Chapter V

Rank and Adjoint of a Matrix

§ 5·01. **Order of a minor.**

Definition. If any r rows and any r columns from an $m \times n$ matrix **A** are retained and remaining $(m-r)$ rows and $(n-r)$ columns removed, then the determinant of the remaining $r \times r$ submatrix of **A** is called **minor of A of order** r.

For example : In the matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{51} & a_{52} & a_{53} & a_{54} \end{bmatrix}$$

elements a_{11}, a_{12}, a_{31}, etc. are minors of order unity ;

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \begin{vmatrix} a_{33} & a_{34} \\ a_{53} & a_{54} \end{vmatrix} \text{ etc.}$$

are minors of order 2 ;

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{41} & a_{43} & a_{44} \\ a_{51} & a_{53} & a_{54} \end{vmatrix} \text{ etc.}$$

are minors of order 3 ;

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{51} & a_{52} & a_{53} & a_{54} \end{bmatrix}, \begin{bmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{51} & a_{52} & a_{53} & a_{54} \end{bmatrix} \text{ etc.}$$

are minors of order 4.

Note. In the above example there cannot be any minor of order higher than 4.

§ 5·02. **Rank of a matrix.

Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$

(Avadh 97; Garhwal 90; Gorakhpur 98; Lucknow 91)

This matrix **A** has only one three-rowed minor *i.e.* minor of order 3, viz. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{vmatrix}$ and its value can easily be calculated to be zero, by expanding with respect to first row.

This matrix **A** has 9 minors of order 2 (or two-rowed minors) and one of them is $\begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix}$ which has the value

$$(3 \times 7) - (5 \times 4) = 21 - 20 = 1 \neq 0.$$

This fact that **A** is a matrix whose every minor of order 3 is zero and there **is at least** one minor of order 2 which is not equal to zero is also expressed as '**the rank** of the matrix **A** is 2'.

****Definition of Rank of a Matrix :**

(Avadh 92 ; Bundelkhand 96, 95, 94; Purvanchal 98, 96; Rohilkhand 92)

If in an $m \times n$ matrix **A**, *at least one of its $r \times r$ minors is different from zero while all the minors of order $(r + 1)$ are zero, then r is defined as the rank of the matrix* **A**.

A number r is defined as the rank of an $m \times n$ matrix **A** provided

(i) **A** has at least one minor of order r which does not vanish and (ii) there is no minor of order $(r + 1)$ which is not equal to zero.

Note 1. The rank of a matrix **A** is also denoted by ρ (**A**).

***Note 2.** The rank of a zero matrix by definition is 0 *i.e.* $\rho(\mathbf{O}) = 0$.

Note 3. The rank of a matrix remains unaltered by the application of elementary row or column operations *i.e.* all equivalent matrices have the same rank.

****Note 4.** From the definition of rank of a matrix we conclude that :—

(a) If a matrix **A** does not possess any minor of order $(r + 1)$ then $\rho(\mathbf{A}) \le r$.

(b) If at least one minor of order r of the matrix **A** is not equal to zero, then $\rho(\mathbf{A}) \ge r$.

Note 5. If every minor of order p of a matrix **A** is zero then every minor of order higher than p is definitely zero.

Solved Examples on § 5·02.

***Ex. 1 (a). Find the rank of the matrix** $A = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & \mathbf{5} & \mathbf{8} \\ \mathbf{4} & \mathbf{10} & \mathbf{18} \end{bmatrix}$ *(Gorakhpur 92)*

Sol. The determinant of order 3 formed by **A**

$$= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 4 & 10 & 18 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 4 & 2 & 6 \end{bmatrix}, \text{ replacing } C_2, C_3, \text{ by } C_2 - 2C_1, C_3 - 3C_1 \text{ respectively.}$$

$$= \begin{vmatrix} 1 & 2 \\ 2 & 6 \end{vmatrix} = 6 - 4 = 2 \neq 0$$

$\therefore$ $$\rho(\mathbf{A}) \ge 3. \qquad \text{...(i)}$$

Also the matrix **A** does not possess any minor of order 4 *i.e.* 3 + 1, so

$$\rho(\mathbf{A}) \le 3 \qquad \text{...(ii)}$$

$\therefore$ From (i) and (ii) we get $\rho(\mathbf{A}) = 3$. **Ans.**

Ex. 1 (b). Find the rank of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 10 & 18 \end{bmatrix}$

Sol. Do as Ex. 1 (a) above. **Ans. 3**

Ex. 1 (c). Find the rank of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 2 & 3 \\ 1 & 5 & 4 \end{bmatrix}$

Hint : Do as Ex. 1 (a) above. **Ans. 3**

Ex. 2 (a). Determine the rank of $\mathbf{A} = \begin{bmatrix} 6 & 1 & 8 & 3 \\ 2 & 1 & 0 & 2 \\ 4 & -1 & -8 & -3 \end{bmatrix}$

Sol. The given matrix **A** possesses a minor of order 3 viz.

$$\begin{vmatrix} 6 & 1 & 8 \\ 2 & 1 & 0 \\ 4 & -1 & -8 \end{vmatrix} = \begin{vmatrix} 10 & 0 & 0 \\ 6 & 0 & -8 \\ 4 & -1 & -8 \end{vmatrix}, \text{ replacing } R_1, R_2 \text{ by } R_1 + R_3, R_2 + R_3$$

$$= 10 \begin{vmatrix} 0 & -8 \\ -1 & -8 \end{vmatrix} = 10\,(0 - 8) = -80 \neq 0$$

$\therefore$ $\rho\,(\mathbf{A}) \geq 3.$ ...(i)

Also **A** does not possess any minor or order 4 *i.e.* 3 + 1, so

$\rho\,(\mathbf{A}) \leq 3.$ ...(ii)

$\therefore$ From (i) and (ii), we get $\rho\,(\mathbf{A}) = 3.$ **Ans.**

Ex. 2 (b). Find the rank of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 5 & 1 \\ 2 & 4 & 8 & 0 \\ 3 & 1 & 7 & 5 \end{bmatrix}$$

Hint : Do as Ex. 2 (a) above. **Ans. 3.**

***Ex. 3 (a). Find the rank of matrix** $\mathbf{A} = \begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{bmatrix}$ *(Kanpur 95)*

Sol. The given matrix **A** possesses a minor of order 3

viz. $$\begin{vmatrix} 1 & 3 & 6 \\ 1 & -3 & -4 \\ 5 & 3 & 11 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 2 \\ 1 & -3 & -4 \\ 6 & 0 & 7 \end{vmatrix}, \text{ replacing } R_1, R_3 \text{ by } R_2 + R_1, R_3 + R_2$$

$$= -3 \begin{vmatrix} 2 & 2 \\ 6 & 7 \end{vmatrix} = -3\,(14 - 12) = -6 \neq 0$$

$\therefore$ $\rho\,(\mathbf{A}) \geq 3.$ ...(i)

Also **A** does not possess any minor of order 4 *i.e.* 3 + 1, so

$\rho\,(\mathbf{A}) \leq 3$ ...(ii)

$\therefore$ From (i) and (ii) we get $\rho\,(\mathbf{A}) = 3.$ **Ans.**

Ex. 3 (b). Find the rank of the matrix $\mathbf{A}=\begin{bmatrix}1&6&8\\2&5&3\\7&9&4\end{bmatrix}$ *(Purvanchal 96)*

Sol. The determinant of order 3 formed by **A**

$$=\begin{vmatrix}1&6&8\\2&5&3\\7&9&4\end{vmatrix}=\begin{vmatrix}1&6&8\\0&-7&-13\\0&-33&-52\end{vmatrix}$$, replacing R_2, R_3 by R_2-2R_1, R_3-7R_1 repectively

$$=\begin{vmatrix}-7&-13\\-33&-52\end{vmatrix}=\begin{vmatrix}-7&-13\\-5&0\end{vmatrix}$$, replacing R_2 by R_2-4R_1

$=-65\neq 0$

$\therefore\quad \rho(\mathbf{A})\geq 3.$ ...(i)

Also **A** does not possess any minor of order 4 *i.e.*3 + 1, so

$$\rho(\mathbf{A})\leq 3$$ ...(ii)

$\therefore$ From (i) and (ii), we get $\rho(A)=3$ **Ans.**

Ex. 3 (c). Find the rank of the matrix $\mathbf{A}=\begin{bmatrix}2&3&8\\5&0&6\\8&9&10\end{bmatrix}$ *(Purvanchal 94)*

Sol. The determinant of order 3 formed by the matrix **A**

$$=\begin{vmatrix}2&3&8\\5&0&6\\8&9&10\end{vmatrix}=\begin{vmatrix}2&3&8\\5&0&6\\2&0&-14\end{vmatrix}$$, replacing R_3 by R_3-3R_1

$$=-3\begin{vmatrix}5&6\\2&-14\end{vmatrix}$$, expanding w.r. to C_2

$=-3(-70-12)=3\times 82=246\neq 0$

$\therefore\quad \rho(\mathbf{A})\geq 3$ ...(i)

Also **A** does not possess any matrix of order 4 *i.e.* 3 + 1 and

$\rho(\mathbf{A})\leq 3.$ ...(ii)

$\therefore$ From (i) and (ii) we get $\rho(\mathbf{A})=3$ **Ans.**

Ex. 4 (a). Find the rank of the matrix $\mathbf{A}=\begin{bmatrix}1&2&3\\2&3&1\\-2&-3&-1\end{bmatrix}$

Sol. The determinant of order 3 formed by this matrix **A**

$$=\begin{vmatrix}1&2&3\\2&3&1\\-2&-3&-1\end{vmatrix}=\begin{vmatrix}1&2&3\\2&3&1\\0&0&0\end{vmatrix}$$, replacing R_3 by R_3+R_2

$=0.$

Also there exists a minor of order 2 of **A**.

viz. $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1 \neq 0$

Hence the rank of the given matrix **A** is 2. **Ans.**

Ex. 4 (b) Find the rank of matrix $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{4} \\ \mathbf{2} & \mathbf{6} & \mathbf{8} \end{bmatrix}$.

Sol. A minor of order 2 formed by this matrix.

$= \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 6 - 6 = 0$. Similarly all minors of order 2 are zero.

Now we are left with minors of order 1 *i.e.* elements of **A which are** not equal to zero.

Hence the rank of the given matrix **A** is 1. **Ans.**

****Ex. 5. Find the rank of the matrix.**

$$\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{1} \\ \mathbf{2} & \mathbf{4} & \mathbf{6} & \mathbf{2} \\ \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{2} \end{bmatrix}$$

(Gorakhpur 96)

Sol. In this matrix, a minor of order 3

$= \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{vmatrix} = 0$, R_1 and R_3 are identical

In a similar way we prove that all the minors of order 3 are zero.

Now a minor of order 2 = $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$.

But another minor of order 2 = $\begin{vmatrix} 3 & 1 \\ 3 & 4 \end{vmatrix} \neq 0$,

Hence rank of the given matrix is 2. **Ans.**

Ex. 6. Find the rank of the matrix $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-3} & \mathbf{2} \\ \mathbf{3} & \mathbf{-9} & \mathbf{6} \\ \mathbf{-2} & \mathbf{6} & \mathbf{-4} \end{bmatrix}$

Sol. The determinant of order 3 formed by this matrix **A**.

$= \begin{vmatrix} 1 & -3 & 2 \\ 3 & -9 & 6 \\ -2 & 6 & -4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ -2 & 0 & 0 \end{vmatrix}$ replacing C_2, C_3 by $C_2 + 3C_1$ and $C_3 - 2C_3$ respectively.

$= 0$.

Also there exists no minor of order 2 of **A** which is not **equal to zero.** (Students can verify for themselves).

Finally all minors of order 1 of the matrix **A** are non-zero, as no **element** of the matrix **A** is 0.

Hence the rank of **A** is 1. **Ans.**

Ex. 7. Find the rank of the matrix

$$\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ -1 & -3 & -4 & -3 \end{bmatrix}$$

Sol. In this matrix, a minor of order 3

$$= \begin{vmatrix} 1 & 3 & 4 \\ 3 & 9 & 12 \\ -1 & -3 & -4 \end{vmatrix} = 3 \begin{vmatrix} 1 & 3 & 4 \\ 1 & 3 & 4 \\ -1 & -3 & -4 \end{vmatrix}, \text{ taking 3 common from } R_2$$

$= 0$, as R_1 and R_2 are identical.

In a similar way we can prove that all minors of order 3 are zero.

Now a minor of order 2

$$= \begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} = 3 \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix}, \text{ taking out 3 common from } R_2$$

$= 0$, as rows are identical.

Similarly all the minors of order 2 are zero.

Hence we are left with minors of order unity, viz. the elements of the given matrix, which are not equal to zero.

Hence rank of the given matrix = 1. **Ans.**

Ex. 8 (a). Find the rank of the matrix

$$\mathbf{A} = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 16 & 4 & 12 & 15 \\ 5 & 3 & 3 & 4 \\ 4 & 2 & 6 & -1 \end{bmatrix}$$

(Kanpur 96)

Sol. The determinant of order 4 formed by **A**

$$= \begin{bmatrix} 6 & 1 & 3 & 8 \\ 16 & 4 & 12 & 15 \\ 5 & 3 & 3 & 4 \\ 4 & 2 & 6 & -1 \end{bmatrix} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -8 & 4 & 0 & -17 \\ -13 & 3 & -6 & -20 \\ -8 & 2 & 0 & -17 \end{vmatrix},$$

, replacing C_1, C_3, C_4 by $C_1 - 6C_2, C_3 - 3C_2$ and $C_4 - 8C_3$ respectively

$$= - \begin{vmatrix} -8 & 0 & -17 \\ -13 & -6 & -20 \\ -8 & 0 & -17 \end{vmatrix} = 0, \text{ as } R_1, R_3 \text{ are indentical.}$$

Also one minor of order 3 *viz.*

$$\begin{vmatrix} 1 & 3 & 8 \\ 3 & 3 & 4 \\ 2 & 6 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 3 & -6 & -20 \\ 2 & 0 & -17 \end{vmatrix} = \begin{vmatrix} -6 & -20 \\ 0 & -17 \end{vmatrix} \neq 0.$$

Hence the rank of given matrix **A** is 3. **Ans.**

Ex. 8 (b). Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 9 & 3 & 4 \\ 3 & 7 & 4 & 6 \end{bmatrix}$$

(Garhwal 93)

Sol. The determinant of order 4 formed by **A**

$$= \begin{vmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 9 & 3 & 4 \\ 3 & 7 & 4 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{vmatrix}$$, replacing R_2, R_3, R_4 by $R_2 - R_1, R_3 - 2R_1,$ $R_4 - 3R_1$ respectively

$= 0$, R_2, R_4 being identical

Also one minor of order 3 viz.

$$\begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 3 & 7 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0, \text{ as above.}$$

But all minor of order 3 are not zero.

e.g. $$\begin{vmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 9 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 2 \\ -1 & 0 & -2 \\ 3 & 0 & -2 \end{vmatrix} = -\begin{vmatrix} -1 & -2 \\ 3 & -2 \end{vmatrix}$$

$$= -[2 + 6] = -8 \neq 0$$

Hence the rank of the given matrix **A** is 3. **Ans.**

Ex. 9 (a). Find the rank of the matrix

$$A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 3 & -3 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

Sol. The determinant of order 4 formed by **A**

$$= \begin{vmatrix} 1 & -2 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 3 & -3 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 2 & -1 & 1 & 1 \\ -2 & 1 & -1 & 1 \end{vmatrix}$$ replacing C_1, C_2, by $C_1 - C_4$ and $C_2 + 2C_4$ respectively

$$= -\begin{vmatrix} 2 & -1 & 1 \\ 2 & -1 & 1 \\ -2 & 1 & -1 \end{vmatrix}$$, expanding w.r. to R_1

$= 0$, R_1, R_2 being identical.

Also one minor of order 3 viz.

$$\begin{vmatrix} 1 & -2 & 1 \\ 2 & -1 & 0 \\ 3 & -3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & -2 \\ 3 & 3 & -2 \end{vmatrix} = 0$$

Similarly all minors of order 3 are zero

Now one minor of order 2 viz $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$

$\therefore$ Rank of given matrix **A** is 2. **Ans.**

****Ex. 9 (b). Find the rank of the matrix**

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 2 \end{bmatrix}$$

(Avadh 92; Bundelkhand 92; Gorakhpur 93; Rohilkhand 98)

Hint : Do exactly as Ex. 9 (a) above. **Ans. 2**

Ex. 9 (c). Find the rank of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{bmatrix}$$

Hint : Do as Ex. 9 (a) above. **Ans. 2**

Ex. 10. Find the rank of the matrix

$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \\ 2 & 3 & -1 & -1 \end{bmatrix}$$

Sol. The determinant of order 4 formed by the given matrix

$$= \begin{vmatrix} 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \\ 2 & 3 & -1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 3 & 4 & 9 & 10 \\ 6 & 9 & 12 & 17 \\ 2 & 5 & 3 & 7 \end{vmatrix}$$, replacing C_2, C_3, C_4 by $C_2 + C_1$ $C_3 + 2C_1, C_4 + 4C_1$ respectively

$$= \begin{vmatrix} 4 & 9 & 10 \\ 9 & 12 & 17 \\ 5 & 3 & 7 \end{vmatrix} = \begin{vmatrix} 4 & 9 & 10 \\ 5 & 3 & 7 \\ 5 & 3 & 7 \end{vmatrix}$$, replacing R_2 by $R_2 - R_1$

$= 0$, as its two rows are identical.

A minor of order 3

$$= \begin{vmatrix} 1 & -1 & -2 \\ 3 & 1 & 3 \\ 6 & 3 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 4 & 9 \\ 6 & 9 & 12 \end{vmatrix}$$, replacing C_2, C_3 by $C_2 + C_1, C_3 + 2C_1$ respectivley

$$= \begin{vmatrix} 4 & 9 \\ 9 & 12 \end{vmatrix} = 48 - 81 = -33 \neq 0$$

Hence the rank of the given matrix is 3. **Ans.**

****Ex. 11. Find the rank of the matrix**

$$\mathbf{A} = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$$

(Avadh 90; Kumaun 90)

Sol. The determinant of order 4 formed by this matrix

$$= \begin{vmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{vmatrix}$$

$$= \begin{vmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 6 & 1 & 3 & 8 \\ 6 & 1 & 3 & 8 \end{vmatrix}$$, replacing R_3 and R_4 by $R_3 - R_2$ and $R_4 - R_3$ respectively.

$= 0$, as its three rows are identical

A minor of order 3

$$= \begin{vmatrix} 6 & 1 & 3 \\ 4 & 2 & 6 \\ 10 & 3 & 9 \end{vmatrix} = \begin{vmatrix} 6 & 1 & 3 \\ 4 & 2 & 6 \\ 6 & 1 & 3 \end{vmatrix}$$, replacing R_3 by $R_3 - R_2$

$= 0$, two rows being identical.

In a similar way we can prove that all the minors of order 3 are zero.

Now a minor of order 2 $= \begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix} = 12 - 4 = 8 \neq 0$

Hence the rank of the given martix = 2. **Ans.**

Ex. 12. Find the rank of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 1 \end{bmatrix}$$

(Gorakhpur 94)

Sol. One minor of order three of **A**

$$= \begin{vmatrix} 1 & 4 & 5 \\ 1 & 6 & 7 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 2 \\ 1 & -4 & -4 \end{vmatrix}$$, replacing C_2, C_3 by $C_2 - 4C_1$ and $C_3 - 5C_1$ respectively.

$$= \begin{vmatrix} 2 & 2 \\ -4 & -4 \end{vmatrix}$$, expending with respect to R_1

$= 2(-4) - 2(-4) = 0.$

In a similar way we can prove that all the minors of order three are zero.

Now a minor of order 2 is $\begin{vmatrix} 2 & 6 \\ 5 & 0 \end{vmatrix} = 2 \cdot 0 - 6 \cdot 5 = -30 \neq 0$

Hence the rank of **A** is 2. **Ans.**

Ex. 13. Find the rank of the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{a} & \mathbf{b} & \mathbf{0} \\ \mathbf{0} & \mathbf{c} & \mathbf{d} & \mathbf{1} \\ \mathbf{1} & \mathbf{a} & \mathbf{b} & \mathbf{0} \\ \mathbf{0} & \mathbf{c} & \mathbf{d} & \mathbf{1} \end{bmatrix}$$

Sol. $|\mathbf{A}| = \begin{vmatrix} 1 & a & b & 0 \\ 0 & c & d & 1 \\ 1 & a & b & 0 \\ 0 & c & d & 1 \end{vmatrix} = 0, \because$ R_1, R_3 are identical

A minor of order 3 of **A**

$$= \begin{vmatrix} a & b & 0 \\ c & d & 1 \\ a & b & 0 \end{vmatrix} = 0, \text{ as } R_1, R_3 \text{ are identical}$$

In a similar way we can show that all the minors of order 3 are zero in value.

A minor of order 2 of $\mathbf{A} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$

Hence the rank of the matrix **A** is 2. **Ans.**

Ex. 14. Find the rank of the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{7} \\ \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{7} & \mathbf{8} \\ \mathbf{5} & \mathbf{6} & \mathbf{7} & \mathbf{8} & \mathbf{9} \\ \mathbf{10} & \mathbf{11} & \mathbf{12} & \mathbf{13} & \mathbf{14} \\ \mathbf{15} & \mathbf{16} & \mathbf{17} & \mathbf{18} & \mathbf{19} \end{bmatrix}$$

(Kumaun 91)

Sol. One minor of order 3 of **A**

$$= \begin{vmatrix} 5 & 7 & 8 \\ 6 & 8 & 9 \\ 16 & 18 & 19 \end{vmatrix} = \begin{vmatrix} 5 & 7 & 8 \\ 1 & 1 & 1 \\ 11 & 11 & 11 \end{vmatrix}, \text{ replacing } R_2, R_3 \text{ by } R_2 - R_1 \text{ and } R_3 - R_1 \text{ respectively.}$$

$$= \begin{vmatrix} 5 & 7 & 8 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix}, \text{ replacing } R_3 \text{ by } R_3 - 11R_2$$

$= 0.$

In a similar way we can prove that all the minors of order 3 of **A** are zero. This shows that all minors of order 4 and $|\mathbf{A}|$ of **A** are automatically zero.

(See Note 5 Page 2 of this chapter)

Now one minor of order 2 of **A**

$$= \begin{vmatrix} 7 & 8 \\ 8 & 9 \end{vmatrix} = (7 \times 9) - (8 \times 8) = 63 - 64 = -1 \neq 0$$

Hence the rank of **A** is 2. **Ans.**

Ex. 15. Find the rank of $\mathbf{A}=\begin{bmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ bc & ca & ab \end{bmatrix}$ *(Kanpur 91)*

Sol. $|\mathbf{A}|=\begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ bc & ca & ab \end{vmatrix}$

$= -(a-b)(b-c)(c-a)$, on evaluating. ...(i)

Now following cases arise :—

Case I. $a=b=c$.

If $a=b=c$, then $\mathbf{A}=\begin{bmatrix} 1 & 1 & 1 \\ 2a & 2a & 2a \\ a^2 & a^2 & a^2 \end{bmatrix}$.

Therefore all minors of order 2 and 3 of **A** vanish.

Also **A** has non-zero minor of order 1, since no element of **A** is zero.

Hence the rank of **A** in this case is 1. **Ans.**

Case II. Two of numbers a, b, c **are equal, but are different from the third.**

Let $a=b\neq c$.

Then $|\mathbf{A}|=\begin{vmatrix} 1 & 1 & 1 \\ a+c & c+a & 2a \\ ac & ca & a^2 \end{vmatrix}=0$, as C_1, C_2 are identical.

Also **A** has a minor of order 2 viz. $\begin{vmatrix} 1 & 1 \\ a+c & 2a \end{vmatrix}$

$=2a-(a+c)=a-c\neq 0, \quad \because a\neq c.$

Hence the rank of **A** in this case is 2.

Similarly we can discuss the cases $b=c\neq a$, $c=a\neq b$. **Ans.**

Case III. a, b, c **are all different.**

In this case $|\mathbf{A}|\neq 0$, as is evident from (i) above.

i.e. **A** has a non-zero minor of order 3 and there exists no minor of order greater than 3.

Hence the rank of **A** in this case in 3. **Ans.**

****Ex. 16. Find the rank of the matrix**

$\mathbf{A}=\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{bmatrix}$, **where** a, b, c **are all real.** *(Rohilkhand 97)*

Sol.

$|\mathbf{A}|=\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}=\begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix}$, replacing C_2, C_3 by C_2-C_1, C_3-C_1

$$= \begin{vmatrix} b-a & c-a \\ b^3-a^3 & c^3-a^3 \end{vmatrix}$$, expanding with respect to R_1

$$= (b-a)(c-a) \begin{vmatrix} 1 & 1 \\ b^2+ab+a^2 & c^2+ca+a^2 \end{vmatrix},$$

taking $(b-a)$, $(c-a)$ common from C_1 and C_2

$$= (b-a)(c-a) \begin{vmatrix} 1 & 0 \\ b^2+ab+a^2 & c^2+ca-b^2-ab \end{vmatrix},$$

replacing C_2 by $C_2 - C_1$

$$= (b-a)(c-a)[(c^2+ca-b^2-ab) - 0]$$

$$= (b-a)(c-a)[(c^2-b^2)+a(c-b)] \qquad \textbf{(Note)}$$

$$= (b-a)(c-a)[(c-b)(c+b+a)]$$

or $|\mathbf{A}| = (a-b)(b-c)(c-a)(a+b+c)$ ...(i)

Now following cases arise :

Case I. $a = b = c$, **then** $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ a & a & a \\ a^3 & a^3 & a^3 \end{bmatrix}$

Therefore all minors of order 3 and 2 of **A** are zero.

Also as no element of **A** is zero, so **A** has non-zero minors of order 1.

Hence in this case the rank of **A** is 1. **Ans.**

Case II. Two of the numbers a, b, c are equal but are different from the third.

Let $a = b \neq c$.

Then $|\mathbf{A}| = \begin{vmatrix} 1 & 1 & 1 \\ a & a & a \\ a^3 & a^3 & a^3 \end{vmatrix} = 0$, as C_1 and C_2 are identical.

Also **A** has a minor of order 2, viz. $\begin{vmatrix} 1 & 1 \\ a & c \end{vmatrix} = c - a \neq 0 \quad \because a \neq c$

Hence in this case the rank of **A** is 2. **Ans.**

Similarly we can discuss the cases $b = c \neq a$, $c = a \neq b$.

Case III. a, b, c are all different but $a + b + c = 0$.

In this case from (i), it is evident that $|\mathbf{A}| = 0$. **(Note)**

Also **A** has a minor of order 2, viz $\begin{vmatrix} 1 & 1 \\ a & b \end{vmatrix} = b - a \neq 0, \because a \neq b$

Hence in this case the rank of **A** is 2. **Ans.**

Case IV. a, b, c are all different but $a + b + c \neq 0$.

In this case from (i), it is evident that $|\mathbf{A}| \neq 0$. **(Note)** *i.e.* **A** has a non-zero minor of order 3.

Also **A** has no minor of order greater than 3.

Hence in this case the rank of **A** is 3. **Ans.**

****Ex. 17. Prove that the points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are collinear if the rank of the matrix is $\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$ is less than 3.** *(Agra 91; Kanpur 95, 93)*

Sol. If the rank of the given matrix is less than 3, then the minor of order 3 of this matrix must be zero. (See § 5·02 Page 1 of this chapter)

i.e.
$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \quad \text{...(i)}$$

Now the area of triangle whose vertices are (x_1, y_1), (x_2, y_2), and (x_3, y_3)

$$= \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$ **(See Authors Co-ordinate Geometry)**

$= 0$, from (i).

Since the area of this triangle is zero, so its vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear. Hence proved.

Ex. 18. Under what conditions the rank of the following matrix A is 3? Is it possible for the rank to be 1 ? Why ?

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 2 \\ 3 & 1 & 2 \\ 1 & 0 & x \end{bmatrix}$$ *(Kanpur 94)*

Sol. If the rank of the matrix **A** is 3, then the minor of order 3 of **A** should be non-zero.

i.e.
$$\begin{vmatrix} 2 & 4 & 3 \\ 3 & 1 & 2 \\ 1 & 0 & x \end{vmatrix} \neq 0, \text{ which is the required condition.}$$

Also the rank of **A** can not be 1 as at least one minor of order 1 of **A** *i.e.*, one element of **A** is zero.

If we are to find the condition under which the rank of **A** is 2, then the same is $|\mathbf{A}| = 0$ *i.e.* minor of order 3 of **A** must be zero.

i.e.
$$\begin{vmatrix} 2 & 4 & 2 \\ 3 & 1 & 2 \\ 1 & 0 & x \end{vmatrix} = 0, \text{ i.e. } \begin{vmatrix} 2 & 4 & 2 \\ 1 & -3 & 0 \\ 1 & 0 & x \end{vmatrix} = 0, \text{ replacing } R_2 \text{ by } R_2 - R_1$$

i.e.
$$\begin{vmatrix} 0 & 10 & 2 \\ 1 & -3 & 0 \\ 1 & 0 & x \end{vmatrix} = 0, \text{ replacing } R_1 \text{ by } R_1 - 2R_2$$

i.e. $\begin{vmatrix} 10 & 2 \\ -3 & 0 \end{vmatrix} - 0 + x \begin{vmatrix} 0 & 10 \\ 1 & -3 \end{vmatrix} = 0$, expanding with respect to R

i.e. $6 - 10x = 0$ *i.e.* $x = 6/10 = 3/5$. **Ans.]**

Ex. 19. Are the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 7 & 9 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 1 & 0 & -5 & 6 \\ 3 & -2 & 1 & 2 \\ 5 & -2 & -9 & 14 \\ 4 & -2 & -4 & 8 \end{bmatrix} \text{ equivalent ?}$$

Sol. Since **A** is a 3×3 matrix and **B** is a 4×4 matrix *i.e.* their dimensions are different, so these can not be equivalent.

Exercises on § 5·02

Find the rank of the following matrices :

Ex. 1. (a) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 4 & 10 & 18 \end{bmatrix}$, **(b)** $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ **(c)** $\begin{bmatrix} 4 & 5 & 6 \\ 5 & 6 & 7 \\ 7 & 8 & 9 \end{bmatrix}$

Ans. (a) 3; (b) 3; (c) 2.

Ex. 2. $\begin{bmatrix} 3 & 11 & 1 & 5 \\ 5 & 13 & -1 & 11 \\ -2 & 2 & 4 & -8 \end{bmatrix}$ **Ans. 2**

Ex. 3. $\begin{bmatrix} 1 & 2 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ 1 & 3 & 4 & 1 \end{bmatrix}$. **Ans. 2**

Ex. 4. $\begin{bmatrix} 13 & 16 & 19 \\ 14 & 17 & 20 \\ 15 & 18 & 21 \end{bmatrix}$ **Ans. 2**

Ex. 5. $\begin{bmatrix} 1 & 0 & -5 & 6 \\ 3 & -2 & 1 & 2 \\ 5 & -2 & -9 & 14 \\ 4 & -2 & -4 & 8 \end{bmatrix}$ *(Lucknow 90)* **Ans. 2**

Ex. 6. $\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}$, where a, b, c are all real. *(Kanpur 90)*

[Hint : See Ex. 16 Page 11 of this chapter]

Ex. 7. $\begin{bmatrix} 0 & c & -b & \alpha \\ -c & 0 & a & \beta \\ b & -a & 0 & \gamma \\ -\alpha & -\beta & -\gamma & 0 \end{bmatrix}$, where a, b, c are, all positive numbers and $a\alpha + b\beta + c\gamma = 0$. **Ans. 2**

Ex. 8. $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 3 & 0 & 3 \\ 1 & -2 & -3 & -3 \\ 1 & 1 & 2 & 3 \end{bmatrix}$ **Ans. 3**

(Agra 96)

§ 5·03. Normal Form of a Matrix.

Every non-zero matrix **A** of order $m \times n$ can be reduced by application of elementary row and column operations into equivalent matrix of one of the following forms :

(i) $\begin{bmatrix} \mathbf{I_r} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$, (ii) $\begin{bmatrix} \mathbf{I_r} \\ \mathbf{O} \end{bmatrix}$, (iii) $[\mathbf{I_r}, \mathbf{O}]$, (iv) $[\mathbf{I_r}]$,

where $\mathbf{I_r}$ is $r \times r$ identity matrix and **O** is null matrix of any order.

These four forms are called **Normal** or **canonical form** of **A**.

Important Theorem (without Proof). *(Avadh 94)*

Th. I. If $m \times n$ matrix **A** is reduced to the canoncial form or normal form $\begin{bmatrix} \mathbf{I_r} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$ by the application of elementary row or column operations, then r, the order of the identity sub-matrix $\mathbf{I}_r$ is the rank of the matrix **A**

Th. II. If a non-singular matrix of order $n \times n$ is reduced to the identity matrix $\mathbf{I_n}$ (which is its canonical or normal form), then the rank of the matrix is n.

Solved Examples on § 5·03.

Ex. 1 (a). Find rank of the matrix $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{4} & \mathbf{5} & \mathbf{6} \end{bmatrix}$ *(Gorakhpur 95)*

Sol. $\mathbf{A} \sim \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 4 & 5 & 1 \end{bmatrix}$, replacing C_3 by $C_3 - C_2$

$\sim \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 4 & 1 & 1 \end{bmatrix}$, replacing C_2 by $C_2 - C_1$

$\sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 3 & 1 & 0 \end{bmatrix}$, replacing C_3 by $C_3 - C_2$ and C_1 by $C_1 - C_2$

$\sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$, replacing R_3 by $R_3 - R_2$

$\sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$, replacing R_2 by $R_2 - R_1$

$$\sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ interchanging } R_1 \text{ and } R_2$$

$$\sim \begin{bmatrix} \mathbf{I}_2 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$$

Hence the rank of **A** is 2. **Ans.**

Ex. 1 (b). Find the rank of the matrix $\mathbf{A} = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 4 & 6 \\ 0 & 6 & 9 \end{bmatrix}$

Sol. $\mathbf{A} \sim \begin{bmatrix} 0 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & 6 & 3 \end{bmatrix}$, replacing C_3 by $C_3 - C_2$

$$\sim \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}, \text{ replacing } C_2 \text{ by } C_2 - 2C_3$$

$$\sim \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ replacing } R_2, R_3 \text{ by } R_2 - 2R_1, R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ interchanging } C_1 \text{ and } C_3$$

$$\sim \begin{bmatrix} \mathbf{I}_1 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$$

Hence the rank of **A** is 1. Ans.

***Ex. 1 (c). Find the rank of the matrix** $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

Hint : Do as Ex. 1 (b) above. Replace C_2, C_3 by $C_2 - C_1$, $C_3 - C_1$ respectively and then from the result so obtained replace R_2, R_3 by $R_2 - 2R_1$, $R_3 - 3R_1$ respectively. Ans. 1.

Ex. 1 (d). Find the rank of the matrix.

$$\mathbf{A} = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$$

Hint : Do Ex. 1 (b) above. **Ans. 1.**

Ex. 2. Find the rank of tthe matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 2 \end{bmatrix}$$

Sol. $A \sim \begin{bmatrix} 1 & 0 & 0 \\ 4 & -3 & -6 \\ 2 & -3 & -4 \end{bmatrix}$, replacing C_2, C_3 by $C_2 - 2C_1$, $C_3 - 3C_1$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & -6 \\ 0 & -3 & -4 \end{bmatrix}$, replacing R_2, R_3 by $R_2 - 4R_1$, $R_3 - 2R_1$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -3 & -4 \end{bmatrix}$, replacing R_2 by $R_2 - R_3$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, replacing C_2, C_3 by $-\frac{1}{3} C_2$ $-\frac{1}{2} C_3$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, replacing R_3 by $R_3 - 2R_2$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, interchanging C_2 and C_3

$\sim [\mathbf{I}_3]$.

Hence the rank of **A** is 3. **Ans.**

Ex. 3 (a) Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$ *(Bundelkhand 94)*

Sol. $A \sim \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & -1 \\ 2 & 2 & -1 \end{bmatrix}$, replacing C_2, C_3 by $C_2 - 2C_1$ and $C_3 - 3C_1$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 0 & 0 \end{bmatrix}$, replacing R_3 by $R_3 - R_2$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, replacing R_2, R_3 by $R_2 - R_1$ and $R_3 - R_1$ respectively

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \text{ replacing } C_2 \text{ by } C_2 + 2C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \text{ replacing } C_3 \text{ by } - C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ interchanging } C_2 \text{ and } C_3$$

$$\sim \begin{bmatrix} \mathbf{I}_2 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$$

Hence the rank of A is 2. **Ans.**

***Ex. 3 (b). Find the rank of the matrix A =** $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$ **after reducing it to the normal form.** *(Avadh 97; Garhwal 90; Meerut 92)*

Sol. $\mathbf{A} \sim \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix}$, replacing C_2 and C_3 by $C_2 - C_1$ and $C_3 - C_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \text{ replacing } R_2 \text{ and } R_3 \text{ by } R_2 - R_1 \text{ and } R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ replacing } C_3 \text{ by } C_3 - C_2$$

$$\sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ replacing } C_1 \text{ by } C_1 - C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ interchanging } C_1 \text{ and } C_2$$

$$\sim \begin{bmatrix} \mathbf{I}_2 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$$

Hence the rank of **A** is 2. **Ans.**

Ex. 3 (c). Reduce matrix A to its normal form and then find its rank, where $\mathbf{A}=\begin{bmatrix}1 & 1 & 1 & -1\\1 & 2 & 3 & 4\\3 & 4 & 5 & 2\end{bmatrix}$ *(Agra 93)*

Sol. $\mathbf{A}\sim\begin{bmatrix}1 & 0 & 0 & 0\\1 & 1 & 2 & 5\\3 & 1 & 2 & 5\end{bmatrix}$, replacing C_2, C_3, C_4 by C_2-C_1, C_3-C_1, C_4+C_1 respectively

$\sim\begin{bmatrix}1 & 0 & 0 & 0\\1 & 1 & 2 & 5\\2 & 0 & 0 & 0\end{bmatrix}$, replacing R_3 by R_3-R_2

$\sim\begin{bmatrix}1 & 0 & 0 & 0\\1 & 1 & 2 & 5\\0 & 0 & 0 & 0\end{bmatrix}$, replacing R_3 by R_3-2R_1

$\sim\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}$, replacing $C_1, C_3\ C_4$ by C_1-C_2, C_3-2C_2, C_4-5C_2 respectively

$\sim\begin{bmatrix}\mathbf{I}_2 & \mathbf{O}\\\mathbf{O} & \mathbf{O}\end{bmatrix}$, which is in the normal form.

Hence the rank of **A** is 2. **Ans.**

Ex. 4. (a). Reduce the matrix A to the normal form.

where $\mathbf{A}=\begin{bmatrix}1 & 1 & 2 & -3\\4 & 1 & 0 & 2\\0 & 3 & 0 & 1\\0 & 1 & 0 & 2\end{bmatrix}$, **hence find the rank of A.** *(Meerut 92 P)*

Sol. $\mathbf{A}\sim\begin{bmatrix}1 & 0 & 2 & 0\\4 & 5 & 0 & 14\\0 & 3 & 0 & 4\\0 & 1 & 0 & 2\end{bmatrix}$, replacing C_2, C_4 by C_2+C_1, C_4+3C_1 respectively,

$\sim\begin{bmatrix}1 & 0 & 1 & 0\\4 & 5 & 0 & 7\\0 & 3 & 0 & 2\\0 & 1 & 0 & 1\end{bmatrix}$, replacing C_3, C_4 by $\frac{1}{2}C_3, \frac{1}{2}C_4$ respectively

$\sim\begin{bmatrix}0 & 0 & 1 & 0\\4 & 5 & 0 & 2\\0 & 3 & 0 & -1\\0 & 1 & 0 & 0\end{bmatrix}$, replacing C_1, C_4 by C_1-C_3, C_4-C_2 respectively

$\sim\begin{bmatrix}0 & 0 & 1 & 0\\1 & 5 & 0 & 2\\0 & 3 & 0 & -1\\0 & 1 & 0 & 0\end{bmatrix}$, replacing C_1 by $\frac{1}{4}C_1$

$$\sim \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \text{ replacing } C_2, C_4 \text{ by } C_2 - 5C_1, C_4 - 2C_1 \text{ respectively}$$

$$\sim \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \text{ replacing } C_2, C_4 \text{ by } C_2 + 3C_4, -C_4 \text{ respectively.}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ rearranging columns}$$

$$\sim [\mathbf{I}_4].$$

∴ Rank of **A** is 4. **Ans.**

Ex. 4 (b). Express the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & -1 & 3 \\ -1 & -4 & -2 & -7 \\ 2 & 1 & 3 & 0 \\ -1 & -2 & 3 & 0 \end{bmatrix}$$

in the normal form and find its rank. *(Purvanchal 93)*

Sol. $\mathbf{A} \sim \begin{bmatrix} 0 & -1 & 0 & 3 \\ 6 & -4 & 2 & -7 \\ 2 & 1 & 2 & 0 \\ -1 & -2 & 5 & 0 \end{bmatrix}$, replacing C_1, C_3 by $C_1 - C_4$, $C_3 - C_2$ respectively

$$\sim \begin{bmatrix} 0 & -1 & 0 & 3 \\ 4 & -4 & 2 & -7 \\ 0 & 1 & 2 & 0 \\ -6 & -2 & 5 & 0 \end{bmatrix}, \text{ replacing } C_1 \text{ by } C_1 - C_3$$

$$\sim \begin{bmatrix} 0 & -1 & 0 & 3 \\ 4 & -5 & 0 & -7 \\ 0 & 1 & 2 & 0 \\ -6 & 0 & 9 & 0 \end{bmatrix}, \text{ replacing } R_2, R_4 \text{ by } R_2 - R_3, R_4 + 2R_3 \text{ respectively}$$

$$\sim \begin{bmatrix} 0 & 0 & 2 & 3 \\ 4 & -5 & 0 & -7 \\ 0 & 1 & 2 & 0 \\ -2 & -5 & 9 & -7 \end{bmatrix}, \text{ replacing } R_1, R_4 \text{ by } R_1 + R_3, R_4 + R_2 \text{ respectively}$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 3 \\ 4 & -5 & 10 & -7 \\ 0 & 1 & 0 & 0 \\ -2 & -5 & 19 & -7 \end{bmatrix}, \text{ replacing } C_3 \text{ by } C_3 - 2C_2$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 3 \\ 4 & 0 & 10 & -7 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 19 & -7 \end{bmatrix}, \text{ replacing } R_2, R_4 \text{ by } R_2 + 5R_3,\ R_4 + 5R_3 \text{ repectively}$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 3 \\ 4 & 0 & 10 & -7 \\ 0 & 1 & 0 & 0 \\ -6 & 0 & 9 & 0 \end{bmatrix}, \text{ replacing } R_4 \text{ by } R_4 - R_2$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 1 \\ 4 & 0 & 10 & -7 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 3 & 0 \end{bmatrix}, \text{ replacing } R_1, R_4 \text{ by } \tfrac{1}{3} R_1, \tfrac{1}{3} R_4 \text{ respectively}$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 16 & -7 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 3 & 0 \end{bmatrix}, \text{ replacing } R_2 \text{ by } R_2 + 2R_4$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 16 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 3 & 0 \end{bmatrix}, \text{ replacing } R_2 \text{ by } R_2 + 7R_1$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 3 & 0 \end{bmatrix}, \text{ replacing } R_2 \text{ by } (1/16) R_2$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix}, \text{ replacing } R_4 \text{ by } R_4 - 3R_2$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \text{ replacing } R_4 \text{ by } -(1/2) R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ interchanging } R_1 \text{ and } R_4 \text{ and interchanging } R_2 \text{ and } R_3$$

or $\mathbf{A} \sim [\mathbf{I}_4]$ $\quad \therefore$ The rank of **A** is 4. **Ans.**

Ex. 5. Find the rank of the matrix

$$A = \begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

(Garhwal 94)

Sol.

$$A \sim \begin{bmatrix} 0 & -1 & -1 & 1 \\ 0 & 2 & 2 & -2 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}, \text{ replacing } R_1, R_2 \text{ by } R_1 + 2R_3, R_2 - R_3 \text{ respectively}$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \text{ replacing } C_2, C_3 \text{ by } C_2 + C_4, C_3 + C_4 \text{ respectively}$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ replacing } R_2, R_3, R_4 \text{ by } R_2 + 2R_1, R_3 - R_1, R_4 + R_1 \text{ respectively}$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ replacing } C_2, C_3 \text{ by } C_2 - C_1, C_3 - 2C_1 \text{ respectively}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ interchanging } R_1, R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ interchanging } C_2, C_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ interchanging } R_2, R_3$$

$$\sim \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}$$

Hence the rank of A is 2. **Ans.**

Ex. 6 (a). Reduce $A = \begin{bmatrix} 1 & -1 & 2 & -1 \\ 4 & 2 & -1 & 2 \\ 2 & 2 & -2 & 0 \end{bmatrix}$ **to normal form.**

(Garhwal 91)

Sol. $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 6 & -9 & 6 \\ 2 & 4 & -6 & 2 \end{bmatrix}$, replacing C_2, C_3, C_4 by $C_2 + C_1, C_3 - 2C_2, C_4 + C_1$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 6 \\ 2 & 2 & 3 & 2 \end{bmatrix}$, replacing C_2, C_3 by $C_2 - C_4, C_3 - (3/2)\, C_2$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 6 \\ 2 & 2 & 0 & 0 \end{bmatrix}$, replacing C_3, C_4 by $C_3 - (3/2)\, C_2, C_4 - C_2$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 3 \\ 1 & 1 & 0 & 0 \end{bmatrix}$, replacing R_2, R_3 by $\frac{1}{2} R_2, \frac{1}{2} R_3$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, replacing R_2, R_3 by $R_3 - 2R_1, R_3 - R_1$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, replacing C_3 by $\frac{1}{3} C_3$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, interchanging C_2, C_4

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, interchaning C_3, C_4

$\sim [\mathbf{I}_3 \ \mathbf{O}]$ which is the required normal form **Ans.**

Ex. 6 (b). **Reduce the matrix A to the normal form and hence find the rank of the matrix A, where**

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 4 & 2 & -1 & 2 \\ 2 & 2 & -2 & 2 \end{bmatrix}.$$

Hint : Do as Ex. 6 (a) above. **Ans. 3**

Ex. 6 (c). **Find the rank of the matrix**

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 4 & 7 \\ 2 & 4 & 5 & 8 \\ 3 & 1 & 2 & 4 \end{bmatrix}$$

Hint : Do as Ex. 6 (a) above. **Ans. 3**

****Ex. 7. By elementary operations, find the rank of the matrix.**

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

(Avadh 95, 93, 91; Garhwal 95, Gorakhpur 90; Kanpur 97; Meerut 98)

Sol. $A \sim \begin{bmatrix} 6 & 3 & 0 & -7 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$, replacing R_1 by $R_1 + R_2 + R_3$

or $\sim \begin{bmatrix} 6 & 3 & 0 & -7 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing R_4 by $R_4 - R_1$

$\sim \begin{bmatrix} 0 & 9 & 12 & 17 \\ 1 & -1 & -2 & -4 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing R_1 and R_3 by $R_1 - 6R_2$, $R_3 - 3R_2$ respectively.

$\sim \begin{bmatrix} 0 & 9 & 12 & 17 \\ 1 & 0 & 0 & 0 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing C_2, C_3 and C_4 by $C_2 + C_1$, $C_3 + 3C_1$ and $C_4 + 4C_1$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 9 & 12 & 17 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, interchanging R_1 and R_2

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing R_2 by $R_2 - 2R_3$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing C_3 and C_4 by $C_3 + 6C_2$ and $C_4 + 3C_2$ respectively.

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing C_3 and C_4 by $\frac{1}{33} C_3$ and $\frac{1}{22} C_2$ respectively.

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing C_2 and C_4 by $C_2 - 4C_3$ and $C_4 - C_3$ respectively.

$\sim \begin{bmatrix} I_3 & O \\ O & O \end{bmatrix}$

Hence the rank of the given matrix = 3. **Ans.**

Ex. 8. Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ 2 & 2 & 8 & 0 \end{bmatrix}$$

Hint : Do as Ex. 7. on Pages 23-24 **Ans. 3**

****Ex. 9 (a). Find the rank of** $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

(Agra 91; Bundelkhand 91; Garhwal 92; Kumaun 96; Lucknow 92; Meerut 90; Purvanchal 98; Rohilkhand 91)

Sol. $A \sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 1 & -3 & -1 \\ 1 & 1 & -3 & -1 \end{bmatrix}$, replacing C_3, C_4 by $C_3 - C_1$ and $C_4 - C_1$ respectively

$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, replacing R_3, R_4 by $R_3 - R_1$ and $R_4 - R_1$ respectively

$\sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, replacing C_3, C_4 by $C_3 - 3C_2$ $C_4 + C_2$ respectively

$\sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing R_3, R_4 by $R_3 - 3R_2$ and $R_4 - R_2$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, interchanging C_1, C_2

$\sim \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}$

Hence the rank of **A** is 2. **Ans.**

Ex. 9 (b). Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$$

(Purvanchal 95)

Sol. $A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix}$, replacing R_2, R_3, R_4 by $R_2 - R_1, R_3 - R_1, R_4 - R_1$ respectively

$\sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix}$, replacing C_2, C_3, C_4 by $C_2 - C_1, C_3 - C_1, C_4 - C_1$ respectively

$\sim \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing R_1, R_3, R_4 by $R_1 - R_2, R_3 - 2R_2, R_4 - 3R_2$ respectively

$\sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing C_3, C_4 by $C_3 - 2C_2, C_4 - 3C_2$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, interchanging C_1 and C_2

$\sim \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}$

Hence the rank of **A** is 2. **Ans.**

***Ex. 10. Find the rank of matrix** $A = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$.

by reducing to canonical form. Also show that it is not equivalent to the matrix $B = \begin{bmatrix} 1 & 0 & -5 & 6 \\ 3 & -2 & 1 & 0 \\ 5 & -2 & -9 & 14 \\ 4 & -2 & -4 & 8 \end{bmatrix}$

Sol. $A \sim \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 4 & 4 & 1 \\ 0 & 2 & 4 & 1 \end{bmatrix}$, replacing R_3 and R_4 by $R_3 - R_1$ and $R_4 - R_3$ respectively

$$\sim \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 2 & 4 & 1 \end{bmatrix}$$, replacing R_2 and R_3 by $R_2 - R_4$ and $R_3 - R_4$ respectively

$$\sim \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 4 & 1 \end{bmatrix}$$, replacing C_1 and R_3 by $\frac{1}{2} C_1$ and $\frac{1}{2} R_3$ respectively

$$\sim \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 1 \end{bmatrix}$$, replacing R_3 by $R_3 - R_2$

$$\sim \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$, interchang R_3 and R_4

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$, replacing C_2, C_3 and C_4 by $C_2 + C_1$, $C_3 - 3C_1$ and $C_4 - 4C_1$ respectively.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$, replacing C_2 and C_3 by $C_2 - 2C_4$ and $C_3 - 4C_4$ respectively.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$, interchaining C_3 and C_4

$$\sim \begin{bmatrix} \mathbf{I}_3 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$$

∴ The required rank of the matrix **A= 3.**

Also we can prove as in Ex. 5 Page 5 Chapter V that the rank of the matrix **B** *i.e.* $\begin{bmatrix} 1 & 0 & -5 & 6 \\ 3 & -2 & 1 & 0 \\ 5 & -2 & -9 & 14 \\ 4 & -2 & -4 & 8 \end{bmatrix}$ is 2.

Since the ranks of the two matrices **A** and **B** are different so these are not equivalent. (See Note 3 of § 5·02 Page 2 Chapter V)

Ex. 11 (a). Find the rank of matrix $\mathbf{A} = \begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

(Gorakhpur 97; Lucknow 92)

Sol. $\mathbf{A} \sim \begin{bmatrix} 0 & 4 & 9 & -7 \\ 0 & 1 & 0 & 0 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$, replacing R_1, R_2 by $R_1 - 3R_3, R_2 - R_4$ respectively

$\sim \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 9 & -7 \\ 0 & 0 & 2 & 1 \end{bmatrix}$, interchanging R_1, R_3 and replacing R_4 by $R_4 - R_2$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 9 & -7 \\ 0 & 0 & 2 & 1 \end{bmatrix}$, replacing C_2, C_3, C_4 by $C_2 + 2C_1$, $C_3 + 3C_1$ and $C_4 - 2C_1$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 9 & -7 \\ 0 & 0 & 2 & 1 \end{bmatrix}$, replacing R_3 by $R_3 - 4R_2$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 23 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$, replacing R_3 by $R_3 + 7R_4$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$, replacing R_3 by $\frac{1}{23} R_3$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, replacing C_3 by $C_3 - 2C_4$

$\sim [\mathbf{I}_4]$

Hence the rank of A is 4. Ans.

Ex. 11 (b). Reduce the following matrix A to the form

$$\begin{bmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$$

and hence determine its rank.

$$A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

(Kumaun 92)

Sol. $A \sim \begin{bmatrix} 1 & 0 & 2 & -1 \\ 4 & 5 & 0 & 2 \\ 0 & 3 & 1 & 5 \\ 0 & 1 & 0 & 2 \end{bmatrix}$, replacing C_2, C_4 by $C_2 + C_1$, $C_4 + C_3$ respectively

$\sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 4 & 5 & 0 & 6 \\ 0 & 3 & 1 & 5 \\ 0 & 1 & 0 & 2 \end{bmatrix}$, replacing C_4 by $C_4 + C_1$

$\sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 4 & 0 & 0 & -4 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 2 \end{bmatrix}$, replacing R_2, R_3 by $R_2 - 5R_4$ $R_3 - 3R_4$ respectively

$\sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 4 & 0 & 0 & -4 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, replacing C_4 by $C_4 - 2C_2$

$\sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, replacing C_4 by $C_4 + C_1 + C_2 + C_3$

$\sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, replacing R_2 by $(1/4)\, R_2$

$\sim \begin{bmatrix} 0 & 0 & 2 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, replacing R_1, C_4 by $R_1 - R_2$ and $C_4 - C_2$ respectively

$\sim \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, replacing R_1, C_4 by $R_1 - 2R_3$ $(1/3)\, C_4$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, interchanging R_1 and R_2, R_3 and R_4

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \text{ interchanging } R_2 \text{ and } R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ interchanging } R_3 \text{ and } R_4$$

$$\sim [\mathbf{I}_4]$$

Hence the rank of **A** is 4. **Ans.**

***Ex. 12. Is the matrix** $\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 0 \\ 2 & -3 & 1 \end{bmatrix}$ **equivalent to** $\mathbf{I}_3$ **?**

Sol. Here we find that the minor of order 3 of **A**

$$= \begin{vmatrix} 1 & 1 & 2 \\ -1 & 2 & 0 \\ 2 & -3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 3 & 2 \\ 2 & -5 & -3 \end{vmatrix}, \text{ replacing } C_2, C_3 \text{ by } C_2 - C_1, C_3 - 2C_1 \text{ respectively}$$

$$= \begin{vmatrix} 3 & 2 \\ -5 & -3 \end{vmatrix}, \text{ expanding with respect to } R_1$$

$= 3(-3) - 2(-5) = -9 + 10 = 1 \neq 0.$

Also from § 5·03 Th. II Paper 15 Chapter V we know that this matrix **A** can be recduced to $\mathbf{I}_3$ by elementary row or column operations.

Hence **A** is equivalent to $\mathbf{I}_3$.

Ex. 13. Determine by reducing to normal form the rank of the matrix

$$\mathbf{A} = \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$$

Sol. $\mathbf{A} \sim \begin{bmatrix} 1 & 1 & 3 & 3 \\ 0 & 3 & 2 & 1 \\ -1 & -1 & -3 & 2 \end{bmatrix}$, replacing C_1 by $\frac{1}{8} C_1$ and C_4 by $\frac{1}{2} C_4$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 1 \\ -1 & 0 & 0 & 5 \end{bmatrix}, \text{ replacing } C_2, C_3 \text{ and } C_4 \text{ by } C_2 - C_1, C_3 - 3C_2\ C_4 - 3C_1 \text{ respectively}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 + R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \text{ replaing } C_2 \text{ by } \tfrac{1}{3} C_2 \text{ and } C_3 \text{ by } \tfrac{1}{2} C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$, replacing C_3 and C_4 by $C_3 - C_2$ and $C_4 - C_2$ respectively

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix}$$, interchanging C_3 and C_4

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$, replacing C_3 by $\frac{1}{3} C_5$

$\sim [\mathbf{I}_3 \ \ \mathbf{O}]$, **(Note)**

Hence the rank of **A** is 3. **Ans.**

Ex. 14. Find the rank of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -4 & 7 \\ -1 & -2 & -1 & 1 \end{bmatrix}$

(Bundellkhand 95; Garhwal 96; Purvanchal 97; Rohilkhand 95

Sol. $\mathbf{A} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & -2 & 1 \\ -1 & 0 & -2 & 4 \end{bmatrix}$, replacing C_2, C_3, C_4 by $C_2 - 2C_1$, $C_3 + C_1$ and $C_4 - 3C_1$ respectively

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & -2 & 1 \\ -3 & 0 & 0 & 3 \end{bmatrix}$$, replacing R_3 by $R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$, replacing R_2, R_3 by $R_2 - 2R_1$ and $R_3 + 3R_1$ respectively

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$, replacing C_3 by $-\frac{1}{2} C_3$ and R_3 by $\frac{1}{3} R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$, replacing C_4 by $C_4 - C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$, interchanging C_2 and C_3

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$, interchanging C_3 and C_4

$\sim [\mathbf{I}_3 \ \ \mathbf{O}]$. **(Note)**

Hence the rank of A is 3.

Ans.

Ex. 15. Use elementary transformations to reduce the following matrix to triangular form and hence find the rank of A.

$$A = \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$$

Sol. $A \sim \begin{bmatrix} 5 & 3 & 8 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 4 & 1 \end{bmatrix}$, replacing C_3, C_4 by $C_3 - 2C_2, C_4 - C_2$

$\sim \begin{bmatrix} 5 & 8 & -12 & -4 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, replacing C_2, C_3, C_4 by $C_2 + C_1, C_3 - 4C_1, C_4 - C_1$ respectively.

$\sim \begin{bmatrix} 5 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, replacing C_2, C_3 by $C_2 + 2C_4, C_3 - 3C_4$ respectively.

$\sim \begin{bmatrix} 5 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, replacing C_1 by $-\frac{1}{4} C_4$

$\sim \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, replacing C_1 by $C_1 - 5C_4$

$\sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, interchanging R_1, R_2

$\sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, interchanging R_2, R_3

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, interchanging R_1, R_2

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, interchanging C_3, C_4

$\sim [I_3 \;\; O]$

Hence the rank A is 3.

Ans.

****Ex. 16. Reduce the matrix A to the normal (or canonical) form and hence obtain its rank.**

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$$

Sol. $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -2 & 1 & 5 \\ -2 & 7 & 2 & 3 \end{bmatrix}$, replacing C_2 and C_4 by $C_2 - 2C_1$ and $C_4 + C_1$ respectively.

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ -2 & 11 & 2 & -7 \end{bmatrix}$, replacing C_2 and C_4 by $C_2 + 2C_3$ and $C_4 - 5C_3$ respectively.

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ 0 & 11 & 2 & 0 \end{bmatrix}$, replacing C_1 by $C_1 + C_3$ and C_4 by $C_4 + \frac{7}{11} C_2$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 11 & 2 & 0 \end{bmatrix}$, replacing R_2 by $R_2 - 4R_1$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 11 & 0 & 0 \end{bmatrix}$, replacing R_3 by $R_3 - 2R_2$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 11 & 0 \end{bmatrix}$, interchanging C_2 and C_3

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, replacing C_3 by $\frac{1}{11} C_3$

$\sim [I_3 \quad O]$ **(Note)**

Hence the rank of **A** is 3. **Ans.**

***Ex. 17. Is rank of** $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **equal to 2 ?** *(Agra 90)*

Sol. Here $A = I_3$, so rank of **A** is 3 and not 2. **Ans.**

Ex. 18. If $A = \begin{bmatrix} -1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & -2 & 6 & -7 \end{bmatrix}$**, find its rank.** *(Rohilkhand 92)*

Sol. $A \sim \begin{bmatrix} -1 & 2 & -1 & 4 \\ 0 & 8 & 1 & 12 \\ 0 & 4 & 2 & 8 \\ 0 & 0 & 5 & -3 \end{bmatrix}$, replacing R_2, R_3, R_4 by $R_2 + 2R_1, R_3 + R_1, R_4 + R_1$ respectively.

$$\sim \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 8 & 1 & 12 \\ 0 & 4 & 2 & 8 \\ 0 & 0 & 5 & -3 \end{bmatrix}, \text{ replacing } C_2, C_3, C_4 \text{ by } C_2 + 2C_1, C_3 - C_1, C_4 + 4C_1 \text{ respectively.}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 12 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 5 & -3 \end{bmatrix}, \text{ replacing } C_1, C_2 \text{ by } -C_1 \text{ and } (1/4)\, C_2 \text{ respectively.}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix}, \text{ replacing } C_3, C_4 \text{ by } C_3 - 2C_2, C_4 - 8C_2 \text{ respectively.}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5/3 & 1 \end{bmatrix}, \text{ replacing } R_2, R_4 \text{ by } R_2 - 2R_3, (-1/3)\, R_4 \text{ respectively}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -29/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5/3 & 1 \end{bmatrix}, \text{ replacing } R_2 \text{ by } R_2 + 4R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5/3 & 1 \end{bmatrix}, \text{ replacing } R_2 \text{ by } -(3/29)\, R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ interchanging } R_2 \text{ and } R_3 \text{ and replacing } R_4 \text{ by } R_4 + (5/3)\, R_3$$

$$\sim [I_4]$$

Hence the rank of A is 4. **Ans.**

Exercises on § 5·03

Find the rank of the following matrices by reducing these to the normal (or canonical) form :—

Ex. 1. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}; \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ **Ans.** 0, 1, 1, 1

Ex. 2. $\begin{bmatrix} 2 & 1 & 3 \\ 4 & 7 & 13 \\ 4 & -3 & -1 \end{bmatrix}$ **Ans.** 2

Ex. 3. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \\ 2 & 0 & -3 & 2 \\ 3 & 3 & 0 & 3 \end{bmatrix}$ **Ans.** 3

Ex. 4. $\begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$ *(Kumaun 93)* **Ans.** 3

***Ex. 5.** $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ **Ans.** 2

Ex. 6. $\begin{bmatrix} 1 & -3 & 2 \\ 3 & -9 & 6 \\ -2 & 6 & -4 \end{bmatrix}$ **Ans.** 1

***Ex. 7.** $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{bmatrix}$ **Ans.** 2

Ex. 8. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \\ 2 & 0 & -3 & 2 \\ 3 & 3 & -3 & 3 \end{bmatrix}$ **Ans.** 3

Ex. 9. $\begin{bmatrix} 1 & 4 & 3 & 2 \\ 1 & 2 & 3 & 4 \\ 2 & 6 & 7 & 5 \end{bmatrix}$ *(Garakhpur 99)* **Ans.** 3

Ex. 10. $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$ *(Meerut 91S)* **Ans.** 3

Ex. 11. $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 3 & 0 & 3 \\ 1 & -2 & -3 & -3 \\ 1 & 1 & 2 & 3 \end{bmatrix}$ **Ans.** 3

***Ex. 12.** $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ *(Garhwal 91)* **Ans.** 2

Ex. 13. $\begin{bmatrix} 6 & 1 & 3 & 8 \\ 5 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$ *(Avadh 98)* **Ans.** 3

Ex. 14. $\begin{bmatrix} 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \end{bmatrix}$ *(Agra 92)* **Ans.** 2

Ex. 15. $\begin{bmatrix} 1 & 3 & 2 & 0 & 1 \\ 9 & 2 & -1 & 6 & 4 \\ 7 & -4 & -5 & 6 & 5 \\ 17 & 1 & -4 & 12 & 7 \end{bmatrix}$ **Ans.** 2

Ex. 16. $\begin{bmatrix} 9 & 7 & 3 & 6 \\ 5 & -1 & 4 & 1 \\ 6 & 8 & 2 & 4 \end{bmatrix}$ *(Rohilkhand 96)* **Ans.** 3

Ex. 17. $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 3 & 2 & 0 & 2 \\ 0 & 1 & 3 & 2 \\ 3 & 3 & -3 & 4 \end{bmatrix}$ *(Lucknow 90)* **Ans.** 4

§ 5·04. Echelon Form of a Matrix.

Definition. If in a matrix.

(i) all the non-zero rows, if any, precede the zero rows,

(ii) the number of zero preceeding the first non-zero element in a row is less than the number of such zero in the succeeding row.

(iii) the first non-zero element in a row is unity, then it is in the Echelon form.

Note. The number of non-zero rows of a matrix given in the Echelon form is its rank. **(Remember)**

Example of a matrix in the Echelon Form :—

$$\begin{bmatrix} 1 & 3 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In this matrix we observe that

(i) the first three non-zero rows precede the fourth row which is a zero row.

(ii) the number of zero in R_4, R_3 and R_2 are 5, 2 and 1 respectively which are in descending order.

(iii) the first non-zero term in each row is unity.

Hence all the three conditions of the Echelon form are satisfied.

Also there being three non-zero rows in this matrix, its rank is 3. This fact can be proved by actually finding the rank of this matrix.

In this matrix, a minor of order 4

$$= \begin{vmatrix} 1 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0, \text{ one row being of zero.}$$

In a similar way we can show that all minors of order 4 are zero.

Now a minor of order 3 $= \begin{vmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix}$

$= \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$, expanding w.r. to C_1

$= 1 \neq 0.$

Hence the rank of this matrix = 3.

Ex. 1. Find the rank of the matrix.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Sol. In the given matrix we observe that

(i) the first two non-zero rows precede the third row which is a zero row,

(ii) the number of zero in R_3, R_2 and R_1 are 4, 2 and 1 respectively which are in descending order, and

(iii) the first non-zero term in each row is unity.

Hence all the three conditions of the Echelon form are satisfied.

Also there being two non-zero rows in this matrix, its rank is 2. **Ans.**

Ex. 2. Reduce the following matrix to its Echelon form and find its rank :

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 4 & 5 \\ 3 & 9 & 12 & 9 \\ -1 & -3 & -4 & -3 \end{bmatrix}$$

(Meerut 93)

Sol. $A \sim \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 2 \end{bmatrix}$, replacing R_2, R_3 by $R_2 - 3R_1, R_3 + R_1$ respectively

$\sim \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, replacing R_2, R_3 by $R_2 + 3R_3, (1/2)\,R_3$ respectively.

$\sim \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, interchanging R_2 and R_3

In the above matrix we observe that.

(i) the first two non-zero rows precede the third row which is a zero row,

(ii) the number of zero in R_3 and R_2 are 4 and 3 respectively which are in descending order, and

(iii) the first non-zero term in each row is unity.

Hence all the three conditions of the Echelon form are satisfied and then the given matrix is reduced to its Echelon form.

Also there being two non-zero rows in this matrix, its rank is 2. **Ans.**

****§ 5·05. Invariance of rank under elementary operations.**

Theorem. *All equivalent matrices have the same ranks i.e. the rank of a matrix remains unaltered by the application of elementary row and column operations.* *(Avadh 99; Bundelkhamd 93)*

Proof. Let r be the rank of $m \times n$ matrix $\mathbf{A} = [a_{ij}]$

Case I. *If ith and jth rows are interchanged (which may be written symbolically as R_{ij} or ($R_i \longleftrightarrow R_j$) then it does not effect the rank.*

Let **B** denote the matrix obtained from the matrix **A** by the elementary operation $R_i \longleftrightarrow R_j$ and let p be the rank of **B**.

Also if **D** be any $(r+1)$ rowed square sub-matrix of **B**, then $|\mathbf{D}| = \pm |\mathbf{C}|$, where **C** is a particular $(r+1)$ rowed sub matrix of **A**.

As r is the rank of the matrix **A** so every $(r+1)$ rowed minor of **A** vanishes and therefore p, the rank of **B**, cannot exceed r, the rank of **A** *i.e.* $p \le r$.

Also we can obtain **A** from **B** by the elementary operation $R_i \longleftrightarrow R_j$, therefore in that case interchanging the roles of **A** and **B** we shall get $r \le p$.

Hence $r = p$.

Case II. *If the elements of a row are multiplied by a non-zero number λ (which may be written symbolically, as $R_i \to \lambda R_i$, $\lambda \neq 0$) then it does not effect the rank.*

Let **B** denote the matrix obtained from the matrix **A** by the elementary operation $R_i \to \lambda R_i$ and let p be the rank of **B**.

Let **D** be any $(r+1)$ rowed square sub-matrix of **B** and let **C** be the sub-matrix of **A** having the same position as **D**. Then either $|\mathbf{D}| = |\mathbf{C}|$ or $|\mathbf{D}| = \lambda |\mathbf{C}|$.

[Here $|\mathbf{D}| = |\mathbf{C}|$ happens if the ith row of **A** is one of those rows which are removed to obtain **D** from **B** and $|\mathbf{D}| = \lambda |\mathbf{C}|$ happens when the ith row is not removed while obtaining **C** from **A**].

Also as r is the rank of the matrix **A** so every $(r+1)$–rowed minor of **A** vanishes and therefore in particular $|\mathbf{C}| = 0$ and consequently in both the above cases $|\mathbf{D}| = 0$.

$\therefore$ p, the rank of **B**, cannot exceed r, the rank of **A**.

i.e. $p \le r$.

Also we can obtain **A** from **B** by the elementary operation $R_i \to \lambda^{-1} R_i$, therefore in that case interchanging the roles of **A** and **B** we shall get $r \le p$.

Hence $r = p$.

Case III. *If to the elements of the ith row are added the products by any non-zero number λ of the corresponding elements of jth row (which may be written symbolically as $R_i \to R_i + \lambda R_j$; $\lambda \neq 0$) then it does not effect the rank.*

Let **B** denote the matrix obtained from the matrix **A** by the elementary operation $R_i \to R_i + \lambda R_j$ and let p be the rank of **B**.

Let **D** be any $(r + 1)$ rowed square submatrix of **B** and let **C** be the submatrix of **A** having the same position as **D**.

Now three sub-cases arise :—

(i) If **A** and **B** differ only in the *i*th row *i.e.* if *i*th row of **B** is one of those rows which have been removed while obtaining **C**.

In this case **D** = **C** and therefore $|\mathbf{D}| = |\mathbf{C}|$.

∴ The rank of **A** is r, so $|\mathbf{C}| = 0$ and consequently $|\mathbf{D}| = 0$

(ii) If *i*th row of **B** has not been removed but *j*th row has been removed while obtaining **D**.

In this case $|\mathbf{D}| = |\mathbf{C}| + \lambda |\mathbf{C}_0|$, where $\mathbf{C}_0$ is an $(r + 1)$ rowed matrix which is obtained from **C** by replacing a_{ik} by a_{jk} *i.e.* $\mathbf{C}_0$ is obtained from **C** by performing the elementary operation R_{ij} or $R_i \longleftrightarrow R_j$ and then removing those rows and columns of the new matrix which are removed to obtain **D** from **B**.

∴ $|\mathbf{C}_0|$ is negative of some $(r + 1)$-rowed minor of **A** and as the rank of **A** is r so every $(r + 1)$-rowed minor of **A** is zero *i.e.* $|\mathbf{C}| = 0, |\mathbf{C}_0| = 0$ and consequently $|\mathbf{D}| = 0$.

(iii) If neither the *i*th row not the *j*th row of **B** has been removed while obtaining **D**.

Here $|\mathbf{D}| = |\mathbf{C}|$ and so as before $|\mathbf{D}| = 0$.

∴ Every $(r + 1)$-rowed minor of **B** vanishes so p, the rank of **B**, cannot exceed r, the rank of **A** *i.e.* $p \leq r$.

Also we can obtain **A** from **B** by the elementary operation $R_i \to R_i - \lambda R_j$, therefore in that case interchanging the roles of **A** and **B** we shall get $r \leq p$.

Hence $r = p$.

Thus we have observed that the rank of a matrix remains invariant under elementary row operations. Similarly it can be shown that the rank of a matrix remains invariant under elementary column operations too.

Note. By the applications of the above theorem we can easily obtain the rank of a matrix for if we can obtain a matrix **B** by elementary operations on a matrix **A** and of the rank of **B** can be easily determined by inspection or simple calculations as given in previous articles in this chapter, then we can determine the rank of **A**.

§ 5·06. Some Important Theorems.

Theorem I. *The rank of a matrix is equal to the rank of the transposed matrix.*

or $\rho(\mathbf{A}) = \rho(\mathbf{A}')$, *where* $\rho(\mathbf{A})$ *denotes rank of* **A**. *(Kanpur 94; Rohilkhand 92)*

Proof. Let $\mathbf{A} = [a_{ij}]$ be any $m \times n$ matrix.

Then the transposed matrix $\mathbf{A}' = [a_{ji}]$ is an $n \times m$ matrix.

Let the rank of **A** be r and let **B** be the $r \times r$ sub-matrix of **A** such that $|\mathbf{B}| \neq 0$.

Also we know that the value of a determinant remains unaltered if its rows and columns are interchanged. (See Prop. II of Determinants)

i.e. $|\mathbf{B}'| = |\mathbf{B}| \neq 0$, where **B** is evidently a $r \times r$ sub-matrix of **A'**.

$\therefore$ The rank of $\mathbf{A}' \geq r$, (See Note 4 (b) Page 2 Ch. V)

Again if **C** be a $(r+1) \times (r+1)$ sub-matrix of **A**, then by definition of rank (See § 5·02 Page 1 Ch. V) we must have all $|\mathbf{C}| = 0$.

Also **C'** is a $(r+1) \times (r+1)$ submatrix of **A'** so we have

$$|\mathbf{C}'| = |\mathbf{C}| = 0, \text{ as explained above.}$$

$\therefore$ We conclude that there cannot be any $(r+1) \times (r+1)$ sub-matrix of **A'** with non-zero determinant.

$\therefore$ The rank of $\mathbf{A}' \geq r$ and it cannot be greater then r as above.

$\therefore$ The rank of **A'** is r which is also the rank of **A**. Hence proved.

Theorem II. *The rank of the product matrix* **AB** *of two matrices* **A** *and* **B** *is less than the rank of either of the matrices* **A** *and* **B**.

Proof. Let r_1 and r_2 be the ranks of the matrices **A** and **B**.

$\because$ r_1 is the rank of **A** therefore $\mathbf{A} \sim \begin{bmatrix} \mathbf{M} \\ \mathbf{O} \end{bmatrix}$, where **M** is a submatrix of rank r_1 and contains r_1 rows.

Post multiplying it by **B**, we get

$$\mathbf{AB} \sim \begin{bmatrix} \mathbf{M} \\ \mathbf{O} \end{bmatrix} \mathbf{B}.$$

But $\begin{bmatrix} \mathbf{M} \\ \mathbf{O} \end{bmatrix} \mathbf{B}$ can have r_1 non-zero rows at the most which are obtained on multiplying r_1 non-zero rows of **M** with columns of **B**.

$\therefore$ Rank of **AB** = Rank of $\begin{bmatrix} \mathbf{M} \\ \mathbf{O} \end{bmatrix} \mathbf{B} \leq \mathbf{r_1}$

i.e. Rank of $\mathbf{AB} \leq$ Rank of **A** ...(i)

In a similar way we get $\mathbf{B} \sim [\mathbf{N} \;\; \mathbf{O}]$, where **N** is a submatrix of rank r_2 and contains r_2 columns.

Pre-multiplying it by **A**, we get

$$\mathbf{AB} \sim \mathbf{A}\,[\mathbf{N} \;\; \mathbf{O}]$$

But $[\mathbf{N} \;\; \mathbf{O}]$ can have r_2 non-zero columns at the most which are obtained on multiplying the rows of **A** with r_2 non-zero columns of $[\mathbf{N} \;\; \mathbf{O}]$

$\therefore$ Rank of **AB** = Rank of **A** [**N** **O**] $\leq r_2$

i.e. Rank of **AB** $\leq$ Rank of **B**. ...(ii)

Hence the theorem from (i) and (ii).

Solved Examples on § 5·05 and § 5·06.

Ex. 1. Show that the rank of a matrix A does not alter by pre or post multiplying it with any non-singular matrix R.

Sol. Let **B** = **RA**.

Then rank of **B** = rank of **RA** $\leq$ rank **A**. ...(i)

...See § 5.06 Th II above

Also **A** = $\mathbf{R}^{-1}$ **B**, where $\mathbf{R}^{-1}$ is the inverse matrix of **R**.

$\therefore$ rank of **A** = rank of ($\mathbf{R}^{-1}\mathbf{B}$) $\leq$ rank of **B**. ...(ii)

$\therefore$ From (i) and (ii) we conclude that

rank of **A** = rank of **B**. Hence proved.

Ex. 2. Show that AA′ has the same rank as A, where A′ is the transpose of A.

Sol. Let **B** = **AA′**, then the rank of **B** $\leq$ rank of **A**. ...(i)

Also $\mathbf{A}^{-1}$ = **A′**, and so we have

rank of **A** = rank of **A′** $\leq$ rank of **B**. ...(ii)

$\therefore$ From (i) and (ii), rank of **A** = rank of **B.**

Ex. 3. Show that AA$^{\Theta}$ has the same rank as A, where A$^{\Theta}$ is the transpose conjugate of A.

[**Hint :** Do as Ex. 2 above]

Ex. 4. Prove that if A is a matrix of order n × n and if B non-singular matrix of order n, then the product P = AB has the same rank as A.

Sol. Here **A** ~ $(m \times n)$, **B** ~ $(n \times n)$

P = **AB** ~ $(m \times n)$

If $m < n$, rank of **A** $\leq m$ but rank of **B** = n

$\therefore$ rank of **A** < rank of **B**,

Now rank (**P**) = rank (**AB**) $\leq$ rank **A** ...(i)

But we can write **A** = $\mathbf{PB}^{-1}$

$\therefore$ rank of **A** = rank of ($\mathbf{PB}^{-1}$) $\leq$ rank of **P**. ...(ii)

$\therefore$ From (i) and (ii) we get rank of **P** = rank of **A**.

§ 5·07. Sweep out method of finding the rank of a matrix.

In the process of evaluation of the rank of a matrix by means of elementary row and column transformations, if certain rows or columns are zero-rows or zero-columns *i.e.* each element of these rows or columns are zero, then we can remove these rows or columns without any effect on the rank of the matrix. (See § 5·05 Page 38 Ch. V). This method is generally called the *Sweep out* method.

Solved Examples on § 5·07.

Ex. 1. Find the rank of the matrix

$$\mathbf{A} = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$$

Sol. $\mathbf{A} \sim \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 4 & 2 & 6 & -1 \\ 4 & 2 & 6 & -1 \end{bmatrix}$, replacing R_3 and R_4 by $R_3 - R_1$ and $R_4 - 2R_1$ respectively.

$\therefore \quad \mathbf{A} \sim \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \end{bmatrix}$ ...See § 5·07 above

Now a minor of order 2 is $\begin{vmatrix} 1 & 8 \\ 2 & -1 \end{vmatrix} = -1 - 16 = -17 \neq 0$.

Hence its rank is 2. **Ans.**

Ex. 2. Find the rank of the matrix

$$\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

Sol. Let $\mathbf{A} = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$.

Now $\mathbf{A} \sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 1 & -2 & 0 \\ 3 & 3 & -6 & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix}$, replacing R_2 and R_3 by $R_2 + R_1$ and $R_3 + 2R_1$ respectively.

$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing R_3 and R_4 by $\frac{1}{3} R_3$ and $R_4 - R_2$ respectively.

$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing R_3 by $R_3 - R_2$ and then C_2 and C_3 by $C_2 - C_1$ and $C_3 + 2C_1$ respectively.

$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. ... See § 5·07 Page 41 Ch. V.

Now a minor of order 2 is

$\sim \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1 \neq 0$. Hence its rank is 2.

Ans.

Ex. 3. Find the rank of the matrix $\mathbf{A} = \begin{bmatrix} 1 & -3 & 4 & 7 \\ 9 & 1 & 2 & 0 \end{bmatrix}$ *(Meerut 95, 94)*

Sol. Here $\mathbf{A} \sim \begin{bmatrix} 28 & -3 & 10 & 7 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, replacing C_1, C_3 by $C_1 - 9C_2$, $C_3 - 2C_2$ respectively.

$\sim \begin{bmatrix} 0 & -3 & 3 & 7 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, replacing C_1, C_3 by $C_1 - 4C_4$, $C_3 - C_4$ respectively.

$\sim \begin{bmatrix} 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, replacing C_2, C_4 by $C_2 + C_3$, $C_4 - 2C_3$ respectively.

$\sim \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, replacing C_3 by $C_3 - 3C_4$

Now a minar of order 2 is $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$

Hence its rank is 2. **Ans.**

Exercises on § 5·07

Find the rank of the following matrices :—

Ex. 1. $\begin{bmatrix} 4 & 3 & 0 & 2 \\ 3 & 4 & -1 & -3 \\ -7 & -7 & 1 & 5 \end{bmatrix}$ **Ans. 3**

Ex. 2. $\begin{bmatrix} 3 & 2 & -2 \\ 2 & 3 & -3 \\ -2 & 4 & 2 \\ 5 & -2 & 4 \end{bmatrix}$ **Ans. 3**

Ex. 3. $\begin{bmatrix} 3 & -2 & 0 & -7 \\ 0 & 2 & 1 & -5 \\ 1 & -2 & -2 & 1 \\ 0 & 1 & 1 & -6 \end{bmatrix}$ **Ans. 4**

§ 5·08. Adjoint of a Matrix. *(Agra 94, 92; Rohilkhand 91, 90)*

Definition. If C_{ij} be the cofactor of the element a_{ij} in $|a_{ij}|$ of the $n \times n$ matrix $\mathbf{A} = [a_{ij}]$, then

$$\text{adjoint of } \mathbf{A} = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

This is also rewritten as Adj. **A**

or adjoint of **A** = transposed of **C**, where $\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \dots & \dots & \dots & \dots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}$.

While solving problems we generally use this definition.

Here students should note carefully that the cofactors of the elements of the first row of $|a_{ij}|$ are the elements of the first column of Adj **A**.

Similarly the cofactors of the elements of the first column of $|a_{ij}|$ are the elements of first row of Adj. **A**.

Solved Examples on § 5·08.

Ex. 1 (a). If $\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 \\ 5 & 7 & 8 \\ 9 & 10 & 12 \end{bmatrix}$**, find Adj. A.** *(Avadh 95)*

Sol. For the given matrix **A**, we have

$$C_{11} = \begin{vmatrix} 7 & 8 \\ 10 & 12 \end{vmatrix} = 4;\ C_{12} = -\begin{vmatrix} 5 & 8 \\ 9 & 12 \end{vmatrix} = 12;\ C_{13} = \begin{vmatrix} 5 & 7 \\ 9 & 10 \end{vmatrix} = -13;$$

$$C_{21} = -\begin{vmatrix} 2 & 4 \\ 10 & 12 \end{vmatrix} = 16;\ C_{22} = \begin{vmatrix} 1 & 4 \\ 9 & 12 \end{vmatrix} = -24;\ C_{23} = -\begin{vmatrix} 1 & 2 \\ 9 & 10 \end{vmatrix} = 8;$$

$$C_{31} = \begin{vmatrix} 2 & 4 \\ 7 & 8 \end{vmatrix} = -12;\ C_{32} = -\begin{vmatrix} 1 & 4 \\ 5 & 8 \end{vmatrix} = 12;\ C_{33} = \begin{vmatrix} 1 & 2 \\ 5 & 7 \end{vmatrix} = -3$$

$$\therefore \quad C = \begin{bmatrix} 4 & 12 & -13 \\ 16 & -24 & 8 \\ -12 & 12 & -3 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } \mathbf{A} = C' = \begin{bmatrix} 4 & 16 & -12 \\ 12 & -24 & 12 \\ -13 & 8 & -3 \end{bmatrix}$$ Ans.

***Ex. 1 (b). Find the adjoint of the matrix** $\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$

Sol. Do as Ex. 1 (a) above. $\begin{bmatrix} 2 & -6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$ **Ans.**

Ex. 2. Find the adjoint of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$ *(Kanpur 96)*

Sol. For the given matrix **A**, we have

$$C_{11} = \begin{vmatrix} 2 & -3 \\ -1 & 3 \end{vmatrix} = 3;\ C_{12} = -\begin{vmatrix} 1 & -3 \\ 2 & 3 \end{vmatrix} = -9;\ C_{13} = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -5;$$

$$C_{21} = -\begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = -4;\ C_{22} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1;\ C_{23} = -\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = 3;$$

$$C_{31} = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -5;\ C_{32} = -\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} = 4;\ C_{33} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$$

$$\therefore \quad C = \begin{bmatrix} 3 & -9 & -5 \\ -4 & 1 & 3 \\ -5 & 4 & 1 \end{bmatrix}$$

$$\therefore \text{ Adj. } \mathbf{A} = C' = \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

Ans.

Ex. 3. Find the adjoint of the matrix A, if

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -1 \\ 2 & 0 & 4 \end{bmatrix}$$

Sol. For the matrix **A**, we have

$$C_{11} = \begin{vmatrix} 1 & -1 \\ 0 & 4 \end{vmatrix} = 4;\ C_{12} = -\begin{vmatrix} 0 & -1 \\ 2 & 4 \end{vmatrix} = -2;\ C_{13} = \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} = -2;$$

$$C_{21} = -\begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} = -4;\ C_{22} = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = -2;\ C_{23} = -\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = 2;$$

$$C_{31} = \begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} = -4;\ C_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} = 1;\ C_{33} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

$$\therefore \quad C = \begin{bmatrix} 4 & -2 & -2 \\ -4 & -2 & 2 \\ -4 & 1 & 1 \end{bmatrix}$$

$$\therefore \text{Adj. } \mathbf{A} = C' = \begin{bmatrix} 4 & -4 & -4 \\ 2 & -2 & 1 \\ -2 & 2 & 1 \end{bmatrix}$$

Ans.

Ex. 4. Find the adjoint of $\mathbf{A} = \begin{bmatrix} 5 & 0 & 0 & 2 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

Sol. For the matrix **A**, we have

$$C_{11} = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2;\ C_{12} = -\begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix} = -\begin{vmatrix} 0 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2;$$

$$C_{13} = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1; \quad C_{14} = - \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} = -2;$$

$$C_{21} = - \begin{vmatrix} 0 & 0 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0; \quad C_{22} = \begin{vmatrix} 5 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2 \begin{vmatrix} 5 & 2 \\ 1 & 1 \end{vmatrix} = 6$$

$$C_{23} = - \begin{vmatrix} 5 & 0 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 0; \quad C_{24} = - \begin{vmatrix} 5 & 0 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 0;$$

$$C_{31} = \begin{vmatrix} 0 & 0 & 2 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 0; \quad C_{32} = - \begin{vmatrix} 5 & 0 & 2 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \end{vmatrix} = 0;$$

$$C_{33} = \begin{vmatrix} 5 & 0 & 2 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 5 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 5 & 2 \\ 1 & 1 \end{vmatrix} = 3; \quad C_{34} = - \begin{vmatrix} 5 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 0;$$

$$C_{41} = - \begin{vmatrix} 0 & 0 & 2 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{vmatrix} = -2 \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = -4;$$

$$C_{42} = \begin{vmatrix} 5 & 0 & 2 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{vmatrix} = -2 \begin{vmatrix} 5 & 2 \\ 1 & 2 \end{vmatrix} = -16;$$

$$C_{43} = - \begin{vmatrix} 5 & 0 & 2 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} 5 & 2 \\ 0 & 1 \end{vmatrix} = -5;$$

$$C_{44} = \begin{vmatrix} 5 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 2 \begin{vmatrix} 5 & 0 \\ 1 & 1 \end{vmatrix} = 10$$

$$\therefore \quad \mathbf{C} = \begin{bmatrix} 2 & 2 & 1 & -2 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ -4 & -16 & -5 & 10 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} 2 & 0 & 0 & -4 \\ 2 & 6 & 0 & -16 \\ 1 & 0 & 3 & -5 \\ -2 & 0 & 0 & 10 \end{bmatrix}$$

Ans.

***Ex. 5. Verify that the adjoint of a diagonal matrix of order 3 is a diagonal matrix.**

Sol. Let A be a diagonal matrix of order 3 given by

$$A=\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

Then for the matrix **A** we have

$$C_{11}=\begin{vmatrix} b & 0 \\ 0 & c \end{vmatrix}=bc;\ C_{12}=-\begin{vmatrix} 0 & 0 \\ 0 & c \end{vmatrix}=0;\ C_{13}=\begin{vmatrix} 0 & b \\ 0 & 0 \end{vmatrix}=0;$$

$$C_{21}=-\begin{vmatrix} 0 & 0 \\ 0 & c \end{vmatrix}=0;\ C_{22}=\begin{vmatrix} a & 0 \\ 0 & c \end{vmatrix}=ac;\ C_{23}=-\begin{vmatrix} a & 0 \\ 0 & 0 \end{vmatrix}=0;$$

$$C_{31}=\begin{vmatrix} 0 & 0 \\ b & 0 \end{vmatrix}=0;\ C_{32}=-\begin{vmatrix} a & 0 \\ 0 & 0 \end{vmatrix}=0;\ C_{33}=\begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}=ab$$

$$\therefore \quad C=\begin{bmatrix} bc & 0 & 0 \\ 0 & ca & 0 \\ 0 & 0 & ab \end{bmatrix}$$

$\therefore$ Adj. $A = C' = \begin{bmatrix} bc & 0 & 0 \\ 0 & ca & 0 \\ 0 & 0 & ab \end{bmatrix}$, which is evidently a diagonal matrix

Hence proved.

Ex. 6. If $A=\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$, find $A^2-2A+\text{Adj. }A$

(Agra 95)

Sol. $A^2=\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}\times\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$

$$=\begin{bmatrix} 1+0+6 & 2+10+12 & 3+0+9 \\ 0+0+0 & 0+25+0 & 0+0+0 \\ 2+0+6 & 4+20+12 & 6+0+9 \end{bmatrix}=\begin{bmatrix} 7 & 24 & 12 \\ 0 & 25 & 0 \\ 8 & 36 & 15 \end{bmatrix} \quad \text{...(i)}$$

Also $C_{11}=\begin{vmatrix} 5 & 0 \\ 4 & 3 \end{vmatrix}=15,\ C_{12}=-\begin{vmatrix} 0 & 0 \\ 2 & 3 \end{vmatrix}=0;\ C_{13}=\begin{vmatrix} 0 & 5 \\ 2 & 4 \end{vmatrix}=-10$

$$C_{21}=-\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix}=6,\ C_{22}=\begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix}=-3;\ C_{23}=-\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}=0;$$

$$C_{33}=\begin{vmatrix} 2 & 3 \\ 5 & 0 \end{vmatrix}=-15,\ C_{32}=-\begin{vmatrix} 1 & 3 \\ 0 & 0 \end{vmatrix}=0;\ C_{33}=\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix}=5$$

$$\therefore \quad C=\begin{bmatrix} 15 & 0 & -10 \\ 6 & -3 & 0 \\ -15 & 0 & 5 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } A = C' = \begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix} \quad \text{...(ii)}$$

$\therefore\quad A^2 - 2A + \text{Adj. } A$

$$= \begin{bmatrix} 7 & 24 & 12 \\ 0 & 25 & 0 \\ 8 & 36 & 15 \end{bmatrix} - 2\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix} + \begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}, \quad \text{from (i) and (ii)}$$

$$= \begin{bmatrix} 7 & 24 & 12 \\ 0 & 25 & 0 \\ 8 & 36 & 15 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 6 \\ 0 & 10 & 0 \\ 4 & 8 & 6 \end{bmatrix} + \begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 7-2+15 & 24-4+6 & 12-6-15 \\ 0-0+0 & 25-10-3 & 0-0+0 \\ 8-4-10 & 36-8+0 & 15-6+5 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 26 & -9 \\ 0 & 12 & 0 \\ -6 & 28 & 14 \end{bmatrix}$$

Ans.

Exercises on § 5·08

Find the adjoint of the following matrices

Ex. 1. $\begin{bmatrix} -1 & -2 & 3 \\ -2 & 2 & 1 \\ 4 & -5 & 2 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 9 & -11 & -8 \\ 8 & -14 & -5 \\ 2 & -13 & -6 \end{bmatrix}$

Ex. 2. $\begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 7 & 9 & -10 \\ 12 & 15 & -17 \\ -1 & -1 & 1 \end{bmatrix}$

Ex. 3. $\begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$. **Ans.** $\begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$

Ex. 4. $\begin{bmatrix} 1 & 5 & 7 \\ 2 & 3 & 1 \\ 4 & 3 & 2 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 3 & 11 & -16 \\ 0 & -26 & 13 \\ -6 & 17 & -7 \end{bmatrix}$

Ex. 5. $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}$

Ex. 6. $\begin{bmatrix} 3 & 3 & 4 \\ 2 & 3 & 4 \\ 0 & 1 & 1 \end{bmatrix}$ **Ans.** $\begin{bmatrix} -7 & 1 & 24 \\ -2 & 3 & -4 \\ 2 & -3 & -15 \end{bmatrix}$

Ex. 7. $\begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & 2 & 2 \end{bmatrix}$

Ex. 8. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 8 & -5 & 2 \\ -4 & -3 & 1 \\ -7 & 3 & -1 \end{bmatrix}$

Ex. 9. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ **Ans.** $\begin{bmatrix} -1 & 2 & -1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{bmatrix}$

Ex. 10. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ **Ans.** $\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

Ex. 11. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 6 & 7 & 9 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 3 & 3 & -3 \\ 0 & -9 & 6 \\ -2 & 5 & -3 \end{bmatrix}$

Ex. 12. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$ *(Agra 95; Bundelkhand 92; Garhwal 92)*

Ans. $\begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}$

Ex. 13. $\begin{bmatrix} -1 & -2 & 3 \\ -2 & 1 & 1 \\ -4 & -5 & 2 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 7 & -11 & -5 \\ 0 & 10 & -5 \\ 14 & 3 & -5 \end{bmatrix}$

***Ex. 14.** $\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{bmatrix}$

§ 5·09. Theorems on Adjoint of a Matrix.

****Theorem I.** *If* $\mathbf{A} = [a_{ij}]$ *be an* $n \times n$ *matrix, then*

$\mathbf{A} \bullet (Adj\ \mathbf{A}) = (Adj\ \mathbf{A}) \bullet \mathbf{A} = |\mathbf{A}| \bullet \mathbf{I}_n$; *where* $\mathbf{I}_n$ *is an* $n \times n$ *identity matrix.*

(Agra 94, 91; Avadh 94, 92, 90; Bundelkhand 94, 93; Garhwal 90; Gorakhpur 97, 92; Kanpur 96; Meerut 91; Purvanchal 95; Rohilkhand 90)

Proof. We know Adj $\mathbf{A} = [C'_{jk}]$,

where C_{kj} is the cofactor of a_{kj} in $|\mathbf{A}|$ and $C'_{jk} = C_{kj}$.

Therefore $\mathbf{A} \bullet (\text{Adj } \mathbf{A}) = [a_{ij}][C'_{jk}]$

$= [B_{ik}]$, say. ...(i)

where $B_{ik} = \sum_{j=1}^{n} a_{ij} C'_{jk} = \sum_{j=1}^{n} a_{ij} C_{kj}$, $\because C'_{jk} = C_{kj}$

$$\left.\begin{array}{l} = |\mathbf{A}|, \text{ if } i = k \\ = 0, \text{ if } i \neq k \end{array}\right\}$$ ... See § 4·05 and § 4·09 in Ch. IV

$\therefore$ From (i), (i, k)th element of $\mathbf{A} \bullet (\text{Adj } \mathbf{A}) = |\mathbf{A}|$, or 0 according as $i = k$ or $i \neq k$.

i.e. All diagonal terms of $\mathbf{A} \bullet (\text{Adj } \mathbf{A})$ are $|\mathbf{A}|$ and non-diagonal terms are zero.

$$\mathbf{A} \bullet (\text{Adj. } \mathbf{A}) = \begin{bmatrix} |\mathbf{A}| & 0 & 0 & \dots & 0 \\ 0 & |\mathbf{A}| & 0 & \dots & 0 \\ 0 & 0 & |\mathbf{A}| & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & |\mathbf{A}| \end{bmatrix}$$

$$= |\mathbf{A}| \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}, \text{ See Chapter I}$$

$$= |\mathbf{A}| \bullet \mathbf{I} \qquad \text{...(ii)}$$

Similarly we can prove that $(\text{Adj. } \mathbf{A}) \bullet \mathbf{A} = |\mathbf{A}| \bullet \mathbf{I}$ ...(iii)

Hence from (ii) and (iii), we get

$$\mathbf{A} \bullet (\text{Adj. } \mathbf{A}) = (\text{Adj. } \mathbf{A}) \bullet \mathbf{A} = |\mathbf{A}| \bullet \mathbf{I}$$

or
$$\mathbf{A} \bullet \frac{(\text{Adj. } \mathbf{A})}{|\mathbf{A}|} = \frac{(\text{Adj. } \mathbf{A})}{|\mathbf{A}|} \bullet \mathbf{A} = \mathbf{I}$$

or
$$\mathbf{A}^{-1} = \frac{(\text{Adj. } \mathbf{A})}{|\mathbf{A}|}, \quad \because \mathbf{A}\mathbf{A}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$$

i.e. The inverse of $\mathbf{A} = \dfrac{\text{Adj. } \mathbf{A}}{|\mathbf{A}|}$ ...(iv)

Note : The result (iv) gives us another method of finding the inverse of a given matrix.

****Theorem II.** *If* $\mathbf{A} = [a_{ij}]$ *be an* $n \times n$ *matrix, then*

$|Adj\ \mathbf{A}| = |\mathbf{A}|^{n-1}$, *if* $|\mathbf{A}| \neq 0$. *(Agra 96; Gorakhpur 92; Rohilkhand 99, 91)*

Proof. We know that $|\mathbf{A}| \bullet |\mathbf{B}| = |\mathbf{AB}|$ ...See Ch. on Determinants

$\therefore \quad |\mathbf{A}| \bullet |Adj\ \mathbf{A}| = |\mathbf{A} \bullet Adj.\ \mathbf{A}|$

$$= \begin{vmatrix} |\mathbf{A}| & 0 & 0 & \dots & 0 \\ 0 & |\mathbf{A}| & 0 & \dots & 0 \\ 0 & 0 & |\mathbf{A}| & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & |\mathbf{A}| \end{vmatrix}, \text{ as proved in Theorem I above}$$

or $\quad |\mathbf{A}| \bullet |Adj\ \mathbf{A}| = (|\mathbf{A}|)^n$ **(Note)**

Dividing both sides by $|\mathbf{A}|$, since $|\mathbf{A}| \neq 0$, we get

$$|Adj\ \mathbf{A}| = |\mathbf{A}|^{n-1}.$$ Hence proved.

****Theorem III.** *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are two* $n \times n$ *matrices, then*

$Adj\ (\mathbf{AB}) = (Adj\ \mathbf{B}) \bullet (Adj\ \mathbf{A})$. *(Agra 93; Rohilkhand 98; Gorakhpur 98)*

Proof. We know $\mathbf{A} \bullet (Adj\ \mathbf{A}) = |\mathbf{A}| \bullet \mathbf{I}$ ...See Th. I Page 49 Ch. V

So we have $(\mathbf{AB}) \bullet (Adj\ \mathbf{AB}) = |\mathbf{AB}| \bullet \mathbf{I}$ ...(i)

Now $(\mathbf{AB}) \bullet (Adj\ \mathbf{B}) \bullet (Adj\ \mathbf{A})$

$= \mathbf{A} \bullet \mathbf{B} \bullet Adj\ \mathbf{B} \bullet Adj\ \mathbf{A}$

$= \mathbf{A} \bullet (\mathbf{B} \bullet Adj\ \mathbf{B}) \bullet (Adj\ \mathbf{A})$ **(Note)**

$= \mathbf{A} \bullet |\mathbf{B}| \bullet \mathbf{I} \bullet Adj\ \mathbf{A}$, $\because \mathbf{B} \bullet Adj\ \mathbf{B} = |\mathbf{B}| \bullet \mathbf{I}$

$= \mathbf{A} \bullet |\mathbf{B}| \bullet Adj\ \mathbf{A}$, $\because \mathbf{I} \bullet Adj\ \mathbf{A} = Adj\ \mathbf{A}$ as $\mathbf{I} \bullet \mathbf{A} = \mathbf{A}$ always

$= |\mathbf{B}| \bullet \mathbf{A} \bullet Adj\ \mathbf{A}$ **(Note)**

$= |\mathbf{B}| \bullet |\mathbf{A}| \bullet \mathbf{I}$ $\because \mathbf{A} \bullet Adj\ \mathbf{A} = |\mathbf{A}| \bullet \mathbf{I}$.

$= |\mathbf{A}| \bullet |\mathbf{B}| \bullet \mathbf{I}$,

$= |\mathbf{AB}| \bullet \mathbf{I}$. $\because |\mathbf{A}| \bullet |\mathbf{B}| = |\mathbf{AB}|$ ...(ii)

$\therefore$ From (i) and (ii) we get

$(\mathbf{AB}) \bullet (Adj\ \mathbf{AB}) = (\mathbf{AB}) \bullet (Adj\ \mathbf{B}) \bullet (Adj\ \mathbf{A})$

or $Adj\ (\mathbf{AB}) = (Adj\ \mathbf{B}) \bullet (Adj\ \mathbf{A})$. Hence proved.

Solved Examples on § 5·09.

****Ex. 1 (a). For the matrix A given in Ex. 2 Page 44 Ch. V verify the theorem A • (Adj A) = (Adj A) • A = | A | • I.**

Sol. In Ex. 2 Page 45 Ch. V. we have proved that if

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}, \text{ then Adj. } \mathbf{A} = \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

$$\therefore \quad \mathbf{A} \bullet (Adj\ \mathbf{A}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} \times \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-9-5 & -4+1+3 & -5+4+1 \\ 3-18+15 & -4+2-9 & -5+8-3 \\ 6+9-15 & -8-1+9 & -10-4+3 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & 0 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & -11 \end{bmatrix} \quad ...(i)$$

$$\text{Also } (Adj\ \mathbf{A}) \bullet \mathbf{A} = \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3-4-10 & 3-8+5 & 3+12-15 \\ -9+1+8 & -9+2-4 & -9-3+12 \\ -5+3+2 & -5+6-1 & -5-9+3 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & 0 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & -11 \end{bmatrix} \quad \text{...(ii)}$$

Also $|\mathbf{A}| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & -4 \\ 2 & -3 & 1 \end{vmatrix}$, replacing C_2, C_3 by $C_2 - C_1, C_3 - C_1$ ...(iii)

$$= \begin{vmatrix} 1 & -4 \\ -3 & 1 \end{vmatrix} = 1 - 12 = -11$$

∴ From (i), (ii) and (iii) we get

$$\mathbf{A} \bullet (\text{Adj } \mathbf{A}) = (\text{Adj. } \mathbf{A}) \bullet \mathbf{A} = -11 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(Note)

$$= (-11)\, \mathbf{I}_3 = |\mathbf{A}| \bullet \mathbf{I}.$$

Hence proved.

Ex. 1 (b). Find the adjoint of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$ and verify the theorem $\mathbf{A} \bullet (\text{Adj } \mathbf{A}) = (\text{Adj } \mathbf{A}) \bullet \mathbf{A} = |\mathbf{A}|\, \mathbf{I}$. *(Bundelkhand 93)*

Sol. For the given matrix **A**, we have

$C_{11} = -5, C_{12} = -3, C_{21} = -2, C_{22} = 1$

$$\therefore \quad \mathbf{C} = \begin{bmatrix} -5 & -3 \\ -2 & 1 \end{bmatrix}$$

And so $\text{Adj } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix}$

$$\therefore \quad \mathbf{A} \bullet (\text{Adj } \mathbf{A}) = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix} \times \begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5-6 & -2+2 \\ -15+15 & -6-5 \end{bmatrix} = \begin{bmatrix} -11 & 0 \\ 0 & -11 \end{bmatrix} \quad \text{...(i)}$$

Also $$(\text{Adj } \mathbf{A}) \bullet \mathbf{A} = \begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} -5-6 & -10+10 \\ -3+3 & -6-5 \end{bmatrix} = \begin{bmatrix} -11 & 0 \\ 0 & -11 \end{bmatrix} \quad \text{...(ii)}$$

Also $|\mathbf{A}| = \begin{vmatrix} 1 & 2 \\ 3 & -5 \end{vmatrix} = -5 - 6 = -11$

∴ From (i) and (ii), we get

$$\mathbf{A} \bullet (\text{Adj } \mathbf{A}) = (\text{Adj } \mathbf{A}) \bullet \mathbf{A} = \begin{bmatrix} -11 & 0 \\ 0 & -11 \end{bmatrix}$$

$$= -11 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |\mathbf{A}|\, \mathbf{I}_2 = |\mathbf{A}|\, \mathbf{I}$$

Hence proved.

Ex. 1 (c). Verify the theorem A • (Adj. A) = (Adj. A) • A

$$= |A| \bullet I \text{ when } A = \begin{bmatrix} 2 & -1 & 3 \\ 5 & 3 & 1 \\ 3 & 2 & 3 \end{bmatrix}$$

Sol. Do as Ex. 1 (a) above.

Ex. 2 (a). Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ *(Agra 91)*

Sol. For the given matrix **A**, we have

$C_{11} = \begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} = -1; C_{12} = -\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} = 3; C_{13} = \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} = -2;$

$C_{21} = -\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = 3; C_{22} = \begin{vmatrix} 1 & 3 \\ 3 & 6 \end{vmatrix} = -3; C_{23} = -\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = 1;$

$C_{31} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -2; C_{32} = -\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} = 1; C_{33} = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$

$$\therefore \quad C = \begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } A = C' = \begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix}$$

Also $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ 3 & -1 & -3 \end{vmatrix}$, replacing C_2, C_3, by $C_2 - 2C_1, C_3 - 3C_1$

$$= \begin{vmatrix} 0 & -1 \\ -1 & -3 \end{vmatrix} = -1$$

$$\therefore \quad A^{-1} = \frac{\text{Adj } A}{|A|}$$

$$= -\begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$ **Ans.**

Ex. 2 (b). Find the inverse of the matrix $A = \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$ *(Agra 96)*

Sol. Here $|A| = \begin{vmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & -1 \\ -7 & 3 & -1 \\ 5 & -2 & 1 \end{vmatrix}$,

replacing C_1, C_2 by $C_1 + 3C_3, C_2 - 2C_1$ respectively.

$$= -\begin{vmatrix} -7 & 3 \\ 5 & -2 \end{vmatrix}, \text{ expanding w. r. to } R_1$$

$$= -[14 - 15] = 1$$

Also for the matrix **A**, we have

$C_{11} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1;\ C_{12} = -\begin{vmatrix} -4 & -1 \\ 2 & 1 \end{vmatrix} = 2;\ C_{13} = \begin{vmatrix} -4 & 1 \\ 2 & 0 \end{vmatrix} = -2;$

$C_{21} = -\begin{vmatrix} -2 & -1 \\ 0 & 1 \end{vmatrix} = 2;\ C_{22} = \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} = 5;\ C_{23} = -\begin{vmatrix} 3 & -2 \\ 2 & 0 \end{vmatrix} = -4;$

$C_{31} = \begin{vmatrix} -2 & -1 \\ 1 & -1 \end{vmatrix} = 3;\ C_{32} = -\begin{vmatrix} 3 & -1 \\ -4 & -1 \end{vmatrix} = 7;\ C_{13} = \begin{vmatrix} 3 & -2 \\ -4 & 1 \end{vmatrix} = -5$

$\therefore \quad \mathbf{C} = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 3 & 7 & -5 \end{bmatrix}$

$\therefore \quad \text{Adj. } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$

$\therefore \quad \mathbf{A}^{-1} = \dfrac{\text{Adj } \mathbf{A}}{|\mathbf{A}|} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$ **Ans.**

Ex. 3 (a). Find the inverse of $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$

(Avadh 98, 91; Purvanchal 96)

Sol. Here $|\mathbf{A}| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{vmatrix},$

replacing R_2, R_3 by $R_2 - R_1, R_3 - R_1$

or $\quad |\mathbf{A}| = \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -2 \quad$ ...(i)

Also for the matrix **A**, we have

$C_{11} = \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} = -7;\ C_{12} = -\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = 1;\ C_{13} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1;$

$C_{21} = -\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} = 6;\ C_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0;\ C_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = -2;$

$C_{31} = \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = -1;\ C_{32} = -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1;\ C_{33} = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1$

$$\therefore \quad C = \begin{bmatrix} -7 & 1 & 1 \\ 6 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } \mathbf{A} = C' = \begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\therefore \quad \mathbf{A}^{-1} = \frac{\text{Adj. } \mathbf{A}}{|\mathbf{A}|}$$

$$= -\frac{1}{2}\begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} & -3 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix}$$

Ans

Ex. 3 (b). Find the inverse of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ *(Avadh 92,*

Hint : Do as Ex. 3 (a) above. **Ans.** $\begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

Ex. 3 (c). Find the adjoint of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$ **and hence evaluate** $\mathbf{A}^{-1}$**.** *(Kumaun 94,*

Hint. Do as Ex. 3 (a). above. **Ans.** $\begin{bmatrix} \frac{11}{3} & -3 & \frac{1}{3} \\ -\frac{7}{3} & 3 & -\frac{2}{3} \\ \frac{2}{3} & -1 & \frac{1}{3} \end{bmatrix}$

Ex. 3 (d). Find the inverse of $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 0 & 3 \end{bmatrix}$

Hint. Do as Ex. 3 (a) above. **Ans.** $-\frac{1}{15}\begin{bmatrix} 15 & -6 & -15 \\ 0 & -3 & 0 \\ -10 & 4 & 5 \end{bmatrix}$

Ex. 4 (a). Find the adjoint of the matrix A and evaluate $\mathbf{A}^{-1}$**, where**

$$\mathbf{A} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 11 \end{bmatrix}$$

Sol. Here for the matrix **A**, we have

$$C_{11} = \begin{vmatrix} 5 & 5 \\ 5 & 11 \end{vmatrix} = 30; C_{12} = -\begin{vmatrix} 2 & 5 \\ 2 & 11 \end{vmatrix} = -12; C_{13} = \begin{vmatrix} 2 & 5 \\ 2 & 5 \end{vmatrix} = 0;$$

$$C_{21} = -\begin{vmatrix} 2 & 2 \\ 5 & 11 \end{vmatrix} = -12; C_{22} = \begin{vmatrix} 2 & 2 \\ 2 & 11 \end{vmatrix} = 18; C_{23} = -\begin{vmatrix} 2 & 2 \\ 2 & 5 \end{vmatrix} = -6;$$

$$C_{31} = \begin{vmatrix} 2 & 2 \\ 5 & 5 \end{vmatrix} = 0; C_{32} = -\begin{vmatrix} 2 & 2 \\ 2 & 5 \end{vmatrix} = -6; C_{33} = \begin{vmatrix} 2 & 2 \\ 2 & 5 \end{vmatrix} = 6$$

$$\therefore \quad \mathbf{C} = \begin{bmatrix} 30 & -12 & 0 \\ -12 & 18 & -6 \\ 0 & -6 & 6 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} 30 & -12 & 0 \\ -12 & 18 & -6 \\ 0 & -6 & 6 \end{bmatrix}$$

Ans.

Also $|\mathbf{A}| = \begin{vmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 11 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 2 & 3 & 3 \\ 2 & 3 & 9 \end{vmatrix}$, applying $C_2 - C_1$, $C_3 - C_1$

$$= 2 \begin{vmatrix} 3 & 3 \\ 3 & 9 \end{vmatrix} = 2\,[27 - 9] = 36$$

$$\therefore \quad \mathbf{A}^{-1} = \frac{\text{Adj. } \mathbf{A}}{|\mathbf{A}|} = \frac{1}{36} \begin{bmatrix} 30 & -12 & 0 \\ -12 & 18 & -6 \\ 0 & -6 & 6 \end{bmatrix}$$

$$= \frac{1}{36} \times 6 \begin{bmatrix} 5 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 5/6 & -1/3 & 0 \\ -1/3 & 1/2 & -1/6 \\ 0 & -1/6 & 1/6 \end{bmatrix}$$

Ans.

Ex. 4 (b). Find the inverse of matrix A, where

$$\mathbf{A} = \begin{bmatrix} \mathbf{4} & \mathbf{3} & \mathbf{3} \\ -\mathbf{1} & \mathbf{0} & -\mathbf{1} \\ -\mathbf{4} & -\mathbf{4} & -\mathbf{3} \end{bmatrix}$$

(Agra 94)

Sol. Here $|\mathbf{A}| = \begin{vmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 3 \\ 0 & 0 & -1 \\ -1 & -4 & -3 \end{vmatrix}$, replacing C_1 by $C_1 - C_3$

$$= \begin{vmatrix} 1 & 3 \\ -1 & -4 \end{vmatrix}, \text{ expanding w.r. to } R_2$$

$$= -4 + 3 = -1 \qquad \text{...(i)}$$

Also for the matrix **A**, we have

$C_{11} = \begin{vmatrix} 0 & -1 \\ -4 & -3 \end{vmatrix} = -4;\ C_{12} = -\begin{vmatrix} -1 & -1 \\ -4 & -3 \end{vmatrix} = 1;\ C_{13} = \begin{vmatrix} -1 & 0 \\ -4 & -4 \end{vmatrix} = 4;$

$C_{21} = -\begin{vmatrix} 3 & 3 \\ -4 & -3 \end{vmatrix} = -3;\ C_{22} = \begin{vmatrix} 4 & 3 \\ -4 & -3 \end{vmatrix} = 0;\ C_{23} = -\begin{vmatrix} 4 & 3 \\ -4 & -4 \end{vmatrix} = 4;$

$C_{31} = \begin{vmatrix} 3 & 3 \\ 0 & -1 \end{vmatrix} = -3;\ C_{32} = -\begin{vmatrix} 4 & 3 \\ -1 & -1 \end{vmatrix} = 1:\ C_{33} = \begin{vmatrix} 4 & 3 \\ -1 & 0 \end{vmatrix} = 3$

$\therefore\quad \mathbf{C} = \begin{bmatrix} -4 & 1 & 4 \\ -3 & 0 & 4 \\ -3 & 1 & 3 \end{bmatrix}$

$\therefore\quad \text{Adj. } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$ ...(ii)

$\therefore\quad \mathbf{A}^{-1} = \frac{\text{Adj } \mathbf{A}}{|\mathbf{A}|} = -\frac{1}{1}\begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$, from (i) and (ii)

$= \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$ **Ans.**

Ex. 5 (a). Find the inverse of $\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ 4 & 1 & 0 \\ 8 & 1 & 1 \end{bmatrix}$

Sol. Here $|\mathbf{A}|$

$= \begin{vmatrix} 1 & -1 & 1 \\ 4 & 1 & 0 \\ 8 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 0 \\ 5 & 1 & 1 \\ 9 & 1 & 2 \end{vmatrix}$, replacing C_1, C_3 by $C_1 + C_2, C_3 + C_2$

or $\quad |\mathbf{A}| = \begin{vmatrix} 5 & 1 \\ 9 & 2 \end{vmatrix} = 10 - 9 = 1 \neq 0$ ...(i)

Also for the matrix **A**, we have

$C_{11} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1;\ C_{22} = -\begin{vmatrix} 4 & 0 \\ 8 & 1 \end{vmatrix} = -4;\ C_{13} = \begin{vmatrix} 4 & 1 \\ 8 & 1 \end{vmatrix} = -4;$

$C_{21} = -\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = 2;\ C_{22} = \begin{vmatrix} 1 & 1 \\ 8 & 1 \end{vmatrix} = -7;\ C_{23} = -\begin{vmatrix} 1 & -1 \\ 8 & 1 \end{vmatrix} = -9:$

$C_{31} = \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = -1,\ C_{32} = -\begin{vmatrix} 1 & 1 \\ 4 & 0 \end{vmatrix} = 4;\ C_{33} = \begin{vmatrix} 1 & -1 \\ 4 & 1 \end{vmatrix} = 5$

$\mathbf{C} = \begin{bmatrix} 1 & -4 & -4 \\ 2 & -7 & -9 \\ -1 & 4 & 5 \end{bmatrix}$

$\therefore$ Adj. $\mathbf{A} = \mathbf{C}' = \begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -9 & 5 \end{bmatrix}$

$\therefore$ $\mathbf{A}^{-1} = \dfrac{\text{Adj. } \mathbf{A}}{|\mathbf{A}|} = \mathbf{C}' = \begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -9 & 5 \end{bmatrix}$, from (i).

Ans.

E. 5 (b). Find the inverse $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{4} \\ \mathbf{5} & \mathbf{7} & \mathbf{8} \\ \mathbf{9} & \mathbf{10} & \mathbf{12} \end{bmatrix}$

Hint : Do as Ex. 5 (a) above. **Ans.** $\begin{bmatrix} -\frac{1}{6} & -\frac{2}{3} & \frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{13}{24} & -\frac{1}{3} & \frac{1}{8} \end{bmatrix}$

Ex. 5 (c). Find the inverse of the matrix $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{3} \\ \mathbf{1} & \mathbf{4} & \mathbf{3} \\ \mathbf{1} & \mathbf{3} & \mathbf{4} \end{bmatrix}$

Hint : Do as Ex, 5 (a) Page 57 **Ans.** $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Ex. 6 (a). Find adj A and $\mathbf{A}^{-1}$ **when** $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{3} \\ \mathbf{1} & \mathbf{4} & \mathbf{3} \\ \mathbf{1} & \mathbf{3} & \mathbf{4} \end{bmatrix}$

(Bundelkhand 94; Kanpur 93)

Sol. Here $|\mathbf{A}| = \begin{vmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$, replacing R_2, R_3 by $R_2 - R_1, R_3 - R_1$ respectively.

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \qquad \text{...(i)}$$

Also for the matrix **A**, we have

$C_{11} = \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} = 7$; $C_{12} = -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1$; $C_{13} = \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = -1$;

$C_{21} = -\begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix} = -3$; $C_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1$; $C_{23} = \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$;

$C_{31} = \begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix} = -3$; $C_{32} = -\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$, $C_{33} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1$.

$\therefore \quad \mathbf{C} = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

$\therefore \quad \text{Adj } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ ...(ii)

$\therefore \quad \mathbf{A}^{-1} = \dfrac{\text{Adj } \mathbf{A}}{|\mathbf{A}|} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, from (i) and (ii) **Ans.**

Ex. 6 (b). Find the inverse of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ *(Lucknow 91)*

Sol. Here $|\mathbf{A}| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$, expanding with respect to C_1

or $\quad |\mathbf{A}| = 1 - 0 = 1.$ ...(i)

Also for the matrix **A**, we have

$C_{11} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1;\ C_{12} = -\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} = 0;\ C_{13} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0;$

$C_{21} = -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1;\ C_{22} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1;\ C_{23} = -\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = 0;$

$C_{31} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0;\ C_{32} = -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1;\ C_{33} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$

$\therefore \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$\therefore \quad \text{Adj. } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ ...(ii)

$\therefore \quad \mathbf{A}^{-1} = \dfrac{\text{Adj. } \mathbf{A}}{|\mathbf{A}|} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. from (i) and (ii) **Ans.**

***Ex. 7 (a). If** $\mathbf{A} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$, **find adj. A and** $\mathbf{A}^{-1}$. *(Garhwal 95, 91; Meerut 95)*

Sol. Here $|\mathbf{A}|$

$$= \begin{vmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 0 \\ 3 & 6 & 1 \\ -1 & -2 & 0 \end{vmatrix}$$, replacing C_1, C_3 by $C_1 + 3C_2, C_2 + C_3$

$$= \begin{vmatrix} 3 & 1 \\ -1 & 0 \end{vmatrix} = 1 \qquad \text{...(i)}$$

Also for the matrix **A**, we have

$C_{11} = \begin{vmatrix} 6 & -5 \\ -2 & 2 \end{vmatrix} = 2$; $C_{12} = -\begin{vmatrix} -15 & -5 \\ 5 & 2 \end{vmatrix} = 5$; $C_{13} = \begin{vmatrix} -15 & 6 \\ 5 & -2 \end{vmatrix} = 0$

$C_{21} = -\begin{vmatrix} -1 & 1 \\ -2 & 2 \end{vmatrix} = 0$; $C_{22} = \begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} = 1$; $C_{23} = -\begin{vmatrix} 3 & -1 \\ 5 & -2 \end{vmatrix} = 1$;

$C_{31} = \begin{vmatrix} -1 & 1 \\ 6 & -5 \end{vmatrix} = -1$; $C_{32} = -\begin{vmatrix} 3 & 1 \\ -15 & -5 \end{vmatrix} = 0$; $C_{33} = \begin{vmatrix} 3 & -1 \\ -15 & 6 \end{vmatrix} = 3$

$$\therefore \quad \mathbf{C} = \begin{bmatrix} 2 & 5 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 3 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\therefore \quad \mathbf{A}^{-1} = \frac{\text{Adj. } \mathbf{A}}{|\mathbf{A}|} = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}, \text{ from (i)}$$

Ans.

Ex. 7 (b). Find the inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-2} \\ \mathbf{-1} & \mathbf{3} & \mathbf{0} \\ \mathbf{0} & \mathbf{-2} & \mathbf{1} \end{bmatrix}$$

(Gorakhpur 97)

Sol. Here $|\mathbf{A}| = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -2 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{vmatrix}$, replacing R_2 by $R_2 + R_1$

$$= \begin{vmatrix} 5 & -2 \\ -2 & 1 \end{vmatrix} = 5 - 4 = 1. \qquad \text{...(i)}$$

Also for the matrix **A**, we have

$C_{11} = \begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} = 3$; $C_{12} = -\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = 1$; $C_{13} = \begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix} = 2$;

$C_{21} = -\begin{vmatrix} 2 & -2 \\ -2 & 1 \end{vmatrix} = 2$; $C_{22} = \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = 1$; $C_{23} = -\begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} = 2$;

$C_{31} = \begin{vmatrix} 2 & -2 \\ 3 & 0 \end{vmatrix} = 6; C_{32} = -\begin{vmatrix} 1 & -2 \\ -1 & 0 \end{vmatrix} = 2; C_{33} = \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = 5$

$\therefore \quad \mathbf{C} = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{bmatrix}$

$\therefore \quad \text{Adj. } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$ ...(ii)

$\therefore \quad \mathbf{A}^{-1} = \dfrac{\text{Adj } \mathbf{A}}{|\mathbf{A}|} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$, from (i) and (ii) **Ans.**

Ex. 7 (c). If $\mathbf{A} = \begin{bmatrix} 1 & 4 & 0 \\ -1 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$, **find** $\mathbf{A}^{-1}$.

Hint : Do as Ex. 7 (a) Page 59. **Ans.** $(1/6)\begin{bmatrix} 2 & -4 & -4 \\ 1 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}$

***Ex. 7 (d). If** $\begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$, **find adj. A and** $\mathbf{A}^{-1}$

(Avadh 94)

Hint : Do as Ex. 7 (a). Page 60. **Ans.** $(1/20)\begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & 8 \\ -18 & 6 & 4 \end{bmatrix}$

Ex. 8 (a). Find the reciprocal or inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

(Kumaun 92)

Sol. Here $|\mathbf{A}| = \begin{vmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 0 & -3 & -2 \\ 0 & -2 & -3 \\ 1 & 2 & 2 \end{vmatrix}$, applying $R_1 - 2R_3, R_2 - 2R_3$

$= \begin{vmatrix} -3 & -2 \\ -2 & -3 \end{vmatrix} = 9 - 4 = 5$

Also we have

$C_{11} = \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = 2; C_{12} = -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = -3; C_{13} = \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = 2;$

$C_{21} = -\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = 2; C_{22} = \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = 2; C_{23} = -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = -3;$

$C_{31} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3;\ C_{32} = -\begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} = 2;\ C_{33} = \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = 2$

$\therefore\ \mathbf{C} = \begin{bmatrix} 2 & -3 & 2 \\ 2 & 2 & -3 \\ -3 & 2 & 2 \end{bmatrix}$

$\therefore\ \text{Adj. } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} 2 & 2 & -3 \\ -3 & 2 & 2 \\ 2 & -3 & 2 \end{bmatrix}$

$\therefore$ Reciprocal of $\mathbf{A} = \mathbf{A}^{-1}$

$$= \frac{\text{Adj. } \mathbf{A}}{|\mathbf{A}|} = \frac{1}{5}\begin{bmatrix} 2 & 2 & -3 \\ -3 & 2 & 2 \\ 2 & -3 & 2 \end{bmatrix}$$

Ans.

Ex. 8 (b). Find the adjoint and inverse of $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{pmatrix}$.

Hint : Do as Ex. 8 (a) above.

Ans. $\begin{bmatrix} -\frac{6}{7} & -\frac{1}{7} & \frac{5}{7} \\ \frac{2}{7} & \frac{5}{7} & -\frac{4}{7} \\ \frac{3}{7} & -\frac{3}{7} & \frac{1}{7} \end{bmatrix}$

Ex. 9 (a). Find the inverse of $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$.

(Avadh 90; Bundelkhand 96, 95; Garhwal 96, 94; Gorakhpur 96; Purvanchal 97)

Sol. For the given matrix **A**, we have

$C_{11} = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 1;\ C_{12} = -\begin{vmatrix} 3 & 3 \\ 1 & 2 \end{vmatrix} = -3;\ C_{13} = \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = 1;$

$C_{21} = -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = -3;\ C_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1;\ C_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1;$

$C_{31} = \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = 4;\ C_{32} = -\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} = 0;\ C_{33} = \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = -4$

$\therefore\ \mathbf{C} = \begin{bmatrix} 1 & -3 & 1 \\ -3 & 1 & 1 \\ 4 & 0 & -4 \end{bmatrix}$

$\therefore\ \text{Adj. } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} 1 & -3 & 4 \\ -3 & 1 & 0 \\ 1 & 1 & -4 \end{bmatrix}$

and $|\mathbf{A}| = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{vmatrix}$, replacing C_3 by $C_3 - C_1$

$= \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = -4$

The inverse of $\mathbf{A} = \dfrac{\text{Adj. } \mathbf{A}}{|\mathbf{A}|}$

$$= -\frac{1}{4}\begin{bmatrix} 1 & -3 & 4 \\ -3 & 1 & 0 \\ 1 & 1 & -4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & -1 \\ \frac{3}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 1 \end{bmatrix}$$ **Ans.**

Ex. 9 (b). Find the inverse of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$ *(Meerut 98)*

Sol. Here $|\mathbf{A}| = \begin{vmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & -5 & 3 \end{vmatrix}$, replacing R_2, R_3 by $R_2 + R_1, R_3 - 2R_1$ respectively.

$= \begin{vmatrix} 3 & 1 \\ -5 & 3 \end{vmatrix} = 9 + 5 = 14$ ...(i)

Also for the matrix **A**, we have

$C_{11} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3$; $C_{12} = -\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} = 5$; $C_{13} = \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} = -1$;

$C_{21} = -\begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} = -1$; $C_{22} = \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 3$; $C_{23} = -\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = 5$;

$C_{31} = \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 5$; $C_{32} = -\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = -1$; $C_{33} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3$

$\therefore$ $\mathbf{C} = \begin{bmatrix} 3 & 5 & -1 \\ -1 & 3 & 5 \\ 5 & -1 & 3 \end{bmatrix}$

$\therefore$ Adj. $\mathbf{A} = \mathbf{C}' = \begin{bmatrix} 3 & -1 & 5 \\ 5 & 3 & -1 \\ -1 & 5 & 3 \end{bmatrix}$ ...(ii)

$\therefore$ $\mathbf{A}^{-1} = \dfrac{\text{Adj } \mathbf{A}}{|\mathbf{A}|} = \dfrac{1}{14}\begin{bmatrix} 3 & -1 & 5 \\ 5 & 3 & -1 \\ -1 & 5 & 3 \end{bmatrix}$, from (i), (ii) **Ans.**

Ex. 9 (c). Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ *(Purvanchal 95)*

Hint : Do as Ex. 9 (a) above.

Ans. $\frac{1}{18}\begin{bmatrix} -5 & 1 & 7 \\ 1 & 7 & -5 \\ 7 & -5 & 1 \end{bmatrix}$.

Ex. 10. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, **find** A^2, **and show that** $A^2 = A^{-1}$.

Sol. $$A^2 = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2+1 & -1+1+0 & 1+0+0 \\ 2-2+0 & -2+1+0 & 2+0+0 \\ 1+0+0 & -1+0+0 & 1+0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \quad ...(i)$$

Also $|A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 1 & 1 & -1 \end{vmatrix}$, replacing C_2, C_3 by $C_2 + C_1, C_3 - C_1$ respectively.

$$= \begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} = -1 + 2 = 1. \quad ...(ii)$$

Also for the matrix **A**, we have

$C_{11} = \begin{vmatrix} -1 & 0 \\ 0 & 0 \end{vmatrix} = 0$; $C_{12} = -\begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} = 0$; $C_{13} = \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} = 1$;

$C_{21} = -\begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix} = 0$; $C_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1$; $C_{23} = -\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = -1$;

$C_{31} = \begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix} = 1$; $C_{32} = -\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = 2$; $C_{33} = \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = 1$

$\therefore \quad C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$

$$\therefore \quad \text{Adj } A = C' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \quad ...(iii)$$

$$\therefore \quad A^{-1} = \frac{\text{Adj. } A}{|A|} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}, \text{ from (ii) and (iii)}$$

$= A^2$, from (i) **Hence proved.**

Ex. 11. Find the adjoint of matrix A and hence find A^{-1}.

$$A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ *(Meerut 96)*

Sol. Here $|A|$

$$= \begin{vmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}, \text{ expanding w.r. to } C_3$$

$$= \cos^2\theta - (-\sin^2\theta) = 1 \quad \text{...(i)}$$

Also we have

$C_{11} = \begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} = \cos\theta$; $C_{12} = -\begin{vmatrix} \sin\theta & 0 \\ 0 & 1 \end{vmatrix} = -\sin\theta$;

$C_{13} = \begin{vmatrix} \sin\theta & \cos\theta \\ 0 & 0 \end{vmatrix} = 0$; $C_{21} = -\begin{vmatrix} -\sin\theta & 0 \\ 0 & 1 \end{vmatrix} = \sin\theta$;

$C_{22} = \begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} = \cos\theta$; $C_{23} = -\begin{vmatrix} \cos\theta & -\sin\theta \\ 0 & 0 \end{vmatrix} = 0$;

$C_{31} = \begin{vmatrix} -\sin\theta & 0 \\ \cos\theta & 0 \end{vmatrix} = 0$; $C_{32} = -\begin{vmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{vmatrix} = 0$;

$C_{33} = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = 1$

$$\therefore \quad C = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } A = C' = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ **Ans.**

And $$A^{-1} = \frac{\text{Adj. } A}{|A|} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ from (i)}$$ **Ans.**

***Ex. 12. How will you use the notion of determinant to compute the inverse of a non-singular square matrix ? Compute the inverse of the matrix**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$$

Sol. For the first part See Theorem I, result (iv) Page 50 of this chapter

For the second part we have for the matrix **A**

$$C_{11} = \begin{vmatrix} 5 & 6 \\ 8 & 10 \end{vmatrix} = 2;\ C_{12} = -\begin{vmatrix} 4 & 6 \\ 7 & 10 \end{vmatrix} = 2;\ C_{13} = \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = -3;$$

$$C_{21} = -\begin{vmatrix} 2 & 3 \\ 8 & 10 \end{vmatrix} = 4;\ C_{22} = \begin{vmatrix} 1 & 3 \\ 7 & 10 \end{vmatrix} = -11;\ C_{23} = -\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = 6;$$

$$C_{31} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = -3;\ C_{32} = -\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = 6;\ C_{33} = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = -3$$

$$\therefore \quad \mathbf{C} = \begin{bmatrix} 2 & 2 & -3 \\ 4 & -11 & 6 \\ -3 & 6 & -3 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} 2 & 4 & -3 \\ 2 & -11 & 6 \\ -3 & 6 & -3 \end{bmatrix} \qquad \text{...(i)}$$

Also $|\mathbf{A}| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 4 & -3 & -6 \\ 7 & -6 & -11 \end{vmatrix}$, replacing C_2, C_3 by $C_2 - 2C_1, C_3 - 3C_1$ respectively.

$$= \begin{vmatrix} -3 & -6 \\ -6 & -11 \end{vmatrix} = 33 - 36 = -3$$

$$\therefore \quad \mathbf{A}^{-1} = \frac{\text{Adj. } \mathbf{A}}{|\mathbf{A}|} = -\frac{1}{3}\begin{bmatrix} 2 & 4 & -3 \\ 2 & -11 & 6 \\ -3 & 6 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2}{3} & -\frac{4}{3} & 1 \\ -\frac{2}{3} & \frac{11}{3} & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

Ans.

***Ex. 13. If $\mathbf{A}^t$ denotes the transpose of a matrix A and**

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}, \textbf{ find } (A^t)^{-1}$$

Sol. $\mathbf{A}' = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$, by definition of transpose of a matrix

$= \mathbf{B}$ (say).

Now $|\mathbf{B}| = \begin{vmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -2 & -1 & -2 \\ 3 & 4 & 7 \end{vmatrix}$, replacing C_3 by $C_3 + 2C_1$

$$= \begin{vmatrix} -1 & -2 \\ 4 & 7 \end{vmatrix}, \text{ expanding with respect to } R_1$$

$$= (-1)(7) - (-2)(4) = -7 + 8 = 1 \neq 0.$$

Also we have

$$C_{11} = \begin{vmatrix} -1 & 2 \\ 4 & 1 \end{vmatrix} = -9; C_{12} = -\begin{vmatrix} -2 & 2 \\ 3 & 1 \end{vmatrix} = 8; C_{13} = \begin{vmatrix} -2 & -1 \\ 3 & 4 \end{vmatrix} = -5;$$

$$C_{21} = -\begin{vmatrix} 0 & -2 \\ 4 & 1 \end{vmatrix} = -8; C_{22} = \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} = 7; C_{23} = -\begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} = -4;$$

$$C_{31} = \begin{vmatrix} 0 & -2 \\ -1 & 2 \end{vmatrix} = -2; C_{32} = -\begin{vmatrix} 1 & -2 \\ -2 & 2 \end{vmatrix} = 2; C_{33} = \begin{vmatrix} 1 & 0 \\ -2 & -1 \end{vmatrix} = -1$$

$$\therefore \quad \mathbf{C} = \begin{bmatrix} -9 & 8 & -5 \\ -8 & 7 & -4 \\ -2 & 2 & -1 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } \mathbf{B} = \mathbf{C}' = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

$$\therefore \quad \mathbf{B}^{-1} = \frac{\text{Adj. } \mathbf{B}}{|\mathbf{B}|} = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

or

$$(\mathbf{A}^t)^{-1} = \mathbf{B}^{-1} = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

Ans.

***Ex. 14. Find the inverse of the matrix A, where**

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -4 \\ -2 & 2 & 5 \\ 3 & -1 & 2 \end{bmatrix}$$

Hint : Do as Ex. 12 Page 65.

Ans. $(1/25)\begin{bmatrix} 9 & 4 & 8 \\ 19 & 14 & 3 \\ -4 & 1 & 2 \end{bmatrix}$

Ex. 15. Find the adjoint and inverse of the matrix

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

(Bundelkhand 92)

Sol. Here $|\mathbf{A}| = \begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix}$

$$= \cos^2\alpha + \sin^2\alpha = 1 \neq 0. \quad \text{...(i)}$$

$C_{11} = \cos\alpha$, $C_{12} = -\sin\alpha$, $C_{21} = -(-\sin\alpha) = \sin\alpha$ and $C_{22} = \cos\alpha$ **(Note)**

$\therefore \quad C = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ ...(ii)

$\therefore \quad \text{Adj. } A = C' = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

$\therefore \quad A^{-1} = \dfrac{\text{Adj. } A}{|A|} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix},$

substituting values from (i) and (ii). **Ans.**

****Ex. 16. Find the inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$** *(Meerut 91S)*

Sol. For the given matrix A, we have

$C_{11} = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1; C_{12} = -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = 8; C_{13} = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5;$

$C_{21} = -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1; C_{22} = \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} = -6; C_{23} = -\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = 3;$

$C_{31} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1; C_{32} = -\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = 2; C_{33} = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -1$

$\therefore \quad C = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix}$

$\therefore \quad \text{Adj. } A = C' = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

and $\quad |A| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 2 & -1 \\ 3 & 1 & -1 \end{vmatrix}$, replacing C_3 by $C_3 - 2C_2$

or $\quad |A| = -\begin{vmatrix} 1 & -1 \\ 3 & -1 \end{vmatrix} = -2 \neq 0$

$\therefore$ Inverse of $A = \dfrac{\text{Adj. } A}{|A|} = -\dfrac{1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix} = \dfrac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{bmatrix}.$ **Ans.**

Ex. 17 (a). Find the inverse of the matrix $A = \begin{bmatrix} i & -1 & 2i \\ 2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$ over the field of the complex numbers.

Sol. Here $|\mathbf{A}| = \begin{vmatrix} i & -1 & 2i \\ 2 & 0 & 2 \\ -1 & 0 & 1 \end{vmatrix} = -\begin{vmatrix} -1 & i & 2i \\ 0 & 2 & 2 \\ 0 & -1 & 1 \end{vmatrix}$, interchanging C_1 and C_2

or $|\mathbf{A}| = \begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix}$, expanding with respect to C_1

$= 2 - (-2) = 4 \neq 0.$ ...(i)

Also for this matrix **A**, we have

$C_{11} = \begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix} = 0;\ C_{12} = -\begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix} = -4;\ C_{13} = \begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix} = 0;$

$C_{21} = -\begin{vmatrix} -1 & 2i \\ 0 & 1 \end{vmatrix} = 1;\ C_{22} = \begin{vmatrix} i & 2i \\ -1 & 1 \end{vmatrix} = 3i;\ C_{23} = -\begin{vmatrix} i & -1 \\ -1 & 0 \end{vmatrix} = 1;$

$C_{31} = \begin{vmatrix} -1 & 2i \\ 0 & 2 \end{vmatrix} = -2;\ C_{32} = -\begin{vmatrix} i & 2i \\ 2 & 2 \end{vmatrix} = 2i,\ C_{33} = \begin{vmatrix} i & -1 \\ 2 & 0 \end{vmatrix} = 2$

$\therefore \quad \mathbf{C} = \begin{bmatrix} 0 & -4 & 0 \\ 1 & 3i & 1 \\ -2 & 2i & 2 \end{bmatrix}$

$\therefore \quad \text{Adj. } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} 0 & 1 & -2 \\ -4 & 3i & 2i \\ 0 & 1 & 2 \end{bmatrix}.$

$\therefore \quad \mathbf{A}^{-1} = \frac{\text{Adj. } \mathbf{A}}{|\mathbf{A}|} = \frac{1}{4}\begin{bmatrix} 0 & 1 & -2 \\ -4 & 3i & 2i \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{4} & -\frac{1}{2} \\ -1 & \frac{3}{4}i & \frac{1}{2}i \\ 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$ **Ans.**

Ex. 17 (b). If $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, then show that $\mathbf{A}^{-1} = \mathbf{A}$** ***(Bundelkhand 91)

Sol. Here $|\mathbf{A}| = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$ ...(i)

Also for the matrix **A**, we have

$C_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0;\ C_{12} = -\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0;\ C_{13} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1;$

$C_{21} = -\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0;\ C_{22} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1;\ C_{23} = -\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0;$

$C_{31} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1;\ C_{32} = -\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0;\ C_{33} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$

$\therefore \quad C = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

$\therefore \quad \text{Adj. } A = C' = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

$\therefore \quad A^{-1} = \dfrac{\text{Adj } A}{|A|} = \dfrac{1}{(-1)} \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = A$

Hence proved.

Ex. 18. Find the inverse of the matrix

$$A = \begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}, \text{ if } a^2+b^2+c^2+d^2 = 1$$

Sol. For this matrix, we have

$C_{11} = a - ib$; $C_{12} = -(-c + id) = c - id$;

$C_{21} = -(c + id)$; $C_{22} = a + ib$

$\therefore \quad C = \begin{bmatrix} a-ib & c-id \\ -c-id & a+ib \end{bmatrix}$

$\therefore \quad \text{Adj. } A = C' = \begin{bmatrix} a-ib & -c-id \\ c-id & a+ib \end{bmatrix}$

Also $|A| = \begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix}$

$= (a+ib)(a-ib) - (c+id)(-c+id)$

$= a^2 - i^2b^2 + c^2 - i^2d^2 = a^2 + b^2 + c^2 + d^2 = 1 \neq 0.$

$\therefore \quad \text{Inverse of } A = \dfrac{\text{Adj. } A}{|A|} = \begin{bmatrix} a-ib & -c-id \\ c-id & a+ib \end{bmatrix}$

Ans.

****Ex. 19. If** $\alpha + i\beta = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$, **verify that**

$$(\alpha + i\beta)^{-1} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}^{-1}$$

Sol. $(\alpha + i\beta)^{-1} = \dfrac{1}{\alpha + i\beta} = \dfrac{(\alpha - i\beta)}{(\alpha + i\beta)(\alpha - i\beta)}$,

multiplying num. and denom. by $\alpha - i\beta$

$= (\alpha - i\beta)/(\alpha^2 + \beta^2).$ ...(i)

Again let $A = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$

Then $|\mathbf{A}| = \begin{vmatrix} \alpha & \beta \\ -\beta & \alpha \end{vmatrix} = \alpha(\alpha) - \beta(-\beta) = \alpha^2 + \beta^2 \neq 0.$ ...(ii)

Also for the matrix **A**, we have

$C_{11} = \alpha;\ C_{12} = \beta;\ C_{21} = -\beta;\ C_{22} = \alpha$

$\therefore\ \mathbf{C} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$ and Adj $\mathbf{A} = \mathbf{C}' = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$

$$\therefore\ \mathbf{A}^{-1} = \frac{\text{Adj. }\mathbf{A}}{|\mathbf{A}|} = \frac{\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}}{\alpha^2 + \beta^2}, \text{ from (ii)}$$

i.e. $$\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}^{-1} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \div (\alpha^2 + \beta^2) \quad \text{...(iii)}$$

Also we are given $\alpha + i\beta = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$

Replacing β by $-\beta$ we get $(\alpha - i\beta) = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$.

$\therefore$ From (i) we have $(\alpha + i\beta)^{-1} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \div (\alpha^2 + \beta^2)$ ...(iv)

Hence from (iii) and (iv), we have

$$(\alpha + i\beta)^{-1} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}^{-1}$$ Hence proved

Ex. 20. If $\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 & 2 \\ -9 & 1 & 0 & 1 \\ 1 & 0 & 2 & -1 \\ -4 & 1 & -3 & 1 \end{bmatrix}$, **find** $\mathbf{A}^{-1}$.

Sol. Here $|\mathbf{A}| = \begin{vmatrix} -1 & 0 & 0 & 2 \\ -9 & 1 & 0 & 1 \\ 1 & 0 & 2 & -1 \\ -4 & 1 & -3 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 & 0 \\ -9 & 1 & 0 & -17 \\ 1 & 0 & 2 & 1 \\ -4 & 1 & -3 & -7 \end{vmatrix}$,

replacing C_4 by $C_4 + 2C_1$

$= -\begin{vmatrix} 1 & 0 & -17 \\ 0 & 2 & 1 \\ 1 & -3 & -7 \end{vmatrix}$, expanding with respect to R_1

$= -\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & -3 & 10 \end{vmatrix}$, replacing C_3 by $C_3 + 17C_1$

$$= -\begin{vmatrix} 2 & 1 \\ -3 & 10 \end{vmatrix} = -[20+3] = -23 \neq 0. \qquad \text{...(i)}$$

Also for the matrix A, we have

$$C_{11} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & -3 & 0 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -3 & 0 \end{vmatrix} = -3;$$

$$C_{12} = -\begin{vmatrix} -9 & 0 & 1 \\ 1 & 2 & -1 \\ -4 & -3 & 1 \end{vmatrix} = -\begin{vmatrix} 0 & 0 & 1 \\ -8 & 2 & -1 \\ 5 & -3 & 1 \end{vmatrix} = -\begin{vmatrix} -8 & 2 \\ 5 & -3 \end{vmatrix} = -14;$$

$$C_{13} = \begin{vmatrix} -9 & 1 & 1 \\ 1 & 0 & -1 \\ -4 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 5 & 0 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & -1 \\ 5 & 0 \end{vmatrix} = -5;$$

$$C_{14} = -\begin{vmatrix} -9 & 1 & 0 \\ 1 & 0 & 2 \\ -4 & 1 & -3 \end{vmatrix} = -\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 5 & 1 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 5 & -3 \end{vmatrix} = -13;$$

$$C_{21} = -\begin{vmatrix} 0 & 0 & 2 \\ 0 & 2 & -1 \\ 1 & -3 & 1 \end{vmatrix} = -\begin{vmatrix} 0 & 2 \\ 2 & -1 \end{vmatrix} = 4;$$

$$C_{22} = \begin{vmatrix} -1 & 0 & 2 \\ 1 & 2 & -1 \\ -4 & -3 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 \\ 1 & 2 & 1 \\ -4 & -3 & -7 \end{vmatrix} = -\begin{vmatrix} 2 & 1 \\ -3 & -7 \end{vmatrix} = 11;$$

$$C_{23} = -\begin{vmatrix} -1 & 0 & 2 \\ 1 & 0 & -1 \\ -4 & 1 & 1 \end{vmatrix} = -\begin{vmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ -4 & 1 & -7 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & -7 \end{vmatrix} = -1;$$

$$C_{24} = \begin{vmatrix} -1 & 0 & 0 \\ 1 & 0 & 2 \\ -4 & 1 & -3 \end{vmatrix} = -\begin{vmatrix} 0 & 2 \\ 1 & -3 \end{vmatrix} = 2;$$

$$C_{31} = \begin{vmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 1 & -3 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 \\ 1 & -3 \end{vmatrix} = -6;$$

$$C_{32} = -\begin{vmatrix} -1 & 0 & 2 \\ -9 & 0 & 1 \\ -4 & -3 & 1 \end{vmatrix} = -\begin{vmatrix} -1 & 0 & 0 \\ -9 & 0 & -17 \\ -4 & -3 & -7 \end{vmatrix} = \begin{vmatrix} 0 & -17 \\ -3 & -7 \end{vmatrix} = -51;$$

$$C_{33} = \begin{vmatrix} -1 & 0 & 2 \\ -9 & 1 & 1 \\ -4 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 \\ -9 & 1 & -17 \\ -4 & 1 & -7 \end{vmatrix} = -\begin{vmatrix} 1 & -17 \\ 1 & -7 \end{vmatrix} = -10;$$

$$C_{34} = -\begin{vmatrix} -1 & 0 & 0 \\ -9 & 1 & 0 \\ -4 & 1 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & -3 \end{vmatrix} = -3;$$

$$C_{41} = -\begin{vmatrix} 0 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{vmatrix} = -2\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = -4;$$

$$C_{42} = \begin{vmatrix} -1 & 0 & 2 \\ -9 & 0 & 1 \\ 1 & 2 & -1 \end{vmatrix} = -2\begin{vmatrix} -1 & 2 \\ -9 & 1 \end{vmatrix} = -2\,(17) = -34;$$

$$C_{43} = -\begin{vmatrix} -1 & 0 & 2 \\ -9 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix} = -\begin{vmatrix} -1 & 0 & 0 \\ -9 & 1 & -17 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -17 \\ 0 & 1 \end{vmatrix} = 1;$$

$$C_{44} = \begin{vmatrix} -1 & 0 & 0 \\ -9 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = -2$$

$$\therefore \quad \mathbf{C} = \begin{bmatrix} -3 & -14 & -5 & -13 \\ 4 & 11 & -1 & 2 \\ -6 & -51 & -10 & -3 \\ -4 & -34 & 1 & -2 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} -3 & 4 & -6 & -4 \\ -14 & 11 & -51 & -34 \\ -5 & -1 & -10 & 1 \\ -13 & 2 & -3 & -2 \end{bmatrix}$$

$$= -\begin{bmatrix} 3 & -4 & 6 & 4 \\ 14 & -11 & 51 & 34 \\ 5 & 1 & 10 & -1 \\ 13 & -2 & 3 & 2 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{\text{Adj. } \mathbf{A}}{|\mathbf{A}|} = \frac{1}{23}\begin{bmatrix} 3 & -4 & 6 & 4 \\ 14 & -11 & 51 & 34 \\ 5 & 1 & 10 & -1 \\ 13 & -2 & 3 & 2 \end{bmatrix}$$

Ans.

***Ex. 21. Porve that $|Adj\,(Adj\,\mathbf{A})| = |\mathbf{A}|^{(n-1)^2}$, if $|\mathbf{A}| \neq 0$ and is any $n \times n$ matrix.** *(Agra 90)*

Sol. We know that

$$|Adj\,\mathbf{A}| = |\mathbf{A}|^{n-1}, \text{ if } |\mathbf{A}| \neq 0. \qquad \text{...(i)}$$

(See Th. II. Page 50 Ch. V)

Replacing **A** by Adj **A** in (i), we get

$$|Adj\,(Adj\,\mathbf{A})| = |Adj\,\mathbf{A}|^{n-1}$$

$$= \{|Adj\,\mathbf{A}|\}^{n-1} \qquad \textbf{(Note)}$$

$$= \{|\mathbf{A}|^{n-1}\}^{n-1}, \text{ from (i)}$$

$$= \{|\mathbf{A}|\}^{(n-1)^2} = |\mathbf{A}|^{(n-1)^2}.$$

Hence proved.

***Ex. 22. Prove that** $Adj\,(Adj\,\mathbf{A}) = |\mathbf{A}|^{n-2} \bullet \mathbf{A}$**, where A is any** $n \times n$ **matrix.** *(Agra 92, 90; Kanpur 90)*

Sol. We know that

$\mathbf{A}\,.\,(Adj\,\mathbf{A}) = |\mathbf{A}| \bullet \mathbf{I}$ (See Th. I Page 49 Ch. V)

or $Adj\,\{\mathbf{A} \bullet (Adj\,\mathbf{A})\} = Adj\,\{|\mathbf{A}| \bullet \mathbf{I}\}$

or $Adj\,(Adj\,\mathbf{A}) \bullet (Adj\,\mathbf{A}) = |\mathbf{A}|^{n-1} \bullet \mathbf{I}$ (See Th. III. Page 50 Ch. V)

or $Adj\,(Adj\,\mathbf{A}) \bullet (Adj\,\mathbf{A})\,\mathbf{A} = |\mathbf{A}|^{n-1} \bullet \mathbf{I} \bullet \mathbf{A}$

or $Adj\,(Adj\,\mathbf{A}) \bullet |\mathbf{A}| \bullet \mathbf{I} = |\mathbf{A}|^{n-1}\,\mathbf{AI},$ See Th. I P. 49 Ch. V

or $Adj\,(Adj\,\mathbf{A}) \bullet |\mathbf{A}| = |\mathbf{A}|^{n-1} \bullet \mathbf{A}$ **(Note)**

or $Adj\,(Adj\,\mathbf{A}) = |\mathbf{A}|^{n-2} \bullet \mathbf{A}.$ Hence proved.

Exercises on § 5·09

Find the inverse of the following matrices

Ex. 1. $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 3 \\ 0 & -1 & 3 \end{bmatrix}$ **Ans.** $\frac{1}{10}\begin{bmatrix} 9 & 1 & 2 \\ -3 & 3 & -4 \\ -1 & 1 & 2 \end{bmatrix}$

Ex. 2. $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Ex. 3. $\begin{bmatrix} 3 & 2 & -1 \\ -1 & 2 & 3 \\ -3 & 1 & 3 \end{bmatrix}$ **Ans.** $\frac{1}{8}\begin{bmatrix} -3 & 7 & -8 \\ 6 & -6 & 8 \\ -5 & 9 & -8 \end{bmatrix}$

Ex. 4. $\begin{bmatrix} 2 & -4 & -2 \\ 4 & 6 & 2 \\ 0 & 10 & -4 \end{bmatrix}$ **Ans.** $-\frac{1}{58}\begin{bmatrix} -11 & -9 & 1 \\ 4 & -2 & -3 \\ 10 & -5 & 7 \end{bmatrix}$

Ex. 5. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ *(Gorakhpur 91; Kanpur 94)* **Ans.** $\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

Ex. 6. $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$ **Ans.** $\frac{1}{18}\begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & 5 \\ -5 & 7 & -1 \end{bmatrix}$

Ex. 7. $\begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ **Ans.** $\frac{1}{4}\begin{bmatrix} -3 & 1 & 7 \\ -1 & -1 & 5 \\ 5 & 1 & -13 \end{bmatrix}$

***Ex. 8.** $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ **Ans.** $\frac{1}{3}\begin{bmatrix} -6 & 5 & -1 \\ 15 & -8 & 1 \\ -6 & 3 & 0 \end{bmatrix}$

Ex. 9. $\begin{bmatrix} 1 & -2 & -1 \\ 2 & 3 & 1 \\ 0 & 5 & -2 \end{bmatrix}$ **Ans.** $\frac{1}{29}\begin{bmatrix} 11 & 9 & -1 \\ -4 & 2 & 3 \\ -10 & 5 & -7 \end{bmatrix}$

Ex. 10. $\begin{bmatrix} 1 & 4 & 0 \\ -1 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ **Ans.** $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$

Ex. 11. $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$ **Ans.** Not possible as $|\mathbf{A}| = 0$.

Ex. 12. $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$

Ex. 13. $\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -1 \\ 2 & 1 & 2 & 1 \\ 3 & -2 & 1 & 6 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 2 & -1 & 1 & -1 \\ -5 & -3 & 1 & 1 \\ 2 & 3 & -1 & 0 \\ -3 & -1 & 0 & 1 \end{bmatrix}$

Ex. 14. $\begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}$ *(Kumaun 90)*

Ex. 15. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$ *(Kumaun 93)* **Ans.** $\begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix}$

Ex. 16. Verify that $\mathbf{A} \bullet (\text{Adj. } \mathbf{A}) = (\text{Adjj. } \mathbf{A}) \bullet \mathbf{A} = |\mathbf{A}|\, \mathbf{I}_3$, where $\mathbf{I}_3$ is the identity matrix of order 3, and $\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Ex. 17. If $\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 0 & 5 & -2 \end{bmatrix}$, verify that

$\mathbf{A} \bullet (\text{Adj. } \mathbf{A}) = (\text{Adj. } \mathbf{A}) \bullet \mathbf{A} = |\mathbf{A}| \bullet \mathbf{I}$ *(Meerut 96P)*

***Ex. 18.** Verify that $\mathbf{A} \bullet (\text{Adj. } \mathbf{A}) = (\text{Adj. } \mathbf{A}) \bullet \mathbf{A} = |\mathbf{A}|\, \mathbf{I}_2$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix} \text{ and } \mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**§ 5·10. Existence of Inverse.

An Important Theorem.

The necessary and sufficient condition that a square matrix may possess an inverse is that it be non-singular. *(Bundelkhand 96, 92; Kumaun 96; Gorakhpur 99; Meerut 92; Purvanchal 98)*

Proof. The condition is necessary.

If $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{B}$ is its inverse then by definition of the inverse we have $\mathbf{AB} = \mathbf{I}_n$ (See Chapter II)

Taking the determinants of both sides we get

$$|\mathbf{AB}| = |\mathbf{I}_n|. \quad \text{...(i)}$$

But $|\mathbf{AB}| = |\mathbf{A}| \bullet |\mathbf{B}|$ .... See Chapter IV

and $|\mathbf{I}_n| = 1$, where $\mathbf{I}_n$ is the $n \times n$ identity matrix

$\therefore$ From (i) we get $|\mathbf{A}| \bullet |\mathbf{B}| = 1$,

which implies that $|\mathbf{A}| \neq 0$.

$\therefore$ The matrix $\mathbf{A}$ is non-singular. .... See Chapter IV

The condition is sufficient.

If $\mathbf{A}$ is an $n \times n$ non-singular matrix and there be another matrix $\mathbf{B}$ defined by

$$\mathbf{B} = \frac{1}{|\mathbf{A}|}(\text{Adj. } \mathbf{A})$$

Then

$$\mathbf{AB} = \mathbf{A}\frac{1}{|\mathbf{A}|}(\text{Adj. } \mathbf{A}) = \frac{1}{|\mathbf{A}|}(\mathbf{A} \bullet \text{Adj. } \mathbf{A})$$

$$= \frac{1}{|\mathbf{A}|} \bullet |\mathbf{A}|\,\mathbf{I}_n \quad \text{... See § 5·09 Th. I Page 49 Ch. V}$$

$$= \mathbf{I}_n$$

Similarly

$$\mathbf{BA} = \frac{1}{|\mathbf{A}|}(\text{Adj } \mathbf{A}) \bullet \mathbf{A} = \frac{1}{|\mathbf{A}|}[(\text{Adj. } \mathbf{A}) \bullet \mathbf{A}]$$

$$= \frac{1}{|\mathbf{A}|} \bullet |\mathbf{A}|\,\mathbf{I}_n \quad \text{...See § 5.09 Th. I Page 49 Ch. V}$$

$$\therefore \quad \mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$$

$\therefore$ $\mathbf{B}$ is the inverse of $\mathbf{A}$ and it exists.

§ 5·11. Some Important Theorems.

***Theorem I.** *If* $\mathbf{A}$ *is a non-singular matrix of order n such that* $\mathbf{AX} = \mathbf{AY}$, *then* $\mathbf{X} = \mathbf{Y}$.

Proof. If A is non-singular matrix, then $\mathbf{A}^{-1}$ exists. ...See § 5·10 above

Given $\mathbf{AX} = \mathbf{AY}$

or $\mathbf{A}^{-1}(\mathbf{AX}) = \mathbf{A}^{-1}(\mathbf{AY})$

or $(\mathbf{A}^{-1}\mathbf{A})\,\mathbf{X} = (\mathbf{A}^{-1}\mathbf{A})\,\mathbf{Y}$

or $\mathbf{IX} = \mathbf{IY}$ $\because \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

or $\quad X = Y$, by left cancellation law. Hence proved.

Theorem II. *The inverse of transpose of a matrix is the transpose of the inverse.*

Proof. Let **A** be the given matrix. Then its inverse is A^{-1}.

Also we have $AA^{-1} = I = A^{-1}A$, by definition.

$\therefore \quad (AA^{-1})' = I' = (A^{-1}A)'$, taking transpose.

or $\quad (A^{-1})' A' = I = A'(A^{-1})'$, $\quad \because (AB)' = B'A'$ and $I' = I$.

Hence **A′** is invertible *i.e.* **A′** possesses inverse

and $\quad (A')^{-1} = (A^{-1})'$

i.e. the inverse of a transpose of a matrix is the transpose of the inverse.

Hence proved.

Theorem III. *If* **A**, **B** *are any two* $n \times n$ *matrices such that* $BA = O$, *where* **O** *is the null matrix, then at least one of them is singular.*

Proof. Since **A**, **B** are two $n \times n$ matrices

so $\quad AB = O$, where **O** is the null matrix

$\Rightarrow \quad |A| \bullet |B| = 0$ **(Note)**

$\Rightarrow \quad \begin{cases} \text{either } |A| = 0, \text{ which means } A \text{ is singular} \\ \text{or } \quad |B| = 0, \text{ which means } B \text{ is singular} \\ \text{or both } |A| \text{ and } |B| \text{ are zero which means both } A \text{ and } B \text{ are singular.} \end{cases}$

Hence at least one of **A** and **B** is singular.

Theorem IV. *The inverse of the inverse of a matrix is the matrix itself i.e.* $(A^{-1})^{-1} = A$, *where* A^{-1} *is the inverse of* **A**.

Proof. Let **A** be the given matrix. Then its inverse is A^{-1}.

Also by definition $AA^{-1} = I = A^{-1}A$.

$\therefore \quad A^{-1}$ is invertible and we have $(A^{-1})^{-1} = A$.

i.e. the inverse of the inverse of **A** is **A** itself. Hence proved.

Theorem V. *If a non singular matrix* **A** *is symmetric, then* A^{-1} *is also symmetric.*

Proof. If **A** is symmetric, then $A = A'$ ...(i)

Also by definition if **A** is non-singular, then

$$\begin{aligned} A^{-1}A &= I \\ &= I', \text{ since } I' = I \\ &= (AA^{-1})', \text{ since } I = A^{-1}A = AA^{-1} \\ &= (A^{-1})'A', \text{ since } (AB') = B'A' \end{aligned}$$

i.e $\quad A^{-1}A = (A^{-1})'A$, since $A = A'$, from (i).

or $\quad A^{-1} = (A^{-1})'$, by right cancellation law.

Hence A^{-1} is symmetric by definition. Hence proved.

Theorem VI. *The inverse of the transposed conjugate of a non-singular matrix* **A** *is the transposed conjugate of the inverse of* **A**

i.e. $$(\mathbf{A}^{\Theta})^{-1} = (\mathbf{A}^{-1})^{\Theta}$$

Proof. If A is a non-singular matrix, then **A** is invertible and we have

$$\mathbf{AA}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$$

or $$(\mathbf{AA}^{-1})^{\Theta} = \mathbf{I}^{\Theta} = (\mathbf{A}^{-1}\mathbf{A})^{\Theta}$$

or $$(\mathbf{A}^{-1})^{\Theta}\,\mathbf{A}^{\Theta} = \mathbf{I} = \mathbf{A}^{\Theta}\,(\mathbf{A}^{-1})^{\Theta}, \text{ since } (\mathbf{AB})^{\Theta} = \mathbf{B}^{\theta}\mathbf{A}^{\Theta}, \mathbf{I}^{\Theta} = \mathbf{I}.$$

$\therefore$ $\mathbf{A}^{\Theta}$ is invertible and we have $(\mathbf{A}^{\Theta})^{-1} = (\mathbf{A}^{-1})^{\Theta}$. Hence proved.

§5·12. Theorem. *If r be the rank of a matrix* **A** *of order* $m \times n$; $\mathbf{A}_r$ *be the normal form of* **A**, **R** *be the product of elementary matrices of order m and* **S** *be the product of elementary matrices of order n, then* $\mathbf{A}_r = \mathbf{RAS}$.

Proof. Since **R** and **S** are non-singular (*i.e.* their inverses exist), therefore

$\mathbf{R}^{-1}\mathbf{A}_r\mathbf{S}^{-1} = \mathbf{A}$, where $\mathbf{R}^{-1}$ and $\mathbf{S}^{-1}$ are the inverses of **R** and **S** respectively.

or $$\mathbf{A} = \mathbf{B}\,\mathbf{A}_r\,\mathbf{C}, \text{ where } \mathbf{B} = \mathbf{R}^{-1}, \mathbf{C} = \mathbf{S}^{-1}$$

or $$\mathbf{A}_r = \mathbf{B}^{-1}\,\mathbf{A}\,\mathbf{C}^{-1}.$$ **(Note)**

Now if **A** is a non-singular matrix of order n, then $r = n$ and

$$\mathbf{A}_r = \mathbf{I}_n$$

Hence $$\mathbf{A} = \mathbf{B}\,\mathbf{I}_n\,\mathbf{C},$$

which is of the form **A** = **B**, since **B** and **C** are the product of elementary matrices.

Cor. *If two matrices* **A** *and* **B** *are of the same order* $m \times n$ *and same rank, then there exists non-singular square matrices* **P, Q** *such that* **B** = **PAQ**.

Proof. From above theorem we find that

$$\mathbf{A} = \mathbf{CA}_r\,\mathbf{D}, \mathbf{B} = \mathbf{C}_1\,\mathbf{A}_r\,\mathbf{D}_1$$

where **C**, C_1 are product of elementary matrices of order m and **D**, $\mathbf{D}_1$ of order n.

From $\mathbf{A} = \mathbf{CA}_r\mathbf{D}$, we get $\mathbf{A}_r = \mathbf{C}^{-1}\,\mathbf{AD}^{-1}$

Substituting this in $\mathbf{B} = \mathbf{C}_1\,\mathbf{A}_r\,\mathbf{D}_1$, we get

$$\mathbf{B} = \mathbf{C}_1\,(\mathbf{C}^{-1}\,\mathbf{A}\,\mathbf{D}^{-1})\,\mathbf{D}_1 = (\mathbf{C}_1\,\mathbf{C}^{-1})\,\mathbf{A}\,(\mathbf{D}^{-1}\,\mathbf{D}_1)$$

which is of the form **B** = **PAQ**.

Solved Examples on § 5·12

Ex. 1 (a). Find the non-singular matrices R and S, such that RAS is the normal form, where $\mathbf{A} = \begin{bmatrix} 2 & 2 & -6 \\ -1 & 2 & 2 \end{bmatrix}$

Sol. Here we find that A is a 2×3 matrix

$\therefore \qquad [\mathbf{A}]_{2\times 3} = \mathbf{I}_2\,\mathbf{A}\,\mathbf{I}_3$

or $\qquad \begin{bmatrix} 2 & 2 & -6 \\ -1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Now we are to bring L.H.S. to the normal form by applying elementary row and column operations.

$\therefore \qquad \begin{bmatrix} 1 & 1 & -3 \\ -1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, by $R_1\,(\frac{1}{2})$

or $\qquad \begin{bmatrix} 1 & 1 & -3 \\ 0 & 3 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, by $R_2 + R_1$

or $\qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, by $C_2 - C_1$ and $C_3 + 3C_1$

or $\qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 1 & -\frac{1}{3} & 3 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, replacing C_2 by $\frac{1}{3}\,C_2$

or $\qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 1 & -\frac{1}{3} & \frac{8}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$, by $C_3 + C_2$

Since L. H. S. is in the normal form, so

$\mathbf{R} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix}$ and $\mathbf{S} = \begin{bmatrix} 1 & -\frac{1}{3} & \frac{8}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$ **Ans.**

Ex. 1 (b). Determine two non-singular matrices P and Q such that PAQ is in the normal form, where

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}$$ *(Garhwal 93)*

Sol. Here we find that A is a 3×4 matrix

$\therefore \qquad [\mathbf{A}]_{3\times 4} = \mathbf{I}_3\,\mathbf{A}\,\mathbf{I}_4$

or $\qquad \begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

or $$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 17 & 9 & 4 & 18 \\ 34 & 18 & 11 & 36 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 2 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

applying $C_1 + 3C_3$, $C_2 + 2C_3$, $C_4 + 5C_3$

or $$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 17 & 9 & 4 & 18 \\ 0 & 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 2 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$ applying $R_3 - 2R_2$

or $$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 17 & 9 & 0 & 18 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 2 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

applying $R_2 + 4R_1$, $R_3 + 3R_1$

or $$\begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & -2 \\ -1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

applying $C_1 - 2C_2$, $C_4 - 2C_2$

or $$\begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & -17 & 0 & -2 \\ -1 & -7 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

applying $C_2 + 9C_1$

or $$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ -4 & -1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & -17 & 0 & -2 \\ -1 & -7 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

applying $-R_1$ and $-R_2$

or $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ -4 & -1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \bullet \mathbf{A} \bullet \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -2 & -17 & -2 \\ 1 & -1 & -7 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

interchanging columns

$\because$ L.H.S. is in the normal form, so we have

$$\mathbf{P} = \begin{bmatrix} -1 & 0 & 0 \\ -4 & -1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \text{ and } \mathbf{Q} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -2 & -17 & -2 \\ 1 & -1 & -7 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Ans.

***Ex. 2. Find two non-singular matrices P and Q such that PAQ is in the normal form, where**

$$\mathbf{A}=\begin{bmatrix}1 & 1 & 1\\ 1 & -1 & -1\\ 3 & 1 & 1\end{bmatrix}$$

(Garhwal 96; Meerut 91)

Sol. Here we find that **A** is a 3×3 metrix

$\therefore$ $[\mathbf{A}]_{3\times 3}=\mathbf{I}_3\,\mathbf{A}\,\mathbf{I}_3$

or $$\begin{bmatrix}1 & 1 & 1\\ 1 & -1 & -1\\ 3 & 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\bullet\mathbf{A}\bullet\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

or $$\begin{bmatrix}1 & 1 & 1\\ 2 & 0 & 0\\ 2 & 0 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}\bullet\mathbf{A}\bullet\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \text{ applying } R_2+R_1, R_3-R_1$$

or $$\begin{bmatrix}1 & 1 & 1\\ 2 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ -2 & -1 & 1\end{bmatrix}\bullet\mathbf{A}\bullet\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \text{applying } R_3-R_2$$

or $$\begin{bmatrix}1 & 1 & 1\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ \frac{1}{2} & \frac{1}{2} & 0\\ -2 & -1 & 1\end{bmatrix}\bullet\mathbf{A}\bullet\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \text{ applying } R_2\left(\tfrac{1}{2}\right)$$

or $$\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & -\frac{1}{2} & 0\\ \frac{1}{2} & \frac{1}{2} & 0\\ -2 & -1 & 1\end{bmatrix}\bullet\mathbf{A}\bullet\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \text{ applying } R_1-R_2$$

or $$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & -\frac{1}{2} & 0\\ -2 & -1 & 1\end{bmatrix}\bullet\mathbf{A}\bullet\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \text{ interchanging } R_1 \text{ and } R_2$$

or $$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}1/2 & 1/2 & 0\\ 1/2 & -1/2 & 0\\ -2 & -1 & 1\end{bmatrix}\bullet\mathbf{A}\bullet\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}, \text{ applying } C_3-C_2$$

(Note)

Since L. H. S. is in the normal form, so we have

$$\mathbf{P}=\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & -\frac{1}{2} & 0\\ -2 & -1 & 1\end{bmatrix}, \mathbf{Q}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}.$$

Ans.

Ex. 3 (a). Using the matrix $A = \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 3 & 1 \\ -1 & 1 & 2 & 0 \end{bmatrix}$**, find two non-singular matrices P and Q such that PAQ is in the normal form.** *(Agra 95)*

Sol. Here we find that A is a 3×4 matrix

$$\therefore \ [A]_{3 \times 4} = I_3 \bullet A \bullet I_4$$

or $$\begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 3 & 1 \\ -1 & 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet A \bullet \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

or $$\begin{bmatrix} 0 & 8 & 24 & 4 \\ 0 & 1 & 3 & 1 \\ -1 & 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet A \bullet \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ applying } R_1 + 5R_3$$

or $$\begin{bmatrix} 0 & 8 & 24 & 4 \\ 0 & 1 & 3 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet A \bullet \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ applying } C_2 + C_1, C_3 + 2C_1$$

(Note)

or $$\begin{bmatrix} 0 & 0 & 0 & -4 \\ 0 & 1 & 3 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -8 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet A \bullet \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ applying } R_1 - 8R_2$$

$$\sim \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -8 & 5 \end{bmatrix} \bullet A \bullet \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ interchanging } R_1 \text{ and } R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1/4 & 2 & -5/4 \end{bmatrix} \bullet A \bullet \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ applying } -R_1 \text{ and } -(1/4) R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1/4 & 2 & -5/4 \end{bmatrix} \bullet A \bullet \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

applying $C_3 - 3C_2, C_4 - C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1/4 & 2 & -5/4 \end{bmatrix} \bullet A \bullet \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

interchanging C_3 and C_4

$\therefore$ L.H.S. is in the normal form, so we have

$$\mathbf{P}=\begin{bmatrix}0 & 0 & -1\\ 0 & 1 & 0\\ -1/4 & 2 & -5/4\end{bmatrix} \text{ and } \mathbf{Q}=\begin{bmatrix}1 & 1 & -1 & -1\\ 0 & 1 & -1 & -3\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}$$

Ans.

Ex. 3. (b). Find non-singular matrices R and S such that RAS is in normal form, where $\mathbf{A}=\begin{bmatrix}1 & 2 & 3\\ 3 & 2 & 1\\ 1 & 3 & 2\\ 2 & 1 & 3\end{bmatrix}$

Sol. Here we find that **A** is a 4×3 matrix

$$\therefore\ [\mathbf{A}]_{4\times 3}=\mathbf{I}_4 \bullet \mathbf{A} \bullet \mathbf{I}_3$$

or
$$\begin{bmatrix}1 & 2 & 3\\ 3 & 2 & 1\\ 1 & 3 & 2\\ 2 & 1 & 3\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\bullet \mathbf{A} \bullet \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

or
$$\begin{bmatrix}1 & 2 & 3\\ 0 & -4 & -8\\ 0 & 1 & -1\\ 0 & -3 & -3\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ -3 & 1 & 0 & 0\\ -1 & 0 & 1 & 0\\ -2 & 0 & 0 & 0\end{bmatrix}\bullet \mathbf{A} \bullet \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},$$

applying $R_2 - 3R_1, R_3 - R_1, R_4 - 2R_1$

or
$$\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 2\\ 0 & 1 & -1\\ 0 & 0 & -6\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ \frac{3}{4} & -\frac{1}{4} & 0 & 0\\ -1 & 0 & 1 & 0\\ -5 & 0 & 3 & 1\end{bmatrix}\bullet \mathbf{A} \bullet \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},$$

applying $R_2 (-\frac{1}{4})$ and $R_4 + 3R_3$

or
$$\begin{bmatrix}1 & 2 & 3\\ 0 & 0 & 3\\ 0 & 1 & -1\\ 0 & 0 & -1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ \frac{7}{4} & -\frac{1}{4} & -1 & 0\\ -1 & 0 & 1 & 0\\ \frac{5}{6} & 0 & -\frac{1}{2} & -\frac{1}{6}\end{bmatrix}\bullet \mathbf{A} \bullet \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},$$

applying $R_2 - R_3, R_4 (-\frac{1}{6})$

or
$$\begin{bmatrix}1 & 2 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\frac{3}{2} & 0 & \frac{3}{2} & \frac{1}{2}\\ \frac{7}{12} & -\frac{1}{12} & -\frac{1}{3} & 0\\ -\frac{1}{6} & 0 & \frac{1}{2} & -\frac{1}{6}\\ \frac{5}{4} & 0 & -\frac{1}{2} & -\frac{1}{6}\end{bmatrix}\bullet \mathbf{A} \bullet \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},$$

applying $R_1 - 3R_4, R_2 (\frac{1}{3}), R_3 + R_4$

or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{7}{6} & 0 & \frac{1}{2} & \frac{5}{6} \\ \frac{7}{12} & -\frac{1}{12} & -\frac{1}{3} & 0 \\ -\frac{1}{6} & 0 & \frac{1}{2} & -\frac{1}{6} \\ \frac{3}{12} & \frac{1}{12} & -\frac{1}{6} & -\frac{1}{6} \end{bmatrix} \bullet A \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix},$

applying $R_1 - 2R_3, R_4 - R_2$

or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{7}{6} & 0 & \frac{1}{2} & \frac{5}{6} \\ \frac{7}{12} & -\frac{1}{12} & -\frac{1}{2} & 0 \\ -\frac{1}{6} & 0 & \frac{1}{2} & -\frac{1}{6} \\ \frac{3}{12} & \frac{1}{12} & -\frac{1}{6} & -\frac{1}{6} \end{bmatrix} \bullet A \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

interchanging C_2 and C_3

$\because$ L.H.S. is in the normal form, so we have

$$R = \begin{bmatrix} -\frac{7}{6} & 0 & \frac{1}{2} & \frac{5}{6} \\ \frac{7}{12} & -\frac{1}{12} & -\frac{1}{2} & 0 \\ -\frac{1}{6} & 0 & \frac{1}{2} & -\frac{1}{6} \\ \frac{1}{4} & \frac{1}{12} & -\frac{1}{6} & -\frac{1}{6} \end{bmatrix}, S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Ans.

Exercise on § 5·12

***Ex. 1. Reduce** $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$ **to normal form N and compute the matrices P and Q, such that PAQ = N.**

Ex. 2. Determine two non-singular matrics P and Q such that PAQ is in the normal form, where

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$$

(Garhwal 94)

MISCELLANEOUS SOLVED EXAMPLES

***Ex. 1. Find the reciprocal (or inverse) of the matrix**

$S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ **and show that the transform of the matrix**

$A = \frac{1}{2}\begin{bmatrix} b+c & c-a & b-a \\ c-b & c+a & a-b \\ b-c & a-c & a+b \end{bmatrix}$ **by S *i.e.* SAS^{-1} is a diagonal matrix.**

Sol. In the usual way we can show that

S^{-1} = inverse of $S = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ (To be proved in the examination)

$$\therefore \quad SA = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \times \frac{1}{2}\begin{bmatrix} b+c & c-a & b-a \\ c-b & c+a & a-b \\ b-c & a-c & a+b \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 2a & 2a \\ 2b & 0 & 2b \\ 2c & 2c & 0 \end{bmatrix}, \text{ multiplying the matrices in the usual way.}$$

or $$SA = \begin{bmatrix} 0 & a & a \\ b & 0 & b \\ c & c & 0 \end{bmatrix}$$

$$\therefore \quad SAS^{-1} = \begin{bmatrix} 0 & a & a \\ b & 0 & b \\ c & c & 0 \end{bmatrix} \times \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2a & 0 & 0 \\ 0 & 2b & 0 \\ 0 & 0 & 2c \end{bmatrix}, \text{ multiplying the two matrices in the usual way.}$$

$$= \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, \text{ which is a diagonal matrix,}$$

[See Chapter II]

Ex. 2. If A is invertible show that $\overline{\mathbf{A}}$ is invertible.

Sol. If **A** is inverible, then we know that

$$\mathbf{AA}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$$

or $$\overline{(\mathbf{AA}^{-1})} = \mathbf{I} = \overline{(\mathbf{A}^{-1}\mathbf{A})}$$ **(Note)**

or $$\overline{\mathbf{A}}\,\overline{(\mathbf{A}^{-1})} = \mathbf{I} = \overline{(\mathbf{A}^{-1})}\,\overline{\mathbf{A}}, \qquad \because \overline{\mathbf{AB}} = \overline{\mathbf{A}} \bullet \overline{\mathbf{B}}$$

Hence $\overline{\mathbf{A}}$ is invertible and we have $(\overline{\mathbf{A}})^{-1} = \overline{(\mathbf{A}^{-1})}$ Hence proved.

Ex. 3. (a). If $\mathbf{A} = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$, where none of a's is zero, then show that A is invertible. Also evaluate $\mathbf{A}^{-1}$

Sol. $|\mathbf{A}| = \begin{vmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{vmatrix} = a_1a_2a_3$, on evaluating

i.e. $|\mathbf{A}| \neq 0$. Hence **A** is invertible. ....Sec Ch. IV

Also

$C_{11} = \begin{vmatrix} a_2 & 0 \\ 0 & a_3 \end{vmatrix} = a_2\, a_3;\; C_{12} = -\begin{vmatrix} 0 & 0 \\ 0 & a_3 \end{vmatrix} = 0;\; C_{13} = \begin{vmatrix} 0 & a_2 \\ 0 & 0 \end{vmatrix} = 0;$

$C_{21} = -\begin{vmatrix} 0 & 0 \\ 0 & a_3 \end{vmatrix} = 0;\; C_{22} = \begin{vmatrix} a_1 & 0 \\ 0 & a_3 \end{vmatrix} = a_1\, a_3;\; C_{23} = -\begin{vmatrix} a_1 & 0 \\ 0 & 0 \end{vmatrix} = 0$

$C_{31} = \begin{vmatrix} 0 & 0 \\ a_2 & 0 \end{vmatrix} = 0;\; C_{32} = -\begin{vmatrix} a_1 & 0 \\ 0 & 0 \end{vmatrix} = 0;\; C_{33} = \begin{vmatrix} a_1 & 0 \\ 0 & a_2 \end{vmatrix} = a_1\, a_2$

$\therefore \quad \mathbf{C} = \begin{bmatrix} a_2a_3 & 0 & 0 \\ 0 & a_3a_1 & 0 \\ 0 & 0 & a_1a_2 \end{bmatrix}$

$\therefore \quad Adj\ \mathbf{A} = \mathbf{C}' = \begin{bmatrix} a_2a_3 & 0 & 0 \\ 0 & a_3a_1 & 0 \\ 0 & 0 & a_1a_2 \end{bmatrix}$

$\therefore \quad \mathbf{A}^{-1} = \frac{Adj\ \mathbf{A}}{|\mathbf{A}|} = \frac{1}{a_1a_2a_3}\begin{bmatrix} a_2a_3 & 0 & 0 \\ 0 & a_3a_1 & 0 \\ 0 & 0 & a_1a_2 \end{bmatrix}$

or $\quad \mathbf{A}^{-1} = \begin{bmatrix} 1/a_1 & 0 & 0 \\ 0 & 1/a_2 & 0 \\ 0 & 0 & 1/a_3 \end{bmatrix}$

Ans.

***Ex. 3 (b). Show that the matrix** $\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$ **is its own inverse.**

Sol. $|\mathbf{A}| = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$, expanding w.r. to C_1

$= -1 \neq 0$ ...(i)

Also $C_{11} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0;\; C_{12} = -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -1;\; C_{13} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0;$

$C_{21} = -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -1;\; C_{22} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0;\; C_{23} = -\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0;$

$C_{31} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0;\; C_{32} = -\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0;\; C_{33} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

$\therefore \quad \mathbf{C} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and adj $\mathbf{A} = \mathbf{C}' = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$$\therefore \quad A^{-1} = \frac{Adj\ A}{|A|} = \frac{1}{(-1)}\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A$$

Hence proved.

Ex. 3 (c). Compute the inverse of the matrix A, if

$$A = \begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

Sol. $|A| = \begin{vmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & -2 \\ 0 & 1 & 2 & 1 \end{vmatrix}$

$= \begin{vmatrix} 0 & 4 & 9 & 5 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & -2 \\ 0 & 1 & 2 & 1 \end{vmatrix}$, replacing R_1 by $R_1 - 3R_2$

$= \begin{vmatrix} 4 & 9 & 5 \\ 2 & 2 & 1 \\ 1 & 2 & 1 \end{vmatrix}$, expanding w.r. to C_1

$= \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{vmatrix}$, applying $R_1 - 4R_3$ and $R_2 - R_3$

$= \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix}$, applying $C_2 - C_3$

$= \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$, expanding w.r. to R_1

$= 1 \neq 0$ ...(i)

Also $C_{11} = \begin{vmatrix} 2 & 2 & 1 \\ -2 & -3 & -2 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -2 & -3 & -2 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} -3 & -2 \\ 2 & 1 \end{vmatrix}$

$= -3 + 4 = 1$

$C_{12} = -\begin{vmatrix} 0 & 2 & 1 \\ 1 & -3 & -2 \\ 0 & 2 & 1 \end{vmatrix} = 0$; $C_{13} = \begin{vmatrix} 0 & 2 & 1 \\ 1 & -2 & -2 \\ 0 & 1 & 1 \end{vmatrix} = -1$;

$$C_{14} = -\begin{vmatrix} 0 & 2 & 2 \\ 1 & -2 & -3 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = 2.$$

$$C_{21} = -\begin{vmatrix} -2 & 0 & -1 \\ -2 & -3 & -2 \\ 1 & 2 & 1 \end{vmatrix} = -\begin{vmatrix} 0 & 4 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{vmatrix} = 1;$$

$$C_{22} = \begin{vmatrix} 3 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & -1 \\ -5 & -3 & -2 \\ 3 & 2 & 1 \end{vmatrix} = -\begin{vmatrix} -5 & -3 \\ 3 & 2 \end{vmatrix} = 1;$$

$$C_{23} = -\begin{vmatrix} 3 & -2 & -1 \\ 1 & -2 & -2 \\ 0 & 1 & 1 \end{vmatrix} = -\begin{vmatrix} 0 & 0 & -1 \\ -5 & 2 & -2 \\ 3 & -1 & 1 \end{vmatrix} = \begin{vmatrix} -5 & 2 \\ 3 & -1 \end{vmatrix} = -1;$$

$$C_{24} = \begin{vmatrix} 3 & -2 & 0 \\ 1 & -2 & -3 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 4 & 9 \\ 1 & -2 & -3 \\ 0 & 1 & 2 \end{vmatrix} = -\begin{vmatrix} 4 & 9 \\ 1 & 2 \end{vmatrix} = 1;$$

$$C_{31} = \begin{vmatrix} -2 & 0 & -1 \\ 2 & 2 & 1 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 2 & 1 \end{vmatrix} = -2;$$

$$C_{32} = -\begin{vmatrix} 3 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{vmatrix} = 0$$

$$C_{33} = \begin{vmatrix} 3 & -2 & -1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 3\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 3;$$

$$C_{34} = -\begin{vmatrix} 3 & -2 & 0 \\ 0 & 2 & 2 \\ 0 & 1 & 2 \end{vmatrix} = -3\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = -6;$$

$$C_{41} = -\begin{vmatrix} -2 & 0 & -1 \\ 2 & 2 & 1 \\ -2 & -3 & -2 \end{vmatrix} = -\begin{vmatrix} 0 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & -1 & -1 \end{vmatrix} = 2\begin{vmatrix} 2 & 0 \\ -1 & -1 \end{vmatrix} = -4;$$

$$C_{42} = \begin{vmatrix} 3 & 0 & -1 \\ 0 & 2 & 1 \\ 1 & -3 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 9 & 5 \\ 0 & 2 & 1 \\ 1 & -3 & -2 \end{vmatrix} = \begin{vmatrix} 9 & 5 \\ 2 & 1 \end{vmatrix} = -1;$$

$$C_{43} = -\begin{vmatrix} 3 & -2 & -1 \\ 0 & 2 & 1 \\ 1 & -2 & -2 \end{vmatrix} = -\begin{vmatrix} 0 & 4 & 5 \\ 0 & 2 & 1 \\ 1 & -2 & -2 \end{vmatrix} = -\begin{vmatrix} 4 & 5 \\ 2 & 1 \end{vmatrix} = 6;$$

$$C_{44} = \begin{vmatrix} 3 & -2 & 0 \\ 0 & 2 & 2 \\ 1 & -2 & -3 \end{vmatrix} = \begin{vmatrix} 0 & 4 & 9 \\ 0 & 2 & 2 \\ 1 & -2 & -3 \end{vmatrix} = \begin{vmatrix} 4 & 9 \\ 2 & 2 \end{vmatrix} = -10$$

$$\therefore \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 1 & 1 & -1 & 1 \\ -2 & 0 & 3 & -6 \\ -4 & -1 & 6 & -10 \end{bmatrix}$$

$$\therefore \quad \text{Adj } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} 1 & 1 & -2 & -4 \\ 0 & 1 & 0 & -1 \\ -1 & -1 & 3 & 6 \\ 2 & 1 & -6 & -10 \end{bmatrix} \quad \text{...(ii)}$$

$$\therefore \quad \mathbf{A}^{-1} = \frac{\text{Adj.} \bullet \mathbf{A}}{|\mathbf{A}|} = \begin{bmatrix} 1 & 1 & -2 & 4 \\ 0 & 1 & 0 & 1 \\ -1 & -1 & 3 & -6 \\ -2 & 1 & -6 & -10 \end{bmatrix}, \text{ from (i) and (ii).}$$

Ans.

****Ex. 4. Find $\mathbf{A}^{-1}$, if $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$, where ω is the cube root of unity.** *(Agra 93)*

Sol. $|\mathbf{A}| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 & 0 \\ 1 & \omega - 1 & \omega^2 - 1 \\ 1 & \omega^2 - 1 & \omega - 1 \end{vmatrix}$; replacing C_2, C_3 by $C_2 - C_1, C_3 - C_1$ respectively.

$= \begin{vmatrix} \omega - 1 & \omega^2 - 1 \\ \omega^2 - 1 & \omega - 1 \end{vmatrix}$; expanding w.r. to R_1

$= (\omega - 1)^2 \begin{vmatrix} 1 & \omega + 1 \\ \omega + 1 & 1 \end{vmatrix}$, taking out common factors

$= (\omega - 1)^2 [1 - (\omega + 1)^2]$

$= -(\omega - 1)^2 (\omega^2 + 2\omega) = -(\omega - 1)^2 (\omega - 1),$

$\because \omega^2 + \omega + 1 = 0 \text{ or } \omega^2 + 2\omega = \omega - 1$

$= -(\omega - 1)^3 \neq 0.$ ...(i)

Also $C_{11} = \begin{vmatrix} \omega & \omega^2 \\ \omega^2 & \omega \end{vmatrix} = \omega^2 - \omega^4 = \omega^2 - \omega, \because \omega^3 = 1$

$$C_{12} = -\begin{vmatrix} 1 & \omega^2 \\ 1 & \omega \end{vmatrix} = -(\omega - \omega^2) = \omega^2 - \omega;$$

$$C_{13} = \begin{vmatrix} 1 & \omega \\ 1 & \omega^2 \end{vmatrix} = \omega^2 - \omega; \; C_{21} = -\begin{vmatrix} 1 & 1 \\ \omega^2 & \omega \end{vmatrix} = \omega^2 - \omega;$$

$$C_{22} = \begin{vmatrix} 1 & 1 \\ 1 & \omega \end{vmatrix} = \omega - 1; \; C_{23} = -\begin{vmatrix} 1 & 1 \\ 1 & \omega^2 \end{vmatrix} = -(\omega^2 - 1);$$

$$C_{31} = \begin{vmatrix} 1 & 1 \\ \omega & \omega^2 \end{vmatrix} = \omega^2 - \omega; \; C_{32} = -\begin{vmatrix} 1 & 1 \\ 1 & \omega^2 \end{vmatrix} = -(\omega^2 - 1);$$

$$C_{33} = \begin{vmatrix} 1 & 1 \\ 1 & \omega \end{vmatrix} = \omega - 1$$

$$\therefore \quad \mathbf{C} = \begin{bmatrix} \omega^2 - \omega & \omega^2 - \omega & \omega^2 - \omega \\ \omega^2 - \omega & \omega - 1 & -(\omega^2 - 1) \\ \omega^2 - \omega & -(\omega^2 - 1) & \omega - 1 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} \omega^2 - \omega & \omega^2 - \omega & \omega^2 - \omega \\ \omega^2 - \omega & \omega - 1 & -(\omega^2 - 1) \\ \omega^2 - \omega & -(\omega^2 - 1) & \omega - 1 \end{bmatrix}$$

$$= (\omega - 1)\begin{bmatrix} \omega & \omega & \omega \\ \omega & 1 & -(\omega + 1) \\ \omega & -(\omega + 1) & 1 \end{bmatrix}$$

$$= (\omega - 1)\begin{bmatrix} \omega & \omega & \omega \\ \omega & 1 & \omega^2 \\ \omega & \omega^2 & 1 \end{bmatrix}, \qquad \because 1 + \omega + \omega^2 = 0$$

or $$\text{Adj. } \mathbf{A} = (\omega - 1)\,\omega\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1/\omega & \omega \\ 1 & \omega & 1/\omega \end{bmatrix}, \text{ where } \frac{1}{\omega} = \frac{\omega^2}{\omega^3} = \frac{\omega^2}{1}$$

$$= \omega\,(\omega - 1)\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}, \qquad \text{...(ii)}$$

$$\therefore \quad \mathbf{A}^{-1} = \frac{\text{Adj. } \mathbf{A}}{|\mathbf{A}|} = \frac{\omega\,(\omega - 1)}{-(\omega - 1)^3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}, \text{ from (i) and (ii)}$$

$$= \frac{\omega}{-(\omega - 1)^2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix} \qquad \text{...(iii)}$$

$$\text{Now} -(\omega-1)^2 = -(\omega^2+1-2\omega)$$
$$= -[(-\omega)-(2\omega)], \qquad \because \omega^2+\omega+1=0 \text{ or } \omega^2+1=-\omega$$
$$= 3\omega$$

$\therefore$ From (iii), we get

$$A^{-1} = \frac{\omega}{3\omega}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}$$ **Ans.**

Ex. 5 (a). Find the rank of the matrix

$$A = \begin{bmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{bmatrix}$$

(Agra 96; Bundelkhand 96)

Sol. Given $A = \begin{bmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 16 & 25 & 36 & 49 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & -7 & -20 & -39 \\ 9 & -20 & -56 & -108 \\ 16 & -39 & -108 & -207 \end{bmatrix}$$ replacing C_2, C_3, C_4 by C_2-4C_3, C_3-9C_1 and C_4-16C_1 respectively

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & -20 & -39 \\ 1 & -6 & -16 & -30 \\ 0 & -11 & -28 & -51 \end{bmatrix}$$ replacing R_2, R_3, R_4 by R_2-4R_1, R_3-2R_2 and R_4-4R_2 respectively

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & -20 & -39 \\ 0 & -6 & -16 & -30 \\ 0 & -4 & -8 & -12 \end{bmatrix}$$ replacing R_3, R_4 by R_3-R_1 and R_4-R_2 respectively

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 9 \\ 0 & 2 & 8 & 18 \\ 0 & 4 & 8 & 12 \end{bmatrix}$$ replacing R_2, R_3, R_4 by $-(R_2-R_3)$, $-(R_3-R_4)$ and $-R_4$ respectively

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -8 & -24 \end{bmatrix}$$ replacing R_3, R_4 by R_3-2R_2, R_4-4R_2 respectively

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -8 & -24 \end{bmatrix}, \text{ replacing } C_3, C_4 \text{ by } C_3 - 4C_2, C_4 - 9C_2 \text{ respectively}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}, \text{ replacing } R_4 \text{ by } -\tfrac{1}{8} R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ interchanging } R_3 \text{ and } R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ replacing } C_4 \text{ by } C_4 - 3C_3$$

$$\sim \begin{bmatrix} \mathbf{I}_3 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$$

$\therefore$ The rank of martix **A** is 3. **Ans.**

Ex. 5 (b). Find the rank of the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{0} \\ \mathbf{2} & \mathbf{4} & \mathbf{3} & \mathbf{2} \\ \mathbf{3} & \mathbf{2} & \mathbf{1} & \mathbf{3} \\ \mathbf{6} & \mathbf{8} & \mathbf{7} & \mathbf{5} \end{bmatrix}$$

(Agra 94; Bundelkhand 93)

Sol. Given

$$\mathbf{A} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 1 & 2 & 0 & 2 \\ 1 & -2 & -2 & 1 \\ 3 & 6 & 6 & 2 \end{bmatrix}, \text{ replacing } R_2, R_3, R_4 \text{ by } R_2 - R_1, R_3 - R_2, R_4 - R_3 \text{ respectively}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & -3 & 2 \\ 1 & -4 & -5 & 1 \\ 3 & 0 & -3 & 2 \end{bmatrix}, \text{ replacing } C_2, C_3 \text{ by } C_2 - 2C_1, C_3 - 3C_1 \text{ respectively}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -5 & 1 \\ 0 & 0 & -3 & 2 \end{bmatrix}, \text{ replacing } R_2, R_3, R_4 \text{ by } R_2 - R_1, R_3 - R_1, R_4 - 3R_1 \text{ respectively}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ replacing } R_3, R_4 \text{ by } R_3 - R_2, R_4 - R_2 \text{ respectively}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ replacing } C_2, C_3 \text{ by } -(1/4)\, C_2 \text{ and } -C_3 \text{ respectively}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ replacing } C_3, C_4 \text{ by } C_3 - 2C_2 \text{ and } C_4 + C_1 \text{ respectiely}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ replacing } C_3 \text{ by } C_3 - C_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ replacing } C_4 \text{ by } C_4 - 2C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ interchanging } R_2 \text{ and } R_3$$

$$\sim \begin{bmatrix} \mathbf{I}_3 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$$

$\therefore$ The rank of the given martix **A** is 3. **Ans.**

***Ex. 6. If** $\mathbf{A} = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, **show that** $\mathbf{A}^3 = \mathbf{A}^{-1}$.

Sol. $|\mathbf{A}| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix}$

or $|\mathbf{A}| = \begin{vmatrix} 1 & 0 & 0 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix}$, replacing R_1 by $R_1 - R_2$

$= \begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} = -3 + 4 = 1 \neq 0.$ ...(i)

Also for the matrix A, we have

$C_{11} = \begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} = 1; C_{12} = -\begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} = -2; C_{13} = \begin{vmatrix} 2 & -3 \\ 0 & -1 \end{vmatrix} = -2;$

$C_{21} = -\begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} = -1; C_{22} = \begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix} = 3; C_{23} = -\begin{vmatrix} 3 & -3 \\ 0 & -1 \end{vmatrix} = 3;$

$C_{31} = \begin{vmatrix} -3 & 4 \\ -3 & 4 \end{vmatrix} = 0; C_{32} = -\begin{vmatrix} 3 & 4 \\ 2 & 4 \end{vmatrix} = -4; C_{33} = \begin{vmatrix} 3 & -3 \\ 2 & -3 \end{vmatrix} = -3;$

$$\therefore \quad \mathbf{C} = \begin{bmatrix} 1 & -2 & -2 \\ -1 & 3 & 3 \\ 0 & -4 & -3 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} \quad \text{...(ii)}$$

$$\therefore \quad \mathbf{A}^{-1} = \frac{\text{Adj. } \mathbf{A}}{|\mathbf{A}|} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}, \text{ from (i), (ii)} \quad \text{...(iii)}$$

Also $\mathbf{A}^2 = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 9-6+0 & -9+9-4 & 12-12+4 \\ 6-6+0 & -6+9-4 & 8-12+4 \\ 0-2+0 & 0+3-1 & 0-4+1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix}$$

$$\therefore \quad \mathbf{A}^3 = \mathbf{A}^2 \bullet \mathbf{A} = \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix} \times \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9-8+0 & -9+12-4 & 12-16+4 \\ 0-2+0 & 0+3+0 & 0-4+0 \\ -6+4+0 & 6-6+3 & -8+8-3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

i.e. $\mathbf{A}^3 = \mathbf{A}^{-1}$, from (iii). Hence proved.

****Ex. 7. If two non-singular symmetric matrices A and B be such that AB = BA (*i.e.* commute under multiplication), then prove that $\mathbf{A}^{-1}\mathbf{B}$ and $\mathbf{A}^{-1}\mathbf{B}^{-1}$ are symmetric.**

Sol. Here we are given that **AB = BA**.

$\therefore$ We have $\mathbf{A}^{-1}\mathbf{AB} = \mathbf{A}^{-1}\mathbf{BA}$, premultiplying by $\mathbf{A}^{-1}$

or $\quad IB = A^{-1}BA, \qquad \because A^{-1}A = I$

or $\quad B = A^{-1}BA, \qquad \because IB = B$

or $\quad BA^{-1} = A^{-1}BAA^{-1}, \qquad$ post multiplying by A^{-1}

$\quad = A^{-1}BI = A^{-1}B, \qquad$ ...(i)

since $\quad AA^{-1} = I$ and $BI = B$.

Again $(A^{-1}B)' = B'(A^{-1})', \qquad \because (AB)' = B'A'$

$= B'(A')^{-1} \qquad \because (A^{-1})' = (A')^{-1}$

...See Th. II Page 77 Chapter V

$= BA^{-1}, \qquad \because A' = A, B' = B$ as A and B are symmetric

i.e. $\quad (A^{-1}B)' = A^{-1}B$, from (i)

Hence $A^{-1}B$ is symmetric.

Similarly $(A^{-1}B^{-1})' = (B^{-1})'(A^{-1})'$, as $(CD)' = D'C'$

or $\quad (A^{-1}B^{-1})' = (B')^{-1}(A')^{-1}, \qquad$ See Th. II Page 77 Ch. V

$= B^{-1}A^{-1} \qquad \because A' = A, B' = B$

$= (AB)^{-1}, \qquad \because (AB)^{-1} = B^{-1}A^{-1}$

$= (BA)^{-1}, \qquad \because AB = BA$ (given)

or $\quad (A^{-1}B^{-1})' = A^{-1}B^{-1}$.

Hence $A^{-1}B^{-1}$ is symmetric.

Ex. 8. Find the rank of $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 1 & 2 & 3 & 2 \end{bmatrix}$

Sol. $A \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, replacing R_2, R_3 by $R_2 - 2R_1$ and $R_3 - R_1$ respectively

$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, replacing R_1 by $R_1 - R_3$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, replacing C_2, C_3 by $C_2 - 2C_1$ and $C_3 - 3C_1$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, interchanging C_2 and C_4

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ interchanging } R_2 \text{ and } R_3$$

$$\sim \begin{bmatrix} \mathbf{I}_2 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$$

$\therefore$ The rank of matrix **A** is 2. **Ans.**

****Ex. 9. Find the rank of an $m \times n$ matrix, every element of which is unity.**

Sol. Let an $m \times n$ matrix be $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix}$

Then we find that every square submatrix of **A** higher than 1×1 will be a matrix each element of which is unity and therefore the value of the determinant will be always zero, since its rows and columns are identical. But the square sub-matrices of order 1×1 are $[\,1\,]$ and the determinants of these are $|\,\mathbf{A}\,| = 1 \neq 0$.

Hence the rank of **A** is 1. **Ans.**

Ex. 10. Show that the matrix $\mathbf{A} = \begin{bmatrix} 1 & a & \alpha & a\alpha \\ 1 & b & \beta & b\beta \\ 1 & c & \gamma & c\gamma \end{bmatrix}$ is of rank 3 provided no two of a, b, c are equal and no two of α, β, γ are equal.

Sol. $\mathbf{A} \sim \begin{bmatrix} 1 & a & \alpha & a\alpha \\ 0 & b-a & \beta-\alpha & b\beta-a\alpha \\ 0 & c-a & \gamma-\alpha & c\gamma-a\alpha \end{bmatrix}$, replacing R_2, R_3 by $R_2 - R_1$, $R_3 - R_1$ respectively

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & b-a & \beta-\alpha & b\beta-a\alpha \\ 0 & c-a & \gamma-\alpha & c\gamma-a\alpha \end{bmatrix}, \text{ replacing } C_2, C_3, C_4 \text{ by } C_2 - aC_1, C_3 - \alpha C_1 \text{ and } C_4 - a\alpha C_1 \text{ respectively}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & b-a & \beta-\alpha & b\beta-b\alpha \\ 0 & c-a & \gamma-\alpha & c\gamma-c\alpha \end{bmatrix}, \text{ replacing } C_4 \text{ by } C_4 - \alpha C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & b-a & \beta-\alpha & 0 \\ 0 & c-a & \gamma-\alpha & c\gamma-c\alpha-b\gamma+b\alpha \end{bmatrix}, \text{ replacing } C_4 \text{ by } C_4 - bC_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & b-a & \beta-\alpha & 0 \\ 0 & c-a & \gamma-\alpha & (c-b)(\gamma-\alpha) \end{bmatrix} = \mathbf{B} \text{ (say);}$$

Now a minor of order 3 of **B**

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & b-a & 0 \\ 0 & c-a & (c-b)(\gamma-\alpha) \end{vmatrix} = \begin{vmatrix} b-a & 0 \\ c-a & (c-b)(\gamma-\alpha) \end{vmatrix},$$

expanding with respect to R_1

$= (b-a)(c-b)(\gamma-\alpha) \neq 0,$ as no two of a, b, c and no two of α, β, γ are equal (given)

$\therefore \qquad \rho(\mathbf{B}) \geq 3$ ...(i)

Also the matrix **B** does not possess any minor of order 4 *i.e.* of order 3 + 1,

so $\qquad \rho(\mathbf{B}) \leq 3.$ ...(ii)

$\therefore$ From (i) and (ii) we get $\rho(\mathbf{B}) = 3$

and therefore $\rho(\mathbf{A}) = 3$, as $\mathbf{A} \sim \mathbf{B}$. Hence proved.

Ex. 11 (a). Find A^{-1} if $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 1 \end{bmatrix}$

Sol. Here $|\mathbf{A}| = \begin{vmatrix} 1 & -1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{vmatrix}$, replacing R_3 by $R_3 - R_1$

$= 0$, since two rows are identical.

Hence the matrix **A** is not non-singular (*i.e.* is singular) and so $\mathbf{A}^{-1}$ does not exist. (See § 5·10 Page 76 Ch. V)

Ex. 11 (b). Find adjoint and inverse of the matrix

$$\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

Sol. Do yourself.

Ans. Adj. $\mathbf{A} = \mathbf{C}' = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$; $\mathbf{A}^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$;

***Ex. 12. If $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$**

then show that $\rho(AB) \neq \rho(BA)$, where ρ denotes its rank.

(Rohilkhand 93)

Sol. $\mathbf{A}\,\mathbf{B} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix} \times \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$

$$= \begin{bmatrix} -1+6-5 & -2+12-10 & -1+6-5 \\ -2-18+20 & -4-36+40 & -2-18+20 \\ -3-12+15 & -6-24+30 & -3-12+15 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$= \mathbf{O}$

$\therefore$ By definition ρ (**AB**) *i.e.* rank of **AB** is 0. ...(i)

... Sec § 5.02 Note 2 Page 2.

Again $\mathbf{BA} = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} -1-4-3 & -1+6+2 & 1-8-3 \\ 6+24+18 & 6-36-12, & -6+48+18 \\ 5+20+15 & 5-30-10 & -5+40+15 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 7 & -10 \\ 48 & -42 & 60 \\ 40 & -35 & 50 \end{bmatrix} = \mathbf{C}, \text{ say} \quad \text{...(ii)}$$

Now $\mathbf{C} \sim \begin{bmatrix} -8 & 7 & -10 \\ 8 & -7 & 10 \\ 40 & -35 & 50 \end{bmatrix}$, replacing R_2 by $R_2 - R_3$

$\sim \begin{bmatrix} -8 & 7 & -10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, replacing R_2, R_3 by $R_2 + R_1$ and $R_3 + 5R_1$ respectively

$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, replacing C_1, C_2, C_3 by $(-1/8)\, C_1$ $(1/7)\, C_2$, $(-1/10)\, C_3$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, replacing C_2, C_3 by $C_2 - C_1$ and $C_3 - C_1$ respectively

$\sim \begin{bmatrix} \mathbf{I}_1 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$.

$\therefore$ $\rho(\mathbf{C}) = 1$ or $\rho(\mathbf{BA}) = 1$, from (ii) ...(iii)

$\therefore$ From (i) and (iii), $\rho(\mathbf{AB}) \neq \rho(\mathbf{BA})$ Hence proved.

Ex. 13. If $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$, then show that

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Sol. Here $\mathbf{A} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 8 \end{vmatrix} = 2$

Also for the matrix **A**, we have

$C_{11} = \begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} = 6; C_{12} = -\begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} = -6; C_{13} = \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 2;$

$C_{21} = -\begin{vmatrix} 1 & 1 \\ 4 & 9 \end{vmatrix} = -5; C_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix} = 8; C_{23} = -\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = -3;$

$C_{31} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1; C_{32} = -\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -2; C_{33} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1;$

$$\therefore \quad \mathbf{C} = \begin{bmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } \mathbf{A} = \mathbf{C}' = \begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

$$\therefore \quad \mathbf{A}^{-1} = \frac{\text{Adj. } \mathbf{A}}{|\mathbf{A}|} = \frac{1}{2}\begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix} \qquad \text{...(i)}$$

Again $|\mathbf{B}| = \begin{vmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 0 & -5 & -1 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -5 & -1 \end{vmatrix} = 4$

Also for the matrix **B**, we have

$C_{11} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3; C_{12} = -\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = -1; C_{13} = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 5;$

$C_{21} = -\begin{vmatrix} 5 & 3 \\ 2 & 1 \end{vmatrix} = 1; C_{22} = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1; C_{23} = -\begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} = 1;$

$C_{31} = \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 7; C_{32} = -\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = 5; C_{33} = \begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix} = -13.$

$$\therefore \quad \mathbf{C} = \begin{bmatrix} -3 & -1 & 5 \\ 1 & -1 & 1 \\ 7 & 5 & -13 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } \mathbf{B} = \mathbf{C}' = \begin{bmatrix} -3 & 1 & 7 \\ -1 & -1 & 5 \\ 5 & 1 & -13 \end{bmatrix}$$

$$\therefore \quad \mathbf{B}^{-1} = \frac{\text{Adj. } \mathbf{B}}{|\mathbf{B}|} = \frac{1}{4}\begin{bmatrix} -3 & 1 & 7 \\ -1 & -1 & 5 \\ 5 & 1 & -13 \end{bmatrix} \qquad \text{...(ii)}$$

Again $\mathbf{AB} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \times \begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2+3+1 & 5+1+2 & 3+2+1 \\ 2+6+3 & 5+2+6 & 3+4+3 \\ 2+12+9 & 5+4+18 & 3+8+9 \end{bmatrix} = \begin{bmatrix} 6 & 8 & 6 \\ 11 & 13 & 10 \\ 23 & 27 & 20 \end{bmatrix} = \mathbf{D} \text{ (say)}$$

Now $|\mathbf{D}| = \begin{vmatrix} 6 & 8 & 6 \\ 11 & 13 & 10 \\ 23 & 27 & 20 \end{vmatrix} = \begin{vmatrix} 6 & 8 & 6 \\ 11 & 13 & 10 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 8 & 6 \\ 1 & 13 & 10 \\ 1 & 1 & 0 \end{vmatrix}$

$$= \begin{vmatrix} 0 & 8 & 6 \\ 0 & 12 & 10 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 8 & 6 \\ 12 & 10 \end{vmatrix} = 80 - 72 = 8 \neq 0.$$

For this matrix **D**, we have

$C_{11} = -\begin{vmatrix} 13 & 10 \\ 27 & 20 \end{vmatrix} = -10$; $C_{12} = -\begin{vmatrix} 11 & 10 \\ 23 & 20 \end{vmatrix} = 10$; $C_{13} = \begin{vmatrix} 11 & 13 \\ 23 & 27 \end{vmatrix} = -2$;

$C_{21} = -\begin{vmatrix} 8 & 6 \\ 27 & 20 \end{vmatrix} = 2$; $C_{22} = \begin{vmatrix} 6 & 6 \\ 23 & 20 \end{vmatrix} = -18$; $C_{23} = -\begin{vmatrix} 6 & 8 \\ 23 & 27 \end{vmatrix} = 22$;

$C_{31} = \begin{vmatrix} 8 & 6 \\ 13 & 10 \end{vmatrix} = 2$; $C_{32} = -\begin{vmatrix} 6 & 6 \\ 11 & 10 \end{vmatrix} = 6$; $C_{33} = \begin{vmatrix} 6 & 8 \\ 11 & 13 \end{vmatrix} = -10$

$$\therefore \quad \mathbf{C} = \begin{bmatrix} -10 & 10 & -2 \\ 2 & -18 & 22 \\ 2 & 6 & -10 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } \mathbf{D} = \mathbf{C}' = \begin{bmatrix} -10 & 2 & 2 \\ 10 & -18 & 6 \\ -2 & 22 & -10 \end{bmatrix}$$

$$\therefore \quad (\mathbf{AB})^{-1} = \mathbf{D}^{-1} = \frac{\text{Adj. } \mathbf{D}}{|\mathbf{D}|} = \frac{1}{8} \begin{bmatrix} -10 & 2 & 2 \\ 10 & -18 & 6 \\ -2 & 22 & -10 \end{bmatrix} \quad \text{...(iii)}$$

From (i) and (ii) we get

$$\mathbf{B}^{-1} \mathbf{A}^{-1} = \frac{1}{4} \begin{bmatrix} -3 & 1 & 7 \\ -1 & -1 & 5 \\ 5 & 1 & -13 \end{bmatrix} \times \frac{1}{2} \begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} -18-6+14 & 15+8-21 & -3-2+7 \\ -6+6+10 & 5-8-15 & -1+2+5 \\ 30-6-26 & -25+8+39 & 5-2-13 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} -10 & 2 & 2 \\ 10 & -18 & 6 \\ -2 & 22 & -10 \end{bmatrix}$$

$= (\mathbf{AB})^{-1}$, from (iii).

Hence proved.

***Ex. 14. Prove that $(\text{Adj. } \mathbf{A})^{-1} = (\text{Adj. } \mathbf{A}^{-1})$, where A is any $\mathbf{n} \times \mathbf{n}$ matrix.**

(Agra 95; Avadh 99; Bundelkhand 92; Purvanchal 96)

Sol. We know from Ex. 22 Page 74 Ch. V (To be proved in the examination) that

$$Adj.\,(Adj.\,\mathbf{A}) = |\,\mathbf{A}\,|^{n-2} \bullet \mathbf{A}$$

Also from Theorems I and II Pages 49–50, Ch. V we know that

$$|\,Adj.\,\mathbf{A}\,| = |\,\mathbf{A}\,|^{n-1} \qquad \text{...(ii)}$$

and $$\mathbf{A}^{-1} = \frac{Adj.\,\mathbf{A}}{|\,\mathbf{A}\,|} \qquad \text{...(iii)}$$

Now $Adj.\,\mathbf{A}^{-1} = Adj.\left\{\dfrac{Adj\,\mathbf{A}}{|\,\mathbf{A}\,|}\right\}$, from (iii)

$= Adj.\left\{\dfrac{1}{|\,\mathbf{A}\,|}\right\} \bullet Adj.\,(Adj.\,\mathbf{A})$ ... See Th. III **Page 50 Ch. V**

$= \dfrac{1}{|\,\mathbf{A}\,|^{n-1}} \bullet |\,\mathbf{A}\,|^{n-2} \bullet \mathbf{A}$, from (i) and (ii)

or $$Adj.\,\mathbf{A}^{-1} = \frac{\mathbf{A}}{|\,\mathbf{A}\,|} \qquad \text{...(iv)}$$

Also $(Adj.\,\mathbf{A})^{-1} = \dfrac{Adj.\,(Adj.\,\mathbf{A})}{|\,Adj.\,\mathbf{A}\,|}$, from (iii). **(Note)**

or $(Adj.\,\mathbf{A})^{-1} = \dfrac{|\,\mathbf{A}\,|^{n-2} \bullet \mathbf{A}}{|\,\mathbf{A}\,|^{n-1}}$, from (i) and (ii)

or $$(Adj.\,\mathbf{A})^{-1} = \frac{\mathbf{A}}{|\,\mathbf{A}\,|}. \qquad \text{...(v)}$$

Hence from (iv) and (v), we get $(Adj.\,\mathbf{A})^{-1} = (Adj.\,\mathbf{A}^{-1})$. **Hence proved.**

Ex. 15. If A is of order $m \times n$, R is a non-singular matrix of order m, show that Rank of RA = Rank of A.

Sol. Let $\mathbf{A} = \mathbf{E}\,\mathbf{A}_r\,\mathbf{F}$ and $\mathbf{R} = \mathbf{E}_1$ See § 5.12 **Page** 78

Then $\mathbf{RA} = \mathbf{E}_1\,(\mathbf{E}\,\mathbf{A}_r\,\mathbf{F}) = \mathbf{E}_1\,\mathbf{E}\,\mathbf{A}_r\,\mathbf{F}$

i.e. **RA** has been expressed as the result of elementary operations **on** $\mathbf{A}_r$.

Thus Rank of (**RA**) = Rank $\mathbf{A}_r$ = Rank **A**.

****Ex. 16. Prove that the rank of a matrix remains unaltered by the application of elementary row and column operations.**

or Prove that two equivalent matrices have the same rank. *(Avadh 99)*

Sol. Let an $m \times n$ matrix **A** be given by

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Let **M** be any minor of order r belonging in the first r rows of $|\,\mathbf{A}\,|$.

Now firstly if we inerchange any two rows or columns of **A**, then the minor **M** either remains unaltered or changes sign.

Secondly if we multiply one row or column of **A** by a number λ, then either the minor **M** remains unaltered or changes into λ**M**.

Thirdly if we replace any row R_i (for column C_i) by $R_i + \lambda R_i$ (or $C_i + \lambda C_i$), then either the minor M remains unaltered or changes into a sum or difference of two of the original minors.

Let **B** be the matrix obtained from **A** by the application of any one of the above three elementary row or column operations.

Thus if all the minor of order r in $|\mathbf{B}|$ are zero, then all the minors of order r in $|\mathbf{B}|$ are also zero.

$\therefore$ rank of $\mathbf{B} \leq$ rank of $\mathbf{A}$. ...(i)

Similarly if all the minors of order r in $|\mathbf{B}|$ are zero, then all the minors of order r in $|\mathbf{A}|$ are also zero.

$\therefore$ rank of $\mathbf{A} \leq$ rank of $\mathbf{B}$ ...(ii)

$\therefore$ From (i) and (ii) we get

rank of $\mathbf{A}$ = rank of $\mathbf{B}$. Hence proved.

Exercises on Chapter V

Ex. 1. Are the following pairs of matrices equivalent ?

$$\begin{bmatrix} 4 & -1 & 2 \\ 3 & 4 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 1 & 4 & 7 \\ 3 & 6 & 2 & 1 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Ans. No.

Ex. 2. Show that the rank of a matrix is not altered if a column of it is multiplied by a non-zero scalar.

Ex. 3. Show that the inverse of

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ is } \begin{bmatrix} 1 & -2 & 1 & 0 \\ 1 & -2 & 2 & -3 \\ 0 & 1 & -1 & -1 \\ -2 & 3 & -2 & 3 \end{bmatrix}$$

Ex. 4. Compute the adjoint and inverse of

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

Ans. $\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Ex. 5. Show that the adjoint and inverse of the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 6 & 7 & 9 \end{bmatrix} \textit{ are } \begin{bmatrix} 1 & 3 & -2 \\ 3 & -9 & 4 \\ -3 & 5 & -2 \end{bmatrix} \text{ and } -\frac{1}{2}\begin{bmatrix} 1 & 3 & -2 \\ 3 & -9 & 4 \\ -3 & 5 & -2 \end{bmatrix}$$

Ex. 6. If $\mathbf{A} = \begin{bmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & a_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_n \end{bmatrix}$, where $a_i \neq 0$ for all $1 \leq i \leq n$, then show that **A** is invertible. Also evaluate $\mathbf{A}^{-1}$

$$\text{Ans. } \mathbf{A}^{-1} = \begin{bmatrix} 1/a_1 & 0 & 0 & \ldots & 0 \\ 0 & 1/a_2 & 0 & \ldots & 0 \\ 0 & 0 & 1/a_3 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1/a_n \end{bmatrix}$$

Ex. 7. Show that the reciprocal (or inverse) of

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \text{ is } \begin{bmatrix} -2 & \frac{5}{3} & -\frac{1}{3} \\ 5 & -\frac{8}{3} & -\frac{1}{3} \\ -2 & 1 & 0 \end{bmatrix}$$

Ex. 8. Show that the inverse of

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \text{ is } \begin{bmatrix} -1 & 1 & 2 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 1 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

Ex. 9. If $\mathbf{A} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, prove that $\mathbf{AA}^{-1} = \mathbf{I}_3$

Ex. 10. Prove that $Adj.\ (\mathbf{A}') = (Adj.\ \mathbf{A})'$.

Ex. 11. Let **A** be a non-singular matrix. Will the adjoint of **A** also be non-singular ?

Ex. 12. Show that $\mathbf{A} = \begin{bmatrix} 3 & 7 & 1 \\ 5 & 9 & -1 \\ 7 & 13 & -5 \end{bmatrix}$ is non-singular.

[**Hint** : Prove that $|\mathbf{A}| \neq 0$].

Ex. 13. Show that if **A** is a square matrix of order n then

$$Adj.\ \mathbf{A}\ \{Adj.\ (Adj.\ \mathbf{A})\} = (\det.\ \mathbf{A})^{n-1}\mathbf{I}.$$

Ex. 14. What is the rank of a non-singular matrix of order n ?

Ex. 15 (a). Show that the inverse of

$$\begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix} \text{ is } \begin{bmatrix} -7 & -9 & 10 \\ -12 & -15 & 17 \\ 1 & 1 & -1 \end{bmatrix}$$

Ex. 15 (b). Find the inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 1 & -1 \\ 2 & 1 & 2 & 1 \\ 3 & -2 & 1 & 6 \end{bmatrix}$$ *(Agra 95)*

$$\textbf{Ans. } \frac{1}{2}\begin{bmatrix} 24 & 10 & -2 & -6 \\ -5 & -3 & 1 & 1 \\ -16 & -6 & 2 & 4 \\ -11 & -5 & 1 & 3 \end{bmatrix}$$

Ex. 16. Compute rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ **Ans. 2**

Ex. 17. Prove that each non-singular matrix has a unique inverse matrix.

***Ex. 18.** Define Rank of a matrix. Determine rank of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & a & b & 0 \\ 0 & c & d & 1 \\ 1 & a & b & 0 \\ 0 & c & d & 1 \end{bmatrix}$$

[Hint : See § 5.02 Pages 1-2 Ex. 13. Page 10 Ch V]

***Ex. 19.** Find the rank of the matrix **A**, given by

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 7 & 1 \\ 4 & 1 & 3 & 2 \\ 1 & -1 & -4 & 1 \end{bmatrix}$$

Ans. 2

Ex. 20. Find an invertible matrix **P** such that $\mathbf{PAP}^{-1}$ is a diagonal matrix, where $\mathbf{A} = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$.

Ex. 21. Prove that every matrix of rank r can be reduced by means of elementary transformation to the form $\begin{bmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$.

Ex. 22. Show that for any matrix **A**, rank $(\mathbf{A'A})$ = rank $(\mathbf{A})$.

Hence or otherwise show that if n be the rank of an $m \times n$ matrix **A**, then **A′A** is a non-singular matrix.

Ex. 23. Find inverse of

(i) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & 1 \\ 3 & 4 & 4 \\ 0 & -4 & -7 \end{bmatrix}$

Ex. 24. If adj. $\mathbf{B} = \mathbf{A}$ and $|\mathbf{P}| = 1 = |\mathbf{Q}|$, then prove that adj. $(\mathbf{Q}^{-1}\mathbf{BP}^{-1}) = \mathbf{PAQ}$. *(Kanpur 95, 93)*

Ex. 25. Prove that the inverse of a matrix is unique.

Ex. 26. Prove that for every matrix **A** there exist two non-singular matrices **P** and **Q** such that **PAQ** is in normal form. *(Rohilkhand 90)*

Ex. 27. Show that every elementary matrix is non-singular *i.e.* it is invertible and its inverse is also an elementary matrix of the same type. *(Purvanchal 93)*

Ex. 28. Find the rank of the matrix **A**, where

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$$

(Rohilkhand 99) **Ans. 1**