CHAPTER I

Addition and Multiplication

§ 1.01. Introduction.

The matrices (formal definition is given in § 1.02 Page 2) were invented about a century ago in connection with the study of simple changes and movements of geometric figures in coordinate geometry.

J. J. Sylvester was the fitst to use the word "matrix" in 1850 and later on in 1858 Arthur Cayley developed the theory of matrices in a systematic way.

"Matrices" is a powreful tool of modern mathematics and its study is becoming important day by day due to its wide applications in almost every branch of science and especially in physics (atomic) and engineering. These are used by Sociologists in the study of dominance within a group, by Demographers in the study of births and deaths, mobility and class structure etc., by Economist in the study of inter-industry economics, by Statisticians in the study of 'design of experiments' and 'multivariate analysis', by Engineers in the study of 'net work analysis' which is used in electrical and communication engineering.

Rectangular Array.

While defining matrix (see § 1.02 Page 2) we use the word 'rectangular array', which should be understood clearly before we come to the formal definition of 'matrices' and to understand the same we consider the following example :

In an inter-university debate, a student can speak either of the five languages : Hindi, English, Bangla, Marathi and Tamil. A certain university (say *A*) sent 25 students of which 8 offered to speak in Hindi, 7 in English, 5 in Bangla, 2 in Marathi and rest in Tamil. Another university (say *B*) sent 20 students of which 10 spoke in Hindi, 7 in English and 3 in Marathi. Out of 25 students from the third university (say *C*), 5 spoke in Hindi, 10 in English, 6 in Bangla and 4 in Tamil.'

The information given in the above example can be put in a compact way if we give them in a tabular form as follows :

University	Number of speakers in				
	Hindi	English	Bangla	Marathi	Tamil
A	8	7	5	2	3
B	10	7	0	3	0
C	5	10	6	0	4

The numbers in the above arrangement form what is known as a **rectangular array.** In this array the lines down the page are called **columns** whereas those across the page are called **rows.** Any particular number in this arrangement is known as an **entry** or **an element.** Thus in the above arrangement we find that there are 3 rows and 5 columns and also we observe that there are 5 elements in each row and so total number of elements $= 3 \times 5$ *i.e.* 15.

If the data given in the above arrangement is written without lines and enclosed by a pair of square brackets *i.e.* in the form

$$\begin{bmatrix} 8 & 7 & 5 & 2 & 3 \\ 10 & 7 & 0 & 3 & 0 \\ 5 & 10 & 6 & 0 & 4 \end{bmatrix},$$

then this is called a matrix.

§ 1.02. Definition of a Matrix.

A system of any mn numbers arranged in a rectangular array of m rows and n columns is called a matrix of order $m \times n$ or an $m \times n$ matrix (which is read as m by n matrix).

Or

A set of mn elements of a set S arranged in a rectangular array of m rows and n columns is called an $m \times n$ matrix over S.

For example : $\begin{bmatrix} 2 & 1 & 3 \\ 3 & -2 & 7 \end{bmatrix}$ is a 2×3 matrix.

and $\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & a_{m3} & \ldots\ldots & a_{mn} \end{bmatrix}$ is an $m \times n$ matrix.

where the symbols a_{ij} represent any numbers (a_{ij} lies in the ith row and jth column).

Note 1. A matrix may be represented by the symbols $[a_{ij}]$, (a_{ij}), $\| a_{ij} \|$ or by a single capital letter **A,** say.

Generally the first system is adopted.

Note 2. Each of the mn numbers constituting an $m \times n$ matrix is known as an **element of the matrix.**

The elements of matrix may be scalar or vector quantities.

Note 3. The plural of 'matrix' is 'matrices'.

Solved Examples on § 1.02.

Ex. 1. Find a_{23} in A = $\begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & 3 & 1 & 5 \\ 5 & 0 & 3 & 6 \end{bmatrix}$

Sol. a_{23} = element in the 2nd row and 3rd column

$= 1$ **Ans.**

Ex. 2. Write down the orders of the matrices :—

(a) $\begin{bmatrix} 2 & 3 & 5 \\ 1 & 0 & 3 \end{bmatrix}$; (b) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$; (c) $[3 \ 4 \ 5]$; (d) $[1]$.

Sol. (a) 2×3 ; (b) 2×1 ; (c) 1×3 ; (d) 1×1. **Ans.**

Ex. 3. How many elements are there in a 5×4 matrix ?

Sol. The required number of elements in 5×4 matrix is 5×4 *i.e.* 20. **Ans.**

Ex. 4. The results of a music competition are given in the following matrix :

$$\begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 3 & 2 & 4 \\ 5 & 0 & 3 & 0 \\ 2 & 1 & 4 & 3 \end{bmatrix}$$

Here the rows represent the teams A, B, C, D in that order and the columns represent the number of wins, first place, second place, third place and fourth place scored by the teams.

From the above matrix find (a) How many events did the team A win ? (b) How many first places did the team B win ? (c) How many third places did the team C win ? What does 0 represent in second row ?

Sol. (a) $\because$ the first row represents the team A, so the required number

$= $ Sum of the elements of first row

$= 3 + 2 + 1 = 6.$ **Ans.**

(b) As first elements of second row (which represents the team B) is zero, so the team B did not win any first place. **Ans.**

(c) The third row represents the team C and third column represents the third place scored by the teams, so the number of third places won by the team C is 3. **Ans.**

(d) The second row represents the team B and the first column represents the first place scored by teams. So 0 in the second row represents that the team B did not score any first place. **Ans.**

Ex. 5. The order of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is

(i) 2×3, (ii) 3×2, (iii) 2×2, (iv) None of these. **Ans. (i)**

Exercises on § 1.02

Ex. 1. In Example 1 above, find (i) a_{32}, (ii) a_{24}. **Ans.** (i) 0, (ii) 5

Ex. 2. Write down the orders of the matrices :—

(a) $\begin{bmatrix} 2 & 3 & 4 & 2 & 1 \\ 3 & 5 & 5 & 3 & 4 \\ 4 & 7 & 6 & 7 & 0 \end{bmatrix}$; (b) $\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$; (c) $[5]$

Ans. (a) 3×5 ; (b) 3×1, (c) 1×1.

§ 1.03. Rectangular Matrices.

The number of rows and columns of a matrix need not be equal ∴ when $m \neq n$ *i.e.* the number of rows and columns of the array are not equal, then the matrix is known as a **rectangular matrix.**

Classifications of rectangular matrices are as follows :—

Square Matrix.

If $m = n$ *i.e.* the number of rows and columns of a matrix are equal, then the matrix is of order $n \times n$ and is called a square matrix of order n.

For example $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 5 & 2 \\ 7 & 6 & 9 \end{bmatrix}$ is a square matrix and $\begin{bmatrix} 1 & 3 & 2 & 3 \\ 2 & 5 & 7 & 9 \end{bmatrix}$ is a rectangular matrix.

Horizontal matrix. If in a matrix the number of columns is more than the number of rows then it is called a horizontal matrix.

For example $\begin{bmatrix} 1 & 3 & 2 & 3 \\ 2 & 5 & 7 & 9 \end{bmatrix}$ is a horizontal matrix.

Row matrix : If in a matrix, there is only one row it is called a row matrix. For example [1, 2, 3]. This is also called a *row vector.*

Vertical matrix. If in a matrix the number of rows is more than the number of columns it is called a vertical matrix.

For example $\begin{bmatrix} 2 & 3 \\ 3 & 5 \\ 4 & 6 \\ 5 & 7 \end{bmatrix}$ is a vertical matrix.

Column matrix : If there is only one column in a matrix, it is called a column matrix.

For example $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$. This is also called *column vector.*

Null (or zero) Matrix : If all the elements of an $m \times n$ matrix are zero, then it is called a null or zero matrix and is denoted by $\mathbf{O}_{m \times n}$ or simply **O.**

For example $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is the 2×3 null matrix.

Unit matrix : A square matrix having unity for its elements in the leading diagonal and all other elements as zero is called an **unit matrix.**

For example $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is a four rowed unit matrix and we denote it by $\mathbf{I}_4$.

∴ an n-rowed square matrix $[a_{ij}]$ is called a unit matrix provided

$$a_{ij} = 1, \text{ whenever } i = j$$
$$= 0, \text{ whenever } i \neq j.$$

Equal matrix : Two matrices are said to be equal if (a) they are of the same type *i.e.* if they have same number of rows and columns and (b) the elements in the corresponding positions of the two matrices are equal.

For example, let two matrices be $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ then the two matrices are said to be equal if $a_{ij} = b_{ij}$, for all values of i and j.

From the definition given above it is evident that

(i) If $\mathbf{A} = \mathbf{B}$, then $\mathbf{B} = \mathbf{A}$ (Symmetry)

(ii) $\mathbf{A} = \mathbf{A}$, where $\mathbf{A}$ is any matrix. (Reflexivity)

(iii) If $\mathbf{A} = \mathbf{B}$ and $\mathbf{B} = \mathbf{C}$, then $\mathbf{A} = \mathbf{C}$ (Transitivity)

i.e. the relation of equality in the set of all matrices is an equivalence relation. **(See Author' Set Theory)**

Matrices over a number field.

A matrix **A** is defined as 'over the field F of numbers' if all the elements of the matrix **A** belong to the field F of the numbers.

Diagonal Element and Principal Diagonal.

Those elements a_{ij} of any matrix $[a_{ij}]$ are called diagonal elements for which $i = j$.

The line along which the above elements lie is called the **Principal diagonal** or the **Diagonal** of the matrix.

Diagonal Matrix : A square matrix in which all elements expect those in the main (or leading) diagonal are zero is known as a diagonal matrix.

For example $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ is a 3–rowed diagonal matrix.

The sum of the diagonal elements of a square matrix **A** (say) is called the trace of the matrix **A**.

Sub-matrix : A matrix which is obtained from a given matrix by deleting any number of rows and number of columns is called a sub-matrix of the given matrix.

For example $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is a sub– matrix of $\begin{bmatrix} 5 & 3 & 2 \\ 1 & 1 & 2 \\ 7 & 3 & 4 \end{bmatrix}$

Exercises on § 1.03

Ex. 1. The unit matrix is

(i) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$; (ii) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$;

(iii) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$; (iv) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **Ans.** (iv)

Ex. 2. What is the type of the matrix $[a\ b\ c]$

(i) column matrix; (ii) unit matrix;

(iii) square matrix; (iv) row matrix. **Ans.** (iv)

Ex. 3. The unit matrix is

(i) $[\ 1\ ]$; (ii) $[\ 0\ ]$;

(iii) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, (iv) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **Ans.** (i)

Ex. 4. The matrix of order $m \times n$ will be a unit matrix if

(i) all its elements are unity;

(ii) $m = n$ and all elements are unity;

(iii) $m = n$, and its diagonal elements are unity;

(iv) $m = n$, diagonal elements are unity and all the remaining elements are zero. **Ans.** (iv)

§ 1.04. Scalar Multiple of a matrix.

If **A** is a matrix and λ is a number then λ**A** is defined as the matrix each element of which is λ times the corresponding element of the matrix **A**.

For example : $2\begin{bmatrix} 3 & 5 & 7 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 10 & 14 \\ 4 & 6 & 8 \end{bmatrix}$

or if $\mathbf{A} = [a_{ij}]$, then $\lambda\mathbf{A} = [\lambda a_{ij}]$, where λ is a number.

§ 1.05. Addition of matrices.

If there be two $m \times n$ matrices given by $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$, then the matrix $\mathbf{A} + \mathbf{B}$ is defined as the matrix each element of which is the sum of the corresponding elements of **A** and **B** *i.e.* $\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$,
where $i = 1, 2, 3, ..., m$ and $j = 1, 2, 3, ..., n$.

For example : If $\mathbf{A} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix}$

then $\mathbf{A} + \mathbf{B} = \begin{bmatrix} a_1 + a_3 & b_1 + b_3 & c_1 + c_3 \\ a_2 + a_4 & b_2 + b_4 & c_2 + c_4 \end{bmatrix}$

§ 1.06. Subtraction of matrices.

If there be two $m \times n$ matrices given by $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$, then the matrix $\mathbf{A} - \mathbf{B}$ is defined as the matrix each element of which is obtained by subtracting the element of **B** from the corresponding element of **A** *i.e.* $\mathbf{A} - \mathbf{B} = [a_{ij} - b_{ij}]$,
where $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$.

For example : If $\mathbf{A} = \begin{Bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{Bmatrix}$ and $\mathbf{B} = \begin{Bmatrix} a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{Bmatrix}$

then $\mathbf{A}-\mathbf{B}=\begin{bmatrix} a_1-a_3 & b_1-b_3 & c_1-c_3 \\ a_2-a_4 & b_2-b_4 & c_2-c_4 \end{bmatrix}$.

***Note.** If the two matrices **A** and **B** are of the same order, then only their addition and subtraction is possible and these matrices are said to be **conformable** for addition or subtraction. On the other hand if the matrices **A** and **B** are of different orders, then their addition and subtraction is not possible and these matrices are called **non-conformable** for addition and subtraction.

§ 1.07. Properties of Matrix addition.

Property I. Addition of matrices is commutative.

i.e. $[a_{ij}]+[b_{ij}]=[b_{ij}]+[a_{ij}]$,

where $[a_{ij}]$ and $[b_{ij}]$ are any two $m \times n$ matrices i.e. matrices of the same order.

(Meerut 95)

Proof : $[a_{ij}]+[b_{ij}]=[a_{ij}+b_{ij}]$, by definition of addition

$=[b_{ij}+a_{ij}]$, $\because$ addition of numbers (elements) is commutative

$=[b_{ij}]+[a_{ij}]$

i.e. $[a_{ij}]+[b_{ij}]=[b_{ij}]+[a_{ij}]$ Hence the theorem.

Property II. Addition of matrices is associative.

i.e. $\{[a_{ij}]+[b_{ij}]\}+[c_{ij}]=[a_{ij}]+\{[b_{ij}]+[c_{ij}]\}$,

where $[a_{ij}]$, $[b_{ij}]$ and $[c_{ij}]$ are any three matrices of the same order $m \times n$, say.

Proof : $\{[a_{ij}]+[b_{ij}]\}+[c_{ij}]$

$=[a_{ij}+b_{ij}]+[c_{ij}]$, by law of addition for matrices

$=[(a_{ij}+b_{ij})+c_{ij}]$, by law of addition for matrices

$=[a_{ij}+(b_{ij}+c_{ij})]$, $\because$ addition of numbers is associative

$=[a_{ij}]+[b_{ij}+c_{ij}]$

$=[a_{ij}]+\{[b_{ij}]+[c_{ij}]\}$. Hence the theorem.

Property III. Addition for matrices obey the distributive law.

i.e. $k([a_{ij}]+[b_{ij}])=k[a_{ij}]+k[b_{ij}]$,

where $[a_{ij}]$ and $[b_{ij}]$ are any two matrices of the same order $m \times n$, say.

Proof : $k([a_{ij}]+[b_{ij}])=k[a_{ij}+b_{ij}]$, by law of addition

$=[k(a_{ij}+b_{ij})]$, by law of scalar multiplication

$=[ka_{ij}+kb_{ij}]$, by distributive law for numbers.

$=[ka_{ij}]+[kb_{ij}]$

$=k[a_{ij}]+k[b_{ij}]$. Hence the theorem.

Property IV. Existence of additive identity.

If $\mathbf{A}=[a_{ij}]$ be any $m \times n$ matrix and $\mathbf{O}$ be the $m \times n$ null matrix then

$$\mathbf{A}+\mathbf{O}=\mathbf{A}=\mathbf{O}+\mathbf{A}$$

Proof : Here $\mathbf{A}=[a_{ij}]_{m \times n}$ and $\mathbf{O}=[0]_{m \times n}$

Then $\mathbf{A}+\mathbf{O}=[a_{ij}]_{m \times n}+[0]_{m \times n}$.

$$= [a_{ij} + 0]_{m \times n}, \text{ by def. of addition}$$

$$= [a_{ij}]_{m \times n} = \mathbf{A} \quad ...(i)$$

Again $\mathbf{O} + \mathbf{A} = [0]_{m \times n} + [a_{ij}]_{m \times n}$

$$= [0 + a_{ij}]_{m \times n}, \text{ by def. of addition}$$

$$= [a_{ij}]_{m \times n} = \mathbf{A} \quad ...(ii)$$

$\therefore$ From (i) and (ii) we get $\mathbf{A} + \mathbf{O} = \mathbf{A} = \mathbf{O} + \mathbf{A}$

Thus we observe that **O** (the null matrix) is the additive identity.

Property V. Existence of addititive inverse.

If $\mathbf{A} = [a_{ij}]$ *be any* $m \times n$ *matrix, there exists another* $m \times n$ *matrix* **B** *such that*

$$\mathbf{A} + \mathbf{B} = \mathbf{O} = \mathbf{B} + \mathbf{A},$$

where **O** *is the* $m \times n$ *null matrix.*

Here the matrix **B** is called the additive inverse of the matrix **A** or the negative of **A**.

Also the (i, j)th element of **B** is $-a_{ij}$ if $\mathbf{A} = [a_{ij}]$

Property VI. Cancellation Law.

If **A, B, C** *are three matrices of the same order* $m \times n$, *say such that* $\mathbf{A} + \mathbf{B} = \mathbf{A} + \mathbf{C}$, *then* $\mathbf{B} = \mathbf{C}$

Proof : Given $\mathbf{A} + \mathbf{B} = \mathbf{A} + \mathbf{C}$

or $-\mathbf{A} + (\mathbf{A} + \mathbf{B}) = -\mathbf{A} + (\mathbf{A} + \mathbf{C})$, adding $-\mathbf{A}$ from left on both sides

or $(-\mathbf{A} + \mathbf{A}) + \mathbf{B} = (-\mathbf{A} + \mathbf{A}) + \mathbf{C}$, by associative law of adition

or $\mathbf{O} + \mathbf{B} = \mathbf{O} + \mathbf{C}$, by def. of additive inverse

or $\mathbf{B} = \mathbf{C}$, by def. of additive identity.

Solved Examples on § 1.04 to § 1.07.

Ex. 1. If $\mathbf{A} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{bmatrix}$ **and** $\mathbf{B} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ **evaluate 3A – 4B.** *(Avadh 90)*

Sol. $3A - 4B = 3\begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 6 & 9 & 3 \\ 0 & -3 & 15 \end{bmatrix} - \begin{bmatrix} 4 & 8 & -4 \\ 0 & -4 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 6-4 & 9-8 & 3-(-4) \\ 0-0 & -3-(-4) & 15-12 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 7 \\ 0 & 1 & 3 \end{bmatrix}$$ **Ans.**

Ex. 2. If $\mathbf{A} = \begin{bmatrix} 1 & 5 & 6 \\ -6 & 7 & 0 \end{bmatrix}$ **and** $\mathbf{B} = \begin{bmatrix} 1 & -5 & 7 \\ 8 & -7 & 7 \end{bmatrix}$

then show that $\mathbf{A} + \mathbf{B} = \begin{bmatrix} 2 & 0 & 13 \\ 2 & 0 & 7 \end{bmatrix}$, $\mathbf{A} - \mathbf{B} = \begin{bmatrix} 0 & 10 & -1 \\ -14 & 14 & -7 \end{bmatrix}$

Sol. Do yourself as Ex. 1 above.

Ex. 3. Determine the matrix A, where

$$A = 2\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 4 & 5 \end{bmatrix} + 3\begin{bmatrix} 3 & 3 & -1 \\ 2 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix}$$

Sol. $$A = \begin{bmatrix} 2 & 4 & 6 \\ 6 & 4 & 8 \\ 2 & 8 & 10 \end{bmatrix} + \begin{bmatrix} 9 & 9 & -3 \\ 6 & 6 & 9 \\ -3 & 9 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2+9 & 4+9 & 6+(-3) \\ 6+6 & 4+6 & 8+9 \\ 2+(-3) & 8+9 & 10+3 \end{bmatrix} = \begin{bmatrix} 11 & 13 & 3 \\ 12 & 10 & 17 \\ -1 & 17 & 13 \end{bmatrix}$$ **Ans.**

Ex. 4. Given $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$**,**

find the matrix C, such that $A + 2C = B$.

Sol. Given that $A + 2C = B$ **or** $2C = B - A$

or $$2C = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-1 & -1-2 & 2-(-3) \\ 4-5 & 2-0 & 5-2 \\ 2-1 & 0-(-1) & 3-1 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 5 \\ -1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$$

or $$C = (1/2)\begin{bmatrix} 2 & -3 & 5 \\ -1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -(3/2) & (5/2) \\ -(1/2) & 1 & (3/2) \\ (1/2) & (1/2) & 1 \end{bmatrix}$$ **Ans.**

Ex. 5. Solve the following equations for A and B;

$$2A - B = \begin{bmatrix} 3 & -3 & 0 \\ 3 & 3 & 2 \end{bmatrix}, \quad 2B + A = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix}$$

Sol. Given $2A - B = \begin{bmatrix} 3 & -3 & 0 \\ 3 & 3 & 2 \end{bmatrix}$

Multiplying both sides by 2, we get

$$4A - 2B = 2\begin{bmatrix} 3 & -3 & 0 \\ 3 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -6 & 0 \\ 6 & 6 & 4 \end{bmatrix} \quad \text{...(i)}$$

Also given that $2B + A = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix}$ ...(ii)

Adding (i) and (ii) we get

$$5A = \begin{bmatrix} 6 & -6 & 0 \\ 6 & 6 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 6+4 & -6+1 & 0+5 \\ 6-1 & 6+4 & 4-4 \end{bmatrix} = \begin{bmatrix} 10 & -5 & 5 \\ 5 & 10 & 0 \end{bmatrix}$$

or $$A = (1/5)\begin{bmatrix} 10 & -5 & 5 \\ 5 & 10 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$ **Ans.**

Again from (ii) we get

$$2\mathbf{B} = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix} - \mathbf{A}$$

or $$2\mathbf{B} = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4-2 & 1+1 & 5-1 \\ -1-1 & 4-2 & -4-0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 4 \\ -2 & 2 & -4 \end{bmatrix}$$

or $$\mathbf{B} = (1/2)\begin{bmatrix} 2 & 2 & 4 \\ -2 & 2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & -2 \end{bmatrix}$$ **Ans.**

Exercises on § 1.04 to § 1.07.

***Ex. 1.** If **X**, **Y** are two matrices given by the equations

$\mathbf{X} + \mathbf{Y} = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$ and $\mathbf{X} - \mathbf{Y} = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$, find **X, Y.**

Ans. $\mathbf{X} = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$, $\mathbf{Y} = \begin{bmatrix} -1 & -2 \\ 2 & 2 \end{bmatrix}$

Ex. 2. If $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 7 \\ 6 & 8 & 9 \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} 2 & 0 & 3 \\ 3 & 0 & 5 \\ 5 & 7 & 0 \end{bmatrix}$ evaluate **2A – 3B.**

Ans. $\begin{bmatrix} -4 & 4 & -3 \\ -9 & 10 & -1 \\ -3 & -5 & 18 \end{bmatrix}$

Ex. 3. If $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$, then **2A** equals

(i) $\begin{bmatrix} 2 & 4 & 6 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$; (ii) $\begin{bmatrix} 2 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 2 & 2 \end{bmatrix}$;

(iii) $\begin{bmatrix} 2 & 4 & 6 \\ 4 & 2 & 6 \\ 6 & 4 & 2 \end{bmatrix}$; (iv) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 6 & 4 & 2 \end{bmatrix}$ **Ans.** (iii)

Ex. 4. If $\mathbf{A} = \begin{bmatrix} \sec^2\theta & \sin^2\theta \\ 1/3 & \operatorname{cosec}^2\theta \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -\tan^2\theta & \cos^2\theta \\ 2/3 & -\cot^2\theta \end{bmatrix}$

then **A + B** is

(i) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; (ii) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$;

(iii) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$; (iv) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **Ans.** (iii)

Ex. 5. If $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ and $\mathbf{A} + \mathbf{B} = \mathbf{0}$, then **B** equals ....

Ans. $\mathbf{B} = \begin{bmatrix} -1 & -2 & -3 \\ -2 & -3 & -1 \\ -3 & -1 & -2 \end{bmatrix}$

*§ 1.08. Multiplication of matrices. *(Gorakhpur 95)*

If **A** and **B** be two matrices such that the number of columns in **A** is equal to the number of rows in **B** *i.e.* if $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{jk}]$ then the product of **A** and **B** denoted by **AB** is defined as matrix $[c_{ik}]$, where $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$ or in other words the product **AB** is defined as the matrix whose element in the ith row and kth column is $a_{i1} b_{1k} + a_{i2} b_{2k} + a_{i3} b_{3k} + \ldots + a_{in} b_{nk}$.

The product matrix will have i rows and k columns.

Thus we conclude that :

'If **A** is an $m \times n$ matrix and **B** is an $n \times k$ matrix then the product matrix **AB** is an $m \times k$ matrix.' **(Remember)**

As an example, consider the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}$$

Here the number of columns in $\mathbf{A} = 3 =$ the number of rows in **B** and thus we can evaluate **AB**.

Let $\mathbf{AB} = [c_{ij}]$, where $[c_{ij}]$ is 2×2 matrix.

Now to write c_{11}, we take the element of the first row of **A** *viz.* 1, 2, 3 in this order and the elements of the first column of **B** *viz.* 7, 9, 11 in this order and form the products 1·7, 2·9, 3·11 and finally add them.

i.e. $c_{11} = 1 \cdot 7 + 2 \cdot 9 + 3 \cdot 11 = 58$

Similarly $c_{12} = 1 \cdot 8 + 2 \cdot 10 + 3 \cdot 12 = 64$;

$c_{21} = 4 \cdot 7 + 5 \cdot 9 + 6 \cdot 11 = 139$

and $c_{22} = 4 \cdot 8 + 5 \cdot 10 + 6 \cdot 12 = 154$

Hence $\mathbf{AB} = [c_{ij}] = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$.

Note. The product **AB** can be calculated only if the number of columns in **A** be equal to the number of rows in **B**. The two matrices **A** and **B** satisfying this condition are called *conformable to multiplication.*

Post-multiplication and Pre-multiplication of matrices.

The matrix **AB** is the matrix **A** post-multiplied by **B** whereas the matrix **BA** is the matrix **A** pre-multiplied by **B**.

In the product **AB**, the matrix **A** is known as the **pre-factor** and the matrix **B** is known as the **post-factor.**

The product in both the above cases *viz.* **AB** and **BA** may or may not exist and may be equal or different,

i.e. we say **AB** $\neq$ **BA** *in general.* (*Bundelkhand 93; Gorakhpur 90*)

The same is discussed below :

Case I. If the matrix **A** is $m \times n$ and the matrix **B** is $n \times k$, then the product **AB** exists whereas **BA** does not exist, since we know that **AB** can be calculated only if the numbers of columns in **A** is equal to the number of rows in **B**.

Case II. If the matrix **A** is $m \times n$ and the matrix **B** is $n \times m$, then both **AB** and **BA** exist, but the matrix **AB** is $m \times m$ while the matrix **BA** is $n \times n$. **(Note)**

Hence **AB** $\neq$ **BA** though **AB** and **BA** exist.

Case III. If both **A** and **B** are square matrices of the same order, then **AB** as well as **BA** exist but are not necessarily equal

i.e. if $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix}$

then $\mathbf{AB} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 1\cdot3+2\cdot4 & 1\cdot1+2\cdot7 \\ 3\cdot3+4\cdot4 & 3\cdot1+4\cdot7 \end{bmatrix}$

$= \begin{bmatrix} 11 & 15 \\ 25 & 31 \end{bmatrix}$

and $\mathbf{BA} = \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3\cdot1+1\cdot3 & 3\cdot2+1\cdot4 \\ 4\cdot1+7\cdot3 & 4\cdot2+7\cdot4 \end{bmatrix}$

$= \begin{bmatrix} 6 & 10 \\ 25 & 36 \end{bmatrix}$

$\therefore$ **AB** $\neq$ **BA.**

But if $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

then $\mathbf{AB} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1\cdot1+0\cdot0 & 1\cdot0+0\cdot4 \\ 0\cdot1-2\cdot0 & 0\cdot0-2\cdot4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & -8 \end{bmatrix}$

and $\mathbf{BA} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1\cdot1+0\cdot0 & 1\cdot0+0\,(-2) \\ 0\cdot1+4\cdot0 & 0\cdot0+4\,(-2) \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & -8 \end{bmatrix}$

$\therefore$ **AB = BA.**

Hence in general **AB** $\neq$ **BA.** (*Gorakhpur 95, 90*)

Note 1. If **AB = BA**, then matrices **A** and **B** are said to commute. If **AB = – BA**, the matrices **A** and **B** are said to anticommute.

****Note 2.** *The product of two non-zero matrices can also be a zero (or null) matrix.* *(Avadh 93; Gorakhpur 91; Meerut 96P)*

Let $\mathbf{A}=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix}1 & 0\\ -1 & 0\end{bmatrix}$,

then $\mathbf{AB}=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\times\begin{bmatrix}1 & 0\\ -1 & 0\end{bmatrix}=\begin{bmatrix}1\cdot1+1\cdot(-1) & 1\cdot0+1\cdot0\\ 1\cdot1+1\cdot(-1) & 1\cdot0+1\cdot0\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=\mathbf{O}$

i.e. **AB** is zero matrix (or null matrix) where neither **A** nor **B** is a zero matrix.

$\therefore$ $\mathbf{AB}=\mathbf{O}$ does not imply that either $\mathbf{A}=\mathbf{O}$ **or** $\mathbf{B}=\mathbf{O}$.

Here $\mathbf{BA}=\begin{bmatrix}1 & 0\\ -1 & 0\end{bmatrix}\times\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$

$=\begin{bmatrix}1\cdot1+0\cdot1 & 1\cdot1+0\cdot1\\ -1\cdot1+0\cdot1 & -1\cdot1+0\cdot1\end{bmatrix}=\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}$

i.e. $\mathbf{BA}\neq\mathbf{O}$

Another Example.

If $\mathbf{A}=\begin{bmatrix}4 & 4\\ 3 & 3\end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}$, then

$\mathbf{AB}=\begin{bmatrix}4 & 4\\ 3 & 3\end{bmatrix}\times\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}$

$=\begin{bmatrix}4(-1)+4(1) & 4(1)+4(-1)\\ 3(-1)+3(1) & 3(1)+3(-1)\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=\mathbf{O}$

i.e. the product of two non-zero square matrices can be a zero matrix.

and $\mathbf{BA}=\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}\times\begin{bmatrix}4 & 4\\ 3 & 3\end{bmatrix}$

$=\begin{bmatrix}(-1)\cdot4+1\cdot3 & (-1)\cdot4+1\cdot3\\ 1\cdot4+(-1)\cdot3 & 1\cdot4+(-1)\cdot3\end{bmatrix}=\begin{bmatrix}-1 & -1\\ 1 & 1\end{bmatrix}\neq\mathbf{O}$

***Note 3.** *The multiplication of matrices generally does not obey the law of cancellation.*

Let $\mathbf{A}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$, $\mathbf{B}=\begin{bmatrix}a & 0\\ 0 & 0\end{bmatrix}$, $\mathbf{C}=\begin{bmatrix}b & 0\\ 0 & 0\end{bmatrix}$,

where $a\neq b$

Then $\mathbf{AB}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}a & 0\\ 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=\mathbf{O}$

and $\mathbf{AC}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}b & 0\\ 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=\mathbf{O}$

$\therefore$ It is evident that here $\mathbf{AB}=\mathbf{AC}$ but $\mathbf{B}\neq\mathbf{C}$.

$\therefore$ Law of cancellation is not obeyed in general.

Solved Examples on § 1.08

Ex. 1 (a). A is any $m \times n$ matrix such that AB and BA are both defined. What is the order of B ?

Sol. The required order of **B** is $\mathbf{n \times m}$. (See Case II Page 12)

Ex. 1 (b). Multiply [3 – 1 4] and $\begin{bmatrix} -2 \\ 6 \\ 3 \end{bmatrix}$

Sol. $[3\ \ -1\ \ 4] \times \begin{bmatrix} -2 \\ 6 \\ 3 \end{bmatrix} = [3(-2) + (-1)\cdot 6 + 4\cdot 3] = [0]$ **Ans.**

Ex. 1 (c). If $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & 4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 4 & 9 \end{bmatrix}$ **find AB.**

Sol. $AB = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & 4 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 4 & 9 \end{bmatrix}$

$$= \begin{bmatrix} 1\cdot 0 + 2\cdot 0 + 0\cdot 1 & 1\cdot 0 + 2\cdot 0 + 0\cdot 4 & 1\cdot 0 + 2\cdot 0 + 0\cdot 9 \\ 1\cdot 0 + 1\cdot 0 + 0\cdot 1 & 1\cdot 0 + 1\cdot 0 + 0\cdot 4 & 1\cdot 0 + 1\cdot 0 + 0\cdot 9 \\ -1\cdot 0 + 4\cdot 0 + 0\cdot 1 & -1\cdot 0 + 4\cdot 0 + 0\cdot 4 & -1\cdot 0 + 4\cdot 0 + 0\cdot 9 \end{bmatrix}$$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O}$, where **O** is the null matrix of order 3. **Ans.**

Ex. 2. If $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & 4 \\ 2 & 2 \\ 1 & 0 \end{bmatrix}$

then find AB. Whether BA exists ? Give reason. *(Purvanchal 89)*

Sol. $AB = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ 2 & 2 \\ 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 3\cdot 1 + 1\cdot 2 + 2\cdot 1 & 3\cdot 4 + 1\cdot 2 + 2\cdot 0 \\ 0\cdot 1 + 1\cdot 2 + 1\cdot 1 & 0\cdot 4 + 1\cdot 2 + 1\cdot 0 \\ 1\cdot 1 + 2\cdot 2 + 0\cdot 1 & 1\cdot 4 + 2\cdot 2 + 0\cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 14 \\ 3 & 2 \\ 5 & 8 \end{bmatrix}$$ **Ans.**

Here **A** is a matrix of order 3×3 and **B** is a matrix of order 3×2.

Hence **BA** does not exist as number of columns in **B** is not equal to the number of rows in **A**.

***Ex. 3 (a).** If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$

find AB and show that AB ≠ BA. (*Rohilkhand 97*)

Sol. $AB = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} \times \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1\cdot2 + (-2)\cdot4 + 3\cdot2 & 1\cdot3 + (-2)\cdot5 + 3\cdot1 \\ -4\cdot2 + 2\cdot4 + 5\cdot2 & -4\cdot3 + 2\cdot5 + 5\cdot1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$$

and $BA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 2\cdot1 + 3(-4) & 2(-2) + 3(2) & 2(3) + 3(5) \\ 4\cdot1 + 5(-4) & 4(-2) + 5(2) & 4(3) + 5(5) \\ 2\cdot1 + 1(-4) & 2(-2) + 1(2) & 2(3) + 1(5) \end{bmatrix}$$

$$= \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}$$

Hence AB ≠ BA.

Ex. 3 (b). If $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

then prove that AB ≠ BA. (*Meerut 97*)

Sol. $AB = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 2\cdot1 + 3(-1) + 4\cdot0 & 2\cdot3 + 3\cdot2 + 4\cdot0 & 2\cdot0 + 3\cdot1 + 4\cdot2 \\ 1\cdot1 + 2(-1) + 3\cdot0 & 1\cdot3 + 2\cdot2 + 3\cdot0 & 1\cdot0 + 2\cdot1 + 3\cdot2 \\ (-1)1 + 1(-1) + 2\cdot0 & (-1)3 + 1\cdot2 + 2\cdot0 & (-1)0 + 1\cdot1 + 2\cdot2 \end{bmatrix}$$

$$= \begin{bmatrix} 2-3+0 & 6+6+0 & 0+3+8 \\ 1-2+0 & 3+4+0 & 0+2+6 \\ -1-1+0 & -3+2+0 & 0+1+4 \end{bmatrix} = \begin{bmatrix} -1 & 12 & 11 \\ -1 & 7 & 8 \\ -2 & -1 & 5 \end{bmatrix} \quad ...(i)$$

and $BA = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1\cdot2 + 3\cdot1 + 0(-1) & 1\cdot3 + 3\cdot2 + 0\cdot1 & 1\cdot4 + 3\cdot3 + 0\cdot2 \\ (-1)2 + 2\cdot1 + 1(-1) & (-1)3 + 2\cdot2 + 1\cdot1 & (-1)4 + 2\cdot3 + 1\cdot2 \\ 0\cdot2 + 0\cdot1 + 2(-1) & 0\cdot3 + 0\cdot2 + 2\cdot1 & 0\cdot4 + 0\cdot3 + 2\cdot2 \end{bmatrix}$$

$$=\begin{bmatrix} 2+3+0 & 3+6+0 & 4+9+0 \\ -2+2-1 & -3+4+1 & -4+6+2 \\ 0+0-2 & 0+0+2 & 0+0+4 \end{bmatrix}=\begin{bmatrix} 5 & 9 & 13 \\ -1 & 2 & 4 \\ -2 & 2 & 4 \end{bmatrix} \quad ...(ii)$$

From (i) and (ii) we find that $\mathbf{AB} \neq \mathbf{BA}$. Hence proved.

****Ex. 4. If** $\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}\mathbf{A}=\begin{bmatrix} -4 & 8 & 4 \\ -1 & 2 & 1 \\ -3 & 6 & 3 \end{bmatrix}$**, find A.**

Sol. From § 1.08 Page 11 we know that if $\mathbf{X}$ is an $m \times n$ matrix, $\mathbf{Y}$ is an $n \times k$ matrix, then the product $\mathbf{XY}$ is an $m \times k$ matrix.

Here $\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$ is 3×1 matrix and $\begin{bmatrix} -4 & 8 & 4 \\ -1 & 2 & 1 \\ -3 & 6 & 3 \end{bmatrix}$

is 3×3 matrix, so $\mathbf{A}$ must be a 1×3 matrix *i.e.* a row matrix. **(Note)**

$\therefore$ Let $\mathbf{A}=[a \;\; b \;\; c]$

Then $\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \times [a \;\; b \;\; c]=\begin{bmatrix} -4 & 8 & 4 \\ -1 & 2 & 1 \\ -3 & 6 & 3 \end{bmatrix}$

which gives $\begin{bmatrix} 4a & 4b & 4c \\ a & b & c \\ 3a & 3b & 3c \end{bmatrix}=\begin{bmatrix} -4 & 8 & 4 \\ -1 & 2 & 1 \\ -3 & 6 & 3 \end{bmatrix}$

Comparing corresponding elements we have

$4a=-4$, $a=-1$, $3a=-3$, $4b=8$, $b=2$, $3b=6$ and $4c=4$, $c=1$, $3c=3$.

All these are satisfied by $a=-1, b=2, c=1$.

Hence from (i) we have $\mathbf{A}=[a \;\; b \;\; c]=[-1, 2, 1]$. **Ans.**

Ex. 5. If $\mathbf{A}=\begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 6 \\ -1 & -1 & -3 \end{bmatrix}$**, show that** $\mathbf{A}^2=\mathbf{O}$

Sol. $\mathbf{A}^2=\begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 6 \\ -1 & -1 & -3 \end{bmatrix}\times\begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 6 \\ -1 & -1 & -3 \end{bmatrix}$

$$=\begin{bmatrix} 1\cdot1+1\cdot2+3\cdot(-1) & 1\cdot1+1\cdot2+3(-1) & 1\cdot3+1\cdot6+3(-3) \\ 2\cdot1+2\cdot2+6\cdot(-1) & 2\cdot1+2\cdot2+6(-1) & 2\cdot3+2\cdot6+6(-3) \\ -1\cdot1-1\cdot2-3\cdot(-1) & -1\cdot1-1\cdot2-3(-1) & -1\cdot3-1\cdot6-3(-3) \end{bmatrix}$$

$=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}=\mathbf{O}$, where $\mathbf{O}$ is 3×3 null matrix.

Hence proved.

Ex. 6. Find the square of the matrix

$$\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

Sol. $\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}^2$

$= \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \times \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$

$$= \begin{bmatrix} (-1)(-1)+1\cdot1+1\cdot1+1\cdot1 & (-1)\,1+1\,(-1)+1\cdot1+1\cdot1 & (-1)\,1+1\cdot1+1\,(-1)+1\cdot1 & (-1)\,1+1\cdot1+1\cdot1+1\,(-1) \\ 1\,(-1)+(-1)\,1+1\cdot1+1\cdot1 & 1\cdot1+(-1)(-1)+1\cdot1+1\cdot1 & 1\cdot1+(-1)\,1+1\,(-1)+1\cdot1 & 1\cdot1+(-1)\,1+1\cdot1+1\,(-1) \\ 1\,(-1)+1\cdot1+(-1)\,1+1\cdot1 & 1\cdot1+1\,(-1)+(-1)\,1+1\cdot1 & 1\cdot1+1\cdot1+(-1)(-1)+1\cdot1 & 1\cdot1+1\cdot1+(-1)\,1+1\,(-1) \\ 1\,(-1)+1\cdot1+1\cdot1+(-1)\,1 & 1\cdot1+1\,(-1)+1\cdot1+(-1)\,1 & 1\cdot1+1\cdot1+1\,(-1)+(-1)\,1 & 1\cdot1+1\cdot1+1\cdot1+(-1)(-1) \end{bmatrix}$$

$= \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = 4 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 4\mathbf{I}.$ **Ans.**

***Ex. 7.**

$\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix};\ \mathbf{B} = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix},\ \mathbf{C} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 2 & -2 \\ -3 & -3 & 3 \end{bmatrix}$

show that AB and CA are null matrices but BA ≠ O, AC ≠ O.

Sol. $\mathbf{AB} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix} \times \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 1\,(-1)+1\cdot6+(-1)\,5 & 1\,(-2)+1\cdot12+(-1)\,10 & 1\,(-1)+1\cdot6+(-1)\,5 \\ 2\,(-1)-3\cdot6+4\cdot5 & 2\,(-2)-3\cdot12+4\cdot10 & 2\,(-1)-3\cdot6+4\cdot5 \\ 3\,(-1)-2\cdot6+3\cdot5 & 3\,(-2)-2\cdot12+3\cdot10 & 3\,(-1)-2\cdot6+3\cdot5 \end{bmatrix}$$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, which is a null matrix.

(See § 1.03 Page 4)

This is known as 'unusual property' of Matrix Multiplication

$\mathbf{CA} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 2 & -2 \\ -3 & -3 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$

$$=\begin{bmatrix} -1\cdot1-1\cdot2+1\cdot3 & -1\cdot1-1(-3)+1(-2) & -1(-1)-1\cdot4+1\cdot3 \\ 2\cdot1+2\cdot2-2\cdot3 & 2\cdot1+2(-3)-2(-2) & 2(-1)+2\cdot4-2\cdot3 \\ -3\cdot1-3\cdot2+3\cdot3 & -3\cdot1-3(-3)+3(-2) & -3(-1)-3\cdot4+3\cdot3 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ which is a null matrix.}$$

Hence proved.

We can prove in a simillar way that **BA ≠ O** and **AC ≠ O.**

Ex. 8. Find the product of the following two matrices

$$\begin{bmatrix} \mathbf{0} & \mathbf{c} & \mathbf{-b} \\ \mathbf{-c} & \mathbf{0} & \mathbf{a} \\ \mathbf{b} & \mathbf{-a} & \mathbf{0} \end{bmatrix} \times \begin{bmatrix} \mathbf{a^2} & \mathbf{ab} & \mathbf{ac} \\ \mathbf{ab} & \mathbf{b^2} & \mathbf{bc} \\ \mathbf{ac} & \mathbf{bc} & \mathbf{c^2} \end{bmatrix}$$

(*Bundelkhand 93; Kanpur 94*)

Sol. The required product

$$=\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \times \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$$

$$=\begin{bmatrix} 0\cdot a^2+c\cdot ab-b\cdot ac & 0\cdot ab+c\cdot b^2-b\cdot bc & 0\cdot ac+c\cdot bc-b\cdot c^2 \\ -c\cdot a^2+0\cdot ab+a\cdot ac & -c\cdot ab+0\cdot b^2+a\cdot bc & -c\cdot ac+0\cdot bc+a\cdot c^2 \\ b\cdot a^2-a\cdot ab+0\cdot ac & b\cdot ab-a\cdot b^2+0\cdot bc & b\cdot ac-a\cdot bc+0\cdot c^2 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Ans.

****Ex. 9. Prove that the product of two matrices**

$$\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \text{ and } \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

is zero when θ and φ differ by an odd multiple of $\frac{1}{2}\pi$.

(*Bundelkhand 92; Meerut 91 S*)

Sol. The required product

$$=\begin{bmatrix} \cos^2\theta\cos^2\phi+\cos\theta\sin\theta\cos\phi\sin\phi & \cos^2\theta\cos\phi\sin\phi+\cos\theta\sin\theta\sin^2\phi \\ \cos\theta\sin\theta\cos^2\phi+\sin^2\theta\cos\phi\sin\phi & \cos\theta\sin\theta\cos\phi\sin\phi+\sin^2\theta\sin^2\phi \end{bmatrix}$$

$$=\begin{bmatrix} \cos\theta\cos\phi(\cos\theta\cos\phi+\sin\theta\sin\phi) & \cos\theta\sin\phi(\cos\theta\cos\phi+\sin\theta\sin\phi) \\ \sin\theta\cos\phi(\cos\theta\cos\phi+\sin\theta\sin\phi) & \sin\theta\sin\phi(\cos\theta\cos\phi+\sin\theta\sin\phi) \end{bmatrix},$$

$$=\begin{bmatrix} \cos\theta\cos\phi\cos(\theta-\phi) & \cos\theta\sin\phi\cos(\theta-\phi) \\ \sin\theta\cos\phi\cos(\theta-\phi) & \sin\theta\sin\phi\cos(\theta-\phi) \end{bmatrix}$$

If $\theta \sim \phi$ = an odd multiple of $\frac{1}{2}\pi$, then $\cos(\theta \sim \phi) = 0$ and consequently the above product is zero (*i.e.* the null matrix of order 2×2).

***Ex. 10. If** $\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$

show that AB = BA. (*Gorakhpur 90*)

Sol. $\mathbf{AB} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \times \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$

$= \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & -\sin\theta\sin\phi + \cos\phi\cos\phi \end{bmatrix}$

$= \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}$ ...(i)

And $\mathbf{BA} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \times \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos\phi\cos\theta - \sin\phi\sin\theta & -\cos\phi\sin\theta - \sin\phi\cos\theta \\ \sin\phi\cos\theta + \cos\phi\sin\theta & -\sin\phi\sin\theta + \cos\phi\cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}$ ...(ii)

$\therefore$ From (i) and (ii) we get **AB = BA.** Hence proved.

****Ex. 11. If A, B, C are three matrices such that**

$\mathbf{A} = [\mathbf{x},\ \mathbf{y},\ \mathbf{z}]$, $\mathbf{B} = \begin{bmatrix} \mathbf{a} & \mathbf{h} & \mathbf{g} \\ \mathbf{h} & \mathbf{b} & \mathbf{f} \\ \mathbf{g} & \mathbf{f} & \mathbf{c} \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix}$ **evaluate ABC.**

(*Gorakhpur 94; Kanpur 93; Kumaun 94; Purvanchal 90*)

Sol. $\mathbf{AB} = [x,\ y,\ z] \times \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

$= [x.a + y.h + z.g \quad x.h + y.b + z.f \quad x.g + y.f + z.c]$

or $\mathbf{ABC} = [ax + hy + gz \quad hx + by + fz \quad gx + fy + cz] \times \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$= [x(ax + hy + gz) + y(hx + by + fz) + z(gx + fy + cz)]$ **(Note)**

$= [ax^2 + by^2 + cz^2 + 2hxy + 2gzx + 2fyz]$. **Ans.**

Ex. 12 If $\mathbf{A} = \begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{1} \\ \mathbf{0} & \mathbf{-1} & \mathbf{5} \end{bmatrix}$ **and** $\mathbf{B} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-6} \\ \mathbf{0} & \mathbf{-1} & \mathbf{3} \end{bmatrix}$

evaluate (a) $\mathbf{A}^2 - \mathbf{B}^2$ **and (b) AB and BA.**

Sol. (a) $\mathbf{A}^2 = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{bmatrix} \times \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{bmatrix}$, which does not exist as number of columns in the first matrix is not equal to number of rows in the second matrix.

Similarly $\mathbf{B}^2$ does not exit.

(b) **AB** and **BA** both do not exist, the reason being the same as in part (a) above.

***Ex. 13. Evalute $\mathbf{A}^3$ if $\mathbf{A} = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}$**

Sol. $\mathbf{A}^2 = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} \times \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}$

$$= \begin{bmatrix} \cosh^2\theta + \sinh^2\theta & \cosh\theta\sinh\theta + \sinh\theta\cosh\theta \\ \sinh\theta\cosh\theta + \cosh\theta\sin\theta & \sinh^2\theta + \cosh^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cosh 2\theta & \sinh 2\theta \\ \sinh 2\theta & \cosh 2\theta \end{bmatrix} \quad \because \ \cosh^2\theta + \sinh^2\theta = \cosh^2\theta, \quad 2\sinh\theta\cosh\theta = \sinh 2\theta$$

$$\therefore \quad \mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \begin{bmatrix} \cosh 2\theta & \sinh 2\theta \\ \sinh 2\theta & \cosh 2\theta \end{bmatrix} \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cosh 2\theta\cosh\theta + \sinh 2\theta\sin\theta & \cosh 2\theta\sinh\theta + \sinh 2\theta\cosh\theta \\ \sinh 2\theta\cosh\theta + \cosh 2\theta\sinh\theta & \sinh 2\theta\sinh\theta + \cosh 2\theta\cosh\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cosh(2\theta + \theta) & \sinh(2\theta + \theta) \\ \sinh(2\theta + \theta) & \cosh(2\theta + \theta) \end{bmatrix},$$

$$\because \ \sinh(A + B) = \sinh A\cosh B + \cosh A\sinh B$$
$$\cosh(A + B) = \cosh A\cosh B + \sinh A\sinh B$$

$$= \begin{bmatrix} \cosh 3\theta & \sinh 3\theta \\ \sinh 3\theta & \cosh 3\theta \end{bmatrix}$$

Ans.

Ex. 14. If $\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$, evaluate $\mathbf{A}^3$.

Sol. $\mathbf{A}^2 = \mathbf{A}\,\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2\cdot2 - 1\cdot0 + 1\cdot1 & 2(-1) - 1\cdot1 + 1\cdot0 & 2\cdot1 - 1\cdot2 + 1\cdot1 \\ 0\cdot2 + 1\cdot0 + 2\cdot1 & 0(-1) + 1\cdot1 + 2\cdot0 & 0\cdot1 + 1\cdot2 + 2\cdot1 \\ 1\cdot2 + 0\cdot1 + 1\cdot1 & 1(-1) + 0\cdot1 + 1\cdot0 & 1\cdot1 + 0\cdot2 + 1\cdot1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 + 0 + 1 & -2 - 1 + 0 & 2 - 2 + 1 \\ 0 + 0 + 2 & 0 + 1 + 0 & 0 + 2 + 2 \\ 2 + 0 + 1 & -1 + 0 + 0 & 1 + 0 + 1 \end{bmatrix} = \begin{bmatrix} 5 & -3 & 1 \\ 2 & 1 & 4 \\ 3 & -1 & 2 \end{bmatrix}$$

$$\therefore \quad \mathbf{A}^3 = \mathbf{A}^2\,\mathbf{A} = \begin{bmatrix} 5 & -3 & 1 \\ 2 & 1 & 4 \\ 3 & -1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 5\cdot2-3\cdot0+1\cdot1 & 5(-1)-3\cdot1+1\cdot0 & 5\cdot1-3\cdot2+1\cdot1 \\ 2\cdot2+1\cdot0+4\cdot1 & 2(-1)+1\cdot1+4\cdot0 & 2\cdot1+1\cdot2+4\cdot1 \\ 3\cdot2-1\cdot0+2\cdot1 & 3(-1)-1\cdot1+2\cdot0 & 3\cdot1-1\cdot2+2\cdot1 \end{bmatrix}$$

$$=\begin{bmatrix} 10-0+1 & -5-3+0 & 5-6+1 \\ 4+0+4 & -2+1+0 & 2+2+4 \\ 6-0+2 & -3-1+0 & 3-2+2 \end{bmatrix}=\begin{bmatrix} 11 & -8 & 0 \\ 8 & -1 & 8 \\ 8 & -4 & 3 \end{bmatrix}$$ **Ans.**

Ex. 15. If $\mathbf{A}=\begin{bmatrix} \mathbf{i} & \mathbf{0} \\ \mathbf{0} & -\mathbf{i} \end{bmatrix}$, $\mathbf{B}=\begin{bmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}$, $\mathbf{C}=\begin{bmatrix} \mathbf{0} & \mathbf{i} \\ \mathbf{i} & \mathbf{0} \end{bmatrix}$,

Prove that

$\mathbf{A}^2=\mathbf{B}^2=\mathbf{C}^2=-\mathbf{I}$ **and** $\mathbf{AB}=-\mathbf{C}=-\mathbf{BA}$, **where** $\mathbf{I}=\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$

(Kumaun 92)

Sol. $\mathbf{A}^2=\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\times\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

$$=\begin{bmatrix} i\cdot i+0\cdot0 & i\cdot0+0(-i) \\ 0\cdot i-i\cdot0 & 0\cdot0+(-i)(-i) \end{bmatrix}=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$=-\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ ...See § 1.04 Page 6

or $\mathbf{A}^2=-\mathbf{I}.$

and $\mathbf{B}^2=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\times\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} 0\cdot0-1\cdot1 & 0(-1)+(-1)\cdot0 \\ 1\cdot0+0\cdot1 & 1(-1)+0\cdot0 \end{bmatrix}$

$$=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}=-\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=-\mathbf{I}$$

Similarly we can prove that $\mathbf{C}^2=-\mathbf{I}$. Hence $\mathbf{A}^2=\mathbf{B}^2=\mathbf{C}^2=-\mathbf{I}.$

Again $\mathbf{AB}=\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\times\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$=\begin{bmatrix} i\cdot0+0\cdot1 & i(-1)+0\cdot0 \\ 0\cdot0-i(1) & 0(-1)-i\cdot0 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}=-\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}=-\mathbf{C}$$

and $\mathbf{BA}=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\times\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

$$=\begin{bmatrix} 0\cdot i-1\cdot0 & 0\cdot0-1(-i) \\ 1\cdot i+0\cdot0 & 1\cdot0+0(-i) \end{bmatrix}=\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}=\mathbf{C}.$$

Hence $\mathbf{AB}=-\mathbf{C}=-\mathbf{BA}.$

***Ex. 16. If** $\mathbf{A}=\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$; $\mathbf{B}=\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix}$ **and** $\mathbf{AB}=\begin{bmatrix} \mathbf{6} \\ \mathbf{3} \\ \mathbf{1} \end{bmatrix}$

find the values of x, y, z.

Sol. $AB = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

or $\begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot x + 2 \cdot y + 3 \cdot z \\ 0 \cdot x + 1 \cdot y + 2 \cdot z \\ 0 \cdot x + 0 \cdot y + 1 \cdot z \end{bmatrix}$

or $6 = x + 2y + 3z,\ 3 = y + 2z,\ 1 = z,$ (Note)

comparing the corresponding elements of the matrices on both sides

Solving these we get $x = 1,\ y = 1,\ z = 1.$ **Ans.**

Ex. 17. Find the values of x, y, z in the following equation

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 0 & -6 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Sol. $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \cdot x + 2 \cdot y + 3 \cdot z \\ 3 \cdot x + 1 \cdot y + 2 \cdot z \\ 2 \cdot x + 3 \cdot y + 1 \cdot z \end{bmatrix}$ ...(i)

And $\begin{bmatrix} 4 & -2 \\ 0 & -6 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \cdot 2 + (-2)\,1 \\ 0 \cdot 2 + (-6)\,1 \\ -1 \cdot 2 + \quad 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \\ 0 \end{bmatrix}$ ...(ii)

With the help of (i) and (ii), the given equation reduces to

$$\begin{bmatrix} x + 2y + 3z \\ 3x + y + 2z \\ 2x + 3y + z \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \\ 0 \end{bmatrix}$$

From this on comparing the corresponding elements on both sides we get $x + 2y + 3z = 6$; $3x + y + 2z = -6$ and $2x + 3y + z = 0.$

Solving these we get $x = -4, y = 2, z = 2.$ **Ans.**

Ex. 18. Given $A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$; $A_2 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}$

$A_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$ **and** $A_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

Show that $A_iA_k + A_kA_i = 2I$ or O according as $i = k$ or $i \neq k$ and I is the unit matrix of order 4 and i and k take the values 1, 2, 3 and 4.

Sol. Let $i = k = 1$ (say). Then $A_iA_k = A_1A_1 = A_kA_i$

$$\therefore \quad A_iA_k = A_1A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+0+1 & 0+0+0+0 & 0+0+0+0 & 0+0+0+0 \\ 0+0+0+0 & 0+0+1+0 & 0+0+0+0 & 0+0+0+0 \\ 0+0+0+0 & 0+0+0+0 & 0+1+0+0 & 0+0+0+0 \\ 0+0+0+0 & 0+0+0+0 & 0+0+0+0 & 1+0+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

$\therefore \quad \mathbf{A_iA_k + A_kA_i = I + I = 2I}$ Hence proved.

If $i \neq k$, let $i = 3$ and $k = 2$

Then $\mathbf{A_iA_k = A_3A_2} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0+0+0+0 & 0+0+i+0 & 0+0+0+0 & 0+0+0+0 \\ 0+0+0+i & 0+0+0+0 & 0+0+0+0 & 0+0+0+0 \\ 0+0+0+0 & 0+0+0+0 & 0+0+0+0 & i+0+0+0 \\ 0+0+0+0 & 0+0+0+0 & 0+i+0+0 & 0+0+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{bmatrix} = i \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

And $\mathbf{A_kA_i = A_2A_3} = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{bmatrix}$, multiplying in the usual way

$$= -i \begin{bmatrix} 0 & 1 & 0. & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\therefore \quad A_iA_k + A_kA_i = i \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} - i \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$= \mathbf{O}$ Hence proved.

We can in a similar way prove the above result by giving i and k other values also.

****Ex. 19. If $A_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then prove that**

(a) $A_\alpha . A_\beta = A_{\alpha+\beta}$ and (b) $A_\alpha . A_{-\alpha}$ is unit matrix.

Sol. (a) $A_\alpha . A_\beta = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \times \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & \cos\alpha\sin\beta + \sin\alpha\cos\beta \\ -\sin\alpha\cos\beta - \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) \\ -\sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} = A_{\alpha+\beta}$$

Hence proved.

(b) Here $A_{-\alpha} = \begin{bmatrix} \cos(-\alpha) & \sin(-\alpha) \\ -\sin(-\alpha) & \cos(-\alpha) \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$\therefore \; A_\alpha . A_{-\alpha} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \times \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & -\cos\alpha\sin\alpha + \sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha + \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which is an unit matrix.

Hence proved.

Exercises on § 1.08.

Ex. 1. Multiply $[4 \;\; 5 \;\; 6]$ and $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ **Ans.** [17]

Ex. 2. Multiply $[1 \;\; 2 \;\; 3]$ and $\begin{bmatrix} 4 & -6 & 9 & 6 \\ 0 & -7 & 10 & 7 \\ 5 & 8 & -11 & -8 \end{bmatrix}$

Ans. $[19 \;\; 4 \;\; -4 \;\; -4]$

Ex. 3. If $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, show that $\mathbf{AB}$ is a null matrix.

Ex. 4. Show that $\begin{bmatrix} -5 & 2 & 3 \\ -5 & 1 & 4 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Ex. 5. Show that $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Ex. 6. If $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, find **AB** and **BA** if they exist.

Ex. 7. If $\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 \\ -2 & 3 & -4 \\ 3 & -2 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$

then prove that $\mathbf{AB} = \mathbf{O}$ but $\mathbf{BA} \neq \mathbf{O}$.

Ex. 8. If $\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$

then prove that $\mathbf{AB} \neq \mathbf{BA}$.

Ex. 9. If $\mathbf{A} = \begin{bmatrix} -2 & 3 & -1 \\ -1 & 2 & -1 \\ -6 & 9 & -4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & -1 \\ 3 & 0 & -1 \end{bmatrix}$

then show that $\mathbf{AB} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\mathbf{BA} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. *(Meerut 94)*

Ex. 10. Show that $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \times \begin{bmatrix} 0 & (1/2) & (1/2) \\ (1/2) & 0 & (1/2) \\ (1/2) & (1/2) & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Ex. 11. Form the products **AB** and **BA**, when

$$\mathbf{A} = [1 \ \ 2 \ \ 3 \ \ 4] \text{ and } \mathbf{B} = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \end{bmatrix}$$

Ans. $\mathbf{AB} = [30]$

Ex. 12. If $\mathbf{A} = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$,

then prove that $\mathbf{AB} - \mathbf{AC} = \mathbf{O}$.

Ex. 13. Show that $\begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 0 \\ 0 & 4 & 1 \\ -2 & 1 & 0 \\ 1 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \end{bmatrix}$

(Bundelkhand 94)

Ex. 14. If $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$, evaluate $\mathbf{A}^2$. **Ans.** $\begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix}$

Ex. 15. If $\mathbf{A}=\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 2 & 4 \end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 1 & -1 \end{bmatrix}$, find **AB** or **BA** whichever exists.

Ans. $\mathbf{AB}=\begin{bmatrix} 1 & -2 \\ 2 & -5 \\ 3 & -8 \end{bmatrix}$ and **BA** does not exist.

Ex. 16. If $\mathbf{A}=\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ 3 & 1 & 2 \end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$ then prove that $\mathbf{AB} \neq \mathbf{BA}$.

****Ex. 17.** If **X, Y** are two matrices given by the equations $\mathbf{X}+\mathbf{Y}=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\mathbf{X}-\mathbf{Y}=\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$, find **XY**. **Ans.** $\mathbf{XY}=\begin{bmatrix} -2 & -4 \\ 3 & 2 \end{bmatrix}$

Ex. 18. In Ex. 11 Page 19 of this chapter, evaluate **A (BC)**.

(*Purvanchal 90*)

Ex. 19. If order of **A** is $m \times n$ and that of **C** is $m \times l$ and $\mathbf{A} \times \mathbf{B} = \mathbf{C}$ then order of **B** will be (i) $l \times n$, (ii) $n \times l$, (iii) 1×3, (iv) 3×1. **Ans.** (ii)

Ex. 20. If **A** is $m \times n$ matrix, **B** is $n \times l$ matrix and **C** is $l \times k$ matrix, then the order of **(AB) C** will be (a) $m \times l$, (b) $n \times p$, (c) $m \times k$, (d) $k \times m$. **Ans.** (c)

§ 1.09. Properties of Multiplication of Matrices.

****Property I Multiplication of matrices is associative.**

(*Agra 96; Avadh 94, 92, 90; Garhwal 91; Gorakhpur 91; Rohilkhand 94*)

Let $\mathbf{A}=[a_{ij}]$; $\mathbf{B}=[b_{jk}]$, $\mathbf{C}=[c_{kr}]$ be three $m \times n$, $n \times p$ and $p \times l$ matrices respectively, then $(\mathbf{AB}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{BC})$.

Proof. Let $\mathbf{AB}=[d_{ik}]$, where $d_{ik}=\sum_{j=1}^{n} a_{ij}\, b_{jk}$ ...(i)

Then $(\mathbf{AB}) \cdot \mathbf{C} = [d_{ik}] \times [c_{kr}] = [e_{ir}]$,

where
$$e_{ir} = \sum_{k=1}^{p} d_{ik} \cdot c_{kr}$$
$$= \sum_{k=1}^{p} \left(\sum_{j=1}^{n} a_{ij}\, b_{jk} \right) \cdot c_{kr}, \text{ from (i)}$$

i.e. $(i \cdot r)$th element of $(\mathbf{AB}) \cdot \mathbf{C} = \sum_{k=1}^{p} \sum_{j=1}^{n} a_{ij}\, b_{jk}\, c_{kr}$ ...(ii)

And let $\mathbf{BC}=[g_{jr}]$, $g_{jr} = \sum_{k=1}^{p} b_{jk}\, c_{kr}$ ...(iii)

Then $\mathbf{A} \cdot \mathbf{BC} = [a_{ij}] \times [g_{jr}] = [h_{ir}]$,

where
$$h_{ir} = \sum_{j=1}^{n} a_{ij}\, g_{jr}$$

$$= \sum_{j=1}^{n} a_{ij} \left(\sum_{k=1}^{p} b_{jk} c_{kr} \right), \text{ from (iii)}$$

i.e. (i, r)th element of $\mathbf{A} \cdot (\mathbf{BC}) = \sum_{k=1}^{p} \sum_{j=1}^{n} a_{ij} b_{jk} c_{kr}$, ...(iv)

since the summation can be interchanged.

$\therefore$ From (iii) and (iv) we can conclude that the (i, r)th elements of $(\mathbf{AB}) \cdot \mathbf{C}$ and $\mathbf{A} \cdot (\mathbf{BC})$ are the same and their orders are also $m \times l$.

Hence $(\mathbf{AB}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{BC})$.

****Property II. Multiplication of matrices is distributive with respect to matrix addition.** *(Bundelkhand 96, 92)*

(a) Let $\mathbf{A} = [a_{ij}]$, $\mathbf{B} = [b_{jk}]$ and $\mathbf{C} = [c_{jk}]$ be three $m \times n$, $n \times p$ and $n \times p$ matrices respectively, then $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$

(Avadh 93; Gorakhpur 93; Rohilkhand 93, 92)

Proof. $\mathbf{A}(\mathbf{B} + \mathbf{C}) = [a_{ij}] \times \{[b_{jk}] + [c_{jk}]\}$

$= [a_{ij}] [b_{jk} + c_{jk}] = [d_{jk}]$, say,

where $$d_{jk} = \sum_{j=1}^{n} a_{ij} (b_{jk} + c_{jk})$$

or (i, k)th element of $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \sum_{j=1}^{n} a_{ij} b_{jk} + \sum_{j=1}^{n} a_{ij} c_{jk}$ ...(i)

Again $\mathbf{AB} = [a_{ij}] [b_{jk}] = [e_{ik}]$, say,

where $e_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$ *i.e.* (i, k)th element of $\mathbf{AB} = \sum_{j=1}^{n} a_{ij} b_{jk}$ ...(ii)

Similarly we can prove

(i, k)th element of $\mathbf{AC} = \sum_{j=1}^{n} a_{ij} c_{jk}$ ...(iii)

$\therefore$ From (ii) and (iii) we have

(i, k)th element of $\mathbf{AB} + \mathbf{AC} = \sum_{j=1}^{n} a_{ij} b_{jk} + \sum_{j=1}^{n} a_{ij} c_{jk}$ ...(iv)

Hence from (i) and (iv) we conclude that $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$.

(b) Let $\mathbf{A} = [a_{ij}]$, $\mathbf{B} = [b_{jk}]$ and $\mathbf{C} = [c_{jk}]$ be three $n \times p$, $m \times n$ and $m \times n$ matrices respectively.

Then $(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$.

(**Note.** If **A** and **B** be $m \times n$ and $n \times p$ matrices then **BA** can not exist whereas **AB** exists).

Proof. Its proof is similar to that of part (a) above.

§ 1.10. Positive integral power of a square matrix.

From § 1.09 we find that if **A** is a square matrix, then only the product **AA** is defined and we write $\mathbf{A}^2$ for **AA**.

Also by associative law

$$\mathbf{A}^2\mathbf{A} = (\mathbf{AA})\,\mathbf{A} = \mathbf{A}\,(\mathbf{AA}) = \mathbf{AA}^2$$

So $\mathbf{A}^2\mathbf{A}$ or $\mathbf{AA}^2$ is written as $\mathbf{A}^3$.

In general $\mathbf{AAA} \ldots \mathbf{A}$ is denoted by $\mathbf{A}^n$ if there are n factors.

Definition. If $\mathbf{A}$ be a square matrix, then $\mathbf{AA} \ldots n$ times $= \mathbf{A}^n$ and $\mathbf{A}^{m+1} = \mathbf{A}^m \,.\, \mathbf{A}$, where m is a positive integer.

Theorem I. *If* $\mathbf{A}$ *be a square matrix* ($n \times n$ *say*), *then*

$\mathbf{A}^p \,.\, \mathbf{A}^q = \mathbf{A}^{p+q}$, *for any pair of positive integers* p *and* q.

Proof. We shall prove this by the method of induction.

From definition we know that $\mathbf{A}^p \,.\, \mathbf{A} = \mathbf{A}^{p+1}$, where p is any positive integer.

$\therefore\ \ \mathbf{A}^p\, \mathbf{A}^q = \mathbf{A}^{p+q}$ holds when $q = 1$, whatever p may be.

We shall now prove that if it holds for a particular value m say of q for all values of p, then it must hold for the value $m + 1$ of q for all values of p.

Now $\mathbf{A}^p\, \mathbf{A}^{m+1} = \mathbf{A}^p \,.\, (\mathbf{A}^m \,.\, \mathbf{A})$, by definition given above

$= (\mathbf{A}^p \,.\, \mathbf{A}^m) \,.\, \mathbf{A}$, by associative law

$= (\mathbf{A}^{p+m})\, \mathbf{A}$, by hypothesis

$= \mathbf{A}^{p+m+1}$, by definition given above

$= \mathbf{A}^{p+(m+1)}$, by associative law of addition of numbers.

i.e. $\mathbf{A}^p \,.\, \mathbf{A}^q = \mathbf{A}^{p+q}$ holds for the value $m + 1$ of q, whatever p may be if it holds for $q = m$.

Hence the proof by mathematical induction.

Theorem II. *If* $\mathbf{A}$ *be a square matrix, then*

$(\mathbf{A}^p)^q = \mathbf{A}^{pq}$, *for every pair of positive integers* p *and* q

Proof is similar to that of Theorem I above.

Solved Examples on § 1.09 — § 1.10.

***Ex. 1. Evaluate $\mathbf{A}^2 - 4\mathbf{A} - 5\mathbf{I}$, where**

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \text{ and } \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(Garhwal 90)

Sol. $\mathbf{A}^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1\cdot1 + 2\cdot2 + 2\cdot2 & 1\cdot2 + 2\cdot1 + 2\cdot2 & 1\cdot2 + 2\cdot2 + 2\cdot1 \\ 2\cdot1 + 1\cdot2 + 2\cdot2 & 2\cdot2 + 1\cdot1 + 2\cdot2 & 2\cdot2 + 1\cdot2 + 2\cdot1 \\ 2\cdot1 + 2\cdot2 + 1\cdot2 & 2\cdot2 + 2\cdot1 + 1\cdot2 & 2\cdot2 + 2\cdot2 + 1\cdot1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$\therefore\ \mathbf{A^2 - 4A - 5I}$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} + \begin{bmatrix} -4 & -8 & -8 \\ -8 & -4 & -8 \\ -8 & -8 & -4 \end{bmatrix} + \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4-5 & 8-8+0 & 8-8+0 \\ 8-8+0 & 9-4-5 & 8-8+0 \\ 8-8+0 & 8-8+0 & 9-4-5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O}, \qquad \textbf{Ans.}$$

where **O** is the null matrix.

Ex. 2. Let f (x) = $x^2 - 5x + 6$, find f (A) if $\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

Sol. $f(\mathbf{A}) = \mathbf{A}^2 - 5\mathbf{A} + 6$

$= \mathbf{A}^2 - 5\mathbf{A} + 6\mathbf{I}$, **where** $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Now proceed is in Ex. 1 above. **Ans.** $\begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$

***Ex. 3. If** $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$ **and** $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$, **show that**

$$(\mathbf{A} + \mathbf{B})^2 = \mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2 \neq \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$$

Sol. $\mathbf{A}^2 = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2\cdot2+(-1)\,0 & 2(-1)-1\cdot1 \\ 0\cdot2+1\cdot0 & 0(-1)+1\cdot1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 0 & 1 \end{bmatrix};$$

$$\mathbf{AB} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2\cdot1-1(-1) & 2\cdot0-1(-1) \\ 0\cdot1+1(-1) & 0\cdot0+1(-1) \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix};$$

$$\mathbf{BA} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \times \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot2+0\cdot0 & 1(-1)+0\cdot1 \\ -1\cdot2-1\cdot0 & -1(-1)-1\cdot1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -2 & 0 \end{bmatrix};$$

$$\mathbf{B}^2 = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot1+0(-1) & 1\cdot0+0(-1) \\ -1\cdot1-1(-1) & -1\cdot0-1(-1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2+1 & -1+0 \\ 0-1 & 1-1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 0 \end{bmatrix};$$

$$(\mathbf{A}+\mathbf{B})^2 = \begin{bmatrix} 3 & -1 \\ -1 & 0 \end{bmatrix} \times \begin{bmatrix} 3 & -1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3\cdot3-1(-1) & 3(-1)-1\cdot0 \\ -1\cdot3+0(-1) & -1(-1)+0\cdot0 \end{bmatrix} = \begin{bmatrix} 10 & -3 \\ -3 & 1 \end{bmatrix} \quad \text{...(i)}$$

Now $\mathbf{A}^2+\mathbf{AB}+\mathbf{BA}+\mathbf{B}^2$

$$= \begin{bmatrix} 4 & -3 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4+3+2+1 & -3+1-1+0 \\ 0-1-2+0 & 1-1+0+1 \end{bmatrix} = \begin{bmatrix} 10 & -3 \\ -3 & 1 \end{bmatrix}$$

$= (\mathbf{A}+\mathbf{B})^2$, from (i) Hence proved.

Also $\mathbf{A}^2+2\mathbf{AB}+\mathbf{B}^2$

$$= \begin{bmatrix} 4 & -3 \\ 0 & 1 \end{bmatrix} + 2\begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 2 \\ -2 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$ See § 1.04 Page 6

$$= \begin{bmatrix} 4+6+1 & -3+2+0 \\ 0-2+0 & 1-2+1 \end{bmatrix} = \begin{bmatrix} 11 & -1 \\ -2 & 0 \end{bmatrix} \neq (\mathbf{A}+\mathbf{B})^2$$

Hence proved.

Ex. 4. If $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, show that

$$(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) \neq \mathbf{A}^2 - \mathbf{B}^2$$

Sol. $\mathbf{A}+\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 1-1 \\ 1+1 & 1+0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$

$$\mathbf{A}-\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0-0 & 1-(-1) \\ 1-1 & 1-0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \quad (\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 0\cdot0+0\cdot0 & 0\cdot2+0\cdot1 \\ 2\cdot0+1\cdot0 & 2\cdot2+1\cdot1 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \quad ...(i)$$

$$A^2=\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\times\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} 0\cdot0+1\cdot1 & 0\cdot1+1\cdot1 \\ 1\cdot0+1\cdot1 & 1\cdot1+1\cdot1 \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$B^2=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\times\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} 0\cdot0-1\cdot1 & -0\cdot1-1\cdot0 \\ 1\cdot0+0\cdot1 & -1\cdot1+0\cdot0 \end{bmatrix}=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\therefore\ A^2-B^2=\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}-\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}=\begin{bmatrix} 1+1 & 1-0 \\ 1-0 & 2+1 \end{bmatrix}=\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \quad ...(ii)$$

Hence from (i) and (ii), $(A+B)(A-B)\neq A^2-B^2$.

***Ex. 5 (a). If A denotes the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, prove that**

$$A^2-(a+d)A+(ad-bc)I=O.$$

Sol. $A^2=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\times\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$=\begin{bmatrix} a\cdot a+b\cdot c & a\cdot b+b\cdot d \\ c\cdot a+d\cdot c & c\cdot b+d\cdot d \end{bmatrix}=\begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & cb+d^2 \end{bmatrix}$$

$\therefore\ A^2-(a+d)A+(ad-bc)I$

$$=\begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & cb+d^2 \end{bmatrix}-(a+d)\begin{bmatrix} a & b \\ c & d \end{bmatrix}+(ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & cb+d^2 \end{bmatrix}+\begin{bmatrix} -a(a+d) & -b(a+d) \\ -c(a+d) & -d(a+d) \end{bmatrix}+\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

$$=\begin{bmatrix} a^2+bc-a(a+d)+ad-bc & b(a+d)-b(a+d)+0 \\ c(a+d)-c(a+d)+0 & cb+d^2-d(a+d)+ad-bc \end{bmatrix}$$

$=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}=O$, where O is the 2×2 null matrix.

Hence proved.

Ex. 5 (b). If $A=\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$, evaluate $6A^2-25A+42I$. *(Agra 94)*

Sol. Here $A^2=\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}\times\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$

$$=\begin{bmatrix} 1\cdot1-2\cdot2-3\cdot3 & -1\cdot2-2\cdot3+3\cdot1 & 1\cdot3+2\cdot1+3\cdot2 \\ 2\cdot1+3\cdot2+1\cdot3 & -2\cdot2+3\cdot3-1\cdot1 & 2\cdot3-3\cdot1-1\cdot2 \\ -3\cdot1+1\cdot2-2\cdot3 & 3\cdot2+1\cdot3+2\cdot1 & -3\cdot3-1\cdot1+2\cdot2 \end{bmatrix}$$

$$=\begin{bmatrix} 1-4-9 & -2-6+3 & 3+2+6 \\ 2+6+3 & -4+9-1 & 6-3-2 \\ -3+2-6 & 6+3+2 & -9-1+4 \end{bmatrix}=\begin{bmatrix} -12 & -5 & 11 \\ 11 & 4 & 1 \\ -7 & 11 & -6 \end{bmatrix}$$

$\therefore\ 6A^2-25A+42I$

$$=6\begin{bmatrix} -12 & -5 & 11 \\ 11 & 4 & 1 \\ -7 & 11 & -6 \end{bmatrix}-25\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}+42\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} -72 & -30 & 66 \\ 66 & 24 & 6 \\ -42 & 66 & -36 \end{bmatrix}-\begin{bmatrix} 25 & -50 & 75 \\ 50 & 75 & -25 \\ -75 & 25 & 50 \end{bmatrix}+\begin{bmatrix} 42 & 0 & 0 \\ 0 & 42 & 0 \\ 0 & 0 & 42 \end{bmatrix}$$

$$=\begin{bmatrix} -72-25+42 & -30+50+0 & 66-75+0 \\ 66-50+0 & 24-75+42 & 6+25+0 \\ -42+75+0 & 66-25+0 & -36-50+42 \end{bmatrix}=\begin{bmatrix} -55 & 20 & -9 \\ 16 & -9 & 31 \\ 33 & 41 & -44 \end{bmatrix}$$

***Ex. 6. If $A=\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, then show that $A^2=2A$ and $A^3=4A$.**

Sol. Given $A=\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ ...(i)

$\therefore\ A^2=A\,.\,A=\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\times\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$=\begin{bmatrix} 1\cdot1+(-1)(-1) & 1(-1)+(-1)1 \\ (-1)1+1(-1) & (-1)(-1)+1\cdot1 \end{bmatrix}=\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$=2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}=2A$, from (i) ...(ii)

Hence proved.

Again $A^3=A\,.\,A^2=A\,.\,(2A)$, from (ii)

$=2A\,.\,A=2A^2=2\,(2A)$, from (ii)

$=4A$ Hence proved.

Ex. 7 (a). If $I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $E=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, prove that

$$(aI+bE)^3=a^3I+3a^2bE.$$

(Avadh 91; Garhwal 96)

Sol. $aI+bE=a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$=\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}+\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} a+0 & 0+b \\ 0+0 & a+0 \end{bmatrix}=\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}=\mathbf{B}\ \text{(say)}$$

$\therefore \quad (a\mathbf{I}+b\mathbf{E})^2=\mathbf{B}^2=\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\times\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$

$=\begin{bmatrix} a\cdot a+b\cdot 0 & a\cdot b+b\cdot a \\ 0\cdot a+a\cdot 0 & 0\cdot b+a\cdot a \end{bmatrix}=\begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix}$

$\therefore \quad (a\mathbf{I}+b\mathbf{E})^3=\mathbf{B}^3=\mathbf{B}^2\mathbf{B}$ **(Note)**

$=\begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix}\times\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$

$=\begin{bmatrix} a^2\cdot a+2ab\cdot 0 & a^2\cdot b+2ab\cdot a \\ 0\cdot a+a^2\cdot 0 & 0\cdot b+a^2\cdot a \end{bmatrix}=\begin{bmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{bmatrix}$ ...(i)

Now $a^3\mathbf{I}+3a^2b\mathbf{E}=a^3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+3a^2b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$=\begin{bmatrix} a^3 & 0 \\ 0 & a^3 \end{bmatrix}+\begin{bmatrix} 0 & 3a^2b \\ 0 & 0 \end{bmatrix}$

$=\begin{bmatrix} a^3+0 & 0+3a^2b \\ 0+0 & a^3+0 \end{bmatrix}=\begin{bmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{bmatrix}$ ...(ii)

$\therefore$ From (i) and (ii) we get $(a\mathbf{I}+b\mathbf{E})^3=a^3\mathbf{I}+3a^2b\mathbf{E}$.

Ex. 7 (b). If $\mathbf{I}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{E}=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Then prove that $(2\mathbf{I}+3\mathbf{E})^3=8\mathbf{I}+36\mathbf{E}$ *(Rohilkhand 95)*

Sol. Do exactly as Ex. 7 (a) above. Here 'a' = 2 and $b=3$.

***Ex. 8. If $\mathbf{A}=\begin{bmatrix} 0 & -\tan(\alpha/2) \\ \tan(\alpha/2) & 0 \end{bmatrix}$, and I is a unit matrix, then prove that $\mathbf{I}+\mathbf{A}=(\mathbf{I}-\mathbf{A})\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$**

Sol. $\mathbf{I}+\mathbf{A}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+\begin{bmatrix} 0 & -\tan(\alpha/2) \\ \tan(\alpha/2) & 0 \end{bmatrix}$

$=\begin{bmatrix} 1+0 & 0-\tan(\alpha/2) \\ 0+\tan(\alpha/2) & 1+0 \end{bmatrix}=\begin{bmatrix} 1 & -\tan(\alpha/2) \\ \tan(\alpha/2) & 1 \end{bmatrix}$ ...(i)

$\mathbf{I}-\mathbf{A}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}-\begin{bmatrix} 0 & -\tan(\alpha/2) \\ \tan(\alpha/2) & 0 \end{bmatrix}$

$=\begin{bmatrix} 1-0 & 0+\tan(\alpha/2) \\ 0-\tan(\alpha/2) & 1-0 \end{bmatrix}=\begin{bmatrix} 1 & \tan(\alpha/2) \\ -\tan(\alpha/2) & 1 \end{bmatrix}$

$\therefore \quad (\mathbf{I}-\mathbf{A})\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$$=\begin{bmatrix} 1 & \tan(\alpha/2) \\ -\tan(\alpha/2) & 1 \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$=\begin{bmatrix} 1\cdot\cos\alpha+\tan(\alpha/2)\cdot\sin\alpha & 1(-\sin\alpha)+\tan(\alpha/2)\cos\alpha \\ -\tan(\alpha/2)\cdot\cos\alpha+1\cdot\sin\alpha & (\sin\alpha)\cdot\tan(\alpha/2)+1\cdot\cos\alpha \end{bmatrix}$$

$$=\begin{bmatrix} \{1-2\sin^2(\alpha/2)\}+2\sin^2(\alpha/2) & -2\sin(\alpha/2)\cos(\alpha/2)+\tan(\alpha/2)\cos\alpha \\ -\tan(\alpha/2)\cos\alpha+2\sin(\alpha/2)\cos(\alpha/2) & 2\sin^2(\alpha/2)+\{1-2\sin^2(\alpha/2)\} \end{bmatrix},$$

writing $\cos\alpha = 1-2\sin^2\frac{1}{2}\alpha$

$$=\begin{bmatrix} 1 & -2\tan(\alpha/2)\cos^2(\alpha/2)+\tan(\alpha/2)\cos\alpha \\ -\tan(\alpha/2)\cos\alpha+2\tan(\alpha/2)\cos^2(\alpha/2) & 1 \end{bmatrix},$$

wrting $\sin\frac{1}{2}\alpha$ as $\tan\frac{1}{2}\alpha\cos\frac{1}{2}\alpha$

$$=\begin{bmatrix} 1 & -\tan(\alpha/2)\,[2\cos^2(\alpha/2)-\{2\cos^2(\alpha/2)-1\}] \\ \tan(\alpha/2)\,[-\{2\cos^2(\alpha/2)-1\}+2\cos^2(\alpha/2)\} & 1 \end{bmatrix},$$

writing $\cos\alpha = 2\cos^2(\alpha/2)-1$

$$=\begin{bmatrix} 1 & -\tan(\alpha/2) \\ \tan(\alpha/2) & 1 \end{bmatrix}=\mathbf{I}+\mathbf{A}, \text{ from (i)}$$

Hence proved.

****Ex. 9 (a). If $\mathbf{A}=\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, show that $\mathbf{A}^n=\begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$**

(*Agra 96; Avadh 92; Garhwal 91; Kanpur 95, Kumaun 95, 93; Meerut 90*)

Sol. $\mathbf{A}^2=\mathbf{A}\,.\,\mathbf{A}=\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}\times\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

$$=\begin{bmatrix} 3\cdot3-4(1) & 3(-4)-4(-1) \\ 1\cdot3-1(1) & 1(-4)-1(-1) \end{bmatrix}=\begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}$$

$$=\begin{bmatrix} 1+2(2) & -4(2) \\ (2) & 1-2(2) \end{bmatrix}$$

$=\mathbf{A}^n$, when $n=2$ **(Note)**

$\therefore\ \mathbf{A}^n=\begin{bmatrix} 1+2n & 4n \\ n & 1-2n \end{bmatrix}$ holds when $n=2$

Now $\mathbf{A}^{n+1}=\mathbf{A}^n\,.\,\mathbf{A}$ ...See def. § 1.10 Page 28.

$$= \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} (1+2n)\cdot 3 - 4n\,(1) & (1+2n)\,(-4) - 4n\,(-1) \\ n\cdot 3 + (1-2n)\,(1) & n\,(-4) + (1-2n)\,(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 3+2n & -4-4n \\ 1+n & -1-2n \end{bmatrix} = \begin{bmatrix} 1+2\,(n+1) & -4\,(n+1) \\ (n+1) & 1-2\,(n+1) \end{bmatrix}$$

i.e. $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$ holds for 'n' $= n+1$.

Also we have shown above that it holds for $n = 2$.

Hence by mathematical induction it is true for all positive integral values of n.

Hence proved.

Ex. 9 (b). If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, prove that $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$, where n is positive integer. *(Kanpur 97, 93)*

Sol. $A^2 = A \bullet A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1.1+1.0 & 1.1+1.1 \\ 0.1+1.0 & 0.1+1.1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$= A^n$, where $n = 2$.

i.e., $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ holds when $n = 2$

Now $A^{n+1} = A^n \bullet A$ ... See def. § 1.10 Page 27

$$= \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.1+n.0 & 1.1+n.1 \\ 0.1+1.0 & 0.1+1.1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & n+1 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ holds for 'n' $= n+1$

Also we have showsn above that it holds for $n = 2$. Hence by mathematical induction it is true for all positive integral values of n.

Hence proved.

Ex. 10. Let $A = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$, where $a \neq 0$. Show that for $n \geq 0$, $A^n = \begin{bmatrix} a^n & \dfrac{b\,(a^n - 1)}{(a-1)} \\ 0 & 1 \end{bmatrix}$

Sol. $A^2 = A \bullet A = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} a.a + b.0 & a.b + b.1 \\ 0.a + 1.0 & 0.b + 1.1 \end{bmatrix} = \begin{bmatrix} a^2 & b\,(a+1) \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a^2 & \dfrac{b\,(a^2 - 1)}{(a-1)} \\ 0 & 1 \end{bmatrix} = A^n \text{, when } n = 2$$

(Note)

$\therefore \quad A^n = \begin{bmatrix} a^n & b\,(a^n - 1)/(a-1) \\ 0 & 1 \end{bmatrix}$ holds when $n = 2$.

Now $A^{n+1} = A^n \bullet A$, See def. § 1·10 Page 27

$$= \begin{bmatrix} a^n & b\,(a^n - 1)/(a-1) \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a^n.a + 0 & a^n b + 1.\{b\,(a^n - 1)/(a-1)\} \\ 0 + 0 & 0 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} a^{n+1} & b\,\{a^n\,(a-1) + (a^n - 1)\}/(a-1) \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a^{n+1} & b\,(a^{n+1} - 1)/(a-1) \\ 0 & 1 \end{bmatrix}$$

$\therefore \quad A^n = \begin{bmatrix} a^n & b\,(a^n - 1)/(a-1) \\ 0 & 1 \end{bmatrix}$ holds for 'n' $= n + 1$.

Also we have shown above that it holds for $n = 2$.

Hence by mathematical induction it is true for all positive integral values of $n \geq 0$.

Hence proved.

***Ex. 11. (a) Show that** $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$,

when n is a positive integer. *(Avadh 95, Gorakhpur 90)*

Sol. Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ ...(i)

Then $(A^2) = A \bullet A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \times \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\theta - \sin^2\theta & -\sin\theta\cos\theta - \sin\cos\theta \\ \sin\theta\cos\theta + \sin\theta\cos\theta & -\sin^2\theta + \cos^2\theta \end{bmatrix}$$

or $(A)^2 = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$ ...(ii)

Similarly $(A)^3 = (A)^2 \bullet A$ ... See def. § 1.10 Page 27

$$=\begin{bmatrix}\cos 2\theta & -\sin 2\theta\\ \sin 2\theta & \cos 2\theta\end{bmatrix}\times\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}, \text{ from (i) and (ii)}$$

$$=\begin{bmatrix}\cos 2\theta\cos\theta-\sin 2\theta\sin\theta & -\cos 2\theta\sin\theta-\sin 2\theta\cos\theta\\ \sin 2\theta\cos\theta+\cos 2\theta\sin\theta & -\sin 2\theta\sin\theta+\cos 2\theta\cos\theta\end{bmatrix}$$

$$=\begin{bmatrix}\cos(2\theta+\theta) & -\sin(2\theta+\theta)\\ \sin(2\theta+\theta) & \cos(2\theta+\theta)\end{bmatrix}$$

or $$(\mathbf{A})^3=\begin{bmatrix}\cos 3\theta & -\sin 3\theta\\ \sin 3\theta & \cos 3\theta\end{bmatrix} \quad ...(iii)$$

In the light of (i), (ii) and (iii) let us assume that

$$(\mathbf{A})^n=\begin{bmatrix}\cos n\theta & -\sin n\theta\\ \sin n\theta & \cos n\theta\end{bmatrix} \quad ...(iv)$$

Now $(\mathbf{A})^{n+1}=(\mathbf{A})^n.(\mathbf{A})$

$$=\begin{bmatrix}\cos n\theta & -\sin n\theta\\ \sin n\theta & \cos n\theta\end{bmatrix}\times\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

$$=\begin{bmatrix}\cos n\theta\cos\theta-\sin n\theta\sin\theta & -\cos n\theta\sin\theta-\sin n\theta\cos\theta\\ \sin n\theta\cos\theta+\cos n\theta\sin\theta & -\sin n\theta\sin\theta+\cos n\theta\cos\theta\end{bmatrix}$$

$$=\begin{bmatrix}\cos(n\theta+\theta) & -\sin(n\theta+\theta)\\ \sin(n\theta+\theta) & \cos(n\theta+\theta)\end{bmatrix}=\begin{bmatrix}\cos(n+1)\theta & -\sin(n+1)\theta\\ \sin(n+1)\theta & \cos(n+1)\theta\end{bmatrix}$$

i.e. (iv) holds for $n+1$ if is true for n.

We have already proved in (ii) and (iii) that (iv) holds for $n=2$ and 3. Hence (iv) holds for all positive integral values of n.

i.e. $$(\mathbf{A})^n=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}^n=\begin{bmatrix}\cos n\theta & -\sin n\theta\\ \sin n\theta & \cos n\theta\end{bmatrix}$$ Hence proved.

****Ex. 11. (b) If $\mathbf{A}=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$, evaulate $\mathbf{A}^n$.** *(Garhwal 94, 92; Meerut 97)*

Sol. $$\mathbf{A}^2=\mathbf{A}\bullet\mathbf{A}=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\times\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

$$=\begin{bmatrix}\cos^2\theta-\sin^2\theta & \cos\theta\sin\theta+\sin\theta\cos\theta\\ -\sin\theta\cos\theta-\cos\theta\sin\theta & -\sin^2\theta+\cos^2\theta\end{bmatrix}$$

or $$(\mathbf{A})^2=\begin{bmatrix}\cos 2\theta & \sin 2\theta\\ -\sin 2\theta & \cos 2\theta\end{bmatrix} \quad ...(i)$$

Similarly $(\mathbf{A})^3=(\mathbf{A})^2\bullet\mathbf{A}$

$$=\begin{bmatrix}\cos 2\theta & \sin 2\theta\\ -\sin 2\theta & \cos 2\theta\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

$$=\begin{bmatrix}\cos 2\theta\cos\theta-\sin 2\theta\sin\theta & \cos 2\theta\sin\theta+\sin 2\theta\cos\theta\\ -\sin 2\theta\cos\theta-\cos 2\theta\sin\theta & -\sin 2\theta\sin\theta+\cos 2\theta\cos\theta\end{bmatrix}$$

$$= \begin{bmatrix} \cos(2\theta+\theta) & \sin(2\theta+\theta) \\ -\sin(2\theta+\theta) & \cos(2\theta+\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos 3\theta & \sin 3\theta \\ -\sin 3\theta & \cos 3\theta \end{bmatrix} \quad \text{...(ii)}$$

In the light of (i), (ii), let us assume that

$$(\mathbf{A})^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix} \quad \text{...(iii)}$$

Now $(\mathbf{A})^{n+1} = (\mathbf{A})^n . \mathbf{A}$

$$= \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix} \times \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos n\theta \cos\theta - \sin n\theta \sin\theta & \cos n\theta \sin\theta + \sin n\theta \cos\theta \\ -\sin n\theta \cos\theta - \cos n\theta \sin\theta & -\sin n\theta \sin\theta + \cos n\theta \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(n\theta+\theta) & \sin(n\theta+\theta) \\ -\sin(n\theta+\theta) & \cos(n\theta+\theta) \end{bmatrix} = \begin{bmatrix} \cos(n+1)\theta & \sin(n+1)\theta \\ -\sin(n+1)\theta & \cos(n+1)\theta \end{bmatrix}$$

$\therefore$ (iii) holds for $n+1$ if it is true for n.

We have already proved in (i) and (ii) that (iii) holds for $n = 2$ and 3. Hence by mathematical induction (iii) holds for all +ve integral values of n and value of $\mathbf{A}^n$ is given by (iii).

***Ex. 12. Show that if** $\mathbf{A} = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}$

then $\mathbf{A}^n = \begin{bmatrix} \cosh n\theta & \sinh n\theta \\ \sinh n\theta & \cosh n\theta \end{bmatrix}$ *(Agra 93)*

Sol. Here $\mathbf{A}^2 = \mathbf{A} \bullet \mathbf{A}$

$$= \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} \times \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cosh^2\theta + \sinh^2\theta & \cosh\theta \sinh\theta + \sinh\theta \cosh\theta \\ \sinh\theta \cosh\theta + \cosh\theta \sinh\theta & \sinh^2\theta + \cosh^2\theta \end{bmatrix}$$

or $$\mathbf{A}^2 = \begin{bmatrix} \cosh 2\theta & \sinh 2\theta \\ \sinh 2\theta & \cosh 2\theta \end{bmatrix}$$

Similarly $\mathbf{A}^3 = \mathbf{A}^2 \bullet \mathbf{A}$ ...(i)

$$= \begin{bmatrix} \cosh 2\theta & \sinh 2\theta \\ \sinh 2\theta & \cosh 2\theta \end{bmatrix} \times \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}, \text{ from (i)}$$

$$= \begin{bmatrix} \cosh 2\theta \cosh\theta + \sinh 2\theta \sinh\theta & \cosh 2\theta \sinh\theta + \sinh 2\theta \cosh\theta \\ \sinh 2\theta \cosh\theta + \cosh 2\theta \sinh\theta & \sinh 2\theta \sinh\theta + \cosh 2\theta \cosh\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cosh(2\theta+\theta) & \sinh(2\theta+\theta) \\ \sinh(2\theta+\theta) & \cosh(2\theta+\theta) \end{bmatrix}$$

or $$A^3 = \begin{bmatrix} \cosh 3\theta & \sinh 3\theta \\ \sinh 3\theta & \cosh 3\theta \end{bmatrix} \qquad \text{...(ii)}$$

In the light of (i), (ii) and the given value of **A**, let us assume that

$$A^n = \begin{bmatrix} \cosh n\theta & \sinh n\theta \\ \sinh n\theta & \cosh n\theta \end{bmatrix} \qquad \text{...(iii)}$$

Now $A^{n+1} = A^n \bullet A$

$$= \begin{bmatrix} \cosh n\theta & \sinh n\theta \\ \sinh n\theta & \cosh n\theta \end{bmatrix} \times \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cosh n\theta \cosh\theta + \sinh n\theta \sinh\theta & \cosh n\theta \sinh\theta + \sinh n\theta \cosh\theta \\ \sinh n\theta \cosh\theta + \cosh n\theta \sinh\theta & \sinh n\theta \sinh\theta + \cosh n\theta \cosh\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cosh(n\theta+\theta) & \sinh(n\theta+\theta) \\ \sinh(n\theta+\theta) & \cosh(n\theta+\theta) \end{bmatrix} = \begin{bmatrix} \cosh(n+1)\theta & \sinh(n+1)\theta \\ \sinh(n+1)\theta & \cosh(n+1)\theta \end{bmatrix}$$

i.e. (iii) holds for $n+1$ if it is true for n.

Also from (i) and (ii) we know that (iii) holds for $n = 2$ and $n = 3$. Hence (iii) holds for all positive integral values of n.

i.e. $$A^n = \begin{bmatrix} \cosh n\theta & \sinh n\theta \\ \sinh n\theta & \cosh n\theta \end{bmatrix}$$ Hence proved.

[Note. See Ex. 13 Page 20 also].

Exercises on § 1·09 – § 1·10

Ex. 1. Show that the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ satisfies the equation $A^2 - 2A - 5I = O$, where **O** is the 2×2 null matrix.

Ex. 2. Evaluate $A^2 - 3A - 13I$, where **I** is the 2×2 unit matrix and $A = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$

Ans. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

Ex. 3. Show that matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ satisfies the equation $A^3 - 3A^2 + 3A - I = O$, where **I** is the unit matrix and **O** the null matrix of order 3.

Ex 4. If $A = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

verify (i) $(AB)C = A(BC)$; (ii) $(A+B)C = AC + BC$.

Ex. 5. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$; $C = \begin{bmatrix} 1 & 1 \\ 7 & 4 \end{bmatrix}$, show that

$$A(B + C) = AB + AC.$$

Ex. 6. If $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, show that

$$(A + B)(A + B) = A^2 + 2AB + B^2.$$

Ex. 8. Show that $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n & \frac{1}{2}n(n+1) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$

for all natural numbers n.

[Hint. See Ex. 10 Page 35]

SOME TYPICAL SOLVED EXAMPLES

****Ex. 1. A manufacturer produces three products A, B, C which he sells in the market. Annual sale volumes are indicated as follows :**

Markets	**Products**		
	A	**B**	**C**
I	**8,000**	**10,000**	**15,000**
II	**10,000**	**2,000**	**20,000**

(i) If unit sale prices of A, B and C are Rs. 2·25, Rs. 1·50 and Rs. 1·25 respectively, find the total revenue in each market with the help of matrices, (ii) if the units costs of the above three products are Rs. 1·60, Rs. 1·20 and Rs. 0·90 respectively, find the gross profit with the help of matrices.

Sol. (i) The total revenue in each market is given by the products matrix.

$$[2{\cdot}25 \quad 1{\cdot}50 \quad 1{\cdot}25] \times \begin{bmatrix} 8,000 & 10,000 \\ 10,000 & 2,000 \\ 15,000 & 20,000 \end{bmatrix} \qquad \textbf{(Note)}$$

$$= [(2{\cdot}25 \times 8,000) + (1{\cdot}50 \times 10,000) + (1{\cdot}25 \times 15,000) \quad (2{\cdot}25 \times 10,000) + (1{\cdot}50 \times 2,000) + (1{\cdot}25 \times 20,000)]$$

$$= [18,000 + 15,000 + 18,750 \qquad 22,500 + 3,000 + 25,000]$$

$$= [51750 \quad 50500]$$

$\therefore$ Total revenue from the market I = Rs. 51,750.

and total revenue from the market II = Rs. 50,500. **Ans.**

(ii) Similarly the total cost of products with the manufacturer sells in the markets are :

$$[1{\cdot}60 \quad 1{\cdot}20 \quad 0{\cdot}90] \times \begin{bmatrix} 8,000 & 10,000 \\ 10,000 & 2,000 \\ 15,000 & 20,000 \end{bmatrix}$$

$$= [(1{\cdot}60 \times 8,000) + (1{\cdot}20 \times 10,000) + (0{\cdot}90 \times 15,000)$$
$$(1{\cdot}60 \times 10,000) + (1{\cdot}20 + 2,000) + (0{\cdot}90 + 20,000)]$$

$$= [12,800 + 12,000 + 13,500 \qquad 16,000 + 2,400 + 18,000]$$

$$= [38,300 \qquad 36,400]$$

∴ Total cost of products which the manufacturer sells in the market I and II are Rs, 38,300 and Rs. 36,400 respectively.

∴ Required gross profit = (Total revenue received frotn both the markets) -- (Total costs of product which the manufactuerr sold in both the markets)

$= (\text{Rs. } 51,750 + \text{Rs, } 50,500) - (\text{Rs. } 38,300 + \text{Rs. } 36,400).$

$= \text{Rs. } 102,250 - \text{Rs. } 74,700 = \text{Rs. } 27,550.$ **Ans.**

Ex. 2. A man buys 8 dozens of mangoes, 10 dozens of apples and 4 dozens of bananas. Mangoes cost Rs. 18 per dozen, apples Rs. 9 per dozen and bananas Rs. 6 per dozen. Represent the quantities bought by a row matrix and the prices by a column matrix and hence obtain the total cost.

(I. C. W. A. Final)

Sol. The quantities bought are represented by 3×1 row matrix $[8 \quad 10 \quad 4]$ and the prices are represented by 3×1 column matrix

$$\begin{bmatrix} 18 \\ 9 \\ 6 \end{bmatrix}$$

∴ The cost of fruits is a single number *i.e.* 1×1 matrix given by the product matrix

$$[8 \quad 10 \quad 4] \times \begin{bmatrix} 18 \\ 9 \\ 6 \end{bmatrix}$$

i.e. $[(8 \times 18) + (10 \times 9) + (4 \times 6)]$ *i.e.* $[144 + 90 + 24]$ *i.e* $[258]$

∴ The required total cost = Rs. 258. **Ans.**

****Ex. 3. A store has in stock 30 dozen shirts, 15 dozen trousers and 25 dozen pairs of socks. If the selling prices are Rs. 50 per shirt, Rs. 90 per trouser and Rs. 12 per pair of socks, then find the total amount the store owner will get after selling all the items in the stock.**

Sol. The stock in the store can be written in the form of a row matrix **A** given by $\mathbf{A} = [20 \times 12 \quad 15 \times 12 \quad 25 \times 12]$

or $\mathbf{A} = [240 \quad 180 \quad 300]$, which is a 1×3 matrix.

The prices can be written in the form of a column matrix **B** given by

$$\mathbf{B} = \begin{bmatrix} 50 \\ 90 \\ 12 \end{bmatrix}, \text{ which is a } 3 \times 1 \text{ matrix.}$$

The required amount is a single number *i.e.* a matrix of order 1×1 and so the same can be obtained by multiplying the matrices **A** and **B**, since their product would be a 1×1 matrix. **(Note)**

Now $\mathbf{AB} = [240 \quad 180 \quad 300] \times \begin{bmatrix} 50 \\ 90 \\ 12 \end{bmatrix}$

$= [(240 \times 50) + (180 \times 90) + (300 \times 12)]$

$= [12000 + 16200 + 36000] = [31800]$

$\therefore$ The required amount received by the store owner

= Rs. 31,800. **Ans.**

Ex. 4. A trust fund has Rs. 50,000 that is to be invested into two types of bonds. The first bond pays 5% interest per year and the second bond pays 6% interest per year. Using matrix multipication, determine how to divide Rs. 50,000 among two types of bonds so as to obtain an annual total interest of Rs. 2780.

Sol. Let Rs. 50,000 be divided into two parts Rs. x and Rs. $(50{,}000 - x)$ out of which first part is invested in first type of bonds and the second part is invested in second type of bonds.

The values of these bonds can be written in the form of a row matrix **A** given by $\mathbf{A} = [x \quad 50{,}000 - x]$, which is a 1×2 matrix.

And the amounts received as interest per rupee annually from these two types of bonds can be written in the form of a column matrix **B** given by

$$\mathbf{B} = \begin{bmatrix} 5/100 \\ 6/100 \end{bmatrix}, \text{ which is a } 2 \times 1 \text{ matrix.}$$

Here the interest has been calculated per rupee annually.

Now the interest to be obtained annually is a single number *i.e.* a matrix of order 1×1 and the same can be obtained by the product matrix **AB**, since this product matrix would be a 1×1 matrix. **(Note)**

Here $\mathbf{AB} = [x \quad 50{,}000 - x] \times \begin{bmatrix} 5/100 \\ 6/100 \end{bmatrix}$

$$= \left[x . \frac{5}{100} + (50{,}000 - x) . \frac{6}{100} \right]$$

$$= \left[3000 - \frac{x}{100} \right]$$

Also we are given that the annual interest = 2,780.

$\therefore$ We must have $\left[3000 - \frac{x}{100}\right] = [2780]$ **(Note)**

or $3000 - \frac{x}{100} = 2780$ or $x = (3000 - 2780) \times 100$

or $x = 220 \times 100 = 22{,}000$

Hence the required amounts are

Rs. 22,000 and Rs. (50,000 – 22,000) *i.e.* Rs. 22,000 and Rs. 28,000 **Ans.**

Ex. 5. A finance company has offices located in every division, every district and every taluka in a certain state in India. Assume that there are five divisions, thirty districts and 200 talukas in the state. Each office has one headclerk, one cashier, one clerk and one peon. A divisional office has, in addition, one office superintendent, two clerks, one typist and one peon. A district office, has in addition, one clerk and one peon, The basic monthly salaries are as follows : office superitendent Rs. 500. Head clerk Rs. 200, cashier Rs. 175, clerks and typists Rs. 150 and peon Rs. 100. Using matrix notation find —

(i) The total number of posts of each kind in all the offices taken together, (ii) the total basic monthly salary bill of each kind of office and (iii) the total basic monthly salary bill of all the offices taken together.

(*C. A. Intermediate*)

Sol. Let us use the symbols Div, Dis, Tal for division, district, taluka respectively and O, H, C, Cl, T and P for office superintendent, Head clerk, cashier, clerk, typist and peon respectively.

Then the number of offices can be arranged as elements of a row matrix **A** (say) given by

$$\begin{matrix} & \text{Div.} & \text{Dis.} & \text{Tal.} \\ \mathbf{A} = (& 5 & 30 & 200) \end{matrix}$$

The composition of staff in various offices can be arranged in a 3×6 matrix **B** (say) given by

$$\begin{matrix} & \text{O} & \text{H} & \text{C} & \text{Cl} & \text{T} & \text{P} \end{matrix}$$
$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 & 2+1 & 1 & 1+1 \\ 0 & 1 & 1 & 1+1 & 0 & 1+1 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

The basic monthly salaries of various types of employees of these offices correspond to the elements of the column matrix **C** (say) given by

$$\mathbf{C} = \begin{matrix} \text{O} \\ \text{H} \\ \text{C} \\ \text{Cl} \\ \text{T} \\ \text{P} \end{matrix} \begin{bmatrix} 500 \\ 200 \\ 175 \\ 150 \\ 150 \\ 100 \end{bmatrix}$$

(i) Total number of posts of each kind in all the offices are the elements of the product matrix **AB**.

i.e. $[5 \quad 30 \quad 200] \times \begin{bmatrix} 1 & 1 & 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$ **(Note)**

i.e. $[5+0+0, \quad 5+30+200, \quad 5+30+200, \quad 15+60+200,$
$5+0+0, \quad 10+60+200]$

i.e.

O	H	C	Cl	T	P
5	235	235	275	5	270

i.e. Required number of posts in all the offices taken together are 5 offices supdts., 235 Head clerks, 235 cashiers, 275 clerks, 5 typists and 270 peons. **Ans.**

(ii) Total basic monthly salary bill of each kind of office are the elements of the product matrix **BC**

i.e. $\begin{bmatrix} 1 & 1 & 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 500 \\ 200 \\ 175 \\ 150 \\ 150 \\ 100 \end{bmatrix}$

$$= \begin{bmatrix} (1 \times 500) + (1 \times 200) + (1 \times 175) + (3 \times 150) + (1 \times 150) + (2 \times 100) \\ (0 \times 500) + (1 \times 200) + (1 \times 175) + (2 \times 150) + (0 \times 150) + (2 \times 100) \\ (0 \times 500) + (1 \times 200) + (1 \times 175) + (1 \times 150) + (0 \times 150) + (1 \times 100) \end{bmatrix}$$

$$= \begin{bmatrix} 500 + 200 + 175 + 450 + 150 + 200 \\ 0 + 200 + 175 + 300 + 0 + 200 \\ 0 + 200 + 175 + 150 + 0 + 100 \end{bmatrix} = \begin{bmatrix} 1675 \\ 875 \\ 625 \end{bmatrix}$$ **Ans.**

i.e. The total basic monthly salary bill of each divisional, district and taluka offices are Rs. 1675, Rs. 875 and Rs. 625 respectively. **Ans.**

(iii) Total basic monthly salary bill of all the officers (*i.e.* of five divisional, 30 district and 200 taluka offices) is the element of the product matrix **ABC**

i.e. $[5 \quad 30 \quad 200] \times \begin{bmatrix} 1675 \\ 875 \\ 625 \end{bmatrix}$ (Note)

i.e. $[(5 \times 1675) + (30 \times 875) + (200 \times 625)]$

i.e. $[8375 + 2650 + 125000]$ *i.e.* $[159625]$

i.e. total basic monthly salary bill of all the offices taken together is Rs. 159,625. **Ans.**

****Ex. 6. In a development plan of a city, a contractor has taken a contract to construct certain houses for which he needs building materials like stones, sand etc. There are three firms A, B, C that can supply him these materials. At one time these firms A B,C, supplied him 40, 35 and 25 truck loads of stones and 10, 5 and 8 truck loads of sand respectively. If the cost of one truck load of stone and sand are Rs. 1,200 and Rs. 500 respectively, then find the total amount paid by the contractor to each of these firms, A, B, C separately.**

Sol. The truck-loads of stone and sand supplied by the firms A, B and C can be written in the from of a matrix **A** (say) given by

$$\mathbf{A} = \begin{matrix} \\ \text{Stone} \\ \text{Sand} \end{matrix} \begin{matrix} \begin{matrix} A & \;B & \;C \end{matrix} \\ \begin{bmatrix} 40 & 35 & 25 \\ 10 & 5 & 8 \end{bmatrix} \end{matrix}, \text{ which is a } 2 \times 3 \text{ matrix}$$

And the cost per truck of stone and sand can be given in the form of a matrix **B** (say) given by

$$\mathbf{B} = \begin{matrix} \text{Stone} & \text{Sand} \\ [1200 & 500] \end{matrix}$$

The required total amount paid to each of the firms A, B and C are given by the product matrix **BA**. [Note **AB** can not be calculated).

$$\text{Now } \mathbf{BA} = [1200 \quad 500] \times \begin{bmatrix} 40 & 35 & 25 \\ 10 & 5 & 8 \end{bmatrix}$$

$$= [(1200 \times 40) + (500 \times 10) \quad (1200 \times 35) + (500 \times 5) \quad (1200 \times 25) + (500 \times 8)]$$

$$= [48000 + 5000 \quad 42000 + 2500 \quad 30000 + 4000]$$

$$= [53,000 \quad 44,500 \quad 34,000]$$

$\therefore$ The amount paid to the firms A, B and C by the contractor are Rs, 53,000, Rs. 44,500 and Rs. 34,000 respectively. **Ans.**

Exercises

Ex. 1. A fruit seller has in stock 20 dozen mangoes, 16 dozen apples and 32 dozen bananas. Suppose the selling prices are Rs. 0.35, Rs. 0.75 and Rs. 0.08 per mango, apple and banana respectively. Find the total amount the fruit seller will get by selling his whole stock. **Ans.** Rs. 258·72

Ex. 2. In Ex. 4 Page 3 write down (i) the row matrix which represents team B's result; (ii) the column matrix which represent the results of first places of various teams. **Ans.** $[0 \quad 3 \quad 2 \quad 4]$ and $\begin{bmatrix} 3 \\ 0 \\ 5 \\ 2 \end{bmatrix}$

MISCELLANEOUS SOLVED EXAMPLES

***Ex. 1. If** $\mathbf{A}=\begin{bmatrix}\mathbf{1} & -\mathbf{1}\\ \mathbf{2} & -\mathbf{1}\end{bmatrix}$, $\mathbf{B}=\begin{bmatrix}\mathbf{a} & \mathbf{1}\\ \mathbf{b} & -\mathbf{1}\end{bmatrix}$

and $(\mathbf{A}+\mathbf{B})^2=\mathbf{A}^2+\mathbf{B}^2$, find a and b. *(Kanpur 96)*

Sol. Here we have

$$\mathbf{A}^2=\begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix}\times\begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix}$$

$$=\begin{bmatrix}1-2 & -1+1\\ 2-2 & -2+1\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$$

$$\mathbf{B}^2=\begin{bmatrix}a & 1\\ b & -1\end{bmatrix}\times\begin{bmatrix}a & 1\\ b & -1\end{bmatrix}=\begin{bmatrix}a^2+b & a-1\\ ab-b & b+1\end{bmatrix}$$

$$\therefore\quad \mathbf{A}^2+\mathbf{B}^2=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}+\begin{bmatrix}a^2+b & a-1\\ ab-b & b+1\end{bmatrix}$$

$$=\begin{bmatrix}-1+a^2+b & 0+a-1\\ 0+ab-b & -1+b+1\end{bmatrix}=\begin{bmatrix}a^2+b-1 & a-1\\ ab-b & b\end{bmatrix}\qquad ...(i)$$

Also $\mathbf{A}+\mathbf{B}=\begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix}+\begin{bmatrix}a & 1\\ b & -1\end{bmatrix}$

$$=\begin{bmatrix}1+a & -1+1\\ 2+b & -1-1\end{bmatrix}=\begin{bmatrix}1+a & 0\\ 2+b & -2\end{bmatrix}$$

$$\therefore\ (\mathbf{A}+\mathbf{B})^2=\begin{bmatrix}1+a & 0\\ 2+b & -2\end{bmatrix}\times\begin{bmatrix}1+a & 0\\ 2+b & -2\end{bmatrix}$$

$$=\begin{bmatrix}(1+a)^2+0 & 0+0\\ (2+b)(1+a)-2(2+b) & 0+4\end{bmatrix}$$

$$=\begin{bmatrix}(1+a)^2 & 0\\ (2+b)(a-1) & 4\end{bmatrix}\qquad ...(ii)$$

Now it is given that $(\mathbf{A}+\mathbf{B})^2=\mathbf{A}^2+\mathbf{B}^2$.

or $\begin{bmatrix}(1+a)^2 & 0\\ (2+b)(a-1) & 4\end{bmatrix}=\begin{bmatrix}a^2+b-1 & a-1\\ ab-b & b\end{bmatrix}$, from (i) and (ii)

or $0=a-1$ and $4=b$, comparing the elements of second column on both sides.

or $a=1$ and $b=4$. **Ans.**

Ex. 2. If $\mathbf{A}=\begin{bmatrix}\mathbf{1} & \mathbf{2}\\ \mathbf{3} & \mathbf{4}\\ \mathbf{5} & \mathbf{6}\end{bmatrix}$ **and** $\mathbf{B}=\begin{bmatrix}-\mathbf{3} & -\mathbf{2}\\ \mathbf{1} & -\mathbf{5}\\ \mathbf{4} & \mathbf{3}\end{bmatrix}$,

find $D = \begin{bmatrix} p & q \\ r & s \\ t & u \end{bmatrix}$, **such that** $A + B - D = O$.

Sol. $A + B - D = O$ or $D = A + B$

or $D = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} -3 & -2 \\ 1 & -5 \\ 4 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1-3 & 2-2 \\ 3+1 & 4-5 \\ 5+4 & 6+3 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 4 & -1 \\ 9 & 9 \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \\ t & u \end{bmatrix}$, given

$\therefore$ We have $p = -2$, $q = 0$, $r = 4$, $s = -1$, $t = 9$, $u = 9$ which gives **D**.

Ex. 3. If $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix}$ **and I is the unit matrix of order 3, show that**

$$A^3 = pI + qA + rA^2.$$

Sol. Here $A^2 = A \cdot A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix}$

$$= \begin{bmatrix} 0+0+0 & 0+0+0 & 0+1+0 \\ 0+0+p & 0+0+q & 0+0+r \\ 0+0+rp & p+0+rq & 0+q+r^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ p & q & r \\ rp & p+rq & q+r^2 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 0 & 0 & 1 \\ p & q & r \\ rp & p+rq & q+r^2 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+p & 0+0+q & 0+0+r \\ 0+0+rp & p+0+rq & 0+q+r^2 \\ 0+0+pq+pr^2 & rp+0+q^2+r^2q & 0+p+rq+rq+r^3 \end{bmatrix}$$

$$= \begin{bmatrix} p & q & r \\ rp & p+rq & q+r^2 \\ pq+pr^2 & rp+q^2+r^2q & p+2rq+r^3 \end{bmatrix} \quad \text{...(i)}$$

And $pI + qA + rA^2$

$$= p\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + q\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix} + r\begin{bmatrix} 0 & 0 & 1 \\ p & q & r \\ rp & p+rq & q+r^2 \end{bmatrix}$$

$$=\begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}+\begin{bmatrix} 0 & q & 0 \\ 0 & 0 & q \\ pq & q^2 & rq \end{bmatrix}+\begin{bmatrix} 0 & 0 & r \\ rp & qr & r^2 \\ r^2p & pr+qr^2 & qr+r^3 \end{bmatrix}$$

$$=\begin{bmatrix} p+0+0 & 0+q+0 & 0+0+r \\ 0+0+rp & p+0+pr & 0+q+r^2 \\ 0+pq+r^2p & 0+q^2+pr+qr^2 & p+2rq+r^3 \end{bmatrix}$$

$= \mathbf{A}^3$, from (i). Hence proved.

Ex. 4. Show that $\mathbf{E^2F + F^2E = E}$, where

$$\mathbf{E}=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ \mathbf{F}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Sol. $\mathbf{E}^2=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\times\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$$=\begin{bmatrix} 0\cdot0+0\cdot0+1\cdot0 & 0\cdot0+0\cdot0+1\cdot0 & 0\cdot1+0\cdot1+1\cdot0 \\ 0\cdot0+0\cdot0+1\cdot0 & 0\cdot0+0\cdot0+1\cdot0 & 0\cdot1+0\cdot1+1\cdot0 \\ 0\cdot0+0\cdot0+0\cdot0 & 0\cdot0+0\cdot0+0\cdot0 & 0\cdot1+0\cdot1+0\cdot0 \end{bmatrix}$$

or $\mathbf{E}^2=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore\ \mathbf{E^2F}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\times\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ..(i)

Again $\mathbf{F}^2=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\times\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$=\begin{bmatrix} 1\cdot0+0\cdot0+0\cdot0 & 1\cdot0+0\cdot1+0\cdot0 & 1\cdot0+0\cdot0+1\cdot0 \\ 1\cdot0+1\cdot0+0\cdot0 & 0\cdot0+1\cdot1+0\cdot0 & 0\cdot0+1\cdot0+0\cdot1 \\ 0\cdot1+0\cdot0+1\cdot0 & 0\cdot0+0\cdot1+1\cdot0 & 0\cdot0+0\cdot0+1\cdot1 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore\ \mathbf{F^2E}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\times\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$$=\begin{bmatrix} 1\cdot0+0\cdot0+0\cdot0 & 1\cdot0+0\cdot0+0\cdot0 & 1\cdot1+0\cdot1+0\cdot0 \\ 0\cdot0+1\cdot0+0\cdot0 & 0\cdot0+1\cdot0+0\cdot0 & 0\cdot1+1\cdot1+0\cdot0 \\ 0\cdot0+0\cdot0+1\cdot0 & 0\cdot0+0\cdot0+1\cdot0 & 0\cdot1+0\cdot1+1\cdot0 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad ...(ii)$$

∴ From (i) and (ii) we get

$$E^2F + F^2E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = E$$

Hence proved.

Ex. 5. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$, find the matrix X such that $A + X + I = O$, where I and O are unit and zero 3×3 matrices respectively.

Sol. Given that $A + X + I = O$ or $X = O - A - I$

$$=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

substituting values of **A, I** and **O**

$$=\begin{bmatrix} 0-1-1 & 0-2-0 & 0-3-0 \\ 0-3-0 & 0+2-1 & 0-1-0 \\ 0-4-0 & 0-2-0 & 0-1-1 \end{bmatrix} = \begin{bmatrix} -2 & -2 & -3 \\ -3 & 1 & -1 \\ -4 & -2 & 2 \end{bmatrix}$$

Ans.

****Ex. 6. Show that**

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & -\tan\frac{1}{2}\theta \\ \tan\frac{1}{2}\theta & 1 \end{bmatrix} \begin{bmatrix} 1 & \tan\frac{1}{2}\theta \\ -\tan\frac{1}{2}\theta & 1 \end{bmatrix}^{-1}$$

Sol. We have

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \times \begin{bmatrix} 1 & \tan\frac{1}{2}\theta \\ -\tan\frac{1}{2}\theta & 1 \end{bmatrix}$$

$$=\begin{bmatrix} \cos\theta + \sin\theta\tan\frac{1}{2}\theta & \cos\theta\tan\frac{1}{2}\theta - \sin\theta \\ \sin\theta - \cos\theta\tan\frac{1}{2}\theta & \sin\theta\tan\frac{1}{2}\theta + \cos\theta \end{bmatrix}$$

$$=\begin{bmatrix} \dfrac{\cos\theta\cos\frac{1}{2}\theta + \sin\theta\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta} & \dfrac{\cos\theta\sin\frac{1}{2}\theta - \sin\theta\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\theta} \\ \dfrac{\sin\theta\cos\frac{1}{2}\theta - \cos\theta\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta} & \dfrac{\sin\theta\sin\frac{1}{2}\theta + \cos\theta\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\theta} \end{bmatrix}$$

$$= \frac{1}{\cos\frac{1}{2}\theta}\begin{bmatrix} \cos\theta\cos\frac{1}{2}\theta + \sin\theta\sin\frac{1}{2}\theta & \cos\theta\sin\frac{1}{2}\theta - \sin\theta\cos\frac{1}{2}\theta \\ \sin\theta\cos\frac{1}{2}\theta - \cos\theta\sin\frac{1}{2}\theta & \cos\theta\cos\frac{1}{2}\theta + \sin\theta\sin\frac{1}{2}\theta \end{bmatrix}$$

$$= (\sec\tfrac{1}{2}\theta)\begin{bmatrix} \cos(\theta - \frac{1}{2}\theta) & -\sin(\theta - \frac{1}{2}\theta) \\ \sin(\theta - \frac{1}{2}\theta) & \cos(\theta - \frac{1}{2}\theta) \end{bmatrix}$$

$$= (\sec\tfrac{1}{2}\theta)\begin{bmatrix} \cos\frac{1}{2}\theta & -\sin\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta & \cos\frac{1}{2}\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\frac{1}{2}\theta\sec\frac{1}{2}\theta & -\sin\frac{1}{2}\theta\sec\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta\sec\frac{1}{2}\theta & \cos\frac{1}{2}\theta\sec\frac{1}{2}\theta \end{bmatrix} = \begin{bmatrix} 1 & -\tan\frac{1}{2}\theta \\ \tan\frac{1}{2}\theta & 1 \end{bmatrix}$$

or $$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & -\tan\frac{1}{2}\theta \\ \tan\frac{1}{2}\theta & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan\frac{1}{2}\theta \\ -\tan\frac{1}{2}\theta & 1 \end{bmatrix}^{-1}$$ Hence proved.

Ex. 7. If A and B be n-rowed square matrices, then show that

(i) $(A + B)^2 = A^2 + AB + BA + B^2$;

(ii) $(A + B)(A - B) = A^2 - AB + BA - B^2$;

(iii) $(A - B)(A + B) = A^2 + AB - BA - B^2$;

and (iv) $(A - B)^2 = A^2 - AB - BA + B^2$.

Sol. As A and B are *n*-rowed square matrices therefore $A + B$ and $A - B$ are also *n*-rowed square matrices and as such distributive law is true.

(i). $(A + B)^2 = (A + B) \times (A + B)$
$= (A + B)A + (A + B)B$, by distributive law
$= AA + BA + AB + BB$, by distributive law
$= A^2 + BA + AB + B^2$.

(ii) $(A + B)(A - B)$
$= (A + B)A + (A + B)(-B)$, by distributive law
$= AA + BA + A(-B) + B(-B)$, by distributive law
$= A^2 + BA - AB - B^2$.

(iii) $(A - B)(A + B) = (A - B)A + (A - B)B$, by distributive law
$= AA - BA + AB - BB$, by distributive law
$= A^2 - BA + AB - B^2$.

(iv) $(A - B)^2 = (A - B) \bullet (A - B)$
$= AA + A(-B) + (-B)A + (-B)(-B)$, by distributive law
$= A^2 - AB - BA + B^2$. Hence proved.

***Ex. 8. If A, B are two n × n matrices and if**

$$C = A + B,\ AB = BA,\ B^2 = O$$

then show that for every integer *m*, $C^{m+1} = A^m[A + (m + 1)B]$.

Sol. We shall prove that $\mathbf{C}^{m+1} = \mathbf{A}^m [\mathbf{A} + (m+1)\mathbf{B}]$, ...(i)

by mathematical induction.

For $m = 1$, from (i) we get $\mathbf{C}^2 = \mathbf{A}[\mathbf{A} + 2\mathbf{B}]$ ...(ii)

Also $\mathbf{C} = \mathbf{A} + \mathbf{B}$, given

$\therefore\ \mathbf{C}^2 = (\mathbf{A}+\mathbf{B})^2 = (\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B})$

$= \mathbf{A}^2 + \mathbf{BA} + \mathbf{AB} + \mathbf{B}^2$, as in Ex. 7 (i) Page 50.

$= \mathbf{A}^2 + 2\mathbf{AB}$, since $\mathbf{AB} = \mathbf{BA}$, $\mathbf{B}^2 = \mathbf{O}$ (given)

or $\mathbf{C}^2 = \mathbf{A}(\mathbf{A} + 2\mathbf{B})$, which in the same as (ii).

Hence (i) is true for $m = 1$.

Let us now assume that (i) holds when $m = k$

i.e. $$\mathbf{C}^{k+1} = \mathbf{A}^k [\mathbf{A} + (k+1)\mathbf{B}] \quad ...(iii)$$

Now $\mathbf{C}^{k+2} = \mathbf{C}^{k+1}\mathbf{C}$, by def. § 1·10 Page 27.

$= \mathbf{A}^k [\mathbf{A} + (k+1)\mathbf{B}].(\mathbf{A}+\mathbf{B})$, from (iii) and $\mathbf{C} = \mathbf{A} + \mathbf{B}$(given)

or $\mathbf{C}^{k+2} = \mathbf{A}^k [\mathbf{A}(\mathbf{A}+\mathbf{B}) + (k+1)\mathbf{B}(\mathbf{A}+\mathbf{B})]$

$= \mathbf{A}^k [\mathbf{A}^2 + \mathbf{AB} + (k+1)\mathbf{BA} + (k+1)\mathbf{B}^2]$

$= \mathbf{A}^k [\mathbf{A}^2 + \mathbf{AB} + (k+1)\mathbf{AB}]$, $\because$ $\mathbf{BA} = \mathbf{AB}$, $\mathbf{B}^2 = \mathbf{O}$

$= \mathbf{A}^k [\mathbf{A}^2 + (1 + k + 1)\mathbf{AB}]$

$= \mathbf{A}^k.\mathbf{A}[\mathbf{A} + \{(k+1)+1\}\mathbf{B}]$

or $\mathbf{C}^{k+2} = \mathbf{A}^{k+1}[\mathbf{A} + \{(k+1)+1\}\mathbf{B}]$.

Hence (i) is true for $m = k+1$ provided (iii) is true *i.e.* for $m = k$. Also we have shown that (i) is true for $m = 1$, so it is true for $m = 1 + 1$ *i.e.* $m = 2$ and so on. Hence by induction (i) is true for all positive integral values of m.

Hence proved.

Ex. 9. If $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ **and** $\mathbf{B} = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}$

then prove that $\mathbf{AB} = 2\mathbf{B}$.

Sol. $\mathbf{AB} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \times \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}$

$$= \begin{bmatrix} 2x_1 + 0 + 0 & 2y_1 + 0 + 0 & 2z_1 + 0 + 0 \\ 0 + 2x_2 + 0 & 0 + 2y_2 + 0 & 0 + 2z_2 + 0 \\ 0 + 0 + 2x_3 & 0 + 0 + 2y_3 & 0 + 0 + 2z_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 & 2y_1 & 2z_1 \\ 2x_2 & 2y_2 & 2z_2 \\ 2x_3 & 2y_3 & 2z_3 \end{bmatrix} = 2\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} = 2\mathbf{B}$$

Hence proved.

Ex. 10. If A, B are two matrices given below, which of the two statements is true AB = BA or AB ≠ BA.

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix};\ \mathbf{B} = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Sol. Do yourself. **Ans. AB ≠ BA.**

Ex. 11. Find a if $[a \quad 4 \quad 1] \times \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 4 \end{bmatrix} \times \begin{bmatrix} a \\ 4 \\ -1 \end{bmatrix} = \mathbf{O}$,

where O is 1 × 1 null matrix.

Sol. $[a \quad 4 \quad 1] \times \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 4 \end{bmatrix}$

$= [2a + 4 + 0 \quad a + 0 + 2 \quad 0 + 8 + 4]$ **(Note)**

$= [2a + 4 \quad a + 2 \quad 12]$

$\therefore \quad [a \quad 4 \quad 1] \times \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 4 \end{bmatrix} \times \begin{bmatrix} a \\ 4 \\ -1 \end{bmatrix}$

$= [2a + 4 \quad a + 2 \quad 12] \times \begin{bmatrix} a \\ 4 \\ -1 \end{bmatrix}$

$= [\{(2a + 4) \times a\} + (a + 2)\, 4 + 12\, (-1)]$ **Ans.**

$= [2a^2 + 4a + 4a + 8 - 12] = [2a^2 + 8a - 4] = \mathbf{O} = [0]$, given

$\therefore \quad 2a^2 + 8a - 4 = 0 \quad \text{or} \quad a^2 + 4a - 2 = 0$

or $\quad a = \frac{1}{2}[-4 \pm \sqrt{(16 + 8)}] = -2 \pm \sqrt{6}.$ **Ans.**

****Ex. 12. Show that if A, B, C are matrices, such that A (BC) is defined, then (AB) C is also defined and A (BC) = (AB) C.**

Sol. Since **A (BC)** is defined so the matrices **A, B, C** are conformable to multipictions and we can take $\mathbf{A} = [a_{ij}]$, $\mathbf{B} = [b_{jk}]$ and $\mathbf{C} = [c_{kl}]$, where **A, B, C** are $m \times n$, $n \times p$, $p \times q$ matrices.

Then $\mathbf{AB} = [a_{ij}]\,[b_{jk}]$ is an $m \times p$ matrix

i.e. (i, k)th element of the product $\mathbf{AB} = \sum_{j=1}^{n} a_{ij}\, b_{jk}$ (Note)

Simmarity (j, l)th element of the product $\mathbf{BC} = \sum_{k=1}^{p} b_{jk}\, c_{kl}$ (Note)

Also (**AB**) **C** is the product of an $m \times p$ and a $p \times q$ matrices and so is conformable to multiplication, hence defined.

$\therefore$ (i, l)th element in the product of (**AB**) and **C**

= sum of products of corresponding elements in the ith row of **AB** and lth column of **C** with k common

$$= \sum_{k=1}^{p} \left[\left(\sum_{j=1}^{n} a_{ij} b_{jk} \right) c_{kl} \right] \qquad \textbf{(Note)}$$

$$= \sum_{k=1}^{p} \sum_{j=1}^{n} a_{ij} b_{jk} c_{kl} \qquad \text{...(i)}$$

Again (i, l)th element in the product of **A** and (**BC**).

= sum of products of corresponding elements in the ith row of **A** and lth column of (**BC**)

$$= \sum_{j=1}^{n} a_{ij} \sum_{k=1}^{p} b_{jk} c_{kl} \qquad \textbf{(Note)}$$

$$= \sum_{k=1}^{p} \sum_{j=1}^{n} a_{ij} b_{jk} c_{kl} \qquad \text{...(ii)}$$

$\therefore$ From (i) and (ii) we conclude that $(\mathbf{AB})\,\mathbf{C} = \mathbf{A}\,(\mathbf{BC})$.

***Ex. 13. If A and B are two matrices such that AB and A + B are both defined, then prove that A and B are square matrices.**

Sol. Let **A** be an $m \times n$ matrix.

Since $\mathbf{A} + \mathbf{B}$ is defined *i.e.* **A** and **B** are conformable to addition, so **B** must also be an $m \times n$ matrix.

Again **AB** is defined *i.e.* **A** and **B** are conformable to multiplication and hence the number of columns in **A** must be equal to number of rows in **B** *i.e.* $n = m$.

Hence **A** and **B** are $m \times m$ matrices *i.e.* square matrices.

****Ex. 14. If AB = BA then prove that $(\mathbf{AB})^n = \mathbf{A}^n\mathbf{B}^n$.**

Sol. We shall prove this by mathematical induction.

If $n = 1$, then $(\mathbf{AB})^n = \mathbf{A}^n \mathbf{B}^n \Rightarrow (\mathbf{AB})^1 = \mathbf{AB}$, which is true.

If $n = 2$, then

$(\mathbf{AB})^n = (\mathbf{AB})^2 = (\mathbf{AB})\,(\mathbf{AB})$

$= (\mathbf{ABA})\,\mathbf{B}$, by associative iaw

$= (\mathbf{AAB})\,\mathbf{B}$, $\because$ $\mathbf{BA} = \mathbf{AB}$, given

$= \mathbf{A}^2\mathbf{B}^2$.

Hence $(\mathbf{AB})^n = \mathbf{A}^n\mathbf{B}^n$ is true for $n = 2$.

Now suppose that it is true for $n = m$ *i.e.* $(\mathbf{AB})^m = \mathbf{A}^m\mathbf{B}^m$

or $(\mathbf{AB})^m\,(\mathbf{AB}) = (\mathbf{A}^m\,\mathbf{B}^m)\,(\mathbf{AB})$

or $(AB)^{m+1} = A^m (B^m A) B$, by associative law

$= A^m (B^{m-1} BA) B, \quad \because \quad B^m = B^{m-1} B$

$= A^m (B^{m-1} AB) B, \quad \because \quad BA = AB$, given

$= A^m (B^{m-2} BAB) B, \quad \because \quad B^{m-1} = B^{m-2} B$

$= A^m (B^{m-2} ABB) B, \quad \because \quad BA = AB$, given

$= A^m (B^{m-2} AB^2) B$

$= A^m (AB^{m-2} B^2) B = (A^m A)(B^{m-2} B^2 B)$

or $(AB)^{m+1} = A^{m+1} B^{m+1}$

i.e. if $(AB)^n = A^n B^n$ is true for $n = m$, it is *true* for $n = m + 1$.

Also we have proved that it is true for $n = 1$ and 2.

Hence by mathematical induction it is true for all +ve integral values of n.

***Ex. 15. If $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, prove that $A^n = I_2, A, -I_2, -A$ according as $n = 4p, 4p+1, 4p+2$ and $4p+3$ respectively.**

Sol. Given $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ ...(i)

$\therefore \quad A^2 = A \bullet A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \times \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$

$= \begin{bmatrix} i.i + 0.0 & i.0 + 0.i \\ 0.i + i.0 & 0.0 + i.i \end{bmatrix} = \begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix}$ ...(ii)

$A^3 = A^2 \bullet A = \begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix} \times \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$

$= \begin{bmatrix} i^2 . i + 0.0 & i^2 .0 + 0.i \\ 0.i + i^2 .0 & 0.0 + i^2 .i \end{bmatrix} = \begin{bmatrix} i^3 & 0 \\ 0 & i^3 \end{bmatrix}$ ...(iii)

From (ii) and (iii) we get $A^2 = \begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix}, A^3 = \begin{bmatrix} i^3 & 0 \\ 0 & i^3 \end{bmatrix}$

Let us assume that $A^n = \begin{bmatrix} i^n & 0 \\ 0 & i^n \end{bmatrix}$ ...(iv)

and also assume that (iv) is true when $n = k$.

i.e. $A^k = \begin{bmatrix} i^k & 0 \\ 0 & i^k \end{bmatrix}$ ...(v)

$\therefore \quad A^{k+1} = A^k \bullet A = \begin{bmatrix} i^k & 0 \\ 0 & i^k \end{bmatrix} \times \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, from (v) and (i)

$$= \begin{bmatrix} i^k.i + 0.0 & i^k.0 + 0.i \\ 0.i + i^k.0 & 0.0 + i^k.i \end{bmatrix} = \begin{bmatrix} i^{k+1} & 0 \\ 0 & i^{k+1} \end{bmatrix}$$

$\because$ (iv) is true for $n = k + 1$ provided (v) is true.

Also we have shown in (ii) and (iii) that (iv) is true for $n = 2$ and 3. So it is true for $3 + 1$ *i.e.* 4 and so on.

Hence (iv) is true for all positive integral values of n.

Also if $n = 4p$ then from (iv) we get

$$\mathbf{A}^n = \begin{bmatrix} i^{4p} & 0 \\ 0 & i^{4p} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ since } i^{4p} = (i^4)^p = (1)^p = 1, \text{ where } i = \sqrt{(-1)}$$

or $\mathbf{A}^n = \mathbf{I}_2$. Hence proved

If $n = 4p + 1$, then $i^n = i^{4p+1} = (i)^{4p}.i = 1.i = i$

$\therefore$ From (iv), we get $\mathbf{A}^n = \begin{bmatrix} i^n & 0 \\ 0 & i^n \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} = \mathbf{A}$. Hence proved.

If $n = 4p + 2$, then $i^n = i^{4p+2} = i^{4p} \times i^2$

$= (1)(-1)$, since $i^{4p} = 1$, $i^2 = -1$

$\therefore$ From (iv), we get

$$\mathbf{A}^n = \begin{bmatrix} i^n & 0 \\ 0 & i^n \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -\mathbf{I}_2$$

Hence proved.

If $n = 4p + 3$, then $i^n = i^{4p+3} = (i^{4p+2}).i = (-1)\,i$, as above

$= -i$

$\therefore$ From (iv), we get

$$\mathbf{A}^n = \begin{bmatrix} i^n & 0 \\ 0 & i^n \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix} = -\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} = -\mathbf{A}$$

Hence proved.

Ex. 16. Evaluate $\begin{bmatrix} \cos\theta + \sin\theta & \sqrt{2}\sin\theta \\ -\sqrt{2}\sin\theta & \cos\theta - \sin\theta \end{bmatrix}^n$

Sol. Let $\mathbf{A} = \begin{bmatrix} \cos\theta + \sin\theta & \sqrt{2}\sin\theta \\ -\sqrt{2}\sin\theta & \cos\theta - \sin\theta \end{bmatrix}$ ...(i)

Then $\mathbf{A}^2 = \mathbf{A}\bullet\mathbf{A}$

$$= \begin{bmatrix} \cos\theta + \sin\theta & \sqrt{2}\sin\theta \\ -\sqrt{2}\sin\theta & \cos\theta - \sin\theta \end{bmatrix} \begin{bmatrix} \cos\theta + \sin\theta & \sqrt{2}\sin\theta \\ -\sqrt{2}\sin\theta & \cos\theta - \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} (\cos\theta + \sin\theta)^2 - 2\sin^2\theta & (\cos\theta + \sin\theta)\sqrt{2}\sin\theta + \sqrt{2}\sin\theta(\cos\theta - \sin\theta) \\ -\sqrt{2}\sin\theta(\cos\theta + \sin\theta) - \sqrt{2}\sin\theta(\cos\theta - \sin\theta) & -\sqrt{2}\sin\theta\sqrt{2}\sin\theta + (\cos\theta - \sin\theta)^2 \end{bmatrix}$$

$$=\begin{bmatrix} (\cos^2\theta-\sin^2\theta)+2\sin\theta\cos\theta & 2\sqrt{2}\sin\theta\cos\theta \\ -2\sqrt{2}\sin\theta\cos\theta & (\cos^2\theta-\sin^2\theta)-2\cos\theta\sin\theta \end{bmatrix}$$

(Note)

or $$A^2=\begin{bmatrix} \cos 2\theta+\sin 2\theta & \sqrt{2}\sin 2\theta \\ -\sqrt{2}\sin 2\theta & \cos 2\theta-\sin 2\theta \end{bmatrix} \quad ...(ii)$$

Looking at (i) and (ii) let us assume that

$$A^n=\begin{bmatrix} \cos n\theta+\sin n\theta & \sqrt{2}\sin n\theta \\ -\sqrt{2}\sin n\theta & \cos n\theta-\sin n\theta \end{bmatrix} \quad ...(iii)$$

Let (iii) be true for $n=k$

i.e. $$A^k=\begin{bmatrix} \cos k\theta+\sin k\theta & \sqrt{2}\sin k\theta \\ -\sqrt{2}\sin k\theta & \cos k\theta-\sin k\theta \end{bmatrix} \quad ...(iv)$$

$$\therefore \quad A^{k+1}=A^k\cdot A$$

$$=\begin{bmatrix} \cos k\theta+\sin k\theta & \sqrt{2}\sin k\theta \\ -\sqrt{2}\sin k\theta & \cos k\theta-\sin k\theta \end{bmatrix}\times\begin{bmatrix} \cos\theta+\sin\theta & \sqrt{2}\sin\theta \\ -\sqrt{2}\sin\theta & \cos\theta-\sin\theta \end{bmatrix}$$

$$=\begin{bmatrix} (\cos k\theta+\sin k\theta)(\cos\theta+\sin\theta)+(\sqrt{2}\sin k\theta)(-\sqrt{2}\sin\theta) & (\cos k\theta+\sin k\theta)(\sqrt{2}\sin\theta)+(\sqrt{2}\sin k\theta)(\cos\theta-\sin\theta) \\ -\sqrt{2}\sin k\theta(\cos\theta+\sin\theta)+(\cos k\theta-\sin k\theta)(-\sqrt{2}\sin\theta) & (-\sqrt{2}\sin k\theta)(\sqrt{2}\sin\theta)+(\cos k\theta-\sin k\theta)(\cos\theta-\sin\theta) \end{bmatrix}$$

$$=\begin{bmatrix} \cos k\theta\cos\theta+\cos k\theta\sin\theta+\sin k\theta\cos\theta+\sin k\theta\sin\theta-2\sin k\theta\sin\theta & \sqrt{2}(\sin k\theta\cos\theta+\cos k\theta\sin\theta) \\ -\sqrt{2}(\sin k\theta\cos\theta+\cos k\theta\sin\theta) & -2\sin k\theta\sin\theta+\cos k\theta\cos\theta-\cos k\theta\sin\theta-\sin k\theta\cos\theta+\sin k\theta\sin\theta \end{bmatrix}$$

$$=\begin{bmatrix} \cos(k\theta+\theta)+\sin(k\theta+\theta) & \sqrt{2}\sin(k\theta+\theta) \\ -\sqrt{2}\sin(k\theta+\theta) & \cos(k\theta+\theta)-\sin(k\theta+\theta) \end{bmatrix}$$

$$=\begin{bmatrix} \cos(k+1)\theta+\sin(k+1)\theta & \sqrt{2}\sin(k+1)\theta \\ -\sqrt{2}\sin(k+1)\theta & \cos(k+1)\theta-\sin(k+1)\theta \end{bmatrix}$$

$\therefore$ (iii) is true for $n=k+1$ provided (iv) is true.

Also we have shown in (ii) that (iii) is true for $n=2$.

Hence it is true for $n=2+1$ *i.e.* 3 and so on.

Hence (iii) is true for all positive integral values of n.

Hence $$A^n=\begin{bmatrix} \cos n\theta+\sin n\theta & \sqrt{2}\sin n\theta \\ -\sqrt{2}\sin n\theta & \cos n\theta-\sin n\theta \end{bmatrix}$$ **Ans.**

***Ex. 17. If** $P(x)=\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$, **then show that**

$$P(x)\cdot P(y)=P(x+y)=P(y)\cdot P(x)$$

Sol. $\mathbf{P}(x).\,\mathbf{P}(y)$

$$=\begin{bmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{bmatrix}\times\begin{bmatrix}\cos y & \sin y\\ -\sin y & \cos y\end{bmatrix}$$

$$=\begin{bmatrix}\cos x\cos y-\sin x\sin y & \cos x\sin y+\sin x\cos y\\ -\sin x\cos y-\cos x\sin y & -\sin x\sin y+\cos x\cos y\end{bmatrix}$$

$$=\begin{bmatrix}\cos(x+y) & \sin(x+y)\\ -\sin(x+y) & \cos(x+y)\end{bmatrix}=\mathbf{P}(x+y)$$

Similarly we can prove (to be proved in the exam) that

$$\mathbf{P}(y)\bullet\mathbf{P}(x)=\mathbf{P}(x+y)$$

Hence $\mathbf{P}(x).\,\mathbf{P}(y)=\mathbf{P}(x+y)=\mathbf{P}(y).\,\mathbf{P}(x)$ Hence proved.

Ex. 18. If $\mathbf{A}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$, find number a, b so that $(\mathbf{aI}+\mathbf{bA})^2=\mathbf{A}$

Sol. $$a\mathbf{I}+b\mathbf{A}=a\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+b\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}a & 0\\ 0 & a\end{bmatrix}+\begin{bmatrix}0 & b\\ -b & 0\end{bmatrix}=\begin{bmatrix}a & b\\ -b & a\end{bmatrix}$$

$$\therefore\quad (a\mathbf{I}+b\mathbf{A})^2=\begin{bmatrix}a & b\\ -b & a\end{bmatrix}\times\begin{bmatrix}a & b\\ -b & a\end{bmatrix}$$

$$=\begin{bmatrix}a^2-b^2 & ab+ba\\ -ab-ab & -b^2+a^2\end{bmatrix}=\begin{bmatrix}a^2-b^2 & 2ab\\ -2ab & a^2-b^2\end{bmatrix}$$

$\therefore$ If $(a\mathbf{I}+b\mathbf{A})^2=\mathbf{A}$, then we have

$$\begin{bmatrix}a^2-b^2 & 2ab\\ -2ab & a^2-b^2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

Equating the corresponding elements, we have

$$a^2-b^2=0,\ 2ab=1\ \Rightarrow\ a=b=1/\sqrt{2}$$ **Ans.**

***Ex. 19. If $\mathbf{e}^{\mathbf{A}}$ is defined as $\mathbf{I}+\mathbf{A}+(\mathbf{A}^2/2\,!)+(\mathbf{A}^3/3\,!)+\ldots$, then show that $\mathbf{e}^{\mathbf{A}}=\mathbf{e}^{\mathbf{x}}\begin{bmatrix}\cosh x & \sinh x\\ \sinh x & \cosh x\end{bmatrix}$, where $\mathbf{A}=\begin{bmatrix}x & x\\ x & x\end{bmatrix}$** *(Budelkhand 95)*

Sol. Given that $\mathbf{A}=\begin{bmatrix}x & x\\ x & x\end{bmatrix}$

$$\therefore\quad \mathbf{A}^2=\begin{bmatrix}x & x\\ x & x\end{bmatrix}\times\begin{bmatrix}x & x\\ x & x\end{bmatrix}=\begin{bmatrix}x.x+x.x & x.x+x.x\\ x.x+x.x & x.x+x.x\end{bmatrix}$$

$$= 2\begin{bmatrix} x^2 & x^2 \\ x^2 & x^2 \end{bmatrix},$$

$$\mathbf{A}^3 = \mathbf{A}^2 \bullet \mathbf{A} = 2\begin{bmatrix} x^2 & x^2 \\ x^2 & x^2 \end{bmatrix}\begin{bmatrix} x & x \\ x & x \end{bmatrix} = 2\begin{bmatrix} x^2 x + x^2 x & x^2 x + x^2 x \\ x^2 x + x^2 x & x^2 x + x^2 x \end{bmatrix}$$

$$= 2^2\begin{bmatrix} x^3 & x^3 \\ x^3 & x^3 \end{bmatrix}.$$

In a similar way we can prove that

$\mathbf{A}^4 = 2^3\begin{bmatrix} x^4 & x^4 \\ x^4 & x^4 \end{bmatrix}$, $\mathbf{A}^5 = 2^4\begin{bmatrix} x^5 & x^5 \\ x^5 & x^5 \end{bmatrix}$, etc.

In general $\mathbf{A}^n = 2^{n-1}\begin{bmatrix} x^n & x^n \\ x^n & x^n \end{bmatrix}$ ...(i)

Now we are given that

$$\mathbf{e}^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + (\mathbf{A}^2/2\,!) + (\mathbf{A}^3/3\,!) + \ldots$$

or $$\mathbf{e}^{\mathbf{A}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} x & x \\ x & x \end{bmatrix} + \frac{2}{2\,!}\begin{bmatrix} x^2 & x^2 \\ x^2 & x^2 \end{bmatrix} + \frac{2^2}{3\,!}\begin{bmatrix} x^3 & x^3 \\ x^3 & x^3 \end{bmatrix} + \ldots + \frac{2^{n-1}}{n\,!}\begin{bmatrix} x^n & x^n \\ x^n & x^n \end{bmatrix}$$

$$+ \ldots\ldots = \begin{bmatrix} u & v \\ v & u \end{bmatrix},$$ ...(ii)

where $u = 1 + x + \dfrac{2x^2}{2\,!} + \dfrac{2^2 x^3}{3\,!} + \ldots + \dfrac{2^{n-1}x^n}{n\,!} + \ldots$

$v = 0 + x + \dfrac{2x^2}{2\,!} + \dfrac{2^3 x^3}{3\,!} + \ldots + \dfrac{2^{n-1}x^n}{n\,!} + \ldots$

or $u = \dfrac{1}{2}\left[2 + 2x + \dfrac{(2x)^2}{2\,!} + \dfrac{(2x)^3}{3\,!} + \ldots + \dfrac{(2x)^n}{n\,!} + \ldots\right]$

$= \dfrac{1}{2}\left[1 + \left\{1 + 2x + \dfrac{(2x)^3}{2\,!} + \dfrac{(2x)^3}{3\,!} + \ldots + \dfrac{(2x)^n}{n\,!} + \ldots\right\}\right]$

$= \dfrac{1}{2}\,[1 + e^{2x}]$

and $v = \dfrac{1}{2}\left[\left\{1 + 2x + \dfrac{(2x)^2}{2\,!} + \ldots + \dfrac{(2x)^n}{n\,!} + \ldots\right\} - 1\right]$, similarly

$= \dfrac{1}{2}\,[e^{2x} - 1]$

$\therefore$ From (ii), we get

$$\mathbf{e}^{\mathbf{A}} = \frac{1}{2}\begin{bmatrix} e^{2x}+1 & e^{2x}-1 \\ e^{2x}-1 & e^{2x}+1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} e^x(e^x+e^{-x}) & e^x(e^x-e^{-x}) \\ e^x(e^x-e^{-x}) & e^x(e^x+e^{-x}) \end{bmatrix} = e^x \begin{bmatrix} (e^x+e^{-x})/2 & (e^x-e^{-x})/2 \\ (e^x-e^{-x})/2 & (e^x+e^{-x})/2 \end{bmatrix}$$

$$= e^x \begin{bmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{bmatrix}$$

Hence proved.

EXERCISES ON CHAPTER I

Ex. 1. Given $\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$

find the matrix **C**, such that **A** + **C** = **B**. **Ans.** $\begin{bmatrix} 2 & -3 & 5 \\ -1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$

Ex. 2. If $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 0 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 & -4 \\ 1 & 5 \\ -2 & 2 \end{bmatrix}$

find **AB** and show that **AB** ≠ **BA**.

Ex. 3. Find **AB** and **BA** if

$$\mathbf{A} = \begin{bmatrix} 3 & 4 & -2 \\ -2 & -1 & -1 \\ -1 & -3 & -1 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} -1 & -1 & -1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

Ans. $\mathbf{AB} = \begin{bmatrix} 7 & 7 & 7 \\ -1 & -1 & -1 \\ -6 & -6 & -6 \end{bmatrix}$, $\mathbf{BA} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Ex. 4. If $\mathbf{A} = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

verify that **AB** = **A** and **BA** = **B**.

Ex. 4. Find **A** and **B**, where

$$\mathbf{A} + 2\mathbf{B} = \begin{bmatrix} 1 & 2 & 0 \\ 6 & -3 & 0 \\ -5 & 3 & 1 \end{bmatrix}, \; 2\mathbf{A} - \mathbf{B} = \begin{bmatrix} 2 & -1 & 5 \\ 2 & -1 & 6 \\ 0 & 1 & 2 \end{bmatrix}$$

Ans. $\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ -1 & 1 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -1 & 0 \\ -2 & 1 & 0 \end{bmatrix}$

Ex. 6. If $\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 1 & 4 \end{bmatrix}$

and $\mathbf{C} = \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix}$, prove that **A** (**BC**) = (**AB**) **C**

Ex. 7. If $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & -3 & 3 & -1 \end{bmatrix}$, show that $\mathbf{A}^2 = \mathbf{I}_4$, where $\mathbf{I}_4$ is 4×4 identity matrix.

Ex. 8. For two matrices **A** and **B**, state the conditions under which (i) $\mathbf{A} = \mathbf{B}$; (ii) **AB** exists and (iii) $(\mathbf{A} + \mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$.

Ex. 9. State true or false in the case of the following statement. Justify your answer.

If **A** and **B** are conformable for addition, then

$$(\mathbf{A} + \mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2.$$

Ex. 10. If $\mathbf{A} = \begin{bmatrix} 3 & -4 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$, then find $[\mathbf{AB}]^2$.

Ex. 11. What is the difference between zero matrix and a unit matrix ?

[Hint : See § 1·03 Page 4]

Ex. 12. Find non-zero matrices **A** and **B** of order 3×3 such that $\mathbf{AB} = \mathbf{O}$, where **O** is the zero matrix of order 3×3.

[Hint : See Ex. 1 (c) Page 14 or Ex. 7 Page 17]