Chapter IX

Quadratic Forms

§ 9.01. Quadratic Form.

Definiton. A homogeneous polynomial of the type

$$q = X' AX = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j \qquad ...(i)$$

whose coefficients a_{ij} are elements of the field F is known as a quadratic form over F in the variables $x_1, x_2, \ldots, x_n$.

For Example. $x_1^2 + 2x_2^2 + 5x_3^2 + 8x_1x_3 - 6x_2x_3$ is a quadratic form in the variables x_1, x_2, x_3. Here the matrix of the form can be written in many ways according as the cross product terms $8x_1 x_3$ and $-6x_2x_3$ are seperated to form the terms $a_{13}x_1x_3$, $a_{31}x_3x_1$ and $a_{23} x_2 x_3$, $a_{32} x_3 x_2$.

Here we shall agree that the matrix **A** of quadratic form be **symmetric** and shall always separate the cross-product terms so that $a_{ij} = a_{ji}$

$$\therefore \quad q = x_1^2 + 2x_2^2 + 5x_3^2 + 8x_1x_3 - 6x_2x_3$$

$$= [x_1\ x_2\ x_3] \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & -3 \\ 4 & -3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$ ...see Ex. 1 (b) Page 223 of this chapter

$$= X' \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & -3 \\ 4 & -3 & 5 \end{bmatrix} X = X'AX$$

The symmetric matrix $A = [a_{ij}]$ is known as the matrix of the quadratic form and the rank of **A** is called the **rank of the quadratic form.**

If the rank of the form is $r < n$, then the quadratic form is called **singular** otherwise **non-singular.**

Transformations.

The linear transformation $X = BY$ over F carries the quadratic form (i) above with symmetric matrix **A** over F into the quadratic form

$$(BY)'A\,(BY) = (Y'B')\,A\,(BY) = Y'\,(B'AB)\,Y \qquad ...(ii)$$

with symmetric matrix $B'AB$.

Equivalent Quadratic Forms.

Definition. Two quadratic forms in the same variables $x_1, x_2, \ldots, x_n$ are called **equivalent** if and only if there exists a non-singular linear transformation $X = BY$ which together with $Y = IX$, where **I** is the identity matrix, carries one of the forms into the other.

As **B′AB** is congruent to **A**, we have

1. The rank of a quadratic form is invariant under a non-singular transformation of the variables.

2. Two quadratic forms over F are equivalent over F iff their matrices are congruent over F.

A quadratic form of rank r can be reduced to the form

$$b_1y_1^2 + b_2y_2^2 + b_3y_3^2 + \dots + b_ry_r^2,\ b_i \neq 0 \qquad \text{...(iii)}$$

in which only terms in the squares of the variables occur by a non-singular linear transformation $\mathbf{X} = \mathbf{BY}$.

§ 9.02. Lagrange's Reduction.

The reduction of a quadratic form to the form (iii) of § 9.01 above can be carried out by a method or procedure called Lagrange's Reduction, which consists of repeated completing of the square.

From example $q = x_1^2 + 2x_2^2 + 5x_3^2 + 8x_1x_3 - 6x_2x_3$ can be reduced to form (iii) of § 9.01 above as follows :—

$$\begin{aligned} q &= x_1^2 + 2x_2^2 + 5x_3^2 + 8x_1x_3 - 6x_2x_3 \\ &= (x_1^2 + 8x_1x_3 + 16x_3^2) + 2x_2^2 - 11x_3^2 - 6x_2x_3 \\ &= (x_1 + 4x_3)^2 + \tfrac{1}{2}(4x_2^2 - 12x_2x_3) - 11x_3^2 \\ &= (x_1 + 4x_3)^2 + \tfrac{1}{2}(4x_2^2 - 12x_2x_3 + 9x_3^2) - \tfrac{9}{2}x_3^2 - 11x_3^2 \\ &= (x_1 + 4x_3)^2 + \tfrac{1}{2}(2x_2 - 3x_3)^2 - \tfrac{31}{2}x_3^2 \\ &= y_1^2 + (1/2)\,y_2^2 - (31/2)\,y_3^2, \end{aligned}$$

where $y_1 = x_1 + 4x_3,\ y_2 = 2x_2 - 3x_3,\ y_3 = x_3$.

§ 9.03. Real Quadratic Forms.

Let a real quadratic form $q = \mathbf{X'AX}$ be reduced by a real non-singular transformation to the form $b_1y_1^2 + b_2y_2^2 + \dots + b_ry_r^2,\ b_i \neq 0$. If one or more of the b_i are negative, then there exists a non-singular transformation $\mathbf{X} = \mathbf{CZ}$, where **C** is obtained from **B** (see § 9.01 Page 220 of this chapter) by a sequence of row and column transformations which carries q into

$$h_1z_1^2 + h_2z_2^2 + \dots + h_pz_p^2 - h_{p+1}z_{p+1}^2 - \dots - h_rz_r^2 \qquad \text{...(i)}$$

in which the terms with positive coefficients precede those with negative coefficients.

Now the non-singular transformation

$$s_i = \sqrt{(h_i)}\, z_i,\ i = 1, 2, \dots, r$$

$$s_j = z_j,\ j = r+1, r+2, \dots, n$$

carries (i) into the canonical form

$$s_2^2 + s_2^2 + s_3^2 + \dots + s_p^2 - s_{p+1}^2 - \dots - s_r^2. \qquad \dots(ii)$$

Thus, as the product of non-singular transformations is a non-singular transformation, we have every real quadratic form can be reduced by a non-singular transformation to the canonical form (ii) above, where p, the number of positive terms is called the index and r is the rank of the given quadratic form.

Example. In Ex. 3 (c) Page 225 the quadratic form $q = x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3$ was reduced to $q_1 = y_1^2 - 2y_2^2 + 9y_3^2$. The non-singular transformation $y_1 = z_1, y_2 = z_3, y_3 = z_2$ carries q_1 into $q_2 = z_1^2 + 9z_2^2 - 2z_3^2$ and the non-singular transformation $z_1 = s_1$, $z_2 = s_2/3, z_3 = s_3/\sqrt{2}$ reduces q_2 to $q_3 = s_1^2 + s_2^3 - s_3^2$.

Also in Ex. 3 (c) Page 225 we have

$$y_1 = x_1 - 2x_2 + 4x_3, y_2 = x_2 - 4x_3, y_3 = x_3$$

or $\quad x_1 = y_1 + 2y_2 + 4y_3, x_2 = y_2 + 4y_3, x_3 = y_3$

or $\quad x_1 = z_1 + 2z_3 + 4z_2, x_2 = z_3 + 4z_2, x_3 = z_2$

or $\quad x_1 = s_1 + 2(s_3/\sqrt{2}) + 4(s_2/3), x_2 = (s_3/\sqrt{2}) + 4(s_2/3), x_3 = s_2/3$

or $\quad x_1 = s_1 + (4/3)s_2 + \sqrt{2}s_3, x_2 = (4/3)s_2 + (1/2)\sqrt{2}s_3, x_3 = (1/3)s_2$

or $$\mathbf{X} = \begin{bmatrix} 1 & 4/3 & \sqrt{2} \\ 0 & 4/3 & (1/2)\sqrt{2} \\ 0 & 1/3 & 0 \end{bmatrix} \mathbf{S}$$

is the non-singular linear transformation that reduces q to $q_3 = s_1^2 + s_2^2 - s_3^2$.

$\therefore$ The quadratic form is of rank 3 and index 2.

§ 9.04. Complex Quadratic Forms

Let the complex quadratic form $\mathbf{X'AX}$ be reduced by a non-singular transformation to the form $b_1y_1^2 + b_2y_2^2 + \dots + b_ry_r^2, b_i \neq 0$.

Evidently the non-singular transformation

$$z_i = \sqrt{(b_i)}\, y_i, \; i = 1, 2, \dots, r$$

$$z_j = y_j, \qquad j = r+1, r+2, \dots, n$$

carries $b_1y_1^2 + b_2y_2^2 + \dots + b_ry_r^2$ into $z_1^2 + z_2^2 + \dots + z_r^2$. ...(i)

Solved Examples on § 9.01 to § 9.04

***Ex. 1. Write the following quadratic forms in matrix notation :**

(a) $x_1^2 + 4x_1x_2 + 3x_2^2$;

(b) $x_1^2 - 2x_2^2 - 3x_3^3 + 4x_1x_2 + 6x_1x_3 - 8x_2x_3$. *(Garhwal 95)*

Sol. Let $x_1^2 + 4x_1x_2 + 3x_2^2 = \mathbf{X'AX} = [x_1 \; x_2] \begin{bmatrix} a_1 & a_2 \\ a_2 & b_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ ...(i) since **A** is a symmetric matrix

i.e. $$x_1^2 + 4x_1x_2 + 3x_2^2 = [a_1x_1 + a_2x_2 \quad a_2x_1 + b_1x_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= [(a_1x_1 + a_2x_2)\, x_1 + (a_2x_1 + b_1x_2)\, x_2]$$

$$\Rightarrow \quad x_1^2 + 4x_1x_2 + 3x_2^2 = a_1x_1^2 + 2a_2x_1x_2 + b_1x_2^2$$

Comparing coefficients of x_1^2, x_1x_2 and x_2^2 on both sides, we get

$$a_1 = 1, a_2 = 2, b_1 = 3.$$

$\therefore$ From (i), $\quad x_1^2 + 4x_1x_2 + 3x_2^2 = \mathbf{X}'\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\mathbf{X}$ **Ans**

(b) Let $x_1^2 - 2x_2^2 - 3x_3^2 + 4x_1x_2 + 6x_1x_3 - 8x_2x_3$

$$= \mathbf{X'AX} = [x_1 \;\; x_2 \;\; x_3]\begin{bmatrix} a_1 & b_1 & c_1 \\ b_1 & b_2 & c_2 \\ c_1 & c_2 & c_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \ldots$$

$\because$ **A** is a symmetric matrix

$$= [a_1x_1 + b_1x_2 + c_1x_3 \quad b_1x_1 + b_2x_2 + c_2x_3 \quad c_1x_1 + c_2x_2 + c_3x_3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

$$= [(a_1x_1 + b_1x_2 + c_1x_3)\, x_1 + (b_1x_1 + b_2x_2 + c_2x_3)\, x_2 + (c_1x_1 + c_2x_2 + c_3x_3)\, x_3]$$

$$\Rightarrow \quad x_1^2 - 2x_2^2 - 3x_3^2 + 4x_1x_2 + 6x_1x_3 - 8x_2x_3$$

$$= a_1x_1^2 + b_2x_2^2 + c_3x_3^2 + 2b_1x_1x_2 + 2c_1x_1x_3 + 2c_2x_2x_3$$

$$\Rightarrow \quad a_1 = 1, b_2 = -2, c_3 = -3, b_1 = 2, c_1 = 3, c_2 = -4.$$

$\therefore$ From (ii), we have $x_1^2 - 2x_2^2 - 3x_3^2 + 4x_1x_2 + 6x_1x_3 - 8x_2x_3$

$$= \mathbf{X}'\begin{bmatrix} 1 & 2 & 3 \\ 2 & -2 & -4 \\ 3 & -4 & -3 \end{bmatrix}\mathbf{X}$$ **Ans.**

Ex. 2 (a). Find the matrix of the quadratic form $\mathbf{x_1^2 + 2x_2^2 - 5x_3^2 - x_1x_2 + 4x_2x_3 - 3x_3x_1}$ and verify that it can be written as a matrix product X′AX.

(Garhwal 94, 93)

Sol. Let $x_1^2 + 2x_2^2 - 5x_3^2 - x_1x_2 + 4x_2x_3 - 3x_3x_1$

$$= \mathbf{X'AX} = [x_1 \;\; x_2 \;\; x_3]\begin{bmatrix} a_1 & b_1 & c_1 \\ b_1 & b_2 & c_2 \\ c_1 & c_2 & c_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \ldots(i)$$

since **A** is a symmetric matrix

$$= [a_1x_1 + b_1x_2 + c_1x_3 \quad b_1x_1 + b_2x_2 + c_2x_3 \quad c_1x_1 + c_2x_2 + c_3x_3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= [(a_1x_1 + b_1x_2 + c_1x_3)\,x_1 + (b_1x_1 + b_2x_2 + c_2x_3)\,x_2 + (c_1x_1 + c_2x_2 + c_3x_3)\,x_3]$$

$$\Rightarrow \quad x_1^2 + 2x_2^2 - 5x_3^2 - x_1x_2 + 4x_2x_3 - 3x_3x_1$$

$$= a_1x_1^2 + b_2x_2^2 + c_3x_3^2 + 2b_1x_1x_2 + 2c_2x_2x_3 + 2c_1x_3x_1$$

Equating coefficients of like terms on both sides, we get

$a_1 = 1, b_2 = 2, c_3 = -5, 2b_1 = -1, 2c_2 = 4, 2c_1 = -3$

$\therefore$ From (i), we have the given quadratic form

$$= [x_1 \quad x_2 \quad x_3]\begin{bmatrix} 1 & -1/2 & -3/2 \\ -1/2 & 2 & 2 \\ -3/2 & 2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\therefore$ The required matrix of the given quadratic form

$$= \begin{bmatrix} 1 & -1/2 & -3/2 \\ -1/2 & 2 & 2 \\ -3/2 & 2 & -5 \end{bmatrix}$$ **Ans.**

Also $[x_1 \quad x_2 \quad x_3]\begin{bmatrix} 1 & -1/2 & -3/2 \\ -1/2 & 2 & 2 \\ -3/2 & 2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$= [x_1 - \tfrac{1}{2}x_2 - \tfrac{3}{2}x_3 \quad -\tfrac{1}{2}x_1 + 2x_2 + 2x_3 \quad -\tfrac{3}{2}x_1 + 2x_2 - 5x_3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= (x_1 - \tfrac{1}{2}x_2 - \tfrac{3}{2}x_3)\,x_1 + (-\tfrac{1}{2}x_1 + 2x_2 + 2x_3)\,x_2 + (-\tfrac{3}{2}x_1 + 2x_2 - 5x_3)\,x_3$$

$= x_1^2 + 2x_2^2 - 5x_3^2 - x_1x_2 + 4x_2x_3 - 3x_3x_1 =$ Given quadratic from.

Hence proved.

Ex. 2 (b). Find the matrix of the quadratic form $G = x^2 + y^2 + 3z^2 + 4xy + 5yz + 6zx$ and express it is the form $G = X'AX$, where $X' = (x, y, z)$

(Garhwal 96)

Sol. Let $G = x^2 + y^2 + 3z^2 + 4xy + 5yz + 6zx$

$$= \mathbf{X'AX} = [x \quad y \quad z]\begin{bmatrix} a_1 & b_1 & c_1 \\ b_1 & b_2 & c_2 \\ c_1 & c_2 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \text{...(i)}$$

since A is a symmetric matrix

$$= [a_1x + b_1y + c_1z \quad b_1x + b_2y + c_2z \quad c_1x + c_2y + c_3y]\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$= [(a_1x + b_1y + c_1z)x + (b_1x + b_2y + c_2z)\, y + (c_1x + c_2y + c_3z)\, z]$

$\Rightarrow\ x^2 + y^2 + 3z^2 + 4xy + 5yz + 6zx$

$$= a_1x^2 + b_2y^2 + c_3z^2 + 2b_1xy + 2c_2yz + 2c_1zx$$

$\Rightarrow\ a_1 = 1, b_2 = 1, c_3 = 3, 2b_1 = 4, 2c_2 = 5, 2c_1 = 6$

$\Rightarrow\ a_1 = 1, b_2 = 1, c_3 = 3, b_1 = 2, c_2 = 5/2, c_1 = 3.$

$\therefore$ From (i), we get $G = \mathbf{X'AX} = [x\ \ y\ \ z]\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 5/2 \\ 3 & 5/2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ ...(ii)

which gives the required matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 5/2 \\ 3 & 5/2 & 3 \end{bmatrix}$ **Ans.**

and G can be expressed in the form $\mathbf{X'\ A\ X}$ by (ii).

Ex. 2 (c). Write out in full the quadratic form in x_1, x_2, x_3 whose matrix is $\begin{bmatrix} 2 & -2 & 5 \\ -2 & 3 & 0 \\ 5 & 0 & 4 \end{bmatrix}$.

Sol. Here $\mathbf{X'AX} = [x_1\ \ x_2\ \ x_3]\begin{bmatrix} 2 & -2 & 5 \\ -2 & 3 & 0 \\ 5 & 0 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$= [2x_1 - 2x_2 + 5x_3\ \ -2x_1 + 3x_2 + 0.x_3\ \ 5x_1 + 0.x_2 + 4x_3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$= [(2x_1 - 2x_2 + 5x_3)\, x_1 + (-2x_1 + 3x_2)\, x_2 + (5x_1 + 4x_3)\, x_3]$

$= [2x_1^2 - 2x_2x_1 + 5x_3x_1 - 2x_1x_2 + 3x_2^2 + 5x_1x_3 + 4x_3^2]$

$= [2x_1^2 + 3x_2^2 + 4x_3^2 - 4x_1x_2 + 10x_1x_3]$

$\therefore$ Required quadratic form is

$$2x_1^2 + 3x_2^2 + 4x_3^2 - 4x_1x_2 + 10x_1x_3.$$ **Ans.**

Ex. 3. Reduce the following by Lagrange's Reduction :

(a) $\mathbf{X'}\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & -2 \\ 4 & -2 & 18 \end{bmatrix}\mathbf{X}$; **(b)** $\mathbf{X'}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix}\mathbf{X}$

(c) $q = x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3$

Sol. (a) $\mathbf{X'}\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & -2 \\ 4 & -2 & 18 \end{bmatrix}\mathbf{X} = [x_1\ x_2\ x_3]\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & -2 \\ 4 & -2 & 18 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$= [x_1 + 2x_2 + 4x_3 \quad 2x_1 + 6x_2 - 2x_3 \quad 4x_1 - 2x_2 + 18x_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= [(x_1 + 2x_2 + 4x_3)\, x_1 + (2x_1 + 6x_2 - 2x_3)\, x_2 + (4x_1 - 2x_2 + 18x_3)\, x_3]$$

$$= [x_1^2 + 6x_2^2 + 18x_3^2 + 4x_1x_2 - 4x_2x_3 + 8x_1x_3]$$

$$\therefore \quad q = x_1^2 + 6x_2^2 + 18x_3^2 + 4x_1x_2 - 4x_2x_3 + 8x_1x_3$$

$$= \{x_1^2 + 4x_1 (x_2 + 2x_3)\} + 6x_2^2 + 18x_3^2 - 4x_2x_3$$

$$= \{x_1^2 + 4x_1 (x_2 + 2x_3) + 4 (x_2 + 2x_3)^2\} + 6x_2^2 + 18x_3^2 - 4x_2x_3 - 4 (x_2 + 2x_3)^2$$

$$= \{x_1 + 2 (x_2 + 2x_3)\}^2 + 6x_2^2 + 18x_3^2 - 4x_2x_3 - 4x_2^2 - 16x_3^2 - 16x_3x_2$$

$$= (x_1 + 2x_2 + 4x_3)^2 + 2x_2^2 + 2x_3^2 - 20x_2x_3$$

$$= (x_1 + 2x_2 + 4x_3)^2 + 2 (x_2^2 - 10x_2x_3) + 2x_3^2$$

$$= (x_1 + 2x_2 + 4x_3)^2 + 2 (x_2^2 - 10x_2x_3 + 25x_3^2) - 48x_3^2$$

$$= (x_1 + 2x_2 + 4x_3)^2 + 2 (x_2 - 5x_3)^2 - 48x_3^2$$

$$= y_1^2 + 2y_2^2 - 48y_3^2.$$

Ans.

where $y_1 = x_1 + 2x_2 + 4x_3$, $y_2 = x_2 - 5x_3$, $y_3 = x_3$.

(b) $$\mathbf{X}' \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix} \mathbf{X} = [x_1 \ x_2 \ x_3] \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= [x_1\, 0 + x_2\, 0 + x_3\, 1 \quad x_1 0 + x_2 1 - 2x_3 \quad x_1 - 2x_2 + 3x_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= [(x_3)\, x_1 + (x_2 - 2x_3)\, x_2 + (x_1 - 2x_2 + 3x_3)\, x_3]$$

$$= [x_2^2 + 3x_3^2 + 2x_1x_3 - 4x_2x_3]$$

$$= [z_1^2 + 3z_2^2 + 2z_3z_1 - 4z_1z_2], \text{ using } x_1 = z_3, x_2 = z_1, x_3 = z_2$$

$$= [\{z_1^2 - 4z_1z_2 + 4z_2^2\} - (z_2^2 - 2z_2z_3)]$$

$$= [(z_1 - 2z_2)^2 - (z_2^2 - 2z_2z_3 + z_3^2) + z_3^2]$$

$$= [(z_1 - 2z_2)^2 - (z_2 - z_3)^2 + z_3^2]$$

$$\therefore \quad q = (z_1 - 2z_2)^2 - (z_2 - z_3)^2 + z_3^2$$

$$= y_1^2 - y_2^2 + y_3^2,$$

where $y_1 = z_1 - 2z_2 = x_2 - 2x_3$, $y_2 = z_2 - z_3 = x_3 - x_1$, $y_3 = z_3 = x_1$

(c) $q = x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3$

$$= \{x_1^2 - 4x_1(x_2 - 2x_3)\} + 2x_2^2 - 7x_3^2$$
$$= \{x_1^2 - 4x_1(x_2 - 2x_3) + 4(x_2 - 2x_3)^2\} + 2x_2^2 - 7x_3^2 - 4(x_2 - 2x_3)^2$$
$$= \{x_1 - 2(x_2 - 2x_3)\}^2 + 2x_2^2 - 7x_3^2 - 4(x_2^2 - 4x_2x_3 + 4x_3^2)$$
$$= (x_1 - 2x_2 + 4x_3)^2 - 2x_2^2 + 16x_2x_3 - 23x_3^2$$
$$= (x_1 - 2x_2 + 4x_3)^2 - 2(x_2^2 - 8x_2x_3 + 16x_3^2) + 9x_3^2 \qquad \textbf{(Note)}$$
$$= (x_1 - 2x_2 + 4x_3)^2 - 2(x_2 - 4x_3)^2 + 9x_3^2$$
$$= y_1^2 - 2y_2^2 + 9y_3^2. \qquad \textbf{Ans.}$$

where $y_1 = x_1 - 2x_2 + 4x_3$, $y_2 = x_2 - 4x_3$, $y_3 = x_3$.

Exercise on § 9.01 – § 9.02

Ex. 1. Write $2x_1^2 - 6x_1x_2 + x_3^2$ in matrix notation

Ans. $\mathbf{X}'\begin{bmatrix} 2 & -3 & 0 \\ -3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\mathbf{X}$

Ex. 2. Write out in full the quadratic form in x_1, x_2, x_3 whose matrix is $\begin{bmatrix} 2 & -3 & 1 \\ -3 & 2 & 4 \\ 1 & 4 & -5 \end{bmatrix}$

Ans. $2x_1^2 - 6x_1x_2 + 2x_1x_3 + 2x_2^2 + 8x_2x_3 - 5x_3^2$

Ex. 3. Reduce the following by Lagrange's reduction :

$$\mathbf{X}'\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix}\mathbf{X}$$

Ans. $y_1^2 - y_2^2 + 8y_3^2$

[Hint : Use $x_1 = z_3, x_2 = z_1, x_3 = z_2$]

§ 9.05. Definite and Semi-definite Forms.

Definition (i) A real non-singular quadratic form $q = \mathbf{X'AX}$, $|\mathbf{A}| \neq 0$, in n variables is known as **positive definite** if its rank and index are equal. Thus, in the real field a positive definite quadratic form can be reduced to the form $y_1^2 + y_2^2 + \ldots + y_n^2$ and for any non-trivial set of values of the x's, $q > 0$.

(ii) A real singluar quadratic form $q = \mathbf{X'AX}$, $|\mathbf{A}| = 0$ is as **positive semi-definite** if its rank and index are equal *i.e.* $r = p < n$.

Thus in the real field a positive semi-definite qudratic form can be reduced to the form $y_1^2 + y_2^2 + \ldots + y_r^2$, $r < n$ and for any non-trivial set of values of the x's, $q \geq 0$.

(iii) A real non-singular quadratic form $q = \mathbf{X'AX}$ is known as **negative defnite** if its index $p = 0$ *i.e.* $r = n$, $p = 0$.

Thus in the real field a negative definite form can be reduced to the form $-y_1^2 - y_2^2 - \ldots - y_n^2$ and for any non-trivial set of values of the x's, $q < 0$.

(iv) A real singular quadratic form $q = \mathbf{X}'\mathbf{A}\mathbf{X}$ is known as **negative semi-definite** if its index $p = 0$ *i.e.* $r < n, p = 0$.

Thus in the real field a negative semi-defnite form can be reduced to the form $-y_1^2 - y_2^2 - \ldots - y_n^2$ and for any non-trivial set of values of the x's, $q \leq 0$.

Note 1. If q is negative defnite (semi-definite), then $-q$ is positive definine (semi-defnite).

Note 2. For positive definite quadratic form, if $q = \mathbf{X}'\mathbf{A}\mathbf{X}$ is positive definite then $|\mathbf{A}| > 0$.

§ 9.06. Definite and Semi-definite Matrices.

Definition. The matrix A of a real quadratic form $q = \mathbf{X}'\mathbf{A}\mathbf{X}$ is known as definite or semi-definite according as the quadratic form is definite or semi-definite. Thus

(i) A real symmetric matrix **A** is positive definite iff there exists a non-singular matrix **C**, such that $\mathbf{A} = \mathbf{C}'\mathbf{C}$.

(ii) A real symmetric matrix of rank r is positive semi-definite iff there exists a matrix **C** of rank r, such that $\mathbf{A} = \mathbf{C}'\mathbf{C}$.

§ 9.07. Principal Minors.

Definition. A minor of matrix **A** is known as **principal** if it is obtained by deleting certain rows and the same numbered columns of the matrix **A**.

Note 1. The diagonal elements of a principal minor of the matrix **A** are diagonal elements of the matrix **A**.

Note 2. Every symmetric matrix of rank r has at least one principal minor of order r different from zero.

Note 3. If the matrix **A** is positive definite, then every principal minor of the matrix **A** is positive.

Note 4. If the matrix **A** is positive semi-definite, then every principal minor of the matrix **A** is non-negative.

Exercises on Chapter IX

Ex. 1. Reduce the square matrix **A** into diagonal form and interpret the result in terms of quadratic form :

$$\mathbf{A} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$ *(Garhwal 94).*

Ex. 2. Reduce the quardative form $2x_1^2 + x_2^2 - 3x_3^2 - 8x_2x_3 - 4x_3x_1 + 12x_1x_2$ to normal form. *(Garhwal 93)*

(A) VERY SHORT AND SHORT ANSWER TYPE QUESTIONS

Ch. V Rank and Adjoint of a Matrix

1. Define rank of a matrix. *(Purvanchal 2000)* [See § 5·02 Pages 1 – 2]

2. Find the rank of the matrix A, where

$$A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

(Meerut 2001)

[**Hint :** Do as Ex. 11 (b) P. 28]

Ans. 4

3. Reduce the matrix $\begin{bmatrix} 1 & 3 & 4 \\ -2 & 1 & -1 \\ 3 & -1 & 2 \end{bmatrix}$ to normal form. *(Kanpur 2001)*

Ans. $\begin{bmatrix} \mathbf{I_2} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$

4. When is a matrix said to be in Echelon form ? *(Purvanchal 98)* [See § 5·04 Page 36]

5. Write down the four normal forms of a matrix. [See §5·03 Page 15]

6. Define adjoint of a matrix. [See § 5·08 Page 43]

7. Show that adjoint of the matrix $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ is $\begin{bmatrix} bc & 0 & 0 \\ 0 & ca & 0 \\ 0 & 0 & ab \end{bmatrix}$.

8. How will you use the notion of determinant to complete the inverse of a non-singular square matrix ? [See Th. I result (iv) Page 50]

9. Find the inverse of the matrix $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ [See Ex. 15 Page 67]

Ans. $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

10. If $a^2 + b^2 + c^2 + d^2 = 1$, then show that the inverse of the matrix

$\begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}$ is $\begin{bmatrix} a-ib & -c-id \\ c-id & a+ib \end{bmatrix}$ [See Ex. 18 Page 70]

11. Find the rank of an $m \times n$ matrix, every element of which is unity.

Ans. 1.

[**Hint :** See Ex. 9 Page 96]

12. **A, B, P** and **Q** are matrices such that adj. $\mathbf{B} = \mathbf{A}, |\mathbf{P}| = |\mathbf{Q}| = 1$, then adj $(\mathbf{Q}^{-1}\,\mathbf{B}\,\mathbf{P}^{-1}) = \mathbf{PAQ}$. *(Kanpur 2001)*

Ch.VI Solution of Linear Equations

13. Express in matrix form the system of equations :

$9x + 7y + 3z = 6,\ 5x + y + 4z = 1;\ 6x + 8y + 2z = 4$ [See Ex. 1 Page 106]

14. Define a homogeneous linear equation. [See § 6·10 Page 144]

15. Solve the simultaneous equations given below :

$x + y + 2z = 3;\ 2x + 2y + 3z = 7;\ 3x - y + 2z = 1,\ 2x - y - z = 2$

(Kanpur 2001) **Ans.** $x = 2, y = 3, z = 1$

Ch. VII Characteristics Equations of a Matrix

16. What do you understand by the characteristic equation of the matrix **A**? [See § 7·02 (iii) Page 160]

17. What is eigen value problem ? [See § 7·02 (v) Page 160]

18. Obtain the characteristic equation of the matrix $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

(Meerut 2001) **Ans.** $\lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0$

19. State Cayely-Hamilton's Theorem.

20. Find latent vectors of the matrix $\begin{bmatrix} a & h & g \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ [See Ex. 25(a) Page 200]

Ch. VIII Linear Dependence of Vectors

21. Define linearly dependent and linearly independent set of vectors.

(Kanpur 2001) [See § 8·03 Page 211]

22. Show that the set of vectors $\mathbf{V_1} = \{1, 2, 3\}, \mathbf{V_2} = \{1, 0, 1\}$ and $\mathbf{V_3} = \{0, 1, 0\}$ are linearly independent. [See Ex. 1 Page 212]

23. Find a linear relation, if any, between the linear forms of the following system $f_1 = x + y + z, f_2 = y - 2z, f_3 = 2x + 3y$.

Ans. $2f_1 + f_2 = f_3$ [See Ex. 3 Page 217]

Ch. IX Quadratic Forms

24. Define a quadratic form. [See § 9·01 Page 220]

25. What do you understand by the rank of a quadratic form.

[See § 9·01 Page 220]

26. Write the quadratic form corresponding to the matrix $\begin{bmatrix} 0 & -3 & -4 \\ 3 & 0 & 2 \\ -4 & -2 & 0 \end{bmatrix}$.

(Kanpur 2001) **Ans.** $-4x_1 x_3$

(B) OBJECTIVE TYPE QUESTIONS

(I) MULTIPLE CHOICE TYPE :

Select (i), (ii), (iii) or (iv) whichever is correct :

Ch. V. Rank and Adjoint of a Martix

1. The rank of the matrix $\begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 2 & 3 & 4 \end{bmatrix}$ is

(i) 0 (ii) 1 (iii) 2 (iv) none of these

(Kanpur 2001)

2. The rank of the matrix $\begin{bmatrix} 0 & 2 & 3 \\ 0 & 4 & 6 \\ 0 & 6 & 9 \end{bmatrix}$ is

(i) 0 (ii) 1 (iii) 2 (iv) 3

3. If **A** be a matrix, which one of the following is a number :

(i) $\mathbf{A}^{-1}$ (ii) adj **A** (iii) rank **A** (iv) none of these

4. If **A**′ be the transpose of the matrix **A**, then

(i) rank **A**′ > rank **A** (ii) rank **A**′ = rank **A**

(iii) rank **A**′ < rank **A** (iv) none of these

5. If by a series of elementary transformations an n-rowed square matrix **A** is reduced to the form $\begin{bmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$, the rank of **A** is

(i) $n+r$ (ii) r (iii) $n-r$ (iv) n

6. The rank of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 9 & 16 \end{bmatrix}$ is

(i) 0 (ii) 1 (iii) 2 (iv) 3

7. The rank of the matrix $\begin{bmatrix} 2 & -1 & 3 & 15 \\ 3 & 2 & 0 & 21 \end{bmatrix}$ is

(i) 1 (ii) 2 (iii) 3 (iv) none of these

8. The rank of the matrix $\begin{bmatrix} 1 & 6 & 5 & 5 \\ 3 & 18 & 15 & 3 \\ 1 & 6 & 5 & 1 \end{bmatrix}$ is.

(i) 4 (ii) 3 (iii) 2 (iv) 1

9. The value of a for which the matrix $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 2 & -1 \\ a & 1 & 3 \end{bmatrix}$ is singular, is

(i) -2 (ii) -1 (iii) 1 (iv) 2

10. The necessary and sufficient condition that a square matrix may possess an inverse is that it be

(i) singular (ii) triangular

(iii) non-singular (iv) none of these

11. If **A** is non-singular matrix, then $(\mathbf{A}^{-1})^{-1}$ is

(i) **I** (ii) $\mathbf{A}^{-1}$ (iii) **A** (iv) $\mathbf{AA}^{-1}$

(Kanpur 2001)

12. If a non-singular matrix **A** is symmetric, then $\mathbf{A}^{-1}$ is

(i) skew-symmetric (ii) Hermitian

(iii) diagonal (iv) symmetric

[**Hint :** See § 5·11 Th. V Page 77]

Ch. VI Solution of Linear Equations

13. The system of equations

$x + 2y + z = 2, 3x + 5y + 5z = 4, 2x + 4y + 3z = 3$ has a

(i) unique solution **(ii)** infinite solution

(iii) trivial solution **(iv)** none of these

[See Ex. 5(a) P.110]

14. The system of equations

$3x - y + z = 0, - 15x + 6y - 5z = 0, 5x - 2y + 2z = 0$ has a

(i) unique solution **(ii)** trivial solution

(iii) infinite solution **(iv)** none of these

15. The theorem 'every square matrix satisfies its characteristic equation' is named after

(i) Cramer **(ii)** Hamilton **(iii)** Newton **(iv)** none of them

Ch. VII Characteristic Equations of a Matrix

16. If **A** be any matrix and **I** the identity matrix, then $\mathbf{A} - \lambda\mathbf{I}$ is known as

(i) characteristic polynomial of **A** **(ii)** spectrum of **A**

(iii) characteristic matrix of **A** **(iv)** none of these

(II) TRUE OR FALSE TYPE :

Write "T" or "F" according as the following statements are true or false :

Ch. V Rank and Adjoint of a Matrix

1. The rank of $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is 2. *(Agra 90)*

2. All equivalent matrices have the same rank.

3. If every minor of order p of a matrix **A** is zero, then every minor of order higher than p is not necessarily zero.

4. If at least one minor of order r of the matrix **A** is not equal to zero, then rank of $\mathbf{A} \geq r$.

5. The rank of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 4 & 10 & 18 \end{bmatrix}$ is 2.

[See Ex. 1(a) P. 2]

6. The matrices $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 7 & 9 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & -5 & 6 \\ 3 & -2 & 1 & 2 \\ 5 & -2 & -9 & 14 \\ 4 & -2 & -4 & 8 \end{bmatrix}$ are equivalent.

7. If the elements of a row of a matrix are multiplied by a non-zero number, then the rank of the matrix remains unaffected.

8. The rank of a matrix is equal to the rank of the transposed matrix.

9. The rank of the product matrix **AB** of two marices **A** and **B** is less than the rank of either of the matrices **A** and **B**.

10. If **A** and **B** are two $n \times n$ matrices, then

$$\text{Adj}(\mathbf{AB}) \neq (\text{Adj } B) \bullet (\text{Adj } \mathbf{A})$$

11. If $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, then $\mathbf{A}^{-1} = \mathbf{A}$

12. The necessary and sufficient condition that a square matrix may possess an inverse is that it be singular.

13. The inverse of transpose of a matrix is not the transpose of the inverse.

14. The inverse of the inverse of a matrix is the matrix itself.

15. If **A** is any square matrix, then $(\text{adj. } \mathbf{A})^{-1} = \text{adj}(\mathbf{A}^{-1})$.

16. Inverse of a matrix **A** exists if **A** is singular.

17. If **A** is a matrix of order $n \times n$, then $\mathbf{A}^{-1}$ is also of the same order.

Ch. VI Solution of Linear Equations

18. A consistent system of equations has no solution.

19. A consistent system of equations has either one solution or infinitely many solutions.

20. A system of m linear equations in n unknowns given by $\mathbf{AX} = \mathbf{K}$ is consistent if the matrix **A** and the augmented matrix $\mathbf{A}^*$ of the system have the same rank.

Ch. VII Characteristic Equation of a Matrix

21. The matrix $\mathbf{A} - \lambda\mathbf{I}$ is known as the characteristics matrix of **A**, when **I** is the identity matrix.

22. Every square matrix satisfies its characteristic equation.

23. The characteristic roots of a Hermitian matrix are either purely imaginary or zero.

24. The characteristic roots of real skew-symmetric matrix are purely imaginary or zero.

25. The characteristic roots of a unitary matrix are of unit modulus.

Ch. VIII Linear Dependence of Vectors

26. The set of vectors $\mathbf{V}_1 = \{1, 2, 3\}$, $\mathbf{V}_2 = \{1, 0, 1\}$ and $\mathbf{V}_3 = \{0, 1, 0\}$ are linearly dependent.

27. If there be n lineariy dependent vectors, then none of these can be expressed as a linear combination of the remaining ones.

[See Th. I § 8·04 Page 214]

Ch. IX Quadratic Forms

28. $x_1^2 + 2x_2^2 + 5x_3^2 - 8x_1 x_2 + 6x_1 x_3$ is a quadratic form in the variables x_1, x_2, x_3.

(III) FILL IN THE BLANKS TYPE :

Fill in the blanks in the following : —

Ch. V Rank and Adjoint of Matrix

1. Rank of the null martix is .... *(Kanpur 2001)*

2. Rank of the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is .... *(Kanpur 2001)*

3. If a matrix A of order $m \times n$ can be expressed as $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$, then rank of A is .... *(Meerut 2001)*

4. All .... matrices have the same rank.

5. If every minor of order p of a matrix A is zero, then every minor of order .... p is definitely zero.

6. If a matrix A does not possess any minor of order $(r + 1)$ then rank of A .... r.

7. The rank of the matrix $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix}$ is ....

8. The rank of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$ is ....

9. The rank of a matrix is .... to the rank of the transposed matrix.

10. The adjoint of the matrix $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ is ....

11. If A be a square matrix of order n, then

$$\mathbf{A}\ (\text{adj}\ \mathbf{A}) = (\text{adj}\ \mathbf{A})\ \mathbf{A} = \text{.............}$$ *(Meerut 2001)*

12. If **A** and **B** are two $n \times n$ matrices, then

$$(\text{Adj}\ \mathbf{B}) \bullet (\text{Adj}\ \mathbf{A}) = \text{..........}$$ [See § 5·09 Th. III P. 50]

13. If **A** be an $n \times n$ matrix and $|\mathbf{A}| \neq 0$, then

$$|\text{Adj}\ \mathbf{A}| = \text{....}$$ [See § 5·09 Th. II P. 50]

14. **A** and **B** are two matrices such that **AB = I**, then

$$\text{adj}\ \mathbf{B} = \text{..........}$$ *(Kanpur 2001)*

[**Hint** : We know $\mathbf{B}^{-1} = \dfrac{\text{adj}\ \mathbf{B}}{|\mathbf{B}|}$ and here $\mathbf{B}^{-1} = \mathbf{A}$, $\because$ $\mathbf{AB} = \mathbf{I}$

$\therefore$ $\text{adj}\ \mathbf{B} = \mathbf{A} \,.\, |\mathbf{B}|$]

15. If A is a non-singular matrix, then $(A^{-1})^{-1}$ = ..., where A^{-1} is the inverse of A.

16. The inverse of a matrix is ....

17. A singular matrix has no ....

18. A matrix when multiplied by its inverse given the .... matrix.

Ch. VI Solution of Linear Equations

19. Inconsistent equations have .... solution.

20. A consistant system of equations has either one solution or .... solutions.

21. A homogeneous system of n linear equations in n unknowns, whose determinants of coefficients does not vanish, has only the .... solution.

22. A set of simultaneous homogeneous equations expressed in matrix form as $\mathbf{AX} = \mathbf{O}$ has non-trivial solution if .... *(Kanpur 2001)*

23. A system of m non-homogeneous linear equation $\mathbf{AX} = \mathbf{B}$ in n unknowns is called .... iff ranks of $\mathbf{A}$ and $[\mathbf{A}, \mathbf{B}]$ are equal. *(Meerut 2001)*

[See § 6·07 P. 119, § 6·06 P. 118]

Ch. VII Characteristic Equation of a Matrix

24. The set of all eigen values of the matrix $\mathbf{A}$ is called the .... of $\mathbf{A}$.

[See § 7·02 (iv) Page 160]

25. The determinant $|\mathbf{A} - \lambda \mathbf{I}|$ is called the characteristic .... of the matrix $\mathbf{A}$, when $\mathbf{I}$ is the identity matrix.

26. Every square matrix .... its characteristic equation.

(Meerut 2001)

27. Characteristic roots of skew-Hermitian matrix are either zero or ....

(Kanpur 2001)

28. All the characteristic roots of a real symmetric matrix are ....

29. The characteristic roots of a Hermitian matrix are all ....

30. The characteristic roots of an orthogonal matrix are of .... modulus.

31. Two .... matrices have the same characteristic roots.

(Kanpur 2001)

[See § 7·05 Th. I Page 167]

Ch. VIII Linear Dependence of Vectors

32. The set of vectors $\mathbf{V}_1 = \{1, 2, 3\}$, $\mathbf{V}_2 = \{1, 0, 1\}$ and $\mathbf{V}_3 = \{0, 1, 0\}$ are linearly .... [See Ex. 1 P. 212]

33. The set of vectors $\mathbf{X}_1 = [2, 3, 1, -1]$, $\mathbf{X}_2 = [2, 3, 1, -2]$ and $\mathbf{X}_3 = [4, 6, 2, -3]$ is linearly .... [See Ex. 1 Page 213]

Ch. IX Quadratic Forms

34. The quadratic form in x_1, x_2, x_3 of the matrix $\begin{bmatrix} 2 & -2 & 5 \\ -2 & 3 & 0 \\ 5 & 0 & 4 \end{bmatrix}$ is ....

ANSWERS TO OBJECTIVE TYPE QUESTIONS

(I) Multiple Choice Type :

1. (iii); **2.** (ii) **3.** (iii); **4.** (ii); **5.** (ii); **6.** (iv

7. (ii) [illegible] (iii) 12. (iv);
13. (i); 14 [illegible] (iii); 16. (iii).

(II) True & False Type :

1.	F;	**2.**	T;	**3.**	F;	**4.**	T;	**5.**	F;	**6.**	F;	**7.**	T;
8.	T;	**9.**	T;	**10.**	F;	**11.**	T;	**12.**	F;	**13.**	[illegible]	**14.**	T;
15.	T;	**16.**	F;	**17.**	T;	**18.**	F;	**19.**	T;	**20.**	T;	**21.**	T;
22.	T;	**23.**	F;	**24.**	T;	**25.**	T;	**26.**	F;	**27.**	F;	**28.**	T.

(III) Fill in the blanks Type :

1. 0; **2.** 2; **3.** r; **4.** equivalent; **5.** higher than; **6.** $\leq$; **7.** 1; **8.** 1; **9.** equal;
10. $\begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix}$;
11. $|A| \bullet I_n$; **12.** Adj (AB); **13.** $|A|^{n-1}$; **14.** $A \bullet |B|$; **15.** A; **16.** unique;
17. inverse; **18.** unit; **19.** no; **20.** infinitely many; **21.** trivial;
22. the rank of A < number of unknowns; **23.** consistent; **24.** spectrum;
25. polynominal; **26.** satisfies; **27.** purely imaginary; **28.** real; **29.** real;
30. unit; **31.** mutually reciprocal; **32.** independent; **33.** dependent;
34. $2x_1^2 + 3x_2^2 + 4x_3^2 - 4x_1 x_2 + 10x_1 x_3$.