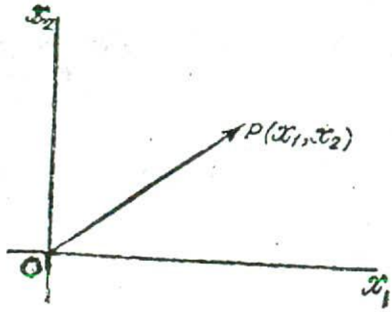
Chapter VIII

Linear Dependence of Vectors

§ 8.01. Two dimensional vector.

We know that the ordered pair of real numbers (x_1, x_2) is used to denote a point P in a plane where Ox_1 and Ox_2 are the coordinate-axes.



(Fig. 1)

A two dimensional vector or **2-vector** OP is denoted by the same pair of numbers written as $[x_1, x_2]$.

If $\mathbf{A}_1 = [x_{11}, x_{12}]$ and $\mathbf{A}_2 = [x_{21}, x_{22}]$ are two distinct two-dimensional vectors, then their sum by parallelogram law of addition is given by

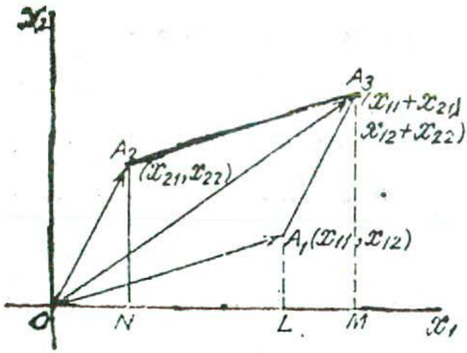
$\mathbf{A}_3 = \mathbf{A}_1 + \mathbf{A}_2$

$= [x_{11} + x_{21}, x_{12} + x_{22}]$

[Here $OM = OL + LM$

$= OL + ON = x_{11} + x_{21}$ etc.]

If we treat $\mathbf{A}_1$ and $\mathbf{A}_2$ as 1×2 matrices, we find that the above is the rule for adding matrices as given in chapter I.



(Fig. 2)

Also we observe that $k \cdot \mathbf{A}_1 = [k\,x_{11}, k\,x_{12}]$, where k is any scalar.

§ 8.02. n-dimensional vector or n-vector.

Definition. An ordered set of n elements x_i of a field F, written as

$$\mathbf{A} = [x_1, x_2, x_3, \ldots, x_n] \qquad \text{...(i)}$$

is called an **n-dimensional vector** or **n-vector A** over F and the elements $x_1, x_2, \ldots, x_n$ are called the first, second, ..., nth **components** of **A**.

We find it more convenient to write the components of a vector in a column as

$$\mathbf{A}' = [x_1, x_2, \ldots, x_n]' = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ \ldots \\ x_n \end{bmatrix} \qquad \text{...(ii)}$$

(i) is called a **row-vector** and (ii) is called a **column-vector.**

Thus we consider the $p \times q$ matrix as defining q column vectors or p row vectors. **(Note)**

Note 1. The sum or difference of two rwo (or column) vectors is formed by the rule governing matrices as given in chapter I.

Note 2. The product of a scalar and a vector is formed by the rule governing matrices as given in chapter I.

Note 3. The vector whose all the components are zero is known as **the null vector** or **zero vector** and is written as **O.**

Solved Examples on § 8.01 – § 8.02

Ex. 1. Given the 3-vectors

$A_1 = [1, 2, 1], A_2 = [2, 1, 4], A_3 = [2, 3, 6]$, evaluate $2A_1 + A_2, 5A_1 - 2A_3$

Solution. $2A_1 + A_2 = 2[1, 2, 1] + [2, 1, 4]$

$= [2, 4, 2] + [2, 1, 4]$

$= [2+2, 4+1, 2+4] = [4, 5, 6]$. **Ans.**

$5A_1 - 2A_3 = 5[1, 2, 1] - 2[2, 3, 6]$

$= [5, 10, 5] - [4, 6, 12]$

$= [5-4, 10-6, 5-12] = [1, 4, -7]$. **Ans.**

Ex. 2. Given the four-dimensional column-vectors

$A_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}$ **and** $A_2 = \begin{bmatrix} 2 \\ 5 \\ 7 \\ 9 \end{bmatrix}$, **evaluate $3A_1 + 2A_2$.**

Solution. $3A_1 + 2A_2 = 3\begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 5 \\ 7 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 6 \\ 9 \end{bmatrix} + \begin{bmatrix} 4 \\ 10 \\ 14 \\ 18 \end{bmatrix} = \begin{bmatrix} 3+4 \\ 0+10 \\ 6+14 \\ 9+18 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ 20 \\ 27 \end{bmatrix}$ **Ans.**

Ex. 3. Given the three-dimensional row vectors

$A_1 = [3, 1, -4], A_2 = [0, -4, 1]; A_3 = [2, 2, -3]$, evaluate $2A_1 - A_2 - 3A_3$.

Solution. $2A_1 - A_2 - 3A_3 = 2[3, 1, -4] - [0, -4, 1] - 3[2, 2, -3]$

$= [6, 2, -8] - [0, -4, 1] - [6, 6, -9]$

$= [0, 0, 0] = \mathbf{O}$. **Ans.**

***§ 8.03. Linear dependence and independence of vectors.**

(Agra 94; Purvanchal 97)

The nm-vectors over the field F,

$A_1 = [x_{11}, x_{12}, \ldots, x_{1m}], A_2 = [x_{21}, x_{22}, \ldots, x_{2m}], \ldots,$

$A_n = [x_{n1}, x_{n2}, \ldots, x_{nm}]$

are called **linearly dependent** over F if there exists a set of n elements $\lambda_1, \lambda_2, \ldots, \lambda_n$ of F, λ's being not all zero, such that

$$\lambda_1 \mathbf{A}_1 + \lambda_2 \mathbf{A}_2 + \ldots + \lambda_n \mathbf{A}_n = \mathbf{O}.$$

Otherwise the n-vectors are called **linearly independent** over F.

For example the 3-vectors given in Ex. 3 above are linearly dependent.

Note : *A vector $\mathbf{A}_{n+1}$ can be expressed as a **linear combination** of the vectors $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$ if there exist elements $\lambda_1, \lambda_2, \ldots, \lambda_n$ of F such that

$$\mathbf{A}_{n+1} = \lambda_1 \mathbf{A}_1 + \lambda_2 \mathbf{A}_2 + \ldots + \lambda_n \mathbf{A}_n$$

Solved Examples on § 8.03.

Ex. 1. Examine whether the set of vector $\mathbf{V}_1 = \{1, 2, 3\}$, $\mathbf{V}_2 = \{1, 0, 1\}$ and $\mathbf{V}_3 = \{0, 1, 0\}$ are linearly dependent or not. *(Purvanchal 94)*

Solution. Let the given set of vectors be linearly dependent, so that

$$\lambda_1 \mathbf{V}_1 + \lambda_2 \mathbf{V}_2 + \lambda_3 \mathbf{V}_3 = \mathbf{O} \quad \ldots(i)$$

or $\lambda_1 \{1, 2, 3\} + \lambda_2 \{1, 0, 1\} + \lambda_3 \{0, 1, 0\} = \mathbf{O} = \{0, 0, 0\}$ **(Note)**

or $\{\lambda_1 + \lambda_2, 2\lambda_1 + \lambda_3, 3\lambda_1 + \lambda_2\} = \{0, 0, 0\}$

$\therefore$ $\lambda_1 + \lambda_2 = 0, 2\lambda_1 + \lambda_3 = 0, 3\lambda_1 + \lambda_2 = 0$

$\Rightarrow$ $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0$ *i.e.* λ's are all zero.

Hence the given set of vectors are not linearly dependent *i.e.* these are linearly independent.

Ex. 2. Examine the following set of vectors of the real field for linear dependence or independence :—

$\mathbf{A}_1 = [2, -1, 3, 2]$; $\mathbf{A}_2 = [1, 3, 4, 2]$; $\mathbf{A}_3 = [3, -5, 2, 2]$

Also express $\mathbf{A}_3$ as a linear combination of $\mathbf{A}_1, \mathbf{A}_2$.

Solution. Suppose the given set of vectors is linearly dependent, so that

$$\lambda_1 \mathbf{A}_1 + \lambda_2 \mathbf{A}_2 + \lambda_3 \mathbf{A}_3 = \mathbf{O}. \quad \ldots(i)$$

or $\lambda_1 [2, -1, 3, 2] + \lambda_2 [1, 3, 4, 2] + \lambda_3 [3, -5, 2, 2] = \mathbf{O} = [0, 0, 0, 0]$ **(Note)**

or $[2\lambda_1 + \lambda_2 + 3\lambda_3, -\lambda_1 + 3\lambda_2 - 5\lambda_3, 3\lambda_1 + 4\lambda_2 + 2\lambda_3, 2\lambda_1 + 2\lambda_2 + 2\lambda_3] = [0, 0, 0, 0]$

$\therefore$ $2\lambda_1 + \lambda_2 + 3\lambda_3 = 0$ ...(i); $-\lambda_1 + 3\lambda_2 - 5\lambda_3 = 0$ ...(ii)

$3\lambda_1 + 4\lambda_2 + 2\lambda_3 = 0$ ...(iii); and $2(\lambda_1 + \lambda_2 + \lambda_3) = 0$ ...(iv)

From (i) and (iv) we get, $\lambda_1 + 2\lambda_3 = 0$ or $\lambda_1 = -2\lambda_3$ ...(v)

$\therefore$ From (iv) we get $\lambda_2 = -\lambda_1 - \lambda_3 = 2\lambda_3 - \lambda_3 = \lambda_3$ ...(vi)

$\therefore$ From (ii), (v) and (vi) we get

$-\lambda_1 + 3\lambda_2 - 5\lambda_3 = -(-2\lambda_3) + 3\lambda_3 - 5\lambda_3 = 0$. Hence (ii) is satisfied.

Again from (iii) we get $3\lambda_1 + 4\lambda_2 + 2\lambda_3$

$= 3(-2\lambda_3) + 4(\lambda_3) + 2\lambda_3$, from (v), (vi)

$= 0$. Hence (iii) is also satisfied.

Thus for $\lambda_1 = -2\lambda_3$ and $\lambda_2 = \lambda_3$ all the equations (i), (ii), (iii) and (iv) are satisfied and therefore the given set of vectors are linearly dependent.

$\therefore$ From (i) we get $-2\lambda_3\mathbf{A}_1 + \lambda_3\mathbf{A}_2 + \lambda_3\mathbf{A}_3 = \mathbf{O}$ or $\mathbf{A}_3 = 2\mathbf{A}_1 - \mathbf{A}_2$, which expresses $\mathbf{A}_3$ as a linear combination of $\mathbf{A}_1$ and $\mathbf{A}_2$.

Ex. 3. Show, using a matrix, that the set of vectors $\mathbf{X}_1 = [2, 3, 1, -1], \mathbf{X}_2 = [2, 3, 1, -2], \mathbf{X}_3 = [4, 6, 2, -3]$ is linearly dependent. *(Agra 96)*

Solution. Let the given set of vectors be linearly dependent, so that

$$\lambda_1\mathbf{X}_1 + \lambda_2\mathbf{X}_2 + \lambda_3\mathbf{X}_3 = \mathbf{O} \quad ...(i)$$

or $\lambda_1 [2, 3, 1, -1] + \lambda_2 [2, 3, 1, -2] + \lambda_3 [4, 6, 2, -3] = \mathbf{O} = [0, 0, 0, 0]$

or $[2\lambda_1 + 2\lambda_2 + 4\lambda_3, 3\lambda_1 + 3\lambda_2 + 6\lambda_3, \lambda_1 + \lambda_2 + 2\lambda_3, -\lambda_1 - 2\lambda_2 - 3\lambda_3] = [0, 0, 0, 0]$

$\therefore$ $2\lambda_1 + 2\lambda_2 + 4\lambda_3 = 0;$ $\quad 3\lambda_1 + 3\lambda_2 + 6\lambda_3 = 0$

$\lambda_1 + \lambda_2 + 2\lambda_3 = 0;$ $\quad -\lambda_1 - 2\lambda_2 - 3\lambda_3 = 0$

which reduce to $\lambda_1 + \lambda_2 + 2\lambda_3 = 0$, $\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0$

whence solving we get $\lambda_1 + \lambda_3 = 0$, $\lambda_2 + \lambda_3 = 0$

which give $\quad \dfrac{\lambda_1}{1} = \dfrac{\lambda_2}{1} = \dfrac{\lambda_3}{-1}$

$\therefore$ From (i) we get $\mathbf{X}_1 + \mathbf{X}_2 - \mathbf{X}_3 = \mathbf{O}$, where $\lambda_1, \lambda_2, \lambda_3$ are not all zero. Hence given set of vectors is linearly dependent.

Ex. 4. Examine the following set of vectors over the real field for linear dependence or independence :—

$\mathbf{A}_1 = [1, 2, 1]; \mathbf{A}_2 = [2, 1, 4]; \mathbf{A}_3 = [4, 5, 6]; \mathbf{A}_4 = [1, 8, -3]$

Solution. Suppose that the given set of vectors is linearly dependent, so that

$$\lambda_1\mathbf{A}_1 + \lambda_2\mathbf{A}_2 + \lambda_3\mathbf{A}_3 + \lambda_4\mathbf{A}_4 = \mathbf{O} \quad ...(i)$$

or $\lambda_1 [1, 2, 1] + \lambda_2 [2, 1, 4] + \lambda_3 [4, 5, 6] + \lambda_4 [1, 8, -3] = \mathbf{O} = [0, 0, 0]$

or $[\lambda_1 + 2\lambda_2 + 4\lambda_3 + \lambda_4, 2\lambda_1 + \lambda_2 + 5\lambda_3 + 8\lambda_4, \lambda_1 + 4\lambda_2 + 6\lambda_3 - 3\lambda_4] = [0, 0, 0]$

$\therefore$ $\lambda_1 + 2\lambda_2 + 4\lambda_3 + \lambda_4 = 0$ ...(ii); $\quad 2\lambda_1 + \lambda_2 + 5\lambda_3 + 8\lambda_4 = 0$ ...(iii)

and $\quad \lambda_1 + 4\lambda_2 + 6\lambda_3 - 3\lambda_4 = 0$ ...(iv)

From (ii) and (iv) we have $2\lambda_2 + 2\lambda_3 - 4\lambda_4 = 0$

or $\quad \lambda_2 + \lambda_3 - 2\lambda_4 = 0$ ...(v)

From (iii) and (v) we have $2\lambda_1 + \lambda_2 + 5\lambda_3 + 4(\lambda_2 + \lambda_3) = 0$

or $\quad 2\lambda_1 + 5\lambda_2 + 9\lambda_3 = 0$ ...(vi)

From (v) and (vi) we have $2\lambda_1 + 5\lambda_2 + 9(2\lambda_4 - \lambda_2) = 0$

or $\quad 18\lambda_4 - 4\lambda_2 + 2\lambda_1 = 0$ or $9\lambda_4 = 2\lambda_2 - \lambda_1$ ...(vii)

(v), (vi) and (vii) are satisfied by $\lambda_1 = 0 = \lambda_2 = \lambda_3 = \lambda_4$

Hence the given set of vectors are linearly independent.

§ 8.04. Basic Theorems on Linear dependence of vectors.

Theorem I. *If there be n linearly dependent vectors, then some one of them can always be expressed as a linear combination of the remaining ones.*

Proof. Let $\mathbf{A}_1 = [x_{11}, x_{12}, ..., x_{1m}]$, $\mathbf{A}_2 = [x_{21}, x_{22}, ..., x_{2m}]$, ... $\mathbf{A}_n = [x_{n1}, x_{n2}, ..., x_{nm}]$ be n m-vectors over the field F, such that

$$\lambda_1\mathbf{A}_1 + \lambda_2\mathbf{A}_2 + \lambda_3\mathbf{A}_3 + \ldots + \lambda_n\mathbf{A}_n = \mathbf{O}, \qquad \text{...(i)}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are elements of F and not all zero.

Let $\lambda_r \neq 0$ then solving (i) we get

$$\mathbf{A}_r = -\frac{1}{\lambda_r}\left[\lambda_1\mathbf{A}_1 + \lambda_2\mathbf{A}_2 + .. + \lambda_{r-1}\mathbf{A}_{r-1} + \lambda_{r+1}\mathbf{A}_{r+1} + \ldots + \lambda_n\mathbf{A}_n\right]$$

or $$\mathbf{A}_r = \mu_1\mathbf{A}_1 + \mu_2\mathbf{A}_2 + .. + \mu_{r-1}\mathbf{A}_{r-1} + \mu_{r+1}\mathbf{A}_{r+1} + \ldots + \mu_n\mathbf{A}_n \qquad \text{...(ii)}$$

Hence proved.

Theorem II. *If there be n linearly independent vectors* $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$, *whereas the set obtained by adding another vector* $\mathbf{A}_{n+1}$ *is linearly dependent, then* $\mathbf{A}_{n+1}$ *can be expressed as linear combination of* $\mathbf{A}_1, \mathbf{A}_2, \ldots \mathbf{A}_n$.

Proof. Given $\lambda_1\mathbf{A}_1 + \lambda_2\mathbf{A}_2 + \ldots + \lambda_n\mathbf{A}_n \neq \mathbf{O}$,. ...(i)

where $\lambda_1, \lambda_2, ..., \lambda_n$ are elements of the field F.

And $(\lambda_1\mathbf{A}_1 + \lambda_2\mathbf{A}_2 + \ldots + \lambda_n\mathbf{A}_n) + \mathbf{A}_{n+1} = \mathbf{O}$ (Note)

or $$\mathbf{A}_{n+1} = -[\lambda_1\mathbf{A}_1 + \lambda_2\mathbf{A}_2 + \ldots + \lambda_n\mathbf{A}_n]$$

Hence proved.

Example. Consider three 3-vectors

$$\mathbf{A}_1 = [4, 5, 6], \mathbf{A}_2 = [2, 1, 4], \mathbf{A}_3 = [1, 2, 1]$$

Let $\mathbf{A}_1$ and $\mathbf{A}_3$ be linearly dependent then we must have

$\lambda_1\mathbf{A}_1 + \lambda_2\mathbf{A}_3 = \mathbf{O}$, where λ_1 and λ_2 are to be determined.

or $\lambda_1 [4, 5, 6] + \lambda_2 [1, 2, 1] = \mathbf{O} = [0, 0, 0]$

∴ $4\lambda_1 + \lambda_2 = 0$; $5\lambda_1 + 2\lambda_2 = 0$ and $6\lambda_1 + \lambda_2 = 0$

Solving first and third of these we get $\lambda_1 = 0 = \lambda_2$ which satisfies the second also. But as all λ's are zero, so $\mathbf{A}_2$ and $\mathbf{A}_3$ are not linearly dependent [See § 8.03 Page 211 of this chapter]

But we find that if we take $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ to be linearly dependent

then $$\lambda_1\mathbf{A}_1 + \lambda_2\mathbf{A}_2 + \lambda_3\mathbf{A}_3 = \mathbf{O}. \qquad \text{...(i)}$$

or $\lambda_1 [4, 5, 6] + \lambda_2 [2, 1, 4] + \lambda_3 [1, 2, 1] = \mathbf{O} = [0, 0, 0]$

∴ $4\lambda_1 + 2\lambda_2 + \lambda_3 = 0$ ...(ii), $5\lambda_1 + \lambda_2 + 2\lambda_3 = 0$ ...(iii)

and $$6\lambda_1 + 4\lambda_2 + \lambda_3 = 0. \qquad \text{...(iv)}$$

From (ii) and (iv) on subtracting we get $\lambda_1 + \lambda_2 = 0$

or $$\lambda_2 = -\lambda_1. \qquad \text{...(v)}$$

From (ii) and (v) we get

$$4\lambda_1 + 2(-\lambda_1) + \lambda_3 = 0 \quad \text{or} \quad \lambda_3 = -2\lambda_1 \qquad \text{...(vi)}$$

Substituting values from (v) and (vi) in (i) we get

$$\lambda_1\mathbf{A}_1 - \lambda_1\mathbf{A}_2 - 2\lambda_1\mathbf{A}_3 = \mathbf{O} \quad \text{or} \quad \mathbf{A}_1 - \mathbf{A}_2 - 2\mathbf{A}_3 = \mathbf{O} \qquad \text{...(vii)}$$

or

$$\mathbf{A}_2 = \mathbf{A}_1 - 2\mathbf{A}_3. \qquad \text{..(viii)}$$

Thus we find that though $\mathbf{A}_1$ and $\mathbf{A}_3$ are not linearly dependent yet $\mathbf{A}_1$ $\mathbf{A}_2$, $\mathbf{A}_3$ satisfy the relation (vii) and from (viii) we find that $\mathbf{A}_2$ can be expressed as a linear combination of $\mathbf{A}_1$ and $\mathbf{A}_3$.

***Theorem III.** *If there be a subset of r linearly dependent vectors among the n vectors $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$ such that $r < n$, then the vectors of the whole set are linearly dependent.*

Proof. Let the subsets $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$ of the given n vectors $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$ be linearly dependent, then we have

$\lambda_1\mathbf{A}_1 + \lambda_2\mathbf{A}_2 + \ldots + \lambda_r\mathbf{A}_r = \mathbf{O}$, where all λ's are not zero

We can rewrite this as

$\lambda_1\mathbf{A}_1 + \lambda_2\mathbf{A}_2 + \ldots + \lambda_r\mathbf{A}_r + 0.\mathbf{A}_{r+1} + 0\,.\,\mathbf{A}_{r+2} + \ldots + 0.\mathbf{A}_n = \mathbf{O}$, **(Note)**

where all λ's are not zero.

Hence the set of vectors $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_r, \mathbf{A}_{r+1}, \ldots, \mathbf{A}_n$ by definition are linearly dependent. Hence proved.

****Theorem IV.** *If the rank of the matrix associated with a set of n m-vector, is r where $r < n$, then there are exactly r vectors which are linearly independent while each of remaining $n - r$ vectos can be expressed as a linear combination of these r vectors.*

****Theorem V.** *A necessary and sufficient condition that the vectors $\mathbf{A}_1 = [x_{11}, x_{12}, \ldots, x_{1m}]$, $\mathbf{A}_2 = [x_{21}, x_{22}, \ldots, x_{2m}] \ldots$, $\mathbf{A}_n = [x_{n1}, x_{n2}, \ldots, x_{nm}]$ be linearly dependent is that the matrix* $\mathbf{A} = \begin{bmatrix} x_{11} & x_{12} & \ldots & \ldots & x_{1m} \\ x_{21} & x_{22} & \ldots & \ldots & x_{2m} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ x_{n1} & x_{n2} & \ldots & \ldots & x_{nm} \end{bmatrix}$ *of the vectors is of rank $r < n$. If the rank is n, the vectors are linearly independent.*

Proofs of Theorem IV and V above are beyond the scope of this book.

§ 8.05. Linear Form.

Definition. A linear form over F in m variables $x_1, x_2, \ldots, x_m$ is a polynomial of the type

$$\sum_{i=1}^{m} a_i x_i = a_1x_1 + a_2x_2 + \ldots + a_mx_m,$$

where the coefficients are in F.

Consider a system of n linear forms in m variables

$$\begin{bmatrix} f_1 = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1m}x_m \\ f_2 = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2m}x_m \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ f_n = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nm}x_m \end{bmatrix} \qquad \ldots(i)$$

and the associated matrix formed by their coefficients is

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}.$$

If there exist elements $\lambda_1, \lambda_2, \ldots, \lambda_n$ in F, λ's being not all zero such that

$$\lambda_1 f_1 + \lambda_2 f_2 + \ldots + \lambda_n f_n = 0,$$

then the forms (i) are said to be **linearly dependent**, otherwise they are said to be **linearly independent**.

Thus, the linear dependence or independence of the forms (i) is equivalent to the linear dependence or independence of the row vectors of the matrix **A**.

More Solved Examples :

Ex. 1. Show that the set of vectors $\mathbf{A}_1 = (1, 1, 1), \mathbf{A}_2 = (1, 2, 3)$, $\mathbf{A}_3 = (2, 3, 8)$ is linearly independent.

Solution. Suppose that the given set of vectors is linearly dependent, so that

$$\lambda_1\mathbf{A}_1 + \lambda_2\mathbf{A}_2 + \lambda_3\mathbf{A}_3 = \mathbf{O}, \qquad \ldots(i)$$

where λ's are to be determined.

or $\lambda_1 (1, 1, 1) + \lambda_2 (1, 2, 3) + \lambda_3 (2, 3, 8) = (0, 0, 0)$

or $(\lambda_1 + \lambda_2 + 2\lambda_3, \lambda_1 + 2\lambda_2 + 3\lambda_3, \lambda_1 + 3\lambda_2 + 8\lambda_3) = (0, 0, 0)$

$\therefore$ $\lambda_1 + \lambda_2 + 2\lambda_3 = 0$ ...(ii); $\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0$ ...(iii)

and $\lambda_1 + 3\lambda_2 + 8\lambda_3 = 0.$ ...(iv)

From (ii) and (iii) we get $\lambda_2 + \lambda_3 = 0.$ ...(v)

From (ii) and (iv) we get $2\lambda_2 + 6\lambda_3 = 0$ or $\lambda_2 + 3\lambda_3 = 0$ ...(vi)

From (v) and (vi) we get $\lambda_3 = 0.$

$\therefore$ From (v) we get $\lambda_2 = 0$ and from (ii) we get $\lambda_1 = 0$, when $\lambda_2 = 0 = \lambda_3$.

Thus all the λ's are zero and hence the given set of vectors is linearly independent.

Ex. 2 (a). Show that set of vectors $\mathbf{A}_1 = [1, 2, 3], \mathbf{A}_2 = [3, 2, 1]$, $\mathbf{A}_3 = [1, 1, 1]$ is linearly dependent. *(Agra 93)*

Solution : Suppose that the given set of vectors is linearly dependent, so that

$$\lambda_1\mathbf{A}_1 + \lambda_2\mathbf{A}_2 + \lambda_3\mathbf{A}_3 = \mathbf{O}$$

or $$\lambda_1 [1, 2, 3] + \lambda_2 [3, 2, 1] + \lambda_3 [1, 1, 1] = \mathbf{O} = [0, 0, 0]$$

or $$[\lambda_1 + 3\lambda_2 + \lambda_3, 2\lambda_1 + 2\lambda_2 + \lambda_3, 3\lambda_1 + \lambda_2 + \lambda_3] = [0, 0, 0]$$

$\therefore$ $\lambda_1 + 3\lambda_2 + \lambda_3 = 0$ ...(i); $2\lambda_1 + 2\lambda_2 + \lambda_3 = 0$ ...(ii)

and $3\lambda_1 + \lambda_2 + \lambda_3 = 0$ ...(iii)

From (i) and (ii) we get $\lambda_1 - \lambda_2 = 0$ or $\lambda_1 = \lambda_2$ ...(iv)

From (iv) and (iii) we get $4\lambda_1 + \lambda_3 = 0$ or $\lambda_3 = -4\lambda_1$ ....(v)

Then we have $\frac{\lambda_1}{1} = \frac{\lambda_2}{1} = \frac{\lambda_3}{-4}$ and from here we do not get all the values of λ as zero.

Hence the given set of vectors is linearly dependent. Hence proved.

Ex. 2 (b). Show that the set of vectors [1, 2, 3], [3, – 2, 1], [1, – 6, – 5] is linearly dependent. *(Agra 92)*

Solution : Let $\mathbf{A}_1 = [1, 2, 3]$, $\mathbf{A}_2 = [3, -2, 1]$ and $\mathbf{A}_3 = [1, -6, -5]$ be a set of linearly dependent vectors.

Then $\lambda_1\mathbf{A}_1 + \lambda_2\mathbf{A}_2 + \lambda_3\mathbf{A}_3 = \mathbf{O}$, where all λ's are not zero.

or $$\lambda_1 [1, 2, 3] + \lambda_2 [3, -2, 1] + \lambda_3 [1, -6, -5] = \mathbf{O} = [0, 0, 0]$$

or $$[\lambda_1 + 3\lambda_2 + \lambda_3, 2\lambda_1 - 2\lambda_2 - 6\lambda_3, 3\lambda_1 + \lambda_2 - 5\lambda_3] = [0, 0, 0].$$

$\therefore$ $\lambda_1 + 3\lambda_2 + \lambda_3 = 0$ ...(i), $2\lambda_1 - 2\lambda_2 - 6\lambda_3 = 0$ ...(ii

and $3\lambda_1 + \lambda_2 - 5\lambda_3 = 0$ ...(iii)

From (i) and (ii), we get $8\lambda_2 + 8\lambda_3 = 0$ or $\lambda_2 + \lambda_3 = 0$ ...(iv

From (ii) and (iii), we get $8\lambda_1 - 16\lambda_3 = 0$ or $\lambda_1 = 2\lambda_3$ ...(v

$\therefore$ From (iv) and (v) we get $\frac{\lambda_1}{2} = \frac{\lambda_2}{-1} = \frac{\lambda_3}{1}$ and this does not give all the valu of λ as zero.

Hence the given set of vectors is linearly dependent. Hence prove

Ex. 3. Find a linear relation, if any, between the linear forms of th following system $f_1 = x + y + z, f_2 = y - 2z, f_3 = 2x + 3y$;

Sollution : Let $\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 = 0.$

Then $\lambda_1 (x + y + z) + \lambda_2 (y - 2z) + \lambda_3 (2x + 3y) = 0$

or $(\lambda_1 + 2\lambda_3)\, x + (\lambda_1 + \lambda_2 + 3\lambda_3)\, y + (\lambda_1 - 2\lambda_2)\, z = 0$

$\Leftrightarrow \lambda_1 + 2\lambda_3 = 0, \lambda_1 + \lambda_2 + 3\lambda_3 = 0, \lambda_1 - 2\lambda_2 = 0$

whence we get $\lambda_3 = -\frac{1}{2}\lambda_1, \lambda_2 = \frac{1}{2}\lambda_1$, which satisfy $\lambda_1 + \lambda_2 + 3\lambda_3 = 0$

Hence from (i) we get $\lambda_1 f_1 + \frac{1}{2}\lambda_1 f_2 - \frac{1}{2}\lambda_1 f_3 = 0$

or $2f_1 + f_2 - f_3 = 0$, which is the required relation.

Ex. 4. Find a linear relation, if any, between the polynomials $f_1 = 2x^3 - 3x^2 + 4x - 2; f_2 = 3x^3 + 2x^2 - 2x + 5; f_3 = 5x^3 - x^2 + 2x + 1.$

Solution. Let $\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 = 0$ ...(i)

Then $\lambda_1 (2x^3 - 3x^2 + 4x - 2) + \lambda_2 (3x^3 + 2x^2 - 2x + 5)$

$+ \lambda_3 (5x^3 - x^2 + 2x + 1) = 0$

or $(2\lambda_3 + 3\lambda_2 + 5\lambda_3) x^3 + (-3\lambda_1 + 2\lambda_2 - \lambda_3) x^2 + (4\lambda_1 - 2\lambda_2 + 2\lambda_3) x$

$+ (-2\lambda_1 + 5\lambda_2 + \lambda_3) = 0$

$\Rightarrow$ $2\lambda_1 + 3\lambda_2 + 5\lambda_3 = 0$ ...(ii); $3\lambda_1 - 2\lambda_2 + \lambda_3 = 0$ ...(iii)

$2\lambda_1 - \lambda_2 + \lambda_3 = 0$ ...(iv); and $2\lambda_1 - 5\lambda_2 - \lambda_3 = 0$ ...(v)

Solving (ii) and (iv) we get $\lambda_2 - \lambda_3 = 0$ ...(vi)

From (iii) and (iv) we get $3\lambda_1 + 3\lambda_3 = 0$ or $\lambda_1 + \lambda_3 = 0$ (vii)

From (v), (vi) and (vii) we get

$2(-\lambda_3) - 5(-\lambda_3) - \lambda_3 = 0$ or $2\lambda_3 = 0$ or $\lambda_3 = 0$

which gives $\lambda_1 = 0 = \lambda_2$.

Hence from (i) no linear relation exists between f_1, f_2 and f_3.

Ex. 5. If the vectors $(0, 1, a), (1, a, 1), (a, 1, 0)$ are linearly dependent, then find the value of a. *(Agra 94)*

Solution : Let $\mathbf{A}_1 = (0, 1, a), \mathbf{A}_2 = (1, a, 1), \mathbf{A}_3 = (a, 1, 0)$ be a set of linearly dependent vectors.

Then $\lambda_1 \mathbf{A}_1 + \lambda_2 \mathbf{A}_2 + \lambda_3 \mathbf{A}_3 = \mathbf{O}$, where λ's are not all zero.

or $\lambda_1 (0, 1, a) + \lambda_2 (1, a, 1) + \lambda_3 (a, 1, 0) = \mathbf{O} = (0, 0, 0)$

or $(\lambda_2 + a\lambda_3, \lambda_1 + a\lambda_2 + \lambda_3, a\lambda_1 + \lambda_2) = (0, 0, 0)$

$\therefore$ $\lambda_2 + a\lambda_3 = 0$ ...(i), $\lambda_1 + a\lambda_2 + \lambda_3 = 0$ ...(ii)

and $a\lambda_1 + \lambda_2 = 0$ ...(iii)

From (i), $\lambda_3 = -(1/a)\lambda_2$

From (iii), $\lambda_1 = -(1/a)\lambda_2$

$\therefore$ From (ii), $-(1/a)\lambda_2 + a\lambda_2 - (1/a)\lambda_2 = 0$

or $[a - (2/a)]\lambda_2 = 0$ or $(a^2 - 2)\lambda_2 = 0$

$\therefore$ Either $a^2 - 2 = 0$ or $\lambda_2 = 0$.

But $\lambda_2 = 0$ gives $\lambda_1 = 0, \lambda_3 = 0$, from (i), (iii)

Then $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ are not linearly dependent.

Hence $a^2 - 2 = 0$ or $a = \pm\sqrt{2}$. Ans.

Exercises on Chapter VIII

Ex. 1. Show that the vectors $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are linearly independent.

Ex. 2. Show that the vectors $[1, 2, 0]$, $[8, 13, 0]$ and $[2, 3, 0]$ are linearly dependent.

Ex. 3. Prove that the set of three vectors

$$[1, 2, -1, 3], [0, -2, 1, -1] \text{ and } [2, 2, -1, 5]$$

is linearly dependent and obtain a relation connecting these vectors.

Ex. 4. Find a linear relation, if any, between the linear forms of the system :—

$$f_1 = 2x_1 - 2x_2 - x_3 + x_4; f_2 = x_1 - x_2 + x_3 + x_4; f_3 = 5x_2 + 3x_3 + x_4.$$

Ex. 5. Prove that any non-empty subset of a linearly independent set is linearly independent. *(Agra 94)*

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