+\tfrac{3}{8}x^4+\tfrac{1}{10}x^5+\tfrac{3}{64}x^8.$$

Again the third approximations are

$$y_3=y_0+\int_0^x f_1(x, y_2, z_2)=1+\int_0^x z_2\,dx$$

$$=1+\int_0^x (\tfrac{1}{2}+\tfrac{3}{8}x^4+\tfrac{1}{10}x^5+\tfrac{3}{64}x^8)$$

$$=1+\tfrac{1}{2}x+\tfrac{3}{40}x^5+\tfrac{1}{60}x^6+\tfrac{3}{576}x^9,$$

$$z_3=z_0+\int_0^x f_2(x, y_2, z_2)=\tfrac{1}{2}+\int_0^x x^3(y_2+z_2)\,dx$$

$$=\tfrac{1}{2}+\int_0^x x^3(1+\tfrac{1}{2}x+\tfrac{3}{40}x^5+\tfrac{1}{2}+\tfrac{3}{8}x^4+\tfrac{1}{10}x^5+\tfrac{3}{64}x^8)\,dx$$

$$=\tfrac{1}{2}+\int_0^x (\tfrac{3}{2}x^3+\tfrac{1}{2}x^4+\tfrac{3}{8}x^7-\tfrac{7}{40}x^8+\tfrac{3}{64}x^{11})\,dx$$

$$=\tfrac{1}{2}+\tfrac{3}{8}x^4+\tfrac{1}{10}x^5+\tfrac{3}{64}x^8+\tfrac{7}{360}x^9+\tfrac{1}{256}x^{12}.$$

Ex. 9. *Use Picard's method to approximate y and corresponding to $x=0.1$ for that particular solution of*

$$\frac{dy}{dx}=x+z,\ \frac{dz}{dx}=x-y^2$$

satisfying $y=2$, $z=1$ *when* $x=0$.

Solution. Let $x+z=f_1(x, y, z)$ and $x-y^2=f_2(x, y, z)$; then first approximations of y and z are given by

$$y_1 = y_0 + \int_0^x f_1(x, y_0, z_0)\, dx = 2 + \int_0^x (x+1)\, dx = 2 + \frac{x^2}{2} + x,$$

$$z_1 = z_0 + \int_0^x f_2(x, y_0, z_0)\, dx = 1 + \int_0^x (x - 2^2)\, dx = 1 + \tfrac{1}{2}x^2 - 4x,$$

The second approximations are now given by

$$y_2 = y_0 + \int_0^x f_1(x, y_1, z_1) = 2 + \int_0^x [x + (1 + \tfrac{1}{2}x^2 - 4x]\, dx$$

$$= 1 + \int_0^x (1 - 3x + \tfrac{1}{2}x^2)\, dx = 2 + x - \tfrac{3}{2}x^2 + \tfrac{1}{6}x^3$$

and $z_2 = z_0 + \int_0^x f_2(x, y_1, z_1)\, dx$

$$= 2 + \int_0^x [x - (2 + \tfrac{1}{2}x^2 + x)^2]\, dx$$

$$= 1 + \int_0^x (-4 - 3x - 3x^2 - x^3 - \tfrac{1}{4}x^4)\, dx$$

$$= 1 - 4x - \tfrac{3}{2}x^2 - x^3 - \tfrac{1}{4}x^4 - \tfrac{1}{20}x^5.$$

Again the third approximations are obtained as

$$y_3 = y_0 + \int_0^x f_1(x, y_2, z_2)\, dx$$

$$= 2 + \int_0^x [x + (1 - 4x - \tfrac{3}{2}x^2 - x^3 - \tfrac{1}{4}x^4 - \tfrac{1}{20}x^5)]\, dx$$

$$= 2 + x - \tfrac{3}{2}x^2 - \tfrac{1}{2}x^3 - \tfrac{1}{4}x^4 - \tfrac{1}{20}x^5 - \tfrac{1}{120}x^6$$

$$z_3 = z_0 + \int_0^x [x - (+ x - \tfrac{3}{2}x^2 + \tfrac{1}{6}x^3)^2]\, dx$$

$$= 1 + \int_0^x (-4 - 3x + 5x^2 + \tfrac{7}{3}x^3 - \tfrac{21}{12}x^4 + \tfrac{1}{2}x^5 - \tfrac{1}{36}x^6)\, dx$$

$$= 1 - 4x - \tfrac{3}{2}x^2 + \tfrac{5}{3}x^3 + \tfrac{7}{12}x^4 - \tfrac{21}{60}x^5 + \tfrac{1}{12}x^6 - \tfrac{1}{252}x^7.$$

Further approximations may similarly be calculated;

When $x = 0 \cdot 1$, $y_1 = 2 \cdot 105$, $z_1 = 0 \cdot 605$,

$y_2 = 2 \cdot 08517$, $z_2 = 0 \cdot 58397$,

$y_3 = 2 \cdot 08447$, $z_3 = 0 \cdot 58672$.

6·3. Taylor Series Method

Given the differential equation

$$y' = \frac{dy}{dx} = f(x, y) \qquad \ldots(1)$$

with initial conditions $y(x_0) = y_0$,

to find solution of the above equation using Taylor's series.

Let $y = y(x)$ be the solution of the differential equation (1), at the point x. If $x = x_0$ is not a singular point of the function, then by Taylor's series

$$y(x) = y(x_0) + (x - x_0)\, y'(x_0) + \frac{1}{2!}(x - x_0)^2\, y''(x_0)$$

$$+\frac{1}{3!}(x-x_0)^3 y'''(x_0)+\ldots \qquad \ldots(2)$$

This converges over some range containing x_0.

Now we determine

$y'(x_0), y''(x_0), y'''(x_0), \ldots$

From the given initial condition

$$y(x_0)=y_0, \qquad \ldots(3)$$

Again from (1), we get

$y'(x)=f(x, y)$.

$$\therefore \quad y'(x_0)=f(x_0, y_0). \qquad \ldots(4)$$

Differentiating (1), we get

$$y''(x)=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}=\frac{\partial f}{\partial x}+f(x, y)\frac{\partial f}{\partial y}. \qquad \ldots(5)$$

$$\therefore \quad y''(x_0)=\left(\frac{\partial f}{\partial x}\right)_0+f(x, y_0)\left(\frac{\partial f}{\partial y}\right)_0.$$

Again

$$y''(x_0)=\frac{d}{dx}\left(\frac{\partial f}{\partial x}+f(x, y)\frac{\partial f}{\partial y}\right)$$

$$=\left(\frac{\partial}{\partial x}+f(x, y)\frac{\partial}{\partial y}\right)\left(\frac{\partial f}{\partial x}+f(x, y)\frac{\partial f}{\partial y}\right)$$

$$=\frac{\partial^2 f}{\partial x^2}+\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}+2f\frac{\partial^2 f}{\partial x\partial y}+f\left(\frac{\partial f}{\partial y}\right)^2+f^2\left(\frac{\partial^2 f}{\partial y^2}\right)$$

Putting x_0 for x and y_0 for y in the righ hand side of this equation, we get

$$y'''(x_0)=\left(\frac{\partial^2 f}{\partial x^2}\right)_0+\left(\frac{\partial f}{\partial x}\right)_0\left(\frac{\partial f}{\partial y}\right)+2f_0\left(\frac{\partial^2 f}{\partial x\partial y}\right)_0+f_0\left(\frac{\partial f}{\partial y}\right)_0^2$$
$$+f_0^2\left(\frac{\partial^2 f}{\partial y^2}\right)_0 \qquad \ldots(6)$$

Putting these values in (2), we get the value of $y(x)$; the first few terms give an approximation for the solution $y(x)$ at x.

As is evident, the evaluation of additional terms becomes increasingly difficult.

Ex 1. *If $dy/dx=3x+y^2$ and $y=1$ when $x=0$, find the Taylor solution and approximate y when $x=0{\cdot}1$.*

Solution. Hence (x_0, y_0) is $(0, 1)$ and $y(x_0)=1$,

Now $\quad y'(x)=(3x+y^2)$; $\quad \therefore \quad y'(x_0)=3x_0+y_0^2=1$.

$$y''(x)=\frac{d}{dy}(3x+y^2)=3+2y\frac{dy}{dx}$$

$=3+2y(x)\,y'(x)$ $\qquad \therefore \quad y''(x_0)=3+2{\cdot}1.1=5$

$y'''(x)=2\{y'(x)\}^2+2y(x)\,y''(x)$ $\qquad \therefore \quad y'''(x_0)=2+2.1.5=12$

$y^{iv}(x) \quad 6y'y''+2yy'''$ $\qquad \therefore \quad y^{iv}(x_0)=54$

$y^{iv}(x)=6\,(y'')^2+8y'y'''+2yy^{iv}$ $\therefore\ \ y^{v}\,(x)=354,$

Thus the Taylor series expansion is

$$y\,(x)=y\,(x_0)+(x-x_0)\,y'\,(x_0)+\frac{1}{2\,!}\,(x-x_0)^2\,y''\,(x_0)+\ldots$$
$$=1+x+\tfrac{5}{2}x^2+2x^3+\tfrac{9}{4}x^4+\tfrac{177}{60}x^5+\ldots$$

Putting $x=0{\cdot}1$,

$$y\,(0{\cdot}1)=1+0{\cdot}1+0{\cdot}025+0{\cdot}002+0{\cdot}00022+0{\cdot}00003$$
$$\approx 1{\cdot}12725.$$

Ex. 2. *Use the Toylor series method to obtain a power series solution of the inittal value problems :*

(*i*) $\dfrac{dy}{dx}=2y+3e^x,\quad y\,(0)=0;$

(*ii*) $\dfrac{dy}{dx}=x+y,\qquad y\,(0)=1;$

(*iii*) $\dfrac{dy}{dx}=x^2+y^2,\qquad y\,(0)=1;$

(*iv*) $\dfrac{dy}{dx}=x+\cos y,\quad y\,(1)=\pi;$

(*v*) $\dfrac{dy}{dx}=\sin x+y^2,\quad y\,(0)=1.$

Proceed yourself.

Answers.

(*i*) $y=3x+\frac{9}{2}x^2+\frac{7}{2}x^3+\frac{15}{8}x^4+\frac{31}{40}x^5+\ldots$

(*ii*) $y=1+x+2\left[\dfrac{x}{1\,!}+\dfrac{x^2}{2\,!}+\ldots\right]$

(*iii*) $y=1+x+x^2+\frac{4}{3}x^3+\frac{7}{6}x^4+\ldots$

(*iv*) $y=\pi+\frac{1}{2}\,(x-1)^2+\frac{1}{40}\,(x-1)^5+\ldots$

(*v*) $y=1+x+\frac{3}{2}x^2+\frac{4}{3}x^3+\frac{11}{8}x^4+\ldots$

7
Legendre's Equation

7·1. Introduction

The Legendre's differential equation is

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0. \qquad \text{...(1)}$$

Sometimes this equation is also written as

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\}+n(n+1)y=0.$$

7·2. Integration in series of Legendre's Equation.

[Punjab 1957]

The Legendre's equation is

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0. \qquad \text{...(1)}$$

$x=0$ is a regular singularity of this differential equation.

Let the series solution of (1) in decending powers of x be

$$y=x^{\alpha}\sum_{0}^{\infty} a_r x^{-r} \qquad \text{...(2)}$$

so that $$\frac{dy}{dx}=\sum_{r=0}^{\infty} a_r(\alpha-r)x^{\alpha-r-1}$$

and $$\frac{d^2y}{dx^2}=\sum_{k=0}^{\infty} a_r(\alpha-r)(\alpha-r-1)x^{\alpha-r-2}.$$

Substituting these in (1), we get

$$(1-x^2)\sum_{r=0}^{\infty} a_r(\alpha-r)(\alpha-r-1)x^{\alpha-r-2}-2x\sum_{r=0}^{\infty} a_r(\alpha-r)x^{\alpha-r-1}$$
$$+n(n+1)\sum_{0}^{\infty} a_r x^{\alpha-r}=0$$

or $$\sum_{r=0}^{\infty}[(\alpha-r)(\alpha-r-1)x^{\alpha-r-2}$$
$$+\{n(n+1)-(\alpha-r)(\alpha-r+1)\}x^{\alpha-r}]a_r=0.$$

Equating to zero coefficient of lowest degree term, *i.e.* of x^{α} from the above, we get

$$a_0\{n(n+1)-\alpha(\alpha+1)\}=0.$$

But $a_0 \neq 0$ as it is the coefficient of the very first term in the series, hence the indicial equation is

$$n(n+1)-\alpha(\alpha+1)=0,$$

which gives $\alpha=n, -n-1.$ ...(3)

Next, equating to zero the coefficient of $x^{\alpha-1}$, we get

$$a_1 [n+1)-(\alpha-1)\alpha]=0$$

or $a_1 [(\alpha+n)(\alpha-n-1)]=0,$

which gives $a_1=0,$

since $(\alpha+n)(\alpha-n-1)\neq 0$ by (3).

Again to find a recurrence relation in successive coefficients a_r, equating the coefficient of $x^{\alpha-r-2}$ to zero, we get

$$(\alpha-r)(\alpha-r-1)a_r+[n(n+1)-(\alpha-r-2)(\alpha-r-1)]a_{r+2}=0$$

or $$a_{r+2}=\frac{(\alpha-r)(\alpha-r-1)}{(\alpha-r+n-1)(\alpha-r-n-2)}a_r. \qquad ...(4)$$

Now since $a_1=0$, $a_3=a_5=a_7=\ldots=0$.

There arise following two cases :

I. *When* $\alpha=n$, we get from (4)

$$a_{r+2}=-\frac{(n-r)(n-r-1)}{(2n-r-1)(r+2)}a_r$$

This gives $a_2=-\dfrac{n(n-1)}{(2n-1).2}a_0,$

$$a_4=\frac{n(n-1)(n-2)(n-3)}{(2n-1)(2n-3).2.4}a_0, \text{ and so on}$$

and since $a_1=0$, $a_1=a_3=a_5\ldots=0$.

Hence the series (2) in this case becomes

$$y=a_0\left[x^n-\frac{n(n-1)}{(2n-1).2}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{(2n-1)(2n-3).2.4}.x^{n-4}-\ldots\right], \qquad ...(5)$$

which for an arbitrary a_0 is a solution of (1).

II. *When* $\alpha=-(n+1)$, (4) gives

$$a_{r+2}=\frac{(n+r+1)(n+r+2)}{(r+2)(2n+r+3)}a_r$$

so that $a_2=\dfrac{(n+1)(n+2)}{2(2n+3)}a_0,$

$$a_4=\frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3).(2n+5)}a_0 \text{ and so on.}$$

Hence the series (2) now becomes

$$y=a_0\left[x^{-n-1}+\frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3}\right.$$
$$\left.+\frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)}x^{-n-5}+\dots\right]. \qquad \dots(6)$$

This is another solution of (1) in a series of decending powers of x.

Legendre's Polynomials $P_n(x)$ and $Q_n(x)$. *Definitions.*
[Agra 66, 55, 53, 52]

The Legendre's equation is

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0. \qquad \dots(1)$$

The two solutions of the above equation in series of descending powers of x are given by (5) and (6) above, where a_0 is an arbitrary constant,

Now, if n is a positive integer and

$$a_0=\frac{1.3.5\dots(2n-1)}{n!},$$

the solution (5) is called Legendre's polynomial $P_n(x)$, so that

$$P_n(x)=\frac{1.3.5\dots(2n-1)}{n!}\left[x^n-\frac{n(n-1)}{(2n-1).2}x^{n-2}+\dots\right]$$

Instead, if we take

$$a_0=\frac{n!}{1.3.5..(2n+1)}, \text{ in (6)}$$

the solution is called $Q_n(x)$, so that

$$Q_n(x)=\frac{n!}{1.3.5\dots(2n+1)}\left[x^{-n-1}+\frac{(n+1)(n+2)}{2.(2n+3)}x^{-n-3}+\dots\right]$$

The series for $Q_n(x)$ is a non-terminating series and converges when $|x|>1$.

Note. Obviously the series for $P_n(x)$ terminates.

When n is even it has $\frac{1}{2}n+1$ terms, and the last term is

$$(-1)^{n/2}\frac{n(n-1)(n-2)\dots1}{(2n-1)(2n-3)\dots(n+1).2.4.6\dots n}.a_0.$$

Again when n is odd it has $\frac{1}{2}(n+1)$ terms and in this case the last term is

$$(-1)^{(1/2)(n-1)}\frac{n(n-1\dots3.2.a_0}{(2n-1)(2n-3)\dots(n+2).2.4\dots n(n-1)}x.$$

$P_n(x)$ is called the *Legendre's function of the first kind and* $Q_n(x)$ of the second kind. **[Agra 66]**

Since $P_n(x)$ and $Q_n(x)$ are two independent solutions of Legendre's equation, therefore the most general solution of Legendre's equation can be written as

$$y=C_1P_n(x)+C_2Q_n(x)$$

where C_1 and C_2 are two arbitrary constants.

Ex. 1. *Show that*

$$P_{2m+1}(0)=0 \text{ and } P_{2m}(0)=(-1)^m\frac{1.3.5...(2m-1)}{2.4.6...2m}$$

$$=(-1)^m.\frac{(2m)!}{2^{2m}(m!)^2}.$$ [Agra 67]

Solution. If n is a positive integer,

$$P_n(x)=\frac{1.3.5...(2n-1)}{n!}\left[x^n-\frac{n(n-1)}{(2n-1.2}x^{n-2}+...\right],$$

is a terminating series.

When n is odd, the last term in the series within the brakets is

$$(-1)^{(1/2)(n-1)}.\frac{n(n-1)...3\ 2}{(2n-1)(2n-3)...(n+2)\ 2.4...(n-1)}x,$$

thus when $n=2m+1$, no term is free from x.

So putting $x=0$, $P_{2m+1}(0)=0$.

Again when n is even, the last term of the series within the brackets is

$$(-1)^{n/2}\frac{n(n-1)(n-2)...1}{(2n-1)(2n-3)...(n+1).2.4.6...n}.$$

This is the only term free from x. Putting $x=0$, we get

$$P_{2m}(0)=\frac{1.3.5...(2n-1)}{n!}(-1)^{n/2}\frac{n(n-1)(n-2)...1}{(2n-1)(n+1)n...4.2}$$

$$=(-1)^m\frac{1.3\cdot5...(2m-1)}{2.4\cdot6...2m} \text{ as } n=2m.$$

7·3. A Generating Function of Legendre's Polynomial

To show that $P_n(x)$ is the coefficient of h^n in the expansion of $(1-2xh+h^2)^{-1/2}$ in ascending powers of h. [Raj. 60; Agra 72, 54]

We have

$$(1-2xh+h^2)^{-1/2}=[1-h(2x-h)]^{-1/2}$$

$$=1+\tfrac{1}{2}.h(2x-h)+\frac{1.3}{2.4}h^2(2x-h)^2+...$$

$$+\frac{1.3\ 5...(2n-1)}{2.4...5n}h^n(2x-h)^n+... \qquad ...(1)$$

We now find coefficients of h^n in various terms of the above expansion and note that

coeff. of h^n in $\frac{1.3.5...(2n-1)}{2.4...2n}h^n(2x-h)^n$ is

$$\frac{1.3.5...(2n-1)}{2^n(1.2.3...n)}.(2x)^n=\frac{1.3.5...(2n-1)}{n!}x^n;$$

and coefficient of h^n in $\frac{1.3.5...(2n-3)}{2.4...(2n-2)}h^{n-1}(2x-h)^{n-1}$ is

$$-\frac{1.3.5\ldots(2n-1)}{n!}\cdot\frac{n(n-1)}{2(2n-1)}x^{n-2}; \text{ and so on.}$$

Thus coefficient of h^n in the expansion (1)

$$=\frac{1.3.5\ldots(2n-1)}{n!}\left[x^n-\frac{n(n-1)}{n(2n-1)}x^{n-2}\right.$$

$$\left.+\frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4}-\ldots\right]$$

$=P_n(x)$.

Thus in the expansion (1), coefficients of h, h^2, h^3, ... are $P_1(x)$, $P_2(x)$, $P_3(x)$...

Therefore,

$$(1-2xh+h^2)^{-1/2}=1+hP_1(x)+h^2P_2(x)+\ldots+h^nP_n(x)+\ldots$$

or $$(1-2xh+h^2)^{-1/2}=\sum_{r=0}^{r=\infty} h^nP_n(x).$$

Cor. 1. *It can be shown that* $P_n(1)=1$.

We have

$$(1-2xh+h^2)^{-1/2}=\sum_{r=0}^{\infty} h^nP_h(x).$$

Putting $x=1$, we have

$$(1-2h+h^2)^{-1/2}=\sum_{0}^{\infty} h^nP_n(1)$$

or $$\sum_{0}^{\infty} h^nP_n(1)=(1-h)^{-1}=\sum_{0}^{\infty} h^n$$

Equating the coefficients of h^n from both the sides, we get $P_n(1)=1$.

Cor. 2. *It can be proved that*

$P_n(-x)=(-1)^n P_n(x)$ *and* $P_n(-1)=(-1)^n$.

We have

$$(1-2xh+h^2)^{-1/2}=\sum_{n=0}^{\infty} h^nP_n(x). \quad \ldots(1$$

Putting $-x$ for x, this gives

$$(1+2xh+h^2)^{-1/2}=\sum_{n=0}^{\infty} h^nP_y(-x). \quad \ldots($$

Next putting $-h$ for h in (1), we have

$$(1+2xh+h^2)^{-1/2}=\sum_{n=0}^{\infty}(-h)^n P_n(x)$$

$$=\sum_{n=0}^{\infty}(-1)^n h^n P_n(x). \qquad \ldots(3)$$

Now equating coefficients of h^n from (2) and (3), we get

$P_n(-x)=(-1)^n P_n(x).$

Taking $x=1$, $P_n(-1)=(-1)^n P_n(1)$

i.e. $P_n(-1)=(-1)^n$ as $P_n(1)=1$.

Note. From above $P_n(-x)=\pm P_n(x)$ according as n is even or odd. This shows $P_n(x)$ is an odd function of x if n is odd.

7·4. Rodrigue's formula

To show that $\mathbf{P_n(x)=\dfrac{1}{2^n\,(n\,!)}\dfrac{d^n}{dx^n}(x^2-1)^n}$.

[Vikram 64, 63 ; Agra 70, 63, 60 ; Raj. 67, 61]

Let $\quad y=(x^2-1)^n; \qquad \ldots(1)$

then $\quad \dfrac{dy}{dx}=n\,(x^2-1)^{n-1}.2x.$

This gives $\quad (x^2-1)\dfrac{dy}{dx}=2nxy.$

Now differentiating it $(n+1)$ times, we have

$$(x^2-1)\frac{d^{n+2}y}{dx^{n+2}}+{}^{n+1}C_1\,(2x)\frac{d^{n+1}y}{dx^{n+1}}+{}^{n+1}C_2.2.\frac{d^ny}{dx^n}$$

$$=2n.\left[x.\frac{d^{n+1}y}{dx^{n+1}}+{}^{n+1}C_1.1.\frac{d^ny}{dx^n}\right]$$

or $\quad (x^2-1)\dfrac{d^{n+2}y}{dx^{n+2}}+2x\dfrac{d^{n+1}y}{dx^{n+1}}-n\,(n+1)\dfrac{d^ny}{dx^n}=0. \qquad \ldots(2)$

Let us now put $V=\dfrac{d^ny}{dx^n}$. Then (2) reduces to

$$(1-x^2)\frac{d^2V}{dx^2}-2x\frac{dV}{dx}+n\,(n+1)\,V=0.$$

Thus $\quad V=\dfrac{d^ny}{dx^n}$ is a solution of the Legendre's equation.

o $\quad C\dfrac{d^ny}{dx^n}=P_n(x), \qquad \ldots(3)$

where C is some constant and we would evalute it.

To evalute C, we have

$y=(x^2-1)^n=(x+1)^n\,(x-1)^n$, so that

$$\frac{d^ny}{dx^n}=(x+1)^n\frac{d^n}{dx^n}(x-1)^n+{}^nC_1.n\,(x+1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}(x-1)^n$$

$$+\ldots(x-1)^n\frac{d^n}{dx^n}(x+1)^n.$$

This gives $\left(\frac{d^n y}{dx^n}\right)_{x=1}=2^n.n\,!.$

other terms vanish as $(x-1)$ is a factor in all the terms except the first. Putting $x=1$ in (3) we get

$$C.2^n\,(n\,!)=P_n(1)=1$$

giving $\quad C=\frac{1}{2^n\,(n\,!)}.$

Thus from (3),

$$P_n(x)=\frac{1}{2^n\,(n\,!)}\frac{d^n y}{dx^n}=\frac{1}{2^n\,(n\,!)}\frac{d^n}{dx^n}(x^2-1)^n.$$

Ex. 1. *Show that* $\int_{-1}^{1} P_n'(x)\,dx=0$ *except when* $n=0$ *in which case the value of the integral is 2.*

Solution. From Rodrigue's formula

$$P_n(x)=\frac{1}{2^n.n\,!}\frac{d^n}{dx^n}(x^2-1)^n.$$

$$\therefore \quad \int_{-1}^{1} P_n(x)\,dx=\frac{1}{2^n.n\,!}\int_{-1}^{1}\frac{d^n}{dx^n}(x^2-1)^n dx$$

$$=\left[\frac{1}{2^n.n\,!}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\right]_{-1}^{1}$$

$$=\frac{1}{2^n.n\,!}\left[\frac{d^{n-1}}{dx^{n-1}}\{(x-1)^n\,(x+1)^n\}\right]_{-1}^{1}$$

$$=\frac{}{2^n.n\,!}[0-0]=0, \qquad \text{if } n\geqslant 1$$

as $\frac{d^{n-1}}{dx^{n-1}}[(x-1)^n\,(x+1)^n]=0$ for $x=\pm 1$ if $n\geqslant 1$.

When $x=0$, $P_0(x)=1$, and therefore

$$\int_{-1}^{1} P_0(x)\,dx=\int_{-1}^{1} dx=\Big[\,x\,\Big]_{-1}^{1}=2$$

7·5. Orthogonality of Legendre Polynomials

To prove that

$$\int_{-1}^{1} \mathbf{P}_m(\mathbf{x})\,\mathbf{P}_n(\mathbf{x})\,\mathbf{dx}=\mathbf{0} \text{ if } \mathbf{n}\neq\mathbf{m}.$$

[Indore 6 ; Vikram 63; Agra 66; Raj. 67]

First Method. The Legendre's equation can be written as

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\}+n\,(n+1)\,y=0.$$

Since P_n is a solution of the equation, we have

$$\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\}+n\,(n+1)P_n=0. \qquad \ldots(1)$$

Similarly

$$\frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\}+m(m+1)P_m=0. \qquad \ldots(2)$$

Multiplying (1) by P_m, (2) by P_n and then subtracting, we get when $n\neq m$,

$$P_m\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\}-P_n\frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\}$$
$$+P_nP_m[n(n+1)-m(m+1)]=0. \qquad \ldots(3)$$

Now $\displaystyle\int_{-1}^{1} P_m\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\}dx$

$$=\left[P_m(1-x^2)\frac{dP_n}{dx}\right]_{-1}^{1}-\int_{-1}^{1}\frac{dP_m}{dx}(1-x^2)\frac{dP_n}{dx}dx$$

integrating by parts

$$=-\int_{-1}^{1}\frac{dP_n}{dx}\cdot\frac{dP_m}{dx}(1-x^2)\,dx$$

and $\displaystyle\int_{-1}^{1} P_n\frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\}dx=-\int_{-1}^{1}\frac{dP_m}{dx}\frac{dP_n}{dx}(1-x^2)\,dx.$

Integrating now (3) w.r.t. x, we get

$$\int P_m\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\}dx-\int P_n\frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\}dx$$
$$+(n-m)(m+n+1)\int P_nP_m\,dx=0$$

or $\displaystyle -\int_{-1}^{1}\frac{dP_m}{dx}\frac{dP_n}{dx}(1-x^2)\,dx+\int_{-1}^{1}\frac{dP_m}{dx}\cdot\frac{dP_n}{dx}(1-x^2)\,dx$

$$+(n-m)(n+m+1)\int_{-1}^{1}P_mP_n\,dx=0,$$

and this gives

$$(n-m)(n+m+1)\int_{-1}^{1}P_mP_n\,dx=0$$

or $\displaystyle\int_{-1}^{1}P_mP_n\,dx=0.$

Second Method. From Rodrigue's formula, we have

$$P_n(x)=\frac{1}{2^n\,(n)\,!}\frac{d^n}{dx^n}(x^2-1)^n$$

and $\displaystyle P_m(x)=\frac{1}{2^m\,(m)\,!}\frac{d^m}{dx^m}(x^2-1)^m.$

Without any loss of generality, we can suppose that $m>n$. Consider now

$$I_{m,\,n}=\int_{-1}^{1}P_m(x)\,P_n(x)\,dx$$

$$=\frac{1}{2^{m+n}\,(m)\,!\,(n)\,!}\int_{-1}^{1}\frac{d^m}{dx^m}(x^2-1)^m.\frac{d}{dx^n}(x^2-1)^n\,dx$$

$$=\frac{1}{2^{m+n}\,(m)\,!\,(n)\,!}\left[\frac{d^{m-1}}{dx^{m-1}}(x^2-1)^m.\frac{d^n}{dx^n}(x^2-1)^n\right]_{-1}^{1}$$

$$-\frac{1}{2^{m+n}\,(m)\,!\,(n)\,!}\int_{-1}^{1}\frac{d^{m-1}}{dx^{m-1}}(x^2-1)^m.\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n\,dx$$

integrating by parts.

Now $\frac{d^{m-1}}{dx^{m-1}}(x^2-1)^m$ in its every term contains factors $(x-1)$ and $(x+1)$ both. Hence in the limits -1 to 1, its every term vanishes.

$$I_{m,\,n}=-\frac{1}{2^{m+n}\,(m)\,!\,(n)\,!}\int_{-1}^{1}\frac{d^{m-1}}{dx^{m-1}}(x^2-1)^m\,\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n\,dx.$$

Integrating now $(n-1)$ times, we get

$$I_{m,\,n}=-\frac{(-1)^n}{2^{m+n}\,(m)\,!\,(n)\,!}\int_{-1}^{1}\frac{d^{m-n}}{dx^{m-n}}(x^2-1)^m\,\frac{d^{2n}}{dx^{2n}}(x^2-1)^n\,dx.$$

But $\qquad \frac{d^{2n}}{dx^{2n}}(x^2-1)^n=(2n)\,!$

$$\therefore\quad I_{m,\,n}=\frac{(-1)^n\,(2n)\,!}{2^{m+n}\,(m)\,!\,(n)\,!}\int_{-1}^{1}(x^2-1)^m\,dx$$

$$=\frac{(-1)^n\,(2n)\,!}{2^{m+n}\,(m)\,!\,(n)\,!}\left[\frac{d^{m-n-1}}{dx^{m-n-1}}(x^2-1)^m\right]_{-1}^{1}$$

$=0.$ This proves the result.

7·6. To show that $\int_{-1}^{1}[P_n(x)]^2\,dx=\frac{2}{2n+1}$.

[Indore 1967; Vikram 63, 62; Agra 66, 55; Raj 67, 59]

First Method. We have

$$(1-2xh+h^2)^{-1/2}=\sum_{n=0}^{n=\infty}h^nP_n(x).$$

Squaring, $(1-2xh+h^{2-1}=\sum_{0}^{\infty}h^{2n}P\,[P^n(x)]^2$

$$+2\sum_{0}^{\infty}h^{m+n}\,P_m(x)\,P_n(x).$$

Now integrating both the sides with respect to x between the limits -1 to $+1$, we get

$$\sum_{0}^{\infty}\int_{-1}^{1}h^{2n}\,[P_n(x)]^2\,dx+2\sum_{0}^{\infty}\int_{-1}^{1}h^{m+n}\,P_m(x)\,P_n(x)\,dx$$

$$=\int_{-1}^{1}(1-2xh+h^2)^{-1}\,dx.$$

Since $\int_{-1}^{1} P_m(x)\, P_n(x)\, dx=0$ when $n \neq m$, this gives

$$\sum_{0}^{\infty} h^{2n} \int_{-1}^{1} [P_n(x)]^2\, dx+0=\int_{-1}^{1} \frac{dx}{(1-2xh+h^2)}$$

or $$\sum_{0}^{\infty} h^{2n} \int_{-1}^{1} [P_n(x)]^2\, dx=-\frac{1}{2h}\Big[\log (1-2xh+h^2)\Big]_{-1}^{1}$$

$$=-\frac{1}{2h}\log\left(\frac{1-2h+h^2}{1+2h+h^2}\right)$$

$$=\frac{1}{h}\log\left(\frac{1+h}{1-h}\right)=\frac{1}{h}[\log(1+h)-\log(1-h)]$$

$$=2\left[1+\frac{h^2}{3}+\frac{h^4}{5}+\ldots+\frac{h^{2n}}{2n+1}+\ldots\right].$$

Equating coefficients of h^{2n} from both the sides, we have

$$\int_{-1}^{1}\Big[P_n(x)\Big]^2 dx=\frac{2}{2n+1}.$$

Second Method Proceeding as in the second method of previous article, we have when $m=n$,

$$\int_{-1}^{1}\Big[P_n(x)\Big]^2 dx=\frac{(-1)^n\,(2n)\,!}{2^{n+n}\,(n)\,!\,(n)\,!}\int_{-1}^{1}\frac{d^{n-n}}{dx^{n-n}}(x^2-1)^n\, dx$$

$$=\frac{(2n)\,!}{2^n\,(n\,!)^2}\int_{-1}^{1}(1-x^2)^n\, dx$$

$$=\frac{(2n)\,!}{2^{2n}\,(n\,!)^2}\cdot 2\int_{0}^{\pi/2}\cos^{2n}\theta . \cos\theta\, d\theta$$

putting $x=\sin\theta$

$$=\frac{(2n)\,!}{2^{2n}\,(n\,!)^2}\cdot 2\cdot\frac{\Gamma(n+1)\,\Gamma(\frac{1}{2})}{2\Gamma\left(\frac{2n+3}{2}\right)}$$

$$=\frac{2}{2n+1}$$ which proves the result.

Note. The above two results can be combined to give

$$\int_{-1}^{+1} P_m(x)\, P_n(x)\, dx=\frac{2}{2n+1}\,\delta_{m\,n}$$

where $\delta_{m,\,n}$ is the Kronecker delta which takes the value 0 if $m \neq n$ and the value 1 if $m=n$. **[Agra 1972, 77]**

Cor. $\int_{0}^{1} [P_n(x)]^2\, dx=\frac{1}{2n+1}.$

7·7. Laplace's Definite Integrals for $P_n(x)$.

I. *To show that*

$$P_n(x)=\frac{1}{\pi}\int_{0}^{\pi}[x \pm \sqrt{(x^2-1)}\cos\phi]^n\, d\phi$$

where n is a positive integer. **[Agra 72; 66; Vikram 62; Raj. 65]**

If $a > b$, we have the result

$$\int_0^\pi \frac{d\phi}{a \pm b \cos\phi} = \frac{\pi}{\sqrt{(a^2-b^2)}}. \quad \ldots(1)$$

Put $a = 1 - xh$ and $b = h\sqrt{(x^2-1)}$.

This gives $a^2 - b^2 = (1-xh)^2 - h^2(x^2-1) = 1 - 2xh + h^2$.

Putting these values of a, b and $a^2 - b^2$ in (1), we get

$$\int_0^\pi \frac{d\phi}{(1-xh) \pm h\sqrt{(x^2-1)} \cos\phi} = \pi(1-2xh+h^2)^{-1/2},$$

or $$\pi(1-2xh+h^2)^{-1/2} = \int_0^\pi \frac{d\phi}{1 - h[x \mp \sqrt{(x^2-1)} \cos\phi]}$$

or $$\pi \sum_n^\infty h^n P_n(x) = \int_0^\pi [1 - h\{x \mp \sqrt{(x^2-1)} \cos\phi\}]^{-1} d\phi,$$

as $(1-2xh+h^2)^{-1/2} = \Sigma h^n P_n(x)$,

or $$\pi \sum_0^\infty h^n P_n(x) = \int_0^\pi \Sigma h^n \{x \mp \sqrt{(x^2-1)} \cos\phi\}^n d\phi,$$

because if $| h(x \mp \sqrt{(x^2-1)} \cos\phi\} | < 1$ and h is a small quanti y,

$$[1 - h\{x \mp \sqrt{(x^2-1)} \cos\phi\}]^{-1} = 1 + t + t^2 + t^3 + \ldots = \Sigma t^n$$

where $t = h\{x \mp \sqrt{(x^2-1)} \cos\phi\}$.

Equating coefficient of h^n from both sides, we get

$$\pi P_n(x) = \int_0^\pi \{x \mp \sqrt{(x^2-1)} \cos\phi\}^n d\phi$$

or $$P_n(x) = \frac{1}{\pi} \int_0^\pi \{x \pm \sqrt{(x^2-1)} \cos\phi\}^n d\phi,$$

which is the Laplace's first definite integral for $P_n(x)$

II. *Laplace's second definite integral is*

$$P_n(x) = \frac{1}{\pi} \int_0^\pi \frac{d\phi}{\{x \pm \sqrt{(x^2-1)} \cos\phi\}^{n+1}},$$

wcere n is a positive integer,

This is obtained by putting

$a = xh - 1$ and $b = h\sqrt{(x^2-1)}$; in (1) so that

$a^2 - b^2 = (xh-1)^2 = h^2(x^2-1) = 1 - 2xh + h^2$

and $$\frac{\pi}{\sqrt{(a^2-b^2)}} = \frac{1}{h} \frac{\pi}{\sqrt{\left(1 - \frac{2x}{h} + \frac{1}{h^2}\right)}} = \frac{\pi}{h}\left(1 - \frac{2x}{h} + \frac{1}{h^2}\right)^{-1/2}$$

$$= \frac{\pi}{h} \sum_{n=0}^{n=\infty} \frac{1}{h^n} P_n(x) \text{ where } n \text{ is an integer}$$

$$= \pi \sum_{n=0}^\infty \frac{1}{h^{n+1}} P_n(x). \quad \ldots(2)$$

With this substitution,

$$\frac{1}{a\pm b\cos\phi}=\frac{1}{h\{x\pm\sqrt{(x^2-1)}\cos\phi\}-1}$$

$$=\frac{1}{u-1}, \text{ where } u=h\{x\pm\sqrt{(x^2-1)}\cos\phi\}$$

$$=\frac{1}{u}\left(1-\frac{1}{u}\right)^{-1}=\frac{1}{u}\left[1+\frac{1}{u}+\frac{1}{n^2}+\dots\right]$$

$$=\sum_{n=0}^{m}\frac{1}{u^{n+1}}=\sum_{n=0}^{\infty}\frac{d\phi}{h^{n+1}\{x\pm\sqrt{(x^2-1)}\cos\phi\}^{n+1}}. \quad \dots(3)$$

The definite integral (1) now gives

$$\pi\sum_{0}^{\infty}\frac{1}{h^{n+1}}P_n(x)=\sum_{n=0}^{\infty}\int_0^{\pi}\frac{d\phi}{h^{n+1}\{x\pm\sqrt{(x^2-1)}\cos\phi\}^{n+1}}.$$

Equating coefficients of $\frac{1}{h^{n+1}}$ from both the sides, we get

$$\pi P_n(x)=\int_0^{\pi}\frac{d\phi}{\{x\pm\sqrt{(x^2-1)}\cos\phi\}^{n+1}}$$

or $$P_n(x)=\frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{\{x\pm\sqrt{(x^2-1)}\cos\phi\}^{n+1}},$$

wgich is the second definite integral for $P_n(x)$.

Important. Comparing two Laplace's integrals for $P_n(x)$, we get

$$\mathbf{P}_n(\mathbf{x})=\mathbf{P}_{-n-1}(\mathbf{x}).$$

Ex. 1. *Show that* $\frac{1-h^2}{(1-2xh+h^2)^{3/2}}=\sum_0^{\infty}(2n+1)h^nP_n.$

[Rajasthan 66, Agra 59]

Solution. Generating function formula is

$$\sum_0^{\infty}h^nP_n=(1-2xh+h^2)^{-1/2}. \quad \dots(1)$$

Differentiating it with respect to h, we get

$$\sum_0^{\infty}nh^{n-1}P_n=\frac{x-h}{(1-2xh+h^2)^{3/2}}$$

or $$\sum_0^{\infty}2nh^nP_n=\frac{2xh-2h^2}{(1-2xh+h^2)^{3/2}}, \quad \dots(2)$$

multiplying by $2h$.

Adding (1) and (2), we get

$$\sum_0^{\infty}(2n+1)h^nP^n=\frac{1-h^2}{(1-2xh+n^2)^{3/2}}.$$

This proves the result.

Ex. 2. *Prove that* $\dfrac{1+h}{h\sqrt{(1-2xh+h^2)}}-\dfrac{1}{h}=\sum_{n=0}^{\infty}\{P_n+P_{n+1}\}\,h^n$

[Agra 63]

Solution. We have

$$\sum_{n=0}^{\infty}\{P_n+P_{n+1}\}\,h^n$$

$$=\sum_{n=0}^{\infty}h^nP_n+\sum_{n=0}^{\infty}h^nP_{n+1}=\sum_{n=0}^{\infty}h^nP_n+\frac{1}{h}\sum_{n=0}^{\infty}h^{n+1}P_{n+1}$$

$$=\sum_{n=0}^{\infty}h^nP_n+\frac{1}{h}\left(-P_0+\sum_{n=0}^{\infty}h^nP_n\right)$$

$$=\left(1+\frac{1}{h}\right)\sum_{n=1}^{\infty}h^nP_n-\frac{P_0}{h}$$

$$=\left(1+\frac{1}{h}\right)(1-2xh+h^2)^{-1/2}-\frac{1}{h}$$

as $\sum_{0}^{\infty}h^nP_n=(1-2xh+h^2)^{-1/2}$ and $P_0=1$

$$=\frac{1+h}{h\sqrt{(1-2xh+h^2)}}-\frac{1}{h}.$$

This proves the result.

Ex. 3. *Show that if* $m<n$,

$$\int_{-1}^{1}x^mP_n(x)\,dx=0$$

and $$\int_{-1}^{1}x^nP_n(x)\,dx=\frac{2^{n+1}(n\,!)^2}{(2n+1)\,!}.$$

Solution. Rodrigue's formula is

$$P_n(x)=\frac{1}{2^n(n)\,!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$\therefore\quad\int_{-1}^{1}x^mP_n(x)\,dx=\frac{1}{2^n(n)\,!}\int_{-1}^{1}x^m\frac{d^n}{dx^n}(x^2-1)^n\,dx$$

$$=\frac{1}{2^n(n)\,!}\left[x^m.\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\right]_{-1}^{1}$$

$$-\frac{1}{2^n(n)\,!}\int_{-1}^{1}mx^{m-1}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,dx$$

integrating by parts

$$=0-\frac{m}{2^n(n)\,!}\int_{-1}^{1}x^{m-1}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,dx$$

$$=(-1)^m \frac{(m)\,!}{2^n\,(n)\,!}\int_{-1}^{1} \frac{d^{n-m}}{dx^{n-m}}\,(x^2-1)^n\,dx \text{ since } m<n \qquad \ldots(1)$$

integrating by parts m times

$$=(-1)^m \frac{(m)\,!}{2^n\,(n)\,!}\left[\frac{d^{n-m-1}}{dx^{n-m-1}}\,(x^2-1)^n\right]_{-1}^{1}=0.$$

Again in the *second part*, $m=n$; therefore proceeding as above integrating by parts n times, we get

$$\int_{-1}^{1} x^n P_n(x)\,dx=\frac{(-1)^n\,(n)\,!}{2^n\,(n)\,!}\int_{-1}^{1}\frac{d^{n-n}}{dx^{n-n}}(x^2-1)^2\,dx$$

$$=\frac{1}{2^n}\int_{-1}^{1}(1-x^2)^n\,dx=\frac{2}{2^n}\int_{0}^{1}(1-x^2)^n\,dx$$

$$=\frac{2}{2^n}\int_{0}^{\pi/2}\cos^{2n}\theta\cos\theta\,d\theta \text{ where } x=\sin\theta$$

$$=\frac{2}{2^n}\,\frac{\Gamma(n+1)\,\Gamma(\tfrac{1}{2})^*}{2\Gamma\left(\frac{2n+3}{2}\right)}$$

$$=\frac{2^{n+1}\,(n\,!).(n\,!)}{2^n.(2n+1)(2n-1)(2n-3)\ldots3.1.n.(n-1)(n-2)\ldots2.1}$$

expanding $\Gamma\left(\dfrac{2n+3}{2}\right)$ and multiplying numerator and denominator by n !

$$=\frac{2^{n+1}\,(n\,!)^2}{(2n+1)\,(2n-1)\,(2n-3)\ldots3.1.2n\,(2n-2)\,(2n-4)\ldots4.2}$$

$$=\frac{2^{n+1}\,(n\,!)^2}{(2n+1)\,!}$$

Ex. 4. *Deduce from Rodrigue's formula*

$$\int_{-1}^{1} f(x)\,P_n(x)\,dx=\frac{(-1)^n}{2^n\,n\,!}\int_{-1}^{1}(x^2-1)^n f^{(n)}(x)\,dx \qquad \textbf{[Raj 67]}$$

Hint. Proceed os above integrating by parts n times.

Ex. 5. *Evaluate $P_2(x)$ using the Laplace's integral formula*

$$P_n(x)=\frac{1}{\pi}\int_{0}^{\pi}[x+\sqrt{(x^2-1)}\cos\phi]^2\,d\phi$$

Solution. Laplace's integral formula gives

$$P_2(x)=\frac{1}{\pi}\int_{0}^{\pi}[x+\sqrt{(x^2-1)}\cos\phi]^2\,d\phi$$

$$=\frac{1}{\pi}\int_{0}^{\pi}[x^2+2x\sqrt{(x^2-1)}\cos\phi+(x^2-1)\cos^2\phi]\,d\phi$$

$$=\frac{1}{\pi}\,[x^2\pi+(x^2-1)\,\tfrac{1}{2}\pi]=\tfrac{1}{2}\,(3x^2-1).$$

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*After this step we can apply the duplication formula of gamma function *viz.*, $\Gamma(\tfrac{1}{2})\,\Gamma(2n)=2^{2n-1}\,\Gamma(n)\,\Gamma(n+\tfrac{1}{2})$ to get the result.

7 8. Recurrence formulae.

I. $nP_n = (2n-1)\,xP_{n-1} - (n-1)\,P_{n-2}$,

[Vikram 69 ; Raj 67 ; Agra 71, 53]

Generating function formula is

$$(1-2xh+h^2)^{-1/2} = \sum_0^{\infty} h^n P_n(x)$$

Differentiating it w.r.t. h, we get

$$(1-2xh+h^2)^{-3/2}\,(x-h) = \sum_0^{\infty} nh^{n-1}\,P_n(x)$$

or $$(x-h)\,(1-2xh+h^2)^{-1/2} = (1-2xh+h^2)\sum_0^{\infty} nh^{n-1}\,P_n(x)$$

or $$(x-h)\sum_0^{\infty} h^n P_n(x) = (1-2xh+h^2)\sum_0^{\infty} nh^{n-1}\,P_n(x). \quad ...(1)$$

Now equating coefficients of h^{n-1} from both sides, we get

$$xP_{n-1} - P_{n-2} = nP_n - 2x\,(n-1)\,P_{n-1} + (n-2)\,P_{n-2}$$

or $$nP_n = (2n-1)\,xP_{n-1} - (n-1)\,P_{n-2} \qquad (n \geqslant 2).$$

This proves the above formula.

Note. If from (1), we equate coefficients of h^n. we get

$$(n+1)\,P_{n+1} = (2n+1)\,xP_n - nP_{n-1}. \qquad \text{[Vikram 63; Rajasthan 62]}$$

which is a form of above formula when $n+1$ is taken for n.

II. $xP_n'(x) - P'_{n-1}(x) = nP_n(x)$

where dashes denote differentiation w.r.t. x.

[Vikram 62 ; Agra 71, 54]

We have

$$(1-2xh+h^2)^{-1/2} = \sum_0^{\infty} h^n P_n(x), \quad ...(1)$$

Differentiating with respect to x, we get

$$h\,(1-2xh+h^2)^{-3/2} = \sum_0^{\infty} h^n P'_n(x), \quad ...(2)$$

Also differentiating (1) w.r.t. h, we get

$$(x-h)\,(1-2xh+h^2)^{-3/2} = \sum_0^{\infty} nh^{n-1}\,P_n(x). \quad ...(3)$$

From (2) and (3),

$$(x-h)\,\Sigma h^n P_n'(x) = h\Sigma nh^{n-1}\,P_n'(x)$$

Now on equating coefficients of h^n from both the sides, we get

$$xP_n'(x) - P'_{n-1}(x) = nP_n(x)$$

which is the above formula.

III. $P'_n - xP'_{n-1} = nP_{n-1}$.

this formula can be obtained formula I and II.

Recurrence formula I is

$$nP_n = (2n-1)\, xP_{n-1} - (n-1)\, P_{n-2}.$$

On differentiating with respect to x, it gives.

$$nP'_n = (2n-1)\, P_{n-1} + (2n-1)\, xP'_{n-1} - (n-1)\, P'_{n-2}$$

or $n\,[P'_n - xP'_{n-1}] - (n-1)\,[xP'_{n-1} - P'_{n-2}] = (2n-1)\, P_{n-1}$

or $n\,[P'_n - xP'_{n-1}] - (n-1)\,[(n-1)\, P_{n-1}] = (2n-1)\, P_{n-1}$

using formula II

i.e. $n\,[P'_n - xP'_{n-1}] = [(n-1)^2 + (2n-1)]\, P_{n-1} = n^2 P_{n-1}$

or $P'_n - xP'_{n-1} = nP_{n-1}$.

IV. $P'_{n+1} - P'_{n-1} = (2n+1)\, P_n$.

[Vikram 62, 63 : Agra 66, 61 ; Raj. 64, 60]

Recurrence formula I on taking $n+1$ for n is

$$(n+1)\, P_{n+1} = (2n+1)\, xP_n - nP_{n-1}.$$

Differentiating it with respect to x, we get

$$(n+1)\, P'_{n+1} = (2n+1)\, P_n + (2n+1)\, xP'_n - nP'_{n-1}. \qquad \ldots(1)$$

Again recurrence formula II gives

$$xP'_n = P'_{n-1} + nP_n. \qquad \ldots(2)$$

Putting this in (1).

$$(n+1)\, P'_{n+1} = (2n+1)\, P_n + (2n+1)\, [nP_n + P'_{n-1}] - nP'_{n-1}$$

or $(n+1)\, P'_{n+1} - (n+1)\, P'_{n-1} = (2n+1)\, (1+n)\, P_n$

or $P'_{n+1} - P'_{n-1} = (2n+1)\, P_n$.

Note. Putting $n-1$ for n, the above formula becomes

$$\frac{dP_n}{dx} - \frac{dP_{n-2}}{dx} = (2n-1)\, P_{n-1}.$$

[Agra 55, 52]

V. $(x^2-1)\, P'_n = n\,(xP_n - P_{n-1})$.

[Vikram 69, 64, 63 ; Raj. 60 ; Agra 59]

Recurrence formula III is

$$P'_n - xP'_{n-1} = nP_{n-1} \qquad \ldots(1)$$

and recurrence formula is

$$xP'_n - P'_{n-1} = nP_n.$$

Multiplying (2) by x and subtracting from (1), we get

$$(1-x^2)\, P'_n = n\,(P_{n-1} - xP_n)$$

Second Method. Laplace's 1st integral is

$$P_n(x) = \frac{1}{\pi} \int_0^{\pi} [x + \sqrt{(x^2-1)} \cos\phi]^n \, d\phi. \qquad \ldots(1)$$

$$\therefore \quad P'_n(x) = \frac{n}{\pi} \int_0^{\pi} [x + \sqrt{(x^2-1)} \cos\phi]^{n-1} \left[1 + \frac{x\cos\phi}{\sqrt{(x^2-1)}}\right] d\phi. \qquad \ldots(2)$$

So $$xP_n - P_{n-1} = \frac{1}{\pi} x \int_0^\pi [x+\sqrt{(x^2-1)}\cos\phi]^n \, d\phi$$
$$-\frac{1}{\pi}\int_0^\pi [x+\sqrt{(x^2-1)}\cos\phi]^{n-1} \, d\phi$$
$$=\frac{1}{\pi}\int_0^\pi [x+\sqrt{(x^2-1)}\cos\phi]^{n-1} [x^2+\sqrt{(x^2-1)}\cos\phi-1] \, d\phi$$
$$=\frac{x^2-1}{\pi}\int_0^\pi [x+\sqrt{(x^2-1)}\cos\phi]^{n-1}\left[1+\frac{x\cos\phi}{\sqrt{(x^2-1)}}\right] d\phi$$
$$=\frac{(x^2-1)}{n} \cdot P'_n(x) \text{ from (2).}$$ Hence the formula.

VI. $(x^2-1)\, P'_n = (n+1)\,(P_{n+1} - xP_n)$. **(Agra 57)**

Formula I is

$$nP_n = (2n-1)\, xP_{n-1} - (n-1)\, P_{n-2}.$$

Putting $(n+1)$ for n, it becomes

$$(n+1)\, P_{n+1} = (2n+1)\, xP_n - nP_{n-1}.$$

This can also be written as

$$(n+1)\,(P_{n+1} - xP_n) = n\,(xP_n - P_{n-1})$$
$$=(x^2-1)\, P'_n \text{ from formula V}$$

or $$(x^2-1)\, P'_n = (n+1)\,(P_{n+1} - xP_n).$$

Second method. The Laplace's second integral is

$$P_n(x) = \frac{1}{\pi}\int_0^\pi \frac{d}{\{x+\sqrt{(x^2-1)}\cos\phi\}^{n+1}}.$$

Hence

$$P'_n(x) = \frac{-(n+1)}{\pi}\int_0^\pi \frac{d\phi}{\{x+\sqrt{(x^2-1)}\cos\phi\}^{n+1}}\left[1+\frac{x\cos\phi}{\sqrt{(x^2-1)}}\right]. \quad \ldots(2)$$

$$\therefore \quad P_{n+1} - xP_n = \frac{1}{\pi}\int_0^\pi \frac{d\phi}{[x+\sqrt{(x^2-1)}\cos\phi]^{n+2}}$$
$$-x\frac{1}{\pi}\int_0^\pi \frac{d\phi}{[x+\sqrt{(x^2-1)}\cos\phi]^{n+1}}$$
$$=\frac{1}{\pi}\int_0^\pi \frac{d\phi}{[x+\sqrt{(x^2-1)}\cos\phi]^{n+2}}[1-x\{x+\sqrt{(x^2-1)}\cos\phi\}]$$
$$=\frac{-(x^2-1)}{\pi}\int_0^\pi \frac{d\phi}{[x+\sqrt{(x^2-1)}\cos\phi]^{n+2}}\left[1+\frac{x\cos\phi}{\sqrt{(x^2-1)}}\right]$$
$$=\frac{(x^2-1)}{n+1} = P'_n(x) \text{ from (2).}$$ Hence the formula.

7·9. Christoffel's expansion.

$$\frac{dP_n}{dx} = (2n-1)\, P_{n-1} + (2n-5)\, P_{n-3} + (2n-9)\, P_{n-5} + \ldots$$

where last term is $3P_1$ *or* P_0 *according as n is even or odd.*

[Raj. 66: 62; Poona 60]

Substituting $n-1$ for n in recurrence formula IV, we have

$$P'_n=(2n-1)\,P_{n-1}+P'_{n-2}. \qquad \ldots(1)$$

Replacing n by $n-2$, $n-4$,...in (1), we get

$$P'_{n-2}=(2n-5)\,P_{n-3}+P'_{n-4},$$
$$P'_{n-4}=(2n-9)\,P_{n-5}+P'_{n-6},$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

If n is even, the last term is

$$P'_2=3P_1+P'_0$$
$$=3P_1 \text{ as } P_0=1 \text{ and } P'_0=0.$$

Adding all the above results when n is even, we obtain

$$P'_n=(2n-1)\,P_{n-1}+(2n-5)\,P_{n-3}+(2n-9)\,P_{n-5}+\ldots+3P_1.$$

But if n odd, the last term is

$$P'_3=5P_2+P'_1=5P_2+P_0 \text{ as } P_1(x)=x,\ P'_1=1=P_0.$$

Hence in case n is odd, the result is

$$P'_n=(2n-1)\,P_{n-1}+(2n-5)\,P_{n-3}+(2n-9)\,P_{n-5}+\ldots+5P_2+P_0.$$

Ex. 1. *Prove that*

$$\int_{-1}^{1}\left(\frac{dP_n}{dx}\right)^2 dx=n\,(n+1).$$

[Agra 65; Raj. 66, 64, 56]

Solution. The Christoffel's expansion formula is

$$\frac{dP_n}{dx}=(2n-1)\,P_{n-1}+(2n-5)\,P_{n-3}+(2n-9)\,P_{n-5}\ldots\ldots$$

where the last term is $3P_1$ or P_0 according as n is even or odd.

$$\therefore\quad \int_{-1}^{1}\left(\frac{dP_n}{dx}\right)^2 dx=\int_{-1}^{1}[(2n-1)\,P_{n-1}+(2n-5)\,P_{n-3}+\ldots]^2\,dx$$

$$=\int_{-1}^{1}(2n-1)^2\,P^2_{n-1}\,dx+\int_{-1}^{1}(2n-5)^2\,P^2_{n-3}\,dx+\ldots$$

other integrals vanish being the integrals of product of different P_n's

$$=(2n-1)^2.\frac{2}{2\,(n-1)+1}+(2n-5)^2.\frac{2}{2\,(n-3)+1}+\ldots\ldots$$

$$\text{as } \int_{-1}^{1}P_n^2\,dx=\frac{2}{2n+1}$$

$$=2\,[(2n-1)+(2n-5)+\ldots].$$

The last term is

$$\int_{-1}^{+1}9P_1^2\,dx=9.\tfrac{2}{3}=2\times 3 \text{ or } \int_{-1}^{+1}P_0^2(x)\,dx=\int_{-1}^{1}dx=2$$

according as n is even or odd ; therefore when n is even

$$\int_{-1}^{1}\left(\frac{dP_n}{dx}\right)^2 dx=2\,[(2n-1)+(2n-5)+\ldots+3]$$

$$=2\frac{n}{2.2}\left[2\times 3-4\left(\frac{n}{2}-1\right)\right]=n\ (n+1)$$

summing the A. P., the number of terms being $\frac{n}{2}$.

And when n is odd,

$$\int_{-1}^{1}\left(\frac{dP_n}{dx}\right)^2 dx=2\ [(2n-1)+(2n-5)+\ldots+1]$$

$$=2.\frac{n+1}{2\ 2}\left[2+4\left(\frac{n+1}{2}-1\right)\right]=n\ (n+1),$$

summing the A.P. of $\frac{1}{2}(n+1)$ terms now.

Ex. 2. *Prove that*

$$(1-x^2)\ P'_n(x)=\frac{n\ (n+1)}{2n+1}[P_{n-1}(x)-P_{n+1}(x)]$$

Solution. Recurrence formula V is

$$(1-x^2)\ P'_n(x)=n[P_{n-1}+xP_n] \qquad \ldots(1)$$

We have to retain P_{n-1} on the right of (1) but have to replace xP_n in terms of P_{n-1} and P_{n+1}; such a relation is given by recurrence formula I from which

$$xP_n(x)=\frac{1}{2n+1}\ [(n+1)\ P_{n+1}+nP_{n-1}].$$

Putting this in (1), we get

$$(1-x^2)\ P_n'(x)=\left[P_{n-1}-\frac{1}{2n+1}\ \{(n+1)\ P_{n+1}+nP_{n+1}\}\right]$$

$$=\frac{n\ (n+1)}{2n+1}\ [P_{n-1}(x)-P_{n+1}(x)]$$

Ex. 3, *Prove that*

$$(2n+1)\ (x^2-1)\ P_n'=n\ (n+1)\ (xP_{n+1}-P_n)$$

Solution. Recurrence farmula VI is

$$(x^2-1)\ P_n'=(n+1)\ (P_{n+1}-P_{n-1}). \qquad \ldots(1)$$

Also recurrence formula V is

$$(x^2-1)\ P_n'=n\ (xP_n-P_{n+1}) \qquad \ldots(2)$$

Eliminating P_n from (1) and (2), we get

$$(x^2-1)\ P_n'=n\left[P_{n+1}-\frac{x^2-1}{n+1}P_n'-P_{n-1}\right]$$

or $\quad (2n+1)\ (x^2-1)\ P_n'=n\ (n+1)\ (P_{n+1}-P_{n-1})$

Ex. 4. *Show that*

$$P'_{n+1}+P'_n=P_0+3P_1+\ldots(2n+1)\ P_n. \qquad \text{[Agra 1963]}$$

Solution. Recurrence formula IV is

$$P'_{n+1}-P'_{n-1}=(2n+1)\ P_n.$$

Replacing n by $n-1,\ n-2,\ \ldots,\ 1$, we get

$$P'_n-P'_{n-2}=(2n-1)\ P_{n-1},$$

$$P'_{n-1}-P'_{n-3}=(2n-3)\,P_{n-2},$$
$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$
$$P'_4-P'_2=7P_3,$$
$$P'_3-P'_1=5P_2,$$
$$P'_2-P'_0=3P_1.$$

Adding these, we get

$$P'_{n+1}+P'_n+P'_1-P'_0=3P_1+5P_2+\ldots+(2n+1)\,P_n$$

or $\quad P'_{n+1}+P'_n=P_0+3P_1+5P_2+\ldots+(2n+1)\,P_n,$

as $P'_0=0$ and $P'_1=1=P_0$.

This proves the result.

Ex. 5. *Show that*

$$P'_{2n+1}=(2n+1)\,P_{2n}+2nxP_{2n-1}+(2n-1)\,x^2P_{2n-2}+\ldots$$
$$\ldots+2x^{2n-1}P_1+x^{2n}.$$

Solution. Recurrence formula III is

$$P'_n=nP_{n-1}+xP'_{n-1}. \qquad \ldots(1)$$

Replacing n by $2n+1$ and $2n$, we get

$$P'_{2n+1}=(2n+1)\,P_{2n}+xP'_{2n}$$

and $\quad P'_{2n}=2nP_{2n-1}+xP'_{2n-1},$

so that $\quad P'_{2n+1}=(2n+1)\,P_{2n}+x\,(2nP_{2n-1}+xP'_{2n-1})$

or $\quad P'_{2n+1}=(2n+1)\,P_{2n}+2nxP_{2n-1}+x^2P'_{2n-1}. \qquad \ldots(2)$

Next putting $2n-1$ for n in (1), we get

$$P'_{2n-1}=(2n-1)\,P_{2n-2}+xP'_{2n-2}.$$

Putting this value of P'_{2n-1} in (2), we get

$$P'_{2n+1}=(2n+1)\,P_{2n}+2nx\,P_{2n-1}+(2n-1)\,x^2P_{2n-2}+x^3P'_{2n-2}$$
$$=(2n+1)\,P_{2n}+2nxP_{2n-1}+(2n-1)\,x^2P_{2n-2}+\ldots x^{2n-1}\,P'_2, \qquad \ldots(3)$$

proceeding step by step.

Lastly from (1)

$$P'_2=2P_1+xP'_1$$
$$=2P_1+x \text{ as } P_1=x \text{ and } P'_1=1.$$

With this value, (3) becomes

$$P'_{2n+1}=(2n+1)\,P_{2n}+2nxP_{2n-1}+\ldots+x^{2n-1}\,(2P_1+x)$$
$$=(2n+1)\,P_{2n}+2nxP_{2n-1}+\ldots+2x^{2n-1}\,P_1+x^{2n}.$$

This is the result.

Ex. 6. *Prove that*

$$1+\tfrac{1}{2}P_1\,(\cos\theta)+\tfrac{1}{3}P_2\,(\cos\theta)+\tfrac{1}{4}P_3\,(\cos\theta)+\ldots=\log\frac{1+\sin\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}.$$

Solution. The generating function formula is

$$\sum^{\infty} h^nP_n(x)=(1-2hx+h^2)^{-1/2}.$$

Integrating it w.r.t. h between the limits 0 to 1, we get

$$\sum_0^\infty \int_0^1 h^n P_n(x)\, dx = \int_0^1 \frac{dh}{\sqrt{(1-2xh+h^2)}}$$

or $$\sum_0^\infty P_n(\cos\theta) \int_0^1 h^n\, dh = \int_0^1 \frac{dh}{\sqrt{(1-2h\cos\theta+h^2)}}$$

or $$\sum_0^\infty \frac{P_n(\cos\theta)}{n+1} = \int_0^1 \frac{dh}{\sqrt{[(h-\cos\theta)^2+\sin^2\theta]}}$$

$$= \Big[\log\{(h-\cos\theta)\} + \sqrt{\{(h-\cos\theta)^2+\sin^2\theta\}}\Big]_0^1$$

$$= \log\frac{(1-\cos\theta)+\sqrt{[(1-\cos\theta)^2+\sin^2\theta}}{1-\cos\theta}$$

$$= \log\frac{\sqrt{(1-\cos\theta)}\,[\sqrt{(1-\cos\theta)}+\sqrt{2}]}{\sqrt{(1-\cos\theta)}\sqrt{(1-\cos\theta)}}$$

$$= \log\frac{\sqrt{(2\sin^2\tfrac{1}{2}\theta)}+\sqrt{2}}{\sqrt{(2\sin^2\tfrac{1}{2}\theta)}} = \log\frac{1+\sin\tfrac{1}{2}\theta}{\sin\tfrac{1}{2}\theta}$$

or $$\log\frac{1+\sin\tfrac{1}{2}\theta}{\sin\tfrac{1}{2}\alpha} = \sum_0^\infty \frac{P_n(\cos\theta)}{n+1}$$

$$= 1 + \tfrac{1}{2}P_1(\cos\theta) + \tfrac{1}{3}P_2(\cos\theta) + \ldots$$

Ex. 7. *Show that*

$$P_n(-\tfrac{1}{2}) = P_0(-\tfrac{1}{2})\, P_{2n}(\tfrac{1}{2}) + P_1(-\tfrac{1}{2})\, P_{2n-1}(\tfrac{1}{2}) + \ldots P_{2n}(-\tfrac{1}{2})\, P_0(\tfrac{1}{2}).$$

[Poona 60]

Solution. The generating function formula is

$$(1-2xh+h^2)^{-1/2} = \sum_0^\infty h^n P_n(x).$$

Putting $x = -\tfrac{1}{2}$ and $+\tfrac{1}{2}$, we get

$$(1+h+h^2)^{-1/2} = \sum_0^\infty h^n P_n(-\tfrac{1}{2}) \qquad \ldots(1)$$

and $$(1-h+h^2)^{-1/2} = \sum_0^\infty h^m P_m(\tfrac{1}{2}). \qquad \ldots(2)$$

Again putting h^2 for h in (1), we get

$$(1+h^2+h^4)^{-1/2} = \sum_0^\infty h^{2n} P_n(-\tfrac{1}{2}).$$

But $(1+h^2+h^4)^{-1/2} = (1+h+h^2)^{-1/2}\, (1-h+h^2)^{-1/2}$

$\therefore \quad \Sigma h^{2n} P_n(-\tfrac{1}{2}) = \Sigma h^n P_n(-\tfrac{1}{2}) \times \Sigma h^m P_m(\tfrac{1}{2}).$

Equating coefficients of h^{2n}, we get

$$P_n(-\tfrac{1}{2}) = P_0(-\tfrac{1}{2})\, P_{2n}(\tfrac{1}{2}) + P_1(-\tfrac{1}{2})\, P_{2n-1}(\tfrac{1}{2}) + \ldots + P_{2n}(-\tfrac{1}{2})\, P_0(\tfrac{1}{2})$$

Ex. 8. (a) *Show that*

$$xP'_n(x) = nP_n(x) + (2n-3)\, P_{n-2}(x) + (2n-7)\, P_{n-4}(x) + \ldots$$

[Agra 1959]

Solution. Recurrence formula II is

$$xP'_n(x)=nP'_n(x)+P'_{n-1}(x) \qquad \ldots(1)$$

and formula IV is

$$P'_{n+1}(x)=(2n+1)\ P_n(x)+P'_{n-1}(x). \qquad \ldots(2)$$

Replacing n by $n-2$, $n-4$, ..., we get

$$P'_{n-1}(x)=(2n-3)\ P_{n-2}(x)+P'_{n-3}(x) \qquad \ldots(3)$$

$$P'_{n-3}(x)=(2n-7)\ P_{n-4}(x)+P'_{n-5}(x) \qquad \ldots(4)$$

......and so on.

Adding (1), (3), (4) ..., we get

$$xP'_n(x)=nP_n(x)+(2n-3)\ P_{n-2}(x)+(2n-7)\ P_{n-4}(x)$$
$$+(2n-11)\ P_{n-6}(x)+\ldots$$

Ex. 8. (b) *Prove that*

$$\int_{-1}^{1} xP_nP'_n\,dx=\frac{2n}{2n+1}$$

Solution. From Ex. 8 (*a*) above, we have

$$xP'_n(x)=nP_n(x)+(2n-3)\ P_{n-2}(x)+\ldots$$

Multiplying this by P_n and integrating between -1 to $+1$, we get

$$\int_{-1}^{1} xP_n(x)\ P'_n(x)\ dx=n\int_{-1}^{1} P_n^2(x)\ dx$$

other integrals vanish as

$$\int_{-1}^{1} P_nP_m\ dx=0,\ n\neq m$$

$$=n.\frac{2}{2n+1}=\frac{2n}{2n+1}.$$

Ex. 9. *Show that, if m and n are integers, the value of*

$$\int_{-1}^{1} xP_n\frac{dP_m}{dx}\,dx \text{ is either } 0,\ 2 \text{ or } \frac{2n}{2n+1}.$$

Solution. From Ex. 8 (*a*), we have

$$x\frac{dP_m}{dx}=mP_m(x)+(2m-3)\ P_{m-2}(x)+(2m-7)\ P_{m-4}(x)+\ldots$$

$$=mP_m(x)+\sum_{r=1}^{\infty}(2m-4r+1)\ P_{m-2r}(x).$$

Now let $n\neq m,\ m-2,\ m-4,\ \ldots$ or $n>m$, then

$$\int_{-1}^{1} xP_n\frac{dP_m}{dx}\,dx$$

$$=m\int_{-1}^{1} P_n(x)\ P_m(x)\ dx+(2m-3)\int_{-1}^{1} P_n(x)\ P_{m-2}(x)\ dx+\ldots$$

$=0$ by § 7·5 page 171.

And if $n=m$.

$$\int_{-1}^{1} xP_n\frac{dP_m}{dx}\,dx=\int_{-1}^{1} xP_nP'_n\,dx.$$

$$=\frac{2n}{2n+1} \text{ as in Ex. 8 (b).}$$

Again if $n=m-2r$, $r=1, 2, \ldots$

$$\int_{-1}^{1} dP_n \frac{dP_m}{dx} dx=(2m-4r+1)\int_{-1}^{1} P^2_{m-2r}\, dx,$$

as other integrals vanish

$$=(2m-4r+1).\frac{2}{2(m-2r)+1}=2.$$

Ex. 10. *Prove that*

$$\int_{-1}^{1} x^2P_n^2(x)\, dx=\frac{1}{8(2n-1)}+\frac{3}{4(2n+1)}+\frac{1}{8(2n+3)}.$$

Solution. We have the recurrence formula I,

$$(2n+1)\, xP_n(x)=(n+1)\, P_{n+1}(x)+nP_{n-1}(x).$$

Squaring it and then integrating between the limits -1 to $+1$, we get

$$(2n+1)^2\int_{-1}^{+1} x^2P_n^2(x)\, dx=\int_{-1}^{+1}[(n+1)\, P_{n+1}(x)+nP_{n-1}(x)]^2\, dx$$

$$=(n+1)^2\int_{-1}^{1} P_{n+1}^2(x)\, dx+n^2\int_{-1}^{1} P_{n-1}^2(x)\, dx$$

other integral vanishes

$$=(n+1)^2\cdot\frac{2}{2n+3}+\frac{n^2\, 2}{(2n-1)} \text{ (using § 7·6)}$$

$$\therefore \int_{-1}^{1} x^2P_n^2(x)\, dx=\frac{2}{(2n+1)^2}\left[\frac{(n+1)^2}{2n+3}+\frac{n^2}{2n-1}\right]$$

$$=\frac{1}{8(2n-1)}+\frac{3}{4(2n+1)}+\frac{1}{8(2n+3)}.$$

Ex. 11. *Prove that*

$$\int_{-1}^{1} x^2P_{n+1}P_{n-1}\, dx=\frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$$

Solution. We have the recurrence formula I,

$$(2n+1)\, xP_n=(n+1)\, P_{n+1}+nP_{n-1}.$$

Putting $(n+1)$ and $(n-1)$ for n, we have

$$(2n+3)\, xP_{n+1}=(n+2)\, P_{n+2}+(n+1)\, P_n$$

and $(2n-1)\, xP_{n-1}=nP_n+(n-1)\, P_{n-2}$.

Multiplying these and then integrating within the limits -1 to $+1$, we get

$$(2n+3)(2n-1)\int_{-1}^{1} x^2P_{n+1}\, P_{n-1}\, dx=n(n+1)\int_{-1}^{1} P_n^2\, dx$$

other integrals vanish

$$=n(n+1).\ \frac{2}{(2n+1)}=\frac{2n(n+1)}{(2n+1)}$$ (using § 7·6)

This gives the result.

Ex. 12. *Solve that*

$$P_p(x)=\sum_{n=0}^{\frac{1}{2}p}(-1)^n\frac{(2p-2n)!}{2^p(n)!(p-n)!(p-2n)!}x^{p-2n}$$

Solution. Rodrigues formula gives

$$P_p(x)=\frac{1}{2^p(p)!}\frac{d^p}{dx^p}(x^2-1)^p$$

$$\text{Now }\frac{d^p}{dx^p}(x^2-1)^p=\sum_{n=0}^{\frac{1}{2}p}(-1)^n\frac{p!}{(n)!(p-n)!}.$$
$$\times(2p-2n)(2p-2n-1)\ldots(p-2n+1)x^{p-2n}$$

$$=\sum_{n=0}^{\frac{1}{2}p}(-1)^n(2p-2n)(2p-2n-1)\ldots(p-2n+1)$$
$$\times\frac{(p-2n)!}{(p-2n)!}\cdot\frac{p!}{n!(p-n)!}x^{p-2n}$$

$$=\sum_{n=0}^{\frac{1}{2}p}(-1)^n\frac{(2p-2n)!(p)!}{(n)!(p-n)!(p-2n)!}x^{p-2n}$$

$$\therefore\ P_p(x)=\frac{1}{2^p(p)!}\sum_{n=0}^{\frac{1}{2}p}(-1)^n\frac{(2p-2n)!(p)!}{(n)!(p-n)!(p-2n)!}x^{p-2n}$$

$$=\sum_{n=0}^{\frac{1}{2}p}(-1)^n\frac{(2p-2n)!}{2^p(n)!(p-n)!(p-2n)!}x^{p-2n}.$$

This proves the result.

Ex. 13. *Solve that*

$$\int_{-1}^{1}(1-x^2)\left(\frac{dP_n}{dx}\right)^2dx=\frac{2n(n+1)}{2n+1}$$ [Agra 67]

Solution. We have

$$\int_{-1}^{1}(1-x^2)P'^2_n\,dx=\int_{-1}^{1}(1-x^2)P'_nP'_n\,dx$$

$$=\Big[(1-x^2)P'_n.P_n\Big]_{-1}^{1}-\int_{-1}^{1}\frac{d}{dx}[(1-x^2)P'_n].\,P_n\,dx$$

integrating by parts

$$=-\int_{-1}^{1}P_n\frac{d}{dx}[(1-x^2)P'_n]\,dx$$

$$=-\int_{-1}^{1}P_n[-n(n+1)P_n]\,dx$$

$$\text{as }\frac{d}{dx}[(1-x^2)P'_n]+n(n+1)P_n=0$$

$$=n(n+1)\int_{-1}^{1}P_n^2\,dx=n(n+1).\ \frac{2}{2n+1}$$

Ex. 14. *Show toat* $\int_{-1}^{1} (1-x^2)\, P'_n P'_m dx=0$ *if* $m \neq n$, **[Agra 67, 59]**

Solution. Integrating by parts, we have

$$\int_{-1}^{1} (1-x^2)\, P'_n P'_m\, dx=n\,(n+1) \int_{-1}^{1} P_n P_m\, dx$$
$$=n\,(n+1)\times 0, \text{ as } m\neq n$$
$$=0.$$

Ex. 15. *Show that*

$$\int_{-1}^{1} (x^2-1)\; P_{n+1}\, P'_n\; dx=-\frac{2n\,(n+1)}{(2n+1)(2n+3)}.$$ **[Raj. 65]**

Solution. We have the recurrence formulae I and IV,

$$(2n+1)\, xP_n=(n+1)\, P_{n+1}+nP_{n-1} \qquad \text{...(1)}$$

and $(x^2-1)\, P'_n=(n+1)\,[P_{n+1}-xP_n]$. ...(2)

Multiplying (2) by P_{n+1} and integrating between -1 to $+1$, we get

$$\int_{-1}^{1} (x^2-1)\; P_{n+1}\; P'_n\; dx$$
$$=(n+1) \int_{-1}^{1} P_{n+1}\, [P_{n+1}-xP_n]\; dx$$
$$=(n+1) \int_{-1}^{1} (P_{n+1})^2\; dx$$
$$=(n+1) \int_{-1}^{1} P_{n+1} \left[\frac{(n+1)\; P_{n+1}-nP_{n-1}}{2n+1}\right] dx$$

from (1)

$$=(n+1)\cdot\frac{2}{2n+1}-\frac{(n+1)^2}{2n+1}\cdot\frac{2}{2n+3}$$
$$=\frac{2n(n+1)}{(2n+1)(2n+3)}.$$

Ex 16. *To show that*

$$P_n\,(\cos\theta)=\frac{1.3.5...(2n-1)}{2^{n-1}\,(n\,!)}\left[\cos n\theta+\frac{1.n}{1.(2n-1)}\cos\,(n-2)\,\theta\right.$$
$$\left.+\frac{1.3n\,(n+1)}{1.2\,(2n-1)\,(2n-3)}\cos\,(n-4)\,\theta+...\right].$$

Solution. We know that

$$\sum_{n=0}^{\infty} h^n P_n\,(x)=(1-2xh+h^2)^{-1/2}.$$

When $x=\cos\theta$, this gives

$$\sum_{0}^{\infty} h^n P_n\,(\cos\theta)=(1-2\cos\theta+h^2)^{-1/2}$$

$=[1-h(e^{i\theta}+e^{-i\theta})+h^2]^{-1/2}.$

$=[(1-he^{i\theta})(1-he^{-i\theta})]^{1/2}.$

Now $(1-x)^{-1/2}=1+\frac{1}{2}x+\frac{1.3}{2.4}x^2+\ldots+\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}x^n+\ldots$

Using this we can write expansions of

$(1-he^{i\theta})^{-1/2}$ and $(1-he^{-i\theta})^{-1/2}$.

In the product of their expansions, coeff. of h^n

$$=\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}\left[(e^{in\theta}+e^{-in\theta})+\frac{1}{2}\cdot\frac{2n}{2n-1}(e^{i(n-2)\theta}+e^{-i(n-2)\theta})\right.$$
$$\left.+\frac{1\cdot3}{2\cdot4}\frac{2n(2n-2)}{(2n-1)(2n-3)}(e^{i(n-4)\theta}+e^{-i(n-4)\theta})+\ldots\right]$$
$$=\frac{1.3\ldots(2n-1)}{2.4\ldots2n}\left[2\cos n\theta+\frac{n}{2n-1}\cos(n-2)\theta\right.$$
$$\left.+\frac{n(n-1)}{(2n-1)(2n-3)}\cdot\frac{1.3}{1.2}.2\cos(n-4)\theta+\ldots\right]$$

But coeff. of h^n is $P_n(\cos\theta)$; therefore equating coefficients of h^n from both the sides, we get the result.

Ex. 17. *Show that*

$$\int_0^\pi P_n(\cos\theta)\cos n\theta\,d\theta=\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}$$ **[Raj. 65]**

Solution. We know that

$$P_n(\cos\theta)=\frac{1.3.5\ldots(2n-1)}{2.4\ldots2n}\left[2\cos n\theta+2.\frac{1.n}{1.(2n-1)}\cos(n-2)\theta\right.$$
$$\left.+2\frac{1.3.n(n-1)}{1.2.(2n-1)(2n-3)}\cos(n-4)\theta+\ldots\right]\quad\ldots(1)$$

Multiplying by $\cos n\theta$ and integrating between the limits 0 to π, we get

$$\int_0^\pi P_n(\cos\theta)\cos n\theta\,d\theta=\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}\int_0^\pi 2\cos^2 n\theta\,d\theta$$

all other terms vanish as

$$\int_0^\pi \cos m\theta\cos n\theta\,d\theta=0\text{ if }m\neq n$$
$$=\frac{1.3.5\ldots(2n-1}{2.4\ 6\ldots2n}\int_0^\pi(1+\cos 2n\theta)\,d\theta$$
$$=\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}\pi.$$

Ex. 18. *If n is a positive integer, show that*

$$\int_0^\pi P_n(\cos\theta)\cos n\theta\,d\theta=B(n+\tfrac{1}{2},\tfrac{1}{2})$$

Solution. As in Ex. 16, we have

$$P_n'(\cos\theta)=\frac{1.3.5\ldots(2n-1)}{2.4\ldots2n}\left[2\cos n\theta+\frac{2.1.n}{1.(2n-1)}\cos(n-2)\theta\ldots\right]$$

Multiplying both the sides by $\cos n\theta$ and integrating between 0 to π, we get

$$\int_0^{\pi} P_n \cos\theta) \cos n\theta \, d\theta$$

$$=\frac{1.3.5...(2n-1)}{2^n.(1.2...n)} \int_0^{\pi} 2\cos^2 n\theta \, d\theta, \text{ other integral vanishes}$$

$$=\frac{\frac{2n-1}{2} \cdot \frac{2n-3}{2} \cdots \frac{3}{2} \cdot \frac{1}{2}}{(n)!} \int_0^{\pi} (1+\cos 2n\theta) \, d\theta$$

$$=\frac{\frac{2n-1}{2} \cdot \frac{2n-1}{2} \cdots \frac{3}{2} \cdot \frac{1}{2} \pi}{\Gamma(n+1)}$$

$$=\frac{\frac{2n-1}{2} \cdot \frac{2n-3}{2} \cdots \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma\frac{1}{2} . \Gamma\frac{1}{2}}{\Gamma(n+1)} = \frac{\Gamma\left(\frac{2n+1}{2}\right)\Gamma(1)}{\Gamma(n+1)}$$

$$=\frac{\Gamma(n+\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(n+1)} = B\,(n+\tfrac{1}{2}, \tfrac{1}{2}),$$

since $B\,(m, n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

7·9. Christoffel's Summation formula

To find the sum of first $(n+1)$ terms of the series

$$\sum_0^{\infty} (2m+1)\, P_m(x)\, P_m(y).$$ [Agra 58]

Recurrence formula I gives

$$(2m+1)\, xP_m(x) = (m+1)\, P_{m+1}(x) + mP_{m-1}(x) \quad ...(1)$$

and $(2m+1)\, yP_m(y) = (m+1)\, P_{m+1}(y) + mP_{m-1}(y).$ ...(2)

Multiplying these by $P_m(y)$ and $P_m(x)$ respectively and subtracting, we get

$$(2m+1)(x-y)\, P_m(x)P_m(y) = (m+1)\,\{P_{m+1}(x)\, P_m(y) - P_m(x)\, P_{m+1}(y)\} - m\,\{P_m(x)\, P_{m-1}(y) - P_{m-1}(x)\, P_m(y)\}.$$

Putting $m=1, 2, 3, \ldots n$ in succession, we get

$$3\,(x-y)\, P_1(x)\, P_1(y) = 2\,\{P_2(x)\, P_1(y) - P_1(x)\, P_2(y)\} - 1\,\{P_1(x)\, P_0(y) - P_0(x) P_1(y)\}$$

$$5\,(x-y)\, P_2(y)\, P_2(y) = 3\,\{P_3(x)\, P_2(y) - P_2(x)\, P_3(y)\} - 2\,\{P_2(x)\, P_1(y) - P_1(x)\, P_2(y)\}$$

... ...

$$(2n+1)(x-y)\, P_n(x)\, P_n(y) = (n+1)\,\{P_{n+1}(x)\, P_n(y) - P_n(x)\, P_{n+1}(y)\} - n\,\{P_n(x)\, P_{n-1}(y) - P_{n-1}(x)\, P_n(y)\}.$$

Adding these, we get

$$(x-y)\,[3P_1(x)\, P_1(y) + 5P_2(x)\, P_2(y) + \ldots (2n+1)\, P_n(x)\, P_n(y)]$$
$$= (n+1)\,[P_{n+2}(x)\, P_n(y) - P_n(x)\, P_{n+1}(y)] - [P_1(x)\, P_0(y) - P_0(x)\, P_1(y)]$$

But $\quad P_1(x)\,P_0(y)-P_0(x)\,P_1(y)=x-y$

$$=(x-y)\,P_0(x)\,P_0(y).$$

as $P_1(x)=x$ and $P_0(x)=1$

$$\therefore\quad (x-y)\,[P_0(x)\,P_0(y)+3P_1(x)\,P_1(y)+5P_2(x)\,P_2(y)+\ldots+(2n+1)\,P_n(x)\,P_n(y)]$$

$$=(n+1)\,[P_{n+1}(x)\,P_n(y)-P_n(x)\,P_{n+1}(y)]$$

or $\quad \sum_0 (2m+1)\,P_m(x)\,P_m(y)$

$$=(n+1)\,\frac{P_{n+1}(x)\,P_n(y)-P_n(x)\,P_{n+1}(y)}{x-y}$$

Ex. 1. *Prove that*

$$P_0^2+3P_1^2+5P_2^2+\ldots(2n+1)\,P_n^2=(n+1)\,\{P_nP'_{n+1}-P_{n+1}P'_n\}.$$

Solution. The Christoffel's summation formula is

$$\sum_{m=0}^{m=n}(2m+1)\,P_m(x)\,P_m(y)=(n+1)\,\frac{P_{n+1}(x)\,P_n(y)-P_{n+1}(y)\,P_n(x)}{x-y}.$$

Now let $y=x+\delta$, where δ is a small quantity.

$$\sum_{m=0}^{n}(2m+1)\,P_m(x)\,P_m(x+\delta)$$

$$=(n+1)\,\frac{P_{n+1}(x)\,P_n(x+\delta)-P_{n+1}(x+\delta)\,P_n(x)}{x-(x+\delta)}$$

or $\quad -\delta\sum_{m=0}^{n}(2m+1)\,P_m(x)\,[P_m(x)+\delta P'_m(x)+\ldots]$

$$=(n+1)\,[P_{n+1}(x)\,\{P_n(x)+\delta P'_n(x)+\tfrac{1}{2}\delta^2P''_n(x)\ldots\}$$

$$-P_n(x)\,\{P_{n+1}(x)+\delta P'_{n+1}(x)+\tfrac{1}{2}\delta^2P''_{n+1}(x)+\ldots\}].$$

Using Taylor's series expansions, dividing by $-\delta$ and proceeding to limits as $\delta\to 0$, we get

$$\sum_{m=0}^{n}(2m+1)\,P_m^2(x)=(n+1)\,[P_n(x)\,P'_{n+1}(x)-P_{n+1}(x)\,P'_n(x)].$$

or $\quad P_0^2+3P_1^2+5P_2^2+\ldots(2n+1)\,P_n^2$

$$=(n+1)\,[P_n(x)\,P'_{n+1}(x)-P_{n+1}(x)\,P'_n(x)].$$

This is the required result.

Ex. 2. *Prove that*

$$P_0^2+3P_1^2+5P_2^2+\ldots+(2n+1)\,P^2_n=(n+1)^2\,[P_n^2-(x^2-1)\,(P'^2_n)].$$

[Agra 58]

Solution. As in the above example, we get

$$P_0^2+3P_1^2+5P_2^2+\ldots+(2n+1)\,P_n^2$$

$$=(n+1)\,[P_n(x)\,P'_{n+1}(x)-P_{n+1}(x)\,P'_n(x)].$$

$$=(n+1)\,[P_n(x)\,P'_{n+1}(x)-xP'_n(x)]+(n+1)\,P'_n(x)\times[xP_n(x)-P_{n+1}(x)]$$

adding and subtracting $(n+1)\,xP_n(x)\,P'_n(x)$

$$=(n+1)\,P_n(x)\,[(n+1)\,P_n]+P'_n(x)\,[(1-x^2)\,P'_n]$$

from recurrence formula III and VI p. 180

$$=(n+1)^2\,P_n^2-(x^2-1)\,(P'_n)^2.$$

This is the required result.

8

Bessel's Equation

8·1. Solution of Bessel's Equation.

[Raj. 66 ; Delhi M.A. (Pre.) 58]

The Bessel's equation of order n is

$$x^2 \frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)\, y=0.$$

Evidently $x=0$ is a regular singularity of this equation.

Let $y=x^{\alpha} \sum_{r=0}^{\infty} a_r x^r$ be a solution of this equation in series, convergent in a neighbourhood of $x=0$.

Then $\frac{dy}{dx}=\sum_{r=1}^{\infty} a_r\, (\alpha+r)\, x^{\alpha+r-1}$

and $\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty} a_r\, (\alpha+r)\, (\alpha+r-1)\, x^{\alpha+r-2}$.

Substituting these values in the equation, we have

$$x^2 \sum_{r=0}^{\infty} a_r\, (\alpha+r)\, (\alpha+r-1)\, x^{\alpha+r-2}+x \sum_{r=0}^{\infty} a_r\, (\alpha+r)\, x^{\alpha+r-1}$$

$$+(x^2-n^2)\sum_{r=0}^{\infty} a_r\, x^{\alpha+r}=0$$

i.e. $$\sum_{r=0}^{\infty} a_r\, \{(\alpha+r)^2-n^2\}\, x^{\alpha+r}+\sum_{r=0}^{\infty} a_r\, x^{\alpha+r+2}=0.$$

Equating to zero the coefficient of x^{α}, we have

$$a_0\, (\alpha^2-n^2)=0.$$

Since $a_0 \neq 0$, the indicial equation is

$$\alpha^2-n^2=0 \text{ giving } \alpha=\pm n. \qquad \text{...(1)}$$

Also equating to zero coefficient of next highest term, i.e. coefficient of $x^{\alpha+1}$, we get

$$a_1\, \{(1+\alpha)^2-n^2\}=0.$$

This gives $a_1=0$ as $(1+\alpha)^2-n^2 \neq 0$.

Equating to zero the coefficient of $x^{\alpha+r+2}$, the recurrence relation in coefficients a_r is given by

$$a_{r+2}\, \{(\alpha+r+2)^2-n^2\}+a_r=0,$$

i.e. $$a_{r+2}=-\frac{1}{(\alpha+r+2)^2-n^2}\, a_r. \qquad \text{...(3)}$$

Therefore $a_3=a_5=\ldots=0$, since $a_1=0$.

When $\alpha=n$, (3) *gives*

$$a_{r+2}=\frac{1}{(n+r+2)^2-n^2}a_r=-\frac{1}{(r+2)(2n+r+2)}a_r.$$

Setting $r=0, 2, \ldots$, we have

$$a_2=-\frac{1}{2.2(n+1)}a_0,$$

$$a_4=-\frac{1}{4.2(n+2)}a_2=\frac{1}{2.4.2^2(n+1)(n+2)}a_0 \text{ ......and so on.}$$

With these values of a's, the solution in the form of the series is

$$y=a_0x^n\left[1-\frac{x^2}{2.2(n+1)}+\frac{x^2}{2.4.2^2(n+1)(n+2)}+\ldots\right.$$
$$\left.+(-1)^r\frac{x^{2r}}{2^r.(r)!\,2^r(n+1)(n+2\ldots(n+r)}+\ldots\right],$$

where a_0 is an arbitrary constant.

Putting $-n$ for n in the above, solution corresponding to $\alpha=-n$ is

$$y=a_0x^{-n}\left[1-\frac{x^2}{2.2(1-n)}+\frac{x^4}{2.4.2^2(1-n)(2-n)}-\ldots\right].$$

8·2. Definition of Bessel's function, $J_n(x)$.

[Agra 55, 54, 52, Final 67]

The Bessel's differential equation of order n is

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0.$$

A solution of this in the form of a series of ascending powers of x is

$$y=a_0x^n\left[1-\frac{x^2}{2.2(n+1)}+\frac{x^4}{2.4.2^2(n+1)(n+2)}+\ldots\right.$$
$$\left.+(-1)^r\frac{x^{2r}}{2^r(r)!\,2^r(n+1)(n+2)\ldots(n+r)}+\ldots\right],$$

where a_0 is an arbitrary constant.

If $a_0=\frac{1}{2^n\,\Gamma(n+1)}$, the above solution is denoted by $J_n(x)$, so that

$$J_n(x)=\frac{1}{2^n\,\Gamma(n+1)}x^n\sum_0^\infty\frac{(-1)^nx^{2r}}{2^r.(r)!\,2^r(n+1)(n+2)\ldots(n+r)}$$

or $$J_n(x)=\sum_{r=0}^\infty\frac{(-1)^r}{(r)!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$J_n(x)$ is usually called the *Bessel's function of the first kind* and of order n.

Ex. 1 *Prove that* $\frac{d}{dx}[J_0(x)]=-J_1(x)$.

Solution. We have by definition

$$J_0(x)=\sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{2r}$$

Differentiating it w.r.t. x, we get

$$\frac{d}{dx}J_0(x)$$

$$=\tfrac{1}{2}\sum_{1}^{\infty}(-1)^r\frac{1}{(r)!\,(r-1)!}\left(\frac{x}{2}\right)^{2r-1}$$

$$=\sum_{j=0}^{\infty}(-1)^{j+1}\frac{1}{(j+1)\,(j)!}\left(\frac{x}{2}\right)^{2j+1}$$

where $j=r-1$

$$=-\sum_{j=0}^{\infty}(-1)^j\frac{1}{(j)!\,(j+1)!}\left(\frac{x}{2}\right)^{2j+1}=-J_1(x).$$

8·3. When n is positive integer,

$$\mathbf{J_{-n}(x)=(-1)^n\,J_n(x)}.$$

Proof. We have

$$J_{-n}(x)=\sum_{r=0}^{\infty}(-1)^r\frac{1}{(r)!\,\Gamma(r-n+1)}\left(\frac{x}{2}\right)^{-n+2r}$$

In this series Γ of a negative number occurs in denominator, for terms when $r=0, 1, 2, \ldots, (n+1)$ and Γ of a negative number being infinite, all these terms are zero.

Thus terms are obtained from $r=n$ onwards

or $$J_{-n}(x)=\sum_{r=n}^{\infty}(-1)^r\frac{1}{(r)!\,\Gamma(r-n+1)}\left(\frac{x}{2}\right)^{-n+2r}$$

Now take $r=n+j$, so that when $r=n$, $j=0$ and

$$J_{-n}(x)=\sum_{j=0}^{\infty}(-1)^{j+n}\frac{1}{(n+j)!\,\Gamma(n+j-n+1)}\left(\frac{x}{2}\right)^{-n+2n+2j}$$

$$=\sum_{j=0}^{\infty}(-1)^{j+n}\frac{1}{(n+j)!\,\Gamma(j+1)}\left(\frac{x}{2}\right)^{n+2j}$$

$$=(-1)^n\sum_{j=0}^{\infty}(-1)^j\frac{1}{(n+j+1).(j)!}\left(\frac{x}{2}\right)^{n+2j}$$

as $(n+j)\,!=\Gamma(n+j+1)$ and $\Gamma(j+1)=(j)\,!$

$$=(-1)^n J_n(x).$$

Note. Because of this relation in $J_n(x)$ and $J_{-n}(x)$, the two functions are not independent when n is an integer. Therefore in this case a second solution of Bessel's equation is to be found.

Ex. *State the Bessel's differeetial equation and prove that its most general solution is $y=AJ_n(x)+BJ_{-n}(x)$ where A and B are arbitrary constants.* **[Agra 1973]**

8·4. Recurrence formulae

I. $\mathbf{xJ_n'=nJ_n-xJ_{n+1}}$.

By definition,

$$J_n(x)=\sum_0^{\infty}(-1)^r\frac{1}{(r)\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

Differentiating it with respect to x, we get

$$J'_n=\sum_{r=0}^{\infty}\frac{(-1)^r\,(n+2r)}{(r)\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1}\cdot\frac{1}{2}.$$

$$\therefore\quad xJ'_n=n\sum_{r=0}^{\infty}\frac{(-1)^r}{(r)\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$+x\sum_{r=0}^{\infty}\frac{(-1)^r.\,2r}{2.(r)\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$=nJ_n+x\sum_{r=1}^{\infty}\frac{(-1)^r}{(r-1)\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1}$$

in the second summation r takes values from 1 to ∞

$$=nJ_n+x\sum_{j=0}^{\infty}\frac{(-)^{j+1}}{(j)\,!\,\Gamma(n+j+2)}\left(\frac{x}{2}\right)^{n+2j+1}$$

taking $j=r-1$

$$=nJ_n-x\sum_{j=0}^{\infty}\frac{(-1)^j}{(j)\,!\,\Gamma\{(n+1)+j+1\}}\left(\frac{x}{2}\right)^{n+1+2j}$$

Thus $xJ'_n=nJ_n-xJ_{n+1}$.

II. $\mathbf{xJ'_n=-nJ_n+xJ_{n-1}}$. [Vikram 63 ; Agra 54]

As above, we have

$$xJ'_n(x)=\sum_{r=0}^{\infty}\frac{(-1)^r\,(n+2r)}{(r)\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$=\sum_{r=0}^{\infty}\frac{(-1)^r\,\{(2n+2r)-n\}}{(r)\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

as $n+2r=(2n+2r)-n$

$$=-n\sum_{r=0}^{\infty}\frac{(-1)^r}{(r)\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$+x\sum_{n=0}^{\infty}\frac{(-1)^r}{(r)\,!\,\Gamma(n+r)}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$=-nJ_n+x\sum_{r=0}^{\infty}\frac{(-1)^r}{(r)\,!\,\Gamma\{(n-1)+r+1\}}\left(\frac{x}{2}\right)^{n-1+2r}$$

or $xJ'_n=-nJ_n+xJ_{n-1}$.

III. $\mathbf{2J'_n=J_{n-1}-J_{n+1}}$.

[Vikram 63 ; Agra 63 ; 58, 56, 52 , Final 66]

This is obtained directly by adding the recurrence relations I and II. We give an alternative proof as follows :

$$J_n=\sum_{r=0}^{\infty}\frac{(-1)^r}{(r)\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

Differentiating w.r.t. x, we get

$$2J'_n=\sum_{r=0}^{\infty}\frac{(-1)^r\,(n+2r)}{(r)!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$=\sum_{r=0}^{\infty}\frac{(-1)^r\,\{(n+r)+r\}}{(r)!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1}$$

taking $n+2r=(n+r)+r$

$$=\sum_{r=1}^{\infty}\frac{(-1)^r}{(r)!\,\Gamma(n+r)}\left(\frac{x}{2}\right)^{n-1+2r}$$

$$+\sum_{r=0}^{\infty}\frac{(-1)^r}{(r-1)!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$=J_{n-1}+\sum_{r=0}^{\infty}\frac{(-1)^{j+1}}{(j)!\,\Gamma(n+1+j+1)}\left(\frac{x}{2}\right)^{n+1-2j}$$

taking $j=r-1$

$=J_{n-1}-J_{n+1}$, This proves the result.

IV. $2nJ_n=x\,(J_{n-1}+J_{n+1})$.

[Agra 72, 56 ; Final 66 ; Raj. 59]

This result is also directly obtained by subtracting formula I and II. A direct proof is given below.

We can write

$$2nJ_n=\sum_{r=0}^{\infty}\frac{(-1)^r}{(r)!\,\Gamma(n+r+1)}(2n).\left(\frac{x}{2}\right)^{n+2r}$$

$$=\sum_{r=0}^{\infty}\frac{(-1)^r}{(r)!\,\Gamma(n+r+1)}(2n+2r-2r)\left(\frac{x}{2}\right)^{n+2r}$$

taking $2n=2n+2r-2r$

$$=x\sum_{r=1}^{\infty}\frac{(-1)^r}{(r)!\,\Gamma(n+r)}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$-\sum_{r=0}^{\infty}\frac{(-1)^r}{(r-1)!\,\Gamma(n+r+1)}\,x\left(\frac{x}{2}\right)^{n+2r-1}$$

$$=xJ_{n-1}-x\sum_{r=0}^{\infty}\frac{(-1)^{j+1}}{(j)!\,\Gamma(n+1+j+1)}\left(\frac{x}{2}\right)^{n+2j+1}$$

taking $j=r-1$

$=xJ_{n-1}+xJ_{n+1}$.

Thus $2nJ_n=x\ J_{n-1}+J_{n+1}]$.

V. $\dfrac{d}{dx}(x^{-n}J_n)=-x^{-n}J_{n+1}$.

Recurrence formula I as obtained above is

$xJ_n'=nJ_n-xJ_{n+1}$, or $xJ'_n-nJ_n=-xJ_{n+1}$.

Multiplying it by x^{-n-1}, we get

$x^{-n}J'_n-nx^{-n-1}J_n=-x^{-n}J_{n+1}$,

This can be written as

$$\frac{d}{dx}(x^{-n}J_n)=-x^{-n}J_{n+1},$$

which is the required result.

Alternatively, we have

$$\frac{d}{dx}(x^{-n}J_n)=\frac{d}{dx}\left[x^{-n}\sum_{r=0}^{\infty}\frac{(-1)^r}{(r)!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}\right]$$

$$=\frac{d}{dx}\left[\sum_{0}^{\infty}\frac{(-1)^r}{(r)!\,\Gamma(n+r+1)}2^{-n}\left(\frac{x}{2}\right)^{2r}\right]$$

$$=\sum_{0}^{\infty}\frac{(-1)^r}{(r)!\,\Gamma(n+r+1)}2^{-n}r\left(\frac{x}{2}\right)^{2r-1}$$

$$=x^{-n}\sum_{r=0}^{\infty}\frac{(-1)^r}{(r-1)!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$=x^{-n}\sum_{j=0}^{\infty}\frac{(-1)^{j+1}}{(j)!\,\Gamma(n+1+j+1)}\left(\frac{x}{2}\right)^{x+1+2j}$$

taking $j=r-1$

$$=-x^{-n}\sum_{j=1}^{\infty}\frac{(-1)^j}{(j)!\,\Gamma\{(n+1)+j+1\}}\left(\frac{x}{2}\right)^{n+1+2j}$$

$$=-x^{-n}J_{n+1}.$$

VI. $\dfrac{d}{dx}(x^nJ_n)=x^nJ_{n-1}.$

[Vikram 62 ; Agra 72, 66, 57 ; Raj. 59]

Recurrence formula II is $xJ'_n=-nJ_n+xJ_{n-1}$

or $xJ_n'+nJ_n=xJ_{n-1}$.

Multiplying it by x^{n-1}, we get

$$x^nJ_n'+nx^{n-1}J_n=x^nJ_{n-1}.$$

This can be written as

$$\frac{d}{dx}(x^nJ_n)=x^nJ_{n-1}$$

which is the required result.

Alternatively, we have

$$\frac{d}{dx}(x^nJ_n)=\left[x^n\sum_{0}^{\infty}\frac{(-1)^r}{(r)!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}\right]$$

$$=\frac{d}{dx}\left[\sum_{0}^{\infty}\frac{(-1)^r}{(r)!\,\Gamma(n+r+1)}.2^n\left(\frac{x}{2}\right)^{2n+2r}\right]$$

$$=2^n\sum_{0}^{\infty}\frac{(-1)^r\,(n+r)}{(r)!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2n+2r-1}$$

$$=x^n\sum_{0}^{\infty}\frac{(-1)^r}{(r)!\,\Gamma(n+r)}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$=x^nJ_{n-1}.$$

8·4. A generating function

To show that when n is a positive integer $J_n(x)$ is the coefficient of t^n in the expansion of $e^{\frac{1}{2}x(t-1/t)}$ in ascending powers of t.

[Vikram 62 ; Raj. 61 ; Agra 72, 70, 66; 63, 62]

We have $e^{\frac{1}{2}xt} = \sum_0^\infty \frac{1}{(r)!}\left(\frac{xt}{2}\right)^r$

and $$e^{-x/2t} = \sum_0^\infty \frac{1}{(s)!}\left(\frac{-x}{2t}\right)^s.$$

Therefore $$e^{\frac{1}{2}x(t-1/t)} = \sum_{r=0}^\infty \frac{1}{(r)!}\left(\frac{xt}{2}\right)^r \sum_{s=0}^\infty \frac{1}{(s)!}\left(-\frac{x}{2t}\right)^s$$
$$= \sum_{r=0}^\infty \sum_{s=0}^\infty \frac{(-1)^s}{(r)!\,(s)!}\left(\frac{x}{2}\right)^{r+s} t^{r-s}. \qquad \ldots(1)$$

We want to collect coefficient of t^n from the right, for that suppose that $n=r-s$, *i.e.*, $r=n+s$. Now give values to s from 0 to infinity, while r takes values such that $r=n+s$.

Thus the coefficient of t^n on the right of (1) is

$$\sum_{s=0}^\infty \frac{(-1)^s}{(n+s)!\,(s)!}\left(\frac{x}{2}\right)^{n+2s}$$
$$= \sum_0^\infty \frac{(-1)^s}{(s)!\,\Gamma(n+s+1)}\left(\frac{x}{2}\right)^{n+2s}$$
as $(n+s)!=\Gamma(n+s+1)$
$$=J_n(x).$$

Again to collect the coefficient of t^{-n} from the right of (1), give values to r from 0 to infinity while s take values such that $s=n+r$.

In this way coefficient of t^{-n} on the right of (1)

$$= \sum_{r=0}^\infty \frac{(-1)^{n+r}}{(r)!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r}$$
$$=(-1)^n J_n(x)$$
$$=J_{-n}(x) \text{ as } J_{-n}(x)=(-1)^n J_n(x).$$

$$\therefore\quad e^{\frac{1}{2}x(t-1/t)} = \sum_{-\infty}^{\infty} t^n J_n(x).$$

8·5. Trigonometric expansions

To show that

$$\cos(x\sin\phi)=J_0(x)+2[\cos 2\phi\, J_2(x)+\cos 4\phi\, J_4(x)+\ldots],$$
$$\sin(x\sin\phi)=2[\sin\phi\, J_1(x)+\sin 3\phi\, J_3(x)+\ldots],$$
$$\cos(x\cos\phi)=J_0(x)-2[\cos 2\phi\, J_2(x)-\cos 4\phi\, J_4(x)+\ldots],$$
$$\sin(x\cos\phi)=2[\cos\phi\, J_1(x)-\cos 3\phi\, J_3(x)+\ldots].$$

Proof. We have

$$e^{\frac{1}{2}x(t-1/t)} = \sum_{-\infty}^{\infty} t^n J_n(x).$$

Putting $t=e^{i\phi}$, so that $\frac{1}{2}(t-1/t)=i\sin\phi$, we get

$$e^{xi\sin\phi}=\sum_{-\infty}^{\infty} e^{in\phi}\, J_n(x)$$

$$=J_0(x)+J_1(x)\, e^{i\phi}+J_{-1}(x).e^{-i\phi}+J_2(x)\, e^{2i\phi}+J_{-2}(x)\, e^{-2i\phi}$$
$$+J_3(x)\, e^{3i\phi}+J_{-3}(x)\, e^{-3i\phi}+\ldots+J_n(x)\, e^{ni\phi}$$
$$+J_{-n}(x)\, e^{-ni\phi}+\ldots$$
$$=J_0(x)+J_1(x)\,[e^{i\phi}-e^{-i\phi}]+J_2(x)\,[e^{2i\phi}+e^{-2i\phi}]$$
$$+J_3(x)\,[e^{3i\phi}-e^{-3i\phi}]+J_4(x)\,[e^{4i\phi}+e^{-4i\phi}]+\ldots,$$
as $J_{-n}(x)=(-1)^n J_n(x)$

i.e. $$e^{ix\sin\phi}=J_0(x)+J_1(x).2i\sin\phi+J_2(x).2\cos 2\phi$$
$$+J_3(x).2i\sin 3\phi+J_4(x).2\cos 4\phi+\ldots \qquad \ldots(1)$$

Equating real and imaginary parts, we get

$$\cos(x\sin\phi)=J_0(x)+2\cos 2\phi.J_2(x)+2\cos 4\phi\, J_4(x)+\ldots \qquad \ldots(2)$$

and $$\sin(x\sin\phi)=2\sin\phi\, J_1(x)+2\sin 3\phi\, J_3(x)+\ldots \qquad \ldots(3)$$

Next putting $\frac{1}{2}\pi-\phi$ for ϕ in (1), (2) and (3), we get

$$e^{ix\cos\phi}=J_0(x)+2i\cos\phi\, J_1(x)-2\cos 2\phi.J_2(x)-2i\cos 3\phi.J_3(x)$$
$$+2\cos 4\phi\, J_4(x)+2i\cos 5\phi.J_5(x)+\ldots \qquad \ldots(4)$$

$$\cos(x\cos\phi)=J_0(x)-2\cos 2\phi\, J_2(x)+2\cos 4\phi J_4(x). \qquad \ldots(5)$$

and $$\sin(x\cos\phi)=2\cos\phi\, J_1(x)-2\cos 3\phi\, J_3(x)$$
$$+2\cos 5\phi\, J_5(x)\ldots \qquad (6)$$

Ex. 1. *Show that*

$$\cos x=J_0(x)-2J_2(x)+2J_4(x)-\ldots$$

and $$\sin x=2J_1(x)-2J_3(x)+2J_5(x)-\ldots$$ [Agra 36]

Solution. As in above article,

$$\cos(x\sin\phi)=J_0(x)+2\cos 2\phi\, J_2(x)+2\cos 4\phi\, J_4(x)+\ldots \qquad \ldots(1)$$

and $$\sin(x\sin\phi)=2\sin\phi\, J_1(x)+2\sin 3\phi\, J_3(x)$$
$$+2\sin 5\phi\, J_5(x)+\ldots \qquad (2)$$

Putting $\phi=\frac{1}{2}\pi$, these give

$$\cos x=J_0(x)-2J_2(x)+2J_4(x)-\ldots$$

and $$\sin x=2J_1(x)-2J_3(x)+2J_5(x)-\ldots$$

Ex. 2. *Prove that*

$$[J_0(x)]^2+2\,[J_1(x)]^2+2\,[J_2(x)]^2+\ldots=1. \qquad \text{[Agra 57]}$$

Solution. We have as in above article

$$J_0(x)+2J_2(x)\cos 2\phi+2J_4(x)\cos 4\phi+\ldots=\cos(x\sin\phi) \qquad ..(1)$$

and $$2J_1(x)\sin\phi+2J_3(x)\sin 3\phi+\ldots=\sin(x\sin\phi). \qquad ..(2)$$

It would be noted that

$$\int_0^\pi \cos^2 n\phi\, d\phi=\int_0^\pi \sin^2 n\phi\, d\phi=\tfrac{1}{2}\int_0^\pi (1\pm\cos 2n\phi)\, d\phi=\tfrac{1}{2}\pi$$

and $$\int_0^\pi \cos n\phi\cos m\phi\, d\phi=\int_0^\pi \sin n\phi\sin m\phi\, d\phi=0,\ m\neq n.$$

Now squaring (1) and integrating w.r.t. ϕ between the limits 0 to π, we get by the use of above integrals

---

$e^{xi\sin\phi}=\cos(x\sin\phi)+i\sin(x\sin\phi)$.

$$[J_0(x)]^2\pi+2\,[J_2(x)]^2\,\pi+2\,[J_4(x)]^2\,\pi+\ldots=\int_0^\pi \sin^2(x\sin\phi)\,d\phi.$$

Again squaring (2) and integrating w.r.t. ϕ between the limits 0 to π, we similarly get

$$[J_1(x)]^2\,\pi+2\,[J_3(x)]^2\pi+\ldots=\int_0^\pi \sin^2(x\sin\phi)\,d\phi.$$

Adding these,

$$\pi\,[\{J_0(x)\}^2+2\,\{J_1(x)\}^2+2\,\{J_2(x)\}^2+\ldots]$$
$$=\int_0^\pi [\cos^2(x\sin\phi)+\sin^2(x\sin\phi)]\,d\phi$$
$$=\int_0^\pi d\phi=\pi,$$

or $\{J_0(x)\}^2+2\,\{J_1(x)\}^2+2\,\{J_2(x)\}^2+\ldots=1.$

Ex. 3. *From Ex. 2 deduce that*

$|J_0(x)|\leqslant 1,\ |J_n(x)|\leqslant 2^{-1/2}\ (n\geqslant 1).$ [Agra 57]

Solution. As in the previous example, we have

$$J_0^2(x)+2\,[J_1(x)]^2+2\,[J_2(x)]^2+\ldots=1, \qquad \ldots(1)$$

i.e. $J_0^2(x)=1-2\,[J_1^2(x)+J_2^2(x)+\ldots]$

$\leqslant 1$ as $J_1^2(x), J_2^2(x)$...are all positive.

or $|J_0(x)|\leqslant 1.$

Also from (1), when $n\geqslant 1$,

$2\,[J_1(x)]^2+2\,[J_2(x)]^2+\ldots 2\,[J_n(x)]^2=1+J_0^2(x)$

or $[J_n(x)]^2=\frac{1}{2}-[\frac{1}{2}J_0^2(x)+J_1^2(x)+J_2^2(x)\ldots]$

$\leqslant\frac{1}{2}.$

This gives $|J_n(x)|\leqslant\dfrac{1}{2^{1/2}}=2^{-1/2}.$

Ex. 4. *Prove that* $\dfrac{d}{dx}(xJ_nJ_{n+1})=x\,(J_n^2-J_{n+1}^2)$

and deduce that $x=2J_0J_1+6J_1J_2+\ldots+2\,(2n+1)\,J_nJ_{n+1}+\ldots$

Solution. We directly have

$$\frac{d}{dx}(xJ_nJ_{n+1})=J_nJ_{n+1}+x\,(J_nJ'_{n+1}+J'_nJ_{n+1})$$
$$=J_nJ_{n+1}+(xJ'_n)\,J_{n+1}+J_n\,(xJ'_{n+1}). \qquad \ldots(1)$$

Also recurrence formulae I and II are

$$xJ'_n=nJ_n-xJ_{n+1} \qquad \ldots(2)$$

and $xJ'_n=-nJ_n+xJ_{n-1}.$

Putting $n+1$ for n. $xJ'_{n+1}=-(n+1)\,J_{n+1}+xJ_n.$ $\qquad\ldots(3)$

Eliminating xJ'_n and xJ'_{n+1} from (2), (3) and (1), we get

$$\frac{d}{dx}(xJ_nJ_{n+1})=J_nJ_{n+1}+(nJ_n-xJ_{n+1})\,J_{n+1}+J_n\{-(n+1)\,J_{n+1}+xJ_n\}$$
$$=x\,(J_n^2-J_{n+1}^2),$$

Deduction. In the above result put $n=0, 1, 2...$respectively and then multiply by 1, 3, 5...etc. This gives

$$\frac{d}{dx}[x(J_0J_1+3J_1J_2+5J_2J_3+...)]$$
$$=x(J_0^2-J_1^2)+3x(J_1^2-J_2^2)+5x(J_2^2-J_3^2)+...$$
$$=x[J_0^2+2J_1^2+2J_2^2+...]$$
$$=x.\ 1 \text{ from example 2.}$$

Now integrating both the sides, we get

$$x(J_0J_1-0J_1J_2+5J_2J_3+...)=\tfrac{1}{2}x^2;$$

constant vanishes as both sides become zero when $x=0$.

Thus $x=2J_0J_1+6J_1J_2+10J_2J_3+...$

Ex. 5. *Show that*

$$\frac{d}{dx}(J^2_n+J^2_{n+1})=2\left(\frac{n}{2}J^2_n-\frac{n+1}{x}J^2_{n+1}\right)$$ [Agra 47]

Solution. Recurrence formula I is

$$xJ'_n=nJ_n-xJ_{n+1} \qquad ...(1)$$

and formula II is $xJ'_n=-nJ_n+xJ_{n-1}$. ...(2)

Putting $n+1$ for n in (2), we get

$$xJ'_{n-1}=-(n+1)J_{n+1}+xJ_n. \qquad ...(3)$$

Now $\frac{d}{dx}(J_n^2+J^2_{n+1})=2J_nJ'_n+2J_{n+1}J'_{n+1}$

$$=2J_n.\frac{1}{x}(nJ_n-xJ_{n+1})+2J_{n+1}.\frac{1}{x}[-(n+1)J_{n+1}+xJ_n]$$

from (1) and (3)

$$=2\left[\frac{n}{x}J_n^2-\frac{n+1}{x}J^2_{n+1}\right].$$

8·6. Bessel's integrals.

To show that

$$\mathbf{J_0(x)=\frac{1}{\pi}\int_0^\pi \cos(x\sin\phi)\,d\phi.}$$

[Vikram 69 ; Raj. 62 ; Agra 55 ; Final 67 ; Delhi 57]

$$\mathbf{J_n(x)=\frac{1}{\pi}\int_0^\pi \cos(n\phi-x\sin\phi)\,d\phi.}$$

[Vikram 69, 63; Raj. 66, 62; Agra 60, 58, 56, 52; Delhi 57]

We have

$$\cos(x\sin\phi)=J_0+2J_2\cos 2\phi+2J_4\cos 4\phi+... \qquad ...(1)$$
$$\sin(x\sin\phi)=2J_1\sin\phi+2J_3\sin 3\phi+... \qquad ...(2)$$

To obtain the first result, we integrate (1) between 0 and π, and get

$$\int_0^\pi \cos(x\sin\phi)\,d\phi=J_0\int_1^\pi d\phi+0+0+...$$

i.e. $J_0=\frac{1}{\pi}\int_0^{\pi} \cos (x \sin \phi)\, d\phi.$

This proves the first result.

Next multiply (1) by $\cos n\phi$ and integrate between 0 and π. This gives

$$\int_0^{\pi} \cos (x \sin \phi) \cos n\phi \, d\phi=0, \quad \text{if } n \text{ is odd} \qquad \ldots(i)$$

and $$\int_0^{\pi} \cos (x \sin \phi) \cos n\phi \, d\phi=\pi J_n (x) \quad \text{if } n \text{ is even.} \qquad \ldots(ii)$$

Again multiply (2) by $\sin n\phi$ and integrate between 0 and π This gives

$$\int_0^{\pi} \sin (x \sin \phi)\, n\phi=\pi J_n (x) \text{ if } n \text{ is odd} \qquad \ldots(a)$$

and $$\int_0^{\pi} \sin (x \sin \phi) \sin n\phi \, d\phi=0 \text{ if } n \text{ is even.} \qquad \ldots(b)$$

Now whether n is even or odd, adding (i) and (a) or (ii) and (b), we get

$$\int_0^{\pi} [\cos (x \sin n\phi) \cos n\phi+\sin (x \sin \phi) \sin n\phi]\, d\phi=\pi J_n(x)$$

or $$J_n(x)=\frac{1}{\pi}\int_0^{\pi} \cos (n\phi-x \sin \phi)\, d\phi.$$

Ex. 1. *Prove that*

(i) $J_{1/2} (x)=\sqrt{\left(\frac{2}{\pi x}\right)} \sin x.$ **[Agra 1971; Raj 66]**

(ii) $J_{-1/2} (x)=\sqrt{\left(\frac{2}{\pi x}\right)} \cos x.$

Solution. We have

$$J_n (x)=\frac{x^n}{2^n \Gamma (n+1)}\left[1-\frac{x^2}{2.2 (n+1)}+\frac{x^4}{2.4.2^2(n+1)(n+2)}-\ldots\right]$$

(i) Putting $n=\frac{1}{2}$, we get

$$J_{1/2} (x)=\frac{x^{1/2}}{2^{1/2} \Gamma(\frac{3}{2})}\left[1-\frac{x^2}{2.3}+\frac{x^4}{2.3.4.5}-\ldots\right]$$

$$=\frac{x^{1/2}}{2^{1/2} \Gamma(\frac{1}{2})}\cdot\frac{1}{x}\left[x-\frac{x^3}{(3)\,!}+\frac{x^5}{(5)\,!}-\ldots\right]$$

$$=\sqrt{\left(\frac{2}{\pi x}\right)} \sin x \text{ as } \Gamma (\tfrac{1}{2})=\sqrt{\pi}.$$

(ii) Again putting $n=-\frac{1}{2}$, we get

$$J_{-1/2} (x)=\frac{x^{-1/2}}{2^{-1/2}\Gamma(\frac{1}{2})}\left[1-\frac{x^2}{2}+\frac{x^4}{2.3.4}-\ldots\right]$$

$$=\sqrt{\left(\frac{2}{\pi x}\right)}\left[1-\frac{x^2}{(2)\,!}+\frac{x^4}{(4)\,!}+\ldots\right]=\sqrt{\left(\frac{2}{\pi x}\right)} \cos x.$$

$\int_0^{\pi} \cos n\phi \, d\phi=\left[\frac{\sin n\phi}{n}\right]_0^{\pi}=0$ when n is an integer.

Ex. 2. *Show that*

(*i*) $\sqrt{(\frac{1}{2}\pi x)}\, J_{3/2}(x)=\frac{\sin x}{x}\, \cos x.$ [Agra 71; Raj. 65]

(*ii*) $\sqrt{(\frac{1}{2}\pi x)}\, J_{-3/2}(x)=-\sin x-\frac{\cos x}{x}.$

Solution. We know that

$$J_n(x)=\frac{x^n}{2^n\, \Gamma(n+1)}\left[1-\frac{x^2}{2.2\,(n+1)}+\frac{x^4}{2.4.2^2\,(n+1)\,(n+2)}-\dots\right]$$

(*i*) Putting $n=\frac{3}{2}$, we get

$$J_{3/2}(x)=\frac{x^{3/2}}{2^{3/2}\,\Gamma(\frac{5}{2})}\left[1-\frac{x^2}{2.5}+\frac{x^4}{2.4.5.7}-\frac{x^6}{2.4.6.5.7.9}+\dots\right]$$

$$=\frac{x^{-1/2}}{2.\sqrt{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}}\left[x^2-\frac{x^4}{2.5}+\frac{x^6}{2.4.5.7}-\frac{x^8}{2.4.6.5.7.9}+\dots\right]$$

$$=\sqrt{\left(\frac{2}{\pi x}\right)}\cdot\left[\frac{x^2}{(2)\,!}-\frac{x^3}{(3)\,!}+\frac{x^4}{(5)\,!}+\frac{x^6}{(6)\,!}-\frac{x^6}{(7)\,!}+\dots\right]$$

$$=\sqrt{\left(\frac{2}{\pi x}\right)}\left[\frac{1}{x}\left\{x-\frac{x^3}{(3)\,!}+\frac{x^5}{(5)\,!}\dots\right\}-\left\{1-\frac{x^2}{(2)\,!}+\frac{x^4}{(4)\,!}+\dots\right\}\right].$$

Hence $\sqrt{(\frac{1}{2}\pi x)}\, J_{3/2}(x)=\frac{\sin x}{x}-\cos x.$

(*ii*) We can write

$$J_n(x)=\frac{x^n\,(n+1)}{2^n\,\Gamma(n+2)}\left[1-\frac{x^2}{2.2\,(n+1)}+\frac{x^4}{2.4.2^2\,(n+1)\,(n+2)}-\dots\right],$$

multiplying numerator and denominator by $(n+1)$.

Putting $n=-\frac{3}{2}$,

$$J_{-3/2}(x)=\frac{x^{-3/2}\,(-\frac{1}{2})}{2^{-3/2}\,\Gamma(\frac{1}{2})}\left[1+\frac{x^2}{2}-\frac{x^4}{2.4.1.1}-\dots\right]$$

$$=-\sqrt{\left(\frac{2}{\pi x}\right)}\,\frac{1}{x}\left[1+\frac{x^2}{2}-\frac{x^4}{2.4}+\dots\right]$$

$$=\sqrt{\left(\frac{2}{\pi x}\right)}\left[-\frac{1}{x}\left\{1+x^2\left(1-\frac{1}{2}\right)-x^4\left(\frac{1}{6}-\frac{1}{24}\right)\dots\right\}\right]$$

$$=\sqrt{\left(\frac{2}{\pi x}\right)}\left[-\frac{1}{x}\left(1-\frac{x^2}{(2)\,!}+\frac{x^4}{(4)\,!}-\dots\right)-\left(x-\frac{x^3}{(3)\,!}+\dots\right)\right]$$

$$=\sqrt{\left(\frac{2}{\pi x}\right)}\left[-\frac{1}{x}\cos x-\sin x\right].$$

Ex. 3. *Show that*

$4J''_n=J_{n-2}-2J_n+J_{n+2}.$ [Vikram 63]

Solution. Formula III is $2J'_n=J_{n-1}-J_{n+1}.$ ...(1)

Differentiating it, we get

$$2J''_n=J'_{n-1}-J'_{n+1},$$

i.e. $4J''_n=2J'_{n-1}-2J'_{n+1}$

$=(J_{n-2}-J_n)-(J_n-J_{n+2})$ applying (1) for $2J'_{n-1}$ and $2J'_{n+1}$

$=J_{n-2}-2J_n+J_{n+2}.$

This proves the result.

***Ex. 4.** *Show that*

$$2^r J^{(r)}(x)=J_{n-r}-rJ_{n-r+2}+\frac{r(r-1)}{(2)!}J_{n-r+4}+\ldots+(-1)^r J_{n+r}.$$

[Agra 63]

Solution. Recurrence formula III is $2J'_n=J_{n-1}-J_{n+1}$.

Differentiating it and multiplying by 2, we get

$$2^2J''_n=2J'_{n-1}-2J'_{n-1}.$$
$$=(J_{n-2}-J_n)-(J_n-J_{n+2}) \text{ from III}$$
$$=J_{n-2}-2J_n+J_{n+2}.$$

Differentiating again and multiplying by 2, we get

$$2^3J'''_n=2J'_{n-2}-4J'_n+2J'_{n+2}$$
$$=J_{n-3}-3J_{n-1}+3J_{n+1}-J_{n+3} \text{ from III}$$
$$=J_{n-3}-{}^3C_1J_{n-1}+{}^3C_2J_{n+1}-{}^3C_3J_{n+3}. \qquad \ldots(1)$$

Differentiating it p times in this manner, we get

$$2^pJ_n^{(p)}=[J_{n-p}-{}^pC_1J_{n-p+2}+{}^pC_2J_{n-p+4}-\ldots+(-1)^pJ_{n+p}] \qquad \ldots(2)$$

Let us prove that the above result is true for next differentiation also.

Differentiating (2) again and multiplying by 2, we get

$$2^{p+1}J_n^{(p+1)}=2J'_{n-p}-2.{}^pC_1J'_{n-p+2}+2\,{}^pC_2J'_{n-p+4}-\ldots$$
$$\ldots+2\cdot(-1)^pJ'_{n+p}$$
$$=(J_{n-p-1}-J_{n-p+1})-{}^pC_1\,(J_{n-p+1}-J_{n-p+3})$$
$$+{}^pC_2\,(J_{n-p+3}-J_{n-p+1})+\ldots(-1)^p\,(J_{n+p-1}-J_{n+p+1})$$
$$=J_{n-(p-1)}-(1+{}^pC_1)\,J_{n-p+1}+({}^pC_1+{}^pC_2)\,J_{n-p+3}-\ldots$$
$$\ldots+(-1)^{p+1}J_{n+(p+1)}$$
$$=J_{n-(p+1)}-{}^{p+1}C_1J_{n-p+1}+{}^{p+1}C_2J_{n-p+3}+\ldots+(-1)^{p+1}J_{n+(p+1)}$$

which is of the form (2). Hence the result is true for $(p+1)$ differentiations also if it is true for p differentiations but in (1) the result is true for 3. Hence it is true for all numerical values of p.

Thus

$$2^rJ_n^{(r)}(x)-J_{n-r}-rJ_{n-r+2}+\frac{r(r-1)}{(2)!}J_{n-r+4}+\ldots(-1)^rJ_{n+r}.$$

This proves the result.

Ex. 5. *Prove that*

$$\tfrac{1}{2}xJ_n=(n+1)\,J_{n+1}-(n+3)\,J_{n+3}+(n+5)\,J_{n+5}-\ldots$$

[Rajasthan 64; Agra 56]

Solution. We know from recurrence formula IV that

$$2nJ_n=x\,(J_{n-1}+J_{n+1})$$

i.e. $\quad 2\,(n+1)\,J_{n+1}=x\,(J_n+J_{n+2})$ putting $n+1$ for n

---

$\cos x=1-\frac{x^2}{(2)!}+\frac{x^4}{(4)!}-\ldots$ and $\sin x=x-\frac{x^3}{(3)!}+\frac{x^5}{(5)!}-\ldots$

i.e. $\frac{1}{2}xJ_n=(n+1)\,J_{n+1}-\frac{1}{2}xJ_{n+2}$. ...(1)

Now $\frac{1}{2}xJ_{n+2}=(n+3)\,J_{n+3}-\frac{1}{2}xJ_{n+4}$ putting $n+2$ for n in (1).

$\therefore$ (1) becomes

$$\tfrac{1}{2}xJ_n=(n+1)\,J_{n+1}-(n+3)\,J_{n+3}+\tfrac{1}{2}xJ_{n+4}\ldots \qquad \ldots(2)$$

Now putti. g $n+4$ for n in (1), we get

$$\tfrac{1}{2}xJ_{n+4}=(n+5)\,J_{n+5}-\tfrac{1}{2}xJ_{n+6}.$$

Putting this value in (2), we get

$$\tfrac{1}{2}xJ_n=(n+1)\,J_{n+1}-(n+3)\,J_{n+3}+(n+5)\,J_{n+5}-\tfrac{1}{2}xJ_{n+6}.$$

Proceeding so on, we get

$$\tfrac{1}{2}xJ_n=(n+1)\,J_{n+1}-(n+3)\,J_{n+3}+(n+5)\,J_{n+5}-\ldots$$

which proves the required result.

Ex. 6 *Prove that*

$$J_{n-1}=2/x\,[nJ_n-(n+2)\,J_{n+2}+(n+4)\,J_{n+4}-\ldots]$$

[Vikram 62; Agra 66, 61]

Solution Proceed as above or put $n-1$ for n in the result of the above example.

Ex. 7. *Prove that*

$$\frac{dJ_n}{dx}=\frac{2}{x}\,[\tfrac{1}{2}nJ_n-(n+2)\,J_{n+2}+(n+4)\,J_{n+4}-\ldots ad.\ inf]$$

[Agra 55]

Solution. The recurrence formula is

$$xJ_n'=-nJ_n+xJ_{n-1}$$

so that $\quad J_n'=-(n/x)\,J_n+J_{n-1}.$ ...(1)

But as found in the last example,

$$J_{n-1}=2/x\,[nJ_n-(n+2)\,J_{n+2}+(n+4)\,J_{n+4}-\ldots]$$

Putting this value of J_{n-1} in (1), we get

$$J_n'=-\frac{n}{x}\,J_n+\frac{2}{x}\,[nJ_n-(n+2)\,J_{n+2}+(n+4)\,J_{n+4}-\ldots]$$

$$=2/x\,[\tfrac{1}{2}nJ_n-(n+3)\,J_{n+2}+(n+4)\,J_{n+4}-\ldots]$$

Ex. 8. *Show that*

$$\int_0^x x^{n+1}J_n(x)\,dx=xJ^{n+1}{}_{n+1},\ n>-1. \qquad \textbf{[Raj. 64]}$$

Solution We know from recurrence formula IV, that

$$\frac{d}{dx}\,[x^nJ_n(x)]=x^nJ_{n-1}(x).$$

Putting $n+1$ for n, we get

$$\frac{d}{dx}\,\{x^{n+1}\,J_{n+1}(x)\}=x^{n+1}J_n(x).$$

Integrating both the sides w.r.t. x between 0 to x, we get

$$x^{n+1}\,J_{n+1}(x)=\int^x x^{n+1}J_n(x),$$

which proves the result.

Ex 9 *Show that*

$$\int_0^x x^{-n}J_{n+1}(x)\,dx=\frac{1}{2^n\,\Gamma(n+1)}-x^nJ_n,$$

n being greater than 1.

Solution. We know from recurrence formula V, that

$$\frac{d}{dx}[x^{-n}J_n(x)]=-x^{-n}J_{n+1}(x).$$

Integrating both the sides w.r.t. x between 0 to x, we get

$$\Big[x^{-n}J_n(x)\Big]_0^x=\int_0^x -x^{-n}J_{n+1}(x)\,dx$$

or $$\int_0^x x^{-n}J_{n+1}(x)\,dx=-x^{-n}J_n(x)+\lim_{x\to 0}\left[\frac{J_n(x)}{x^n}\right]$$

$$=-x^nJ_n(x)+\frac{1}{2^n\,\Gamma(n+1)}\cdot$$

since $$J_n(x)=\frac{x^n}{2^n\,\Gamma(n+1)}\left[1-\frac{x^2}{2.2\,(n+1)}+\ldots\right],$$

and $$\lim_{x\to 0}\frac{J_n(x)}{x^n}=\frac{1}{2^n\,\Gamma(n+1)}\cdot$$

Ex. 10. *Show that*

$$J_0(x)=\frac{x^n}{2^{n-1}\,\Gamma(\tfrac{1}{2})\,\Gamma(n+\tfrac{1}{2})}\int_0^1(1-t^2)^{n-\frac{1}{2}}\cos(xt)\,dt.$$

Solution.

$$I=\int_{-1}^1(1-t^2)^{n-\frac{1}{2}}e^{ixt}\,dt \qquad n>-\tfrac{1}{2}$$

$$=\int_{-1}^1(1-t^2)^{n-\frac{1}{2}}\left[1+(ix)\,t+\frac{(ix)^2}{2\,!}\,t^2+\frac{(ix)^3}{3\,!}\,t^3+\ldots\right]dt$$

$$=\sum_{s=0}^{\infty}\frac{(ix)^s}{s\,!}\int_{-1}^1(1-t^2)^{n-\frac{1}{2}}\,t^s\,dt$$ for odd integers s, the integrals vanish

$$=\sum_{r=0}^{\infty}\frac{(-1)^r\,x^{2r}}{(2r)\,!}\int(1-u)^{n-\frac{1}{2}}\,u^{r-\frac{1}{2}}\,du$$ taking $s=2r$ and $u=t^2$

$$=\sum_{r=0}^{\infty}\frac{(-1)^r\,x^{2r}}{(2r)\,!}\,\frac{\Gamma(n+\tfrac{1}{2})\,\Gamma(r+\tfrac{1}{2})}{\Gamma(n+r+1)}$$

$$=\Gamma(\tfrac{1}{2})\,\Gamma(n+\tfrac{1}{2})\sum_{r=0}^{\infty}\frac{(-1)^r\,x^{2r}}{(r)\,!\,\Gamma(n+r+1).2^{2r}}$$

as $\Gamma(\tfrac{1}{2})\,(2r)\,!=2^{2r}\,(r)\,!\,\Gamma(r+\tfrac{1}{2})$ the duplication formula

$$=\Gamma(\tfrac{1}{2})\,\Gamma(n+\tfrac{1}{2})\left(\frac{2}{x}\right)^n\sum_{r=0}^{\infty}\frac{(-1)^r}{(r)\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$=\Gamma(\tfrac{1}{2})\,\Gamma(n+\tfrac{1}{2})\left(\frac{2}{x}\right)^n J_n(x).$$

$$\therefore \quad J_n(x)=\frac{x^n}{2^n\,\Gamma(\frac{1}{2})\,\Gamma(n+\frac{1}{2})}\int_{-1}^{1}(1-t^2)^{n-\frac{1}{2}}\,e^{ixt}\,dt$$

$$=\frac{x^n}{2^n\,\Gamma(\frac{1}{2})\,\Gamma(n+\frac{1}{2})}\left[\int_{-1}^{0}(1-t^2)^{n-\frac{1}{2}}\,e^{ixt}\,dt+\int_{-1}^{1}(1-t^2)^{n-\frac{1}{2}}\,e^{ixt}\,dt\right]$$

$$=\frac{x^n}{2^n\,\Gamma(\frac{1}{2})\,\Gamma(n+\frac{1}{2})}\left[\int_{0}^{1}(1-t^2)^{n-\frac{1}{2}}\,e^{-ixt}\,dt+\int_{0}^{1}(1-t^2)^{n-\frac{1}{2}}e^{ixt}\,dt\right]$$

putting $-t$ for t in the first integral

$$=\frac{x^n}{2^{n-1}\,\Gamma(\frac{1}{2})\,\Gamma(n+\frac{1}{2})}\int_{0}^{1}(1-t^2)^{n-\frac{1}{2}}\,\frac{e^{ixt}+e^{-ixt}}{2}\,dt$$

$$=\frac{x^n}{2^{n-1}\,\Gamma(\frac{1}{2})\,\Gamma(n+\frac{1}{2})}\int_{0}^{1}(1-t^2)^{n-\frac{1}{2}}\cos(xt)\,dt,$$

which proves the result. In particular when $n=0$,

$$J_0(x)=\frac{2}{\pi}\int_0^1\frac{\cos(xt)}{\sqrt{(1-t^2)}}\,dt.$$

***Ex. 11.** *Show that*

$$J_n(x)=\frac{1}{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\left(\frac{x}{2}\right)^n\int_0^{\pi}\cos(x\sin\phi)\cos^{2n}\phi\,d\phi.$$

Solution. We know that **[Agra 65]**

$$\cos\alpha=1-\frac{\alpha^2}{(2)\,!}+\frac{\alpha^4}{(4)\,!}-\ldots+(-1)^r\frac{\alpha^{2r}}{(2r)\,!}+\ldots$$

Putting $\alpha=x\sin\phi$, we get

$$\cos(x\sin^r\phi)=\sum_0^\infty(-1)^r\frac{x^{2r}}{(2r)\,!}\sin^{2r}\phi.$$

Now $$(-1)^r\frac{x^{2r}}{(2r)\,!}\int_0^{\pi}\sin^{2r}\phi.\cos^{2n}\phi\,d\phi$$

$$=2\,(-1)^r\frac{x^{2r}}{(2r)\,!}\int_0^{\frac{1}{2}\pi}\sin^{2r}\phi\cos^{2n}\phi\,d\phi$$

$$=2\,(-1)^r\frac{x^{2r}}{(2r)\,!}\,\frac{\Gamma\left(\frac{2r+1}{2}\right)\Gamma\left(\frac{2n+1}{2}\right)}{2\Gamma(n+r+1)}$$

Hence $$\frac{1}{\sqrt{\pi}\Gamma(n+\frac{1}{2})}\left(\frac{x}{2}\right)^n\int_0^{\pi}\cos(x\sin\phi)\cos^{2n}\phi\,d\phi$$

$$=\frac{1}{\sqrt{\pi}\Gamma\left(\frac{2n+1}{2}\right)}\left(\frac{x}{2}\right)^n\int_0^{\pi}\sum_{r=0}^{\infty}(-1)^r\frac{x^{2r}}{(2r)\,!}\sin^{2r}\phi\cos^{2n}\phi\,d\phi$$

$$=\sum_{r=0}^{\infty}\frac{1}{\sqrt{x}\Gamma\left(\frac{2n+1}{2}\right)}\left(\frac{x}{2}\right)^n(-1)^r\,\frac{\Gamma\left(\frac{2r+1}{2}\right)\Gamma\left(\frac{2n+1}{2}\right)}{\Gamma(n+r+1)}$$

*This result can be obtained by putting $t=\sin\phi$ in Ex. 10.

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{(r)!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$
$$=J_n(x).$$

This proves the result.

Ex. 12. *Prove that*

$$J_n(x)=\frac{2}{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\left(\frac{x}{2}\right)^n \int_0^{\pi/2} \cos(x\cos\phi)\,(\sin\phi)^{2n}\,d\phi.$$

[Raj. 61]

Proceed as in the above example or put $t=\cos\phi$ in Ex. 10.

Ex. 13. *Prove that*

$$J_n=(-2)^n x^n \left[\frac{d}{d(x^2)}\right](J_0).$$

[Raj. 64, 62 ; Agra 58]

Solution. Recurrence formula V is

$$\frac{d}{dx}(x^{-n}J_n)=-x^{-n}J_{n+1}. \quad \text{...(1)}$$

When $n=0$,

$$\frac{d}{dx}(J_0)=-J_1$$

or $\frac{1}{x}\frac{d}{dx}(J_0)=-x^{-1}J_1.$

Differentiating again w.r t. x, we get

$$\frac{d}{dx}\left[\frac{1}{x}\frac{d}{dx}(J_0)\right]=-\frac{d}{dx}(x^{-1}J_1)$$
$$=(-1)^2 x^{-2}J_2.$$

so that $\left(\frac{1}{x}\frac{d}{dx}\right)^2(J_0)=(-1)^2 x^{-2}J_2.$

Proceeding similarly n times, we get

$$\left(\frac{1}{x}\frac{d}{dx}\right)^n(J_0)=(-1)^n x^{-n}J_n$$

i.e., $J_n=(-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n(J_0)$

$$=(-1)^n x^n \left(2\frac{d}{d(x^2)}\right)^n(J_0)$$
$$=(-2)^n x^n \left(\frac{d}{d(x^2)}\right)^n(J_0)$$

This proves the result.

Ex. 14. *Prove that*

(*i*) $\left(\frac{1}{x}\frac{d}{dx}\right)^m\{x^nJ_n(x)\}=x^{n-m}J_{n-m}(x)$

where $m<n$ and m is an integer.

(ii) $\left(\frac{1}{x}\frac{d}{dx}\right)^m \{x^{-n}J_n'(x)\}=(-1)^m x^{-n-m} J_{n+m}$.

Solution. (*i*) From recurrence formula VI, we have

$$\frac{d}{dx}\{x^n J_n'(x)\}=x^n J_{n-1}(x),$$

i.e., $\frac{1}{x}\frac{d}{dx}\{x^n J_n(x)\}=x^{n-1}J_{n-1}(x)$.

Also $\left(\frac{1}{x}\frac{d}{dx}\right)^2 \{x^n J_n(x)\}=\left(\frac{1}{x}\frac{d}{dx}\right)\{x^{n-1}J_{n-1}(x)$

$$=x^{n-2}J_{n-2}.$$

Thus if m is a positive integer less than n, we have

$$\left(\frac{1}{x}\frac{d}{dx}\right)^m \{x^n J_n(x)\}=x^{n-m}J_{n-m}(x).$$

(*ii*) It can be obtained as (*i*) from recurrence formula V.

8·7. To prove that

$$J_nJ_{-n}'-J_n'J_{-n}=-\frac{2\sin n\pi}{\pi x}. \qquad \text{[Agra 59]}$$

or $$\frac{d}{dx}\left(\frac{J_{-n}}{J_n}\right)=-\frac{2\sin n\pi}{\pi x J_n^2}. \qquad \text{[Agra 72]}$$

We know that J_n and J_{-n} are both solutions of Bessel's equation,

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{n^2}{x^2}\right)y=0$$

or $y''+\frac{1}{x}y'+\left(1-\frac{n^2}{x^2}\right)y=0.$

$\therefore J_n''+\frac{1}{x}J_n'+\left(1+\frac{n^2}{x^2}\right)J_n=$ ...(1)

and $J_{-n}''+\frac{1}{x}J_{-n}'+\left(1+\frac{n^2}{x^2}\right)J_{-n}=0.$ ...(2)

Multipiying (1) by J_{-n} and (2) by J_n and subtracting, we get

$(J_{-n}J_n''-J_nJ_{-n}'')+(1/x)(J_{-n}J_n'-J_nJ_{-n}')=0$ ...(3)

Now let $v=J_{-n}J_n'-J_nJ_{-n}'$,

$v'=J_{-n}J_n''+J_{-n}'J_n'-J_n'J_{-n}'-J_nJ_{-n}''$.

Hence (3) becomes

$$v'+\frac{1}{x}v=0 \quad \text{or} \quad \frac{v'}{v}=-\frac{1}{x}$$

Integrating, $\log v-\log\frac{c}{x}$ *i.e.* $v=\frac{c}{x}$,

where c is an arbitrary constant,

i.e. $J_n'J_{-n}-J_nJ_{-n}'=c/x$, ...(4)

Now comparing coefficient of lowest degree term from both the sides of (4), we get

$$\frac{1}{2^n\,\Gamma(n+1)}+\frac{1}{2^{-n}\,\Gamma(1-n)}\,[n-(-n)]=c$$

or $$c=\frac{2}{\Gamma(n+1)\,\Gamma(1-n)}=\frac{2}{\Gamma(n)\,\Gamma(1-n)}$$

or $$\frac{2}{\pi/\sin n\pi}=\frac{2\sin n\pi^*}{\pi}$$

Thus (4) gives

$$J_n'J_{-n}-J_nJ_{-n}'=\frac{2\sin n\pi}{\pi x}$$

or $$J_nJ_{-n}'-J_n'J_{-n}=-\frac{2\sin n\pi}{\pi x}.$$

This proves the required result.

Note. Because of the relation $J_{-n}(x)=(-1)^n J_n(x)$ when n i an integer, the functions $J_n(x)$ and $J_{-n}(x)$ are not independent Therefore in this case a second solution of Bessel's equation ca be found.

$^*\Gamma(n)\,\Gamma(1-n)=\frac{\pi}{\sin n\pi}$; see Author's Integral Caclulus for post-gra students, § 21 p. 67.