Chapter III

Equivalence

§ 3·01. Elementary Row operations.

Consider the matrices

$$\mathbf{A}=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}4&5&6\\1&2&3\\7&8&9\end{bmatrix},\ \mathbf{C}=\begin{bmatrix}3&6&9\\4&5&6\\7&8&9\end{bmatrix},\ \mathbf{D}=\begin{bmatrix}1&2&3\\6&9&12\\7&8&9\end{bmatrix}.$$

Here we observe that the matrices **B**, **C**, **D** are related to the matrix **A** in as much as :

(a) **B** can be obtained from **A** by interchanging first and second rows of **A** ;

(b) **C** can be obtained from **A** by multiplying the first row of **A** by 3 and

(c) **D** can be obtained from **A** by adding two times the first row to the second row of **A**.

Such operations on the rows of a matrix are known as elementary row operations. Formal definition is given below :

Definition. Let $\mathbf{A}_i$ denote the ith row of the matrix $\mathbf{A}=[a_{ij}]$. then the elementary row operations on the marix **A** are defined as :

(i) the interchanging of any two rows $\mathbf{A}_i$ and $\mathbf{A}_j$ (*i.e.* ith and jth rows). The symbols R_{ij} or $R_i \longleftrightarrow R_j$ are generally employed for this operation.

(ii) the multiplication of every element of $\mathbf{A}_i$ by a non-zero scalar c *i.e.* replacing the ith row $\mathbf{A}_i$ by $c\mathbf{A}_i$. The symbols $R_i(c)$ or $R_i \to cR_i$ are employed for this operation.

(iii) the addition to the elements of row $\mathbf{A}_i$ of c (a scalar) times the corresponding elements of the row $\mathbf{A}_k$ *i.e.* replacing the row $\mathbf{A}_i$ by $\mathbf{A}_i + c\mathbf{A}_k$.

The symbols $R_{ik}(c)$ or $R_i \to R_i + cR_k$ are used for this operation.

Note : The above operation do not change the order of the matrix.

Example : Let $\mathbf{A}=\begin{bmatrix}1&2&3\\3&4&5\\5&6&7\end{bmatrix}$

The effect of the elementary row operation $R_2 - R_1$ or $R_{21}(-1)$ is to produce the matrix

$$\mathbf{B}=\begin{bmatrix}1&2&3\\3-1&4-2&5-3\\1&6&7\end{bmatrix}=\begin{bmatrix}1&2&3\\2&2&2\\5&6&7\end{bmatrix}$$

Again the effect of elementary row operation R_2+R_1 or $R_{21}(1)$ is to produce the matrix

$$\mathbf{B}=\begin{bmatrix} 1 & 3 & 3 \\ 2+1 & 2+2 & 2+3 \\ 5 & 6 & 7 \end{bmatrix}=\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix} \text{ i.e. the matrix } \mathbf{A}.$$

Thus the above two operations are the inverse elementary row operations.

§ 3·02. Row equivalent Matrices.

Definition. If an $m \times n$ matrix **B** can be obtained from an $m \times n$ matrix **A** by a finite number of elementary row operations, then **B** is called the row equivalent to **A** and is written as

$$\mathbf{B} \overset{row}{\sim} \mathbf{A}.$$

Note : Equivalent matrices have the same order.

Example : $\begin{bmatrix} 1 & 3 & 4 & 7 \\ 2 & -3 & 5 & 6 \\ 1 & 0 & 3 & 2 \end{bmatrix} \overset{row}{\sim} \begin{bmatrix} 2 & -3 & 5 & 6 \\ 1 & 3 & 4 & 7 \\ 1 & 0 & 3 & 2 \end{bmatrix}$

(interchanging first and second rows).

§ 3·03. Elementary Row Matrix.

Definition. The matrix obtained by the application of one elementary row operation to the identity matrix $\mathbf{I}_n$ is called an elementary row matrix.

Example. Examples of elementary matrices obtained from $\mathbf{I}_3$, where

$$\mathbf{I}_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(i) $\mathbf{I}_3 \sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{E}_a$ (say),

obtained by interchanging first two rows.

(ii) $\mathbf{I}_3 \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{E}_b$ (say),

obtained by multiplying the elements of second row by c.

(iii) $\mathbf{I}_3 \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{E}_c$ (say),

obtained by adding two times the elements of second row to the corresponding elements of first row *i.e.* replacing R_1 by R_i+2R_2 *i.e.* $R_{12}(2)$.

§ 3·04. Types of Elementary Row Matrices and their symbols.

(i) $\mathbf{E}_{ij}$ denotes the elementary matrix obtained by interchanging the ith and jth rows (or columns) of an identity (or unit) matrix.

(ii) $\mathbf{E}_i(c)$ denotes the elementary matrix obtained by multiplying the ith row (or column) of the identity matrix by c.

(iii) $\mathbf{E}_{ik}(c)$ denotes the elementary matrix obtained by adding to the elements of the ith row of the identity matrix c times the corresponding elements of the kth row.

(iv) $\mathbf{E}'_{ik}(c)$ denotes the transpose of $\mathbf{E}_{ik}(c)$ and can be obtained by adding to the elements of the ith column of the identity matrix c times the corresponding elements of the kth column.

§ 3·05. Theorem. *Each elementary row operation on $m \times n$ matrix can be effected by premultiplying it by the corresponding elemntary matrix.*

Example : Let $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$

(i) Interchanging the first and third rows, we have

$$\mathbf{A} \sim \begin{bmatrix} a_{31} & a_{32} & a_{33} & a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix} = \mathbf{B} \text{ (say)}$$

The corresponding elementary matrix (obtained by interchanging first and third row of $\mathbf{I}_2$) is given by

$$\mathbf{E}_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

[Here students should note that as we are to premultiply **A** therefore the number of columns of $\mathbf{E}_{13}$ should be 3, the number of rows of **A**].

$$\text{Now } \mathbf{E}_{13}.\,\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

$$= \begin{bmatrix} a_{31} & a_{32} & a_{33} & a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix} = \mathbf{B}$$

This shows that **B** can be obtained from **A** by pre-multiplying it by $\mathbf{E}_{13}$, the corresponding elementary matrix.

(ii) Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Multiplying the elements of second row by 2, we get

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 8 & 10 & 12 \\ 7 & 8 & 9 \end{bmatrix} = B \text{ (say)}$$

The corresponding elementary matrix (obtained by multilying the elements of second row of I_3 by 2) is given by

$$E_2(2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\text{Then } E_3(2) \times A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot 1 + 0 \cdot 4 + 0 \cdot 7 & 1 \cdot 2 + 0 \cdot 5 + 0 \cdot 8 & 1 \cdot 3 + 0 \cdot 6 + 0 \cdot 9 \\ 0 \cdot 1 + 2 \cdot 4 + 0 \cdot 7 & 0 \cdot 2 + 2 \cdot 5 + 0 \cdot 8 & 0 \cdot 3 + 2 \cdot 6 + 0 \cdot 9 \\ 0 \cdot 1 + 0 \cdot 4 + 1 \cdot 7 & 0 \cdot 2 + 0 \cdot 5 + 1 \cdot 8 & 0 \cdot 3 + 0 \cdot 6 + 1 \cdot 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 8 & 10 & 12 \\ 7 & 8 & 9 \end{bmatrix} = B$$

i.e. **B** can be obtained fromA by pre multiplying it by $E_2(2)$.

(iii) Let $A = \begin{bmatrix} 1 & -2 & 3 \\ -3 & 4 & 5 \\ 5 & 6 & -7 \end{bmatrix}$

Replacing R_1 by $R_1 + 2R_2$ *i.e.* adding two times the elements of second row to the corresponding elements of first row, we get

$$A \sim \begin{bmatrix} -5 & 6 & 13 \\ -3 & 4 & 5 \\ 5 & 6 & -7 \end{bmatrix} = B \text{ (say)}$$

The corresponding elementary matrix (obtained by adding two times the elements of second row of I_3 to the corresponding elements of the first row) is given by

$$E_{12}(2) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Then } E_{12}(2) \times A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -2 & 3 \\ -3 & 4 & 5 \\ 5 & 6 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot 1 + 2(-3) + 0 \cdot 5 & 1 \cdot (-2) + 2 \cdot 4 + 0 \cdot 6 & 1 \cdot 3 + 2 \cdot 5 + 0 \cdot (-7) \\ 0 \cdot 1 + 1(-3) + 0 \cdot 5 & 0 \cdot (-2) + 1 \cdot 4 + 0 \cdot 6 & 0 \cdot 3 + 1 \cdot 5 + 0 \cdot (-7) \\ 0 \cdot 1 + 0(-3) + 1 \cdot 5 & 0 \cdot (-2) + 0 \cdot 4 + 1 \cdot 6 & 0 \cdot 3 + 0 \cdot 5 + 1 \cdot (-7) \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 6 & 13 \\ -3 & 4 & 5 \\ 5 & 6 & -7 \end{bmatrix} = \mathbf{B}$$

i.e. **B** can be obtained from **A** by pre-multiplying it by $\mathbf{E}_{12}(2)$.

COROLLARY of Theorem given in 3·05 Page 105.

If the matrix **B** is row equivalent to the matrix **A**, then $\mathbf{B} = \mathbf{S} \bullet \mathbf{A}$, where **S** is a product of the elementary matrices.

§ **3·06. Theorem.** *The elementary matrices* $\mathbf{E}_{ij}$, $\mathbf{E}_i(c)$, $\mathbf{E}_{jk}(1)$ *are non-singular.* **(See § 2·18 Page 91)**

Proof : (i) The elementary matrix $\mathbf{E}_{ij}$ is obtained by interchanging the *i*th and *j*th rows of **I**. We shall get back **I** if we now apply the same row operation upon $\mathbf{E}_{ij}$ which can also be effected by pre-multiplying $\mathbf{E}_{ij}$ by $\mathbf{E}_{ij}$ (See § 3·05 Page 105).

$$\therefore \mathbf{E}_{ij} \bullet \mathbf{E}_{ij} = \mathbf{I}.$$

i.e. $\mathbf{E}_{ij}$ its own inverse *i.e.* $\mathbf{E}_{ij}$ is non-singular.

(ii) The elementary matrix $\mathbf{E}_i(c)$ is obtained by multiplying the *i*th row of the identity matrix by c (where $c \neq 0$). We shall get back **I** if we now multiply the elements of *i*th row of $\mathbf{E}_i(c)$ by $1/c$ which can also be effected by pre-multiplying $\mathbf{E}_i(c)$ with the corresponding elementary matrix which is obtained from **I** by multiplying its *i*th row by $1/c$, which is therefore the inverse of $\mathbf{E}_i(c)$.

For example, let $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{E}_3(c) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{bmatrix}$

Then $\{\mathbf{E}_3(c)\}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/c \end{bmatrix}$, where $\{\mathbf{E}_3(c)\}^{-1}$ is the inverse of $\mathbf{E}_3(c)$

(iii) The elementary matrix $\mathbf{E}_{ik}(1)$ obtained from **I** by replacing its *j*th row by (*j*th row + *k*th row).

We shall get back **I** if we not replace the *j*th row of $\mathbf{E}_{ij}(1)$ by (*j*th row − *k*th row). **(Note)**

Hence the inverse of $\mathbf{E}_{jk}(1)$ is the elementary matrix obtained from **I** by replacing its *j*th row by (*j*th row − *k*th row).

For example, let $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{E}_{13}(1) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

obtained from **I** by replacing its 1st row by (1st row + 3rd row).

Then $\{E_{13}(1)\}^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, obtained from I by replacing its first row by (1st row – 3rd row)

§ 3·07. Theorem : *If the matrix* **B** *is row equivalent to the matrix* **A**, *then* **B** = **SA** *where* **S** *is non-singular.*

From Cor. of § 3·05 Page 107 we know that

if $B \overset{row}{\sim} A$, then **B** = **SA**, where S is the product of the elementary matrices and in § 3·06 above we have proved that elementary matrices are non-singular and hence their product is also non-singular.

This proves the above theorem.

§ 3·08. Theorem : *If a square matrix* **A** *of order n is row equivalent to the identity matrix* I_n, *then* **A** *is non-singular.*

Proof : From § 3·07 above we know that

$A \overset{row}{\sim} I_n$, then $A = S.I_n$ wherre S is non-singular.

Now $S \bullet I_n$ being the product of two non-singular matrices is non-singular. Therefore **A** is non-singular.

Note. The converse of this theorem is also true.

§ 3·09. Theorem : *If a sequence of row operations applied to a square matrix* **A** *reduces it to the identity matrix* **I**, *then the same sequence of row operations applied to the identity matrix gives the inverse of* **A** *(i.e.* A^{-1}*).*

Proof : From Cor. of § 3·05 Page 107 we know that **SA** = **I**, where **S** is the product of the elementary matrices.

i.e. $(E_k \ldots E_3 E_2 E_1) A = I$, where E_i denotes the elementary matrices

or $(E_k \ldots E_3 E_2 . E_1) AA^{-1} = IA^{-1}$

or $(E_k \ldots E_3 . E_2 . E_1) I = A^{-1}$, since $AA^{-1} = I$ and $IA^{-1} = A^{-1}$

(See § 2·18 Page 91 and Ex. 1 Page 64)

Hence the theorem

Note. With the help of the above theorem we shall find the inverse of the given non-singular matrix A.

In the following examples we shall show the successive matrices row equivalent to **A** and **I** in the left hand and right hand columns respectively. When ultimately **A** is reduced to **I** in the left hand column, **I** is reduced to A^{-1} in the right hand column.

Also R_1, R_2, R_3 ,... etc. stand for first row, second row, third row, etc.

Solved Examples on § 3·09.

***Ex. 1 Find the inverse of the matrix** $A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 0 & 0 \\ 1 & 4 & 1 \end{bmatrix}$

Sol.

$$\overset{\mathbf{A}}{\begin{bmatrix} 1 & -3 & 2 \\ 2 & 0 & 0 \\ 1 & 4 & 1 \end{bmatrix}} \Bigg| \sim \overset{\mathbf{I}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\sim \begin{bmatrix} 1 & -3 & 2 \\ 1 & 0 & 0 \\ 1 & 4 & 1 \end{bmatrix} \Bigg| \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(Replacing R_2 by $\frac{1}{2} R_2$)

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 1 & -3 & 2 \\ 1 & 4 & 1 \end{bmatrix} \Bigg| \sim \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(Interchanging R_1 and R_2)

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 2 \\ 0 & 4 & 1 \end{bmatrix} \Bigg| \sim \begin{bmatrix} 0 & (1/2) & 0 \\ 1 & -(1/2) & 0 \\ 0 & -(1/2) & 1 \end{bmatrix}$$

(Replacing R_2 by $R_2 - R_1$ and R_3 by $R_3 - R_1$)

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -11 & 0 \\ 0 & 4 & 1 \end{bmatrix} \Bigg| \sim \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 1 & \frac{1}{2} & -2 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}$$

(Replacing R_2 by $R_2 - 2R_3$)

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \Bigg| \sim \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{11} & -\frac{1}{22} & \frac{2}{11} \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}$$

(Replacing R_2 by $-\frac{1}{11} R_2$)

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Bigg| \sim \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{11} & -\frac{1}{22} & \frac{2}{11} \\ \frac{4}{11} & -\frac{7}{22} & \frac{3}{11} \end{bmatrix}$$

(Replacing R_3 by $R_3 - 4R_2$)

$$= \mathbf{I} \qquad | \qquad = \mathbf{A}^{-1}$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{11} & -\frac{1}{22} & \frac{2}{11} \\ \frac{4}{11} & -\frac{7}{22} & \frac{3}{11} \end{bmatrix}$$

Ans.

***Ex. 2.** $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$, **evaulate** A^{-1},

Sol.

$$\overset{A}{\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}} \Bigg| \sim \overset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 0 \\ 0 & -1 & 1 \end{bmatrix} \Bigg| \sim \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

(Replacing R_2 by $R_2 - 3R_1$ and R_3 by $R_3 - R_1$)

$$\sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \Bigg| \sim \begin{bmatrix} 2 & 0 & -1 \\ \frac{3}{4} & -\frac{1}{4} & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

(Replacing R_1 by $R_1 - R_3$ and R_2 by $-\frac{1}{4} R_2$)

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Bigg| \sim \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & -1 \\ \frac{3}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 1 \end{bmatrix}$$

(Replacing R_1 by $R_1 - 3\, R_2$ and R_3 by $R_3 + R_2$)

$$= I \quad | \quad = A^{-1}$$

$$\therefore \quad A^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & -1 \\ \frac{3}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 1 \end{bmatrix}$$

Ans.

Ex.3. Find the inverxe of the matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

Sol.

$$\overset{A}{\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}} \Bigg| \sim \overset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \Bigg| \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(Replacing R_1 by $R_1 + 2R_2$)

$$\sim\left[\begin{array}{ccc}1 & -2 & 0\\ 0 & 1 & 0\\ 0 & -2 & 1\end{array}\right] \Bigg\| \sim\left[\begin{array}{ccc}1 & 0 & 2\\ 1 & 1 & 2\\ 0 & 0 & 1\end{array}\right]$$

(Replacing R_2 by $R_2 + R_1$)

$$\sim\left[\begin{array}{ccc}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right] \Bigg\| \sim\left[\begin{array}{ccc}3 & 2 & 6\\ 1 & 1 & 2\\ 2 & 2 & 5\end{array}\right]$$

(Replacing R_1 by $R_1 + 2R_2$ and R_3 by $R_3 + 2R_2$)

$$= \mathbf{I} \qquad | \qquad = \mathbf{A}^{-1}$$

$$\therefore \quad \mathbf{A}^{-1} = \left[\begin{array}{ccc}3 & 2 & 6\\ 1 & 1 & 2\\ 2 & 2 & 5\end{array}\right]$$

Ans.

Exercises on § 3·09

Ex. 1. Find $\mathbf{A}^{-1}$ if $\mathbf{A} = \left[\begin{array}{ccc}2 & 4 & 3\\ 0 & 1 & 1\\ 2 & 2 & -1\end{array}\right]$ **Ans.** $\frac{1}{4}\left[\begin{array}{ccc}3 & -10 & -1\\ -2 & 8 & 2\\ 2 & -4 & -2\end{array}\right]$

Ex. 2. Find the reciprocal matrix of $\left[\begin{array}{ccc}1 & 1 & 1\\ 2 & 2 & 3\\ 2 & 4 & 9\end{array}\right]$ **Ans.** $\frac{1}{3}\left[\begin{array}{ccc}-6 & 5 & -1\\ 15 & -8 & 1\\ -6 & 3 & 0\end{array}\right]$

Ex. 3. Find $\mathbf{A}^{-1}$, if $\mathbf{A} = \left[\begin{array}{ccc}1 & 2 & 1\\ 3 & 1 & 2\\ 0 & 0 & 2\end{array}\right]$ **Ans.** $\frac{1}{9}\left[\begin{array}{ccc}0 & 3 & -3\\ 6 & -2 & -1\\ -3 & 1 & 5\end{array}\right]$

§ 3·10. Elementary Column Operation and Column Equivalent Matrices.

In § 3·01 Page 103, if the word *row* is replaced by the word *column* we get the definition of the elementary column operation.

Similarly in § 3·02 Page 104 repacing the word *row* by the word *column* we get definition of column equivalent matrices.

$\mathbf{B} \overset{col}{\sim} \mathbf{A}$ means the matrix **B** is column equivalent to the matrix **A**.

Symbols for column operations are similar to those given for row operations in § 3·01 Page 103. Here the letter R in the symbols are to be replaced by C *e.g.* C_{ij}, $C_{ij}(c)$, $C_{ik}(c)$, where C_{ij} stands for the interchange of ith and jth columns etc or $C_i \longleftrightarrow C_j$; $C_i \to cC_i$, $C_i \to cC_k$.

§ 3·11. Theorem. *Each elemntary column operation on an* $m \times n$ *matrix* **A** *can be effected by* **post multiplying A** *by the* $n \times n$ *matrix obtained from the* $n \times n$ *identity matrix* $\mathbf{I}_n$ *by the same elementary column operation.*

Proof : If $\mathbf{B} \overset{col}{\sim} \mathbf{A}$

then $\mathbf{B}' \overset{row}{\sim} \mathbf{A}'$ where $\mathbf{B}'$ and $\mathbf{A}'$ are the transposed matrices of **B** and **A**.

(See § 2·08 Page 69)

Since if **B** is obtained from **A** by elementary column operation, then $\mathbf{B}'$ can be obtained from $\mathbf{A}'$ by an elementary row operation.

Hence $\mathbf{B}' = \mathbf{E}\mathbf{A}'$, where **E** is the elementary matrix obtained from $\mathbf{I}_n$ by an elementary row operation. (See § 3·05 Page 105)

Therefore $\mathbf{B} = \mathbf{A}\mathbf{E}'$, **(Note)**

where $\mathbf{E}'$, the transposed matrix of **E**, can be obtained from $\mathbf{I}_n$ by the same elementary column operation.

Hence the theorem.

§ 3·12. Theorem. If there be two $m \times n$ matrices **A** and **B**, then $\mathbf{B} \overset{col}{\sim} \mathbf{A}$ if $\mathbf{B} = \mathbf{AT}$, where **T** is an $n \times n$ non-singular matrix

Proof : If $\mathbf{B} \overset{col}{\sim} \mathbf{A}$, then $\mathbf{B}' \overset{col}{\sim} \mathbf{A}'$, where $\mathbf{B}'$ and $\mathbf{A}'$ are the transposed matrices of **B** and **A** respectively.

Therefore $\mathbf{B}' = \mathbf{S}\mathbf{A}'$, where **S** is an $n \times n$ non-singular matrix.

(See § 3·07 Page 108)

Consequently $\mathbf{B} = \mathbf{A}\mathbf{S}'$, where $\mathbf{S}'$ is the transposed matrix of **S**

$= \mathbf{AT}$, where $\mathbf{T} = \mathbf{S}'$, an $n \times n$ singular matrix.

Hence the theorem.

§ 3·13. Equivalent Matrices (General Definition). *(Avadh 95)*

Definition. Two $m \times n$ matrices **A** and **B** are called equivalent if one can be obtained from the other by a finite number of row and column operations (or elementary operations) and written as $\mathbf{B} \sim \mathbf{A}$.

§ 3·14. Triangular Matrix.

Definition. A matrix $[a_{ij}]$ is called a triangular matrix if

$$a_{ij} = 0 \text{ for } i > j.$$

For Example $\begin{bmatrix} 2 & 3 & 1 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & 7 \end{bmatrix}$ or $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

Note 1. Triangular matrix need not be square. If it is square, then it is called upper triangular matrix. (See § 2·01 (a) Page 61)

Note 2. The elements a_{ij} for which $i \leq j$ are not necessarily zero.

***§ 3·15. Theorem.** *Every matrix can be reduced to triangular form by elementary row operations.*

Proof : We shall prove this theorem by Mathematical intduction.

Assume that this theorem holds for all matrices containing $n-1$ rows and let $\mathbf{A}=[a_{ij}]$ be an $n \times m$ matrix given below —

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & \cdots & a_{nm} \end{bmatrix},$$

Now the following cases arise :—

Case I. If $a_{11} \neq 0$, then replacing R_1 by $(1/a_{11})\ R_1$ (*i.e.* by applying elementary row operation) the matrix **A** reduces to an $n \times m$ matrix

$$\mathbf{B}=[b_{ij}]=\begin{bmatrix} b_{11} & b_{12} & \cdots & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & \cdots & b_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & \cdots & b_{nm} \end{bmatrix},$$

where $b_{11}=1$.

Now applying elementary row-operation $R_k - b_{ki}R_1$ to R_k where $k = 1, 2, \ldots, n$ *i.e.* subtract b_{ki} times R_1 from R_k, where k takes values from 1 to n. This reduces the matrix **B** to matrix $\mathbf{C}=[c_{ij}]$ where $c_{k1}=0$ whenever $k>1$ and we have

$$\mathbf{C}=\begin{bmatrix} 1 & c_{12} & c_{13} & \cdots & c_{1m} \\ 0 & c_{22} & c_{23} & \cdots & c_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & c_{n2} & c_{n3} & \cdots & c_{nm} \end{bmatrix}.$$

Now by our assumption that the theorem which we are going to prove holds for matrices containing $(n-1)$ rows we find that $(n-1)$ rowed matrix

$$\begin{bmatrix} 0 & c_{22} & c_{23} & \cdots & c_{2m} \\ 0 & c_{32} & c_{33} & \cdots & c_{3m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & c_{n2} & c_{n3} & \cdots & c_{nm} \end{bmatrix}$$

can always be reduced to triangular form by elementary row operations and hence from (i) the matrix **C** will reduce to triangular form when the same elementary row operations are applied to **C**.

Case II. If $a_{11}=0$ but $a_{k1} \neq 0$ for some value of k then interchanging R_1 and R_k the matrix **A** reduces to the matrix $\mathbf{D}=[d_{ij}]$ where $d_{11} \neq 0$.

Then the matrix **D** can always be reduced to the triangular form as in case I above.

Case III. If $a_{k1}=0$ for all values of k then we have

$$\mathbf{A}=\begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1m} \\ 0 & a_{22} & a_{23} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & a_{n2} & a_{n3} & \cdots & a_{nm} \end{bmatrix}$$

By hypothesis (inductive) the $(n-1)$ rowed matrix

$$\begin{bmatrix} 0 & a_{22} & a_{23} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & a_{n2} & a_{n3} & \dots & a_{nm} \end{bmatrix}$$

as in case I above can be reduced to triangular form by elementary row operations and the same elementary operations when applied on **A** will reduce **A** to triangular form.

Hence the matrix **A** can always be reduced to triangular form and the proof is complete by mathematical induction.

Solved Examples on § 3·15.

Ex. 1. Reduce the matrix $\begin{bmatrix} 3 & 1 & 4 \\ 1 & 2 & -5 \\ 0 & 1 & 2 \end{bmatrix}$ **to triangular form.**

Sol. Let $\mathbf{A} = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 2 & -5 \\ 0 & 1 & 2 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & \frac{1}{3} & \frac{4}{3} \\ 1 & 2 & -5 \\ 0 & 1 & 2 \end{bmatrix}$, replacing R_1 by $\frac{1}{4} R_1$

$\sim \begin{bmatrix} 1 & \frac{1}{3} & \frac{4}{3} \\ 0 & \frac{5}{3} & -\frac{19}{3} \\ 0 & 1 & 2 \end{bmatrix}$, replacing R_3 by $R_2 - R_1$

$\sim \begin{bmatrix} 1 & \frac{1}{3} & \frac{4}{3} \\ 0 & \frac{5}{3} & -\frac{19}{3} \\ 0 & 0 & \frac{29}{5} \end{bmatrix}$, replacing R_3 by $R_3 - \frac{3}{5} R_2$

This is the required triangular form.

Aliter $\mathbf{A} = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 2 & -5 \\ 0 & 1 & 2 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & -5 \\ 3 & 1 & 4 \\ 0 & 1 & 2 \end{bmatrix}$, interchanging R_1 and R_2

$\sim \begin{bmatrix} 1 & 2 & -5 \\ 0 & -5 & 19 \\ 0 & 1 & 2 \end{bmatrix}$, replacing R_2 by $R_2 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & -5 \\ 0 & -5 & 19 \\ 0 & 5 & 10 \end{bmatrix}, \text{ replacing } R_3 \text{ by } 5R_3$$

$$\sim \begin{bmatrix} 1 & 2 & -5 \\ 0 & -5 & 19 \\ 0 & 0 & 29 \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 + R_2$$

This is also a triangular matrix.

Note. The above shows that reduction of a matrix to triangular form is not unique.

Ex. 2. Reduce $\mathbf{A} = \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 3 & 1 \\ -1 & 1 & 2 & 0 \end{bmatrix}$ **to triangular form.** *(Agra 95)*

Sol. Let $\mathbf{A} = \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 3 & 1 \\ -1 & 1 & 2 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} -1 & 1 & 2 & 0 \\ 0 & 1 & 3 & 1 \\ 5 & 3 & 14 & 4 \end{bmatrix}, \text{ interchanging } R_1 \text{ and } R_3$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 1 & 3 & 1 \\ 5 & 3 & 14 & 4 \end{bmatrix}, \text{ replacing } R_1 \text{ by } -R_1$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 8 & 24 & 4 \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 - 5R_1$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & -4 \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 - 8R_2$$

This is a triangular matrix as here $a_{ij} = 0$ for $i > j$. [See definition § 3·14 Page 112]

Exercises on § 3·15

Ex. 1. Reduce the matrix $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 4 \\ 3 & -1 & 4 \end{bmatrix}$ to the triangular form.

Ans. $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Ex. 2. Reduce the matrix $\begin{bmatrix} -1 & 2 & 1 & 8 \\ 2 & 1 & -1 & 0 \\ 3 & 2 & 1 & 7 \end{bmatrix}$ to the triangular form.

Ans. $\begin{bmatrix} 1 & -2 & -1 & -8 \\ 0 & 5 & 1 & 16 \\ 0 & 0 & 12 & 27 \end{bmatrix}$

MISCELLANEOUS SOLVED EXAMPLES

Ex. 1. Apply successively the row transformations (or operation) R_{23}, $R_3(-2)$ and $R_{12}(4)$ to the matrix $\begin{bmatrix} 3 & 1 & 2 & 1 \\ 2 & 0 & 3 & 2 \\ 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 1 \end{bmatrix}$

Sol. (i). Applying R_{23} opertion to the given matrix we have

$\begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 0 & 3 & 2 \\ 3 & 1 & 4 & 1 \end{bmatrix}$ [Here we have interchanged second and third rows].

(ii) Applying $R_3(-2)$ operation to the given matrix we have

$\begin{bmatrix} 3 & 1 & 2 & 1 \\ 2 & 0 & 3 & 2 \\ -2 & -4 & -6 & -8 \\ 3 & 1 & 4 & 1 \end{bmatrix}$ [Here we have replaced third row R_3 by $-2R_3$]

(iii) Applying $R_{12}(4)$ operation to the given matrix we have

$\begin{bmatrix} 7 & 9 & 14 & 17 \\ 2 & 0 & 3 & 2 \\ 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 1 \end{bmatrix}$ [Here we have replaced the first row R_1 by $R_1 + 4R_3$]

Ex. 2. Apply successively the column operation C_{13}; $C_2(-4)$ and $C_{23}(-2)$ to the mâtrix $\begin{bmatrix} 1 & -1 & 2 & 3 & 4 \\ 2 & 1 & -2 & 1 & 3 \\ 3 & 2 & 1 & -2 & 5 \\ 4 & 5 & 6 & 7 & 8 \end{bmatrix}$

Sol. (i) Applying C_{13} operation to the given matrix, we have

$\begin{bmatrix} 2 & -1 & 1 & 2 & 4 \\ -2 & 1 & 2 & 1 & 3 \\ 1 & 2 & 3 & -2 & 5 \\ 6 & 5 & 4 & 7 & 8 \end{bmatrix}$ [Here we have interchanged C_1 and C_3 *i.e.* first and third columns.]

(ii) Applying $C_2(-4)$ operation to the given matrix, we have

$$\begin{bmatrix} 1 & 4 & 2 & 3 & 4 \\ 2 & -4 & -2 & 1 & 3 \\ 3 & -8 & 1 & -2 & 5 \\ 4 & -20 & 6 & 7 & 8 \end{bmatrix}$$ [Here we have replaced second column C_2 by $-4C_2$].

(iii) Applying $C_{23}(-2)$ operation to the given matrix, we have

$$\begin{bmatrix} 1 & -5 & 2 & 3 & 4 \\ 2 & 5 & -2 & 1 & 3 \\ 3 & 0 & 1 & -2 & 5 \\ 4 & -7 & 6 & 7 & 8 \end{bmatrix}$$ [Here we have replaced second column C_2 by $C_2 - 2C_3$].

Ex. 3. Compute the following elementary matrices of order 4 E_{23}, $E_2(4)$, $E_{34}(-2)$, $E'_{34}(-2)$. **(Refer § 3·04 Page 105)**

Sol. The identity (or unit) matrix of order four is given by

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(i) $$E_{23} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ [Interchanging R_2 and R_3 or C_2 and C_3] **(Note)**

(ii) $$E_2(4) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ [Replacing R_2 by $4R_2$ or C_2 by $4C_2$].

(iii) $$E_{34}(-2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ [Replacing R_3 by $R_3 - 2R_4$].

(iv) $$E'_{34}(-2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$ [Replacing C_3 by $C_3 - 2C_4$].

[Here students should note that $E'_{34}(-2)$ is nothing but the **transpose** matrix of $E_{34}(-2)$].

Ex. 4. Evaluate the inverse of the following elementary matrices of order four : $E_3(-2)$, $E_{23}(4)$ **(Refer § 3·06 Page 107-108)**

Sol. The identity matrix of order four is given by

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(i) Then $\mathbf{E_3}(-2)=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-2&0\\0&0&0&1\end{bmatrix}$, replacing R_3 by $-2R_3$

$\therefore$ Inverse of $\mathbf{E_3}(-2)$ *i.e.* $\{\mathbf{E_3}(-2)\}^{-1}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-\frac{1}{2}&0\\0&0&0&1\end{bmatrix}$ **Ans.**

(obtained by replacing R_3 of $\mathbf{I_4}$ by $-\frac{1}{2}R_3$). **(Note)**

(ii) $\mathbf{E_{23}}(4)=\begin{bmatrix}1&0&0&0\\0&1&4&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$, replacing R_2 of $\mathbf{I_4}$ by R_2+4R_3.

Then the inverse of $\mathbf{E_{23}}(4)$ *i.e.* $\{\mathbf{E_{23}}(4)\}^{-1}$ is given by

$\begin{bmatrix}1&0&0&0\\0&1&-4&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$, replacing R_2 of $\mathbf{I_4}$ by R_2-4R_3. **(Note)**

****Ex. 5. Find the inverse of the matrix** $\mathbf{A}=\begin{bmatrix}i&-1&2i\\2&0&2\\-1&0&1\end{bmatrix}$

Sol.

$$\overset{\mathbf{A}}{\begin{bmatrix}i&-1&2i\\2&0&2\\-1&0&1\end{bmatrix}}\ \Bigg|\ \sim\overset{\mathbf{I}}{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}}$$

$$\sim\begin{bmatrix}i&-1&2i\\1&0&1\\-1&0&1\end{bmatrix}\ \Bigg|\ \sim\begin{bmatrix}1&0&0\\0&1/2&0\\0&0&1\end{bmatrix}$$

(Replacing R_2 by $\frac{1}{2}R_2$)

$$\sim\begin{bmatrix}i&-1&2i\\1&0&1\\0&0&2\end{bmatrix}\ \Bigg|\ \sim\begin{bmatrix}1&0&0\\0&1/2&0\\0&1/2&1\end{bmatrix}$$

(Replacing R_3 by R_3+R_2)

$$\sim\begin{bmatrix}i&-1&2i\\1&0&1\\0&0&1\end{bmatrix}\ \Bigg|\ \sim\begin{bmatrix}1&0&0\\0&1/2&0\\0&1/4&1/2\end{bmatrix}$$

(Replacing R_3 by $\frac{1}{2}R_3$)

$$\sim \begin{bmatrix} i & -1 & 2i \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Bigg| \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & -1/2 \\ 0 & 1/4 & 1/2 \end{bmatrix}$$

(Replacing R_2 by $R_2 - R_3$)

$$\sim \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Bigg| \sim \begin{bmatrix} 1 & -\frac{3}{4}i & -\frac{1}{2}i \\ 0 & 1/4 & -1/2 \\ 0 & 1/4 & 1/2 \end{bmatrix}$$

(Replacing R_1 by $R_1 - iR_2 - 2i\,R_3$) **(Note)**

$$\sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Bigg| \sim \begin{bmatrix} -1 & \frac{3}{4}i & \frac{1}{2}i \\ 0 & 1/4 & -1/2 \\ 0 & 1/4 & 1/2 \end{bmatrix}$$

(Replacing R_1 by $-R_1$)

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I} \Bigg| \sim \begin{bmatrix} 0 & 1/4 & -1/2 \\ -1 & \frac{3}{4}i & \frac{1}{2}i \\ 0 & 1/4 & 1/2 \end{bmatrix} = \mathbf{A}^{-1}$$

(Interchanging R_1 and R_2)

Therefore $\mathbf{A}^{-1} = \begin{bmatrix} 0 & 1/4 & -1/2 \\ -1 & \frac{3}{4}i & \frac{1}{2}i \\ 0 & 1/4 & 1/2 \end{bmatrix}$ **Ans.**

EXERCISES ON CHAPTER III

Ex. 1. Apply the row operation $R_4(-3)$ and $R_{21}(4)$ to the matrix

$$\begin{bmatrix} 4 & -1 & 2 & 3 \\ -1 & 8 & -3 & -4 \\ 2 & 3 & 4 & -1 \\ -3 & -4 & -1 & 8 \end{bmatrix}$$

(Hint ; See Ex. 1 Page 116)

Ans. $\begin{bmatrix} 4 & -1 & 2 & 3 \\ -1 & 8 & -3 & -4 \\ 2 & 3 & 4 & -1 \\ 9 & 12 & 3 & -24 \end{bmatrix}$ and $\begin{bmatrix} 4 & -1 & 2 & 3 \\ 15 & 4 & 5 & 8 \\ 2 & 3 & 4 & -1 \\ -3 & -4 & -1 & 8 \end{bmatrix}$

Ex. 2. Apply the column operation $C_3(4)$ and $C_{12}(-3)$ to the matrix

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 3 & 4 & 0 & 1 & 2 \\ 2 & 0 & 1 & 3 & 4 \end{bmatrix}$$

[Hint : See Ex. 2 Page 116]

Ans. $\begin{bmatrix} 0 & 1 & 8 & 3 & 4 \\ 1 & 2 & 12 & 4 & 0 \\ 3 & 4 & 0 & 1 & 2 \\ 2 & 0 & 4 & 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} -3 & 1 & 2 & 3 & 4 \\ -5 & 2 & 3 & 4 & 0 \\ -9 & 4 & 0 & 1 & 2 \\ 2 & 0 & 1 & 3 & 4 \end{bmatrix}$

Ex. 3. Compute E_{23}, $E_2(-2)$ and $E_{34}(-1)$ for the identity matrix of order 4. **(Hint. See Ex. 3 Page 117)**

Ans. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Ex. 4. Evaluate the inverse of the following elementary matrices of order 4 :

$$E_{14}, E_4(3), E_{22}(2).$$

(Hint : See Ex. 4 Page 117).

Ans. E_{14}, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Ex. 5. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & -4 \\ 2 & 3 & 5 & -5 \\ 3 & -4 & -5 & 8 \end{bmatrix} \quad \textbf{Ans. } \frac{1}{18}\begin{bmatrix} 2 & 16 & 6 & 4 \\ 22 & 41 & -30 & -1 \\ -10 & -44 & 30 & -2 \\ 4 & -13 & 6 & -1 \end{bmatrix}$$

(Hint : See Ex. 5 Page 118).

***Ex. 6.** Find the inverse of the matrix $\begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$.

(Hint : See Ex. 5 Page 118).

Ans. $\frac{1}{14}\begin{bmatrix} 3 & -1 & 5 \\ 5 & 3 & -1 \\ -1 & 5 & 3 \end{bmatrix}$

Ex. 7. Find the inverse of the matrix $\begin{bmatrix} 1 & 3 & 3 & 2 & 1 \\ 1 & 4 & 3 & 3 & -1 \\ 1 & 3 & 4 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 \\ 1 & -2 & -1 & 2 & 2 \end{bmatrix}$

(Hint : See Ex. 5 Page 118).

Ans. $\frac{1}{15}\begin{bmatrix} 30 & -20 & -15 & 25 & -5 \\ 30 & -11 & -18 & 7 & -8 \\ -30 & 12 & 21 & -9 & 6 \\ -15 & 2 & 6 & -9 & 6 \\ 15 & -7 & -6 & -1 & -1 \end{bmatrix}$

Ex. 8. Has the following matrix an inverse ?

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 2 & -1 & 4 \\ 3 & 3 & 2 & 5 \\ 1 & -1 & 4 & -1 \end{bmatrix}$$

(Hint : It can not be reduced to I_4). **Ans.** No.