Chapter II

Some Types of Matrices

§ 2·01. Triangular Matrices. *(Bundelkhand 94)*

(a) Upper Triangular Matrix. A square matrix **A** whose elements $a_{ij} = 0$ for $i > j$ is called an upper triangular matrix.

For example $$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots\dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots\dots & a_{2n} \\ 0 & 0 & a_{33} & \dots\dots & a_{3n} \\ \dots & \dots & \dots & \dots\dots & \dots \\ 0 & 0 & 0 & \dots\dots & a_{nn} \end{bmatrix}$$

(b) Lower Triangular Matrix. A square matrix **A** whose elements $a_{ij} = 0$ for $i < j$ is called a lower triangular matrix.

For example $$\begin{bmatrix} a_{11} & 0 & 0 \dots\dots & 0 \\ a_{21} & a_{22} & 0 \dots\dots & 0 \\ a_{31} & a_{32} & a_{33} \dots\dots & 0 \\ \dots & \dots & \dots \dots\dots & 0 \\ a_{n1} & a_{n2} & a_{n3} \dots\dots & a_{nn} \end{bmatrix}$$

§ 2·02. Diagonal Matrix.

Definition. A square matrix which is both upper and lower triangular is called a diagonal matrix. *(Bundelkhand 94)*

For example $$\begin{bmatrix} a_{11} & 0 & 0 \dots\dots & 0 \\ 0 & a_{22} & 0 \dots\dots & 0 \\ 0 & 0 & a_{33} \dots\dots & 0 \\ \dots & \dots & \dots \dots\dots & \dots \\ 0 & 0 & 0 \dots\dots & a_{nn} \end{bmatrix}$$ (See § 1·03 Page 4 also)

Theorem I. *Any two diagonal matrices of the same order commute under multiplication.* *(Bundelkhand 95, 94)*

Proof. Let any two diagonal matrices be

$$\mathbf{A} = \begin{bmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_n \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} b_1 & 0 & 0 & \dots & 0 \\ 0 & b_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & b_n \end{bmatrix}$$

Then we have

$$\mathbf{AB} = \begin{bmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_n \end{bmatrix} \times \begin{bmatrix} b_1 & 0 & 0 & \dots & 0 \\ 0 & b_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & b_n \end{bmatrix}$$

or $$\mathbf{AB} = \begin{bmatrix} a_1b_1 & 0 & 0 & \dots & 0 \\ 0 & a_2b_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_nb_n \end{bmatrix} \quad \text{...(i)}$$

and $$\mathbf{BA} = \begin{bmatrix} b_1 & 0 & 0 & \dots & 0 \\ 0 & b_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & b_n \end{bmatrix} \times \begin{bmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_n \end{bmatrix}$$

$$= \begin{bmatrix} b_1a_1 & 0 & 0 & \dots & 0 \\ 0 & b_2a_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & b_na_n \end{bmatrix} \quad \text{...(ii)}$$

$\therefore$ From (i) and (ii), we find that $\mathbf{AB} = \mathbf{BA}$ and each one of them is a diagonal matrix of order n. **(Note)**

Hence proved.

Theorem II. *Product of any two diagonal matrices of order n is a diagonal matrix of order n.*

Proof. The same as of Theorem I above.

Theorem III. *Sum of any two diagonal matrices of order n is a diagonal matrix of order n and commute under addition.*

Proof. Let any two diagonal matrices be

$$\mathbf{A} = \begin{bmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_n \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} b_1 & 0 & 0 & \dots & 0 \\ 0 & b_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & b_n \end{bmatrix}$$

$$\therefore \quad \mathbf{A} + \mathbf{B} = \begin{bmatrix} a_1 + b_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 + b_2 & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & a_n + b_n \end{bmatrix} \quad \text{...(i)}$$

and $$\mathbf{B} + \mathbf{A} = \begin{bmatrix} b_1 + a_1 & 0 & 0 & \dots & 0 \\ 0 & b_2 + a_2 & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & b_n + a_n \end{bmatrix} \quad \text{...(ii)}$$

$\therefore$ From (i) and (ii), we get $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ and each one of them is a diagonal matrix of order n.

§ 2·03. Scalar matrix.

Definition. If in a square matrix **A** all the diagonal elements are equal to a (where $a \neq 0$) and all the remaining elements are equal to zero then it is called a scalar matrix.

For example $\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix}$ is a scalar matrix of order 4×4.

Commutative Matrices

Definition. If **A** and **B** are two square matrices such that **AB** = **BA**, then **A** and **B** are called **commutative** matrices or are said to **commute.**

If **AB** = – **BA**, the matrices **A** and **B** are said to **anti-commute.**

Solved Examples on § 2·03.

Ex. 1. If $\mathbf{A} = \begin{bmatrix} \mathbf{a} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{a} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{a} \end{bmatrix}$ **and** $\mathbf{B} = \begin{bmatrix} \mathbf{a_{11}} & \mathbf{a_{12}} & \mathbf{a_{13}} \\ \mathbf{a_{21}} & \mathbf{a_{22}} & \mathbf{a_{23}} \\ \mathbf{a_{31}} & \mathbf{a_{32}} & \mathbf{a_{33}} \end{bmatrix}$

Then prove that AB = BA = aB.

Sol. $\mathbf{AB} = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \times \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$= \begin{bmatrix} aa_{11} & aa_{12} & aa_{13} \\ aa_{21} & aa_{22} & aa_{23} \\ aa_{31} & aa_{32} & aa_{33} \end{bmatrix} = a \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$= a\mathbf{B}$.

Similarly $\mathbf{BA} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \times \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$

$$= \begin{bmatrix} aa_{11} & aa_{12} & aa_{13} \\ aa_{21} & aa_{22} & aa_{23} \\ aa_{31} & aa_{33} & aa_{33} \end{bmatrix} = a \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a\mathbf{B}$$

Hence $\mathbf{AB} = \mathbf{BA} = a\mathbf{B}$.

Ex. 2. Show that the matrices A and B anti-commute, where

$$\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-1} \\ \mathbf{2} & \mathbf{-1} \end{bmatrix} \textbf{ and } \mathbf{B} = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{4} & \mathbf{-1} \end{bmatrix}$$

Sol. Here $\mathbf{AB} = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 1 \cdot 1 - 1 \cdot 4 & 1 \cdot 1 + (-1).(-1) \\ 2 \cdot 1 - 1 \cdot 4 & 2 \cdot 1 + (-1).(-1) \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -2 & 3 \end{bmatrix} \quad \text{...(i)}$$

And $\mathbf{BA} = \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 1 \cdot 1 + 1 \cdot 2 & 1.(-1) + 1.(-1) \\ 4 \cdot 1 + (-1).2 & 4.(-1) + (-1)(-1) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & -3 \end{bmatrix}$$

$$= -\begin{bmatrix} -3 & 2 \\ -2 & 3 \end{bmatrix} \qquad \text{...(ii)}$$

$\therefore$ From (i) and (ii) we find that **AB** = – **BA**.

Hence **A** and **B** anti-commute.

Exercise on § 2·03

Ex. 1. Show that the matrices $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ anti-commute.

Ex. 2. Show that the matrices $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $\begin{bmatrix} 5 & 7 \\ 7 & 5 \end{bmatrix}$ commute.

§ 2·04. Unit Matrix or Identity Matrix.

Definition. If in a scalar matrix the diagonal element $a = 1$, then the matrix is called the unit matrix or identity matrix and is denoted by $\mathbf{I_n}$ in the case of $n \times n$ matrix.

For example $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Solved Examples on § 2·04.

***Ex. 1. If A be any $n \times n$ matrix and $\mathbf{I_n}$ is the identity matrix of order $n \times n$, then prove that A $\mathbf{I_n}$ = $\mathbf{I_n}$ A = A**

Sol. Let us suppose that

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \text{ and } \mathbf{I_n} = \begin{bmatrix} 1 & 0 & \dots 0 \\ 0 & 1 & \dots 0 \\ \dots & \dots & \dots 0 \\ 0 & 0 & \dots 1 \end{bmatrix}$$

$$\therefore \quad \mathbf{A} \bullet \mathbf{I_n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \times \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}.1 + a_{12}.0 + \dots + a_{1n}.0 & a_{11}.0 + a_{12}.1 + \dots + a_{1n}.0 & \dots & \dots & a_{11}.0 + a_{12}.0 + \dots + a_{1n}.1 \\ a_{21}.1 + a_{22}.0 + \dots + a_{2n}.0 & a_{21}.0 + a_{22}.1 + \dots + a_{2n}.0 & \dots & \dots & a_{21}.0 + a_{22}.0 + \dots + a_{2n}.1 \\ \dots & & \dots & \dots & \\ a_{n1}.1 + a_{n2}.0 + \dots + a_{nn}.0 & a_{1n}.0 + a_{n2}.1 + \dots + a_{nn}.0 & \dots & \dots & a_{n1}.0 + a_{n2}.0 + \dots + a_{nn}.1 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} = \mathbf{A}$$

Similarly we can show that $\mathbf{I_n} \bullet \mathbf{A} = \mathbf{A}$.

Hence we have $\mathbf{A} \bullet \mathbf{I_n} = \mathbf{I_n} \bullet \mathbf{A} = \mathbf{A}$.

***Ex. 2. Prove that $\mathbf{I^m} = \mathbf{I^{m-1}} = \ldots = \mathbf{I^2} = \mathbf{I}$, where m is any positive integer and $\mathbf{I_n}$ is the unit matrix of order $n \times n$.**

Sol. Let **A** be any $n \times n$ matrix and **I** be the unit matrix of order $n \times n$ *i.e.* $\mathbf{I} = \mathbf{I_n}$.

Now we know that $\mathbf{AI_n} = \mathbf{I_nA} = \mathbf{A}$ (See Ex. 1 above)

But $\mathbf{I_n} = \mathbf{I}$. ...(i)

$\therefore$ $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$

Taking $\mathbf{A} = \mathbf{I}$, we have $\mathbf{I} \bullet \mathbf{I} = \mathbf{I}$ or $\mathbf{I^2} = \mathbf{I}$ ...(ii)

Again from (i), taking $\mathbf{A} = \mathbf{I^2}$, where $\mathbf{I^2} = \mathbf{I}$ (proved), we get

$\mathbf{I^2} \bullet \mathbf{I} = \mathbf{I^2}$ or $\mathbf{I^3} = \mathbf{I^2} = \mathbf{I}$, from (ii).

Proceeding in this way, we can prove that

$\mathbf{I^m} = \mathbf{I^{m-1}} = \ldots = \mathbf{I^2} = \mathbf{I}$, where m is any positive integer.

Exercise on § 2·04

Ex. If $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -3 & 3 & 1 \end{bmatrix}$, show that $\mathbf{A^2} = \mathbf{I}$, where **I** is the unit matrix.

§ 2·05. Periodic Matrix.

Definition. A square matrix **A** is called periodic, if $\mathbf{A^{k+1}} = \mathbf{A}$, where k is a positive integer.

If k is the least positive integer for which $\mathbf{A^{k+1}} = \mathbf{A}$, then **A** is said to be of period k.

Idempotent matrix.

Definition. A square matrix **A** is called idempotent provided it satisfies the relation $\mathbf{A^2} = \mathbf{A}$.

Symmetric Idempotent Matrix.

Definition. A square matrix **A** is called symmetric idempotent if $\mathbf{A} = \mathbf{A'}$ and $\mathbf{A^2} = \mathbf{A}$, where $\mathbf{A'}$ is the transposed matrix of **A**, (See § 2·08 Page 69).

Solved Examples on § 2·05.

Ex. 1 (a) **Show that the matrix** $\mathbf{A} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ **is idempotent.**

(*Rohilkhand 96*)

Sol. $A^2 = A \bullet A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \times \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 2\cdot2-2(-1)-4\cdot1 & 2(-2)-2\cdot3-4(-2) & 2(-4)-2\cdot4-4(-3) \\ -1\cdot2+3(-1)+4\cdot1 & -1(-2)+3\cdot3+4(-2) & -1(-4)+3\cdot4+4(-3) \\ 1\cdot2-2(-1)-3\cdot1 & 1(-2)-2\cdot3-3(-2) & 1(-4)-2\cdot4-3(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

Hence the matrix **A** is idempotent.

Ex. 1 (b) Show that the matrix $A = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$ **is idempotent.** *(Avadh 91)*

Sol $A^2 = A \bullet A = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix} \times \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$

$$= \begin{bmatrix} 4+3-5 & -6-12+15 & -10-15+20 \\ -2-4+5 & 3+16-15 & 5+20-20 \\ 2+3-4 & -3-12+12 & -5-15+16 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix} = A$$

Hence the matrix **A** is idempotent.

***Ex. 2. If A and B are idempotent matrices, then show that AB is idempotent if A and B commute.**

Sol. If **A** is the idempotent, then $A^2 = A$ and if **B** is idempotent then

$$B^2 = B. \quad ...(i)$$

And if **A** and **B** commute, then $AB = BA$ ...(ii)

Now $(AB)^2 = (AB).(AB)$

$= A(BA)B$, by associative law

$= A(AB)B$, from (ii)

$= (AA)(BB)$, by associative law

$= A^2B^2$

$= AB$ by (i)

Hence **AB** is idempotent

Ex. 3. If A is an idempotent matrix, then the matrix $B = I - A$ is idempotent and $AB = O = BA$.

Sol. We know $IA = AI = A$. (See Ex. 1 Page 64) ...(i)

Also $\mathbf{A}$ being an idempotent matrix, we have $\mathbf{A}^2 = \mathbf{A}$. ...(ii)

Since $\mathbf{I}$ and $\mathbf{A}$ are square matrices, so $\mathbf{I} - \mathbf{A}$ is also a square matrix and therefore we have

$$(\mathbf{I} - \mathbf{A})^2 = (\mathbf{I} - \mathbf{A})(\mathbf{I} - \mathbf{A})$$
$$= (\mathbf{I} - \mathbf{A})\mathbf{I} - (\mathbf{I} - \mathbf{A})\mathbf{A}, \text{ by distributive law}$$
$$= \mathbf{I}^2 - \mathbf{AI} - \mathbf{IA} + \mathbf{A}^2$$
$$= \mathbf{I} - \mathbf{A} - \mathbf{A} + \mathbf{A}, \text{ from (i), (ii) and } \mathbf{I}^2 = \mathbf{I}$$

or $(\mathbf{I} - \mathbf{A})^2 = \mathbf{I} - \mathbf{A}$, *i.e.* $\mathbf{I} - \mathbf{A}$ or $\mathbf{B}$ is an idempotent matrix by definition.

Again $\mathbf{AB} = \mathbf{A}(\mathbf{I} - \mathbf{A}) = \mathbf{AI} - \mathbf{A}^2$, by distributive law

$= \mathbf{A} - \mathbf{A}$, from (i) and (ii)

i.e $\quad \mathbf{AB} = \mathbf{O}$.

And $\quad \mathbf{BA} = (\mathbf{I} - \mathbf{A})\mathbf{A} = \mathbf{IA} - \mathbf{A}^2$, by distributive law

$= \mathbf{A} - \mathbf{A} = \mathbf{O}$.

Ex. 4. Show that if A and B are matrices of order $n \times n$ and such that AB = A and BA = B, then A and B are idempotent matrices.

Sol. We have $\mathbf{ABA} = (\mathbf{AB})\mathbf{A} = (\mathbf{A})\mathbf{A}$, $\quad \because \mathbf{AB} = \mathbf{A}$ (given)

or $\quad \mathbf{ABA} = \mathbf{A}^2$ ...(i)

Also $\quad \mathbf{ABA} = \mathbf{A}(\mathbf{BA}) = \mathbf{A}(\mathbf{B})$, $\quad \because \mathbf{BA} = \mathbf{B}$ (given)

$= \mathbf{AB} = \mathbf{A}$ $\quad \because \mathbf{AB} = \mathbf{A}$ (given)

or $\quad \mathbf{ABA} = \mathbf{A}$ ...(ii)

From (i) and (ii), we have $\mathbf{A}^2 = \mathbf{A}$ *i.e.* $\mathbf{A}$ is idempotent.

In a similar manner, we can prove that

$\mathbf{BAB} = \mathbf{B}(\mathbf{AB}) = \mathbf{B}(\mathbf{A})$, $\quad \because \mathbf{AB} = \mathbf{A}$ (given)

$= \mathbf{BA} = \mathbf{B}$, $\quad \because \mathbf{BA} = \mathbf{B}$ (given)

or $\quad \mathbf{BAB} = \mathbf{B}$ ...(iii)

Also $\quad \mathbf{BAB} = (\mathbf{BA})\mathbf{B} = (\mathbf{B})\mathbf{B}$, $\quad \because \mathbf{BA} = \mathbf{B}$ (given)

or $\quad \mathbf{BAB} = \mathbf{B}^2$ ...(iv)

From (iii) and (iv), we have $\mathbf{B}^2 = \mathbf{B}$ *i.e.* $\mathbf{B}$ is idempotent. Hence proved.

Exercises on § 2·05

Ex. If $\mathbf{A}$ and $\mathbf{B}$ are idempotent, then $\mathbf{A} + \mathbf{B}$ will be idempotent if $\mathbf{AB} = \mathbf{BA} = \mathbf{O}$, where $\mathbf{O}$ is the null matrix.

[**Hint :** $(\mathbf{A} + \mathbf{B})^2 = \mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2 = \mathbf{A} + \mathbf{O} + \mathbf{O} + \mathbf{B}$]

§ 2·06. Involutory Matrix.

Definition. A square matrix $\mathbf{A}$ is called Involutory provided it satisfies the relation $\mathbf{A}^2 = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix.

For example, the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is involutory matrix, since $A^2=\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\times\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$

$$=\begin{bmatrix}1\cdot1+0\cdot0 & 1\cdot0+0.(-1)\\0\cdot1+(-1).0 & 0\cdot0+(-1).(-1)\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=I.$$

Solved Examples on § 2·06.

Ex. 1. Show that the matrix $A=\begin{bmatrix}-5 & -8 & 0\\3 & 5 & 0\\1 & 2 & -1\end{bmatrix}$ is involutory. *(Rohilkhand 91)*

Sol. $A^2=\begin{bmatrix}-5 & -8 & 0\\3 & 5 & 0\\1 & 2 & -1\end{bmatrix}\times\begin{bmatrix}-5 & -8 & 0\\3 & 5 & 0\\1 & 2 & -1\end{bmatrix}$

$$=\begin{bmatrix}(-5)(-5)+(-8).3+0\cdot1 & (-5)\cdot(-8)+(-8)\cdot5+0\cdot2 & (-5)\cdot0+(-8)\cdot0+0\cdot(-1)\\3\cdot(-5)+5\cdot3+0\cdot1 & 3\cdot(-8)+5\cdot5+0\cdot2 & 3\cdot0+5\cdot0+0\cdot(-1)\\1.(-5)+2\cdot3+(-1)\cdot1 & 1\cdot(-8)+2\cdot5+(-1)\cdot2 & 1\cdot0+2\cdot0+(-1)(-1)\end{bmatrix}$$

$$=\begin{bmatrix}25-24+0 & 40-40+0 & 0+0+0\\-15+15+0 & -24+25+0 & 0+0+0\\-5+6-1 & -8+10-2 & 0+0+1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}=I$$

Hence the given matrix A is involutory.

Ex. 2. If A is any square matrix of order n and I_n is the identity matrix of order n, such that $(I_n-A)(I_n+A)=O$, then show that A is involutory matrix.

Sol. Given that $(I_n-A)(I_n+A)=O$

or $\quad I_n^2+I_n\bullet A-A\bullet I_n-A^2=O$

or $\quad I_n+A-A-A^2=O,\qquad \because\ I_n^2=I_n,\ I_n\bullet A=A=A\bullet I_n.$

(See Ex. 1. Page 64)

or $\quad I_n-A^2=O$ or $A^2=I_n$ *i.e.* A is involutory by definition.

§ 2·07. Nilpotent Matrix. *(Avadh 93)*

Definition. A square matrix A is called Nilpotent matrix of order m, provided it satisfies the relation $A^m=O$ and $A^{m-1}\neq O$, where m is a positive integer and O is the null matrix.

For example, the matrix $A=\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$ is a nilpotent matrix,

since $\quad A=\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\neq O,$

$$A^2=\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\times\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}=\begin{bmatrix}0\cdot0+1\cdot0 & 0\cdot1+1\cdot0\\0\cdot0+0\cdot0 & 0\cdot1+0\cdot0\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}=\mathbf{O},$$

$$\mathbf{A}^3=\mathbf{A}^2\bullet\mathbf{A}=\mathbf{O}\bullet\mathbf{A}=\mathbf{O}.$$

i.e. **A** is a matrix which is not itself a zero matrix though its powers are zero matrices and so it is a nilpotent matrix (**Another definition of nilpotent matrix**).

Solved Examples on § 2·07.

Ex. Show that $\mathbf{A}=\begin{bmatrix}1 & 2 & 3\\1 & 2 & 3\\-1 & -2 & -3\end{bmatrix}$ **is a nilpotent matrix of order 2.**

Sol. Given $\mathbf{A}=\begin{bmatrix}1 & 2 & 3\\1 & 2 & 3\\-1 & -2 & -3\end{bmatrix}\neq\mathbf{O}$

$$\therefore\quad \mathbf{A}^2=\begin{bmatrix}1 & 2 & 3\\1 & 2 & 3\\-1 & -2 & -3\end{bmatrix}\times\begin{bmatrix}1 & 2 & 3\\1 & 2 & 3\\-1 & -2 & -3\end{bmatrix}$$

$$=\begin{bmatrix}1\cdot1+2\cdot1+3(-1) & 1\cdot2+2\cdot2+3(-2) & 1\cdot3+2\cdot3+3(-3)\\1\cdot1+2\cdot1+3(-1) & 1\cdot2+2\cdot2+3(-2) & 1\cdot3+2\cdot3+3(-3)\\-1\cdot1-2\cdot1-3(-1) & -1\cdot2-2\cdot2-3(-2) & -1\cdot3-2\cdot3-3(-3)\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}=\mathbf{O},$$ where **O** is the null matrix of order 3.

i.e. $\mathbf{A}^2=\mathbf{O}$ but $\mathbf{A}\neq\mathbf{O}$. Hence **A** is a nilpotent matrix of order 2.

Exercises on § 2·07

Ex. 1. Show that the matrix $\begin{bmatrix}ab & b^2\\-a^2 & -ab\end{bmatrix}$ is nilpotent.

Ex. 2. Show that $\begin{bmatrix}1 & 1 & 3\\5 & 2 & 6\\-2 & -1 & -3\end{bmatrix}$ is a nilpotent matrix of order 3. *(Avadh 93, 90)*

[**Hint** : Prove that $\mathbf{A}^3=\mathbf{O},\ \mathbf{A}^2\neq\mathbf{O}$]. *(Agra 94)*

**§ 2·08. Transposed Matrix.

Definition. The matrix of order $n\times m$ obtained by interchanging the rows and columns of a matrix **A** of order $m\times n$ is called the *transposed matrix of* **A** or *transpose of the matrix* **A** and is denoted by $\mathbf{A}'$ or $\mathbf{A}^t$ (read as **A transpose**).

Another Definition. If $\mathbf{A}=[a_{ij}]$ be a matrix of order $m\times n$, **then the** matrix $\mathbf{B}=[b_{ij}]$ of order $n\times m$, such that $b_{ij}=a_{ji}$ is known as *transposed matrix of* **A** *or the transpose of the matrix* **A** and is denoted by $\mathbf{A}'$ or $\mathbf{A}^t$.

For example : If $\mathbf{A}=\begin{bmatrix}1 & 3 & 5\\ 2 & 4 & 6\end{bmatrix}$ then $\mathbf{A}'=\begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix}$

Note 1. The element a_{ij} in the ith row and jth column of **A** stands in jth row and ith column of **A′**.

Note 2. The transpose of an $m \times n$ matrix is an $n \times m$ matrix.

***§ 2·09. Some Important Theorems on Transposed Matrices.**

Theorem I. *The transpose of the sum of two matrices is the sum of their transpose i.e.* $(\mathbf{A}+\mathbf{B})'=\mathbf{A}'+\mathbf{B}'$.

Proof. Let $\mathbf{A}=[a_{ij}]$ and $\mathbf{B}=[b_{ij}]$.

Then $\mathbf{A}+\mathbf{B}=[a_{ij}+b_{ij}], =[c_{ij}]$, say

then $c_{ij}=a_{ij}+b_{ij}$

$\therefore\ (\mathbf{A}+\mathbf{B})'=[d_{ij}]$, where $d_{ij}=c_{ij}$ for all $1 \le i \le m,\ 1 \le j \le n$

i.e. $d_{ij}=a_{ij}+b_{ij}$, for all $1 \le i \le m,\ 1 \le j \le n$

or $(\mathbf{A}+\mathbf{B})'=[c_{ij}]=[a_{ij}+b_{ij}]$ ...(i)

Also $\mathbf{A}'=[f_{ji}]$, where $f_{ji}=a_{ij}$, for all $1 \le i \le m,\ 1 \le j \le n$

and $\mathbf{B}'=[g_{ji}]$, where $g_{ji}=b_{ij}$ for all $1 \le i \le m,\ 1 \le j \le n$

$\therefore\ \mathbf{A}'+\mathbf{B}'=[f_{ji}]+[g_{ji}]=[f_{ji}+g_{ji}]$

$=[a_{ij}+b_{ij}]$ ...(ii)

$\therefore$ From (i) and (ii) we get $(\mathbf{A}+\mathbf{B})'=\mathbf{A}'+\mathbf{B}'$

***Theorem II.** *The transpose of the transpose of a matrix is the matrix itself i.e.* $(\mathbf{A}')'=\mathbf{A}$. *(Meerut 95, 94)*

Proof. Let $\mathbf{A}=[a_{ij}]$ be an $m \times n$ matrix. Then $\mathbf{A}'$ *i.e.* the tranpose of **A** is $n \times m$ matrix and $(\mathbf{A}')'$ *i.e.* the transpose of $\mathbf{A}'$ (or the transpose of **A**) is an $m \times n$ matrix.

Therefore the matrices **A** and $(\mathbf{A}')$ are both $m \times n$ matrices and hence comparable. ...(i)

Also, the element in the ith row and jth column of $(\mathbf{A}')'$.

= the element in the jth row and ith column of $\mathbf{A}'$

= the element in the ith row and jth column of **A**

i.e. the corresponding elements of $(\mathbf{A}')'$ and **A** are equal ...(ii)

$\therefore$ From (i) and (ii), we conclude that $(\mathbf{A}')'=\mathbf{A}$. Hence proved.

Theorem III. *If* **A** *is any* $m \times n$ *matrix, then* $(k\mathbf{A})'=k\mathbf{A}'$, *where k is any number.*

Proof. Let $\mathbf{A}=[a_{ij}]$ be any $m \times n$ matrix. Then $k\mathbf{A}$ is also $m \times n$ matrix and therefore $(k\mathbf{A})'$ *i.e.* the transpose of the matrix $k\mathbf{A}$ is an $n \times m$ matrix.

Also $\mathbf{A}'$, the transpose of the matrix **A**, is $n \times m$ matrix and $k\mathbf{A}'$ is also an $n \times m$ matrix.

Thus we find that the matrices $(k\mathbf{A})'$ and $k\mathbf{A}'$ are both $n \times m$ matrices and hence comparable. ...(i)

Again the element in ith row and jth column of $(k\mathbf{A})'$

= the element in jth row and ith column of $k\mathbf{A}$
= k times the element in jth row and ith column of **A** (Note)
= k times the element in ith row and jth column of **A′** (Note)
= ka_{ij} (Note)
= the element in ith row and jth column of $k\mathbf{A}'$
i.e. the corresponding elements of $(k\mathbf{A})'$ and $k\mathbf{A}'$ are equal ...(ii)
From (i) and (ii), we conclude that $(k\mathbf{A})' = k\mathbf{A}'$. Hence proved.

****Theorem IV.** *The transpose of the product of two matrices is the product in reverse order of their transpose i.e.* $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$.

(*Garhwal 95, 93; Gorakhpur 96, Rohilkhand 94*)

Proof. Let $\mathbf{A} = [a_{ik}]$ and $\mathbf{B} = [b_{kj}]$ be the two matrices of orders $m \times n$ and $n \times p$ respectively.

Let $\mathbf{C} = \mathbf{AB} = [g_{ik}] \times [b_{kj}] = [c_{ij}]$, say

where **C** is a matrix of order $m \times p$.

$\therefore$ The element in the ith row and jth column of **AB** is $c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$.

This is also the element in the ith row and jth column of $(\mathbf{AB})'$. ...(i)

The elements in the jth row of $\mathbf{B}'$ are $b_{1j}, b_{2j}, b_{3j}, ..., b_{nj}$ and elements in the ith column of $\mathbf{A}'$ are $a_{i1}, a_{i2}, a_{i3}, ..., a_{in}$. Then the element in the jth row and ith column of $\mathbf{B}'\mathbf{A}'$ is

$$\sum_{k=1}^{n} b_{kj}\, a_{ik} = \sum_{k=1}^{n} a_{ik}\, b_{kj} = c_{ij} \qquad ...(ii)$$

Hence from (i) and (ii) we conclude that $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$.

Note. The statement of theorem IV is called the *reversal rule for the transpose of a product.*

Solved Examples on § 2·08 to § 2·09.

Ex. 1. Write down the transpose of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \end{bmatrix}$

Sol. Let **A′** be the required transpose of the matrix **A**. Then **A′** = matrix obtained by interchanging the rows and columns of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 6 \\ 2 & 8 \\ 4 & 1 \end{bmatrix}$. **Ans.**

Ex. 2. Verify that $(\mathbf{B})^t (\mathbf{A})^t = (\mathbf{AB})^t$, **when**

(a) $\mathbf{A} = [2 \quad 1]$, $\mathbf{B} = \begin{bmatrix} 1 & -2 & 0 \\ 4 & 5 & -3 \end{bmatrix}$ (*Budenkhand 91*)

(b) $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}$

(c) $\mathbf{A}=\begin{bmatrix} 2 & 4 & -1 \\ -1 & 0 & 2 \end{bmatrix}$, $\mathbf{B}=\begin{bmatrix} 3 & 4 & 5 \\ -1 & 2 & 7 \\ 2 & 1 & 0 \end{bmatrix}$ *(Avadh 92)*

Sol. (a) Here $\mathbf{A}^t=\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{B}^t=\begin{bmatrix} 1 & 4 \\ -2 & 5 \\ 0 & -3 \end{bmatrix}$

$$\therefore \quad \mathbf{B}^t\mathbf{A}^t=\begin{bmatrix} 1 & 4 \\ -2 & 5 \\ 0 & -3 \end{bmatrix}\times\begin{bmatrix} 2 \\ 1 \end{bmatrix}=\begin{bmatrix} 1\cdot2+4\cdot1 \\ -2\cdot2+5\cdot1 \\ 0\cdot2-3\cdot1 \end{bmatrix}=\begin{bmatrix} 6 \\ 1 \\ -3 \end{bmatrix} \quad \text{...(i)}$$

Also $\mathbf{AB}=[2 \quad 1]\times\begin{bmatrix} 1 & -2 & 0 \\ 4 & 5 & -3 \end{bmatrix}$

$$=[2\cdot1+1\cdot4 \quad 2(-2)+1\cdot5 \quad 2\cdot0+1(-3)]$$

$$=[6 \quad 1 \quad -3].$$

$\therefore$ $(\mathbf{AB})^t$ = transposed matrix of **AB**

$$=\begin{bmatrix} 6 \\ 1 \\ -3 \end{bmatrix}=\mathbf{B}^t\mathbf{A}^t, \text{ from (i)}$$

Hence proved.

(b) Here $\mathbf{A}^t=\begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 3 & 1 \end{bmatrix}$, $\mathbf{B}^t=\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix}$

$$\therefore \quad \mathbf{B}^t\mathbf{A}^t=\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix}\times\begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 3 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 1\cdot1+2\cdot2-1\cdot3 & 1\cdot3+2(-2)-1\cdot1 \\ 2\cdot1+0\cdot2+1\cdot3 & 2\cdot3+0(-2)+1\cdot1 \end{bmatrix}$$

$$=\begin{bmatrix} 2 & -2 \\ 5 & 7 \end{bmatrix} \quad \text{...(ii)}$$

Also $\mathbf{AB}=\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \end{bmatrix}\times\begin{bmatrix} 1 & 2 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}$

$$=\begin{bmatrix} 1\cdot1+2\cdot2+3(-1) & 1\cdot2+2\cdot0+3\cdot1 \\ 3\cdot1-2\cdot2+1(-1) & 3\cdot2-2\cdot0+1\cdot1 \end{bmatrix}=\begin{bmatrix} 2 & 5 \\ -2 & 7 \end{bmatrix}$$

$\therefore$ $(\mathbf{AB})^t$ = transposed matrix of **AB**

$$=\begin{bmatrix} 2 & -2 \\ 5 & 7 \end{bmatrix}=\mathbf{B}^t\mathbf{A}^t, \text{ from (ii)},$$

Hence proved.

(c) Here $A^t = \begin{bmatrix} 2 & -1 \\ 4 & 0 \\ -1 & 2 \end{bmatrix}$ and $B^t = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 1 \\ 5 & 7 & 0 \end{bmatrix}$

$\therefore \quad B^t A^t = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 1 \\ 5 & 7 & 0 \end{bmatrix} \times \begin{bmatrix} 2 & -1 \\ 4 & 0 \\ -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 3\cdot2 - 1\cdot4 + 2(-1) & 3(-1) - 1\cdot0 + 2\cdot2 \\ 4\cdot2 + 2\cdot4 + 1(-1) & 4(-1) + 2\cdot0 + 1\cdot2 \\ 5\cdot2 + 7\cdot4 + 0(-1) & 5(-1) + 7\cdot0 + 0\cdot2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 15 & -2 \\ 38 & -5 \end{bmatrix}$ ...(iii)

Also $AB = \begin{bmatrix} 2 & 4 & -1 \\ -1 & 0 & 2 \end{bmatrix} \times \begin{bmatrix} 3 & 4 & 5 \\ -1 & 2 & 7 \\ 2 & 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 2\cdot3 + 4(-1) - 1\cdot2 & 2\cdot4 + 4\cdot2 - 1\cdot1 & 2\cdot5 + 4\cdot7 - 1\cdot0 \\ -1\cdot3 + 0(-1) + 2\cdot2 & -1\cdot4 + 0\cdot2 + 2\cdot1 & -1\cdot5 + 0\cdot7 + 2\cdot0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 15 & 38 \\ 1 & -2 & -5 \end{bmatrix}$

$\therefore \quad (AB)^t$ = transposed matrix of **AB**

$= \begin{bmatrix} 0 & 1 \\ 15 & -2 \\ 38 & -5 \end{bmatrix} = B^t A^t$, from (iii).

Hence proved.

***Ex. 3. If $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 & 0 \\ 2 & -3 & 1 \\ 1 & 1 & -1 \end{bmatrix}$**

then verify $(AB)^t = B^t A^t$. *(Meerut 93, 91)*

Sol. $AB = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \\ 4 & 1 & 8 \end{bmatrix} \times \begin{bmatrix} 4 & 1 & 0 \\ 2 & -3 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$= \begin{bmatrix} 1\cdot4 - 1\cdot2 + 0\cdot1 & 1\cdot1 + 1\cdot3 + 0\cdot1 & 1\cdot0 - 1\cdot1 - 0\cdot1 \\ 2\cdot4 + 1\cdot2 + 3\cdot1 & 2\cdot1 - 1\cdot3 + 3\cdot1 & 2\cdot0 + 1\cdot1 - 3\cdot1 \\ 4\cdot4 + 1\cdot2 + 8\cdot1 & 4\cdot1 - 1\cdot3 + 8\cdot1 & 4\cdot0 + 1\cdot1 - 8\cdot1 \end{bmatrix}$

or $\quad AB = \begin{bmatrix} 2 & 4 & -1 \\ 13 & 2 & -2 \\ 26 & 9 & -7 \end{bmatrix}$ and so $(AB)^t = \begin{bmatrix} 2 & 13 & 26 \\ 4 & 2 & 9 \\ -1 & -2 & -7 \end{bmatrix}$ ...(i)

Again $A^t = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 1 & 1 \\ 0 & 3 & 8 \end{bmatrix}$ and $B^t = \begin{bmatrix} 4 & 2 & 1 \\ 1 & -3 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

$$\therefore \quad B^tA^t = \begin{bmatrix} 4 & 2 & 1 \\ 1 & -3 & 1 \\ 0 & 1 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 4 \\ -1 & 1 & 1 \\ 0 & 3 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 4\cdot1 - 2\cdot1 + 1\cdot0 & 4\cdot2 + 2\cdot1 + 1\cdot3 & 4\cdot4 + 2\cdot1 + 1\cdot8 \\ 1\cdot1 + 3\cdot1 + 1\cdot0 & 1\cdot2 - 3\cdot1 + 1\cdot3 & 1\cdot4 - 3\cdot1 + 1\cdot8 \\ 0\cdot1 - 1\cdot1 - 1\cdot0 & 0\cdot2 + 1\cdot1 - 1\cdot3 & 0\cdot4 + 1\cdot1 - 1\cdot8 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 13 & 26 \\ 4 & 2 & 9 \\ -1 & -2 & -7 \end{bmatrix} = (AB)^t, \text{ from (i)}$$

Hence proved.

Ex. 4. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, verify that $AA' = I_2 = A'A$.

Sol. Here $A' = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$$\therefore \quad AA' = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \times \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & -\cos\alpha\sin\alpha + \sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha + \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

Similarlly we can prove that

$$A'A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \times \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & \cos\alpha\sin\alpha - \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha - \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

Hence $AA' = I_2 = A'A$.

Exercises on § 2·08 – 2·09

Ex. 1. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$, verify that $(AB)' = B'A'$, where A', B' are transposes of A and B.

Ex. 2. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 & 0 \\ 2 & -3 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

then verify that $(AB)' = B'A'$.

Ex. 3. If $\mathbf{A}=\begin{bmatrix} 2 & 4 & -1 \\ -1 & 0 & 2 \end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix} 3 & 4 & 5 \\ -1 & 2 & 7 \\ 2 & 1 & 0 \end{bmatrix}$

prove that $(\mathbf{AB})'$ and $\mathbf{B'A'}$ are equal.

Ex. 4. If $\mathbf{A}=\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$, $\mathbf{B}=\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$, then verify that $[\mathbf{AB}]^t=\mathbf{B}^t\mathbf{A}^t$

Ex. 5. If $\mathbf{A}=\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 2 & 4 & 9 \end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix} 0 & 1 & -1 \\ -3 & 2 & 4 \\ 1 & 1 & 0 \end{bmatrix}$

then verify that $(\mathbf{AB})^t=\mathbf{B}^t\mathbf{A}^t$.

*§ 2·10. Complex conjugate (or conjugate) of a Matrix.

Definition. The matrix obtained from any given matrix **A** of order $m \times n$ with complex elements a_{ij} by replacing its elements by the corresponding conjugate complex numbers is called the complex conjugate or conjugate of **A** denoted by $\overline{\mathbf{A}}$ and is read as '**A** conjugate.'

or If $\mathbf{A}=[a_{ij}]$ and $\overline{a_{ij}}$ is the complex conjugate of the element a_{ij} then $\overline{\mathbf{A}}=[\overline{a_{ij}}]$, for all $1 \le i \le m$, $1 \le j \le n$.

For example : If $\mathbf{A}=\begin{bmatrix} 1+i & 2+3i \\ 2 & 3i \end{bmatrix}$,

then $$\overline{\mathbf{A}}=\begin{bmatrix} 1-i & 2-3i \\ 2 & -3i \end{bmatrix}$$

Real Matrix. *(Avadh 93)*

Definition. A matrix **A** is called real provided it satisfies the relation

$$\mathbf{A}=\overline{\mathbf{A}}$$

Imaginary Matrix. *(Avadh 93)*

Definition. A matrix **A** is called imaginary provided it satisfies the relation $\mathbf{A}=-\overline{\mathbf{A}}$

**§ 2·11. Theorems on complex conjugate of a matrix.

Theorem I. *If* $\mathbf{A}=[a_{ij}]$ *be any* $m \times n$ *matrix with complex elements* a_{ij}, *then the complex conjugate of* $\overline{\mathbf{A}}$ *is the matrix* **A** *itself.*

Proof : By definition (given in § 2·10 above) we know that $\overline{\mathbf{A}}=[\overline{a_{ij}}]$, for all $1 \le i \le m$, $1 \le j \le n$ and $\overline{a_{ij}}$ is the complex conjugate of a_{ij}.

i.e. the element in the *i*th row and *j*th column of complex conjugate of **A** *i.e.* $\overline{\mathbf{A}}$.

= the complex conjugate of element in *i*th row and *j*th column of **A**.

∴ The element in the *i*th row and *j*th column of the complex conjugate of $\overline{\mathbf{A}}$ *i.e.* $\overline{\overline{\mathbf{A}}}$.

= the complex conjugate of the element in *i*th row and *j*th column of $\overline{\mathbf{A}}$

= the complex conjugate of $\overline{a_{ij}}$ **(Note)**

$= a_{ij}$ *i.e.* the element in the *i*th row and *j*th column of **A**. **(Note)**

i.e. the corresponding elements of **A** and the complex conjugate of $\overline{\mathbf{A}}$ are equal. ...(i)

Also it is evident that **A**, $\overline{\mathbf{A}}$ and its conmplex conjugate are $m \times n$ matrices and hence comparable. ...(ii)

$\therefore$ From (i) and (ii), we conclude that the complex conjugate of $\overline{\mathbf{A}}$ is equal to **A** or $\overline{\overline{\mathbf{A}}} = \mathbf{A}$.

Theorem II. *If* $\mathbf{A} = [a_{ij}]$ *be any* $m \times n$ *matrix with complex elements* a_{ij}, *then* $\overline{\lambda \mathbf{A}} = \bar{\lambda}\, \overline{\mathbf{A}}$.

Proof : By definition, we know

$\overline{\mathbf{A}} = [\bar{a}_{ij}]$, for all $1 \le i \le m$, $1 \le j \le n$ and $\bar{a}_{ij}$ is the complex conjugate of a_{ij}.

Also $\lambda \mathbf{A} = [\lambda a_{ij}]$, for all $1 \le i \le m$, $1 \le j \le n$.

$\therefore$ $\overline{\lambda \mathbf{A}} = [\overline{\lambda a_{ij}}] = [\bar{\lambda}\, \bar{a}_{ij}]$, for all $1 \le i \le m$, $1 \le i \le n$ ...(i)

and we know that $\overline{z_1 z_2} = \overline{z_1} . \overline{z_2}$, where z_1, z_2 are any two complex numbers,

Again $\bar{\lambda}\, \overline{\mathbf{A}} = [b_{ij}]$, where $b_{ij} = \overline{\lambda\, a_{ij}}$ for all $1 \le i \le m$, $1 \le j \le n$

$= [\bar{\lambda}\, \overline{a_{ij}}]$, for all $1 \le i \le m$, $1 \le j \le n$. ...(ii)

$\therefore$ From (i) and (ii) we conclude that the corresponding elements of $\overline{\lambda \mathbf{A}}$ and $\bar{\lambda}\, \overline{\mathbf{A}}$ are equal. Also it is evident that $\overline{\lambda \mathbf{A}}$ and $\bar{\lambda}\, \overline{\mathbf{A}}$ are matrices of the same order. Hence we conclude that $\overline{\lambda \mathbf{A}} = \bar{\lambda}\, \overline{\mathbf{A}}$.

Theorem III. *If* **A** *and* **B** *are two matrices conformable to addition, then*

$$\overline{\mathbf{A} + \mathbf{B}} = \overline{\mathbf{A}} + \overline{\mathbf{B}}.$$

Proof : Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be any two matrices of order $m \times n$. Then as these matrices are given as conformable to addition, so we have $\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$, for all $1 \le i \le m$, $1 \le j \le n$. ...(i)

Also $\overline{\mathbf{A}} = [\bar{a}_{ij}]$ and $\overline{\mathbf{B}} = [\bar{b}_{ij}]$, by definition.

$\therefore$ $\overline{\mathbf{A}} + \overline{\mathbf{B}} = [\bar{a}_{ij} + \bar{b}_{ij}] = [\overline{a_{ij} + b_{ij}}]$, ...(ii)

for all $1 \le i \le m$, $1 \le j \le n$

and also as $\overline{z_1} + \overline{z_2} = \overline{z_1 + z_2}$, where z_1, z_2 are any two complex numbers.

Again from (i), we have

$\overline{\mathbf{A} + \mathbf{B}}$ = complex conjugate of $[a_{ij} + b_{ij}]$

= complex conjugate of $[c_{ij}]$, where $c_{ij} = a_{ij} + b_{ij}$

$= [\overline{c_{ij}}] = [\overline{a_{ij} + b_{ij}}]$, for all $1 \le i \le m$, $1 \le j \le n$

i.e. $\overline{\mathbf{A} + \mathbf{B}} = [\overline{a_{ij} + b_{ij}}]$, for all $1 \le i \le m$, $1 \le j \le n$ ...(iii)

$\therefore$ from (ii) and (iii) we conclude that the corresponding elements of $\overline{A} + \overline{B}$ and $\overline{A + B}$ are equal. Also it is evident that both $\overline{A} + \overline{B}$ and $\overline{A + B}$ are matrices of order $m \times n$ as **A** and **B** are given as conformable to addition. Hence we conclude that $\overline{A + B} = \overline{A} + \overline{B}$

Theorem IV. *If* $A = [a_{ij}]$ *be any* $m \times n$ *matrix and* $B = [b_{jk}]$ *be any* $n \times p$ *matrix i.e. if* **A** *and* **B** *are conformable to the product* **AB** *then* $\overline{AB} = \overline{A}\,\overline{B}$.

Proof : Since **A** and **B** are conformable to the product **AB**, so $AB = [a_{ij}] \times [b_{jk}] = [c_{ik}]$, where $c_{ik} = a_{ij}\, b_{jk}$, for all $1 \le i \le m$, $1 \le k \le p$ and there is summation on j, where $j = 1, 2, 3, \ldots\ldots, n$.

Also $\overline{A} = [\overline{a}_{ij}]$, for all $1 \le i \le m$, $1 \le j \le n$

and $\qquad \overline{B} = [\overline{b}_{jk}]$, for $1 \le j \le n$, $1 \le k \le p$

$\overline{A}\,\overline{B}$ is defined and we have $\overline{A}\,\overline{B} = [\overline{a}_{ij}] \times [\overline{b}_{jk}] = [\overline{d}_{ik}]$, ...(i)

where $\overline{d}_{ik} = \overline{a}_{ij}\, \overline{b}_{jk}$ for all $1 \le i \le m$, $1 \le k \le p$ and $j = 1, 2, \ldots\ldots, n$.

Again $\overline{AB}$ = complex conjugate of **AB** *i.e.* $[c_{ik}]$

or $\qquad \overline{AB} = [\overline{c}_{ik}]$, where $c_{ik} = a_{ij}\, b_{jk}$

$= [\overline{a_{ij}\, b_{jk}}] = [\overline{a}_{ij}\, \overline{b}_{jk}]$, $\because \overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$,

for any complex numbers z_1 and z_2

$= [\overline{d}_{ik}]$, since $\overline{d}_{ik} = \overline{a}_{ij}\, \overline{b}_{jk}$ for all $1 \le i \le m$,

$1 \le k \le p$ and $j = 1, 2, \ldots, n$ ...(ii)

From (i) and (ii), we conclude the $\overline{AB} = \overline{A}\,\overline{B}$.

§ 2·12. Transposed Conjugate of a Matrix.

Definition. The transpose of conjugage of a matrix **A** *i.e.* $(\overline{A})'$ is defined as transposed conjugage or tranjugate **A** and is denoted by A^{Θ} *i.e.* $A^{\Theta} = (\overline{A})'$.

For example : If $A = \begin{bmatrix} 1+i & 2+3i \\ 2 & 3i \end{bmatrix}$,

then $$\overline{A} = \begin{bmatrix} 1-i & 2-3i \\ 2 & -3i \end{bmatrix}$$

$$\therefore \quad A^{\Theta} = \text{transpose of } \overline{A} = (\overline{A})'$$
$$= \begin{bmatrix} 1-i & 2 \\ 2-3i & -3i \end{bmatrix}$$

***§ 2·13. Theorems on Transposed conjugate of a matrix.**

Theorem I. *For any matrix* **A**, $(\overline{A})' = (\overline{A'})$

i.e. the transposed conjugate of a matrix is equal to conjugate of its transpose.

Proof : Let $A = [a_{ij}]$ be any $m \times n$ matrix.

Then by definition, $\overline{A} = [\overline{a}_{ij}]$, for all $1 \le i \le m$ and $1 \le j \le n$.

$\therefore (\overline{A})'$ = transpose of $\overline{A}$,

i.e. $\qquad (\overline{A})' = [b_{ji}]$, where $[b_{ji}]$ is $n \times m$ matrix and $b_{ji} = \overline{a}_{ij}$

for all $1 \le i \le m$, $1 \le j \le n$, ...(i)

Again A' = transpose of **A** *i.e.* $[a_{ij}]$

$= [c_{ji}]$, where $c_{ji} = a_{ij}$ and $[c_{ji}]$ is $n \times m$ matrix for all $1 \le i \le m,\ 1 \le j \le n$.

$\therefore \quad (\bar{A})' =$ complex conjugate of A'.

$= \overline{[c_{ji}]}$, by definition.

$= \overline{[a_{ij}]}$, since $c_{ji} = a_{ij}$

or $\quad (\overline{A})' = [b_{ji}]$, since $b_{ji} = \overline{a_{ij}}$, where $[b_{ji}]$ is $n \times m$ matrix for all $1 \le i \le m,\ 1 \le j \le n$ ...(ii)

$\therefore$ From (i) and (ii), we conclude that $(\overline{A})' = (\overline{A'})$.

Theorem II. *For any matrix* $A, (A^{\Theta})^{\Theta} = A$.

Proof : Let $A^{\Theta} = B$ *i.e.* $B = (\overline{A})'$

Then $B' =$ transpose of B

$=$ transpose of $(\overline{A})'$

$= \overline{A}$, since we know $(A')' = A$ ...See Th. II Page 70

$\therefore \quad (\overline{B})' =$ complex conjugate of B' ...See Th. I above

$=$ complex conjugate of $\overline{A}$ **(Note)**

$= A$, since we know $\overline{\overline{A}} = A$ ...See Th. I Page 75

i.e. $\quad B^{\Theta} = A$, since $B^{\Theta} = (\overline{B'}) = (\overline{B})'$ ...See Th. I above

i.e. $\quad (A^{\Theta})^{\Theta} = A$, since $A^{\Theta} = B$. Hence proved.

Theorem III. (a). *For any matrix* $A, (kA)^{\Theta} = kA^{\Theta}$, *where k is a scalar.*

Proof : By definition, we know that

$(kA)^{\Theta} = (\overline{kA})'$

$= (\bar{k}\,\overline{A})'$, by Th. II Page 76

$= \bar{k}\,(\overline{A})'$, by Th. III Page 70

$= kA^{\Theta}$, since k is a scalar. Hence proved.

Theorem III (b). *For any matrix* $A, (k\,A)^{\Theta} = \bar{k}\,A^{\Theta}$, *where k is any complex number.*

Proof : By definition, we know that

$(kA)^{\Theta} = (\overline{kA})'$

$= (\bar{k}\,\overline{A})'$, by Th. II Page 76

$= (\bar{k}\,\overline{A})'$, by Th. III Page 70

$= \bar{k}\,A^{\Theta}$, $\because \quad (\overline{A})' = A^{\Theta}$, by definition. Hence proved.

Theorem IV. *If* A *and* B *are two matrices conformable to addition, then*

$$(A + B)^{\Theta} = A^{\Theta} + B^{\Theta}.$$ *(Meerut 90)*

Proof : By definition, we have

$(A + B)^{\Theta} = (\overline{A + B})' = (\overline{A} + \overline{B})', \quad \because \quad \overline{A + B} = \overline{A} + \overline{B}$ ...See Th. III Page 76

$= (\overline{A})' + (\overline{B})'$ by Th. I Page 70

$= A^{\Theta} + B^{\Theta}$, by definition. Hence proved.

****Theorem V.** *If* **A** *and* **B** *are two matrices conformable to the product* **AB**, *then* $(\mathbf{AB})^{\Theta} = \mathbf{B}^{\Theta}\mathbf{A}^{\Theta}$

Proof : $(\mathbf{AB})^{\Theta} = (\overline{\mathbf{AB}})'$, by definition
$= (\overline{\mathbf{A}}\,\overline{\mathbf{B}})'$, by Th. IV Page 77
$= (\overline{\mathbf{B}})'(\overline{\mathbf{A}})'$, by Th. IV Page 71 **(Note)**
$= \mathbf{B}^{\Theta}\mathbf{A}^{\Theta}$, by definition. Hence proved

Example : Find $\{\mathbf{A}^{\Theta}\}^{\Theta}$, $\overline{\mathbf{A}}'$ **and** $(\overline{\mathbf{A}})'$ **for the matrix**

$$\mathbf{A} = \begin{bmatrix} 1+i & 3-5i \\ 2i & 5 \end{bmatrix}$$

Solution. $\overline{\mathbf{A}} = \begin{bmatrix} 1-i & 3+5i \\ -2i & 5 \end{bmatrix}$ ... See § 2·10 Page 75

$(\overline{\mathbf{A}})' = \text{Transpose of } \overline{\mathbf{A}} = \begin{bmatrix} 1-i & -2i \\ 3+5i & 5 \end{bmatrix}$ ... See § 2·08 Page 69

$\mathbf{A}' = \text{Transpose of } \mathbf{A} = \begin{bmatrix} 1+i & 2i \\ 3-5i & 5 \end{bmatrix}$ ...See § 2·08 Page 69

$\overline{\mathbf{A}}' = \text{conjugate of } \mathbf{A}' = \begin{bmatrix} 1-i & -2i \\ 3+5i & 5 \end{bmatrix} = (\overline{\mathbf{A}})'$

$\mathbf{A}^{\Theta} = \text{conjugate transpose of } \mathbf{A} = \begin{bmatrix} 1-i & -2i \\ 3+5i & 5 \end{bmatrix} = \overline{\mathbf{A}}'$

and $\{\mathbf{A}^{\Theta}\}^{\Theta} = \text{conjugate transpose of } \mathbf{A}^{\Theta}$

$$= \begin{bmatrix} 1+i & 3-5i \\ 2i & 5 \end{bmatrix} = \mathbf{A}$$

****§ 2·14. Symmetric and skew-symmetric matrices.**

(a) Symmetric Matrix. *(Agra 94; Avadh 92)*

Definition. A square matrix $\mathbf{A} = [a_{ij}]$ is called symmetric provided $a_{ij} = a_{ji}$, for all values of i and j.

For example : $\mathbf{A} = \begin{bmatrix} 1 & -3 & 5 \\ -3 & 2 & 7 \\ 5 & 7 & 3 \end{bmatrix}$

Note. $k\mathbf{A}$ is also symmetric, if k is scalar.

(b) Skew-symmetric Matrix. *(Agra 94; Avadh 92)*

Definition. A square matrix $\mathbf{A} = [a_{ij}]$ is called skew-symmetric provided $a_{ij} = -a_{ji}$, for all values of i and j.

For example : $\mathbf{A} = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & -5 & 0 \end{bmatrix}$

Note. k**A** is also skew-symmetric, if k is scalar.

§ 2·15. Theorems on Symmetric and Skew-symmetric matrices.

Theorem I. *A square matrix* **A** *is symmetric iff* $\mathbf{A} = \mathbf{A}'$. (*Kanpur 90*)

Proof : Let **A** be an $n \times n$ square matrix *i.e.* $\mathbf{A} = [a_{ij}]$, for all $1 \le i \le n$ and $1 \le j \le n$.

If **A** is symmetric matrix, then by definition, we have

$$[a_{ij}] = [a_{ji}], \text{ for all } 1 \le i \le n \text{ and } 1 \le j \le n \qquad ...(i)$$

Also, by definition,

$\mathbf{A}' = [b_{ij}]$ such that $b_{ij} = a_{ji}$ for all $1 \le i \le n$, $i \le j \le n$ ...See § 2·08 Page 69

or $\mathbf{A}' = [a_{ji}]$, for all $1 \le i \le n$, $1 \le j \le n$

$= [a_{ij}]$, from (i).

Hence $\mathbf{A}' = \mathbf{A}$.

Conversely if $\mathbf{A} = \mathbf{A}'$. Then **A** must be a square matrix

Also $\mathbf{A} = \mathbf{A}' \Rightarrow [a_{ij}] = [a_{ji}]$, for all $1 \le i \le n$, $1 \le j \le n$

$\Rightarrow a_{ij} = a_{ji}$, for all $1 \le i \le n$, $1 \le j \le n$

$\Rightarrow$ **A** is a symmetric matrix. Hence proved.

Theorem II. *A square matrix* **A** *is skew-symmetric iff* $\mathbf{A}' = -\mathbf{A}$.

Proof : Let **A** be an $n \times n$ square matrix *i.e* $\mathbf{A} = [a_{ij}]$ for all $1 \le i \le n$ and $1 \le j \le n$.

If **A** is a skew-symmetric matrix, then by definition, we have

$$[a_{ij}] = [-a_{ji}], \text{ for all } 1 \le i \le n,\ 1 \le j \le n \qquad ...(i)$$

Also, by definition, $\mathbf{A}' = [b_{ij}]$, such that $b_{ij} = a_{ji}$.

for all $1 \le i \le n$, $1 \le j \le n$. ...See § 2·08 Page 69

or $\mathbf{A}' = [a_{ji}]$ for all $1 \le i \le n$, $1 \le j \le n$

$= -[-a_{ji}] = -[a_{ij}]$, from (i).

Hence $\mathbf{A}' = -\mathbf{A}$.

Conversely if $\mathbf{A}' = -\mathbf{A}$, then **A** must be a square matrix.

Also $\mathbf{A}' = -\mathbf{A} \Rightarrow [a_{ji}] = -[a_{ij}]$, for all $1 \le i \le n$, $1 \le j \le n$

$\Rightarrow a_{ji} = -a_{ij}$

$\Rightarrow a_{ij} = -a_{ji}$, for all $1 \le i \le n$, $1 \le j \le n$

$\Rightarrow$ **A** is a skew-symmetric matrix. Hence proved.

*****Theorem III.** *Every square matrix can be uniquely expressed as the sum of a symmetric and a skew-symmetric matrices.*

(*Avadh 94, 92, 90; Bundelkhand 95; Meerut 93*)

Proof : Let **A** be a square matrix, then we can write

$$\mathbf{A} = \tfrac{1}{2}\mathbf{A} + \tfrac{1}{2}\mathbf{A} = \tfrac{1}{2}(\mathbf{A} + \mathbf{A}') + \tfrac{1}{2}(\mathbf{A} - \mathbf{A}') \qquad ...(i)$$

since $\frac{1}{2}\mathbf{A}$, $\frac{1}{2}\mathbf{A}'$ are conformable to addition, **A** being a square matrix. **(Note)**

Now $\{\frac{1}{2}(\mathbf{A} + \mathbf{A}')\}' =$ transpose of $\frac{1}{2}(\mathbf{A} + \mathbf{A}')$

$= \frac{1}{2}(\mathbf{A} + \mathbf{A}')'$ ...by § 2·09 Th. III Page 70

$= \frac{1}{2}\{\mathbf{A}' + (\mathbf{A}')'\}$ ...by § 2·09 Th. I Page 70

$= \frac{1}{2}(A' + A)$ ...by § 2·09 Th. II Page 70

or $\{\frac{1}{2}(A + A')\}' = \frac{1}{2}(A + A')$, as matrix addition is commutative.

Therefore, by definition, $\frac{1}{2}(A + A')$ is a symmetric matrix ...(ii)

Again $\{\frac{1}{2}(A - A')\}' = \frac{1}{2}(A - A')'$ ...by § 2·09 Th. III Page 70

$= \frac{1}{2}\{A + (-1)A'\}'$ **(Note)**

$= \frac{1}{2}\{A' + \{(-1)(A')'\}$ ...by § 2·09 Th. I Page 70

$= \frac{1}{2}\{A' + (-1)(A')'\}$ ...by § 2·09 Th. III Page 70

$= \frac{1}{2}\{A' + (-1)A\}$ ...by § 2·09 Th. II Page 70

$= \frac{1}{2}(A' - A) = \frac{1}{2}\{(-1)^2 A' + (-1)A\}$ **(Note)**

$= (-1).\frac{1}{2}(-A' + A)$

or $\{\frac{1}{2}(A - A')\}' = -\frac{1}{2}(A - A')$, as matric addition is commutative.

Therefore, by definition, $\frac{1}{2}(A - A')$ is a skew-symmetric matric. ...(iii)

Hence from (i), (ii) and (iii), we find that the matrix **A** can be expressed as the sum of a symmetric and a skew symmetric matrices.

To prove that the representation (i) is unique, let

$$A = A_1 + A_2. \quad ...(iv)$$

where A_1 is symmetric and A_2 is skew-symmetric.

Then $$A_1 = A_1' \quad ...(v)$$

and $$A_2 = -A_2' \quad ...(vi)$$

From (iv), we have $A' = (A_1 + A_2)'$

$= A_1' + A_2'$ ... by Th. I § 2·09 Page 70

or $A' = A_1 - A_2$, from (v), (vi) ...(vii)

Adding and subtracting (iv) and (vii), we get

$$A + A' = 2A_1 \text{ and } A - A' = 2A_2$$

or $$A_1 = \frac{1}{2}(A + A') \text{ and } A_2 = \frac{1}{2}(A - A')$$

∴ From (iv), we get $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$, which is the same as (i).

Hence the representation (i) is unique. Hence proved.

Solved Examples on § 2·14 and § 2·15

Ex. 1. Show that the matric $A = \begin{bmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ -7 & -8 & 0 \end{bmatrix}$ **is skew-symmetric.** *(Meerut 94)*

Sol. In the given matrix, we find that

$a_{11} = 0, a_{12} = 6 = -a_{21}, a_{13} = 7 = -a_{31}, a_{22} = 0, a_{23} = 8 = -a_{32}, a_{33} = 0$

i.e. $a_{ij} = -a_{ji}$ for all $1 \le i \le 3, 1 \le j \le 3$.

Hence by definition [See § 2·14 (b) Page 79] the given matrix **A** is skew-symmetric.

Ex. 2. If $A = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$**, then show that AA′ and A′A are both symmetric matrices.**

Sol. Here $A' = \begin{bmatrix} 3 & 0 \\ 1 & 1 \\ -1 & 2 \end{bmatrix}$

$$\therefore \quad AA' = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} 3 & 0 \\ 1 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3\cdot3 + 1\cdot1 - 1\,(-1) & 3\cdot0 + 1\cdot1 - 1\cdot2 \\ 0\cdot3 + 1\cdot1 + 2\,(-1) & 0\cdot0 + 1\cdot1 + 2\cdot2 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & -1 \\ -1 & 5 \end{bmatrix}, \text{ which is a symmetric matrix.}$$

[See § 2·14 (a) Page 79]

Similarly $A'A = \begin{bmatrix} 3 & 0 \\ 1 & 1 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} 3 & 1 & -1 \\ 0 & 1 & .2 \end{bmatrix}$

$$= \begin{bmatrix} 3\cdot3 + 0\cdot0 & 3\cdot1 + 0\cdot1 & 3\,(-1) + 0\cdot2 \\ 1\cdot3 + 1\cdot0 & 1\cdot1 + 1\cdot1 & 1\,(-1) + 1\cdot2 \\ -1\cdot3 + 2\cdot0 & -1\cdot1 + 2\cdot1 & -1\,(-1) + 2\cdot2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 3 & -3 \\ 3 & 2 & 1 \\ -3 & 1 & 5 \end{bmatrix}, \text{ which is a symmetric matrix}$$

[See § 2·14 (a) Page 79]

***Ex. 3. (a). If A and B are both skew-symmetric matrices of same order such that AB = BA, then show that AB is symmetric.**

Sol. If **A** and **B** are both skew-symmetric matrices,

then $$A = -A' \text{ and } B = -B' \qquad \text{...(i)}$$

Also given that $AB = BA$

$= (-B')(-A')$, from (i)

$= B'A' = (AB)'$ ...See Th. IV § 2·09 Page 71

or $AB = (AB)'$ *i.e.* **AB** is a symmetric matrix. Hence proved.

Ex. 3 (b). If A is a symmetric matrix, then show that kA is also symmetric for any scalar k.

Sol. Here $(kA)' = kA'$, See. § 2·09 Th. III Page 70

$= kA, \quad \because \quad A' = A$, **A** being symmetric

Hence kA is symmetric, if **A** is so.

****Ex. 4 (a). Find the symmetric and skew-symmetric parts of the matrix**

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix}$$

Sol. (Refer Theorem III § 2·15 Pages 80 – 81)

Here $\mathbf{A}'$ = transpose of $\mathbf{A}$

$$= \begin{bmatrix} 1 & 6 & 3 \\ 2 & 8 & 5 \\ 4 & 1 & 7 \end{bmatrix}$$

The symmetric part of $\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}')$

$$= \frac{1}{2}\left(\begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 6 & 3 \\ 2 & 8 & 5 \\ 4 & 1 & 7 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 1+1 & 2+6 & 4+3 \\ 6+2 & 8+8 & 1+5 \\ 3+4 & 5+1 & 7+7 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 8 & 7 \\ 8 & 16 & 6 \\ 7 & 6 & 14 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & \frac{7}{2} \\ 4 & 8 & 3 \\ \frac{7}{2} & 3 & 7 \end{bmatrix}$$ **Ans.**

And the skew-symmetric part of $\mathbf{A} = \frac{1}{2}(\mathbf{A} - \mathbf{A}')$

$$= \frac{1}{2}\left(\begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 6 & 3 \\ 2 & 8 & 5 \\ 4 & 1 & 7 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 1-1 & 2-6 & 4-3 \\ 6-2 & 8-8 & 1-5 \\ 3-4 & 5-1 & 7-7 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -4 \\ -1 & 4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2 & \frac{1}{2} \\ 2 & 0 & -2 \\ -\frac{1}{2} & 2 & 0 \end{bmatrix}$$ **Ans.**

***Ex. 4 (b) Express given matrix A as sum of a symmetric and skew-symmetric matrices.** $\mathbf{A} = \begin{bmatrix} 6 & 8 & 5 \\ 4 & 2 & 3 \\ 1 & 7 & 1 \end{bmatrix}$ *(Agra 93)*

Sol. From Theorem III § 2·15 Pages 80 – 81 we find that the symmetric and skew-symmetric parts of a matrix $\mathbf{A}$ are $\frac{1}{2}(\mathbf{A} + \mathbf{A}')$ and $\frac{1}{2}(\mathbf{A} - \mathbf{A}')$ respectively whose sum is eveidently A. **(Note)**

i.e. $\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}') + \frac{1}{2}(\mathbf{A} - \mathbf{A}')$ ...(i)

Now $\mathbf{A}'$ = transpose of $\mathbf{A} = \begin{bmatrix} 6 & 4 & 1 \\ 8 & 2 & 7 \\ 5 & 3 & 1 \end{bmatrix}$

$$\therefore \quad A + A' = \begin{bmatrix} 6 & 8 & 5 \\ 4 & 2 & 3 \\ 1 & 7 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 4 & 1 \\ 8 & 2 & 7 \\ 5 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6+6 & 8+4 & 5+1 \\ 4+8 & 2+2 & 3+7 \\ 1+5 & 7+3 & 1+1 \end{bmatrix} = \begin{bmatrix} 12 & 12 & 6 \\ 12 & 4 & 10 \\ 6 & 10 & 2 \end{bmatrix}$$

$$\therefore \quad \frac{1}{2}(A + A)' = \frac{1}{2}\begin{bmatrix} 12 & 12 & 6 \\ 12 & 4 & 10 \\ 6 & 10 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 6 & 3 \\ 6 & 2 & 5 \\ 3 & 5 & 1 \end{bmatrix},$$

which is evidently a symmetric matrix as $a_{ij} = a_{ji}$ for all values of i and j

$$\text{And } A - A' = \begin{bmatrix} 6 & 8 & 5 \\ 4 & 2 & 3 \\ 1 & 7 & 1 \end{bmatrix} - \begin{bmatrix} 6 & 4 & 1 \\ 8 & 2 & 7 \\ 5 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6-6 & 8-4 & 5-1 \\ 4-8 & 2-2 & 3-7 \\ 1-5 & 7-3 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 4 \\ -4 & 0 & -4 \\ -4 & 4 & 0 \end{bmatrix}$$

$$\therefore \frac{1}{2}(A - A') = \frac{1}{2}\begin{bmatrix} 0 & 4 & 4 \\ -4 & 0 & -4 \\ -4 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 \\ -2 & 0 & -2 \\ -2 & 2 & 0 \end{bmatrix},$$

which is evidently a skew-symmetric matrix as $a_{ij} = -a_{ji}$ for all values of i, j

$\therefore$ From (i), we get

$$A = \begin{bmatrix} 6 & 6 & 3 \\ 6 & 2 & 5 \\ 3 & 5 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 2 \\ -2 & 0 & -2 \\ -2 & 2 & 0 \end{bmatrix}$$

= sum of a symmetric and skew-symmetric matrices, as proved above.

****Ex. 5. If A is any square matrix, show that AA′ is a symmetric matrix.**

Sol. $(AA')' =$ transpose of AA'

$= (A')'A'$ ...See Th. IV § 2·09 Page 71

$= AA'$ ... See Th. II § 2·09 Page 70

i.e. $AA' = (AA')'$. Hence AA' is a symmetric matrix by definition.

***Ex. 6. If A be a square matrix, show that A + A′ is symmetric and A − A′ is a skew-symmetric matrix.** *(Meerut 99)*

Sol. If **A** is a square matrix, then

$(A + A')' = A' + (A')'$ ...See § 2.09 Th. I Page 70

$= A' + A$ ...See § 2.09 Th. II Page 70

$= A + A'$, by commutative law of addition

Hence by definition $A + A'$ is symmetric.

Again $(\mathbf{A}-\mathbf{A}')'=\mathbf{A}'-(\mathbf{A}')'$, ...See § 2.09 Th. I Page 70

$=\mathbf{A}'-\mathbf{A}$ ...See § 2.09 Th. II Page 70

$=-(\mathbf{A}-\mathbf{A}')$

Hence by definition $\mathbf{A}-\mathbf{A}'$ is skew-symmetric.

***Ex. 7. If A is skew-symmetric matrix, then show that $\mathbf{AA}'=\mathbf{A}'\mathbf{A}$ and $\mathbf{A}^2$ is symmetric.**

Sol. If **A** is a skew-symmetric matrix, then we know that

$$\mathbf{A}'=-\mathbf{A} \quad ...(i)$$

Pre-multiplying both sides of (i) by **A**, we get

$$\mathbf{AA}'=-\mathbf{AA}=-\mathbf{A}^2 \quad ...(ii)$$

Post-multiplying both sides of (i) by **A**, we get

$$\mathbf{A}'\mathbf{A}=-\mathbf{AA}=-\mathbf{A}^2 \quad ...(iii)$$

From (ii) and (iii) we conclude that $\mathbf{AA}'=\mathbf{A}'\mathbf{A}$

Further we can prove (as in Ex. 5 Page 84) that $\mathbf{AA}'$ and $\mathbf{A}'\mathbf{A}$ are symmetric matrices. Hence from (ii) and (iii) we find that $-\mathbf{A}^2$ is a symmetric matrix or $\mathbf{A}^2$ is a symmetric matrix, as we know that $k\mathbf{A}$ is also symmetric if k is scalar and **A** is symmetric. **Hence proved.**

Exercises on § 2.14 – § 2.15

***Ex. 1.** If **A** and **B** are symmetric (or skew-symmetric) matrices, then so is $\mathbf{A}+\mathbf{B}$.

Ex. 2. If **A** and **B** are symmetric matrices, then prove that $\mathbf{AB}+\mathbf{BA}$ is symmetric and $\mathbf{AB}-\mathbf{BA}$ is skew-symmetric.

Ex. 3. Show that all positive integral powers of a symmetric matrix are symmetric.

Ex. 4. If **A** is any matrix, then show that $\mathbf{A}'\mathbf{A}$ is a symmetric matrix.

(Hint : See Ex. 5 Page 84)

Ex. 5. If **A** is a symmetric matrix, then show that $\mathbf{AA}'=\mathbf{A}'\mathbf{A}$ and $\mathbf{A}^2$ is symmetric.

(Hint : See Ex. 7 above)

Ex. 6. What is the main diagonal of a skew symmetric matrix ? *(Kanpur 90)*

[Hint : See § 2.14 (b) Page 79. Each element is zero].

Ex. 7. What is the transpose of a symmetric matrix ? *(Kanpur 90)*

[Hint : See Th. I § 2.15 Page 80]. **Ans.** The matrix itself.

Ex. 8. **A** is a skew symmetric matrix. How will be $\mathbf{A}^n$? n is any positive integer.

***Ex. 9.** Prove that every diagonal element of a skew-symmetric matrix is necessarily zero. *(Garhwal 91; Kanpur 94)*

[**Hint :** In the case of skew-symmetric matrix, we know

$$a_{ij}=-a_{ji} \text{ for all values of } i \text{ and } j$$

$\therefore$ If $i=j$, then $a_{ii}=-a_{ii}$ for all i

i.e. $a_{ii}+a_{ii}=0$ or $2a_{ii}=0$ or $a_{ii}=0$

i.e. all diagonal element of a skew symmetric matrix are necessarily zero.]

**§2.16. Hermitian and Skew-Hermitian Matrices.

(a) Hermitian Matrix. *(Avadh 95, 91, 90)*

Definition. A square matrix **A** such that $\overline{\mathbf{A}}' = \mathbf{A}$ is called Hermitian *i.e.* the matrix $[a_{ij}]$ is Hermitian provided $a_{ij} = \overline{a}_{ji}$, for all values of i and j.

For example : $\mathbf{A} = \begin{bmatrix} l & \alpha + i\beta & \gamma + i\delta \\ \alpha - i\beta & m & x + iy \\ \gamma - i\delta & x - iy & n \end{bmatrix}$

(b) Skew-Hermitian Matrix. *(Avadh 91, 90)*

Definition. A square matrix **A** such that $\overline{\mathbf{A}}' = -\mathbf{A}$ is called skew-Hermitian *i.e.* the matrix $[a_{ij}]$ is skew-Hermitian provided $a_{ij} = -\overline{a}_{ji}$ for all values of i and j.

For example : $\mathbf{A} = \begin{bmatrix} 2i & -\alpha - i\beta & -3 + i \\ \alpha - i\beta & -i & -\gamma + i\delta' \\ 3 + i & \gamma + i\delta & 0 \end{bmatrix}$

§ 2.17. Theorems on Hermitian and Skew-Hermitian Matrices.

***Theorem I.** *The diagonal elements of a Hermitian matrix are necessarily real.* *(Avadh 95)*

Proof : Let $[a_{ij}]$ be a $n \times n$ Hermitian matrix, then according to definition [as given in § 2.16 (a) above], we have

$$a_{ij} = \overline{a}_{ji}, \text{ for all } 1 \leq i \leq n, \ 1 \leq j \leq n \qquad \text{...(i)}$$

Now the diagonal elements are a_{ii}, where $1 \leq i \leq n$.

$\therefore$ From (i), we have $a_{ii} = \overline{a}_{ii}$, for all $1 \leq i \leq n$ ...(ii)

If $a_{ii} = \alpha + i\beta$ where α and β are real,

then $\overline{a}_{ii} = \alpha - i\beta$

$\therefore$ From (ii), we get $\alpha + i\beta = \alpha - i\beta$

or $2i\beta = 0$ or $\beta = 0$

$\therefore$ $a_{ii} = \alpha + i\,(0) = \alpha$, which is purely real.

Hence the diagonal elements of a Hermitian matrix are necessarily real.

Hence proved.

***Theorem II.** *The diagonal elements of a skew-Hermitian matrix are either purely imaginary or zero.* *(Avadh 90)*

Proof : Let $[a_{ij}]$ be an $n \times n$ skew-Hermitian matrix, then according to definition [as given in § 2.16 (b) above] we have

$$a_{ij} = -\overline{a}_{ji}, \text{ for all } 1 \leq i \leq n, \ 1 \leq j \leq n. \qquad \text{...(i)}$$

Now the diagonal elements are a_{ii}, where $1 \leq i \leq n$.

$\therefore$ From (i), we have $a_{ii} = -\overline{a}_{ii}$, for all $1 \leq i \leq n$. ...(ii)

If $a_{ii} = \alpha + i\beta$, where α and β are real,

then $\bar{a}_{ii} = \alpha - i\beta$.

$\therefore$ From (ii), we get $\alpha + i\beta = -(\alpha - i\beta)$

or $\alpha + i\beta = -\alpha + i\beta$ or $2\alpha = 0$ or $\alpha = 0$

$\therefore a_{ii} = 0 + i\beta = i\beta$, which is purely imaginary and can be zero if $\beta = 0$.

Hence the diagonal elements of a skew-Hermitian matrix **are either** purely imaginary or zero.

****Theorem III.** *Every square matrix (with complex elements)* ***can be*** *uniquely expressed as the sum of a Hermitian and a skew-Hermitian* ***matrices.***

(*Garhwal* 92)

Proof. Let **A** be a square matrix. Then we can write

$$\mathbf{A} = \tfrac{1}{2}(\mathbf{A} + \mathbf{A}^{\Theta}) + \tfrac{1}{2}(\mathbf{A} - \mathbf{A}^{\Theta}) \qquad \text{...(i)}$$

Now $\left(\overline{\mathbf{A} + \mathbf{A}^{\Theta}}\right) = \overline{\mathbf{A}} + \overline{\mathbf{A}^{\Theta}}$ ...See § 2.11 Th. III Page 76

$\therefore \left\{\left(\overline{\mathbf{A} + \mathbf{A}^{\Theta}}\right)\right\}' = \left\{\overline{\mathbf{A}} + \overline{\mathbf{A}^{\Theta}}\right\}' = (\overline{\mathbf{A}})' + (\overline{\mathbf{A}^{\Theta}})'$, ...See § 2.09 Th. I Page 70

$= \mathbf{A}^{\Theta} + (\overline{\mathbf{A}^{\Theta}})'$, by def. $(\overline{\mathbf{A}})' = \mathbf{A}^{\Theta}$, See § 2.12 Page 77

$= \mathbf{A}^{\Theta} + (\overline{\mathbf{A}^{\Theta}})'$. ...(ii)

Now $(\overline{\mathbf{A}^{\Theta}})'$ = transposed conjugate of $\mathbf{A}^{\Theta}$

= transposed conjugate of $(\overline{\mathbf{A}})'$, ...See § 2.12 P. 77

= transposed matrix of $(\mathbf{A})'$,

since conjugate of $\overline{\mathbf{A}}$ is **A** ...See Th. I Page 75

$= \mathbf{A}, \quad \because \quad (\mathbf{A}')' = \mathbf{A}$ ...See § 2.09 Th. II Page 70

$\therefore$ From (ii) we get, $\left\{\left(\overline{\mathbf{A} + \mathbf{A}^{\Theta}}\right)\right\}' = \mathbf{A}^{\Theta} + \mathbf{A} = \mathbf{A} + \mathbf{A}^{\Theta}$,

as addition of matrices obey commutative law.

$\therefore$ By definition (See § 2.16 (a) Page 86) we find that $\mathbf{A} + \mathbf{A}^{\Theta}$ is a Hermitian matrix.

Again $\left\{\left(\overline{\mathbf{A} - \mathbf{A}^{\Theta}}\right)\right\}' = (\overline{\mathbf{A}} - \overline{\mathbf{A}^{\Theta}})' = (\overline{\mathbf{A}})' - \left(\overline{\mathbf{A}^{\Theta}}\right)'$

$= \mathbf{A}^{\Theta} - \mathbf{A}$, as above

$= -(\mathbf{A} - \mathbf{A}^{\Theta})$.

$\therefore$ By definition (See § 2.16 (b) Page 86) we find that $\mathbf{A} - \mathbf{A}^{\Theta}$ is a skew-Hermitian matrix.

$\therefore$ From (i) we conclude that the square matrix **A** is the sum of a Hermitian and a skew-Hermitian matrices.

Solved Examples on § 2.16 – § 2.17.

Ex. 1 (a). Is $\mathbf{A} = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3-i \\ -2-5i & 3+i & 5 \end{bmatrix}$

a hermitian matrix ?

Sol. $\mathbf{A}' = \begin{bmatrix} 3 & 7+4i & -2-5i \\ 7-4i & -2 & 3+i \\ -2+5i & 3-i & 5 \end{bmatrix}$

$\therefore \quad \overline{\mathbf{A}}' = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3-i \\ -2-5i & 3+i & 5 \end{bmatrix} = \mathbf{A}$

Hence by definition [See § 2.16 (a) Page 86], the given matrix **A** is hermitian.

Ex. 1 (b). Prove that the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1-i & 2 \\ 1+i & 3 & i \\ 2 & -i & 0 \end{bmatrix}$

is Hermitian. *(Avadh 91; Rohilkhand 97)*

Sol. $\mathbf{A}' = \begin{bmatrix} 1 & 1+i & 2 \\ 1-i & 3 & -i \\ 2 & i & 0 \end{bmatrix}$

$\therefore \quad \overline{\mathbf{A}}' = \begin{bmatrix} 1 & 1-i & 2 \\ 1+i & 3 & i \\ 2 & -i & 0 \end{bmatrix} = \mathbf{A}$

$\therefore$ **A** is Hermitian. ...See § 2.16 (a) Page 86

Ex. 2. If $\mathbf{A} = \begin{bmatrix} 3 & 2-3i & 3+5i \\ 2+3i & 5 & i \\ 3-5i & -i & 7 \end{bmatrix}$,

then prove that $\overline{\mathbf{A}}$ **is Hermitian.** *(Meerut 96)*

Sol. $\overline{\mathbf{A}} = \begin{bmatrix} 3 & 2+3i & 3-5i \\ 2-3i & 5 & -i \\ 3+5i & i & 7 \end{bmatrix} = \mathbf{B}$ (say)

Then $\mathbf{B}' = \begin{bmatrix} 3 & 2-3i & 3+5i \\ 2+3i & 5 & i \\ 3-5i & -i & 7 \end{bmatrix}$

$$\therefore \quad \overline{\mathbf{B}}' = \begin{bmatrix} 3 & 2+3i & 3-5i \\ 2-3i & 5 & -i \\ 3+5i & i & 7 \end{bmatrix} = \mathbf{B}$$

$\therefore$ **B** *i.e.* $\overline{\mathbf{A}}$ is Hermitian. ...See § 2.16 (a) Page 86

Ex. 3. Show that $\mathbf{A} = \begin{bmatrix} \mathbf{i} & \mathbf{3+2i} & \mathbf{-2-i} \\ \mathbf{-3+2i} & \mathbf{0} & \mathbf{3-4i} \\ \mathbf{2-i} & \mathbf{-3-4i} & \mathbf{-2i} \end{bmatrix}$,

is skew-Hermitian Matrix. (*Rohilkhand 95*)

Sol. Here $\mathbf{A}' = \begin{bmatrix} i & -3+2i & 2-i \\ 3+2i & 0 & -3-4i \\ -2-i & 3-4i & -2i \end{bmatrix}$

$$\therefore \quad \overline{\mathbf{A}}' = \begin{bmatrix} -i & -3-2i & 2+i \\ 3-2i & 0 & -3+4i \\ -2+i & 3+4i & 2i \end{bmatrix} \qquad \textbf{(Note)}$$

$$= -\begin{bmatrix} i & 3+2i & -2-i \\ -3+2i & 0 & 3-4i \\ 2-i & -3-4i & -2i \end{bmatrix} = -\mathbf{A}$$

Hence by definition [See § 2.16 (b) Page 86], the given matrix **A** is skew-Hermition.

Ex. 4. If A and B are Hermitian, then show that AB is Hermitian if and only if A and B commute.

Sol. If **A** and **B** are Hermitian matrices, then we have

$$\mathbf{A} = (\overline{\mathbf{A}})' = \mathbf{A}^{\Theta} \text{ and } \mathbf{B} = (\overline{\mathbf{B}})' = \mathbf{B}^{\Theta} \qquad ...(i)$$

Then $(\mathbf{AB})^{\Theta} = \mathbf{B}^{\Theta}\mathbf{A}^{\Theta}$, by § 2.13 Th. V Page 79

$= \mathbf{BA}$, by (i) above

$= \mathbf{AB}$, if **A** and **B** commute

i.e. $(\mathbf{AB})^{\Theta} = \mathbf{AB}$ or $(\overline{\mathbf{AB}})' = \mathbf{AB}$, $\because \ \mathbf{A}^{\Theta} = (\overline{\mathbf{A}})'$

Hence by definition **AB** is Hermitian.

Converse of this can be proved to be true by reversing the above calculations.

Ex. 5 (a). If A is a Hermitian matrix, then show that iA is skew-Hermitian. (*Kanpur 90*)

Sol. If **A** is a Hermitian matrix, then

we have $\mathbf{A} = \overline{\mathbf{A}}'$ ...See § 2.16 (a) Page 86

Also $\overline{\mathbf{A}}' = \mathbf{A}^{\Theta}$ ...See § 2.12 Page 77

$\therefore$ Here $\mathbf{A} = \overline{\mathbf{A}}' = \mathbf{A}^{\Theta}$ ...(i)

Now $(i\mathbf{A})^{\Theta} = -i\mathbf{A}^{\Theta}$, $\because \bar{i} = -i$

...See § 2.13 Th. III (a) Page 78

$$= -\left(i\mathbf{A}^{\Theta}\right)$$

or $(i\mathbf{A})^{\Theta} = -(i\mathbf{A})$, from (i) ...(ii)

Also from § 2.16 (b) Page 86 we know that if **A** is a skew-Hermitian matrix, then $\overline{\mathbf{A}}' = -\mathbf{A} = \mathbf{A}^{\Theta}$, from (i)

And from (ii), we find that $-(i\mathbf{A}) = (i\mathbf{A})^{\Theta}$, hence $(i\mathbf{A})$ is a skew-Hermitian-matrix.

Ex, 5 (b). If A is a skew-Hermitian matrix, then show that iA is Hermitian.

Sol. If **A** is a skew-Hermitian matrix, then we have

$-\mathbf{A} = \overline{\mathbf{A}}'$ ...See § 2.16 (b) Page 86

Also $\overline{\mathbf{A}}' = \mathbf{A}^{\Theta}$ ...See § 2.12 Page 77

$\therefore$ $-\mathbf{A} = \overline{\mathbf{A}}' = \mathbf{A}^{\Theta}$ ...(i)

Now $(i\mathbf{A})^{\Theta} = -i\mathbf{A}^{\Theta}$, $\because \bar{i} = -i$

...See § 2.13 Th. III (a) Page 78

$$= -i(-\mathbf{A}), \text{ from (i)}$$

or $(i\mathbf{A})^{\Theta} = i\mathbf{A}$ ...(ii)

Also from § 2.16 (a) Page 86 we know that if **A** is a Hermitian matrix, then $\overline{\mathbf{A}}' = \mathbf{A} = \mathbf{A}^{\Theta}$, from (i).

And from (ii) we find that $(i\mathbf{A}) = (i\mathbf{A})^{\Theta}$, hence $i\mathbf{A}$ is a Hermitian matrix.

Ex. 6. If A is any square matrix, show that $\mathbf{AA}^{\Theta}$ and $\mathbf{A}^{\Theta}\mathbf{A}$ are Hermitian.

Sol. $(\mathbf{AA}^{\Theta})^{\Theta} = (\mathbf{A}^{\Theta})^{\Theta}\,\mathbf{A}^{\Theta}$ ...by § 2.13 Th. V Page 79

$= \mathbf{AA}^{\Theta}$ ...by § 2.13 Th. II Page 78

$\therefore$ By definition (See § 2.16 (a) Page 86), $\mathbf{AA}^{\Theta}$ is Hermitian.

Similarly $(\mathbf{A}^{\Theta}\,\mathbf{A})^{\Theta} = \mathbf{A}^{\Theta}\,(\mathbf{A}^{\Theta})^{\Theta}$ ...by § 2.13 Th. V Page 79

$= \mathbf{A}^{\Theta}\,\mathbf{A}$ ...by § 2.13 Th. II Page 78

$\therefore$ By definition (See § 2.16 (a) Page 86), $\mathbf{A}^{\Theta}\,\mathbf{A}$ is Hermitian.

Ex. 7. Show that A is Hermitian iff $\overline{\mathbf{A}}$ is Hermitian.

Sol. Let A be Hermitian, then $\mathbf{A} = \mathbf{A}^{\Theta}$ ...(i)

Now $(\overline{\mathbf{A}})^{\Theta}$ = transposed conjugate of $\overline{\mathbf{A}}$

= transposed matrix of **A**, since $(\overline{\overline{\mathbf{A}}}) = \mathbf{A}$

...See § 2.11 Th. I Page 75

$= A' = (A^{\Theta})'$, by (i)

= transpose of transposed conguate of **A**

= conguate of **A**, $\because (B')' = B$

i.e., $(\overline{A})^{\Theta} = \overline{A}$

Hence by definition, $\overline{A}$ is a Hermitian matrix

Again if $\overline{A}$ is Hermitian, then we have

$$\overline{A} = (\overline{A})^{\Theta}$$

= transposed conjuage of $\overline{A}$

= transponse of **A** ... by § 2·11 Th. I Page 75

or $\overline{A} = A'$ ...(ii)

Now $A^{\Theta} = (\overline{A})'$, by definition

$= (A')'$, by (ii)

i.e., $A^{\Theta} = A$, ... by § 2·09 Th. II Page 70

Hence by definition **A** is Hermitian. Hence proved

Exercises on § 2.16 – § 2.17

Ex.1. If $A = \begin{bmatrix} i & 1+i & 2-3i \\ -1+i & 2i & 1 \\ -2-3i & -1 & 0 \end{bmatrix}$, then slow that $\overline{A}$ is skew–Hermitian

Ex.2. Show that $A = \begin{bmatrix} 0 & 2-3i & -2-i \\ -2-3i & 0 & -3+4i \\ 2-i & 3+4i & 0 \end{bmatrix}$ is skew–Hermitian.

Ex. 3. Show that **A** is skew-Hermitian iff $\overline{A}$ is skew-Hermitian.

[**Hint** : See Ex. 7. Page 90]

Ex. 4. Give an example of matrix which is skew symmetric but not skew-Hermitian.

Ex. 5. If **A** and **B** are Hermitian matrices, show that **AB + BA** is Hermitian and **AB – BA** is skew-Hermitian.

Ex. 6. Show that every square matrix can be uniquely expressed as $P + iQ$. where P, Q are Hermitian. (*Garhwal 95; Rohilkhand 91*)

[**Hint** : See Th. III Page 87, Ex 5(a) Page 89].

***§ 2.18. The inverse of a matrix.**

(*Avadh 91; Bundelkhand 93; Garhwal 91*)

If for a given square matrix **A**, there exists a matrix **B** such that $AB = BA = I$ where **I** is an unit matrix, then **A** is called **non-singular** or **Invertible** and **B** is called **inverse of A** and we write $B = A^{-1}$ (read as **B** equals **A** inverse).

Here **A** is the inverse of **B** and we can write $A = B^{-1}$

If **B** *i.e.*, A^{-1} does not exist, then **A** is called **singular.**

Note 1. If **AB** and **BA** are both defined and equal then the matrices **A** and **B** should both be square matrices of the same order.

Note 2. Non-square matrix has no inverse.

For example : $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=I$

Each matrix in the product is the inverse of the other.

§ 2.19. Theorems on Inverse of a matrix.

****Theorem I.** *If a given square matrix* **A** *has an inverse, then it is unique or there exists one and only one inverse matrix to a given matrix.*

(*Bundelkhand 93, 91*)

Proof. Let us suppse that **B** and **C** are two possible inverses of **A**. Then we must have (See § 2·18 above).

$$AB = BA = I \qquad ...(i)$$

and $$AC = CA = I \qquad ...(ii)$$

$\therefore$ From (i) and (ii), we get $AB = AC$, each being equal to **I**

or $$B\,(AB) = B\,(AC)$$

or $$(BA)\,B = (BA)\,C \qquad \text{... See § 1·09 Prop. I Page 26}$$

or $$IB = IC, \text{ from (i)}$$

or $$B = C \qquad \text{...See Ex. 1 Page 64}$$

Hence there cannot be two inverses of **A**.

****Theorem II.** *If* **A** *and* **B** *be two non-singluar or invertible matrices of the same order then* **AB** *is also non-singular and*

$$(AB)^{-1} = B^{-1}A^{-1}$$

(*Avadh 91; Bundelhkand 95; Garhwal 92; Gorakhpur 97; Purvanchal 97, 94*)

Or

The inverse of a product is the product of the inverse taken in the reverse order.

This is also known as the **Reciprocal Law for the inverse of a product.**

Proof. A^{-1} and B^{-1} exist since **A** and **B** are non-singluar.

$\therefore (AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$, by associative law

$$= AIA^{-1} = AA^{-1}, \qquad \text{...See Ex. 1. Page 64}$$

$$= I \qquad \text{...See § 2·18 Page 91}$$

And $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$, by associative law

$$= B^{-1}(I)B, \quad \because A^{-1}A = I$$

$$= B^{-1}(IB) = B^{-1}B, \qquad \text{...See Ex. 1. Page 64}$$

$$= I. \qquad \text{...See § 2·18 Page 91}$$

$\therefore (B^{-1}A^{-1})(AB) = (AB)(B^{-1}A^{-1}) = I$

i.e., $B^{-1} A^{-1}$ is the inverse of **AB** or $(AB)^{-1} = B^{-1} A^{-1}$ and as such **AB** is also non-singluar.

Note : For more details on inverse of matrices see chapter V of this book.

****§ 2·20. Orthogonal Matrix.**

Definition. A square matrix **A** is called an orthogonal matrix if $AA' = I$, where **I** is an identity matrix and **A′** is the transposed matrix of **A**. (*Kanpur 97*)

Theorems on Orthogonal Matrices.

Theorem I. *For any square matrix* **A**, *if* $AA' = I$, *then* $A'A = I$.

Proof : Since $AA' = I$, so **A** is invertible (*i.e.* **A** possesses an inverses) and there exists another matrix **B** such that

$$AB = BA = I \quad ...(i)$$

(See § 2·18 Page 91)

Now $B = BI = B(AA')$, $\because AA' = I$ (given)

$= (BA)A' = IA'$, from (i)

i.e. $B = A'$

$\therefore$ From (i), we get $AA' = A'A = I$. Hence proved.

Theorem II. *If* **A** *is an orthogonal matrix, then* **A′** *is also orthogonal.*

Proof : By definition if **A** is an orthogonal matrix, then

$$AA' = A'A = I$$

or $(AA')' = (A'A)' = I$, transposing and remembering $I' = I$

or $(A')' A' = A' (A')' = I$. by Th. IV § 2·09 Page 71

or **A′** is orthogonal by definition. Hence proved.

i.e. Transpose of an orthogonal matrix is also orthogonal.

Theorem III. *If* **A** *is an orthogonal matrix, then* A^{-1} *is also orthogonal.*

Proof : By definition if **A** is orthogonal, then

$$AA' = A'A = I$$

or $(AA')^{-1} = (A'A)^{-1} = I$,

taking inverse and remembering $I^{-1} = I$

or $(A')^{-1} A^{-1} = A^{-1} (A')^{-1} = I$. by Th. II § 2·19 Page 92

or $(A^{-1})' A^{-1} = A^{-1} (A^{-1})' = I$ (Note)

or A^{-1} is orthogonal by definition. Hence proved.

i.e. Inverse of an orthogonal matrix is also orthogonal.

Theorem IV. *For any orthogonal matrices,* **A** *and* **B**, *show that* **AB** *is an orthogonal matrix.*

Proof : If **A** and **B** are orthogonal matrices, then by definition we have

$$AA' = A'A = I \quad ...(i)$$

and $$BB' = B'B = I \quad ...(ii)$$

$\therefore (AB)(AB)' = (AB)(B'A')$ by Th. IV § 2·09 Page 71

$= \mathbf{AB\,B'A'} = \mathbf{A\,(BB')\,A'}$ **(Note)**

$= \mathbf{AIA'}$, from (ii).

$= \mathbf{AA'} = \mathbf{I}$, from (i).

Similarly, we can prove that

$(\mathbf{AB})'\,(\mathbf{AB}) = \mathbf{B'A'\,AB}$, by Th, IV § 2·09 Page 71

$= \mathbf{B'IB}$, from (i).

$= \mathbf{B'B} = \mathbf{I}$, from (ii).

Hence **AB** is an orthogonal matrix by definition.

§ 2·21. Unitary Matrix.

Definition. A square matrix **A** is called an unitary matrix if $\mathbf{A}^{\Theta} = \mathbf{I}$, where **I** is an identity matrix and $\mathbf{A}^{\Theta}$ is the transposed conjuage of**A**.

Theorems on Unitary matrices.

Theorem I. *For any square matrix, if* $\mathbf{AA}^{\Theta} = \mathbf{I}$, *then* $\mathbf{A}^{\Theta}\,\mathbf{A} = \mathbf{I}$.

Proof : Since $\mathbf{AA}^{\Theta} = \mathbf{I}$, where **I** is the unit matrix, so we find that **A** is invertible and there exists another matrix **B** such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I} \qquad \text{...(i)}$$

Now $\mathbf{B} = \mathbf{BI} = \mathbf{B}\,(\mathbf{AA}^{\Theta})$, $\because \mathbf{AA}^{\Theta} = \mathbf{I}$ (given)

$= (\mathbf{BA})\,\mathbf{A}^{\Theta} = \mathbf{IA}^{\Theta}$, from (i).

i.e. $\mathbf{B} = \mathbf{A}^{\Theta}$

$\therefore$ From (i), we get $\mathbf{AA}^{\Theta} = \mathbf{A}^{\Theta}\,\mathbf{A} = \mathbf{I}$ Hence proved.

Theorem II. *If* **A** *is an unitary matrix, then* **A′** *is also unitary.*

Proof : By definition if **A** is an unitary matrix, then

$$\mathbf{AA}^{\Theta} = \mathbf{A}^{\Theta}\,\mathbf{A} = \mathbf{I}$$

or $(\mathbf{AA}^{\Theta})^{\Theta} = (\mathbf{A}^{\Theta}\,\mathbf{A})^{\Theta} = \mathbf{I}$, taking transposed conguate and remembering that $\mathbf{I}^{\Theta} = \mathbf{I}$ **(Note)**

or $(\mathbf{A}^{\Theta})^{\Theta}\,\mathbf{A}^{\Theta} = \mathbf{A}^{\Theta}\,(\mathbf{A}^{\Theta})^{\Theta} = \mathbf{I}$, using § 2·09 Th. IV Page 71

or $\mathbf{AA}^{\Theta} = \mathbf{A}^{\Theta}\,\mathbf{A} = \mathbf{I}$, since $(\mathbf{A}^{\Theta})^{\Theta} = \mathbf{A}$

or $(\mathbf{AA}^{\Theta})' = (\mathbf{A}^{\Theta}\,\mathbf{A})' = \mathbf{I}$, taking transpose of each side

or $(\mathbf{A}^{\Theta})'\,\mathbf{A}' = \mathbf{A}'\,(\mathbf{A}^{\Theta})' = \mathbf{I}$, using § 2·09 Th. IV Page 71

or $(\mathbf{A}')^{\Theta}\,\mathbf{A}' = \mathbf{A}'\,(\mathbf{A}')^{\Theta} = \mathbf{I}$ **(Note)**

or **A′** is an unitary matrix. Hence proved.

Theorem III. *If* **A** *is an unitary matrix then* $\mathbf{A}^{-1}$ *is also unitary.*

Proof : By definition if **A** is an unitary matrix, then

$$\mathbf{AA}^{\Theta} = \mathbf{A}^{\Theta}\,\mathbf{A} = \mathbf{I}$$

or $(\mathbf{AA}^{\Theta})^{-1} = (\mathbf{A}^{\Theta}\,\mathbf{A})^{-1} = \mathbf{I}$, taking inverse

or $(\mathbf{A}^{\Theta})^{-1}\,\mathbf{A}^{-1} = \mathbf{A}^{-1}\,(\mathbf{A}^{\Theta})^{-1} = \mathbf{I}$, by Th. II § 2·19 Page 92 or $$(A^{-1})^{\Theta} A^{-1} = A^{-1} (A^{-1})^{\Theta} = I \qquad \textbf{(Note)}$$

or A^{-1} is an unitary matrix by definition. Hence proved.

Theorem IV. *For any two unitary matrices* **A** *and* **B** *show that* **AB** *is an unitary matrix.* (*Bundelkhand 91*)

Proof : If **A** an **B** are unitary matrices then by definition we have

$$AA^{\Theta} = A^{\Theta} A = I \qquad ...(i)$$

and $$BB^{\Theta} = B^{\Theta} B = I \qquad ...(ii)$$

$\therefore$ $(AB)(AB)^{\Theta} = (AB)(B^{\Theta} A^{\Theta})$; by Th. V § 2·13 Page 79

$= A (BB^{\Theta}) A^{\Theta} = AIA^{\Theta}$, from (ii)

$= AA^{\Theta} = I$, from (i)

Similarly $(AB)^{\Theta} (AB) = B^{\Theta} A^{\Theta} AB$, by Th. V. § 2·13 Page 79

$= B^{\Theta} IB$, from (i)

$= B^{\Theta} B = I$, from (ii)

Hence **AB** is an unitary matrix. Hence proved.

Solved Examples on § 2·20 and § 2·21.

Ex. 1. Show that the matrix $\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ **is orthogonal.** (*Bundelkhand 95*)

Sol. Let $A = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

Then $A' = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

$$\therefore \quad A'A = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \times \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} (-1).(-1)+2.2+2.2 & (-1).2+2.(-1)+2.2 & (-1).2+2.2+2(-1) \\ 2.(-1)+(-1).2+2.2 & 2.2+(-1).(-1)+2.2 & 2.2+(-1)2+2.(-1) \\ 2.(-1)+2.2+(-1).2 & 2.2+2.(-1)+(-1).2 & 2.2+2.2+(-1)(-1) \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence the given matrix **A** is orthogonal.

Ex. 2. Verify that the matrix

$$\mathbf{A}=\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix} \text{ is orthogonal.}$$

Sol. Here $A'=\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$

$$\therefore \quad A'A=\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \times \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{1}{3}+\frac{1}{3}+\frac{1}{3} & \frac{1}{3\sqrt{2}}-\frac{2}{3\sqrt{2}}+\frac{1}{3\sqrt{2}} & \frac{-1}{\sqrt{6}}+0+\frac{1}{\sqrt{6}} \\ \frac{1}{3\sqrt{2}}-\frac{2}{3\sqrt{2}}+\frac{1}{3\sqrt{2}} & \frac{1}{6}+\frac{4}{6}+\frac{1}{6} & \frac{-1}{2\sqrt{3}}+0+\frac{1}{2\sqrt{3}} \\ \frac{-1}{\sqrt{6}}+0+\frac{1}{\sqrt{6}} & \frac{-1}{2\sqrt{3}}+0+\frac{1}{2\sqrt{3}} & \frac{1}{2}+0+\frac{1}{2} \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=\mathbf{I}.$$

Hence **A** is orthogonal.

****Ex. 3. Show that the matrix** $\mathbf{A}=\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ **is orthogonal.**

(*Bundelkhand 91; Kanpur 97*)

Sol. $A'=\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$$\therefore \quad A'A=\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$=\begin{bmatrix} \cos^2\alpha+\sin^2\alpha & \cos\alpha\sin\alpha-\sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha-\cos\alpha\sin\alpha & \sin^2\alpha+\cos^2\alpha \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\mathbf{I}$$

Hence **A** is orthogonal.

Ex. 4. Prove that the matrix $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ **is unitary.**

(*Meerut 96*)

Sol. Let $A=\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

Then $A^{\Theta}=\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$$\therefore \quad \mathbf{A}^{\Theta}\mathbf{A} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \times \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$= \tfrac{1}{3}\begin{bmatrix} 1.1+(1+i).(1-i) & 1.(1+i)+(1+i)(-1) \\ (1-i).1+(-1)(1-i) & (1-i)(1+i)+(-1)(-1) \end{bmatrix}$$

$$= \tfrac{1}{3}\begin{bmatrix} 1+1-i^2 & 0 \\ 0 & 1-i^2+1 \end{bmatrix} = \tfrac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

Hence **A** is an unitary matrix.

Exercises on § 2·20 – § 2·21

Ex. 1. Show that the matrix $\mathbf{A} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$ is unitary.

Ex. 2. Show that the matrix $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$ is orthogonal.

Ex. 3. For any two orthogonal matrices **A** and **B**, show that **BA** is an orthogonal matrix.

Ex. 4. For any two unitary matrices A and **B**, show that **BA** is an **unitary** matrix.

Ex. 5. Prove that the following matrix is unitary :—

$$\begin{bmatrix} \frac{1}{2}(1+i) & \frac{1}{2}(-1+i) \\ \frac{1}{2}(1+i) & \frac{1}{2}(1-i) \end{bmatrix}$$

Ex. 6. Prove that a real matrix is unitary if it is orthogonal.

(Rohilkhand 93)

§ 2·22. Partitioning of Matrices.

Submatrix.

Definition. A matrix obtained by striking off some of the rows and columns of another matrix **A** is defined as a sub-matrix of **A**.

For example if $\mathbf{A} = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 5 & 7 \end{bmatrix}$, then

[2], [3], [5] etc.

$\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 5 & 7 \end{bmatrix}$ etc. are all sub– matrices of **A**

It is sometimes found useful to subdivide a matrix into sub-matrices by drawing lines parallel to its rows and columns and to consider these sub-matrices as the elements of the original matrix.

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} x_1\, y_1\, z_1 : \alpha_1\, \beta_1 \\ x_2\, y_2\, z_2 : \alpha_2\, \beta_2 \\ x_3\, y_3\, z_3 : \alpha_3\, \beta_3 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ p_1\, q_1\, r_1 : a_1\, b_1 \\ p_2\, q_2\, r_2 : a_2\, b_2 \end{bmatrix}$$

Let $\mathbf{A}_{11} = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}$; $\mathbf{A}_{12} = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix}$;

$\mathbf{A}_{21} = \begin{bmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{bmatrix}$; $\mathbf{A}_{22} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$

Then we may write $\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$

The matrix **A** is then said to have been **partitioned** and the dotted lines indicate the partitions. Here it is obvious that a matrix can be partitioned in several ways. The elements $\mathbf{A}_{11}$, $\mathbf{A}_{12}$, $\mathbf{A}_{21}$ and $\mathbf{A}_{22}$ are themselves matrices and are the sub-matrices of A.

Identically partitioned matrices.

Two matrices of the same size are known as identically partitioned matrices if when expressed as matrices of matrices (*i.e.* when partitioned) they are of the same order and the corresponding submatrices (or elements) are also of the same size. Such matrices are said to be **additively coherent.**

For example :

$$\begin{bmatrix} 1\,2\,3 : 7\,5 \\ 4\,5\,6 : 9\,8 \\ \cdots\cdots\cdots \\ 2\,3\,4 : 2\,3 \\ 5\,6\,7 : 4\,5 \\ 4\,5\,8 : 6\,7 \end{bmatrix} \text{ and } \begin{bmatrix} 1\,2\,4 : 3\,0 \\ 2\,0\,5 : 4\,6 \\ \cdots\cdots\cdots \\ 1\,0\,2 : 1\,2 \\ 2\,5\,4 : 3\,4 \\ 2\,6\,2 : 5\,6 \end{bmatrix}$$

Two matrices **A** and **B**, which are conformable to the product **AB**, are called **multiplicative coherent** if **A** and **B** are partioned in such a way that columns of **A** are partitioned in the same way as the rows of **B** are partitioned. Here the rows of **A** and columns of **B** can be partitioned in any way.

For example :

Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 3 & 1 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 7 & 1 \\ 4 & 0 & 2 \\ 2 & 5 & 1 \end{bmatrix}$

Here **A** is a 3×4 matrix and **B** is a 4×3 matrix, so these are conformable to the product **AB** (*i.e.* the product **AB** exits). Now if write

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & : & 0 \\ 2 & 1 & 0 & : & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 2 & 3 & 1 & : & 1 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 2 & : & 3 & 5 \\ 3 & : & 7 & 1 \\ 4 & : & 0 & 2 \\ \cdots & \cdots & \cdots & \cdots \\ 2 & : & 5 & 1 \end{bmatrix}$$

then the partitioning of the columns of **A** is in the same way as the partitioning of the rows of **B**. (Here we note that after third column in **A** the partitioning has been done and in **B** the partitioning has been done after third row). Thus according to definition given above the matrices **A** and **B** are called multiplicative coherent.

Exercise on § 2·22

Ex. Compute **AB** using partitioning

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}; \ \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 3 & 1 & 2 \end{bmatrix}.$$

MISCELLANEOUS SOLVED EXAMPLES

Ex. 1. Show that $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$ **is the inverse of** $\begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$

Sol. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix} \times \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1.3 + 2(-4) + 3.2 & 1(-2) + 2.1 + 3.0 & 1(-1) + 2(-1) + 3.1 \\ 2.3 + 5(-4) + 7.2 & 2(-2) + 5.1 + 7.0 & 2(-1) + 5(-1) + 7.1 \\ -2.3 - 4(-4) - 5.2 & -2(-2) - 4.1 - 5.0 & -2(-1) - 4(-1) - 5.1 \end{bmatrix}$$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$, where **I** is an unit matrix.

Hence $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$ is the inverse of $\begin{bmatrix} 3 & -2 & 1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$

***Ex. 2. If A is a non-singular matrix, then prove that $AB = AC \Rightarrow B = C$, where B and C are square matrices of the same order.**

(*Kanpur 96*)

Sol. Since **A** is non-singular matrix, so $\mathbf{A}^{-1}$ exists.

Now $AB = AC \Rightarrow A^{-1}(AB) = A^{-1}(AC)$,

premultiplying both sides by A^{-1}

$\Rightarrow (A^{-1}A)B = (A^{-1}A)C$,

by associative law of multiplication

$\Rightarrow IB = IC, \because A^{-1}A = I$

$\Rightarrow B = C, \because IB = B$ etc. Hence proved.

****Ex. 3. If product of two non-zero square matrices is a zero matrix, then prove that both of them are singular matrices.**

Sol. Let **A** and **B** be two non-zero $n \times n$ matrices.

Given that $AB = O$, where **O** is the $n \times n$ null matrix.

Let us suppose that **B** is non-singluar matrix then B^{-1} exists.

Then $AB = O \Rightarrow (AB)B^{-1} = OB^{-1}$ post multiplying both sides by B^{-1},

$\Rightarrow A(BB^{-1}) = O$, by associative law of multiplication. **(Note)**

$\Rightarrow AI = O, \because BB^{-1} = I$

$\Rightarrow A = O$,

which is against hypothesis as **A** is a non-zero matrix.

Hence **B** is not a non-singular matrix *i.e.* **B** is a singluar matrix.

Similarly we can prove that **A** is also a singluar matrix.

****Ex. 4. Express the following matrix as the sum of a hermitian and a skew hermitian matrix :**

$$A = \begin{bmatrix} 2+3i & 1-i & 2+i \\ 3 & 4+3i & 5 \\ 1 & 1+i & 2i \end{bmatrix}$$ (*Kumaun 92*)

Sol. From § 2·17 Theorem III Page 87 we know that

$$A = \tfrac{1}{2}(A + A^{\Theta}) + \tfrac{1}{2}(A - A^{\Theta}) \quad \text{...(i)}$$

i.e. the hermitian and skew-hermitian parts of the matrix **A** are $\frac{1}{2}(A + A^{\Theta})$ and $\frac{1}{2}(A - A^{\Theta})$ respectively.

Now we know that $A^{\Theta} = (\overline{A})'$, ...(ii)

where $\overline{A} = \begin{bmatrix} 2-3i & 1+i & 2-i \\ 3 & 4-3i & 5 \\ 1 & 1-i & -2i \end{bmatrix}$ **(Note)**

$\therefore$ From (ii) we have $A^{\Theta} = (\overline{A})' =$ transpose of $\overline{A}$

$$= \begin{bmatrix} 2-3i & 3 & 1 \\ 1+i & 4-3i & 1-i \\ 2-i & 5 & -2i \end{bmatrix} \quad \text{...(iii)}$$

$$\therefore \quad \mathbf{A} + \mathbf{A}^{\Theta} = \begin{bmatrix} 2+3i & 1-i & 2+i \\ 3 & 4+3i & 5 \\ 1 & 1+i & 2i \end{bmatrix} + \begin{bmatrix} 2-3i & 3 & 1 \\ 1+i & 4-3i & 1-i \\ 2-i & 5 & -2i \end{bmatrix}$$

$$= \begin{bmatrix} 2+3i+2-3i & 1-i+3 & 2+i+1 \\ 3+1+i & 4+3i+4-3i & 5+1-i \\ 1+2-i & 1+i+5 & 2i-2i \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4-i & 3+i \\ 4+i & 8 & 6-i \\ 3-i & 6+i & 0 \end{bmatrix}$$

$\therefore$ Hermitian part of the given matrix **A**

$$= \tfrac{1}{2}(\mathbf{A} + \mathbf{A}^{\Theta}) = \tfrac{1}{2}\begin{bmatrix} 4 & 4-i & 3+1 \\ 4+i & 8 & 6-i \\ 3-i & 6+i & 0 \end{bmatrix}$$

Again $\mathbf{A} - \mathbf{A}^{\Theta} = \begin{bmatrix} 2+3i & 1-i & 2+i \\ 3 & 4+3i & 5 \\ 1 & 1+i & 2i \end{bmatrix} - \begin{bmatrix} 2-3i & 3 & 1 \\ 1+i & 4-3i & 1-i \\ 2-i & 5 & -2i \end{bmatrix}$

$$= \begin{bmatrix} 2+3i-2+3i & 1-i-3 & 2+i-1 \\ 3-1-i & 4+3i-4+3i & 5-1+i \\ 1-2+i & 1+i-5 & 2i+2i \end{bmatrix}$$

$$= \begin{bmatrix} 6i & -2-i & 1+i \\ 2-i & 6i & 4+i \\ -1+i & -4+i & 4i \end{bmatrix}$$

$\therefore$ Skew-hermitian part of the given matrix **A**

$$= \tfrac{1}{2}(\mathbf{A} - \mathbf{A}^{\Theta}) = \tfrac{1}{2}\begin{bmatrix} 6i & -2-i & 1+i \\ 2-i & 6i & 4+i \\ -1+i & -4+i & 4i \end{bmatrix}$$

Hence from (i), we have the given matrix **A**

$$= \tfrac{1}{2}\begin{bmatrix} 4 & 4-i & 3+i \\ 4+i & 8 & 6-i \\ 3-i & 6+i & 0 \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} 6i & -2-i & 1+i \\ 2-i & 6i & 4+i \\ -1+i & -4+i & 4i \end{bmatrix},$$

which is the sum of a hermitian and a shew-hermitian matrix (as proved above).

EXERCISES ON CHAPTER II

Ex. 1. Show that

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 \\ -2 & 3 & 1 & 1 \end{bmatrix} \text{ is the inverse of } \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 8 & -1 & -1 & 1 \end{bmatrix}$$

(Hint. See Ex. 1 Page 99)

Ex. 2. If **A** be any square matrix, then show that $\mathbf{A} + \mathbf{A}^{\Theta}$ is Hermitian.

Ex. 3. If **A** and **B** are symmetric and they commute, then $\mathbf{A}^{-1}\mathbf{B}$ and $\mathbf{A}^{-1}\mathbf{B}^{-1}$ are symmetric.

Ex. 4. Show that every square matrix can be expressed in one and only one way as $P + iQ$, where P and Q are Hermitian.

Ex. 5. If **B** is any square matrix, show that **B′AB** is symmetric or skew-symmetric according as **A** is symmetric or skew-symmetric provided **B′AB** is defined.

Ex. 6. If **A** and B are two non-singular square matrices of the same order, which of the following statements is true :—

(i) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$;

(ii) $(\mathbf{AB})' = \mathbf{A}'\mathbf{B}'$;

(iii) $(\mathbf{AB})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$;

(iv) $\mathbf{A} \,.\, \mathbf{A}' = \mathbf{I} \Rightarrow \mathbf{A}' = \mathbf{A}^{-1}$

(v) $\mathbf{A} + \mathbf{A}'$ is a symmetric matrix,

Ex. 7. If **A** is Hermitian, such that $\mathbf{A}^2 = \mathbf{O}$, show that $\mathbf{A} = \mathbf{O}$, where **O** is the zero matrix.

Ex. 8. Show that every skew-symmetric matrix of odd order is singular.

Ex. 9. When is a matrix said to be invertible ?

[Hint : See § 2·18 Page 91].

Ex. 10. If $\mathbf{D} = \text{diag}\,[d_1, d_2, \ldots, d_n]$,

$d_1 d_2 \ldots d_n \neq 0$, what will be $\mathbf{D}^{-1}$?

Ex. 11. If non-singular matrices **A** and **B** commute, then

(i) $\mathbf{A}^{-1}$ and **B** and (ii) $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$

also commute.