Chapter VII

Characteristic Equation of a Matrix

§ 7.01. Zero Divisors.

We have previously read in § 5.10 Page 76 of Chapter V that the necessary and sufficient condition for a square matrix to possess an inverse is that it must be non-singular. From this result, we get a very important result which does not hold for ordinary multiplication of numbers viz. 'If **A** and **B** are two singular matrices, it is possible to obtain the result **AB** = **O**, where neither **A** = **O** nor **B** = **O**, **O** being the null matrix'. In such a case **A** and **B** are called **proper divisors of zero.**

If however, **A** and **B** be two square matrices of order n such that

$$\mathbf{AB} = \mathbf{O}, \qquad \text{...(i)}$$

then if **A** is non-singular, $\mathbf{A} \neq \mathbf{O}, \mathbf{A}^{-1}$ exists and $\mathbf{A}^{-1} \neq \mathbf{O}$.

Pre-multiplying (i) with $\mathbf{A}^{-1}$, we get

$$\mathbf{A}^{-1}\mathbf{AB} = \mathbf{A}^{-1}\mathbf{O} \text{ or } \mathbf{IB} = \mathbf{O}, \because \ \mathbf{A}^{-1}.\mathbf{O} = \mathbf{O} \text{ and } \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

or $\qquad \mathbf{B} = \mathbf{O}$.

Hence we conclude that

If $\qquad \mathbf{A} \neq \mathbf{O}$, then $\mathbf{AB} = \mathbf{O} \Rightarrow \mathbf{B} = \mathbf{O}$

Similarly if $\mathbf{B} \neq \mathbf{O}$, then $\mathbf{AB} = \mathbf{O} \Rightarrow \mathbf{A} = \mathbf{O}$

Hence non-singular matrices are not proper divisiors of zero.

Note. If $\mathbf{AB} = \mathbf{O}$ and $\mathbf{B} \neq \mathbf{O}$ then **A** is called a **left zero divisor** and if $\mathbf{AB} = \mathbf{O}$ and $\mathbf{A} \neq \mathbf{O}$, then **B** is called a **right zero divisor**.

§ 7.02. Characteristic equation and roots of a matrix.

(Agra 94; Rohilkhand 92)

Let $\mathbf{A} = [a_{ij}]$ be an $n \times n$ matrix.

(i) Characteristic Matrix of A :— The matrix $\mathbf{A} - \lambda\mathbf{I}$ is called the **characteristic matrix of** A, where **I** is the identity matrix.

(ii) Characteristic polynomial of A :— The determinant $|\mathbf{A} - \lambda\mathbf{I}|$ is called the **characteristic polynomial of** A.

(iii) Characteristic equation of A :— The equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$ is known as the characterisctic equation of **A** and its roots are called the **characteristic roots** or **latent roots** or **eigenvalues** or **characteristic values** or **latent values** or **proper values of A**. *(Avadh 99)*

(iv) Spectrum of A :— The set of all eigen values of the matrix **A** is called the **spectrum of A.**

(v) Eigen value problems :— The problem of finding the eigen values of a matrix is known as an **eigen-value problem.**

Solved Examples on § 7.02.

***Ex. 1. Find the characteristic roots of the matrix**

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{bmatrix}$$

Sol. Here $\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{bmatrix}$ and $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore \quad |\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} \cos\theta - \lambda & -\sin\theta - 0 \\ -\sin\theta - 0 & -\cos\theta - \lambda \end{vmatrix}$$

$$= (\cos\theta - \lambda)(-\cos\theta - \lambda) - (\sin^2\theta)$$

$$= -(\cos^2\theta - \lambda^2) - \sin^2\theta = \lambda^2 - 1.$$

$\therefore$ The characteristic equaiton of the matrix **A** is $\lambda^2 - 1 = 0$ and its roots *i.e.* characterisctic roots are ± 1. **Ans.**

Ex. 2. (a) Find the characteristic roots of the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$ ***(Kanpur 96)***

Sol. Here $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ and $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\therefore \quad |\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 2-0 & 3-0 \\ 0-0 & 2-\lambda & 3-0 \\ 0-0 & 0-0 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 2-\lambda & 3 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 2-\lambda & 3 \\ 0 & 2-\lambda \end{vmatrix}, \text{ expanding with respect to } C_1$$

$$= (1-\lambda)(2-\lambda)^2.$$

$\therefore$ The characteristic equation of the matrix **A** is

$(1-\lambda)(2-\lambda)^2 = 0$ .... See § 7.02 (iii) Page 160 Ch. VII and its roots *i.e.* required characteristic roots are 1, 2, 2. **Ans.**

Ex. 2 (b). Find the characteristic roots of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

Hint : Do as Ex. 2 (a) above. **Ans.** $-2, 4, -2$

Ex. 3. Find the characteristic roots of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Sol. Here $\mathbf{A} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ and $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\therefore \quad |\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 2-\lambda & 2-0 & 1-0 \\ 1-0 & 3-\lambda & 1-0 \\ 1-0 & 2-0 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} -\lambda & 0 & 1 \\ -1 & 1-\lambda & 1 \\ -3+2\lambda & -2+2\lambda & 2-\lambda \end{vmatrix}, \text{ applying } C_1 - 2C_3 \text{ and } C_2 - 2C_3$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ \lambda - 1 & 1-\lambda & 1 \\ -3+4\lambda-\lambda^2 & -2+2\lambda & 2-\lambda \end{vmatrix}, \text{ applying } C_1 - \lambda C_3$$

$$= \begin{vmatrix} \lambda - 1 & 1-\lambda \\ (\lambda-1)(3-\lambda) & -2(1-\lambda) \end{vmatrix} = (\lambda-1)^2 \begin{vmatrix} 1 & -1 \\ 3-\lambda & 2 \end{vmatrix}$$

$$= (\lambda-1)^2 (2+3-\lambda) = (\lambda-1)^2 (5-\lambda).$$

$\therefore$ The characteristic equation of the matrix **A** is $(\lambda-1)^2(5-\lambda) = 0$ and its roots (or characteristic roots of **A**) are 1, 5. **Ans.**

Ex. 4. Obtain the characteristic roots of the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{b} & \mathbf{c} & \mathbf{a} \\ \mathbf{c} & \mathbf{a} & \mathbf{b} \\ \mathbf{a} & \mathbf{b} & \mathbf{c} \end{bmatrix}$$

Sol. Here

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} b-\lambda & c-0 & a-0 \\ c-0 & a-\lambda & b-0 \\ a-0 & b-0 & c-\lambda \end{vmatrix} = \begin{vmatrix} b-\lambda & c & a \\ c & a-\lambda & b \\ a & b & c-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} b-\lambda+c+a & c & a \\ c+a-\lambda+b & a-\lambda & b \\ a+b+c-\lambda & b & c-\lambda \end{vmatrix}, \text{ applying } C_1 + C_2 + C_3$$

$$= (a+b+c-\lambda) \begin{vmatrix} 1 & c & a \\ 1 & a-\lambda & b \\ 1 & b & c-\lambda \end{vmatrix}$$

$$= (a+b+c-\lambda) \begin{vmatrix} 1 & c & a \\ 0 & a-\lambda-c & b-a \\ 0 & b-c & c-\lambda-a \end{vmatrix}, \text{applying } R_2 - R_1, R_3 - R_1$$

$$= (a+b+c-\lambda)\,[(a-c-\lambda)(c-a-\lambda) - (b-a)(b-c)]$$

$$= (a+b+c-\lambda)\,[\,(a-c)(c-a) - \lambda\,\{(c-a)+(a-c)\} + \lambda^2 - (b-a)(b-c)]$$

$$= (a+b+c-\lambda)(\lambda^2 + 2ac - a^2 - c^2 - b^2 + bc + ab - ac]$$

$$= (a+b+c-\lambda)(\lambda^2 - a^2 - b^2 - c^2 + ab + bc + ca).$$

$\therefore$ The characteristic equation of **A** is $|\mathbf{A} - \lambda\mathbf{I}| = 0$

i.e. $(a+b+c-\lambda)(\lambda^2 - a^2 - b^2 - c^2 + ab + bc + ca) = 0$

and the characteristic roots of A are

$$a+b+c, \pm\sqrt{(a^2+b^2+c^2-ab-bc-ca)}$$ **Ans.**

Ex. 5. Find the eigenvalues (or latent roots) of

$$\mathbf{A} = \begin{bmatrix} 8 & 6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

(Avadh 99; Kumaun 91, Lucknow 90)

Sol. Here $|\mathbf{A} - \lambda\,\mathbf{I}| = \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix}$

$$= (8-\lambda)\{(7-\lambda)(3-\lambda) - 16\} + 6\{-6(3-\lambda) + 8\} + 2\{24 - 2(7-\lambda)\}$$

$$= (8-\lambda)(5 - 10\lambda + \lambda^2) + 6(6\lambda - 10) + 2(10 - 2\lambda)$$

$$= -45\lambda + 18\lambda^2 - \lambda^3.$$

$\therefore$ The characteristic equation of **A** is $-45\lambda + 18\lambda^2 - \lambda^3 = 0$

which gives

$$\lambda(\lambda^2 - 18\lambda + 45) = 0 \text{ or } \lambda(\lambda-3)(\lambda-15) = 0 \text{ or } \lambda = 0, 3, 15.$$

$\therefore$ The required latent roots or eigenvalues of **A** are 0, 3 and 15.

Ex. 6. If $a_1, a_2, a_3, \ldots, a_n$ are the characteristic roots of the n-square matrix A and μ is a scalar, then show that the characteristic roots of $A - \mu I$ are $a_1 - \mu, \ldots, a_n - \mu$.

Sol. Since $a_1, a_2, \ldots, a_n$ are the characteristic roots of the matrix **A**, so form § 7.02 Page 162 we have

$$|\mathbf{A} - \lambda\mathbf{I}| = (\lambda - a_1)(\lambda - a_2)\ldots(\lambda - a_n). \qquad \ldots(i)$$

Now the characteristic function of $\mathbf{A} - \mu\mathbf{I}$

$$= |(\mathbf{A} - \mu\mathbf{I}) - \lambda\mathbf{I}| = |\mathbf{A} - (\mu+\lambda)\,\mathbf{I}|$$

$$= \{(\mu+\lambda) - a_1\}\{(\mu+\lambda) - a_2\}\{(\mu+\lambda) - a_3\}\ldots\{(\mu+\lambda) - a_n\}, \text{ from (i)}$$

$$= \{\lambda - (a_1 - \mu)\}\{\lambda - (a_2 - \mu)\}\ldots\{(\lambda - (a_n - \mu)\},$$

rearranging the terms in each bracket.

$\therefore$ The characteristic roots of $\mathbf{A} - \mu\mathbf{I}$ are $(a_1 - \mu), (a_2 - \mu)$, etc.

Exercises on § 7.02

Ex. 1. Find the characteristic roots of $\mathbf{A} = \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}$ **Ans.** a_1, b_2, c_3

Ex. 2. Find the eigenvalues of the matrix $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$ **Ans.** $2, -1 \pm \sqrt{3}$.

Ex. 3. Find the eigen values of the matrix $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ **Ans.** $5, -3, -3$.

***Ex. 4.** Find the eigenvalues of the matrix $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$ **Ans.** $1, -1 \pm \sqrt{2}$

Ex. 5. Find the characteristic roots of the matrix

$$\begin{bmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Ans. $1, 3, -4$

Ex. 6. Determine the characteristic equation and roots of the matrix

$$\begin{bmatrix} 1 & -1 & 4 \\ 0 & 3 & 7 \\ 0 & 0 & 5 \end{bmatrix}$$

Ans. $(\lambda - 1)(\lambda - 3)(\lambda - 5) = 0$; 1, 3, 5.

Ex. 7. Find the eigenvalues of the matrix

$$\begin{bmatrix} -3 & 2 & 2 \\ -6 & 5 & 2 \\ -7 & 4 & 4 \end{bmatrix}$$

Ans. 1, 2, 3.

Ex. 8. Find the eigenvalues of the matrix

$$\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

(Agra 92; Lucknow 92) **Ans.** 2,3,5

Ex. 9. Find the latent roots of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & \sin\alpha & \cos\alpha\sin\beta \\ -\sin\alpha & 0 & \cos\alpha\cos\beta \\ -\cos\alpha\sin\beta & \cos\alpha\cos\beta & 0 \end{bmatrix}$$

Ans. $0, i, -i$.

Ex. 10. Show that the matrices

$$\begin{bmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{bmatrix}, \begin{bmatrix} 0 & a & c \\ a & 0 & b \\ c & b & 0 \end{bmatrix}, \begin{bmatrix} 0 & b & a \\ b & 0 & c \\ a & c & 0 \end{bmatrix}$$

have the same characteristic equation. *(Kumaun 90)*

§ 7.03. †Cayely Hamilton Theorem *(Agra 95, 93, 91; Avadh 99; Garhwal 96; Kanpur 97, 90; Kumaun 92; Lucknow 92; Meerut 98, 92, 91; Rohilkhand 94, 93, 92, 90)*

Statement : *Every square matrix satisfies its characteristic equation or if* $|\mathbf{A} - \lambda\mathbf{I}| = (-1)^n [\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n]$ *be the characteristic polynomial of* $n \times n$ *matrix* $\mathbf{A} = [a_{ij}]$, *then the matrix equation* $\mathbf{X}_n + a_1 \mathbf{X}^{n-1} + \ldots + a_n \mathbf{I} = \mathbf{O}$ *is satisfied by* $\mathbf{X} = \mathbf{A}$

i.e.
$$\mathbf{A}^n + a_1 \mathbf{A}^{n-1} + \ldots + a_n \mathbf{I} = \mathbf{O}.$$

Proof. $\because$ the elements $(\mathbf{A} - \lambda\mathbf{I})$ are at the most of first degree in λ.

$\therefore$ The elements of Adj $(\mathbf{A} - \lambda\mathbf{I})$ are at the most of degree $(n-1)$ in λ and the coefficients of various powers of λ being polymomials in the a_{ij}.

$\therefore$ Adj $(\mathbf{A} - \lambda\mathbf{I})$ can be written as

$$\mathbf{B} = \mathbf{B}_0 \lambda^{n-2} + \mathbf{B}_1 \lambda^{n-1} + \ldots + \mathbf{B}_{n-1},$$

where $\mathbf{B}_0, \mathbf{B}_1, \ldots, \mathbf{B}_{n-1}$ are $n \times n$ matrices, their elements being polynomials in a_{ij}.

Also from § 5.09 Page 49 Ch. V we know that if $\mathbf{A} = [a_{ij}]$ be an $n \times n$ matrix, then $\mathbf{A} \bullet (\text{Adj } \mathbf{A}) = (\text{Adj } \mathbf{A}) \bullet \mathbf{A} = |\mathbf{A}| \bullet \mathbf{I}_0$, where $\mathbf{I}$ is an $n \times n$ identity matrix.

Therefore $(\mathbf{A} - \lambda\mathbf{I}) \bullet \text{Adj } (\mathbf{A} - \lambda\mathbf{I}) = |\mathbf{A} - \lambda\mathbf{I}| \bullet \mathbf{I}$

or
$$(\mathbf{A} - \lambda\mathbf{I}) \bullet \mathbf{B} = |\mathbf{A} - \lambda\mathbf{I}| \bullet \mathbf{I}, \quad \because \mathbf{B} = \text{Adj } (\mathbf{A} - \lambda\mathbf{I})$$

or
$$(\mathbf{A} - \lambda\mathbf{I})(\mathbf{B}_0 \lambda^{n-1} + \mathbf{B}_1 \lambda^{n-2} + \ldots + \mathbf{B}_{n-1})$$
$$= (-1)^n [\lambda^n + a_1\lambda^{n-1} + a_2 \lambda^{n-2} + \ldots a_n] \mathbf{I}.$$

Comparing coefficients of like powers of λ on both sides, we get

$$-\mathbf{I}\mathbf{B}_0 = (-1)^n \mathbf{I};$$
$$\mathbf{A}\mathbf{B}_0 - \mathbf{I}\mathbf{B}_1 = (-1)^n a_1 \mathbf{I};$$
$$\mathbf{A}\mathbf{B}_1 - \mathbf{I}\mathbf{B}_2 = (-1)^n a_2 \mathbf{I};$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\mathbf{A}\mathbf{B}_{n-1} = (-1)^n a_n \mathbf{I}.$$

Now pre-multiplying these equations by $\mathbf{A}^n, \mathbf{A}^{n-1}, \ldots, \mathbf{A}, \mathbf{I}$ respectively and adding the results so obtained we get

$$\mathbf{A}^n (-\mathbf{I}\mathbf{B}_0) + \mathbf{A}^{n-1} (\mathbf{A}\mathbf{B}_0 - \mathbf{I}\mathbf{B}_1) + \mathbf{A}^{n-2} (\mathbf{A}\mathbf{B}_1 - \mathbf{I}\mathbf{B}_2) + \ldots + \mathbf{I} (\mathbf{A}\mathbf{B}_{n-1})$$
$$= (-1)^n [\mathbf{I}\mathbf{A}^n + a_1 \mathbf{I}\mathbf{A}^{n-1} + a_2 \mathbf{I}\mathbf{A}^{n-2} + a_n \mathbf{I} \bullet \mathbf{I}]$$

†This theorem was first established by Hamilton in 1883 for a particular type of matrices and was later on stated by Cayley in 1885.

or $\quad \mathbf{O} = (-1)^n [\mathbf{A}^n + a_1 \mathbf{A}^{n-1} + a_2 \mathbf{A}^{n-2} + \ldots + a_n \mathbf{I}]$,

where **O** is the null matrix.

Hence $\mathbf{A}^n + a_1 \mathbf{A}^{n-1} + a_2 \mathbf{A}^{n-2} + \ldots + a_n \mathbf{I} = \mathbf{O}$. Hence the theorem.

Cor. I. Multiplying the result of § 7.03 above by $\mathbf{A}^{m-n}$, where $m \geq n$ and m is a positive integer, we get

$$\mathbf{A}^m + a_1 \mathbf{A}^{m-1} + a_2 \mathbf{A}^{m-2} + \ldots + a_n \mathbf{A}^{m-n} = \mathbf{O}$$

i.e. any positive integral power $\mathbf{A}^m$ of **A** can be linearly expressed in terms of $\mathbf{I}, \mathbf{A}, \ldots, \mathbf{A}^{n-1}$.

Cor. II. In § 7.03 above we have proved that

$$\mathbf{A}^n + a_1 \mathbf{A}^{n-1} + a_2 \mathbf{A}^{n-2} + \ldots + a_n \mathbf{I} = \mathbf{O} \qquad \ldots(i)$$

or $\quad -a_n \mathbf{I} = \mathbf{A} [\mathbf{A}^{n-1} + a_1 \mathbf{A}^{n-2} + a_2 \mathbf{A}^{n-3} + \ldots + a^{n-1} \mathbf{I}]$ **(Note)**

or $\quad -a_n \mathbf{A}^{-1} \mathbf{I} = \mathbf{A}^{n-1} + a_1 \mathbf{A}^{n-2} + \ldots + a_{n-1} \mathbf{I} \qquad \ldots(ii)$

or $\quad (-1)^n \mathbf{A}\mathbf{B}_{n-1} \mathbf{A}^{n-1} = -\mathbf{A}^{n-1} - a_1 \mathbf{A}^{n-2} + \ldots - a_{n-1} \mathbf{I}$,

$\because \mathbf{A}\mathbf{B}_{n-1} = (-1)^n a_n \mathbf{I}$ (§ 7.03 above)

or $\quad \mathbf{B}_{n-1} = (-1)^n [-\mathbf{A}^{n-1} - a_1 \mathbf{A}^{n-2} - \ldots - a_{n-1} \mathbf{I}] = \text{Adj } \mathbf{A} \qquad \ldots(iii)$

Cor. III. From result (ii) of cor. II above we have

$$\mathbf{A}^{-1} = -\frac{1}{a_n} [\mathbf{A}^{n-1} + a_1 \mathbf{A}^{n-2} + \ldots + a_{n-1} \mathbf{I}] \qquad \ldots(iv)$$

which show that $\mathbf{A}^{-1}$ can be expressed linearly in terms of $\mathbf{A}^{n-1}, \mathbf{A}^{n-2}, \ldots, \mathbf{I}$.

§ 7.04. Characteristic vectors (or Eigenvectors). *(Agra 94)*

Let us consider the linear transformation

$$\mathbf{K} = \mathbf{A}\mathbf{X} \qquad \ldots(i)$$

which transforms a column vector **X** by means of a square matrix A into another column vector **K**.

If **X** be a vector which transforms to its multiple $\mu \mathbf{X}$ by the above transformation (i), then we have $\mu \mathbf{X} = \mathbf{A}\mathbf{X}$ ...(ii)

or $\quad \mathbf{A}\mathbf{X} - \mu \mathbf{I}\mathbf{X} = \mathbf{O}$ or $(\mathbf{A} - \mu \mathbf{I}) \mathbf{X} = \mathbf{O}$. ...(iii)

This equation (iii) when written in full gives n homogeneous equations in $x_1, x_2, \ldots, x_n$ which are n unknowns. The n equations will have a non-zero solution only if $|\mathbf{A} - \mu \mathbf{I}| = 0$ *i.e.* the coefficients matrix is singular. (Note)

This equation is called the characteristic equation of transformation and is the same as the characteristic equation of the matrix **A**. (See § 7.02 Page 160 Ch, VII). This equation has n roots and corresponding to each root, the equation (iii) has a non-zero solution

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_n \end{bmatrix}$$

which is defined as **characteristic vector** or **Eigenvector** or **latent vector** or **invariant vector.**

****§ 7.05. Theorems on latent roots (or characteristics roots).**

Theorem I. *If square matrix* **A** *of order n has latent roots* $\lambda_1, \lambda_2, ..., \lambda_n$ ***then*** **A′** *has also the same latent roots.* ***(Kumaun 95)***

Proof. Let $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots\cdots & a_{nn} \end{bmatrix}$

∴ The characteristic equation of **A** is

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0 \quad ...(i)$$

Also $\mathbf{A}' = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots\cdots & a_{n1} \\ a_{12} & a_{22} & a_{32} & \cdots\cdots & a_{n2} \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots\cdots & a_{nn} \end{bmatrix}$

∴ The characteristic equation of **A′** is

$$|\mathbf{A}' - \lambda \mathbf{I}| = \begin{vmatrix} a_{11} - \lambda & a_{21} & \cdots\cdots & a_{n1} \\ a_{12} & a_{22} - \lambda & \cdots\cdots & a_{n2} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ a_{1n} & a_{2n} & \cdots\cdots & a_{nn} - \lambda \end{vmatrix} = 0 \quad ...(ii)$$

Also we know that the value of a determinant remains unaltered if rows are changed into columns and thus we find that the determinants given by (i) and (ii) are the same, the diagonal elements being the same.

Hence from (i) and (ii) we conclude that the characteristic equations of **A** and **A′** are the same. Consequently the latent roots of **A** and **A′** are the same.

Theorem II. *If* **A** *is an* $n \times n$ *triangular matrix, then the elements of the principal diagonals are the characteristic roots of* **A**.

Proof. Let $A' = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots\dots\dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots\dots\dots & a_{2n} \\ 0 & 0 & a_{33} & \dots\dots\dots & a_{3n} \\ \dots & \dots & \dots & \dots\dots\dots & \dots \\ 0 & 0 & 0 & \dots\dots\dots & a_{nn} \end{bmatrix}$

(Here we have taken upper triangular matrix).

$\therefore$ **The characteristic equation of A is**

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22}-\lambda & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33}-\lambda & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{nn}-\lambda \end{vmatrix} = 0$$

or $$(a_{11}-\lambda)\begin{vmatrix} a_{22}-\lambda & a_{23} & \dots & a_{2n} \\ 0 & a_{33}-\lambda & \dots & a_{3n} \\ \dots\dots & \dots\dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn}-\lambda \end{vmatrix} = 0,$$ expanding with respect to C_1

or $$(a_{11}-\lambda)(a_{22}-\lambda)\begin{vmatrix} a_{33}-\lambda & a_{34} & \dots & a_{3n} \\ 0 & a_{44}-\lambda & \dots & a_{4n} \\ \dots\dots & \dots\dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn}-\lambda \end{vmatrix} = 0,$$ expanding with respect to C_2

or $(a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda)\dots(a_{nn}-\lambda) = 0$, proceeding in this way

or $$\lambda = a_{11}, a_{22}, a_{33}, \dots, a_{nn}$$

i.e. the element roots (or characteristic roots) of **A** are the elements of the principal diagonal of **A**. Hence proved.

Theorem III. *The characteristic roots of a hermitian matrix are all real.*

(Agra 95, Kanpur 95, 94)

Proof. Let **A** be the hermitian matrix. Then from § 7.04 (ii) Page 166 Ch. VII] we know that $\mathbf{AX} = \lambda\mathbf{X}$, ...(i)

where λ is a characteristic root of **A** and **X** the corresponding characteristic vector.

From (i) we get $\mathbf{X}^{\Theta}\mathbf{AX} = \mathbf{X}^{\Theta}\lambda\mathbf{X}$, premultiplying both sides by $\mathbf{X}^{\Theta}$

or $$\mathbf{X}^{\Theta}\mathbf{AX} = \lambda\mathbf{X}^{\Theta}\mathbf{X}, \quad \text{...(ii)}$$

where $\mathbf{X}^{\Theta}$ is the transponsed conjugate of **X** (See chapter II)

Also if **A** is a hermitian matrix, then by definition we have

$$\mathbf{A}^{\Theta} = \mathbf{A} \text{ (See Chapter II)}. \quad \text{...(iii)}$$

Now taking transposed conjugate of both sides of (ii) we get

$$X^{\Theta}AX = \bar{\lambda}\, X^{\Theta} X, \text{ using (iii) also} \quad ...(iv)$$

$\therefore$ From (ii) and (iv) we get $\lambda X^{\Theta} X = \bar{\lambda}\, X^{\Theta} X$

or $\quad (\lambda - \bar{\lambda})\, X^{\Theta} X = O \quad$ or $\quad \lambda - \bar{\lambda} = 0, \qquad \because X^{\Theta} X \neq O$

or $\quad \lambda = \bar{\lambda}$ or λ is real. Hence proved.

****Theorem IV.** *The characteristic roots of a real symmetric matrix are all real.*

Proof. Do as Theorem III above. Here all the elements of **A** are real and as such it is particular case of Theorem III above.

****Theorem V.** *The characteristic roots of a skew-hermitian matrix are either purely imaginary or zero.*

Proof. Let **A** be a skew-hermitian matrix, then (see Chapter II) we know that $i\mathbf{A}$ is hermitian.

If λ be a charcteristic root of **A**, then $|\mathbf{A} - \lambda \mathbf{I}| = 0$

or $\quad i\,|\mathbf{A} - \lambda\, \mathbf{I}| = 0$ or $|\, i\mathbf{A} - (i\lambda)\, \mathbf{I}| = 0$, where $i\mathbf{A}$ is hermitian.

or . $(i\lambda)$ is real, since the characteristic roots of a hermitian matrix are all real.

(See Theorem III Page 168 Ch. VII)

or λ is either purely imaginary or zero.

i.e. the characteristic roots of a skew hermitian matrix **A** are either purely imaginary or zero. Hence proved.

Theorem VI. *The characteristic roots of real skew-symmetric matrix are purely imaginary or zero.*

Proof. Do as Theorem **V** above. Here all the elements of **A** are real and as such it is a particular case of Theorem **V** above.

****Theorem VII.** *The characteristic roots of a unitary matrix are of unit modulus.*

Proof. Let **A** be a unitary matrix (See Chapter II). Let λ be a characteristic root of **A** and **X** the corresponding characteristic vector.

Then $\quad AX = \lambda X \quad ...(i) \quad$ (See § 7.04 (ii) Page 166 Ch. VII)

Taking transposed conjugate of both sides of (i), we get

$$(AX)^{\theta} = (\lambda X)^{\theta} \text{ or } \quad X^{\theta} A^{\theta} = \bar{\lambda}\, X^{\theta} \quad ...(ii)$$

$\therefore$ From (i) and (ii) we get $X^{\theta} A^{\theta} AX = \bar{\lambda}\, X^{\theta} \lambda X$ **(Note)**

or $\quad X^{\theta} (A^{\theta} A)\, X = \bar{\lambda} \lambda\, X^{\theta} X$

or $\quad X^{\theta} (I)\, X = \bar{\lambda} \lambda\, X^{\theta} X, \qquad \because$ **A** is unitary (See Chapter II)

or $\quad X^{\theta} X\, (1 - \bar{\lambda} \lambda) = O$

or $\quad 1 - \lambda \bar{\lambda} = 0, \qquad \because X^{\theta} X \neq O$

or $\quad \lambda \bar{\lambda} = 1 \quad$ or $|\lambda|^2 = \lambda \bar{\lambda} = \mathbf{I}$.

i.e the characteristic roots of **A** are of unit modulus. Hence proved.

****Theorem VIII.** *The characterise roots of an orthogonal matrix are of unit modulus.*

Proof. Do as Theorem VII above remembering that if all the elements of the unitary matrix **A** are real then **A** is an orthogonal matrix.

***§ 7.06. An Important Theorem.**

The scalar λ is a characteristic root of the matrix **A** *if and only if the matrix* $\mathbf{A} - \lambda\mathbf{I}$ *is singular.*

Proof. Let $\mathbf{A} = [a_{ij}]_{n \times n}$ $\mathbf{X} = [x_1, x_2, \ldots, x_n]$.

then $$\mathbf{AX} = \lambda\mathbf{X}$$ ...See § 7.04 Page 166

reduces to
$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$$

$$= \lambda \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$$

or
$$\left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \mathbf{O}$$

or
$$\begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \mathbf{O}$$

or $$(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{X} = \mathbf{O}$$

i.e. $$\mathbf{AX} = \lambda\mathbf{X} \Rightarrow (\mathbf{A} - \lambda\,\mathbf{I})\,\mathbf{X} = \mathbf{O},$$

which is a homogeneous system of linear equations whose coefficient matrix is $\mathbf{A} - \lambda\mathbf{I}$.

Now as we require a vector $\mathbf{X} \neq \mathbf{O}$, so we must have
$$|\mathbf{A} - \lambda\mathbf{I}| = 0$$

i.e. the matrix $\mathbf{A} - \lambda\mathbf{I}$ must be singular.

Solved Examples on § 7.03 to § 7.06

Ex. 1. (a). Verify Cayley Hamilton's Theorem for the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

(Agra 93)

Sol. Here $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\therefore \quad |\mathbf{A} - \lambda\,\mathbf{I}| = \begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix}, \text{ expanding w.r. to } C_1$$

$$= (1-\lambda)^3$$

$\therefore$ Characteristic equation of $\mathbf{A}$ is $(1-\lambda)^3 = 0$

or $$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0 \qquad \text{...(i)}$$

Now $$\mathbf{A}^2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}^3 = \mathbf{A}^2 . \mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad \mathbf{A}^3 - 3\mathbf{A}^2 + 3\mathbf{A} - \mathbf{I}$$

$$= \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 6 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-3+3-1 & 0-0+0-0 & 3-6+3-0 \\ 0-0+0-0 & 1-3+3-1 & 0-0+0-0 \\ 0-0+0-0 & 0-0+0-0 & 1-3+3-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O},$$

where $\mathbf{O}$ is the null matrix.

Hence the given matrix $\mathbf{A}$ satisfies its characteristics equation given by (i).

Hence proved.

Ex. 1 (b) Show that the matrix $\mathbf{A}=\begin{bmatrix}1 & 2 & 1\\-1 & 0 & 3\\2 & -1 & 1\end{bmatrix}$ **satisfies Cayley Hamilton Theorem.** *(Meerut 92 P)*

Sol. Here $\mathbf{I}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ and $\mathbf{A}=\begin{bmatrix}1 & 2 & 1\\-1 & 0 & 3\\2 & -1 & 1\end{bmatrix}$

$$\therefore \quad |\mathbf{A}-\lambda\mathbf{I}|=\begin{vmatrix}1-\lambda & 2 & 1\\-1 & -\lambda & 3\\2 & -1 & 1-\lambda\end{vmatrix} \qquad \textbf{(Note)}$$

$$=\begin{vmatrix}1-\lambda & 2 & 1\\-1 & -\lambda & 3\\0 & -1-2\lambda & 7-\lambda\end{vmatrix}, \text{ adding } 2R_2 \text{ to } R_3$$

$$=(1-\lambda)\begin{vmatrix}-\lambda & 3\\-1-2\lambda & 7-\lambda\end{vmatrix}+\begin{vmatrix}2 & 1\\-1-2\lambda & 7-\lambda\end{vmatrix}$$

$$=(1-\lambda)[-7\lambda+\lambda^2+3+6\lambda]+[14-2\lambda+1+2\lambda]$$

$$=(1-\lambda)(\lambda^2-\lambda+3)+15=-\lambda^3+2\lambda^2-4\lambda+18$$

$\therefore$ Characteristic equation of **A** is

$$\lambda^3-2\lambda^2+4\lambda-18=0$$

Now $$\mathbf{A}^2=\begin{bmatrix}1 & 2 & 1\\-1 & 0 & 3\\2 & -1 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 1\\-1 & 0 & 3\\2 & -1 & 1\end{bmatrix}=\begin{bmatrix}1 & 1 & 8\\5 & -5 & 2\\5 & 3 & 0\end{bmatrix}$$

Now $$\mathbf{A}^3=\mathbf{A}^2\mathbf{A}=\begin{bmatrix}1 & 1 & 8\\5 & -5 & 2\\5 & 3 & 0\end{bmatrix}\begin{bmatrix}1 & 2 & 1\\-1 & 0 & 3\\2 & -1 & 1\end{bmatrix}=\begin{bmatrix}16 & -6 & 12\\14 & 8 & -8\\2 & 10 & 14\end{bmatrix}$$

$$\therefore \quad \mathbf{A}^3-2\mathbf{A}^2+4\mathbf{A}-18\mathbf{I}$$

$$=\begin{bmatrix}16 & -6 & 12\\14 & 8 & -8\\2 & 10 & 14\end{bmatrix}-2\begin{bmatrix}1 & 1 & 8\\5 & -5 & 2\\5 & 3 & 0\end{bmatrix}+4\begin{bmatrix}1 & 2 & 1\\-1 & 0 & 3\\2 & -1 & 1\end{bmatrix}-18\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}16 & -6 & 12\\14 & 8 & -8\\2 & 10 & 14\end{bmatrix}+\begin{bmatrix}-2 & -2 & -16\\-10 & 10 & -4\\-10 & -6 & 0\end{bmatrix}+\begin{bmatrix}4 & 8 & 4\\-4 & 0 & 12\\8 & -4 & 4\end{bmatrix}+\begin{bmatrix}-18 & 0 & 0\\0 & -18 & 0\\0 & 0 & -18\end{bmatrix}$$

$$=\begin{bmatrix}16-2+4-18 & -6-2+8+0 & 12-16+4+0\\14-10-4+0 & 8+10+0-18 & -8-4+12+0\\2-10+8+0 & 10-6-4+0 & 14+0+4-18\end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O} \text{ where } \mathbf{O} \text{ is the null matrix}$$

Hence the given matrix **A** satisfies its characteristic equaiton given by (i).

Ex. 2 (a). Use Cayley Hamilton Theorem to find the inverse of the matrix $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{1} & \mathbf{3} & \mathbf{5} \\ \mathbf{1} & \mathbf{5} & \mathbf{12} \end{bmatrix}$ *(Rohilkhand 95)*

Sol. Here $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$

$$\therefore \quad |\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 1 & 3-\lambda & 5 \\ 1 & 5 & 12-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 1-\lambda & 2 & 3 \\ 1 & 3-\lambda & 5 \\ 0 & 2+\lambda & 7-\lambda \end{vmatrix}, \text{ applying } R_3 - R_2$$

$$= (1-\lambda)[(3-\lambda)(7-\lambda) - 5(2+\lambda)] - [2(7-\lambda) - 3(2+\lambda)],$$

expanding w.r. to C_1

$$= (1-\lambda)[21 - 10\lambda + \lambda^2 - 10 - 5\lambda] - [14 - 2\lambda - 6 - 3\lambda]$$

$$= (1-\lambda)(11 - 15\lambda + \lambda^2) - (8 - 5\lambda)$$

$$= 3 - 21\lambda + 16\lambda^2 - \lambda^3, \text{ on simplifying.}$$

$\therefore$ The characteristic equation of **A** is

$$\lambda^3 - 16\lambda^2 + 21\lambda - 3 = 0. \qquad \text{...(i)}$$

Now as **A** must satisfy its characteristic equation (i), so we have

$$\mathbf{A}^3 - 16\mathbf{A}^2 + 21\mathbf{A} - 3\mathbf{I} = \mathbf{0}$$

or

$$3\mathbf{I} = \mathbf{A}^3 - 16\mathbf{A}^2 + 21\mathbf{A}$$

Multiplying both sides by $\mathbf{A}^{-1}$, we get

$$3\mathbf{A}^{-1} = \mathbf{A}^2 - 16\mathbf{A} + 21\mathbf{I}, \qquad \because \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

or

$$\mathbf{A}^{-1} = (1/3)\mathbf{A}^2 - (16/3)\mathbf{A} + 7\mathbf{I}. \qquad \text{...(ii)}$$

Now $\mathbf{A}^2 = \mathbf{A} \cdot \mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$

$$= \begin{bmatrix} 1+2+3 & 2+6+15 & 3+10+36 \\ 1+3+5 & 2+9+25 & 3+15+60 \\ 1+5+12 & 2+15+60 & 3+25+144 \end{bmatrix} = \begin{bmatrix} 6 & 23 & 49 \\ 9 & 36 & 78 \\ 18 & 77 & 172 \end{bmatrix}.$$

$\therefore$ From (ii), we get

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 6 & 23 & 49 \\ 9 & 36 & 78 \\ 18 & 77 & 172 \end{bmatrix} - \frac{16}{3}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-(16/3)+7 & (23/6)-(32/3)+0 & (49/3)-(48/3)+0 \\ 3-(16/3)+0 & 12-16+7 & 26-(80/3)+0 \\ 6-(16/3)+0 & (77/3)-(80/3)+0 & (172/3)-(192/3)+7 \end{bmatrix}$$

$$= \begin{bmatrix} 11/3 & -3 & 1/3 \\ -7/3 & 3 & -2/3 \\ 2/3 & -1 & 1/3 \end{bmatrix} = (1/3)\begin{bmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$ **Ans.**

Ex. 2 (b). Verify Cayley Hamilton's Theorem for the matrix

$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$. **Hence compute** $\mathbf{A}^{-1}$

(Agra 94; Kanpur 96; Meerut 96)

Sol. Here $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$

$$\therefore \quad |\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 0-\lambda & 0 & 1 \\ 3 & 1-\lambda & 0 \\ -2 & 1 & 4-\lambda \end{vmatrix}$$

$$= -\lambda\{(1-\lambda)(4-\lambda)\} + 1\{3+2(1-\lambda)\}$$

$$= -\lambda(4-5\lambda+\lambda^2) + 5 - 2\lambda$$

$$= 5 - 6\lambda + 5\lambda^2 - \lambda^3$$

$\therefore$ The characteristic equation of **A** is

$$\lambda^3 - 5\lambda^2 + 6\lambda - 5 = 0 \quad \text{...(i)}$$

Also we have

$$\mathbf{A}^2 = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ -5 & 5 & 14 \end{bmatrix} \quad \text{...(ii)}$$

and $$\mathbf{A}^3 = \mathbf{A}^2 . \mathbf{A} = \begin{bmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ -5 & 5 & 14 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} -5 & 5 & 14 \\ -3 & 4 & 15 \\ -13 & 19 & 51 \end{bmatrix}$$

$$\therefore \quad \mathbf{A}^3 - 5\mathbf{A}^2 + 6\mathbf{A} - 5\mathbf{I}$$

$$= \begin{bmatrix} -5 & 5 & 14 \\ -3 & 4 & 15 \\ -13 & 19 & 51 \end{bmatrix} - 5\begin{bmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ -5 & 5 & 14 \end{bmatrix} + 6\begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 5 & 14 \\ -3 & 4 & 15 \\ -13 & 19 & 51 \end{bmatrix} - \begin{bmatrix} -10 & 5 & 20 \\ 15 & 5 & 15 \\ -25 & 25 & 70 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 6 \\ 18 & 6 & 0 \\ -12 & 6 & 24 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -5+10+0-5 & 5-5+0-0 & 14-20+6-0 \\ -3-15+18-0 & 4-5+6-5 & 15-15+0-0 \\ -13+25-12-0 & 19-25+6-0 & 51-70+24-5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O}, \text{ where } \mathbf{O} \text{ is the null matrix.}$$

Hence the matrix **A** satisfies its characterstic equation given by (i). Hence Cayley-Hamilton's Theorem is satisfied by the matrix **A**.

Again $$\mathbf{A}^3 - 5\mathbf{A}^2 + 6\mathbf{A} - 5\mathbf{I} = \mathbf{O}$$

or $$5\mathbf{I} = \mathbf{A}^3 - 5\mathbf{A}^2 + 6\mathbf{A}$$

Multiplying both sides by $\mathbf{A}^{-1}$, we get

$$5\mathbf{A}^{-1} = \mathbf{A}^2 - 5\mathbf{A} + 6\mathbf{I}, \qquad \because \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

$$= \begin{bmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ -5 & 5 & 14 \end{bmatrix} - 5 \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} + 6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ from (ii)}$$

$$= \begin{bmatrix} -2 & 1 & 4 \\ 3 & 1 & 3 \\ -5 & 5 & 14 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -5 \\ -15 & -5 & 0 \\ 10 & -5 & -20 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -2+0+6 & 1+0+0 & 4-5+0 \\ 3-15+0 & 1-5+6 & 3+0+0 \\ -5+10+0 & 5-5+0 & 14-20+6 \end{bmatrix} = \begin{bmatrix} 4 & 1 & -1 \\ -12 & 2 & 3 \\ 5 & 0 & 0 \end{bmatrix}$$

or $$\mathbf{A}^{-1} = \frac{1}{5} \begin{bmatrix} 4 & 1 & -1 \\ -12 & 2 & 3 \\ 5 & 0 & 0 \end{bmatrix}$$ **Ans.**

Ex. 3 (a). Verify Cayley-Hamilton Theorem for the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ **and hence find** $\mathbf{A}^{-1}$.

(Kanpur 97)

Sol. Here we have

$$[\mathbf{A} - \lambda\mathbf{I}] = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{bmatrix}$$

$$\therefore \quad |\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix}$$

$$= (-1-\lambda)\,[(1-\lambda)(-1-\lambda) - 4], \text{ expanding w.r. to } C_3$$

$$= (-1-\lambda)[-1+\lambda^2-4] = (1+\lambda)(5-\lambda^2)$$
$$= 5 + 5\lambda - \lambda^2 - \lambda^3.$$

∴ The characteristic equation of **A** is $\lambda^3 + \lambda^2 - 5\lambda - 5 = 0$ ...(i)

Also we have

$$\mathbf{A}^2 = \mathbf{A}.\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}^3 = \mathbf{A}^2.\mathbf{A} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

∴ $\mathbf{A}^3 + \mathbf{A}^2 - 5\mathbf{A} - 5\mathbf{I}$

$$= \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5+5-5-5 & 10+0-10-0 & 0+0-0-0 \\ 10+0-10-0 & -5+5+5-5 & 0+0-0-0 \\ 0+0-0-0 & 0+0-0-0 & -1+1+5-5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O}, \text{ where } \mathbf{O} \text{ is the null matrix.}$$

Hence the matrix **A** satisfies its characteristic equation given by (i). Hence Cayley Hamilton theorem is verified by the matrix **A**.

Again $\mathbf{A}^3 + \mathbf{A}^2 - 5\mathbf{A} - 5\mathbf{I} = \mathbf{O}$ gives

$$5\mathbf{I} = \mathbf{A}^3 + \mathbf{A}^2 - 5\mathbf{A}$$

or $5\mathbf{A}^{-1} = \mathbf{A}^2 + \mathbf{A} - 5\mathbf{I}$, multiplying both sides by $\mathbf{A}^{-1}$ and $\mathbf{AA}^{-1} = \mathbf{I}$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

∴
$$\mathbf{A}^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

Ans.

Ex. 3 (b). Show that the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ **satisfies Cayley-Hamilton Theorem and hence compute** $\mathbf{A}^{-1}$**.**

Hint : Do as Ex. 3(a) above. **Ans.** $(1/9)\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -1 \\ 3 & -7 & -1 \end{bmatrix}$

Ex. 4 (a). Determine the characteristic equation of the matrix $= \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ **and verify that A satisfies its characteristic equation.**

(Garhwal 96)

Sol. Here we have $|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 1-\lambda & 3 & 7 \\ 4 & 2-\lambda & 3 \\ 1 & 2 & 1-\lambda \end{vmatrix}$

$$= (1-\lambda)\begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} - 3\begin{vmatrix} 4 & 3 \\ 1 & 1-\lambda \end{vmatrix} + 7\begin{vmatrix} 4 & 2-\lambda \\ 1 & 2 \end{vmatrix},$$

expanding w.r. to R_1

$= (1-\lambda)[(2-\lambda)(1-\lambda) - 6] - 3[4 - 4\lambda - 3] + 7[8 - 2 + \lambda]$

$= (1-\lambda)[-3\lambda + \lambda^2 - 4] - 3[1 - 4\lambda] + 7[6 + \lambda]$

$= 35 + 20\lambda + 4\lambda^2 - \lambda^3$, on simplifying.

Characteristic equation of matrix **A** is

$$\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0. \quad \text{...(i)}$$

Also we have

$$\mathbf{A}^2 = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$$\mathbf{A}^3 = \mathbf{A}^2 . \mathbf{A} = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 20+92+23 & 60+46+46 & 140+69+23 \\ 15+88+37 & 45+44+74 & 105+66+37 \\ 10+36+14 & 30+18+28 & 70+27+14 \end{bmatrix}$$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}$$

$$\mathbf{A}^3 - 4\mathbf{A}^2 - 20\mathbf{A} - 35\,\mathbf{I}$$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - 4\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 20\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 35\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O}, \text{ where } \mathbf{O} \text{ is the null matrix}$$

Hence the matrix A satisfies its characteristic equation (i). Hence proved

Ex. 4 (b). Find the characteristic equaiton of the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ and show that A satisfies this equation.

Hint : Do as Ex. 4 (a) above.

Ans. $\lambda^3 = 0$

Ex. 4 (c). Verify that matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ **satisfies its characteristic equation.**

(Rohilkhand 99)

Hint : Do as Ex. 4 (a) above.

Ex. 4 (d). Determine the characteristic equation of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$ and verify that A satisfies its characteristic equation.

(Garhwal 94)

Hint : Do as Ex. 4(a) above.

Ans. $\lambda^3 - 6\lambda + 4 = 0$

****Ex. 5. Find the characteristic vectors of the matrix A given in Ex. 4 (c) above.**

Sol. As in Ex. 4 (a) above we can find that the characteristic equation of A is $(\lambda - 1)^2 (5 - \lambda) = 0$ and so the characteristic roots of A are 1, 1, 5.

The equation $(\mathbf{A} - \lambda \mathbf{I}) \mathbf{X} = \mathbf{O}$ of the matrix **A** is

$$\begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{...(i)}$$

...See § 7.04 Page 166 Chapter VII

Putting $\lambda = 1$ in the above equation, we get

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The corresponding characteristic vector is given by the equation

$$x_1 + 2x_2 + x_3 = 0.$$

∴ The characteristic vector corresponding to $\lambda = 1$ may be taken as (1, − 1, 1).

Ans.

Putting $\lambda = 5$ in (i), we get

$$\begin{bmatrix} 2-5 & 2 & 2 \\ 1 & 3-5 & 1 \\ 1 & 2 & 2-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or
$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The corresponding characteristic vector is given by the equations $-3x_1 + 2x_2 + x_3 = 0$, $x_1 - 2x_2 + x_3 = 0$ and $x_1 + 2x_2 - 3x_3 = 0$.

Solving these we get $\dfrac{x_1}{1} = \dfrac{x_2}{1} = \dfrac{x_3}{1}$.

∴ The characteristic vector corresponding to $\lambda = 5$ may be taken as (1, 1, 1).

Ex. 6 (a). Verify Cayley Hamilton's Theorem for the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and hence find A^{-1}.

(Agra 96; Garhwal 93; Kanpur 95, 93; Kumaun 95; Lucknow 91; Meerut 98, 97; Rohilkhand 97)

or find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and verify that it is satisfied by A. *(Kanpur 91)*

Sol. Here $|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix}$

$$= \begin{vmatrix} 2-\lambda & 0 & 1 \\ -1 & 1-\lambda & -1 \\ 1 & 1-\lambda & 2-\lambda \end{vmatrix}, \text{ replacing } C_2 \text{ by } C_2 + C_3$$

$$= (1-\lambda) \begin{vmatrix} 2-\lambda & 0 & 1 \\ -1 & 1 & -1 \\ 1 & 1 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda) \left\{ (2-\lambda) \begin{vmatrix} 1 & -1 \\ 1 & 2-\lambda \end{vmatrix} + \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} \right\}$$

$$= (1-\lambda) \{(2-\lambda)(3-\lambda) + (-1-1)\}$$

$$= (1-\lambda) \{4 - 5\lambda + \lambda^2\} = 4 - 5\lambda + \lambda^2 - 4\lambda + 5\lambda^2 - \lambda^3$$

$$= -\lambda^3 + 6\lambda^2 - 9\lambda + 4.$$

∴ The characteristic equation of the matrix **A** is

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0 \qquad \text{...(i)}$$

Now $A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$

and $A^3 = A^2 . A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$\therefore$ $A^3 - 6A^2 + 9A - 4I$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} + \begin{bmatrix} -36 & 30 & -30 \\ 30 & -36 & 30 \\ -30 & 30 & -36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} + \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 22-36+18-4 & -21+30-9+0 & 21-30+9+0 \\ -21+30-9+0 & 22-36+18-4 & -21+30-9+0 \\ 21-30+9+0 & -21+30-9+0 & 22-36+18-4 \end{bmatrix}$$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$, where O is the null matrix

Hence A satisfies its characteristic equation given by (i).

Hence Cayley Hamilton theorem is satisfied by the matrix A.

Again $A^3 - 6A^2 + 9A - 4I = O$ gives $4I = A^3 - 6A^2 + 9A$

Multiplying both sides by A^{-1}, we get

$$4A^{-1} = A^2 - 6A + 9I, \quad \because AA^{-1} = I$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6-12+9 & -5+6+0 & 5-6+0 \\ -5+6+0 & 6-12+9 & -5+6+0 \\ 5-6+0 & -5+6+0 & 6-12+9 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

or $$A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$ **Ans.**

Ex. 6 (b). Find the characteristic equation of the matrix $A = \begin{bmatrix} 0 & 0 & 2 \\ 2 & -3 & 0 \\ 1 & 1 & -1 \end{bmatrix}$ and verify that it is satisfied by A and hence obtain A^{-1} *(Garhwal 95)*

Hint : Do as Ex. 6 (a) above.

Ans. $\lambda^3 + 4\lambda^2 + \lambda - 10 = 0$, $\frac{1}{10} \begin{bmatrix} 3 & 2 & 6 \\ 2 & -2 & 4 \\ 5 & 0 & 0 \end{bmatrix}$

Ex. 6 (c). Verify Cayley Hamilton Theorem for the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 1 & -2 \end{bmatrix}$. Find out the inverse if possible. *(Rohilkhand 94)*

Hint : Do as Ex. 6 (a) above.

Ans. $\frac{1}{2} \begin{bmatrix} -6 & 4 & -6 \\ 4 & -2 & 4 \\ -1 & 1 & -2 \end{bmatrix}$

Ex. 7. Find the characteristic roots of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and verify Cayley Hamilton theorem for the matrix.

Sol. Do as Ex. 6 (a) above. **Ans. 5, − 1**

Ex. 8. Find the characteristic root and inverse of the matrix $A = \begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix}$

Sol. Here $$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 6 \\ 1 & 2-\lambda \end{vmatrix}$$

$$= (5-\lambda)(2-\lambda) - 6 = 4 - 7\lambda + \lambda^2$$

$\therefore$ The characteristic equation of **A** is $\lambda^2 - 7\lambda + 4 = 0$ ...(i)

Now as **A** must satisfy Cayley Hamilton's Theorem, so we get

$$A^2 - 7A + 4I = O, \text{ where } O \text{ is the null matrix}$$

or $$I = -\tfrac{1}{4} A^2 + \tfrac{7}{4} A.$$

Multiplying both sides by A^{-1} we get

$$A^{-1} = -\tfrac{1}{4}A + \tfrac{7}{4}I \qquad \because AA^{-1} = I$$

$$= -\frac{1}{4}\begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix} + \frac{7}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{5}{4} & -\frac{3}{2} \\ -\frac{1}{4} & -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{7}{4} & 0 \\ 0 & \frac{7}{4} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{5}{4}+\frac{7}{4} & -\frac{3}{2}+0 \\ -\frac{1}{4}+0 & -\frac{1}{2}+\frac{7}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{4} & \frac{5}{4} \end{bmatrix} = (1/4)\begin{bmatrix} 2 & -6 \\ -1 & 5 \end{bmatrix} \qquad \textbf{Ans.}$$

Also the characteristic roots of **A** are the roots of (i).

i.e. $\lambda = \frac{1}{2}[7 \pm \sqrt{(49-16)}] = \frac{1}{2}\sqrt{[7 \pm \sqrt{(33)}]}$. **Ans.**

Ex. 9. Using Cayley Hamilton's Theorem find A^{-2} where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$ *(Agra 95)*

Sol. Here $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix}$

$$= (1-\lambda\begin{vmatrix} -1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 0 \\ 0 & -1-\lambda \end{vmatrix}$$

$$= (1-\lambda)(1+\lambda)^2 + 4(1+\lambda) = (1+\lambda)[(1-\lambda^2)+4]$$

$$= (1+\lambda)(5-\lambda^2) = 5 - \lambda^2 + 5\lambda - \lambda^3.$$

$\therefore$ The characteristic equation of **A** is

$$\lambda^3 + \lambda^2 - 5\lambda - 5 = 0 \qquad \text{...(i)}$$

Now $$A^2 = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And $$A^3 = A^2 . A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

By Cayley Hamilton's Theorem from (i) we have

$$A^3 + A^2 - 5A - 5I = O$$

or $$5I = A^3 + A^2 - 5A. \qquad \text{...(ii)}$$

Multiplying both sides by A^{-1}, we get

$$5A^{-1} = A^2 + A - 5I, \qquad \because AA^{-1} = I$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5+1-5 & 0+2+0 & 0+0+0 \\ 0+2+0 & 5-1-5 & 0+0+0 \\ 0+0+0 & 0+0+0 & 1-1-5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

or $$\mathbf{A}^{-1} = \frac{1}{5} \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix} \qquad ...(iii)$$

Again multiplying both sides of (ii) by $\mathbf{A}^{-2}$ we get

$$5\mathbf{A}^{-2} = \mathbf{A} + \mathbf{I} - 5\mathbf{A}^{-1}. \qquad \textbf{(Note)}$$

or $$5\mathbf{A}^{-2} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}, \text{ from (iii)}$$

$$= \begin{bmatrix} 1+1-1 & 2+0-2 & 0+0+0 \\ 2+0-2 & -1+1+1 & 0+0+0 \\ 0+0+0 & 0+0+0 & -1+1+5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

or $$\mathbf{A}^{-2} = \frac{1}{5} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Ans.

Ex. 10. Verify Cayley Hamilton's Theorem for the matrix $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{2} \\ \mathbf{0} & \mathbf{2} & \mathbf{1} \\ \mathbf{2} & \mathbf{0} & \mathbf{3} \end{bmatrix}$

Hence of otherwise compute $\mathbf{A}^{-1}$. *(Garhwal 92; Kumaun 94, 92; Lucknow 92; Meerut 96 P; Rohilkhand 98)*

Sol. Here $$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix}$$

$$= (1-\lambda)\{(2-\lambda)(3-\lambda) - 0\} + 2\{0 - 2(2-\lambda)\}$$

$$= (1-\lambda)(2-\lambda)(3-\lambda) - 4(2-\lambda)$$

$$= (2-\lambda)\{(1-\lambda)(3-\lambda) - 4\} = (2-\lambda)\{3 - 4\lambda + \lambda^2 - 4\}$$

$$= (2-\lambda)(\lambda^2 - 4\lambda - 1) = -\lambda^3 + 6\lambda^2 - 7\lambda - 2$$

∴ The characteristic equation of **A** is

$$\lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0. \qquad ...(i)$$

Now $$\mathbf{A}^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

and $$A^3 = A^2 . A = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$$

$\therefore$ $$A^3 - 6A^2 + 7A + 2I$$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6 \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} + \begin{bmatrix} -30 & 0 & -48 \\ -12 & -24 & -30 \\ -48 & 0 & -78 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 14 \\ 0 & 14 & 7 \\ 14 & 0 & 21 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 21-30+7+2 & 0+0+0+0 & 34-48+14+0 \\ 12-12+0+0 & 8-24+14+2 & 23-30+7+0 \\ 34-48+14+0 & 0+0+0+0 & 55-78+21+2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O}, \text{ where } \mathbf{O} \text{ is the null matrix.}$$

Hence **A** satisfies its characteristic equation given by (i)

Hence Cayley Hamilton's Theorem is satisfied by **A** *i.e.*

$$A^3 - 6A^2 + 7A + 2I = O \text{ or } 2I = -A^3 + 6A^2 - 7A.$$

Multiplying both sides by A^{-1}, we get

$$2A^{-1} = -A^2 + 6A - 7I, \qquad \because A A^{-1} = I, IA^{-1} = A^{-1}$$

$$= - \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 6 \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} - 7 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 0 & -8 \\ -2 & -4 & -5 \\ -8 & 0 & -13 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 12 \\ 0 & 12 & 6 \\ 12 & 0 & 18 \end{bmatrix} + \begin{bmatrix} -7 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} -5+6-7 & 0+0+0 & -8+12-0 \\ -2+0+0 & -4+12-7 & -5+6-0 \\ -8+12+0 & 0+0+0 & -13+18-7 \end{bmatrix} = \begin{bmatrix} -6 & 0 & 4 \\ -2 & 1 & 1 \\ 4 & 0 & -2 \end{bmatrix}$$

or $$A^{-1} = (1/2) \begin{bmatrix} -6 & 0 & 4 \\ -2 & 1 & 1 \\ 4 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 2 \\ -1 & 1/2 & 1/2 \\ 2 & 0 & -1 \end{bmatrix}$$ Ans

Ex. 11 (a). Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

and hence find A^{-1}. Also verify Cayley Hamilton's Theorem for A.

(Kanpur 90)

Sol. Here $|A - \lambda I|$

$$= \begin{vmatrix} 1-\lambda & 3 & 7 \\ 4 & 2-\lambda & 3 \\ 1 & 2 & 1-\lambda \end{vmatrix} = \begin{vmatrix} -\lambda & 1 & 6+\lambda \\ 0 & -6-\lambda & -1+4\lambda \\ 1 & 2 & 1-\lambda \end{vmatrix}, \text{ applying } R_1 - R_3, R_2 - 4R_3$$

$$= -\lambda \begin{vmatrix} -6-\lambda & -1+4\lambda \\ 2 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 1 & 6+\lambda \\ -6-\lambda & -1+4\lambda \end{vmatrix}$$

$$= -\lambda[-(6+\lambda)(1-\lambda) - 2(4\lambda - 1)] + [(4\lambda - 1) + (6+\lambda)^2]$$

$$= -\lambda(-6 + 6\lambda - \lambda + \lambda^2 - 8\lambda + 2) + (\lambda^2 + 16\lambda + 35)$$

$$= -\lambda^3 + 4\lambda^2 + 20\lambda + 35.$$

$\therefore$ The characteristic equation of the matrix A is

$$\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0 \quad \text{...(i)}$$

$\therefore$ By Cayley Hamilton's Theorem, we have

$$A^3 - 4A^2 - 20A - 35I = O, \text{ where } O \text{ is the null matrix.}$$

or $$35I = A^3 - 4A^2 - 20A.$$

Multiplying both sides by A^{-1}, we get

$$35A^{-1} = A^2 - 4A - 20I, \quad \because AA^{-1} = I \quad \text{...(ii)}$$

Now $$A^2 = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+12+7 & 3+6+14 & 7+9+7 \\ 4+8+3 & 12+4+6 & 28+6+3 \\ 1+8+1 & 3+4+2 & 7+6+1 \end{bmatrix} = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$\therefore$ From (ii) we get

$$35A^{-1} = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 4\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 20\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 20-4-20 & 23-12-0 & 23-28-0 \\ 15-16-0 & 22-8-20 & 37-12-0 \\ 10-4-0 & 9-8-0 & 14-4-20 \end{bmatrix} = \begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix} \textbf{ Ans.}$$

Verify Cayley Hamilton's Theorem for yourself.

Ex. 11 (b). Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 1 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$ **and hence find A^{-1}.**

(Meerut 91 S)

Sol. Do as Ex. 11 (a) above.

Ex. 11 (c). Using the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

find A^{-1}.

Sol. Do as Ex. 11 (a) above. **Ans.** $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Ex. 12. If $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -8 \\ 2 & -4 & 3 \end{bmatrix}$, **find the characteristic roots of A.**

Verify Cayley Hamilton's Theorem and hence find A^{-1}.

Sol. Here $|A - \lambda I| = \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -8 \\ 2 & -4 & 3-\lambda \end{vmatrix}$

$$= (8-\lambda)\{(7-\lambda)(3-\lambda) - 32\} + 6\{-6(3-\lambda) + 16\} + 2\{24 - 2(7-\lambda)\}$$

$$= (8-\lambda)\{-11 - 10\lambda + \lambda^2\} + 6\{6\lambda - 2\} + 2\{10 + 2\lambda\}$$

$$= -88 - 69\lambda + 18\lambda^2 - \lambda^3 + 36\lambda - 12 + 20 + 4\lambda = -\lambda^3 + 18\lambda^2 - 29\lambda - 80$$

∴ The characteristic equation of the matrix **A** is

$$\lambda^3 - 18\lambda^2 + 29\lambda + 80 = 0. \qquad ...(i)$$

Its roots are the required characteristic roots of **A** (students can calculate it f they have read solution of cubic equations, so left as an exercise for the students).

Now $A^2 = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -8 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -8 \\ 2 & -4 & 3 \end{bmatrix} = \begin{bmatrix} 104 & -98 & 70 \\ -106 & 117 & -92 \\ 46 & -52 & 45 \end{bmatrix}$

And $A^3 = A^2 . A = \begin{bmatrix} 104 & -98 & 70 \\ -106 & 117 & -92 \\ 46 & -52 & 45 \end{bmatrix} \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -8 \\ 2 & -4 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1560 & -1590 & 1202 \\ -1734 & 1823 & -1424 \\ 770 & -820 & 643 \end{bmatrix}$, on evaluating

∴ $A^3 - 18A^2 + 29A + 80I$

$$= \begin{bmatrix} 1560 & -1590 & 1202 \\ -1734 & 1823 & -1424 \\ 770 & -820 & 643 \end{bmatrix} - 18 \begin{bmatrix} 104 & 98 & 70 \\ -106 & 117 & -92 \\ 46 & -52 & 45 \end{bmatrix}$$

$$+ 29 \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -8 \\ 2 & -4 & 3 \end{bmatrix} + 80 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1560 & -1590 & 1202 \\ -1734 & 1823 & -1424 \\ 760 & -820 & 643 \end{bmatrix} + \begin{bmatrix} -1872 & 1764 & -1260 \\ 1908 & -2160 & 1656 \\ -828 & 936 & -810 \end{bmatrix}$$

$$+ \begin{bmatrix} 232 & -174 & 58 \\ -174 & 203 & -232 \\ 58 & 116 & 87 \end{bmatrix} + \begin{bmatrix} 80 & 0 & 0 \\ 0 & 80 & 0 \\ 0 & 0 & 80 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O}, \text{ where } \mathbf{O} \text{ is the null matrix.}$$

Hence **A** satisfies the characteristic equation given by (i).

Hence Cayley Hamilton's Theorem is satisfied by the given matrix **A** *i.e.* from (i) we get $\mathbf{A}^3 - 18\mathbf{A}^2 + 29\mathbf{A} + 80\mathbf{I} = \mathbf{O}$

or $$80\mathbf{I} = -\mathbf{A}^3 + 18\mathbf{A}^2 - 29\mathbf{A}$$

or $$80\,\mathbf{A}^{-1} = -\mathbf{A}^2 + 18\mathbf{A} - 29\mathbf{I}, \text{ multiplying both sides by } \mathbf{A}^{-1}$$

$$= -\begin{bmatrix} 104 & -98 & 70 \\ -106 & 117 & -92 \\ 46 & -52 & 45 \end{bmatrix} + 18\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -8 \\ 2 & -4 & 3 \end{bmatrix} - 29\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -104+144-29 & 98-108+0 & -70+36+0 \\ 106-108+0 & -117+126-29 & 92-144+0 \\ -46+36+0 & 52-72+0 & -45+54-29 \end{bmatrix}$$

or $$80\mathbf{A}^{-1} = \begin{bmatrix} 11 & -10 & -34 \\ -2 & -20 & -52 \\ -10 & -20 & -20 \end{bmatrix}$$

or $$\mathbf{A}^{-1} = (1/80)\begin{bmatrix} 11 & -10 & -34 \\ -2 & -20 & -52 \\ -10 & -20 & -20 \end{bmatrix}$$

Ans.

Ex. 13 (a) Verify that $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ **satisfies its own characteristic equation. (b) Is it true of every square matrix ? (c) State the theorem that applies here. (d) find** $\mathbf{A}^{-1}$. *(Meerut 93, 90; Rohilkhand 91)*

Sol. (a). Here as in Ex. 9 Page 182 Ch. VII we can prove that the characteristic equation of the matrix **A** is

$$\lambda^3 + \lambda^2 - 5\lambda - 5 = 0 \quad ...(i)$$

Now $$\mathbf{A}^2 = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And $A^3 = A^2 . A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$\therefore$ $$A^3 + A^2 - 5A - 5I$$

$$= \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5+5-5-5 & 10+0-10+0 & 0+0+0+0 \\ 10+0-10+0 & -5+5+5-5 & 0+0+0+0 \\ 0+0+0+0 & 0+0+0+0 & -1+1+5-5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$= \mathbf{O}$, where $\mathbf{O}$ is the null matrix.

Hence the matrix **A** satisfies its characteristic equation given by (i).

(b) Every square matrix satisfies its characteristic equation.

(c) Cayley Hamilton's Theorem. **(d)** Do yourself

Ex. 14. Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$ and hence compute its cube.

Sol. Here we have

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 1-\lambda & 2 \\ 1 & 2 & 0-\lambda \end{vmatrix}$$

$$= (1-\lambda)\{-\lambda(1-\lambda) - 4\} + 1\{-2(1-\lambda)\}$$

$$= -\lambda(1-\lambda)^2 - 6(1-\lambda) = -\lambda^3 + 2\lambda^2 + 5\lambda - 6$$

$\therefore$ The characteristic equation of the matrix **A** is

$$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0. \qquad ...(i)$$

$\therefore$ By Cayley-Hamilton theorem we have

$$A^3 - 2A^2 - 5A + 6I = O \qquad ...(ii)$$

Now $A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 2 \\ 2 & 5 & 2 \\ 1 & 2 & 6 \end{bmatrix}$

$\therefore$ From (ii) we have $A^3 = 2A^2 + 5A - 6I$

$$= 2\begin{bmatrix} 3 & 4 & 2 \\ 2 & 5 & 2 \\ 1 & 2 & 6 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 8 & 4 \\ 4 & 10 & 4 \\ 2 & 4 & 12 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 10 \\ 0 & 5 & 10 \\ 5 & 10 & 0 \end{bmatrix} + \begin{bmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 8 & 14 \\ 4 & 9 & 14 \\ 7 & 14 & 6 \end{bmatrix}$$ **Ans.**

Ex. 15 (a). Using Cayley Hamilton Theorem calculate $2A^5 + 3A^4 + A^2 - 11I$, where $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$. *(Rohilkhand 93)*

Sol. Here $|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix}$

$$= (3-\lambda)(2-\lambda) - (-1)(1) = \lambda^2 - 5\lambda + 7.$$

$\therefore$ The characteristic equation of the matrix **A** is $\lambda^2 - 5\lambda + 7 = 0$. ...(i)

Now $A^2 = A \cdot A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$

$\therefore \quad A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{O},$$

where **O** is the null matrix.

Hence **A** satisfies the characteristic equation of **A** given by (i). Thus we have

$$A^2 - 5A + 7I = O. \quad \text{...(ii)}$$

Now $\quad 2A^5 + 3A^4 - A^2 - 11I$

$= 2A^3(A^2 - 5A + 7I) + 13A^4 - 14A^3 - A^2 - 11I$

$= 2A^3(O) + 13A^2(A^2 - 5A + 7I) + 51A^3 - 92A^2 - 11I,$ from (i)

$= 13A^2(O) + 51A(A^2 - 5A + 7I) + 163A^2 - 355A - 11I$

$= 51A(O) + 163(A^2 - 5A + 7I) + 460A - 1152I$

$= 163(O) + 460A - 1152I = 460A - 1152I$

$= 460\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} - 1152\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1380-1152 & 460-0 \\ -460-0 & 920-1152 \end{bmatrix}$

$= \begin{bmatrix} 228 & 460 \\ -460 & -232 \end{bmatrix} = 4\begin{bmatrix} 57 & 115 \\ -115 & -58 \end{bmatrix}.$ **Ans.**

Ex. 15 (b). Verify Cayley Hamilton's Theorem for the following and compute $2A^8 - 3A^5 + A^4 + A^2 - 4I$.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$ *(Kumaun 90)*

Sol. Here $|A - \lambda I|$

$$= \begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & 1 \\ 0 & 1 & 0-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix}$$

$$= (1-\lambda)\,[\lambda + \lambda^2 - 1] = -\lambda^3 + 2\lambda - 1$$

$\therefore$ The characteristic equation of the matrix **A** is

$$\lambda^3 - 2\lambda + 1 = 0. \qquad \text{...(i)}$$

Now $A^2 = A \cdot A$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$\therefore$ $A^3 = A^2 \cdot A$

$$= \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & -3 & 2 \\ 0 & 2 & -1 \end{bmatrix}$$

$\therefore$ $A^3 - 2A + I$

$$= \begin{bmatrix} 1 & 0 & 4 \\ 0 & -3 & 2 \\ 0 & 2 & -1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2+1 & 0+0+0 & 4-4+0 \\ 0+0+0 & -3+2+1 & 2-2+0 \\ 0+0+0 & 2-2+0 & -1+0+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$= \mathbf{O}$, where **O** is the null matrix.

Hence **A** satisfies the characteristic equation of **A** given by (i) and so Cayley Hamilton's Theorem for the matrix **A** is verified.

Thus we have $$A^3 - 2A + I = O \qquad \text{...(ii)}$$

Now $$2A^8 - 3A^5 + A^4 + A^2 - 4I$$

$$= 2A^5(A^3 - 2A + I) + 4A^6 - 5A^5 + A^4 + A^2 - 4I$$

$$= 2A^5(O) + 4A^3(A^3 - 2A + I) - 5A^5 + 9A^4 - 4A^3 + A^2 - 4I,$$ from (ii)

$$= 4A^3(O) - 5A^2(A^3 - 2A + I) + 9A^4 - 14A^3 + 6A^2 - 4I$$

$$= -5A^2(O) + 9A(A^3 - 2A + I) - 14A^3 + 24A^2 - 9A - 4I$$

$$= 9A(O) - 14(A^3 - 2A + I) + 24A^2 - 37A + 10I$$

$$= -14(\mathbf{O}) + 24\begin{bmatrix}1 & 2 & 2\\0 & 2 & -1\\0 & -1 & 1\end{bmatrix} - 37\begin{bmatrix}1 & 0 & 2\\0 & -1 & 1\\0 & 1 & 0\end{bmatrix} + 10\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}24-37+10 & 48+0+0 & 48-74+0\\0+0+0 & 48+37+10 & -24-37+0\\0+0+0 & -24-37+0 & 24+0+10\end{bmatrix}$$

$$= \begin{bmatrix}-3 & 48 & -26\\0 & 85 & -61\\0 & -61 & 34\end{bmatrix}$$ **Ans.**

Ex. 16 Evaluate the matrix $2A^4 - 7A^3 - 4A^2$, where $A = \begin{bmatrix}1 & 0 & 2\\0 & 1 & 2\\1 & 2 & 0\end{bmatrix}$.

Sol. As in Ex. 10 Page 185 Ch. VII it can be calculated that the matrix A satisfies

$$\mathbf{A}^3 - 2\mathbf{A}^2 - 5\mathbf{A} + 6\mathbf{I} = \mathbf{O} \quad \ldots$$

Now $2\mathbf{A}^4 - 7\mathbf{A}^3 - 4\mathbf{A}^2 = (\mathbf{A}^3 - 2\mathbf{A}^2 - 5\mathbf{A} + 6\mathbf{I})(2\mathbf{A} - 3\mathbf{I}) - 27\mathbf{A} + 1$

$= -27\mathbf{A} + 18\mathbf{I}$, from (i), Now calculate.

Ex. 17. If $A = \begin{bmatrix}1 & 0 & 0\\1 & 0 & 1\\0 & 1 & 0\end{bmatrix}$, show that for every integer $n \geq 4$, $A^n = A^{n-2} + A^3 - A$. Hence evaluate A^{20}.

Sol. Here we have

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix}1-\lambda & 0 & 0\\1 & 0-\lambda & 1\\0 & 1 & 0-\lambda\end{vmatrix}$$

$$= (1-\lambda)\{(-\lambda)(-\lambda) - 1\} = (1-\lambda)(\lambda^2 - 1)$$

$$= -\lambda^3 + \lambda^2 + \lambda - 1$$

$\therefore$ The characteristic equation of **A** is $\lambda^3 - \lambda^2 - \lambda + 1 = 0$.

And so by Cayley-Hamilton's theorem we have

$$\mathbf{A}^3 - \mathbf{A}^2 - \mathbf{A} + \mathbf{I} = \mathbf{O}$$

or $$\mathbf{A}(\mathbf{A}^2 - \mathbf{I}) = \mathbf{A}^2 - \mathbf{I} \quad \ldots(i) \quad \textbf{(Note)}$$

Premultiplying both sides of (i) by $\mathbf{A}^{r-3}$, we have

$$\mathbf{A}^{r-2}(\mathbf{A}^2 - \mathbf{I}) = \mathbf{A}^{r-3}(\mathbf{A}^2 - \mathbf{I}). \quad \ldots(ii)$$

Putting $r = n, n-1, n-2, \ldots, 4$ in (ii) we get

$$\mathbf{A}^{n-2}(\mathbf{A}^2 - \mathbf{I}) = \mathbf{A}^{n-3}(\mathbf{A}^2 - \mathbf{I})$$

$$\mathbf{A}^{n-3}(\mathbf{A}^2 - \mathbf{I}) = \mathbf{A}^{n-4}(\mathbf{A}^2 - \mathbf{I})$$

....

....

$$\mathbf{A}^2(\mathbf{A}^2-\mathbf{I}) = \mathbf{A}(\mathbf{A}^2-\mathbf{I})$$

Multiplying these $n-3$ identities we have

$$\mathbf{A}^{n-2}(\mathbf{A}^2-\mathbf{I}) = \mathbf{A}(\mathbf{A}^2-\mathbf{I}).$$ **(Note)**

or $\mathbf{A}^n - \mathbf{A}^{n-2} = \mathbf{A}^3 - \mathbf{A}$ or $\mathbf{A}^n = \mathbf{A}^{n-2} + \mathbf{A}^3 - \mathbf{A}$, for all $n \geq 4$.

Now we have $\mathbf{A}^n - \mathbf{A}^{n-2} = \mathbf{A}(\mathbf{A}^2-\mathbf{I})$ ...(iii)

Putting $n = 20, 18, 16, \ldots, 4$ in (iii) we get **(Note)**

$$\mathbf{A}^{20} - \mathbf{A}^{18} = \mathbf{A}(\mathbf{A}^2-\mathbf{I})$$

$$\mathbf{A}^{18} - \mathbf{A}^{16} = \mathbf{A}(\mathbf{A}^2-\mathbf{I})$$

....

....

$$\mathbf{A}^{4} - \mathbf{A}^{2} = \mathbf{A}(\mathbf{A}^2-\mathbf{I})$$

Adding these nine identities, we get

$$\mathbf{A}^{20} - \mathbf{A}^{2} = 9\mathbf{A}(\mathbf{A}^2-\mathbf{I})$$ **(Note)**

$$= 9\mathbf{A}^3 - 9\mathbf{A} = 9(\mathbf{A}^2 + \mathbf{A} - \mathbf{I}) - 9\mathbf{A}, \text{ from (i)}$$

or $$\mathbf{A}^{20} = 10\mathbf{A}^2 - 9\mathbf{I}.$$ ...(iv)

Now $$\mathbf{A}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$\therefore$ From (iv) we have

$$\mathbf{A}^{20} = 10\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 0 & 0 \\ 10 & 10 & 0 \\ 10 & 0 & 10 \end{bmatrix} + \begin{bmatrix} -9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 10 & 1 & 0 \\ 10 & 0 & 1 \end{bmatrix}$$ **Ans.**

Ex. 18. Compute $\mathbf{A}^{-2}$ in Example 17 above.

Sol. In Ex. 17 above we have already proved that

$$\mathbf{A}^3 - \mathbf{A}^2 - \mathbf{A} + \mathbf{I} = \mathbf{O}.$$ ...(i)

Also in Cor. III Page 168 Ch. VII we have proved that

$$\mathbf{A}^{-1} = -\frac{1}{a_n}\left\{\mathbf{A}^{n-1} + a_1\mathbf{A}^{n-2} + \ldots + a_{n-1}\mathbf{I}\right\},$$ ...(ii)

provided $\mathbf{A}^n + a_1\mathbf{A}^{n-1} + a_2\mathbf{A}^{n-2} + \ldots + a_{n-1}\mathbf{A} + a_n\mathbf{I} = \mathbf{O}.$ ...(iii)

Comparing (i) and (iii) we have

$a_n = -1; a_{n-1} = 1, a_{n-2} = -1$ etc. or $a_1 = -1, a_2 = 1$ etc.

∴ From (ii) we have

$$\mathbf{A}^{-1} = -\frac{1}{(-1)}\left\{\mathbf{A}^{n-1} + (-1)\,\mathbf{A}^{n-2} + \ldots + (1)\,\mathbf{I}\right\}, \text{ where } n = 3$$

$$= \{\mathbf{A}^2 - \mathbf{A} + \mathbf{I}\}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From Ex. 17 Pages 191-192

or
$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \quad \text{...(iv)}$$

Also from (i) pre-multiplying each term by $\mathbf{A}^{-2}$ we have

$$\mathbf{A} - \mathbf{I} - \mathbf{A}^{-1} + \mathbf{A}^{-2} = \mathbf{O}. \quad \textbf{(Note)}$$

or
$$\mathbf{A}^{-2} = -\mathbf{A} + \mathbf{I} + \mathbf{A}^{-1}$$

$$= -\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix}$$

Ans.

Ex. 19 (a). Determine the eigen values and the corresponding eigen vectors of the matrix $\mathbf{A} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ *(Garhwal 95)*

Sol. Here $|\mathbf{A} - \lambda\mathbf{I}|$

$$= \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 0 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & -2+2\lambda & 2-\lambda \end{vmatrix}, \text{ applying } C_2 - 2C_3$$

$$= (2-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 2\lambda-2 & 2-\lambda \end{vmatrix} + \begin{vmatrix} 1 & 1-\lambda \\ 1 & 2\lambda-2 \end{vmatrix}$$

$$= (2-\lambda)\,[(2-3\lambda+\lambda^2) - (2\lambda-2)] + [(2\lambda-2) - (1-\lambda)]$$

$$= -\lambda^3 + 7\lambda^2 - 11\lambda + 5 = -(\lambda-1)^2(\lambda-5)$$

∴ The characteristic equation of the matrix **A** is $(\lambda-1)^2(\lambda-5) = 0$

Its roots *i.e.* required eigen values of **A** are 1, 5. **Ans.**

Now the equation $(\mathbf{A}-\lambda\mathbf{I})\,\mathbf{X}=\mathbf{O}$, for the matrix **A** is

$$\begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{...(i)}$$

...See §7.04 P. 166 Ch. VII

Puting $\lambda = 1$ in (i), we get

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The corresponding eigen-vector are given by the equations

$x_1 + 2x_2 + x_3 = 0$, which does not give any non-zero solution.

∴ The eigen vector corresponding to $\lambda = 1$ cannot be evaluated.

Putting $\lambda = 5$ in (i), we get

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 0 & 2 \\ 1 & 2 & 1 \\ 0 & 4 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ applying } R_1 \to R_1 + R_2,\ R_3 \to R_3 - R_2$$

$$\Rightarrow -2x_1 + 2x_3 = 0,\ x_1 - 2x_2 + x_3 = 0,\ 4x_2 - 4x_3 = 0$$

$$\Rightarrow x_1 = x_3,\ x_1 - 2x_2 + x_3 = 0,\ x_2 = x_3$$

$$\Rightarrow x_1 = x_2 = x_3,\ x_1 - 2x_1 + x_1 = 0$$

$\Rightarrow x_1 = x_2 = x_3$ and x_1 can take any value.

∴ Corresponding eigen vector is (x_1, x_1, x_1), where x_1 can take any non-zero value.

Ex. 19 (b). Find the eigen-values and eigen vectors of the matrix

$$\mathbf{A}=\begin{bmatrix} 3 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

Sol. Here $|\mathbf{A}-\lambda\mathbf{I}|$

$$= \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 5-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & 0 & 1 \\ 1 & 4-\lambda & 1 \\ 1 & \lambda-2 & 3-\lambda \end{vmatrix}, \text{ applying } C_2 - C_3.$$

$$= (3-\lambda)\begin{vmatrix} 4-\lambda & 1 \\ \lambda-2 & 3-\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & 4-\lambda \\ 1 & \lambda-2 \end{vmatrix}$$

$$= (3-\lambda)\,[\,(4-\lambda)(3-\lambda)-(\lambda-2)\,] + [(\lambda-2)-(4-\lambda)]$$

$$= (3-\lambda)\,[\,\lambda^2 - 8\lambda + 14] + (2\lambda - 6)$$

$= (3-\lambda)[\lambda^2 - 8\lambda + 14 - 2] = (3-\lambda)(\lambda^2 - 8\lambda + 12)$

$= (3-\lambda)(\lambda-6)(\lambda-2) = -(\lambda-3)(\lambda-2)(\lambda-6)$

$\therefore$ The characteristic equation of **A** is $(\lambda-2)(\lambda-3)(\lambda-6)=0$.

Its roots *i.e.* required eigen values of **A** are 2, 3, 6. **Ans.**

Now the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{X} = \mathbf{O}$, for the matrix A is

$$\begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 5-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{O} \quad \text{...(i)}$$

(See § 7.04 Page 166 Ch. VII)

Putting $\lambda = 2$ in (i), we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{O} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The corresponding eigen-vector is given by the equations

$$x_1 + x_2 + x_3 = 0, x_1 + 3x_2 + x_3 = 0$$

These give $\frac{x_1}{2} = \frac{x_2}{0} = \frac{x_3}{2}$ or $\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{1}$

$\therefore$ The characteristic vector corresponding to $\lambda = 2$ may be taken as (1, 0, 1). **Ans.**

Similarly calculate for $\lambda = 3$ and 6 also.

Ex. 20. Find the eigen-values and eigen vectors of the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{3} & \mathbf{-5} & \mathbf{-4} \\ \mathbf{-5} & \mathbf{-6} & \mathbf{-5} \\ \mathbf{-4} & \mathbf{-5} & \mathbf{3} \end{bmatrix}$$

Sol. Here $|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 3-\lambda & -5 & -4 \\ -5 & -6-\lambda & -5 \\ -4 & -5 & 3-\lambda \end{vmatrix}$

$$= (3-\lambda)\begin{vmatrix} -6-\lambda & -5 \\ -5 & 3-\lambda \end{vmatrix} + 5\begin{vmatrix} -5 & -5 \\ -4 & 3-\lambda \end{vmatrix} - 4\begin{vmatrix} -5 & -6-\lambda \\ -4 & -5 \end{vmatrix}$$

$$= (3-\lambda)[-(6+\lambda)(3-\lambda) - 25] + 5[-15 + 5\lambda - 20] - 4[25 + 4(-6-\lambda)]$$

$$= -\lambda^3 + 93\lambda - 308 = -(\lambda-4)(\lambda^2 + 4\lambda - 77)$$

$$= -(\lambda-4)(\lambda-7)(\lambda+11).$$

$\therefore$ Characteristic equation of **A** is $(\lambda-4)(\lambda-7)(\lambda+11)=0$

Its roots *i.e.* required eigen values of **A** are 4, 7, − 11.

Now the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{X} = \mathbf{O}$, for the matrix **A** is

$$\begin{bmatrix} 3-\lambda & -5 & -4 \\ -5 & -6-\lambda & -5 \\ -4 & -5 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{O} \quad \text{...(ii)}$$

...See § 7.04 Page 166 Ch. VII

Putting $\lambda = 4$ in (i), we get

$$\begin{bmatrix} -1 & -5 & -4 \\ -5 & -10 & -5 \\ -4 & -5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{O}$$

The corresponding eigen-vectors is given by the equations

$$x_1 + 5x_2 + 4x_3 = 0,\ 5x_1 + 10x_2 + 5x_3 = 0$$

and $4x_1 + 5x_2 + x_3 = 0$, which give $x_1 = 0 = x_2 = x_3$.

These being all zero, the eigen vector corresponding to $\lambda = 4$ cannot be evalutated.

Putting $\lambda = 7$ in (i), we get

$$\begin{bmatrix} -4 & -5 & -4 \\ -5 & -13 & -5 \\ -4 & -5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{O}$$

The corresponding eigen vectors is given by the equations

$$4x_1 + 5x_2 + 4x_3 = 0 \text{ and } 5x_1 + 13x_2 + 5x_3 = 0$$

which give $x_2 = 0$ and $x_3 = -x_1$

These give $\dfrac{x_1}{1} = \dfrac{x_2}{0} = \dfrac{x_3}{-1}$.

$\therefore$ The characteristic vector corresponding to $\lambda = 7$ may be taken as $(1, 0, -1)$.

Ans.

Similarly calculate for $\lambda = -11$ also.

Ex. 21. Find the eigen-vectors of $\mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Sol. Here we can calculate that

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix}$$

and the eigen values are given by $\lambda = 5, -3, -3$.

(Students are to find these in exam.)

Now the equation $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{X} = \mathbf{O}$, for the matrix $\mathbf{A}$ is

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{O} \quad \text{...(i)}$$

(See § 7.04 Page 166 Ch. VII)

Putting $\lambda = 5$, in (i) we get

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{O}$$

The corresponding eigen-vector is given by the equations

$$-7x_1 + 2x_2 - 3x_3 = 0,\ 2x_1 - 4x_2 - 6x_3 = 0,\ -x_1 - 2x_2 - 5x_3 = 0$$

These give $\dfrac{x_1}{1} = \dfrac{x_2}{2} = \dfrac{x_3}{-1}$

$\therefore$ The eigen-vector corresponding to $\lambda = 5$ may be taken as $(1, 2, -1)$.

Ans.

Putting $\lambda = -3$, in (i) we get $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{O}$

The corresponding eigen-vector is given by the equations

$$x_1 + 2x_2 - 3x_3 = 0,\ 2x_1 + 4x_2 - 6x_3 = 0,\ -x_1 - 2x_2 + 3x_3 = 0.$$

These reduce to $x_1 + 2x_2 - 3x_3 = 0$ only and so non-zero solution of this cannot be found, hence no eigen-vector can be derived for $\lambda = -3$.

Ex. 22 (a). Determine the characteristic roots and characteristic vector of the matrix $\mathbf{A} = \begin{bmatrix} \mathbf{5} & \mathbf{4} \\ \mathbf{1} & \mathbf{2} \end{bmatrix}$ *(Garhwal 93, 92)*

Sol. Here $|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix}$

$$= (5-\lambda)(2-\lambda) - 4 = 6 - 7\lambda + \lambda^2$$

$\therefore$ The characteristic equation of the matrix **A** is

$$\lambda^2 - 7\lambda + 6 = 0 \quad \text{or} \quad (\lambda - 1)(\lambda - 6) = 0 \quad \text{or} \quad \lambda = 1, 6.$$

$\therefore$ The characteristic roots of the matrix **A** are 1, 6. **Ans.**

Now the equation $(\mathbf{A} - \lambda \mathbf{I})\mathbf{X} = \mathbf{O}$, for the matrix **A** is

$$\begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{O} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{...(i)}$$

Putting $\lambda = 1$ in (i), we get

$$\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{-1}$$

$\therefore$ The eigen-vector corresponding to $\lambda = 1$ may be taken as $(1, -1)$. **Ans.**

Putting $\lambda = 6$ in (i), we get

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 - 4x_2 = 0$$

$$\Rightarrow \frac{x_1}{-4} = \frac{x_2}{1}$$

$\therefore$ The eigen-vector corresponding to $\lambda = 6$ may be taken as $(-4, 1)$. **Ans.**

Ex. 22 (b). Find the characteristic vector of $\mathbf{A}=\begin{bmatrix}\mathbf{1} & \mathbf{2} & \mathbf{3}\\ \mathbf{0} & \mathbf{2} & \mathbf{3}\\ \mathbf{0} & \mathbf{0} & \mathbf{2}\end{bmatrix}$ *(Agra 90)*

Sol. Here we get $|\mathbf{A}-\lambda\mathbf{I}|=\begin{vmatrix}1-\lambda & 2 & 3\\ 0 & 2-\lambda & 3\\ 0 & 0 & 2-\lambda\end{vmatrix}$

and the eigen-values are given by $\lambda = 1, 2, 2$.

(Students are to find these in the exam.)

Now the equaiton $(\mathbf{A}-\lambda\mathbf{I})\,\mathbf{X}=\mathbf{O}$, for the matrix $\mathbf{A}$ is

$$\begin{bmatrix}1-\lambda & 2 & 3\\ 0 & 2-\lambda & 3\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\mathbf{O}$$, see § 7.04 Page 166 Ch. VII.

Putting $\lambda = 1$ in (i) we get ...(i)

$$\begin{bmatrix}0 & 2 & 3\\ 0 & 1 & 3\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\mathbf{O}$$

The corresponding characteristic vector is given by the equations $2x_2+x_3=0, x_2+x_3=0, x_3=0$ which do not give non-zero solution of these equations and hence no characteristic vector can be derived for $\lambda = 1$.

Putting $\lambda = 2$ in (i) we get

$$\begin{bmatrix}-1 & 2 & 3\\ 0 & 0 & 3\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\mathbf{O}$$

The corresponding characteristic vector is given by the equations

$$-x_1+2x_2-3x_3=0,\ 3x_3=0.$$

These gives $\dfrac{x_1}{2}=\dfrac{x_2}{1}=\dfrac{x_3}{0}$.

$\therefore$ The characteristic vector corresponding to $\lambda = 2$ may be taken as (2, 1, 0).

Ans.

Ex. 23. Find the eigen-vector of the matrix $\mathbf{A}=\begin{bmatrix}\mathbf{3} & \mathbf{1} & \mathbf{4}\\ \mathbf{0} & \mathbf{2} & \mathbf{6}\\ \mathbf{0} & \mathbf{0} & \mathbf{5}\end{bmatrix}$ *(Agra 92)*

Sol. Here we have $|\mathbf{A}-\lambda\mathbf{I}|=\begin{vmatrix}3-\lambda & 1 & 4\\ 0 & 2-\lambda & 6\\ 0 & 0 & 5-\lambda\end{vmatrix}$

and the characteristic roots (or eigen values) are given by $\lambda = 2, 3, 5$.

(Students are to find these in the exam.)

Now the equation $(\mathbf{A}-\lambda\mathbf{I})\,\mathbf{X}=\mathbf{O}$, for the matrix $\mathbf{A}$ is

$$\begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{O} \quad ...(i)$$

(See § 7.04 Page 166 Ch. VII)

Putting $\lambda = 5$ in (i) we get

$$\begin{vmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{O}$$

The corresponding eigen-vector is given by the equations

$$-2x_1 + x_2 + 4x_3 = 0, \ -3x_2 + 6x_3 = 0,$$

which give $\quad 3x_2 = 6x_3$ or $\dfrac{x_2}{2} = \dfrac{x_3}{1}$. ...(ii)

Now $\dfrac{x_2}{2} = \dfrac{x_3}{1} = k$ (say), then from $-2x_1 + x_2 + 4x_3 = 0$ we get

$$2x_1 = x_2 + 4x_3 = 2k + 4k = 6k \quad \text{or} \quad x_1 = 3k$$

or $\quad \dfrac{x_1}{3} = k$. So we get $\dfrac{x_1}{3} = \dfrac{x_2}{2} = \dfrac{x_3}{1}$.

$\therefore$ The characteristic vector corresponding to $\lambda = 5$ may be taken as (3, 2, 1).

For $\lambda = 2, 3$ we find that $|\mathbf{A} - \lambda \mathbf{I}| = 0$ and so non-zero solutions of (i) cannot be evaluated in these cases *i.e.* eigen-vectors cannot be calculated.

***Ex. 24. If a + b + c = 0, find the characteristic roots of the matrix**

$$\mathbf{A} = \begin{bmatrix} \mathbf{a} & \mathbf{c} & \mathbf{b} \\ \mathbf{c} & \mathbf{b} & \mathbf{a} \\ \mathbf{b} & \mathbf{a} & \mathbf{c} \end{bmatrix}$$

(Garhwal 96)

Sol. Here we have

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} a-\lambda & c-0 & b-0 \\ c-0 & b-\lambda & a-0 \\ b-0 & a-0 & c-\lambda \end{vmatrix} = \begin{vmatrix} a-\lambda & c & b \\ c & b-\lambda & a \\ b & a & c-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c-\lambda & c & b \\ c+b+a-\lambda & b-\lambda & a \\ b+a+c-\lambda & a & c-\lambda \end{vmatrix}, \text{ replacing } C_1 \text{ by } C_1 + C_2 + C_3$$

$$= \begin{vmatrix} -\lambda & c & b \\ -\lambda & b-\lambda & a \\ -\lambda & a & c-\lambda \end{vmatrix}, \text{ since } a+b+c=0 \text{ (given)}$$

$$= \begin{vmatrix} -\lambda & c & b \\ 0 & b-\lambda-c & a-b \\ 0 & a-c & c-\lambda-b \end{vmatrix}, \text{ replacing } R_2, R_3 \text{ by } R_2 - R_1 \text{ and } R_3 - R_1$$

$$= -\lambda \begin{vmatrix} b-c-\lambda & a-b \\ a-c & c-b-\lambda \end{vmatrix}, \text{ expanding with respect to } C_1$$

$$= -\lambda [(b-c-\lambda)(c-b-\lambda) - (a-b)(a-c)]$$

$$= -\lambda [bc - b^2 - b\lambda - c^2 + cb + c\lambda - \lambda c + b\lambda + \lambda^2 - a^2 + ac + ba - bc]$$

or $$|\mathbf{A} - \lambda \mathbf{I}| = \lambda [(a^2 + b^2 + c^2 - ab - bc - ca) - \lambda^2]. \quad ...(i)$$

Also $a + b + c = 0 \Rightarrow (a + b + c)^2 = 0$

$$\Rightarrow a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = 0$$

$$\Rightarrow 2(ab + bc + ca) = -(a^2 + b^2 + c^2) \quad \textbf{(Note)}$$

$$\Rightarrow -(ab + bc + ca) = \frac{1}{2}(a^2 + b^2 + c^2)$$

$\therefore$ From (i) we get

$$|\mathbf{A} - \lambda \mathbf{I}| = \lambda [\{(a^2 + b^2 + c^2) + \frac{1}{2}(a^2 + b^2 + c^2)\} - \lambda^2] \quad \textbf{(Note)}$$

$$= \lambda [(3/2)(a^2 + b^2 + c^2) - \lambda^2]$$

$\therefore$ The characteristic equation of **A** is $\lambda [(3/2)(a^2 + b^2 + c^2) - \lambda^2] = 0$ which gives $\lambda = 0$ or $\lambda^2 = (3/2)(a^2 + b^2 + c^2)$.

The required roots are $0, \pm [(3/2)(a^2 + b^2 + c^2)]^{1/2}$. **Ans.**

Ex. ?~ (a). Find latent roots and latent vectors of the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{a} & \mathbf{h} & \mathbf{g} \\ \mathbf{0} & \mathbf{b} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{c} \end{bmatrix}$$ *(Kanpur 94)*

Sol. Here $|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} a-\lambda & h & g \\ 0 & b-\lambda & 0 \\ 0 & 0 & c-\lambda \end{vmatrix}$

$$= (a-\lambda)(b-\lambda)(c-\lambda)$$

$\therefore$ The characteristic equation of the matrix A is

$$(\lambda - a)(\lambda - b)(\lambda - c) = 0$$

and the characteristic or latent roots of **A** are a, b, c. **Ans.**

Again the equation $(\mathbf{A} - \lambda \mathbf{I})\mathbf{X} = \mathbf{O}$ for the matrix **A** is

$$\begin{bmatrix} a-\lambda & h & g \\ 0 & b-\lambda & 0 \\ 0 & 0 & c-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad ...(i)$$

Putting $\lambda = a$ in the above equation we get

$$\begin{bmatrix} 0 & h & g \\ 0 & b-a & 0 \\ 0 & 0 & c-a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The corresponding characteristic vector is given by the equations

$$0x_1 + hx_2 + gx_3 = 0,\ (b-a)\,x_2 = 0,\ (c-a)\,x_3 = 0.$$

$\therefore$ The characteristic vector corresponding $\lambda = a$ may be taken as $(\alpha, 0, 0)$.

Ans.

Similarly we can find the characteristic vector corresponding to $\lambda = b$ and $\lambda = c$ as $(-h, a-b, 0)$ and $(-h, 0, a-c)$. **Ans.**

Ex. 25 (b). Find the latent roots and latent vector of the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{a} & \mathbf{h} & \mathbf{g} \\ \mathbf{0} & \mathbf{b} & \mathbf{f} \\ \mathbf{0} & \mathbf{0} & \mathbf{c} \end{bmatrix}$$

Sol. Do as Ex. 25 (a), above.

Ex. 25 (c). Find the latent roots and latent vectors of the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{2} & \mathbf{5} & \mathbf{1} \\ \mathbf{0} & \mathbf{3} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{4} \end{bmatrix}$$ *(Agra 95)*

Hint : Do as Ex. 25(a) above.

Ex. 26. If $\mathbf{B} = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}$ then find the characteristic equation of B and verify that the matrix B satisfies the equation. Also find the characteristic roots and the corresponding characteristic vectors of B. *(Garhwal 94)*

Sol. Here $\mathbf{B} = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}$ and $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore \quad |\mathbf{B} - \lambda \mathbf{I}| = \begin{vmatrix} 2-\lambda & \sqrt{2} \\ \sqrt{2} & 1-\lambda \end{vmatrix}$$

$$= [(2-\lambda)(1-\lambda) - \sqrt{2}\sqrt{2}]$$

$$= 2 - 2\lambda - \lambda + \lambda^2 - 2$$

$$= \lambda^2 - 3\lambda = \lambda(\lambda - 3).$$

$\therefore$ The characteristic equation of the matrix **B** is $\lambda(\lambda - 3) = 0$ **Ans.**

and the characteristic roots of **B** are 0 and 3. **Ans.**

Also $$\mathbf{B}^2 = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix} \times \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.2 + \sqrt{2}.\sqrt{2} & 2.\sqrt{2} + \sqrt{2}.1 \\ \sqrt{2}.2 + 1.\sqrt{2} & \sqrt{2}.\sqrt{2} + 1.1 \end{bmatrix} = \begin{bmatrix} 6 & 3\sqrt{2} \\ 3\sqrt{2} & 3 \end{bmatrix}$$

$$\therefore \quad \mathbf{B}^2 - 3\mathbf{B} = \begin{bmatrix} 6 & 3\sqrt{2} \\ 3\sqrt{2} & 3 \end{bmatrix} - 3\begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 3\sqrt{2} \\ 3\sqrt{2} & 3 \end{bmatrix} + \begin{bmatrix} -6 & -3\sqrt{2} \\ -3\sqrt{2} & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 6-6 & 3\sqrt{2}-3\sqrt{2} \\ 3\sqrt{2}-3\sqrt{2} & 3-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$= \mathbf{O}$, where $\mathbf{O}$ is the null matrix.

Hence the matrix **B** satisfies its characteristic equation given by

$$\lambda(\lambda-3)=0 \text{ or } \lambda^2-3\lambda=0.$$

The equation $(\mathbf{B}-\lambda\mathbf{I})=\mathbf{O}$ for the matrix **B** is

$$\begin{bmatrix} 2-\lambda & \sqrt{2} \\ \sqrt{2} & 1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\mathbf{O} \quad \text{...(i)}$$

(see § 7.04 Page 166 chapter VII)

Putting $\lambda=0$ in (i), we get $\begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\mathbf{O}$

The corresponding characteristic vector is given by the equations

$$2x_1+\sqrt{2}x_2=0 \text{ and } \sqrt{2}x_1+x_2=0.$$ **(Note)**

Taking any one of them we get

$$\frac{x_1}{\sqrt{2}}=\frac{x_2}{-2} \text{ or } \frac{x_1}{1}=\frac{x_2}{-\sqrt{2}}.$$

$\therefore$ The characteristic vector corresponding to $\lambda=0$ may be taken as $(1,-\sqrt{2})$.

Putting $\lambda=3$ in (i), we get

$$\begin{bmatrix} -1 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\mathbf{O}$$

The corresponding characteristic vector is given by the equations $-x_1+\sqrt{2}x_2=0$ and $\sqrt{2}x_1-2x_2=0$.

Taking any one of them we get $\frac{x_1}{2}=\frac{x_2}{\sqrt{2}}$ or $\frac{x_1}{\sqrt{2}}=\frac{x_2}{1}$

$\therefore$ The characteristic vector corresponding to $\lambda=3$ can be taken as $(\sqrt{2},1)$. **Ans.**

****Ex. 27. Show that the matrix** $\mathbf{A}=\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ **satisfies its characteristic equation. Also find** $\mathbf{A}^{-1}$. *(Agra 91; Kumaun 91; Rohilkhand 92)*

Sol. Here $|\mathbf{A}-\lambda\mathbf{I}| = \begin{vmatrix} 0-\lambda & c-0 & -b-0 \\ -c-0 & 0-\lambda & a-0 \\ b-0 & -a-0 & 0-\lambda \end{vmatrix}$

$$=-\lambda\begin{bmatrix} -\lambda & a \\ -a & -\lambda \end{bmatrix}-c\begin{bmatrix} -c & a \\ b & -\lambda \end{bmatrix}-b\begin{bmatrix} -c & -\lambda \\ b & -a \end{bmatrix}$$

$$=-\lambda(\lambda^2+a^2)-c(c\lambda-ab)-b(ca+b\lambda)$$

$$= -\lambda^3 - \lambda (a^2 + b^2 + c^2)$$

$\therefore$ The characteristic equation of **A** is

$$\lambda^3 + \lambda (a^2 + b^2 + c^2) = 0. \quad \text{...(i)}$$

Also we have $\mathbf{A}^2 = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$

$$= \begin{bmatrix} -(c^2 + b^2) & ab & ca \\ ab & -(c^2 + a^2) & bc \\ ac & bc & -(a^2 + b^2) \end{bmatrix}$$

$$\therefore \quad \mathbf{A}^3 = \mathbf{A}^2 . \mathbf{A}$$

$$= \begin{bmatrix} -(b^2 + c^2) & ab & ca \\ ab & -(c^2 + a^2) & bc \\ ac & bc & -(a^2 + b^2) \end{bmatrix} \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -c (b^2 + c^2 + a^2) & b (b^2 + c^2 + a^2) \\ c (c^2 + a^2 + b^2) & 0 & -a (b^2 + c^2 + a^2) \\ -b (c^2 + a^2 + b^2) & a (c^2 + a^2 + b^2) & 0 \end{bmatrix}$$

$$= -(a^2 + b^2 + c^2) \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix},$$

taking out $-(a^2 + b^2 + c^2)$ common. **(Note)**

or $$\mathbf{A}^3 = -(a^2 + b^2 + c^2)\, \mathbf{A} \quad \text{...(ii)}$$

or $\mathbf{A}^3 + (a^2 + b^2 + c^2)\, \mathbf{A} = \mathbf{O}$, which shows that the matrix **A** satisfies its characteristic equation given by (i).

Again multiplying both sides of (ii) by $\mathbf{A}^{-2}$, we get

$$\mathbf{A} = -(a^2 + b^2 + c^2)\, \mathbf{A}^{-1} \quad \textbf{(Note)}$$

which gives $$\mathbf{A}^{-1} = -\frac{1}{(a^2 + b^2 + c^2)} \mathbf{A}$$

$$= -\frac{1}{(a^2 + b^2 + c^2)} \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \quad \textbf{Ans.}$$

Exercises on § 7.03–7.06

Ex. 1. Show that the characteristic equation of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \text{ is } (\lambda - 1)(\lambda + 2)(\lambda - 3) = 0$$

Ex. 2. Show that the matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ satisfies Cayley-Hamilton Theorem.

Ex. 3. Let $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$. Find the characteristic equation of **A** and verify that matrix **A** satisfies this equation. Also find the characteristic roots and the corresponding vectors of **A**.

Ex. 4. Using Cayley-Hamilton Theorem or otherwise determine the inverse of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$

Ex. 5. Verify Cayley-Hamilton Theorem in the case of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & -3 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. Hence find A^{-1}.

***Ex. 6.** If matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, find the characteristic roots of **A**. Verify Cayley-Hamilton's Theorem and hence compute A^{-1}.

Ans. $1, 1, 4, \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

Ex. 7. Verify the Cayley-Hamilton Theorem and find the characteristic roots, where $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$

(Garhwal 91)

Ex. 8. Show that the matrix $A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & 0 & 4 \end{bmatrix}$ satisfies Cayley Hamilton Theorem.

Ex. 9. Using Cayley-Hamilton Theorem find the inverse of

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 0 & -1 & 1 \\ 2 & 2 & -1 \end{bmatrix}$$

(Rohilkhand 96, 90)

Ex. 10. Verify Cayley-Hamilton Theorem and verify it for the matrix **A** and hence find A^{-1}, where

$$A = \begin{bmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(Lucknow 92, 90)

****Ex. 11.** Using Cayley-Hamilton's Theorem, compute the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ **Ans.** $\frac{1}{4}\begin{bmatrix} 3 & -2 & 1 \\ 2 & 0 & -2 \\ -3 & 2 & 3 \end{bmatrix}$

Ex. 12. Show that $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ satisfies the matrix equation $A^2 - 4A - 5I = O$, where **I** is the unit matirx, Deduce A^{-1}.

Ans. $\frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$

Ex. 13. Obtain the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$ and hence evaluate A^{-1}. *(Kumaun 93)*

Ans. $\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$; $A^{-1} = \begin{bmatrix} 4 & -4 & 2 \\ -2 & 2 & 2 \\ 1 & 2 & -1 \end{bmatrix}$

Ex. 14 If $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, calculate A^{-1} with the help of Cayley-Hamilton's Theorem.

Ans. $\frac{1}{6}\begin{bmatrix} -3 & 2 \\ 4 & -1 \end{bmatrix}$

Ex. 15. If $A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$, calculate A^{-1} with the help of Cayley-Hamilton's Theorem.

Ans. $\frac{1}{4}\begin{bmatrix} 5 & 0 & 0 \\ -3 & 4 & 0 \\ 2 & -1 & 1 \end{bmatrix}$

Ex. 16. Find the eigen-values and eigen-vectors of the matrix

$$\begin{bmatrix} 2 & 1 & 1 \\ -11 & 4 & 5 \\ -1 & 1 & 0 \end{bmatrix}$$

Ans. $-1, \frac{1}{2}[7 \pm \sqrt{(-39)}]$, no eigen vectors.

Ex. 17. Find the eigen-vectors of the following matrices :—

(a) $\begin{bmatrix} -3 & 2 & 2 \\ -6 & 5 & 2 \\ -7 & 4 & 4 \end{bmatrix}$; (b) $\begin{bmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$; (c) $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ *(Agra 92)*

Ex. 18. Find the characteristic vectors of the matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$ *(Kumaun 91)*

Ex. 19. Find the eigen values and eigen-vectors for the matrix

$$A = \begin{bmatrix} 3 & -5 & -4 \\ -5 & -6 & -5 \\ -4 & -5 & 3 \end{bmatrix}$$

Ex. 20. Find the characteristic roots of

$$\begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$ *(Kumaun 96)*

Ex. 21. Find the invariant vector of the matrix

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Ex. 22. Find the characteristic roots and vectors of the matrix

$$A = \begin{bmatrix} -2 & -1 \\ 5 & 4 \end{bmatrix}$$ **Ans.** $-1, 3; (1, -1), (1 - 5)$

Ex. 23. Determine the characteristic roots and associated invariant vectors, given

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$ *(Lucknow 91)*

MISCELLANEOUS SOLVED EXAMPLES

***Ex. 1 Prove that matrices A and $B^{-1}AB$ have the same latent roots.**

Sol. We know that two matrices have the same latent roots (or characteristic roots) if their characteristic equations are the same. (See definition of latent roots in § 7.02 Page 162).

Let $B^{-1}AB = C$, then $C - \lambda I = B^{-1}AB - \lambda I$. ...(i)

Also $B^{-1}\lambda IB = B^{-1}\lambda B = \lambda B^{-1}B = \lambda I$

$\therefore$ From (i) we get $C - \lambda I = B^{-1}AB - B^{-1}\lambda IB$

$$= B^{-1}(A - \lambda I)B$$

or $$|C - \lambda I| = |B^{-1}||A - \lambda I||B|$$

$$= |A - \lambda I||B^{-1}||B|$$

$$= |\mathbf{A} - \lambda \mathbf{I}|\,|\mathbf{B}^{-1}\mathbf{B}|$$
$$= |\mathbf{A} - \lambda \mathbf{I}|\,|\mathbf{I}| = |\mathbf{A} - \lambda \mathbf{I}|$$
$$\therefore \quad |\mathbf{C} - \lambda \mathbf{I}| = 0 \Rightarrow |\mathbf{A} - \lambda \mathbf{I}| = 0$$

Hence the characteristic equation of **C** and **A** are the same *i.e.* **C** and **A** or $\mathbf{B}^{-1}$ **AB** and **A** have the same latent roots. Hence proved.

Ex. 2. Prove that the eigen values of a diagonal matrix are given by its diagonal elements.

Sol. Let
$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

be a diagonal matrix with $a_{11}, a_{22}, \dots, a_{nn}$ as diagonal elements.

Then the characteristic equation is

$$|\mathbf{A} - \lambda \mathbf{I}| = 0 \text{ or } \begin{vmatrix} a_{11} - \lambda & 0 & \dots & 0 \\ 0 & a_{22} - \lambda & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} - \lambda \end{vmatrix} = 0$$

or $\quad (a_{11} - \lambda)(a_{22} - \lambda) \dots (a_{nn} - \lambda) = 0$

or $\quad \lambda = a_{11}, a_{22}, \dots, a_{nn}$ are the eigen values of the matrix **A** and are given by the diagonal elements of the diagonal matrix **A**. Hence proved

Ex. 3. Find the spectrum of the matrix

$$\mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Sol. We can find that the eigen values of the matrix **A** are 5, − 3, − 3.

Also we know spectrum of **A** is the set of eigen values of **A**.

[See § 7.02 (iv) Page 160]

$\therefore$ Required spectrum of $\mathbf{A} = \{5, -3, -3\} = \{5, -3\}$ **Ans.**

Ex. 4. The equation $\mathbf{AK} = \lambda \mathbf{X}$ has non-trivial solution X iff λ is a characteristic value of A.

Sol. Let λ_1 be a characteristic value of **A** and $\mathbf{X}_1$ be the corresponding characteristic vector of **A**, then

$$\mathbf{AX} = \lambda_1 \mathbf{X}_1 = \lambda_1 \mathbf{I} X_1 = (\lambda_1 \mathbf{I})\mathbf{X}_1$$

or $\quad \mathbf{AX}_1 - (\lambda_1 \mathbf{I})\mathbf{X}_1 = \mathbf{O}$, where **O** is the null matrix

or $\quad (\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{X}_1 = \mathbf{O} \quad$ or $\quad (\mathbf{A} - \lambda_1 \mathbf{I}) = \mathbf{O}, \because \mathbf{X}_1 \neq \mathbf{O}$

or $\quad |\mathbf{A} - \lambda_1| = 0$

Hence every characteristic value λ of **A** is a root of its characteristic equation.

Conversely if λ_1 be any root of the characteristic equation $|\mathbf{A} - \lambda_1| = 0$, then the equation $(\mathbf{A} - \lambda_1\mathbf{I})\,\mathbf{X} = \mathbf{O}$ must possess a non-zero vector $\mathbf{X}_1$,

such that $$\mathbf{AX}_1 = \lambda_1\,\mathbf{IX}_1 = \lambda_1\,\mathbf{X}_1.$$

Hence every root λ of the characteristic equation of **A** is a characteristic value of **A**.

EXERCISES ON CHAPTER VII

Ex. 1. Evaluate the matrix $\mathbf{A}^5 - 27\mathbf{A}^3 + 65\mathbf{A}^2$, where

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} \qquad \textbf{Ans.} \begin{bmatrix} 40 & 2 & 48 \\ 128 & -3 & 0 \\ 86 & -43 & -132 \end{bmatrix}$$

Ex. 2. Evaluate $\mathbf{A}^{50}$ in Ex. 17 Page 193 Ch. VII.

Ex. 3. If the matrix $\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, find the characteristic roots of **A**.

Verify Cayley-Hamilton Theorem and hence compute $\mathbf{A}^{-1}$.

****Ex. 4.** When do you say that two matrices **A** and **B** are smilar ? Prove that the similar matrices have the same characteristic roots.

Ex. 5. Find the characteristic polynomials of the matrix **A** and hence compute $2\mathbf{A}^6 - 3\mathbf{A}^5 + \mathbf{A}^4 + \mathbf{A}^2 - 4\mathbf{I}$, where **A** is the matrix $\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

Also determine one of the characteristic roots corresponding the characteristic vectors.

Ex. 6. Verify the Cayley Hamilton's Theorem and find the latent roots where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

Ex. 7. Determine the characteristic roots of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Ex. 8. Let **A** and **B** be two square matrices over the field of real numbers, and let **B** be non singular, obtain the characteristic roots of **A**, if

$$\mathbf{B}^{-1}\mathbf{AB} = -\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}.$$

Ex. 9. Use Cayley-Hamilton Theorem to find $\mathbf{A}^{-1}$ if

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{bmatrix}$$

Ex. 10. Find the characteristic roots and characteristic vectors of the following matrix :— $\mathbf{A} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 2 \\ -2 & 0 & 4 \end{bmatrix}$.

Ex. 11. Find eigen-vectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -8 \\ 2 & -4 & 1 \end{bmatrix}$$

****Ex. 12.** If $\lambda_1, \lambda_2, ..., \lambda_n$ are the characteristic roots of a square matrix **A** of order n, then show that $1/\lambda_1, 1/\lambda_2, ..., 1/\lambda_n$ are the characteristic roots of the matrix $\mathbf{A}^{-1}$. *(Agra 92; Kumaun 94)*

Ex. 13. Does the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$ satisfy Cayley Hamilton Theorem ?

Find eigen-values and eigen vectors of **A**. *(Agra 90)*

Ex. 14. Prove that one characteristic root of **A** is 2, and find the corresponding characteristic vectors where $\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Ex. 15. Find the characteristic roots and characteristic vectors of the matrix

$$\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

Ex. 16. Show that if $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are the latent roots of a matrix **A**, then $\mathbf{A'}_3$ has the latent roots

$$\lambda_1^3, \lambda_2^3, \lambda_3^3, \ldots \lambda_n^3.$$ *(Agra 96)*

Ex. 17. Find the characteristic roots and characteristic vectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$ *(Agra 96)*

Ex. 18. Show that the characteristic roots of an idempotent matix are either zero or unity. *(Bundelkhand 91)*