CHAPTER IV.

Determinants

§ 4·01. Permutations.

Def. The operation of rearranging n distinct elements of a set among themselves is called permutation.

Let S be a set defined by

$$S = \{i_1, i_2, i_3, \ldots, i_n\} \; ; \; i_m \neq i_k \text{ for } m \neq k.$$

Let P be the transformation on S, such that $P(i_1) = a_1$, $P(i_2) = a_2$, $P(i_3) = a_3, \ldots, P(i_n) = a_n$, where $a_1, a_2, \ldots, a_n$ is some arrangement of the elements $i_1, i_2 \ldots, i_n$ of S.

Then a two line notation for the permutation is

$$P = \begin{pmatrix} i_1 & i_2 & i_3 \ldots i_n \\ a_1 & a_2 & a_3 \ldots a_n \end{pmatrix}$$

(**Note :** The order of columns in this notation is immaterial).

If in a given permutation a larger integer precedes a smaller one then there is an inversion. For example in 452 we see that 5 precedes 2.

If in a given permutation the number of inversions is odd, the permutation is known as odd. For example the permutation 5312 is odd as in this permutation we observe that 5 precedes 3, 5 precedes 1, 5 precedes 2, 3 precedes 1 and 3 precedes 2 *i.e.*, there are five (*i.e.* odd) inversion in 5312

Similarly if in a given permutation the number of inversions is even the permutation is known as even. For example the permutation 5314 is even as in this permutation we observe that 5 precedes 3, 5 precedes 1, 5 precedes 4 and 3 precedes 1.

Note : If there is no inversion the permutation is even, for example 345.

§ 4·02. Determinant of a square matrix.

Let us consider a square matrix **A** of order $n \times n$ given by

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & \ldots & a_{nn} \end{bmatrix}$$

The product of the elements in the principal diagonal is

$$a_{11}\, a_{22}\, a_{33} \ldots a_{nn}.$$

This is also called the trace of the matrix.

Now obtain $n\,!$ terms of the above type by operating on the row-subscripts of the elements of the above expression by $n\,!$ permutations $p = \begin{pmatrix} 1 & 2 & 3 \ldots n \\ i_1 & i_2 & i_3 \ldots i_n \end{pmatrix}$, where $i_1, i_2 \ldots, i_n$ are one of the $n\,!$ permutations of the integers 1, 2, 3, ..., n.

The sum of n ! signed terms thus obtained is defined as the **determinants of the matrix A** and is denoted by $|\mathbf{A}|$ or $|a_{ij}|$.

Therefore the determinant of the square matrix $\mathbf{A}=[a_{ij}]$ of order $n \times n$ is given by $$|a_{ij}| = \Sigma \pm a_{\alpha_1}\, a_{\beta_2}\, a_{\gamma_3}\, a_{\delta_4} \cdots a_{k_n}$$
where +or −sign is taken when $\alpha, \beta, \gamma, \delta, ..., k$ is an even or odd permutation of $1, 2, 3, ..., n$, and the summation extends over n ! permutations of the row subscripts 1, 2, 3, ...

Note : The determinant of a square matrix of order n is known as a determinant of order n.

§ 4·03. Determinant of order two.

Let us consider asquare matrix $\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ of order 2×2.

Then 2 ! permutation on two symbols 1 and 2 are
$$I=\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix},\ p=\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

The product of the elements of the principal diagonal are $a_{11}\, a_{22}$.

Operating on the row subscripts of $a_{11}\, a_{22}$ by the permutation I we get $+a_{11}\, a_{22}$, prefixing + sign as the permutation I is even (See § 4·01 Note) and by the permutation p we get $-a_{21}\, a_{12}$, prefixing − sign as the permutation is odd.

Hence $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\, a_{22} - a_{21}\, a_{12}$

§ 4·04. Determinant of order three.

Let us consider a sqaure matrix $\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ of order 3×3.

Then 3 ! permutations on three symbols 1, 2 and 3 are
$$i=\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix};\ p_1=\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix};\ p_2=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix};$$
$$p_3=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix};\ p_4=\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \text{ and } p_5=\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

Operating on the row subscripts of $a_{11}\, a_{22}\, a_{33}$ by the permutation $I, p_1, p_2, ... p_5$ we have successively

(a) $+ a_{11}\, a_{22}\, a_{33}$, prefixing + as permutation I is even.

(b) $- a_{11}\, a_{32}\, a_{23}$, prefixing − as permutation p_1 is odd

(c) $+ a_{21}\, a_{32}\, a_{13}$, prefixing + sign as permutation p_2 is even.

(d) $- a_{21}\, a_{12}\, a_{33}$, prefixing − sign as permutation p_3 is odd.

(e) $+ a_{31}\, a_{12}\, a_{23}$, prefixing + sign as permutation p_4 is even.

(f) $-a_{31}\,a_{22}\,a_{13}$, prefixing $-$ sign as permutation p_5 is odd.

Hence we have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\,a_{22}\,a_{33} - a_{11}\,a_{32}\,a_{23} + a_{21}\,a_{32}\,a_{13} - a_{21}\,a_{12}\,a_{33} + a_{31}\,a_{12}\,a_{23} - a_{31}\,a_{22}\,a_{13}$$

$$= a_{11}\,(a_{22}\,a_{33} - a_{23}\,a_{32}) - a_{12}\,(a_{21}\,a_{33} - a_{23}\,a_{31}) + a_{13}\,(a_{21}\,a_{32} - a_{22}\,a_{31})$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Solved Examples on § 4·02 to § 4·04.

Ex. 1. Evaluate $\begin{vmatrix} -5 & 0 \\ 7 & -2 \end{vmatrix}$

Sol. $\begin{vmatrix} -5 & 0 \\ 7 & -2 \end{vmatrix} = (-5)\,(-2) - 0 \times 7 = 10 - 0 = 10.$ **Ans.**

Ex. 2. Evaluate $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

Sol. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\,\{5\cdot9 - 6\cdot8\} - 2\,(4\cdot9 - 6\cdot7) + 3\,\{4\cdot8 - 5\cdot7\},$ See § 4·04 Page 123

$$= 1\,\{45 - 48\} - 2\,\{36 - 42\} + 3\,\{32 - 35)$$
$$= -3 + 12 - 9 = 0$$ **Ans.**

Exercise

Ex. Show that $\begin{vmatrix} 2 & 5 \\ -3 & 7 \end{vmatrix} = 29$

§ 4·05. Cofactor of an element.

Definition. If in the expansion of a determinant $|a_{ij}|$, all the terms containing a_{ij} as a factor are collected and their sum be denoted by $a_{ij}\,C_{ij}$, then the factor C_{ij} is defined as the cofactor of the element a_{ij}.

From the above definition we find that if $[a_{ij}]$ be the $n \times n$ matrix whose determinant is $|a_{ij}|$ then if from $[a_{ij}]$ the element of its ith row and jth column are removed, the terms of C_{ij} are then composed of elements from the remaining $(n-1) \times (n-1)$sub-matrix M_{ij} of $[a_{ij}]$.

Hence the determinant $|a_{ij}|$ can be expressed as a function of the elements of ith row, by collecting all the terms containing $a_{i1}, a_{i2}, a_{i3}, \ldots, a_{ij}$ and finding their sum

i.e. $$|a_{ij}| = a_{i1}\,C_{i1} + a_{i2}\,C_{i2} + \ldots + a_{in}\,C_{in}$$
$$= \sum_{j=1}^{n} a_{ij}\,C_{ij},$$

which is the expansion of the determinant $|a_{ij}|$ by the elements of the ith row and their cofactors.

In a similar manner we can expand the determinant $|a_{ij}|$ by the elements of the kth column and their cofactors and write as

$$|a_{ij}| = \sum_{k=1}^{n} a_{ik} C_{ik}$$

Example : Expand the determinant $\begin{vmatrix} \mathbf{a} & \mathbf{h} & \mathbf{g} \\ \mathbf{h} & \mathbf{b} & \mathbf{f} \\ \mathbf{g} & \mathbf{f} & \mathbf{c} \end{vmatrix}$ **by the elements of 1st row.**

Sol. Elements of first row are a, h, g.

Let A, H and G denote the cofactors of a, h, g.

Then $A = \begin{vmatrix} b & f \\ f & c \end{vmatrix}$; $H = -\begin{vmatrix} h & f \\ g & c \end{vmatrix}$ and $G = \begin{vmatrix} h & b \\ g & f \end{vmatrix}$

Hence $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = aA + hH + gG$ See. § 4·04 above

$$= a \begin{vmatrix} b & f \\ f & c \end{vmatrix} - h \begin{vmatrix} h & f \\ g & c \end{vmatrix} + g \begin{vmatrix} h & b \\ g & f \end{vmatrix}$$

$$= a(bc - f^2) - h(ch - fg) + g(hf - bg)$$

$$= abc + 2fgh - af^2 - bg^2 - ch^2.$$ **Ans.**

***§ 4·06. Properties of Determinants.**

Prop I. *If the elements of ith row (or ith column) of the determinant* $|a_{ij}|$ *are multiplied by a scalar c then the resulting determinant is* $c\,|a_{ij}|$.

(Gorakhpur 93)

Proof : We can write $|a_{ij}| = \sum_{j=1}^{n} a_{ij} C_{ij}$ ... See § 4·05 above.

In this case, the resulting determinant (when the elements of ith row are multiplied by c)

$$= \sum_{j=1}^{n} ca_{ij} C_{ij} = c \sum_{j=1}^{n} a_{ij} C_{ij} = c\,|a_{ij}|.$$

Similarly we can prove that statement when elements of kth column are multiplied by c.

Prop II. *If* $\mathbf{A} = [a_{ij}]$ *is an* $n \times n$ *matrix then* $|\mathbf{A}'| = |\mathbf{A}|$, *where* $\mathbf{A}'$ *is the transpose of the matrix* $\mathbf{A}$. (See § 2·08 Page 69)

Proof : If $\mathbf{A} = [a_{ij}]$, then $\mathbf{A}' = [a'_{ij}]$, where $a'_{ij} = a_{ji}$.

Now the product of elements of the principal diagonal of $\mathbf{A}'$

$$= a'_{11}\, a'_{22}\, a'_{33} \ldots a'_{nn}.$$

Operating on the row subscripts of the elements of this product by the permutation $p=\begin{pmatrix}1 & 2 & 3\ldots n\\ i_1 & i_2 & i_3\ldots i_n\end{pmatrix}$, where $i_1, i_2, i_3, \ldots$ are 1, 2, 3, ... in some order, we have $\pm a'_{i_11}\, a'_{i_22}\, a'_{i_33}\ldots a'_{i_nn}$ as a term of $|\mathbf{A}'|$ plus or minus sign to be taken according as p is even or odd.

Now as $a'_{ij}=a_{ji}$, so we have

$a'_{i_11}\, a'_{i_22}\, a'_{i_33}\ldots a'_{i_nn}=a_{1i_1}\, a_{2i_2}\, a_{3i_3}\ldots a_{ni_n}$

The term $a_{1i_1}\, a_{2i_2}\ldots a_{ni_n}$ can be obtained from the term $a_{i_1i_1}\, a_{i_2i_2}\, a_{i_3i_3}\ldots a_{i_ni_n}$ by opperating on its row subscripts the permutation

$$p'=\begin{pmatrix}i_1 & i_2 & i_3\ldots i_n\\ 1 & 2 & 3\ldots n\end{pmatrix}=p^{-1}.$$

Hence p' is even or odd according as p is even or odd. Therefore the term $\pm a'_{i_11}\, a'_{i_22}\, a'_{i_33}\ldots a'_{i_nn}$ of $|\mathbf{A}'|$ is also a term of $|\mathbf{A}|$.

We can thus prove that every one of the n ! terms of $|\mathbf{A}'|$ is a term of $|\mathbf{A}|$. Hence the property.

For example, if $|\mathbf{A}|=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$

$$=a_1\begin{vmatrix}b_2 & b_3\\ c_2 & c_3\end{vmatrix}-a_2\begin{vmatrix}b_1 & b_3\\ c_1 & c_3\end{vmatrix}+a_3\begin{vmatrix}b_1 & b_2\\ c_1 & c_2\end{vmatrix}$$

$$=a_1b_2c_3-a_1b_3c_2-a_2b_1c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1.$$

Then $|\mathbf{A}'|=\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}$

$$=a_1\begin{vmatrix}b_2 & c_2\\ b_3 & c_3\end{vmatrix}-b_1\begin{vmatrix}a_2 & c_2\\ a_3 & c_3\end{vmatrix}+c_1\begin{vmatrix}a_2 & b_2\\ a_3 & b_3\end{vmatrix}$$

$$=a_1b_2c_3-a_1b_3c_2-a_2b_1c_3+a_3b_1c_2+a_2b_3c_1-a_3b_2c_1$$

$\therefore\ |\mathbf{A}'|=|\mathbf{A}|.$ *(Gorakhpur 95)*

Prop. III. *If* **B** *is obtained from* **A** *by interchanging two rows (or columns) then* $|\mathbf{B}|=-|\mathbf{A}|$.

Proof : Let us suppose that sth and tth columns of the determinant **A** are interchanged where $s<t$.

$$\text{Let } \mathbf{A}=\begin{bmatrix}a_{11} & a_{12} & \ldots & a_{1s} & \ldots & a_{1t} & \ldots & a_{1n}\\ a_{21} & a_{22} & \ldots & a_{2s} & \ldots & a_{2t} & \ldots & a_{2n}\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ a_{i1} & a_{i1} & \ldots & a_{is} & \ldots & a_{it} & \ldots & a_{in}\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ a_{n1} & a_{n2} & \ldots & a_{ns} & \ldots & a_{nt} & \ldots & a_{nn}\end{bmatrix}$$

Then the trace of **A** *i.e.* the product of the elements of the principal diagonal of A = $a_{11}\, a_{22}\ldots a_{ss}\ldots a_{tt}\, ..\, a_{nn}$.

If the sth and tth columns are interchanged the product of the elements of the principal diagonal of **B**

$$= a_{11}\, a_{22} \ldots a_{ss} \ldots a_{tt} \ldots a_{nn} \qquad \textbf{(Note)}$$

In order to have a term of | **A** |, let us operate on the row subscripts of the trace of **A** by the permutation

$$p = \begin{pmatrix} 1 & 2 & \ldots & s & \ldots & t & \ldots & n \\ i_1 & i_2 & \ldots & i_s & \ldots & i_t & \ldots & i_n \end{pmatrix}$$

Then we have $a_{i_1 1}\, a_{i_2 2} \ldots a_{i_s s} \ldots a_{i_t t} \ldots a_{i_n n}$.

This terms can also be obtained from the trace of **B** by operating on its row subscripts by the permutation

$$p' = \begin{pmatrix} 1 & 2 & \ldots & s & \ldots & t & \ldots & n \\ i_1 & i_2 & \ldots & i_t & \ldots & i_s & \ldots & i_n \end{pmatrix}$$

Here we observe that $p' = p\,(i_s\, i_t)$, since $(i_s\, i_t)$ is a transposition. Therefore p' is odd or even according as p is even or odd. Hence the term $\pm\, a_{i_1 1}\, a_{i_2 2} \ldots a_{i_s s} \ldots a_{i_t t} \ldots a_{i_n n}$ is also a term of | **B** | but with its sign changed. Thus every one of the n ! terms of | **A** | is a term of | **B** | but with sign changed.

Hence $|\mathbf{B}| = -|\mathbf{A}|$.

Example : Let $|\mathbf{A}| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Let the determinant | **B** | be formed by interchanging second and third columns.

Then $|\mathbf{B}| = \begin{vmatrix} a_1 & a_3 & a_2 \\ b_1 & b_3 & b_2 \\ c_1 & c_3 & c_2 \end{vmatrix}$

Exapanding | **B** | by the elements of its first row, we get

$$|\mathbf{B}| = a_1 \begin{vmatrix} b_3 & b_2 \\ c_3 & c_2 \end{vmatrix} - a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} + a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}$$

$$= a_1\,(b_3c_2 - b_2c_3) - a_3\,(b_1c_2 - b_2c_1) + a_2\,(b_1c_3 - b_3c_1)$$

$$= -\,[a_1\,(b_2c_3 - b_3c_2) - a_2\,(b_1c_3 - b_3c_1) + a_3\,(b_1c_2 - b_2c_1)]$$

$$= -\left[a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\right]$$

$$= -\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -|\mathbf{A}|.$$

Prop. IV. *If two rows or two columns of a determinant* | **A** | *are identical then* $|\mathbf{A}| = 0$.

Proof : In Prop. III above we have proved that if any two rows or columns of a determinant are interchanged then the value of the determinant changes in sign only.

Thus if the two *identical* columns (or rows) of a determinant are interchanged, then the determinant does not change but its sign only changes.

Hence $|A| = -|A|$ *i.e.* $|A| + |A| = 0$ *i.e.* $|A| = 0$.

Example : Evaluate $\begin{vmatrix} a_1 & a_2 & a_1 \\ b_1 & b_2 & b_1 \\ c_1 & c_2 & c_1 \end{vmatrix}$

Expanding the determinant with respect to the first row we have the given determinant

$$= a_1 \begin{vmatrix} b_2 & b_1 \\ c_2 & c_1 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_1 \\ c_1 & c_1 \end{vmatrix} + a_1 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1 (b_2c_1 - b_1c_2) - a_2 (b_1c_1 - b_1c_1) + a_1 (b_1c_2 - b_2c_1)$$

$$= a_1b_2c_1 - a_1b_1c_2 - a_2 (0) + a_1b_1c_2 - a_1b_2c_1 = 0.$$

***§ 4·07. Minor of an element.**

Definition : If M_{ij} be the $(n-1) \times (n-1)$ sub-matrix of the matrix $A = [a_{ij}]$ obtained by removing the ith row and jth column, then the determinant $|M_{ij}|$ is defined as the minor of the element a_{ij} in the determinant $|a_{ij}|$ of order n.

§ 4·08. Theorem. *The cofactor C_{ij} of the element a_{ij} in the determinant $|a_{ij}|$ is given by $C_{ij} = (-1)^{i+j} |M_{ij}|$.*

Proof : Let us first of all prove the case $C_{11} = (-1)^2 |M_{11}|$. *i.e.* $C_{11} = |M_{11}|$.

The tems in C_{11} are composed of elements taken from the $(n-1) \times (n-1)$ sub-matrix M_{11} of A.

The general term of $a_{11} C_{11} = \pm a_{11} a_{i_2 2} a_{i_3 3} \ldots a_{i_n n}$, where $i_2, i_3, \ldots, i_n$ are 2, 3, ..., n in some order.

This term can also be obtained from the trace of matrix A *i.e.* the product of the elements of the diagonal of the matrix **A** *i.e.* $a_{11} a_{22} a_{33} \ldots a_{nn}$, by operating on its row subscripts by the permutation $p = \begin{pmatrix} 1 & 2 & 3 \ldots n \\ i_1 & i_2 & i_3 \ldots i_n \end{pmatrix}$, where $i_2, i_3, \ldots, i_n$ are defined as above.

Thus the permutation p may be regarded as a permutation on the symbols 2, 3, ..., n. Hence all the terms of $a_{11} C_{11}$ can be obtained by running p through the $(n-1)$! permutations of the symbols 2, 3, ..., n keeping 1 fixed.

Thus the terms of C_{11} can be obtained by operating on the row subscripts of the elements of the product $a_{22} a_{33} \ldots a_{nn}$, which is the product of the elements of the diagonal of M_{11} *i.e.* the trace of M_{11}.

Hence $C_{11} = |\mathbf{M}_{11}|$.

Now let us prove $C_{ij} = (-1)^{i+j} |\mathbf{M}_{ij}|$.

Move the jth column of the matrix **A** to the first column by performing $(j-1)$ successive interchanges of adjacent columns and move the ith row of the matrix **A** to the first row by performing $(i-1)$ successive interchanges of adjacent rows. Then the element a_{ij} is in the first row and first column of the resulting matrix **B**, say. The sub-matrix of **B** obtained by removing the first row and first column is the sub-matrix $\mathbf{M}_{ij}$ of the matrix **A**. Hence $a_{ij} |\mathbf{M}_{ij}|$ is the term of $|\mathbf{B}|$ containing a_{ij}.

Also we know that if two rows or two columns of determinant $|\mathbf{A}|$ are interchanged, the new determinant $= -|\mathbf{A}|$.

$\therefore$ We have $|\mathbf{B}| = (-1)^{i-1+j-1} |\mathbf{A}|$ **(Note)**

$$= (-1)^{i+j} (-1)^{-2} |\mathbf{A}|$$

$$= (-1)^{i+j} |\mathbf{A}|, \quad \because \quad (-1)^{-2} = 1$$

or $\quad |\mathbf{A}| = (-1)^{i+j} |\mathbf{B}|$ **(Note)**

Equating the coefficients of a_{ij} from both sides, we have

$$C_{ij} = (-1)^{i+j} |\mathbf{M}_{ij}|.$$

*§ 4.09. **Theorem.** If C_{ij} is the cofactor of a_{ij} in the determinant $|\mathbf{A}| = |a_{ij}|$ of order n, then (i) the sum of the products of the elements of the ith row with the cofactors of the corresponding elements of the kth row is zero provided $i \neq k$.

(ii) Also the sum of the products of the elements of the jth column with the cofactors of the corresponding elements of the kth column is zero provided $j \neq k$.

i.e. (i) $\sum_{j=1}^{n} a_{ij} C_{kj} = 0$, if $i \neq k$.

and (ii) $\sum_{j=1}^{n} a_{ij} C_{ik} = 0$, if $j \neq k$.

Proof : (i) The given determinant $|\mathbf{A}| = \sum_{j=1}^{n} a_{kj} C_{kj}$.

Now eplace the kth row by ith row, then we have the new determinant

$$= \sum_{j=1}^{n} a_{ij} C_{kj}.$$

But the kth and ith rows of the new determinant are identical, hence its value is zero.

$$\therefore \quad \sum_{i=1}^{n} a_{ij} C_{kj} = 0.$$

(ii) The given determinant $|A| = \sum_{i=1}^{n} a_{ik} C_{ik}$.

Now replace the kth column by jth column, then we have the new determinant $= \sum_{i=1}^{n} a_{ik} C_{ij}$.

But the kth and jth columns of the new determinant are identical, hence its value is zero.

$$\therefore \sum_{i=1}^{n} a_{ij} C_{ik} = 0.$$

****Example. In the determinant $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ prove that**

$a_1A_2 + b_1B_2 + c_1C_2 = 0$, $b_1C_1 + b_2C_2 + b_3C_3 = 0$ and $c_1B_1 + c_2B_2 + c_3B_3 = 0$, where capital letters dentoe the cofactors of the corresponding small letters.

Also prove that

$a_1A_1 + b_1B_1 + c_1C_1 = |A| = a_2A_2 + b_2B_2 + c_2C_2 = a_3A_3 + b_3B_3 + c_3C_3$.

Sol. In the det. $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ we have

$$A_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}; A_2 = -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}; B_1 = -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}; B_2 = \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix};$$

$$B_3 = -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}; C_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}; C_2 = -\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}; A_3 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$C_3 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$\therefore a_1A_2 + b_1B_2 + c_1C_2$

$$= -a_1 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_1 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_1 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

$= -a_1 (b_1c_3 - b_3c_1) + b_1 (a_1c_3 - a_3c_1) - c_1 (a_1b_3 - a_3b_1)$

$= 0$, on simplifying.

$$b_1C_1 + b_2C_2 + b_3C_3 = b_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} - b_2 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} + b_3 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$= b_1 (a_2b_3 - a_3b_2) - b_2 (a_1b_3 - a_3b_1) + b_3 (a_1b_2 - a_2b_1)$

$= 0$, on simplifying.

In a similar way the remaining part can also be proved.

Also $|\mathbf{A}| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}, \quad ...(i)$$

expanding with respect to first row.

Again $a_1A_1 + b_1B_1 + c_1C_1$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1 \left\{ - \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} \right\} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$= |\mathbf{A}|$, from (i)

Similarly $a_2A_2 + b_2B_2 + c_2C_2$

$$= a_2 \left\{ - \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} \right\} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} + c_2 \left\{ - \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \right\}$$

$$= - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \quad ...(ii)$$

Also $|\mathbf{A}| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$= - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}, \quad ...(iii)$$

expanding with respect to second row.

$\therefore$ From (ii) and (iii), we get $a_2A_2 + b_2B_2 + c_2C_2 = |\mathbf{A}|$.

In a similar way we can prove that $a_3A_3 + b_3B_3 + c_3C_3 = |\mathbf{A}|$, by expanding $|\mathbf{A}|$ with respect to third row.

§ 4·10. An Important Property of the Determinant.

(a) *If A_i, the ith row of a determinant $|\mathbf{A}| = |a_{ij}|$ of order n, be replaced by $A_i + cA_k$, where c is a scalar and A_k denotes the kth row of the determinant $|\mathbf{A}|$, then the value of the determinant remains unaltered.* (*Gorakhpur 94*)

Proof : The determinant $|\mathbf{A}| = \sum_{j=1}^{n} a_{ij} C_{ij}$.

Replacing A_i by $A_i + cA_k$ we get the new determinant $|\mathbf{B}|$, say.

Then $|\mathbf{B}| = \sum_{j=1}^{n} (a_{ij} + ca_{kj}) C_{ij}$

$$= \sum_{j=1}^{n} a_{ij} C_{ij} + c \sum_{j=1}^{n} a_{kj} C_{ij}$$

$= |A| + c.0$ ... See § 4·09 Pages 129-131

i.e. $|B| = |A|$.

(b) *If C_i, the ith column of determinant $|A| = |a_{ij}|$ of order n be replaced by $C_i + \lambda C_k$, where λ is a scalar and C_k dentoes the kth column of $|A|$, then the value of the determinant remains unaltered.*

Proof is similar to part (a) above.

Solved Examples on the Evaluation of Determinants :

In the following examples $R_1, R_2, R_3, \dots$ stand for first, second, third, ... rows and $C_1, C_2, C_3, \dots$ stand for first, second, third... columns.

Ex. 1. Without expanding the determinants, prove that

$$\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix}$$

Sol. $\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = -\begin{vmatrix} x & y & z \\ a & b & c \\ p & q & r \end{vmatrix}$, interchanging R_1 and R_2

$= \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix}$, interchanging R_2 and R_3

Hence proved.

Again $\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} a & x & p \\ b & y & q \\ c & z & r \end{vmatrix}$, interchanging rows and columns

$= -\begin{vmatrix} x & a & p \\ y & b & q \\ z & c & r \end{vmatrix}$, interchanging C_1 and C_2

$= \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix}$, interchanging R_1 and R_2

Hence proved.

Ex. 2 (a). Evaluate $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

Sol. The given determinant

$= \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix}$, replacing C_3 by $C_3 + C_2$

$= (a+b+c) \begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix}$, taking out $(a+b+c)$ common from C_3

$= 0$, since two columns are identical. **Ans.**

Ex. 2. (b). Evaluate $\begin{vmatrix} 13 & 16 & 19 \\ 14 & 17 & 20 \\ 15 & 18 & 21 \end{vmatrix}$

Sol. The given determinant

$$= \begin{vmatrix} 13 & 16 & 3 \\ 14 & 17 & 3 \\ 15 & 18 & 3 \end{vmatrix}, \text{ replacing } C_3 \text{ by } C_3 - C_2$$

$$= \begin{vmatrix} 13 & 3 & 3 \\ 14 & 3 & 3 \\ 15 & 3 & 3 \end{vmatrix}, \text{ replacing } C_2 \text{ by } C_2 - C_1$$

= 0, since two columns are identical. **Ans.**

Ex. 3. Evaluate $\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$ *(Kanpur 90)*

Sol. Given determinant

$$= \begin{vmatrix} 25 & 21 & 219 \\ 15 & 27 & 198 \\ 21 & 17 & 181 \end{vmatrix}, \text{ replacing } C_1, C_2 \text{ by } C_1 - C_2, C_2 - C_3 \text{ respectively}$$

$$= \begin{vmatrix} 4 & 21 & 9 \\ -12 & 27 & -72 \\ 4 & 17 & 11 \end{vmatrix}, \text{ replacing } C_1, C_3 \text{ by } C_1 - C_2, C_3 - 10C_2 \text{ respectively}$$

$$= \begin{vmatrix} 0 & 4 & -2 \\ 0 & 78 & -39 \\ 4 & 17 & 11 \end{vmatrix}, \text{ replacing } R_1, R_2 \text{ by } R_1 - R_3, R_2 + 3R_3 \text{ respectively}$$

$$= 4 \begin{vmatrix} 4 & -2 \\ 78 & -39 \end{vmatrix}, \text{ expanding with respect to } C_1$$

$$= 4 \begin{vmatrix} 0 & -2 \\ 0 & -39 \end{vmatrix}, \text{ replacing } C_1 \text{ by } C_1 + 2C_2.$$

$= 4 \times 0 = 0$ **Ans.**

Ex. 4. Evaluate $\begin{vmatrix} \mathbf{a} & -\mathbf{a} & -\mathbf{a} & -\mathbf{a} \\ \mathbf{b} & -\mathbf{b} & -\mathbf{b} & -\mathbf{b} \\ \mathbf{c} & -\mathbf{c} & -\mathbf{c} & -\mathbf{c} \\ \mathbf{d} & -\mathbf{d} & -\mathbf{d} & -\mathbf{d} \end{vmatrix}$

Sol. Since three columns of the given determinant are identical, so the value of the determinant is zero. **Ans.**

Ex. 5. (a) Evaluate $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}$.

Sol. The given determinant

$$= \begin{vmatrix} 10 & 2 & 3 & 4 \\ 10 & 3 & 4 & 1 \\ 10 & 4 & 1 & 2 \\ 10 & 1 & 2 & 3 \end{vmatrix}, \text{ replacing } C_1 \text{ by } C_1 + C_2 + C_3 + C_4$$

$$= 10\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 1 \\ 1 & 4 & 1 & 2 \\ 1 & 1 & 2 & 3 \end{vmatrix}, \text{ taking out 10 common from } C_1$$

$$= 10\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & -3 \\ 0 & 2 & -2 & -2 \\ 0 & -1 & -1 & -1 \end{vmatrix}, \text{ replacing } R_2, R_3, R_4 \text{ by } R_2 - R_1, R_3 - R_1 \text{ and } R_4 - R_1 \text{ respectively}$$

$$= 10\begin{vmatrix} 1 & 1 & -3 \\ 2 & -2 & -2 \\ -1 & -1 & -1 \end{vmatrix}, \text{ expanding with respect to } C_1$$

$$= -20\begin{vmatrix} 1 & 1 & -3 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{vmatrix}, \text{ taking out 2 common from } R_2 \text{ and } -1 \text{ from } R_3$$

$$= -20\begin{vmatrix} 1 & 1 & -3 \\ 0 & -2 & 2 \\ 0 & 0 & 4 \end{vmatrix}, \text{ replacing } R_2 \text{ and } R_3 \text{ by } R_2 - R_1 \text{ and } R_3 - R_1 \text{ respectively}$$

$$= -20\begin{vmatrix} -2 & 2 \\ 0 & 4 \end{vmatrix}, \text{ expanding with respect to } C_1$$

$= -20\,[(-2)\cdot 4 - 0\cdot 2] = -20\,[-8] = 160.$ **Ans.**

Ex. 5 (b) Evaluate $\begin{vmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{vmatrix}$

Sol. The given determinant

$$= \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & -4 & 1 & 1 & 1 \\ 0 & 1 & -4 & 1 & 1 \\ 0 & 1 & 1 & -4 & 1 \\ 0 & 1 & 1 & 1 & -4 \end{vmatrix}, \text{ replacing } C_1 \text{ by } C_1 + C_2 + C_3 + C_4 + C_5$$

(Note)

$= 0$, expanding with respect to elements of first column. **Ans.**

***Ex. 6. Evaluate** $\begin{vmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$ *(Meerut 96)*

Sol. The given determinant

$$= -\begin{vmatrix} \cos\theta & 0 & -\sin\theta \\ \sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \end{vmatrix}, \text{ interchanging } C_2 \text{ and } C_3$$

$= \begin{vmatrix} 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \\ 1 & 0 & 0 \end{vmatrix}$, interchanging C_1 and C_2

$= \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$, expanding with respect to C_1

$= \cos\theta . \cos\theta - (-\sin\theta)\sin\theta = \cos^2\theta + \sin^2\theta =$ **Ans.**

Ex. 7. Evaluate $\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1+x} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1+y} \end{vmatrix}$

Sol. The given determinant

$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & x & 0 \\ 0 & 0 & y \end{vmatrix}$, replacing R_2 by $R_2 - R_1$ and R_3 by $R_3 - R_1$

$= 1 \times \begin{vmatrix} x & 0 \\ 0 & y \end{vmatrix}$, expanding with respect to the first column,

$= xy.$

****Ex. 8. Evaluate** $\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$ *(Meerut 97; Purvanchal 97)*

Sol. The given determinant

$= \begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 16 & 25 & 36 & 49 \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 4 & -7 & -20 & -39 \\ 9 & -20 & -56 & -108 \\ 16 & -39 & -108 & -207 \end{vmatrix}$, replacing C_2, C_3, C_4 by $C_2 - 4C_1$, $C_3 - 9C_1$ and $C_4 - 16C_1$ respectively.

$= (-1)^3 \begin{vmatrix} 7 & 20 & 39 \\ 20 & 56 & 108 \\ 39 & 108 & 207 \end{vmatrix}$, expanding with respect to first row

$= - \begin{vmatrix} 7 & -1 & -1 \\ 20 & -4 & -4 \\ 39 & -9 & -9 \end{vmatrix}$, replacing C_2 and C_3 by $C_2 - 3C_1$ and $C_3 - 2C_2$ repsectively.

$= 0$, as two columns are identical. **Ans.**

Ex. 9. Evaluate $\begin{vmatrix} \mathbf{5} & \mathbf{7} & \mathbf{10} & \mathbf{14} \\ \mathbf{2} & \mathbf{3} & \mathbf{7} & \mathbf{6} \\ \mathbf{3} & \mathbf{3} & \mathbf{6} & \mathbf{9} \\ \mathbf{5} & \mathbf{6} & \mathbf{11} & \mathbf{20} \end{vmatrix}$

Sol. The given determinant

$$= \begin{vmatrix} 0 & 1 & -1 & -6 \\ 2 & 3 & 7 & 6 \\ 1 & 0 & -1 & 3 \\ 5 & 6 & 11 & 20 \end{vmatrix}, \text{ replacing } R_1 \text{ and } R_3 \text{ by } R_1 - R_4 \text{ and } R_3 - R_2 \text{ repsectively.}$$

$$= \begin{vmatrix} 0 & 1 & 0 & 0 \\ 2 & 3 & 10 & 24 \\ 1 & 0 & -1 & 3 \\ 5 & 6 & 17 & 56 \end{vmatrix}, \text{ replacing } C_3 \text{ and } C_4 \text{ by } C_2 + C_3 \text{ and } C_4 + 6C_2 \text{ respectively.}$$

Now expand with respect to 1st row and proceed as in Ex. 8 Page 135

Ans. 96

Ex. 10. Show that $\begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \beta+\gamma & \gamma+\delta & \delta+\alpha & \alpha+\beta \\ \delta & \alpha & \beta & \gamma \end{vmatrix} = 0$

Sol. The given determinant

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \beta+\gamma & \gamma+\delta & \delta+\alpha & \alpha+\beta \\ \alpha+\beta+\gamma+\delta & \alpha+\beta+\gamma+\delta & \alpha+\beta+\gamma+\delta & \alpha+\beta+\gamma+\delta \end{vmatrix},$$

replacing R_4 by $R_2 + R_3 + R_4$

$$= (\alpha+\beta+\gamma+\delta) \begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \beta+\gamma & \gamma+\delta & \delta+\alpha & \alpha+\beta \\ 1 & 1 & 1 & 1 \end{vmatrix},$$

taking out $(\alpha+\beta+\gamma+\delta)$ common from R_4.

$= 0$, since two rows are identical.

Hence proved.

Ex. 11. Evaluate $\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 69 \end{vmatrix}$

Sol. The given determinant

$$= \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 & 4 \\ 1 & 2 & 5 & 9 & 14 \\ 1 & 3 & 9 & 19 & 34 \\ 1 & 4 & 14 & 34 & 68 \end{vmatrix}, \text{ replacing } C_2, C_3, C_4 \text{ and } C_5 \text{ by } C_2 - C_1, C_3 - C_1, C_4 - C_1 \text{ and } C_5 - C_1 \text{ respectively.}$$

$$= \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 9 & 14 \\ 3 & 9 & 19 & 34 \\ 4 & 14 & 34 & 68 \end{vmatrix}, \text{ expanding with respect to first row.}$$

$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 3 & 6 \\ 3 & 3 & 10 & 22 \\ 4 & 6 & 22 & 52 \end{vmatrix}$, replacing C_2, C_3 and C_4 by $C_2 - 2C_1$, $C_3 - 3C_1$ and $C_4 - 4C_1$ repsectively.

$= \begin{vmatrix} 1 & 3 & 6 \\ 3 & 10 & 22 \\ 6 & 22 & 52 \end{vmatrix}$, expanding with respect to first row

$= \begin{vmatrix} 1 & 0 & 0 \\ 3 & 1 & 4 \\ 6 & 4 & 16 \end{vmatrix}$, replacing C_2 and C_3 by $C_2 - 3C_1$ and $C_3 - 6C_1$ respectively.

$= \begin{vmatrix} 1 & 4 \\ 4 & 16 \end{vmatrix}$, expanding with respect to first row.

$= 1.(16) - 4.4 = 0.$ **Ans.**

Ex. 12. Show that $\begin{vmatrix} \mathbf{a+b+2c} & \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{b+c+2a} & \mathbf{b} \\ \mathbf{c} & \mathbf{a} & \mathbf{c+a+2b} \end{vmatrix} = \mathbf{2\,(a+b+c)^3}$

(Garhwal 90; Meerut 92)

Sol. The given determinant

$= \begin{vmatrix} 2a+2b+2c & a & b \\ 2a+2b+2c & b+c+2a & b \\ 2a+2b+2c & a & c+a+2b \end{vmatrix}$, replacing C_1 by $C_1 + C_2 + C_3$

$= (2a+2b+2c) \begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$, taking out $2a+2b+2c$ common.

$= 2\,(a+b+c) \begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & c+a+b \end{vmatrix}$, replacing R_2 and R_3 by $R_2 - R_1$ and $R_3 - R_1$ respectively

$= 2\,(a+b+c) \begin{vmatrix} b+c+a & 0 \\ 0 & c+a+b \end{vmatrix}$, expanding with respect to 1st column.

$= 2\,(a+b+c)\,[(b+c+a)\,(c+a+b)] = 2\,(a+b+c)^3.$

***Ex. 13. Prove that** $\begin{vmatrix} \mathbf{b+c} & \mathbf{c+a} & \mathbf{a+b} \\ \mathbf{q+r} & \mathbf{r+p} & \mathbf{p+q} \\ \mathbf{y+z} & \mathbf{z+x} & \mathbf{x+y} \end{vmatrix} = \mathbf{2} \begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{p} & \mathbf{q} & \mathbf{r} \\ \mathbf{x} & \mathbf{y} & \mathbf{z} \end{vmatrix}$ *(Meerut 90)*

Sol. L.H.S. $= \begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix}$

$= \begin{vmatrix} b & c+a & a+b \\ q & r+p & p+q \\ y & z+x & x+y \end{vmatrix} + \begin{vmatrix} c & c+a & a+b \\ r & r+p & p+q \\ z & z+x & x+y \end{vmatrix}$ **(Note)**

$$= \begin{vmatrix} b & c+a & a \\ q & r+p & p \\ y & z+x & x \end{vmatrix} + \begin{vmatrix} b & c+a & b \\ q & r+p & q \\ y & z+x & y \end{vmatrix} + \begin{vmatrix} c & c & a+b \\ r & r & p+q \\ z & z & x+y \end{vmatrix} + \begin{vmatrix} c & a & a+b \\ r & p & p+q \\ z & x & x+y \end{vmatrix}$$

$$= \begin{vmatrix} b & c & a \\ q & r & p \\ y & z & x \end{vmatrix} + \begin{vmatrix} b & a & a \\ q & p & p \\ y & x & x \end{vmatrix} + \begin{vmatrix} c & a & a \\ r & p & p \\ z & x & x \end{vmatrix} + \begin{vmatrix} c & a & b \\ r & p & q \\ z & x & y \end{vmatrix},$$

second and third determinants vanish as two columns in each are identical.

$$= \begin{vmatrix} b & c & a \\ q & r & p \\ y & z & x \end{vmatrix} + \begin{vmatrix} c & a & b \\ r & p & q \\ z & x & y \end{vmatrix}$$, second and third determinants vanish as two columns in each are identical.

$$= -\begin{vmatrix} b & a & c \\ q & p & r \\ y & x & z \end{vmatrix} - \begin{vmatrix} a & c & b \\ p & r & q \\ x & z & y \end{vmatrix}$$, interchanging C_2 and C_3 in first and C_1 and C_2 in second determinant

$$= \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} + \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$, interchanging C_1 and C_2 in first and C_2 and C_3 in second determinant

$$= 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = \text{R.H.S.}$$

Hence proved.

Ex. 14. Evaluate $\begin{vmatrix} \mathbf{y+z} & \mathbf{x} & \mathbf{y} \\ \mathbf{z+x} & \mathbf{z} & \mathbf{x} \\ \mathbf{x+y} & \mathbf{y} & \mathbf{z} \end{vmatrix}$

Hint : Add all the rows to first, take out $(x+y+z)$ common from first row and then subtract sum of second and third columns from first. Then expand.

Ans. $(x+y+z)(x-z)^2$

Ex. 15. Evaluate $\begin{vmatrix} \mathbf{b+c} & \mathbf{a+b} & \mathbf{a} \\ \mathbf{c+a} & \mathbf{b+c} & \mathbf{b} \\ \mathbf{a+b} & \mathbf{c+a} & \mathbf{c} \end{vmatrix}$

Sol. The given determinant

$$= \begin{vmatrix} 2a+2b+2c & 2a+2b+2c & a+b+c \\ c+a & b+c & b \\ a+b & c+a & c \end{vmatrix}$$, replacing R_1 by $R_1+R_2+R_3$

$$= (a+b+c)\begin{vmatrix} 2 & 2 & 1 \\ c+a & b+c & b \\ a+b & c+a & c \end{vmatrix}$$, taking out $(a+b+c)$ common

$$= (a+b+c)\begin{vmatrix} 0 & 2 & 1 \\ a-b & b+c & b \\ b-c & c+a & c \end{vmatrix}$$, replacing C_1 by $C_1 - C_2$

$$= (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ a-b & c-b & b \\ b-b & a-c & c \end{vmatrix}$$, replacing C_2 by $C_2 - 2C_3$

$= (a+b+c)\,[(a-b)(a-c)-(b-c)(c-b)]$

$= (a+b+c)\,[a^2-ac-ba+bc+b^2+c^2-2bc]$

$= (a+b+c)(a^2+b^2+c^2-ab-bc-ca) = a^3+b^3+c^3-3abc.$ **Ans.**

***Ex. 16. Evaluate** $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

(Kanpur 96; Kumaun 93; Meerut 95)

Sol. The given determinant

$= \begin{vmatrix} a+b+c & b+c+a & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$, replacing R_1 by $R_1+R_2+R_3$

$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$, taking out $(a+b+c)$ common

$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -a-b-c & 0 \\ 2c & 0 & -a-b-c \end{vmatrix}$, replacing C_2 and C_3 by C_2-C_1 and C_3-C_1 respectively.

$= (a+b+c)\begin{vmatrix} -a-b-c & 0 \\ 0 & -a-b-c \end{vmatrix}$, expanding with respect to first row.

$= (a+b+c)(-a-b-c)(-a-b-c) = (a+b+c)^3.$ **Ans.**

Ex. 17. Evaluate $\begin{vmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix}$

Sol. The given determinant

$= \begin{vmatrix} a+3 & 1 & 1 & 1 \\ a+3 & a & 1 & 1 \\ a+3 & 1 & a & 1 \\ a+3 & 1 & 1 & a \end{vmatrix}$, replacing C_1 by $C_1+C_2+C_3+C_4$

Now proceed as in Ex. 5 (a) Page 133. **Ans.** $(a-1)^3(a+3)$

***Ex. 18. Evaluate** $\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+a & 1 & 1 \\ 1 & 1 & 1+a & 1 \\ 1 & 1 & 1 & 1+a \end{vmatrix}$

Sol. The given determinant

$= \begin{vmatrix} 4+a & 1 & 1 & 1 \\ 4+a & 1+a & 1 & 1 \\ 4+a & 1 & 1+a & 1 \\ 4+a & 1 & 1 & 1+a \end{vmatrix}$, replacing C_1 by $C_1+C_2+C_3+C_4$

$$= (4+a)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+a & 1 & 1 \\ 1 & 1 & 1+a & 1 \\ 1 & 1 & 1 & 1+a \end{vmatrix}$$, taking out $(4+a)$ common from first column.

$$= (4+a)\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & a & 0 & 0 \\ 1 & 0 & a & 0 \\ 1 & 0 & 0 & a \end{vmatrix}$$, replacing C_2, C_3 and C_4 by $C_2 - C_1$, $C_3 - C_1$ and $C_4 - C_1$

$$= (4+a)\begin{vmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix}$$, expanding with respect to R_1

$$= (4+a)\,a\begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix}$$, expanding with respect to R_1

$= (4+a)\,a\,(a^2) = (4+a)\,a^3$. **Ans.**

Ex. 19. Prove that

$$\begin{vmatrix} 1+a_1 & a_2 & a_3 & a_4 \\ a_1 & 1+a_2 & a_3 & a_4 \\ a_1 & a_2 & 1+a_3 & a_4 \\ a_1 & a_2 & a_3 & 1+a_4 \end{vmatrix} = 1 + a_1 + a_2 + a_3 + a_4$$

Sol. The given determinant

$$= \begin{vmatrix} 1+a_1+a_2+a_3+a_4 & a_2 & a_3 & a_4 \\ 1+a_1+a_2+a_3+a_4 & 1+a_2 & a_3 & a_4 \\ 1+a_1+a_2+a_3+a_4 & a_2 & 1+a_3 & a_4 \\ 1+a_1+a_2+a_3+a_4 & a_2 & a_3 & 1+a_4 \end{vmatrix}$$, replacing C_1 by $C_1 + C_2 + C_3 + C_4$

Now proceed further as in Ex. 18 above.

***Ex. 20. Evaluate** $\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix}$

Sol. The given determinant

$$= \begin{vmatrix} x+3a & a & a & a \\ x+3a & x & a & a \\ x+3a & a & x & a \\ x+3a & a & a & x \end{vmatrix}$$, replacing C_1 by $C_1 + C_2 + C_3 + C_4$

$$= (x+3a)\begin{vmatrix} 1 & a & a & a \\ 1 & x & a & a \\ 1 & a & x & a \\ 1 & a & a & x \end{vmatrix}$$, taking out $(x+3a)$ common from C_1

$$=(x+3a)\begin{vmatrix} 1 & a & a & a \\ 0 & x-a & 0 & 0 \\ 0 & 0 & x-a & 0 \\ 0 & 0 & 0 & x-a \end{vmatrix}, \text{ replacing } R_2, R_3 \text{ and } R_4 \text{ by } R_2-R_1, R_3-R_1 \text{ and } R_4-R_1 \text{ respectively.}$$

$$=(x+3a)\begin{vmatrix} x-a & 0 & 0 \\ 0 & x-a & 0 \\ 0 & 0 & x-a \end{vmatrix}, \text{ expanding with respect to } C_1$$

$$=(x+3a)(x-a)\begin{vmatrix} x-a & 0 \\ 0 & x-a \end{vmatrix}, \text{ expanding with respect to } C_1$$

$$=(x+3a)(x-a)(x-a)(x-a)=(x-3a)(x-a)^3.$$ **Ans.**

Ex. 21. Evaluate $\begin{vmatrix} \mathbf{x+a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\ \mathbf{a} & \mathbf{x+b} & \mathbf{c} & \mathbf{d} \\ \mathbf{a} & \mathbf{b} & \mathbf{x+c} & \mathbf{d} \\ \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{x+d} \end{vmatrix}$ *(Meerut 96)*

Sol. The given determinant

$$=\begin{vmatrix} x+a+b+c+d & b & c & d \\ x+a+b+c+d & x+b & c & d \\ x+a+b+c+d & b & x+c & d \\ x+a+b+c+d & b & c & x+d \end{vmatrix}, \text{ replacing } C_1 \text{ by } C_1+C_2+C_3+C_4$$

$$=(x+a+b+c+d)\begin{vmatrix} 1 & b & c & d \\ 1 & x+b & c & d \\ 1 & b & x+c & d \\ 1 & b & c & x+d \end{vmatrix}, \text{ taking out } (x+a+b+c+d) \text{ common from } C_1$$

$$=(x+a+b+c+d)\begin{vmatrix} 1 & b & c & d \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{vmatrix}, \text{ replacing } R_2, R_3, R_4 \text{ by } R_2-R_1, R_3-R_1 \text{ and } R_4-R_1 \text{ respectively.}$$

$$=(x+a+b+c+d)\begin{vmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix}, \text{ expanding with respect to } C_1$$

$$=x(x+a+b+c+d)\begin{vmatrix} x & 0 \\ 0 & x \end{vmatrix}, \text{ expanding with respect to } C_1$$

$$=x(x+a+b+c+d)x^2=x^3(x+a+b+c+d).$$ **Ans.**

Ex. 22. Evaluate $\begin{vmatrix} \mathbf{b+c} & \mathbf{a} & \mathbf{a} \\ \mathbf{b} & \mathbf{c+a} & \mathbf{b} \\ \mathbf{c} & \mathbf{c} & \mathbf{a+b} \end{vmatrix}$ *(Garhwal 90)*

Sol. The given determinant

$$=\begin{vmatrix} b+c & 0 & a \\ b & c+a-b & b \\ c & c-a-b & a+b \end{vmatrix}, \text{ replacing } C_2 \text{ by } C_2-C_3$$

$$= (b+c)\begin{vmatrix} c+a-b & b \\ c-a-b & a+b \end{vmatrix} + a\begin{vmatrix} b & c+a-b \\ c & c-a-b \end{vmatrix},$$

expanding with respect to R_1

$$= (b+c)\begin{vmatrix} c+a-b & b \\ -2a & a \end{vmatrix} + a\begin{vmatrix} b & c+a-b \\ c-b & -2a \end{vmatrix},$$

replacing R_2 by $R_2 - R_1$ in each determinant.

$= (b+c)\,[a\,(c+a-b) - b\,(-2a)] + a\,[b\,(-2a) - (c-b)\,(c+a-b)]$

$= (b+c)\,(ac + a^2 + ab) + a\,(-2ab - c^2 - ca + bc + bc + ab - b^2)$

$= abc + a^2b + ab^2 + ac^2 + a^2c + abc - 2a^2b - ac^2 - ca^2 + 2abc + a^2b - ab^2$

$= 4abc.$ **Ans.**

Ex. 23. Evaluate $\begin{vmatrix} \mathbf{b+c} & \mathbf{a-c} & \mathbf{a-b} \\ \mathbf{b-c} & \mathbf{c+a} & \mathbf{b-a} \\ \mathbf{c-b} & \mathbf{c-a} & \mathbf{a+b} \end{vmatrix}$

Sol. The given determinant

$$= \begin{vmatrix} 2b & 2a & 0 \\ b-c & c+a & b-a \\ 0 & 2c & 2b \end{vmatrix},$$ replacing R_1 and R_3 by $R_1 + R_2$ and $R_3 + R_2$ respectively.

$$= 4\begin{vmatrix} b & a & 0 \\ b-c & c+a & b-a \\ 0 & c & b \end{vmatrix},$$ taking out 2 common from R_1 and R_3

$$= 4\begin{vmatrix} b & a & 0 \\ -c & c & b-a \\ 0 & c & b \end{vmatrix},$$ replacing R_2 by $R_2 - R_1$

$$= 4\left[b\begin{vmatrix} c & b-a \\ c & b \end{vmatrix} + c\begin{vmatrix} a & 0 \\ c & b \end{vmatrix}\right],$$ expanding with respect to C_1

$$= 4\left[b\begin{vmatrix} c & b-a \\ 0 & a \end{vmatrix} + c\begin{vmatrix} a & 0 \\ c & b \end{vmatrix}\right],$$ replacing R_2 by $R_2 - R_1$ in first determinant.

$= 4\,[b\,(ca) + c\,(ab)] = 4\,(2abc) = 8abc.$ **Ans.**

Ex. 24. Evaluate $\begin{vmatrix} \mathbf{x} & \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{1} \\ \mathbf{d} & \mathbf{x} & \mathbf{f} & \mathbf{h} & \mathbf{1} \\ \mathbf{d} & \mathbf{e} & \mathbf{x} & \mathbf{k} & \mathbf{1} \\ \mathbf{d} & \mathbf{e} & \mathbf{g} & \mathbf{x} & \mathbf{1} \\ \mathbf{d} & \mathbf{e} & \mathbf{g} & \mathbf{m} & \mathbf{1} \end{vmatrix}$

Sol. The given determinant

$$= \begin{vmatrix} x-d & a-x & b-f & c-h & 0 \\ 0 & x-e & f-x & h-k & 0 \\ 0 & 0 & x-g & k-x & 0 \\ 0 & 0 & 0 & x-m & 0 \\ d & e & g & m & 1 \end{vmatrix},$$

replacing R_1, R_2, R_3 and R_4 by $R_1 - R_2$, $R_2 - R_3$, $R_3 - R_4$ and $R_4 - R_5$ resplactively.

$$= (x-m) \begin{vmatrix} x-d & a-x & b-f & 0 \\ 0 & x-e & f-x & 0 \\ 0 & 0 & x-g & 0 \\ d & e & g & 1 \end{vmatrix}$$, exapanding with respect to R_4. **(Note)**

$$= (x-m) \begin{vmatrix} x-d & a-x & b-f \\ 0 & x-e & f-x \\ 0 & 0 & x-g \end{vmatrix}$$, expanding with respect to C_4

$$= (x-m)(x-d) \begin{vmatrix} x-e & f-x \\ 0 & x-g \end{vmatrix}$$, expanding with respect to C_1

$$= (x-m)(x-d)(x-e)(x-g).$$ **Ans**

Ex. 25. Evaluate $\begin{vmatrix} 1 & x & 1 & y \\ x & 1 & y & 1 \\ 1 & y & 1 & x \\ y & 1 & x & 1 \end{vmatrix}$

Sol. The given determinant

$$= \begin{vmatrix} x+y+2 & x+y+2 & x+y+2 & x+y+2 \\ x & 1 & y & 1 \\ 1 & y & 1 & x \\ y & 1 & x & 1 \end{vmatrix}$$, replacing R_1 by $R_1 + R_2 + R_3 + R_4$

$$= (x+y+2) \begin{vmatrix} 1 & 1 & 1 & 1 \\ x & 1 & y & 1 \\ 1 & y & 1 & x \\ y & 1 & x & 1 \end{vmatrix}$$, taking out $(x+y+2)$ common from R_1

$$= (x+y+2) \begin{vmatrix} 1 & 0 & 0 & 0 \\ x & 1-x & y-x & 1-x \\ 1 & y-1 & 0 & x-1 \\ y & 1-y & x-y & 1-y \end{vmatrix}$$, replacing C_2, C_3 and C_4 by $C_2 - C_1$, $C_3 - C_1$ and $C_4 - C_1$ respectively.

$$= (x+y+2) \begin{vmatrix} 1-x & y-x & 1-x \\ y-1 & 0 & x-1 \\ 1-y & x-y & 1-y \end{vmatrix}$$, expanding with respect to R_1

$$= (x+y+2) \begin{vmatrix} 0 & y-x & 1-x \\ y-x & 0 & x-1 \\ 0 & x-y & 1-y \end{vmatrix}$$, replacing C_1 by $C_1 - C_3$

$$= -(x+y+2)(y-x) \begin{vmatrix} y-x & 1-x \\ x-y & 1-y \end{vmatrix}$$, expanding with respect to C_1

$$= -(x+y+2)(y-x)(y-x) \begin{vmatrix} 1 & 1-x \\ -1 & 1-y \end{vmatrix}$$,

taking out $(y-x)$ common from C_1

$= -(x+y+2)(y-x)^2[(1-y)-(-1)(1-x)]$

$= (x+y+2)(x-y)^2(x+y-2).$ **Ans.**

Ex. 26. Evaluate $\begin{vmatrix} a & x & y & a \\ x & 0 & 0 & y \\ y & 0 & 0 & x \\ a & y & x & a \end{vmatrix}$

Sol. Then given determinant

$= \begin{vmatrix} 0 & x+y & y & a \\ x-y & 0 & 0 & y \\ y-x & 0 & 0 & x \\ 0 & y+x & x & a \end{vmatrix}$, replacing C_1 and C_2 by $C_1 - C_4$ and $C_2 + C_3$ respectively.

$= (x-y)(x+y)\begin{vmatrix} 0 & 1 & y & a \\ 1 & 0 & 0 & y \\ -1 & 0 & 0 & x \\ 0 & 1 & x & a \end{vmatrix}$, taking out $x-y$ and $x+y$ common from C_1 and C_2 respectively.

$= (x^2-y^2)\begin{vmatrix} 0 & 1 & y & a \\ 1 & 0 & 0 & y \\ 0 & 0 & 0 & x+y \\ 0 & 0 & x-y & 0 \end{vmatrix}$, resplacing R_3 and R_4 by $R_2 + R_3$ and $R_4 - R_1$ rspectively.

$= -(x^2-y^2)\begin{vmatrix} 1 & y & a \\ 0 & 0 & x+y \\ 0 & x-y & 0 \end{vmatrix}$, expanding with respect to C_1

$= -(x^2-y^2)\begin{vmatrix} 0 & x+y \\ x-y & 0 \end{vmatrix}$, expanding with respect to C_1

$= -(x^2-y^2)[-(x+y)(x-y)] = (x^2-y^2)^2.$ **Ans**

Ex. 27 (a) Evaluate $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$

Sol. The given determinant

$= \begin{vmatrix} 1 & bc & a(b+c) \\ 0 & c(a-b) & c(b-a) \\ 0 & b(a-c) & b(c-a) \end{vmatrix}$, replacing R_2 and R_3 by $R_2 - R_1$ and $R_3 - R_1$ respectively.

$= \begin{vmatrix} c(a-b) & c(b-a) \\ b(a-c) & b(c-a) \end{vmatrix}$, expanding with respect to C_1

$= (a-b)(a-c)\begin{vmatrix} c & -c \\ b & -b \end{vmatrix}$, taking out $(a-b)$ and $(a-c)$ common from R_1 and R_2 respectively

$= -(a-b)(a-c)\begin{vmatrix} c & c \\ b & b \end{vmatrix} = 0$, two columns being identical.

Ex. 27 (b) Evaluate $\begin{vmatrix} \mathbf{1/a} & \mathbf{a^2} & \mathbf{bc} \\ \mathbf{1/b} & \mathbf{b^2} & \mathbf{ca} \\ \mathbf{1/c} & \mathbf{c^2} & \mathbf{ab} \end{vmatrix}$ (*Kanpur 92*)

Sol. The given determinant

$= \dfrac{1}{abc} \begin{vmatrix} 1 & a^3 & abc \\ 1 & b^3 & bca \\ 1 & c^3 & cab \end{vmatrix}$, multiplying R_1, R_2 and R_3 by a, b and c respectively. **(Note)**

$= \dfrac{1}{abc} \cdot abc \begin{vmatrix} 1 & a^3 & 1 \\ 1 & b^3 & 1 \\ 1 & c^3 & 1 \end{vmatrix}$, taking out abc common from C_3

$= 0$, since C_1 and C_3 are identical. **Ans.**

***Ex. 28. Evaluate** $\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a^2} & \mathbf{b^2} & \mathbf{c^2} \end{vmatrix}$ (*Meerut 96P*)

Sol. The given determinant

$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}$, replacing C_2 and C_3 by $C_2 - C_1$ and $C_3 - C_1$ respectively.

$= \begin{vmatrix} b-a & c-a \\ b^2-a^2 & c^2-a^2 \end{vmatrix}$, expanding with respect to R_1

$= (b-a)(c-a) \begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix}$, taking common factors out

$= (b-a)(c-a)[(c+a)-(b+a)] = (a-b)(c-a)(b-c)$. **Ans.**

***Ex. 29. Show that** $\begin{vmatrix} \mathbf{1} & \mathbf{a} & \mathbf{bc} \\ \mathbf{1} & \mathbf{b} & \mathbf{ca} \\ \mathbf{1} & \mathbf{c} & \mathbf{ab} \end{vmatrix} = \mathbf{(b-c)(c-a)(a-b)}$ (*Meerut 96P; Purvanchal 96*)

Sol. The given determinant

$= \begin{vmatrix} 1 & a & bc \\ 0 & b-a & ca-bc \\ 0 & c-a & ab-bc \end{vmatrix}$, replacing R_2 and R_3 by $R_2 - R_1$ and $R_3 - R_1$ respectively.

$= \begin{vmatrix} b-a & -c(b-a) \\ c-a & -b(c-a) \end{vmatrix}$, expanding with respect to C_1

$= (b-a)(c-a) \begin{vmatrix} 1 & -c \\ 1 & -b \end{vmatrix}$, taking common factors out

$= (b-a)(c-a)(-b+c) = (a-b)(b-c)(c-a)$. Hence proved.

Ex. 30. Show that $\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a^2} & \mathbf{b^2} & \mathbf{c^2} \\ \mathbf{a^3} & \mathbf{b^3} & \mathbf{c^3} \end{vmatrix} = \mathbf{abc(a-b)(b-c)(c-a)}$

Sol. The given determinant

$$= abc \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$, taking out a, b and c common from C_1, C_2 and C_3 respectively.

Now proceed as in Ex. 28 Page 145.

Ex. 31. Evaluate $\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a^3} & \mathbf{b^3} & \mathbf{c^3} \end{vmatrix}$

Sol. The given determinant

$$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix}$$, replacing C_2, C_3 by $C_2 - C_1$ and $C_3 - C_1$ respectively.

$$= \begin{vmatrix} b-a & c-a \\ b^3-a^3 & c^3-a^3 \end{vmatrix}$$, expanding with respect to R_1

$$= (b-a)(c-a) \begin{vmatrix} 1 & 1 \\ b^2+ab+a^2 & c^2+ac+a^2 \end{vmatrix},$$

taking common factors out from C_1 and C_2

$= (b-a)(c-a)[(c^2+ac+a^2)-(b^2+ab+a^2)]$

$= (b-a)(c-a)[(c^2-b^2)+a(c-b)]$

$= (b-a)(c-a)(c-b)[(c+b)+a]$

$= (a-b)(b-c)(c-a)(a+b+c)$. **Ans.**

***Ex. 32. If a, b, c are all different and**

$$\begin{vmatrix} \mathbf{a} & \mathbf{a^2} & \mathbf{1+a^3} \\ \mathbf{b} & \mathbf{b^2} & \mathbf{1+b^3} \\ \mathbf{c} & \mathbf{c^2} & \mathbf{1+c^3} \end{vmatrix} = \mathbf{0},$$ **prove that abc = – 1.**

(Meerut 91 S)

Sol. The given determinant

$$= \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$$, breaking into two determinants. **(Note)**

$$= \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$$, interchanging rows and columns in each determinant. **(Note)**

Now proceed as in Ex. 28 Page 145 and Ex. 30 above.

The value of given determinant $= (1 + abc)[(a-b)(b-c)(c-a)]$

Ex. 33 (a). Evaluate $\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{a^2} & \mathbf{b^2} & \mathbf{c^2} \\ \mathbf{a^3} & \mathbf{b^3} & \mathbf{c^3} \end{vmatrix}$

Sol. The given determinant

$$= \begin{vmatrix} 1 & 0 & 0 \\ a^2 & b^2-a^2 & c^2-a^2 \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix}$$, replacing C_2 and C_3 by $C_2 - C_1$ and $C_3 - C_1$ respectively.

$$= \begin{vmatrix} b^2-a^2 & c^2-a^2 \\ b^3-a^3 & c^3-a^3 \end{vmatrix}$$, expanding with respect to R_1

$$= (b-a)(c-a) \begin{vmatrix} b+a & c+a \\ b^2+ab+a^2 & c^2+ac+a^2 \end{vmatrix},$$

taking out common factors of C_1, C_2

$$= (b-a)(c-a) \begin{vmatrix} b-c & c+a \\ b^2+ab-c^2-ca & c^2+ca+a^2 \end{vmatrix},$$

replacing C_1 by $C_1 - C_2$

$$= (b-a)(c-a) \begin{vmatrix} b-c & c+a \\ (b^2-c^2)+a(b-c) & c^2+ca+a^2 \end{vmatrix}$$

$$= (b-a)(c-a)(b-c) \begin{vmatrix} 1 & c+a \\ b+c+a & c^2+ca+a^2 \end{vmatrix}$$

$$= -(a-b)(b-c)(c- \text{[illegible]} 1(c^2+ca+a^2)-(c+a)(b+c+a)]$$
$$= -(a-b)(b-c)(c- \text{[illegible]}^2+ca+a^2-cb-c^2-ac-ab-ac-a^2]$$
$$= -(a-b)(b-c)(c \text{[illegible]} -ab-bc-ca]$$
$$= (a-b)(b-c)(c-a)(ab+bc+ca)$$ **Ans.**

Ex. 33. (b) If $\begin{vmatrix} \mathbf{a} & \mathbf{a^3} & \mathbf{a^4-1} \\ \mathbf{b} & \mathbf{b^3} & \mathbf{b^4-1} \\ \mathbf{c} & \mathbf{c^3} & \mathbf{c^4-1} \end{vmatrix} = 0$**, then prove that**

$$\mathbf{abc\,(ab+bc+ca) = a+b+c.}$$

Sol. The given equation is

$$\begin{vmatrix} a & a^3 & a^4 \\ b & b^3 & b^4 \\ c & c^3 & c^4 \end{vmatrix} - \begin{vmatrix} a & a^3 & 1 \\ b & b^3 & 1 \\ c & c^3 & 1 \end{vmatrix} = 0$$

or $$abc \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} - \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = 0$$

or $$abc\,[(a-b)(b-c)(c-a)(ab+bc+ca)] - [(a-b)(b-c)(c-a)(a+b+c)] = 0$$

interchaing rows and columns of the determinant and evaluating determinants as in Ex. 31 Page 146 and Ex. 33 (a) above

or $$(a-b)(b-c)(c-a)\,[abc\,(ab+bc+ca)-(a+b+c)] = 0$$

or $$abc\,(ab+bc+ca)-(a+b+c) = 0, \because \quad a \neq b \neq c$$

or $$abc\,(ab+bc+ca) = a+b+c.$$ Hence proved.

Ex. 34. Prove that

$$\begin{vmatrix} 1 & a^2+bc & a^3 \\ 1 & b^2+ca & b^3 \\ 1 & c^2+ab & c^3 \end{vmatrix} = -(b-c)(c-a)(a-b)(a^2+b^2+c^2)$$

Sol. The given determinant

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} + \begin{vmatrix} 1 & bc & a^3 \\ 1 & ca & b^3 \\ 1 & ab & c^3 \end{vmatrix}$$, breaking into two determinants **(Note)**

$$= \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ bc & ca & ab \\ a^3 & b^3 & c^3 \end{vmatrix}$$, interchanging rows and columns. **(Note)** ...(i)

The first determinant $= (a-b)(b-c)(c-a)(ab+bc+ca)$. ...(ii)

[See Ex. 33 (a) Page 146]

The second determinant

$$= \begin{vmatrix} 1 & 0 & 0 \\ bc & ca-bc & ab-bc \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix}$$, replacing C_2 and C_3 by $C_2 - C_1$ and $C_3 - C_1$

$$= \begin{vmatrix} c(a-b) & b(a-c) \\ (b-a)(b^2+ab+a^2) & (c-a)(c^2+ac+a^2) \end{vmatrix},$$

expanding with respect to R_1

$$= (a-b)(a-c) \begin{vmatrix} c & b \\ -(b^2+ab+a^2) & -(c^2+ac+a^2) \end{vmatrix},$$

taking out common factors from C_1 and C_2

$$= (a-b)(a-c) \begin{vmatrix} (c-b) & b \\ -(b^2-c^2+ab-ac) & -(c^2+ac+a^2) \end{vmatrix},$$

replacing C_1 by $C_1 - C_2$

$$= (a-b)(a-c) \begin{vmatrix} (c-b) & b \\ (c-b)(c+b+a) & -(c^2+ca+a^2) \end{vmatrix}$$

$$= (a-b)(a-c)(c-b) \begin{vmatrix} 1 & b \\ a+b+c & -(c^2+ca+a^2) \end{vmatrix},$$

taking $(c-b)$ common from C_1.

$$= (a-b)(b-c)(c-a)[-(c^2+ca+a^2) - b(a+b+c)]$$

$$= -(a-b)(b-c)(c-a)(a^2+b^2+c^2+ab+bc+ca) \quad ...(iii)$$

Substituting values from (ii) and (iii) in (i), we find the given determinant

$$= (a-b)(b-c)(c-a)(ab+bc+ca)$$

$$- (a-b)(b-c)(c-a)(a^2+b^2+c^2+ab+bc+ca)$$

$= (a-b)(b-c)(c-a)[(ab+bc+ca)-(a^2+b^2+c^2+ab+bc+ca)]$

$= -(a-b)(b-c)(c-a)(a^2+b^2+c^2).$ Hence proved.

Ex. 35. Show that

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = (y-z)(z-x)(x-y)(yz+zx+xy).$$

(*Gorakhpur 93; Kumaun 96; Meerut 94*)

Sol.

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = \frac{1}{xyz}\begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ xyz & xyz & xyz \end{vmatrix}$$, multiplying C_1, C_2, C_3 by x, y, z respectively. **(Note)**

$$= \frac{1}{xyz} xyz \begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ 1 & 1 & 1 \end{vmatrix}$$, taking out xyz common from R_3

$$= \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$$, interchanging R_2 and R_3 and then R_1 and R_2

For the 2nd part do as Ex. 33 (a) Pages 146–47. **(Note)**

Ex. 36. Evaluate
$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1+z \end{vmatrix}$$

Sol. The given determinant

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & z \end{vmatrix}$$, replacing R_2, R_3 and R_4 by $R_2 - R_1, R_3 - R_1$ and $R_4 - R_1$ respectively

$$= \begin{vmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{vmatrix}$$, expanding with respect to C_1

$$= x\begin{vmatrix} y & 0 \\ 0 & z \end{vmatrix}$$, expanding with respect to R_1

$= xyz.$ **Ans.**

Ex. 37. Evaluate
$$\begin{vmatrix} a^3 & 3a^2 & 3a & 1 \\ a^2 & a^2+2a & 2a+1 & 1 \\ a & 2a+1 & a+2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix}$$

Sol. The given determinant

$$= \begin{vmatrix} a^3-3a^2+3a-1 & 3a^2 & 3a & 1 \\ 0 & a^2+2a & 2a+1 & 1 \\ 0 & 2a+1 & a+2 & 1 \\ 0 & 3 & 3 & 1 \end{vmatrix}, \text{ replacing } C_1 \text{ by } C_1-C_2+C_3-C_4$$ (Note)

$$= (a^3-3a^2+3a-1)\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}, \text{ expanding with respect to } C_1$$

$$= (a-1)^3\begin{vmatrix} (a^2+2a)-(2a+1)+1 & 2a+1 & 1 \\ (2a+1)-(a+2)+1 & a+2 & 1 \\ 3-3+1 & 3 & 1 \end{vmatrix}, \text{ replacing } C_1 \text{ by } C_1-C_2+C_3$$

$$= (a-1)^3\begin{vmatrix} a^2 & 2a+1 & 1 \\ a & a+2 & 1 \\ 1 & 3 & 1 \end{vmatrix}, \text{ on simplying}$$

$$= (a-1)^3\begin{vmatrix} a^2-1 & 2a-2 & 0 \\ a-1 & a-1 & 0 \\ 1 & 3 & 1 \end{vmatrix}, \text{ replacing } R_1 \text{ and } R_2 \text{ by } R_1-R_3 \text{ and } R_2-R_3 \text{ respectively.}$$

$$= (a-1)^3\begin{vmatrix} a^2-1 & 2(a-1) \\ a-1 & a-1 \end{vmatrix}, \text{ expanding with respect to } C_3$$

$$= (a-1)^3(a-1)(a-1)\begin{vmatrix} a+1 & 2 \\ 1 & 1 \end{vmatrix}, \text{ taking out common factors.}$$

$$= (a-1)^5[(a+1)-2] = (a-1)^6.$$ **Ans.**

Ex.38. Evaluate $\begin{vmatrix} \mathbf{1+a} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1+b} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1+c} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1+d} \end{vmatrix}$

(*Gorakhpur 90; Kanpur 97; Meerut 91; Purvanchal 93*)

Sol. The given determinant

$$= abcd\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 & \frac{1}{c} \\ \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & \frac{1}{d}+1 \end{vmatrix}, \text{ taking out } a, b, c \text{ and } d \text{ common from } R_1\, R_2, R_3 \text{ and } R_4 \text{ respectively.}$$

$$= abcd\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+1\right)\begin{vmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 & \frac{1}{c} \\ \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & \frac{1}{d}+1 \end{vmatrix}$$

replacing, R_1 by $R_1+R_2+R_3+R_4$ and taking $\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+1\right)$ **common from R_1**

$$= abcd\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+1\right)\begin{vmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{b} & 1 & 0 & 0 \\ \frac{1}{c} & 0 & 1 & 0 \\ \frac{1}{d} & 0 & 0 & 1 \end{vmatrix},$$

replacing C_2, C_3, and C_4 by $C_2 - C_1$, $C_3 - C_1$ and $C_4 - C_1$ respectively.

$$= abcd\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+1\right)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$, expanding with respect to R_1

$$= abcd\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+1\right)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$, expanding with respect to R_1

$$= abcd\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+1\right)$$ **Ans.**

Ex. 39. Prove that $\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{b} & \mathbf{b} \\ \mathbf{a} & \mathbf{b} & \mathbf{a} & \mathbf{a} \\ \mathbf{a} & \mathbf{a} & \mathbf{b} & \mathbf{a} \\ \mathbf{b} & \mathbf{b} & \mathbf{b} & \mathbf{a} \end{vmatrix} = -(\mathbf{b}-\mathbf{a})^4$.

Sol. The given determinant

$$= \begin{vmatrix} a & b-a & b-a & b-a \\ a & b-a & 0 & 0 \\ a & 0 & b-a & 0 \\ b & 0 & 0 & a-b \end{vmatrix}$$, replacing C_2, C_3 and C_4 by $C_2 - C_1$, $C_3 - C_1$ and $C_4 - C_1$ respectively.

$$= -(b-a)\begin{vmatrix} a & b-a & 0 \\ a & 0 & b-a \\ b & 0 & 0 \end{vmatrix} + (a-b)\begin{vmatrix} a & b-a & b-a \\ a & b-a & 0 \\ a & 0 & b-a \end{vmatrix},$$

expanding with respect to C_4

$$= -(b-a)\,b\begin{vmatrix} b-a & 0 \\ 0 & b-a \end{vmatrix} + (a-b)\begin{vmatrix} 0 & 0 & b-a \\ a & b-a & 0 \\ a & 0 & b-a \end{vmatrix},$$

expanding first determinant with respect to R_3 and in the second determinant replacing R_1 by $R_1 - R_2$.

$$= -b\,(b-a)^3 + (a-b)\,(b-a)\begin{vmatrix} a & b-a \\ a & 0 \end{vmatrix},$$

expanding the second determinant with respect to R_1

$$= -b\,(b-a)^3 + (a-b)\,(b-a)\,[0 - a\,(b-a)]$$

$$= -b\,(b-a)^3 + a\,(b-a)^3 = -(b-a)^3\,(b-a)$$

$$= -(b-a)^4$$

Hence proved

****Ex. 40. Show that** $\begin{vmatrix} (\mathbf{b}+\mathbf{c})^2 & \mathbf{a}^2 & \mathbf{a}^2 \\ \mathbf{b}^2 & (\mathbf{c}+\mathbf{a})^2 & \mathbf{b}^2 \\ \mathbf{c}^2 & \mathbf{c}^2 & (\mathbf{a}+\mathbf{b})^2 \end{vmatrix} = \mathbf{2abc}\,(\mathbf{a}+\mathbf{b}+\mathbf{c})^3$

(Kumaun 94; Meerut 93; Purvanchal 95)

Sol. The given determinant

$$= \begin{vmatrix} (b+c)^2 - a^2 & 0 & a^2 \\ 0 & (c+a)^2 - b^2 & b^2 \\ c^2 - (a+b)^2 & c^2 - (a+b)^2 & (a+b)^2 \end{vmatrix},$$ replacing C_1 and C_2 by $C_1 - C_3$ & $C_2 - C_3$ respectively.

$$= \begin{vmatrix} (b+c+a)\,(b+c-a) & 0 & a^2 \\ 0 & (c+a+b)\,(c+a-b) & b^2 \\ (c+a+b)\,(c-a-b) & (c+a+b)\,(c-a-b) & (a+b)^2 \end{vmatrix}$$

$$= (a+b+c)^2\begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix},$$ taking out the common factors from C_1 and C_2.

$$= (a+b+c)^2\begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix},$$ replacing R_3 by $R_3 - R_1 - R_2$.

$$= (a+b+c)^2\begin{vmatrix} b+c & a^2/b & a^2 \\ b^2/a & c+a & b^2 \\ 0 & 0 & 2ab \end{vmatrix},$$ replacing C_1 and C_2 by $C_1 + \frac{1}{a}C_3$ and $C_2 + \frac{1}{b}C_3$ respectively.

(Note)

$$= 2ab\,(a+b+c)^2\begin{vmatrix} b+c & a^2/b \\ b^2/a & c+a \end{vmatrix},$$ expanding with respect to R_3

$$= 2ab\,(a+b+c)^2\,[(b+c)\,(c+a) - (b^2/a)\,(a^2/b)]$$

$= 2ab\,(a+b+c)^2\,[bc+ba+c^2+ca-ab]$

$= 2ab\,(a+b+c)^2\,c\,[b+a+c] = 2abc\,(a+b+c)^3.$ Hence proved.

Ex. 41. Evaluate $\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{ax+by} \\ \mathbf{b} & \mathbf{c} & \mathbf{bx+cy} \\ \mathbf{ax+by} & \mathbf{bc+cy} & \mathbf{0} \end{vmatrix}$ *(Kanpur 90)*

Sol. The given determinant

$$= \begin{vmatrix} a & b & 0 \\ b & c & 0 \\ ax+by & bx+cy & -x\,(ax+by) \\ & & -y\,(bx+cy) \end{vmatrix}$$, replacing C_3 by $C_3 - xC_1 - yC_2$. **(Note)**

$= -(ax^2+2bxy+cy^2)\begin{vmatrix} a & b \\ b & c \end{vmatrix}$, expanding with respect to C_3

$= -(ax^2+2bxy+cy^2)\,(ac-b^2).$ **Ans.**

Ex. 42. Prove that

$$\begin{vmatrix} \mathbf{1} & \mathbf{0} & \mathbf{x} & \mathbf{0} & \mathbf{x} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{x} & \mathbf{0} \\ \mathbf{x} & \mathbf{0} & \mathbf{x+1} & \mathbf{0} & \mathbf{x} \\ \mathbf{0} & \mathbf{x} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{x} & \mathbf{0} & \mathbf{x} & \mathbf{0} & \mathbf{1} \end{vmatrix} = \mathbf{(x-1)^2\,(x+1)\,(1+2x-x^2)}$$

Sol. The given determinat

$$= \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & x & 0 \\ x & 0 & x+1-x^2 & 0 & -1 \\ 0 & x & 0 & 1 & 0 \\ x & 0 & x-x^2 & 0 & 1-x \end{vmatrix}$$, replacing C_3 and C_5 by $C_3 - xC_1$ and $C_5 - C_3$ respectively.

$$= \begin{vmatrix} 1 & 0 & x & 0 \\ 0 & x+1-x^2 & 0 & -1 \\ x & 0 & 1 & 0 \\ 0 & x-x^2 & 0 & 1-x \end{vmatrix}$$, expanding with respect to R_1

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & x+1-x^2 & 0 & -1 \\ x & 0 & 1-x^2 & 0 \\ 0 & x-x^2 & 0 & 1-x \end{vmatrix}$$, replacing C_3 by $C_3 - xC_1$

$$= \begin{vmatrix} x+1-x^2 & 0 & -1 \\ 0 & 1-x^2 & 0 \\ x\,(1-x) & 0 & 1 \end{vmatrix}$$, expanding with respect to R_1

$$= -(1-x)\begin{vmatrix} 0 & 1-x^2 & 0 \\ x+1-x^2 & 0 & -1 \\ x & 0 & 1 \end{vmatrix}$$, interchanging R_1 and R_2 and taking out $(1-x)$ common from R_3

$$= (1-x)(1-x^2)\begin{vmatrix} x+1-x^2 & -1 \\ x & 1 \end{vmatrix}$$, expanding with respect to R_1

$$= (1-x)^2(1+x)[(x+1-x^2).1-(-1)x]$$
$$= (x-1)^2(1+x)(2x+1-x^2).$$

Hence proved.

Ex. 43. Show that $\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{bc\,(b+c)} & \mathbf{ca\,(c+a)} & \mathbf{ab\,(a+b)} \\ \mathbf{b^2c^2} & \mathbf{c^2a^2} & \mathbf{a^2b^2} \end{vmatrix}$

$= \mathbf{abc\,(a-b)\,(b-c)\,(c-a)\,(a+b+c)}$

Sol. The given determinant

$$= \begin{vmatrix} 1 & 1 & 1 \\ b^2c+bc^2 & c^2a+ca^2 & a^2b+ab^2 \\ b^2c^2 & c^2a^2 & a^2b^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ b^2c+bc^2 & c\,(a-b)\,(a+b+c) & b\,(a-c)\,(a+b+c) \\ b^2c^2 & c^2\,(a-b)\,(a+b) & b^2\,(a-c)\,(a+c) \end{vmatrix},$$

replacing C_2, C_3 by $C_2 - C_1$, $C_3 - C_1$

$$= \begin{vmatrix} c\,(a-b)\,(a+b+c) & b\,(a-c)\,(a+b+c) \\ c^2\,(a-b)\,(a+b) & b^2\,(a-c)\,(a+c) \end{vmatrix}$$, expanding with respect to R_1

$$= c\,(a-b)\,b\,(a-c)\begin{vmatrix} a+b+c & a+b+c \\ c\,(a+b) & b\,(a+c) \end{vmatrix}$$, taking out common factors from C_1, C_2

$$= bc\,(a-b)\,(a-c)\,(a+b+c)\begin{vmatrix} 1 & 1 \\ ca+cb & ba+bc \end{vmatrix}$$, taking out $a+b+c$ common from R_1

$$= bc\,(a-b)\,(a-c)\,(a+b+c)\begin{vmatrix} 1 & 0 \\ ca+cb & ba-ca \end{vmatrix}$$, replacing C_2 by $C_2 - C_1$

$$= bc\,(a-b)\,(a-c)\,(a+b+C)\,(ab-ca)$$
$$= -abc\,(a-b)\,(b-c)\,(c-a)\,(a+b+c).$$

Hence proved.

***Ex. 44. Evaluate** $\begin{vmatrix} \mathbf{1} & \mathbf{bc+ad} & \mathbf{b^2c^2+a^2d^2} \\ \mathbf{1} & \mathbf{ca+bd} & \mathbf{c^2a^2+b^2d^2} \\ \mathbf{1} & \mathbf{ab+cd} & \mathbf{a^2b^2+c^2d^2} \end{vmatrix}$

Sol. The given determinant

$$= \begin{vmatrix} 1 & bc+ad & b^2c^2+a^2d^2 \\ 0 & ca+bd-bc-ad & c^2a^2+b^2d^2-b^2c^2-a^2d^2 \\ 0 & ab+cd-bc-ad & a^2b^2+c^2d^2-b^2c^2-a^2d^2 \end{vmatrix},$$

replacing R_2 and R_3 by $R_2 - R_1$ and $R_3 - R_1$ respectively

$$= \begin{vmatrix} ca - bc + bd - ad & c^2a^2 - b^2c^2 + b^2d^2 - a^2d^2 \\ ab - bc + cd - ad & a^2b^2 - b^2c^2 + c^2d^2 - a^2d^2 \end{vmatrix},$$

expanding with respect to C_1

$$= \begin{vmatrix} (c-d)(a-b) & (c^2-d^2)(a^2-b^2) \\ (b-d)(a-c) & (b^2-d^2)(a^2-c^2) \end{vmatrix}, \text{ factorising the elements}$$

$$= (c-d)(a-b)(b-d)(a-c) \begin{vmatrix} 1 & (c+d)(a+b) \\ 1 & (b+d)(a+c) \end{vmatrix},$$

taking out the common factors

$$= (c-d)(a-b)(b-d)(a-c) \begin{vmatrix} 1 & ca + bc + da + db \\ 0 & ba + dc - ca - db \end{vmatrix},$$

replacing R_2 by $R_2 - R_1$

$$= (c-d)(a-b)(b-d)(a-c)(ba + dc - ca - db)$$
$$= (c-d)(a-b)(b-d)(a-c)(a-d)(b-c).$$ **Ans.**

***Ex. 45. Prove that**

$$\begin{vmatrix} \mathbf{a}^2+\lambda & \mathbf{ab} & \mathbf{ac} & \mathbf{ad} \\ \mathbf{ba} & \mathbf{b}^2+\lambda & \mathbf{bc} & \mathbf{bd} \\ \mathbf{ca} & \mathbf{cb} & \mathbf{c}^2+\lambda & \mathbf{cd} \\ \mathbf{da} & \mathbf{db} & \mathbf{dc} & \mathbf{d}^2+\lambda \end{vmatrix} = \lambda^3 (\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2 + \mathbf{d}^2 + \lambda)$$

(*Kanpur* 90)

Sol. The given determinant

$$= abcd \begin{vmatrix} a+\frac{\lambda}{a} & a & a & a \\ b & b+\frac{\lambda}{b} & b & b \\ c & c & c+\frac{\lambda}{c} & c \\ d & d & d & d+\frac{\lambda}{d} \end{vmatrix},$$ taking out a, b, c, d common from C_1, C_2, C_3 and C_4 respectively.

$$= abcd \begin{vmatrix} a+\frac{\lambda}{a} & -\frac{\lambda}{a} & -\frac{\lambda}{a} & -\frac{\lambda}{a} \\ b & \frac{\lambda}{b} & 0 & 0 \\ c & 0 & \frac{\lambda}{c} & 0 \\ d & 0 & 0 & \frac{\lambda}{d} \end{vmatrix},$$ replacing C_2, C_3 and C_4 by $C_2 - C_1$, $C_3 - C_1$ and $C_4 - C_1$ respectively.

$$= abcd \begin{vmatrix} a & 0 & 0 & -\frac{\lambda}{a} \\ b & \frac{\lambda}{b} & 0 & 0 \\ c & 0 & \frac{\lambda}{c} & 0 \\ d+\frac{\lambda}{d} & -\frac{\lambda}{d} & -\frac{\lambda}{d} & \frac{\lambda}{d} \end{vmatrix}$$

, replacing C_1, C_2 and C_3 by $C_1 + C_4$, $C_2 - C_4$ and $C_3 - C_4$ respectively.

$$= a^2bcd \begin{vmatrix} \frac{\lambda}{b} & 0 & 0 \\ 0 & \frac{\lambda}{c} & 0 \\ -\frac{\lambda}{d} & -\frac{\lambda}{d} & \frac{\lambda}{d} \end{vmatrix} + \lambda\, bcd \begin{vmatrix} b & \frac{\lambda}{b} & 0 \\ c & 0 & \frac{\lambda}{c} \\ d+\frac{\lambda}{d} & -\frac{\lambda}{d} & -\frac{\lambda}{d} \end{vmatrix},$$

expanding with respect to R_1

$$= a^2bcd\, \frac{\lambda}{b} \begin{vmatrix} \frac{\lambda}{c} & 0 \\ -\frac{\lambda}{d} & \frac{\lambda}{d} \end{vmatrix} + \lambda bcd.b \begin{vmatrix} 0 & \frac{\lambda}{c} \\ -\frac{\lambda}{d} & \frac{\lambda}{d} \end{vmatrix} - \lambda\, bcd \cdot \frac{\lambda}{b} \begin{vmatrix} c & \frac{\lambda}{c} \\ d+\frac{\lambda}{d} & -\frac{\lambda}{d} \end{vmatrix},$$

expanding each determinant with respect to R_1

$$= \lambda\, a^2\, cd \left(\frac{\lambda^2}{cd}\right) + \lambda\, b^2 cd \left(\frac{\lambda^2}{cd}\right) - \lambda^2\, cd \left(-\frac{\lambda c}{d} - \frac{\lambda d}{c} - \frac{\lambda^2}{cd}\right)$$

$$= \lambda^3 a^2 + \lambda^3 b^2 + \lambda^2 cd \left(\frac{\lambda c^2 + \lambda d^2 + \lambda^2}{cd}\right)$$

$= \lambda^3 a^2 + \lambda^3 b^2 + \lambda^3 (c^2 + d^2 + \lambda) = \lambda^3 (a^2 + b^2 + c^2 + d^2 + \lambda)$ Hence proved.

Ex. 46. Evaluate $\begin{vmatrix} \mathbf{a^2} & \mathbf{a^2 - (b-c)^2} & \mathbf{bc} \\ \mathbf{b^2} & \mathbf{b^2 - (c-a)^2} & \mathbf{ca} \\ \mathbf{c^2} & \mathbf{c^2 - (a-b)^2} & \mathbf{ab} \end{vmatrix}$

Sol. The given determinant

$$= \begin{vmatrix} a^2 & -(b-c)^2 & bc \\ b^2 & -(c-a)^2 & ca \\ c^2 & -(a-b)^2 & ab \end{vmatrix}$$

, replacing C_2 by $C_2 - C_1$

$$= -\begin{vmatrix} a^2 & (b^2+c^2)-2bc & bc \\ b^2 & (c^2+a^2)-2ca & ca \\ c^2 & (a^2+b^2)-2ab & ab \end{vmatrix}$$ **(Note)**

$$= -\begin{vmatrix} a^2 & (b^2+c^2) & bc \\ b^2 & (c^2+a^2) & ca \\ c^2 & (a^2+b^2) & ab \end{vmatrix}$$, replacing C_2 by $C_2 + 2C_3$

$$= -\begin{vmatrix} a^2 & b^2+c^2+a^2 & bc \\ b^2 & c^2+a^2+b^2 & ca \\ c^2 & a^2+b^2+c^2 & ab \end{vmatrix}$$, replacing C_2 by $C_2 + C_1$

$$= -(a^2+b^2+c^2)\begin{vmatrix} a^2 & 1 & bc \\ b^2 & 1 & ca \\ c^2 & 1 & ab \end{vmatrix}$$, taking out the common factor from C_2

$$= -(a^2+b^2+c^2)\begin{vmatrix} a^2 & 1 & bc \\ b^2-a^2 & 0 & ca-bc \\ c^2-a^2 & 0 & ab-bc \end{vmatrix}$$, replacing R_2 and R_3 by $R_2 - R_1$ and $R_3 - R_1$ respectively

$$= (a^2+b^2+c^2)\begin{vmatrix} b^2-a^2 & c\,(a-b) \\ c^2-a^2 & b\,(a-c) \end{vmatrix}$$, expanding with respect to C_2

$$= (a-b)\,(a-c)\,(a^2+L^2+c^2)\begin{vmatrix} -(b+a) & c \\ -(c+a) & b \end{vmatrix}$$, taking out the common factors

$$= (a-b)\,(a-c)\,(a^2+b^2+c^2)\,[-b\,(b+a)+c\,(c+a)]$$
$$= (a-b)\,(a-c)\,(a^2+b^2+c^2)\,[-b^2-ab+c^2+ac]$$
$$= (a-b)\,(a-c)\,(a^2+b^2+c^2)\,[(c^2-b^2)+a\,(c-b)]$$
$$= (a-b)\,(a-c)\,(a^2+b^2+c^2)\,(c-b)\,(a+b+c).$$
$$= (a-b)\,(b-c)\,(c-a)\,(a+b+c)\,(a^2+b^2+c^2).$$ **Ans.**

****Ex. 47. Prove that**

$$\begin{vmatrix} \mathbf{0} & -\mathbf{c} & \mathbf{b} & -\mathbf{l} \\ \mathbf{c} & \mathbf{0} & -\mathbf{a} & -\mathbf{m} \\ -\mathbf{b} & \mathbf{a} & \mathbf{0} & -\mathbf{n} \\ \mathbf{x} & \mathbf{y} & \mathbf{z} & \mathbf{0} \end{vmatrix} = \mathbf{(al + bm + cn)\,(ax + by + cy)}$$

Sol. The given determinant

$$= \frac{1}{a}\begin{vmatrix} 0 & -ac & ab & -al \\ c & 0 & -a & -m \\ -b & a & 0 & -n \\ x & y & z & 0 \end{vmatrix}$$, taking $1/a$ common from R_1 **(Note)**

$$=\frac{1}{a}\begin{vmatrix} 0 & 0 & 0 & -al-bm-cn \\ c & 0 & -a & -m \\ -b & a & 0 & -n \\ x & y & z & 0 \end{vmatrix}, \text{ replacing } R_1 \text{ by } R_1+b\,R_2+c\,R_3$$

$$=\frac{(al+bm+cn)}{a}\begin{vmatrix} c & 0 & -a \\ -b & a & 0 \\ x & y & z \end{vmatrix}, \text{ expanding with respect to } R_1$$

$$=\frac{(al+bm+cn)}{a^2}\begin{vmatrix} ac & 0 & -a \\ -ab & a & 0 \\ ax & y & z \end{vmatrix}, \text{ taking } 1/a \text{ common from } C_1$$ **(Note)**

$$=\frac{(al+bm+cn)}{a^2}\begin{vmatrix} 0 & 0 & -a \\ 0 & a & 0 \\ ax+by+cz & y & z \end{vmatrix}, \text{ replacing } C_1 \text{ by } C_1+bC_2+cC_3$$

$$=\frac{(al+bm+cn)\,(ax+by+cz)}{a^2}\begin{vmatrix} 0 & -a \\ a & 0 \end{vmatrix}$$

$= (al+bm+cn)\,(ax+by+cz).$ Hence proved.

Exercises on Evaluation of Determinants

Ex. 1. Evaluate $\begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix}$ **Ans.** -8

Ex. 2. Show that $\begin{vmatrix} 29 & 26 & 22 \\ 25 & 31 & 27 \\ 65 & 54 & 46 \end{vmatrix} = 132$

Ex. 3. Evaluate $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

[Hint : Replace C_3 by $C_3 + C_2$]. **Ans.** 0.

Ex. 4. Show that $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0.$

[Hint : Replace C_1 by $C_1 + C_2 + C_3$].

Ex. 5. Evaluate $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$, where ω is one of the imaginary cube roots of unity. **Ans.** 0

[Hint : Replace C_1 by $C_1 + C_2 + C_3$ remembering $1 + \omega + \omega^2 = 0$]

Ex. 6. Prove that $\begin{vmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{vmatrix} = 0$

[Hint : Replace C_1 by $C_1 + C_2 + C_3 + C_4 + C_5$].

Ex. 7. Show that $\begin{vmatrix} 21 & 17 & 7 & 10 \\ 24 & 22 & 6 & 10 \\ 6 & 8 & 2 & 3 \\ 5 & 7 & 1 & 2 \end{vmatrix} = 0$

Ex. 8. Calculate the value of the determinant $\begin{vmatrix} 7 & 13 & 10 & 6 \\ 5 & 9 & 7 & 4 \\ 8 & 12 & 11 & 7 \\ 4 & 10 & 6 & 3 \end{vmatrix}$ **Ans.** 0

Ex. 9. Evaluate $\begin{vmatrix} x+a & x+2a & x+3a \\ x+2a & x+3a & x+4a \\ x+4a & x+5a & x+6a \end{vmatrix}$

[Hint : Replace C_2 and C_3 by $C_2 - C_1$ and $C_3 - C_2$]. **Ans.** 0

Ex. 10. Prove that

$$\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 2\,(3\,abc - a^3 - b^3 - c^3).$$

[Hint : See Ex. 13 Page 137].

***Ex. 11.** Prove that $\begin{vmatrix} -2a & a+b & c+a \\ b+a & -2b & c+b \\ c+a & c+b & -2c \end{vmatrix} = 4\,(a+b)\,(b+c)\,(c+a)$

Products of Determinants

§ 4·11. Theorem. *If* **A** *is an* $n \times n$ *matrix and* **E** *is an elementary matrix obtained from the identity matrix* $\mathbf{I_n}$, *then*

$$|\mathbf{EA}| = |\mathbf{AE}| = |\mathbf{E}| \bullet |\mathbf{A}| = |\mathbf{A}| \bullet |\mathbf{E}| \qquad (\textit{Purvanchal 94})$$

Proof : $\because$ $\mathbf{I}_n$ is the $n \times n$ identity matrix.

$\therefore\ |\mathbf{I}_n| = 1.$ ...(i)

Let $\mathbf{E}_a$, $\mathbf{E}_b$, $\mathbf{E}_c$ be three elementary matrices as defined in § 3·03 Page 104.

Then $|\mathbf{E}_a| = -|\mathbf{I}_n|$ ...See § 4·06 Prop. III Page 126.

$\therefore$ From (i) we get $|\mathbf{E}_a| = -1$ ...(ii)

Again $|\mathbf{E}_b| = c\,|\mathbf{I}_n|$ ...See § 4·06 Prop. II, Page 125.

$\therefore$ From (i) we get $|\mathbf{E}_b| = c$ ...(iii)

Similarly $|\mathbf{E}_c| = |\mathbf{I}_n|$ ... See § 4·10 Page 131.

$\therefore$ From (i) we get $|\mathbf{E}_c| = 1$ ...(iv)

Now as the matrix $\mathbf{E}_a\,\mathbf{A}$ can be obtained by interchanging two rows of the matrix **A,** so we have from (ii)

$$|\mathbf{E}_a A| = -|\mathbf{A}| = |\mathbf{E}_a| \cdot |\mathbf{A}| \qquad \text{...(v)}$$

Similarly the product $\mathbf{E}_b\,\mathbf{A}$ can be obtained by applying second elementary row operation (as given in § 3·01 Page 103) on the matrix **A**, so we have from (iii)

$$|\mathbf{E}_b\,\mathbf{A}| = c\,|\mathbf{A}| = |\mathbf{E}_b| \cdot |\mathbf{A}| \qquad \text{...(vi)}$$

Again the product $\mathbf{E}_c\,\mathbf{A}$ can be obtained by applying third elementary row operation (as given in § 3·01 Page 103) on the matrix **A**, so we have from (iv)

$$|\mathbf{E}_c\,\mathbf{A}| = |\mathbf{A}| = |\mathbf{E}_c|\,|\mathbf{A}| \qquad \text{...(vii)}$$

Hence from (v), (vi) and (vii) we conclude that

$$|\mathbf{EA}| = |\mathbf{E}| \cdot |\mathbf{A}|,$$

where **E** is any one of the elementary matrices as defined in § 3·01 Page 103.

In a similar way we can prove that

$$|\mathbf{AE}| = |\mathbf{A}| \cdot |\mathbf{E}|.$$ Hence the theorem.

§ 4·12. Canonical Form (or Normal Form) of a matrix.

Every non-zero $m \times n$ matrix **A** can be reduced by means of elementary transformations (*i.e.* elementary row and column operations) to the form

$$\begin{bmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix},$$

where $\mathbf{I}_r$ is the $r \times r$ identity matrix and the remaining sub-matrices are zero matrices.

The above form is called the canonical form or orthogonal form or normal form of the matrix **A.**

Solved Examples on § 4·12.

Ex. 1. Reduce $\mathbf{A} = \begin{vmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{vmatrix}$ **to the canonical form.**

Sol. $\mathbf{A} \sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix}$, replacing R_2, R_3 and R_4 by $R_2 - 2R_1$, $R_3 - R_1$ and $R_4 + R_1$ respectively.

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix}$, replacing C_2, C_3 and C_4 by $C_2 - 2C_1$, $C_3 + C_1$ and $C_4 - 4C_1$ respectively.

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, replacing R_3 and R_4 by $\frac{1}{4} R_3$ and $R_4 - R_2$ respectively.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ interchanging } C_2 \text{ and } C_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ replacing } R_2 \text{ by } R_2 - 5R_2.$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ replacing } R_2 \text{ by } -\tfrac{1}{3} R_2.$$

$$\sim \begin{bmatrix} \mathbf{I}_3 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}, \text{ which is the required canonical form.}$$

Ex. 2. Reduce $\mathbf{A} = \begin{bmatrix} \mathbf{13} & \mathbf{16} & \mathbf{19} \\ \mathbf{14} & \mathbf{17} & \mathbf{20} \\ \mathbf{15} & \mathbf{18} & \mathbf{21} \end{bmatrix}$ **to the canonical form.**

Sol. $A \sim \begin{bmatrix} 13 & 16 & 19 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, replacing R_2 and R_3 by $R_2 - R_1$ and $R_3 - R_2$ respectively.

$$\sim \begin{bmatrix} 13 & 3 & 3 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \text{ replacing } C_2 \text{ and } C_3 \text{ by } C_2 - C_1 \text{ and } C_3 - C_2 \text{ respectively.}$$

$$\sim \begin{bmatrix} 13 & 3 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \text{ replacing } C_3 \text{ by } C_3 - C_2$$

$$\sim \begin{bmatrix} 13 & 3 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ replacing } R_3 \text{ by } R_3 - R_2$$

$$\sim \begin{bmatrix} 3 & 13 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ interchanging } C_1 \text{ and } C_2.$$

$$\sim \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ replacing } C_2 \text{ by } C_2 - (13/3)\, C_1.$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ replacing } R_1 \text{ by } \tfrac{1}{3} R_1.$$

$\sim\begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}$; which is the required canonical form.

Exercise on § 4·12

Ex. Reduce $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 3 & 2 & 0 & 2 \\ 0 & 1 & 3 & 2 \\ 3 & 3 & 3 & 4 \end{bmatrix}$ to the canonical form.

***§ 4·13. Definition.** *If* **A** *and* **B** *be two $m \times n$ matrices, then* $\mathbf{B} \sim \mathbf{A}$ *if and only if* $\mathbf{B} = \mathbf{SAT}$, *where* **S** *is an $m \times m$ non-singular matrix and* **T** *is an $n \times n$ singular matrix.*

With the help of § 3·03 Page 104 and § 3·11 Page 111 it can be proved.

The following two properties of the above relation are fundamental.

1. Symmetry. If $\mathbf{A} \sim \mathbf{B}$, then $\mathbf{B} \sim \mathbf{A}$ for if $\mathbf{A} = \mathbf{PBQ}$.

then $\quad \mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{Q}^{-1}$, where $\mathbf{P}^{-1}$ and $\mathbf{Q}^{-1}$ are non-singular matrices.

2. Reflexivity. Every matrix **A** is equivalent to itself since we can write $\mathbf{A} = \mathbf{IAI}$, so that $\mathbf{P} = \mathbf{I} = \mathbf{Q}$.

***§ 4·14. Theorem.** *An $n \times n$ matrix* **A** *is non-singular (or invertible) if and only if the determinant* $|\mathbf{A}| \neq 0$.

Proof : If **C** be the canonical form of the matrix **A**, then $\mathbf{C} \sim \mathbf{A}$

Therefore $$\mathbf{C} = \mathbf{S\,A\,T},$$

where **S** and **T** are non singular. (See § 4·13 above)

Hence $$\mathbf{A} = \mathbf{S}^{-1}\,\mathbf{C}\,\mathbf{T}^{-1}$$

or $$\mathbf{A} = \mathbf{E}_r \ldots \mathbf{E}_2\,\mathbf{E}_1\,\mathbf{C}\,\mathbf{D}_1\,\mathbf{D}_2 \ldots \mathbf{D}_s$$

where $\mathbf{E}_i$ and $\mathbf{D}_i$ are elementary matrices. ...See § 3·09 Page 108

By the successive application of § 4·11 Page 159, we have

$$|\mathbf{A}| = |\mathbf{E}_r| \ldots |\mathbf{E}_2|\,|\mathbf{E}_1|\,|\mathbf{C}|\,|\mathbf{D}_1|\,|\mathbf{D}_2| \ldots |\mathbf{D}_s|$$

$\therefore$ If $\quad |\mathbf{A}| = 0$, then $|\mathbf{C}| = 0$, as $|\mathbf{E}_i| \neq 0$ and $|\mathbf{D}_i| \neq 0$.

If $|\mathbf{C}| = 0$, then it has at least one row of zero.

$\therefore$ The rank of matrix **A** is less than n (see next chapter) *i.e.* the matrix **A** is singular.

If the matrix **A** is non-singular, then

$$\mathbf{C} = \mathbf{I_n}, \text{ where } \mathbf{I_n} \text{ is the } n \times n \text{ identity matrix.}$$

i.e. $$|\mathbf{C}| = |\mathbf{I_n}| = 1$$

$\therefore$ From (i) above, we have $|\mathbf{A}| \neq 0$. Hence the theorem.

***§ 4·15. Theorem.** $|\mathbf{A}_1\,\mathbf{A}_2| = |\mathbf{A}_1| \bullet |\mathbf{A}_2|$, *where* $|\mathbf{A}_1|$ *and* $|\mathbf{A}_2|$ *are two determinants.* *(Purvanchal 94)*

Proof : Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be the canonical form of the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ *i.e.* $\mathbf{A}_1 \sim \mathbf{C}_1$ and $\mathbf{A}_2 \sim \mathbf{C}_2$.

If $\mathbf{A}_1 \sim \mathbf{C}_1$ then from § 4·13 above we have

$A_1 \sim S\,C_1\,T$, where S and T are non-singular matrices.

or $A_1 = E_r \ldots E_2 E_1 C_1 D_1 D_2 \ldots D_s$,

where E_i and D_i are elementary matrices. (See § 3·09 Page 108)

Similarly $A_2 = F_t \ldots F_2 F_1 C_2 K_1 K_2 \ldots K_s$,

where F_i and K_i are elementary matrices.

$\therefore A_1 A_2 = E_r \ldots E_2 E_1 C_1 D_1 D_3 \ldots D_s F_t \ldots F_2 F_1 C_2 K_1 K_2 \ldots K_s$

Hence by § 4·11 Page 159, we have

$$|A_1 A_2| = |E_r \ldots E_2 E_1| \bullet |C_1 D_1 D_2 \ldots D_s F_t \ldots F_2 F_1 C_2| \bullet |K_1 K_2 \ldots K_s| \quad \ldots(i)$$

Now the following cases ariese.

Case I. Let A_1 be a singular matrix. (See § 2·18 Page 91)

Then C_1 has at least one row of zero.

$\therefore |C_1 D_1 D_2 \ldots D_s F_t \ldots F_2 F_1 C_2| = 0$,

since the matrix $C_1 D_1 D_2 \ldots D_s F_t \ldots F_2 F_1 C_2$ has a row of zero.

$\therefore$ From (i) above we have $|A_1 A_2| = 0$.

Case II. If A_2 is a singular matrix. Then C_2 has at least one column of zero, hence as in Case I above

$$|C_1 D_1 D_2 \ldots D_s F_t \ldots F_1 C_2| = 0$$

$\therefore$ From (i) above we have $|A_1 A_2| = 0$.

Case III. If either A_1 or A_2 is singular, then $|A| \bullet |A_2| = 0$

Hence $|A_1 A_2| = 0 = |A_1| \bullet |A_2|$.

Case IV. If A_1 and A_2 are non-singular matrices . Then C_1 and C_2 are identity matrices. Hence from § 4·11 Page 159, we have

$$|A_1 A_2| = |E_r \ldots E_2 E_1 C_1 D_1 D_2 \ldots D_s|\ |F_t \ldots F_2 F_1 C_2 K_1 \ldots K_s|$$
$$= |A_1| \bullet |A_2|$$

$\therefore$ From all the above cases it is clear that

$$|A_1 A_2| = |A_1| \bullet |A_2|.$$

Cor. $|A_1 A'_2| = |A_1| \bullet |A_2|$, where A'_2 is the transpose of A_2.

Proof : $|A'_2| = |A_2|$, $\because A'_2$ is the transpose of A_2

$\therefore\ |A_1 A'_2| = |A_1|\,|A'_2|$, from § 4·14 above.

$$= |A_1|\,|A_2|, \quad \because |A'_2| = |A_2|$$

The corolllary leads to the row by row rule of multiplication of the determinants as given in Examples below :

Solved Examples on Multiplication of Determinants.

***Ex. 1. Evaluate** $\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2$

Sol. $\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} \times \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$

$$= \begin{vmatrix} 0\cdot 0 + c\cdot c + b\cdot b & 0\cdot c + c\cdot 0 + b\cdot a & 0\cdot b + c\cdot a + b\cdot 0 \\ c\cdot 0 + 0\cdot c + a\cdot b & c\cdot c + 0\cdot 0 + a\cdot a & c\cdot b + 0\cdot a + a\cdot 0 \\ b\cdot 0 + a\cdot c + 0\cdot b & b\cdot c + a\cdot 0 + 0\cdot a & b\cdot b + a\cdot a + 0\cdot 0 \end{vmatrix}$$

$$= \begin{vmatrix} c^2 + b^2 & ba & ca \\ ab & c^2 + a^2 & bc \\ ac & bc & b^2 + a^2 \end{vmatrix}$$

Ans.

Ex. 2. Evaluate $\begin{vmatrix} 0 & \cos x & -\sin x \\ \sin x & 0 & \cos x \\ \cos x & \sin x & 0 \end{vmatrix}^2$

Sol. The required product

$$= \begin{vmatrix} 0 & \cos x & -\sin x \\ \sin x & 0 & \cos x \\ \cos x & \sin x & 0 \end{vmatrix} \times \begin{vmatrix} 0 & \cos x & -\sin x \\ \sin x & 0 & \cos x \\ \cos x & \sin x & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0\cdot 0 + \cos x\cdot\cos x + \sin x\cdot\sin x & 0\cdot\sin x + \cos x\cdot 0 - \sin x\cdot\cos x & 0\cdot\cos x + \cos x\cdot\sin x - \sin x\cdot 0 \\ \sin x\cdot 0 + 0\cdot\cos x - \cos x\cdot\sin x & \sin x\cdot\sin x + 0\cdot 0 + \cos x\cdot\cos x & \sin x\cdot\cos x + 0\cdot\sin x + \cos x\cdot 0 \\ \cos x\cdot 0 + \sin x\cdot\cos x + 0\cdot(-\sin x) & \cos x\cdot\sin x + \sin x\cdot 0 + 0\cdot\cos x & \cos x\cdot\cos x + \sin x\cdot\sin x + 0\cdot 0 \end{vmatrix}$$

$$= \begin{vmatrix} \cos^2 x + \sin^2 x & -\sin x\cos x & \cos x\sin x \\ -\cos x\sin x & \sin^2 x + \cos^2 x & \sin x\cos x \\ \sin x\cos x & \cos x\sin x & \cos^2 x + \sin^2 x \end{vmatrix}$$

$= \begin{vmatrix} 1 & -\lambda & \lambda \\ -\lambda & 1 & \lambda \\ \lambda & \lambda & 1 \end{vmatrix}$, where $\lambda = \sin x\cos x$

Ans.

Ex. 3. Evaluate $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

Sol. The required product

$$= \begin{vmatrix} a_1x_1 + b_1y_1 + c_1z_1 & a_1x_2 + b_1y_2 + c_1z_2 & a_1x_3 + b_1y_3 + c_1z_3 \\ a_2x_1 + b_2y_1 + c_2z_1 & a_2x_2 + b_2y_2 + c_2z_2 & a_2x_3 + b_2y_3 + c_2z_3 \\ a_3x_1 + b_3y_1 + c_3z_1 & a_3x_2 + b_3y_2 + c_3z_2 & a_3x_3 + b_3y_3 + c_3z_3 \end{vmatrix}$$

****Ex. 4. Prove that**

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} 2bc - a^2 & c^2 & b^2 \\ c^2 & 2ac - b^2 & a^2 \\ b^2 & a^2 & 2ab - c^2 \end{vmatrix}$$

$$= (a^3 + b^3 + c^3 - 3abc)^2$$ (*Gorakhpur 91; Kanpur 95*)

Sol. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$= -\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix}$, interchanging C_2, C_3 of the second determinant. **(Note)**

$= \begin{vmatrix} -a & b & c \\ -b & c & a \\ -c & a & b \end{vmatrix} \times \begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix}$, multiplying C_1 of first determinant by -1. **(Note)**

$$= \begin{vmatrix} -aa + bc + cb & -ab + ba + cc & -ac + bb + ca \\ -ba + cc + ab & -bb + ca + ac & -bc + cb + aa \\ -ca + ac + bb & -cb + aa + bc & -cc + ab + ba \end{vmatrix}$$

$$= \begin{vmatrix} 2bc - a^2 & c^2 & b^2 \\ c^2 & 2ac - b^2 & a^2 \\ b^2 & a^2 & 2ab - c^2 \end{vmatrix}$$ Hence proved.

(See Ex. 9 Page 167 also)

Also $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = a \begin{vmatrix} c & a \\ a & b \end{vmatrix} - b \begin{vmatrix} b & a \\ c & b \end{vmatrix} + c \begin{vmatrix} b & c \\ c & a \end{vmatrix}$

$= a(cb - a^2) - b(b^2 - ac) + c(ab - c^2)$

$= abc - a^3 - b^3 + abc + abc - c^3$

$= -(a^3 + b^3 + c^3 - 3abc)$

$\therefore \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = (a^3 + b^3 + c^3 - 3abc)^2$ Hence proved.

***Ex. 5. If $u = ax + by + cz$, $v = ay + bz + cx$, $w = az + bx + cy$,**

prove that $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix} = u^3 + v^3 + w^3 - 3uvw$

Sol. By row-by-row multiplication, the product of the given determinants

$$= \begin{vmatrix} ax + by + cz & ay + bz + cx & az + bx + cy \\ bx + cy + az & by + cz + ax & bz + cx + ay \\ cx + ay + bz & cy + az + bx & cz + ax + by \end{vmatrix}$$

$= \begin{vmatrix} u & v & w \\ w & u & v \\ v & w & u \end{vmatrix}$, since $ax + by + cz = u$ etc. (given)

$$= u \begin{vmatrix} u & v \\ w & u \end{vmatrix} - v \begin{vmatrix} w & v \\ v & u \end{vmatrix} + w \begin{vmatrix} w & u \\ v & w \end{vmatrix}$$

$= u(u^2 - vw) - v(uw - v^2) + w(w^2 - uv)$

$= u^3 + v^3 + w^3 - 3uvw.$ Hence proved.

Ex. 6. Express $\begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix}$

as the product of two determinants. (Gorakhpur 99; *Purvanchal 95*)

Sol. The given determinant.

$$= \begin{vmatrix} a^2 - 2ax + x^2 & b^2 - 2bx + x^2 & c^2 - 2cx + x^2 \\ a^2 - 2ay + y^2 & b^2 - 2by + y^2 & c^2 - 2cy + y^2 \\ a^2 - 2az + z^2 & b^2 - 2bz + z^2 & c^2 - 2cz + z^2 \end{vmatrix}$$

The element in the first row and first column is $a^2 - 2ax + x^2$, which can be written as $1(a^2) + (-2x)(a) + x^2(1)$. **(Note)**

This suggests that the first row of the required determinants are $1 - 2x, x^2$ and $a^2, a, 1$.

Hence proceeding in this way we may write the given determinant

$$= \begin{vmatrix} 1 & -2x & x^2 \\ 1 & -2y & y^2 \\ 1 & -2z & z^2 \end{vmatrix} \times \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$$

We can verify by multiplying with the help of row-by-row rule that above two determinants are the required ones.

[**Note :** Such questions are acutally done by trial and error].

Ex. 7. Express $\begin{vmatrix} (1+ax)^2 & (1+ay)^2 & (1+az)^2 \\ (1+bx)^2 & (1+by)^2 & (1+bz)^2 \\ (1+cx)^2 & (1+cy)^2 & (1+cz)^2 \end{vmatrix}$

as the product of two determinants. (*Gorakhpur 95; Purvanchal 96*)

Sol. The given determinant

$$= \begin{vmatrix} 1 + 2ax + a^2x^2 & 1 + 2ay + a^2y^2 & 1 + 2az + a^2z^2 \\ 1 + 2bx + b^2x^2 & 1 + 2by + b^2y^2 & 1 + 2bz + b^2z^2 \\ 1 + 2cx + c^2x^2 & 1 + 2cy + c^2y^2 & 1 + 2cz + c^2z^2 \end{vmatrix}$$

The element in the first row and first column is $1 + 2ax + a^2x^2$ which can be written as $(1)(1) + (2a).(x) + (a^2).(x^2)$

This suggests that the first rows of the two required determinants are $1, 2a, a^2$ and $1, x, x^2$

Hence the given determinant may be written as

$$\begin{vmatrix} 1 & 2a & a^2 \\ 1 & 2b & b^2 \\ 1 & 2c & c^2 \end{vmatrix} \times \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix},$$ **Ans.**

***Ex. 8. Express** $\begin{vmatrix} b^2+c^2 & ab & ca \\ ab & c^2+a^2 & bc \\ ca & bc & b^2+b^2 \end{vmatrix}$

as the square of a determinant.

Hence evaluate. *(Purvanchal 94)*

Sol. The element in first row and first column is $b^2 + c^2$ which can be written as $0 \cdot 0 + c \cdot c + b \cdot b$. **(Note)**

So by trial and error method, we get the given determinant

$$= \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} \times \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2$$

Now $\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$

$= -c \begin{vmatrix} c & a \\ b & 0 \end{vmatrix} + b \begin{vmatrix} c & 0 \\ b & a \end{vmatrix}$, expanding with respect to R_1.

$= -a\,[c \cdot 0 - a \cdot b] + b\,[c \cdot a - b \cdot 0] = 2abc.$

$\therefore$ The given determinat

$= \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = (2abc)^2 = 4a^2b^2c^2.$ **Ans.**

****Ex. 9. Express** $\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$

as the product of two determiants.

Sol. The elements in the first row and first column is $2bc - a^2$ which can be written as $a(-a) + b(c) + c(b)$ ...(i)

The element in the first row and second column is c^2 which can be written as $a(-b) + b(a) + c(c)$ ...(ii)

The element in the first row and third column is b^2 which can be written as $a(-c) + b(b) + c(a)$ ...(iii) (i), (ii) and (iii) suggest that the given determinant can be written tentatively as

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix}$$

But actually multiplying these two determinants we get the given determinant. Hence these determinants are the required ones.

Ex. 10. Show that

$$\begin{vmatrix} 2bc - a^2 & c^2 & b^2 \\ c^2 & 2ac - b^2 & a^2 \\ b^2 & a^2 & 2ac - b^2 \end{vmatrix} = (a^3 + b^3 + c^3 - 3abc)^2$$

Sol. As in Ex. 9. above, we can show that the given determinant

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}, \qquad \text{(Note)}$$

taking – sign common from C_1 and interchanging C_2, C_3 in 2nd determinant

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = (a^3 + b^3 + c^3 - 3abc)^2,$$

on expanding the determinant. Hence proved.

Ex. 11. Find the product of determinants of different orders

or evaluate $\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} \times \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

Sol. Here the two given determinants are of different orders, so we adopt the following method :

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} \times \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$= \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} \times \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \qquad \text{(Note)}$$

making the two determinats of the same order

$$= \begin{vmatrix} \alpha_1 a_1 + \beta_1 b_1 + \gamma_1 \cdot 0 & \alpha_1 a_2 + \beta_1 b_2 + \gamma_1 \cdot 0 & \alpha_1 \cdot 0 + \beta_1 \cdot 0 + \gamma_1 \cdot 1 \\ \alpha_2 a_1 + \beta_2 b_1 + \gamma_2 \cdot 0 & \alpha_2 a_2 + \beta_2 b_2 + \gamma_2 \cdot 0 & \alpha_2 \cdot 0 + \beta_2 \cdot 0 + \gamma_2 \cdot 1 \\ \alpha_3 a_1 + \beta_3 b_1 + \gamma_3 \cdot 0 & \alpha_3 a_2 + \beta_3 b_2 + \gamma_3 \cdot 0 & \alpha_3 \cdot 0 + \beta_3 \cdot 0 + \gamma_3 \cdot 1 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_2\alpha_1 + b_2\beta_1 & \gamma_1 \\ a_1\alpha_2 + b_1\beta_2 & a_2\alpha_2 + b_2\beta_2 & \gamma_2 \\ a_1\alpha_3 + b_1\beta_3 & a_2\alpha_3 + b_2\beta_3 & \gamma_3 \end{vmatrix}$$

Ans.

Exercises on Multiplication of Determinants

Ex. 1. Show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}^2 = \begin{vmatrix} 3 & a+b+c & a^2+b^2+c^2 \\ a+b+c & a^2+b^2+c^2 & a^3+b^3+c^3 \\ a^2+b^2+c^2 & a^3+b^3+c^3 & a^4+b^4+c^4 \end{vmatrix}.$$

Ex. 2. Show that

$$\begin{vmatrix} a^2+\lambda^2 & ab+c\lambda & ca-b\lambda \\ ab-c\lambda & b^2-\lambda^2 & bc+a\lambda \\ ac+b\lambda & bc-a\lambda & c^2+\lambda^2 \end{vmatrix} \times \begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix}$$

$= \lambda^3 (\lambda^2 + a^2 + b^2 + c^2)$

Ex. 3. If ω is one of the imaginary cube roots of unity, show that

$$\begin{vmatrix} 1 & \omega & \omega^2 & \omega^3 \\ \omega & \omega^2 & \omega^3 & 1 \\ \omega^2 & \omega^3 & 1 & \omega \\ \omega^3 & 1 & \omega & \omega^2 \end{vmatrix}^2 = \begin{vmatrix} 1 & 1 & -2 & 1 \\ 1 & 1 & 1 & -2 \\ -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \end{vmatrix} = -27$$

(Gorakhpur 96, 93)

***Ex. 4.** Prove that the determinant

$$\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$$ is a perfect square and find its value.

(Gorakhpur 92)

Ex. 5. Express the product of the following determinats as a single determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \text{ and } \begin{vmatrix} u & r \\ v & s \end{vmatrix}$$

****§ 4·16. Theorem.** *If C_i be the cofactor of a_{ij} in the $n \times n$ matrix* $\mathbf{A} = [a_{ij}]$ *then* $|C_{ij}| = \{|a_{jk}|\}^{n-1}$

[**Note.** $|C_{ij}|$ is known as reciprocal of the determinant $|a_{ij}|$],

Proof. If $\mathbf{A} = [a_{ij}]$, then $\mathbf{A}' = [a'_{ki}]$, where $\mathbf{A}'$ is the transpose of $\mathbf{A}$ and

$$a'_{ki} = a_{ik}$$

Now $\mathbf{A}'.\ [C_{ij}] = [a'_{ki}]\ [C_{ij}] = [b_{kj}]$, say ...(i)

where $$b_{kj} = \sum_{i=1}^{n} a'_{ki}\, C_{ij} = \sum_{i=1}^{n} a_{ik}\, C_{ij}, \because a'_{ki} = a_{ik}$$

Also by § 4·09 Page 129 we know that

$$b_{kj} = \sum_{i=1}^{n} a_{ik} C_{ij} = 0, \text{ if } j \neq k$$
$$= |\mathbf{A}|, \text{ if } j = k.$$

∴ From (i) we conclude that for the product $\mathbf{A}'. [C_{ij}]$ *i.e.* $[b_{kj}]$ all the diagonal terms (for which $j = k$) are $|\mathbf{A}|$, whereas the nondiagonal terms (for which $j \neq k$) are zero.

i.e. $$\mathbf{A}'. [C_{ij}] = \begin{bmatrix} |\mathbf{A}| & 0 & \ldots & 0 \\ 0 & |\mathbf{A}| & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ .. & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & |\mathbf{A}| \end{bmatrix}$$

Hence $$|\mathbf{A}'. [C_{ij}]| = \begin{vmatrix} |\mathbf{A}| & 0 & \ldots & 0 \\ 0 & |\mathbf{A}| & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ .. & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & |\mathbf{A}| \end{vmatrix}$$

or $$|\mathbf{A}'|.|C_{ij}| = \{|\mathbf{A}|\}^n, \because |\mathbf{A_1}.\mathbf{A_2}| = |\mathbf{A_1}|.|\mathbf{A_2}|$$

or $$|\mathbf{A}|.|C_{ij}| = \{|\mathbf{A}|\}^n, \because |\mathbf{A}'| = |\mathbf{A}|$$

r $$|C_{ij}| = \{|\mathbf{A}|\}^{n-1}$$

§ 4·17. Complementary Minor of a Determinant.

Definition. If **B** is $r \times r$ submatrix of an $n \times n$ matrix **A**, then the determinant **E′** of **A** formed by removing the rows and columns of **A** containing the elements of **B** is called the completementary minor of **B**.

For example : In the matrix $\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix}$

the completemtary minor of the det $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is $\begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix}$; **(Note)**

the completementary minor of $\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$ is a_4

and the complementary minor of $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ is $\begin{vmatrix} a_1 & d_1 \\ a_4 & d_4 \end{vmatrix}$ **(Note)**

§ 4·18. Laplace's Expansion of a determinant by the minors of first r columns.

If $|\mathbf{B}_i|$ is $r \times r$ minor of an $n \times n$ matrix **A** formed by the elements of the first r columns of **A** and $|\mathbf{B}'_i|$ is the complementary minor of $|\mathbf{B}_i|$, then

$$|\mathbf{A}| = \Sigma \pm |\mathbf{B_i}|.|\mathbf{B'_i}|,$$

where the summation is extended over all the possible $r \times r$ minors of **A** whcḥ can be formed from the elements of the first r columns and + or −sign taken according as an even or odd number of interchanges of adjacent rows of **A** is required to bring the submatrix $\mathbf{B_i}$ into the first r rows of **A**.

The following solved examples explain the above theorem.

Ex. 1. Expand $\begin{vmatrix} a & x & y & a \\ x & 0 & 0 & y \\ y & 0 & 0 & x \\ a & y & x & a \end{vmatrix}$ **Laplace's expansion by the minors of the first two columns. Hence evaluate it.**

Sol. All the possible minors of the first two columns and their complementary minors are given by

$$|B_1| = \begin{vmatrix} a & x \\ x & 0 \end{vmatrix}; |B'_1| = \begin{vmatrix} 0 & x \\ x & a \end{vmatrix};$$

$$|B_2| = \begin{vmatrix} a & x \\ y & 0 \end{vmatrix}; |B'_2| = \begin{vmatrix} 0 & y \\ x & a \end{vmatrix};$$

$$|B_3| = \begin{vmatrix} a & x \\ a & y \end{vmatrix}; |B'_3| = \begin{vmatrix} 0 & y \\ 0 & x \end{vmatrix};$$

$$|B_4| = \begin{vmatrix} x & 0 \\ y & 0 \end{vmatrix}; |B'_4| = \begin{vmatrix} y & a \\ x & a \end{vmatrix};$$

$$|B_5| = \begin{vmatrix} x & 0 \\ a & y \end{vmatrix}; |B'_5| = \begin{vmatrix} y & a \\ 0 & x \end{vmatrix};$$

and $$|B_6| = \begin{vmatrix} y & 0 \\ a & y \end{vmatrix}; |B'_6| = \begin{vmatrix} y & a \\ 0 & y \end{vmatrix};$$

Therefore the given determinant

$$= \begin{vmatrix} a & x \\ x & 0 \end{vmatrix} . \begin{vmatrix} 0 & x \\ x & a \end{vmatrix} - \begin{vmatrix} a & x \\ y & 0 \end{vmatrix} . \begin{vmatrix} 0 & y \\ x & a \end{vmatrix} + \begin{vmatrix} a & x \\ a & y \end{vmatrix} . \begin{vmatrix} 0 & y \\ 0 & x \end{vmatrix}$$

$$+ \begin{vmatrix} x & 0 \\ y & 0 \end{vmatrix} . \begin{vmatrix} y & a \\ x & a \end{vmatrix} - \begin{vmatrix} x & 0 \\ a & y \end{vmatrix} . \begin{vmatrix} y & a \\ 0 & x \end{vmatrix} + \begin{vmatrix} y & 0 \\ a & y \end{vmatrix} . \begin{vmatrix} y & a \\ 0 & y \end{vmatrix} \quad ...(i)$$

The submatrix $\mathbf{B_2}$ requires one interchange of rows viz. of second and third rows to bring it into the first two rows therefore −sign is put before the product $|\mathbf{B_2}||\mathbf{B'_2}|$. Again the submatrix $\mathbf{B_3}$ requires two interchanges of rows to bring fourth row to the position of second row *i.e.* to bring $\mathbf{B_3}$ into first two rows, therefore + sign is put before the product $|\mathbf{B_3}||\mathbf{B'_3}|$.

Similarly $\mathbf{B_4}$ requires two interchanges, $\mathbf{B_5}$ requires three interchanges and $\mathbf{B_6}$ requires four interchanges, hence +, − and + sign are put before $|\mathbf{B_4}|.|\mathbf{B'_4}|$, $|\mathbf{B_5}|.|\mathbf{B'_5}|$ and $|\mathbf{B_6}|.|\mathbf{B'_6}|$ respectively.

Hence from (i) we have (expanding the determinants) the given determinants.

$$= (-x^2)(-x^2) - (-xy)(-xy) + (ay - ax)(0) + (0)(ay - ax) - (xy)(xy) + (y^2)(y^2)$$

$$= x^4 - 2x^2y^2 + y^4 = (x^2 - y^2)^2$$ **Ans.**

Ex. 2. Expand $\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ 0 & 0 & j & k \\ 0 & 0 & 1 & m \end{vmatrix}$ **by Laplace's expansion by the minors of the first two columns.**

Sol. All the possible minors of the first two columns and their complementary minor are given by :

$$|\mathbf{B}| = \begin{vmatrix} a & b \\ e & f \end{vmatrix}, \; |\mathbf{B}'_1| = \begin{vmatrix} j & k \\ l & m \end{vmatrix}$$

$|\mathbf{B_2}| = \begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0$, hence $|\mathbf{B}'_2|$ need not be calculated

Similarly $|\mathbf{B_3}| = \begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0$; $|\mathbf{B_4}| = \begin{vmatrix} e & f \\ 0 & 0 \end{vmatrix} = 0$;

$|\mathbf{B_5}| = \begin{vmatrix} e & f \\ 0 & 0 \end{vmatrix} = 0$ and $|\mathbf{B_6}| = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0$ and therefore their complementary minors need not be calculated.

Then the given determinant by Laplace's Expansion method

$$= \begin{vmatrix} a & b \\ e & f \end{vmatrix} \cdot \begin{vmatrix} j & k \\ l & m \end{vmatrix}$$ **Ans.**

Ex. 3. Expand $\begin{vmatrix} 3 & 2 & 1 & 4 \\ 15 & 29 & 2 & 14 \\ 16 & 19 & 3 & 17 \\ 33 & 39 & 8 & 38 \end{vmatrix}$ **by Laplace's expansion by the minors of the first two columns.**

Sol. All the possible minors of the first two columns and their complementary minors are given by :

$$|\mathbf{B_1}| = \begin{vmatrix} 3 & 2 \\ 15 & 29 \end{vmatrix}, \; |\mathbf{B}'_1| = \begin{vmatrix} 3 & 17 \\ 8 & 38 \end{vmatrix}$$

$$|\mathbf{B_2}| = \begin{vmatrix} 3 & 2 \\ 16 & 19 \end{vmatrix}, \; |\mathbf{B}'_2| = \begin{vmatrix} 2 & 14 \\ 8 & 38 \end{vmatrix}$$

$$|\mathbf{B_3}| = \begin{vmatrix} 3 & 2 \\ 33 & 39 \end{vmatrix}, \; |\mathbf{B}'_3| = \begin{vmatrix} 2 & 14 \\ 3 & 17 \end{vmatrix}$$

$$|\mathbf{B_4}| = \begin{vmatrix} 15 & 29 \\ 16 & 19 \end{vmatrix}, \; |\mathbf{B}'_4| = \begin{vmatrix} 1 & 4 \\ 8 & 38 \end{vmatrix}$$

$$|\mathbf{B_5}| = \begin{vmatrix} 15 & 29 \\ 33 & 39 \end{vmatrix}, \ |\mathbf{B'_5}| = \begin{vmatrix} 1 & 4 \\ 3 & 17 \end{vmatrix}$$

$$|\mathbf{B_6}| = \begin{vmatrix} 16 & 19 \\ 33 & 39 \end{vmatrix}, \ |\mathbf{B'_6}| = \begin{vmatrix} 1 & 4 \\ 2 & 14 \end{vmatrix}$$

∴ The given determinat

$$= \begin{vmatrix} 3 & 2 \\ 15 & 29 \end{vmatrix} \bullet \begin{vmatrix} 3 & 17 \\ 8 & 38 \end{vmatrix} - \begin{vmatrix} 3 & 2 \\ 16 & 19 \end{vmatrix} \bullet \begin{vmatrix} 2 & 14 \\ 8 & 38 \end{vmatrix}$$

$$+ \begin{vmatrix} 3 & 2 \\ 33 & 39 \end{vmatrix} \bullet \begin{vmatrix} 2 & 14 \\ 3 & 17 \end{vmatrix} + \begin{vmatrix} 15 & 29 \\ 16 & 19 \end{vmatrix} \bullet \begin{vmatrix} 1 & 4 \\ 8 & 38 \end{vmatrix}$$

$$- \begin{vmatrix} 15 & 29 \\ 33 & 39 \end{vmatrix} \bullet \begin{vmatrix} 1 & 14 \\ 3 & 17 \end{vmatrix} + \begin{vmatrix} 16 & 19 \\ 33 & 39 \end{vmatrix} \bullet \begin{vmatrix} 1 & 4 \\ 2 & 14 \end{vmatrix}$$

Ex. 4. Expand $\begin{vmatrix} a & 1 & 0 & 0 & 0 \\ b & a & 1 & 0 & 0 \\ 0 & b & a & 1 & 0 \\ 0 & 0 & b & a & 1 \\ 0 & 0 & 0 & b & a \end{vmatrix}$ **by Laplace's expansion by the minors of the first two columns. Hence evaluate it.**

Sol. All the possible minors of the first two columns and their complementary minors are given by :

$$|\mathbf{B_1}| = \begin{vmatrix} a & 1 \\ b & a \end{vmatrix}, \ |\mathbf{B'_1}| = \begin{vmatrix} a & 1 & 0 \\ b & a & 1 \\ 0 & b & a \end{vmatrix}$$

$$|\mathbf{B_2}| = \begin{vmatrix} a & 1 \\ 0 & b \end{vmatrix}, \ |\mathbf{B'_2}| = \begin{vmatrix} 1 & 0 & 0 \\ b & a & 1 \\ 0 & b & a \end{vmatrix}$$

$$|\mathbf{B_3}| = \begin{vmatrix} b & a \\ 0 & b \end{vmatrix}, \ |\mathbf{B'_3}| = \begin{vmatrix} 0 & 0 & 0 \\ b & a & 1 \\ 0 & b & a \end{vmatrix} = 0$$

All other minors of the first two columns are equal to zero as they have at least one row of zero.

Hence the given determinant

$$= \begin{vmatrix} a & 1 \\ b & a \end{vmatrix} \bullet \begin{vmatrix} a & 1 & 0 \\ b & a & 1 \\ 0 & b & a \end{vmatrix} - \begin{vmatrix} a & 1 \\ 1 & b \end{vmatrix} \bullet \begin{vmatrix} 1 & 0 & 0 \\ b & a & 1 \\ 0 & b & a \end{vmatrix} \qquad \text{...(i)}$$

(Note)

Now $\begin{vmatrix} a & 1 & 0 \\ b & a & 1 \\ 0 & b & a \end{vmatrix} = \begin{vmatrix} a & 1 \\ b & a \end{vmatrix} a - \begin{vmatrix} a & 1 \\ 0 & b \end{vmatrix} \bullet 1 + \begin{vmatrix} b & a \\ 0 & b \end{vmatrix} \cdot 0,$

expanding by the minors of first two columns.

$$= (a^2 - b)\, a - (ab) = a^3 - 2ab.$$

∵ From (i), the given determinant

$= \begin{vmatrix} a & 1 \\ b & a \end{vmatrix} \cdot (a^3 - 2ab) - \begin{vmatrix} a & 1 \\ 0 & b \end{vmatrix} \cdot \begin{vmatrix} a & 1 \\ b & a \end{vmatrix}$,

expanding the last determinant with respect to R_1.

$= (a^2 - b)(a^3 - 2ab) - (ab)(a^2 - b) = (a^2 - b)[a^3 - 3ab]$

$= a(a^2 - b)(a^2 - 3b)$. **Ans.**

Exercises on § 4·18

Ex. 1. Expand $\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$ by Laplace's method of expansion by the minors of the first two columns.

Ex. 2. Use Laplace's method of expansion of a determinant by means of its second minors to expand

$$\begin{vmatrix} -1 & 0 & 0 & l \\ 0 & -1 & 0 & m \\ 0 & 0 & -1 & n \\ p & q & r & -1 \end{vmatrix}$$

**§ 4·19. Solution of Linear Equations. (Cramer's Rule)

Let the n simultaneous equations in n unknown quantities $x_1, x_2, \ldots, x_n$ be

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots + a_{1n}x_n = k_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2j}x_j + \ldots + a_{2n}x_n = k_2$$
$$a_{31}x_1 + a_{32}x_2 + \ldots + a_{3j}x_j + \ldots + a_{3n}x_n = k_3$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n = k_i$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nj}x_j + \ldots + a_{nn}x_n = k_n$$

These equations can be written as

$$\sum_{j=1}^{n} a_{ij}\, x_j = k_i, \quad i = 1, 2, \ldots, n \qquad \ldots(i)$$

Let the determinant of the coefficients, $|\mathbf{A}| = |a_{ij}| \neq 0$.

Multiplying (i) by the cofactor of a_{ij} in $|a_{ij}|$ viz. C_{ij}, $i = 1, 2, 3, \ldots, n$ and summing with respect to i, we have

$$\sum_{j=1}^{n} a_{ij}\, C_{ij}\, x_j = \sum_{j=1}^{n} k_i\, C_{ij}$$

or $\quad |\mathbf{A}|.\, x_j = \sum_{j=1}^{n} k_i\, C_{ij}, \because |\mathbf{A}| = \sum_{j=1}^{n} a_{ij}\, C_{ij}$

= determinant formed by replacing jth column of the det. $|\mathbf{A}|$ by the constants $k_1, k_2, \ldots, k_n$. (Note)

$$= \begin{vmatrix} a_{11} & a_{12} \ldots a_{1j-1} & k_1 & a_{1j+1} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2j-1} & k_2 & a_{2j+1} \ldots a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{i1} & a_{i2} \ldots a_{ij-1} & k_i & a_{ij+1} \ldots a_{in} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} \ldots a_{nj-1} & k_n & a_{nj+1} \ldots a_{nn} \end{vmatrix}$$

or $$x_j = \frac{1}{|\mathbf{A}|} \cdot \begin{vmatrix} a_{11} & a_{12} \ldots a_{1j-1} & k_1 & a_{1j+1} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2j-1} & k_2 & a_{2j+1} \ldots a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} \ldots a_{nj-1} & k_2 & a_{nj+1} \ldots a_{nn} \end{vmatrix}$$

Solved Examples on § 4·19.

Ex. 1 (a). Solve the following equations by Cramer's Rule.

$$\mathbf{x + y + z = 1,\ ax + by + cz = k,\ a^2x + b^2y + c^2z = k^2.}$$

Sol. The given equations are $x + y + z = 1$

$$ax + by + cz = k$$

$$a^2x + b^2y + c^2z = k^2$$

$\therefore$ By Cramer's Rule we have

$$\frac{x}{\begin{vmatrix} 1 & 1 & 1 \\ k & b & c \\ k^2 & b^2 & c^2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 1 & 1 & 1 \\ a & k & c \\ a^2 & k^2 & c^2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 1 & 1 \\ a & b & k \\ a^2 & b^2 & k^2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}}$$

or $$\frac{x}{(k-b)(b-c)(c-k)} = \frac{y}{(a-k)(k-c)(c-a)} = \frac{z}{(a-b))(b-k)(k-a)}$$

$$= \frac{1}{(a-b)(b-c)(c-a)}$$ ...See Ex. 28 Page 145

$\therefore$ $$x = \frac{(k-b)(b-c)(c-k)}{(a-b)(b-c)(c-a)} = \frac{(k-b)(c-k)}{(b-c)(c-a)}$$

Similarly $y = \dfrac{(a-k)(k-c)}{(a-b)(b-c)}$ and $z = \dfrac{(b-k)(k-a)}{(b-c)(c-a)}$ **Ans.**

Ex. 1. (b) Solve the following equations by the method of determinants —

$$\mathbf{a_1x + b_1y + c_1z + d_1 = 0}$$

$$\mathbf{a_2x + b_2y + c_2z + d_2 = 0}$$

$$\mathbf{a_3x + b_3y + c_3z + d_3 = 0}$$

Sol. Do exacity as Ex. 1 (a) above

Ex. 2. Solve the equations :

$$\mathbf{x + y + z = 7;\ x + 2y + 3z = 16;\ x + 3y + 4z = 22}$$

Sol. Solving the equations by Cramer's Rule we get

$$\frac{x}{\begin{vmatrix} 7 & 1 & 1 \\ 16 & 2 & 3 \\ 22 & 3 & 4 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 1 & 7 & 1 \\ 1 & 16 & 3 \\ 1 & 22 & 4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 1 & 7 \\ 1 & 2 & 16 \\ 1 & 3 & 22 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{vmatrix}}. \quad \text{...(i)}$$

Now $\begin{vmatrix} 7 & 1 & 1 \\ 16 & 2 & 3 \\ 22 & 3 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & 7 & 1 \\ 2 & 16 & 3 \\ 3 & 22 & 4 \end{vmatrix}$, interchanging C_1 and C_2.

$= -\begin{vmatrix} 1 & 7 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix}$, replacing R_2, R_3 by $R_2 - 2R_1$ and $R_3 - 3R_1$ respectively.

$= -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$, expanding w.r to C_1

$= -(2-1) = -1$;

$\begin{vmatrix} 1 & 7 & 1 \\ 1 & 16 & 3 \\ 1 & 22 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 7 & 1 \\ 0 & 9 & 2 \\ 0 & 15 & 3 \end{vmatrix}$, replacing R_2, R_3 by $R_2 - R_1, R_3 - R_1$ respectively.

$= \begin{vmatrix} 9 & 2 \\ 15 & 3 \end{vmatrix}$, expanding w.r. to C_1.

$= 27 - 30 = -3$;

$\begin{vmatrix} 1 & 1 & 7 \\ 1 & 2 & 16 \\ 1 & 3 & 22 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 7 \\ 0 & 1 & 9 \\ 0 & 2 & 15 \end{vmatrix}$, replacing R_2, R_3 by $R_2 - R_1, R_3 - R_1$ respectively.

$= \begin{vmatrix} 1 & 9 \\ 2 & 15 \end{vmatrix}$

$= 15 - 18 = -3$

and $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix}$, replacing R_2, R_3 by $R_2 - R_1, R_3 - R_2$ respectively.

$= \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}$, expanding w.r. to C_1.

$= 1 - 2 = -1$.

$\therefore$ From (i) we have $\dfrac{x}{-1} = \dfrac{y}{-3} = \dfrac{z}{-3} = \dfrac{1}{-1}$

which gives $x = 1, y = 3, z = 3$. **Ans.**

***Ex. 3. Solve the equations (with the help of determinants)**

$$\mathbf{x + y + z = 1;\ x + 2y + 3z = 2;\ x + 4y + 9z = 4.}$$

Sol. The given equations are $x + y + z = 1$.

$$x + 2y + 3z = 2$$

$$x + 4y + 9z = 4$$

$\therefore$ By Cramer's Rule we have

$$\frac{x}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 4 & 4 & 3 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 4 & 4 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}}$$

or $\quad \frac{x}{0} = \frac{y}{D} = \frac{z}{0} = \frac{1}{D}$, where $D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$

This gives $x = 0.\left(\frac{1}{D}\right) = 0;\ y = D.\left(\frac{1}{D}\right) = 1,\ z = 0.\left(\frac{1}{D}\right) = 0.$

i.e. $\quad x = 0,\ y = 1,\ z = 0,$ **Ans.**

***Ex. 4. Solve the equations by determinants $3x + 5y - 7z = 13$, $4x + y - 12z = 6$, $2x + 9y - 3z = 20$** (*Purvanchal 97*)

Sol. The given equation are

$$3x + 5y - 7z = 13$$
$$4x + y - 12z = 6$$
$$2x + 9y - 3z = 20$$

$\therefore$ By Cramer's Rule, we have

$$\frac{x}{\begin{vmatrix} 13 & 5 & -7 \\ 6 & 1 & -12 \\ 20 & 9 & -3 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 3 & 13 & -7 \\ 4 & 6 & -12 \\ 2 & 20 & -3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 3 & 5 & 13 \\ 4 & 1 & 6 \\ 2 & 9 & 20 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 3 & 5 & -7 \\ 4 & 1 & -12 \\ 2 & 9 & -3 \end{vmatrix}} \quad \text{...(i)}$$

Now $\begin{vmatrix} 13 & 5 & -7 \\ 6 & 1 & -12 \\ 20 & 9 & -3 \end{vmatrix} = \begin{vmatrix} -17 & 0 & 53 \\ 6 & 1 & -12 \\ -34 & 0 & 105 \end{vmatrix}$, adding $-5R_2$ to R_1 and $-9R_2$ to R_3

$= \begin{vmatrix} -17 & 53 \\ -34 & 105 \end{vmatrix}$, expanding w.r. to C_2

$= \begin{vmatrix} -17 & 53 \\ 0 & -1 \end{vmatrix}$, adding $-2R_1$ to R_2

$= (-17)(-1) - (0)(53) = 17;$

$\begin{vmatrix} 3 & 13 & -7 \\ 4 & 6 & -12 \\ 2 & 20 & -3 \end{vmatrix} = \begin{vmatrix} 1 & -7 & -4 \\ 0 & -34 & -6 \\ 2 & 20 & -3 \end{vmatrix}$, adding $-2R_3$ to R_2 and $-R_3$ to R_1

$= \begin{vmatrix} 1 & -7 & -4 \\ 0 & -34 & -6 \\ 0 & 34 & 5 \end{vmatrix}$, adding $-2R_1$ to R_3

$= \begin{vmatrix} -34 & -6 \\ 34 & 5 \end{vmatrix}$, expanding w.r. to C_1

$$= \begin{vmatrix} 0 & -1 \\ 34 & 5 \end{vmatrix}, \text{ adding } R_2 \text{ to } R_1$$

$$= 0\cdot 5 - (-1)\cdot 34 = 34;$$

$$\begin{vmatrix} 3 & 5 & 13 \\ 4 & 1 & 6 \\ 2 & 9 & 20 \end{vmatrix} = \begin{vmatrix} -17 & 0 & -17 \\ 4 & 1 & 6 \\ -34 & 0 & -34 \end{vmatrix}, \text{ adding } -5R_2 \text{ to } R_1 \text{ and } -9R_2 \text{ to } R_3$$

$$= \begin{vmatrix} -17 & -17 \\ -34 & -34 \end{vmatrix}, \text{ expanding w.r. to } C_2$$

$$= 0$$

And
$$\begin{vmatrix} 3 & 5 & -7 \\ 4 & 1 & -12 \\ 2 & 9 & -3 \end{vmatrix} = \begin{vmatrix} -17 & 0 & 53 \\ 4 & 1 & -12 \\ -34 & 0 & 105 \end{vmatrix}, \text{ adding } -5R_2 \text{ to } R_1 \text{ and } -9R_2 \text{ to } R_3$$

$$= \begin{vmatrix} -17 & 53 \\ -34 & 105 \end{vmatrix}, \text{ expanding w.r. to } C_2$$

$$= \begin{vmatrix} -17 & 53 \\ 0 & -1 \end{vmatrix}, \text{ adding } -2R_1 \text{ to } R_2$$

$$= 17$$

$\therefore$ From (i), we get $\frac{x}{17} = \frac{y}{34} = \frac{z}{0} = \frac{1}{17}$

which gives $x = 1, y = 2, z = 0$ **Ans.**

Ex. 5. Solve the eqations x + y + z = 3, x + 2y + 3z = 4, x + 4y + 9z = 6. (*Purvanchal 94*)

Sol. Given equation are

$$x + y + z = 3$$
$$x + 2y + 3z = 4$$
$$x + 4y + 9z = 6$$

$\therefore$ By Cramer's Rule, we get

$$\frac{x}{\begin{vmatrix} 3 & 1 & 1 \\ 4 & 2 & 3 \\ 6 & 4 & 9 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 1 & 3 & 1 \\ 1 & 4 & 3 \\ 1 & 6 & 9 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 4 & 6 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}}. \quad \text{...(i)}$$

Now
$$\begin{vmatrix} 3 & 1 & 1 \\ 4 & 2 & 3 \\ 6 & 4 & 9 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ -2 & 2 & 1 \\ -6 & 4 & 5 \end{vmatrix}, \text{ replacing } C_1, C_3, \text{ by } C_1 - 3C_2, C_3 - C_1 \text{ respectively.}$$

$$= -\begin{vmatrix} -2 & 1 \\ -6 & 5 \end{vmatrix} = -[-10 + 6] = 4, \quad \text{...(ii)}$$

$$\begin{vmatrix} 1 & 3 & 1 \\ 1 & 4 & 3 \\ 1 & 6 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{vmatrix}, \text{ replacing } R_2, R_3 \text{ by } R_2 - R_1, R_3 - R_1 \text{ respectively.}$$

$$= \begin{vmatrix} 1 & 2 \\ 3 & 8 \end{vmatrix} = 8 - 6 = 2; \qquad \text{...(iii)}$$

$$\begin{vmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 4 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \end{vmatrix}, \text{ replacing } R_2, R_3 \text{ by } R_2 - R_1, R_3 - R_1 \text{ respectively.}$$

$$= \begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} = 0 \qquad \text{...(iv)}$$

And

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 8 \end{vmatrix}, \text{ replacing } C_2, C_3 \text{ by } C_2 - C_1, C_3 - C_1 \text{ respectively.}$$

$$= \begin{vmatrix} 1 & 2 \\ 3 & 8 \end{vmatrix} = 8 - 6 = 2 \qquad \text{...(v)}$$

$\therefore$ From (i), (ii), (iii), (iv) and (v) we have

$$\frac{x}{4} = \frac{y}{2} = \frac{z}{0} = \frac{1}{2} \quad \text{or} \quad x = 2, y = 1, z = 0 \qquad \textbf{Ans.}$$

Ex. 6 (a). Using determinants, solve the simultaneous equations :
$x + 2y + 3z = 6$; $2x + 4y + z = 7$, $3x + 2y + 9z = 14$. (*Purvanchal 90*)

Sol. By Cramer's Rule, we get

$$\frac{x}{\begin{vmatrix} 6 & 2 & 3 \\ 7 & 4 & 1 \\ 14 & 2 & 9 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & 6 & 3 \\ 2 & 7 & 1 \\ 3 & 14 & 9 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 2 & 6 \\ 2 & 4 & 7 \\ 3 & 2 & 14 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{vmatrix}} \qquad \text{...(i)}$$

Now

$$\begin{vmatrix} 6 & 2 & 3 \\ 7 & 4 & 1 \\ 14 & 2 & 9 \end{vmatrix} = -\begin{vmatrix} 2 & 6 & 3 \\ 4 & 7 & 1 \\ 2 & 14 & 9 \end{vmatrix}, \text{ interchanging } C_1 \text{ and } C_2$$

$$= -\begin{vmatrix} 2 & 6 & 3 \\ 0 & -5 & -5 \\ 0 & 8 & 6 \end{vmatrix}, \text{ replacing } R_2, R_3 \text{ by } R_2 - 2R_1 \text{ and } R_3 - R_1 \text{ respectively.}$$

$$= -2 \begin{vmatrix} -5 & -5 \\ 8 & 6 \end{vmatrix}, \text{ expanding with respect to } C_1$$

$$= -2[-30 + 40] = -20;$$

$$\begin{vmatrix} 1 & 6 & 3 \\ 2 & 7 & 1 \\ 3 & 14 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 6 & 3 \\ 0 & -5 & -5 \\ 0 & -4 & 0 \end{vmatrix}, \text{ replacing } R_2, R_3 \text{ by } R_2 - 2R_1, R_3 - 3R_1$$

$$= \begin{vmatrix} -5 & -5 \\ -4 & 0 \end{vmatrix}, \text{ expanding with respect to } C_1$$

$$= -20;$$

$$\begin{vmatrix} 1 & 2 & 6 \\ 2 & 4 & 7 \\ 3 & 2 & 14 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 6 \\ 0 & 0 & -5 \\ 0 & -4 & -4 \end{vmatrix}, \text{ replacing } R_2, R_3 \text{ by } R_2 - 2R_1, R_3 - 3R_1.$$

$= \begin{vmatrix} 0 & -5 \\ -4 & -4 \end{vmatrix}$, expanding with respect to C_1

$= -20$

and $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 0 & -5 \\ 3 & -4 & 0 \end{vmatrix}$, replacing C_2, C_3 by $C_2 - 2C_1$, $C_3 - 3C_1$ respectively.

$= \begin{vmatrix} 0 & -5 \\ -4 & 0 \end{vmatrix}$, expanding with respect to R_1

$= -20$

$\therefore$ From (i) we get $\dfrac{x}{-20} = \dfrac{y}{-20} = \dfrac{z}{-20} = \dfrac{1}{-20}$

or $x = 1, y = 1, z = 1.$ **Ans.**

Ex. 6 (b). Solve the following equation with the help of determinants $2x + y + z = 1$, $x - 2y - 3z = 1$, $3x + 2y - z = 5$. *(Purvanchal 96)*

Sol. By Carmer's Rule, we get

$$\frac{x}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & -3 \\ 5 & 2 & -1 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & -3 \\ 3 & 5 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & 1 \\ 3 & 2 & 5 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & -3 \\ 3 & 2 & -1 \end{vmatrix}} \quad \ldots(i)$$

Now $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & -3 \\ 5 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -3 & -4 \\ 5 & -3 & -6 \end{vmatrix}$, replacing C_2, C_3, by $C_2 - C_1$ and $C_3 - C_1$ respectively

$= \begin{vmatrix} -3 & -4 \\ -3 & -6 \end{vmatrix}$, expanding w.r. to R_1

$= (-3)(-6) - (-3)(-4) = 18 - 12 = 6;$

$\begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & -3 \\ 3 & 5 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 \\ 7 & 4 & 0 \\ 5 & 6 & 0 \end{vmatrix}$, replacing R_2, R_3 by $R_2 + 3R_1$ and $R_3 + R_1$ respectively

$= \begin{vmatrix} 7 & 4 \\ 5 & 6 \end{vmatrix}$, expanding w.r. to C_3

$= (7)(6) - (5)(4) = 42 = 20 = 22;$

$\begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & 1 \\ 3 & 2 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 \\ -1 & -3 & 0 \\ -7 & -3 & 0 \end{vmatrix}$, replacing R_2, R_3 by $R_2 - R_1$, $R_3 - 5R_1$ respectively.

$= \begin{vmatrix} -1 & -3 \\ -7 & -3 \end{vmatrix}$, expanding w.r. to C_3

$= (-1)(-3) - (-7)(-3) = 3 - 21 = -18$

Also $\begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & -3 \\ 3 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 \\ 7 & 1 & 0 \\ 5 & 3 & 0 \end{vmatrix}$, replacing R_2, R_3 by $R_2 + 3R_1$ and $R_3 + R_1$ respectively

$$= \begin{vmatrix} 7 & 1 \\ 5 & 3 \end{vmatrix}, \text{ expanding w.r. to } C_3$$

$$= 7 \cdot 3 - 5 \cdot 1 = 21 - 5 = 16$$

$\therefore$ From (i), we have $\frac{x}{6} = \frac{y}{22} = \frac{z}{-18} = \frac{1}{16}$

which gives $x = \frac{6}{16} = \frac{3}{8}$, $y = \frac{22}{16} = \frac{11}{8}$, $z = -\frac{18}{16} = -\frac{9}{8}$ **Ans.**

Ex. 7. Solve the equations :

$x + y + z + u = 1$, $ax + by + cz + du = k$,

$a^2x + b^2y + c^2z + d^2u = k^2$ and $a^3x + b^3y + c^3z + d^3u = k^3$

Sol. The given equations are

$$x + y + z + u = 1$$
$$ax + by + cz + du = k,$$
$$a^2x + b^2y + c^2z + d^2u = k^2,$$

and $$a^3x + b^3y + c^3z + d^3u = k^3$$

Solving these by Cramer's Rule, we get

$$\frac{x}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ k & b & c & d \\ k^2 & b^2 & c^2 & d^2 \\ k^3 & b^3 & c^3 & d^3 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & k & c & d \\ a^2 & k^2 & c^2 & d^2 \\ a^3 & k^3 & c^3 & d^3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & k & d \\ a^2 & b^2 & k^2 & d^2 \\ a^3 & b^3 & k^3 & d^3 \end{vmatrix}}$$

$$= \frac{u}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & k \\ a^2 & b^2 & c^2 & k^2 \\ a^3 & b^3 & c^3 & k^3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix}} \quad \text{...(i)}$$

Now $$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ a & b-a & c-a & d-a \\ a^2 & b^2-a^2 & c^2-a^2 & d^2-a^2 \\ a^3 & b^3-a^3 & c^3-a^3 & d^3-a^3 \end{vmatrix}$$

replacing C_2, C_3 and C_4 by $C_2 - C_1$, $C_3 - C_1$ and $C_4 - C_1$ respectively.

$$= \begin{vmatrix} b-a & c-a & d-a \\ b^2-a^2 & c^2-a^2 & d^2-a^2 \\ b^3-a^3 & c^3-a^3 & d^3-a^3 \end{vmatrix}, \text{ expanding with respect to } R_1$$

$$= (b-a)(c-a)(d-a) \begin{vmatrix} 1 & 1 & 1 \\ b+a & c+a & d+a \\ b^2+ab+a^2 & c^2+ac+a^2 & d^2+ad+a^2 \end{vmatrix}$$

$$= (b-a)(c-a)(d-a)\begin{vmatrix} 1 & 0 & 0 \\ b+a & c-b & d-b \\ b^2+ab+a^2 & c^2+ac-b^2-ab & d^2+ad-b^2-ab \end{vmatrix},$$

replacing C_2 and C_3 by $C_2 - C_1$ and $C_3 - C_1$ respectively.

$$= (b-a)(c-a)(d-a)\begin{vmatrix} c-b & d-b \\ (c-b)(a+b+c) & (d-b)(a+b+d) \end{vmatrix},$$

expanding with respect to R_1,

$$= (a-b)(a-c)(d-a)(c-b)(d-b)\begin{vmatrix} 1 & 1 \\ a+b+c & a+b+d \end{vmatrix}$$

$$= (a-b)(a-c)(a-d)(b-c)(d-b)[(a+b+d)-(a+b+c)]$$

$$= (a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$$

Similarly we can have $\begin{vmatrix} 1 & 1 & 1 & 1 \\ k & b & c & d \\ k^2 & b^2 & c^2 & d^2 \\ k^3 & b^3 & c^3 & d^3 \end{vmatrix} = (k-b)(k-c)(k-d)(b-c)(b-d)(c-d)$

$\therefore$ From (i), $x = \dfrac{(k-b)(k-c)(k-d)(b-c)(b-d)(c-d)}{(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)}$.

or $$x = \frac{(k-b)(k-c)(k-d)}{(a-b)(a-c)(a-d)}$$ **Ans.**

Similarly from (i) we can find the values of y, z and u.

Exercises on § 4·19

Solve the equations by Cramer's Rule :—

Ex. 1. $x-2y+z=-1,\ 3x+y-2z=4,\ y-z=1.$ **Ans.** $x=1,\ y=1,\ z=0,$

Ex. 2. $2x+3y-4z=2,\ 3x-2y+5z=5,\ x+2y+3z=11.$

Ans. $x=\frac{10}{19},\ y=\frac{50}{19},\ z=\frac{33}{19}.$

Ex. 3. $x+2y-z=3;\ 3x-y+z=8,\ x+y+z=0.$

Ans. $x=\frac{18}{7},\ y=-\frac{5}{7},\ z=-\frac{13}{7}.$

Ex. 4. $x+y+z=3;\ 2x+3y+4z=9,\ x+2y-4z=-1.$

Ans. $x=1,\ y=1,\ z=1$

Ex. 5. $2x-y+3z=9,\ x+y+z=6,\ x-y+z=2.$

Ans. $x=1,\ y=2,\ z=3.$

Ex. 6. $3x+y+2z=3,\ 2x-3y-z=-3,\ x+2y+z=4.$

Ans. $x=1,\ y=2,\ z=-1$

Ex. 7. $x_1+2x_2+3x_3+5=0,\ 2x_1+x_2+x_3+7=0.\ x_1+x_2+x_3=0.$

Ans. $x_1=-7,\ x_2=19,\ x_3=-12.$

Ex. 8. $6x+y+2z=7,\ 3x-y+4z=14,\ 5x+2y-3z=-7.$

(*Purvanchal 91*)

Ans. $x=1, y=-3, z=2$

*Ex. 9. $x+y+z=9,\ 2x+5y+7z=52,\ 2x+y-z=0$

Ans. $x=1,\ y=3,\ z=5$.

§ 4·20. Derivative of a determinant.

If some elements of the $n \times n$ matrix $\mathbf{A}=[a_{ij}]$ are differentiable functions of a variable x, then the derivative of $|\mathbf{A}|$ with respect to x *i.e.* $\frac{d}{dx}|\mathbf{A}|$ is the sum of n determinants formed by replacing in all possible ways the elements of one row (or column) of the det. $|\mathbf{A}|$ by their differential coefficients with respect to x.

The above procedure will be illustrated by the following examples —

Ex. 1. Find the derivative of the det. $\begin{vmatrix} x^3 & 2x+3 \\ 3x^2 & x^4 \end{vmatrix}$.

Sol. $\frac{d}{dx}\begin{vmatrix} x^3 & 2x+3 \\ 3x^2 & x^4 \end{vmatrix} = \begin{vmatrix} 3x^2 & 2 \\ 3x^2 & x^4 \end{vmatrix} + \begin{vmatrix} x^3 & 2x+3 \\ 6x & 4x^3 \end{vmatrix}$,

differentiating the elements of R_1 in the first det. whereas differentiating the elements of R_2 in the second det.

$$= [3x^6 - 6x^2] + [4x^6 - 6x(2x+3)]$$

$$= 3x^6 - 6x^2 + 4x^6 - 12x^2 - 18x = 7x^6 - 18x^2 - 18x.$$ **Ans.**

***Ex. 2. Find the derivative of** $\begin{vmatrix} x^2 & x^3 & 2 \\ 2x & 3x+1 & x^3 \\ 0 & 3x-2 & x^2+1 \end{vmatrix}$

Sol. The derivative of the given determinant

$$= \begin{vmatrix} 2x & 3x^2 & 0 \\ 2x & 3x+1 & x^3 \\ 0 & 3x-2 & x^2+1 \end{vmatrix} + \begin{vmatrix} x^2 & x^3 & 2 \\ 2 & 3 & 3x^2 \\ 0 & 3x-2 & x^2+1 \end{vmatrix} + \begin{vmatrix} x^2 & x^3 & 2 \\ 2x & 3x+1 & x^3 \\ 0 & 3 & 2x \end{vmatrix}$$

$$= \begin{vmatrix} 2x & 3x^2 & 0 \\ 0 & 3x+1-3x^2 & x^3 \\ 0 & 3x-2 & x^2+1 \end{vmatrix} + \begin{vmatrix} x^2 & x^3 & 2 \\ 2 & 5-3x & 2x^2-1 \\ 0 & 3x-2 & x^2+1 \end{vmatrix}$$

$$+ \begin{vmatrix} x^2 & x^3 & 2 \\ 2x & 3x+1 & x^3 \\ 0 & 3 & 2x \end{vmatrix}$$, replacing R_2 of 1st det. by $R_2 - R_1$ and R_2 of 2nd det. by $R_2 - R_3$

$$= 2x\begin{vmatrix} 3x+1-3x^2 & x^3 \\ 3x-2 & x^2+1 \end{vmatrix} + x^2\begin{vmatrix} 5-3x & 2x^2-1 \\ 3x-2 & x^2+1 \end{vmatrix}$$

$$-2\begin{vmatrix} x^3 & 2 \\ 3x-2 & x^2+1 \end{vmatrix}$$

$$+x^2\begin{vmatrix}3x+1 & x^3\\ 3 & 2x\end{vmatrix}-2x\begin{vmatrix}x^3 & 2\\ 3 & 2x\end{vmatrix}$$, expanding det. w.r.t. to C_1

$= 2x\,[(x^2+1)(3x+1-3x^2)-x^3(3x-2)]+x^2\,[(5-3x)(x^2+1)$
$-(3x-2)(2x^2-1)]-2\,[x^3(x^2+1)-2(3x-2)]$
$+x^2\,[2x(3x+1)-3x^3]-2x\,[2x^4-6],$

$= -30x^5+25x^4-4x^3+9x^2+26x-8$, on similifying. **Ans.**

Exercise on § 4·20

Ex. Find the derivative of $\begin{vmatrix}x^2-1 & x-1 & 1\\ x^4 & x^3 & 2x+5\\ x+1 & x^2 & x\end{vmatrix}$

Ans. $6x^5-5x^4-28x^3+9x^2+20x-2$

MISCELLANEOUS SOLVED EXAMPLES

***Ex. 1. Solve** $\begin{vmatrix}1+x & 2 & 3 & 4\\ 1 & 2+x & 3 & 4\\ 1 & 2 & 3+x & 4\\ 1 & 2 & 3 & 4+x\end{vmatrix}$

Sol. The given determinant

$$=\begin{vmatrix}x+10 & 2 & 3 & 4\\ x+10 & 2+x & 3 & 4\\ x+10 & 2 & 3+x & 4\\ x+10 & 2 & 3 & 4+x\end{vmatrix}$$, replacing C_1 by $C_1+C_2+C_3+C_4$

$$=(x+10)\begin{vmatrix}1 & 2 & 3 & 4\\ 1 & 2+x & 3 & 4\\ 1 & 2 & 3+x & 4\\ 1 & 2 & 3 & 4+x\end{vmatrix}$$, taking out $(x+10)$ common from C_1

$$=(x+10)\begin{vmatrix}1 & 2 & 3 & 4\\ 0 & x & 0 & 0\\ 0 & 0 & x & 0\\ 0 & 0 & 0 & x\end{vmatrix}$$, replacing R_2, R_3 and R_4 by R_2-R_1, R_3-R_1 and R_4-R_1 respectively.

$$=(x+10)\begin{vmatrix}x & 0 & 0\\ 0 & x & 0\\ 0 & 0 & x\end{vmatrix}$$, expanding with respect to C_1

$$=(x+10)\,x\begin{vmatrix}x & 0\\ 0 & x\end{vmatrix}$$, expanding with respect to R_1

$= (x+10)\,x\,(xx) = x^3\,(x+10).$ **Ans.**

Ex. 2. Show that $-(a+b+c)$ is a root of the equation

$$\begin{vmatrix} x+a & b & c \\ b & x+c & a \\ c & a & x+b \end{vmatrix} = 0$$

Sol. The given equation can be written as

$$\begin{vmatrix} x+a+b+c & b & c \\ b+x+c+a & x+c & a \\ c+a+x+b & a & x+b \end{vmatrix} = 0, \text{ replacing } C_1 \text{ by } C_1+C_2+C_3 \text{ in the det.}$$

or $$(x+a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & x+c & a \\ 1 & a & x+b \end{vmatrix} = 0, \text{ taking out the common factor from } C_1$$

or $$(x+a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & x+c-b & a-c \\ 0 & a-b & x+b-c \end{vmatrix} = 0, \text{ replacing } R_2, R_3 \text{ by } R_2-R_1, R_3-R_1$$

or $$(x+a+b+c)\begin{vmatrix} x+c-b & a-c \\ a-b & x+b-c \end{vmatrix} = 0, \text{ expanding with respect to } C_1$$

or $$(x+a+b+c)\,[(x+c-b)(x+b-c)-(a-c)(a-b)] = 0$$

or $$(x+a+b+c)(x^2+ab+bc+ca-a^2-b^2-c^2) = 0$$

This gives $x = -(a+b+c), \pm\sqrt{(a^2+b^2+c^2-ab-bc-ca)}$

Hence $-(a+b+c)$ is a root of the given equation.

Ex. 3. Show that $$\begin{vmatrix} x & l & m & 1 \\ \alpha & x & n & 1 \\ \alpha & \beta & x & 1 \\ \alpha & \beta & \gamma & 1 \end{vmatrix} = (x-\alpha)(x-\beta)(x-\gamma)$$

Sol. The given determinant

$$= \begin{vmatrix} x & l & m & 1 \\ \alpha-x & x-l & n-m & 0 \\ \alpha-x & \beta-l & x-m & 0 \\ \alpha-x & \beta-l & \gamma-m & 0 \end{vmatrix}, \text{ replacing } R_2, R_3, R_4 \text{ by } R_2-R_1, R_3-R_1, R_4-R_1 \text{ respectively.}$$

$$= -\begin{vmatrix} \alpha-x & x-l & n-m \\ \alpha-x & \beta-l & x-m \\ \alpha-x & \beta-l & \gamma-m \end{vmatrix}, \text{ expanding with respect to } C_4$$

$$= -(\alpha-x)\begin{vmatrix} 1 & x-l & n-m \\ 1 & \beta-l & x-m \\ 1 & \beta-l & \gamma-m \end{vmatrix}, \text{ taking out } (\alpha-x) \text{ common}$$

$$= (x-\alpha)\begin{vmatrix} 1 & x-l & n-m \\ 0 & \beta-x & x-n \\ 0 & \beta-x & \gamma-n \end{vmatrix}, \text{ replacing } R_2 \text{ and } R_3 \text{ by } R_2-R_1 \text{ and } R_3-R_1 \text{ respectively.}$$

$$= (x-\alpha)\begin{vmatrix} \beta-x & x-n \\ \beta-x & \gamma-n \end{vmatrix}, \text{ expanding with respect to } C_1$$

$$= (x-\alpha)(\beta-x)\begin{vmatrix} 1 & x-n \\ 1 & \gamma-n \end{vmatrix} = (x-\alpha)(\beta-x)\left[(\gamma-n)-(x-n)\right]$$

$$= (x-\alpha)(\beta-x)(\gamma-x) = (x-\alpha)(x-\beta)(x-\gamma)$$ Hence proved.

****Ex. 4. Evaluate** $\begin{vmatrix} 0 & x & y & z \\ -x & 0 & c & b \\ -y & -c & 0 & a \\ -z & -b & -a & 0 \end{vmatrix}$

Sol. The given determinant

$$= \frac{1}{a}\begin{vmatrix} 0 & ax-by+cz & y & z \\ -x & 0-bc+cb & c & b \\ -y & -ac+0+ca & 0 & a \\ -z & -ab+ba+0 & -a & 0 \end{vmatrix}$$, replacing C_2 by $aC_2 - bC_3 + cC_4$. Here $(1/a)$ has been taken common due to aC_2 **(Note)**

$$= \frac{1}{a}\begin{vmatrix} 0 & ax-by+cz & y & z \\ -x & 0 & c & b \\ -y & 0 & 0 & a \\ -z & 0 & -a & 0 \end{vmatrix}$$

$$= -(ax-by+cz)(1/a)\begin{vmatrix} -x & c & b \\ -y & 0 & a \\ -z & -a & 0 \end{vmatrix}$$, expanding with respect to C_2.

$$= (ax+by+cz)(1/a)\begin{vmatrix} x & c & b \\ y & 0 & a \\ z & -a & 0 \end{vmatrix}$$, taking out -1 common from C_1

$$= \frac{(ax-by+cz)}{a.a}\begin{vmatrix} ax-by+cz & ac-0-ca & ba-ab+0 \\ y & 0 & a \\ z & -a & 0 \end{vmatrix}$$,

replacing R_1 by $aR_1 - bR_2 + cR_3$ and taking out $(1/a)$ common as before

$$= \frac{(ax-by+cz)}{a^2}\begin{vmatrix} ax-by+cz & 0 & 0 \\ y & 0 & a \\ z & -a & 0 \end{vmatrix}$$

$$= \frac{(ax+by+cz)}{a^2}(ax-by+cz)\begin{vmatrix} 0 & a \\ -a & 0 \end{vmatrix}$$, expanding with respect to R_1

$$= \frac{(ax-by+cz)^2}{a^2}\left[0\cdot 0 - a.(-a)\right] = \frac{(ax-by+cz)^2}{a^2}\cdot\left[a^2\right]$$

$$= (ax-by+cz)^2$$ **Ans.**

Ex. 5. Show that $\begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+cda \\ 1 & c & c^2 & c^3+dab \\ 1 & d & d^2 & d^3+abc \end{vmatrix} = 0$ *(Gorakhpur 92; Kumaun 95)*

given that abcd ≠ 0.

Sol. The given determinant

$$=\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} + \begin{vmatrix} 1 & a & a^2 & bcd \\ 1 & b & b^2 & cda \\ 1 & c & c^2 & dab \\ 1 & d & d^2 & abc \end{vmatrix} \qquad \text{...(i)}$$

Now $\begin{vmatrix} 1 & a & a^2 & bcd \\ 1 & b & b^2 & cda \\ 1 & c & c^2 & dab \\ 1 & d & d^2 & abc \end{vmatrix}$

$=\frac{1}{abcd}\begin{vmatrix} a & a^2 & a^3 & abcd \\ b & b^2 & b^3 & bcda \\ c & c^2 & c^3 & cdab \\ d & d^2 & d^3 & dabc \end{vmatrix}$, multiplying R_1, R_2, R_3, R_4 by a, b, c, d respectively and dividing the result by $abcd$, where $abcd \neq 0$. **(Note)**

$=\frac{1}{abcd}(abcd)\begin{vmatrix} a & a^2 & a^3 & 1 \\ b & b^2 & b^3 & 1 \\ c & c^2 & c^3 & 1 \\ d & d^2 & d^3 & 1 \end{vmatrix}$, taking out $abcd$ common from C_4

$=-\begin{vmatrix} a & a^2 & 1 & a^3 \\ b & b^2 & 1 & b^3 \\ c & c^2 & 1 & c^3 \\ d & d^2 & 1 & d^3 \end{vmatrix}$, interchanging C_3 and C_4

$=(-1)^2\begin{vmatrix} a & 1 & a^2 & a^3 \\ b & 1 & b^2 & b^3 \\ c & 1 & c^2 & c^3 \\ d & 1 & d^2 & d^3 \end{vmatrix}$, interchanging C_2 and C_2

$=-\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}$, interchanging C_1 and C_2, also $(-1)^3 = -1$.

Substituting this value in (i), the value of the given determinant is zero.

Ex. 6. Show that $\begin{vmatrix} -1 & 0 & 0 & a \\ 0 & -1 & 0 & b \\ 0 & 0 & -1 & c \\ x & y & z & -1 \end{vmatrix} = 1 - ax - by - cz$ *(Gorakhpur 91)*

Sol. The given determinant

$$= \begin{vmatrix} -1 & 0 & 0 & -a+0+0+a \\ 0 & -1 & 0 & 0-b+0+b \\ 0 & 0 & -1 & 0+0-c+c \\ x & y & z & ax+by+cz-1 \end{vmatrix}, \text{ replacing } C_4 \text{ by } aC_1+bC_2+cC_3+C_4$$

$$= \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ x & y & z & ax+by+cz-1 \end{vmatrix}$$

$$= (ax+by+cz-1) \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \text{ expanding with respect to } C_4$$

$$= -(ax+by+cz-1) \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix}, \text{ expanding with respect to } C_1$$

$$= (1-ax-by+cz)\,[(-1)(-1)-0 \cdot 0]$$

$$= (1-ax-by-cz)\,[1] = (1-ax-by-cz).$$ Hence proved.

***Ex. 7. Prove that**

$$\begin{vmatrix} \mathbf{yz-x^2} & \mathbf{zx-y^3} & \mathbf{xy-z^2} \\ \mathbf{zx-y^2} & \mathbf{xy-z^2} & \mathbf{yz-x^2} \\ \mathbf{xy-z^2} & \mathbf{yz-x^2} & \mathbf{zx-y^2} \end{vmatrix} = \begin{vmatrix} \mathbf{v^2} & \mathbf{u^2} & \mathbf{u^2} \\ \mathbf{u^2} & \mathbf{v^2} & \mathbf{u^2} \\ \mathbf{u^2} & \mathbf{u^2} & \mathbf{v^2} \end{vmatrix} = \begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \mathbf{y} & \mathbf{z} & \mathbf{x} \\ \mathbf{z} & \mathbf{x} & \mathbf{y} \end{vmatrix}^2,$$

where $\mathbf{v^2 = x^2+y^2+z^2,\ u^2 = yz+zx+xy}$. *(Gorakhpur 90)*

Sol. $$\begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}^2 = \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix} \times \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}$$

$$= \begin{vmatrix} x^2+y^2+z^2 & xy+yz+zx & xz+yx+zy \\ yx+zy+xz & y^2+z^2+x^2 & yz+zx+xy \\ zx+xy+yz & zy+xz+xy & z^2+x^2+y^2 \end{vmatrix}$$

$$= \begin{vmatrix} v^2 & u^2 & u^2 \\ u^2 & v^2 & u^2 \\ u^2 & u^2 & v^2 \end{vmatrix}, \quad \because\ u^2 = yz+zx+xy,\ v^2 = x^2+y^2+z^2, \text{ (given)} \qquad \text{...(i)}$$

Again from § 4·16 Page 169 we knwo that if C_{ij} be the cofactor of a_{ij} in the $n \times n$ matrix $\mathbf{A} = [a_{ij}]$, then

$$|C_{ij}| = \{|\mathbf{A}|\}^{n-1}$$

Here $n = 3$, so $|C_{ij}| = \{|\mathbf{A}|\}^2$, ...(ii)

where $$\mathbf{A} = \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}$$

$\therefore \; |C_{ij}| = \begin{vmatrix} zy-x^2 & xz-y^2 & xy-z^2 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix}$, where C_{ij} is the cofactor of a_{ij} in $|\mathbf{A}|$,

$\therefore$ From (ii), $\begin{vmatrix} zy-x^2 & zx-y^2 & xy-z^2 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix} = \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}^2$

From (i) and this we have the required result.

Ex. 8. Write down as a determinant the product

$$\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{c} & \mathbf{a} & \mathbf{b} \\ \mathbf{b} & \mathbf{c} & \mathbf{a} \end{vmatrix} \cdot \begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \mathbf{z} & \mathbf{x} & \mathbf{y} \\ \mathbf{y} & \mathbf{z} & \mathbf{x} \end{vmatrix}$$

Sol. Multiplying by the 'row-by-row' rule we get

$$\begin{vmatrix} ax+by+cz & az+bx+cy & ay+bz+cz \\ cx+ay+bz & cz+ax+by & cy+az+bx \\ bx+cy+az & bz+cx+ay & by+cz+ax \end{vmatrix}$$

$$= (a+b+c)(x+y+z) \begin{vmatrix} 1 & 1 & 1 \\ cx+ay+bz & cz+ax+by & cy+az+bx \\ bx+cy+az & bz+cx+ay & by+cz+ax \end{vmatrix}$$

replacing R_1 by $R_1 + R_2 + R_3$ and taking the common factor out.

Ex. 9. Expand $\begin{vmatrix} \mathbf{0} & \mathbf{1} & \mathbf{x} & \mathbf{y} \\ \mathbf{0} & \mathbf{0} & \mathbf{y} & \mathbf{x} \\ \mathbf{z} & \mathbf{w} & \mathbf{0} & \mathbf{0} \\ \mathbf{w} & \mathbf{z} & \mathbf{0} & \mathbf{0} \end{vmatrix}$ **by Laplace's Expansion's by the minors of the fi-st two columns.**

Sol. All the possible minors of the frist two columns and their complementary minors are given below :

$|\mathbf{B}_1| = \begin{vmatrix} 0 & 1 \\ z & w \end{vmatrix}$, $|\mathbf{B}'_1| = \begin{vmatrix} y & x \\ 0 & 0 \end{vmatrix} = 0$

$|\mathbf{B}_2| = \begin{vmatrix} 0 & 1 \\ w & z \end{vmatrix}$, $|\mathbf{B}'_2| = \begin{vmatrix} y & x \\ 0 & 0 \end{vmatrix} = 0$

$|\mathbf{B}_3| = \begin{vmatrix} z & w \\ w & z \end{vmatrix}$, $|\mathbf{B}'_3| = \begin{vmatrix} x & y \\ y & x \end{vmatrix}$

$\therefore$ The given determinat $= \begin{vmatrix} z & w \\ w & z \end{vmatrix} \cdot \begin{vmatrix} x & y \\ y & x \end{vmatrix}$,

the remaing minors or complementary minors are zero.

$$= (z^2 - w^2)(x^2 - y^2).$$

Ans.

****Ex. 10 (a). Show that** $\begin{vmatrix} \mathbf{a_1} & \mathbf{b_1} & \mathbf{c_1} \\ \mathbf{a_2} & \mathbf{b_2} & \mathbf{c_2} \\ \mathbf{a_3} & \mathbf{b_3} & \mathbf{c_3} \end{vmatrix}^2 = \begin{vmatrix} \mathbf{A_1} & \mathbf{B_1} & \mathbf{C_1} \\ \mathbf{A_2} & \mathbf{B_2} & \mathbf{C_2} \\ \mathbf{A_3} & \mathbf{B_3} & \mathbf{C_3} \end{vmatrix}$

where the capital letters denote the cofactors of the corresponding small letters. (*Gorakhpur 96, 92; Kanpur 93; Purvanchal 97*)

Sol. Let $|\mathbf{A}| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$

Then $\Delta \times |\mathbf{A}|$

$$= \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \times \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1A_1 + b_1B_1 + c_1C_1 & a_2A_1 + b_2B_1 + c_2C_1 & a_3A_1 + b_3B_1 + c_3C_1 \\ a_1A_2 + b_1B_2 + c_1C_2 & a_2A_2 + b_2B_2 + c_2C_2 & a_3A_2 + b_3B_2 + c_3C_2 \\ a_1A_3 + b_1B_3 + c_1C_3 & a_2A_3 + b_2B_3 + c_2C_3 & a_3A_3 + b_3B_3 + c_3C_3 \end{vmatrix}$$

$$= \begin{vmatrix} |\mathbf{A}| & 0 & 0 \\ 0 & |\mathbf{A}| & 0 \\ 0 & 0 & |\mathbf{A}| \end{vmatrix}$$, since $a_1A_1 + b_1B_1 + c_1C_1 = |\mathbf{A}|$ etc. and $a_1A_2 + b_1B_2 + c_1C_2$ etc.

(See example on Page 117)

Note. Students are to prove these in the examination.

$$= |\mathbf{A}| \bullet \begin{vmatrix} |\mathbf{A}| & 0 \\ 0 & |\mathbf{A}| \end{vmatrix}$$, expanding with respect to first row.

$$= |\mathbf{A}| [|\mathbf{A}| \bullet |\mathbf{A}| - 0 \cdot 0] = [|\mathbf{A}|]^3$$

or $\quad \Delta \times |\mathbf{A}| = \{|\mathbf{A}|\}^3$ or $\Delta = |\mathbf{A}|^2$

or $\quad \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$

Hence proved.

***Ex. 10 (b). Prove that**

$$\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{c} & \mathbf{a} & \mathbf{b} \\ \mathbf{b} & \mathbf{c} & \mathbf{a} \end{vmatrix}^2 = \begin{vmatrix} \mathbf{a^2 - bc} & \mathbf{b^2 - ca} & \mathbf{c^2 - ab} \\ \mathbf{c^2 - ab} & \mathbf{a^2 - bc} & \mathbf{b^2 - ac} \\ \mathbf{b^2 - ca} & \mathbf{c^2 - ab} & \mathbf{a^2 - bc} \end{vmatrix}$$ (*Gorakhpur 90*)

Sol. We know [See Ex. 10 (a) above] that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2 = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix},$$

where capital letters denote the cofactors of the corresponding small letters.

$$\therefore \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}^2 = \begin{vmatrix} A & B & C \\ C & A & B \\ B & C & A \end{vmatrix}, \quad \ldots(i)$$

where A, B, C are the cofactors of a, b, c respectively in the determinant on the left.

$\therefore A = \begin{vmatrix} a & b \\ c & a \end{vmatrix} = a^2 - bc,\ B = -\begin{vmatrix} c & b \\ b & a \end{vmatrix} = b^2 - ac$

and $C = \begin{vmatrix} c & a \\ b & c \end{vmatrix} = c^2 - ab$ **(Note)**

Hence from (i) we get

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}^2 = \begin{vmatrix} a^2-bc & b^2-ca & c^2-ab \\ c^2-ab & a^2-bc & b^2-ac \\ b^2-ca & c^2-ab & a^2-bc \end{vmatrix}$$ Hence proved.

Ex. 11. Solve the equation $\begin{vmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0.$ *(Meerut 97)*

Sol. Given that $\begin{vmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} 3x-2 & 3 & 3 \\ 3x-2 & 3x-8 & 3 \\ 3x-2 & 3 & 3x-8 \end{vmatrix} = 0$, replacing C_1 by $C_1 + C_2 + C_3$

$\Rightarrow (3x-2)\begin{vmatrix} 1 & 3 & 3 \\ 1 & 3x-8 & 3 \\ 1 & 3 & 3x-8 \end{vmatrix} = 0$, taking out $(3x-2)$ common.

$\Rightarrow (3x-2)\begin{vmatrix} 1 & 3 & 3 \\ 0 & 3x-11 & 0 \\ 0 & 0 & 3x-11 \end{vmatrix} = 0$, replacing R_2, R_3 by $R_2 - R_1$, $R_3 - R_1$ respectively.

$\Rightarrow (3x-2)\begin{vmatrix} 3x-11 & 0 \\ 0 & 3x-11 \end{vmatrix} = 0$, expanding with respect to C_1

$\Rightarrow (3x-2)(3x-11)^2 = 0$

$\Rightarrow x = 2/3$ or $11/3$. **Ans.**

Ex. 12. Prove that the value of determinant

$\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$ **is independent of x.**

Sol. The given determinant

$= \begin{vmatrix} x+1 & 1 & a-1 \\ x+2 & 1 & b-2 \\ x+3 & 1 & c-3 \end{vmatrix}$, replacing C_2, C_3 by $C_2 - C_1$ and $C_3 - C_1$ respectively

$= \begin{vmatrix} x+1 & 1 & a-1 \\ 1 & 0 & b-a-1 \\ 2 & 0 & c-a-2 \end{vmatrix}$, replacing R_2, R_3 by $R_2 - R_1$, $R_3 - R_1$ respectively.

$$= -\begin{vmatrix} 1 & x+1 & a-1 \\ 0 & 1 & b-a-1 \\ 0 & 2 & c-a-2 \end{vmatrix}$$, interchanging C_1 and C_2

$$= -\begin{vmatrix} 1 & b-a-1 \\ 2 & c-a-2 \end{vmatrix} = -[(c-a-2) - 2(b-a-1)]$$

$= -[c-a-2-2b+2a+2] = -a+2b-c,$ which is independent of x.

Hence proved.

***Ex. 13. Give correct answer to the following :**

The value of the determinant $\begin{vmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{vmatrix}$

is (A) − 1, (B) 1, (C) 0, (D) 4.

Sol. The correct answer is (*C*) *i.e.* 0, since replacing C_1 by $C_1 + C_2 + C_3 + C_4$ we find that all the elements of C_1 are zero. Hence the value of the given determinant is zero.

Ex. 14. Show that $(a+b+c)$ and $(a^2+b^2+c^2)$ are factors of determinant $\begin{vmatrix} a^2 & (b+c)^2 & bc \\ b^2 & (c+a)^2 & ca \\ c^2 & (a+b)^2 & ab \end{vmatrix}$ **and find the remaining factors.**

Sol. The given determinant

$$= \begin{vmatrix} a^2 & (b^2+c^2+2bc)+a^2 & bc \\ b^2 & (c^2+a^2+2ca)+b^2 & ca \\ c^2 & (a^2+b^2+2ab)+c^2 & ab \end{vmatrix}$$, replacing C_2 by $C_2 + C_1$

$$= \begin{vmatrix} a^2 & b^2+c^2+a^2 & bc \\ b^2 & c^2+a^2+b^2 & ca \\ c^2 & a^2+b^2+c^2 & ab \end{vmatrix}$$, replacing C_2 by $C_2 - 2C_3$

$$= (a^2+b^2+c^2)\begin{vmatrix} a^2 & 1 & bc \\ b^2 & 1 & ca \\ c^2 & 1 & ab \end{vmatrix}$$, taking out $(a^2+b^2+c^2)$ common from C_2

$$= \frac{a^2+b^2+c^2}{abc}\begin{vmatrix} a^3 & a & abc \\ b^3 & b & bca \\ c^3 & c & cab \end{vmatrix}$$, taking $1/a$ $1/b$, $1/c$ common from R_1, R_2 and R_3 respectively.

(Note)

$$= \frac{a^2+b^2+c^2}{abc} \times abc \begin{vmatrix} a^3 & a & 1 \\ b^3 & b & 1 \\ c^3 & c & 1 \end{vmatrix}$$, taking out abc common from C_3.

Now proceed as in Ex. 31 Page 146.

Ex. 15. Prove that $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2.$

Sol. The given determinant

$$= abc \begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$, taking out a, b, c common from C_1, C_2, C_3 respectively

$$= abc \begin{vmatrix} a+c & c & a+c \\ a+2b & b & a \\ 2b+c & b+c & c \end{vmatrix}$$, replacing C_1 by $C_1 + C_2$

$$= abc \begin{vmatrix} 0 & c & a+c \\ 2b & b & a \\ 2b & b+c & c \end{vmatrix}$$, replacing C_1 by $C_1 - C_3$

$$= abc \begin{vmatrix} 0 & c & a+c \\ 2b & b & a \\ 0 & c & c-a \end{vmatrix}$$, replacing R_3 by $R_3 - R_2$

$$= -2ab^2c \begin{vmatrix} c & a+c \\ c & c-a \end{vmatrix}$$, expanding with respect to C_1

$$= -2ab^2c \begin{vmatrix} 0 & 2a \\ c & c-a \end{vmatrix}$$, replacing R_1 by $R_1 - R_2$

$= -2ab^2c\,(-2ac)$, expanding the determinant.

$= 4a^2b^2c^2.$ **Hence proved.**

***Ex. 16. Solve** $\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix} = 0$

Sol. The given equation is

$$\begin{vmatrix} x+2 & -1 & -2 \\ 2x+3 & -x-2 & -2x-4 \\ 3x+5 & -x-2 & x+2 \end{vmatrix} = 0$$, replacing C_2, C_3 by $C_2 - 2C_1$, $C_3 - 3C_1$ respectively.

or $$\begin{vmatrix} x+2 & -1 & -2 \\ 2x+3 & -x-2 & -2x-4 \\ x+2 & 0 & 3x+6 \end{vmatrix} = 0$$, replacing R_3 by $R_3 - R_2$

or $$\begin{vmatrix} x+2 & -1 & 0 \\ 2x+3 & -x-2 & 0 \\ x+2 & 0 & 3x+6 \end{vmatrix} = 0$$, replacing C_3 by $C_3 - 2C_2$

or $$(3x+6) \begin{vmatrix} x+2 & -1 \\ 2x+3 & -x-2 \end{vmatrix} = 0$$, expanding with respect to C_3

or $(3x+6)\,[-(x+2)^2 + (2x+3)] = 0$

or $\quad (3x+6)(x^2+2x+1)=0 \quad$ or $\quad (3x+6)(x+1)^2=0$

or $\quad x=-1,-2$ **Ans.**

***Ex. 17. Prove that** $\begin{vmatrix} \alpha & x & x & x \\ x & \beta & x & x \\ x & x & \gamma & x \\ x & x & x & \delta \end{vmatrix} = f(x) - x f'(x)$,

where $f(x) = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$ and $f'(x)$ is the first derivative of $f(x)$ with respect to x.

Sol. The given determinant

$= \begin{vmatrix} \alpha & x & 0 & 0 \\ x & \beta & x-\beta & x-\beta \\ x & x & \gamma-x & 0 \\ x & x & 0 & \delta-x \end{vmatrix}$, replacing C_3 and C_4 by $C_3 - C_2$ and $C_4 - C_2$ respectively.

$= \alpha \begin{vmatrix} \beta & x-\beta & x-\beta \\ x & \gamma-x & 0 \\ x & 0 & \delta-x \end{vmatrix} - x \begin{vmatrix} x & x-\beta & x-\beta \\ x & \gamma-x & 0 \\ x & 0 & \delta-x \end{vmatrix}$, expanding w.r. to R_1

$= \alpha \begin{vmatrix} \beta & x-\beta & 0 \\ x & \gamma-x & x-\gamma \\ x & 0 & \delta-x \end{vmatrix} - x^2 \begin{vmatrix} 1 & x-\beta & 0 \\ 1 & \gamma-x & x-\gamma \\ 1 & 0 & \delta-x \end{vmatrix}$, replacing C_3 by $C_3 - C_2$ in each determinant

$= \alpha\beta \begin{vmatrix} \gamma-x & x-\gamma \\ 0 & \delta-x \end{vmatrix} - \alpha(x-\beta) \begin{vmatrix} x & x-\gamma \\ x & \delta-x \end{vmatrix} - x^2 \begin{vmatrix} \gamma-x & x-\gamma \\ 0 & \delta-x \end{vmatrix}$

$+ x^2 (x-\beta) \begin{vmatrix} 1 & x-\gamma \\ 1 & \delta-x \end{vmatrix}$, expanding each det. w.r. to R_1

$= (\alpha\beta - x^2)(\gamma-x)(\delta-x) - (x-\beta)\, x\, (\alpha-x)\, [(\delta-x)-(x-\gamma)]$

$= (\alpha\beta - x^2)(x-\gamma)(x-\delta) - x(x-\alpha)(x-\beta)(x-\delta)$

$- x(x-\alpha)(x-\beta)(x-\gamma)$

$= (x^2 - \alpha x - \beta x + \alpha\beta - x^2 + \beta x - x^2 + \alpha x)(x-\gamma)(x-\delta)$

$- x(x-\alpha)(x-\beta)(x-\delta) - x(x-\alpha)(x-\beta)(x-\gamma)$ (Note)

$= [(x-\alpha)(x-\beta) - x(x-\beta) - x(x-\alpha)](x-\gamma)(x-\delta)$

$- x(x-\alpha)(x-\beta)(x-\delta) - x(x-\alpha)(x-\beta)(x-\gamma)$ (Note)

$= (x-\alpha)(x-\beta)(x-\gamma)(x-\delta) - x[(x-\beta)(x-\gamma)(x-\delta)$

$+ (x-\alpha)(x-\gamma)(x-\delta) + (x-\alpha)(x-\beta)(x-\delta)$

$+ (x-\alpha)(x-\beta)(x-\gamma)]$

$= f(x) - x f'(x)$, where $f(x) = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$ Hence proved.

Ex. 18. Prove that $\begin{vmatrix} 1 & \omega^3 & \omega^2 \\ \omega^3 & 1 & \omega \\ \omega^2 & \omega & 1 \end{vmatrix}$ **= 3, where ω is one of the imaginary cube roots of unity.**

Sol. If ω be one of the imaginary cube roots of unity, then

$$\omega^3 = 1 \quad \text{and} \quad 1 + \omega + \omega^2 = 0 \qquad \text{...(i)}$$

Now the given determinant

$$= \begin{vmatrix} 1 & 1 & \omega^2 \\ 1 & 1 & \omega \\ \omega^2 & \omega & 1 \end{vmatrix}, \text{ from (i) using } \omega^3 = 1$$

$$= \begin{vmatrix} 0 & 1 & \omega^2 \\ 0 & 1 & \omega \\ \omega^2 - \omega & \omega & 1 \end{vmatrix}, \text{ replacing } C_1 \text{ by } C_1 - C_2$$

$$= (\omega^2 - \omega) \begin{vmatrix} 1 & \omega^2 \\ 1 & \omega \end{vmatrix}, \text{ expanding with respect to } C_1$$

$$= (\omega^2 - \omega)(\omega - \omega^2) = \omega^3 - \omega^4 - \omega^2 + \omega^3 = 1 - \omega - \omega^2 + 1, \quad \because \omega^3 = 1$$

$$= 2 - (\omega + \omega^2) = 2 - (-1), \because 1 + \omega + \omega^2 = 0 \text{ or } \omega + \omega^2 = -1$$

$$= 2 + 1 = 3.$$ Hence proved

****Ex. 19. Prove that the determinant**

$$\begin{vmatrix} 1 & \cos(\beta - \alpha) & \cos(\gamma - \alpha) \\ \cos(\alpha - \beta) & 1 & \cos(\gamma - \beta) \\ \cos(\alpha - \gamma) & \cos(\beta - \gamma) & 1 \end{vmatrix}$$

is a perfect square (of a determinant) and find its value. (*Gorakhpur 94*)

Sol. The given determinant

$$= \begin{vmatrix} \cos\alpha\cos\alpha + \sin\alpha\sin\alpha & \cos\alpha\cos\beta + \sin\alpha\sin\beta & \cos\alpha\cos\gamma + \sin\alpha\sin\gamma \\ \cos\beta\cos\alpha + \sin\beta\sin\alpha & \cos\beta\cos\beta + \sin\beta\sin\beta & \cos\beta\cos\gamma + \sin\beta\sin\gamma \\ \cos\gamma\cos\alpha + \sin\gamma\sin\alpha & \cos\gamma\cos\beta + \sin\gamma\sin\beta & \cos\gamma\cos\gamma + \sin\gamma\sin\gamma \end{vmatrix}$$

(Note)

The element in the first row and first column is $\cos\alpha\cos\alpha + \sin\alpha\sin\alpha$, which can be written as

$$(\cos\alpha)(\cos\alpha) + (\sin\alpha)(\sin\alpha) + 0 \cdot 0$$ **(Note)**

Similarly the element in the first row and second column is $\cos\alpha\cos\beta + \sin\alpha\sin\beta$, which can be writter as

$$(\cos\alpha)(\cos\beta) + (\sin\alpha)(\sin\beta) + 0 \cdot 0,$$ **(Note)**

Proceeding in this way we can write the given determinant

$$= \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix} \times \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix}$$

$$= \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix}^2, \text{ hence a perfect square of a determinant}$$

= 0, since the value of this determinant is zero as all the elements of one of its columns are zero.

***Ex. 20. (a) Show that** $\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix} \times \begin{vmatrix} \alpha+i\beta & \gamma+i\delta \\ -\gamma+i\delta & \alpha-i\beta \end{vmatrix}$

can be expressed as $\begin{vmatrix} A+iB & C+iD \\ -C+iD & A-iB \end{vmatrix}$

Hence prove the Euler's Theorem, 'the product of two sums of four squares each is equal to the sum of the four squares'.

Sol. $\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix} \times \begin{vmatrix} \alpha+i\beta & \gamma+i\delta \\ -\gamma+i\delta & \alpha-i\beta \end{vmatrix}$

$$= \begin{vmatrix} (a+ib)(\alpha+i\beta) + (c+id)(\gamma+i\delta) & (-c+id)(\alpha+i\beta) + (a-ib)(\gamma+i\delta) \\ (a+ib)(-\gamma+i\delta) + (c+id)(\alpha-i\beta) & (-c+id)(-\gamma+i\delta) + (a-ib)(\alpha-i\beta) \end{vmatrix}$$

$$= \begin{vmatrix} (a\alpha-b\beta+c\gamma-d\delta) + i(a\beta+b\alpha+c\delta+d\gamma) & (-c\alpha-d\beta+a\gamma+b\delta) + i(-c\beta+d\alpha+a\delta-b\gamma) \\ (-a\gamma-b\delta+c\alpha+d\beta) + i(a\delta-b\gamma-c\beta+d\alpha) & (a\alpha-b\beta+c\gamma-d\delta) + i(-a\beta-b\alpha-c\delta-d\gamma) \end{vmatrix}$$

$$= \begin{vmatrix} A+iB & C+iD \\ -C+iD & A-iB \end{vmatrix}, \qquad \text{...(i)}$$

where $A = a\alpha - b\beta + c\gamma - d\delta$, $B = a\beta + b\alpha + c\delta + d\gamma$,

$C = a\gamma + b\delta - c\alpha - d\beta$, $D = a\delta - b\gamma - c\beta + d\alpha$. ...(ii)

Now $\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix}$

$= (a+ib)(a-ib) - (c+id)(-c+id)$

$= (a^2 - i^2b^2) - (i^2d^2 - c^2) = a^2 + b^2 + c^2 + d^2$

Similaryly $\begin{vmatrix} \alpha+i\beta & \gamma+i\delta \\ -\gamma+i\delta & \alpha-i\beta \end{vmatrix} = \alpha^2 + \beta^2 + \gamma^2 + \delta^2$

and $\begin{vmatrix} A+iB & c+iD \\ -C+iD & A-iB \end{vmatrix} = A^2 + B^2 + C^2 + D^2$

$\therefore$ From (i) we have

$(a^2 + b^2 + c^2 + d^2)(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) = (A^2 + B^2 + C^2 + D^2)$ ...(iii)

i.e. product of two sums of four squares each is equal to the sum of four squares.

Hence proved.

Ex. 20 (b). With the help of determinants express the following as a sum of four squares

$$(1^2 + 2^2 + 3^2 + 4^2)(5^2 + 6^2 + 7^2 + 8^2)$$

Sol. As in Ex. 20 (a) above we can show that,

$(a^2 + b^2 + c^2 + d^2)(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) = (A^2 + B^2 + C^2 + D^2)$ ...(i) where A, B, C, D are given by Ex. 20 (a) result (ii).

Now let $a=2,\ b=2,\ c=3,\ d=4,\ \alpha=5,\ \beta=6,\ \gamma=7,\ \delta=8$

Then $A = a\alpha - b\beta + c\gamma - d\delta = 1\,(5) - 2\,(6) + 3\,(7) - 4\,(8)$

$= 5 - 12 + 21 - 32 = -18.$

$B = a\beta + b\alpha + c\delta + d\gamma = 1\,(6) + 2\,(5) + 3\,(8) + 4\,(7)$

$= 6 + 10 + 24 + 28 = 68,$

$C \equiv a\gamma + b\delta - c\alpha - d\beta = 1\,(7) + 2\,(8) - 3\,(5) - 4\,(6)$

$= 7 + 6 - 15 - 24 = -16$

$D = a\delta - b\gamma - c\beta + d\alpha = 1\,(8) - 2\,(7) - 3\,(6) + 4\,(5)$

$= 8 - 14 - 18 + 20 = -4.$

From (I) above we have $(1^2 + 2^2 + 3^2 + 4^2)(5^2 + 6^2 + 7^2 + 8^2)$

$$= (-18)^2 + (68)^2 + (-16)^2 + (-4)^2$$

$$= (18)^2 + (68)^2 + (16)^2 + (4)^2.$$ **Ans.**

Ex. 21. Evaluate $\begin{vmatrix} \mathbf{a} & -\mathbf{a} & -\mathbf{a} & -\mathbf{a} \\ \mathbf{b} & \mathbf{b} & -\mathbf{b} & -\mathbf{b} \\ \mathbf{c} & \mathbf{c} & \mathbf{c} & -\mathbf{c} \\ \mathbf{d} & \mathbf{d} & \mathbf{d} & \mathbf{d} \end{vmatrix}$

Sol. The given determinant

$= \begin{vmatrix} a & 0 & 0 & 0 \\ b & 2b & 0 & 0 \\ c & 2c & 2c & 0 \\ d & 2d & 2d & 2d \end{vmatrix}$, replacing C_2, C_3 and C_4 by $C_2 + C_1, C_3 + C_1$ and $C_4 + C_1$ respectively

$= a \begin{vmatrix} 2b & 0 & 0 \\ 2c & 2c & 0 \\ 2d & 2d & 2d \end{vmatrix}$, expanding with respect to R_1

$= 2ab \begin{vmatrix} 2c & 0 \\ 2d & 2d \end{vmatrix}$, expanding with respect to R_1

$= 2ab\,[2c \times 2d - (2d) \times 0] = 8abcd.$ **Ans.**

***Ex. 22. Prove that** $\begin{vmatrix} \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{x} \\ \mathbf{5} & \mathbf{6} & \mathbf{7} & \mathbf{y} \\ \mathbf{6} & \mathbf{7} & \mathbf{8} & \mathbf{z} \\ \mathbf{x} & \mathbf{y} & \mathbf{z} & \mathbf{0} \end{vmatrix} = (\mathbf{x} - \mathbf{2y} + \mathbf{z})^2$ *(Gorakhpur 94; Kanpur 94)*

Sol. The given determinant

$= \begin{vmatrix} 10 & 5 & 6 & x \\ 12 & 6 & 7 & y \\ 14 & 7 & 8 & z \\ x+z & y & z & 0 \end{vmatrix}$, replacing C_1 by $C_1 + C_3$

$= \begin{vmatrix} 0 & 5 & 6 & x \\ 0 & 6 & 7 & y \\ 0 & 7 & 8 & z \\ x - 2y + z & y & z & 0 \end{vmatrix}$, replacing C_1 by $C_1 - 2C_2$

$$= -(x-2y+z)\begin{vmatrix} 5 & 6 & x \\ 6 & 7 & y \\ 7 & 8 & z \end{vmatrix}$$, expanding with respect to C_1

$$= -(x-2y+z)\begin{vmatrix} 12 & 14 & x+z \\ 6 & 7 & y \\ 7 & 8 & z \end{vmatrix}$$, replacing R_1 by $R_1 + R_2$

$$= -(x-2y+z)\begin{vmatrix} 0 & 0 & x+z-2y \\ 6 & 7 & y \\ 7 & 8 & z \end{vmatrix}$$, replacing R_1 by $R_1 - 2R_2$

$$= -(x-2y+z)^2\begin{vmatrix} 6 & 7 \\ 7 & 8 \end{vmatrix}$$, expanding with respect to R_1

$= -(x-2y+z)^2\,[48-49]$, expanding the det.

$= (x-2y+z)^2$.

Hence proved.

Ex. 23. Evaluate $\begin{vmatrix} 0 & \alpha & \beta & \gamma \\ l & 0 & c & -b \\ m & -c & 0 & a \\ n & b & -a & 0 \end{vmatrix}$ *(Gorakhpur 95; Kanpur 90)*

Sol. The given determinant

$$= \frac{1}{a}\begin{vmatrix} 0 & \alpha & \beta & \gamma \\ al & 0 & ac & -ab \\ m & -c & 0 & a \\ n & b & -a & 0 \end{vmatrix}$$, taking $(1/a)$ common from R_2 **(Note)**

$$= \frac{1}{a}\begin{vmatrix} 0 & \alpha & \beta & \gamma \\ al+bm+cm & 0 & 0 & 0 \\ m & -c & 0 & a \\ n & b & -a & 0 \end{vmatrix}$$, replacing R_2 by $R_2 + bR_3 + cR_1$

$$= -\frac{1}{a}(al+bm+cn)\begin{vmatrix} \alpha & \beta & \gamma \\ -c & 0 & a \\ b & -a & 0 \end{vmatrix}$$, expanding with respect to R_2

$$= -\frac{1}{a^2}(al+bm+cn)\begin{vmatrix} a\alpha & \beta & \gamma \\ -ac & 0 & a \\ ab & -a & 0 \end{vmatrix}$$, taking out $1/a$ common from C_1

$$= -\frac{1}{a^2}(al+bm+cn)\begin{vmatrix} a\alpha+b\beta+c\gamma & \beta & \gamma \\ 0 & 0 & a \\ 0 & -a & 0 \end{vmatrix}$$,

replacing C_1 by $C_1 + bC_2 + cC_3$

$$= -\frac{1}{a^2}(al+bm+cn)(a\alpha+b\beta+c\gamma)\begin{vmatrix} 0 & a \\ -a & 0 \end{vmatrix}$$, expanding with respect to C_1

$= -(1/a^2)(al + bm + cn)(a\alpha + b\beta + c\gamma)\, a^2$,

$= -(al + bm + cn)(a\alpha + b\beta + c\gamma)$. **Ans.**

Ex. 24. Prove that $\begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix}$

$= -(\beta - \gamma)(\gamma - \alpha)(\alpha - \beta)(\alpha - \delta)(\beta - \delta)(\gamma - \delta)$

Hence evaluate $\begin{vmatrix} s_0 & s_1 & s_2 & s_3 \\ s_1 & s_2 & s_3 & s_4 \\ s_2 & s_3 & s_4 & s_5 \\ s_3 & s_4 & s_5 & s_6 \end{vmatrix}$, **where** $s_r = a^r + \beta^r + \gamma^r + \delta^r$

Sol. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ \alpha & \beta - \alpha & \gamma - \alpha & \delta - \alpha \\ \alpha^2 & \beta^2 - \alpha^2 & \gamma^2 - \alpha^2 & \delta^2 - \alpha^2 \\ \alpha^3 & \beta^3 - \alpha^3 & \gamma^3 - \alpha^3 & \delta^3 - \alpha^3 \end{vmatrix}$, replacing C_2, C_3, C_4 by $C_2 - C_1$, $C_3 - C_1$ and $C_4 - C_1$

$= (\beta - \alpha)(\gamma - \alpha)(\delta - \alpha) \begin{vmatrix} 1 & 1 & 1 \\ \beta + \alpha & \gamma + \alpha & \delta + \alpha \\ \beta^2 + \alpha^2 + \alpha\beta & \gamma^2 + \alpha^2 + \alpha\gamma & \delta^2 + \alpha^2 + \alpha\delta \end{vmatrix}$,

expanding w.r. to R_1 and taking out common factors.

$= (\alpha - \beta)(\gamma - \alpha)(\alpha - \delta) \begin{vmatrix} 1 & 0 & 0 \\ \beta + \alpha & \gamma - \beta & \delta - \beta \\ \beta^2 + \alpha^2 + \alpha\beta & \gamma^2 + \alpha\gamma - \beta^2 - \alpha\beta & \delta^2 + \alpha\delta - \beta^2 - \alpha\beta \end{vmatrix}$,

replacing C_2, C_3 by $C_2 - C_1$, $C_3 - C_1$

$= (\alpha - \beta)(\alpha - \delta)(\gamma - \alpha) \begin{vmatrix} 1 & 0 & 0 \\ \beta + \alpha & \gamma - \beta & \delta - \beta \\ \beta^2 + \alpha^2 + \alpha\beta & (\gamma - \beta)(\alpha + \beta + \gamma) & (\delta - \beta)(\alpha + \beta + \delta) \end{vmatrix}$.

$= (\alpha - \beta)(\alpha - \delta)(\gamma - \alpha)(\beta - \gamma)(\beta - \delta) \begin{vmatrix} 1 & 1 \\ \alpha + \beta + \gamma & \alpha + \beta + \delta \end{vmatrix}$,

expanding w.r.t. to R_1

$= (\alpha - \beta)(\alpha - \delta)(\gamma - \alpha)(\beta - \gamma)(\beta - \delta)\,[(\alpha + \beta + \delta) - (\alpha + \beta + \gamma)]$

$$= -(\alpha-\beta)(\alpha-\delta)(\gamma-\alpha)(\beta-\gamma)(\beta-\delta)(\gamma-\delta)\,. \qquad ...(i)$$

Hence proved.

Squaring both sides of (i), we get

$$(\alpha-\beta)^2(\alpha-\delta)^2(\gamma-\alpha)^2(\beta-\gamma)^2(\beta-\delta)^2(\gamma-\delta)^2$$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix} \qquad \textbf{(Note)}$$

$$= \begin{vmatrix} 1+1+1+1 & \alpha+\beta+\gamma+\delta & \alpha^2+\beta^2+\gamma^2+\delta^2 & \alpha^3+\beta^3+\gamma^3+\delta^3 \\ \alpha+\beta+\gamma+\delta & \alpha^2+\beta^2+\gamma^2+\delta^2 & \alpha^3+\beta^3+\gamma^3+\delta^3 & \alpha^4+\beta^4+\gamma^4+\delta^4 \\ \alpha^2+\beta^2+\gamma^2+\delta^2 & \alpha^3+\beta^3+\gamma^3+\delta^3 & \alpha^4+\beta^4+\gamma^4+\delta^4 & \alpha^5+\beta^5+\gamma^5+\delta^5 \\ \alpha^3+\beta^3+\gamma^3+\delta^3 & \alpha^4+\beta^4+\gamma^4+\delta^4 & \alpha^5+\beta^5+\gamma^5+\delta^5 & \alpha^6+\beta^6+\gamma^6+\delta^6 \end{vmatrix}$$

$$= \begin{vmatrix} s_0 & s_1 & s_2 & s_3 \\ s_1 & s_2 & s_3 & s_4 \\ s_2 & s_3 & s_4 & s_5 \\ s_3 & s_4 & s_5 & s_6 \end{vmatrix}, \text{ where } s_r = \alpha^r+\beta^r+\gamma^r+\delta^r$$

and $s_0 = \alpha^0+\beta^0+\gamma^0+\delta^0 = 1+1+1+1$, etc.

Hence proved.

***Ex. 25. If ω is the cube root of unity, then one root of the equation**

$$\begin{vmatrix} \mathbf{x+1} & \omega & \omega^2 \\ \omega & \mathbf{x}+\omega^2 & \mathbf{1} \\ \omega^2 & \mathbf{1} & \mathbf{x}+\omega \end{vmatrix} = 0 \textbf{ is } 0.$$

(MNR 90)

Sol. Adding all the rows of the given determinant to first, the given equation reduces to

$$\begin{vmatrix} x+1+\omega+\omega^2 & x+\omega+\omega^2+1 & \omega^2+1+x+\omega \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix} = 0$$

or $$\begin{vmatrix} x & x & x \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix} = 0, \quad \because \ 1+\omega+\omega^2 = 0$$

or $$x\begin{vmatrix} 1 & 1 & 1 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix} = 0 \Rightarrow x = 0$$

Hence proved.

***Ex. 26. The value of θ which lies between $\theta = 0$ and $\theta = \pi/2$ and satisfy the equation**

$$\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0 \text{ is}$$

(a) $7\pi/24$, (b) $5\pi/24$, (c) $11\pi/24$, (d) $\pi/24$. *(I.I.T)*

Sol. Given equation is

$$\begin{vmatrix} 1+\sin^2\theta+\cos^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta+1+\cos^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta+\cos^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0,$$

adding 2nd col. to 1st.

or $$\begin{vmatrix} 2 & \cos^2\theta & 4\sin 4\theta \\ 2 & 1+\cos^2\theta & 4\sin 4\theta \\ 1 & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0, \quad \because \quad \cos^2\theta+\sin^2\theta = 1$$

or $$\begin{vmatrix} 2 & \cos^2\theta & 4\sin 4\theta \\ 0 & 1 & 0 \\ 1 & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0, \text{ replacing } R_2 \text{ by } R_2 - R_1$$

or $$\begin{vmatrix} 2 & 4\sin 4\theta \\ 1 & 1+4\sin 4\theta \end{vmatrix} = 0, \text{ expanding w.r. to } R_2$$

or $2(1+4\sin 4\theta) - 4\sin 4\theta = 0$ or $4\sin 4\theta + 2 = 0$

or $\sin 4\theta = -1/2 = \sin 210^\circ$ or $\sin 330^\circ$

$= \sin(7\pi/6)$ or $\sin(11\pi/6)$

or $4\theta = 7\pi/6$ or $11\pi/6$ or $\theta = 7\pi/24$ or $11\pi/24$

Hence the required values of θ are given by (a), (c). **Ans.**

Ex. 27. If $\Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix}$ are the given determinants then show that $\dfrac{d}{dx}\Delta_1 = 3\Delta_2$ *(M.N.R.)*

Sol. $$\frac{d}{dx}\Delta_1 = \begin{vmatrix} 1 & 0 & 0 \\ a & x & b \\ a & a & x \end{vmatrix} + \begin{vmatrix} x & b & b \\ 0 & 1 & 0 \\ a & a & x \end{vmatrix} + \begin{vmatrix} x & b & b \\ a & x & b \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x & b \\ a & x \end{vmatrix} + \begin{vmatrix} x & b \\ a & x \end{vmatrix} + \begin{vmatrix} x & b \\ a & x \end{vmatrix},$$

expanding 1st, 2nd and 3rd determinants w.r. to 1st, 2nd and 3rd row respectively.

$$= 3\begin{vmatrix} x & b \\ a & x \end{vmatrix} = 3\Delta_2$$

Hence proved.

Ex. 28. The value of determinant $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix}$ is zero if

(a) a, b, c are in H.P.

(b) a, b, c, are in G.P.

(c) α is a root of the equation $ax^2+bx+c=0$

(d) $(x-\alpha)$ is a factor of $ax^2+2bx+c$ *(I.I.T.)*

Sol. If $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix}=0,$

then $\begin{vmatrix} a & b & 0 \\ b & c & 0 \\ a\alpha+b & b\alpha+c & -\alpha(a\alpha+b)-(b\alpha+c) \end{vmatrix}=0,$

replacing C_3 by $C_3-\alpha C_1-C_2$

or $\quad -(a\alpha^2+2b\alpha+c)\begin{vmatrix} a & b \\ b & c \end{vmatrix}=0$

or $\quad (a\alpha^2+2b\alpha+c)(ac-b^2)=0$

i.e. $\quad$ either $b^2=ac$ or $a\alpha^2+2b\alpha+c=0$

If $b^2=ac$, then a, b, c are in G.P.

Hence result (b) is true.

If $a\alpha^2+2b\alpha+c=0$, then α is a root of the equation $ax^2+2bx+c=0$ or $(x-\alpha)$ is a factor of $ax^2+2bx+c$. Hence result (d) is true.

Ans. (b) and (d).

EXERCISES ON CHAPTER IV

***Ex. 1.** If $2s=a+b+c$, prove that

$$\begin{vmatrix} a^2 & (s-a)^2 & (s-a)^2 \\ (s-b)^2 & b^2 & (s-b)^2 \\ (s-c)^2 & (s-c)^2 & c^2 \end{vmatrix}=2s^3(s-a)(s-b)(s-c)$$

Ex. 2. If $a+b+c=0$, solve the equation

$$\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix}=0$$

[Hint : Add all the rows or columns]

Ex. 3. Show that $\begin{vmatrix} a & a^2 & a^3+bc \\ b & b^2 & b^3+ca \\ c & c^2 & c^3+ab \end{vmatrix}=(a-b)(b-c)(c-a)(ab+bc+ca+abc).$

***Ex. 4.** Show that there are three values of t for which the system of equations :

$(a-t)\,x + by + cz = 0,\ bx + (c-t) + az = 0,\ cx + ay + (b-t)\,z = 0$

have a common non-zero solution. If the three values of t are t_1, t_2, t_3 show that $t_1\, t_2\, t_3 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Ex. 5. Prove that $\begin{vmatrix} 0 & x & y & z \\ -x & 0 & c & -b \\ -y & -c & 0 & a \\ -z & b & -a & 0 \end{vmatrix} = (ax + by + cz)^2$

Ex. 6. Show that $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1+a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$

Ex. 7. Solve $\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 3x & x+2 \end{vmatrix} = 0$

(Meerut 92)

Ans. $x = -3, 2, 1$

Ex. 8. Show that the roots of the following equations are all real :

$$\begin{vmatrix} a+x & h & g \\ h & b+x & f \\ g & f & c+x \end{vmatrix} = 0,$$

where a, b, c, f, g, h are all real numbers.

Ex. 9. Solve $\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+3 \end{vmatrix} = 0$

***Ex. 10.** By the product of determinants estabilish Eulier's theorem that the product of any two sums each of four squares is expressible as the sum of four squares. Does the theorem hold for $(3^2+4^2)(1^2+2^2+3^2)$?

Hence express $(9^2+2^3+3^2+4^2)(5^2+6^2+7^2+8^2)$ as sum of four squares.

(**Hint.** See Ex. 20 (a) and (b) Pages 196–97.

Also $(3^2+4^2)(1^2+2^2+3^2) = (3^2+4^2+0^2+0^2)(1^2+2^2+3^2+0^2)$

Now proceed as in Ex. 20 (b) Page 197.

Ans. $(3^2+4^2)(1^2+2^2+3^2) = 5^2 + (10)^2 + 9^2 + (12)^2$

and $(9^2+2^2+3^2+4^2)(5^2+6^2+7^2+8^2)$

$= (22)^2 + (116)^2 + (40)^2 + (60)^2.$

Four possible answers for the following questions are given. Choose the correct answer :—

Ex. 11. The value of the determinant $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \\ 8 & 15 & 20 \end{vmatrix}$ is

(a) 20, (b) 10, (c) 2, (d) 5.

(*M.N.R. 91*)

Ans. (c)

Ex. 12. The value of the determinant $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$ is

(a) $a+b+c$, (b) 1, (c) 0, (d) abc

Ans. (c)

Ex. 13. If $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, then Δ is equal to

(a) $-b_1B_1 - b_2B_2 - b_3B_3$; (b) $-b_1B_1 + b_2B_2 - b_3B_3$;

(c) $b_1B_1 - b_2B_2 + b_3B_3$; (d) $b_1B_1 + b_2B_2 + b_3B_3$

Ans. (d)

Ex. 14. The value of the determinant $\begin{vmatrix} 3 & 6 & 12 \\ 5a & 5b & 5c \\ a & b & c \end{vmatrix}$ is

(a) 15, (b) 2, (c) 0, (d) 4.

Ans. (c)

Ex. 15. The cofactor of a in the determinant $\begin{vmatrix} 3 & 4 & 5 \\ 7 & 8 & 9 \\ a & b & c \end{vmatrix}$ is

(a) $\begin{vmatrix} 4 & 5 \\ 8 & 9 \end{vmatrix}$, (b) $\begin{vmatrix} 3 & 4 \\ 7 & 8 \end{vmatrix}$, (c) $\begin{vmatrix} 3 & 5 \\ 7 & 9 \end{vmatrix}$, (d) None of These

Ans. (a)

●

OBJECTIVE TYPE QUESTIONS
CH. I TO IV

(A) SHORT & VERY SHORT ANSWER TYPE QUESTIONS

1. Define matrix. *(Kanpur 2001)* [See § 1·02 Page 2]

2. Define a rectangular and a square matrix. [See § 1·03 P. 4]

3. What are horizontal and vertical matrices ? [See § 1·03 P. 4]

4. What are row and column vectors ? [See § 1·03 P. 4]

5. What is $\mathbf{I}_4$? [See § 1·03 P. 4]

6. Define an unit matrix. [See § 1·03 P. 4]

7. Define a diagonal matrix. [See § 1·03 P. 5]

8. What do you understand by a sub-matrix ? [See § 1·03 P. 5]

9. Write down the properties of matrix addition. [See § 1·07 P. 7]

10. If **A, B, C** be three matrices of the same order and are such that $\mathbf{A} + \mathbf{B} = \mathbf{A} + \mathbf{C}$, then show that $\mathbf{B} = \mathbf{C}$. [See Prop. VI Page 8]

11. When are the two matrices conformable to multiplication ? [See § 1·08 Page 11]

12. Give an example to show that the product of two non-zero matrices can be a zero matrix. [See § 1·08 Note 2 Page 13]

13. If $\mathbf{A} = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 4 \\ 2 & 2 \\ 1 & 0 \end{bmatrix}$, then does **BA** exist ?

Ans. No.

14. If $\mathbf{A} = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 6 \\ -1 & -1 & -3 \end{bmatrix}$, then show that $\mathbf{A}^2 = \mathbf{O}$.

15. If $\mathbf{A} = [\,x, y, z\,]$, $\mathbf{B} = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, then show that

$\mathbf{B}\,\mathbf{C}\,| = ax^2 + by^2 + cz^2 + 2hxy + 2gzx + 2fyz.$

16. If $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix}$, calculate integral powers of **A**. *(Purvanchal 97)*

17. When are two matrices said to be equal ? *(Purvanchal 2001)*

18. Define null matrix. *(Purvanchal 2001)* [See § 1·03 Page 4]

19. Define transposed matrix. [See § 2·08 Page 69]

20. Define Nilpotent matrix. [See § 2·07 Page 68]

1o2/VOO

21. State reversal rule for the inverse of product of two matrices.
(Purvanchal 98) [See § 2·19 Th. II Page 92]

22. Show that **AB** = **BA**, where

$$\mathbf{A}=\begin{bmatrix}\cos\theta_1 & -\sin\theta_1\\ \sin\theta_1 & \cos\theta_1\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}\cos\theta_2 & -\sin\theta_2\\ \sin\theta_2 & \cos\theta_2\end{bmatrix}.$$

(Meerut 2001)

23. State reversal rule for the transpose of a product.
[See § 2·09 Th. IV Page 71]

24. Describe elementary row operations on a matrix.
(Purvanchal 99) [See § 3·01 Page 103]

25. Define cofactor of an element of a determinant.
[See § 4·05 P. 124]

26. Define minor of an element of a determinant.
[See § 4·07 P. 128]

27. Show that $\begin{vmatrix}1 & a & b+c\\ 1 & b & c+a\\ 1 & c & a+b\end{vmatrix}=0$ [See Ex. 2(a) P. 132]

28. Evaluate $\begin{vmatrix}-2 & 1 & 1\\ 1 & -2 & 1\\ 1 & 1 & -2\end{vmatrix}$ **Ans.** 0

29. Show that $\begin{vmatrix}1 & 1 & 1\\ 1 & 1+x & 1\\ 1 & 1 & 1+y\end{vmatrix}=xy.$ [See Ex. 7. Page 135]

30. Evaluate $\begin{vmatrix}x & a & a\\ a & x & a\\ a & a & x\end{vmatrix}.$ **Ans.** $(x+2a)(x-a)^2$
[See Ex. 20. P. 140]

31. Show that $\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^2 & b^2 & c^2\end{vmatrix}=(a-b)(b-c)(c-a).$
[See Ex. 28. P. 145]

32. Evaluate $\begin{vmatrix}1 & 1 & 1 & 1\\ 1 & 1+a & 1 & 1\\ 1 & 1 & 1+b & 1\\ 1 & 1 & 1 & 1+c\end{vmatrix}.$ **Ans.** abc.
[See Ex. 36 P. 149]

33. Show that $\begin{vmatrix}0 & c & b\\ c & 0 & a\\ b & a & 0\end{vmatrix}^2=\begin{vmatrix}c^2+b^2 & ba & ca\\ ab & c^2+a^2 & bc\\ ac & bc & b^2+a^2\end{vmatrix}.$
[See Ex. 1. P. 163]

34. Express $\begin{bmatrix}0 & c & b\\ c & 0 & a\\ b & a & 0\end{bmatrix}$ as a determinant. *(Purvanchal 97)*

(B) OBJECTIVE TYPE QUESTIONS

(I) MULTIPLE CHOICE TYPE

Select the correct answer of the following :

1. The order of the matrix $\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$ is

(i) 3×2 ; (ii) 2×3 ; (iii) 3 ; (iv) none of these.

2. A matrix $\mathbf{A} = [a_{ij}]_{m \times n}$ is called a square matrix, if

(i) $m \leq n$; (ii) $m \geq n$; (iii) $m = n$; (iv) none of these.

3. A matrix $\mathbf{A} = [a_{ij}]_{m \times n}$ is called a vertical matrix, if

(i) $m > n$; (ii) $m = n$; (iii) $m < n$; (iv) none of these.

4. A matrix $\mathbf{A} = [a_{ij}]_{m \times n}$ is called a horizontal matrix, if

(i) $m > n$; (ii) $m = n$; (iii) $m < n$; (iv) none of these.

5. If a is the number of elements in a row and b is the number of elements in a column of a matrix **A**, then the order of **A** is

(i) $a \times b$; (ii) $b \times a$; (iii) $a \times a$; (iv) $b \times b$.

6. A matrix $\mathbf{A} = [a_{ij}]_{m \times n}$ is called a row matrix, if

(i) $m < n$; (ii) $m = n$; (iii) $m = 1$; (iv) $n = 1$.

7. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called a

(i) column matrix; (ii) row matrix;
(iii) null matrix; (iv) unit matrix.

8. The matrix $[a_{ij}]_{m \times n}$ will be an unit matrix, if

(i) all its mn elements are unity;
(ii) $m = n$ and all elements are unity;
(iii) $m = n$, diagonal elements are unity and other elements are zero;
(iv) none of these.

9. The negative of a matrix **A** is

(i) zero ; (ii) $-\mathbf{A}$; (iii) $+\mathbf{A}$; (iv) non-existant.

(Kanpur 2001)

10. If **A** is of order 2×2 and **B** is of order 2×3, then **BA** is of order

(i) 2×3 ; (ii) 3×2 ; (iii) 2×2 ; (iv) none of these.

11. **A** is any $m \times n$ matrix such that **AB** and **BA** are both defined, then the order of **B** is

(i) $m \times m$; (ii) $n \times n$; (iii) $n \times m$; (iv) none of these.

12. If $\mathbf{A} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} b & 0 \\ -a & 0 \end{bmatrix}$, then **AB** is

(i) null matrix ; (ii) unit matrix ;
(iii) vertical matrix ; (iv) none of these.

13. If $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 5 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 0 & 7 \\ 5 & 4 \end{bmatrix}$, then

(i) $AB = BA$; (ii) AB does not exist but BA exists

(iii) AB exists but BA does not exist ;

(iv) $AB \neq BA$.

14. If $A = \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix}$, then A^2 is a

(i) diagonal matrix ; (ii) unit matrix ;

(iii) null matrix ; (iv) none of these.

15. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, the value of $A^2 - 4A$ is

(i) I, (ii) $2I$, (iii) $4I$, (iv) $5I$.

16. Transpose of a column matrix is

(i) square matrix ; (ii) column matrix ;

(iii) row matrix ; (iv) none of these.

17. If $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 5 \\ 2 & -5 & 0 \end{bmatrix}$, then $A' =$

(i) A ; (ii) $-A$; (iii) $2A$; (iv) none of these.

(Kanpur 2001)

18. If A is a square matrix, then $A + A'$ will be

(i) diagonal ; (ii) symmetric ;

(iii) skew-symmetric ; (iv) identity matrix. *(Kanpur 2001)*

19. The matrix $\begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is

(i) unitary ; (ii) idempotent ;

(iii) nilpotent ; (iv) identity.

20. In an upper triangular matrix, the elements $a_{ij} = 0$ for

(i) $i > j$; (ii) $i = j$; (iii) $i < j$; (iv) $i \leq j$.

21. A matrix A is called involutory if

(i) $A^2 = A$; (ii) $A^2 = O$; (iii) $A^2 = I$; (iv) none of these.

22. Which of the following is a scalar matrix ?

(i) $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix}$; (ii) $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$; (iii) $\begin{bmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & a \end{bmatrix}$;

(iv) none of these.

23. If ω be the cube root of unity, then the value of the determinant

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{vmatrix} \text{ is}$$

(i) 1; (ii) 0; (iii) −1; (iv) none of these.

24. If $\omega^3 = 1$, then the value of the determinant

$$\begin{vmatrix} 1 & \omega^3 & \omega^2 \\ \omega^3 & 1 & \omega \\ \omega^2 & \omega & 1 \end{vmatrix} \text{ is :}$$

(i) 0 ; (ii) 1 ; (iii) 2 ; (iv) 3.

25. The determinant $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$ is divisible by

(i) $a - b$; (ii) $a + b$; (iii) ab ; (iv) a/b.

26. If $\begin{vmatrix} 2 & x \\ 3 & 7 \end{vmatrix} = 2$, the value of x is

(i) 1 ; (ii) 2 ; (iii) 3 ; (iv) 4.

27. If two rows of a determinant are proportional, the value of the determinant is

(i) infinite ; (ii) not zero ; (iii) negative ; (iv) zero.

28. One of the roots of $\begin{vmatrix} 2-x & 3 & 3 \\ 3 & 4-x & 5 \\ 3 & 5 & 4-x \end{vmatrix} = 0$ is

(i) −1 ; (ii) 0 ; (iii) 1 ; (iv) none of these.

29. If each element in the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is doubled, the value of the determinant of the matrix is

(i) doubled ; (ii) unchanged ;
(iii) multiplied by 4 ; (iv) none of these.

30. $\Delta = | a_{ij} |$ is a determinant of order three and A^{ij} denotes the cofactors of a_{ij} in Δ, then which of the following is not correct ?

(i) $a_{ij} A^{ij} = \Delta$; (ii) $a_{i1} A^{i2} = 0$;
(iii) $a_{i3} A^{i2} = 0$; (iv) $a_{2j} A^{2j} = 0$. *(Kanpur 2001)*

31. If two rows of a determinant are interchanged, the value of the determinant

(i) remains unchanged ;
(ii) is negative of the value of original determinant ;

(iii) doubles ; (iv) none of these.

32. The system of linear equations can be solved easily by the rule of

(i) Newton ; (ii) Bessel ; (iii) Cramer ; (iv) none of these.

(II) TRUE AND FALSE TYPE

Write "T" or "F" according as the statement is true or false :

1. The elements of matrix may be scalar or vector quantities.
2. Matrix denotes a number.
3. The order of the matrix [2] is 2 × 1.
4. If in a matrix, the number of columns is more than the number of rows, then it is called a horizontal matrix.
5. In a row matrix there is only one column.
6. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is an unit matrix.
7. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -5 \end{bmatrix}$ is called a diagonal matrix.
8. It is not necessary for the two matrices **A** and **B** to be of the same order so as to be conformable for addition and subtraction.
9. Commutative law holds but associative law does not for addition of matrices.
10. Addition for matrices obeys the distributive law.
11. The product of two non-zero matrices can be a zero matrix.
12. Matrices of different orders can be subtracted.
13. Matrix multiplication in general is commutative.
14. Commutative law does not hold for addition of matrices.
15. The transpose of the transpose of a matrix is the matrix itself.
16. A square matrix $\mathbf{A} = [a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ for all values of i and j.
17. If $\mathbf{A} = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$, then show that $\mathbf{AA}'$ is a symmetric matrix. [See Ex. 2 Page 82]
18. The inverse of a matrix is not unique. [See § 2·19 Th. I P. 92]
19. If **A** is an orthogonal matrix, then **A**′ is also orthogonal.
20. If **A** is an unitary matrix, then $\mathbf{A}^{-1}$ is not an unitary matrix.
21. A matrix $[a_{ij}]$ is called a triangular matrix if $a_{ij} = 0$ for $i > j$.
22. Non-square matrix has no inverse.
23. If two rows of a determinant $|\mathbf{A}|$ are identical, then $|\mathbf{A}| = 0$.
24. A square matrix **A** is singular if and only if $|\mathbf{A}| \neq 0$.

25. $\begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix} = \begin{vmatrix} 1 & -2x & x^2 \\ 1 & -2y & y^2 \\ 1 & -2z & z^2 \end{vmatrix} \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$.

26. A determinant can be expanded using any row or column of the determinant.

27. The value of a determinant changes if the elements of a row are added to or subtracted from the corresponding elements of another row.

28. If each element of a column of a determinant be multiplied by some constant, then the determinant is multiplied by that constant.

29. The value of a determinant changes if its rows and columns are interchanged.

30. The value of a determinant changes in sign if two consecutive rows (or columns) are interchanged.

(III) FILL IN THE BLANKS TYPE

Fill in the blanks in the following :

1. Each of the mn numbers constituting an $m \times n$ matrix is known as an .... of the matrix.

2. The plural of the word 'matrix' is ....

3. The order of the matrix [3] is ....

4. A matrix which is not a square matrix is known as a .... matrix.

5. If $m = 1$ in the matrix $\mathbf{A} = [a_{ij}]_{m \times n}$, then it is called a .... matrix.

6. If $m > n$ in the matrix $\mathbf{A} = [a_{ij}]_{m \times n}$, then it is called a .... matrix.

7. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called .... matrix.

8. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ is called a .... matrix.

9. Two matrices are conformable for addition and subtraction if they are of the .... order.

10. Matrices of different orders .... be added.

11. Additive identity .... for addition of matrices.

12. If **B** be the additive inverse of the matrix $\mathbf{A} = [a_{ij}]_{m \times n}$ then (i, j)th element of **B** is ....

13. If **A** is an $m \times n$ matrix and **B** is an $n \times k$ matrix, then **AB** is an .... matrix.

14. If **AB** = – **BA**, the matrices **A** and **B** are said to ....

15. Multiplication of matrices is .... with respect to matrix addition.

16. Negative matrix is obtained by multiplying it by ....

17. A matrix when added to its negative gives the .... matrix.

18. A matrix when multiplied by .... gives the null matrix.

19. If **A** is a skew-Hermitian matrix, then i**A** is ....

20. If a square matrix **A** is idempotent then ....

21. The square matrix $\mathbf{A} = [a_{ij}]$ is skew-symmetric, if $a_{ij} =$ ...., for all values of i and j.

22. If **A** is a Hermitian matrix, then i**A** is .... *(Meerut 2001)*

23. If **A** is an orthogonal matrix, then $\mathbf{A}^{-1}$ is ....

24. $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} =$ .... $(a-b)(b-c)(c-a)$.

25. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} =$ .... $(a-b)(b-c)(c-a)$.

26. If any two columns of a determinant are .... , then value of the determinant is zero.

27. The system of linear equations can easily be solved by the rule of ....

28. A determinant can be expanded using .... row of it.

Answers to Objective Questions

(I) Multiple Choice Type :

1. (ii) ; **2.** (iii) ; **3.** (i) ; **4.** (iii) ; **5.** (ii) ; **6.** (iii) ; **7.** (iv) ; **8.** (iii) ; **9.** (ii) ; **10.** (iv) ; **11.** (iii) ; **12.** (i) ; **13.** (iv) ; **14.** (iii) ; **15.** (iv) ; **16.** (iii) ; **17.** (ii) ; **18.** (ii) ; **19.** (iv) ; **20.** (i) ; **21.** (iii) ; **22.** (ii) ; **23.** (ii) ; **24.** (iv) ; **25.** (i) ; **26.** (iv) ; **27.** (iv) ; **28.** (i) ; **29.** (iii) ; **30.** (iv) ; **31.** (ii) ; **32.** (iii).

(II) True and False Type :

1. T ; **2.** F ; **3.** F ; **4.** T ; **5.** F ; **6.** F ; **7.** T ; **8.** F ; **9.** F ; **10.** T ; **11.** T ; **12.** F ; **13.** F ; **14.** F ; **15.** T ; **16.** T ; **17.** T ; **18.** F ; **19.** T ; **20.** F ; **21.** T ; **22.** T ; **23.** T ; **24.** F ; **25.** T ; **26.** T ; **27.** F ; **28.** T ; **29.** F ; **30.** T.

(III) Fill in the blanks type :

1. element ; **2.** matrices ; **3.** 1×1 ; **4.** rectangular ; **5.** row ; **6.** vertical ; **7.** an unit ; **8.** diagonal ; **9.** same ; **10.** cannot ; **11.** exists ; **12.** $-a_{ij}$; **13.** $m \times k$; **14.** anticommute ; **15.** distributive ; **16.** -1 ; **17.** null ; **18.** zero ; **19.** Hermitian ; **20.** $\mathbf{A}^2 = \mathbf{A}$; **21.** $-a_{ji}$; **22.** skew-Hermitian ; **23.** also orthogonal ; **24.** abc ; **25.** $(a+b+c)$; **26.** identical ; **27.** Cramer ; **28.** any.