PART III

DIFFERENTIAL EQUATIONS

1

Linear Partial Differential Equations of Order One

1·1. Introduction.

In the theory of partial differential equations, a variable z is function of more than one independent variables. In case there are n independent variables, we take these as $x_1, x_2, x_3 \ldots x_n$. However the study is generally confined to when z is a function of two independent variables x and y; we write

$$z = f(x, y),$$

$\frac{\partial z}{\partial x}$, the partial differential coefficient of z w.r.t. x, is denoted by p, so that

$$p \equiv \frac{\partial z}{\partial x}.$$

Similarly $\frac{\partial z}{\partial y}$, the partial differential coefficient of z with respect to y, is denoted by q, so that

$$q \equiv \frac{\partial z}{\partial y}.$$

The second partial derivatives of z with respect to x and y are denoted by r, s and t, so that

$$r \equiv \frac{\partial^2 z}{\partial x^2},\ s \equiv \frac{\partial^2 z}{\partial x \partial y} \text{ and } t \equiv \frac{\partial^2 x}{\partial y^2}.$$

A partial differential equation is a relation between dependent variable, independent variables and partial derivatives of dependent variable with respect to the independent variables. For example,

$$x^2 p + y^2 q = z^2 \qquad \ldots(1)$$

$$r + (a+b)\, s + abt = xy \qquad \ldots(2)$$

are partial differential equations.

The order of a partial differential equation is determined by the highest order partial derivative in it. Thus (1) is a partial differential equation of order one and (2) is of order 2.

1·2. Origin of partial differentiale quations

The partial differential equations may be obtained in the following two ways :

I. Elimination of Arbitrary Constants. Let a function z of x, y be such that

$$\phi(x, y, z, a, b)=0.$$

Differentiating it partially w.r.t. x and y and then eliminating constants a, b differential equation is obtained.

Ex. 1. *Eliminate a, b from* $z=(x^2+a)(y^2+b)$. ...(1)

Solution. Differentiating w.r.t. x and y, we get

$$p=\frac{\partial z}{\partial x}=2x(y^2+b),\ q=\frac{\partial z}{\partial y}=(x^2+a)\,2y,$$

so that $pq=4xy(x^2+a)(y^2+b)=4xyz$.

Hence $pq=4xyz$ is the partial differential equation.

Note. In case the number of arbitrary constants are more than two, then three relations namely, the given relation and the two relations obtained by partially differentiating with respect to x and y, are not sufficient to eliminate these constants. Therefore, in this case we have to take relations involving higher derivatives and the differential equation would not be of order one. The following example would illustrate it.

Ex. 2. *Eliminate the constants a, b, and c, from the relation*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \qquad ...(1)$$

Solution. Differentiating partially with regard to x and y, we get

$$\frac{x}{a^2}+\frac{z}{c^2}p=0, \qquad ...(2)$$

and $$\frac{y}{b^2}+\frac{z}{c^2}q=0. \qquad ...(3)$$

There being three constants a, b, c these cannot be eliminated from (1), (2) and (3). Therefore we need one more relation. Differentiating (2) again partially with respect to x, we get

$$\frac{1}{a^2}+\frac{p^2}{c^2}+\frac{z}{c^2}r=0. \qquad ...(4)$$

Multiplying it by x and subtracting (2) from it, we get

$$\frac{1}{c^2}\{pz-xp^2-xzr\}=0,$$

or $pz=xp^2+xzr$.

This is the partial differential equation obtained after eliminating a, b and c and is of order 2.

Note. Other partial differential equations can also be obtained; for example, if we differentiate (4) with respect to y, we get

$$\frac{1}{b^2}+\frac{z}{c^2}\,t+\frac{1}{c^2}\,q^2=0.$$

Multiplying this by y and then subtracting from (3), we get

$$qz=yq^2+yzt.$$

II. Elimination of Arbitrary Functions. Let $u=u\ (x, y, z)$, $v=v\ (x, y, z)$ be two functions of x, y, z connected by the relation

$$\phi\ (u, v)=0. \qquad \ldots(1)$$

Regarding z as dependent variable and differentiating (1), partially w.r.t. x and y, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\,\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\,\frac{\partial v}{\partial z}\right)=0,$$

and $$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y}+q\,\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\,\frac{\partial v}{\partial z}\right)=0.$$

Eliminating $\frac{\partial \phi}{\partial u}, \frac{\partial \phi}{\partial v}$ from these, we get

$$\begin{vmatrix} \frac{\partial u}{\partial x}+p\,\frac{\partial u}{\partial z} & \frac{\partial v}{\partial x}+p\,\frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial y}+q\,\frac{\partial u}{\partial z} & \frac{\partial v}{\partial y}+q\,\frac{\partial v}{\partial z} \end{vmatrix}=0$$

or $$p\left(\frac{\partial u}{\partial z}\frac{\partial u}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right)+q\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}\right)$$
$$+\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)=0.$$

Denoting the expressions under the above three brackets by λP, λQ, and $-\lambda R$, this can be written as

$$Pp+Qq=R.$$

Ex. 1. *Find the differential equations from*

$$\phi\ (x+y+z,\ x^2+y^2-z^2)=0.$$

Solution. Let $u=x+y+z$, $v=x^2+y^2-z^2$.

Then the given equation is $\phi\ (u, v)=0$.

Differentiating it w.r.t., x partially, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\,\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\,\frac{\partial v}{\partial z}\right)=0$$

i.e., $$\frac{\partial \phi}{\partial u}\,(1+p)+\frac{\partial \phi}{\partial v}\,(2x-2zp)=0. \qquad \ldots(1)$$

Again differentiating w.r.t. y partially, we get

$$\frac{\partial \phi}{\partial u}\,(1+q)+\frac{\partial \phi}{\partial v}\,(2y-2zq)=0. \qquad \ldots(2)$$

Eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$, we get

$$(1+p)(2y-2zq)-(1+q)(2x-2zp)=0$$

i.e., $(y+z)p-(x+z)q=x-y$.

Exercises

1. Eliminate the constants a and b from the following equations :

(a) $z=(x+a)(y+b)$. **Ans.** $z=pq$

(b) $2z=(ax+y)^2+b$. **Ans.** $px+qy=q^2$

(c) $ax^2+by^2+z^2=1$. **Ans.** $z(px+qy)=z^2-1$

(d) $z=ax+by+cxy$ (c also).

Ans. $r=0$, $t=0$ or $z=px+qy-xys$.

2. Eliminate the arbitrary functions f and g from the following :

(a) $z=e^{mx}f(x+y)$. **Ans.** $p-q=mz$.

(b) $lx+my+nz=f(x^2+y^2+z^2)$.

Ans. $(l+np)y+(lq-mp)z=(m+nq)x$.

1·3. Linear Partial Differential Equations of Order One.

A differential equation involving partial derivatives p and q only and no higher is called of order one.

If, in addition, the degree (or power) of p and q is unity, then it is a *linear* partial differential equation of order one.

Thus $3xp+9yq=z$ and $px^3+qy^4=z^2$ are both linear partial differential equations of order one.

On the other hand, equations

$$p^2+q=z, \quad x+e^q=z^3$$

are not linear, although these are of order one.

Equation Pp+Qq=R is the standard form of the linear partial differential equation of order one.

1·4. Lagrange's Method. [Vikram 64]

The general solution of the linear partial differential equation

$$Pp+Qq=R \qquad \ldots(1)$$

is $\phi(u, v)=0$,

where ϕ *is an arbitrary function and* $u(x, y, z)=c_1$ *and* $v(x, y, z)=c_2$ *are solutions of equations*

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R},$$

where P, Q, R *are functions of* x, y *and* z.

We have seen in II on page 5 that equation (1) can be obtained by eliminating arbitrary function ϕ from $\phi(u, v)=0$.

And we have

$$\left.\begin{aligned} \lambda P&=\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}=\frac{\partial(u, v)}{\partial(y, z)} \\ \lambda Q&=\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}=\frac{\partial(u, v)}{\partial(z, x)} \\ \text{and } \lambda R&=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}=\frac{\partial(u, v)}{\partial(x, y)} \end{aligned}\right\} \qquad \ldots(2)$$

where $u=a$ and $v=b$ are two integrals of (1).

Differentiating these integrals we get

$$\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz=0$$

and $$\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial v}{\partial z}dz=0.$$

Solving these simultaneously for dx, dy, dz, we get

$$\frac{dx}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}}=\frac{dy}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}}=\frac{dz}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}$$

i.e., $\frac{dx}{\lambda P}=\frac{dy}{\lambda Q}=\frac{dz}{\lambda R}$ rom (2)

or $$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}. \quad ...(3)$$

These are called Lagrange's auxiliary equations. If solutions of (3) are $u=a$, $v=b$, then the solution of the given equation is

$\phi(u, v)=0$, $u=\phi(v)$.

Cor. If the equation is

$$P\frac{\partial z}{\partial x}+Q\frac{\partial z}{\partial y}+R\frac{\partial z}{\partial t}=S, \quad ...(4)$$

having three independent variables x, y, t, then the Lagrange's auxiliary equations are

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dt}{R}=\frac{dz}{S}. \quad ...(5)$$

If $u=a$, $v=b$, $w=c$ are three independent solutions of (5) then general solution of (4) is $\phi(u, v, w)=0$. Thus can be generalized to any number of independent variables.

Ex. 1. *Solve* $xzp+yzq=xy$. [Karnatak M.Sc. 61; Vikram 64]

Solution. Auxiliary equations are

$$\frac{dx}{xz}=\frac{dy}{yz}=\frac{dz}{xy}.$$

From first two, $\frac{dx}{x}=\frac{dy}{y}.$

Integrating $\log x=\log y+\log c_1$ or $x/y=c_1$.

Similarly from the last two, $y/z=c_2$.

Hence the general solution is

$\phi(x/y, y/z)=0$.

Ex. 2. *Solve* $\frac{y^2z}{x}p+xzq=y^2$. [Agra 67, 54; Raj. 55]

Solution. The equation is $y^2zp+x^2zq=xy^2$.

Auxiliary equations are $\frac{dx}{y^2z}=\frac{dy}{x^2z}=\frac{dz}{xy^2}$.

From first two, $x^2\,dx=y^2\,dy$, *i.e.* $x^3-y^3=c$.

From first and third, $\frac{dx}{z}=\frac{dz}{x}$,

i.e., $x\,dx=z\,dz$ or $x^2-z^2=c_2$.

Hence $\phi(x^3-y^3,\ x^2-z^2)=0$ is the complete solution.

Ex. 3. *Solve* $(x^2-yz)\,p+(y^2-zx)\,q=(z^2-xy)$.

[Agra 65 ; Karnatak 63]

Solution. Auxiliary equations are

$$\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy}.$$

This gives

$$\frac{dx-dy}{x-y}=\frac{dy-dz}{y-z}=\frac{dz-dx}{z-x}.$$

The first two give

$$\frac{x-y}{y-z}=c_1$$

and from the last two, we get

$$\frac{y-z}{z-x}=c_2.$$

Hence $\phi\left(\frac{x-y}{y-z},\ \frac{y-z}{z-x}\right)=0$ is the complete solution.

***Ex. 4.** *Solve* $(y+z)\,p+(z+x)\,q=x+y$. [Agra 61, 78]

Solution. Lagrange's auxiliary equations are

$$\frac{dx}{y+z}=\frac{dy}{z+x}=\frac{dz}{x+y}=\frac{dx-dy}{y-x}=\frac{dy-dz}{z-y}=\frac{dx+dy+dz}{2\,(x+y+z)}. \quad \ldots(1)$$

Now from these, the two solutions are

$$\frac{y-x}{z-y}=c_1,\ (y-z)^2\,(x+y+z)=c_2$$

Therefore solution of the given equation is

$$\phi\left[\frac{y-x}{z-y},\ (y-z)^2\,(x+y+z)\right]=0.$$

Ex. 5. *Solve* $(mz-ny)\,p+(nx-lz)\,q=ly-mx$. [Delhi Hons. 68]

Solution. The auxiliary equations are

$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx},$$

using multipliers x, y, z, we get

$$x\,dx+y\,dy+z\,dz=0 \text{ giving } x^2+y^2+z^2=c_1.$$

Again using multipliers l, m, n, we get

$l\,dx+m\,dy+n\,dz=0$ giving $lx+my+nz=c_2$.

Hence $\phi\,(lx+my+nz,\ x^2+y^2+z^2)=0$

is the complete solution.

Ex. 6. $(z^2-2yz-y^2)\,p+(xy+xz)\,q=xy-xz$. [Agra 49]

Solution. Auxiliary equations are

$$\frac{dx}{z^2-2yz-y^2}=\frac{dz}{x\,(y+z)}=\frac{dz}{x\,(y-z)}.$$

Using multipliers x, y, z respectively, we get

$x\,dx+y\,dy+z\,dz=0$.

Integrating, $x^2+y^2+z^2=c_1$.

Also $\dfrac{dy}{x\,(y+z)}=\dfrac{dz}{x\,(y-z)}$ gives $\dfrac{dy}{y+z}=\dfrac{dz}{y-z}$,

i.e. $y\,dy-(z\,dy+y\,dz)-z\,dz=0$; integrating, $y^2-2yz-z^2=c_2$.

Hence the general integral is

$\phi(x^2+y^2+z^2,\ y^2-2yz-z^2)=0$.

Ex. 7 (a). *Solve* $\dfrac{y-z}{yz}\,p+\dfrac{z-x}{zx}\,q=\dfrac{x-y}{xy}$ [Agra 69, 66, 56]

Solution. The equation after multiplying by xyz is

$x\,(y-z)\,p+y\,(z-x)\,q=(x-y)\,z$.

Auxiliary equations are $\dfrac{dx}{x\,(y-z)}=\dfrac{dy}{y\,(z-x)}=\dfrac{dz}{z\,(x-y)}$

$$=\frac{dx+dy+dz}{0}=\frac{dx/x+dy/y+dz/z}{0}.$$

Now the two integrals are

$x+y+z=c_1,\ xyz=c_2$.

$\therefore$ $\phi\,x+y+z,\ xyz)=0$ is the solution.

Ex. 7. (b). *Solve* $x\,(y-z)\,p+y\,(z-x)\,q=z\,(x-y)$. [Raj. 70]

Just the above example.

Ex. 8. *Solve* $x(y^2+z)\,p-y\,(x^2+z)\,q=z\,(x^2-y^2)$.

Solution. The auxiliary equations are

$$\frac{dx}{x\,(y^2+z)}=\frac{dy}{-y\,(x^2+z)}=\frac{dz}{z\,(x^2-y^2)}.$$

Using multipliers, x, y, -1 and $1/x$, $1/y$, $1/z$, we get

$$x\,dx+y\,dy-z\,dz=0 \text{ and } \frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0.$$

Integrating, we get

$x^2+y^2-2y=c_1$ and $xyz=c_2$.

$\therefore$ $\phi(x^2+y^2-2z,\ xyz)=0$ is the solution.

***Ex. 9.** *Solve* $(y^2+z^2-x^2)\,p-2xyq+2xz=0$.

[Guru Nanak 73; Meerut 68; Delhi Hons. 68; Agru 57; Vikram 62]

Solution. Auxiliary equations are

$$\frac{dx}{y^2+z^2-x^2}=\frac{dz}{-2xy}=\frac{dz}{-2xz}.$$

From the last two, we get

$$\frac{dy}{y}=\frac{dz}{z}.$$

On integration, one solution of auxiliary equations is

$$y/z=c_1.$$

Next using x, y, z as multipliers, we get

$$\frac{dx}{x^2-y^2-z^2}=\frac{dy}{2xy}=\frac{dz}{2xz}=\frac{x\,dx+y\,dy+z\,dz}{x\,(x^2+y^2+z^2)}$$

∴ from the last two of these

$$\frac{dz}{z}=\frac{2\,(x\,dx+y\,dy+z\,dz)}{x^2+y^2+z^2}.$$

Integrating, we get

$$x^2+y^2+z^2=c_2 z.$$

Hence $\phi\left(\frac{y}{z}, \frac{x^2+y^2+z^2}{z}\right)=0$ is the required solution.

***Ex. 10.** *Solve $p\cos(x+y)+q\sin(x+y)=z$.*

[Agra 63; Raj. 63, 58, 64]

Solution. Lagrange's auxiliary equations are

$$\frac{dx}{\cos(x+y)}=\frac{dy}{\sin(x+y)}=\frac{dz}{z}. \quad ...(1)$$

Now to find two integrals of (1), we have

$$\frac{dx+dy}{\cos(x+y)+\sin(x+y)}=\frac{dx-dy}{\cos(x+y)-\sin(x+y)}$$

or $$\frac{[-\sin(x+y)+\cos(x+y)]\,(dx+dy)}{\cos(x+y)+\sin(x+y)}=dx-dy.$$

Now numerator on the left is defferential of the denominator : integrating, $\log[\cos(x+y)+\sin(x+y)]=x-y+\log a$

or $$[\cos(x+y)+\sin(x+y)]\,e^{y-x}=a. \quad ...(2)$$

Again $$\frac{dx+dy}{\cos(x+y)+\sin(x+y)}=\frac{dz}{z}$$

i.e., $$\frac{dx+dy}{\sqrt{2}\sin(x+y+\frac{1}{4}\pi)}=\frac{dz}{z}$$

or $$\frac{1}{\sqrt{2}}\operatorname{cosec}(x+y+\tfrac{1}{4}\pi)\,d\,(x+y)=\frac{dz}{z}.$$

Integrating, $$\frac{1}{\sqrt{2}}\log\tan[\tfrac{1}{2}(x+y)+\tfrac{1}{8}\pi]=\log z+\log b.$$

$$\therefore\ \tan[\tfrac{1}{2}(x+y)+\tfrac{1}{8}\pi]\,z^{-\sqrt{2}}=b^{\sqrt{2}}. \quad ...(3)$$

Hence the solution of given equation is

$$\phi[\{\cos(x+y)+\sin(x+y)\}\, e^{y-x},\ z^{-\sqrt{2}}\tan\{\tfrac{1}{2}(x+y)+\tfrac{1}{8}\pi\}]=0.$$

***Ex. 11.** *Solve* $(y^3x-2x^4)\,p+(2y^4-x^3y)\,q=9z\,(x^3-y^3).$

[Agra 60, 61 ; Raj 52]

Solution. Auxiliary equations are

$$\frac{dx}{y^3x-2x^4}=\frac{dy}{2y^4-x^3y}=\frac{dz}{9z\,(x^3-y^3)}$$

$$=\frac{dx/x+dy/y+dz/3z}{0}.$$

Hence $\dfrac{dx}{x}+\dfrac{dy}{y}+\dfrac{dz}{3z}=0.$

Integrating, $xyz^{1/3}=c_1,$ ...(1)

is one solution of auxiliary equations.

Next from the first two terms of auxiliary equations, we have

$$(2y^4+x^3y)\,dx-(y^3x-2x^4)\,dy=0$$

or $y^3\,(2y\,dx-x\,dy)-x^3\,(y\,dx-2x\,dy)=0.$

By trial the

$$\text{I.F.}=\frac{1}{x^3y^3}.$$

Hence multiplying by $\dfrac{1}{x^3y^3}$, the equation becomes

$$\left(\frac{2y}{x^3}-\frac{1}{y^2}\right)dx-\left(\frac{1}{x^2}-\frac{2x}{y^3}\right)dy=0.$$

This is exact now.

Its integral is

$$\frac{y}{x^2}-\frac{x}{y^2}=c_2.$$

Hence $\phi\left(xyz^{1/3},\ \dfrac{y}{x^2}-\dfrac{x}{y^2}\right)=0$ is the solution.

Ex. 12. *Solve* $z-xp-yq=a\sqrt{(x^2+y^2+z^2)}.$

[Agra 70; Raj. 49; Nag. 61]

Solution. The equation can be written as

$$xp+yq=z-a\sqrt{(x^2+y^2+z^2)}.$$

The auxiliary equations are

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z-a\sqrt{(x^2+y^2+z^2)}}$$

$$=\frac{x\,dx+y\,dy+z\,dz}{x^2+y^2+z^2-az\sqrt{(x^2+y^2+z^2)}}.$$

Putting $x^2-y^2=z^2+t^2$ in

$$\frac{dz}{z-a\sqrt{(x^2+y^2+z^2)}}=\frac{x\,dx+y\,dy+z\,dz}{x^2+y^2+z^2-az\sqrt{(x^2+y^2+z^2)}},$$

we get $\dfrac{dz}{z-at}=\dfrac{t\,dt}{t^2-azt}$ or $\dfrac{dz}{z-at}=\dfrac{dt}{t-az}$.

Thus $\dfrac{dx}{x}=\dfrac{dy}{y}=\dfrac{dz}{z-at}=\dfrac{dt}{t-az}=\dfrac{dz+dt}{(1-a)(t+z)}$,

so that $\dfrac{dx}{x}=\dfrac{dy}{y}$ gives $y=c_1x$, ...(1)

and $\dfrac{dx}{x}=\dfrac{dz+dt}{(1-a)(z+t)}$ gives $x^{1-a}=c_2(z+t)$,

i.e., $x^{1-a}=c_2[z+\sqrt{(x^2+y^2+z^2)}]$. ...(2)

Therefore from (1) and (2), the general solution is

$$\phi\left(\frac{y}{x}, \frac{x^{1-a}}{z+\sqrt{(x^2+y^2+z^2)}}\right)=0.$$

Ex. 13. $(2x^2+y^2+z^2-2yz-zx-xy)\,p$
$+(x^2+2y^2+z^2-yz-2zx-xy)\,q=x^2+y^2+2z^2-yz-2xy.$

Solution. Auxiliary equations are

$$\frac{dx}{2x^2+y^2+z^2-2yz-zx-xy}=\frac{dy}{x^2+2y^2+z^2-yz-2zx-xy}$$
$$=\frac{dz}{x^2+y^2+2z^2-yz-zx-2xy},$$

so that $$\frac{dx-dy}{x^2-y^2-yz+zx}=\frac{dy-dz}{y^2-z^2-zx+xy}$$

i.e., $$\frac{dx-dy}{(x-y)(x+y+z)}=\frac{dy-dz}{(y-z)(x+y+z)}$$

or $$\frac{dx-dy}{x-y}=\frac{dy-dz}{y-z}.$$

Integrating, $\log(x-y)=\log(y-z)+\log c_1$ or $\dfrac{x-y}{y-z}=c_1$.

Similarly, $\dfrac{z-x}{y-z}=c_2$.

Hence $\phi\left(\dfrac{x-y}{y-z}, \dfrac{z-x}{y-z}\right)=0$ is the solution.

Ex. 14. *Solve* $x^2p+y^2q=z^2$. **[Raj. 51]**

Solution. Auxiliary equations are $\dfrac{dx}{x^2}=\dfrac{dy}{y^2}=\dfrac{dz}{z^2}$.

From first two, $\dfrac{1}{x}=\dfrac{1}{y}+c_1$ or $\dfrac{1}{x}-\dfrac{1}{y}=c_1$, From last two, $\frac{1}{y}-\frac{1}{z}=c_2$.

Hence $\phi\left(\frac{1}{x}-\frac{1}{y}, \frac{1}{y}-\frac{1}{z}\right)=0$ is the solution.

Ex. 15. *Solve* $pz-qz=z^2+(x+y)^2$. **[Lucknow 54]**

Solution. Auxiliary equations are $\frac{dx}{z}=\frac{dy}{-z}=\frac{dz}{z^2+(x+y)^2}$.

Now $\frac{dx}{z}=\frac{dy}{-z}$ gives $dx+dy=0$ *i.e.* $x+y=c_1$,

Also $\frac{dx}{z}=\frac{(x+y)[dx+dy]+z\,dz}{(x+y)[z-z]+z[z^2+(x+y)^2]}$

i.e., $\frac{dx}{z}=\frac{(x+y)\;dx+dy]+z\,dz}{z\,[z^2+(x+y)^2]}$

or $2\,dx=\frac{2(x+y)(dx+dy)+2z\,dz}{z^2+(x+y)^2}$.

Integrating, $2x+c_2=\log[(x+y)^2+z^2]$. ...(2)

Hence the complete integral is

$\phi\,[x+y, 2x-\log\{(x+y)^2+z^2\}]=0$.

Ex. 16. *Solve* $x^2(y-z)p+y^2(z-x)q=z^2(x-y)$.

Solution. Auxiliary equations are

$$\frac{dx}{x^2(y-z)}=\frac{dy}{y^2(z-x)}=\frac{dz}{z^2(x-y)}$$

Using multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, we get

$$\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0 \text{ or } xyz=c_1. \qquad ...(1)$$

Again using multipliers $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$, we get

$$\frac{dx}{x^2}+\frac{dy}{y^2}+\frac{dz}{z^2}=0, \text{ i.e., } \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=c_2.$$

Hence $\phi\left(xyz, \frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=0$ is the solution.

Ex. 17. $(x^3+3xy^2)p+(y^3+3x^2y)q=2(x^2+y^2)z$.

Solution. Auxiliary equations are

$$\frac{dx}{x^3+3xy^2}=\frac{dy}{y^3+3x^2y}=\frac{dz}{2(x^2+y^2)z},$$

so that $\frac{dx/x+dy/y}{4(x^2+y^2)}=\frac{dz}{2(x^2+{}^2)\;z}$ o $\frac{dx}{x}+\frac{dy}{y}=\left(\frac{dz}{2(x^2+y^2)z}\right)$.

Integrating, $\log xy=\log z^2+\log c_1$ *i.e.* $\frac{xy}{z^2}=c_1$

Again $\frac{dx+dy}{(x+y)^3}=\frac{dx-dy}{(x-y)^3}$ or $\frac{1}{(x+y)^2}-\frac{1}{(x-y)^2}=c_2$.

Hence $\phi\left(\frac{xy}{z^2}, \frac{1}{(x+y)^2}-\frac{1}{(x-y)^2}\right)=0$ is the solution.

Ex. 18. $(x+2z)\,p+(4zx-y)\,y-2x^2+y$.

Solution. The auxiliary equations are

$$\frac{dx}{x+2z}=\frac{dy}{4zx-y}=\frac{dz}{2x^2+y}.$$

Using multipliers y, x and $-2z$, we get

$$y\,dx+x\,dy-2z\,dz=0, \text{ i.e. } xy-z^2=c_1.$$

Again using multipliers $2x$, -1, -1, we get

$$2x\,dx-dy-dz=0, \text{ i.e. } x^2-y-z=c_2.$$

Hence $\phi(xy-z^2, x^2-y-z)=0$ is the soluion.

Exercises

Solve the following differential equations :

1. $x^2p+y^2q=(x+y)\,z$.

Ans. $\phi\left(\frac{xy}{z}, \frac{x-y}{z}\right)=0$.

2. $(3x+y-z)\,p+(x+y-z)\,q=2\,(z-y)$.

Ans. $\phi\left(\frac{x-y+z}{\sqrt{(x+y-z)}}, x-3y-z\right)=0$

3. $(y^2+z^2)\,p-xyz+xz=0$

Ans. $\phi(yz, x^2y^2+y^4)=0$.

4. $z\,(z^2-xy)\,(px-qy)=x^4$.

Ans. $\phi\,((z^2-xy)^2-x^4, xy)=0$.

5 $x\,(y^n-z^n)\,p+y\,(z^n-x^n)\,q=z\,(x^n-y^n)$.

Ans. $\phi\,\{xyz, x^n+y^n+z^n\}=0$.

6. $(x+y-z)\,(p-q)+a\,(px-qy+x-y)=0$. **[Delhi Hons. 69]**

7. $z-px-qy=a\sqrt{(x^2+y^2+z^2)}$. **[Agra 1970]**

1·5. Lagrange's method for more than two independent variables.

We have developed Lagrange's method for two independent variables. This can be extended to the case of n independent variables also in a simple way. Let z be a function of n independent variables $x_1, x_2, \ldots x_n$. We have the notations

$$\frac{dz}{dx_i}=p_i,\ i=1, 2, \ldots n.$$

Lagrange's equation can be now written as

$$P_1p_1+P_2p_2+\ldots+P_np_n=R, \qquad \ldots(1)$$

where $P_1, P_2, \ldots P_n, R$ are functions of $z, x_1, \ldots x_n$.

To solve (1), we find n independent solutions of auxiliary equations,

$$\frac{dx_1}{P_1}=\frac{dx_2}{P_2}=\ldots=\frac{dx_n}{P_n}=\frac{dz}{R}.$$

If these solutions are $u_1=c_1, \ldots, u_n=c_n$, then the complete solution of (1) is

$$\phi(u_1, u_2, \ldots, u_n)=0,$$

where ϕ is an arbitrary function.

Ex. 1. *Solve* $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}+t\frac{\partial z}{\partial t}=az+\frac{xy}{t}$. **[Agra 62]**

Solution. The auxiliary equations are

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dt}{t}=\frac{dz}{az+xy/t}.$$

Now $\frac{dx}{x}=\frac{dy}{y}$ and $\frac{dx}{x}=\frac{dt}{t}$ give

$y=c_1x$, ...(1) and $t=c_2x$. ...(2)

Again $\frac{dx}{x}=\frac{dz}{az+xy/t}$ gives $\frac{dz}{dx}-a\frac{z}{x}=\frac{c_1}{c_2}$.

Linear, I.F. $=e^{-\int a/x\, dx}=x^{-a}$.

Hence $zx^{-a}=c_3+\int\frac{c_1}{c_2}x^{-a}\,dx$

$$=c_3+\frac{c_1}{c_2}\frac{x^{1-a}}{1-a}$$

or $\frac{z}{x^a}=c_3+\frac{y}{t}\frac{x^{1-a}}{1-a}$ ...(3) as $\frac{c_1}{c_2}=\frac{y}{t}$

(1), (2) and (3) are three integrals of auxiliary equations; hence the solution of the given equation is

$$\phi\left(\frac{y}{x}, \frac{t}{x}, \frac{z}{x^a}-\frac{y}{t}\frac{x^{1-a}}{1-a}\right)=0.$$

Ex. 2. *Solve* $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=xyz$.

Solution. The auxiliary equations are

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}=\frac{du}{xyz}.$$

From first two, we have $x/y=c_1$. ...(1)

From second and third, we have $y/z=c_2$

Again, we have $\frac{yz\,dx+xz\,dy+xy\,dz}{3xyz}=\frac{du}{xyz}$

i.e., $yz\,dx+xz\,dy+xy\,dz=3\,du$.

Integrating $xyz=3u+c_3$. ...(3)

Hence the general integral is

Ex. 3. $(t+y+z)\frac{\partial t}{\partial x}+(t+z+x)\frac{\partial t}{\partial y}+(t+x+y)\frac{\partial t}{\partial z}=x+y+z$.

Solution. The auxiliary equations are

$$\frac{dx}{t+y+z}=\frac{dy}{t+z+x}=\frac{dz}{t+x+y}=\frac{dt}{x+y+z}.$$

$$\therefore \quad \frac{dx-dy}{-(x-y)}=\frac{dy-dz}{-(y-z)}=\frac{dz-dt}{-(z-t)}=\frac{dx+dy+dz+dt}{3(x+y+z+t)}.$$

First two terms give

$$-\log(x-y)=-\log(y-z)+\log c_1.$$

$$\therefore \quad \frac{y-z}{x-y}=c_1. \qquad \ldots(1)$$

Similarly from second and third terms, we get

$$\frac{t-z}{y-z}=c_2 \qquad \ldots(2)$$

The last two terms give

$$\log c_3-\log(t-z)=\tfrac{1}{3}\log(x+p+z+t)$$

i.e., $(x+y+z+t)^{1/3}\ t-z)=c_3.$ ...(3)

Therefore the general integral is

$$\phi\left[\frac{y-z}{x-y}, \frac{t-z}{y-z}, (x+y+z+t)^{1/3}(t-z)\right]=0.$$

Exercises

Solve the following partial differential equations :

1. $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}+\frac{\partial z}{\partial t}\{1+\sqrt{\{x+y+t+z\}}\}+3=0.$

Ans, $\phi(z+3x, z+3y, z+6\sqrt{\{x+y+t+z\}}=0.$

2. $(x_3-x_2)\, p_1+x_2p_2-x_3p_3=x_2(x_1+x_3)-x_2^2.$

Ans. $\phi(x_1+x_2+x_3, z-x_1x_2, x_2x_3\}=0.$

Integral surfaces through a given curve.

Ex. *Find the integral surface of the linear partial differential equation*

$$x(y^2+z)\,p-y(x^2+z)\,q=(x^2-y^2)\,z, \qquad \ldots(1)$$

which contains the line $x+y=0,\ z=1.$ ...(2)

Solution. The auxiliary equations are

$$\frac{dx}{x(y^2+z)}=\frac{dy}{-y(x^2+z)}=\frac{dz}{z(x^2-y^2)}.$$

The two solutions are $xyz=c_1$ (3)

and $x^2+y^2-2z=c_2.$ ...(4)

From (2), $z=1$. $\therefore$ (3) and (4) give $xy=c_1,$

and $x^2+y^2-2=c_2.$

We have to determine relation in c_1 and c_2 so that $x+y=0$

Now $(x+y)^2=0=x^2+y^2+2xy$

$$=2+c_2+2c_1 \quad \text{or} \quad 2c_1+c_2+2=0.$$

Therefore the required surface is

$$2xyz+x^2+y^2-2z+2=0.$$

Exercises

1. Find the equation of the integral surface of the dtfferential equation $2y(z-3)\,x+(2x-z)\,q=y\,(2x-3)$
which passes through the circle $z=0$. $x^2+y^2=2x$.

Ans. $x^2+y^2-2x=z^2-4z.$

2. Find the integral surface of the equation
$(x-y)\,y^2p+(y-x)\,x^2q+(x^2+y^2)z$
which passes through the curve $xz=a^3, y=0$.

Ans. $z^3\,(x^3+y^3)^2=a^9\,(a-y)^3.$

3. Find the integral surface of the equation
$(x-y)\,p+(y-x-z)\,q=z$
which contains the circle $z=1, x^2+y^2=1$.

Ans. $(x-y+z)^2+z^4\,(x+y+z)^2-2z^2\,(x-y+z)$
$-2z^4\,(x+y+z)=0.$

2

Non-Linear Partial Differential Equations of Order One

2·1. Introduction :

Before coming to integration of partial differential equations of order one, we give a few definitions and general theory.

Classification of Integrals. [Delhi Hon's 70]

As is already understood, the integration of a differential equation is the derivation of all values of z in terms of independent variables which identically satisfy the differential equation.

We now come to the various classes of integrals of a partial differential equation. Definitions and proofs are given for an equation involving only two independent variables. However, the results can be easily generalized for an equation involving n independent variables.

Complete Integral. Suppose that a relation between z, x and y be written as

$$f(z, x, y, a, b)=0, \quad \text{...(1)}$$

where a and b are arbitrary constants, it being free from differential coefficients of z.

Differentiating (1) partially w.r.t. x and y respectively, we get

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\,p=0 \text{ and } \frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\,q=0. \quad \text{...(2)}$$

There being only two arbitrary constants in above three equations, these can be eliminated and single relation

$$\phi(z, x, y, p, q)=0. \quad \text{...(3)}$$

can be obtained involving z, x, y and the derivatives p and q and free from a and b.

It is evident that (1) satisfies identically the partial differential equation (3) or that (1) is an integral of (3), having greatest number of arbitrary constants which can be expected in a solution of (3). Here (1) is complete integral of (3).

Complete integral of a partial differential equation of the form $\phi(z, x, y, p, q)=0$ is a relation between the variables z, x and y which includes as many arbitrary constants as there are independent variables.

A **particular integral** of a differential equation is obtained by giving particular values of arbitrary constants in complete integral.

2·2. Singular Integral.

As earlier let there be a relation in variables z, x and y given by

$$f(z, x, y, a, b)=0 \quad \ldots(1)$$

If a and b are constants, then differentiating partially w.r.t. x and y,

$$\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}=0 \text{ and } \frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}=0. \quad \ldots(2)$$

and the differential equation satisfied by (1) is

$$\phi(z, x, y, p, q)=0, \quad \ldots(3)$$

which is free from a and b. Now let us suppose that a and b are not constants but functions of x and y such that equations (2) which have been derived from (1) still hold; then on elimination of a and b we shall again get (3). This is because the algebraic elimination takes no cognisance of values of a and b but only their form.

But when a and b are functions of x and y, we get on differentiating (1) partially w.r.t. x and y respectively,

$$\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}+\frac{\partial f}{\partial a}\frac{\partial a}{\partial x}+\frac{\partial f}{\partial b}\frac{\partial b}{\partial x}=0,$$

and $$\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}+\frac{\partial f}{\partial a}\frac{\partial a}{\partial y}+\frac{\partial f}{\partial b}\frac{\partial b}{\partial y}=0.$$

Since a and b are such that (2) hold, these give

$$\left.\begin{aligned}\frac{\partial f}{\partial a}\frac{\partial a}{\partial x}+\frac{\partial f}{\partial b}\frac{\partial b}{\partial x}=0\\ \frac{\partial f}{\partial a}\frac{\partial a}{\partial y}+\frac{\partial f}{\partial b}\frac{\partial b}{\partial x}=0\end{aligned}\right\} \quad \ldots(4)$$

Solving these for $\frac{\partial f}{\partial a}$ and $\frac{\partial f}{\partial b}$, we get

$$\left(\frac{\partial a}{\partial x}\frac{\partial b}{\partial y}-\frac{\partial b}{\partial x}\frac{\partial a}{\partial y}\right)\frac{\partial f}{\partial a}=0,$$

and $$\left(\frac{\partial a}{\partial x}\frac{\partial b}{\partial y}-\frac{\partial b}{\partial x}\frac{\partial a}{\partial y}\right)\frac{\partial f}{\partial b}=0.$$

Now if $$\frac{\partial a}{\partial x}\frac{\partial b}{\partial y}-\frac{\partial b}{\partial x}\frac{\partial a}{\partial y}=\begin{vmatrix}\frac{\partial a}{\partial x} & \frac{\partial a}{\partial y}\\ \frac{\partial b}{\partial x} & \frac{\partial b}{\partial y}\end{vmatrix}\neq 0.$$

then these give

$$\left.\begin{aligned}\frac{\partial f}{\partial a}=0\\ \frac{\partial f}{\partial b}=0\end{aligned}\right\} \quad \ldots(5)$$

If these equations determine the values of a and b in any of the possible forms (constants or functions of x and y), the relation (1) would still be a solution of (3). Since a and b are not arbitrary constants now the new solution which has no arbitrary constants would in general be different from the complete integral which has two arbitrary constants and is called singular solution of differential equation (this in general cannot be obtained for any particular value of arbitrary constants in complete integral).

Thus singular solution is obtained by eliminating a and b from (1) and (5)

Note. Sometimes singular integral also occurs as a particular integral from the complete integral.

General Integral. As in singular integral, (1) would satisfy (3) if

$$R=\begin{vmatrix} \frac{\partial a}{\partial x} & \frac{\partial a}{\partial y} \\ \frac{\partial b}{\partial x} & \frac{\partial b}{\partial y} \end{vmatrix}=0.$$

But $R=0$ implies that there is a functional dependence in a and b and this dependence can be arbitrary.

If $\quad b=\psi(a)$ $\quad$ ...(6)

is the functional dependence relation in a and b, then multiplying equations in (4) by dx and dy and adding, we get

$$\frac{\partial f}{\partial a}da+\frac{\partial f}{\partial b}db=0. \quad ...(7)$$

Also from (6) $\quad db=\frac{\partial \phi}{\partial a}da$

$$\therefore \frac{\partial f}{\partial a}+\frac{\partial f}{\partial b}\frac{\partial \psi}{\partial a}=0.$$

This determines a involving the arbitrary function ψ. Thus b can be determined by (6). Eliminating a and b thus from these relations and (1), we get a solution of (3), which is in general different from complete integral as well as singular integral. This solution is called General Integral, and it is a relation between the variables involving one (one less than the number of independent variables) independent funtion of those variables together with an arbitrary function of this one function.

Note. Usually, but not universally, the above three classes of integrals namely, complete, singular and general integrals exhaust all possible solutions of a given differential equation. In case there exists an integral which is none of the above types, it is called **special integral.**

2·3. Theorem. *Every solution (which is not special) of the differential equation*

$$\phi(z, x, y, p, q)=0 \quad ...(1)$$

is included in one or other of three classes of solutions which are complete, singular and general integrals. [Delhi 70]

Proof. Let the complete integral of (1) be

$f(z, x, y, a, b)=0.$

Then the singular integral is given by

$$\frac{\partial f}{\partial a}=0, \quad \frac{\partial f}{\partial b}=0,$$

and the general integral is given by

$b=\psi(a).$

We shall show that no other integral exists. Let us suppose that there exists one given by

$g(z, x, y)=0.$

A value of z derived from (2) would be denoted by Z and that derived by (5) would be denoted by ξ.

Now suppose that it is possible to select values of a, b whether variable or constant, so that Z while satisfying the partial differential equation, is equal to ξ in terms of x and y. In that case of p and q derived from Z and ξ are the same and we get

$$\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}=0, \quad \frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}=0.$$

as well as

$$\frac{\partial g}{\partial x}+p\frac{\partial g}{\partial z}=0, \quad \frac{\partial g}{\partial y}+q\frac{\partial g}{\partial z}=0.$$

Equating values of p and q from these, we get

$$\left.\begin{aligned}&\frac{\partial f}{\partial x}\frac{\partial g}{\partial z}-\frac{\partial f}{\partial z}\frac{\partial g}{\partial x}=0\\ \text{and}\quad &\frac{\partial f}{\partial y}\frac{\partial g}{\partial z}-\frac{\partial f}{\partial z}\frac{\partial g}{\partial y}=0\end{aligned}\right\} \quad \ldots(6)$$

Now there arise two cases :

I. When equations (6) do not determine values of a and b, we cannot proceed further and Z cannot be modified to take value equal to ξ. The integral ξ is then called special.

II. When the two equations determine values of a and b, we modify Z as follows. Since (5) is a solution of (1), we have

$\phi(\xi, x, y, p, q)=0,$

and since (2) is a solution,

$\phi(Z, x, y, p, q)=0.$

The last equation is also satisfied, when the quantities a and b instead of being arbitrary constants, are functions of the variables satisfying (3) or (4). We may therefore replace a and b by the functions of x and y obtained as their values from the equations (6), provided, the necessary conditions be satified. In this case, the values of p and q are the same for the two forms of the equation (2) : and then from a comparison of these two forms, we always get $\xi=Z.$

where in the integral equation for Z, the constants a and b are changed into the values that have been derived from them.

The forms of p and q for the new values of a and b, would remain unchanged provided in addition to (6) two equations

$$\frac{\partial f}{\partial z}\frac{\partial g}{\partial x}=-\frac{\partial f}{\partial z}\frac{\partial g}{\partial z}p$$

$$=\frac{\partial g}{\partial z}\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial a}\frac{\partial a}{\partial x}+\frac{\partial f}{\partial b}\cdot\frac{\partial b}{\partial x}\right)$$

and $$\frac{\partial f}{\partial z}\frac{\partial g}{\partial y}=\frac{\partial g}{\partial z}\left(\frac{\partial f}{\partial y}+\frac{\partial f}{\partial a}\frac{\partial a}{\partial y}+\frac{\partial f}{\partial b}\frac{\partial b}{\partial y}\right)$$

are also satisfied. Therefore values of a and b are such as to satisfy the equations :

$$\frac{\partial f}{\partial a}\frac{\partial a}{\partial x}+\frac{\partial f}{\partial b}\frac{\partial b}{\partial x}=0,$$

$$\frac{\partial f}{\partial a}\frac{\partial a}{\partial y}+\frac{\partial f}{\partial b}\frac{\partial b}{\partial y}=0.$$

But these are the equations (compare from (4) under singular solutions) which help us in going from complete integral to singular or general integral. Therefore the values of a and b are the same which give singular or general integral according as $R\neq 0$, $R=0$.

Thus we necessarily have

$$\xi=Z,$$

i.e., value of z derived from (5) is always included in complete, singular or general integrals.

This proves the theorem.

2·4. Charpit's Method. [Pb. 60, 62; Agra 60; Raj. 62; Delhi Hon's 71; Vikarm 62; Sagar 63; Karnatak 62]

There is a general method of solving the partial differential equations of order 1, due to Charpit. This is as follows.

Let the partial differential equation be given by

$$f(x, y, z, p, q)=0 \qquad ...(1)$$

also we have $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{dy}dy,$

i.e., $$dz=pdx+qdy. \qquad ...(2)$$

Let us suppose that a relation $F(x, y, z, p, q)=0$ ...(3) exists such that after solving (1) and (3) simultaneously for p and q and putting these values in (2), (2) becomes integrable.

Thus z, p, q may be expressed as functions of x and y.

Since these values identically satisfy (1) and (3) both, their differentiating coefficients with respect to x and y vanish.

Differentiating (1) and (3) w.r.t. x, we get

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}p+\frac{\partial f}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial x}=0, \qquad ...(4)$$

and $$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}p+\frac{\partial F}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial x}=0. \quad ...(5)$$

Again differentiating (1) and (3) w.r.t. y, we get

$$\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}q+\frac{\partial f}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial y}=0, \quad ...(6)$$

and $$\frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}q+\frac{\partial F}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial y}=0. \quad ...(7)$$

Eliminating $\partial p/\partial x$ from (4) and (5), we get

$$\frac{\dfrac{\partial f}{\partial x}+\dfrac{\partial f}{\partial z}p+\dfrac{\partial f}{\partial q}\dfrac{\partial q}{\partial x}}{\dfrac{\partial f}{\partial p}}=\frac{\dfrac{\partial F}{\partial q}+\dfrac{\partial F}{\partial z}p+\dfrac{\partial F}{\partial q}\dfrac{\partial q}{\partial x}}{\dfrac{\partial F}{\partial p}},$$

i.e. $$\left(\frac{\partial f}{\partial x}\frac{\partial F}{\partial p}-\frac{\partial F}{\partial x}\frac{\partial f}{\partial p}\right)+p\left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial p}-\frac{\partial F}{\partial z}\frac{\partial f}{\partial p}\right)$$
$$+\frac{\partial q}{\partial x}\left(\frac{\partial f}{\partial q}\frac{\partial F}{\partial p}-\frac{\partial F}{\partial z}\frac{\partial f}{\partial p}\right)=0 \quad ...(8)$$

In the same way, eliminating $\partial q/\partial y$ from (6) and (7), we get

$$\left(\frac{\partial f}{\partial y}\frac{\partial F}{\partial q}-\frac{\partial F}{\partial y}\frac{\partial f}{\partial q}\right)+q\left(\frac{\partial F}{\partial q}\frac{\partial f}{\partial z}-\frac{\partial f}{\partial q}\frac{\partial F}{\partial z}\right)$$
$$+\frac{\partial p}{\partial y}\left(\frac{\partial f}{\partial p}\frac{\partial F}{\partial q}-\frac{\partial F}{\partial p}\frac{\partial F}{\partial q}\right)=0. \quad ...(9)$$

But $\dfrac{\partial p}{\partial y}=\dfrac{\partial q}{\partial x}=\dfrac{\partial^2 z}{\partial x\,\partial y}$.

Hence adding (8) and (9), we get after rearranging the terms,

$$\left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial y}+\left(\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial q}+\left(-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial z}$$
$$+\left(-\frac{\partial f}{\partial p}\right)\frac{\partial F}{\partial x}+\left(-\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial y}=0.$$

This is a linear equation of order one with x, y, z, p, q as independent variables and F as dependent variable.

Therefore as in Lagrangian Method, the auxiliary equations* are

$$\frac{dp}{\dfrac{\partial f}{\partial x}+p\dfrac{\partial f}{\partial z}}=\frac{dq}{\dfrac{\partial f}{\partial y}+q\dfrac{\partial f}{\partial z}}=\frac{dz}{-p\dfrac{\partial f}{\partial p}-q\dfrac{\partial f}{\partial q}}=\frac{dx}{-\dfrac{\partial f}{\partial p}}=\frac{dy}{-\dfrac{\partial f}{\partial q}}=\frac{dF}{0} \quad ...(10)$$

Any integral of (10) will satisfy (9). The simplest relation involving at least one of p and q may be taken as $F=0$. Now from $F=0$ and $f=0$ the values of p and q should be found in terms of x and y and should be substituted in (2) which on integration gives the solution.

*Remember very carefully the equation in (10).

Ex. 1 *Find the complete integral of the equation*
$px+qy=pq.$ **[Agra 53; Luck. 54]**

Solution. Here the differential equation $f(x, y, z, p, q)=0$ is
$f=px+qy-pq=0.$

$$\therefore \quad \frac{\partial f}{\partial x}=p, \frac{\partial f}{\partial y}=q, \frac{\partial f}{\partial z}=0, \frac{\partial f}{\partial p}=x-q, \frac{\partial f}{\partial q}=y-p.$$

The Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial q}-p\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dp}{-\frac{\partial f}{\partial q}}=\frac{dF}{0}$$

These in the present case become

$$\frac{dp}{p}=\frac{dq}{q}=\frac{dz}{-p(x-q)-q(y-p)}=\frac{dx}{-(x-q)}=\frac{dy}{-(y-p)}=\frac{dF}{0}.$$

The first two give $\frac{dp}{p}=\frac{dq}{q}$.

Integrating, $\log p=\log q+\log a$, *i.e.* $p=aq$,
where a is an arbitrary constant.

Putting aq for p in the given equation, we get

$$q(ax+y)=aq^2 \quad \text{or} \quad q=\frac{y+ax}{a} \text{ and thus } p=aq=y+ax.$$

Putting these values of p and q in $dz=p\,dx+q\,dy$, we get

$$dz=(y+ax)\,dx+\frac{1}{x}(y+ax)\,dy$$

or $\quad a\,dz=(y+ax)(dy+a\,dx).$

Integrating, $az=\frac{1}{2}(y+ax)^2+b$ which is the complete integral,
where a and b are arbitrary constants.

Ex. 2. *Solve* $\quad (p^2+q^2)\,y=qz$
[Delhi Hons. 68 ; Indore 67; Agra 51]

Solution. Here the differential equation is
$f\equiv(p^2+q^2)\,y-qz=0.$ ...(1)

$$\therefore \quad \frac{\partial f}{\partial p}=2py, \frac{\partial f}{\partial q}=2qy-z, \frac{\partial f}{\partial x}=0, \frac{\partial f}{\partial y}=(p^2+q^2), \frac{\partial f}{\partial z}=-q.$$

Now the Charpit's auxiliary equations become

$$\frac{dp}{-pq}=\frac{dq}{(p^2+q^2)-q^2}=\frac{dz}{-p.2py-q.(qy-z)}=\frac{dx}{-2py}$$

$$=\frac{dy}{-2qy+z}=\frac{dF}{0}.$$

The first two given $p\,dp+q\,dq=0.$

Integrating, $\quad p^2+q^2=a^2$ (say).

Now putting $p^2+q^2=a^2$ in the given equation (1), we get
$a^2y=qz \quad$ or $\quad q=a^2y/z,$
so that from $p^2+q^2=a^2,$

$$p=\sqrt{\left(a^2-\frac{a^4y^2}{z^2}\right)}=\frac{a}{z}\sqrt{(z^2-a^2y^2)}.$$

Now putting these values of p and q in $dz=p\,dx+q\,dy$, we get

$$dz=\frac{a}{z}\sqrt{(z^2-a^2y^2)}\,dx+\frac{a^2y}{z}\,dy.$$

or $$\frac{z\,dz-a^2y\,dy}{\sqrt{(z^2-a^2y^2)}}=a\,dx.$$

Integrating, $(z^2-a^2y^2)^{1/2}=ax+b$,
where b is an arbitrary constant

or $z^2-a^2y^2=(ax+b)^2$ ...(2)

which is the complete solution.

Singular Integral Differentiating (2) w.r.t. a and b partially, we get $-2ay^2=2x\,(ax+b)$, ...(3)

and $0=2\,(a+b)$. ...(4)

Eliminating a and b between (2), (3) and (4), we get $z=0$ which clearly satisfies the given differential equation and therefore is the singular integral.

General Integral. Writing $\phi(a)$ for b, (2) becomes

$z^2-a^2y^2=(ax+\phi(a))]^2$. ...(5)

Now differentiating it w r.t. a, we get

$-2ay^2=2\,\{ax+\phi(a)\}\,\{x+\phi'(a)\}$. ...(6)

Eliminating a between (5) and (6), we get general integral.

Ex. 3. *Solve* $2xz-px^2-\partial qxy+pq=0$. **[Meerut 70; Delhi Hons. 69; Raj. 64: Agra 54; Nag. 57; Pb. 63, 64]**

Solution. Here equation is

$f\equiv 2xz-px^2-2qxy+pq=0$. ...(1)

The Charpit's auxiliary equations are

$$\frac{dp}{2z-2qy}=\frac{dq}{0}=\frac{dx}{x^3-q}=\frac{dy}{2xy-p}=\frac{dz}{px^2+2xyq-2pq}=\frac{dF}{0}.$$

The second gives an integral $q=a$, ...(2)

where a is an arbitrary constant.

Putting $q=a$ in (1), we get $p=\dfrac{2x\,(z-ay)}{x^2-a}$

$$\therefore\quad dz=p\,dx+q\,dy=\frac{2x\,(z-ay)\,dx}{x^2-a}+a\,dy$$

or $$\frac{dz-a\,dy}{z-ay}=\frac{2x\,dx}{x^2-a}.$$

Integrating, $\log(z-ay)=\log(x^2-a)+\text{const.}$

or $z-ay-b\,(x^2-a)$ where b is a constant.

This is complete integral.

Ex. 4. *Apply Charpit's method to solve*

$z-px-qy=p^2+q^2$. **[Agra 59]**

The Charpit's auxiliary equations are

$$\frac{dp}{-p+p}=\frac{dq}{-q+q}=\ldots$$

Here $dp=0$. $dq=0$ give $p=a$, $q=b$,
where a and b are arbitrary constants.

Putting these values in the given equation, we get
$$r=ax+by+a^2+b^2,$$
which is the complete integral.

Ex. 4 (b). $z=px+qy+pq$. **[Karnatak 63]**

Solution. Proceed as above.

Ex. 5. *Solve* $p(1+q^2)=q(z-a)$. **[Pp. 60; Agra 60]**

Solution. Here $f\equiv p(1+q^2)-q(z-a)=0$.

$\therefore$ the Charpit's auxiliary equations are
$$\frac{dp}{pq}=\frac{dq}{q^2}=\frac{dz}{3pq^2+p+(a-z)q}=\frac{dx}{q^2+1}=\frac{dy}{-z+a+2pq}.$$

From the first two, we get $\dfrac{dp}{p}=\dfrac{dq}{q}$,
which on integration gives $q=cp$. Putting $q=cp$ in the given equation, we get
$$p=\frac{\sqrt{[c(z-a)-1]}}{c}\text{ and thus } q=\sqrt{[c(z-a)-1]}.$$

Putting these values in $dz=p\,dx+q\,dy$, we get
$$dz=\sqrt{[c(z-a)-1]}\left[\frac{dx}{c}+dy\right]$$
or
$$\frac{c\,dz}{\sqrt{[c(z-a)-1]}}=(dx+c\,dy).$$

Integrating, $2\sqrt{[c(z-a)-1]}=x+cy+d$,
where c and d are arbitrary constants.

Ex. 6. *Find the complete integral of the equation*
$$p^2+q^2-2px-2qy+2xy=0.$$

Solution. We have $f\equiv p^2+q^2-2px-2qy+2xy=0$. ...(1)

The Charpit's auxiliary equations are
$$\frac{dp}{-2p+2y+p.0}=\frac{dq}{-2q+2x+q.0}=\frac{dx}{2x-2p}=\frac{dy}{2y-2q},$$
i.e. $$\frac{dp}{-p+y}=\frac{dq}{-q+x}=\frac{dx}{x-p}=\frac{dy}{y-q},$$
so that $dp+dq=dx+dy$.

Integrating, $p+q=x+y+a$ or $(p-x)+(q-y)=a$. ...(2)
where a is an arbitrary constant.

Again the given equation (1) can be put as
$$(p-x)^2+(q-y)^2=(x-y)^2.$$

In (2) and (3) let us put $p-x=P$, $q-y=Q$, so that
$$P+Q=a \text{ and } P^2+Q^2=(x-y)^2.$$

Now $(P-Q)^2=P^2+Q^2-2PQ$

$$=P^2+Q^2-[(P+Q)^2-(P^2+Q^2)]$$
$$=2\,(x-y)^2-a^2,$$

i.e., $P-Q=\sqrt{[2\,(x-y)^2-a^2]}$; also $P+Q=a$.

$\therefore$ $P=p-x=\frac{1}{2}a+\frac{1}{2}\sqrt{[2\,(x-y)^2-a^2]}$

and $Q=q-y=\frac{1}{2}a-\frac{1}{2}\sqrt{[2\,(x-y)^2-a^2]}$.

Putting these values of p and q in $dz=p\,dx+q\,dy$, we get $dz=(x+\frac{1}{2}a)\,dx+(y+\frac{1}{2}a)\,dy+\frac{1}{2}\sqrt{[2\,(x-y)^2-a^2]}\,(dx-dy)$.

Integrating it, we get

$$2z=x^2+ax+y^2+ay+\frac{1}{\sqrt{2}}\left[\frac{u}{2}\sqrt{(u^2-a^2)}-\frac{1}{2}a^2\log\{u+\sqrt{(u^2-a^2)}\}\right]+b$$

where $u=\sqrt{2}(x-y)$.

This forms the complete integral.

Ex. 7. *Solve* $p^2+q^2-2px-2qy+1=0$. [Agra 72, 60]

Solution. Here the equation is

$$f\equiv p^2+q^2-2px-2qy+1=0.$$

$$\therefore\quad \frac{\partial f}{\partial p}=2p-2x,\ \frac{\partial f}{\partial q}=2q-2y,\ \frac{\partial f}{\partial z}=0,\ \frac{\partial f}{\partial x}=-2p,\ \frac{\partial f}{\partial y}=-2q.$$

The Charpit's auxiliary equations are

$$\frac{dp}{-2p+0}=\frac{dq}{-2q+0}=\ldots\text{etc.}$$

so that $\dfrac{dp}{p}=\dfrac{dq}{q}$.

Integrating, $\log p=\log q+\log a$ or $p=aq$.

Now putting $p=aq$, the given equation becomes

$$q^2\,(a^2+1)-2q\,(ax+y)+1=0$$

$$\therefore\quad q=\frac{2\,(ax+y)\pm\sqrt{\{4\,(ax+y)^2-4\,(a^2+1)\}}}{2\,(a^2+1)}$$

Also $p=aq=a\dfrac{2\,(ax+y)\pm\sqrt{\{4\,(ax+y)^2-4\,(a^2+1)\}}}{2\,(a^2+1)}$

Putting these values of p and q in $dz=p\,dx+q\,dy$, we get

$$dz=\frac{2\,(ax+y)\pm\{4\,(ax+p)^2-4\,(a^2+1)\}}{2\,(a^2+1)}\,(a\,dx+dy).$$

Let $ax+y=u$, so that $a\,dx+dy=du$; then

$$dz=\frac{u\pm\sqrt{\{u^2-a^2+1)\}}}{(a^2+1)}\,du$$

or $(a^2+1)\,dz=u\,du\pm\sqrt{\{u^2-(a^2+1)\}}\,du$.

$$\therefore\quad (a^2+1)\,z=\tfrac{1}{2}u^2\pm\sqrt{[\tfrac{1}{2}u\sqrt{\{u^2-(a^2+1)}}-\tfrac{1}{2}\,(a^2+1)\log\{u+\sqrt{u^2-(a^2+1)}\}].$$

Putting $u=ax+y$ in this, we get the complete integral.

Ex. 8. *Solve* $q=-xp+p^2$. [Guru Nanak 73 ; Vikram 62]

Solution. The Charpit's auxiliary equations are

$\therefore \quad \dfrac{dp}{p+p0}=\dfrac{dq}{0+0}.$

$\therefore$ $q=a$. Putting $q=a$ in the given equation, we get

$p^2-px=a$ or $p=\frac{1}{2}[x\pm\sqrt{(x^2+4a)}]$.

Putting these values of p and q in

$$dz=p\,dx+q\,dy$$
$$=\tfrac{1}{2}[x\pm\sqrt{(x^2+4a)}]\,dx+a\,dy.$$

Integrating $z=\frac{1}{4}[x^2\pm\{x\sqrt{(x^2+4a)}+a\log(x+\sqrt{x^2+4a})\}]+ay+c$ which is the complete integral.

Ex. 9. *Solve* $q=px+p^2$. [Agra 55]

Solution. Charpit's auxiliary equations are

$\dfrac{dp}{p+0}=\dfrac{dq}{0}$ or $q=a$, then $p=\frac{1}{2}[-x\pm\sqrt{(x^2+4a)}]$.

$\therefore$ $dz=p\,dx+q\,dy=\frac{1}{2}[-x\pm\sqrt{(x^2+4a)}]\,dx+a\,dy$ etc.

Ex. 10. *Solve* $z=pq$. [Agra 57]

Solution. The Charpit's auxiliary equations are

$\dfrac{dp}{0+p}=\dfrac{dq}{0+q}$; $\therefore$ $p=aq$.

Putting $p=aq$ in the given equation, we get $z=aq^2$

or $q=\sqrt{\left(\dfrac{z}{a}\right)}$ and $p=aq=\sqrt{(az)}$.

$$\therefore \quad dz=p\,dx+q\,dy$$
$$=\sqrt{(az)}\,dx+\sqrt{(z/a)}\,dy$$

or $\sqrt{(a/z)}\,dz=a\,dx+dy$

Integrating, $2\sqrt{(az)}=ax+y+c$.

Ex. 11. *Solve* $px+qy=z(1+pq)^{1/2}$.

Solution. The Charpit's auxiliary equations are

$\dfrac{dp}{p-p(1+pq)^{1/2}}=\dfrac{dp}{q-q(1+pq)^{1/2}}$ or $\dfrac{dp}{p}=\dfrac{dq}{q}$.

Integrating $\log p=\log q+\log a$ or $p=aq$.

Putting $p=aq$ in the given equation, we get

$q(ax+y)=z(1+aq^2)^{1/2}$ or $q^2[(ax+y)^2-z^2a]=z^2$

$\therefore$ $q=\dfrac{z}{[(ax+y)^2-az^2]^{1/2}}$ and $p=aq=\dfrac{az}{[(ax+y)^2-az^2]^{1/2}}$.

Now substituting these values in $dz=p\,dx+q\,dy$,

$$dz=\frac{z}{[(ax+y)^2-az^2]^{1/2}}[a\,dx+dy]$$

or $\dfrac{dz}{z}=\dfrac{dt}{[t-z^2]^{1/2}}$.

where $\sqrt{a}t=ax+y$, $\sqrt{a}\,dt=a\,dx+dy$.

This is a simple homogeneous equation.

To solve it put $t=uz$, $\dfrac{dt}{dz}=u+z\dfrac{du}{dz}$.

$\therefore$ we get $u+z\dfrac{du}{dz}=\dfrac{[t^2-z^2]^{1/2}}{z}=\sqrt{(u^2-1)}$

or $\dfrac{dz}{z}=\dfrac{dz}{\sqrt{(u^2-1)}-u}=\dfrac{\sqrt{(u^2-1)}+u}{[\sqrt{(u^2-1)}-u]\,[\sqrt{(u^2-1)}+u]}du$

$=-[\sqrt{(u^2-1)}+u]\,du.$

Integrating, $\log z+\frac{1}{2}u^2+\frac{1}{2}u\sqrt{(u^2-1)}+\frac{1}{2}\log[u+\sqrt{(u^2-1)}]=c$

which is the complete integral where $u=\dfrac{t}{z}=\dfrac{ax+y}{z\sqrt{a}}$.

Ex. 12. *Solve* $p=(qy+z)^2$.

Solution. Charpit's auxiliary equations are

$$\frac{dp}{2p\,(qy+z)}=\frac{dp}{4q\,(qy+z)}=\frac{dy}{-2y\,(qy-z)}.$$

First and third fractions give $\dfrac{dp}{p}=-\dfrac{dy}{y}$.

Integrating, $\log p+\log y=\log a$ or $py=a$.

Putting $p=a/y$ in the given equation, we get

$$q=\frac{1}{y}\left[\sqrt{\left(\frac{a}{y}\right)}-z\right].$$

$\therefore$ $dz=p\,dx+q\,dy$

$$=\frac{a}{y}dx+\frac{1}{y}\left[\sqrt{\left(\frac{a}{y}\right)}-z\right]dy.$$

or $(y\,dz+z\,dy)=a\,dx+\sqrt{\left(\frac{a}{y}\right)}\,dy$.

Integrating, $yz=ax+2\sqrt{(ay)}+c$ is the complete integral.

Ex. 13. *Solve* $(p+q)(px+qy)-1=0$.

Solution. The Charpit's auxiliary equations are

$$\frac{dp}{p^2+pq}=\frac{dq}{pq+q^2}\quad\text{or}\quad\frac{dp}{p}=\frac{dq}{q}.$$

Integrating $p=aq$. Putting $p=aq$ in the given equation, we get

$$(aq+q)(aqx+qy)-1=0,\ \therefore\ q=\frac{1}{(1+a)\,(ax+y)^{1/2}}.$$

$\therefore$ $dz=p\,dx+q\,dy$ gives

$$dz=\frac{1}{[(1+a)\,(ax+y)]^{1/2}}[a\,dx+dy].$$

Integrating, $z=\dfrac{2}{\sqrt{(1+a)}}\sqrt{(ax+y)}+c$.

Ex. 14. *Solve* $pxy+pq+qy-yz=0$. **[Raj. 63]**

Solution. The Charpit's auxiliary equations are

$$\frac{dp}{py-py}=\frac{dq}{px+q-qy}.$$

The first gives $dp=0$ or $p=a$.

Then from the given equation $q=\dfrac{yz-axy}{y+a}$.

$\therefore \quad dz=p\,dx+q\,dy$

$$=a\,dx+\frac{yz-axy}{y+a}\,dy$$

or $$\frac{dz-a\,dx}{z-ax}=\frac{y}{y+a}\,dy=\left(1-\frac{a}{y+a}\right)dy.$$

Integrating, $\log(z-ax)=y-a\log(y+a)+\log c$,

$(z-ax)(y+a)^a=ce^y$.

Ex. 15. *Apply Charpit's method to solve the differential equations.* $2(pq+py+qx)+x^2+y^2=0$. **[Saugar 63]**

Solution The Charpit's auxiliary equations are

$$\frac{dp}{2(q+x)}=\frac{dq}{2(p+y)}=\frac{dx}{-2(q+y)}=\frac{dy}{-2(p+x)}$$

$$=\frac{dz}{-2p(q+y)-2q(p+x)}=\frac{dF}{0}$$

$$=\frac{dp+dq+dx+dy}{0}$$

last relation on integration gives

$p+q+x+y=0$ or $(p+y)+(q+x)=0$.

The given equation cannot be written as

$(p+y)^2+(q+x)^2=(p-q)^2$.

Proceeding now as in Ex. 6, we get

$$2z=ax-x^2+ay-y^2+\tfrac{1}{2}(x-y)\sqrt{\{(x-y)^2+a^2\}}$$
$$+\frac{a^2}{2^{3/2}}\log\left[\sqrt{\{2(x-y)\}}+\sqrt{\{2(x-y)^2+a^2\}}\right]+b.$$

Exercises

Find complete integrals of following equatians by Charpit's method.

1. $z^2=pqxy$
2. $(p^2+q^2)y=pz$. **Ans.** $z=bx^a y^{1/a}$
3. $p^2+q^2-2pq\tanh 2y=\text{sech}^2\,2y$. **[Delhi Hons. 70]**

   Ans. $z+b=ax+\tfrac{1}{2}a\log\cosh 2y+\sqrt{(1-a^2)}\,(\tan^{-1}e^{2y})$.
4. $px^5-4q^3x^2+6x^2z-2=0$.

   Ans. $z=\tfrac{2}{3}(y+a)^{3/2}+\tfrac{1}{8}+\dfrac{1}{3x^2}+be^{3/x^2}$
5. $(x^2-yz)p+(y^2-zx)q=z^2-xy$. **[Vikram 69]**
6. $2z+p^2+qy+qy^2=0$. **[Delhi Hons. 72]**
7. $p^2x+q^2y=z$. **[Delhi Hons. 71]**
8. $2(z+xp+yq)=yp^2$. **Ans.** $\dfrac{ax}{y^2}+\dfrac{b}{y}+\dfrac{a^2}{4y^3}$.

9. $(p^2+q^2)^n (qx-py)=1.$

Ans. $z+b=a^{2n}\tan^{-1}(y/x)=1\int(ua^{-2}-a^{4n})^{1/2}\dfrac{du}{u}$

where $u=x^2+y^2$.

2·5. Particular Methods

The general method of solving partial differential equations of order one has been discussed (Charpit's method). There can be some shorter methods for special forms of differential equations. We give below some of these special methods of solving these equations.

2·6. Type 1. Equation of the form f (p, q)=0,

i.e., equation involving p and q only and not x, y and z.

In this case, Charpit's auxiliary equations become (see § 2·4)

$$\frac{dx}{\partial f/\partial p}=\frac{dy}{\partial f/\partial q}=\frac{dz}{p\,(\partial f/\partial p)+q\,(\partial f/\partial q)}=\frac{dp}{0}=\frac{dq}{0}.$$

Obviously from $dp=0$ and $dq=0$, we get

$$p=a \text{ and } q=b,$$

where a and b are arbitrarily constants. Again replacing p by a and q by b in $f(p, q)=0$, a, b satisfy the condition

$f(a, b)=0$ which suppose gives $b=\phi(a)$.

Therefore putting in $dz=p\,dx+q\,dy$,

we get $$dz=a\,dx+b\,dy.$$

Integrating $$z=ax+by+c=ax+\phi(a)\,y+c$$

where $$f(a, b)=0$$

is the complete integral. This has two arbitrary constants a and c.

General Integral To obtain general integral take

$c=\psi(a)$

where ψ is an arbitrary function.

Now the general integral is obtained by eliminating a between

$$z=ax+\phi(a)\,y+\psi(a)$$

or $$\frac{\partial z}{\partial z}=0=x+\phi'(a)\,y+\psi'(a)$$

Singular Integral. Also to obtain singular solution, if it exists we would be required to eliminate a and c from the equations

$$z=ax+\phi(a)\,y+c$$

$$\frac{\partial z}{\partial a}=0=x+\phi'(a)\,y$$

and $$\frac{\partial z}{\partial c}=0=1.$$

Apparently $1=0$ is inconsistent, therefore in this case singular integrals would just not exist.

Ex. 1. *Solve* $p^2-q^2=1$.

Solution. The equation is of the form $f(p, q)=0$.

The solution is $z=ax+by+c$

where $a^2-b^2=1$ *i.e.* $b=\pm\sqrt{(a^2-1)}$.

Hence the complete solution is $z=ax+\sqrt{(a^2-1)}\,y+c$.

A different form is obtained by puttin

$a=\sec\alpha$, so that $\sqrt{(a^2-1)}=\tan\alpha$,

and the solution now becomes $z=x\sec\alpha+y\tan\alpha+c$,

Ex 2. *Solve* $p^2+q^2=n^2$.

Solution. The equation is of the form $f(p, q)=0$.

Hence complete solution is $z=ax+by+c$

where $a^2+b^2=n^2$ or $b^2=\sqrt{(n^2-a^2)}$.

$\therefore$ complete solution is $z=ax+\sqrt{(n^2-a^2)}\,y+c$.

General Integral. Let $c=\phi(a)$; then

$$z=ax+\sqrt{(n^2-a^2)}\,y+\phi(a). \qquad ...(1)$$

Differentiating it w.r.t. h, we get

$$0=x-\frac{a}{\sqrt{(n^2-a^2)}}\,y+\phi'(a). \qquad ...(2)$$

The general integral is obtained by eliminating a from (1) and (2).

Ex. 3. *Solve* $p^2+q^2=npq$.

Solution. The equation is of the form $f(p, q)=0$.

Therefore the solution is $z=ax+by+c$

where $a^2+b^2=nab$ or $b^2-nab+a^2=0$

or $b=\frac{na\pm\sqrt{(n^2a^2-4a^2)}}{2}=\frac{n\pm\sqrt{(n^2-4)}}{2}\,a.$

Hence $z=ax+\frac{n\pm\sqrt{(n^2-4)}}{2}\,ay+c$ is the solution.

Ex. 4. *Solve* $q=e^{-p/\alpha}$.

Solution. The complete solution is $z=ax+by+c$

where $b=e^{-a/\alpha}$.

$\therefore$ $z=ax+e^{-a/\alpha}\,y+c$ is the complete integral.

Ex. 5. *Solve* $\sqrt{p}+\sqrt{q}=1$.

Solution. The solution is $z=ax+by+c$, where $\sqrt{a}+\sqrt{b}=1$

or $z=ax+(1-\sqrt{a})^2+c$.

Ex. 6. *Solve* $pq=k$. **[Raj 62]**

Solution. $z=ax+by+c$ where $ab=k$.

$\therefore$ $z=ax+(k/a)\,y+c$ is the complete solution.

Exercises

Find a complete integral for each of the following equations .

1. $p+q=pq$.

Ans. $z=ax+\frac{ay}{a-1}+b$.

2. $p=q^2$ **Ans.** $z=a^2x+ax+b$.
3. $p^2+p=q^2$. **Ans.** $z=ax+\sqrt{(a+a^2)}y+b$.

Type II. Equation $z=px+qy+f(p, q)$ ...(1)

i.e. equation analogous to Clairaut's form in ordinary differential equations.

In this case Charpit's auxiliary equations reduce to

$$\frac{dp}{-p+p}=\frac{dq}{-q+q}=\ldots \text{or } \frac{dp}{0}=\frac{dq}{0}$$

giving $p=\text{const.}=a$ (say),

$q=\text{const.}=b$ (say).

Putting in (1), the *complete solution is*

$$z=ax+by+f(a, b).$$ ...(2)

General integral. To find general integral let $b=\phi(a)$.

$$\therefore \quad z=ax+y\phi(a)+f\{a, \phi(a)\}.$$ ...(3)

Differentiating it w.r.t. a, we get

$$0=x+y\phi'(a)+f'\{a, \phi(a)\}.$$ ...(4)

Eliminating a from (1) and (3), we get the general integral.

Singular integral. Differentiating (1) w.r.t. a and b partially, we get

$$0=x+\frac{\partial f}{\partial a}, \quad ...(4) \qquad 0=y+\frac{\partial f}{\partial b}.$$ ...(5)

Eliminating a and b between (1), (4) and (5), we get the singular solution.

Ex. 1. *Solve* $z=px+qy+c\sqrt{(1+p^2+q^2)}$.

[**Delhi Hons's 71 ; Agra 71, 62, 53; Nag. 58**]

Solution. This is of the form

$$z=px+yq+f(p, q)$$

Hence the complete solution is

$$z=ax+by+c\sqrt{(1+a^2+b^2)}.$$ ...(1)

Singular soution. Differentiating (1) partially w.r.t. a and b respectively, we get

$$0=x+\frac{ac}{\sqrt{(1+a^2+b^2)}}, \ldots(2) \quad 0=y+\frac{bc}{\sqrt{(1+a^2+b^2)}}$$ ...(3)

so that $$x^2+y^2=\frac{a^2c^2+b^2c^2}{1+a^2+b^2},$$

i.e., $$c^2-x^2-y^2=\frac{c^2}{1+a^2+b^2}.$$

i.e., $$1+a^2+b^2=\frac{c^2}{c^2-x^2-y^2}.$$

$\therefore$ From (2) $a=-\frac{x}{c}\sqrt{(1+a^2+b^2)}=-\frac{x}{\sqrt{(c^2-x^2-y^2)}}$.

From (3) $b=-\frac{y}{c}\sqrt{(1+a^2+b^2)}=-\frac{y}{\sqrt{(c^2-x^2-y^2)}}$.

Putting these values of a, b in (1), the singular solution is

$$z=\frac{-x^2}{\sqrt{(c^2-x^2-y^2)}}-\frac{y^2}{\sqrt{(c^2-x^2-y^2)}}+c\,\frac{c}{\sqrt{(c^2-x^2-y^2)}},$$

or $z=\frac{c^2-x^2-y^2}{\sqrt{(c^2-x^2-y^2)}}=\sqrt{(c^2-x^2-y^2)}$,

so that $z^2=c^2-x^2-y^2$ or $x^2+y^2+z^2=c^2$.

Ex. 2. *Solve* $z=px+qy+\sqrt{(\alpha p^2+\beta q^2+\gamma)}$.

Solution. The complete integral is

$z=ax+by+\sqrt{(\alpha q^2+\beta b^2+\gamma)}$.

Singular integral is $\frac{x^2}{\alpha}+\frac{y^2}{\beta}+\frac{z^2}{\gamma}=1$, as in Ex. 1.

Ex. 3. *Solve* $z=px+qy-2\sqrt{(pq)}$,

Solution. This is of the form

$z=px+qy+f(p, q)$.

$\therefore$ the complete integral is

$z=ax+by-2\sqrt{(ab)}$. ...(1)

Singular Integral. Differentiating (1) partially w.r.t. a and b, respectively, we get

$0=x-2\sqrt{b}\,.\,\frac{1}{2}a^{-1/2}$, $0=y-2\sqrt{a}\,.\,\frac{1}{2}b^{-1/2}$

or $\sqrt{\left(\frac{b}{a}\right)}=x$ and $\left(\frac{a}{b}\right)=y$.

Multiplying these to eliminate a and b, the singular solution is

$xy=1$

Ex. 4. *Solve* $z=px+qy+pq$. [Saugar 62]

Solution Complete integral is

$z=ax+by+ab$. ...(1)

Singular Integral. Differentiating (1) partially w.r.t a and b respectively, we get

$0=x+b$, $0=y+a$, *i e.* $a=-y$, $b=-x$, $z=-xy$.

Exercises

Solve the following differential equations :

1. $z=px+qy+\log pq$. **Ans.** $z=ax+by+\log ab$.
2. $z=px+qy+(p^2+q^2)$. **Ans.** $z=ax+by+a^2+b^2$.
   S.S. is $z=\frac{1}{4}(x^2+y^2)$.
3. $z=px+qy+\sin(pq)$. **Ans.** $z=ax+by-\sin(ab)$
4. $z=px+qy-2\sqrt{pq}$. **Ans.** $z=ax+qy-2\sqrt{ab}$.
   Singular : $(x-z)(y-z)=1$.

$z=px+qy+3(pq)^{1/3}$. **Ans.** $z=ax+by+3(ab)^{1/3}$.
Singular : $xyz=1$.

Type III. Equation f (z, p, q)=0. *i.e. differential equation not containing independent variables x and y.*

In this case, Charpit's equations take the form

$$\frac{dx}{\partial f/\partial p}=\frac{dy}{\partial f/\partial y}=\frac{dz}{p\ \partial f/\partial p+q\ \partial f/\partial q}=\frac{dp}{-p\ \partial f/\partial z}=\frac{dq}{-q\ \partial f/\partial z}$$

the last two of which lead to the relation

$$ap=q, \qquad ...(1)$$

where a is an arbitrary constant.

Solving now (1) and $f(z, p, q)=0$, ...(2)

we get expressions for p and q which when put in

$$dz=p\ dx+q\ dy,$$

give $dz=qd\ (y+ax)$

$=p\ dX$ where $X=y+ax$

or $p=\frac{dz}{dX}$.

The equation (2) now becomes

$$f\left(z, \frac{dz}{dX}, a\frac{dz}{dX}\right)=0.$$

This now being an ordinary (not partial) differential equation of first order may be easily solved.

Procedure In an equation of the type $f(z, p, q)=0$, put

(*i*) $\frac{dz}{dX}$ for p, $a\frac{dz}{dX}$ for q.

(*ii*) Integrate the resulting ordinary differential equation,

(*iii*) Put $x+ay$ for X.

This gives a complete solution.

Note. General integral and singular integral are both determined in the usual way.

Ex. 1. *Solve* $z=pq$. **[Agra 57 ; Raj. 60]**

Solution. This is of the form $f(z, p, q)=0$,

Putting $\frac{dz}{dX}$ for p, $a\frac{dz}{dX}$ for q, the equation becomes $z=a\left(\frac{dz}{dX}\right)^2$

or $dX=\sqrt{a}\ \frac{dz}{\sqrt{z}}$ where $X=x+ay$.

Integrating, we get $X=c+2\sqrt{(az)}$.

Now putting $x+ay$ for X, the required solution is

$$x+ay=c+2\sqrt{(az)} \text{ or } (x+ay-c)^2=4az.$$

Ex. 2. *Solve* $9\ (p^2z+q^2)=4$. **[Agra 67, 59]**

Solution. The equation does not contain x and y and is of the form $f(z, p, q)=0$.

Putting $\frac{dz}{dX}$ for p, $a\frac{dz}{dX}$ for q, the equation becomes

$$9\left[z\left(\frac{dz}{dX}\right)^2+a^2\left(\frac{dz}{dX}\right)^2\right]=4 \text{ where } X=x+ay$$

or $\left(\frac{dz}{dX}\right)^2.9(z+a^2)=4$ or $\frac{dz}{dX}=\frac{2}{3\sqrt{(z+a^2)}}$

or $$dX=\tfrac{3}{2}.\sqrt{(z+a^2)}\,dz.$$

Integrating, $X+c=(z+a^2)^{3/2}$.

The complete solution is

$x+ay+c=(z+a)^{3/2}$, putting $X=x+ay$,

or $(x+ay+c)^2=(z+a^2)^3$. ...(1)

To obtain *general integral* take

$c=\psi(a)$,

where ψ is an arbitrary function.

Substituting in (1), we get

$(z+a^2)^3=\{x+ay+\psi(a)\}^2$ ...(2)

Differentiating it w.r.t. a, we get

$3a\,(z+a^2)^2=\{x+ay+\psi(a)\}\,\{y+\psi'(a)\}$. ...(3)

General solution is obtained by eliminating a from (2) and (3).

Again singular integral is obtained by eliminating a and c from (1) and

$3a\,(z+a^2)^2=(x+ay+c)\,y$,

and $(x+ay+c)=0$.

From these it is evident that the singular solution does not exist.

Ex. 3. *Solve* $pz=1+q^2$. **[Agra 65; 58]**

Solution. This is of the form $f(z, p, q)=0$.

Putting $p=\frac{dz}{dX}$, $q=a\frac{dz}{dX}$ where $X=x+ay$, the equation becomes

$$\frac{dz}{dX}=1+a^2\left(\frac{dz}{dX}\right)^2$$

or $a^2\left(\frac{dz}{dX}\right)^2-z\frac{dz}{dX}+1=0$

$\therefore$ $\frac{dz}{dX}=\frac{z\pm\sqrt{(z^2-4a^2)}}{2a}$ or $\frac{dz}{z\pm\sqrt{(z^2-4a^2)}}=\frac{dX}{2a^2}$,

or $\frac{dz\,[z\mp\sqrt{(z^2-4a^2)}]}{[z\pm\sqrt{(z^2-4a^2)}]\,[z\mp\sqrt{(z^2-4a^2)}]}=\frac{dX}{2a^2}$,

$\frac{z\mp\sqrt{(z-4a^2)}}{4a^2}\,dz=\frac{dX}{2a^2}$ or $[z\pm\sqrt{(z^2-4a^2)}]\,dz=2\,dX$.

Integrating,

$$\tfrac{1}{2}z^2\pm[\tfrac{1}{2}z\sqrt{(z^2-4a^2)}-\tfrac{1}{2}\,4a^2\log\{z+\sqrt{(z^2-4a^2)}\}]=2X+c$$

or $z^2 \mp [z\sqrt{(z^2-4a^2)} - 4a^2 \log \{z+\sqrt{(z^2-4a^2)}\}] = 4(x+ay)+2c$
is the complete solution (on putting $X=x+ay$).

Ex. 4. *Sovle* $p^2 = z^2(1-pq)$.

Solution. Putting $p=\dfrac{dz}{dX}$, $q=a\dfrac{dz}{dX}$, the equation becomes

$$\left(\frac{dz}{dX}\right)^2 = z^2\left[1-a\left(\frac{dz}{dX}\right)^2\right]$$

or $\dfrac{dz}{dX} = \dfrac{z}{\sqrt{(1+az^2)}}$ or $\dfrac{\sqrt{(1+az^2)}}{z}\,dz = dX$

or $dX = \dfrac{1+z^2}{z\sqrt{(1+az^2)}}\,dz = \dfrac{1}{z\sqrt{(1+az^2)}}\,dz + \dfrac{az}{\sqrt{(1+az^2)}}\,dz.$

Integrating, $X+c = \dfrac{1}{\sqrt{a}} \log [z\sqrt{a} + \sqrt{(1+az^2)}] + \sqrt{(1+az^2)}.$

Putting $X=x+ay$ we get the complete solution.

Ex. 5. *Solve* $p(1+q^2) = q(z-\alpha)$.

Solution. Putting $p=\dfrac{dz}{dX}$, $q=a\dfrac{dz}{dX}$, the equation becomes

$$\frac{dz}{dX}\left[1+a^2\left(\frac{dz}{dX}\right)^2\right] = a\frac{dz}{dX}(z-\alpha)$$

or $a\dfrac{dz}{dX} = \sqrt{(z-\alpha-1)}$ or $\dfrac{a\,dz}{\sqrt{(z-\alpha-1)}} = dX$ etc.

Exercises

Find complete solutions of the following equations :

1. $p^3+q^3 = 3pqz$. Ans. $(1+a^3)\log z = 3a(x+ay+c)$
2. $p^3+q^3 = 27z$. Ans. $z^2(1+a^3) = 8(x+ay+c)^2$
3. $z^2(p^2+q^2+1) = k^2$. [Agra 66]
   Ans. $(a^2+1)(k^2-z^2) = (x+ay+c)$
4. $p(1+q) = qz$. Ans. $\log(az-1) = (x+ay+c)$
5. $p(1+q^2) = q(z-a)$. Ans. $4c(z-a) = (x+ay+b)^2+4$
6. $p+q=z$. Ans. $x+ay+c = (1+a)\log z$
7. $p^2 = qz$ [Meerut 68]
8. $z^2(p^2z^2+q^2)+1$. [Delhi 72]

Type IV. Equation of the form

$$f_1(x, p) = f_2(y, q) \qquad \ldots(1)$$

i.e., separable equations.

In this case Clairaut's auxiliary equations are

$$\frac{dp}{\partial f_1/\partial x} = \frac{dq}{-\partial f_2/\partial y} = \frac{dx}{-\partial f_1/\partial p} = \frac{dy}{\partial f_1/\partial q}$$

giving $\dfrac{\partial f_1}{\partial p}\,dp + \dfrac{\partial f_1}{\partial x}\,dx = 0$ or $df_1 = 0,$

or $f_1 = \text{constant} = (c \text{ say})$.

Thus (1) gives

$f_1(x, p) = f_2(y, q) = c$;

these give p and q. The values of p and q so obtained are put in

$dz = p\,dx + q\,dy$.

On integrating it, the complete integral is obtained.

Singular and general integrals are obtained in the usual way.

Ex. 1. *Solve* $q = px + p^2$. [Agra 55]

Solution. Let $px + p^2 = q = c$ (say)

[of the form $f_1(p, x) = f_2(q, y)$]

Then $p^2 + px = c$ gives $p = \frac{1}{2}[-x + \sqrt{(x^2 + 4c)}$,

and $q = c$ gives $q = c$.

Putting these in $dz = p\,dx + q\,dy$, we get

$$dz = \tfrac{1}{2}[-x + \sqrt{(x^2 + 4c)}]\,dx + c\,dy.$$

Integrating,

$$z = \tfrac{1}{4}[-x^2 + x\sqrt{(x^2 + 4c)} + c \log [x + \sqrt{(x^2 + 4c)}] + cy + a.$$

Ex. 2. *Solve* $p^2 + q^2 = x + y$.

[Vikram 64; Agra 59; 54; Sagar 62]

Solution. The equation can be written as

$p^2 - x = y - q^2 = c$ (say) [form $f_1(x, p) = f_2(y, q)$].

Then $p^2 - x = c$ gives $p = \sqrt{(c + x)}$

and $y - q^2 = c$ gives $q = \sqrt{(y - c)}$.

Putting these in $dz = p\,dx + q\,dy$, we get

$$dz = \sqrt{(c + x)}dx + \sqrt{(y - c)}dy.$$

Integrating, $z + a = \frac{2}{3}(c + x)^{3/2} + \frac{2}{3}(y - c)^{3/2}$ is the solution.

Ex. 3. *Solve* $\sqrt{p} + \sqrt{q} = 2x$ [Agra 56]

Solution. Write the equation as $\sqrt{p} - 2x = -\sqrt{q} = c$ (say).

Then $\sqrt{p} - 2x = c$ gives $p = (c + 2x)^2$,

and $-\sqrt{q} = c$ gives $q = c^2$.

Putting these values in $dz = p\,dx + q\,dy$, we get

$$dz = (c + 2x)^2\,dx + c^2\,dy.$$

Integrating, $z = \frac{1}{6}(c + 2x)^3 + c^2 y + a$.

Ex. 4. *Solve* $x^2p^2 = yq^2$.

Solutium. Let $x^2p^2 = yq^2 = c^2$ (say).

Then $p = c/x$ and $q = c/\sqrt{y}$.

$\therefore\ dz = p\,dx + q\,dy = (c/x)\,dx + (c/\sqrt{y})\,dy$.

or $z = c \log x + 2c\sqrt{y} + a$.

Ex. 5. *Integrate*

$pe^y = qe^x$. [Meerut 68]

Solution. This can be written as

$pe^{-x} = qe^{-y} = c$, say.

Then $p = ce^x$ and $y = ce^y$.

Putting these values in $dz = p\,dx + q\,dy$,

we get $dz = c\,(e^x\,dx + e^y\,dy)$.

Integrating, the complete solution is $z + a = c\,(e^x + e^y)$.

Ex. 6. *Solve* $pq = xy$. **[Karnatak 63; I.A.S. 60]**

Solution. We have $p/x = y/q = c$ (say).

$\therefore$ $p = cx$, $q + y/c$.

$\therefore$ $dz = p\,dx + q\,dy = cx\,dx + (y/c)\,dy$.

Integrating, $z = \frac{1}{2}cx^2 + \frac{1}{2}(y^2/c) + a$ or $2cz = c^2x^2 + y^2 + b$.

Exercises

Find complete integral for the following equations :

1. $q = 2yp^2$. **Ans.** $z = cx + c^2y^2 + a$
2. $p^2 - y^3q = x^2 - y^2$.

Ans. $z = \frac{1}{2}x\sqrt{\{x^2 + c\}} + \frac{1}{2}c \log\{x + \sqrt{(x^2 + c)}\} - \frac{1}{2}\frac{c}{y^2} + \log y + a$

3. $q = xyp^2$. **Ans.** $z + a = 2\sqrt{(cx)} + \frac{1}{2}cy^2$.

2·7. Use of Transformation.

We now take some of those examples which can be reduced to standard forms by using some transformations.

Examples which can be reduced to the form f(p, q)=0

Ex. 1. *Solve* $x^2p^2 + y^2q^2 = z^2$.

[Raj. 63, 51; Karnatak M.Sc. 61]

Solution. The equation can be put as

$$\left(\frac{x}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z}\frac{\partial z}{\partial y}\right)^2 = 1.$$

Now, let us put $\dfrac{dz}{z} = dZ$, i.e. $z = e^Z$,

$$\frac{dy}{y} = dZ, \quad \text{i.e.} \quad y = e^Y$$

and $$\frac{dx}{x} = dX, \quad \text{i.e.} \quad x = e^X$$

Then the given equation becomes

$$\left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = 1$$

which is of the above form.

Hence complete solution is $Z = aX + bY + c_1$

where $a^2 + b^2 = 1$,

i.e., $Z = aX + \sqrt{(1 - a^2)}\log Y + c_1$.

or $\log z = a \log x + \sqrt{(1 - a^2)} \log y + c_1$.

If we put $a=\cos\alpha$, so that $\sqrt{(1-a^2)}=\sin\alpha$, the complete solution can be written as

$$\log z=\cos\alpha\log x+\sin\alpha\log y+\log c,$$

or $$z=cx^{\cos\alpha}y^{\sin\alpha}.$$

Ex. 2. *Solve* $(y-x)(qy-px)=(p-q)^2$. **[Raj. 54; Agra 65]**

Solution. Let us put $x+y=X$, $xy=Y$, so that

$$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x}+\frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial x}=\frac{\partial z}{\partial X}.1+\frac{\partial z}{\partial Y}y,$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial X}\frac{\partial X}{\partial y}+\frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial y}=\frac{\partial z}{\partial X}.1+\frac{\partial z}{\partial X}.x.$$

Substituting these values of p and q, the given equation becomes

$$(y-x)\left[\left(\frac{\partial z}{\partial X}+\frac{\partial z}{\partial Y}.x\right)y-\left(\frac{\partial z}{\partial X}+\frac{\partial z}{\partial Y}\cdot y\right)x\right]=\left(\frac{\partial z}{\partial Y}\right)^2(y-x)^2$$

i.e., $$(y-x)^2\left(\frac{\partial z}{\partial X}\right)=(y-x)^2\left(\frac{\partial z}{\partial Y}\right)^2$$

or $$\frac{\partial z}{\partial X}=\left(\frac{\partial z}{\partial Y}\right)^2$$ which is of the above form.

$\therefore$ Complete solution is $z=aX+bX+c$, where $a=b^2$ or $b=\sqrt{a}$.

$\therefore$ $z=aX+\sqrt{a}Y+c=a(x+y)+\sqrt{(axy)}+c.$

***Ex. 3.** $(x+y)(p+q)^2+(x-y)(p-q)^2=1$. **[Agra 58]**

Solution. Let us put $x+y=X^2$, $x-y=Y^2$. Then

$$\left.\begin{aligned}p&=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}\frac{\partial X}{\partial x}+\frac{\partial z}{\partial Y}\frac{\partial Y}{\partial x}\\&=\tfrac{1}{2}\left(\frac{1}{X}\frac{\partial z}{\partial X}+\frac{1}{X}\frac{\partial z}{\partial Y}\right)\\q&=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial X}\frac{\partial X}{\partial Y}=\frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y}\\&=\tfrac{1}{2}\left(\frac{1}{X}\frac{\partial z}{\partial X}-\frac{1}{Y}\frac{\partial z}{\partial Y}\right)\end{aligned}\right\}\text{ so that }\begin{aligned}p+q&=\frac{1}{X}\frac{\partial z}{\partial X},\\p-q&=\frac{1}{Y}\frac{\partial z}{\partial Y}\end{aligned}$$

Putting these values, the given equation becomes

$$\left(\frac{\partial z}{\partial X}\right)^2+\left(\frac{\partial z}{\partial Y}\right)^2=1$$ of the standard form $f(p,q)=0$.

$\therefore$ complete integral is $z=aX+bY+c$,

where $a^2+b^2=1$ or $b=\sqrt{(1-a^2)}$.

$\therefore$ $z=aX+\sqrt{(1-a^2)}\,Y+c,$

$z=a\sqrt{(x+y)}+\sqrt{(1-a^2)}\sqrt{(x-y)}+c.$

Ex. 4. *Solve* $pq=x^my^nz^{2l}$.

Solution. The equation is $\dfrac{pz^{-l}}{x^m}\cdot\dfrac{qz^{-l}}{y^n}=1.$

Put $Z=\dfrac{z^{1-l}}{1-l}$, $X=\dfrac{x^{m-1}}{m+1}$, $Y=\dfrac{y^{n+1}}{n+1}$.

Then $\frac{\partial Z}{\partial X}=\frac{\partial Z}{\partial x}\frac{dx}{dX}=z^{-l}\,p.\frac{1}{x^m}$, $\frac{\partial Z}{\partial Y}=\frac{\partial Z}{\partial y}\cdot\frac{dy}{dY}=z^{-l}q.\frac{1}{y^n}$.

$\therefore$ the equation becomes $\frac{\partial Z}{\partial X}\cdot\frac{\partial Z}{\partial Y}=1$

or $PQ=1$. (Type 1)

$\therefore$ solution is $Z=aX+bY+c$, where $ab=1$

$$Z=aX+\frac{1}{a}Y+c$$

or $$\frac{z^{1-l}}{1-l}=a\frac{x^{m+1}}{m+1}+\frac{y^{n+1}}{a\,(n+1)}+c.$$

Ex. 5. $p^m \sec^{2m} x+z^l q^n \operatorname{cosec}^{2n} y=z^{lmn/(m-n)}$.

Solution. The equation can be written as

$$\left(\frac{z^{-l/(m-n)}}{\cos^2 x}\frac{\partial z}{\partial x}\right)^m+\left(\frac{z^{-l/(m-n)}}{\sin^2 y}\frac{\partial z}{\partial y}\right)^n=1.$$

Put $z^{-l/(m-n)}\,dz=dZ$, *i.e.* $Z=\frac{m-n}{m-n-l}\,z^{(m-n-l)/(m-n)}$;

$\cos^2 x\,dx=dX$, *i.e.* $X=\frac{1}{2}(x+\frac{1}{2}\sin 2x)$;

$\sin^2 y\,dy=dY$ *i.e.* $Y=\frac{1}{2}(y-\frac{1}{2}\sin 2y)$.

Then the equation becomes

$$\left(\frac{\partial Z}{\partial X}\right)^m+\left(\frac{\partial Z}{\partial Y}\right)^n=1.$$

Complete solution is $Z=aX+bY+c$,

where $a^m+b^n=1$ or $b=(1-a^m)^{1/n}$.

$\therefore$ $Z=aX+(1-a^m)^{1/n}\,Y+c$ is the complete solution where X, Y, Z are as given above.

Examples which transform to type II

Ex. 6. *Solve* $4xyz=pq+2px^2y+2qxy^2$.

Solution. Put $x=X^{1/2}$, $y=Y^{1/2}$; then

$$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}\cdot\frac{dX}{dx}=2X^{1/2}\frac{\partial z}{\partial X},\quad q=2Y^{1/2}\frac{\partial z}{\partial Y}.$$

$\therefore$ the equation becomes

$$z=X\frac{\partial z}{\partial X}+Y\frac{\partial z}{\partial Y}+\frac{\partial z}{\partial X}\cdot\frac{\partial z}{\partial Y}.\qquad\text{(Type II)}$$

$\therefore$ complete solution is $z=aX+bY+ab$

or $z=ax^2+by^2+ab$. ...(1)

Singular Solution. Differentiating (1) partially w.r.t. a and b, we get $0=x^2+b$, $0=y^2+a$.

Eliminating a, b the singular solution is

$$z+x^2y^2=0.$$

Examples which transform to type III

Ex. 7. *Solve* $q^2y^2=z\,(z-px)$. [Delhi Hons. 72, 69 ; Raj. 64]

Solution. Putting $u=\log x$, $v=\log y$, the equation becomes

$$\left(\frac{\partial z}{\partial v}\right)^2=z\left[z-\frac{\partial z}{\partial u}\right].$$

Now put $\frac{\partial z}{\partial u}=\frac{dz}{\partial X}$, $\frac{\partial z}{\partial v}=a\frac{dz}{dX}$, where $X=u+av$.

The equation then becomes $a^2\left(\frac{dz}{dX}\right)^2=z\left[z-\frac{dz}{dX}\right]$

i.e. $a^2\left(\frac{dz}{dX}\right)^2+z\frac{dz}{dX}-z^2=0,$

so that $\frac{dz}{dx}=\frac{-z\pm\sqrt{(z^2+4a^2z^2)}}{2a^2}=\frac{z}{2a^2}[-1\pm\sqrt{(1+4a^2)}].$

Integrating, $2a^2z=[-1\pm\sqrt{(1+4a^2)}](X+\log c)].$

or $z^{2a^2/[-1\pm\sqrt{(1+4a^2)}]}=cxy^a$ as $X=u+av$ etc.

Ex. 8. *Solve* $p^2x^2=z(z-qy)$.

Solution. Put $u=\log x\cdot v-\log y$ and proceed as above.

Ex. 9. *Solve* $pq=x^my^nz^l$. **[Raj. 61; Agra 57; Lucknow 56]**

Solution Put $\frac{x^{m+1}}{m+1}=u$, $\frac{y^{n+1}}{n+1}=v$,

so that $p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}=x^m\frac{\partial z}{\partial u}$

and $q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}=y^n\frac{\partial z}{\partial v}$.

Hence the equation becomes

$\frac{\partial z}{\partial u}\frac{\partial z}{\partial v}=z^l$ or $PQ=z^l$.

This is of the form $f(P,Q,z)=0$.

Put $\frac{\partial z}{\partial u}=\frac{dz}{dX}$ and $\frac{\partial z}{\partial v}=a\frac{dz}{dX}$, where $X=u+av$; then equation is $a\left(\frac{dz}{dX}\right)^2=z^l$.

or $\sqrt{a}z^{-l/2}\,dz=dX$.

Integrating, $\frac{z^{l-(1/2)l}}{1-\frac{1}{2}l}=\frac{X}{\sqrt{a}}+c=\frac{u+cv}{\sqrt{a}}+c$

or $\frac{z^{1-(1/2)l}}{1-\frac{1}{2}l}=\frac{1}{\sqrt{a}}\left[\frac{x^{m+1}}{m+1}+a\frac{y^{n+1}}{n+1}\right]+c$ from (1).

Examples which after substitution take the form of type IV

Ex. 10 *Solve* $z^2(p^2+q^2)=x^2+y^2$. **[Agra 65]**

Solution. If we put $z\,dz=dZ$, *i.e.*, $Z=\frac{1}{2}z^2$,

$z\frac{\partial z}{\partial x}=\frac{\partial Z}{\partial x}=P$ (say) and $z\frac{\partial z}{\partial y}=\frac{\partial Z}{\partial y}=Q$ (say), then the equation becomes

$$P^2+Q^2=x^2+y^2 \quad \text{or} \quad P^2-x^2=y^2-Q^2=c^2 \text{ (say)}.$$

$$\therefore \quad P=\sqrt{(c^2+x^2)},\ Q=\sqrt{(y^2-c^2)}.$$

$$\therefore \quad dZ=P\,dx+Q\,dy$$

$$=\sqrt{(c^2+x^2)}\,dx+\sqrt{(y^2-c^2)}\,dy.$$

Integrating $Z=\frac{1}{2}x\sqrt{(c^2+x^2)}+\frac{1}{2}c^2\log\{x+\sqrt{(c^2+x^2)}\}$
$+\frac{1}{2}y\sqrt{(y^2-c^2)}-\frac{1}{2}c^2\log\{y+\sqrt{(y^2-c^2)}\}+a.$

Replace Z by $\frac{1}{2}z^2$ to get the complete solution.

Ex. 11. *Solve* $x^2y^3p^2q=z^3$.

Solution. The equation can be written as

$$x^2y^3\left(\frac{1}{z}\frac{\partial z}{\partial x}\right)^2\left(\frac{1}{z}\frac{\partial z}{\partial y}\right)=1.$$

Put $\frac{1}{z}\,dz=dZ$, i.e., $Z=\log z$, then

$$\frac{1}{z}\frac{\partial z}{\partial x}=\frac{\partial Z}{\partial x}=P \text{ (say) and } \frac{1}{z}\frac{\partial z}{\partial y}=\frac{\partial Z}{\partial y}=Q.$$

$\therefore$ the equation becomes $x^2y^3P^2Q=1$,

or $\quad x^2P^2=\dfrac{1}{Qy^3}=c^2$ (say).

Then $\quad P=\dfrac{c}{x},\ Q=\dfrac{1}{c^2y^3}.$

$$\therefore \quad dZ=P\,dx+Q\,dy=\frac{c}{x}\,dx+\frac{1}{c^2y^3}\,dy,$$

Integrating, $\quad Z=c\log x-\dfrac{1}{2c^2y^2}+a$

Now put $Z=\log z$ and simplify.

Ex. 12. *Solve* $(x^2+y^2)(p^2+q^2)=1$. [Vikram 63]

Solution. Putting $x=r\cos\theta$, $y=r\sin\theta$,

i.e., $\theta=\tan^{-1}y/x$, $r=\sqrt{(x^2+y^2)}$,

$$\frac{\partial r}{\partial x}=\frac{x}{r}=\cos\theta,\ \frac{\partial r}{\partial y}=\frac{y}{r}=\sin\theta.$$

$$\frac{\partial\theta}{\partial x}=\frac{1}{1+y^2/x^2}\cdot\left(-\frac{y}{x^2}\right)=-\frac{\sin\theta}{r},\ \frac{\partial\theta}{\partial y}=\frac{1}{1+y^2/x^2}\left(\frac{1}{x}\right)=\frac{\cos\theta}{r}.$$

$$\therefore \quad p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial z}{\partial\theta}\cdot\frac{\partial\theta}{\partial x}$$

$$=\cos\theta\,\frac{\partial z}{\partial r}-\frac{\sin\theta}{r}\,\frac{\partial z}{\partial\theta}.$$

Also $\quad q=\dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial r}\cdot\dfrac{\partial r}{\partial y}+\dfrac{\partial z}{\partial\theta}\cdot\dfrac{\partial\theta}{\partial y}$

$$=\sin\theta\,\frac{\partial z}{\partial r}+\frac{\cos\theta}{r}\,\frac{\partial z}{\partial\theta}.$$

$\therefore \quad (x^2+y^2)(p^2+q^2)=1$ becomes

$$r^2\left[\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)\right]=1 \quad \text{or} \quad r^2\left(\frac{\partial z}{\partial r}\right)=1-\left(\frac{\partial z}{\partial \theta}\right)^2=c^2 \text{ (say)}$$

$$\therefore \quad \frac{\partial z}{\partial r}=\frac{c}{r},\ \frac{\partial z}{\partial \theta}=\sqrt{(1-c^2)}.$$

$\therefore \quad dz+\frac{\partial z}{\partial r}\,dr+\frac{\partial z}{\partial \theta}\,d\theta$ gives,

$$dz=\frac{c}{r}\,dr+\sqrt{(1-c^2)}\,d\theta,\ z=c\log r+\sqrt{\{(1-c^2)\}}\,\theta+a$$

or $\quad z=\frac{1}{2}c\log(x^2+y^2)+\sqrt{\{(1-c^2)\}}\tan^{-1}(y/x)+a.$

Ex. 13. *Solve* $z(p^2-q^2)=x-y.$ **[Agra 61]**

Solution. The equation is

$$\left[\left(\sqrt{z}\frac{\partial z}{\partial x}\right)-\left(\sqrt{z}\frac{\partial z}{\partial y}\right)^2\right]=x-y.$$

Let us put $\sqrt{z}\,dz=dZ$ *i.e.* $Z=\frac{2}{3}z^{3/2}$.

Then the equation is

$$\left(\frac{\partial Z}{\partial x}\right)^2-\left(\frac{\partial Z}{\partial y}\right)^2=x-y$$

or $P^2-Q^2=x-y$ or $P^2-x=C^2-y=c$, say,

so that $P=\sqrt{(c+x)},\quad Q=\sqrt{(c+y)}.$

Now $dZ=P\,dx+Q\,dy=\sqrt{(c+x)}\,dx+\sqrt{(c+y)}\,dy.$

Integrat ng, $Z=\frac{2}{3}(c+x)^{3/2}+a+\frac{2}{3}(c+y)^{3/2}$

or $z^{3/2}=(c+x)^{3/2}+(c+y)^{3/2}+b$ as $Z=\frac{2}{3}z^{3/2}$

Exercises

Find complete integral of following examples :

1. $(1-x^2)\,yp^2+x^2q=0$ **Ans.** $(2z-ay^2-b)^2=a(1-x^2).$
2. $(p^2+q^2)=z^2(x+y).$ **Ans.** $\frac{3}{2}\log z=(x+a)^{3/2}-(y-a)^{3/2}+b.$
3. $py+qx+pq=0.$ **Ans.** $2z=ax^2+by^2+c,$
4. $\frac{p^2}{x}-\frac{q^2}{y}=\frac{1}{z}\left(\frac{1}{x}+\frac{1}{y}\right).$ **Ans.** $az^{3/2}=(1+ax)^{3/2}+(ay-1)^{3/2}+b.$

Hint. Put $\frac{2}{3}z^{3/2}=u$ etc.

Solutions satisfying given conditions.

Ex. *Find a complete integral of the partial differential equation* $(p^2+q^2)\,x=pz,$ *and deduce the solution which passes through the curve* $x=0,\ z^2=4y.$

Solution. Let $f=(p^2+q^2)\,x-pz.$ ...(1)

The Charpit's auxiliary equations are

$$\frac{dp}{(p^2+q^2)-p^2}=\frac{dq}{-pq}=\frac{dq}{-p(2px-z)-q\,2px}=\frac{dx}{-2px+z}=\frac{dy}{-2qx}.$$

The first two give $p\,dp++q\,dq=0.$

Integrating, $p^2+q^2=a^2$ (say).

Now putting $p^2+q^2=a^2$ in (1), this reduces to

$$a^2x-pz=0 \quad \text{or} \quad p=\frac{a^2x}{z},$$

so that $q^2=a^2-p^2=\frac{a^2}{z^2}(z^2-a^2x^2)$ or $q=\frac{a}{z}\sqrt{(z^2-a^2x^2)}$,

Now putting these in $dz=p\,dx+q\,dy$, we get

$$dz=\frac{a^2x}{z}\,dx+\frac{a}{z}\sqrt{(z^2-a^2x^2)}\,dy$$

or $$\frac{z\,dz-a^2x\,dx}{\sqrt{(z^2-a^2x^2)}}=a\,dx.$$

Integrating it, $(z^2-a^2x^2)^{1/2}=ax+b$,

so that the complete integral is

$$z^2=a^2x^2+(ay+b)^2. \qquad \ldots(2)$$

We have to determine values of a and b so that it passes through $x=0$, $z^2=4y$. ...(3)

Setting $x=0$ and $z^2=4y$, (2) gives

$$4y=(ay+b)^2 \quad \text{or} \quad 4y=a^2y^2+2aby+b^2$$

or $a^2y^2+(2ab-4)\,y+b^2=0$.

This will have real roots if $(ab-2)^2=a^2b^2$ *i.e.* if $ab=1$.

Therefore the appropriate one-parameter family is

$$z^2=a^2x^2+\left(ay+\frac{1}{a}\right)^2$$

or $a^4(x^2+y^2)+a^2(2y-z^2)+1=0$,

and this has its envelope the surface

$$(2y-z^2)^2=4\,(x^2+y^2)$$

which is the required solution.

Exercises

1. Find a complete integral of the equation

$$p^2x+qy=z$$

and hence derive the equation of an integral surface of which the line $y=1$, $x+z=0$ is a generator.

Ans. $(x+ay-z+b)^2=4bx$, $xy=z\,(y-2)$,

2. Show that the equation

$$xpq+yq^2=1$$

has complete integrals

(*a*) $(z+b)^2=4\,(ax+b)$

(*b*) $kx\,(z+h)=k^2y+x^2$

and deduce (b) from (*a*).

3

Linear Partial Differential Equations

3·1. Linear partial Differential equation with constant coefficients

In this chapter we shall consider partial differential equations in which higher partial differential coefficients of z occur with respect to x and y but power of each differential that occurs or z is one. Such an equation is called linear partial differential equation. If the coefficients of various terms are constant quantities, then it is called the linear differential equation with constant coefficients. Thus it is in general of the form

$$\left(A_0\frac{\partial^n z}{\partial x^n}+A_1\frac{\partial^n z}{\partial x^{n-1}\partial y}\cdots+A_n\frac{\partial^n z}{\partial y^n}\right)+\left(B_0\frac{\partial^{n-2}z}{\partial x^{n-1}}+B_1\frac{\partial^{n-2}z}{\partial x^{n-2}\partial y}\right.$$
$$\left.+\ldots+B_{n-1}\frac{\partial^{n-1}z}{\partial y^{n-1}}\right)+\ldots+\left(K_0\frac{\partial z}{\partial x}+K_1\frac{\partial z}{\partial y}\right)+Lz=f(x,y),$$

where $A_0, A_1, \ldots A_n, B_0, B_1 \ldots B_{n-1}, \ldots K_0, K_1, L$ are all constants.

For convenience the operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are denoted by D and D' and now the above equation can be written as

$$[(A_0D^n+A_1D^{n-1}D'+\ldots+A_nD'^n)+(B_0D^{n-1}+B_1D^{n-2}D'$$
$$+\ldots+B_{n-1}D'^{n-1})+\ldots(K_0D+K_1D')+L]\,z=f(x,y)$$

or more briefly as $F(D, D')\,z=f(y, y)$,

where $F(D, D')=(A_0D^n+\ldots+A_nD'^n)+\ldots+(K_0D+K_1D')+L.$

3·2. Homogeneous linear partial differential equation with constant coefficients

If $F(D, D')$ is homogeneous in D and D'

i.e. $F(D, D')=A_0D^n+A_1D^{n-1}D'+\ldots+A_nD'^n$,

then equation $F(D, D')\,z=f(x, y)$

or $(A_0D^n+A_1D^{n-1}D'+A_2D^{n-2}D'^2+\ldots+A_nD'^n)\,z=f(x, y)$

is called the linear homogeneous equation.

3·3. Solution of a partial differential equation.

There are in general two parts of the complete solution, namely *complementary function* (C. F.) and the *particular integral* (P.I.).

The most general solution of $F(D, D')\,z=0$ is called complementary function and any particular solution if $F(D, D')\,z=f(x, y)$ is called a particular solution ; and

Complete Solution = C.F. + P.I.

3·4. To find complementary function.

Complementary function is solution of $F(D, D')\, z=0$.

Let $F(D, D') \equiv (D-m_1D')(D-m_2D')\ldots(D-m_nD')$, where $m_1, m_2, \ldots, m_n$ are some constants.

Consider $(D-m_rD')\, z=0$,

i.e., $\dfrac{\partial z}{\partial x}-m_r\dfrac{\partial z}{\partial y}=0$ or $p-m_rq=0$,

which is of Lagrange's form. For it Lagrange's subsidiary equations* are $\dfrac{dx}{1}=\dfrac{dy}{-m_r}=\dfrac{dz}{0}$.

First two relations give $dy+m_r\, dx=0$

or $y+m_rx=c_1$ (on integration).

Last relation gives $dz=0$ or $z=c_2$.

Hence $z=\phi_r(y+m_rx)$ is solution of $(D-m_rD')\, z=0$ where ϕ_r is an arbitrary function

Similarly solutions corresponding to all factors of $F(D, D')$ can be obtained.

Hence if $m_1, m_2, \ldots, m_n$ are all **distinct**, the complementary solution is given by

$$z=\phi_1(y+m_1x)+\phi_2(y+m_2x)+\ldots+\phi_n(y+m_nx).$$

Cor. Let us suppose that $z=\phi(y+mx)$ be a solution of $F(D, D')\, z=0$; then

$$D^rz=m^r\phi^{(r)}(y+mx),$$

and $$D'^rz=\phi^{(r)}(y+mx).$$

Thus $F(D, D')\, z=(A_0D^n+A_1D^{n-1}D'+\ldots A_nD'^n)\, z$

$$=(A_0m^n+A_1m^{n-1}+\ldots A_n)\phi^{(n)}(y+mx).$$

The equation $A_0m^n+A_1m^{n-1}+\ldots+A_n=0$

is called the *auxiliary equation*; $m_1, m_2, \ldots m_n$ as considered above are the n roots of the auxiliary equation. It will be noted that the auxiliary equation is simply obtained by putting $D=m$, $D'=1$ in $F(D, D')=0$.

Ex. 1. *Solve* $r=a^2t$. **[Agra 62. 59]**

Solution. We know that $r=\dfrac{\partial^2 z}{\partial x^2}=D^2z$.

and $t=\dfrac{\partial^2 z}{\partial y^2}=D'^2z$.

Hence the equation can be written as $(D^2-a^2D'^2)\, z=0$.

The auxiliary equation is $m^2-a^2=0$,

giving $m_1=a$, $m_2=-a$.

---

* If an equation is $Pp+Qq=R$, then Lagrange's subsidiary equations are

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}.$$

Therefore the solution is $z=\phi_1(y+m_1x)+\phi_2(y+m_2x)$

$$z=\phi_1(y+ax)+\phi_2(y-ax)$$

Ex. 2. *Solve* $(D^3-7DD'^2+6D'^3)\,z=0$.

Solution. The auxiliary equation is

$$m^3-7m+6=0,$$

giving $m_1=1,\ m_2=2,\ m_3=-3$.

Therefore the solution is

$$z=\phi_1(y+m_1x)+\phi_2(y+m_2x)+\phi_3(y+m_3x)$$

or $z=\phi_1(y+x)+\phi_2(y+2x)+\phi_3(y-3x)$.

Ex. 3. *Solve* $(D^3-3D^2D'+2D'^2D)\,z=0$.

Solution. The auxiliary equation is

$$m^3-3m^2+2m=0,$$

giving $m_1=0,\ m_2=1,\ m_2=2$.

Therefore the solution is

$$z=\phi_1(y)+\phi_2(y+x)+\phi_3(y+2x)$$

Ex. 4. *Solve* $\dfrac{\partial^2 z}{\partial x^2}+a^2\dfrac{\partial^2 z}{\partial y^2}=0$.

Solution. The equation is $(D^2+a^2D'^2)\,z=0$.

The auxiliary equation is $m^2+a^2=0$

giving $m_1=ai,\ m_2=-ai$.

Therefore the solution is $z=\phi_1(y+aix)+\phi_2(y-aix)$.

Ex. 5. *Solve* $\dfrac{\partial^4 z}{\partial x^4}-\dfrac{\partial^4 z}{\partial y^4}=0$.

Solution. The equation is $(D^4-D'^4)\,z=0$.

The auxiliary equation is $m^4-1=0$,

or $(m^2+1)(m^2-1)=0$, giving $m=\pm 1,\ \pm i$.

Therefore the solution is

$$z=\phi_1(y+x)+\phi_2(y-x)+\phi_3(y+ix)+\phi_4(y-ix),$$

Exercises

Solve the following differential equations :

1. $2r+5s+2t=6$. **Ans.** $y=\phi_1(2y-x)+\phi_2(y-2x)$
2. $(2D^2D'-3DD'^2+D'^3)\,z=0$.
   Ans. $y=\phi_1(y)+\phi_2(x+y)+\phi_3(x+2y)$
3. $(D^2-3aDD'+2a^2D'^2)\,z=0$. **Ans.** $z=\phi_1(y+ax)+\phi_2(y+2ax)$
4. $(D^3-6D^2D'+11DD'^2-6D'^3)\,z=0$.
   Ans. $z=\phi_1(y+x)+\phi_2(y+2x)+\phi_3(y+3x)$

3·5. When auxiliary equation has repeated roots

Let a root m of the auxiliary equation be repeated twice. Consider

$$(D-mD')(D-mD')\,z=0. \qquad \ldots(1)$$

Putting $(D-mD')\,z=u,$ ...(2)

equation (1) becomes

$$(D-mD')\,u=0$$

and its solution is $u=\phi_1(y+mx)$,

or $(D-mD')\,z=\phi_1(y+my)$ putting the value of u in (2).

This can be written as

$$p-mq=\phi_1(y+mx),$$

which is of Lagrange's form. Lagrange's subsidiary equations for this are

$$\frac{dx}{1}=\frac{dy}{-m}=\frac{dz}{\phi_1(y+mx)}.$$

The first two relations give $dy+m\,dx=0$ or $y+mx=c_1$.

Again from the relations $dx=\dfrac{dz}{\phi_1(y+mx)}$,

we get $dz=\phi_1(c_1)\,dx$ as $y+mx=c_1$

or $z=x\phi_1(c_1)+c_2$.

Therefore the general solution of (1) is

$$z=x\phi_1(y+mx)+\phi_2(y+mx).$$

In general if a root m repeats r times,

i.e., $(D-mD')^r\,z=0$,

then $z=\phi_1(y+mx)+x\phi_2\,(y+mx)+\ldots+x^{r-1}\,\phi_r(y+mx)$.

Ex. 1. *Solve* $25r-40s+16t=0$.

Solution. The equation can be written as

$$(25D^2-40DD'+16D'^2)\,z=0.$$

The auxiliary equation is $25m^2-40m+16=0$,

or $(5m-4)^2=0$, $m=\frac{4}{5}, \frac{4}{5}$.

Therefore solution is $z=\phi_1(5y+4x)+x\phi_2(5y+4x)$.

Ex. 2. *Solve* $(D^4-2D^3D'+2DD'^3-D'^4)\;z=0$.

Solution. The auxiliary equation is

$$M^4-2m^3+2m-1=0,$$

or $(m-1)^3\,(m+1)=0$, $m=1, 1, 1, -1$.

Therefore the solution is

$$z=\phi_1(y-x)+\phi_2(y+x)+x\phi_3(y+x)+x^2\phi_4(y+x).$$

Ex. 3. *Solve* $r-4s+4t=0$. [Raj. 66]

Solution. The equation can be written as

$$(D^2-4DD'+4D'^2)\,z=0.$$

The auxiliary equation is $m^2-4m-4=0$

giving $m=2, 2$.

Therefore the solution is $z=\phi_1(y+2x)+x\phi_2(y+2x)$.

Exercises

Solve the following equations :

1. $(D^3-3D^2D'+3DD'^2-D'^3)\, z=0,$

Ans $z=\phi_1(y+x)+x\phi_2(y+x)+x^2\phi_3(y+x)$

2. $(D^4+D'^4-2D^2D'^2)\, z=0.$

Ans. $z=\phi_1(y+x)+x\phi_2(y+x)+\phi_3(y-x)+x\phi_2(y-x)$

3. $(D^3-2D^2D'+DD'^2)\, z=0.$

Ans. $z=\phi_1(y)+\phi_2(y+x)+x\phi_3(y+x)$

4. $(4D^2+12DD'+9D'^2)\, z=0.$

Ans. $z=\phi_1(2y-3x)+x\phi_2\,(2y-3x)$

3·6. Particular Integral.

Given the partial differential equation

$F(D, D')=f(x, y),$

any solution of it free from arbitrary constants gives a particular integral. Now consider

$$\frac{1}{F(D, D')} f(x, y).$$

This identically satisfies the given equation. Therefore,

$$\text{particular integral}=\frac{1}{F(D, D')} f(x, y).$$

The symbolic function $F(D, D')$ can be treated as an algebraic function of D and D' and can be factorized or expanded in ascending powers of D or D'*.

Ex. 1. *Solve* $(D'-6DD'+9D'^2)\, z=12x^2+36xy.$

Solution. The auxiliary equation is $m^2-6m+9=0$ giving $m=3, 3.$

Therefore C.F. $=\phi_1(y+3x)+x\phi_2(y+3x).$

$$\text{Now P.I.}=\frac{1}{(D^2-6DD'+9D'^2)}(12x^2+36xy)$$

$$=\frac{1}{(D-3D')^2}(12x^2+36xy)$$

$$=\frac{1}{D^2}\left(1-\frac{3D'}{D}\right)^{-2}(12x^2+36xy)$$

$$=\frac{1}{D^2}\left[\left(1+\frac{6D'}{D}+27\frac{D'^2}{D^2}+\ldots\ldots\right)\right](12x^2+36xy)$$

$$=\frac{1}{D^2}[(12x^2+36xy)]+\frac{6}{D^3}(36x)$$

---

* $\frac{1}{D}$ means integration w.r.t. x, $\frac{1}{D'}$ means integration w. r. t. y, and so on and P.I. would be different if $F(D, D')$ is expanded in ascending powers of D or D'.

$$=x^4+6x^3y+6\times 36\frac{x^6}{2.3.4}=10x^4+6x^3y.$$

Therefore the complete solution is

$$z=\phi_1(y+3x)+x\phi_2(y+3x)+10x^4+6x^3y.$$

***Ex. 2.** *Solve* $r+(a+b)s+abt=xy.$

[Raj. 1961 ; Agra 1958]

Solution. The equation can be written as

$$[D^2+(a+b)DD'+abD'^2]z=xy.$$

The auxiliary equation is $m^2+(a+b)m+ab=0$,

or $(m+a)(m+b)=0$, giving $m=-a, -b$.

Hence the C.F,$=\phi_1(y-ax)+\phi_2(y-bx)$.

$$\text{P.I.}=\frac{1}{D^2+(a+b)DD'+abD'^2}(xy)$$

$$=\frac{1}{D^2}\left[1+(a+b)\frac{D'}{D}+ab\frac{D'^2}{D^2}\right]^{-1}(xy)$$

$$=\frac{1}{D^2}\left[1-(a+b)\frac{D'}{D}\cdots\right](xy)$$

$$=\frac{1}{D^2}xy-\frac{(a+b)}{D^3}x=\frac{x^3y}{6}-(a+b)\frac{x^4}{24}.$$

Hence the complete solution is

$$z=\phi_1(y-ax)+\phi_2(y-bx)+\tfrac{1}{6}x^3y-\tfrac{1}{24}(a+b)x^4.$$

Ex. 3. *Solve* $\frac{\partial^3 z}{\partial x^3}-\frac{\partial^3 z}{\partial y^3}=x^3y^3.$

Solution. The equation can be written as

$$(D^3-D'^3)z=x^3y^3.$$

Auxiliary equation is $m^3-1=0$ or $m^3=1$,

giving $m=1, \omega, \omega^2$, where ω is a cube root of unity.

$$\text{C.F.}=\phi_1(y+x)+\phi_2(y+\omega x)+\phi_3(y+\omega^2x).$$

$$\text{P.I.}=\frac{x^3y^3}{D^3-D'^3}=\frac{1}{D^3}\left(1-\frac{D'^3}{D^3}\right)^{-1}(x^3y^3)$$

$$=\frac{1}{D^3}\left(1+\frac{D'^3}{D^3}+\ldots\right)(x^3y^3)$$

$$=\frac{1}{D^3}x^3y^3+\frac{1}{D^6}6x^3$$

$$=\frac{x^6.y^3}{4.5.6}+\frac{6x^9}{4.5.6.7.8.9}=\frac{x^6y^3}{120}+\frac{x^9}{10080}.$$

Therefore the complete solution is

$$z=\phi_1(y+x)+\phi_2(y+\omega x)+\phi_3(y+\omega^2x)+\frac{x^6y^3}{120}+\frac{x^9}{10080}$$

Ex. 4. *Solve* $\log s=x+y.$

Solution. The equation can be written as

$s=e^{x+y}$ or $DD'z=e^{x+y}$.

For complementary function we have to consider

$DD'z=0$.

This gives C.F. $=\phi_1(x)+\phi_2(y)$.

Now P.I. $=\dfrac{1}{DD'}e^{x+y}=e^{x+y}$.

Therefore the complete integral is

$z=\phi_1(x)+\phi_2(y)+e^{x+y}$.

Ex. 5. *Solve* $\dfrac{\partial^2 z}{\partial x^2}-\dfrac{\partial^2 z}{\partial y^2}=x-y$.

Solution. The auxiliary equation is

$m^2-1=0$ giving $m=1, -1$

$\therefore$ C.F. $=\phi_1(y+x)+\phi_2(y-x)$.

Also P.I. $=\dfrac{1}{D^2-D'^2}(x-y)=\dfrac{1}{D^2}\left(1-\dfrac{D'^2}{D^2}\right)^{-1}(x-y)$

$=\dfrac{1}{D^2}\left(1+\dfrac{D'^2}{D^2}+\ldots\right)(x-y)=\dfrac{1}{D^2}(x-y)=\frac{1}{6}x^3-\frac{1}{2}x^2y$.

Therefore the complete solution is

$y=\phi_2(y+x)+\phi_2(y-x)+\frac{1}{6}x^3-\frac{1}{2}x^2y$.

Exercises

Solve the following equations :

1. $(D^2-a^2D'^2)\,z=x$. **Ans.** $z=\phi_1(y+ax)+\phi_2(y-ax)+\frac{1}{6}x^3$.
2. $(D^2-DD'-6D'^2)\,z=xy$.
   Ans. $z=\phi_1(y+3x)+\phi_2(y-2x)+\frac{1}{6}x^3y+\frac{1}{24}x^4$.
3. $(D^2-2DD'+D'^2)\,r=12xy$.
   Ans. $z=\phi_1(y+x)+x\phi_2(y+x)+2x^3y+x^4$.
4. $(D^3-7DD'^3-6D'^3)\,z=x^2+xy^2+y^3$.
   Ans. $z=\phi_1(y-x)+\phi_2(y-2x)+\phi_3(y+3x)$
   $+\frac{5}{72}x^6+\dfrac{x^5}{60}(1+21y)+\frac{1}{24}x^4y^2+\frac{1}{6}x^3y^3$
5. $(D^2D'-2DD'^2+D'^3)\,z=\dfrac{1}{x^3}$.
   Ans. $z=\phi_1(y)+\phi_2(x+y)+y\phi_3(x+y)+\dfrac{1}{2x}y$.

3·7. A Short Method

When $f(x, y)$ is a function of $ax+by$, we have a shorter method for determining the particular integral.

Consider $f(x, y)=\phi(ax+by)$.

Then $D^r\phi(ax+by)=a^r\phi^{(r)}(ax+by)$

and $\quad D'^r \phi(ax+by) = b^r \phi^{(r)}(ax+by)$,

where $\phi^{(r)}$ is rth differential of ϕ with respect to $ax+by$ as a whole.

Since $F(D, D')$ is homogeneous in D and D' of order n,

$$F(D, D')\, \phi(ax+by) = F(a, b)\, \phi^{(n)}(ax+by),$$

or
$$\frac{1}{F(D, D')} \phi^{(n)}(ax+by) = \frac{1}{F(a, b)} \phi(ax+by),$$

provided that $F(a, b) \neq 0$.

Further let $ax+by=t$; this gives

$$\frac{1}{F(D, D')} \phi^{(n)}(t) = \frac{1}{F(a, b)}\, \phi(t).$$

Integrating both the sides n times with respect to t, we get

$$\frac{1}{F(D, D')}\, \phi(t) = \frac{1}{F(a, b)} \int \int \ldots \int \phi(t)\, dt \ldots dt,$$

where $t=ax+by$.

Working Rule. To get the particular integral of an equation $F(D, D')\, z = \phi(ax+by)$, where $F(D, D')$ is a homogeneous function of D, D' of degree n, proceed as follows :

(*i*) Put $ax+by=t$; and integrate $\phi(t)$, n times with respect to t.

(*ii*) Put a for D and b for D' to get $F(a, b)$ in $F(D, D')$.

(*iii*) Now $\text{P.I.} = \dfrac{1}{F(a, b)} \times n$th integral of $\phi(t)$ with respect to t,

where $t=ax+by$.

Ex. 1. *Solve* $(D^2+2DD'+D'^2)\, z = e^{2x+3y}$.

Solution. The auxiliary equation is

$$m^2+2m+1=0 \text{ giving } m=-1, -1.$$

Hence $\quad \text{C.F.} = \phi_1(y-x) + x\phi_2(y-x)$.

Also $\quad \text{P.I.} = \dfrac{1}{D^2+2DD'+D'^2}\, e^{2x+3y}$.

Here e^{2x+3y} is a function of the form $\phi(ax+by)$ and $D^2+2DD' +D'^2$ is a homogeneous function of D, D' of degree 2.

Integrating e^{2x+3y} twice with respect to $(2x+3y)$, we get e^{2x+3y}.

Also putting 2 for D and 3 for D',

$$\text{P.I.} = \frac{1}{2^2+2.2.3+3^2}\, e^{2x+3y} = \frac{e^{2x+3y}}{25}.$$

Therefore the complete solution is

$$z = \phi_1(y-x) + n\phi_2(y-x) + \tfrac{1}{25} e^{x+3y}$$

Ex. 2. *Solve* $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \cos mx \cos ny$.

Solution Auxiliary equation is $m^2+1=0$, giving $m = \pm i$.

$\therefore \quad \text{C.F.} = \phi_1(y+ix) + \phi_2(y-ix)$.

$$\text{P.I.}=\frac{1}{D^2+D'^2}\cos mx\cos ny$$

$$=\tfrac{1}{2}\,\frac{1}{D^2+D'^2}\,[\cos(mx+ny)+\cos(mx-ny)]$$

$$=\tfrac{1}{2}\,\frac{-\cos(mx+ny)}{m^2+n^2}+\tfrac{1}{2}\,\frac{-\cos(mx-ny)}{m^2+n^2}$$

integrating $\cos t$ twice with respect to t where $t=mx\pm ny$

$$=-\frac{\cos mx\cos ny}{(m^2+n^2)}.$$

Therefore the complete solution is

$$z=\phi_1(y+ix)+\phi_2(y-ix)=\frac{\cos mx\cos ny}{m^2+n^2}.$$

Ex. 3. *Solve* $(D^2+3DD'+2D'^2)\,z=x+y.$

[Delhi Hons. 72, 68 ; Agra 52]

Solution. The auxiliary equation is

$m^2+3m+2=0$ giving $m=-2,\,-1.$

$\therefore$ $\text{C.F.}=\phi_1(y-x)+\phi_2(y-2x),$

$$\text{P.I.}=\frac{1}{D^2+3DD'+2D'^2}(x+y)$$

$$=\frac{1}{1^2+3.1.1+2.1^2}\iint t\,dt\,dt,\text{ where } t=x+y$$

$$=\frac{(x+y)^3}{36}.$$

Therefore the complete solution is

$$z=\phi_1(y-x)+\phi_2(y-2x)+\frac{(x+y)^3}{36}.$$

Ex. 4. *Solve* $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}=30\,(2x+y).$

Solution. Auxiliary equation is $m^2+1=0$, giving $m=\pm i.$

Hence $\text{C.F.}=\phi_1(y+ix)+\phi_2(y-ix).$

$$\text{P.I.}=\frac{30\,(2x+y)}{D^2+D'^2}$$

$$=\frac{1}{2^2+1^2}\,30\iint t\,dt\,dt,\text{ where } t=2x+y$$

$$=(2x+y)^3.$$

Therefore the complete solution is

$$z=\phi_1(y+ix)+\phi_2(y-ix)+(2x+y)^3.$$

Ex. 5. *Solve* $(D-aD')^2\,z=(\phi x)+\psi(y)+X\,x+by).$

Solution. The auxiliary equation is

$(m-a)^2=0$, giving $m=a,\,a.$

Hence $\text{C.F.}=\phi_1(y+ax)+x\phi_2(y+ax).$

$$\text{P.I.}=\frac{1}{(D-aD')^2}[\phi(x)+\psi(y)+X(x+by)]$$

$$=\frac{1}{(D-aD')^2}[\phi(x+0.y)+\psi(0.x+y)+X(x+by)]$$

$$=\iint \phi(x)\,dx+\frac{1}{a^2}\iint \psi(y)\,dy\,dy+\frac{1}{(1-ab)^2}\iint X(t)\,dt\,dt,$$

where $t=x+dy$.

Therefore the complete solution is

$z=$C.F.$+$P.I.

Exercises

Solve the following equations :

1. $(D^2+DD'-2D'^2)\,z=\sqrt{(2x+y)}$.

   Ans. $z=\phi_1(p+x)+\phi_2(y-2x)+\frac{2}{15}(2x+y)^{5/2}$.

2. $(r-2s+t)=\sin(2x+3y)$.

   Ans. $z=\phi_1(y+x)+x\phi_2(y+x)-\sin(2x+3y)$.

3. $(2D^2-5DD'+2D'^2)\,z=24\,(y-x)$.

   Ans. $z=\phi_1(y+2x)+\phi_2(2y+x)+\frac{4}{9}(y-x)^3$.

4. $(D+D')\,z=\sin x$.

   Ans. $z=\phi(y-x)-\cos x$.

5. $(D^2-2DD'+D'^2)\,z=e^{x+2y}$.

   Ans. $z=\phi_1(y+x)+x\varphi_2(y+x)+e^{x+2y}$.

3·8. Exceptional case when F(a, b)=0

If $F(a, b)=0$, then our method of previous article fails.

Now $F(a, b)=0$ if and only if

$(bD-aD')$ is factor of $F(D, D')$.

Therefore we may write

$F(D, D')=(bD-aD')\,G(D, D')$.

Consider now $(bD-aD')\,z=\phi(ax+by)$. The subsidiary equations for this are

$$\frac{dx}{b}=\frac{dy}{-a}=\frac{dz}{\phi(ax+by)},$$

The first two relations give $ax+by=c$ (const.)

Also when $\dfrac{dx}{b}=\dfrac{dz}{\phi(ax+by)}$, we have

$$dz=\frac{1}{b}\phi(c)\,dx,\text{ as } ax+by=c.$$

Integrating, $z=\dfrac{x}{b}\phi(c)$ or $z=\dfrac{x}{b}\phi(ax+by)$.

$\therefore$ The particular integral is now given by

$$z=\frac{1}{(bD-aD')}\cdot\frac{1}{G(D, D')}\phi(ax+by)$$

$$=\frac{x}{b}\,\frac{1}{G(D, D')}\,\phi(ax+by)$$

$$=\frac{x}{b}\,\frac{1}{G(a, b)}\,\psi(ax+by)$$

where $\psi(t)$ is obtained after integrating $\phi(t)$ as many times as is the degree of $G(D, D')$ and $G(a, b)\neq 0$.

Next consider the relation

$$F(D, D')=(bD-aD')\,G(D, D').$$

Differentiating it w.r.t. D, this gives

$$F'(D, D')=bG(D, D')+(bD-aD')\,G'(D, D'),$$

so that $$F'(a, b)=bG(a, b)+0.$$

Therefore the particular integral can be written as

$$z=x\,\frac{1}{F'(a, b)}\,\psi(ax+by).$$

Working Rule. To evaluate $\frac{1}{F(D, D')}\phi(ax+by)$ when $F(a, b)=0$, proceed as follows :

(*i*) differentiate $F(D, D')$ with respect to D partially and multiply the expression by x, so that

$$\frac{1}{F(D, D')}\,\phi(ax+by)=x\,\frac{1}{F'(D, D')}\,\phi(ax+by).$$

(*ii*) If $F'(a, b)$ is also zero, differentiate $F'(D, D')$ with respect to D partially and multiply by x again, so that

$$\frac{1}{F(D, D')}\,\phi(ax+by)=x^2\,\frac{1}{F''(D, D')}\,\phi(ax+by).$$

Proceed with this type of differentiation and every time multiply by x as long as the derivative of $F(D, D')$ vanishes when $D=a$ and $D'=b$.

(*iii*) If $F^{(r)}(a, b)\neq 0$, $\frac{1}{F^{(r)}(D, D')}\phi(ax+by)$ can be evaluated as in § 3·7.

***Ex. 1.** *Solve* $4(r-s)+t=16\log(x+2y)$. [Agra 61]

Solution. The differential equation can be written as

$$(4D^2-4DD'+D'^2)\,z=16\log(x+2y).$$

The auxiliary equation is

$$4m^2-4m+1=0, \text{ giving } m=\tfrac{1}{2}, \tfrac{1}{2}.$$

Hence $$\text{C.F.}=\phi_1(2y+x)+x\phi_2(2y+x),$$

$$\text{P.I.}=\frac{1}{4D^2-4DD'+D'^2}\cdot 16\log(x+2y).$$

The denominator vanishes when $D=1$ and $D'=2$. So differentiating $F(D, D')$ w.r.t. D and multiplying the expression by x,

$$\text{P.I.}=x\,\frac{1}{8D-4D'}\cdot 16\log(x+2y).$$

The denominator again vanishes when $D=1, D'=2$.

Hence again differentiating the denominator w.r.t. D and multiplying by x,

$$\text{P.I.}=x^2.\tfrac{1}{8}.16 \log (x+2y)$$
$$=2x^2 \log (x+2y).$$

Hence the complete solution is

$$z=\phi_1(2y+x)+x\phi_2(2y+x)+2x^2 \log (x+2y).$$

Ex. 2. *Solve* $(D^3-4D^2D'+4DD'^2)\, z=4 \sin (2x+y)$.

[Delhi Hons 69]

Solution. The auxiliary equation is

$$m^3-4m^2+4m=0 \text{ or } m(m-2)^2=0.$$

This gives $m=0, 2, 2$.

$$\therefore \quad \text{C.F.}=\phi_1(y)+\phi_2(y+2x)+x\phi_3(y+2x).$$

Now $$\text{P.I.}=\frac{1}{D^3-4D^2D'+4DD'^2}.4 \sin (2x+y).$$

The denominator becomes zero when $D=2, D'=1$.

Differentiating the deno. w.r.t D and multiplying by x,

$$\text{P.I.}=x\frac{1}{3D^2-8DD'+4D'^2}4 \sin (2x+y).$$

The denominator again vanishes when $D=2, D'=1$.

Therefore again diff. the denominator w.r.t. D and multiplying by x,

$$\text{P.I.}=x^2 \frac{1}{6D-8D'}4 \sin (2x+y).$$

The denominator, which is of order 1, does not vanish when $D=2, D'=1$.

$$\therefore \quad \text{P. I.}=\frac{4x^2}{6.2-8.1}\int \sin t\, dt \text{ where } t=(2x+y)$$
$$=-x^2 \cos t=-x^2 \cos (2x+y).$$

Therefore the complete solution is

$$z=\phi_1(y)+\phi_2(y+2x)+x\phi_3(y+2x)-x^2 \cos (2x+y).$$

Ex. 3. *Solve the equation*

$$\frac{\partial^3 u}{\partial x^3}-4\frac{\partial^3 u}{\partial x^2\partial y}+4\frac{\partial^3 u}{\partial x\partial y^2}=\cos (2x+y).$$

[Agra 1967]

Solution, The auxiliary equation

$m^3-4m^2+4m=4$, gives $m=0, 2, 2$.

$$\therefore \quad \text{C.F.}=\phi_1(y)+\phi_2(y+2x)+x\phi_3(y+2x)$$

$$\text{P.I.}=\frac{\cos (2x+y)}{D^3-4D^2D'+4DD'^2}$$

the denominator vanishes when $D=2, D'=1$

$$=x^2 \frac{1}{6D-8D'} \cos(2x+y)$$

differentiating twice and multiplying by x^2

$$=\frac{x^2}{4} \sin(2x+y)$$

integrating cos $(2x+y)$ once w.r.t. $2x+y$.

Therefore the complete solution is

$$z=\phi_1(y)+\phi_2(y+2x)+x\phi_3(y+2x)+\tfrac{1}{8}x^2 \sin(2x+y).$$

Ex. 4. *Solve* $(D^3-2D^2D'-DD'^2+2D'^3)\, z=e^{x+y}$.

Solution. The auxiliary equation is

$$m^3-2m^2-m+2=0,$$

i.e. $(m-1)(m+1)(m-2)=0,\ m=1,\ -1,\ 2.$

$\therefore$ C.F. $=\phi_1(y+x)+\phi_2(y-x)+\phi_3(y+2x)$.

$$\text{P.I.}=\frac{1}{D^3-2D^2D'-DD'^2+2D'^3} e^{x+y}$$

here deno. vanishes when $D=1,\ D'=1$

$$=x \frac{1}{3D^2-4DD'+D'^2} e^{x+y}$$

differentiating the deno. w.r.t. D and multiplying by x,

$$=x \frac{1}{3.1^2-4.1.1-1^2} \iint e^t\, dt\, dt \text{ where } t=x+y$$

integrating twice as $3D^2-4DD'-D'^2$ is of order 2.

$$=-\tfrac{1}{2}xe^{x+y}.$$

The complete solution is

$$z=\phi_1(y+x)+\phi_2(y-x)+\phi_3(y+2x)-\tfrac{1}{2}xe^{x+y}.$$

Ex. 5. *Solve* $\dfrac{\partial^2 z}{\partial x^2}-2a\dfrac{\partial^2 z}{\partial x \partial y}+a^2\dfrac{\partial^2 z}{\partial y^2}=f(y+ax)$.

Solution. The auxiliary equation is

$m^2-2am+a^2=0$ giving $m=a,\ a$.

Hence the C.F. $=\phi_1(y+ay)+x\phi_2(y+ax)$.

Now $\text{P.I.}=\dfrac{1}{D^2-2aDD'+a^2D'^2} f(y+ax)$

multiplying by x and diff. the deno. w.r.t. D

$$=x \frac{1}{2D-2aD'} f(y-ax)$$

$$=x^2.\tfrac{1}{2} f(y+ax)=\tfrac{1}{2}x^2.f(y+ax)$$

multiplying again by x and diff. the deno. w.r.t. D.

Hence the complete solution is

$$z=\phi_1(y+ax)+x\phi_2(y+ax)+\tfrac{1}{2}x^2 f(y+ax).$$

Ex. 6. *Solve* $(D-D')^2\, z=x+\phi(x+y)$.

Solution. The auxiliary equation is $(m-1)^2=0$.

$\therefore \quad m=1, 1.$

$\therefore \quad$ C.F.$=\phi_1(y+x)+x\phi_2(y+x).$

$$\text{P.I.}=\frac{1}{(D-D')^2}x+\frac{1}{(D-D')^2}\phi(x+y)$$

second is a case of failure

$$=\frac{1}{D^2}\left(1-\frac{D'}{D}\right)^{-2}x+x\frac{1}{2(D-D')}\phi(x+y)$$

multiplying second by x and diff. its deno. w.r.t. D

$$=\frac{1}{D^2}\left(1-\frac{2D'}{D}+\ldots\right)x+x^2.\tfrac{1}{2}\phi(x+y)$$

$$=\tfrac{1}{6}x^3+\tfrac{1}{2}x^2\phi(x+y).$$

Hence the complete solution is

$$z=\phi_1(y+x)+x\phi_2(y+x)+\tfrac{1}{6}x^3+\tfrac{1}{2}x^2\phi(x+y).$$

Exercises

Solve the following equations :

1. $(2D^2-DD'-3D'^2)\,z=5e^{x+y}.$

   Ans. $z=\phi_1(y-x)+\phi_2(2y+3x)+xe^{x-y}.$

2. $(D^2-5DD'+4D'^2)\,z=\sin(4x+y),$

   Ans. $z=\phi_1(y+x)+\phi_2(y+4x)-\frac{1}{3}x\cos(4x+y).$

3. $(D^2-6DD'+9D'^2)\,z=6x+2y.$

   Ans. $z=\phi_1(y+3x)+x\phi_2(y+3x)+x^2(3x+y).$

3 9. A general method of finding the P.I.

Consider the equation

$$(D-mD')\,z=f(x,y).$$

This can be written as

$$p-mq=f(x,y).$$

Lagrange's subsidiary equations for this are

$$\frac{dx}{1}=\frac{dy}{-m}=\frac{dz}{f(x,y)}.$$

The first two relations give

$$y+mx=c \text{ (constant).} \qquad \ldots(1)$$

Taking $\dfrac{dx}{1}=\dfrac{dz}{f(x,y)}$, we get on integration

$$z=\int f(x,y)\,dx$$

$$=\int f(x, c-mx)\,dx \text{ as putting } y=c-mx \text{ from (1).}$$

Thus $\quad z=\dfrac{1}{D-mD'}f(x,y)=\displaystyle\int f(x,c-mx)\,dx,$

where the constant c is to be replaced by $y+mx$ after integration, as the particular integral is not to contain an arbitrary constant.

Now if the equation is $F(D, D')\ z=f(x, y)$,

where $F(D, D')=(D-m_1D')\ (D-m_2D')...(D-m_nD')$,

then $\text{P.I.}=\dfrac{1}{D-m_1D'}\cdot\dfrac{1}{D-m_2D'}\cdots\dfrac{1}{D-m_nD'}f(x, y).$

This can now be evaluated by the repeated application of the above method.

Ex. 1. *Solve* $r+s-6t=y\cos x$ **[Raj. 66 ; Agra 63, 61]**

Solution. The equation can be written as

$(D^2+DD'-6D'^2)\ z=y\cos x.$

The auxiliary equation is $m^2+m-6=0$, giving

$m=2, -3.$

$\text{C.F.}=\phi_1(y+2x)+\phi_2(y-3x).$

For finding particular integral, we use the general method.

$$\text{P.I.}=\frac{1}{(D-2D')\ (D+3D')}\ y\cos x$$

$$=\frac{1}{D-2D'}\int(c+3x)\cos x\, dx,$$

because corresponding to $(D+3D')\ z=0,\ y-3x=c$

$$=\frac{1}{D-2D'}[c\sin x+3x\sin x+3\cos x]$$

$$=\frac{1}{D-2D'}\cdot[(y-3x)\sin x+3x\sin x+3\cos x]$$

replacing c by $y-3x$

$$=\frac{1}{D-2D'}[y\sin x+3\cos x].$$

Again when $(D-2D')\ z=0,\ y+2x=c'$.

$$\therefore\quad \text{P.I.}=\int[(c'-2x)\sin x+3\cos x]\ dx$$

$$=-c'\cos x-2\ (-x\cos x+\sin x)+3\sin x$$

$$=-(y+2x)\cos x+2x\cos x+\sin x \text{ as } c'=y+2x$$

$$=-y\cos x+\sin x.$$

Therefore the complete solution is

$$z=\phi_1(y+2x)+\phi_2(y-3x)-y\cos x+\sin x.$$

Ex. 2. *Solve* $(D^2+2DD'+D'^2)\ z=2\cos y-x\sin y.$

Solution. The auxiliary equatioa is $m^2+2m+1=0$,

i.e., $(m+1)^2=0$ or $m=-1, -1$,

$\therefore\ \text{C.F.}=\phi_1(y-x)+x\phi_2(y-x).$

Now $\text{P.I.}=\dfrac{1}{D^2+2DD'+D'^2}(2\cos y-x\sin y)$

$$=\frac{1}{D+D'}\cdot\frac{1}{D+D'},\ (2\cos y-x\sin y)$$

$$=\frac{1}{D+D'}\int[2\cos(c+x)-x\sin(c+x)]\,dx$$

as for $(D+D')\,z=0,\ y-x=c$

$$=\frac{1}{D+D'}[2\sin(c+x)+x\cos(c+x)-\sin(c+x)]$$

$$=\frac{1}{D+D'}(\sin y+x\cos y)\text{ replacing } c \text{ by } y-x$$

$$=\int[\sin(c+x)+x\cos(c+x)]\,dx\text{ as again } y-x=c$$

$$=-\cos(c+x)+x\sin(c+x)+\cos(c+x).$$

on integratio·

$$=x\sin(c+x)=x\sin y\text{ as } c=y-x.$$

Therefore the complete solution is

$$z=\phi_1(y-x)+x\phi_2(y-x)+x\sin y.$$

Ex. 3. *Solve* $r-t=\tan^3 x\tan y-\tan x\tan^3 y.$ **[Agra 72]**

Solution. The given equation can be written as

$$(D^2-D'^2)\,z=\tan^3 x\tan y-\tan x\tan^3 y.$$

A.E. is $m^2-1=0$ or $m=\pm 1,$

$$\therefore\ \text{C.F.}=\phi_1(y+x)+\phi_2(y-x)$$

and $\text{P.I.}=\frac{1}{D-D'^2}\tan x\tan y\,(\tan^2 x-\tan^2 y)$

$$=\frac{1}{(D+D')}\frac{1}{(D-D')}\tan x\tan y\,(\sec^3 x-\sec^2 y)$$

$$=\frac{1}{D+D'}\int[\tan x\sec x^2\tan(c-x)$$

$$-\tan x\tan(c-x)\sec^2(c-x)]\,dx$$

as corresponding to $(D-D')\,z=0,\ y+x=c$

$$=\frac{1}{D+D'}\Big[\tfrac{1}{2}\tan^2 x\tan(c-x)+\tfrac{1}{2}\int\tan^2 x\sec^2(c-x)\,dx$$

$$+\tfrac{1}{2}\tan x\tan^2(c-x)-\tfrac{1}{2}\int\tan^2(c-x)\sec^2 x\,dx\Big]$$

$$=\tfrac{1}{2}\frac{1}{D+D'}\Big[\tan^2 x\tan(c-x)+\tan x\tan^2(c-x)$$

$$+\int\{\sec^2 x-\sec^2(c-x)\}\,dx\Big]$$

$$=\tfrac{1}{2}\frac{1}{D+D'}[\tan^2 x\tan y+\tan x\tan^2 x+\tan x+\tan y]$$

replacing c by $y+x$

$$=\frac{1}{2}\int [\tan x \sec^2 (c'+x)+\tan (c'+x) \sec^2 x]\, dx$$

as corresponding to $(D+D')\, z=0, y-x=c'$

$$=\frac{1}{2} \tan x \tan (c'+x)$$

$$=\frac{1}{2} \tan x \tan y \text{ as } c'=y-x.$$

Therefore the complete solution is

$$y=\phi_1(y+x)+\phi_2(y-x)+\tfrac{1}{2} \tan x \tan y.$$

Ex. 4. *Solve* $r-s-2t=(2x^2+xy-y^2) \sin (xy)-\cos (xy)$.

Solution. The equation can be written as

$$(D^2-DD'-2D'^2)\, z=(2x^2+xy-y^2) \sin xy-\cos xy.$$

A.E. is $m^2-m-2=0$, giving $m=+2, -1$.

$\therefore$ C.F. $=\phi_1(y+2x)+\phi_2(y-x)$.

$$\text{P.I.}=\frac{1}{(D+D')(D-2D')}[(2x-y)(x+y) \sin xy-\cos xy]$$

$$=\frac{1}{D+D'}\int [(4x-c)(c-x) \sin \{x(c-2x)\}-\cos \{x(c-2x)\}]\, dx$$

as corresponding to $(D-2D')\, z=0, y+2x=c$

$$=\frac{1}{D+D'}\left[(c-x) \cos \{x(c-2x)\}+\int \cos \{x(c-2x)\}\, dx - \int \cos \{x(c-2x)\}\, dx\right]$$

integrating the first integral by parts

$$=\frac{1}{D+D'}(c-x) \cos \{x(c-2x)\}$$

$$=\frac{1}{D+D'}(y+x) \cos xy \text{ replacing } c \text{ by } y+2x$$

$$=\int (c'+2x) \cos \{x(c'+x)\}\, dx$$

as corresponding to $(D+D')\, z=0, y-x=c'$

$$=\sin x(c'+x)=\sin xy.$$

Thus the complete solution is

$$z=\phi_1(y+2x)+\phi_2(y-x)+\sin xy$$

Ex. 5. *Solve* $(D^2-4D'^2)\, z=\dfrac{4x}{y^2}-\dfrac{y}{x^2}$.

Solution. A.E. is $m^2-4=0$, giving $m=\pm 2$

C.F. $=\phi_1(y+2x)+\phi_2(y-2x)$.

$$\text{P.I.}=\frac{1}{(D+2D')(D-2D')}\left[\frac{4x}{y^2}-\frac{y}{x^2}\right]$$

$$=\frac{1}{D+2D'}\int\left[\frac{4x}{(c-2x)^2}-\frac{c-2x}{x^2}\right] dx$$

as corresponding to $(D-2D')\, z=0, y+2x=c$

$$=\frac{1}{D+2D'}\int\left[-\frac{2}{c-2x}-\frac{-2c}{(c-2x)^2}-\frac{c}{x^2}+\frac{2}{x}\right]dx$$

$$=\frac{1}{D+2D'}\left[\log(c-2x)+\frac{c}{(c-2x)}+\frac{c}{x}+2\log x\right]$$

$$=\frac{1}{D+2D'}\left[\log y+\frac{y+2x}{y}+\frac{y+2x}{x}+2\log x\right]$$

replacing c by $y+2x$

$$=\int\left[\log(c'+2x)+\frac{c'+4x}{c'+2x}+\frac{c'+4x}{x}+2\log x\right]dx$$

as corresponding to $(D+2D')\,z=0,\ y-2x=c'$

$$=\int\left[\log(c'+2x)+2-\frac{c'}{c'+2x}+\frac{c'}{x}+4+2\log x\ dx\right]$$

$$=\int\left[1.\log(c'+2x)+6-\frac{c'}{c'+2x}+\frac{c'}{x}+2\log x\right]dx$$

$$=x\log(c'+2x)-\int\frac{2x\,dx}{c'+2x}+6x-\tfrac{1}{2}c'\log(c'+2x)+c'\log x$$

$$+2x\log x-\int 2\,dx$$

$$=x\log(c'+2x)+(c'+2x)\log x+4x-\int\frac{c'+2x-c'}{c'+2x}dx$$

$$-\tfrac{1}{2}c'\log(c'+2x)$$

$$=x\log(c'+2x)+(c'+2x)\log x+4x-\int dx+\int\frac{c'}{c'+2x}dx$$

$$-\tfrac{1}{2}c'\log(c'+2x)$$

$$=x\log(c'+2x)+(c'+2x)\log x+3x$$

$$=x\log y+y\log x+3x \quad \text{as} \quad c'=y-2x.$$

Hence the complete solution is

$$z=\phi_1(y+2x)+\phi_2(y-2x)+x\log y+y\log x+3x.$$

Exercises

Solve the following differential equations .

1. $(D^3+D^2D'-DD'^2-D'^3)\,z=e^y\cos 2x.$

Ans. $z=\phi_1(y+x)+\phi_2(y-x)+x\phi_3(y-x)$
$-\frac{1}{25}e^y\cos 2x-\frac{2}{25}e^y\sin 2x.$

2. $(D^2+5DD'+5D'^2)\,z=x\sin(3x-2y).$

Ans. $z=\phi_1\{y+\frac{1}{2}(-5+\sqrt{5})\,x\}+\phi_2\{y+\frac{1}{2}(-5-\sqrt{5})\,x\}$
$+x\sin(3x-y)+4\cos(3x-2y).$

3. $(D^2-3DD'+2D^2)\,z=e^{2x-y}+e^{x+y}+\cos(x+2y).$

Ans. $z=\phi_1(y+2x)+\phi_2(y+x)+\frac{1}{12}e^{2x-y}-xe^{2x+y}-\frac{1}{3}\cos(x+2y)$

4. $(D^3-7DD'^2-6D'^3)\,z=\cos(x-y)+x^2+xy^2+y^3.$

Ans. $z=\phi_1(y-x)+\phi_2(y+3x)+\phi_3(y-2x)-\frac{1}{4}x\cos(x-y)$
$+\frac{5}{72}x^6+\frac{1}{60}x^5+\frac{7}{20}x^5y+\frac{1}{24}x^4y^2+\frac{1}{6}x^3y^3.$

5. $(D^3-4D^2D'+5DD'^2-2D'^3)\ z=e^{y+2x}+(y+x)^{1/2}$.

Ans. $z=\phi_1(y+2x)+\phi_2(y+x)+x\phi_3(y+x)+xe^{y+2x}-\frac{4}{15}x^2(y+x)^{3/2}$.

6. $(D^2+6DD'+6D'^2)\ z=\dfrac{1}{y-2x}$.

Ans. $z=\phi_1(y-2x)+\phi_2(y-3x)+x-(y-3x)\log(y-2x)$.

7. $(D^2-DD'-2D'^2)\ z=(y-1)\ e^x$.

Ans. $z=\phi_1(y-x)+\phi_2(y+2x)+ye^x$.

3·10. Non-Homogeneous Linear Equations

A linear partial differential equation which is not homogeneous is called a non-homogeneous, linear equation. Consider the differential equation $\quad F(D, D')\ z=f(x, y)$
where $F(D, D')$ is now not necessarily homogeneous. While $F(D, D')$ when it is homogeneous, is always resolvable into linear factors, the same is not always true when $F(D, D')$ is non-homogeneous. Therefore we classify linear differential operators $F(D, D')$ into two main types, which we shall treat separately. These are :

(*i*) $F(D, D')$ is *reducible* if it can be expressed as product of linear factors of the form $D+aD'+b$. where a and b are constants.

(*ii*) $F(D, D')$ is *irreducible, i.e.* when $F(D, D')$ is not reducible for example D^2-D'.

We first take up case of reducible $F(D, D')$ and it can be simply verified that the order in which linear factors occur is important.

3·11. Complementary functions corresponding to linear factors

Let $\alpha D+\beta D'+\gamma$ be a factor of $F(D, D')$. To find C.F. corresponding to this factor, consider the most simple non-homogeneous equation $\quad (\alpha D+\beta D'+\gamma)\ z=0.$

This can be written as $\alpha p+\beta q=-\gamma z$.

The Lagrange's subsidiary equations for it are

$$\frac{dx}{\alpha}=\frac{dy}{\beta}=\frac{dz}{-\gamma z}$$

The first to relations give

$$\alpha\ dy=\beta\ dx \quad \text{or} \quad \beta x-\alpha y=c.$$

Also $\dfrac{dx}{\alpha}=\dfrac{dz}{-\gamma z}$ gives $\log z=-\dfrac{\gamma}{\alpha}x+\text{const.}$

or $\qquad z=\text{const}.\ e^{(-\gamma x/\alpha)}$

Thus the complementary function is

$$z=e^{(-\gamma x/\alpha)}\ \phi(\beta x-\alpha y)$$

where ϕ is an arbitrary function.

Note. If the linear factor is $D-mD'-\gamma$, then the corresponding C F. is $e^{\gamma x}\phi(y+mx)$.

We now come to the various cases that arise :

I. *F(D, D') has repeated linear factors.*

If $F(D, D')=(\alpha_1 D+\beta_1 D'+\gamma_1)\ldots(\alpha_n D+\beta_n D'+\gamma_n)$ when all the factors are distinct, then the C.F. of $F(D, D')\,z=0$ is

$$z=e^{(-\gamma_1 x/\alpha_1)}\phi_1(\beta_1 x-\alpha_1 y)+\ldots+e^{(-\gamma_n x/\alpha_n)}\phi_n(\beta_n x-\alpha_n y).$$

II. *F(D, D') has repeated roots.*

Let a factor $\alpha D+\beta D'+\gamma$ occur twice in $F(D, D')$.

Consider $(\alpha D+\beta D'+\gamma)^2 z=0$ ...(1)

Take $(\alpha D+\beta D'+\gamma)\; z=Z.$ ...(2)

Equating (1) now becomes

$$(\alpha D+\beta D'+\gamma)\, Z=0.$$

This gives $Z=e^{(-\gamma x/\alpha)}\;\phi_1(\beta x-\alpha y)$ as above.

And (2) now becomes

$$(\alpha D+\beta D'+\gamma)\; z=e^{(\gamma x/\alpha)}\phi_1(\beta x-\alpha y).$$

This can be written as

$$\alpha p+\beta q=-\gamma z+e^{(-\gamma x/\alpha)}\;\phi_1(\beta x-\alpha y).$$

The Lagrange's subsidiary equations for this are

$$\frac{dx}{\alpha}=\frac{dy}{\beta}=\frac{dz}{-\gamma z+e^{(-\gamma x/\alpha)}\phi_1\,(\beta x-\alpha y)}.$$

The first two relations give $\beta x-\alpha y=c.$

Again first and last, give

$$\frac{dx}{\alpha}=\frac{dz}{-\gamma z+e^{(-\gamma x/\alpha)}\phi(c)}$$

or $$\frac{dz}{dx}+\frac{\gamma}{\alpha}\,z=\frac{1}{\alpha}\,e^{(-\gamma x/\alpha)}\phi(c)$$

a linear equation of first with integrating factor

$$e^{\int P\,dx}=e^{(\gamma x/\alpha)}$$

Therefore,

$$ze^{(\gamma x/\alpha)}=c_1+\int\frac{1}{\alpha}\;\phi(c)\;dx$$

$$=c_1+\frac{1}{\alpha}\;x\phi(c)$$

or $$z=\phi_1(\beta x-\alpha y)\;e^{-\gamma x/\alpha}+xe^{-\gamma x/\alpha}\phi_2(\beta x-\alpha y)$$

taking $c_1=\phi_1(c)=\phi_1(\beta x-\alpha y)$ etc.

Thus C.F. is $e^{-\gamma x/\alpha}\,[\phi_1(\beta x-\alpha y)+x\phi_2(\beta x-\alpha y)]$.

In general if $\alpha D+\beta D'+\gamma$ occurs n times in $F(D, D')$, then the corresponding point of C.F. is

$$e^{(-\gamma x/\alpha)}\,[\phi_1(\beta x-\alpha y)+x\phi_2(\beta x-\alpha y)+x^{n-1}\,\phi_n(\beta x-\alpha y)].$$

Note. If the factor $D-mD'-\gamma$ repeats n times ... corresponding to it is

$e^{\gamma x}\,[\phi_1(y+mx)+x\,\phi_2(y+mx)+\ldots+x^{n-1}\phi_n(y+mx)]$.

Cor. If a factor of $F(D, D')$ is $\beta D'+\gamma$, and occurs only once, then corresponding to it,

C.F. is $e^{(-\gamma y/\beta)}\,\phi(\beta x)$.

Next if $\beta D'+\gamma$ repeats n times, then its contribution in C.F. is

$e^{(-\gamma y/\beta)}\,[\phi_1(\beta x)+x\phi_2(\beta x)+\ldots+x^{n-1}\,\phi_n(\beta x)]$.

Ex. 1. *Solve* $(D^2-a^2D'^2+2abD+2a^2bD')\,z=0$.

Solution. The equation can be written as

$(D+aD')\,(D-aD'+2ab)z=0$;

there being linear distinct factors, the solution is

$z=\phi_1(y-ax)+e^{-2ab}\phi_2(y+ax)$.

Ex. 2. *Solve* $(D-2D'+5)^2\,z=0$.

Solution. The equation can be written as

$[D-2D'-(-5)]^2\,z=0$.

There are repeated linear factors.

Hence the solution is

$z=e^{-5x}\,\phi_1(y+2x)+xe^{-5x}\,\phi_2(y+2x)$.

Exercises

Solve the following differential equations :

1. $(D+D'-1)\,(D+2D'-2)\,z=0$.

Ans. $z=e^x\phi_1(y-x)+e^{2x}\phi_2(y-2x)$

2. $r+2s+t+2p+2q+z=0$.

Ans. $z=e^{-x}\,[\phi_1(y-x)+x\phi_1(y-x)]$

3. $r-t+p-q=0$. Ans. $z=\phi_1\,(y+x)+e^{-x}\phi_2(y-x)$

3·12. Complete Solution.

The complete solution of

$F(D, D')=f\,(x, y)$

is $z=\text{C.F.}+\text{P.I.}$

where $\text{P.I.}=\dfrac{1}{F(D, D')}f\,(x, y)$.

Now the particular integral of non-homogeneous partial differential equation can be found in a very simple way in some of the cases. We discuss these below.

3·13. Particular Integral

Particular integral of non-homogeneous partial differential equation can be found in a way similar to those of ordinary differential equations. We give some cases of finding the particular equations.

Case I. When $f\,(x, y)=e^{ax+by}$.

We have $De^{ax+by}=ae^{ax+by}$

$D^r e^{ax+by}=a^r e^{ax+by}$, etc.

and $D'^r e^{ax+by}=b^r e^{ax+by}$.

$$\therefore \quad F(D, D')\, e^{ax+by} = F(a, b)\, e^{ax+by}.$$

Operating both the sides by $\frac{1}{F(D, D')}$, we get

$$\frac{1}{F(D, D')} F(D, D')\, e^{ax+by} = \frac{1}{F(D, D')} F(a, b)\, e^{ax+by}$$

or
$$e^{ax+by} = F(a, b) \frac{1}{F(D, D')} e^{ax+by}$$

or dividing by $F(a, b)$, we get $\frac{1}{F(a, b)} e^{ax+by} = \frac{1}{F(D, D')} e^{ax+by}$.

Thus
$$\boxed{\frac{1}{F(D, D')} = e^{ax+by} \frac{1}{F(a, b)} e^{ax+by}}$$

provided that $F(a, b) \neq 0$.

Case II When $f(x, y) = \sin(ax+by)$.

We know that $D \sin(ax+by) = a \cos(ax+by)$,

$D^2 \sin(ax+by) = (-a^2) \sin(ax+by)$,

$DD' \sin(ax+by) = (-ab) \sin(ax+by)$

and $D'^2 \sin(ax+by) = (-b^2) \sin(ax+by)$.

From the results, we see then

$$\frac{1}{F(D, D')} \sin(ax+by) \frac{1}{F(D^2, DD', D'^2, D, D')} \sin(ax+by)$$

$$= \frac{1}{F(-a^2, -ab, -b^2, D, D')} \sin(ax+by),$$

putting $D^2 = -a^2$, $DD' = -ab$ and $D'^2 = -b^2$.

This can be evaluated further.

Case III When $f(x, y) = x^m y^n$.

Here as usual $\frac{1}{F(D, D')} x^m y^n = F[(D, D')]^{-1} x^m y^n$,

which can be evaluated after expanding $F(D, D')^{-1}$, in powers of D and D'.

Case IV. **To evaluate** $\frac{1}{F(D, D')}(e^{ax-by}V)$

where V is a function of x and y.

Here also it can be checked up that we have

$$\frac{1}{F(D, D')} (e^{ax+by}V) = e^{ax+by} \frac{1}{F(D+a, D+b)} V.$$

The following solved examples will illustrate the procedure.

Solution. The complementary function is

$e^x\phi_1(y+x) + e^{2x}\phi_2(y+x)$.

$$\text{P.I.} = \frac{1}{(D-D'-1)(D-D'-2)} e^{2x-y}$$

$$=\frac{1}{\{2-(-1)-1\}\{2-(-1)-2\}}e^{2x-y}$$

writting 2 for D and -1 for D'

$$=\tfrac{1}{2}e^{2x-y}.$$

Hence the complete solution is

$$z=e^x\phi_1(y+x)+e^{2x}\phi_2(y+x)+\tfrac{1}{2}e^{2x-y}.$$

Ex. 2. *Solve* $(D^2+DD'+D'-1)\ z=\sin(x+2y)$.

[Rajasthan 66]

Solution. The given equation is

$$(D+1)(D+D'-1)\ z=\sin(x+2y).$$

$\therefore$ $\quad$ $\text{C.F.}=e^{-x}\phi_1(y)+e^x\phi_2(y-x)$

Now $\quad \text{P.I.}=\dfrac{1}{D^2+DD'+D'-1}\sin(x+2y)$

$$=\frac{1}{-1^2-1.2+D'-1}\sin(x+2y)$$

writting -1^2 for D^2 and -1.2 for DD'

$$=\frac{1}{D'-4}\sin(x+2y)=\frac{D'+4}{D'^2-16}\sin(x+2y)$$

$$=(D'+4)\frac{1}{2^2-16}\sin(x+2y)$$

$$=-\tfrac{1}{20}(D'+4)\sin(x+2y)$$

$$=-\tfrac{1}{20}[2\cos(x+2y)+4\sin(x+2y)].$$

Therefore the complete solution is

$$z=e^{-x}\phi_1(y)+e^x\phi_2(y-x)-\tfrac{1}{20}[2\cos(x+2y)+4\sin(x+2y)].$$

***Ex. 3.** *Solve*

$$\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial x\,\partial y}+\frac{\partial z}{\partial y}-z=\cos(x+2y)+e^x.$$

[Agra 72, 65, 66]

Solution. This equation can be written as

$$(D-1)(D-D'+1)\ z=\cos(x+2y)+e^y.$$

$\therefore$ $\text{C.F.}=e^x\phi_1(y)+e^{-x}\phi_1(y+x).$

Now $\quad \text{P.I.}=\dfrac{1}{D^2-DD'+D'-1}\cos(x+2y)$

$$+\frac{1}{D^2-DD'+D'-1}e^y$$

We consider these separately. So

$$\frac{1}{D^2-DD'+D'-1}e^y=x\,\frac{1}{2D-D'},\ e^y,$$

multiplying by x and differentiating the deno. w.r.t. D.

as $D^2-DD'+D'-1=0$ when coeff. $D=0$, $D'=1$

$=-xe^y$ pntting $D=0$, $D=1$

and $\dfrac{1}{D^2-DD'+D'-1}\cos(x+2y)$

$=\dfrac{1}{-1^2+1.2+D'-1}\cos(x+2y)=\dfrac{1}{D'}\cos(x+2y)$

$=\frac{1}{2}\sin(x+2y).$

$\therefore$ P I. $=\frac{1}{2}\sin(x+2y)-xe^y.$

Therefore the complete solution is

$z=e^x\phi_1(y)+e^{-x}\phi_2(y+x)+\frac{1}{2}\sin(x+2y)-xe^y,$

Ex. 4. *Solve* $(D+D'-1)(D+2D'-3)z=2x+3y.$

Solution. C.F. $=e^x\phi_1(y-x)+e^{3x}\phi_2(y-2x)$

and P.I. $=\dfrac{1}{(D+D'-1)(D+2D'-3)}(2x+3y)$

$=\frac{1}{3}[1-(D+D')]^{-1}[1-\frac{1}{3}(D+2D')]^{-1}(2x+3y)$

$=\frac{1}{3}[1+D+D'+...][1+\frac{1}{3}(D+2D')+...](2x+3y)$

$=\frac{1}{3}(1+\frac{4}{3}D+\frac{5}{3}D'+...)(2x+3y)$

$=\frac{1}{3}(2x+3y+\frac{8}{3}+5)=\frac{2}{3}x+y+\frac{23}{9}.$

Hence the complete solution is

$z=e^x\phi_1(y-x)+e^{3x}\phi_2(x-2x)+\frac{2}{3}x+y+\frac{23}{9}.$

Ex. 5. *Solve* $(D^2+DD'+D'-1)z=x^2y.$

Solution $(D+1)(D+D'-1)z=x^2y.$

$\therefore$ C.F. $=e^{-x}\phi_1(y)+e^x\phi_2(y-x).$

P.I. $=\dfrac{1}{(D+1)(D+D'-1)}x^2y$

$=-(1+D)^{-1}-[1-(D+D')]^{-1}x^2y$

$=-(1-D+D^2...)[1+D+D'+(D+D')^2+(D+D')^3...]x^2y$

$=-[1-D+D^2+...][1+D'+D^2+2DD'+D'^2+3D^2D'...]x^2y$

$=-[1+D^2+DD'+D'+4D^2D']x^2y$

$=-(x^2y+2y+2x+x^2+8).$

Hence the complete solution is

$z=r^{-x}\phi_1(y+e^x\phi_2(y-x)-(x^2y+2y+2x+x^2+8).$

Ex. 6. *Solve* $(D^2-D'^2-3D+3D')z=xy+e^{x+2y}.$

[Delhi Hons. 71; Rajasthan 64; Agra 58]

Solution. The equation can be written as

$(D-D')(D+D'-3)z=xy+e^{x+2y}.$

$\therefore$ C.F. $=\phi_1(x+y)+e^{3x}\phi_2(y-x).$

Part of the particular integral corresponding to e^{x+2y} is

$\dfrac{1}{(D-D')(D+D'-3)}(e^{x+2y})$ case of failure

$=e^x\dfrac{1}{(1-D')(D'-2)}e^{2y}$ putting $D=1$

$=e^{x+2y}\dfrac{1}{(1-D'-2)(D'+2-2)}\cdot 1$

$$=-e^{x+2y}\frac{1}{D'(D'+1)}.1=-e^{(x+2y)}\frac{1}{D'}(1+D')^{-1}.1$$

$$=-e^{x+2y}\frac{1}{D'}(1-D'+...)1=-ye^{x+2y}.$$

Also part of the particular integral corresponding to xy is

$$\frac{1}{(D-D')(D+D'-3)}xy$$

$$=-\tfrac{1}{3}\frac{1}{D-D'}[1+\tfrac{1}{3}(D+D')+\{\tfrac{1}{3}(D+D')\}^2+...]xy$$

$$=-\frac{1}{3D}\left(1+\frac{D'}{D}\right)(xy+\tfrac{1}{3}x+\tfrac{1}{3}y+\tfrac{2}{9})$$

$$=-\frac{1}{3D}(xy+\tfrac{1}{3}x+\tfrac{1}{3}y+\tfrac{2}{9}+\tfrac{1}{2}x^2+\tfrac{1}{3}x)$$

$$=-(\tfrac{1}{6}x^2y+\tfrac{1}{9}x^2+\tfrac{1}{9}xy+\tfrac{2}{27}x+\tfrac{1}{18}x^3).$$

Thus P. I. $=-ye^{x+2y}-(\tfrac{1}{6}x^2y+\tfrac{1}{9}x^2+\tfrac{1}{9}xy+\tfrac{2}{27}x+\tfrac{1}{18}x^3)$.

Hence complete solution is

$$z=\phi_1(y+x)+e^{3x}\phi_2(y+x)-ye^{x+2y}$$
$$-(\tfrac{1}{6}x^2y+\tfrac{1}{9}x^2+\tfrac{1}{9}xy+\tfrac{2}{27}x+\tfrac{1}{18}x^3)$$

Ex. 7. *Solve* $(D-D'-1)(D-D'-2)z=e^{2x-y}+x$.

[Rajasthan 59]

Solution. The C.F. $=e^x\phi_1(y+x)+e^{2x}\phi_2(y+x)$.

Part of the P.I. corresponding to e^{2x-y} is

$$\frac{1}{(D-D'-1)(D-D'-2)}e^{2x-y}$$

$$=\frac{1}{(2-(-1)-1)(2-(-1)-2)}e^{2x-y}$$

$$=\tfrac{1}{2}e^{2x-y}.$$

Also part of P.I, corresponding to x is

$$\frac{1}{(D-D'-1)(D-D'-2)}x$$

$$=\tfrac{1}{2}[1-(D-D')]^{-1}[1-\tfrac{1}{2}(D-D')]^{-1}x$$

$$=\tfrac{1}{2}[1+D-D'+...][1+\tfrac{1}{2}(D-D')+...]x$$

$$=\tfrac{1}{2}(1+\tfrac{3}{2}D-\tfrac{3}{2}D'+...)x$$

$$=\tfrac{1}{2}x+\tfrac{3}{4}.$$

Thus P.I. $=\tfrac{1}{2}^{2x-y}+\tfrac{1}{2}x+\tfrac{3}{4}$.

Thus the complete solution is

$$z=e^x\phi_1(y+x)+e^{2x}\phi_2(y+x)+\tfrac{1}{2}e^{2x-y}+\tfrac{1}{2}x+\tfrac{3}{4}.$$

Ex. 8. *Solve* $\dfrac{\partial^2 y}{\partial x^2}-\dfrac{\partial^2 z}{\partial y^2}+\dfrac{\partial z}{\partial y}+3\dfrac{\partial z}{\partial y}-2z=e^{x-y}-x^2y.$

[Rajasthan 60 ; Agra 57]

Solution. The equation can be written as

$$(D^2-D'^2+D+3D'-2)\,z=e^{x-y}-x^2y.$$

or $(D-D'+2)(D+D'-1)\,z=e^{x-y}-x^2y.$

$\therefore$ $\text{C.F.}=e^{-2x}\phi_1(y+x)+e^{x}\phi_2(y-x).$

Now the part of P. I. corresponding to e^{x-y} is

$$\frac{1}{(D^2-D'^2+D+3D'-2)}\,e^{x-y}$$

$$=\frac{1}{1^2-(-1)^2+1+3(-1)-2}\,e^{x-y}=-\tfrac{1}{4}e^{x-y}$$

and part of P.I. corresponding to $-x^2y$,

$$\frac{1}{(D^2-D'^2+D+3D'-2)}\,(-x^2y)$$

$$=\tfrac{1}{2}\left[1-\tfrac{1}{2}(3D'+D-D'^2\times D^2)\right]^{-1}x^2y$$

$$=\tfrac{1}{2}\left[1+\tfrac{1}{2}(3D'+D-D'^2+D^2)+\tfrac{1}{4}(3D'+D-D'^2+D^2)^2+\tfrac{1}{8}(3D'+D-D'^2+D^2)^3+\ldots\right]x^2y$$

$$=\tfrac{1}{2}\left[1+\tfrac{1}{2}D+\tfrac{3}{2}D'+\tfrac{1}{2}D^2+\tfrac{1}{4}D^2+\tfrac{3}{2}DD'+\tfrac{3}{2}D^2D'+\tfrac{3}{2}D^2D'+\tfrac{3}{4}D^2D'+\ldots\right]x^2y$$

$$=\tfrac{1}{2}\left[x^2y+xy+\tfrac{3}{2}x^2+y+\tfrac{1}{2}y+3x+3+\tfrac{3}{2}+\tfrac{3}{4}\right].$$

$\therefore$ $\text{P.I.}=\tfrac{1}{4}e^{x-y}+\tfrac{1}{2}\left[x^2y+xy+\tfrac{3}{2}x^2+\tfrac{3}{2}y+3x+\tfrac{21}{4}\right].$

The complete solution is

$$z=e^{-2x}\phi_1(y+x)+e^{x}\phi_2(y-x)-\tfrac{1}{4}e^{x-y}+\tfrac{1}{2}\left[x^3+xy+\tfrac{3}{2}x^2+\tfrac{3}{2}y+3x+\tfrac{21}{4}\right]$$

***Ex. 9.** *Solve*

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial^2 z}{\partial x\,\partial y}-2\frac{\partial^2 z}{\partial y^2}+2\frac{\partial z}{\partial x}+2\frac{\partial z}{\partial y}=e^{2x+3y}+\sin(2x+y)+xy.$$

[Delhi Hons 61 ; Vikram 52 ; Agra 66, 63, 56, 54]

Solution. The differential equation can be written as

$$(D+D')(D-2D'+2)\,z=e^{2x+3y}+\sin(2x+y)+xy$$

$$\text{C.F.}=\phi_1(y-x)+e^{-2x}\phi_2(2x+y).$$

Now P.I. corresponding to e^{2x+3y}

$$=\frac{1}{(D+D')(D-2D'+2)}\,e^{2x+3y}$$

$$=\frac{1}{(2+3)(2-6+2)}\,e^{2x+3y}=-\tfrac{1}{10}e^{2x+3y}$$

Also P.I. corresponding to $\sin(2x+y)$

$$=\frac{1}{D^2-DD'-2D'^2+2D+2D'}\sin(2x+y)$$

$$=\frac{1}{-2+2.1-2(-1^2)+2D+2D'}\sin(2x+y)$$

$$=\frac{1}{2(D+D')}\sin(2x+y)=\tfrac{1}{2}\,\frac{D-D'}{D^2-D'^2}\sin(2x+y)$$

$$=\frac{1}{2}\frac{(D-D')}{-2^2-(-1^2)}\sin(2x+y)$$

$$=-\frac{1}{6}\cos(2x+y).$$

Again P.I. corresponding to xy

$$=\frac{1}{(D+D')(D-2D'+2)}xy$$

$$=\frac{1}{2D}\left(1-\frac{D'}{D}\right)[1-\tfrac{1}{2}(D-2D')+\tfrac{1}{4}(D-2D')^2+\ldots]\,xy$$

$$=\frac{1}{2D}\left(1-\frac{D'}{D}+\ldots\right)(xy-\tfrac{1}{2}y+x-1)$$

$$=\frac{1}{2D}(xy-\tfrac{1}{2}y-\tfrac{1}{2}x^2+\tfrac{3}{2}x-1).$$

$$=\tfrac{1}{2}(\tfrac{1}{2}x^2y-\tfrac{1}{2}xy-\tfrac{1}{6}x^3+\tfrac{3}{4}x^2-x).$$

Thus

$$\text{P.I.}=-\tfrac{1}{10}e^{2x+3y}-\tfrac{1}{6}\cos(2x+y)+\tfrac{1}{2}(\tfrac{1}{2}x^2y-\tfrac{1}{2}xy-\tfrac{1}{6}x^3+\tfrac{3}{4}x^2-x)$$

and the complete solution is $z=\text{C.F.}+\text{P.I.}$

Ex. 10. *Solve* $(D-3D'-2)^2 z=2e^{2x}\tan(y+3x)$.

Solution. We have

$$\text{C.F.}=e^{2x}[\phi_1(x+3x)+x\phi_2(y+3x)].$$

Now $$\text{P.I.}=\frac{1}{(D-3D'-2)^2}2e^{2x}\tan(y+3x)$$

$$=2e^{2x}\frac{1}{(D-3D')^2}\tan(y+3x)$$

putting $D+2$ for D and taking e^{2x} outside.

This is now a case when operator is homogeneous in D and D' and function is of the kind $\phi(ax-by)$. Also since when $D=3$, and $D'=1$, $D-3D'=0$, it is case of failure.

Thus $$\text{P.I.}=2e^{2x}\,x\,\frac{1}{2(D-3D')}\tan(y+3x)$$

multiplying by x and differentiating the denominator w.r.t. D

$$=2e^{2x}.x^2.\tfrac{1}{2}\tan(y+3x)$$

again diff. the deno. w.r.t. D and multiplying by x

$$=x^2e^{2x}\tan(y+3x).$$

Therefore the complete solution is

$$z=e^{2x}[\phi_1(y+3x)+x\phi_2(y+3x)]+x^2e^{2x}\tan(y+3x).$$

Ex. 11. *Solve* $(D^2+DD'-6D'^2)z=x^2\sin(x+y)$.

Solution. The equation is

$$(D-2D')(D+3D')z=x^2\sin(x+y),$$

$$\therefore\ \text{C.F.}=\phi_1(y+2x)+\phi_2(y-3x).$$

Now evaluate the P.I., we take

$\sin(x+y)=$ imaginary part of $e^{i(x+y)}$

and P.I. $=$ Im. part of $\dfrac{1}{D^2+DD'-6D'^2}x^2e^{i(x+y)}$

$=$ Im. part of $e^{i(x+y)}\dfrac{1}{(D+i)^2+i(D+i)+6}x^2$

putting $D+i$ for D and i for D'

$=$ Im. part of $e^{i(x+y)}\dfrac{1}{D^2+3iD+4}x^2$

$=$ Im. part of $\dfrac{e^{i(x+y)}}{4}\left[1+\dfrac{3iD}{4}+\dfrac{D^2}{4}\right]^{-1}x^2$

$=$ Im. part of $\dfrac{e^{i(x+y)}}{4}\left[1-\dfrac{3iD}{4}-\dfrac{D^2}{4}-\dfrac{9D^2}{16}\dots\right]x^2$

$=$ Im. part of $\dfrac{e^{i(x+y)}}{4}\left[x^2-\frac{3}{2}ix-\frac{13}{8}\right]$

$=\frac{1}{4}\sin(x+y)\left[x^2-\frac{13}{8}\right]-\frac{3}{8}x\cos(x+y)$.

Therefore the complete solution is

$z=\phi_1(y+2x)+\phi_2(y-3x)+\frac{1}{4}\sin(x+y)(x^2-\frac{13}{8})-\frac{3}{8}x\cos(x+y)$.

Exercises

Solve the following differential equations :

1. $(D^2-D'^2-3D')z=e^{x-2y}$.

Ans. $z=\phi_1(y-x)+e^{3x}\phi_2(y-x)-\frac{1}{12}e^{x-2y}$.

2. $(D+D')(D+D'-2)z=\sin(2x+2y)$.

Ans. $z=\phi_1(y+x)+e^{2x}\phi_2(y-x)$
$+\frac{1}{117}[6\cos(x+2y)-9\sin(x+2y)]$

3. $(3DD'-2D'^2-D')z=\sin(2x+3y)$.

Ans. $z=\phi_1(x)+e^{x/3}\phi_2(2x+3y)+\frac{1}{3}\cos(2x+3y)$

4. $(D^2-DD'-2D'+2D+2D')z=e^{2x+3y}+\sin(2x+y)$.

[Agra 68; Delhi Hons. 68]

Ans. $z=\phi_1(y-x)+e^{-2x}\phi_2(2x-y)-\frac{1}{10}e^{2x+3y}-\frac{1}{8}\cos(2x+2y)$.

5. $(D^3-DD'^2-D^2+DD')z=\dfrac{x+2}{x^2}$.

Ans. $z=\phi_1(y)+\phi_2(x+y)+e^x\phi_3(y-x)+\log x$

6. $r-s+p=1$. Ans. $z=\phi_1(y)+e^{-x}\phi_2(y+x)+x$

7. $(D+D'-1)(D+2D'-3)z=4+3x+6y$.

Ans. $z=e^x\phi_1(y-x)+e^{3x}\phi_2(y-2x)+6+x+2y$

8. $s+p-q=xy$.

Ans. $z=e^x\phi_1(y-x)+e^{-y}\phi_2(x)-xy-y+x+1$

9. $(DD'+aD+bD'+ab)\, z=e^{mx+ny}$.

Ans. $z=e^{-bx}\phi_1(y)e^{-ay}\phi_2(x)+\dfrac{e^{mx+ny}}{(m+b)(n+a)}$

10. $(D^2-DD'+D'-1)\, z=\cos(x+2y)+e^y+xy+1$.

Ans. $z=e^x\phi_1(y)+e^y\phi_2(x+y)+\frac{1}{2}\sin(x+2y)+ye^y-x\,(y+1)$

11. $(3D^2-2D'^2+D-1)\, z=4e^{x+y}\cos(x+y)$.

Ans. $z=\sum_{i=1}^{\infty} c_i e^{a_i x+b_i y}+\frac{4}{3}e^{x+y}\sin(x+y)$

where $3a_i^2-2b_i^2+a_i-1=0$

12. $D\,(D-2D'\,(D+D')\, z=e^{x+2y}\,(x^2+4y^2)$.

Ans. $z=\phi_1(y)+\phi_2(y+2x)+\phi_3\,(y-x)$

$-\frac{1}{81}(9x^2+36y^2-18x-72y+76)\, e^{x+2y}$

3·14. Equations reducible to linear form with constant coefficients

A differential equation having variable coefficients can sometimes be reduced to equations with constant coefficients by suitable substitutions. One such form is $F(xD, yD')=f(x, y)$.

Substitution in this case is

$$x=e^u \text{ and } y=e^v,$$

so that $u=\log x$ and $v=\log y$.

Denoting $D\equiv\dfrac{\partial}{\partial u}$ ahd $D'\equiv\dfrac{\partial}{\partial v}$,

it can be easily shown that

$$x\frac{\partial}{\partial y}=D'\; x^2\frac{\partial^2}{\partial x^2}=D\,(D-1),$$

$$y\frac{\partial}{\partial y}=D',\; y^2\frac{\partial^2}{\partial y^2}=D'\,(D'-1),$$

and in general,

$$x^m y^n\frac{\partial^{m+n}}{\partial x^m\,\partial y^n}=x^m\frac{\partial^m}{\partial x^m},\, y^n\frac{\partial^n}{\partial y^n}$$

$$=D\,(D-1)\ldots(D-m+1)\; D'\,(D'-1)\ldots(D'-n+1).$$

These substitutions reduce the equation to an equation having constant cofficients, and can be easily solved by methods discussed in this chapter.

Following examples illustrate the procedure.

Ex. 1. *Solve* $x^2\dfrac{\partial^2 z}{\partial x^2}-y^2\dfrac{\partial^2 z}{\partial y^2}=xy$. **[Agra 72; Indore 67]**

Solution. Put $x=e^u$ and $y=e^v$.

Now if $\dfrac{\partial}{\partial u}\equiv D$ and $\dfrac{\partial}{\partial v}\equiv D'$,

$$x^2\frac{\partial^2}{\partial x}=D\,(D-1) \text{ and } y^2\frac{\partial^2}{\partial y^2}=D'\,(D'-1).$$

With these substitutions the equation becomes

$$[D\,(D-1)-D'-1)]\; z=e^u.e^v$$

or $\quad (D-D')(D+D'-1)\,z=e^{u+v}$.

This is a linear equation with constant coefficients ; independent variable being z and independent variables being u and v.

The C.F. $=\phi_1(u+v)+e^u\phi_2(u-v)$

$=\phi_1(\log x+\log y)+x\phi_2(\log x-\log y)$

$=f_1(xy+xf_2(y/x)$.

Now P.I. $=\dfrac{1}{(D-D')(D+D'-1)}\,e^{u+v}$

$=\dfrac{1}{D-D'}\cdot\dfrac{1}{1+1-1}\,e^{u+v}$

$=\dfrac{1}{D-D'}\,e^{u+v}=u\,\dfrac{1}{1}\,e^{u+v}$

diff. the deno. w.r.t. D and multiplying by u

$=ue^{u+v}=(\log x).xy$ as $x=e^u,\ y=e^v$

$=xy\log x$.

Hence the complete solution is

$z=f_1(xy)+xf_2(y/x)+xy\log x$.

Ex. 2. *Solve* $x^2\dfrac{\partial^2 z}{\partial x^2}-4xy\dfrac{\partial^2 z}{\partial x\,\partial y}+4y^2\dfrac{\partial^2 z}{\partial y^2}+6y\dfrac{\partial z}{\partial y}=x^3y^4$.

[Agra 65, 57]

Solution. Put $x=e^u,\ y=e^v$.

Now if $D\equiv\dfrac{\partial}{\partial u}$ and $D'\equiv\dfrac{\partial}{\partial v}$, we have

$x\dfrac{\partial}{\partial x}=D,\ y\dfrac{\partial}{\partial y}=D',\ x^2\dfrac{\partial^2}{\partial x^2}=D(D-1)$,

$xy\dfrac{\partial^2}{\partial x\,\partial y}=DD',\ y^2\dfrac{\partial^2}{\partial y^2}=D'(D'-1)$.

With these substitutions, the given equation becomes

$[D(D-1)-4DD'+4D'(D'-1)+6D']\,z=e^{3u}.e^{4v}$

or $\quad (D-2D')(D-2D'-1)\,z=e^{3u+4v}$.

This is a linear equation with constant coefficients and dependent variable z and independent variables u and v.

The C.F. $=\phi_1(v+2u)+e^u\phi_2(v+2u)$

$=\phi_1(\log y+2\log x)+x\phi_2(\log y+2\log x)$

$=\phi_1(\log x^2y)+x\phi_2(\log yx^2)$

$=f_1(x^2y)+xf_2(x^2y)$

and P.I. $=\dfrac{1}{(D-2D')(D-2D'-1)}\,e^{3u+4v}$.

$=\dfrac{1}{(3-8)(3-8-1)}\,e^{3u+4v}=\frac{1}{30}e^{3u}.e^{4v}$

$=\frac{1}{30}x^3y^4$, as $e^u=x$ and $e^v=y$.

Therefore the complete solution is

$$z=f_1(x^2y)+xf_2(x^2y)+\tfrac{1}{30}x^3y^4.$$

Ex. 3. *Solve*

$$\left(x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\,\partial y}-x\frac{\partial z}{\partial x}\right)=\frac{x^3}{y^2}.$$

Solution Put $x=e^u$, $y=e^v$.

Now if $D\equiv\dfrac{\partial}{\partial u}$, $D'\equiv\dfrac{\partial}{\partial v}$, then $x\dfrac{\partial}{\partial x}=D$,

$$x^2\frac{\partial^2}{\partial x^2}=D\,(D-1),\ xy\frac{\partial^2}{\partial x\,\partial y}=DD'.$$

With those substitutions the given equation becomes

$$[D\,(D-1)+2DD'-D]\,z=e^{(3u-2v)},$$

$$D\,(D+2D'-2)\,z=e^{(3u-2v)}.$$

$$\text{C.F.}=f_1(v)+e^{2u}f_2(v-2u)$$

$$=f_1\,(\log y)+x^2f_2(\log y-2\log x)$$

$$=\phi_1(y)+x^2\phi_2\left(\frac{y}{x^2}\right).$$

$$\text{P.I.}=\frac{e^{3u-2v}}{D(D+2D'-2)}=\frac{e^{3u-2v}}{3\,(3-4-2)}=-\tfrac{1}{9}e^{3u-2v}$$

$$=\tfrac{1}{9}x^3/y^2.$$

Therefore the complete solution becomes

$$z=\phi_1(y)+x^2\phi_2\left(\frac{y}{x^2}\right)-\frac{1}{9}\frac{x^3}{y^2}.$$

Ex. 4. *Solve* $x^2r-y^2t+xp-yp=\log x.$ **[Delhi Hons. 70]**

Solution. Put $x=e^u$ and $y=e^v$.

Now if $\dfrac{\partial}{\partial u}\equiv D$ and $\dfrac{\partial}{\partial v}\equiv D'$, then $xp=Dz$, $yq=D'z$.

$$x^2r=D\,(D-1)\,z,\ y^2t=D'\,(D'-1)\,z.$$

Therefore the equation becomes

$$[D\,(D-1)-D'\,(D'-1)+D-D']\,z=u$$

or $$(D^2-D'^2)\,z=u.$$

$$\text{C.F.}=\phi_1(u+v)+\phi_2(u-v)$$

$$=\phi(\log x+\log y)+\phi_2(\log x-\log y)$$

$$=f_1(xy)+f_2(y/x),$$

$$\text{P.I.}=\frac{1}{D^2-D'^2}\,u=\frac{1}{D^2}\left[1+\frac{D'^2}{D^2}\cdots\right]u$$

$$=\frac{1}{D^2}\,u=\frac{u^3}{6}=\frac{(\log x)^3}{6}.$$

Therefore the complete integral is

$$z=f_1(xy)+f_2(y/x)+\tfrac{1}{6}\,(\log x)^3.$$

Ex. 5. *Solve*

$$x^2 \frac{\partial^2 z}{\partial x^2}+2xy \frac{\partial^2 z}{\partial x \partial y}+y^2 \frac{\partial^2 z}{\partial y^2}=(x^2+y^2)^{n/2}.$$

Solution. Put $x=e^u$ and $y=e^v$.

Now if $D\equiv\frac{\partial}{\partial u}$ and $D' \frac{\partial}{\partial v}$,

then $x^2 \frac{\partial^2}{\partial x^2}=D(D-1)$, $xy \frac{\partial^2}{\partial x \partial y}=DD'$

and $y^2 \frac{\partial^2 z}{\partial y^2}=D'(D'-1)$.

With these substitutions the given equation becomes

$$(D+D')(D+D'-1)\, z=(e^{2u}+e^{2v})^{n/2}.$$

The C.F. $=\phi_1(v-u)+e^u\phi_2(v-u)$

$$f_1=\left(\frac{y}{x}\right)+xf_2\left(\frac{y}{x}\right)$$

and $$\text{P.I.}=\frac{1}{(D+D')(D+D'-1)}(e^{2u}+e^{2v})^{n/2}$$

$$=\frac{1}{(D+D')(D+D'-1)}e^{nu}[1+e^{2(v-u)}]^{n/2}$$

$$=\frac{1}{(D+D')(D+D'-1)}\left[e^{nu}+\tfrac{1}{2}ne^{(n-2)u+2v}+\frac{\tfrac{1}{2}n(\tfrac{1}{2}n-1)}{(2)!}e^{(n-4)u+4v}+\ldots\right]$$

$$=\frac{e^{nu}}{(n^2-n)}[1+\tfrac{1}{2}ne^{2(v-u)}+\ldots]$$

$$=\frac{e^{nu}[1+e^{2(v-u)}]^{n/2}}{n(n-1)}=\frac{(x^2+y^2)^{n/2}}{n(n-1)}.$$

Therefore the complete solution is

$$z=f_1\left(\frac{y}{x}\right)+xf_2\left(\frac{y}{x}\right)+\frac{(x^2+y^2)^{n/2}}{n(n-1)}.$$

Ex. 5. *Solve*

$$\frac{1}{x^2}\frac{\partial^2 z}{\partial x^2}-\frac{1}{x^3}\frac{\partial z}{\partial x}=\frac{1}{y^2}\frac{\partial^2 z}{\partial y^2}-\frac{1}{y^3}\frac{\partial z}{\partial y}.$$ [Agra 58]

Solution. Put $\frac{1}{2}x^2=u$ and $\frac{1}{2}y^2=v$,

so that $x\,dx=du$ and $y\,dy=dv$.

Hence $$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial u}\cdot\frac{\partial x}{\partial u}=\frac{1}{x}\frac{\partial z}{\partial x}$$

and $$\frac{\partial^2 z}{\partial u^2}=\frac{\partial}{\partial u}\left(\frac{1}{x}\frac{\partial z}{\partial u}\right)=\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial u}$$

$$=-\frac{1}{x^3}\frac{\partial z}{\partial x}+\frac{1}{x^2}\frac{\partial^2 z}{\partial x^2}$$

i.e. $\frac{1}{x^2}\frac{\partial^2 z}{\partial x^2}-\frac{1}{x^3}\frac{\partial z}{\partial x}=\frac{\partial^2 z}{\partial u^2}$.

Similarly $\frac{1}{y^2}\frac{\partial^2 z}{\partial y^2}-\frac{1}{y^3}\frac{\partial z}{\partial y}=\frac{\partial^2 z}{\partial y^2}$.

Hence the given equation reduces to

$$\frac{\partial^2 z}{\partial u^2}=\frac{\partial^2 z}{\partial v^2} \quad \text{or} \quad \frac{\partial^2 z}{\partial u^2}-\frac{\partial^2 z}{\partial v^2}=0.$$

Therefore the solution is

$$z=\phi_1(v+u)+\phi_2(v-u)$$
$$=\phi_1\left(\frac{x^2+y^2}{2}\right)+\phi_2\left(\frac{y^2-x^2}{2}\right)$$
$$=f_1(x^2+y^2)+f_2(y^2+x^2).$$

Exercises

Solve the following partial differential equations :

1. $x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\,\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=0.$ [Raj. 60]

   Ans. $z=f_1(y/x)+xf_2(y/x)$

2. $x^2\frac{\partial^2 z}{\partial x^2}-y^2\frac{\partial^2 z}{\partial y^2}-y\frac{\partial z}{\partial y}+x\frac{\partial z}{\partial x}=0.$ [Agra 53]

   Ans. $z=f_1(xy)+f_2(y/x)$

3. $x^2\frac{\partial^2 z}{\partial x^2}-y^2\frac{\partial^2 z}{\partial y^2}=x^2y.$

   Ans. $z=f_1(xy)+xf_2(y/x)-\frac{1}{2}x^2y$

4. $x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\,\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=x^my^n.$

   Ans. $z=f_1(y/x)+xf_2(y/x)+\frac{x^my^n}{(m+n)(m+n-1)}$

5. $x^2\frac{\partial^2 z}{\partial x^2}-3xy\frac{\partial^2 z}{\partial x\,\partial y}+2y^2\frac{\partial^2 z}{\partial y^2}+5y\frac{\partial z}{\partial y}-2z=0.$

   Ans. $z=x^2f_1(yx)+xf_2(x^2y)$

6. $x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\,\partial y}+y^2\frac{\partial^2 z}{\partial x^2}-nx\frac{\partial z}{\partial x}-xy\frac{\partial z}{\partial y}+nz=0.$

   Ans. $z=x^nf_1(y/x)+xf_2(y/x)$

7. $x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\,\partial y}+y^2\frac{\partial^2 z}{\partial x^2}+x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}-z=0.$

   Ans. $z=x^{-1}f_1(y/x)+xf_2(y/x)$

3·15. Case when F(D, D′) cannot be resolved into linear factors.

In case $F(D, D')$ is irreducible, *i.e.* it cannot be factorised into linears factors in D and D', the methods discussed above of finding the complementary function fail. A solution by trial is found. Following few examples will illustrate the method.

Ex. 1. *Solve* $(D-D'^2)\,z=0.$

Solution. Let the solution of the above equation be

$$z=Ae^{hx+ky}\;; \qquad \text{...(1)}$$

then $Dz=Ahe^{hx+ky},\; D'^2z=Ak^2e^{hx+ky}.$

Putting these in the differential equation, we get

$$(h-k^2)\,Ae^{hx+ky}=0.$$

Thus (1) would be solution if $h=k^2$.

Putting k^2 for h, in (1), the solution is given by

$$z=Ae^{k^2x+ky}.$$

Since all values of k satisfy the given equation, a more general solution of the given equation is

$$z=\Sigma\, Ae^{k^2x+ky}.$$

Ex. 2. *Solve*

$$\frac{\partial^2 z}{\partial x^2}=\frac{1}{k}\frac{\partial z}{\partial t}\,. \qquad \textbf{[Delhi Hons. 70]}$$

Solution. Let

$$\frac{\partial}{\partial x}=D,\;\frac{\partial}{\partial t}=D',$$

then equation is

$$(kD^2-D')\,z=0.$$

Let a solution of this be

$$z=Ae^{c_1x+c_2t}, \qquad \text{...(1)}$$

so that $D^2z=Ac_1^2e^{c_1x+c_2t},$

and $D'z=Ac_2e^{c_1x+c_2t}.$

Therefore (1) would be solution of the given equation if

$$kc_1^2-c_2=0$$

$$c_2=kc_1^2.$$

Putting this value of c_2 in (1), the solution is

$$z=Ae^{c_1x+kc_1^2t}.$$

Since all the values of c_1 satisfy the given equation, a more general solution is given by

$$z=\Sigma\, Ae^{c_1x+kc_1^2t}.$$

Ex. 3. *Solve* $(D^2-D')\,z=\cos(3x-y).$

Solution. For C F., $(D^2-D')\,z=0.$

So let C.F. be given by $z=Ae^{hx+ky}$.

$\therefore\; D^2z=Ah^2e^{hx+ky}$ and $D'z=Ake^{hx+ky},$

so that $(D^2-D')\,z\equiv A\,(h^2-k)\,e^{hx+ky}=0$

which holds if $h^2-k=0$ or $k=h^2$.

Hence in general the C.F.$=\Sigma\ Ae^{hx+h^2y}$.

$$\text{P.I.}=\frac{1}{D^2-D'}\cos(3x-y)=\frac{1}{-9-D'}\cos(3x-y)$$

putting -3^2 for D^2

$$=-\frac{D'-9}{D'^2-81}\cos(3x-y)$$

$$=-\frac{D'-9}{-1-81}\cos(3x-y)$$

$$=\tfrac{1}{82}[\sin(3x-y)-9\cos(3x-y)].$$

Hence the complete solution is

$$z=\Sigma\ Ae^{hx+h^2y}+\tfrac{1}{82}[\sin(3x-y)-9\cos(3x-y)].$$

Exercises

1. $(D^2-2D'^2-1)\ z=0$ Ans. $z=\Sigma Ae^{(2k+1)+ky}$
2. $(D^2+D'^2-n^2)\ z=0.$ Ans. $z=Ae^{n(x\cos\theta+y\sin\theta)}$.

Solutions under given conditions.

Ex. 1. *Find a surface passing through the two lines $z=x=0$, $z-1=x-y=0$, satisfying $r-4s+4t=0$.* [Agra 63]

Solution. The differential equation can be put as

$$(D^2-4DD'+4D'^2)\ z=0$$

or

$$(D-2D')^2\ z=0.$$

Therefore the general solution is

$$z=\phi_1(2x+y)+x\phi_2(2x+y). \quad \text{...(1)}$$

We wish to determine the arbitrary functions ϕ_1 and ϕ_2 under given conditions. For this we know that this passes through

$$z=x=0.$$

$\therefore$ $0=\phi_1(y)$ and so $\phi_1(2x+y)=0.$ ...(2)

(1) now reduces to

$$z=x\phi_2(2x+y). \quad \text{...(3)}$$

Again (3) passes through

$$z-1=x-y=0.$$

$\therefore$ Putting $z=1$ and $x=y$, (3) gives

$$1=x\phi_2(3x) \quad \text{i.e.} \quad \frac{1}{x}=\phi_2(3x)$$

or $\phi_2(2x+y)=\dfrac{3}{2x+y}$. ...(4)

We have thus determined the arbitrary functions.

Putting in (1) values of ϕ_1 and ϕ_2 as obtained in (2) and (4), the solution is

$$z=x\frac{3}{2x+y}$$

or $3x = z(2x+y)$,
which is the required surface.

Ex. 2. *Find a surface satisfying $r+s=0$ and touching the elliptic paraboloid $z=4x^2+y^2$ along its section by the plane $y=2x+1$.* [Agra 67, 65]

Solution. The differential equation can be put as

$$(D^2+DD')\,z=0 \quad \text{or} \quad D(D+D')\,z=0.$$

The general solution is

$$z=\phi_1(y)+\phi_2(y-x). \quad \ldots(1)$$

It is given that the surface

$$z=4x^2+y^2 \quad \ldots(2)$$

and (1) touch each other along the section by the plane $y=2x+1$. We shall use this condition to determine the arbitrary functions ϕ_1 and ϕ_2. The condition requires that values of p and q from (1) and (2) must be equal at $y=2x+1$. Equating values of p from (1) and (2) at $y=2x-1$

$$p=\frac{\partial z}{\partial x}=-\phi_2'(y-x)=8x \text{ at } y=2x-1$$

i.e. $\phi_2'(x-1)=-8x$.

Integrating $\phi_2(x-1)=-4x^2$.

This gives $\phi_2(y-x)=-4(y-x+1)^2$. ...(3)

Again $q=\dfrac{\partial z}{\partial y}=\phi_1'(y)+\phi_2'(y-x)=2y$ at $y=2x-1$,

i.e $\phi_1'(y)-8x=2y$ as $\phi_2'(y-x)=-8x$, from (3)

or $\phi_1'(y)=2y+8x=6y+4$ as $2x=y+1$.

Integrating, $\phi_1(y)=3y^2+4y$. ...(4)

Thus the required surface is

$$z=3y^2+4y-4(y-x-1)^2.$$

Ex. 3. *Find a surface satisfying $t=6x^3y$ containing the two lines $y=0=z$, $y=1=z$.* [Agra 70]

Solution. The equation can be written as

$$D'^2z=x^3y.$$

For this C.F. $=\phi_1(x)+y\phi_2(x)$

and P.I. $=\dfrac{1}{D'^2}\,6x^3y=x^3y^3$.

Therefore the solution is

$$z=\phi_1(x)+y\phi_2(x)=x^3y^3. \quad \ldots(1)$$

Therefore when $y=0$, $z=0$ $\quad\therefore\ 0=\phi_1(x)$

and when $y=1$, $z=1$. $\quad\therefore\ 1=\phi_2(x)+x^3$,

because $\phi_1(x)=0$.

Thus $\phi_2(x)=1-x^3$ and $\phi_1(x)=0$. ...(2)

Putting these values in (1), the required surface is

$$z=y(1-x^3)+x^3y^3.$$

Ex. 4. *Find a surface satisfying*

$$2x^2r-5xys+2y^2t+2(px+qy)=0$$

and touching the hyperbolic paraboloid $z=x^2-y^2$ *along its section by the plane* $y=1$. **[Agra 67, 60]**

Solution. The given equation can be written as

$$2x^2\frac{\partial^2 z}{\partial x^2}-5xy\frac{\partial^2 z}{\partial x\,\partial y}+y^2\frac{\partial^2 z}{\partial u^2}+2\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)=0.$$

Put $x=e^u$ and $y=e^v$, so that $u=\log x$ and $v=\log y$.

Denote $D\equiv\frac{\partial}{\partial u}$ and $D'\equiv\frac{\partial}{\partial v}$;

then $x\frac{\partial}{\partial v}=D$, $x^2\frac{\partial^2}{\partial x^2}=D(D-1)$, $xy\frac{\partial^2}{\partial x\,\partial y}=DD'$

$y\frac{\partial}{\partial y}=D'$ and $y^2\frac{\partial^2}{\partial y^2}=D'(D'-1)$.

With this substitution given equation becomes

$$(2D^2-5DD'+2D'^2)z=0$$

or $(2D-D')(D-2D')z=0$.

Hence the solution is

$$z=\phi_1(2v+u)+\phi_2(v+2u)$$
$$=\phi_1(2\log y+\log x)+\phi_2(\log y+2\log x)$$
$$=\phi_1(\log y^2x)+\phi_2(\log yx^2)$$

or $z=f_1(y^2x)+f_2(yx^2)$. ...(1)

The other given surface is

$z=x^2-y^2$. ...(2)

Since the two surfaces touch each other along the section by $y=1$, the value of p and q for two surfaces must be equal at $y=1$.

Equating values of p and q from (1) and (2),

$y^2f_1'(y^2x)+2xyf_2'(x^2y)=2x$ (3)

and $2xyf_1'(y^2x)+x^2f_2'(x^2y)=-2y$. ...(4)

These give $f_1'(y^2x)=-\frac{2x}{3y^2}-\frac{4}{3x}$

and $f_2'(x^2y)=\frac{2y}{3x^2}+\frac{4}{3y}$.

Putting $y=1$, $f_1'(x)=-\frac{2x}{3}-\frac{4}{3x}$

and $f_2'(x^2)=\frac{2}{3x^2}+\frac{4}{3}$.

Integrating $f_1(x)=-\frac{1}{3}x^2-\frac{4}{3}\log x$ ...(5)

and $f_2(x^2)=\frac{2}{3}\log x^2+\frac{4}{3}x^2$ (integrating w.r.t. x^2)

$=\frac{4}{3}\log x+\frac{4}{3}x^2$. ...(6)

From (5) and (6),

$$f_1(y^2x) = -\tfrac{1}{3}y^4x^2 + \tfrac{4}{3}\log y^2x$$
$$= -\tfrac{1}{3}y^4x^2 - \tfrac{4}{3}\log x - \tfrac{4}{3}\log y,$$
$$f_2(x^2y) = \tfrac{4}{3}\log x\sqrt{y} + \tfrac{4}{3}x^2y$$
$$= \tfrac{4}{3}\log x + \tfrac{2}{3}\log y + \tfrac{4}{3}x^2y.$$

Putting these values in (1), the required surface is

$$z = -\tfrac{1}{3}y^4x^2 - 2\log y + \tfrac{4}{3}x^2y + C$$

or $3z = 4x^2y - y^2y^4 - 6\log y + 3C$, where C is a constant.

But when $y=1$, $3z = 4x^2 - x^2 + 3C = 3x^2 + 3C$

and from (2), $z = x^2 - 1$.

These must be same, hence $C = -1$.

Therefore the required surface is

$$3z = 4x^2y - x^2y^4 - 6\log y - 3.$$

Exercises

1. Show that a surface of resolution satisfying the differential equation

$$\frac{\partial^2 z}{\partial x^2} = 12x^2 + 4y^2$$

and touching the surface $z=0$ is $z=(x^2+y^2)^2$. [Agra 69]

2. Solve the equation $r+t=2s$, and determine the arbitrary function by the condition that $bz=y^2$ which $x=0$ and $az=x^2$ when $y=0$. Ans. $z=(x+y)(x/a+y/b)$

3. Find a surface satisfying

$$(D^2 - 2DD' + D'^2)\, z = 6$$

and touching the hyperbolic paraboloid $z=xy$ along its section by the plane $y=x$. Ans $z=x^2-xy+y^2$

4. A surface satisfies $(D^2+D'^2)\, z=0$ and touches $x^2+z^2=1$ along its section $y=0$, obtain is equation.

Ans. $z^2(x^2+z^2-1)=y^2(x^2+z^2)$

5. Find a surface satisfying the equation $D^2z=6x+2$ and touching $z=x^3+y^3$ along its section by the plane

$x+y+1=0$. Ans. $z=x^3+y^3+(x+y+1)^2$

6. Find the surface passing through the parabolas $z=0$, $y^2=4ax$ and $z=1$, $y^2=-4ax$ and satisfying the differential equation

$$x\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial z}{\partial x} = 0.$$

[Agra 71]

4

Second Order Partial Differential Equations with Variable Coefficients

4·1. Introduction.

All equations of second order contains at least one of the second order partial differential coefficients r, s and t but not of higher order. The differentials p and q may also appear in the equation. Thus the general form of a second order partial differential equation is

$$F(x, y, z, p, q, r, s, t)=0 \quad \ldots(1)$$

The most general relation between x, y, z satisfying the given differential equation is the complete integral of the equation. An *intermediate integral* is a relation in the form of a partial differential equation of first order such that the given differential equation may be deduced from it. It is not in general unique and the complete integral can be deduced as a solution of this intermediate integral.

It is only in special cases that a partial differential equation (1) can be intergrated. The most important method of solution, due to Monge. is applicable to a wide class of such equations but by no means to all. We shall next discuss Monge's methods, which depends on establishing one or two intermediate integrals (first integrals) of the form

$$u=f(v) \quad \ldots(2)$$

where u and v are functions of x, y, z, p, q and f is some arbitrary function. We give the two Monge's methods below.

4·2. Monge' Method of Integrating Rr+Ss+Tt=V, *where R, S, T and V are functions of x, y, z, p and q.*

[Meerut 70, 78; Poona 60; Agra 67, 65, 56, 54, 52]

We have the equation

$$Rr+Ss+Tt=V. \quad \ldots(1)$$

Now $\quad ap=\dfrac{\partial p}{\partial x}\,dx+\dfrac{\partial p}{\partial y}\,dy=r\,dx+s\,dy$

giving $\quad r=\dfrac{dp-s\,dy}{dx}$

and $dq = \frac{\partial q}{\partial x} dx + \frac{\partial q}{\partial y} dx = s\ dy + t\ dy,$

giving $t = \frac{dq - s\ dx}{dy}.$

Putting values of r and t in (1), the equation becomes

$$R\left(\frac{dp - s\ dy}{dx}\right) + Ss + T\left(\frac{dq - s\ dx}{dy}\right) = V$$

or $R\ dp\ dy + T\ dq\ dx - V\ dx\ dy - s\ (R\ dy^2 - S\ dx\ dy + T\ dx^2) = 0.$ ...(2)

Any relation between x, y, z, p, q that satisfies (2) must necessarily satisfy the two simultaneous equations

$R\ dp\ dy + T\ dp\ dx - V\ dx\ dy = 0$ ...(3)

and $R\ dy^2 - S\ dx\ dy + T\ dx^2 = 0.$ ...(4)

These are *the Monge's Subsidiary Equations for equation* (1).

Therefore the complete solution of (1) also satisfies (3) and (4) and vice versa. We therefore proceed to obtain solutions of (3) and (4).

We first note the relation

$dz = p\ dx + q\ dy.$ ...(5)

In general, (4) can be resolved into two equations of the type $dy - m_1\ dx = 0$ and $dy - m_2\ dx = 0$.

There now arise two cases namely (*i*) when m_1 and m_2 are distinct, and (*ii*) when m_1 and m_2 are equal. If m_1 and m_2 are distinct then $dy - m_1\ dx = 0$ and equation (3), if necessary [illegible] use of (5), leads to two integrals of the type $u_1 = a$ and $v_1 = b$. These give

$u_1 = f_1(v_1),$ ...(6)

where f_1 is an arbitrary function. This is an *intermediate integral* of (1).

Next $dy - m_2\ dx = 0$ and (3) similarly lead to another intermediate integral

$u_2 = f_2(v_2).$ ...(7)

Values of p and q in general can be determined in terms of x and y from (6) and (7). Substituting these values in (5), we get an equation containing x, y, z and dx, dy, dz which on integration gives the complete integral of the equation (1).

However in case $m_1 = m_2$ *i.e.*, (4) is a perfect square, we get only one intermediate solution containing p, q and x, y, z of the form $Pq + Qp = R.$

The solution can now be obtained by forming Lagrange's subsidiary equations.

Note. Sometimes in the first case also it is convenient to find the complete solution with the help of one intermediate solution only, as in the case when (4) is a perfect square.

***Ex. 1.** *Solve* $r+(a+b)s+abt=xy.$ **[Agra 58, 55]**

Solution. We have $dp=r\,dx+s\,dy$ and $dq=s\,dx+t\,dy$.

These give $r=\dfrac{dp-s\,dy}{dx}$ and $t=\dfrac{dq-s\,dx}{dy}$.

Putting these values of r and t, the given equation becomes

$$\frac{dp-s\,dy}{dx}+(a+b)s+ab\frac{dq-s\,dx}{dy}=xy.$$

or $(dp\,dy+ab\,dq\,dx-xy\,dx\,dy)-s\{dy^2-(a+b)\,dx\,dy+ab\,dx^2\}=0.$

Monge's subsidiary equations are

$$dp\,dy+ab\,dq\,dx-xy\,dx\,dy=0 \quad ...(1)$$

and $dy^2-(a+b)\,dx\,dy+ab\,dx^2=0.$ ...(2)

Two factors of (2) are $dy-a\,dx=0$ giving $y-ax=c_1$ ...(3)

and $dy-b\,dx=0$ giving $y-bx=c_2$. ...(4)

Combining (3) with (1), we get

$$a\,dp+ab\,dq-ax(c_1+ax)\,dx=0,$$

i.e., $dp+b\,dq-x(c_1+ax)\,dx=0.$

Integrating, $p+bq-(c_1.\frac{1}{2}x^2+\frac{1}{3}ax^3)=A,$

i.e., $p+bq+\frac{1}{2}x^2(y-ax)-\frac{1}{3}ax^3=f_1(y-ax),$

or $p+bq+\frac{1}{6}ax^3-\frac{1}{2}x^2y=f_1(y-ax),$ ...(5)

where f_1 is an arbitrary function.

Similarly (replacing a by b and b by a). the other intermediate integral obtained by combining (4) and (1) is

$$p+aq+\tfrac{1}{6}bx^3-\tfrac{1}{2}x^2y=f_2(y-bx), \quad ...(6)$$

where f_2 is an arbitrary function.

Now (5) and (6) give

$$p=\tfrac{1}{2}yx^2-\tfrac{1}{6}(a+b)x^3-[1/(b-a)][af_1(y-ax)-bf_2(y-bx)]$$

and $q=\frac{1}{6}x^3+[1/(b-a)][f_1(y-ax)-f_2(y-bx)]$

Putting these values in the relation $dz=p\,dx+q\,dy$, we get

$$dz=\tfrac{1}{2}x^2y\,dx+\tfrac{1}{6}x^3\,dy-(a+b)\tfrac{1}{6}x^3\,dx$$
$$+\frac{1}{b-a}[f_1(y-ax)(dy-a\,dx)-f_2(y-bx)(dy-b\,dx)].$$

Integrating it the complete solution of given differential equation is

$$z=\tfrac{1}{6}x^3y-(a+b)\frac{x^4}{24}+F_1(y-ax)+F_2(y-bx),$$

where F_1 and F_2 are arbitrary functions.

Ex. 2. *Solve* $t-r\sec^4 y=2q\tan y.$ **[Agra 68, 67]**

Solution. We have $dp=r\,dx+s\,dy$, $dq=s\,dx+t\,dy$.

These give $r=\dfrac{dp-s\,dy}{dx}$ and $t=\dfrac{dq-s\,dx}{dy}$.

Putting these values of r and t in the given equation, we get

$$\frac{dq-s\,dx}{dy}-\frac{dp-s\,dy}{dx}\sec^4 y=2q\tan y$$

or $dq\,dx-dp\,dy\sec^4 y-2q\tan y\,dx\,dy-s\,(dx^2-dy^2\sec^4 y)=0.$

The Monge's subsidiary equations are

$$dq\,dx-dp\,dy\sec^4 y-2q\tan y\,dy\,dx=0 \qquad \text{...(1)}$$

and $dx^2-\sec^4 y\,dy^2=0.$ ...(2)

The two factors of (2) are

$$dx-\sec^2 y\,dy=0 \text{ giving } x-\tan y=A \qquad \text{...(3)}$$

and $dx+\sec^2 y\,dy=0$ giving $x+\tan y=B.$ ...(4)

Now (3) and (1) give

$$dq-dp\sec^2 y-2q\tan y\,dy=0$$

or $dp-dq\cos^2 y+2a\sin y\cos y\,dy=0.$

Integrating, $p-q\cos^2 y=c=f_1(A).$

Hence $p-q\cos^2 y=f_1(x-\tan y)$

is an intermediate integral.

Similarly the other intermediate integral obtained from (4) and (1) is

$$p+q\cos^2 y=f_2(x+\tan y) \qquad \text{...(6)}$$

Adding and subtracting (5) and (6), we get

$$p=\tfrac{1}{2}\,[f_1(x-\tan y)+f_2(x+\tan y)]$$

and $q=\tfrac{1}{2}\sec^2 y\,[f_2(x+\tan y)-f_1(x-\tan y)].$

Thus $dz=p\,dx+q\,dy$

$$=\tfrac{1}{2}\,[f_1(x-\tan y)+f_2(x+\tan y)]\,dx+\tfrac{1}{2}\sec^2 y\,[f_2(x+\tan y)-f_1(x-\tan y)]\,dy$$

$$=\tfrac{1}{2}\,[dx+\sec^2 y\,dy]\,f_2(x+\tan y)+\tfrac{1}{2}\,[dx-\sec^2 y\,dy]\,f_1(x-\tan y).$$

Integrating, the complete integral is

$$z=F_1(x-\tan y)+F_2(x+\tan y),$$

where F_1 and F_2 are arbitrary functions.

***Ex. 3.** *Solve* $q\,(1+q)\,r-(p+q+2pq)\,s+p\,(1+p)\,t=0.$

[Agra 65, 57]

Solution. We have $dp=r\,dx+s\,dy$, $dq=s\,dx+t\,dy$

These give $r=\dfrac{dp-s\,dy}{dx}$ and $t=\dfrac{dq-s\,dx}{dy}.$

Putting these values of r and t in the given equation, we get

$$q\,(1+q)\frac{dp-s\,dy}{dx}-(p+q+2pq)\,s+p\,(1+p)\frac{dq-s\,dx}{dy}=0$$

or $(q+q^2)\,dp\,dy+(p+p^2)\,dq\,dx$

$$-s\,[(q+q^2)\,dy^2+(p+q+2pq)\,dx\,dy+(p+p^2)\,dx^2]=0.$$

The Monge's subsidiary equations are

$$(q+q^2)\,dp\,dy+(p+p^2)\,dq\,dx=0 \qquad ...(1)$$

and $(q+q^2)\,dy^2+(p+q+2pq)\,dy\,dx+(p+p^2)\,dx^2=0.$ ...(2)

The two factors of (2) are

$$p\,dx+q\,dy=0 \qquad ...(3)$$

and $(1+x)\,dx+(1+q)\,dy=0$ ...(4)

Also $dz=p\,dx+q\,dy.$ ...(5)

Combining (3) and (1), we get

$$-(1+q)\,dp+(1+p)\,dq=0 \quad \text{or} \quad \frac{dp}{1+p}-\frac{dq}{1+q}=0.$$

Integrating, $\dfrac{1+p}{1+q}=A$ or $1+p=A\,(1+q).$

Now from (3) and (5), $dz=0$, *i.e.* $z=B$.

Therefore the intermediate integral from (3) and (1) is

$$(1+p)=(1+q)\,f_1\,(z). \qquad ...(6)$$

Next combining (4) and (1), we get

$$-q\,dp+p\,dq=0 \quad \text{or} \quad \frac{dp}{p}-\frac{dq}{q}=0$$

or $p=Cq.$

We can write (4) as

$$dx+dy+p\,dx+q\,dy=0$$

or $dx+dy+dz=0$ as $dz=p\,dx+q\,dy.$

Integrating, $x+y+z=D.$

Therefore another intermediate integral is

$$p=qf_2\,(x+y+z). \qquad ...(7)$$

Now solving (6) and (7) for p and q, we get

$$p=\frac{(f_1-1)\,f_2}{f_2-f_1} \quad \text{and} \quad q=\frac{(f_1-1)}{f_2-f_1}.$$

Putting these values in $dz=p\,dx+q\,dy$, we get

$$dz\,(f_2-f_1)=(f_1-1)\,f_2\,dx+(f_1-1)\,dy$$

$$=(f_1-1)\,f_2\,dx+(f_1-1)\,(dx+dy+dz)-(f_1-1)\,(dx+dz)$$

or $f_2\,dz=(f_1-1)\,f_2dx+(f_1-1)\,(dx+dy+dz)-(f_1-1)\,dx+dz$

cancelling $-f_1\,dz$ from both the sides,

i.e. $(f_2-1)\,dz=(f_1-1)\,(f_2-1)\,dx+(f_1-1)\,(dx+dy+dz)$

i.e. $\dfrac{dz}{f_1\,(z)-1}=dx+\dfrac{dx+dy+dz}{f_2\,(x+y+z)-1}$

dividing by $(f_2-1)\,(f_1-1)$

which on integration gives the complete solution.

$$F_1\,(z)=x+F_2\,(x+y+z).$$

*Ex. 4. *Solve* $(b+cq)^2\,r-2\,(b+cq)\,(a+cp)\,s+(a+cp)^2\,t=0.$

[Poona 60: Agra 62, 59, 56]

Solution. We have $dp=r\,dx+s\,dy$ and $dq=s\,dx+t\,dy$.

This gives $r=\dfrac{dp-s\,dy}{dx}$ and $t=\dfrac{dq-s\,dx}{dy}$.

Putting these values of r and t in the given equation, we get

$$(b+cq)^2\frac{dp-s\,dy}{dx}-2\,(b+cq)\,(a+cp)\,s+(a+cp)^2\frac{dp+s\,dx}{dy}=0.$$

The Monge's subsidiary equations are

$$(b+cq)^2\,dp\,dy+(a+cp)^2\,dq\,dx=0 \qquad ...(1)$$

and $(b+cq)^2\,dy^2+2\,(b+cq)\,(a+cp)\,dx\,dy+(a+cp)^2\,dx^2=0$ ...(2)

Here (2) is a perfect square of

$$(b+cq)\,dy+(a+cp)\,dx=0. \qquad ...(3)$$

Therefore we shall be getting only one intermediate integral, (3) can be written as

$$b\,dy+a\,dx+c\,(p\,dx+q\,dy)=0$$

or $a\,dx+b\,dy+c\,dz=0$ as $dz=p\,dx+q\,dy$.

Integrating $ax+by+cz=A$. ...(4)

Combining (3) with (1), we get

$$(b+cq)\,dp-(a+cp)\,dq=0$$

i.e. $\dfrac{dp}{a+cp}-\dfrac{dq}{b+cq}=0.$

Integrating, $\dfrac{a+cp}{b+cq}=B$ or $a+cp=B\,(b+cq)$.

From (3) and (4) the intermediate integral is

$$a+cp=(b+cq)\,f\,(ax+by+cz)$$

or $cp-cf\,(ax+by+cz)\,q=-a+bf\,(ax+by+cz)$,

For this the Lagrange's* subsidiary equations are

$$\frac{dx}{c}=\frac{dy}{-cf(ax+by+cz)}=\frac{dz}{-a-bf\,(ax+by+cz)}$$

$$=\frac{a\,dx+b\,dy+c\,dz}{0}$$

One integral is $ax+by+cz=C$ (const).

Again from the first two relations, we have

$$\frac{dx}{c}=\frac{dy}{-cf(C)}$$

or $dy+f(C)\,dx=0.$

Then other integral is $y+xf(C)=\text{const.}=\psi(C)$, say.

Therefore the complete solution is

$$y+xf(ax+by+cz)=\psi(ax+by+cz).$$

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*In solving $Pp+Qq=R$, the Lagrange's subsidiary equations are

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}.$$

***Ex. 5.** *Solve* $(1+q)^2 r-2(1+p+q+pq)s+(1+p)^2 t=0.$
[Agra 72, 52]

Solution. We have $dp=r\,dx+s\,dy$ and $dp=s\,dx+t\,dy$

so that $r=\dfrac{dp-s\,dy}{dx}$ and $t=\dfrac{dq-s\,dx}{dy}$.

Putting these values of r and t in the given equation, we get

$$(1+q)^2\frac{dp-s\,dy}{dx}-2(1+p+q+pq)s+(1+p)^2\frac{dq-s\,dx}{dy}=0.$$

The Monge's subsidiary equations are

$$(1+q)^2\,dp\,dy+(1+p)^2\,dq\,dx=0 \quad ...(1)$$

and $[(1+q)\,dy+(1+p)\,dx]^2=0.$ ...(2)

(2) gives only one relation

$$(1+q)\,dy+(1+p)\,dx=0. \quad ...(3)$$

We shall get only one intermediate integral. To get it we combine (3) with (1), to get

$$(1+q)\,dp-(1+p)\,dq=0 \quad \text{or} \quad \frac{dp}{1+p}-\frac{dq}{1+q}=0.$$

Integrating, $\dfrac{1+p}{1+q}=A.$

We may write (3) as

$$dx+dy+(p\,dx+q\,dy)=0$$

i.e. $dx+dy+dz=0$ as $dz=p\,dx+q\,dy$.

This gives $x+y+z=C.$

Therefore the intermediate integral is

$$(1+p)=(1+q)f(x+y+z)$$

or $p-qf(x+y+z)=[f(x+y+z)-1].$

This is of the form $Pp+Qq=R$, and the Lagrange's subsidiary equations for this are

$$\frac{dx}{1}=\frac{dy}{-f(x+y+z)}=\frac{dz}{f(x+y+z)-1}=\frac{dx+dy+dz}{0}.$$

One integral of it is $x+y+z=B.$

Again first two relations give

$$\frac{dx}{1}+\frac{dy}{-f(B)} \quad \text{or } y+xf(B)=\text{const.}=\phi(B), \text{ say.}$$

Thus the complete integral is

$$y+xf(x+y+z)=\phi(x+y+z).$$

Ex. 6. *Solve* $2x^2r-5xys+2y^2t+2(px+qy)=0.$
[Meerut 70]

Solution. Putting $r=\dfrac{dp-s\,dy}{dx}$ and $t=\dfrac{dq-s\,dx}{dy}$

the given equation becomes

$$2x^2\frac{dp-s\,dy}{dx}-5xys+2y^2\frac{dq-s\,dx}{dy}+2(px+qy)=0.$$

Hence the Monge's subsidiary equations are

$$2x^2\,dp\,dy+2y^2\,dq\,dx+2\,(pq+qy)\,dx\,dy=0 \qquad \ldots(1)$$

and $2x^2\,dy^2+5xy\,dx\,dy+2y^2\,dp^2=0.$ ...(2)

Two factors of (3) are $x\,dy+2y\,dx=0$ ...(3)

and $2x\,dy+y\,dx=0.$ ...(4)

(3) can be written as

$$\frac{dy}{y}+\frac{2dx}{x}=0 \text{ or } x^2y=A.$$

Now combining (3) with (1), we get

$$2x\,dp-y\,dq+2p\,dx-q\,dy=0$$

which on integration gives $2px-qy=\text{const}.$

Hence the intermediate integral is

$$2px-qy=f(x^2y) \qquad \ldots(5)$$

which is of Lagrange's form; hence Lagrange's subsidiary equations are

$$\frac{dx}{2x}=\frac{dy}{-y}=\frac{dz}{f(x^2y)}.$$

From the first two relations, we have

$$\frac{dx}{x}+\frac{2\,dy}{y}=0$$

or $xy^2=c.$

From last two relations, we have

$$\frac{dy}{-y}=\frac{dz}{f(c^2/y^4.y)}=\frac{dz}{f(c^2/y^3)}$$

or $dz=-\frac{1}{y}f\left(\frac{c^2}{y^3}\right)dy=-\frac{y^2}{y^3}f\left(\frac{c^2}{y^3}\right)dy=-y^2f_1\left(\frac{c^2}{y^3}\right)dy$

Integration, $z=F_1\left(\frac{c^2}{y^3}\right)+\text{const}.$

or $z=F_1(x^2y)+F_2(xy^2)$ as $c=xy^2$

which is the complete solution.

Ex. 7. *Solve the equation* $x^2r+2xys+y^2t=0.$ [Agra 54]

Solution. We have $dp=r\,dx+r\,dy,\ dq=s\,dx+t\,dy,$

which give $r=\dfrac{dp-s\,dy}{dx}$ and $t=\dfrac{dp-s\,dx}{dy}.$

Putting these values of r and t in the given equation, we have

$$x^2\left(\frac{dp-s\,dy}{dx}\right)+2xys+y^2\left(\frac{dq-s\,dx}{dy}\right)=0$$

or $x^2\,dp\,dy+y^2\,dq\,dx-s\,(x^2\,dy^2-2xy\,dx\,dy+y^2\,dx^2)=0.$

Thus Monge's subsidiary equations are

$$x^2\,dp\,dy+y^2\,dq\,dx=0 \qquad \ldots(1)$$

*For alternate solution of Ex. 7, see Ex. 5, p. 77 or Ex. 1, P. 79.

and $x^2 \, dy^2 - 2xy \, dx \, dy + y^2 \, dx^2 = 0.$ ...(2)

(2) gives $(x \, dy - y \, dx)^2 = 0$, *i.e.* $x \, dy - y \, dx = 0.$ ...(3)

Combining (3) with (1), we have $x \, dp + y \, dq = 0$

or $x \, dp + p \, dx + y \, dq + q \, dy = p \, dx + q \, dy$

or $x \, dp + p \, dx + y \, dq + q \, dy = dz.$

Integrating, $px + qy = z + B.$

Also integrating (3), we get $\frac{y}{x} + A.$

Thus the intermediate integral is

$$px + qy = z + f(A).$$

Hence by Lagrange's method, the subsidiary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z + f(A)}.$$

The first two relations give

$$\frac{y}{x} = A$$

and last two relations give $\log Cy = \log [z + f(A)]$

or $z + f(A) = Cy.$

Hence the complete solution is

$$z = y f_1 \left(\frac{y}{x}\right) + f_2 \left(\frac{y}{x}\right).$$

Ex. 8. *Solve* $y^2 r + 2xys + x^2 t + px + qy = 0.$

Solution. Putting $r = \frac{dp - s \, dy}{dx}$ and $t = \frac{dq - s \, dx}{dy}$,

the Monge's subsidiary equations are

$$y^2 \, dp \, dy + x^2 \, dq \, dx + (px + qy) \, dx \, dy = 0 \quad ...(1)$$

and $y^2 \, dy^2 - 2xy \, dx \, dy + x^2 \, dx^2 = 0.$ ...(2)

(2) gives $(y \, dx - x \, dx)^2 = 0$ or $y \, dy - x \, dx = 0.$ ...(3)

Integrating (3), we get $x^2 - y^2 = A.$

Combining (3) with (1), we get

$$y \, dp + x \, dq + p \, dy + q \, dx = 0.$$

Integrating, $py + qx = \text{const.}$

Hence the intermediate integral is

$$py + qx = f(x^2 - y^2).$$

This is of Lagrange's subsidiary equations are

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{f(x^2 - y^2)}.$$

From first two relations, we get $x^2 - y^2 = A.$

Also from the relation $\frac{dy}{x} = \frac{dz}{f(x^2 - y^2)}$, we get

$$\frac{dz}{\sqrt{(A+y^2)}}=\frac{dz}{f(A)}$$

or $$dz=f(A)\frac{dy}{\sqrt{(A+y^2)}}$$

Integrating, $z=f(A)\frac{1}{A}\log[y+\sqrt{(A+y^2)}]+\text{const.}$

or $$z=F_1(x^2-y^2)\log(y+x)+F_2(x^2-y^2),$$
which is the complete integral.

Ex. 9. *Solve* $y^2r-2ys+t=p+6y.$

Solution. $dp=r\,dx+s\,dy$ and $dq=s\,dx+t\,dy$

so that $$r=\frac{dp-s\,dx}{dx} \text{ and } t=\frac{dq-s\,dx}{dy}$$

Putting these values of r and t in the given equation, the Monge's subsidiary equations are

$$y^2\,dp\,dy+dq\,dx-(p+6y)\,dx\,dy=0 \qquad ...(1)$$

and $$y^2\,dy^2+2y\,dy\,dx+dx^2=0. \qquad ...(2)$$

(2) gives $(y\,dy+dx)^2=0$

or $$y\,dy+dx=0. \qquad ...(3)$$

Combining (3) and (1), we get

$$y\,dp-dq+(p+6y)\,dy=0 \text{ or } (y\,dp+p\,dy)-dq+6y\,dy=0.$$

Integrating, $py-q+3y^2=A.$

Also integrating (3), we get $y^2+2x=B.$

Thus the intermediate integral is

$$py-q+3y^2=f(y^2+2x). \qquad ...(4)$$

Other intermediate integral cannot be found ; hence we proceed to solve with the help of Lagrage's method.

(4) can be written as

$$py-q=f(y^2+2x)-3y^2.$$

Hence Lagrange's subsidiary equations are

$$\frac{dx}{y}=\frac{dy}{-1}=\frac{dz}{f(y^2+2x)-3y}$$

From second and third relations, we have

$$dz+[f(y^2+2x)-3y^2]\,dy=0$$

or $$dz+[f(B)-3y^2]\,dy=0,$$

as from two relations, $\frac{dx}{y}=\frac{dy}{-1}$, $y^2+2x=B$

or $z+yf(B)-y^3=C=\phi(B).$

Hence the complete solution is

$$z=y^3-yf(y^2+2x)+\phi(y^2+2x).$$

Ex. 10. *Solve* $xy(t-r)+(x^2-y^2)(s-2)=py-qx.$

Solution. The equation can be written as

$$xyr-(x^2-y^2)s-xyt+2(x^2-y^2)+py-qx=0.$$

Putting $r=\dfrac{dp-s\,dy}{dx}$ and $t=\dfrac{dq-s\,dx}{dy}$, the Monge's equations are

$$xy\,dp\,dy-xy\,dq\,dx-[2(x^2-y^2)+py-qx]\,dx\,dy=0 \quad \text{..(1)}$$

and $xy\,dy^2+(x^2-y^2)\,dy\,dx-xy\,dx^2=0$ ...(2)

(2) gives $x\,dx+y\,dy=0$ ...(3)

and $x\,dy-y\,dx=0.$ ...(4)

Integrating (3), we get $x^2+y^2=a_1$.

Combining (3) with (1), we get

$x\,dp+y\,dq-2x\,dy-2y\,dx+p\,dx+q\,dy=0.$

Integrating, $xp+yq-2xy=$constant.

Hence one intermediate integral is

$xp+yq-2xy=f(x^2+y^2).$

This is of the Lagrange's form.

Lagrange's subsidiary equations are

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{2xy+f(x^2+y^2)}.$$

First two relations give $\dfrac{y}{x}=c$, *i.e.* $y=cx$.

Also from first and last $\dfrac{dx}{x}=\dfrac{dz}{2cx^2+f(x^2+c^2x^2)}$ as $y=cx$

or $dz=\dfrac{1}{x}[2cx^2+f(x^2+c^2x^2)]\,dx$

$$=2cx\,dx+\frac{1}{x^2}f(x^2+c^2x^2)\,x\,dx$$

$$=2cx\,dx+f_1(x^2+c^2x^2)\,x\,dx.$$

Integrating $z=cx^2+F_1(x^2+c^2x^2)+$const.

or $$z=xy+F_1(x^2+y^2)+F_2\left(\frac{y}{x}\right), \text{ as } c=y/x$$

which gives the complete integral.

Ex. 11. *Solve* $r+ka^2t-2as=0.$

Solution. $r=\dfrac{dp-s\,dy}{dx}$ and $t=\dfrac{dq-s\,dx}{dy}$.

Putting these values of r and t in the given equation, we get

$$\frac{dp-s\,dy}{ds}+ka^2\frac{dq-s\,dx}{dy}-2as=0.$$

Hence Monge's subsidiary equations are

$dp\,dy+ka^2\,dq\,dx=0$ ...(1)

and $dy^2+ka^2\,dx^2+2a\,dx\,dy=0.$ ...(2)

From (2), $dy=\dfrac{-2a\pm\sqrt{(4a^2-4ka^2)}}{2}\,dx$

or $$dy+a\{1+\sqrt{1-k}\}\,dx=0$$

or $\quad dy-a(1\pm l)\,dx=0$ where $l=\sqrt{(1-k)}$

or $\quad dy+a(1+l)\,dx=0$ ...(3)

or $\quad dy+a(1-l)\,dx=0.$ ...(4)

Combining (3) and (1), we get

$$(1+l)\,dp-ka\,dq=0$$

or $\quad (1+l)\,p-kaq=B.$

Also integrating (3), we get $y+a(1+l)\,x=A$.

Thus one in intermediate integral is

$$(1+l)\,p-kaq=f_1[y+a(1-l)\,x]. \quad ...(6)$$

Similarly combining (4) with (1), the other intermediate integral is

$$(1-l)\,p-kaq=f_2\,[y+a(1-l)\,x]. \quad ...(6)$$

Solving (5) and (6), we get

$$p=\frac{1}{2l}\,[f_1\{y+a(1+l)\,x\}-f_2\{y+a(1-l)\,x\}]$$

and $q=\dfrac{1}{2kla}\,[(1-l)f_1\{y+a(1+l)\,x\}-(1+l)f_2\{y+a(1-l)\,x\}]$.

Putting these values in the relation $dz=p\,dx+q\,dy$ and integrating, we get

$$r=F_1\,[y+a(1+l)\,x]+F_2\,[y+a(1-l)\,x],$$

which is the complete solution.

Ex. 2. *Solve* $q^2r-2pqs+p^2t=0.$

[Raj. 66; Agra 71, 61; Delhi Hons. 68]

Solution. We have $dp=r\,dx+s\,dy$, $dq=s\,dx+t\,dy$

which give $\quad r=\dfrac{dp-s\,dy}{dx}$ and $t=\dfrac{dq-s\,dx}{dy}$.

Putting these values of r and t in the given equation, we have

$$q^2\,\frac{dp-s\,dy}{dx}-2pqs+p^2\,\frac{dq-s\,dx}{dy}=0$$

or $\quad q^2\,dp\,dy+p^2\,dq\,dx-s\,(q^2\,dy^2+2pq\,dy\,dx+p^2\,dx)=0.$

Hence Monge's subsidiary equations are

$$q^2\,dp\,dy+p^2\,dq\,dx=0 \quad ...(1)$$

and $\quad q^2\,dy^2+2pq\,dy\,dx+p^2\,dx^2=0.$ ...(2)

(2) gives $\quad (q\,dy+p\,dx)^2=0$

or $\quad q\,dy+p\,dx=0.$ ...(3)

Also $\quad dz=p\,dx+q\,dy,$

i.e. $\quad dz=0$ by (3) or $z=A$.

Now combining (3) with (1), we get

$$q\,dp-p\,dq=0 \text{ or } \frac{dp}{p}-\frac{dq}{q}=0.$$

or $\quad \frac{p}{q}=B.$

Hence the intermediate integral is

$$\frac{p}{q}=f(z) \quad \text{or} \quad p-qf(A)=0.$$

So Lagrange's subsidiary equations are

$$\frac{dx}{1}=\frac{dy}{-f(A)}=\frac{dz}{0},$$

which gives $\quad y+xf(A)=C,\ z=A.$

Hence the complete solution is

$$y+xf(z)=F(z) \text{ where } C=F(A)=F(z).$$

Ex. 13. *Solve* $pt-qs=q^3$. **[Delhi Hons. 71; Agra 70]**

Solution. Putting $t=\frac{dq-s\,dx}{dy}$, the equation becomes

$$p\,\frac{dq-s\,dx}{dy}-qs=q^3.$$

Hence Monge's subsidiary equations are

$$p\,dq-q^3\,dy=0. \quad ...(1)$$

and $\quad p\,dx-q\,dy=0. \quad ...(2)$

Since $dz=p\,dx+q\,dy$, from (2), $dz=0$ *i.e.* $z=A$ (const).

Again combining (2) with (1), we get

$$dp+q^2\,dx=0$$

or $\quad \frac{dq}{q^2}+dx=0$, *i.e.* $-\frac{1}{q}+x=B$

or $\quad -\frac{1}{q}+x=f(z)$

or $\quad -\frac{\partial y}{\partial z}+x=f(z)$ as $q=\frac{\partial z}{\partial y}$

or $\quad \frac{\partial y}{\partial z}=x-f(z).$

Integrating, $\quad y=xz-\int f(z)\,dz+C.$

or $\quad y=xz-F_1(z)+F_2(x),$

since C is a function of x which is regarded constant at the time of integration.

Ex. 14. *Solve*

$$q\,(yq+z)\,r-p\,(2yq+z)\,s+yp^2t+p^2q=0.$$

Solution. Putting $r=\frac{dp-s\,dy}{dx}$ and $t=\frac{dq-s\,dx}{dy}$, the Monge's subsidiary equations are

$$q\,(yq+z)\,dp\,dy+yp^2\,dq\,dx+p^2q\,dx\,dy=0 \quad ...(1)$$

and $\quad q(yq+z)\,dy^2+p(2yq+z)\,dx\,dy+yp^2\,dx^2=0.$ ...(2)

The two factors of (2) are

$$q\,dy+p\,dx=0 \quad ...(3)$$

and $\quad (yq+z)\,dy+yp\,dx=0.$ ...(4)

From (3), $dz=0$ *i.e.* $z=A$ (const).

Now combining (3) and (1), we get

$$(yq+z)\,dp-yp\,dq-dq\,dy=0$$

or $\quad (yq+z)\,dp-pd\,(yq)=0$

or $\quad (yq+z)\,dp-pd\,(yq)=0$ as $dz=0$

or $\quad \dfrac{dp}{p}-\dfrac{d\,(yq+z)}{yp+z}=0$ or $yq+z=pB.$

Thus $\quad yq+z=pf_1(z)$ ...(5)

is one intermediate integral, where f_1 is arbitrary function.

Next from (4), $y\,dz+z\,dy=0$ or $dz=p\,dx+q\,dy$

or $\quad yz=\text{const.}=C.$

Now combining (4) with (1), we get

$$\frac{dp}{p}-\frac{dq}{q}-\frac{dy}{y}=0 \text{ or } \frac{qy}{q}=\text{const.}$$

Therefore another intermediate integral is

$qy=pf_2(yz),$ ...(6)

where f_2 is an arbitrary function.

Solving (5) and (6) for p and q, we get

$$p=\frac{z}{f_1(z)-f_2(yz)},\ q=\frac{zf_2(yz)}{y\,\{f_1(z)-f_2(yz)\}}.$$

Substituting these in $dz=p\,dx+q\,dy$, we get

$$dz=\frac{z}{f_1(z)-f_2(yz)}\left[dx+\frac{1}{y}f_2(yz)\,dy\right]$$

or $\quad \dfrac{f(z)\,dz}{z}=dx+\dfrac{f_2(yz)}{yz}\,d(yz).$

Integrating now, the complete solution is

$F_1(z)=x+F_2(yz),$

where F_1 and F_2 are arbitrary functions.

Ex. 15. *Solve*

$$(x-2y)\,[2xr-(x+2y)\,s+yt]=(x+2y)\,(2p-q).$$

Solution. Putting $r=\dfrac{dp-s\,dy}{dx}$ and $t=\dfrac{dq-s\,dx}{dy}$

Monge's subsidiary equations are

$$2x\,dp\,dy+y\,dq\,dx-\frac{x+2y}{x-2y}(2p-q)\,dy\,dx=0 \quad ...(1)$$

and $\quad 2x\,dy^2+(x+2y)\,dy\,dx+y\,dx^2=0.$ ...(2)

(2) gives $\quad x\,dy+y\,dx=0$ ...(3)

and $$2\,dy+dx=0. \qquad ...(4)$$

Integrating (3), we get $xy=A$.

Combining (3) with (1), we get

$$\frac{2\,dp-dq}{2p-q}=\frac{dx-2\,dy}{x-2y}.$$

Integrating, $2p-q=B\,(x-2y)$

Hence the intermediate integral is

$$2p-q=(x-2y)\,f(xy). \qquad ...(5)$$

This is of Lagrange's form. The Lagrange's subsidiary equations are

$$\frac{dx}{2}=\frac{dy}{-1}=\frac{dz}{(x-2y)\,f(xy)}=\frac{yf(xy)\,dx+xf(xy)\,dy+dz}{0}.$$

The last relation gives $yf(xy)\,dx+xf(xy)\,dy+dz=0$

or $$dz+f(xy)\,d(xy)=0.$$

Integrating, $z=F_1(xy)+C$.

Also from first two relations, we have $\frac{dx}{2}=\frac{dy}{-1}$.

i.e. $2y+x=$constants.

Hence the complete solution is

$$z=F_1(xy)+F_2(2y+x).$$

Ex. 16. *Solve* $(x-y)\,(xr-xs-ys-yt)=(x+y)\,(p-q)$.

[Agra 63, 54]

Solution. Putting $r=\frac{dp-s\,dy}{dx}$ and $t=\frac{dq-s\,dx}{dy}$ in the given equation, the Monge's subsidiary equations are

$$x\,(x-y)\,dp\,dy+y\,(x-y)\,dq\,dx-(x+y)\,(p-q)\,dx\,dy=0 \qquad ...(1)$$

and $$x\,dy^2+(x+y)\,dx\,dy+y\,dx^2=0. \qquad ...(2)$$

(2) gives $$x\,dy+y\,dx=0 \qquad ...(3)$$

and $$dx+dy=0. \qquad ...(4)$$

Integrating (3), we get $xy=A$.

Combining (3) with (1), we get

$$-y\,(x-y)\,dp\,dx+y\,(x-y)\,dq\,dx-(p-q)\,(-y\,dx^2+y\,dx\,dy)=0,$$

i.e., $(x-y)\,(dp+dq)-(p-q)\,dx-dy)=0$

or $$\frac{dp-dq}{p-q}=\frac{dx-dy}{x-y}.$$

Integrating, $p-q=B\,(x-y)$.

Hence the intermediate integral is

$$p-q=(x-y)\,f(xy),$$

which is of the Lagrange's form. Hence the Lagrange's subsidiary equations are

$$\frac{dx}{1}=\frac{dy}{-1}=\frac{dz}{(x-y)\,f(xy)}=\frac{f(xy)\,[y\,dx-x\,dy]+dz}{0}.$$

From the first two relations, we get

$$x+y=C.$$

And from the last relation, we get

$$dz+f(xy)\,d(xy)=0$$

or $z=F_1(xy)+\text{const.}$

Hence the complete solution is

$$z=F_1(xy)+F_2(x+y).$$

Ex. 17. *Solve* $x^2r-y^2t-2xp+2z=0.$

Solution. Putting $r=\dfrac{dp-s\,dy}{dx}$ and $t=\dfrac{dq-s\,dx}{dy}$, the given equation becomes

$$x^2\,\frac{dp-s\,dy}{dx}-y^2\,\frac{dq-s\,dx}{dq}-2xp-2z=0.$$

The Monge's subsidiary equations are

$$x^2\,dp\,dy-y^2\,dq\,dx-(2xp-2z)\,dx\,dy=0 \qquad ...(1)$$

and $$x^2\,dy^2-y^2\,dx^2=0. \qquad ...(2)$$

Two factors of (2) are

$$x\,dy-y\,dx=0 \qquad ...(3)$$

and $$x\,dy+y\,dx=0. \qquad ...(4)$$

From (3), $\dfrac{dy}{y}=\dfrac{dx}{x}$ or $\dfrac{y}{x}=A$, const.

Combining (3) and (1), we get

$$x\,dp-q\,dq-2\,(xp-z)\,\frac{dx}{x}=0$$

or $$d(xp-z)+dz-p\,dx-y\,dq-2\,(xp-z)\,\frac{dx}{x}=0$$

or $$d(xp-z)+(p\,dx+q\,dy)-p\,dx-y\,dq-2\,(xp-z)\,\frac{dx}{x}=0$$

or $$d(xp-z)-d(yq)-2\,(xp-yq-z)\,\frac{dx}{x}=0$$

or $$\frac{d(xp-yq-z)}{xp-yq-z}=\frac{2\,dx}{x}.$$

Integrating it, we get $(xp-yq-z)=Bx^2$, where B is a constant. Thus one intermediate integral is

$$xp-yq-z=f_1(y/x)\,x^2$$

or $$xp-yq=x^2 f_1(y/x)+z. \qquad ...(5)$$

We may find another intermediate integral or else find the complete solution from (5) alone as follows. From (5) the Lagrange's auxiliary equations are

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This problem can also be solved by substituting $z=e^u$, $y=e^v$ etc.

$$\frac{dx}{x}=\frac{dy}{-y}=\frac{dz}{x^2 f_1(y/x)+z}.$$

The first two give, $xy=$const , C (say). ...(6)

Also taking $\dfrac{dy}{-y}=\dfrac{dz}{x^2 f_1(y/x)+z}$, we get

$$\frac{dz}{dy}+\frac{z}{y}=-\frac{x^2}{y} f_1\left(\frac{y}{x}\right)$$

$$=-\frac{c^2}{y^3} f_1\left(\frac{y^2}{c}\right) \text{ from (6).}$$

This is linear equation, with integrating factor y.

$$\therefore \quad yz=-\int \frac{c^2}{y^2} f_1\left(\frac{y^2}{c}\right) dy+D \text{ (const.)}$$

$$=-\frac{c^{3/2}}{2}\int u^{3/2} f_1(u)\, du+\text{const.}$$

when $\dfrac{y^3}{c}=u$

$$=c^{3/2} F_1(u)+\text{const.}$$

$$=x^{3/2} y^{3/2} F_1(y/x)+F_2(xy).$$

Ex. 18. *Solve*

$$(x-z)\,[xq^2 r-q\,(x+z+2px)\,s+(z+px+pz+p^2x)\,t]$$
$$=(1+p)\,q^2\,(x+z).$$

Solution. Putting $r=\dfrac{dp-s\,dy}{dx}$ and $t=\dfrac{dq-s\,dx}{dy}$, the Monge's subsidiary equations are

$$xq^2\,(x-z)\,dp\,dy+(x-z)\,(z+px+pz+p^2x)\,dq\,dx$$
$$-(1+p)\,q^2\,(x+z)\,dx\,dy=0 \quad ...(1)$$

$$xq^2\,dy^2+q\,(x+z+2px)\,dx\,dy+(z+px+pz+p^2x)\,dx^2=0. \quad ...(2)$$

The two factors of (2) are

$$q\,dy+(1+p)\,dx=0 \quad ...(3)$$

and $\quad xq\,dy+(z+px)\,dx=0.$ ...(4)

From (3), we get

$x+z=A$, const., as $dz=p\,dx+q\,dy$.

Also combining (3) and (1), we get

$$xq\,(2x-A)\,dp-(2x-A)\,(A-x+px)\,dq-(1+p)\,qA\,dx=0. \quad ...(5)$$

To solve it let $x=$const., $dx=0$; then it becomes

$$xq\,dp-(A-x+px)\,dq=0.$$

Integrating, $\dfrac{A-x+px}{q}=f(x)$, ...(6)

where $f(x)$ is to be determined so that (6) satisfies (5).

From (6), we get on differentiation,

$$q^2 df(x)=q\,(x\,dp+p\,dx-dx)-(A-x+px)\,dq$$

or $$(2x-A)\,q^2 df(x)=(2x-A)\,xq\,dp+(2x-A)\,q\,(p-1)\,dx$$
$$-(2x-A)\,(A-x+px)\,dq. \quad \ldots(7)$$

(5) and (7) on subtraction give

$$(2x-A)\,q^2\,df(x)=[(1+p)\,qA+(2x-A)\,q\,(p-1)]\,dx$$

or $$\frac{df(x)}{f(x)}=\frac{2\,dx}{2x-A} \text{ using (6).}$$

Integrating now $f(x)=\text{const.}\,(2x-A)$

Thus $$\frac{A-x+px}{q}=(2x-A).\text{ const.}$$

or $$\frac{z+px}{q}=(x-z).\text{ const.},$$

putting value of $A=x+z$.

Thus the first intermediate integral is

$$z+px=q\,(x+z)\,f_1(x-z). \quad \ldots(8)$$

Next (4) can be written as $x\,dz+z\,dx=0$

giving $xz=\text{const.}=B$ (say).

Combining now (4) with (1), we get

$$xq\,(x^2-B)\,dp-x\,(x^2-B)\,(1+p)\,dq$$
$$-(1+p)\,(x^2+B)\,dx=0. \quad \ldots(9)$$

To solve this we take $x=\text{const.}$ *i.e.* $dx=0$

(9) then reduces to

$$xq\,dp-x\,(1+p)\,dq=0 \quad \text{or} \quad q\,dp-(1+p)\,dq=0.$$

Integrating this, we get

$$\frac{1+p}{q}=g(z), \quad \ldots(10)$$

where $g(z)$ is to be determined as before.

Differentiating (10), we get

$$q^2\,dg(z)=q\,dp-(1+p)\,dq$$

or $$x\,(x^2-B)\,q^2\,dg\,(z)=xq\,(x^2-B)\,dp-x\,(x^2-B)\,(1+p)\,dq.$$

Comparing it with (9), we get

$$x\,(x^2-B)\,q^2\,dg(z)=(1+p)\,q\,(x^2+B)\,dx$$

or $$\frac{dg(z)}{g(z)}=\frac{x^2+B}{x\,(x^2-B)}\,dx=\left\{-\frac{1}{x}+\frac{2x}{x^2-B}\right\}dx.$$

Integrating, $$\frac{xg(z)}{x^2-B}=\text{const.}$$

Substituting these values of $g(z)$ and B, we get

$$\frac{1+p}{q\,(x-z)}=\text{const.}$$

So that another intermediate integral is

$$1+p=q\,(x-z)\,f_2(xz). \quad \ldots(11)$$

From (8) and (11) now

$$p=\frac{f_1(x+z)-z\,f_2(zx)}{xf_2(xz)-f_1(x+q)},\quad q=\frac{1}{xf_2(xz)-f_1(x+z)}.$$

Putting these values in $dz=p\,dx+q\,dy$, we get

$$dz=\frac{\{f_1(x+z)-z\,f_2(zx)\}\,dx+dy}{xf_2(xz)-f_1(x+z)}$$

or $f_2(xz)\,\{x\,dz+z\,dx\}-f_1(x+z)\,\{dz+dx\}=dy.$

Integrating, the complete solution is

$$F_2(xz)+F_1(x+z)=y,$$

where F_1 and F_2 are arbitrary functions.

Ex. 9. *Solve*

$$(1+pq+q^2)\,r+s\,(q^2-p^2)-(l+pq+p^2)\,t=0.$$

Solution. Putting $r=\dfrac{dp-s\,dy}{dx}$ and $t=\dfrac{dq-s\,dx}{dy}$, the given equation becomes

$$(1+pq+q^2)\,\frac{dp-s\,dy}{dx}+s\,(q^2-p^2)-(1+pq+p^2)\,\frac{dq-s\,dx}{dy}=0.$$

The Monge's subsidiary equations are

$$(1+pq+q^2)\,dp\,dy-(1+pq+p^2)\,dq\,dx=0 \quad ...(1)$$

and $(1+pq+q^2)\,dy^2-(q^2-p^2)\,dx\,dy-(1+pq+p^2)\,dx^2=0.$ ...(2)

The two factors of (2) are

$$dy-dx=0 \quad ...(3)$$

and $(1+pq+q^2)\,dy+(1+pq+p^2)\,dx=0.$ ...(4)

Now from (3), $y-x=A$, const.

Combining (3) with (1), we obtain

$$(1+pq+q^2)\,dp-(1+pq+p^2)\,dq=0$$

or $$(1+pq)\,(dp-dq)+(q^2\,dp-p^2\,dq)=0$$

or $$(1+pq)\,d(p-q)+p^2q^2\left(\frac{dp}{p^2}-\frac{dq}{q^2}\right)=0$$

or $$(1+pq)\,d(p-q)+p^2q^2\,d\left(\frac{1}{p}-\frac{1}{q}\right)=0$$

or $$\frac{d\,(p-q)}{p-q}-\frac{d(pq)}{1+2pq}=0.$$

Integrating, this gives

$$(p-q)\,(1+2pq)^{-1/2}=\text{const.}$$

Therefore one intermediate integral is

$$(p-q)\,(1+2pq)^{-1/2}=f_1(y-x). \quad ...(5)$$

where f_1 is an arbitrary function.

Next, (4) can be written as

$$(dy+dx)+(p+q)\,(q\,dy+p\,dx)=0$$

or $d(y+x)+(p+q)\,dz=0$ as $dz=p\,dx+q\,dy.$ ...(6)

Also combining (4) with (1), we get $dp+dq=0$

giving $p+q=B$.

$\therefore$ (6) gives $y+x+Bz=\text{const.}$

Therefore another intermediate integral is

$$x+y+(p+q)\,z=f_2\,(p+q). \qquad ...(7)$$

Let us take

$$p+q=l, \qquad ...(8)$$

so that (7) becomes $x+y=f_2(l)-lz.$ ...(9)

Also Lagrange's auxiliary equations from (8) are

$$\frac{dx}{1}=\frac{dy}{1}=\frac{dz}{l}=\frac{dx+dy}{2},$$

giving $x-y=\text{const.}$ and $\dfrac{dz}{l}=\dfrac{d\,(x+y)}{2}$

or $\dfrac{dz}{l}=\dfrac{d\,[\,f_2(l)-lz]}{2}$ from (9)

or $\dfrac{dz}{dl}+\dfrac{l}{2+l^2}\,z+\dfrac{f_2'(l)\,dl}{2+l^2}.$

This is a linear ordinary differential equation with integrating factor $\sqrt{(2+l^2)}$.

$\therefore$ Solution is

$$z\,(2+l^2)^{1/2}=\int l\,(2+l^2)^{-1/2}\,f_2'(l)\,dl+C\ (\text{const.})$$

Therefore the complete solution of the given equation is

$$z\,(2+l^2)^{1/2}=\int l\,(2+l^2)^{-1/2}\,f_2'(l)\,dl+F\,(x-y)$$

$$=F_1(l)+F_2(x-y)$$

or $z\,[2+(x+y)^2]^{1/2}=F_1\,(x+y)+F_2(x-y).$

Exercises

Solve the following differential equations by Monge's method :

1. $r=a^2t.$ [Agra 62, 59]

   Ans. $z=F_1(y+ax)+F_2(y-ax).$

2. $r-t\cos^2 x+p\tan x=0.$

   Ans $z=F_1(y+\sin x)+F_2(y-\sin x).$

3. $rq^2-2pqs+tp^2=pt-qs.$ [Delhi Hons. 70]

   Ans. $y=f_1(x+z)+f_2(z)$

4. $z\,(qs-pt)=pq^2.$ **Ans.** $y=f_1(z)+zf_2(x).$

5. $x^2r-2xs+t+q=0.$

   Ans. $z=F_1(y+\log x)+xF_2(y+\log x).$

6. $(r-s)\,y+(s-t)\,x+q-p=0.$

   Ans. $z=f_1(x+y)+f_2(x^2-y^2).$

7. $x^2r-y^2t=xy.$ **Ans.** $z=xy\log x+xF_1(y/x)+F_2(xy).$

8. $q(1+q)r-(1+2q)(1+p)s+(1+p)^2t=0.$

Ans. $x=F_1(x+y+z)+F_2(x+z)$

9. $(e^x-1)(qr-ps)=pqe^x.$ Ans. $x=F_1(z)+F_2(y)+e^x.$

10. $x^{-2}r-y^{-2}t=x^{-3}p-y^{-3}q.$

Ans. $z=F_1(x^2+y^2)+F(x^2-y^2).$

4·3. Monge's method of integrating

$$Rr+Ss+Tt+U(rt-s^2)=V,$$

where r, s, t have their usual meanings and R, S, T, U, V are functions of x, y, z, p, q.

Substituting $r=\dfrac{dp-s\,dy}{dx}$ and $t=\dfrac{dq-s\,dx}{dy}$,

the given differential equation becomes

$$R\,dp\,dy+T\,dq\,dx+U\,dp\,dq-V\,dx\,dy$$
$$-s\,(R\,dy^2-S\,dx\,dy+T\,dx^2+U\,dp\,dx+V\,dq\,dy)=0.$$

The Monge's subsidiary equations are

$$L\equiv R\,dp\,dy+T\,dq\,dx+U\,dp\,pq-V\,dx\,dy=0. \quad ...(1)$$

and $M\equiv R\,dy^2-S\,dx\,dy+T\,dx^2+U\,dp\,dx+V\,dq\,dy=0,$ ...(2)

Here (2) cannot be factorised into linear factors on account of the terms $U\,dp\,dx+V\,dq\,dy$ in it.

However we try to factorise $M+\lambda L$,

where λ is some multiplier to be determined later.

Now $M+\lambda L=R\,dy^2+T\,dx^2+(S+\lambda V)\,dx\,dy+U\,dp\,dx$
$+U\,dq\,dy+\lambda R\,dp\,dy+vT\,dq\,dx+\lambda U\,dp\,dq.$

Also let

$$M+\lambda L\equiv(R\,dy+mT\,dx+KU\,dp)\left(dy+\frac{1}{m}\,dx+\frac{\lambda}{K}\,dq\right)=0 \quad ...(3)$$

Comparing coefficients, we have

$$\frac{R}{m}+mT=-(S+\lambda V),\ K=m,\ \frac{R\lambda}{K}=V.$$

From the last two relations is obtained

$$m=\frac{R\lambda}{U}.$$

Putting this value of m in the first of these relations, we get the quadratic relations in λ given by

$$\lambda^2(UV+RT)+\lambda SU+U^2=0.$$

Let λ_1, λ_2 be two values of λ, which are in general distinct.

For $\lambda=\lambda_1$, *i.e.* $m=R\lambda_1/U$, the factors from (3) are

$$\left(R\,dy+\frac{R\lambda_1}{U}\,T\,dx+R\lambda_1\,dp\right)\left(dy+\frac{U}{R\lambda_1}\,dx+\frac{u}{R}\,dq\right)=0 \quad ...(4)$$

or $(U\,dy+\lambda_1T\,dx+\lambda_1U\,dp)(U\,dx+\lambda_1R\,dy+\lambda_1U\,dq)=0.$

Similarly for $\lambda=\lambda_2$, (3) can be written as

$$(U\,dy+\lambda_2T\,dx+\lambda_2U\,dp)(U\,dx+\lambda_2R\,dy+\lambda_2U\,dq)=0 \quad ...(5)$$

Now one factor of (4) is combined with one factor of (5) to give an intermediate integral and similarly other pair gives another intermediate integral. This cannot be obtained if we combine first of (4) with first of (5) and second of (4) with second of (5).

However the pair,

$$\left.\begin{array}{l} U\,dy+\lambda_1 T\,dx+\lambda_1 U\,dp=0 \\ \text{and } U\,dx+\lambda_2 R\,dx+\lambda_2 U\,dp=0 \end{array}\right\} \qquad \ldots(\text{I})$$

gives two integrals $u_1=a,\ v_1=b$

$$\left.\begin{array}{ll} \text{and pair} & U\,dx+\lambda_1 R\,dy+\lambda_1 U\,dq=0 \\ \text{and} & U\,dy+\lambda_2 T\,dx+\lambda_2 U\,dp=0 \end{array}\right\} \qquad \ldots(\text{II})$$

gives two integrals $u_2=c$ and $v_2=d$.

Thus the two intermediate integrals are

$$u_1=f_1(v_1) \text{ and } u_2=f_2(v_2).$$

Find from these two intermediate integrals, the values of p and q and substitute these values in the relation

$$dz=p\,dx+q\,dy,$$

which after integration gives the general solution.

Note. When two values of λ are equal, we proceed with one intermediate integral only.

Ex. 1. *Solve* $2s+(rt-s^2)=V$. [Raj. 61, 59]

Solution. Comparing this with the equation

$$Rr+Ss+Tt+U(rt-s^2)=V.$$

we find that $R=0,\ S=2,\ T=0,\ U=1,\ V=1$.

The λ-equation,

$$\lambda^2(UV+RT)+\lambda SU+U^2=0$$

becomes $\lambda^2+2\lambda+1=0,\ (\lambda+1)^2=0$.

This gives $\lambda_1=-1,\ \lambda_2=-1$.

Since both the values of λ are equal, there would be only one intermediate integral and the same is given by

$$U\,dy+\lambda_1 T\,dx+\lambda_1 U\,dp=0$$

and $U\,dx+\lambda_2 R\,dy+\lambda_2 U\,dq=0$,

or by $dy-dp=0$ which gives $y-p=a$

and $dx-dq=0$ which gives $x-q=b$,

where a and b are arbitrary constants.

Thus the intermediate integral is

$$x-q=f(y-p).$$

Substituting $p=y-a$ and $q=x-b$ in the relation

$$dz=p\,dx+q\,dy,$$

we get $dz=(y-a)\,dx+(x-b)\,dy$

$$=x\,dy+y\,dx-a\,dx-b\,dy.$$

Integrating $z=xy-ax-by+c$,

or $$z=xy-ax-\phi(a)\,y+\psi\,(a), \quad \ldots(1)$$
which is the general solution.

General integral would be obtained by eliminating a from (1) and $0=-x-\phi'\,(a)\,y+\psi'\,(a)$.
where a is an arbitrary constant.

Ex. 1. *Solve* $r+3s+t+(rt-s^2)=1$. [Agra 72; Raj. 66]

Solution. Here $R=1,\ S=3,\ T=1,\ U=1,\ V=1$.

Hence λ-equation, $\lambda^2\,(UV+RT)+\lambda SU+U^2=0$ gives
$$2\lambda^2+3\lambda+1=0,\ (2\lambda+1)\,(\lambda+1)=0,$$
so that $\lambda_1=-1,\ \lambda_2=-\frac{1}{2}$.

Hence the first system of integral is given by
$$\left.\begin{array}{l} dy-dx-dp=0,\ i.e.\ y-x-p=\text{const.} \\ \text{and}\quad dy-2\,dx+dq=0,\ i.e.\ y-2x+q=\text{const.}\end{array}\right\} \quad \ldots(1)$$
So another intermediate integral is
$$y-x-p=f_1\,(y-2x+q)=f_1\,(\alpha), \quad \ldots(2)$$
where $\alpha=y-2x+q$.

The second system of integrals is given by
$$\left.\begin{array}{l} 2\,dy-dx-dp=0,\ i.e.\ 2y-x-p=\text{const.} \\ \text{and}\quad dy-dx-dq=0,\ i.e.\ \ y-x-q=\text{const.}\end{array}\right\} \quad \ldots(3)$$
So one intermediate integral is
$$2y-x-p=f_2\,(x-y-q)=f_2\,(\beta),$$
where $\beta=x-y-q$

From these relations, we get
$$x=-\beta-\alpha,\ y=f_2\,(\beta)-f_1\,(\alpha),$$
$$p=y-x-f_1\,(\alpha),\ q=x-y-\beta.$$
Substituting these values in $dz=p\,dx+q\,dy$,
$$dz=[y-x-f_1\,(\alpha)]\,dx+(x-y-\beta)\,dy$$
$$=-(x-y)\,(dx-dy)-f_1\,(\alpha)\,[-d\beta-d\alpha]$$
$$-\beta\,[f_2'\,(\beta)\,d\beta-f_1'\,(\alpha)\,d\alpha].$$
Integrating,
$$z=-\tfrac{1}{2}\,(x-y)^2+\int f_1\,(\alpha)\,d\alpha-\int \beta f_2'(\beta)\,d\beta+\beta f_1(\alpha)$$
$$=-\tfrac{1}{2}\,(x-y)^2+F_1\,(\alpha)+F_2(\beta)-\beta f_2\,(\beta)+\beta f_1(\alpha),$$
which is the required solution.

Ex. 3. *Solve*
$$z\,(1+q^2)\,r-2pqzs+z\,(1+p^2)\,t-z^2\,(s^2-rt)+1+p^2+q^2=0.$$
[Vikram 62; Agra 66, 53]

Solution. Comparing the given equation with
$$Rr+Ss+Tt+U\,(rt-s^2)=V,$$
we get $R=z\,(1+q^2),\ S=-2pqz,\ T=z\,(1+p^2)$,
$U=z^2$ and $V=-(1+p^2+q^2)$.

The λ-equation

$$\lambda^2 (UV+RT)+\lambda SU+U^2=0,$$

becomes $\lambda^2p^2q^2-2\lambda zpq+z^2=0.$

This gives $\lambda_1+\lambda_2=z/(pq).$

Therefore the intermediate integral is given by

$$U\,dy+\lambda_1T\,dx+\lambda_1U\,dp=0$$

and $U\,dx+\lambda_2R\,dy+\lambda_2U\,dp=0,$

or by $pq\,dy+(1+p^2)\,dx+z\,dp=0$ ...(1)

and $pq\,dx+(1+q^2)\,dy+z\,dq=0$ ...(2)

Also, we have $dz=p\,dx+q\,dy.$ ...(3)

From (1) and (3), we get

$dx+z\,dp+p\,dz=0,$ *i.e.* $x+zp=a,$ ...(4)

where a is an arbitrary constant.

Also from (2) and (3), we get

$dy+z\,dq+q\,dz=0,$ *i.e.* $y+zq=b,$ ...(5)

where b is an arbitrary constant.

From (4) and (5), $p=\dfrac{a-x}{z}$ and $q=\dfrac{b-y}{z}.$

Putting these values of p and q in (3), we get

$$dz=\frac{a-x}{z}\,dx+\frac{b-y}{z}\,dy.$$

or $z\,dz=(a-x)\,dx+(b-y)\,dy.$

Integrating, $z^2=-(a-x)^2-(b-y)^2,\ a=f(b)$

is the general solution of the given equation.

Ex. 4. *Solve* $qr+(p+x)\,s+yt+y\,(rt-s^2)+q=0.$ [Raj. 62]

Solution. Comparing it with

$$Rr+Ss+Tt+U\,(rt+s^2)=V,$$

we get $R=q,\ S=p+x,\ T=y,\ U=y,\ V=-q,$

Hence λ-equation, $\lambda^2\,(UV+RT)+\lambda SU+U^2=0$

becomes $\lambda^2\,(-yq+yq)+\lambda\,(p+x)\,y+y^2=0$

this gives $\lambda_1=-y/(p+x)$ and $\lambda_2=\infty.$

For intermediate integral we have the pair

$$U\,dy+\lambda_1T\,dx+\lambda_1\,U\,dp=0$$

and $U\,dx+\lambda_2R\,dy+\lambda_2\,U\,dq=0$

or $y\,dx-\dfrac{y^2}{p+x}\,dx+\dfrac{y^2}{p+x}\,dp=0$

and $0+q\,dy+y\,dq=0,$ as $1/\lambda_2=0$

These give $(p+x)/y=a$ and $qy=b,$

where a and b are arbitrary constants.

Intermediate integral is

$$qy=f_1\left(\frac{p+x}{y}\right). \quad \ldots(1)$$

The other intermediate integral is give by the pair,

$U\,dy+\lambda_2 T\,dx=\lambda_2 U\,dp=0$

and $U\,dx+\lambda_1 R\,dy+\lambda_1 U\,dq=0$

or $y\,dx+y\,dp=0$

and $y\,dx-\dfrac{qy}{p+x}\,dy-\dfrac{y^2\,dq}{p+x}=0$ giving $p+x=c$, ...(2)

where c is an arbitrary constant.

Now from (2), $p=c-x$

and from (1) $q=\dfrac{1}{y}f_1\left(\dfrac{p+x}{y}\right)=f\left(\dfrac{c}{y}\right)$

$=\dfrac{af(a)}{c}$ as $(p+x)/y=a$.

Putting these values in the relation $dz=p\,dx+q\,dy$, we get

$dz=(c-x)\,dx+(a/c)\,f(a)\,dy.$

Integrating, $z=cx-\frac{1}{2}x^2+(a/c)\,f(a)\,y+\text{const.}$

$=cx-\frac{1}{2}x^2+f(c/y)+F(c)$, ...(3)

which is the required general solution.

General integral is obtained by eliminating c between (3) and

$0=x+(1/y)\,f'(c/y)+F'(c)$.

Ex. 6. *Solve* $ar+bs+ct+e\,(rt-s^2)=h$,

a, b, c, e, h *being constants.* [Raj. 64; Agra 52]

Solution. Here $R=a$, $S=b$, $T=c$, $U=e$, $V=h$.

Hence the λ-equation is

$\lambda^2\,(ac+eh)+\lambda eb+e^2=0$

or if we write $\lambda m+e=0$, the equation which gives m is

$m^2-bm+ac+eh=0.$

Let m_1, m_2 be its two roots; then first system of integral is given by

$$\left.\begin{aligned} c\,dx+e\,dp-m_1\,dy=0 \\ a\,dy+e\,dq-m_2\,dx=0 \end{aligned}\right\},$$

which gives $cx+ep-m_1y=\text{const.}$

and $ay+eq-m_2x=\text{const.}$

Hence one intermediate integral is

$(cx+ep-m_1y=\phi_1(ay+eq-m_2x)$. ...(1)

The second system of integrals is given by

$a\,dy+e\,dq-m_1\,dx=0$, giving $ay+eq-m_1x=\text{const.}$

and $c\,dx+e\,dp-m_2\,dy=0$, giving $ex=ep-m_2y=\text{const.}$

Therefore the other intermediate integral is

$cx+ep-m_2y=\phi_2(ay+eq-m_1x)$. ...(2)

Here p and q cannot be directly found out ; so we combine any particular integral of the second with the general of the first system. Thus, we take

$cx+ep-m_2y=\alpha.$ ...(3)

$\therefore$ from (1), $\phi_1(ay+eq-m_2x)=(m_2-m_1)\,y+\alpha.$

If ψ be the inverse function of ϕ_1, we have

$ay+eq=m_2x+\psi\{(m_2-m_1)\,y+\alpha\}.$ ...(4)

Thus (3) and (4) give p and q.

Substituting these values in the relation

$dz=p\,dx+q\,dy,$

$e\,dz=cx\,dx-ay\,dy+(m_2y+\alpha)\,dx+[m_2x+\phi\{(m_2-m_1)\,y+\alpha\}]\,dy.$

Integrating it, we get

$ez+\frac{1}{2}cx^2+\frac{1}{2}ay^2=m_2xy+\alpha x+F[(m_2-m_1)\,y+\alpha]+\beta,$

where F is an arbitrary function given by

$$(m_2-m_1)\,F(z)=\int\psi(z)\,dz$$

and β is an arbitrary constant,

Ex. 6. *Solve* $s^2-rt=a^2.$ **[Delhi Hons. 69]**

Solution. Comparing the given equation with

$Rr+Ss+Tt+U\,(rt-s^2)=V,$

we get $R=0,\ S=0,\ T=0,\ U=1,\ V=-a^2.$

Hence λ-equation is

$\lambda^2\,(-a^2)+\lambda\,(0)+1=0$ or $a^2\lambda^2-1=0,$

which gives $\lambda_1=\dfrac{1}{a},\ \lambda_2=-\dfrac{1}{a}.$

Therefore the two intermediate integrals are given by

$$\left.\begin{aligned}&-dy-\frac{1}{a}\,dp=0,\text{ giving } p+ay=\alpha',\\&-dx+\frac{1}{a}\,dq=0,\text{ giving } q-ax=\alpha.\end{aligned}\right\}\quad ...(1)$$

and
$$\left.\begin{aligned}&-dy+\frac{1}{a}\,dp=0,\text{ giving } p-ay=\beta',\\&-dx-\frac{1}{a}\,dq=0\text{ giving } q+ax=\beta.\end{aligned}\right\}\quad ...(2)$$

Thus the two intermediate integrals are

$p+ay=f_1(q-ax)$ or $p+ay=f_1(\alpha)$ ...(3)

and $p-ay=f_2(q+ax)$ or $p-ay=f_2(\beta).$ ...(4)

But it is not possible to find the values of p and q from (3) and (4). Hence we proceed as follows :

Solving (1) and (2), we get

$x=\dfrac{\beta-\alpha}{2a}$ and $q=\dfrac{\alpha+\beta}{2}.$

Also $p=\frac{1}{2}(\alpha'+\beta')=\frac{1}{2}[f_1(\alpha)+f_2(\beta)]$,

$$y=\frac{\alpha'-\beta'}{2a}=\frac{1}{2a}[f_1(\alpha)-f_2(\beta)].$$

Let us regard α and β as parameters.

Putting these values in the relation $dz=p\,dx+q\,dy$, we get

$$dz=\tfrac{1}{2}[f_1(\alpha)+f_2(\beta)]\,d\left(\frac{\beta-\alpha}{2a}\right)+\tfrac{1}{2}(\alpha-\beta)\,d\left[\frac{1}{2a}\{f_1(\alpha)-f_2(\beta)\}\right]$$

$$=\frac{1}{4a}[\{f_1(\alpha)\,d\beta-\beta f_1'(\alpha)\,d\alpha\}-\{f_2(\beta)\,d\alpha+\beta f_2'(\beta)\,d\beta\}]$$

$$\times[\{f_1(\alpha)\,d\alpha+\alpha f_1'(\alpha)\,d\alpha\}-f_2(\beta)\,d\beta+df_2'(\beta)\,d\beta$$
$$+2f_2(\beta)\,d\beta-2f_1(\alpha)\,d\alpha].$$

Integrating, we get

$$z=\frac{1}{4a}\left[\beta f_1(\alpha)-\alpha f_2(\beta)+\beta f_2(\beta)+\alpha f_1(\alpha)\right.$$
$$\left.+2\int f_2(\beta)\,d\beta-2\int f_1(\alpha)\,d\alpha\right]$$

$$=\frac{1}{4a}[(\alpha+\beta)f_1(\alpha)-(\alpha+\beta)f_2(\beta)]+\frac{2}{4a}\phi_1(\beta)-\frac{2}{4a}\phi_2(\alpha)$$

$$=\tfrac{1}{2}(\alpha+\beta)\left[\frac{f_1(\alpha)-f_2(\beta)}{2a}\right]+\frac{1}{2a}\phi_1(\beta)-\frac{1}{2a}\phi_2(\alpha)$$

$$=qy+\frac{1}{2a}\phi_1(\beta)-\frac{1}{2a}\phi_2(\alpha)$$

or $\quad z-qy=\frac{1}{2a}\psi_1(\beta)-\frac{1}{2a}\psi_2(\alpha)=F_1(q+ax)+F_2(q-ax)$

from (1) and (2),

Hence the complete solution is

$$z-qy=F_1(q+ax)+F_2(q-ax),$$

where $\quad -y=F_1'(q+ax)+F_2'(q-ax)$.

Ex. 7. *Solve* $2r+te^x-(rt-s^2)=2e^x$.

Solution. In this equation, we proceed directly by putting

$$z=\frac{dp-s\,dy}{dx}\text{ and }t=\frac{dq-s\,dx}{dy}.$$

Then the Monge's equations are

$$2\,dp\,dy+e^x\,dq\,dx-dp\,dq-2e^x\,dx\,dy=0. \quad ...(1)$$

and $\quad 2\,dy^2+e^x\,dx^2-dp\,dx-dq\,dy=0. \quad ...(2)$

(1) gives $\quad (2\,dy-dq)(dp-e^x\,dx)=0$,

i.e. $\quad 2\,dy-dq=0$ giving $2y-q=a \quad ...(3)$

and $\quad dp-e^x\,dx=0$ giving $p-e^x=b, \quad ...(4)$

where a and b are arbitrary constants.

(3) and (4) give $b=2y-a$, $p=b+e^x$.

Substituting these values in the relation

$dz = p\,dx + q\,dy;$

we get $dz = (b + e^x)\,dx + (2y - a)\,dy.$

Integrating, we get

$z = e^x + bx + y^x - ay + C.$

Ex. 8. *Solve* $rt - s^2 - s(\sin x + \sin y) = \sin x \sin y.$

Solution. Comparing it with $Rr + Ss + Tt + U(rt - s^2) = V$, we find that $R = 0$, $T = 0$, $S = -(\sin x + \sin y)$, $U = 1$, $V = \sin x \sin y$.

The λ-equation is

$$\lambda^2 (\sin x \sin y) - \lambda (\sin x + \sin y) + 1 = 0$$

This gives $\lambda_1 = \sin x$, $\lambda_2 = \sin y$.

One of the intermediate integrals is given by

$\sin x\,dy + dp = 0$, $\sin y\,dx + dq = 0$.

This is not integrable.

The other intermediate integral is given by

$\sin y\,dy + dp = 0$, $\sin x\,dx + dq = 0$.

This gives on integration,

$p - \cos y = a$, $q - \cos x = b$

where a and b are arbitrary constants.

Therefore the intermediate integral is

$$p - \cos y = f(q - \cos x). \qquad \text{..(1)}$$

From (1) Charpit's auxiliary equations are

$$\frac{dp}{-\sin x f'(q - \cos x)} = \frac{dq}{\sin y} = \frac{dx}{-1} = \frac{dy}{-f'(q - \cos x)}.$$

These cannot be integrated. Let us therefore suppose that the arbitrary function f is linear, *i.e.*

$$p - \cos y = \mu (q - \cos x) + \nu \qquad \text{..(2)}$$

where μ and ν are constants.

Lagrange's auxiliary equations from (2) are

$$\frac{dx}{1} = \frac{dy}{-\mu} = \frac{dz}{\cos y - \mu \cos x + \nu}$$

These gives $y + \mu x = C$ (const.).

Also $\dfrac{dx}{1} = \dfrac{dz}{\cos y - \mu \cos x + \nu}$ gives

$dz = [\cos (c - \mu x) - \mu \cos x + \nu]\,dx.$

Integrating,

$$z = -\frac{1}{\mu} \sin (c - \mu x) - \mu \sin x = \nu x + d' \text{ (const.)}$$

or $\mu z + \sin y + \mu^2 \sin x - \mu \nu x = \mu d'.$

Thus in this case the most general integral is

$$\mu z+\sin y+\mu^2 \sin x-\mu\nu x=\mu\phi\,(y+\mu x)$$

Exercises

Solve the following examples by Monge's method :

1 $5r+6s+3t+2\,(rt-s^2)+3=0.$

Ans. $4z=6xy-3x^2-2ax-5y^2-2by+c$

2 $3r+s+t+(rt-s^2)+9=0.$

Ans. $z=cy-2xy-\frac{1}{2}x^2-\frac{3}{2}y^2+\phi\,(c-5x)+\psi(c),$

$0=y+\phi'(c-5x)+\psi'(c)$

3. $xqr+ypt+xy\,(s^2-rt)=pq.$

Ans. $z+c=x^2\phi(c)+y^2\psi(c),\ 1=x^2\phi'(c)+y^2\psi'(c)$

4. $yr-ps+t+y\,(rt-s^2)+1=0.$

Ans. $6c^2z=2y^3-3c^2y^2+6cxy+6\,dy+\phi\,(cx+\frac{1}{2}y^2).$

5. $2yr+(px+qy)\,s+xt-xy\,(rt-s^2)=2-pq.$

Hint. $\lambda_1=y/p,\ \lambda_2=x/q$

6. $(1+q^2)\,r-2pqs+(1+p^2)\,t+(1+p^2+q^2)^{-1/2}\,(rt-s^2)$

$+(1+p^2+q^2)^{3/2}=0$

Hint. $\lambda_1=\lambda_2=1/qp\sqrt{(1+p^2+q^2)}$

4·4. Canonical Forms (*Method of Transformations*)

We shall now consider equations of the type

$$Rr+Ss+Tt+f(x,\,y,\,z,\,p,\,q)=0 \qquad ...(1)$$

in which R, S, T are continuous functions of x and y possessing continuous partial derivatives of as high order as necessary.

We shall show that by suitable change of independent variab'es, the equation (1) can be transformed into one of three canonical forms, which can be easily integrated.

Let us change independent variables x and y to u and v through the transformation equations

$$u=u\,(x,\,y), \qquad ...(2)$$

$$v=v\,(x,\,y). \qquad ...(3)$$

We shall start with these general transformations and later by suitable conditions determine their form.

Now from (2) and (3),

$$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x},\quad q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y},$$

$$r=\frac{\partial p}{\partial x}=\frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2+2\,\frac{\partial^2 z}{\partial u\,\partial v}\,\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2$$

$$+\frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2}+\frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2},$$

$$t=\frac{\partial q}{\partial y}=\frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2+2\frac{\partial^2 z}{\partial u\,\partial v}\frac{\partial u}{\partial y}\frac{\partial u}{\partial y}+\frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2$$
$$+\frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y^2}+\frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y^2},$$

$$s=\frac{\partial p}{\partial y}=\frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial^2 z}{\partial u\,\partial v}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)$$
$$+\frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y\,\partial x}+\frac{\partial z}{\partial v}\frac{\partial^2 n}{\partial y\,\partial x^2}.$$

Putting these values in (1), the equation reduces to

$$A\frac{\partial^2 z}{\partial u^2}+2B\frac{\partial^2 z}{\partial u\,\partial v}+C\frac{\partial^2 z}{\partial v^2}+F\left(u,v,z,\frac{\partial z}{\partial u},\frac{\partial z}{\partial v}\right)=0, \quad ...(4)$$

where $F\left(u,v,z,\frac{\partial z}{\partial u},\frac{\partial z}{\partial v}\right)$

is the transformed form of $f(x, y, z, p, q)$ and

$$A=R\left(\frac{\partial u}{\partial x}\right)^2+S\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+T\left(\frac{\partial u}{\partial y}\right)^2, \quad ...(5)$$

$$B=R\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\tfrac{1}{2}S\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)+T\frac{\partial u}{\partial y}\frac{\partial u}{\partial y}, \quad ...(6)$$

$$C=R\left(\frac{\partial v}{\partial x}\right)^2+S\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+T\left(\frac{\partial v}{\partial y}\right)^2. \quad ...(7)$$

The problem is now to determine u and v so that equation (4) takes the simplest possible form. The procedure is simple when the discriminant S^2-4RT of the quadratic equation

$$R\alpha^2+S\alpha+T=0 \quad ...(8)$$

is either positive, negative or zero everywhere. We discuss these cases separately.

Case I. When $S^2-4RT>0$. The two roots α_1 and α_2 of (8) would be real and distinct in this case.

Let us take $$\frac{\partial u}{\partial x}=\alpha_1\frac{\partial u}{\partial y} \quad ...(9)$$

and $$\frac{\partial y}{\partial x}=\alpha_2\frac{\partial v}{\partial y}. \quad ...(10)$$

Under these conditions and the fact that α_1 and α_2 are roots of (8), it is found that

$$A=(R\alpha_1^2+S\alpha_1+T)(\partial u\,\partial y)^2=0,$$

Similarly because of (10), $C=0$.

For differential equations (9) and (10), we determine the form of u and v as functions of x and y.

For this, from (9), the auxiliary equations are

$$\frac{dx}{1}=\frac{dy}{-\alpha_1}=\frac{du}{0}.$$

From the last relation, we get $u=$const.

Also from the first two,

$$\frac{dy}{dx}+\alpha_1=0. \qquad ...(11)$$

Let $f_1(x, y)=$const, be solution* of (11), then the solution of (9) is $u=f_1(x, y)$. ...(12)

Similarly, if $f_2(x, y)$ is a solution of

$$\frac{dy}{dx}+\alpha_2=0, \qquad ...(13)$$

then solution of (10) is

$$v=f_2(x, y). \qquad ...(14)$$

Relations (12) and (14) are the desired transformation relations to change the independent variables.

Now it can be easily shown that

$$AC-B^2=\tfrac{1}{4}(4RT-S^2)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)^2$$

i.e. $$4B^2=(-4RT+S^2)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)^2 \qquad ...(15)$$

as $A=C=0$

i.e. $B^2>0$ as $S^2-4RT>0$.

And therefore we may divide both sides of the equation by it. The equation is finally reduced to the form

$$\frac{\partial^2 z}{\partial u\,\partial v}=\phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right) \qquad ...(16)$$

which is the canonical form in this case.

Case II. When $S^2-4RT=0$. In this case two roots of (8) are equal;

We define the function u as in case I and take v to be any function of x and y, which is independent of u.

As before $A=0$.

Since $S^2-4RT=0$, from (15)

$$B=0.$$

Putting A and B equal to zero, and dividing by $C\ (\neq 0)$, the equation takes the canonical form

$$\frac{\partial^2 z}{\partial v^2}=\phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right). \qquad ...(17)$$

Case III. When $R^2-4ST<0$. This is practically the same ase as case I except that now the roots of (8) are complex.

The equation (1) would reduce to equation (16) if we proceed as in case I, but the variables u and v are not real but conjugate omplex.

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*If α_1 is a constant, then the solution of (11) is $y+\alpha_1 x=$const.

To get a real canonical form we further make the transformation

$$\alpha=\tfrac{1}{2}(u+v),\ \beta=\tfrac{1}{2}i(v-u),$$

so that

$$\frac{\partial z}{\partial u}=\tfrac{1}{2}\left(\frac{\partial z}{\partial \alpha}-i\frac{\partial z}{\partial \beta}\right),\ \frac{\partial z}{\partial v}=\tfrac{1}{2}\left(\frac{\partial z}{\partial \alpha}+i\frac{\partial z}{\partial \beta}\right),$$

$$\frac{\partial^2 z}{\partial u\,\partial v}=\tfrac{1}{4}\left(\frac{\partial^2 z}{\partial \alpha^2}+i\frac{\partial^2 z}{\partial \beta\,\partial \alpha}\right)-\tfrac{1}{4}\left(\frac{\partial^2 z}{\partial \alpha\,\partial \beta}+i\frac{\partial^2 z}{\partial \beta^2}\right)$$

$$=\tfrac{1}{4}\left(\frac{\partial^2 z}{\partial \alpha^2}+\frac{\partial^2 z}{\partial \beta^2}\right).$$

And the equation reduces to the canonical form

$$\frac{\partial^2 z}{\partial \alpha^2}+\frac{\partial^2 z}{\partial \beta^2}=\psi\left(\alpha,\beta,z,\frac{\partial z}{\partial \alpha},\frac{\partial z}{\partial \beta}\right).$$

4·5. Classification of second order Partial Diff. Equations

Depending on their canonical forms, the partial differential equation

$$Rr+Ss+Tt+f(x,y,z,p,q)=0$$

is called.

(*i*) *Hyperbolic* if $S^2-4RT>0$,

(*ii*) *Parabolic* if $S^2-4RT=0$,

and (*iii*) *Elliptic* if $S^2-4RT<0$.

Ex. 1. *Reduce the equation*

$$xyr-(x^2-y^2)s-xyt+py-qx=2(x^2-y^2)$$

into canonical form and hence solve it.

Solution. Comparing it with

$$Rr+Ss+Tt=f(x,y,z,p,q),$$

we find that

$$R=xy,\ S=-(x^2+y^2),\ T=-xy.$$

The quadratic α equation

$$R\alpha^2+S\alpha-T=0$$

therefore becomes

$$xy\alpha^2-(x^2+y^2)+T=0.$$

This gives $\alpha_1=\dfrac{x}{y}$ and $\alpha_2=-\dfrac{y}{x}$ as two roots.

The equations $\dfrac{dy}{dx}+\alpha_1=0$ and $\dfrac{dy}{dx}+\alpha_2=0$ are

$$\frac{dy}{dx}+\frac{x}{y}=0 \text{ and } \frac{dy}{dx}-\frac{y}{x}=0$$

which on integration give

$$y^2+x^2=\text{const. and } y/x=\text{const.}$$

Thus the transformation of independent variables from x, y to u, v is made by

$$u=x^2+y$$

and $$v=y/x.$$

$$\therefore \quad p=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}=\left(2x\frac{\partial z}{\partial u}-\frac{y}{x^2}\frac{\partial z}{\partial v}\right),$$

$$q=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\left(2y\frac{\partial z}{\partial u}+\frac{1}{x}\frac{\partial z}{\partial v}\right),$$

$$r=4x^2\frac{\partial^2 z}{\partial u^2}+4x\left(-\frac{y}{x^2}\right)\frac{\partial^2 z}{\partial u\,\partial v}+\left(-\frac{y}{x^2}\right)^2\frac{\partial^2 z}{\partial v^2}+\frac{2y}{x^3}\frac{\partial z}{\partial v}+\frac{\partial z}{\partial u},$$

$$t=\frac{\partial^2 z}{\partial u^2}+2\,\frac{1}{x}(2y)\frac{\partial^2 z}{\partial u\,\partial v}+\frac{1}{x^2}\frac{\partial^2 z}{\partial v^2}+2\,\frac{\partial z}{\partial u},$$

$$s=4xy\frac{\partial^2 z}{\partial u^2}+\left\{2y\left(-\frac{y}{x^2}\right)+2x\,\frac{1}{x}\right\}\frac{\partial^2 z}{\partial u\,\partial v}+\left(-\frac{y}{x^2}\right)\left(\frac{1}{x}\right)\frac{\partial^2 z}{\partial v^2}-\frac{1}{x^2}\frac{\partial z}{\partial v}.$$

Substituting these in the given equation, that reduces to

$$(x^2+y^2)^2\frac{\partial^2 z}{\partial u\,\partial v}=(y^2-x^2)\,x^2 \quad \text{or} \quad \frac{\partial^2 z}{\partial u\,\partial v}=\frac{v^2-1}{(v^2+1)^2}.$$

This is canonical form.

Integrating w.r.t. v, we get

$$\frac{\partial z}{\partial u}=\int\frac{v^2-1}{(v^2+1)^2}\,dv+\phi_1(u)$$

$$=\int\frac{dv}{v^2+1}-2\int\frac{dv}{(v^2+1)^2}+\phi_1(u),$$

where ϕ_1 is an arbitrary f-.nction of u. ...(1)

Now consider $I=\int\frac{dv}{(v^2+1)}$.

Integrating it by parts treating 1 as first function and $\frac{1}{v^2+1}$ as the second,

$$I=\frac{e^v}{(v^2+1)}+\int\frac{2v^2}{(v^2+1)^2}\,dv$$

$$=\frac{v}{v^2+1}+\int\frac{2dv}{(v^2+1)}-2\int\frac{dv}{(v^2+1)^2}$$

or $$-I=\frac{v}{v^2+1}-2\int\frac{dv}{(v^2+1)^2}$$

or $\int \frac{dv}{(v^2+1)}-2\int \frac{dv}{v^2+1}=-\frac{v}{v^2+1}$.

Thus (1) becomes

$$\frac{dz}{du}=-\frac{v}{v^2+1}+\phi_1(u).$$

Now integrating it w.r.t. u, we get

$$z=-\frac{uv}{v^2+1}+\psi_1(u)+\psi_2(v),$$

where ψ_2 is an arditrary function of v and ψ_1 is integral of ϕ_1.

Thus $z=-xy+\psi_1(x^2+y^2)+\psi_2(y/x)$

is the complete solution.

Ex. 2. *Reduce into canonical form the equation*

$$(y-1)\,r-(y^2-1)\,s+y\,(y-1)\,t+p-q=2ye^{2x}\,(1-y)^3$$

and hence solve it.

Solution. Comparing it with

$$Rr+Ss+Tt+f(x, y, z, p, q)=0,$$

we have $R=y-1,\ S=-(y^2-1),\ T=y\,(y-1)$.

The quadratic equation

$$R\alpha^2+S\alpha+T=0$$

is $(y-1)\,\alpha^2-(y^2-1)\,\alpha+y\,(y-1)=0$

or $(y-1)\,[\alpha^2-(y-1)\,\alpha+y]=0$.

This gives $\alpha_1=1,\ \alpha_2=y$.

The equations $\frac{dy}{dx}+\alpha_1=0$ and $\frac{dy}{dx}+\alpha_2=0$ become

$$\frac{dy}{dx}+1=0 \text{ and } \frac{dy}{dx}+y=0.$$

These on integration give

$x+y=$const. and $ye^x=$const.

So to change the independent variables from x, y to u, v, we take

$$u=x+y,\ v=ye^x,$$

giving
$$p=\frac{\partial z}{\partial u}+ye^x\frac{\partial z}{\partial v},$$
$$q=\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v},$$
$$r=\frac{\partial^2 z}{\partial u^2}+2ye^x\frac{\partial^2 z}{\partial u\,\partial v}+y^2e^{2x}\frac{\partial^2 z}{\partial v^2}+ye^x\frac{dz}{\partial v},$$
$$t=\frac{\partial^2 z}{\partial u^2}+2e^x\frac{\partial^2 z}{\partial u\,\partial v}+e^{2x}\frac{\partial^2 z}{\partial v^2},$$
$$s=\frac{\partial^2 z}{\partial u^2}+e^x\,(y+1)\frac{\partial^2 z}{\partial u\,\partial v}+ye^{2x}\frac{\partial^2 z}{\partial v^2}+e^x\frac{\partial z}{\partial v}.$$

Substituting these in the given equation the equation reduces to

$$(1-y)^3 e^x \frac{\partial^2 z}{\partial u\, \partial v}=2ye^{2x}(1-y)^3.$$

$$\therefore \quad \frac{\partial^2 z}{\partial u\, \partial v}=2v.$$

This is canonical form.

Integrating w.r.t. u, we get

$$\frac{\partial z}{\partial u}=v^2+\phi_1(u),$$

where $\phi_1(u)$ is an arbitrary function of u.

Integrating again w.r.t. u, we get

$$z=v^2u+\psi_1(n)+\psi_2(v),$$

where ψ_1 is integral of ϕ_1 and ψ_2 is an arbitrary function. This can be written as

$$z=y^2e^{2x}(x+y)+\psi_1(x+y)+\psi_2(ye^x).$$

Ex. 3. *Reduce into canonical form the equation*

$$x^2r-2xys+y^2t-xp+3yq=8y/x$$

and hence solve it.

Solution. Comparing the given equation with

$$Rr+Ss+Tt+f(x, y, z, p, q)=0.$$

we find that $R=x^2$, $S=-2xy$, $T=y^2$.

The quadratic α-equations

$R\alpha^2+S\alpha+T=0$ becomes

$$x^2\alpha^2-2xy\alpha+y^2=0,\ (x\alpha-y)^2=0.$$

This gives only one value of $\alpha=y/x$.

The equation $\frac{dy}{dx}+\alpha=0$ becomes

$$\frac{dy}{dx}+\frac{y}{x}=0 \quad \text{or} \quad \frac{dy}{y}+\frac{dx}{x}=0.$$

This on integration gives $xy=$const.

Therefore we take

$$u=xy$$

and choose v to be any function of x, y which is indepenaent of v. Hence there can be many choices

Let $z=y/x$.

Then $p=y\frac{\partial z}{\partial u}+\left(-\frac{y}{x^2}\right)\frac{\partial z}{\partial v}$

$$q=x\frac{\partial z}{\partial u}+\frac{1}{x}\frac{\partial z}{\partial v}$$

$$r=y^2\frac{\partial^2 z}{\partial u^2}+2y\left(-\frac{y}{x^2}\right)\frac{\partial^2 z}{\partial u\, \partial v}+\left(-\frac{y}{x^2}\right)\frac{\partial^2 z}{\partial v^2}+\frac{2y}{x^2}\frac{\partial z}{\partial v}$$

$$t=x^2\frac{\partial^2 z}{\partial u^2}+2x.\frac{1}{x}\frac{\partial^2 z}{\partial u\,\partial v}+\left(\frac{1}{x}\right)^2\frac{\partial^2 z}{\partial v^2}$$

$$s=xy\frac{\partial^2 z}{\partial u^2}+\left\{y.\frac{1}{x}+\left(-\frac{y}{x^2}\right)x\right\}\frac{\partial^2 z}{\partial u\,\partial v}+\left(-\frac{y}{x^2}\right)\left(\frac{1}{x}\right)\frac{\partial^2 z}{\partial v^2}$$
$$+\frac{\partial z}{\partial u}+\left(-\frac{1}{x^2}\right)\frac{\partial z}{\partial v}.$$

Putting these in the given equation, we get

$$v\frac{\partial^2 z}{\partial v^2}+2\frac{\partial z}{\partial v}=2$$

or $$\frac{\partial Z}{\partial v}+\frac{2}{v}Z=\frac{2}{v} \text{ where } Z=\frac{\partial z}{\partial v}.$$

This is a linear equation with integrating factor

$$e^{\int(2/v)dx}=v^3.$$

$$\therefore\quad v^2\frac{\partial z}{\partial u}=v^2+\phi_1(u)$$

where ϕ_1 is an arbitrary function.

This gives

$$\frac{\partial z}{\partial v}=1+\frac{1}{v^2}\phi_1(u).$$

Integrating this again w.r.t. v, we get

$$z=v-\frac{1}{v^2}\phi_1(u)+\phi_2(u)$$

or $$z=\frac{y}{x}-\frac{x}{y}\phi_1(xy)+\phi_2(xy)$$

$$=\frac{y}{x}-\frac{x^2}{xy}\phi_1(xy)+\phi_2(xy)$$

or $$z=\frac{y}{x}+x^2\psi_1(xy)+\phi_2(xy)$$

where $$\psi_1(xy)=-\frac{1}{xy}\phi_1(xy)$$

is the complete solution.

Ex. 4. *Reduce the equation*

$$\frac{d^2 z}{\partial x^2}+2\frac{\partial^2 z}{dx\,\partial y}+\frac{\partial^2 z}{\partial y^2}=0$$

to canonical form and hence solve it.

Solution. Comparing it with

$$Rr+Ss+Tt+f(x, y, z, y, p, q)=0,$$

we find that $R=1$, $S=2$, $T=1$.

$\therefore$ α-equation $R\alpha^2+S\alpha+T=0$ becomes $\alpha^2+2\alpha+1=0$ giving $\alpha=-1, -1$.

The equation $\frac{dy}{dx}+\alpha=0$ becomes

$$\frac{dy}{dx}-1=0$$

which on integration gives

$$x-y=\text{const.}$$

To change independent variables. we take

$$u=x-y.$$

We have to take v as some function of x and y independent of u, let $v=x+y$.

We now determine values of p, q, r, s and t and putting these in the given equation, the given equation reduces to

$$\frac{\partial^2 z}{\partial v^2}=0.$$

This is canonical form.

Integrating it, $\frac{\partial z}{\partial v}=\phi_1(u)$.

Integrating again $z=v\phi_1(u)+\phi_2(u)$,
where ϕ_1 and ϕ_2 are arbitrary functions of u.

Thus the solution is

$$z=(x+y)\,\phi_1(x-y)+\phi_2(x-y).$$

Ex. 5. *Reduce the equation*

$$\frac{\partial^2 z}{\partial x^2}+x^2\,\frac{\partial^2 z}{\partial y^2}=0$$

Solution. Comparing it with

$$Rr+Ss+Tt+f(x, y, z, p, q)=0$$

we find that $R=1$, $S=0$, $T=x^2$.

Thus quadratic α-equation

$$R\alpha^2+S\alpha+T=0$$

becomes $\alpha^2+x^2=0$ giving $\alpha_1=ix$, $\alpha_2=-ix$.

The equations

$$\frac{dy}{dx}+\alpha_1=0 \text{ and } \frac{dy}{dx}+\alpha_2=0$$

becomes $\frac{dy}{dx}+ix=0$ and $\frac{dy}{dx}-ix=0$.

Integrating these, we get

$$y+\tfrac{1}{2}ix^2=\text{const. and } y-\tfrac{1}{2}ix=\text{const.}$$

We take $u=iy+\frac{1}{2}x^2$

and $v=-iy+\frac{1}{2}x^2$.

Next we use the transformation

$$\alpha=\tfrac{1}{2}(u+v) \text{ and } \beta=\tfrac{1}{2}i\,(v-u).$$

$$\therefore\quad \alpha=\tfrac{1}{2}x^2 \quad\text{and } \beta=y.$$

We now find p, q, r, s, t and substitute in the given equation which reduces it to

$$\frac{\partial^2 z}{\partial \alpha^2}+\frac{\partial^2 z}{\partial \beta^2}=-\frac{1}{2\alpha}\frac{\partial z}{\partial \alpha}$$

which is the canonical form.

Exercises

Reduce the following to canonical forms and hence solve them :

$y\,(x+y)\,(r-s)-xp-yq-z=0.$

$$\textbf{Ans. } \frac{\partial^2 z}{\partial u\,\partial v}+\frac{1}{v}\frac{\partial z}{\partial u}+\frac{1}{uv}\frac{1}{4}\frac{\partial z}{\partial v}+z=0$$

$$z=\frac{1}{y}\,\phi_1(x+y)+\frac{1}{y(x+y)}\phi_2(y)$$

2. $x\,(y-x)\,r-(y^2-x^2)\,s+y\,(y-x)\,t+(y+x)\,(p-q)=2x+2y+2.$

$$\textbf{Ans. } \frac{\partial^2 z}{\partial u\,\partial v}=\frac{2\,(v+1)}{\sqrt{\{(v^2-4u)^3\}}}$$

$$z=\phi_1\,(x+y)+\phi_2\,(xy)+x+y+\log\,(2x)$$

3. $x\,(xy-1)\,r-(x^2y^2-1)\,s+y\,(xy-1)\,t+(x-1)p+(y-1)\,q=0.$

$$\textbf{Ans. } \frac{\partial^2 z}{\partial u\,\partial v}=0,\ z=\phi_1\,(ye^x)+\phi_2(xe^y)$$

4. $x^2\,(y-1)\,r-x\,(y^2-1)\,s+y\,(y-1)\,t+xyp-q=0.$

$$\textbf{Ans. } \frac{\partial^2 z}{\partial u\,\partial v}=0,\ z=\phi_1\,(xy)+\phi_2\,(xe^y)$$

5. $y^2r-2xys+x^2t=\dfrac{y^2}{x}\,p+\dfrac{x^2}{y}\,q$ **Ans.** Choose $v=x^2-y^2$, $\dfrac{\partial^2 z}{\partial v^2}=0.$

$$z=(x^2-y^2)\,\phi_1\,(x^2+y^2)+\phi_2\,(x^2+y^2)$$

4·6. Special types of Partial Differential Equations of second order.

We have already discussed the different methods that can be applicable in a large number of situations. However some simple methods can also work if the equation is of a given type.

4·7. Type I.

An equation consisting of **only one** *of the derivatives r, s or t and not p and q.*

Thus these equations are of the form

$$r=f_1\,(x, y) \quad\text{or}\quad s=f_2\,(x, y) \quad\text{or}\quad t=f_3\,(x, y).$$

Following examples illustrate the methods.

Ex. 1. *Solve* $t=x^2\cos\,(xy).$

Solution. The equations can be written as

$$\frac{\partial^2 z}{\partial y^2}=x^2 \cos (xy).$$

Integrating w.r.t. y, we get

$$\frac{\partial z}{\partial y}=\frac{x^2 \sin (xy)}{x}+\phi_1 (x)=x \sin (xy)+\phi_1 (x),$$

where ϕ_1 is an arbitrary function of x.

Again integrating w.r.t. y, we get

$$z=-\cos (xy)+y\phi_1(x)+\phi_2(x),$$

which is the complete solution.

Note. This method of integrating directly usually works under such situations.

Exercises

1. $r=\sin xy$ Ans. $z=-\frac{1}{y^2}\sin (xy)+x\phi_1 (y)+\phi_2 (y)$
2. $r=x^2e^y$. Ans. $z=\frac{1}{12}x^4e^y+x\phi_1 (y)+\phi_2 (y)$
3. $s=x^2-y^2$. Ans. $z=\frac{1}{3} (x^3y-xy^3)+\phi_1 (x)+\phi_2 (y)$.

4·8. Type II.

Equations containing one second order partial derivative and one of order one.

Such equations can be written as

$$R \frac{\partial p}{\partial x}+Pp=f_1 (x, y),$$

$$S \frac{\partial p}{\partial y}+Pp=f_2 (x, y)$$

$$S \frac{\partial q}{\partial y}+Qq=f_3 (x, y)$$

$$T \frac{\partial q}{\partial y}+Qq=f_4 (x, y).$$

These can be solved as ordinary linear differential equations for p and q and thereafter directly.

Ex. 1. *Solve* $ys+p=\cos (x+y)-y \sin (x+y)$.

Solution. The equation can be written as

$$\frac{\partial p}{\partial y}+\frac{p}{y}=\frac{1}{y} \cos (x+y)-\sin (x+y).$$

Its integrating factor $=e^{\int 1/y\, dy}=y$.

$\therefore \quad py=y \cos (x+y)+\phi_1 (x)$.

This can further be written as

$$y\frac{\partial z}{\partial x}=y \cos (x+y)+\phi_1 (x).$$

Now integrating w.r.t. x we get

$$yz = -y \sin (x+y) + \phi_1 (x) + \phi_2 (y),$$

where ϕ_2 is arbitrary and ϕ_1 is integral of ϕ_1.

Exercises

1. $xr + p = 9x^3y^2$ **Ans.** $z = x^3y^2 + \phi_1(y) + \phi_2(y) \log x$.
2. $t - xq = x^2$. **Ans.** $z = -xy + \phi_1(x) + e^{xy} \phi_2 (x)$.
3. $ys - p = xy^2 \cos xy$. **Ans.** $z = -\cos xy + y\phi_1(x) + \phi_2(y)$.

4·9. Type III

In this type there came equations which are of the form

$$Rr + Pp + Zz = f_1 (x, y)$$

or $$Tt + Qq + Rz = f_2 (x, y).$$

Ex. 1 *Solve*

$$r - p - \frac{1}{y}\left(\frac{1}{y} - 1\right) z = x^2y - x^2y^2 + 2xy^3 - 2y^3.$$

Solution. We write the equation as

$$\left\{D^2 - D - \frac{1}{y}\left(\frac{1}{y} - 1\right)\right\} z = x^2y - x^2y^2 + 2xy^3 - 2y^3.$$

A.E. is $\left(D - \frac{1}{y}\right)\left(D + \frac{1}{y} - 1\right) = 0.$

$\therefore$ C.F. $= e^{x/y} \phi_1(y) + e^{x-(x/y)} \phi_2 (y)$.

where ϕ_1 and ϕ_2 are arbitrary functions.

Assuming the P.I. to be

$$z = ax^2 + bx + c,$$

we substitute from it values of p and r in the given equation and get

$$a = -y^3, \ b = c = 0.$$

$\therefore$ P.I. is $-x^2y^3$.

Hence the solution is

$$z = e^{x/y} \phi_1 (y) + e^{x-(x/y)} \phi_2 (y) - x^2y^3.$$

Exercises

Solve the following differential equation :

1. $r - 2yp + y^2z = (y-2) e^{2x+3y}$.

Ans. $z = \frac{e^{2x+3y}}{y-2} + e^{xy}\phi_1(y) + xe^{xy}\phi_2(y)$.

4·10. Type IV

In this type are equations of the form

$$Rr + Ss + Pp = f_1 (x, y)$$

or $$Ss + Tt + Qq = f_2 (x, y)$$

These can be made linear in p or q and integrated.

Ex. 1. *Solve* $sy-2xr-2p=6xy$.

Solution. The equation can be written as

$$y\frac{\partial p}{\partial y}-2x\frac{\partial p}{\partial x}=6xy+2p.$$

Auxiliary equations of this are

$$\frac{dx}{-2x}=\frac{dy}{y}=\frac{dp}{6xy+2p}=\frac{-2y^2\,dx+(2yp+2xy^2)\,dy-y^2dp}{0}$$

The first two give $xy^2=$const.

and the last gives $\frac{2dy}{y}-\frac{d(p+2xy)}{p+2xy}=0$

i.e. $p+2xy=y^2$ (const.)

$$\therefore\quad \frac{\partial z}{\partial x}+2xy=y^2\phi_1(xy)^2.$$

Integrating now w.r.t. x, we get

$$z=\psi_1(xy^2)+\psi_2(y)-x^2y.$$

Exercises

Solve the following differential equations :

1. $xr+ys+p=10xy^3$. Ans. $z=x^2y^3+\phi_1(y)+\phi_2(x/y)$
2. $xyr+x^2s-yp=x^2y''$. Ans. $z=\frac{1}{2}x^2e^y+\phi_1(y)+\phi_1(x^2-y^2)$.