Chapter VI

Solution of Linear Equations

§ 6.01. Matrix of coefficients of a system of equations.

Definiton. Let the system of m simultaneous equations in n unknowns $x_1, x_2, \ldots, x_n$ be

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= k_1,\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= k_2,\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n &= k_3,\\
\ldots \ldots \ldots \ldots \ldots \ldots &\ldots \ldots\\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n &= k_m
\end{aligned}$$

or written in a compact form

$$\sum_{j=1}^{n} a_{ij}\, x_j = k_i, \text{ where } i = 1, 2, \ldots m \qquad \ldots\text{(i)}$$

Then the matrix $\mathbf{A} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$

of order $m \times n$ is known as the **matrix of coefficients of the system of equations** given by (i).

The determinant of the matrix $\mathbf{A}$, [if there be n equations in (i)] viz.

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{vmatrix}$$

is called the **determinant of coefficients of the system of equations** given by (i).

Note. If all the k's are zero, then the system of equations given by (i) is said to be **homogeneous** and if at least one of k's is not zero, then the above system of equations is said to be **non-homogeneous.**

§ 6.02. System of equations in the Matrix Form.

The system of equations given by (i) in § 6.01 above can be written in the matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1j} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2j} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{i1} & a_{i2} & \ldots & a_{ij} & \ldots & a_{in} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mj} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ \ldots \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ \ldots \\ \ldots \\ \ldots \\ k_m \end{bmatrix} \qquad \textbf{(Note)} \ \ldots\text{(i)}$$

or in more compact form it may be written as

$$\mathbf{AX} = \mathbf{K}.$$

where $\mathbf{A} = [a_{ij}]$ *i.e.* the matrix of coefficients of the system of equations given by (i) in § 6.01 on Page 105;

$\mathbf{X}$ = the transposed matrix of $[x_1, x_2, x_3, \ldots x_n]$ and

$\mathbf{K}$ = the transposed matrix of $[k_1, k_2, k_3, \ldots, k_m]$.

Here students should note that the product $\mathbf{AX}$ is a matrix of order $m \times 1$, as $\mathbf{A}$ is a matrix of order $m \times n$ and $\mathbf{X}$ is a matrix of order $n \times 1$. And $\mathbf{X}$ is also a matrix of order $m \times 1$.

§ 6.03. Consistent and Inconsistent Equations.

Consider the system of equations given by (i) of § 6.01 Page 105;

If the above system has a solution (*i.e.* a set of values of $x_1, x_2, x_3, \ldots, x_n$ satisfy simultaneously these m equations), then the equations are said to be **consistent** otherwise the equations are **said to be inconsistent.**

A consistent system of equations has either one solution or infinitely many solutions.

§ 6.04. Solution of non-homogeneous Simultaneous equations.

Solution of equation given by (i) *of* § 6.01 *Page* 105 *Ch. VI when* $m = n$ *and the matrix* $\mathbf{A}$ *is non-singular.*

We know from § 6.02 Page 105 Ch. VI that the matrix form of the given equations is

$$\mathbf{AX} = \mathbf{K} \qquad \text{...(i)}$$

Also we know that if $\mathbf{A}$ is non-singular, its inverse matrix *i.e.*, $\mathbf{A}^{-1}$ exists such that

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}, \qquad \text{...(ii)}$$

where $\mathbf{I}$ is the identity matrix.

Hence by multiplying both sides of (i) by $\mathbf{A}^{-1}$, we have

$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{K}$$

or $\mathbf{IX} = \mathbf{A}^{-1}\mathbf{K}$, from (ii)

or $\mathbf{X} = \mathbf{A}^{-1}\mathbf{K}$, which is the required solution of the given equations and is unique.

Solved Examples on § 6.02—§ 6.04.

Ex. 1. Express in matrix form the system of equations

$$\mathbf{9x + 7y + 3z = 6;\ 5x + y + 4z = 1;\ 6x + 8y + 2z = 4.}$$ *(Gorakhpur 97, 94)*

Sol. The given equations are

$$9x + 7y + 3z = 6$$
$$5x + y + 4z = 1$$
$$6x + 8y + 2z = 4$$

$\therefore$ The required matrix form of these equations is

$$\mathbf{AX} = \mathbf{K},$$

where $A = \begin{bmatrix} 9 & 7 & 3 \\ 5 & 1 & 4 \\ 6 & 8 & 2 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $K = \begin{bmatrix} 6 \\ 1 \\ 4 \end{bmatrix}$

Ex. 2 (a). Find the matrix X from the equations AX = B,

where $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}$.

Sol. Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, then from **AX = B**, we have

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}$$

$\Rightarrow$ $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}$, by the elementary row operation $R_3 \to R_3 + R_2$

$\Rightarrow$ $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 12 \end{bmatrix}$, by the elementary row operations $R_3 \to R_3 + 2R_1$

$\Rightarrow$ $x - y = 2, y - z = 1, 3x = 12$ **(Note)**

$\Rightarrow$ $y = x - 2, z = y - 1, x = 4$

$\Rightarrow$ $y = 4 - 2 = 2, z = 2 - 1 = 1, x = 4$

$\Rightarrow$ $x = 4, y = 2, z = 1.$

$\therefore$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ **Ans.**

Ex. 2 (b). Solve by matrix method

$$x - 2y + 3z = 2, 2x - 3z = 0, x + y + z = 0.$$

Sol. Do as Ex. 2 (a) above. **Ans.** $x = (6/19), y = -(10/19), z = (4/19)$

****Ex. 3 (a). Solve by matrix method :**

$$x + y + z = 6, x - y + z = 2, 2x + y - z = 1$$

(Gorakhpur 96; Kanpur 95; Rohilkhand 95)

Sol. Given equations are

$$x + y + z = 6$$
$$x - y + z = 2$$
$$2x + y - z = 1$$

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$, $K = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$ and assume that there exists a matrix

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that **AX = K**

Then $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & -1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ -11 \end{bmatrix}$, by the elementary row operations $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$

$\Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & -2 & 0 \\ 0 & -1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -4 \\ -11 \end{bmatrix}$, by the elementary row operations $R_1 \to R_1 + R_3$

$\Rightarrow \quad x - 2z = -5, -2y = -4, -y - 3z = -11$

$\Rightarrow \quad x - 2z = -5, y = 2, 2 + 3z = 11$

$\Rightarrow \quad x = 2z - 5, y = 2, z = 3$

$\Rightarrow \quad x = 6 - 5, y = 2, z = 3 \Rightarrow x = 1, y = 2, z = 3.$ **Ans.**

***Ex. 3 (b). Solve by matrix method**

$$x + 2y + 3z = 14, 3x + y + 2z = 11, 2x + 3y + z = 11.$$

(Bundelkhand 92; Purvanchal 93; Rohilkhand 98)

Sol. The given equations are

$$x + 2y + 3z = 14$$
$$3x + y + 2z = 11$$
$$2x + 3y + z = 11$$

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$ and $K = \begin{bmatrix} 14 \\ 11 \\ 11 \end{bmatrix}$ and assume that there exists a matrix

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that **AX = K.**

Then $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 11 \\ 11 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 5 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 25 \\ 11 \end{bmatrix}$, by the elementary row operation $R_2 \to R_1 + R_2$

$\Rightarrow \begin{bmatrix} 2 & 4 & 6 \\ 2 & 0 & 4 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 28 \\ 14 \\ 11 \end{bmatrix}$, by the elementary row operation $R_1 \to 2R_1$, $R_2 \to R_2 - R_3$

$$\Rightarrow \begin{bmatrix} 0 & 4 & 2 \\ 2 & 0 & 4 \\ 0 & 3 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 14 \\ -3 \end{bmatrix}, \text{ by the elementary row operation } R_1 \to R_1 - R_2,\ R_3 \to R_3 - R_2$$

$$\Rightarrow \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ -1 \end{bmatrix}, \text{ by the elementary row operation } R_1 \to \tfrac{1}{2} R_1,\ R_2 \to \tfrac{1}{2} R_2,\ R_3 \to \tfrac{1}{3} R_3$$

$$\Rightarrow \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ -1 \end{bmatrix}, \text{ by the elementary row operations } R_1 \to R_1 + R_3$$

$\Rightarrow \quad 3y = 6,\ x + 2z = 7,\ y - z = -1$ **(Note)**

$\Rightarrow \quad y = 2,\ z = y + 1 = 2 + 1 = 3,\ x = 7 - 2z = 7 - 6 = 1$

$\Rightarrow \quad x = 1,\ y = 2,\ z = 3.$ **Ans.**

Ex. 3 (c). Solve the following equations by matrix method :

$x + 2y + 3z = 4,\ 2x + 3y + 8z = 7,\ x - y + 9z = 1$. *(Agra 96, 93)*

Sol. The given equations are

$$x + 2y + 3z = 4$$
$$2x + 3y + 8z = 7$$
$$x - y + 9z = 1$$

Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 8 \\ 1 & -1 & 9 \end{bmatrix}$; $\mathbf{K} = \begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix}$ and assume that there exists a matrix $\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

such that $\mathbf{AX} = \mathbf{K}$.

Then $$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 8 \\ 1 & -1 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 0 & -3 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix}, \text{ by elementary row operations } R_2 \to R_2 - 2R_1 \text{ and } R_3 \to R_3 - R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}, \text{ by elementary row operations } R_3 \to R_3 - 3R_2$$

$\Rightarrow \quad x + 2y + 3z = 4,\ -y + 2z = -1$

$\Rightarrow \quad x + 7z = 2,\ -y + 2z = -1$

$\Rightarrow \quad x = 2 - 7z,\ y = 2z + 1$ and z can take any finite value. **Ans.**

****Ex. 4 (a). Using matrix method, solve the following equations—**

$2x - y + 3z = 9,\ x + y + z = 6$ and $x - y + z = 2$.

(Avadh 98; Agra 95; Garhwal 95; Gorakhpur 99; Kanpur 90; Meerut 92P, 91; Rohilkhand 97)

Sol. The given equations are

$$2x - y + 3z = 9$$
$$x + y + z = 6$$
$$x - y + z = 2$$

Let $\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$; $\mathbf{K} = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$, and assume that there exists a matrix $\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $\mathbf{AX} = \mathbf{K}$.

Then $\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$ *(Kanpur 90)*

$\Rightarrow$ $\begin{bmatrix} 3 & 0 & 4 \\ 1 & 1 & 1 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 6 \\ 8 \end{bmatrix}$, by the elementary row operations $R_1 \to R_1 + R_2$, $R_3 \to R_3 + R_2$

$\Rightarrow$ $3x + 4z = 15$, $x + y + z = 6$, $2x + 2z = 8$

$\Rightarrow$ $3x + 4z = 15$, $x + y + z = 6$, $x + z = 4$

$\Rightarrow$ $y = 6 - (x + z) = 6 - 4$, $\quad \because x + z = 4$

or $y = 2$

Also $3x + 4z = 15$ gives $3x + 4(4 - x) = 15$, $\quad \because x + z = 4$

or $3x + 16 - 4x = 15$ or $x = 1$

$\therefore$ $z = 4 - x = 4 - 1 = 3$

$\Rightarrow$ $x = 1, y = 2, z = 3$.

***Ex. 4 (b). Solve by matrix method only the equations :**

$$x + y + z = 6; x + 2y + 3z = 14, x + 4y + 9z = 36.$$

(Gorakhpur 98, 91; Kanpur 94; Rohilkhand 99)

Hint : Do as Ex. 4 (a) above. **Ans.** $x = 1, y = 2, z = 3$.

****Ex. 5 (a). Solve the following equations by matrix method :**

$$x + 2y + z = 2, 3x + 5y + 5z = 4, 2x + 4y + 3z = 3.$$ *(Bundelkhand 91)*

Sol. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 5 & 5 \\ 2 & 4 & 3 \end{bmatrix}$; $\mathbf{K} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$ and assume that there exists a matrix $\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, such that $\mathbf{AX} = \mathbf{K}$.

Then $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 5 & 5 \\ 2 & 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$, by the elementary row operations $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 2R_1$

$\Rightarrow \quad x + 2y + z = 2, -y + 2z = -2, z = -1$

$\Rightarrow \quad x + 2y - 1 = 2, -y - 2 = -2, z = -1$

$\Rightarrow \quad x + 2y = 3, y = 0, z = -1$

$\Rightarrow \quad x + 0 = 3, y = 0, z = -1$

$\Rightarrow \quad x = 3, y = 0, z = -1.$ **Ans**

****Ex. 5 (b). Solve the equations :**

$$x + y + z = 9, 2x + 5y + 7z = 52, 2x + y - z = 0. \qquad \textit{(Gorakhpur 92)}$$

Sol. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}$; $\mathbf{K} = \begin{bmatrix} 9 \\ 52 \\ 0 \end{bmatrix}$ and assume that there exists a matrix

$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $\mathbf{AX} = \mathbf{K}$.

Then $\quad \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 52 \\ 0 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 5 \\ 0 & -1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 34 \\ -18 \end{bmatrix}$, by the elementary row operations $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 2R_1$

$\Rightarrow \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -4 \\ 0 & -1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -20 \\ -18 \end{bmatrix}$, by the elementary row operations $R_2 \to R_2 + 3R_3$

$\Rightarrow \quad x + y + z = 9; -4z = -20; -y - 3z = -18$

$\Rightarrow \quad z = 5; y = 18 - 3z = 18 - 3(5) = 3$

and $\quad x = 9 - y - z = 9 - 3 - 5 = 1$

$\Rightarrow \quad x = 1, y = 3, z = 5.$ **Ans.**

Ex. 6. Solve the following equations by matrix method—

$$2x - y + 3z = 8, -x + 2y + z = 4, 3x + y - 4z = 0. \qquad \textit{(Rohilkhand 91)}$$

Sol. The given equations are

$$2x - y + 3z = 8$$
$$-x + 2y + z = 4$$
$$3x + y - 4z = 0$$

Sol. Let $\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{bmatrix}$; $\mathbf{K} = \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}$ and assume that there exists a matrix

$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $\mathbf{AX} = \mathbf{K}$.

Then $$\begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $$\begin{bmatrix} 0 & 3 & 5 \\ -1 & 2 & 1 \\ 0 & 7 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ 4 \\ 12 \end{bmatrix}$$, by the elementary row operations $R_1 \to R_1 + 2R_2$ $R_3 \to R_3 + 3R_2$

$\Rightarrow$ $$\begin{bmatrix} 0 & 3 & 5 \\ -1 & 2 & 1 \\ 0 & 1 & -11 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ 4 \\ -20 \end{bmatrix}$$, by the elementary row operation $R_3 \to R_3 - 2R_1$

$\Rightarrow$ $$\begin{bmatrix} 0 & 0 & 38 \\ -1 & 2 & 1 \\ 0 & 1 & -11 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 76 \\ 4 \\ -20 \end{bmatrix}$$, by the elementary row operation $R_1 \to R_1 - 3R_3$

$\Rightarrow$ $38z = 76, -x + 2y + z = 4, y - 11z = -20$

$\Rightarrow$ $z = 2, -x + 2y = 4 - z = 2, y = -20 + 11z = -20 + 22 = 2$

$\Rightarrow$ $-x = 2 - 2y = 2 - 4 = -2, y = 2, z = 2$

$\Rightarrow$ $x = 2, y = 2, z = 2$ **Ans.**

Ex. 7. Solve the equations—

$$x_1 + 2x_2 + x_3 = 4, x_1 - x_2 + x_3 = 5, 2x_1 + 3x_2 - x_3 = 1$$

Sol. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{bmatrix}$ and $\mathbf{K} = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}$ and assume that there exists a matrix $\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that $\mathbf{AX} = \mathbf{K}$

Then $$\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}$$

$\Rightarrow$ $$\begin{bmatrix} 0 & 3 & 0 \\ 1 & -1 & 1 \\ 0 & 5 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ -9 \end{bmatrix}$$, by the elementary row operations $R_1 \to R_1 - R_2$; $R_3 \to R_3 - 2R_2$

$\Rightarrow$ $3x_2 = -1, x_1 - x_2 + x_3 = 5, 5x_2 - 3x_3 = -9$

$\Rightarrow$ $x_2 = -1/3, x_1 + (1/3) + x_3 = 5, 5(-1/3) - 3x_3 = -9$

$\Rightarrow$ $x_2 = -1/3, x_1 + x_3 = 14/3, x_3 = 22/9.$

$\Rightarrow \quad x_2 = -1/3, x_1 + (22/9) = 14/3, x_3 = 22/9.$

$\Rightarrow \quad x_1 = 20/9, x_2 = -1/3, x_3 = 22/9.$ **Ans.**

Ex. 8. Solve the following equations by matrix method :

$2x + 3y + z = 9; x + 2y + 3z = 6, 3x + y + 2z = 8.$ *(Gorakhpur 95)*

Sol. Let $\mathbf{A} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}; \mathbf{K} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$ and assume that there exists a matrix

$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, such that $\mathbf{AX} = \mathbf{K}$

Then $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 0 & -1 & -5 \\ 1 & 2 & 3 \\ 0 & -5 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ -10 \end{bmatrix}$, **by the elementaty row operations $R_1 \rightarrow R_1 - 2R_2$ and $R_3 \rightarrow R_3 - 3R_2$**

$\Rightarrow \begin{bmatrix} 0 & -1 & -5 \\ 1 & 2 & 3 \\ 0 & 0 & 18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ 5 \end{bmatrix}$, **by the elementaty row operation $R_3 \rightarrow R_3 - 5R_1$**

$\Rightarrow \quad -y - 5z = -3, x + 2y + 3z = 6, 18z = 5$

$\Rightarrow \quad -y - \frac{25}{18} = -3, x + 2y + \frac{5}{6} = 6, z = \frac{5}{18}$

$\Rightarrow \quad y = \frac{29}{18}, x + 2\left(\frac{29}{18}\right) = \frac{31}{6}, z = \frac{5}{18}$

$\Rightarrow \quad x = \frac{35}{18}, y = \frac{29}{18}, z = \frac{5}{18}$ **Ans.**

Exercises on § 6.02—§ 6.04

Solve the following equations by the matrix method —

Ex. 1. $3x + y - z = 2, x + 2y + z = 3, -x + y + 4z = 9$ *(Purvanchal 98)*

Ans. $x = 2, y = -1, z = 3$

Ex. 2. $x + 2y - z = 1, x + y + 2z = 9, 2x + y - z = 2$ **Ans.** $x = 2, y = 1, z = 3$

Ex. 3. $x + 2y + 3z = 14, 2x - y + 5z = 15, 3x - 2y + 4z = -13$

Ans. $x = -17\frac{6}{17}, y = \frac{10}{17}, z = 10\frac{1}{17}$

***Ex. 4.** $x + y + z = 3, x + 2y + 3z = 4, x + 4y + 9z = 6.$ **Ans.** $x = 2, y = 1, z = 0$

Ex. 5. $4x + 3y + 6z = 25, x + 5y + 7z = 13, 2x + 9y + z = 1$

Ans. $x = 4, y = -1, z = 2$

Ex. 6. $x + y = 5, 2x - y = 1.$ **Ans.** $x = 2, y = 3$

Ex. 7. $x - y + 2z = 3, 2x + z = 1, 3x + 2y + z = 4.$ **Ans.** $x = -1, y = 2, z = 3$

Ex. 8. $x - 2y + 3z = 11, 3x + y - z = 2, 5x + 3y + 2z = 3.$

Ans. $x = 2, y = -3, z = 1$

Ex. 9. $x + y + z = 9, 2x + 5y + 7z = 50, 2x + y - z = 2.$ **Ans.** $x = 1, y = 4, z = 4$

Ex. 10. $x + y + z = 4, 2x - y + 2z = 5, x - 2y - z = -3.$ *(Agra 92)*

Ans. $x = 1, y = 1, z = 2$

Ex. 11. $x - y + 2z = 4, 3x + y + 4z = 6, x + y + z = 1.$ *(Meerut 94)*

Ans. $x = 0, y = -2/3, z = 5/3$

Ex. 12. $x + 2y - z = 3, 3x - y + 2z = 1, 2x - 2y + 3z = 2.$ *(Gorakhpur 93)*

Ans. $x = -1, y = 4, z = 4$

Ex. 13. $x + 2y + 3z = 6, 2x + 4y + z = 7, 3x + 2y + 9z = 14.$ *(Meerut 97)*

Ans. $x = 1, y = 1, z = 1$

Ex. 14. $x - 2y + 3z = 6, 3x + y - 4z = -7, 5x - 3y + 2z = 5.$ *(Meerut 95)*

Ans. $x = -8/7, y = -25/7, z = 0$

Ex. 15. $5x - 6y + 4z = 15, 7x + 4y - 3z = 19; 2x + y + 6z = 46.$ *(Meerut 93)*

Ans. $x = 3; y = 4, z = 6.$

****§ 6.05. To compute inverse of a square matrix with the help of the linear equations.**

Let a system of n linear equations in n unknowns $x_1, x_2, x_3, \dots, x_n$ be

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= k_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= k_2 \\ a_{31}x_1 + a_{32}x_2 + \dots + a_{3n}x_n &= k_3 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= k_n. \end{aligned}$$

Then this system can be written in the matrix from as

$$\mathbf{AX} = \mathbf{K}, \quad \dots(i)$$

where $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$; $\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$ and $\mathbf{K} = \begin{bmatrix} k_1 \\ k_2 \\ \dots \\ k_n \end{bmatrix}$

...See § 6.02 Page 105 Ch. VI

If $|\mathbf{A}| \neq 0$, the matrix $\mathbf{A}$ is non-singular and the inverse of $\mathbf{A}$ exists,

...(See Ch. V and Ch. IV)

Hence premultiplying (i) by $\mathbf{A}^{-1}$, we have

$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{K} \quad \text{or} \quad (\mathbf{A}^{-1}\mathbf{A})\,\mathbf{X} = \mathbf{A}^{-1}\mathbf{K}$$

or $\mathbf{IX} = \mathbf{A}^{-1}\mathbf{K}, \quad \because \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

or $\mathbf{X} = \mathbf{A}^{-1}\mathbf{K},$ which gives the value of $\mathbf{A}^{-1}$

Solved Examples on § 6.05.

Ex.1. If $\mathbf{A}=\begin{bmatrix}2 & -1 & 3\\ 1 & 1 & 1\\ 1 & -1 & 1\end{bmatrix}$**, then find** $\mathbf{A}^{-1}$ **and hence solve the equations**

$\mathbf{2x - y + 3z = 9, x + y + z + 6}$ **and** $\mathbf{x - y + z = 2.}$ *(Kanpur 97)*

Sol. The matrix equation $\mathbf{AX}=\mathbf{K}$ is here equivalent to the equations

$$2x - y + 3z = k_1 \quad ...(i)$$

$$x + y + z = k_2 \quad ...(ii)$$

$$x - y + z = k_3 \quad ...(iii)$$

Adding (i) and (ii) we get $3x + 4z = k_1 + k_2$ ...(iv)

Adding (ii) and (iii) we get $2x + 2z = k_2 + k_3$ ...(v)

Multiplying (v) by 2, we get $4x + 4z = 2k_2 + 2k_3$ ...(vi)

Subtracting (iv) from (vi), we get $x = -k_1 + k_2 + 2k_3$ ...(vii)

$\therefore$ From (vi), (vii), we get $4z = 2k_2 + 2k_3 - 4(-k_1 + k_2 + 2k_3)$

or $$4z = 4k_1 - 2k_2 - 6k_3$$

or $$z = k_1 - \frac{1}{2}k_2 - \frac{3}{2}k_3 \quad ...(viii)$$

From (iii), $y = x + z - k_3$

$$= (-k_1 + k_2 + 2k_3) + (k_1 - \tfrac{1}{2}k_2 - \tfrac{3}{2}k_3) - k_3$$

or $$y = 0\,k_1 + \frac{1}{2}k_2 - \frac{1}{2}k_3 \quad ...(ix)$$

$\therefore$ From (vii), (viii) and (ix) we get

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}-1 & 1 & 2\\ 0 & \frac{1}{2} & -\frac{1}{2}\\ 1 & -\frac{1}{2} & -\frac{3}{2}\end{bmatrix}\begin{bmatrix}k_1\\ k_2\\ k_3\end{bmatrix} \text{ i.e. } \mathbf{X}=\mathbf{A}^{-1}\mathbf{K}$$

$\therefore$ $$\mathbf{A}^{-1}=\begin{bmatrix}-1 & 1 & 2\\ 0 & \frac{1}{2} & -\frac{1}{2}\\ 1 & -\frac{1}{2} & -\frac{3}{2}\end{bmatrix} \quad ...(x) \textbf{ Ans.}$$

Also given equations can be written in the matrix from as $\mathbf{AX}=\mathbf{K}$, ...(xi)

where $\mathbf{A}=\begin{bmatrix}2 & -1 & 3\\ 1 & 1 & 1\\ 1 & -1 & 1\end{bmatrix}$, $\mathbf{X}=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ and $\mathbf{K}=\begin{bmatrix}9\\ 6\\ 2\end{bmatrix}$

Now from (xi), we also have $\mathbf{X}=\mathbf{A}^{-1}\mathbf{K}$.

or $$\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}-1 & 1 & 2\\ 0 & 1/2 & -1/2\\ 1 & -1/2 & -3/2\end{bmatrix}\begin{bmatrix}9\\ 6\\ 2\end{bmatrix}$$

$\Rightarrow \quad x = (-1) . 9 + 1.6 + 2.2 = 1$

$y = (0) . 9 + (1/2) . 6 + (-1/2). 2 = 2$

$z = (1) . 9 + (-1/2) .6 + (-3/2) . 2 = 3$

$\Rightarrow \quad x = 1, y = 2, z = 3.$ **Ans.**

***Ex. 2. Find the invesre of the matrix** $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$

Sol. The matrix equation $\mathbf{AX = K}$ is here eqaivalent to the equations

$$x_1 + 2x_2 + 3x_3 = k_1 \quad ...(i)$$

$$0.x_1 + 5x_2 + 0.x_3 = k_2 \quad ...(ii)$$

$$2x_1 + 4x_2 + 3x_3 = k_3 \quad ...(iii)$$

From (ii) we get $x_2 = \frac{1}{5} k_2 = 0.k_1 + \frac{1}{5} k_2 + 0 . k_3$ ...(iv)

Subtracting (i) from (iii), we get

$$x_1 + 2x_2 = k_3 - k_1$$

or $\quad x_1 = k_3 - k_1 - 2x_2 = k_3 - k_1 - 2 . \frac{1}{5} k_2$, from (iv)

or $\quad x_1 = -k_1 - (2/5)\, k_2 + k_3$ ...(v)

Also from (i), $3x_3 = k_1 - x_1 - 2x_2$

$= k_1 + k_1 + \frac{2}{5} k_2 - k_3 - \frac{2}{5} k_2$, from (iv) and (v)

or $\quad x_3 = \frac{2}{3} k_1 + 0 . k_2 - \frac{1}{3} k_2$ ...(vi)

From (iv), (v) and (vi) we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & -\frac{2}{5} & 1 \\ 0 & \frac{1}{5} & 0 \\ \frac{2}{3} & 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \qquad i.e.\ \mathbf{X = A^{-1} K}$$

$\therefore$

$$\mathbf{A}^{-1} = \begin{bmatrix} -1. & -\frac{2}{5} & 1 \\ 0 & \frac{1}{5} & 0 \\ \frac{2}{3} & 0 & -\frac{1}{3} \end{bmatrix}$$ **Ans.**

Ex. 3. Solve the equations by finding the inverse of the coefficient matrix :

$\mathbf{5x - 6y + 4z = 15; 7x + 4y - 3z = 19, 2x + y + 6z = 46.}$ *(Gorakhpur 90)*

Sol. The coefficient matrix $= \begin{bmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{bmatrix} = \mathbf{A}$ (say).

The matrix equation $\mathbf{AX = K}$ is here equivalent to the equations

$$5x - 6y + 4z = k_1 \quad ...(i)$$

$$7x + 4y - 3z = k_2 \qquad \text{...(ii)}$$

$$2x + y + 6z = k_3 \qquad \text{...(iii)}$$

Multiplying (iii) by 6 and adding to (i) we get

$$17x + 40z = k_1 + 6k_3 \qquad \text{...(iv)}$$

Multiplying (iii) by -4 and adding to (ii) we get

$$x + 27z = 4k_3 - k_2 \qquad \text{...(v)}$$

Multiplying (v) by 17 and subtracting from (iv) we get

$$z = -\frac{1}{419}(k_1 + 17k_2 - 62k_3) \qquad \text{...(vi)}$$

Substituting this values of z in (v) and simplifying we get

$$x = \frac{1}{419}(27k_1 + 40k_2 + 2k_3) \qquad \text{...(vii)}$$

Substituting values of x and z in (iii) we get on simplifying,

$$y = \frac{1}{419}(-48k_1 + 22k_2 + 43k_3) \qquad \text{...(viii)}$$

$\therefore$ From (vi), (vii) and (viii) we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{419}\begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

i.e. $\mathbf{X} = \mathbf{A}^{-1}\mathbf{K}$

$\therefore$ $$\mathbf{A}^{-1} = (1/419)\begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}$$ **Ans.**

Also the given equations can be written in the matrix form as $\mathbf{AX} = \mathbf{K}$ or $\mathbf{X} = \mathbf{A}^{-1}\mathbf{K}$, where

$$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix};\ \mathbf{A}^{-1} = \frac{1}{419}\begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix};\ \mathbf{K} = \begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix}$$

or $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{419}\begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}\begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix}$$

$\Rightarrow$ $x = (1/419)\,[27\,(15) + 40\,(19) + 2\,(46)] = 3;$

$y = (1/419)\,[-48\,(15) + 22\,(19) + 43\,(46)] = 4;$

$z = (1/419)\,[-1\,(15) - 17\,(19) + 62\,(46)] = 6;$

$\Rightarrow$ $x = 3, y = 4, z = 6.$ **Ans.**

Exercises on § 6.05

Find the inverses of the following matrices :

Ex. 1. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$ **Ans.** $\frac{1}{3}\begin{bmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{bmatrix}$

Ex. 2. $\begin{bmatrix} 5 & -2 & 4 \\ -2 & 1 & 1 \\ 4 & 1 & 0 \end{bmatrix}$. **Ans.** $\frac{1}{37}\begin{bmatrix} 1 & -4 & 6 \\ -4 & 16 & 13 \\ 6 & 13 & -1 \end{bmatrix}$

Ex. 3. $\begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}$ **Ans.** $\frac{1}{5}\begin{bmatrix} -17 & -3 & 15 & 7 \\ 9 & 1 & -5 & -4 \\ -10 & -5 & 10 & 5 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

Ex. 4. Find the inverse of the coefficient matrix of the following system of equations—

$$x + y + z = 1,\ x + 2y + 2z = 1,\ x + 2y + 3z = 0$$

and hence solve them. **Ans.** $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$; $x = 1, y = 1, z = -1$

Ex. 5. Solve the following equations by finding the inverse of coefficient matrix :

$x + y + z = 9,\ 2x + 5y + 7z = 52,\ 2x + y - z = 0$ **Ans.** $x = 1, y = 3, z = 5$

Ex. 6. Find the inverse of the matrix $\mathbf{A} = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$ and apply the results to solve the equations.

$2x + 5y + 3z = 9,\ 3x + y + 2z = 3;\ x + 2y - z = 6.$ **Ans.** $x = 1, y = 2, z = -1$

§ 6.06. Augmented Matrix.

Definition : The matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$

augmented by the matrix $\mathbf{K} = \begin{bmatrix} k_1 \\ k_2 \\ \dots \\ k_m \end{bmatrix}$ is called augmented matrix of **A** and is written as $\mathbf{A}^*$ or $[\mathbf{A}, \mathbf{K}]$.

$$\therefore \quad \mathbf{A}^* = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & k_1 \\ a_{21} & a_{22} & \dots & a_{2n} & k_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & k_m \end{bmatrix}$$

Also it is evident that the order of the matrix $\mathbf{A}^*$ or $[\mathbf{A}, \mathbf{K}]$ is $m \times (n + 1)$.

§ 6.07. Fundamental Theorem.

A system of m linear equations in n unknowns given by **AX = K** *is consistent (i.e. has a solution) if and only if the matrix of coefficients* **A** *and the augmented matrix* **A*** *of the system have the same rank.* (Agra 94, 92)

[If the above common rank is r then r of the unknowns can be expressed as linear combinations of the remaining $n-r$ unknowns. When these $n-r$ unknowns are assigned arbitrary values, the system has an infinite number of solutions out of which $(n-r+1)$ are linearly independent whereas the rest are linear combinations of them.]

Proof. Consider m non-homogeneous linear equations in n unknowns given by $\mathbf{AX} = \mathbf{K}$, where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \text{ and } \mathbf{K} = \begin{bmatrix} k_1 \\ k_2 \\ \dots \\ k_n \end{bmatrix}$$

Let r be the rank of the matrix **A** and $C_1, C_2, C_2 ..., C_n$ be the column vectors of the matrix **A**, then $\mathbf{A} = [C_1, C_2, \dots C_n]$ and so $\mathbf{AX} = \mathbf{K}$ reduces to

$$[\mathbf{C_1}, \mathbf{C_2}, \dots, \mathbf{C_n}] \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \mathbf{K}$$

or $$\mathbf{C_1}x_1 + \mathbf{C_2}x_2 + \dots + \mathbf{C_n}x_n = \mathbf{K}. \qquad \text{...(i)}$$

Necessary Condition. Let the given system of equations possess a solution (*i.e.* be consistent), then there must exist n scalars $b_1, b_2, ..., b_n$ which satisfy (i)

i.e. $$\mathbf{C}_1 b_1 + \mathbf{C}_2 b_2 + \dots + \mathbf{C}_n b_n = \mathbf{K}. \qquad \text{...(ii)}$$

Since rank of **A** is r, so each $n-r$ columns viz. $\mathbf{C_{r+1}}, \mathbf{C_{r+2}}, \dots, \mathbf{C_n}$ is a linear combination of $\mathbf{C_1}, \mathbf{C_2}, ..., \mathbf{C_n}$.

∴ From (ii) we find that **K** is a linear combination of $\mathbf{C_1}, \mathbf{C_2}, .., \mathbf{C}_r$, since $\mathbf{C}_{r+1}, \mathbf{C}_{r+2}, \dots \mathbf{C}_n$. in (ii) can be expressed in terms of $\mathbf{C_1}, \mathbf{C_2}, \dots\dots, \mathbf{C}_n$

∴ The maximum number of linearly independent columns of the augmented matrix [**A, K**] or $\mathbf{A}^*$ is also r. Hence the rank of $\mathbf{A}^*$ is r.

Thus **A** and $\mathbf{A}^*$ are of the same rank r.

Sufficient Condition. Let the matrices **A** and $\mathbf{A}^*$ be of the same rank r. Then the number of linearly independent columns of the matrix $\mathbf{A}^*$ is r. But the column vectors $\mathbf{C_1}, \mathbf{C_2}, \dots \mathbf{C_r}$ of the matrix $\mathbf{A}^*$ already form a linearly independent set and thus the matrix **K** can be expressed as a linear combination of the columns $\mathbf{C_1}, \mathbf{C_2}, \dots, \mathbf{C}_r$.

$\therefore$ There exist r scalars $b_1, b_2, \ldots, b_r$ such that

$$b_1\mathbf{C}_1 + b_2\mathbf{C}_2 + \ldots + b_r\mathbf{C}_r + 0\mathbf{C}_{r+1} + \ldots + 0\mathbf{C}_n = \mathbf{K} \qquad \text{....(iii)}$$

From (i) and (ii) on comparing, we get

$x_1 = b_1, x_2 = b_2, \ldots, x_r = b_r, x_{r+1} = 0, \ldots, x_n = 0$ and these are the solutions of the system of equations given by $\mathbf{AX} = \mathbf{K}$.

***§ 6.08. Theorem.**

If **A** *be an* $n \times n$ *matrix,* **X** *and* **K** *be* $n \times 1$ *matrices, then the system of equations* $\mathbf{AX} = \mathbf{K}$ *possess a unique solution if matrix* **A** *is non-singular.*

Proof. Let $\mathbf{A} = [a_{ij}]$ and $|\mathbf{A}| \neq 0$.

Then rank of **A** and augmented matrix $[\mathbf{A}, \mathbf{K}]$ or $\mathbf{A}^*$ are both n. Thus from §6.07 Page 120 Ch. VI we conclude that the system of the equations $\mathbf{AX} = \mathbf{K}$ is consistent.

From $\mathbf{AX} = \mathbf{K}$, we have

$$\mathbf{A}^{-1}(\mathbf{AX}) = \mathbf{A}^{-1}\mathbf{K}, \text{ premultiplying both sides by } \mathbf{A}^{-1}$$

or $\quad (\mathbf{A}^{-1}\mathbf{A})\mathbf{X} = \mathbf{A}^{-1}\mathbf{K}$ or $\mathbf{IX} = \mathbf{A}^{-1}\mathbf{K}$, $\because \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

or $\quad \mathbf{X} = \mathbf{A}^{-1}\mathbf{K}$ is the solution of the given system of equations.

Now let $\mathbf{X}_1$ and $\mathbf{X}_2$ be two sets of solutions of $\mathbf{AX} = \mathbf{K}$.

then $\quad \mathbf{AX}_1 = \mathbf{K}; \mathbf{AX}_2 = \mathbf{K}$

$\Rightarrow \mathbf{AX}_1 = \mathbf{AX}_2$, as each is equal to **K**

$\Rightarrow \mathbf{A}^{-1}(\mathbf{AX}_1) = \mathbf{A}^{-1}(\mathbf{AX}_2)$, premultiplying both sides by $\mathbf{A}^{-1}$

$\Rightarrow (\mathbf{A}^{-1}\mathbf{A})\mathbf{X}_1 = (\mathbf{A}^{-1}\mathbf{A})\mathbf{X}_2$

$\Rightarrow \mathbf{IX}_1 = \mathbf{IX}_2, \quad \because \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

$\Rightarrow \mathbf{X}_1 = \mathbf{X}_2$

$\Rightarrow$ the solution is unique.

§ 6.09. Reduced Echelon Form of a Matrix.

Definition. If in an Echelon Form matrix (See § 5.04 Page 36 Ch. V) the first non-zero element in the *i*th row lies in *j*th column and all other elements in the *j*th column are zero, then the matrix is said to be in reduced Echelon form.

For example : In $\begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ the first non-zero element in the second row lies in the second column and all other elements in the second column are zero.

Solved Examples on § 6.07—§ 6.09.

Ex. 1. Solve the system of equations :

$$x + 2y - 3z - 4w = 6$$
$$x + 3y + z - 2w = 4$$

$$2x + 5y - 2z - 5w = 10$$

Sol. The given equations in the matrix form $\mathbf{AX} = \mathbf{K}$ is

$$\begin{bmatrix} 1 & 2 & -3 & -4 \\ 1 & 3 & 1 & -2 \\ 2 & 5 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 10 \end{bmatrix}$$

The augmented matrix $\mathbf{A}^* = \begin{bmatrix} 1 & 2 & -3 & -4 & 6 \\ 1 & 3 & 1 & -2 & 4 \\ 2 & 5 & -2 & -5 & 10 \end{bmatrix}$

or $\quad \mathbf{A}^* \sim \begin{bmatrix} 1 & 2 & -3 & -4 & 6 \\ 0 & 1 & 4 & 2 & -2 \\ 0 & 1 & 4 & 3 & -2 \end{bmatrix}$, replacing R_2 and R_3 by $R_2 - R_1$ and $R_3 - 2R_1$ respectively

$\sim \begin{bmatrix} 1 & 0 & -11 & -8 & 10 \\ 0 & 1 & 4 & 2 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$, replacing R_1, R_2 by $R_1 - 2R_2$, $R_3 - R_2$ respectively

This is a matrix in the reduced Echelon form having three non-zero row and hence the rank of $\mathbf{A}^*$ is 3.

Simultaneously we get the reduced Echelon from of **A** viz.

$\begin{bmatrix} 1 & 0 & -11 & -8 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ having three non–zero rows and hence the rank of **A** is also 3

Thus we observe that **A** and $\mathbf{A}^*$ have the same rank and as such the giv equations have solutions which can be obtained as follows :

The matrix equation is

$$\begin{bmatrix} 1 & 0 & -11 & -8 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 10 \\ -2 \\ 0 \end{bmatrix}$$

or $\quad x + 0y - 11z - 8w = 10;$

$\quad y + 4z + 2w = -2$ and $w = 0$

or $\quad y = -4z - 2;\ w = 0z;\ x = 11z + 10.$ **(Note)**

Thus we find (See § 6.07 Page 119 Ch. VI) that as the rank of **A** and $\mathbf{A}^*$ is 3, so three of the unknowns viz. x, y and w are expressed as a linear function of the remaining $4 - 3$ *i.e.* one unknown viz. z.

By assigning arbitrary values to z, an infinite number of corresponding values of x, y and w can be obtained. Hence the system of equations has infinite number of solutions.

Now we can show that the system has only $n - r + 1$ *i.e.* $4 - 3 + 1$ *i.e.* 2 linearly independent solutions. (See § 6.07 Page 119 Ch. VI)

Assigning two arbitrary values 0, 1 to z, we have two sets of solutions of the given equations as

x	10	21
y	-2	-6
z	0	1
w	0	0

Let any other solution of the given equations be

$$x = -1, y = 2, z = -1, w = 0,$$

corresponding to the value -1 of z.

If this third solution is a linear combination of the first two solutions then a, b can be found as follows :

$$\left.\begin{aligned} 10a + 21b &= -1 \\ -2a - 6b &= 2 \\ 0.a + 1.b &= -1 \\ 0.a + 0.b &= 0 \end{aligned}\right\} \textbf{(Note)}$$

or $$10a + 21b = -1 \quad \text{...(i)}$$

$$-2a - 6b = 2 \quad \text{...(ii)}$$

and $$b = -1. \quad \text{...(iii)}$$

Solving (i) and (iii) we get $a = 2, b = -1$.

These values of a and b satisfy (ii) also. Hence the third solution is a linear combination of the first two solutions.

Ex. 2 (a). Examine if the following equations are consistent ? If yes, solve it :

$$\mathbf{x + y + 4z = 6, 3x + 2y - 2z = 9, 5x + y + 2z = 13.}$$ *(Meerut 96)*

Sol. The given equations in the matrix from $\mathbf{AX} = \mathbf{K}$ can be written as

$$\begin{bmatrix} 1 & 1 & 4 \\ 3 & 2 & -2 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 13 \end{bmatrix}$$

The augmented matrix

$$\mathbf{A}^* = \begin{bmatrix} 1 & 1 & 4 & 6 \\ 3 & 2 & -2 & 9 \\ 5 & 1 & 2 & 13 \end{bmatrix}$$

or $$\mathbf{A}^* \sim \begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & -1 & -14 & -9 \\ 4 & 0 & -2 & 7 \end{bmatrix}, \text{ replacing } R_2, R_3 \text{ by } R_2 - 3R_1, R_3 - R_1 \text{ respectively}$$

$$\sim \begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & -1 & -14 & -9 \\ 0 & -4 & -18 & -17 \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 - 4R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & -1 & -14 & -9 \\ 0 & 0 & 38 & 19 \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 - 4R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & -1 & -14 & -9 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}, \text{ replacing } R_3 \text{ by } (1/38)\, R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}, \text{ replacing } R_1, R_2 \text{ by } R_1 - 4R_3,\ R_2 + 14R_3 \text{ respectively}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}, \text{ replacing } R_1 \text{ by } R_1 + R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}, \text{ replacing } R_2 \text{ by } -R_2$$

This is a matrix in the reduced Echelon form having three non-zero rows and hence the rank of $\mathbf{A}^*$ is 3.

Simultaneously we get the reduced Echelon form of $\mathbf{A}$ viz $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ which is equal to $\mathbf{I}_3$ and so the rank of $\mathbf{A}$ is also 3.

Thus we observe that $\mathbf{A}$ and $\mathbf{A}^*$ have the same rank and as such the given equations are consistent *i.e.* have solutions which can be obtained as follows :

The matrix equation is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1/2 \end{bmatrix}$$

$\Rightarrow\ x = 2, y = 2, z = 1/2.$ **Ans.**

Ex. 2 (b). Examine, if the system of following equations is consistent. If consistent find the solution :

$$x + y + z = 6,\ 2x + 3y - 2z = 2,\ 5x + y + 2z = 13.$$

Sol. Do as Ex. 2 (a) above.

Ans. Given equations are consistent. $x = 1, y = 2, z = 3$

Ex. 2 (c). Apply rank test to examine if the following system of equations is consistent and if consistent then find the complete solution :

$$2x - y + 3z = 8,\ -x + 2y + z = 4,\ 3x + y - 4z = 0.$$ *(Garhwal 92)*

Sol. The given equations in the matrix form can be written as

$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}$$

The augmented matrix $\mathbf{A}^* = \begin{bmatrix} 2 & -1 & 3 & 8 \\ -1 & 2 & 1 & 4 \\ 3 & 1 & -4 & 0 \end{bmatrix}$

or $\mathbf{A}^* \sim \begin{bmatrix} 0 & 3 & 5 & 16 \\ -1 & 2 & 1 & 4 \\ 0 & 7 & -1 & 12 \end{bmatrix}$, replacing R_1, R_3 by $R_1 + 2R_2$ and $R_3 + 3R_2$ respectively

$\sim \begin{bmatrix} 0 & 3 & 5 & 16 \\ -1 & 2 & 1 & 4 \\ 0 & 1 & -11 & -20 \end{bmatrix}$, replacing R_3 by $R_3 - 2R_1$

$\sim \begin{bmatrix} 0 & 0 & 38 & 76 \\ -1 & 2 & 1 & 4 \\ 0 & 1 & -11 & -20 \end{bmatrix}$, replacing R_1 by $R_1 - 3R_2$

$\sim \begin{bmatrix} 0 & 0 & 1 & 2 \\ -1 & 2 & 1 & 4 \\ 0 & 1 & -11 & -20 \end{bmatrix}$, replacing R_1 by $(1/38)R_1$

$\sim \begin{bmatrix} 0 & 0 & 1 & 2 \\ -1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{bmatrix}$, replacing R_2, R_3 by $R_2 - R_1$ $R_3 + 11R_1$ respectively

$\sim \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$, replacing R_2 by $-(R_2 - R_3)$

$\sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$, rearranging rows

$\sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$, replacing R_1 by $R_1 + R_2$

This is a matrix in the reduced Echelon form having three non-zero rows and hence the rank of $\mathbf{A}^*$ is 3.

Simultaneously we get the reduced Echelon from of $\mathbf{A}$ viz.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ i.e } \mathbf{I}_3$$

and so the rank of $\mathbf{A}$ is also 3.

Thus we find that the ranks of **A** and $\mathbf{A}^*$ are the same and so the given equations are consistent *i.e.* have solutions which can be obtained as follows—

The matrix equation is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

or $x = 2, y = 2, z = 2.$ **Ans.**

***Ex. 3 (a). Apply rank test to examine if the following system of equations is consistent and if consistent, find the complete solution :**

$$x + 2y - z = 6, 3x - y - 2z = 3, 4x + 3y + z = 9.$$ *(Meerut 98)*

Sol. Given equations can be written in the matrix form as

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & -2 \\ 4 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix}$$

$\therefore$ The augmented matrix $\mathbf{A}^* = \begin{bmatrix} 1 & 2 & -1 & 6 \\ 3 & -1 & -2 & 3 \\ 4 & 3 & 1 & 9 \end{bmatrix}$

or $\mathbf{A}^* \sim \begin{bmatrix} 1 & 2 & -1 & 6 \\ 1 & -5 & 0 & -9 \\ 5 & 5 & 0 & 15 \end{bmatrix}$, replacing R_2, R_3 by $R_2 - 2R_1$ and $R_3 + R_1$ respectively

$\sim \begin{bmatrix} 0 & 7 & -1 & 15 \\ 1 & -5 & 0 & -9 \\ 1 & 1 & 0 & 3 \end{bmatrix}$, replacing R_1, R_3 by $R_1 - R_2$ $(1/5)\, R_3$ respectively

$\sim \begin{bmatrix} 0 & 7 & -1 & 15 \\ 0 & -6 & 0 & -12 \\ 1 & 1 & 0 & 3 \end{bmatrix}$, replacing R_2 by $R_2 - R_3$

$\sim \begin{bmatrix} 0 & 1 & -1/7 & 15/7 \\ 0 & 1 & 0 & 2 \\ 1 & 1 & 0 & 3 \end{bmatrix}$, replacing R_1, R_2 by $(1/7)\, R_1$ $(-1/6)\, R_2$ respectively

$\sim \begin{bmatrix} 0 & 1 & -1/7 & 15/7 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 1/7 & 6/7 \end{bmatrix}$, replacing R_3 by $R_3 - R_1$

$\sim \begin{bmatrix} 0 & 0 & -1/7 & 1/7 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 1/7 & 6/7 \end{bmatrix}$, replacing R_1 by $R_1 - R_2$

$\sim \begin{bmatrix} 1 & 0 & 1/7 & 6/7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1/7 & 1/7 \end{bmatrix}$, rearranging rows

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \text{ replacing } R_1 \text{ by } R_1 + R_3 \text{ and } R_3 \text{ by } -7R_3$$

This is a matrix in the reduced Echelon form having three non-zero rows and hence the rank of $\mathbf{A}^*$ is 3.

Simultaneoausly we get the reduced Echelon form of $\mathbf{A}$ viz. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *i.e.* $\mathbf{I}_3$ and so the rank of $\mathbf{A}$ is also 3.

Thus we find that the ranks of $\mathbf{A}$ and $\mathbf{A}^*$ are the same and so the given system of equations is consistent *i.e.* have solutions which can be obtained as follows—

The matrix equation is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$\Rightarrow$ $x = 1, y = 2, z = -1$. **Ans.**

***Ex. 3 (b). Apply rank test to examine if the following system of equations is consistent, solve them :**

$$\mathbf{2x + 4y - z = 9,\ 3x - y + 5z = 5,\ 8x + 2y + 9z = 12.}$$

Sol. The given equations in the matrix form $\mathbf{AX} = \mathbf{K}$ can be written as

$$\begin{bmatrix} 2 & 4 & -1 \\ 3 & -1 & 5 \\ 8 & 2 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 19 \end{bmatrix}$$

$\therefore$ The augmented matrix $\mathbf{A}^* = \begin{bmatrix} 2 & 4 & -1 & 9 \\ 3 & -1 & 5 & 5 \\ 8 & 2 & 9 & 19 \end{bmatrix}$

or $\mathbf{A}^* = \begin{bmatrix} 2 & 4 & -1 & 9 \\ 3 & -1 & 5 & 5 \\ 2 & 4 & -1 & 9 \end{bmatrix}$, replacing R_3 by $R_3 - 2R_2$

$$\sim \begin{bmatrix} 2 & 4 & -1 & 9 \\ 1 & -5 & 6 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ replacing } R_2, R_3 \text{ by } R_2 - R_1 \text{ and } R_3 - R_1 \text{ respectively}$$

$$\sim \begin{bmatrix} 2 & 4 & -1 & 9 \\ 0 & -7 & 13/2 & -17/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ replacing } R_2 \text{ by } R_2 - \tfrac{1}{2} R_1$$

$$\sim \begin{bmatrix} 2 & 0 & 19/7 & 29/7 \\ 0 & 1 & -13/14 & 17/14 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ replacing } R_2 \text{ by } -(1/7)\, R_2 \text{ and then } R_1 \text{ by } R_1 - 4R_2$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{19}{14} & \frac{29}{14} \\ 0 & 1 & -\frac{13}{14} & \frac{17}{14} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ replacing } R_1 \text{ by } \frac{1}{2} R_1$$

This is a matrix in the reduced Echelon form having two non-zero rows and hence the rank of $\mathbf{A}^*$ is 2.

Simultaneously we get the reduced Echelon form of $\mathbf{A}$ viz. $\begin{bmatrix} 1 & 0 & (19/14) \\ 0 & 1 & -(13/14) \\ 0 & 0 & 0 \end{bmatrix}$ which also has two non-zero rows and as such the rank of $\mathbf{A}$ is also 2.

Thus we observe that the rank of $\mathbf{A}$ and $\mathbf{A}^*$ are the same and as such the given equations are consistent *i.e.* have solutions which can be obtained as follows—

The matrix equation is

$$\begin{bmatrix} 1 & 0 & (19/14) \\ 0 & 1 & -(13/14) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (29/14) \\ (17/14) \\ 0 \end{bmatrix}$$

or $\qquad x + (19/14) z = (29/14); \; y - (13/14) z = (17/14)$

or $\qquad y = \dfrac{13}{14} z + \dfrac{17}{14}; \; x = \dfrac{29}{14} - \dfrac{19}{14} z$

or $\qquad x = -\dfrac{19}{14} z + \dfrac{29}{14}; \; y = \dfrac{13}{14} z + \dfrac{17}{14}.$

Thus we find that as the rank of $\mathbf{A}^*$ and $\mathbf{A}$ is 2, so two of the unknowns viz x and y are expressed as a linear function of the remaining $3 - 2$ *i.e.* one unknown viz. z.

By assining arbitrary values to z, an infinite number of corresponding values of x and y can be obtained. Hence the given system of equations has an infinite number of solutions.

Now we can show that the system has only $n - r + 1$ *i.e.* $3 - 2 + 1$ *i.e.* 2 linearly independent solution (See § 6.07 Page 120 Ch VI).

Assigning two arbitrary values 0, 1 to z, we have two sets of solutions of the given equations as

x	$\frac{29}{14}$	$\frac{10}{14}$
y	$\frac{17}{14}$	$\frac{30}{14}$
z	0	1

Let any other solution of the given equation be $x = (24/7)$, $y = (2/7)$, $z = -1$, corresponding to the value -1 of z.

If this third solution is a linear combination of the first two solutions then a and b can be found as follows —

$$\left.\begin{aligned} (29/14)\,a + (10/14)\,b &= (24/7) \\ (17/14)\,a + (30/14)\,b &= (2/7) \\ 0.a + 1.b &= -1 \end{aligned}\right\} \qquad \textbf{(Note)}$$

or
$$29a + 10b = 48 \qquad \text{...(i)}$$
$$17a + 30b = 4 \qquad \text{...(ii)}$$
$$b = -1. \qquad \text{...(iii)}$$

Solving (i) and (iii) we get $a = 2, b = -1$, which satisfy (ii) also. Hence the third solution is a linear combination of the first two solutions.

***Ex. 4. Apply rank test to examine if the following system of equations is consistent and if consistent, find the complete solution.**

$$\mathbf{x + y + z = 6, x + 2y + 3z = 0, x + 2y + 4z = 1.}$$

Sol. The given equations in the matrix form $\mathbf{AX} = \mathbf{K}$ can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 1 \end{bmatrix}$$

The augmented matrix $\mathbf{A}^* = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & 4 & 1 \end{bmatrix}$

or
$$\mathbf{A}^* \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 0 & 0 & 1 & -9 \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & -9 \end{bmatrix}, \text{ replacing } R_2 \text{ by } R_2 - R_1$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & -9 \end{bmatrix}, \text{ replacing } R_1 \text{ by } R_1 - R_2$$

or
$$\mathbf{A}^* \sim \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & 22 \\ 0 & 0 & 1 & -9 \end{bmatrix}, \text{ replacing } R_1, R_2 \text{ by } R_1 + R_3 \text{ and } R_2 - 2R_3 \text{ respectively}$$

This is a matrix in the reduced Echelon form having three non-zero rows and hence the rank of $\mathbf{A}^*$ is 3.

Simultaneously we get the reduced Echelon form of the matrix $\mathbf{A}$ viz.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *i.e.* $\mathbf{I}_3$ and hence the rank of **A** is 3.

Thus we find that the ranks of **A** and $\mathbf{A}^*$ are the same and so the given equations are consistent.

$\therefore$ The matrix equation is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -7 \\ 22 \\ -9 \end{bmatrix}$

or $\quad x = -7, y = 22, z = -9.$ **Ans.**

Ex. 5. Are the following equations consistent ?

$$x + y + 2z + w = 5$$
$$2x + 3y - z - 2w = 2$$
$$4x + 5y + 3z = 7.$$

(Agra 91)

Sol. The given equations in the matrix form **AX** = **K** can be written as

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & -1 & -2 \\ 4 & 5 & 3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 7 \end{bmatrix}$$

(Note)

$\therefore$ The augmented matrix $\mathbf{A}^* = \begin{bmatrix} 1 & 1 & 2 & 1 & 5 \\ 2 & 3 & -1 & -2 & 2 \\ 4 & 5 & 3 & 0 & 7 \end{bmatrix}$

or $\quad \mathbf{A}^* \sim \begin{bmatrix} 1 & 1 & 2 & 1 & 5 \\ 0 & 1 & -5 & -4 & -8 \\ 0 & 1 & -5 & -4 & -13 \end{bmatrix}$, replacing R_2 and R_3 by $R_2 - 2R_1$ and $R_3 - 4R_1$ respectively

$\sim \begin{bmatrix} 1 & 0 & 7 & 5 & 13 \\ 0 & 1 & -5 & -4 & -8 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix}$, replacing R_1, R_3 by $R_1 - R_2$, $R_3 - R_2$ respectively

This is a matrix in the reduced Echelon form having three non-zero rows, hence its rank is 3.

Simultaneously we get the reduced Echelon form of **A** viz. $\begin{bmatrix} 1 & 0 & 7 & 5 \\ 0 & 1 & -5 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ which has two non-zero rows hence its rank is 2.

Thus we find that the ranks of **A** and $\mathbf{A}^*$ are not the same, hence the given equations are not consistent *i.e.* they cannot have any solutions.

Ex. 6. Discuss the consistency and find the solution set of the following equations :—

$$x + 2y + 2z = 1, 2x + y + z = 2, 3x + 2y + 2z = 3, y + z = 0.$$

Sol. The given equations in the matrix form $\mathbf{AX} = \mathbf{K}$ can be written as

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$$

$\therefore$ The augmented matrix $\mathbf{A}^* = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 \\ 3 & 2 & 2 & 3 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

or $\mathbf{A}^* \sim \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 0 & 0 & 2 \\ 2 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix}$, replacing R_2, R_3 by $R_2 - R_4$ and $R_3 - R_1$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$, replacing R_1, R_2, R_3 by $R_1 - 2R_4$ $\frac{1}{2} R_2, \frac{1}{2} R_3$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$, replacing R_2, R_3 by $R_2 - R_1$ $R_3 - R_1$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, interchanging R_2 and R_4

This is a matrix in the reduced Echelon form having two non-zero rows, hence its rank is 2.

Simultaneously we get the reduced Echelon form of $\mathbf{A}$ viz.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, which has two non-zero rows and hence its rank is also 2.

Thus we find that the ranks of $\mathbf{A}$ and $\mathbf{A}^*$ are the same and so the given equations are consistent.

$\therefore$ The matrix equation reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

or $$x = 1, y + z = 0. \qquad ...(i)$$

Also here we find that the rank of **A** and **A*** is each 2 *i.e.* less than the number of unknowns viz. x, y and z. So the number of solutions of given equations will be **infinite** given by (i) above, which gives $x = 1$ and $y + z = 0$ can be satisfied by an infinite number of values *e.g.* 0, 0; 1, − 1; 2, − 2; etc.

Ex. 7. Show that the equations $5x + 3y + 7z = 4$, $3x + 26y + 2z = 9$, $7x + 2y + 10z = 5$ are consistent and solve them. *(Bundelkhand 96)*

Sol. The given equations in the matrix form **AX** = **K** can be written as

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

$\therefore$ The augmented matrix $\mathbf{A}^* = \begin{bmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{bmatrix}$

or $$\mathbf{A}^* \sim \begin{bmatrix} -4 & -75 & 1 & -23 \\ 3 & 26 & 2 & 9 \\ -8 & -128 & 0 & -40 \end{bmatrix}, \text{ replacing } R_1, R_3 \text{ by } R_1 - 3R_2 \text{ and } R_3 - 5R_2 \text{ respectively.}$$

$$\sim \begin{bmatrix} 4 & 75 & -1 & 23 \\ 3 & 26 & 2 & 9 \\ 0 & 22 & -2 & 6 \end{bmatrix}, \text{ replacing } R_1, R_3 \text{ by } -R_1 \text{ and } R_3 - 2R_1 \text{ respectively}$$

$$\sim \begin{bmatrix} 4 & 75 & -1 & 23 \\ 3 & 26 & 2 & 9 \\ 0 & 11 & -1 & 3 \end{bmatrix}, \text{ replacing } R_3 \text{ by } \tfrac{1}{2} R_2$$

$$\sim \begin{bmatrix} 4 & 64 & 0 & 20 \\ 3 & 48 & 0 & 15 \\ 0 & 11 & -1 & 3 \end{bmatrix}, \text{ replacing } R_1, R_2 \text{ by } R_1 - R_3 \; R_2 + 2R_3 \text{ respectively.}$$

$$\sim \begin{bmatrix} 1 & 16 & 0 & 5 \\ 1 & 16 & 0 & 3 \\ 0 & 1 & -(1/11) & (3/11) \end{bmatrix}, \text{ replacing } R_1, R_2 \text{ and } R_3 \text{ by } \tfrac{1}{4} R_1, \tfrac{1}{3} R_2 \text{ and } (1/11) R_3 \text{ respectively.}$$

$$\sim \begin{bmatrix} 1 & 0 & (16/11) & (7/11) \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -(1/11) & (3/11) \end{bmatrix}, \text{ replacing } R_2 \text{ by } R_2 - R_1 \text{ and then } R_1 \text{ by } R_1 - 16R_3$$

$$\sim \begin{bmatrix} 1 & 0 & (16/11) & (7/11) \\ 0 & 1 & -(1/11) & (3/11) \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ interchanging } R_2 \text{ and } R_3$$

This is a matrix in the reduced Echelon form having two non-zero rows, hence its rank is 2.

Simultaneously we get the reduced Echelon form of A

viz. $\begin{bmatrix} 1 & 0 & (16/11) \\ 0 & 1 & -(1/11) \\ 0 & 0 & 0 \end{bmatrix}$ which also has two non zero rows and so its rank is also 2.

Thus we find that the rank of A and A^* are the same and as such the given equations are consistent. And so the matrix equation reduces to

$$\begin{bmatrix} 1 & 0 & (16/11) \\ 0 & 1 & -(1/11) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (7/11) \\ (3/11) \\ 0 \end{bmatrix}$$

or $$x + (16/11)\, z = (7/11);\ y - (1/11)\, z = (3/11)$$

or $$11x = 7 - 16z,\ 11y = z + 3.$$

Thus we find that as the ranks of A and A^* is 2, so two of the unknowns viz. x and y are expressed as a linear function of remaining $3 - 2$ *i.e.*, one known viz. z.

By assigning arbitrary values to z, an infinite number of corresponding values of x and y can be obtained. Thus given system of equations has an infinite number of solutions. Now we can show that the system has only $(n - r + 1)$ *i.e.* $(3 - 2 + 1)$ *i.e.* 2 linearly independent solutions (See § 6.07 Page 119 Ch. VI).

Assigning two arbitrary values 0, 1 to z, we have two sets of solutions of the given equations as

x	$\frac{7}{11}$	$-\frac{9}{11}$
y	$\frac{3}{11}$	$\frac{4}{11}$
z	0	1

Let any other solution of the given equation be $x = (23/11)$, $y = (2/11)$, $z = -1$ corresponding to the value -1 of z.

If this third solution is a linear combination of the first two solutions, then a and b can be found as follows :—

$$(7/11)\, a - (9/11)\, b = (23/11) \qquad \text{...(i)}$$

$$(3/11)\, a + (4/11)\, b = (2/11) \qquad \text{...(ii)}$$

$$0.a + 1.b = -1. \qquad \text{...(iii)}$$

From (i) and (iii) we get $a = 2$, $b = -1$, which satisfy (ii) also. Hence the third solution is a linear combination of the first two solutions.

***Ex. 8. Apply test of rank to examine if the equations $x + y + z = 6$, $x + 2y + 3z = 14$, $x + 4y + 7z = 30$ are consistent and if consistent find the complete solution.** *(Kumaun 91; Meerut 96P)*

Sol. In the matrix form $AX = K$, the given equations can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$$

$\therefore$ The augmented matrix $\mathbf{A}^* = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{bmatrix}$

or $\mathbf{A}^* \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{bmatrix}$, replacing R_2 and R_3 by $R_2 - R_1$ and $R_3 - R_1$ respectively.

$\sim \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing R_1 and R_3 by $R_1 - R_2$ and $R_3 - 3R_2$ respectively.

This is a matrix in the reduced Echelon form having two non-zero rows, hence its rank is 2.

Simultaneously we get the reduced Echelon form A viz.

$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, which has two non-zero rows and hence its rank is 2.

Thus we find that the ranks of **A** and **A** are the same and as such the given equations are consistent.

Now the matrix form of the given equations reduce to

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -2 \\ 8 \\ 0 \end{bmatrix}$$

which is equivalent to

$$1 \,.\, x + 0.\, y - 1 \,.\, z = -2;\ 0 \,.\, x + 1 \,.\, y + 2 \,.\, z = 8;\ 0.x + 0.y + 0.z = 0$$

or $$x - z = -2,\ y + 2z = 8$$

$\therefore$ As the rank of **A** and $\mathbf{A}^*$ is 2, so two of the unknowns viz. x and y are expressed as a linear function of the remaining unknown z viz. $x = -2 + z$, $y = 8 - 2z$, where z is arbitrary.

By assigning arbitrary values to z, an infinite number of corresponding values of x and y can be obtained. Hence the system of equations has infinite number of solutions.

And $x = -2 + k$, $y = 8 - 2k$, $z = k$ forms the general solution of the given equations.

In the matrix form the solution can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -2 \\ 8 \\ 0 \end{bmatrix}+k\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

***Ex. 9. Find the values of λ so that the equations**

$$\mathbf{ax + hy + g = 0,\ hx + by + f = 0,\ gx + fy + c = \lambda}$$

are consistent.

Sol. The given equations in the matrix form $\mathbf{AX = K}$ can be written as

$$\begin{bmatrix} a & h \\ h & b \\ g & f \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} -g \\ -f \\ \lambda - c \end{bmatrix}$$

$\therefore$ The augmented matrix $\mathbf{A}^* = \begin{bmatrix} a & h & -g \\ h & b & -f \\ g & f & \lambda - c \end{bmatrix}$

or $\mathbf{A}^* \sim \begin{bmatrix} 1 & h/a & -g/a \\ 1 & b/h & -f/h \\ 1 & f/g & \{(\lambda - c)/g\} \end{bmatrix}$, replacing R_1, R_2 and R_3 by R_1/a, R_2/h and R_3/g respectively. (Note)

$$\sim \begin{bmatrix} 1 & h/a & -g/a \\ 0 & (b/h)-(h/a) & -(f/h)+(g/a) \\ 0 & (f/g)-(h/a) & \{(\lambda - c)/g\}+(g/a) \end{bmatrix}$$, replacing R_2 and R_3 by $R_2 - R_1$ and $R_3 - R_1$

$$\sim \begin{bmatrix} 1 & h/a & -g/a \\ 0 & (ba-h^2)/ha & (gh-af)/ah \\ 0 & (af-gh)/ga & \{(\lambda a + g^2 - ac)/ag\} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & h/a & -g/a \\ 0 & (ba-h^2)/ha & (gh-af)/ah \\ 0 & \dfrac{(ba-h^2)}{ha} & \dfrac{(\lambda a + g^2 - ac)(ba-h^2)}{ah(af-gh)} \end{bmatrix},$$

replacing R_2 by $\dfrac{g}{h}\dfrac{(ba-h^2)R_2}{(af-gh)}$

$$\sim \begin{bmatrix} 1 & h/a & -g/a \\ 0 & (ba-h^2)/ha & (gh-af)/ah \\ 0 & 0 & \{\lambda(ab-h^2)-(abc+2fgh \\ & & -af^2-bg^2-ch^2)\}/\{h(af-gh)\} \end{bmatrix}$$, replacing R_3 by $R_3 - R_2$

$$\sim \begin{bmatrix} 1 & h/a & -g/a \\ 0 & 1 & (gh-af)/(ba-h^2) \\ 0 & 0 & \mu \end{bmatrix}$$, replacing R_2 by $ah\,R_2/(ba-h^2)$ ...(i)

Here $\mu = \dfrac{\lambda\,(ab - h^2) - (abc + 2fgh - af^2 - bg^2 - ch^2)}{h\,(af - gh)}$ ...(ii)

Simultaneously we get the reduced Echelon form of **A** viz.

$\begin{bmatrix} 1 & h/a \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, which has two non-zero rows hence its rank is 2 ...(iii)

From (i) and (iii) we conclude that if the given equations have solution then the ranks of **A** and $\mathbf{A}^*$ must be the same viz. 2 and from (i) if the rank of $\mathbf{A}^*$ is 2, then it must have two non-zero rows *i.e.* $\mu = 0$.

i.e. $\lambda = (abc + 2fgh - af^2 - bg^2 - ch^2)/(ab - h^2)$, from (i). **Ans.**

****Ex. 10. For what values of λ, the equations $x + y + z = 1$, $x + 2y + 4z = \lambda$, $x + 4y + 10z = \lambda^2$ have a solution and solve completely in each case.** *(Garhwal 90; Kanpur 97, 93, 91; Rohilkhand 92)*

Sol. The given equation in the matrix form $\mathbf{AX} = \mathbf{K}$ can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix}$$

$\therefore$ The augmented matrix $\mathbf{A}^* = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & \lambda \\ 1 & 4 & 10 & \lambda^2 \end{bmatrix}$

or $\mathbf{A}^* \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda - 1 \\ 0 & 3 & 9 & \lambda^2 - 1 \end{bmatrix}$, replacing R_2 and R_3 by $R_2 - R_1$ and $R_3 - R_1$ respectively.

$\sim \begin{bmatrix} 1 & 0 & -2 & 2 - \lambda \\ 0 & 3 & 9 & 3\lambda - 3 \\ 0 & 3 & 9 & \lambda^2 - 1 \end{bmatrix}$, replacing R_1 by $R_1 - R_2$ and then R_2 by $3R_2$

$\sim \begin{bmatrix} 1 & 0 & -2 & 2 - \lambda \\ 0 & 1 & 3 & \lambda - 1 \\ 0 & 0 & 0 & \lambda^2 - 3\lambda + 2 \end{bmatrix}$, replacing R_3 by $R_3 - R_2$ and R_2 by $\frac{1}{3} R_2$ ...(i)

Simultaneously we get the reduced Echelon form of **A** viz.

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ which has two non-zero rows hence its rank is 2. ...(ii)

From (i), (ii) we conclude that if the given equations have solution then the ranks of **A** and $\mathbf{A}^*$ must be the same viz. 2 and from (i) if the rank $\mathbf{A}^*$ is 2, then it must have two non-zero rows *i.e.* $\lambda^2 - 3\lambda + 2 = 0$ or $\lambda = 1, 2.$ **Ans.**

The matrix form of the given equations reduces to

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$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2-\lambda \\ \lambda - 1 \\ \lambda^2 - 3\lambda + 2 \end{bmatrix}$$

which is equivalent to

$$\left.\begin{aligned} &1 . x + 0. y - 2 . z = 2 - \lambda, 0. x + 1. y + 3 . z = \lambda - 1 \\ \text{and} \quad &0 . x + 0 . y + 0 . z = \lambda^2 - 3\lambda + 2 \end{aligned}\right\}$$

If $\lambda = 1$, then these are $x - 2z = 1, y + 3z = 0$.

As the rank of **A** and **A*** is 2, so two of the unknowns viz, x and y are expressed as a linear function of the remaining unknown z viz $x = 2z + 1, y = -3z$.

By assigning arbitrary values to z, an infinite number of corresponding values of x and y can be obtained. Hence the system of equations has infinite number of solutions.

Assigning two arbitrary values 0, 1 to z, we have two sets of solutions of the given equations as

x	1	3
y	0	-3
z	0	1

Let any other solution of the given equations be $x = -1, y = 3, z = -1$ corresponding to the value -1 of z.

If this third solution is a linear combination of the first two solutions then a, b can be found as follows :

$$\left.\begin{aligned} 1.a + 3.b &= -1 \\ 0.a - 3.b &= 3 \\ 0.a + 1.b &= -1 \end{aligned}\right\} \qquad \text{...(iv)}$$

or $a + 3b = -1, 3b = -3$ or $b = -1$

i.e. $b = -1, a = 2$. These values of a and b satisfy all the three equations given by (iv). Hence the third solution is a linear combination of the first two solutions.

We can similarly solve for $\lambda = 2$ also.

***Ex. 11. Express the following system of equations into the matrix equations** $\mathbf{AX = K}$

$$\mathbf{4x - y + 6z = 16, x - 4y - 3z = -16.}$$
$$\mathbf{2x + 7y + 12z = 48, 5x - 5y + 3z = 0}$$

Determine if the system of equations is consistent and if so find its solution.

Sol. In the given system of equations, we observe that the number of unknowns are not equal to the number of equations.

The single matrix equation of these is

$$\begin{bmatrix} 4 & -1 & 6 \\ 1 & -4 & -3 \\ 2 & 7 & 12 \\ 5 & -5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ -16 \\ 48 \\ 0 \end{bmatrix} \quad \textit{i.e.}\ \mathbf{AX} = \mathbf{K}$$

$\therefore$ The augmented matrix $\mathbf{A}^* = \begin{bmatrix} 4 & -1 & 6 & 16 \\ 1 & -4 & -3 & -16 \\ 2 & 7 & 12 & 48 \\ 5 & -5 & 3 & 0 \end{bmatrix}$

or $\mathbf{A}^* \sim \begin{bmatrix} 0 & 15 & 18 & 80 \\ 1 & -4 & -3 & -16 \\ 0 & 15 & 18 & 80 \\ 0 & 15 & 18 & 80 \end{bmatrix}$ replacing R_1, R_3 and R_4 by $R_1 - 4R_2$, $R_3 - 2R_2$ and $R_4 - 5R_2$ respectively.

or $\mathbf{A}^* \sim \begin{bmatrix} 1 & -4 & -3 & -16 \\ 0 & 15 & 18 & 80 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ interchanging R_1 and R_2 and then replacing R_3, R_4 by $R_3 - R_2$, $R_4 - R_2$

$\sim \begin{bmatrix} 1 & 0 & (9/5) & (16/3) \\ 0 & 1 & (6/5) & (16/3) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ replacing R_2 by $(1/15)\,R_2$ and then R_1 by $R_1 + 4R_2$

This is a matrix in the reduced Echelon form having two non-zero rows hence its rank is 2.

Simultaneously we get the reduced Echelon form of **A** viz.

$\begin{bmatrix} 1 & 0 & (9/5) \\ 0 & 1 & (6/5) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ which has two non-zero rows and hence its rank is also 2.

Thus we find that the ranks of **A** and $\mathbf{A}^*$ are the same and as such the given equations are consistent.

Now the matrix form of the given equations reduce to :

$$\begin{bmatrix} 1 & 0 & (9/5) \\ 0 & 1 & (6/5) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (16/3) \\ (16/3) \end{bmatrix}$$

which is equivalent to

$$1.x + 0.y + (9/5)\,z = (16/3);\ 0.x + 1.y + (6/5)z = (16/3)$$

or $\quad 15x + 27z = 80;\ 15y + 18z = 80.$

As the rank of **A** and $\mathbf{A}^*$ is 2, so two of the unknowns viz. x and y are expressed as a linear function of the remaining unknown z viz.

$$15x = 80 - 27z \text{ and } 15y = 80 - 18z$$

or $$x = -(9/5)z + (16/3) \text{ and } y = -(6/5)z + (16/3),$$

where z is arbitrary.

By assigning arbitrary values to z, an infinite number of corresponding values of x and y can be obtained. Hence the given equations have infinite number of solutions.

Also $x = (16/3) - (9/5)k$, $y = (16/3) - (6/5)k$, $z = k$ forms the general solutions of the given equations.

In the matrix form the solutions can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (16/3) \\ (16/3) \\ \end{bmatrix} + k \begin{bmatrix} -(9/5) \\ -(6/5) \\ 1 \end{bmatrix}$$ **Ans.**

***Ex. 12. Examine if the system of equations $x + y + 4z = 6$, $3x + 2y - 2z = 9$, $5x + y + 2z = 13$ is consistent ? Find also the solution if it is consistent.**

Sol. The given equations are

$$x + y + 4z = 6$$
$$3x + 2y - 2z = 9$$
$$5x + y + 2z = 13$$

In the matrix form $\mathbf{AX} = \mathbf{K}$, these can be written as

$$\begin{bmatrix} 1 & 1 & 4 \\ 3 & 2 & -2 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 13 \end{bmatrix}$$

$\therefore$ The arugmented matrix $\mathbf{A}^* = \begin{bmatrix} 1 & 1 & 4 & 6 \\ 3 & 2 & -2 & 9 \\ 5 & 1 & 2 & 13 \end{bmatrix}$

or $$\mathbf{A}^* \sim \begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & -1 & -14 & -9 \\ 0 & -4 & -18 & -17 \end{bmatrix}, \text{ replacing } R_2, R_3 \text{ by } R_2 - 3R_1 \text{ and } R_3 - 5R_1 \text{ respectively.}$$

$$\sim \begin{bmatrix} 1 & 0 & -10 & -3 \\ 0 & -1 & -14 & -9 \\ 0 & 0 & 38 & 19 \end{bmatrix}, \text{ replacing } R_1, R_2 \text{ by } R_1 + R_2, R_3 - 4R_2 \text{ respectively.}$$

$$\sim \begin{bmatrix} 1 & 0 & -10 & -3 \\ 0 & 1 & 14 & 9 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}, \text{ replacing } R_2, R_3 \text{ by } -R_2, (1/38)R_3 \text{ respectively.}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 14 & 9 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}, \text{ replacing } R_1 \text{ by } R_1 + 10R_3$$

This is a matrix having three non-zero rows and in the reduced Echelon form, hence its rank is 3.

Simultaneously we get the reduced Echelon form of **A** viz.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 14 \\ 0 & 0 & 1 \end{bmatrix}$, which also has three non-zero rows and hence its rank is also 3.

Thus we find that the ranks of **A** and $\mathbf{A}^*$ are the same and as such the given equations are consistent.

Now the matrix form of the given equations reduces to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 14 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 2 \\ 9 \\ \frac{1}{2} \end{bmatrix}$$

which is equivalent to

$$1.x+0.y+0.z=2;\ 0.x+1.y+14.z=9;\ 0.x+0.y+1.z=\tfrac{1}{2}$$

or $\quad x=2;\ y+14z=9;\ z=\frac{1}{2} \quad$ or $\quad x=2;\ y=9-14z;\ z=\frac{1}{2}$

or $\quad x=2,\ y=9-7=2,\ z=\frac{1}{2}$.

Ans.

****Ex. 13. Show the equations $-2x+y+z=a$, $x-2y+z=b$, $x+y-2z=c$ have no solution unless $a+b+c=0$ in which case they have infinitely many solutions. Find the solution when $a=1$, $b=1$, $c=-2$.** *(Lucknow 90)*

Sol. The given equations are

$$-2x+y+z=a$$
$$x-2y+z=b$$
$$x+y-2z=c$$

In the matrix form **AX = K**, these can be written as

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$\therefore$ The augmented matrix $\mathbf{A}^*=\begin{bmatrix} -2 & 1 & 1 & a \\ 1 & -2 & 1 & b \\ 1 & 1 & -2 & c \end{bmatrix}$

or $\quad \mathbf{A}^* \sim \begin{bmatrix} 0 & 0 & 0 & a+b+c \\ 1 & -2 & 1 & b \\ 1 & 1 & -2 & c \end{bmatrix}$, replacing R_1 by $R_1+R_2+R_3$

$\sim \begin{bmatrix} 1 & -2 & 1 & b \\ 0 & 0 & 0 & a+b+c \\ 0 & 1 & -1 & \frac{1}{3}(c-b) \end{bmatrix}$, interchanging R_1 and R_2 and then replacing R_3 by $\frac{1}{3}(R_3-R_1)$

$$\sim \begin{bmatrix} 1 & 0 & -1 & \frac{1}{3}(2c+b) \\ 0 & 1 & -1 & \frac{1}{3}(c-b) \\ 0 & 0 & 0 & a+b+c \end{bmatrix}, \text{ interchanging } R_2 \text{ and } R_3 \text{ and then replacing } R_1 \text{ by } R_1 + 2R_2 \quad ...(i)$$

This is a matrix in the reduced Echelon form and has three non-zero rows, hence its rank is 3.

Simultaneously we get the reduced form of **A** viz.

$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, which has two non-zero rows and hence its rank is 2.

Thus we find that the rank of $\mathbf{A}^*$ and **A** are not the same and as such the given equations are inconsistent *i.e.* have no solution.

But in case $a+b+c=0$, the augmented matrix $\mathbf{A}^*$ has two non-zero rows *i.e.* the rank of $\mathbf{A}^*$ is also 2 and thus **A** and $\mathbf{A}^*$ have the same rank 2. Consequently the given equations have solutions if $a+b+c=0$ and in this case from (i) we have

$$\mathbf{A}^* = \begin{bmatrix} 1 & 0 & -1 & \frac{1}{3}(2c+b) \\ 0 & 1 & -1 & \frac{1}{3}(c-b) \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$\therefore$ The matrix form of given equations reduce to,

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{3}(2c+b) \\ \frac{1}{3}(c-b) \\ 0 \end{bmatrix}, \text{ which is equivalent to}$$

$$1.x + 0.y - 1.z = \frac{1}{3}(2c+b);\ 0.x + 1.y - 1.z = \frac{1}{3}(c-b)$$

or $$x - z = \frac{1}{3}(2c+b),\ y - z = \frac{1}{3}(c-b). \quad ...(ii)$$

As a, b, c can take different values we shall get different solutions rather infinitely many solutions (as will be evident below also) of the given equations.

If $a=1, b=1, c=-2$, then from (ii) we get

$$x - z = -1,\ y - z = -1$$

or $$x = z - 1,\ y = z - 1$$

$\therefore$ By assigning arbitrary values to z, an infinite number of corresponding values of x and y can be obtained. Hence in this case the given system of equations has infinite number of solutions.

Assigning two arbitrary values 0, 1 to z, we have two sets of solutions of the equations as

x	-1	0
y	-1	0
z	0	1

Let any other solution of the given equations be $x = -2$, $y = -2$, $z = -1$, corresponding to the value -1 of z.

If this third solution is a linear combination of the first two solutions, then two constants λ, μ can be found as follows —

$$\left.\begin{aligned} -1.\lambda + 0.\mu &= -2 \\ -1.\lambda + 0.\mu &= -2 \\ 0.\lambda + 1.\mu &= -1 \end{aligned}\right\} \qquad \text{...(iii)}$$

$$\Rightarrow \; -\lambda = -2, \mu = -1, \Rightarrow \lambda = 2, \mu = -1.$$

These values of λ, μ satisfy all the equations of (iii) and as such third solution corresponding to $z = -1$ is a linear solution of the first two solutions.

***Ex. 14. Solve the equations with the help of matrices considering specially the case when $\lambda = 2$:—**

$$\lambda x + 2y - 2z = 1, \; 4x + 2\lambda y - z = 2, \; 6x + 6y + \lambda z = 3. \qquad \textit{(Kumaun 90)}$$

Sol. The given equations in the matrix form $\mathbf{AX} = \mathbf{K}$ can be written as

$$\begin{bmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

$\therefore$ The augmented matrix $\mathbf{A}^* = \begin{bmatrix} \lambda & 2 & -2 & 1 \\ 4 & 2\lambda & -1 & 2 \\ 6 & 6 & \lambda & 3 \end{bmatrix}$

or $\quad \mathbf{A}^* \sim \begin{bmatrix} 6\lambda & 12 & -12 & 6 \\ 12\lambda & 6\lambda^2 & -3\lambda & 6\lambda \\ 6\lambda & 6\lambda & \lambda^2 & 3\lambda \end{bmatrix}$, replacing R_1, R_3 by $6R_1, 3\lambda R_2, \lambda R_3$ respectively.

or $\quad \mathbf{A}^* \sim \begin{bmatrix} 6\lambda & 12 & -12 & 6 \\ 0 & 6\lambda^2 - 24 & 24 - 3\lambda & 6\lambda - 12 \\ 0 & 6\lambda - 12 & \lambda^2 + 12 & 3\lambda - 6 \end{bmatrix}$, replacing R_2, R_3 by $R_2 - 2R_1, R_3 - R_1$ respectively.

This is a matrix having three non-zero rows and in the reduced Echelon form. Hence the rank of $\mathbf{A}^*$ is 3.

Simultaneously we get reduced Echelon form of $\mathbf{A}$ viz.

$\begin{bmatrix} 6\lambda & 12 & -12 \\ 0 & 6\lambda^2 - 24 & 24 - 3\lambda \\ 0 & 6\lambda - 12 & \lambda^2 + 12 \end{bmatrix}$ which also has three non-zero rows and so its rank is also 3.

Thus the ranks of $\mathbf{A}^*$ and $\mathbf{A}$ are the same and as such the given equations have solutions.

The matrix form of the given equations then reduce to

$$\begin{bmatrix} 6\lambda & 12 & -12 \\ 0 & 6\lambda^2 - 24 & 24 - 3\lambda \\ 0 & 6\lambda - 12 & \lambda^2 + 12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 6\lambda - 12 \\ 3\lambda - 6 \end{bmatrix}$$

This gives $\qquad 6\lambda x + 12y - 12z = 6;$

$$(6\lambda^2 - 24)\, y + (24 - 3\lambda)\, z = 6\lambda - 12$$

and $\qquad (6\lambda - 12)\, y + (\lambda^2 + 12)\, z = 3\lambda - 6$

If $\lambda = 2$, then these equations reduce to

$$12x + 12y - 12z = 6;\ 18z = 0$$

which gives $\qquad z = 0$ and $2x + 2y = 1$

i.e. $\qquad x = -y + \frac{1}{2},\ z = 0.y$ **(Note)**

By assigning arbitrary values to y, an infinite number of corresponding values of x and z can be obtained. Hence the given system of equations has an infinite number of solutions. (This can be ascertained from the ranks of $\mathbf{A}^*$ and $\mathbf{A}$ also in the case when $\lambda = 2$).

Assigning two arbitrary values 0, 1 to y we have two sets of solutions of the given equations as

x	$\frac{1}{2}$	$-\frac{1}{2}$
y	0	1
z	0	0

Let any other solution of the given equation be $x = 3/2,\ y = -1,\ z = 0$ corresponding to the value -1 of y.

If this third solution is a linear combination of the first two solutions, then a and b can be found as follows :

$$\frac{1}{2}a - \frac{1}{2}b = 3/2;\ 0.a + 1.b = -1;\ 0.a + 0.b = 0.$$

Solving the first two of these we get $a = 1,\ b = -1$ which satisfy the third also.

Hence this solution is a linear combination of the first two solutions. In this way we can get two linearly independent solutions of the given set of equations for $\lambda = 2$.

****Ex. 15. Solve the system of linear equations :**

$$\mathbf{2x - 3y + 4z = 3,\ x - 3z = -2.}$$

Sol. The given equations in the matrix form $\mathbf{AX = K}$ can be written as

$$\begin{bmatrix} 2 & -3 & 4 \\ 1 & 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$\therefore$ The augmented matrix $\mathbf{A}^* = \begin{bmatrix} 2 & -3 & 4 & 3 \\ 1 & 0 & -3 & -2 \end{bmatrix}$

or $\quad \mathbf{A}^* \sim \begin{bmatrix} 0 & -3 & 10 & 7 \\ 1 & 0 & -3 & -2 \end{bmatrix}$, replacing R_1 by $R_1 - 2R_2$

$\sim \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & -(10/3) & (-7/3) \end{bmatrix}$, interchanging R_1 and R_2 and replacing R_2 by $\frac{1}{3} R_3$

This is a matrix having two non-zero rows, hence its rank is 2.

Simultaneously we get $\mathbf{A} \sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -(10/3) \end{bmatrix}$, which is also in the reduced Echelon form having two non-zero rows. Hence its rank is 2.

Thus we find that $\mathbf{A}$ and $\mathbf{A}^*$ have the same rank 2 and then two of th unknowns can be expressed as a linear function of the remaining unknown.

Now the matrix form of the given equations reduce to

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -(10/3) \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -2 \\ (-7/3) \end{bmatrix}$$

which is equivalent to $1.x + 0.y - 3.z = -2;\ 0.x + 1.y - \frac{10}{3} z = -\frac{7}{3}$

i.e. $x - 3z = -2,\ -3y + 10z = 7$ *i.e.* $x = 3z - 2,\ y = \frac{10}{3} z - \frac{7}{3}$, for all z

i.e. $x = 3k - 2,\ y = (10/3)k - (7/3),\ z = k$, for all k. **Ans.**

Hence the complete solution of the given system of equations is $x = 3k - 2$, $y = (10/3)k - (7/3),\ z = k$, for all k. **Ans.**

Exercises on § 6.07 – § 6.09

Ex. 1. Show that the following equations are consistant :

$3x - 4y = 2,\ 5x - 2y = 12,\ -x + 3y = 1.$

Ex. 2. Show that the equations

$2x + 6y + 11 = 0,\ 6x + 20y - 6z + 3 = 0,\ 6y - 13z + 1 = 0$

are not consistent.

Show that if in the following problems the given equations are consistent, then solve them.

Ex. 3. $5x + 3y + 7z = 4,\ 3x + 20y + 2z = 9,\ 7x + 2y + 10z = 5.$

(Kanpur 84; Meerut 86) **Ans.** $x = 5,\ y = 0,\ z = -3$

Ex. 4. $x_1 + 2x_2 + x_3 = 2,\ 2x_1 + 4x_2 + 3x_3 = 3,\ 3x_1 + 6x_2 + 5x_3 = 4.$

Ex. 5. $x_1 - x_2 + x_3 = 2,\ 3x_1 - x_2 + 2x_3 = -6,\ 3x_1 + x_2 + x_3 = -18.$

Ex. 6. $x_1 - 3x_2 + x_3 = 2,\ 2x_1 + x_2 + 3x_3 = 3,\ x_1 + 5x_2 + 5x_3 = 2.$

Ans. $x_1 = 1,\ x_2 = -(1/5),\ x_3 = 2/5$

Ex. 7. Can the following equations have solutions ?

(i) $x - y + 3z + 2w = 3,\ 3x + 2y + z + w = 1,\ 4x + y + 2z + 2w = 3.$

(ii) $x - 4y + 7z = 8,\ 3x + 8y - 2z = 6,\ 7x - 8y + 16z = 31.$

Ex. 8. Prove, without actually solving that the following system of equations have a unique solution—

$$5x + 3y + 14z = 4,\ y + 2z = 1,\ x - y + 2z = 0.$$

Ex. 9. Can the following equations have solutions ?

(i) $x + 2y + 3z = 4,\ 2x + 3y + 8z = 7,\ x - y + 9z = 1.$ **Ans.** Yes

(ii) $x + 2y + 3z = 2,\ 2x + 3y + 4z = 5,\ 3x + 4y + 5z = 9.$ *(Agra 92)* **Ans.** No.

Ex. 10. Show that the following equations

$$x + 2y - z = 3,\ 3x - y + 2z = 1,\ 2x - 2y + 3z = 2,\ x - y + z = -1$$

are consistent and solve them by the use of matrices.

(Rohilkhand 94; Garhwal 93)

Ex. 11. Show that the following equations are consistent and find their solutions by matrix method.

$$x_1 + x_2 + x_3 = 2,\ 4x_1 - x_2 + 2x_3 = -6,\ 3x_1 + x_2 + x_3 = -18.$$

Ans. $x_1 = -10,\ x_2 = -10/3,\ x_3 = 46/3$

Ex. 12. Solve $3x - 4y = 2,\ 5x + 2y = 12,\ -x + 3y = 1.$ **Ans.** $x = 2,\ y = 1$

Solution of Homogeneous Linear Equations

§ 6.10. Definition : A linear equation of the type

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + \ldots + a_nx_n = 0$$

is called a homogeneous linear equation.

§ 6.11. Definition : A system of homogeneous linear equations is given in the matrix form by $\mathbf{AX} = \mathbf{O}$, where $\mathbf{A}$ and $\mathbf{X}$ are the same notations as used in §6.02 Page 105 Chapter VI.

Note 1 : Here $\mathbf{K}$ is zero matrix.

Note 2 : The matrix of coefficients $\mathbf{A}$ and the augmented matrix $\mathbf{A}^*$ being the same have equal ranks and thus the system is always consistent.

Note 3 : $x_1 = 0 = x_2 = \ldots = x_n$ is always a solution and is called the **trivial solution.**

An Important Theorem (Without Proof).

A homogeneous system of n linear equations in n unknowns, whose determinants of coefficients does not vanish, has only the trivial solution.

§ 6.12. Theorem (Without Proof)

A system m of m homogeneous equations in n unknowns $x_1, x_2, \ldots, x_n$ has a solution other than the trivial solution viz. $x_1 = 0 = x_2 = \ldots = x_n$ if and only if the rank r of the matrix of coefficients $\mathbf{A}$ is less than n, the number of unknowns.

If $r = n$, then n of the equations can be solved by Cramer's Rule for the unique solution $x_1 = 0 = x_2 = \ldots = x_n$ and the given system has non-trival solutions.

If r is the rank of **A**, then r of the n unknowns $x_1, x_2, \ldots, x_n$ can be expressed as linear combination of the remaining $n-r$ unknowns to which arbitrary values may be assigned. Hence the system will have an infinite number of solutions out of which $n-r$ are linearly independent and the remaining can be expressed as linear combination of these $n-r$.

The above theorem is illustrated in the following examples :—

Ex. 1. Solve

$$\begin{aligned} x+y+z+w&=0\\ x+3y+2z+4w&=0\\ 2x+z-w&=0\end{aligned}$$

Sol. The given equations in the matrix form $AX = O$ is given by

$$\begin{bmatrix}1&1&1&1\\1&3&2&4\\2&0&1&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\\w\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

The matrix **A** of coefficients $=\begin{bmatrix}1&1&1&1\\1&3&2&4\\2&0&1&-1\end{bmatrix}$

or $$A \sim \begin{bmatrix}1&1&1&1\\0&2&1&3\\0&-2&-1&-3\end{bmatrix}, \text{ replacing } R_2 \text{ and } R_3 \text{ by } R_2-R_1 \text{ and } R_3-2R_1 \text{ respectively}$$

$$\sim \begin{bmatrix}1&1&1&1\\0&2&1&3\\0&0&0&0\end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3+R_2$$

$$\sim \begin{bmatrix}1&1&1&1\\0&1&1/2&(3/2)\\0&0&0&0\end{bmatrix}, \text{ replacing } R_2 \text{ by } \frac{1}{2}R_2$$

or $$A \sim \begin{bmatrix}1&0&1&1\\0&1&1/2&(3/2)\\0&0&0&0\end{bmatrix}, \text{ replacing } C_2 \text{ by } C_2-C_1$$

This is a matrix in the reduced Echelon form having two non-zero rows, hence the rank of **A** is 2 and is less than the number of unknowns x, y, z and w *i.e.* 4.

$\therefore$ The matrix form of the given equations reduces to

$$\begin{bmatrix}1&0&1&1\\0&1&\frac{1}{2}&(3/2)\\0&0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\\w\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

or
$$\begin{aligned}&1.x+0.y+1.z+1.w=0;\\ &0.x+1.y+\tfrac{1}{2}.z+\tfrac{3}{2}.w=0;\\ &0.x+0.y+0.z+0.w=0.\end{aligned}$$

From the first two equations we have

$$\left.\begin{aligned} x &= -z-w \\ y &= -\tfrac{1}{2}z-(3/2)\,w \end{aligned}\right\} \quad \text{...(i)}$$

i.e. two of the unknowns viz. x and y have been expressed as linear combinations of the remaining two unknowns viz. z and w.

An infinite number of solutions of the given equations can be obtained by assigning arbitrary values to z and w.

Also according to § 6.11 Page 144 Ch. VI we know that the system has $n-r$ *i. e.* $4-2$ *i.e.* 2 linearly independent solutions.

Take any two solutions of the system by assigning the following arbitrary values to z and w

$$z = 2, 4$$
$$w = 0, 2$$

Then the solutions are given in the tabular form as

x	-2	-6
y	-1	-5
z	2	4
w	0	2

(**Note :** The corresponding values of x and y are calculated from the equations (i) above)

Let any other solution be

$$x=-2,\ y=-3,\ z=0,\ w=2, \quad \text{...(ii)}$$

obtained by assigning z and w the value 0 and 2 in equations (i).

If this solution is a linear combination of the first two solutions (given in the above table) then we can always find two constants λ and μ such that

$$\left.\begin{aligned} -2\lambda-6\mu &= -2 && \text{...(iii)} \\ -\lambda-5\mu &= -3 && \text{...(iv)} \\ 2\lambda+4\mu &= 0 && \text{...(v)} \\ 0.\lambda+2\mu &= 2 && \text{(vi)} \end{aligned}\right\} \quad \textbf{(Note)}$$

From (vi) we get $\mu = 1$.

From (v) we get $\lambda = -2$.

These values of λ and μ satisfy (iii) and (iv) also.

Hence the third solution [given by (ii) above] is a linear combination of the first two solutions (given in the tabular form above).

Ex. 2. Find the solution of the following equations by the matrix method :

$$\mathbf{2x_1 - x_2 + x_3 = 0,\ 3x_1 + 2x_2 + x_3 = 0,\ x_1 - 3x_2 + 5x_3 = 0.}$$

Sol. The given equations in the matrix form $\mathbf{AX} = \mathbf{O}$ is given by

$$\begin{bmatrix} 2 & -1 & 1 \\ 3 & 2 & 1 \\ 1 & -3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The matrix A of coefficients

$$= \begin{bmatrix} 2 & -1 & 1 \\ 3 & 2 & 1 \\ 1 & -3 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & 1 \\ -9 & 2 & 5 \end{bmatrix}, \text{ replacing } C_1, C_2 \text{ by } C_1 - 2C_3 \text{ and } C_2 + C_3 \text{ respectively}$$

$$\sim \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 9 & 29 & 14 \end{bmatrix}, \text{ replacing } C_2, C_3 \text{ by } C_2 - 3C_1 \text{ and } C_3 - C_1 \text{ respectively}$$

$$\sim \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 29 & 14 \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 - 9R_2$$

$$\sim \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 - 14R_1 \text{ and } C_2 \text{ by } (1/29)\, C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \text{ interchanging } R_1 \text{ and } R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ interchanging } R_2 \text{ and } R_3$$

$$= I_3$$

The rank of A is 3 and is equal to the number of unknowns viz. x_1, x_2, x_3.

Hence $x_1 = 0 = x_2 = x_3$. (See § 6.12 Page 144 Chapter V

Ex. 3. Show by considering rank of an appropriate matrix, that the following system of equations, possesses no solution other than the trivial solutions $x = 0 = y = z$:—

$$3x - y + z = 0,\ -15x + 6y - 5z = 0,\ 5x - 2y + 2z = 0$$

Sol. The give equations in the matrix form $AX = O$ is given by

$$\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The matrix A of coefficients

$$= \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -5 \\ -1 & 0 & 2 \end{bmatrix}, \text{ replacing } C_1, C_2 \text{ by } C_1 - 3C_3, C_2 + C_3 \text{ respectively}$$

$$\sim \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix}, \text{ replacing } R_2, R_3 \text{ by } R_2 + 5R_1, -R_3 \text{ respectively}$$

$$\sim \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 + 2R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ interchanging } R_1 \text{ and } R_3,$$

$$= \mathbf{I}_3$$

$\therefore$ The rank of A is 3 and is equal to the number of unknowns viz. x, y, z. Hence $x = 0 = y = z$ and the given system has no non-trivial solutions. (See § 6.12 Page 144 Chapter VI).

Ex. 4. Solve :

$$\begin{aligned} 2x - 2y + 5z + 3w &= 0 \\ 4x - y + z + w &= 0 \\ 3x - 2y + 3z + 4w &= 0 \\ x - 3y + 7z + 6w &= 0 \end{aligned}$$

(Kumaun 94)

Sol. The given equations in the matrix form $\mathbf{AX} = \mathbf{O}$ is given by

$$\begin{bmatrix} 2 & -2 & 5 & 3 \\ 4 & -1 & 1 & 1 \\ 3 & -2 & 3 & 4 \\ 1 & -3 & 7 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The matrix A of coefficients

$$= \begin{bmatrix} 2 & -2 & 5 & 3 \\ 4 & -1 & 1 & 1 \\ 3 & -2 & 3 & 4 \\ 1 & -3 & 7 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 4 & -9 & -9 \\ 0 & 11 & -27 & -23 \\ 0 & 7 & -18 & -14 \\ 1 & -3 & 7 & 6 \end{bmatrix}, \text{ replacing } R_1, R_2 \text{ and } R_4 \text{ by } R_1 - 2R_4, R_2 - 4R_4 \text{ and } R_3 - 3R_4 \text{ respectively}$$

$$\sim \begin{bmatrix} 1 & -3 & 7 & 6 \\ 0 & 11 & -27 & -23 \\ 0 & 7 & -18 & -14 \\ 0 & 4 & -9 & -9 \end{bmatrix}, \text{ interchanging } R_1 \text{ and } R_4$$

$$\sim\begin{bmatrix}1 & -3 & 7 & 6\\ 0 & 4 & -9 & -9\\ 0 & 7 & -18 & -14\\ 0 & 4 & -9 & -9\end{bmatrix}, \text{ replacing } R_2 \text{ by } R_2 - R_3$$

$$\sim\begin{bmatrix}1 & -3 & 7 & 6\\ 0 & 4 & -9 & -9\\ 0 & 7 & -18 & -14\\ 0 & 0 & 0 & -0\end{bmatrix}, \text{ replacing } R_4 \text{ by } R_4 - R_2$$

$$\sim\begin{bmatrix}1 & 0 & 7 & 6\\ 0 & 4 & -9 & -9\\ 0 & 7 & -18 & -14\\ 0 & 0 & 0 & -0\end{bmatrix}, \text{ replacing } C_2 \text{ by } C_2 + 3C_1$$

$$\sim\begin{bmatrix}1 & 0 & 7 & 6\\ 0 & 4 & -9 & -9\\ 0 & 0 & -9/4 & 7/4\\ 0 & 0 & 0 & 0\end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 - \tfrac{7}{4} R_2$$

$$\sim\begin{bmatrix}1 & 0 & 7 & 6\\ 0 & 1 & -9/4 & -9/4\\ 0 & 0 & 1 & -7/9\\ 0 & 0 & 0 & 0\end{bmatrix}, \text{ replacing } R_2 \text{ by } \tfrac{1}{4}R_2 \text{ and } R_3 \text{ by } -\tfrac{4}{9} R_3$$

This is a matrix in the reduced Echelon form having three non-zero rows, hence the rank of **A** is 3 and is less than the number of unknowns viz. 4.

∴ The matrix form of the given equations reduces to

$$\begin{bmatrix}1 & 0 & 7 & 6\\ 0 & 1 & -9/4 & -9/4\\ 0 & 0 & 1 & -7/9\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\\ w\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}$$

or $$x + 7z + 6w = 0; \quad \text{...(i)}$$

$$y - \frac{9}{4} z - \frac{9}{4} w = 0 \quad \text{...(ii)}$$

and $$z - \frac{7}{9} w = 0 \quad \text{...(iii)}$$

From (iii) we get $z = (7/9)\, w$

∴ From (i) $x = -7z - 6w = [-(49/9) - 6]\, w = -(103/9) w$

and $y = (9/4)\, z + (9/4\, w) = [(7/4) + (9/4)]\, w = 4w.$

Thus we find that the three of the unknowns viz. x, y and z are expressed in terms of the 4th unknown viz. w.

An infinite number of solutions of the given equations can be obtained by assigning arbitrary values to w.

Also we know that the system has $n - r$ *i. e.* $4 - 3$ *i. e.* 1 linearly independent solution.

Assigning w one arbitrary value 9, we have a set of solution as $x = -103, y = 36, z = 7, w = 9$.

Let another solution (by assigning 18 to w) be

$$x = -206, y = 72, z = 14 \text{ and } w = 18.$$

It is evident that this second set of values are nothing but double of the first set of values. Hence the theorem of § 6.12 Page 144 Chapter VI is fully verified.

Ex. 5. Solve completely the system of equations :

$$\begin{aligned} x - 2y + z - w &= 0 \\ x + y - 2z + 3w &= 0 \\ 4x + y - 5z + 8w &= 0 \\ 5x - 7y + 2z - w &= 0 \end{aligned}$$

(Kumaun 93)

Sol. The given equation in the matrix form $\mathbf{AX} = \mathbf{O}$ is given by

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The matrix A of coefficients

$$= \begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 3 & -3 & 4 \\ 0 & 9 & -9 & 12 \\ 0 & 3 & -3 & 4 \end{bmatrix}, \text{ replacing } R_2, R_3 \text{ and } R_4 \text{ by } R_2 - R_1, R_3 - 4R_1 \text{ and } R_4 - 5R_1 \text{ respectively}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 5/3 \\ 0 & 1 & -1 & 4/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ replacing } R_3 \text{ and } R_4 \text{ by } R_3 - 3R_2 \text{ and } R_4 - R_2 \text{ respectively, then } R_2 \text{ by } \tfrac{1}{3} R_2 \text{ and finally } R_1 \text{ by } R_1 + 2R_2.$$

This is a matrix in the reduced Echelon form having two non-zerorows, hence the rank of A is 2 and is less than the number of unknowns viz. 4.

$\therefore$ The matrix form of the given equations reduces to

$$\begin{bmatrix} 1 & 0 & -1 & (5/3) \\ 0 & 1 & -1 & (4/3) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which is equivalent to

$$x - z + (5/3)\, w = 0,\ y - z + (4/3) w = 0$$

i.e. $$y = z - (4/3)\, w,\ x = z - (5/3)\, w.$$

$$\therefore \mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} z - (5/3)\, w \\ z - (4/3)\, w \\ z \\ w \end{bmatrix}$$

By multiplication we find that

$$\mathbf{AX} = \begin{bmatrix} 1 & 0 & -1 & (5/3) \\ 0 & 1 & -1 & (4/3) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z - (5/3)\, w \\ z - (4/3)\, w \\ z \\ w \end{bmatrix}$$

$$= \begin{bmatrix} z - (5/3)\, w - z + (5/3)\, w \\ 0 + z - (4/3)\, w - z + (4/3)\, w \\ 0 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{O},$$

whatever the values of z and w may be.

$\therefore$ We have $x = \lambda + (5/3)\, \mu$, $y = \lambda - (4/3)\, \mu$, $z = \lambda$, $w = \mu$, where λ and μ can take any values, as the complete solution of the given system of equations.

****Ex. 6. Find the general solution of the matrix :**

$$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 2 & 9 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ v \\ t \end{bmatrix} = \mathbf{O}$$

Sol. The given equation in the matrix form $\mathbf{AX} = \mathbf{O}$ is given by

$$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 2 & 9 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ v \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The matrix $\mathbf{A}$ of coefficients

$$= \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 2 & 9 & -7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 5 & 3 & 7 \\ 1 & -1 & -2 & -4 \\ 0 & 4 & 9 & 10 \\ 0 & 8 & 21 & 17 \end{bmatrix}$$ replacing R_1, R_2 and R_4 by $R_1 - 2R_2$, $R_3 - 3R_2$ and $R_4 - 6R_2$ respectively.

$$\sim \begin{bmatrix} 0 & 1 & -6 & -3 \\ 1 & -1 & -2 & -4 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 3 & -3 \end{bmatrix}$$ replacing R_1, R_4 by $R_1 - R_2$ and $R_4 - 2R_2$ respectively.

$$\sim \begin{bmatrix} 0 & 1 & -6 & -3 \\ 1 & 0 & -8 & -7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 3 & -3 \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 - 4R_1 \text{ and } R_2 \text{ by } R_2 + R_1.$$

$$\sim \begin{bmatrix} 0 & 1 & -9 & -0 \\ 1 & 0 & -8 & -7 \\ 0 & 0 & 0 & 55 \\ 0 & 0 & 3 & -3 \end{bmatrix}, \text{ replacing } R_1 \text{ and } R_3 \text{ by } R_1 - R_4 \text{ and } R_3 - 11\,R_4 \text{ respectively.}$$

$$\sim \begin{bmatrix} 0 & 1 & 0 & -9 \\ 1 & 0 & 0 & -15 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \text{ replacing } R_1, R_2, R_3 \text{ and } R_4 \text{ by } R_1 + 3\,R_4, R_2 + (8/3)\,R_4, (1/55)\,R_3 \text{ and } (1/3)\,R_4 \text{ respectively.}$$

$$\sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \text{ replacing } R_1, R_2, \text{ and } R_4 \text{ by } R_1 + 9\,R_3, R_2 + 15\,R_3 \text{ and } R_4 + R_3 \text{ respectively.}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ rearranging the rows}$$

$$\sim [\mathbf{I}_4].$$

$\therefore$ The rank of the matrix **A** is 4 and is equal to the number or unknowns viz. x, y, v, t.

Hence $x = 0, y = 0, v = 0, t = 0$ [See § 6.12 Page 144 Ch. VI]

Exercises on § 6.10 – § 6.12

Ex. 1. Solve the following equations :—

$x_1 - x_2 + x_3 = 0, x_1 + 2x_2 - x_3 = 0, 2x_1 + x_2 + 3x_3 = 0.$ *(Lucknow 92)*

Ex. 2. Find the rank of the coefficient matrix for the following system of homogeneous equations over the field of real numbers and compute all the solutions :—

$$x_1 + 2x_2 - 3x_3 + 4x_4 = 0, x_1 + 3x_2 - x_3 = 0, 6x_1 + x_2 + 2x_3 = 0.$$

Ex. 3. Solve completely the following equations using matrices :—

$x + 3y - 2z = 0, 2x - y + 4z = 0, x - 11y + 14z = 0.$ *(Lucknow 90)*

Ex. 4. Solve completely the following equations with the help of matrices :

(i) $x - y - z + t = 0, x - y + 2z - t = 0, 3x + y + t = 0.$

(ii) $2w + 3x - y - z = 0, 4w - 6x - 2y + 2z = 0, -6w + 12\,x + 3y - 4z = 0.$

MISCELLANEOUS SOLVED EXAMPLES

***Ex. 1. Show that the only real value of λ for which the following equations have non-zero solution is 6 :**

$$\mathbf{x + 2y + 3z = \lambda x,\ 3x + y + 2z = \lambda y,\ 2x + 2y + z = \lambda z.}$$ *(Kanpur 95)*

Sol. The given equations can be rewritten as

$$(1-\lambda)x + 2y + 3z = 0;$$
$$3x + (1-\lambda)y + 2z = 0;$$
and
$$2x + 3y + (1-\lambda)z = 0.$$

The equations in the matrix form $\mathbf{AX} = \mathbf{K}$ can be rewritten as

$$\begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If the given system of equations has a non-zero solution then the matrix A must have a rank < the number of unknown quantities x, y, z *i.e.* 3 and $|\mathbf{A}| = 0$

$$i.e. \begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

or $$\begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0, \text{ replacing } R_1 \text{ by } R_1 + R_2 + R_3$$

or $$(6-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0, \text{ taking out } (6-\lambda) \text{ common from } R_1$$

or $$(6-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 3 & -2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} = 0, \text{ applying } C_2 - C_1 \text{ and } C_3 - C_1$$

or $$(6-\lambda)\begin{vmatrix} -2-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} = 0$$

or $$(6-\lambda)[(2+\lambda)(1+\lambda)+1] = 0$$

or $$(6-\lambda)[\lambda^2 + 3\lambda + 3] = 0 \text{ or } \lambda = 6, \tfrac{1}{2}[-3 \pm \sqrt{(9-12)}]$$

or $\lambda = 6$ (the other roots being imaginary) is the only real value of λ for which the given system of equations has a non-zero solution.

Ex. 2. Prove that if the system of equations

$$\mathbf{x = ay + z,\ y = z + ax,\ z = x + y}$$

is consistent (having non-zero solutions) then a + 1 = 0.

Sol. The given equations can be rewritten as

$$x - ay - z = 0$$
$$ax - y + z = 0$$
$$x + y - z = 0$$

These equations in the matrix form $\mathbf{AX} = \mathbf{K}$ can be rewritten as

$$\begin{bmatrix} 1 & -a & -1 \\ a & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If the given system of equations has a non-zero solution then the matrix **A** have a rank < the number of unknown quantities x, y, z *i.e.* 3 and $|\mathbf{A}| = 0$. Here we have

$$\mathbf{A} = \begin{bmatrix} 1 & -a & -1 \\ a & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

or

$$\mathbf{A} \sim \begin{bmatrix} 0 & -a-1 & -1 \\ a+1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \text{replacing } C_1, C_2 \text{ by } C_1 + C_3, C_2 + C_3 \text{ respectively}$$

$$\sim \begin{bmatrix} 0 & -(a+1) & 0 \\ a+1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \text{replacing } R_1, R_2 \text{ by } R_1 - R_3, R_2 + R_3 \text{ respectively} \quad \text{...(i)}$$

$$\sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{replacing } C_1, C_2, C_3 \text{ by } C_1/(a+1), -C_2/(a+1), -C_3 \text{ respectively}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{interchanging } R_1 \text{ and } R_2$$

$$= \mathbf{I}_3$$

i.e. the rank of **A** is 3 *i.e.* equal to the number of unknowns viz. x, y, z.

But if $a + 1 = 0$, then from (i) we get

$$\mathbf{A} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \text{which has one non- zero row and so the rank of } \mathbf{A} \text{ is } 1$$

i.e. < 3, the number of unknowns viz. x, y, z.

Also
$$|\mathbf{A}| = \begin{vmatrix} 1 & -a & -1 \\ a & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} a & -a-1 & -1 \\ a+1 & 0 & 1 \\ 0 & 0 & -1 \end{vmatrix}, \text{adding } C_3 \text{ to } C_1 \text{ and } C_2$$

$$= \begin{vmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{vmatrix}, \text{if } a + 1 = 0$$

$$= 0, \text{ two rows being identical}$$

Hence the given equations are consistent if $a + 1 = 0$.

***Ex. 3. Investigate for what values of λ, μ the simultaneous equations :**

$$x + 2y + z = 8,\ 2x + y + 3z = 13,\ 3x + 4y - \lambda z = \mu$$

have (i) no solution (ii) a unique solution and (iii) infinitely many solutions.

Sol. The given equations in the matrix form $\mathbf{AX} = \mathbf{K}$ can be written as

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 3 & 4 & -\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 13 \\ \mu \end{bmatrix}$$

$\therefore$ The augmented matrix $\mathbf{A}^* = \begin{bmatrix} 1 & 2 & 1 & 8 \\ 2 & 1 & 3 & 13 \\ 3 & 4 & -\lambda & \mu \end{bmatrix}$

or $\mathbf{A}^* \sim \begin{bmatrix} 1 & 2 & 1 & 8 \\ 0 & -3 & 1 & -3 \\ 0 & -2 & -\lambda-3 & \mu-24 \end{bmatrix}$, replacing R_2, R_3, by $R_2 - 2R_1, R_3 - 3R_1$ respectively;

$\sim \begin{bmatrix} 1 & 5 & 0 & 11 \\ 0 & -6 & 2 & -6 \\ 0 & -6 & -3\lambda-9 & 3\mu-72 \end{bmatrix}$, replacing R_1, R_2, R_3 by $R_1 - R_2$, $2R_2$ and $3R_3$ respectively;

$\sim \begin{bmatrix} 1 & 5 & 0 & 11 \\ 0 & 1 & -\frac{1}{3} & 1 \\ 0 & 0 & -3\lambda-11 & 3\mu-66 \end{bmatrix}$, replacing R_3 and R_2 by $R_3 - R_2$, and $-\frac{1}{6}R_2$ respectively ...(i)

Case I If $3\lambda + 11 \neq 0$, $3\mu - 66 \neq 0$ *i.e.* $\lambda \neq -(11/3)$, $\mu \neq 22$, then from (i) matrix $\mathbf{A}^*$ has three non-zero rows and is in the reduced Echelon form. Thus the matrix $\mathbf{A}^*$ is of rank 3. Also then the matrix $\mathbf{A} = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & -1/3 \\ 0 & 0 & -3\lambda-11 \end{bmatrix}$ is also in reduced Echelon form having three non-zero rows and thus its rank is also 3. Also there are three unknown quantities x, y, z so, in the case $\lambda \neq -(11/3)$, $\mu \neq 22$, the given equations have a **unique solution.**

Case II. If $3\lambda + 11 = 0$, $3\mu - 66 \neq 0$ *i.e.* $\lambda = -(11/3)$, $\mu \neq 22$, then the rank of the matrix $\mathbf{A}^*$ is 3 but that of $\mathbf{A}$ is 2, since in this case both $\mathbf{A}^*$ and $\mathbf{A}$ are in the reduced Echelon form but $\mathbf{A}^*$ has three non-zero rows, wheares $\mathbf{A}$ has two non-zero rows. Thus the ranks of $\mathbf{A}^*$ and $\mathbf{A}$ are not the same and so there is **no solution** of the given equations.

Case III. If $3\lambda + 11 = 0$, $3\mu - 66 = 0$ *i.e.* $\lambda = -(11/3)$, $\mu = 22$, then the ranks of $\mathbf{A}$ as well as $\mathbf{A}^*$ are the same and each is 2 *i.e.* less than the number of unknowns viz. x, y and z. Hence in this case two unknowns will be expressed in terms of the third and thus we shall have an **infinite number of solutions.**

***Ex. 4. Investigate for what values of λ and μ, the simultaneous equations : $x + y + z = 6$, $x + 2y + 3z = 10$ and $x + 2y + \lambda z = \mu$ have (i) no solution, (ii) unique solution (iii) infinite solutions.**

(Agra 96, 93, 91; Garhwal 91, 90; Kanpur 96, 94; Meerut 91 S, 90; Rohilkhand 96, 90)

Sol. The given equations in the matrix form **AX** = **K** can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

$\therefore$ The augmented matrix $\mathbf{A}^* = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$

or $\mathbf{A}^* = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{bmatrix}$, replacing R_2, R_3 by $R_2 - R_1, R_3 - R_2$ respectively and then R_1 by $R_1 - R_2$ ...(i)

Now following cases arise :—

Case I. If $\lambda - 3 = 0, \mu - 10 \neq 0$ *i.e.* $\lambda = 3, \mu \neq 10$, then from (i), $\mathbf{A}^*$ is in the reduced Echelon form having three non-zero rows and **A** is in the reduced Echelon form having two non-zero rows.

$\therefore$ The ranks of $\mathbf{A}^*$ and **A** are 3 and 2 respectively which being different, the given equations have **no solution**.

Case II. If $\lambda - 3 \neq 0, \mu - 10 \neq 0$ *i.e.* $\lambda \neq 3, \mu \neq 10$, then from (i) we find that both $\mathbf{A}^*$ and **A** are in the reduced Echelon form having three non-zero rows and hence the ranks of $\mathbf{A}^*$ and **A** are each 3 and these being the same the given equations are consistent. Also there are three unknowns viz. x, y, z so the solution is **unique** in this case.

Case III. If $\lambda - 3 = 0, \mu - 10 = 0$ *i.e.* $\lambda = 3, \mu = 10$, then from (i) we find that the matrices $\mathbf{A}^*$ and **A** are in the reduced Echelon form having two non-zero rows each. Hence the ranks of $\mathbf{A}^*$ and **A** are the same each being 2, which is less than the number of unknowns x, y, z. Therefore in this case two unknowns will be expressed in terms of the third and thus we shall have an **infinite** number of solutions.

Ex. 5. Show that the following equations are consistent $x + y + z = -3$, $x + y - 2z = -2$, $2x + 4y + 7z = 7$. *(Kumaun 96)*

Sol. Given equations can be written in the matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \\ 2 & 4 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -3 \\ -2 \\ 7 \end{bmatrix}$$

$\therefore$ The augmented matrix $\mathbf{A}^* = \begin{bmatrix} 1 & 1 & 1 & -3 \\ 1 & 1 & -2 & -2 \\ 2 & 4 & 7 & 7 \end{bmatrix}$

or $\mathbf{A}^* \sim \begin{bmatrix} 1 & 1 & 1 & -3 \\ 0 & 0 & -3 & 1 \\ 1 & 3 & 6 & 10 \end{bmatrix}$, replacing R_2, R_3 by $R_2 - R_1$, $R_3 - R_1$ respectively

$\sim \begin{bmatrix} 1 & 1 & 1 & -3 \\ 0 & 0 & -3 & 1 \\ 1 & 3 & 0 & 12 \end{bmatrix}$, replacing R_3 by $R_3 + 2R_2$

$\sim \begin{bmatrix} 1 & 1 & 1 & -3 \\ 0 & 0 & -1 & 1/3 \\ 0 & 2 & -1 & 15 \end{bmatrix}$, replacing R_2, R_3 by $(1/3)\, R_2$, $R_3 - R_1$ respectively

$\sim \begin{bmatrix} 1 & 1 & 0 & -8/3 \\ 0 & 0 & -1 & 1/3 \\ 0 & 2 & 0 & 44/3 \end{bmatrix}$, replacing R_1, R_3 by $R_1 + R_2$ $R_3 - R_2$ respectively

$\sim \begin{bmatrix} 1 & 1 & 0 & -8/3 \\ 0 & 0 & 1 & -1/3 \\ 0 & 1 & 0 & 22/3 \end{bmatrix}$, replacing R_2, R_3 by $-R_2$, $(1/2)\, R_3$ respectively

$\sim \begin{bmatrix} 1 & 0 & 0 & -10 \\ 0 & 0 & 1 & 1/3 \\ 0 & 1 & 0 & 22/3 \end{bmatrix}$, replacing R_1 by $R_1 - R_3$

$\sim \begin{bmatrix} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & 22/3 \\ 0 & 0 & 1 & -1/3 \end{bmatrix}$, interchanging R_2 and R_3

This is a matrix in the reduced Echelon form having three non-zero rows, so its rank is 3.

Simultaneously we get the reduced form of **A** viz. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *i.e.* $\mathbf{I}_3$ and so the rank of **A** is also 3.

Thus we find that the ranks of $\mathbf{A}^*$ and **A** are the same and so the given equations are consistent i.e. have solutions given by $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -10 \\ 22/3 \\ -1/3 \end{bmatrix}$,

which gives

$$x = -10,\ y = 22/3,\ z = -1/3.$$

Ans.

EXERCISES ON CHAPTER VI

Ex. 1. Examine whether the following linear equations are consistent, and if consistent solve them :—

$$x_1 + x_2 + x_3 + x_4 = 0,\ 2x_1 - x_2 + 3x_3 + 4x_4 = 4$$

and $$3x_1 + 4x_3 + 5x_4 = 1.$$

Ex. 2. Solve the equations by matrix method :—

$$x + 2y + 3z = 14;\ x + y + z = 6,\ x + 3y + 6z = 25.$$

Ex. 3. Solve by matrix method :—

$$x - 2y + 3z = 2,\ 2x - 3z = 3;\ x + y + z = 0.$$

Ex. 4. Solve by matrix method :—

$x + 2y - z = 3,\ 3x - y + 2z = 1,\ 2x - 2y + 3z = 2.$ **Ans.** $x = -1,\ y = 4,\ z = 4$

Ex. 5. Solve $x + 2y + 3z = 14,\ 2x + y + 2z = 11,\ 2x + 3y + z = 11$ with the help of matrices. **Ans.** $x = 18/11,\ y = 17/11,\ z = 34/11$

Ex. 6. Investigate k such that the following system of linear equations is consistent and obtain its solution :—

$$2x + y - z = 12,\ x - y - 2z = -3;\ 3y + 3z = k.$$

Ex. 7. Investigate for which values of λ and μ the following system of equations will have

(i) No solution and (ii) a unique solution :—

$$x + 2y + 3z = 5,\ 3x - y + 2z = 12,\ 3x - y + \lambda z = \mu.$$

Ex. 8. Investigate for what values of λ and μ, the simultaneous equations $x + y + z = 16,\ x + 2y + 5z = \mu,\ x + 2y + \lambda z = 10$ have unique solution. *(Agra 90)*

Ex. 9. Does the following system of linear equations possess a unique solution ? If so, then solve them. If not, why ?

$$x + 2y + 3z = 6,\ 2x + 4y + z = 7,\ 3x + 2y + 9z = 14.$$

Ex. 10. Find the inverse of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$ and use it to solve the equations $x + 2y + 3z = 3,\ 2x + 3y + 2z = 0,\ 3x + 3y + 4z = 5.$

Ex. 11. Show that the following system of equations have unique solution :

$$x + 2z = 0,\ y + 2z = 1;\ 5x + 3y + 14z = 4.$$

Ex. 12. Solve : $2x_1 - 3x_2 + 4x_3 + x_4 = 0;\ x_1 + x_2 - x_4 = 0;$

$$3x_1 - 3x_2 + 5x_3 = 0;\ 4x_1 - 3x_2 + 6x_3 - x_4 = 0.$$

(Hint. See Ex. 1 Page 145 Chapter VI).

Ex. 13. State the conditions under which a system of non-homogeneous equations will have (i) no solution, (ii) a unique solution, (iii) infinity of solutions.

Ex. 14. For what values of a, b do the system of equations $x + 2y + 3z = 6,\ x + 3y + 5z = 9;\ 2x + 5y + az = b$ have (i) no solution; (ii) a unique solution; (iii) more than one solutions ?

*Ex. 15. Solve $x + 2y - z = 3; 3x - y + 2z = 1; 2x - 2y + 3z = 2; x - y + z = -1$.

Ex. 16. Find the inverse of the coefficient matrix and hence solve the following equations :—

$$x + 2y - 3z = 1,\ 3x + y + z = 8,\ x - 2y = 0.$$

Ex. 17. Apply rank-test to examine if the following system of equations is consistent and if consistent, then only find the complete solution :—

$$x + y + z = 6,\ 4x + 3y + z = 9,\ 2x + 2y + 3z = 8.$$

Ex. 18. Apply rank test to find whether

$$x + 2y + 3z + 4w = 0,\ 8x + 5y + z + 4w = 0,\ 5x + 6y + 8z + w = 0,$$
$$8x + 3y + 7z + 2w = 0$$

have any solution other than $x = y = z = w = 0$.

Ex. 19. Solve the following equations by the use of matrices

(a) $x - y - 2z - 4t = 0,\ 2x + 3y - z - t = 0,\ 6x + 3y - 7t = 0,$
$3x + y + 3z - 2t = 0$.

(b) $x + y + z = 0,\ 2x - y - 3z = 0,\ 3x - 5y + 4z = 0,\ x + 17y + 4z = 0$

(c) $x + 2y + 3z = 0, 3x + 4y + 4z = 0,\ 7x + 10y + 12z = 0$.

Ex. 20. Solve the equations by matrix inversion :

$$x + y + z = 4,\ 2x - y + 3z = 1,\ 3x + 2y - z = 1.$$

Ex. 21. Show by matrix method that the following system of equations is consistent and solve it

$$4x_1 + 3x_2 + 3x_3 + x_4 = 9,\ x_1 + 2x_2 + x_3 + x_4 = 2;$$
$$3x_1 + 4x_2 + 2x_3 - x_4 = 8;\ 2x_1 + 3x_2 + 4x_3 + 5x_4 = 5;$$
$$x_1 - x_2 + x_3 - x_4 = 4.$$

Ex. 22. Show that the equations $x - 3y - 8z + 10 = 0,\ 3x + y - 4z = 0,$ $2x + 5y + 6z - 13 = 0$ are consistent and solve them. *(Meerut 92)*

Ex. 23. Solve by matrix method :—

$$x + y + z = 4,\ x - y + z = 5,\ 2x + 3y - z = 1.$$

Ex. 24. Show that the system of equations is consistent

$$2x + 6y = -11,\ 6x + 20y - 6z = -3,\ 6y - 18z = -1.$$

Ex. 25. Show by matrix method, the following equations are consistent and have infinite number of solutions

$$x_1 + x_2 + x_3 = 0,\ 2x_1 + 5x_2 + 6x_3 = 0.$$

Ex. 26. Solve by matrix method ;

$$x + y + z + w = 1,\ 2x - y + z - 2w = 2,\ 3x + 2y - z - w = 3.$$

Ex. 27. Slove the following equations :—

$$2x_1 - 2x_2 + 3x_3 = 3,\ x_1 + 2x_2 - x_3 - 5x_4 = 4,\ x_1 + 2x_2 - 2x_3 + 7x_4 = 5$$

(Rohilkhand 93)

Ex. 28. If x, y, z are not all zero and if $ax + by + cz = 0$,
$bx + cy + az = 0,\ cx + ay + bz = 0$, prove that,

$$x : y : z = 1 : 1 : 1 \quad \text{or} \quad 1 : \omega : \omega^2 \quad \text{or} \quad 1 : \omega^2 : \omega$$

where ω is a complex cube root of unity. *(Purvanchal 98)*