Chapter 1

Complex Numbers

THE REAL NUMBER SYSTEM

The number system as we know it today is a result of gradual development as indicated in the following list.

1. **Natural numbers** $1, 2, 3, 4, \ldots,$ also called *positive integers,* were first used in counting. The symbols varied with the times, e.g. the Romans used I, II, III, IV, ... If a and b are natural numbers, the *sum* $a+b$ and *product* $a \cdot b$, $(a)(b)$ or ab are also natural numbers. For this reason the set of natural numbers is said to be *closed* under the operations of *addition* and *multiplication* or to satisfy the *closure property* with respect to these operations.

2. **Negative integers and zero,** denoted by $-1, -2, -3, \ldots$ and 0 respectively, arose to permit solutions of equations such as $x + b = a$ where a and b are any natural numbers. This leads to the operation of *subtraction,* or *inverse of addition,* and we write $x = a - b$.

   The set of positive and negative integers and zero is called the set of *integers* and is closed under the operations of addition, multiplication and subtraction.

3. **Rational numbers** or *fractions* such as $\frac{3}{4}, -\frac{8}{3}, \ldots$ arose to permit solutions of equations such as $bx = a$ for all integers a and b where $b \neq 0$. This leads to the operation of *division* or *inverse of multiplication,* and we write $x = a/b$ or $a \div b$ [called the *quotient* of a and b] where a is the *numerator* and b is the *denominator*.

   The set of integers is a part or *subset* of the rational numbers, since integers correspond to rational numbers a/b where $b = 1$.

   The set of rational numbers is closed under the operations of addition, subtraction, multiplication and division, so long as division by zero is excluded.

4. **Irrational numbers** such as $\sqrt{2} = 1.41423\cdots$ and $\pi = 3.14159\cdots$ are numbers which are not rational, i.e. cannot be expressed as a/b where a and b are integers and $b \neq 0$.

The set of rational and irrational numbers is called the set of *real* numbers. It is assumed that the student is already familiar with the various operations on real numbers.

GRAPHICAL REPRESENTATION OF REAL NUMBERS

Real numbers can be represented by points on a line called the *real axis,* as indicated in Fig. 1-1. The point corresponding to zero is called the *origin.*

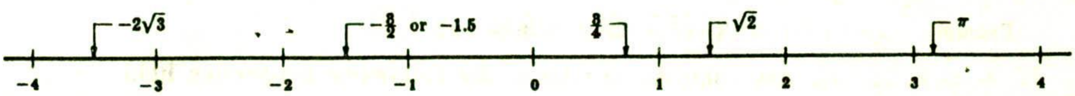

Fig. 1-1

Conversely, to each point on the line there is one and only one real number. If a point A corresponding to a real number a lies to the right of a point B corresponding to a real number b, we say that a is *greater than* b or b is *less than* a and write respectively $a > b$ or $b < a$.

The set of all values of x such that $a < x < b$ is called an *open interval* on the real axis while $a \leqq x \leqq b$, which also includes the endpoints a and b, is called a *closed interval.* The symbol x, which can stand for any of a set of real numbers, is called a *real variable.*

The *absolute value* of a real number a, denoted by $|a|$, is equal to a if $a > 0$, to $-a$ if $a < 0$ and to 0 if $a = 0$. The distance between two points a and b on the real axis is $|a - b|$.

THE COMPLEX NUMBER SYSTEM

There is no real number x which satisfies the polynomial equation $x^2 + 1 = 0$. To permit solutions of this and similar equations, the set of *complex numbers* is introduced.

We can consider a *complex number* as having the form $a + bi$ where a and b are real numbers and i, which is called the *imaginary unit,* has the property that $i^2 = -1$. If $z = a + bi$, then a is called the *real part* of z and b is called the *imaginary part* of z and are denoted by $\text{Re}\{z\}$ and $\text{Im}\{z\}$ respectively. The symbol z, which can stand for any of a set of complex numbers, is called a *complex variable.*

Two complex numbers $a + bi$ and $c + di$ are *equal* if and only if $a = c$ and $b = d$. We can consider real numbers as a subset of the set of complex numbers with $b = 0$. Thus the complex numbers $0 + 0i$ and $-3 + 0i$ represent the real numbers 0 and -3 respectively. If $a = 0$, the complex number $0 + bi$ or bi is called a *pure imaginary number.*

The *complex conjugate,* or briefly *conjugate,* of a complex number $a + bi$ is $a - bi$. The complex conjugate of a complex number z is often indicated by $\bar{z}$ or z^*.

FUNDAMENTAL OPERATIONS WITH COMPLEX NUMBERS

In performing operations with complex numbers we can proceed as in the algebra of real numbers, replacing i^2 by -1 when it occurs.

1. *Addition*

$$(a + bi) + (c + di) = a + bi + c + di = (a + c) + (b + d)i$$

2. *Subtraction*

$$(a + bi) - (c + di) = a + bi - c - di = (a - c) + (b - d)i$$

3. *Multiplication*

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$$

4. *Division*

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{ac - adi + bci - bdi^2}{c^2 - d^2 i^2}$$

$$= \frac{ac + bd + (bc - ad)i}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} i$$

ABSOLUTE VALUE

The *absolute value* or *modulus* of a complex number $a + bi$ is defined as $|a + bi| = \sqrt{a^2 + b^2}$.

Example: $|-4 + 2i| = \sqrt{(-4)^2 + (2)^2} = \sqrt{20} = 2\sqrt{5}$

If $z_1, z_2, z_3, \ldots, z_m$ are complex numbers, the following properties hold.

1. $|z_1 z_2| = |z_1||z_2|$ or $|z_1 z_2 \cdots z_m| = |z_1||z_2| \cdots |z_m|$
2. $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ if $z_2 \neq 0$
3. $|z_1 + z_2| \leqq |z_1| + |z_2|$ or $|z_1 + z_2 + \cdots + z_m| \leqq |z_1| + |z_2| + \cdots + |z_m|$
4. $|z_1 + z_2| \geqq |z_1| - |z_2|$ or $|z_1 - z_2| \geqq |z_1| - |z_2|$

EULER'S FORMULA

By assuming that the infinite series expansion $e^x = 1 + x + x^2/2! + x^3/3! + \cdots$ of elementary calculus holds when $x = i\theta$, we can arrive at the result

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad e = 2.71828\ldots \tag{7}$$

which is called *Euler's formula*. It is more convenient, however, simply to take *(7)* as a definition of $e^{i\theta}$. In general, we define

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y) \tag{8}$$

In the special case where $y = 0$ this reduces to e^x.

Note that in terms of *(7)* De Moivre's theorem essentially reduces to $(e^{i\theta})^n = e^{in\theta}$.

POLYNOMIAL EQUATIONS

Often in practice we require solutions of polynomial equations having the form

$$a_0z^n + a_1z^{n-1} + a_2z^{n-2} + \cdots + a_{n-1}z + a_n = 0 \tag{9}$$

where $a_0 \neq 0, a_1, \ldots, a_n$ are given complex numbers and n is a positive integer called the *degree* of the equation. Such solutions are also called *zeros* of the polynomial on the left of *(9)* or *roots of the equation*.

A very important theorem called the *fundamental theorem of algebra* [to be proved in Chapter 5] states that every polynomial equation of the form *(9)* has at least one root which is complex. From this we can show that it has in fact n complex roots, some or all of which may be identical.

If $z_1, z_2, \ldots, z_n$ are the n roots, *(9)* can be written

$$a_0(z - z_1)(z - z_2)\cdots(z - z_n) = 0 \tag{10}$$

which is called the *factored form* of the polynomial equation. Conversely if we can write *(9)* in the form *(10)*, we can easily determine the roots.

THE *n*th ROOTS OF UNITY

The solutions of the equation $z^n = 1$ where n is a positive integer are called the nth *roots of unity* and are given by

$$z = \cos 2k\pi/n + i\sin 2k\pi/n = e^{2k\pi i/n} \qquad k = 0, 1, 2, \ldots, n-1 \tag{11}$$

If we let $\omega = \cos 2\pi/n + i\sin 2\pi/n = e^{2\pi i/n}$, the n roots are $1, \omega, \omega^2, \ldots, \omega^{n-1}$. Geometrically they represent the n vertices of a regular polygon of n sides inscribed in a circle of radius one with centre at the origin. This circle has the equation $|z| = 1$ and is often called the *unit circle*.

VECTOR INTERPRETATION OF COMPLEX NUMBERS

A complex number $z = x + iy$ can be considered as a vector OP whose *initial point* is the origin O and whose *terminal point* P is the point (x, y) as in Fig. 1-4. We sometimes call $OP = x + iy$ the *position vector* of P. Two vectors having the same *length* or *magnitude* and *direction* but different initial points, such as OP and AB in Fig. 1-4, are considered equal. Hence we write $OP = AB = x + iy$.

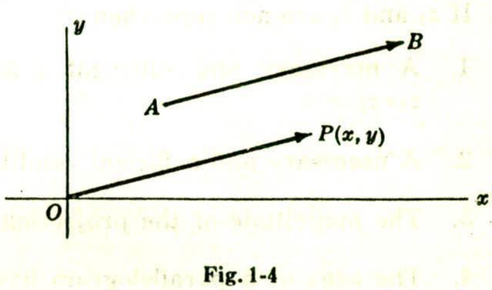

Fig. 1-4

Addition of complex numbers corresponds to the *parallelogram law* for addition of vectors [see Fig. 1-5]. Thus to add the complex numbers z_1 and z_2, we complete the parallelogram $OABC$ whose sides OA and OC correspond to z_1 and z_2. The diagonal OB of this parallelogram corresponds to $z_1 + z_2$. See Problem 5.

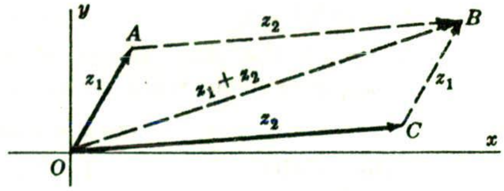

Fig. 1-5

SPHERICAL REPRESENTATION OF COMPLEX NUMBERS. STEREOGRAPHIC PROJECTION

Let $\mathcal{P}$ [Fig. 1-6] be the complex plane and consider a unit sphere $\mathcal{S}$ [radius one] tangent to $\mathcal{P}$ at $z = 0$. The diameter NS is perpendicular to $\mathcal{P}$ and we call points N and S the *north* and *south poles* of $\mathcal{S}$. Corresponding to any point A on $\mathcal{P}$ we can construct line NA intersecting $\mathcal{S}$ at point A'. Thus to each point of the complex plane $\mathcal{P}$ there corresponds one and only one point of the sphere $\mathcal{S}$, and we can represent any complex number by a point on the sphere. For completeness we say that the point N itself corresponds to the "point at infinity" of the plane. The set of all points of the complex plane including the point at infinity is called the *entire complex plane*, the *entire z plane*, or the *extended complex plane*.

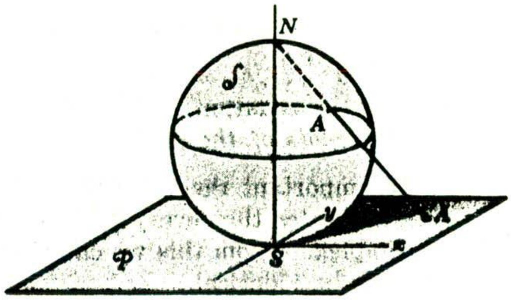

Fig. 1-6

The above method for mapping the plane on to the sphere is called *stereographic projection*. The sphere is sometimes called the *Riemann sphere*.

DOT AND CROSS PRODUCT

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers [vectors]. The *dot product* [also called the *scalar product*] of z_1 and z_2 is defined by

$$z_1 \circ z_2 = |z_1|\,|z_2| \cos\theta = x_1x_2 + y_1y_2 = \operatorname{Re}\{\bar{z}_1 z_2\} = \tfrac{1}{2}\{\bar{z}_1 z_2 + z_1 \bar{z}_2\} \qquad (12)$$

where θ is the angle between z_1 and z_2 which lies between 0 and π.

The *cross product* of z_1 and z_2 is defined by

$$z_1 \times z_2 = |z_1|\,|z_2| \sin\theta = x_1y_2 - y_1x_2 = \operatorname{Im}\{\bar{z}_1 z_2\} = \frac{1}{2i}\{\bar{z}_1 z_2 - z_1 \bar{z}_2\} \qquad (13)$$

Clearly,

$$\bar{z}_1 z_2 = (z_1 \circ z_2) + i(z_1 \times z_2) = |z_1|\,|z_2|\,e^{i\theta} \qquad (14)$$

If z_1 and z_2 are non-zero, then

1. A necessary and sufficient condition that z_1 and z_2 be perpendicular is that $z_1 \circ z_2 = 0$.
2. A necessary and sufficient condition that z_1 and z_2 be parallel is that $z_1 \times z_2 = 0$.
3. The magnitude of the projection of z_1 on z_2 is $|z_1 \circ z_2|/|z_2|$.
4. The area of a parallelogram having sides z_1 and z_2 is $|z_1 \times z_2|$.

COMPLEX CONJUGATE COORDINATES

A point in the complex plane can be located by rectangular coordinates (x, y) or polar coordinates (r, θ). Many other possibilities exist. One such possibility uses the fact that $x = \frac{1}{2}(z + \bar{z})$, $y = \frac{1}{2i}(z - \bar{z})$ where $z = x + iy$. The coordinates $(z, \bar{z})$ which locate a point are called *complex conjugate coordinates* or briefly *conjugate coordinates* of the point [see Problems 43 and 44].

POINT SETS

Any collection of points in the complex plane is called a (*two-dimensional*) point set, and each point is called a *member* or *element* of the set. The following fundamental definitions are given here for reference.

1. **Neighbourhoods.** A *delta, or* δ, *neighbourhood* of a point z_0 is the set of all points z such that $|z - z_0| < \delta$ where δ is any given positive number. A *deleted* δ *neighbourhood* of z_0 is a neighbourhood of z_0 in which the point z_0 is omitted, i.e. $0 < |z - z_0| < \delta$.

2. **Limit Points.** A point z_0 is called a *limit point, cluster point,* or *point of accumulation* of a point set S if every deleted δ neighbourhood of z_0 contains points of S.

   Since δ can be any positive number, it follows that S must have infinitely many points. Note that z_0 may or may not belong to the set S.

3. **Closed Sets.** A set S is said to be *closed* if every limit point of S belongs to S, i.e. if S contains all its limit points. For example, the set of all points z such that $|z| \leqq 1$ is a closed set.

4. **Bounded Sets.** A set S is called *bounded* if we can find a constant M such that $|z| < M$ for every point z in S. An *unbounded set* is one which is not bounded. A set which is both bounded and closed is sometimes called *compact*.

5. **Interior, Exterior and Boundary Points.** A point z_0 is called an *interior point* of a set S if we can find a δ neighbourhood of z_0 all of whose points belong to S. If every δ neighbourhood of z_0 contains points belonging to S and also points not belonging to S, then z_0 is called a *boundary point*. If a point is not an interior or boundary point of a set S, it is an *exterior point* of S.

6. **Open Sets.** An *open set* is a set which consists only of interior points. For example, the set of points z such that $|z| < 1$ is an open set.

7. **Connected Sets.** An open set S is said to be *connected* if any two points of the set can be joined by a path consisting of straight line segments (i.e. a *polygonal path*) all points of which are in S.

8. **Open Regions or Domains.** An open connected set is called an *open region* or *domain*.

9. **Closure of a Set.** If to a set S we add all the limit points of S, the new set is called the *closure* of S and is a closed set.

10. **Closed Regions.** The closure of an open region or domain is called a *closed region*.

11. **Regions.** If to an open region or domain we add some, all or none of its limit points, we obtain a set called a *region*. If all the limit points are added, the region is *closed*; if none are added, the region is *open*. In this book whenever we use the word *region* without qualifying it, we shall mean *open region* or *domain*.

12. **Union and Intersection of Sets.** A set consisting of all points belonging to set S_1 or set S_2 or to both sets S_1 and S_2 is called the *union* of S_1 and S_2 and is denoted by $S_1 + S_2$ or $S_1 \cup S_2$.

    A set consisting of all points belonging to both sets S_1 and S_2 is called the *intersection* of S_1 and S_2 and is denoted by S_1S_2 or $S_1 \cap S_2$.

13. **Complement of a Set.** A set consisting of all points which do not belong to S is called the *complement* of S and is denoted by $\tilde{S}$.

14. **Null Sets and Subsets.** It is convenient to consider a set consisting of no points at all. This set is called the *null set* and is denoted by $\emptyset$. If two sets S_1 and S_2 have no points in common (in which case they are called *disjoint* or *mutually exclusive sets*), we can indicate this by writing $S_1 \cap S_2 = \emptyset$.

    Any set formed by choosing some, all or none of the points of a set S is called a *subset* of S. If we exclude the case where all points of S are chosen, the set is called a *proper subset* of S.

15. **Countability of a Set.** If the members or elements of a set can be placed into a one to one correspondence with the natural numbers $1, 2, 3, \ldots$, the set is called *countable* or *denumerable*; otherwise it is *non-countable* or *non-denumerable*.

The following are two important theorems on point sets.

1. **Weierstrass-Bolzano Theorem.** Every bounded infinite set has at least one limit point.

2. **Heine-Borel Theorem.** Let S be a compact set each point of which is contained in one or more of the open sets $A_1, A_2, \ldots$ [which are then said to *cover* S]. Then there exists a finite number of the sets $A_1, A_2, \ldots$ which will cover S.

Solved Problems

FUNDAMENTAL OPERATIONS WITH COMPLEX NUMBERS

1. Perform each of the indicated operations.

(*a*) $(3+2i) + (-7-i) = 3 - 7 + 2i - i = -4 + i$

(*b*) $(-7-i) + (3+2i) = -7 + 3 - i + 2i = -4 + i$

The results (*a*) and (*b*) illustrate the *commutative law of addition.*

(*c*) $(8-6i) - (2i-7) = 8 - 6i - 2i + 7 = 15 - 8i$

(*d*) $(5+3i) + \{(-1+2i) + (7-5i)\} = (5+3i) + \{-1+2i+7-5i\} = (5+3i) + (6-3i) = 11$

(*e*) $\{(5+3i) + (-1+2i)\} + (7-5i) = \{5+3i-1+2i\} + (7-5i) = (4+5i) + (7-5i) = 11$

The results (*d*) and (*e*) illustrate the *associative law of addition.*

(*f*) $(2-3i)(4+2i) = 2(4+2i) - 3i(4+2i) = 8 + 4i - 12i - 6i^2 = 8 + 4i - 12i + 6 = 14 - 8i$

(*g*) $(4+2i)(2-3i) = 4(2-3i) + 2i(2-3i) = 8 - 12i + 4i - 6i^2 = 8 - 12i + 4i + 6 = 14 - 8i$

The results (*f*) and (*g*) illustrate the *commutative law of multiplication.*

(*h*) $(2-i)\{(-3+2i)(5-4i)\} = (2-i)\{-15+12i+10i-8i^2\}$
$= (2-i)(-7+22i) = -14 + 44i + 7i - 22i^2 = 8 + 51i$

(*i*) $\{(2-i)(-3+2i)\}(5-4i) = \{-6+4i+3i-2i^2\}(5-4i)$
$= (-4+7i)(5-4i) = -20 + 16i + 35i - 28i^2 = 8 + 51i$

The results (*h*) and (*i*) illustrate the *associative law of multiplication.*

(j) $(-1+2i)\{(7-5i)+(-3+4i)\} = (-1+2i)(4-i) = -4+i+8i-2i^2 = -2+9i$

Another method. $(-1+2i)\{(7-5i)+(-3+4i)\} = (-1+2i)(7-5i)+(-1+2i)(-3+4i)$
$= \{-7+5i+14i-10i^2\}+\{3-4i-6i+8i^2\}$
$= (3+19i)+(-5-10i) = -2+9i$

This illustrates the *distributive law.*

(k) $\dfrac{3-2i}{-1+i} = \dfrac{3-2i}{-1+i}\cdot\dfrac{-1-i}{-1-i} = \dfrac{-3-3i+2i+2i^2}{1-i^2} = \dfrac{-5-i}{2} = -\dfrac{5}{2}-\dfrac{1}{2}i$

Another method. By definition, $(3-2i)/(-1+i)$ is that number $a+bi$, where a and b are real, such that $(-1+i)(a+bi) = -a-b+(a-b)i = 3-2i$. Then $-a-b = 3$, $a-b = -2$ and solving simultaneously, $a = -5/2$, $b = -1/2$ or $a+bi = -5/2-i/2$.

(l) $\dfrac{5+5i}{3-4i}+\dfrac{20}{4+3i} = \dfrac{5+5i}{3-4i}\cdot\dfrac{3+4i}{3+4i}+\dfrac{20}{4+3i}\cdot\dfrac{4-3i}{4-3i}$

$= \dfrac{15+20i+15i+20i^2}{9-16i^2}+\dfrac{80-60i}{16-9i^2} = \dfrac{-5+35i}{25}+\dfrac{80-60i}{25} = 3-i$

(m) $\dfrac{3i^{30}-i^{19}}{2i-1} = \dfrac{3(i^2)^{15}-(i^2)^9i}{2i-1} = \dfrac{3(-1)^{15}-(-1)^9i}{-1+2i}$

$= \dfrac{-3+i}{-1+2i}\cdot\dfrac{-1-2i}{-1-2i} = \dfrac{3+6i-i-2i^2}{1-4i^2} = \dfrac{5+5i}{5} = 1+i$

2. If $z_1 = 2+i$, $z_2 = 3-2i$ and $z_3 = -\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i$, evaluate each of the following.

(a) $|3z_1-4z_2| = |3(2+i)-4(3-2i)| = |6+3i-12+8i|$
$= |-6+11i| = \sqrt{(-6)^2+(11)^2} = \sqrt{157}$

(b) $z_1^3-3z_1^2+4z_1-8 = (2+i)^3-3(2+i)^2+4(2+i)-8$
$= \{(2)^3+3(2)^2(i)+3(2)(i)^2+i^3\}-3(4+4i+i^2)+8+4i-8$
$= 8+12i-6-i-12-12i+3+8+4i-8 = -7+3i$

(c) $(\bar{z}_3)^4 = \left(\overline{-\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i}\right)^4 = \left(-\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}i\right)^4 = \left[\left(-\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}i\right)^2\right]^2$

$= \left[\dfrac{1}{4}+\dfrac{\sqrt{3}}{2}i+\dfrac{3}{4}i^2\right]^2 = \left(-\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i\right)^2 = \dfrac{1}{4}-\dfrac{\sqrt{3}}{2}i+\dfrac{3}{4}i^2 = -\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}i$

(d) $\left|\dfrac{2z_2+z_1-5-i}{2z_1-z_2+3-i}\right|^2 = \left|\dfrac{2(3-2i)+(2+i)-5-i}{2(2+i)-(3-2i)+3-i}\right|^2$

$= \left|\dfrac{3-4i}{4+3i}\right|^2 = \dfrac{|3-4i|^2}{|4+3i|^2} = \dfrac{(\sqrt{(3)^2+(-4)^2})^2}{(\sqrt{(4)^2+(3)^2})^2} = 1$

3. Find real numbers x and y such that $3x+2iy-ix+5y = 7+5i$.

The given equation can be written as $3x+5y+i(2y-x) = 7+5i$. Then equating real and imaginary parts, $3x+5y = 7$, $2y-x = 5$. Solving simultaneously, $x = -1$, $y = 2$.

4. Prove: (a) $\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$, (b) $|z_1z_2| = |z_1|\,|z_2|$.

Let $z_1 = x_1+iy_1$, $z_2 = x_2+iy_2$. Then

(a) $\overline{z_1+z_2} = \overline{x_1+iy_1+x_2+iy_2} = \overline{x_1+x_2+i(y_1+y_2)}$
$= x_1+x_2-i(y_1+y_2) = x_1-iy_1+x_2-iy_2 = \overline{x_1+iy_1}+\overline{x_2+iy_2} = \bar{z}_1+\bar{z}_2$

(b) $|z_1z_2| = |(x_1+iy_1)(x_2+iy_2)| = |x_1x_2-y_1y_2+i(x_1y_2+y_1x_2)|$
$= \sqrt{(x_1x_2-y_1y_2)^2+(x_1y_2+y_1x_2)^2} = \sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)} = \sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2} = |z_1|\,|z_2|$

Another method.

$|z_1z_2|^2 = (z_1z_2)(\overline{z_1z_2}) = z_1z_2\bar{z}_1\bar{z}_2 = (z_1\bar{z}_1)(z_2\bar{z}_2) = |z_1|^2|z_2|^2$ or $|z_1z_2| = |z_1|\,|z_2|$

where we have used the fact that the conjugate of a product of two complex numbers is equal to the product of their conjugates (see Problem 55).

GRAPHICAL REPRESENTATION OF COMPLEX NUMBERS. VECTORS

5. Perform the indicated operations both analytically and graphically:

(a) $(3+4i)+(5+2i)$, (b) $(6-2i)-(2-5i)$, (c) $(-3+5i)+(4+2i)+(5-3i)+(-4-6i)$.

(a) *Analytically.* $(3+4i)+(5+2i) = 3+5+4i+2i = 8+6i$

Graphically. Represent the two complex numbers by points P_1 and P_2 respectively as in Fig. 1-7 below. Complete the parallelogram with OP_1 and OP_2 as adjacent sides. Point P represents the sum, $8+6i$, of the two given complex numbers. Note the similarity with the parallelogram law for addition of vectors OP_1 and OP_2 to obtain vector OP. For this reason it is often convenient to consider a complex number $a+bi$ as a vector having *components* a and b in the directions of the positive x and y axes respectively.

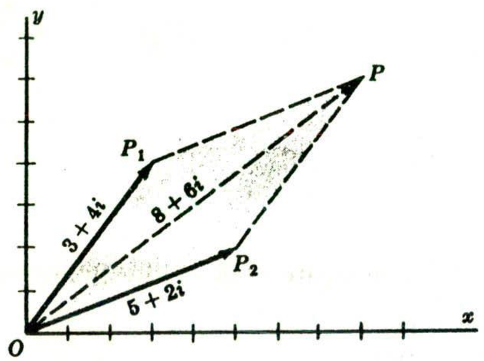

Fig. 1-7

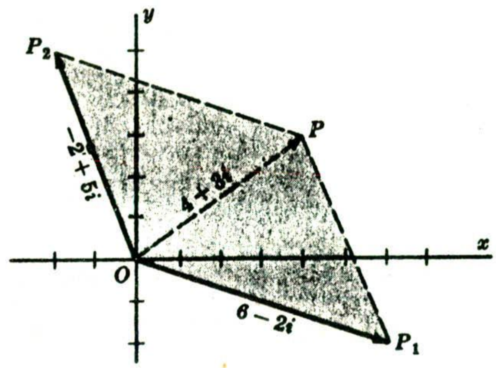

Fig. 1-8

(b) *Analytically.* $(6-2i)-(2-5i) = 6-2-2i+5i = 4+3i$

Graphically. $(6-2i)-(2-5i) = 6-2i+(-2+5i)$. We now add $6-2i$ and $(-2+5i)$ as in part (a). The result is indicated by OP in Fig. 1-8 above.

(c) *Analytically.*

$$(-3+5i)+(4+2i)+(5-3i)+(-4-6i) = (-3+4+5-4)+(5i+2i-3i-6i) = 2-2i$$

Graphically. Represent the numbers to be added by z_1, z_2, z_3, z_4 respectively. These are shown graphically in Fig. 1-9. To find the required sum proceed as shown in Fig. 1-10. At the terminal point of vector z_1 construct vector z_2. At the terminal point of z_2 construct vector z_3, and at the terminal point of z_3 construct vector z_4. The required sum, sometimes called the *resultant*, is obtained by constructing the vector OP from the initial point of z_1 to the terminal point of z_4, i.e. $OP = z_1+z_2+z_3+z_4 = 2-2i$.

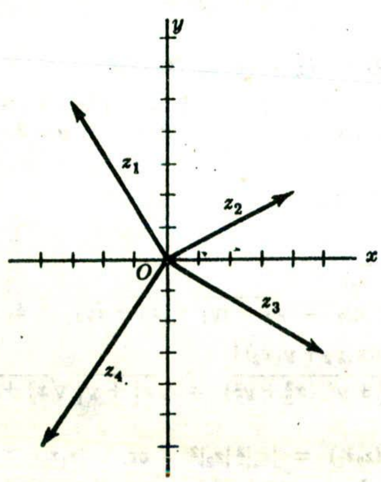

Fig. 1-9

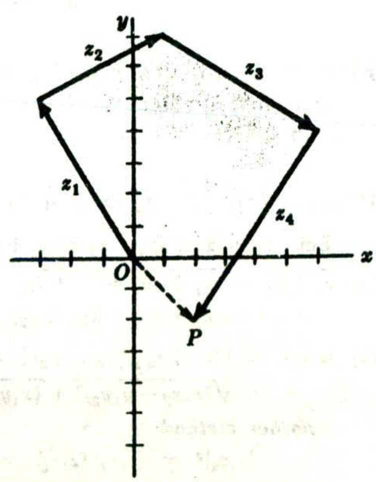

Fig. 1-10

6. If z_1 and z_2 are two given complex numbers (vectors) as in Fig. 1-11, construct graphically
(a) $3z_1 - 2z_2$ (b) $\frac{1}{2}z_2 + \frac{5}{3}z_1$

(a) In Fig. 1-12 below, $OA = 3z_1$ is a vector having length 3 times vector z_1 and the same direction.

$OB = -2z_2$ is a vector having length 2 times vector z_2 and the opposite direction.

Then vector $OC = OA + OB = 3z_1 - 2z_2$.

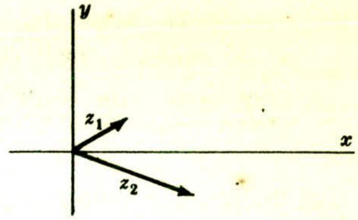

Fig. 1-11

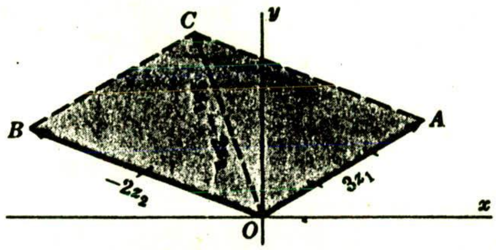

Fig. 1-12

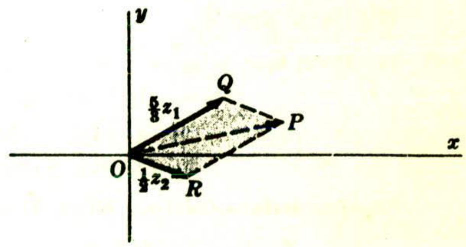

Fig. 1-13

(b) The required vector (complex number) is represented by OP in Fig. 1-13 above.

7. Prove (a) $|z_1 + z_2| \leqq |z_1| + |z_2|$, (b) $|z_1 + z_2 + z_3| \leqq |z_1| + |z_2| + |z_3|$, (c) $|z_1 - z_2| \geqq |z_1| - |z_2|$ and give a graphical interpretation.

(a) *Analytically.* Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Then we must show that

$$\sqrt{(x_1+x_2)^2 + (y_1+y_2)^2} \leqq \sqrt{x_1^2+y_1^2} + \sqrt{x_2^2+y_2^2}$$

Squaring both sides, this will be true if

$$(x_1+x_2)^2 + (y_1+y_2)^2 \leqq x_1^2 + y_1^2 + 2\sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)} + x_2^2 + y_2^2$$

i.e. if
$$x_1x_2 + y_1y_2 \leqq \sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$$

or if (squaring both sides again)

$$x_1^2x_2^2 + 2x_1x_2y_1y_2 + y_1^2y_2^2 \leqq x_1^2x_2^2 + x_1^2y_2^2 + y_1^2x_2^2 + y_1^2y_2^2$$

or
$$2x_1x_2y_1y_2 \leqq x_1^2y_2^2 + y_1^2x_2^2$$

But this is equivalent to $(x_1y_2 - x_2y_1)^2 \geqq 0$ which is true. Reversing the steps, which are reversible, proves the result.

Graphically. The result follows graphically from the fact that $|z_1|, |z_2|, |z_1+z_2|$ represent the lengths of the sides of a triangle (see Fig. 1-14) and that the sum of the lengths of two sides of a triangle is greater than or equal to the length of the third side.

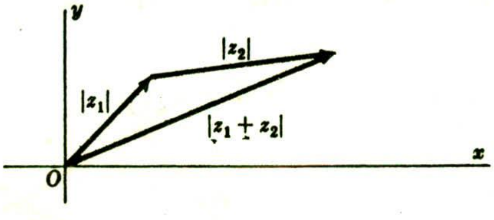

Fig. 1-14

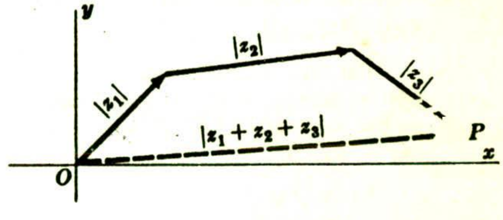

Fig. 1-15

(b) *Analytically.* By part (a),

$$|z_1+z_2+z_3| = |z_1 + (z_2+z_3)| \leqq |z_1| + |z_2+z_3| \leqq |z_1| + |z_2| + |z_3|$$

Graphically. The result is a consequence of the geometric fact that in a plane a straight line is the shortest distance between two points O and P (see Fig. 1-15).

(c) *Analytically.* By part (a), $|z_1| = |z_1 - z_2 + z_2| \leqq |z_1 - z_2| + |z_2|$. Then $|z_1 - z_2| \geqq |z_1| - |z_2|$. An equivalent result obtained on replacing z_2 by $-z_2$ is $|z_1 + z_2| \geqq |z_1| - |z_2|$.

Graphically. The result is equivalent to the statement that a side of a triangle has length greater than or equal to the difference in lengths of the other two sides.

8. Let the position vectors of points $A(x_1, y_1)$ and $B(x_2, y_2)$ be represented by z_1 and z_2 respectively. (a) Represent the vector AB as a complex number. (b) Find the distance between points A and B.

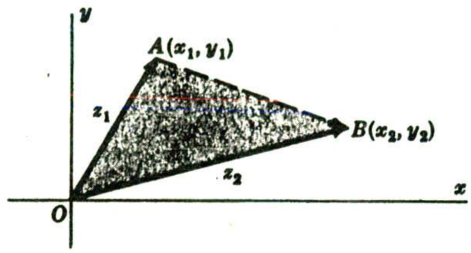

Fig. 1-16

(a) From Fig. 1-16, $OA + AB = OB$ or

$$\begin{aligned} AB &= OB - OA = z_2 - z_1 \\ &= (x_2 + iy_2) - (x_1 + iy_1) \\ &= (x_2 - x_1) + i(y_2 - y_1) \end{aligned}$$

(b) The distance between points A and B is given by

$$|AB| = |(x_2 - x_1) + i(y_2 - y_1)| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

9. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ represent two non-collinear or non-parallel vectors. If a and b are real numbers (scalars) such that $az_1 + bz_2 = 0$, prove that $a = 0$ and $b = 0$.

The given condition $az_1 + bz_2 = 0$ is equivalent to $a(x_1 + iy_1) + b(x_2 + iy_2) = 0$ or $ax_1 + bx_2 + i(ay_1 + by_2) = 0$. Then $ax_1 + bx_2 = 0$ and $ay_1 + by_2 = 0$. These equations have the simultaneous solution $a = 0, b = 0$ if $y_1/x_1 \neq y_2/x_2$, i.e. if the vectors are non-collinear or non-parallel vectors.

10. Prove that the diagonals of a parallelogram bisect each other.

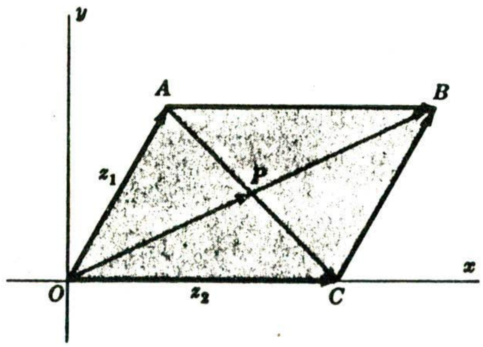

Fig. 1-17

Let $OABC$ [Fig. 1-17] be the given parallelogram with diagonals intersecting at P.

Since $z_1 + AC = z_2$, $AC = z_2 - z_1$. Then $AP = m(z_2 - z_1)$ where $0 \leqq m \leqq 1$.

Since $OB = z_1 + z_2$, $OP = n(z_1 + z_2)$ where $0 \leqq n \leqq 1$.

But $OA + AP = OP$, i.e. $z_1 + m(z_2 - z_1) = n(z_1 + z_2)$ or $(1 - m - n)z_1 + (m - n)z_2 = 0$. Hence by Problem 9, $1 - m - n = 0$, $m - n = 0$ or $m = \frac{1}{2}$, $n = \frac{1}{2}$ and so P is the midpoint of both diagonals.

11. Find an equation for the straight line which passes through two given points $A(x_1, y_1)$ and $B(x_2, y_2)$.

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be the position vectors of A and B respectively. Let $z = x + iy$ be the position vector of any point P on the line joining A and B.

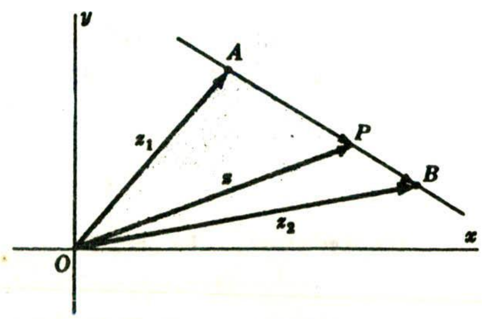

Fig. 1-18

From Fig. 1-18,

$OA + AP = OP$ or $z_1 + AP = z$, i.e. $AP = z - z_1$

$OA + AB = OB$ or $z_1 + AB = z_2$, i.e. $AB = z_2 - z_1$

Since AP and AB are collinear, $AP = tAB$ or $z - z_1 = t(z_2 - z_1)$ where t is real, and the required equation is

$$z = z_1 + t(z_2 - z_1) \quad \text{or} \quad z = (1 - t)z_1 + tz_2$$

Using $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ and $z = x + iy$, this can be written

$$x - x_1 = t(x_2 - x_1), \quad y - y_1 = t(y_2 - y_1) \qquad \text{or} \qquad \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$$

The first two are called *parametric equations* of the line and t is the parameter; the second is called the equation of the line in *standard form*.

Another method. Since AP and PB are collinear, we have for real numbers m and n:

$$mAP = nPB \qquad \text{or} \qquad m(z - z_1) = n(z_2 - z)$$

Solving,
$$z = \frac{mz_1 + nz_2}{m + n} \qquad \text{or} \qquad x = \frac{mx_1 + nx_2}{m + n}, \quad y = \frac{my_1 + ny_2}{m + n}$$

which is called the *symmetric form*.

12. Let $A(1, -2)$, $B(-3, 4)$, $C(2, 2)$ be the three vertices of triangle ABC. Find the length of the median from C to the side AB.

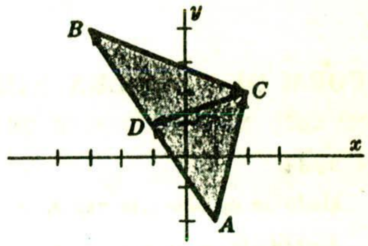

Fig. 1-19

The position vectors of A, B and C are given by $z_1 = 1 - 2i$, $z_2 = -3 + 4i$ and $z_3 = 2 + 2i$ respectively. Then from Fig. 1-19,

$$AC = z_3 - z_1 = 2 + 2i - (1 - 2i) = 1 + 4i$$
$$BC = z_3 - z_2 = 2 + 2i - (-3 + 4i) = 5 - 2i$$
$$AB = z_2 - z_1 = -3 + 4i - (1 - 2i) = -4 + 6i$$
$$AD = \tfrac{1}{2}AB = \tfrac{1}{2}(-4 + 6i) = -2 + 3i \quad \text{since } D \text{ is the midpoint of } AB.$$

$$AC + CD = AD \quad \text{or} \quad CD = AD - AC = -2 + 3i - (1 + 4i) = -3 - i.$$

Then the length of median CD is $|CD| = |-3 - i| = \sqrt{10}$.

13. Find an equation for (*a*) a circle of radius 4 with centre at $(-2, 1)$, (*b*) an ellipse with major axis of length 10 and foci at $(-3, 0)$ and $(3, 0)$.

(*a*) The centre can be represented by the complex number $-2 + i$. If z is any point on the circle [Fig. 1-20], the distance from z to $-2 + i$ is

$$|z - (-2 + i)| = 4$$

Then $|z + 2 - i| = 4$ is the required equation. In rectangular form this is given by

$$|(x + 2) + i(y - 1)| = 4, \quad \text{i.e.} \quad (x + 2)^2 + (y - 1)^2 = 16$$

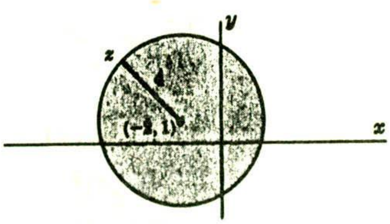

Fig. 1-20

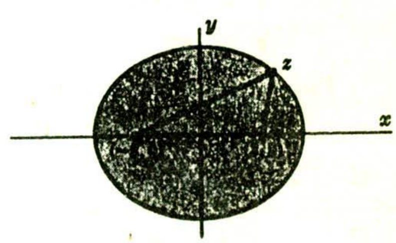

Fig. 1-21

(*b*) The sum of the distances from any point z on the ellipse [Fig. 1-21] to the foci must equal 10. Hence the required equation is

$$|z + 3| + |z - 3| = 10$$

In rectangular form this reduces to $x^2/25 + y^2/16 = 1$ (see Problem 74).

AXIOMATIC FOUNDATIONS OF COMPLEX NUMBERS

14. Use the definition of a complex number as an ordered pair of real numbers and the definitions on Page 3 to prove that $(a, b) = a(1, 0) + b(0, 1)$ where $(0, 1)(0, 1) = (-1, 0)$.

From the definitions of sum and product on Page 3, we have

$$(a,b) = (a,0)+(0,b) = a(1,0)+b(0,1)$$

where $$(0,1)(0,1) = (0\cdot 0 - 1\cdot 1,\ 0\cdot 1 + 1\cdot 0) = (-1,0)$$

By identifying $(1,0)$ with 1 and $(0,1)$ with i, we see that $(a,b) = a+bi$.

15. If $z_1 = (a_1,b_1)$, $z_2 = (a_2,b_2)$ and $z_3 = (a_3,b_3)$, prove the distributive law: $z_1(z_2+z_3) = z_1z_2 + z_1z_3$.

We have
$$\begin{aligned} z_1(z_2+z_3) &= (a_1,b_1)\{(a_2,b_2)+(a_3,b_3)\} = (a_1,b_1)(a_2+a_3,\ b_2+b_3) \\ &= \{a_1(a_2+a_3)-b_1(b_2+b_3),\ a_1(b_2+b_3)+b_1(a_2+a_3)\} \\ &= (a_1a_2-b_1b_2+a_1a_3-b_1b_3,\ a_1b_2+b_1a_2+a_1b_3+b_1a_3) \\ &= (a_1a_2-b_1b_2,\ a_1b_2+b_1a_2) + (a_1a_3-b_1b_3,\ a_1b_3+b_1a_3) \\ &= (a_1,b_1)(a_2,b_2) + (a_1,b_1)(a_3,b_3) = z_1z_2 + z_1z_3 \end{aligned}$$

POLAR FORM OF COMPLEX NUMBERS

16. Express each of the following complex numbers in polar form.

(a) $2+2\sqrt{3}\,i$

Modulus or absolute value, $r = |2+2\sqrt{3}\,i| = \sqrt{4+12} = 4$.

Amplitude or argument, $\theta = \sin^{-1} 2\sqrt{3}/4 = \sin^{-1}\sqrt{3}/2 = 60° = \pi/3$ (radians).

Then

$$2+2\sqrt{3}\,i = r(\cos\theta + i\sin\theta) = 4(\cos 60° + i\sin 60°) = 4(\cos\pi/3 + i\sin\pi/3)$$

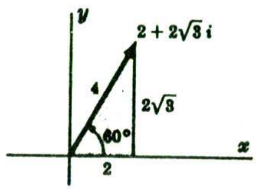

Fig. 1-22

The result can also be written as $4\ \text{cis}\ \pi/3$ or, using Euler's formula, as $4e^{\pi i/3}$.

(b) $-5+5i$

$$r = |-5+5i| = \sqrt{25+25} = 5\sqrt{2}$$
$$\theta = 180° - 45° = 135° = 3\pi/4 \text{ (radians)}$$

Then $$-5+5i = 5\sqrt{2}(\cos 135° + i\sin 135°) = 5\sqrt{2}\ \text{cis}\ 3\pi/4 = 5\sqrt{2}\,e^{3\pi i/4}$$

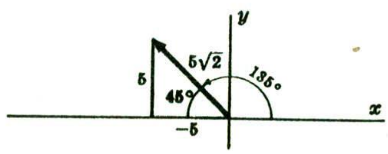

Fig. 1-23

(c) $-\sqrt{6}-\sqrt{2}\,i$

$$r = |-\sqrt{6}-\sqrt{2}\,i| = \sqrt{6+2} = 2\sqrt{2}$$
$$\theta = 180° + 30° = 210° = 7\pi/6 \text{ (radians)}$$

Then $$-\sqrt{6}-\sqrt{2}\,i = 2\sqrt{2}(\cos 210° + i\sin 210°) = 2\sqrt{2}\ \text{cis}\ 7\pi/6 = 2\sqrt{2}\,e^{7\pi i/6}$$

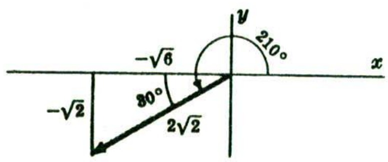

Fig. 1-24

(d) $-3i$

$$r = |-3i| = |0-3i| = \sqrt{0+9} = 3$$
$$\theta = 270° = 3\pi/2 \text{ (radians)}$$

Then $$-3i = 3(\cos 3\pi/2 + i\sin 3\pi/2) = 3\ \text{cis}\ 3\pi/2 = 3e^{3\pi i/2}$$

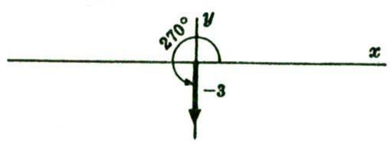

Fig. 1-25

17. Graph each of the following: (a) $6(\cos 240° + i\sin 240°)$, (b) $4e^{3\pi i/5}$, (c) $2e^{-\pi i/4}$.

(a) $$6(\cos 240° + i\sin 240°) = 6\ \text{cis}\ 240° = 6\ \text{cis}\ 4\pi/3 = 6\,e^{4\pi i/3}$$

can be represented graphically by OP in Fig. 1-26 below.

If we start with vector OA, whose magnitude is 6 and whose direction is that of the positive x axis, we can obtain OP by rotating OA counterclockwise through an angle of 240°. In general, $re^{i\theta}$ is equivalent to a vector obtained by rotating a vector of magnitude r and direction that of the positive x axis, counterclockwise through an angle θ.

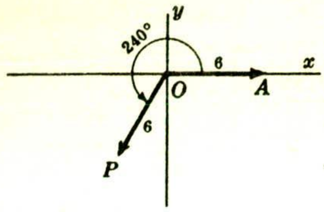

Fig. 1-26

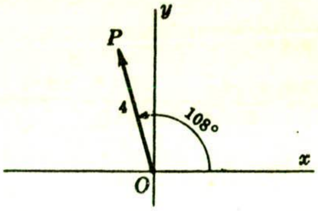

Fig. 1-27

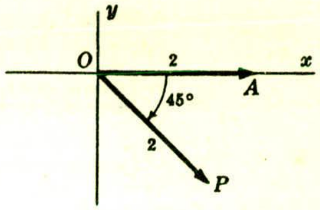

Fig. 1-28

(*b*) $$4\,e^{3\pi i/5} = 4(\cos 3\pi/5 + i\sin 3\pi/5) = 4(\cos 108^\circ + i\sin 108^\circ)$$
is represented by OP in Fig. 1-27 above.

(*c*) $$2\,e^{-\pi i/4} = 2\{\cos(-\pi/4) + i\sin(-\pi/4)\} = 2\{\cos(-45^\circ) + i\sin(-45^\circ)\}$$

This complex number can be represented by vector OP in Fig. 1-28 above. This vector can be obtained by starting with vector OA, whose magnitude is 2 and whose direction is that of the positive x axis, and rotating it counterclockwise through an angle of -45° (which is the same as rotating it *clockwise* through an angle of 45°).

18. A man travels 12 miles northeast, 20 miles 30° west of north, and then 18 miles 60° south of west. Determine (*a*) analytically and (*b*) graphically how far and in what direction he is from his starting point.

P—Q
2 miles

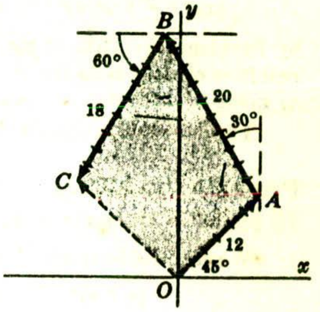

Fig. 1-29

(*a*) *Analytically.* Let O be the starting point (see Fig. 1-29). Then the successive displacements are represented by vectors OA, AB and BC. The result of all three displacements is represented by the vector

$$OC = OA + AB + BC$$

Now
$$OA = 12(\cos 45^\circ + i\sin 45^\circ) = 12\,e^{\pi i/4}$$
$$AB = 20\{\cos(90^\circ + 30^\circ) + i\sin(90^\circ + 30^\circ)\} = 20\,e^{2\pi i/3}$$
$$BC = 18\{\cos(180^\circ + 60^\circ) + i\sin(180^\circ + 60^\circ)\} = 18\,e^{4\pi i/3}$$

Then
$$\begin{aligned} OC &= 12\,e^{\pi i/4} + 20\,e^{2\pi i/3} + 18\,e^{4\pi i/3} \\ &= \{12\cos 45^\circ + 20\cos 120^\circ + 18\cos 240^\circ\} + i\{12\sin 45^\circ + 20\sin 120^\circ + 18\sin 240^\circ\} \\ &= \{(12)(\sqrt{2}/2) + (20)(-1/2) + (18)(-1/2)\} + i\{(12)(\sqrt{2}/2) + (20)(\sqrt{3}/2) + (18)(-\sqrt{3}/2)\} \\ &= (6\sqrt{2} - 19) + (6\sqrt{2} + \sqrt{3})i \end{aligned}$$

If $r(\cos\theta + i\sin\theta) = 6\sqrt{2} - 19 + (6\sqrt{2} + \sqrt{3})i$, then $r = \sqrt{(6\sqrt{2}-19)^2 + (6\sqrt{2}+\sqrt{3})^2} =$ 14.7 approximately, and $\theta = \cos^{-1}(6\sqrt{2} - 19)/r = \cos^{-1}(-.717) = 135^\circ 49'$ approximately.

Thus the man is 14.7 miles from his starting point in a direction $135^\circ 49' - 90^\circ = 45^\circ 49'$ west of north.

(*b*) *Graphically.* Using a convenient unit of length such as PQ in Fig. 1-29 which represents 2 miles, and a protractor to measure angles, construct vectors OA, AB and BC. Then by determining the number of units in OC and the angle which OC makes with the y axis, we obtain the approximate results of (*a*).

DE MOIVRE'S THEOREM

19. If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, prove:

(*a*) $z_1 z_2 = r_1 r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$

(*b*) $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}$.

(*a*)
$$\begin{aligned} z_1 z_2 &= \{r_1(\cos\theta_1 + i\sin\theta_1)\}\{r_2(\cos\theta_2 + i\sin\theta_2)\} \\ &= r_1 r_2\{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\} \\ &= r_1 r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\} \end{aligned}$$

$$(b)\quad \frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}\cdot\frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1}{r_2}\left\{\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}\right\}$$

$$= \frac{r_1}{r_2}\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}$$

In terms of Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, the results state that if $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, then $z_1z_2 = r_1r_2\,e^{i(\theta_1+\theta_2)}$ and $\dfrac{z_1}{z_2} = \dfrac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \dfrac{r_1}{r_2}\,e^{i(\theta_1-\theta_2)}$.

20. Prove De Moivre's theorem: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ where n is any positive integer.

We use the *principle of mathematical induction.* Assume that the result is true for the particular positive integer k, i.e. assume $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$. Then multiplying both sides by $\cos\theta + i\sin\theta$, we find

$$(\cos\theta + i\sin\theta)^{k+1} = (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta) = \cos(k+1)\theta + i\sin(k+1)\theta$$

by Problem 19. Thus *if* the result is true for $n = k$, then it is also true for $n = k+1$. But since the result is clearly true for $n = 1$, it must also be true for $n = 1+1 = 2$ and $n = 2+1 = 3$, etc., and so must be true for all positive integers.

The result is equivalent to the statement $(e^{i\theta})^n = e^{ni\theta}$.

21. Prove the identities: (*a*) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$; (*b*) $(\sin 5\theta)/(\sin\theta) = 16\cos^4\theta - 12\cos^2\theta + 1$, if $\theta \neq 0, \pm\pi, \pm 2\pi, \ldots$.

We use the *binomial formula*

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + b^n$$

where the coefficients $\binom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$, also denoted by ${}_nC_r$, are called the *binomial coefficients.* The number $n!$ or *factorial n*, is defined as the product $1\cdot2\cdot3\cdots n$ and we define $0! = 1$.

From Problem 20, with $n = 5$, and the binomial formula,

$$\begin{aligned}\cos 5\theta + i\sin 5\theta &= (\cos\theta + i\sin\theta)^5\\ &= \cos^5\theta + \binom{5}{1}(\cos^4\theta)(i\sin\theta) + \binom{5}{2}(\cos^3\theta)(i\sin\theta)^2\\ &\qquad + \binom{5}{3}(\cos^2\theta)(i\sin\theta)^3 + \binom{5}{4}(\cos\theta)(i\sin\theta)^4 + (i\sin\theta)^5\\ &= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta\\ &\qquad - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta\\ &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta\\ &\qquad + i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)\end{aligned}$$

Hence

(*a*)
$$\begin{aligned}\cos 5\theta &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta\\ &= \cos^5\theta - 10\cos^3\theta\,(1-\cos^2\theta) + 5\cos\theta\,(1-\cos^2\theta)^2\\ &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\end{aligned}$$

and

(*b*)
$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

or

$$\begin{aligned}\frac{\sin 5\theta}{\sin\theta} &= 5\cos^4\theta - 10\cos^2\theta\sin^2\theta + \sin^4\theta\\ &= 5\cos^4\theta - 10\cos^2\theta\,(1-\cos^2\theta) + (1-\cos^2\theta)^2\\ &= 16\cos^4\theta - 12\cos^2\theta + 1\end{aligned}$$

provided $\sin\theta \neq 0$, i.e. $\theta \neq 0, \pm\pi, \pm2\pi, \ldots$.

22. Show that (*a*) $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$, (*b*) $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$.

We have (*1*) $e^{i\theta} = \cos\theta + i\sin\theta$, (*2*) $e^{-i\theta} = \cos\theta - i\sin\theta$

(a) Adding (1) and (2), $e^{i\theta} + e^{-i\theta} = 2\cos\theta$ or $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$

(b) Subtracting (2) from (1), $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$ or $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$

23. Prove the identities (a) $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$, (b) $\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$.

(a)
$$\sin^3\theta = \left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^3 = \frac{(e^{i\theta}-e^{-i\theta})^3}{8i^3} = -\frac{1}{8i}\{(e^{i\theta})^3 - 3(e^{i\theta})^2(e^{-i\theta}) + 3(e^{i\theta})(e^{-i\theta})^2 - (e^{-i\theta})^3\}$$
$$= -\frac{1}{8i}(e^{3i\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-3i\theta}) = \frac{3}{4}\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right) - \frac{1}{4}\left(\frac{e^{3i\theta}-e^{-3i\theta}}{2i}\right)$$
$$= \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$$

(b)
$$\cos^4\theta = \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^4 = \frac{(e^{i\theta}+e^{-i\theta})^4}{16}$$
$$= \frac{1}{16}\{(e^{i\theta})^4 + 4(e^{i\theta})^3(e^{-i\theta}) + 6(e^{i\theta})^2(e^{-i\theta})^2 + 4(e^{i\theta})(e^{-i\theta})^3 + (e^{-i\theta})^4\}$$
$$= \frac{1}{16}(e^{4i\theta} + 4e^{2i\theta} + 6 + 4e^{-2i\theta} + e^{-4i\theta}) = \frac{1}{8}\left(\frac{e^{4i\theta}+e^{-4i\theta}}{2}\right) + \frac{1}{2}\left(\frac{e^{2i\theta}+e^{-2i\theta}}{2}\right) + \frac{3}{8}$$
$$= \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$$

24. Given a complex number (vector) z, interpret geometrically $ze^{i\alpha}$ where α is real.

Let $z = re^{i\theta}$ be represented graphically by vector OA in Fig. 1-30. Then

$$ze^{i\alpha} = re^{i\theta}\cdot e^{i\alpha} = re^{i(\theta+\alpha)}$$

is the vector represented by OB.

Hence multiplication of a vector z by $e^{i\alpha}$ amounts to rotating z counterclockwise through angle α. We can consider $e^{i\alpha}$ as an *operator* which acts on z to produce this rotation.

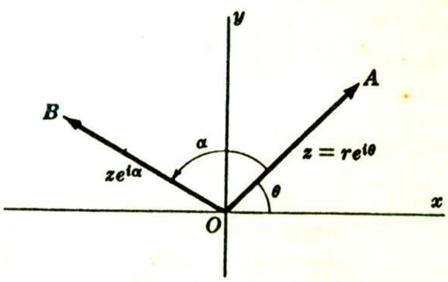

Fig. 1-30

25. Prove: $e^{i\theta} = e^{i(\theta+2k\pi)}$, $k = 0, \pm 1, \pm 2, \ldots$

$$e^{i(\theta+2k\pi)} = \cos(\theta+2k\pi) + i\sin(\theta+2k\pi) = \cos\theta + i\sin\theta = e^{i\theta}$$

26. Evaluate each of the following.

(a)
$$[3(\cos 40^\circ + i\sin 40^\circ)][4(\cos 80^\circ + i\sin 80^\circ)] = 3\cdot 4[\cos(40^\circ+80^\circ) + i\sin(40^\circ+80^\circ)]$$
$$= 12(\cos 120^\circ + i\sin 120^\circ)$$
$$= 12\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -6 + 6\sqrt{3}\,i$$

(b)
$$\frac{(2\text{ cis }15^\circ)^7}{(4\text{ cis }45^\circ)^3} = \frac{128\text{ cis }105^\circ}{64\text{ cis }135^\circ} = 2\text{ cis }(105^\circ - 135^\circ)$$
$$= 2[\cos(-30^\circ) + i\sin(-30^\circ)] = 2[\cos 30^\circ - i\sin 30^\circ] = \sqrt{3} - i$$

(c)
$$\left(\frac{1+\sqrt{3}\,i}{1-\sqrt{3}\,i}\right)^{10} = \left\{\frac{2\text{ cis }(60^\circ)}{2\text{ cis }(-60^\circ)}\right\}^{10} = (\text{cis }120^\circ)^{10} = \text{cis }1200^\circ = \text{cis }120^\circ = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

Another method.

$$\left(\frac{1+\sqrt{3}\,i}{1-\sqrt{3}\,i}\right)^{10} = \left(\frac{2e^{\pi i/3}}{2e^{-\pi i/3}}\right)^{10} = (e^{2\pi i/3})^{10} = e^{20\pi i/3}$$
$$= e^{6\pi i}\,e^{2\pi i/3} = (1)[\cos(2\pi/3) + i\sin(2\pi/3)] = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

27. Prove that (a) $\arg(z_1z_2) = \arg z_1 + \arg z_2$, (b) $\arg(z_1/z_2) = \arg z_1 - \arg z_2$, stating appropriate conditions of validity.

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then $\arg z_1 = \theta_1$, $\arg z_2 = \theta_2$.

(a) Since $z_1z_2 = r_1r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}$, $\arg(z_1z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$.

(b) Since $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\{\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\}$, $\arg\left(\dfrac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2$.

Since there are many possible values for $\theta_1 = \arg z_1$ and $\theta_2 = \arg z_2$, we can only say that the two sides in the above equalities are equal for *some* values of $\arg z_1$ and $\arg z_2$. They may not hold even if principal values are used.

ROOTS OF COMPLEX NUMBERS

28. (a) Find all values of z for which $z^5 = -32$, and (b) locate these values in the complex plane.

(a) In polar form, $-32 = 32\{\cos(\pi+2k\pi) + i\sin(\pi+2k\pi)\}$, $k = 0, \pm1, \pm2, \cdots$.

Let $z = r(\cos\theta + i\sin\theta)$. Then by De Moivre's theorem,

$$z^5 = r^5(\cos 5\theta + i\sin 5\theta) = 32\{\cos(\pi+2k\pi) + i\sin(\pi+2k\pi)\}$$

and so $r^5 = 32$, $5\theta = \pi + 2k\pi$, from which $r = 2$, $\theta = (\pi+2k\pi)/5$. Hence

$$z = 2\left\{\cos\left(\frac{\pi+2k\pi}{5}\right) + i\sin\left(\frac{\pi+2k\pi}{5}\right)\right\}$$

If $k = 0$, $z = z_1 = 2(\cos\pi/5 + i\sin\pi/5)$.

If $k = 1$, $z = z_2 = 2(\cos 3\pi/5 + i\sin 3\pi/5)$.

If $k = 2$, $z = z_3 = 2(\cos 5\pi/5 + i\sin 5\pi/5) = -2$.

If $k = 3$, $z = z_4 = 2(\cos 7\pi/5 + i\sin 7\pi/5)$.

If $k = 4$, $z = z_5 = 2(\cos 9\pi/5 + i\sin 9\pi/5)$.

By considering $k = 5, 6, \ldots$ as well as negative values, $-1, -2, \ldots$, repetitions of the above five values of z are obtained. Hence these are the only *solutions* or *roots* of the given equation. These five roots are called the *fifth roots of −32* and are collectively denoted by $(-32)^{1/5}$. In general, $a^{1/n}$ represents the nth roots of a and there are n such roots.

(b) The values of z are indicated in Fig. 1-31. Note that they are equally spaced along the circumference of a circle with centre at the origin and radius 2. Another way of saying this is that the roots are represented by the vertices of a regular polygon.

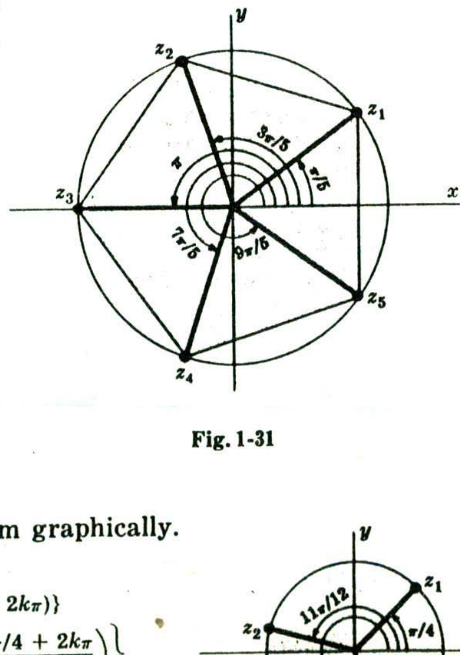

Fig. 1-31

29. Find each of the indicated roots and locate them graphically.

(a) $(-1+i)^{1/3}$

$$-1+i = \sqrt{2}\{\cos(3\pi/4 + 2k\pi) + i\sin(3\pi/4 + 2k\pi)\}$$

$$(-1+i)^{1/3} = 2^{1/6}\left\{\cos\left(\frac{3\pi/4 + 2k\pi}{3}\right) + i\sin\left(\frac{3\pi/4 + 2k\pi}{3}\right)\right\}$$

If $k = 0$, $z_1 = 2^{1/6}(\cos\pi/4 + i\sin\pi/4)$.

If $k = 1$, $z_2 = 2^{1/6}(\cos 11\pi/12 + i\sin 11\pi/12)$.

If $k = 2$, $z_3 = 2^{1/6}(\cos 19\pi/12 + i\sin 19\pi/12)$.

These are represented graphically in Fig. 1-32.

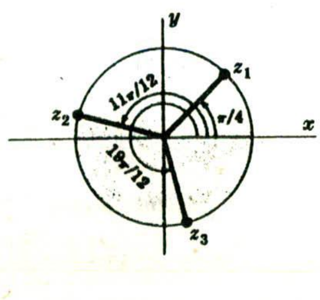

Fig. 1-32

(*b*) $(-2\sqrt{3}-2i)^{1/4}$

$$-2\sqrt{3}-2i \;=\; 4\{\cos(7\pi/6+2k\pi) + i\sin(7\pi/6+2k\pi)\}$$

$$(-2\sqrt{3}-2i)^{1/4} \;=\; 4^{1/4}\left\{\cos\left(\frac{7\pi/6+2k\pi}{4}\right) + i\sin\left(\frac{7\pi/6+2k\pi}{4}\right)\right\}$$

If $k=0$, $z_1 = \sqrt{2}(\cos 7\pi/24 + i\sin 7\pi/24)$.

If $k=1$, $z_2 = \sqrt{2}(\cos 19\pi/24 + i\sin 19\pi/24)$.

If $k=2$, $z_3 = \sqrt{2}(\cos 31\pi/24 + i\sin 31\pi/24)$.

If $k=3$, $z_4 = \sqrt{2}(\cos 43\pi/24 + i\sin 43\pi/24)$.

These are represented graphically in Fig. 1-33.

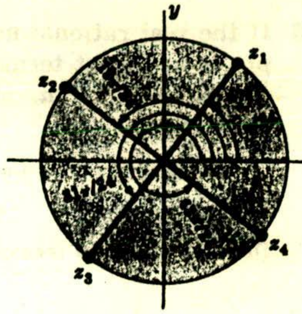

Fig. 1-33

30. Find the square roots of $-15-8i$.

Method 1.

$$-15-8i \;=\; 17\{\cos(\theta+2k\pi) + i\sin(\theta+2k\pi)\} \quad \text{where } \cos\theta = -15/17,\ \sin\theta = -8/17.$$

Then the square roots of $-15-8i$ are

$$\sqrt{17}(\cos\theta/2 + i\sin\theta/2) \tag{1}$$

and
$$\sqrt{17}\{\cos(\theta/2+\pi) + i\sin(\theta/2+\pi)\} \;=\; -\sqrt{17}(\cos\theta/2 + i\sin\theta/2) \tag{2}$$

Now
$$\cos\theta/2 \;=\; \pm\sqrt{(1+\cos\theta)/2} \;=\; \pm\sqrt{(1-15/17)/2} \;=\; \pm 1/\sqrt{17}$$
$$\sin\theta/2 \;=\; \pm\sqrt{(1-\cos\theta)/2} \;=\; \pm\sqrt{(1+15/17)/2} \;=\; \pm 4/\sqrt{17}$$

Since θ is an angle in the third quadrant, $\theta/2$ is an angle in the second quadrant. Hence $\cos\theta/2 = -1/\sqrt{17}$, $\sin\theta/2 = 4/\sqrt{17}$ and so from (*1*) and (*2*) the required square roots are $-1+4i$ and $1-4i$. As a check note that $(-1+4i)^2 = (1-4i)^2 = -15-8i$.

Method 2.

Let $p+iq$, where p and q are real, represent the required square roots. Then

$$(p+iq)^2 \;=\; p^2-q^2+2pqi \;=\; -15-8i \qquad \text{or} \qquad (3)\ p^2-q^2 = -15, \quad (4)\ pq = -4$$

Substituting $q=-4/p$ from (*4*) into (*3*), it becomes $p^2-16/p^2 = -15$ or $p^4+15p^2-16 = 0$, i.e. $(p^2+16)(p^2-1) = 0$ or $p^2=-16$, $p^2=1$. Since p is real, $p=\pm 1$. From (*4*) if $p=1$, $q=-4$; if $p=-1$, $q=4$. Thus the roots are $-1+4i$ and $1-4i$.

POLYNOMIAL EQUATIONS

31. Solve the *quadratic equation* $az^2+bz+c = 0$, $a \neq 0$.

Transposing c and dividing by $a \neq 0$,
$$z^2 + \frac{b}{a}z \;=\; -\frac{c}{a}$$

Adding $\left(\frac{b}{2a}\right)^2$ [completing the square],
$$z^2 + \frac{b}{a}z + \left(\frac{b}{2a}\right)^2 \;=\; -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

Then
$$\left(z+\frac{b}{2a}\right)^2 \;=\; \frac{b^2-4ac}{4a^2}$$

Taking square roots,
$$z + \frac{b}{2a} \;=\; \frac{\pm\sqrt{b^2-4ac}}{2a}$$

Hence
$$z \;=\; \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

32. Solve the equation $z^2+(2i-3)z+5-i = 0$.

From Problem 31, $a=1$, $b=2i-3$, $c=5-i$ and so the solutions are

$$z \;=\; \frac{-b\pm\sqrt{b^2-4ac}}{2a} \;=\; \frac{-(2i-3)\pm\sqrt{(2i-3)^2-4(1)(5-i)}}{2(1)} \;=\; \frac{3-2i\pm\sqrt{-15-8i}}{2}$$

$$=\; \frac{3-2i\pm(1-4i)}{2} \;=\; 2-3i \quad\text{or}\quad 1+i$$

using the fact that the square roots of $-15-8i$ are $\pm(1-4i)$ [see Problem 30]. These are found to satisfy the given equation.

33. If the real rational number p/q (where p and q have no common factor except ± 1, i.e. p/q is in lowest terms) satisfies the polynomial equation $a_0 z^n + a_1 z^{n-1} + \cdots + a_n = 0$ where $a_0, a_1, \ldots, a_n$ are integers, show that p and q must be factors of a_n and a_0 respectively.

Substituting $z = p/q$ in the given equation and multiplying by q^n yields

$$a_0 p^n + a_1 p^{n-1} q + \cdots + a_{n-1} p q^{n-1} + a_n q^n = 0 \qquad (1)$$

Dividing by p and transposing the last term,

$$a_0 p^{n-1} + a_1 p^{n-2} q + \cdots + a_{n-1} q^{n-1} = -\frac{a_n q^n}{p} \qquad (2)$$

Since the left side of (2) is an integer, so also is the right side. But since p has no factor in common with q, it cannot divide q^n and so must divide a_n.

Similarly on dividing (1) by q and transposing the first term, we find that q must divide a_0.

34. Solve $6z^4 - 25z^3 + 32z^2 + 3z - 10 = 0$.

The integer factors of 6 and -10 are respectively $\pm 1, \pm 2, \pm 3, \pm 6$ and $\pm 1, \pm 2, \pm 5, \pm 10$. Hence by Prob. 33 the possible rational solutions are $\pm 1, \pm 1/2, \pm 1/3, \pm 1/6, \pm 2, \pm 2/3, \pm 5, \pm 5/2, \pm 5/3, \pm 5/6, \pm 10, \pm 10/3$.

By trial we find that $z = -1/2$ and $z = 2/3$ are solutions, and so the polynomial $(2z+1)(3z-2) = 6z^2 - z - 2$ is a factor of $6z^4 - 25z^3 + 32z^2 + 3z - 10$, the other factor being $z^2 - 4z + 5$ as found by long division. Hence

$$6z^4 - 25z^3 + 32z^2 + 3z - 10 = (6z^2 - z - 2)(z^2 - 4z + 5) = 0$$

The solutions of $z^2 - 4z + 5 = 0$ are [see Problem 31]

$$z = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

Then the solutions are $-1/2, 2/3, 2+i, 2-i$.

35. Prove that the sum and product of all the roots of $a_0 z^n + a_1 z^{n-1} + \cdots + a_n = 0$ where $a_0 \neq 0$, are $-a_1/a_0$ and $(-1)^n a_n/a_0$ respectively.

If $z_1, z_2, \ldots, z_n$ are the n roots, the equation can be written in factored form as

$$a_0(z - z_1)(z - z_2) \cdots (z - z_n) = 0$$

Direct multiplication shows that

$$a_0\{z^n - (z_1 + z_2 + \cdots + z_n)z^{n-1} + \cdots + (-1)^n z_1 z_2 \cdots z_n\} = 0$$

It follows that $-a_0(z_1 + z_2 + \cdots + z_n) = a_1$ and $a_0(-1)^n z_1 z_2 \cdots z_n = a_n$, from which

$$z_1 + z_2 + \cdots + z_n = -a_1/a_0, \qquad z_1 z_2 \cdots z_n = (-1)^n a_n/a_0$$

as required.

36. If $p + qi$ is a root of $a_0 z^n + a_1 z^{n-1} + \cdots + a_n = 0$ where $a_0 \neq 0, a_1, \ldots, a_n$, p and q are real, prove that $p - qi$ is also a root.

Let $p + qi = re^{i\theta}$ in polar form. Since this satisfies the equation,

$$a_0 r^n e^{in\theta} + a_1 r^{n-1} e^{i(n-1)\theta} + \cdots + a_{n-1} r e^{i\theta} + a_n = 0$$

Taking the conjugate of both sides

$$a_0 r^n e^{-in\theta} + a_1 r^{n-1} e^{-i(n-1)\theta} + \cdots + a_{n-1} r e^{-i\theta} + a_n = 0$$

we see that $re^{-i\theta} = p - qi$ is also a root. The result does not hold if $a_0, \ldots, a_n$ are not all real (see Problem 32).

The theorem is often expressed in the statement: The zeros of a polynomial with real coefficients occur in conjugate pairs.

THE nth ROOTS OF UNITY

37. Find all the 5th roots of unity.

$$z^5 = 1 = \cos 2k\pi + i \sin 2k\pi = e^{2k\pi i} \quad \text{where } k = 0, \pm 1, \pm 2, \ldots$$

Then
$$z = \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5} = e^{2k\pi i/5}$$

where it is sufficient to use $k = 0, 1, 2, 3, 4$ since all other values of k lead to repetition.

Thus the roots are $1, e^{2\pi i/5}, e^{4\pi i/5}, e^{6\pi i/5}, e^{8\pi i/5}$. If we call $e^{2\pi i/5} = \omega$, these can be denoted by $1, \omega, \omega^2, \omega^3, \omega^4$.

38. If $n = 2, 3, 4, \ldots$, prove that

(a) $$\cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \cos\frac{6\pi}{n} + \cdots + \cos\frac{2(n-1)\pi}{n} = -1$$

(b) $$\sin\frac{2\pi}{n} + \sin\frac{4\pi}{n} + \sin\frac{6\pi}{n} + \cdots + \sin\frac{2(n-1)\pi}{n} = 0$$

Consider the equation $z^n - 1 = 0$ whose solutions are the nth roots of unity,

$$1, \; e^{2\pi i/n}, \; e^{4\pi i/n}, \; e^{6\pi i/n}, \; \ldots, \; e^{2(n-1)\pi i/n}$$

By Problem 35 the sum of these roots is zero. Then

$$1 + e^{2\pi i/n} + e^{4\pi i/n} + e^{6\pi i/n} + \cdots + e^{2(n-1)\pi i/n} = 0$$

i.e.,

$$\left\{1 + \cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \cdots + \cos\frac{2(n-1)\pi}{n}\right\} + i\left\{\sin\frac{2\pi}{n} + \sin\frac{4\pi}{n} + \cdots + \sin\frac{2(n-1)\pi}{n}\right\} = 0$$

from which the required results follow.

DOT AND CROSS PRODUCT

39. If $z_1 = 3 - 4i$ and $z_2 = -4 + 3i$, find (a) $z_1 \circ z_2$, (b) $z_1 \times z_2$.

(a) $z_1 \circ z_2 = \operatorname{Re}\{\bar{z}_1 z_2\} = \operatorname{Re}\{(3+4i)(-4+3i)\} = \operatorname{Re}\{-24 - 7i\} = -24$

Another method. $z_1 \circ z_2 = (3)(-4) + (-4)(3) = -24$

(b) $z_1 \times z_2 = \operatorname{Im}\{\bar{z}_1 z_2\} = \operatorname{Im}\{(3+4i)(-4+3i)\} = \operatorname{Im}\{-24 - 7i\} = -7$

Another method. $z_1 \times z_2 = (3)(3) - (-4)(-4) = -7$

40. Find the acute angle between the vectors in Problem 39.

From Problem 39(a), we have $\cos\theta = \dfrac{z_1 \circ z_2}{|z_1|\,|z_2|} = \dfrac{-24}{|3-4i|\,|-4+3i|} = \dfrac{-24}{25} = -.96.$

Then the acute angle is $\cos^{-1} .96 = 16°16'$ approximately.

41. Prove that the area of a parallelogram having sides z_1 and z_2 is $|z_1 \times z_2|$.

Area of parallelogram [Fig. 1-34]

$= \text{(base)(height)}$
$= (|z_2|)(|z_1| \sin\theta)$
$= |z_1|\,|z_2| \sin\theta = |z_1 \times z_2|$

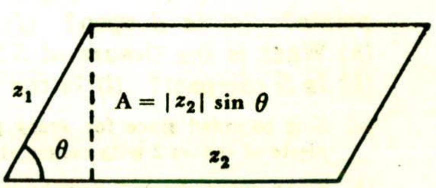

Fig. 1-34

42. Find the area of a triangle with vertices at $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$.

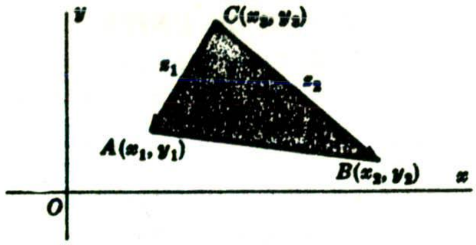

Fig. 1-35

The vectors from C to A and B [Fig. 1-35] are respectively given by

$$z_1 = (x_1 - x_3) + i(y_1 - y_3),$$
$$z_2 = (x_2 - x_3) + i(y_2 - y_3)$$

Since the area of a triangle with sides z_1 and z_2 is half the area of the corresponding parallelogram, we have by Problem 41:

$$\begin{aligned} \text{Area of triangle} &= \tfrac{1}{2}|z_1 \times z_2| = \tfrac{1}{2}|\operatorname{Im}\{[(x_1 - x_3) - i(y_1 - y_3)][(x_2 - x_3) + i(y_2 - y_3)]\}| \\ &= \tfrac{1}{2}|(x_1 - x_3)(y_2 - y_3) - (y_1 - y_3)(x_2 - x_3)| \\ &= \tfrac{1}{2}|x_1y_2 - y_1x_2 + x_2y_3 - y_2x_3 + x_3y_1 - y_3x_1| \\ &= \tfrac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right| \end{aligned}$$

in determinant form.

COMPLEX CONJUGATE COORDINATES

43. Express each equation in terms of conjugate coordinates: (*a*) $2x + y = 5$, (*b*) $x^2 + y^2 = 36$.

(*a*) Since $z = x + iy$, $\bar{z} = x - iy$, $x = \dfrac{z + \bar{z}}{2}$, $y = \dfrac{z - \bar{z}}{2i}$. Then $2x + y = 5$ becomes

$$2\left(\frac{z+\bar{z}}{2}\right) + \left(\frac{z-\bar{z}}{2i}\right) = 5 \qquad \text{or} \qquad (2i+1)z + (2i-1)\bar{z} = 10i$$

The equation represents a straight line in the z plane.

(*b*) *Method 1.* The equation is $(x + iy)(x - iy) = 36$ or $z\bar{z} = 36$.

Method 2. Substitute $x = \dfrac{z+\bar{z}}{2}$, $y = \dfrac{z-\bar{z}}{2i}$ in $x^2 + y^2 = 36$ to obtain $z\bar{z} = 36$.

The equation represents a circle in the z plane of radius 6 with centre at the origin.

44. Prove that the equation of any circle or line in the z plane can be written as $\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0$ where α and γ are real constants while β may be a complex constant.

The general equation of a circle in the xy plane can be written

$$A(x^2 + y^2) + Bx + Cy + D = 0$$

which in conjugate coordinates becomes

$$Az\bar{z} + B\left(\frac{z+\bar{z}}{2}\right) + C\left(\frac{z-\bar{z}}{2i}\right) + D = 0 \qquad \text{or} \qquad Az\bar{z} + \left(\frac{B}{2} + \frac{C}{2i}\right)z + \left(\frac{B}{2} - \frac{C}{2i}\right)\bar{z} + D = 0$$

Calling $A = \alpha$, $\dfrac{B}{2} + \dfrac{C}{2i} = \beta$ and $D = \gamma$, the required result follows.

In the special case $A = \alpha = 0$, the circle degenerates into a line.

POINT SETS

45. Given the point set $S: \{i, \frac{1}{2}i, \frac{1}{3}i, \frac{1}{4}i, \ldots\}$ or briefly $\{i/n\}$. (*a*) Is S bounded? (*b*) What are its limit points, if any? (*c*) Is S closed? (*d*) What are its interior and boundary points? (*e*) Is S open? (*f*) Is S connected? (*g*) Is S an open region or domain? (*h*) What is the closure of S? (*i*) What is the complement of S? (*j*) Is S countable? (*k*) Is S compact? (*l*) Is the closure of S compact?

(*a*) S is bounded since for every point z in S, $|z| < 2$ [for example], i.e. all points of S lie inside a circle of radius 2 with centre at the origin.

(*b*) Since every deleted neighbourhood of $z = 0$ contains points of S, a limit point is $z = 0$. It is the only limit point.

Note that since S is bounded and infinite the Weierstrass-Bolzano theorem predicts *at least one* limit point.

(c) S is not closed since the limit point $z=0$ does not belong to S.

(d) Every δ neighbourhood of any point i/n [i.e. every circle of radius δ with centre at i/n] contains points which belong to S and points which do not belong to S. Thus every point of S, as well as the point $z=0$, is a boundary point. S has *no* interior points.

(e) S does not consist of any interior points. Hence it cannot be open. Thus S is neither open nor closed.

(f) If we join any two points of S by a polygonal path, there are points on this path which do not belong to S. Thus S is not connected.

(g) Since S is not an open connected set, it is not an open region or domain.

(h) The closure of S consists of the set S together with the limit point zero, i.e. $\{0, i, \frac{1}{2}i, \frac{1}{3}i, \ldots\}$.

(i) The complement of S is the set of all points not belonging to S, i.e. all points $z \neq i, i/2, i/3, \ldots$.

(j) There is a one to one correspondence between the elements of S and the natural numbers $1, 2, 3, \ldots$ as indicated below.

$$\begin{array}{ccccc} i & \frac{1}{2}i & \frac{1}{3}i & \frac{1}{4}i & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 1 & 2 & 3 & 4 & \cdots \end{array}$$

Hence S is countable.

(k) S is bounded but not closed. Hence it is not compact.

(l) The closure of S is bounded and closed and so is compact.

46. Given the point sets $A = \{3, -i, 4, 2+i, 5\}$, $B = \{-i, 0, -1, 2+i\}$, $C = \{-\sqrt{2}\,i, \frac{1}{2}, 3\}$. Find (a) $A+B$ or $A \cup B$, (b) AB or $A \cap B$, (c) AC or $A \cap C$, (d) $A(B+C)$ or $A \cap (B \cup C)$, (e) $AB + AC$ or $(A \cap B) \cup (A \cap C)$, (f) $A(BC)$ or $A \cap (B \cap C)$.

(a) $A+B = A \cup B$ consists of points belonging either to A or B or both and is given by $\{3, -i, 4, 2+i, 5, 0, -1\}$.

(b) AB or $A \cap B$ consists of points belonging to both A and B and is given by $\{-i, 2+i\}$.

(c) AC or $A \cap C = \{3\}$, consisting of only the member 3.

(d) $B+C$ or $B \cup C = \{-i, 0, -1, 2+i, -\sqrt{2}\,i, \frac{1}{2}, 3\}$.

Hence $A(B+C)$ or $A \cap (B \cup C) = \{3, -i, 2+i\}$, consisting of points belonging to both A and $B+C$.

(e) $AB = \{-i, 2+i\}$, $AC = \{3\}$ from parts (b) and (c). Hence $AB + AC = \{-i, 2+i, 3\}$.

From this and the result of (d) we see that $A(B+C) = AB + AC$ or $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, which illustrates the fact that A, B, C satisfy the *distributive law*. We can show that sets exhibit many of the properties valid in the algebra of numbers. This is of great importance in theory and application.

(f) $BC = B \cap C = \emptyset$, the *null set*, since there are no points common to both B and C. Hence $A(BC) = \emptyset$ also.

MISCELLANEOUS PROBLEMS

47. A number is called an *algebraic number* if it is a solution of a polynomial equation $a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0$ where $a_0, a_1, \ldots, a_n$ are integers. Prove that (a) $\sqrt{3} + \sqrt{2}$ and (b) $\sqrt[3]{4} - 2i$ are algebraic numbers.

(a) Let $z = \sqrt{3} + \sqrt{2}$ or $z - \sqrt{2} = \sqrt{3}$. Squaring, $z^2 - 2\sqrt{2}\,z + 2 = 3$ or $z^2 - 1 = 2\sqrt{2}\,z$. Squaring again, $z^4 - 2z^2 + 1 = 8z^2$ or $z^4 - 10z^2 + 1 = 0$, a polynomial equation with integer coefficients having $\sqrt{3} + \sqrt{2}$ as a root. Hence $\sqrt{3} + \sqrt{2}$ is an algebraic number.

(b) Let $z = \sqrt[3]{4} - 2i$ or $z + 2i = \sqrt[3]{4}$. Cubing, $z^3 + 3z^2(2i) + 3z(2i)^2 + (2i)^3 = 4$ or $z^3 - 12z - 4 = i(8 - 6z^2)$. Squaring, $z^6 + 12z^4 - 8z^3 + 48z^2 + 96z + 80 = 0$, a polynomial equation with integer coefficients having $\sqrt[3]{4} - 2i$ as a root. Hence $\sqrt[3]{4} - 2i$ is an algebraic number.

Numbers which are not algebraic, i.e. do not satisfy any polynomial equation with integer coefficients, are called *transcendental numbers*. It has been proved that the numbers $\pi = 3.14159\ldots$ and $e = 2.71828\ldots$ are transcendental. However, it is still not yet known whether numbers such as $e\pi$ or $e + \pi$, for example, are transcendental or not.

48. Represent graphically the set of values of z for which (a) $\left|\frac{z-3}{z+3}\right| = 2$, (b) $\left|\frac{z-3}{z+3}\right| < 2$.

(a) The given equation is equivalent to $|z-3| = 2|z+3|$ or, if $z = x+iy$, $|x+iy-3| = 2|x+iy+3|$, i.e.,

$$\sqrt{(x-3)^2+y^2} = 2\sqrt{(x+3)^2+y^2}$$

Squaring and simplifying, this becomes

$$x^2+y^2+10x+9 = 0 \quad \text{or} \quad (x+5)^2+y^2 = 16$$

i.e. $|z+5| = 4$, a circle of radius 4 with centre at $(-5,0)$ as shown in Fig. 1-36.

Geometrically, any point P on this circle is such that the distance from P to point $B(3,0)$ is twice the distance from P to point $A(-3,0)$.

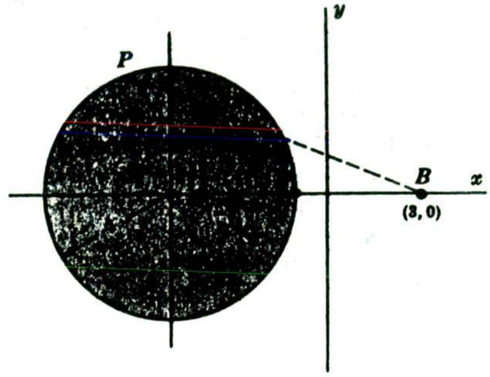

Fig. 1-36

Another method.

$\left|\frac{z-3}{z+3}\right| = 2$ is equivalent to

$$\left(\frac{z-3}{z+3}\right)\left(\frac{\bar{z}-3}{\bar{z}+3}\right) = 4 \quad \text{or} \quad z\bar{z}+5\bar{z}+5z+9 = 0$$

i.e. $(z+5)(\bar{z}+5) = 16$ or $|z+5| = 4$.

(b) The given inequality is equivalent to $|z-3| < 2|z+3|$ or $\sqrt{(x-3)^2+y^2} < 2\sqrt{(x+3)^2+y^2}$. Squaring and simplifying, this becomes $x^2+y^2+10x+9 > 0$ or $(x+5)^2+y^2 > 16$, i.e. $|z+5| > 4$.

The required set thus consists of all points external to the circle of Fig. 1-36.

49. Given the sets A and B represented by $|z-1| < 3$ and $|z-2i| < 2$ respectively. Represent geometrically (a) $A \cap B$ or AB, (b) $A \cup B$ or $A+B$.

The required sets of points are shown shaded in Figures 1-37 and 1-38 respectively.

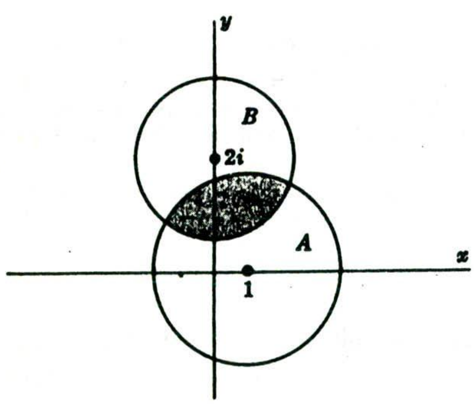

Fig. 1-37

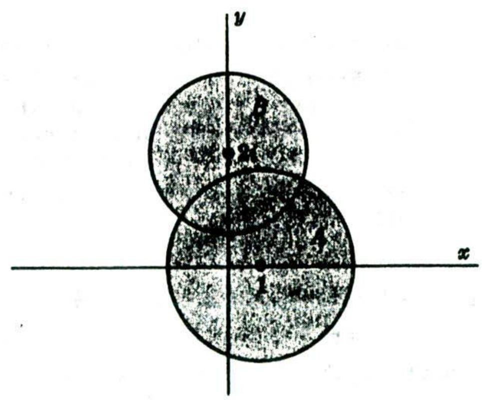

Fig. 1-38

50. Solve $z^2(1-z^2) = 16$.

Method 1. The equation can be written $z^4-z^2+16 = 0$, i.e. $z^4+8z^2+16-9z^2 = 0$, $(z^2+4)^2-9z^2 = 0$ or $(z^2+4+3z)(z^2+4-3z) = 0$. Then the required solutions are the solutions of $z^2+3z+4 = 0$ and $z^2-3z+4 = 0$, or $-\frac{3}{2} \pm \frac{\sqrt{7}}{2}i$ and $\frac{3}{2} \pm \frac{\sqrt{7}}{2}i$.

Method 2. Letting $w = z^2$, the equation can be written $w^2-w+16 = 0$ and $w = \frac{1}{2} \pm \frac{3}{2}\sqrt{7}\,i$. To obtain solutions of $z^2 = \frac{1}{2} \pm \frac{3}{2}\sqrt{7}\,i$, the methods of Problem 30 can be used.

51. If z_1, z_2, z_3 represent vertices of an equilateral triangle, prove that

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

From Fig. 1-39 we see that

$$z_2 - z_1 = e^{\pi i/3}(z_3 - z_1)$$
$$z_1 - z_3 = e^{\pi i/3}(z_2 - z_3)$$

Then by division, $\dfrac{z_2 - z_1}{z_1 - z_3} = \dfrac{z_3 - z_1}{z_2 - z_3}$ or

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

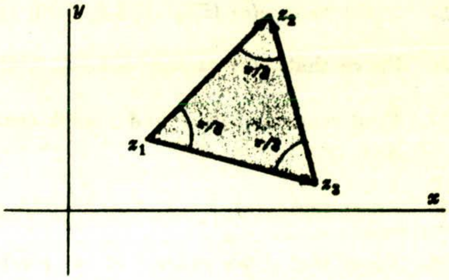

Fig. 1-39

52. Prove that for $m = 2, 3, \ldots$

$$\sin\frac{\pi}{m}\sin\frac{2\pi}{m}\sin\frac{3\pi}{m}\cdots\sin\frac{(m-1)\pi}{m} = \frac{m}{2^{m-1}}$$

The roots of $z^m = 1$ are $z = 1, e^{2\pi i/m}, e^{4\pi i/m}, \ldots, e^{2(m-1)\pi i/m}$. Then we can write

$$z^m - 1 = (z-1)(z - e^{2\pi i/m})(z - e^{4\pi i/m})\cdots(z - e^{2(m-1)\pi i/m})$$

Dividing both sides by $z - 1$ and then letting $z = 1$ [realizing that $(z^m - 1)/(z-1) = 1 + z + z^2 + \cdots + z^{m-1}$] we find

$$m = (1 - e^{2\pi i/m})(1 - e^{4\pi i/m})\cdots(1 - e^{2(m-1)\pi i/m}) \tag{1}$$

Taking the complex conjugate of both sides of (1) yields

$$m = (1 - e^{-2\pi i/m})(1 - e^{-4\pi i/m})\cdots(1 - e^{-2(m-1)\pi i/m}) \tag{2}$$

Multiplying (1) by (2) using $(1 - e^{2k\pi i/m})(1 - e^{-2k\pi i/m}) = 2 - 2\cos(2k\pi/m)$, we have

$$m^2 = 2^{m-1}\left(1 - \cos\frac{2\pi}{m}\right)\left(1 - \cos\frac{4\pi}{m}\right)\cdots\left(1 - \cos\frac{2(m-1)\pi}{m}\right) \tag{3}$$

Since $1 - \cos(2k\pi/m) = 2\sin^2(k\pi/m)$, (3) becomes

$$m^2 = 2^{2m-2}\sin^2\frac{\pi}{m}\sin^2\frac{2\pi}{m}\cdots\sin^2\frac{(m-1)\pi}{m} \tag{4}$$

Then taking the positive square root of both sides yields the required result.

Supplementary Problems

FUNDAMENTAL OPERATIONS WITH COMPLEX NUMBERS

53. Perform each of the indicated operations:

(a) $(4 - 3i) + (2i - 8)$

(b) $3(-1 + 4i) - 2(7 - i)$

(c) $(3 + 2i)(2 - i)$

(d) $(i - 2)\{2(1 + i) - 3(i - 1)\}$

(e) $\dfrac{2 - 3i}{4 - i}$

(f) $(4 + i)(3 + 2i)(1 - i)$

(g) $\dfrac{(2 + i)(3 - 2i)(1 + 2i)}{(1 - i)^2}$

(h) $(2i - 1)^2\left\{\dfrac{4}{1 - i} + \dfrac{2 - i}{1 + i}\right\}$

(i) $\dfrac{i^4 + i^9 + i^{16}}{2 - i^5 + i^{10} - i^{15}}$

(j) $3\left(\dfrac{1 + i}{1 - i}\right)^2 - 2\left(\dfrac{1 - i}{1 + i}\right)^3$

Ans. (a) $-4 - i$ (b) $-17 + 14i$ (c) $8 + i$ (d) $-9 + 7i$ (e) $11/17 - (10/17)i$ (f) $21 + i$ (g) $-15/2 + 5i$ (h) $-11/2 - (23/2)i$ (i) $2 + i$ (j) $-3 - 2i$

54. If $z_1 = 1 - i$, $z_2 = -2 + 4i$, $z_3 = \sqrt{3} - 2i$, evaluate each of the following:

(a) $z_1^2 + 2z_1 - 3$

(b) $|2z_2 - 3z_1|^2$

(c) $(z_3 - \bar{z}_3)^5$

(d) $|z_1\bar{z}_2 + z_2\bar{z}_1|$

(e) $\left|\dfrac{z_1 + z_2 + 1}{z_1 - z_2 + i}\right|$

(f) $\dfrac{1}{2}\left(\dfrac{z_3}{\bar{z}_3} + \dfrac{\bar{z}_3}{z_3}\right)$

(g) $\overline{(z_2 + z_3)(z_1 - z_3)}$

(h) $|z_1^2 + \bar{z}_2^2|^2 + |\bar{z}_3^2 - z_2^2|^2$

(i) $\text{Re}\{2z_1^3 + 3z_2^2 - 5z_3^2\}$

(j) $\text{Im}\{z_1z_2/z_3\}$

Ans. (a) $-1 - 4i$ (b) 170 (c) $1024i$ (d) 12 (e) $3/5$ (f) $-1/7$ (g) $-7 + 3\sqrt{3} + \sqrt{3}\,i$ (h) $765 + 128\sqrt{3}$ (i) -35 (j) $(6\sqrt{3} + 4)/7$

55. Prove that (a) $(\overline{z_1 z_2}) = \bar{z}_1 \bar{z}_2$, (b) $(\overline{z_1 z_2 z_3}) = \bar{z}_1 \bar{z}_2 \bar{z}_3$. Generalize these results.

56. Prove that (a) $(\overline{z_1/z_2}) = \bar{z}_1/\bar{z}_2$, (b) $|z_1/z_2| = |z_1|/|z_2|$ if $z_2 \neq 0$.

57. Find real numbers x and y such that $2x - 3iy + 4ix - 2y - 5 - 10i = (x + y + 2) - (y - x + 3)i$.
Ans. $x = 1,\ y = -2$

58. Prove that (a) $\text{Re}\{z\} = (z + \bar{z})/2$, (b) $\text{Im}\{z\} = (z - \bar{z})/2i$.

59. Prove that if the product of two complex numbers is zero then at least one of the numbers must be zero.

60. If $w = 3iz - z^2$ and $z = x + iy$, find $|w|^2$ in terms of x and y.
Ans. $x^4 + y^4 + 2x^2y^2 - 6x^2y - 6y^3 + 9x^2 + 9y^2$

GRAPHICAL REPRESENTATION OF COMPLEX NUMBERS. VECTORS.

61. Perform the indicated operations both analytically and graphically.

(a) $(2 + 3i) + (4 - 5i)$ (c) $3(1 + 2i) - 2(2 - 3i)$ (e) $\frac{1}{2}(4 - 3i) + \frac{3}{2}(5 + 2i)$

(b) $(7 + i) - (4 - 2i)$ (d) $3(1 + i) + 2(4 - 3i) - (2 + 5i)$

Ans. (a) $6 - 2i$, (b) $3 + 3i$, (c) $-1 + 12i$, (d) $9 - 8i$, (e) $19/2 + (3/2)i$

62. If z_1, z_2 and z_3 are the vectors indicated in Fig. 1-40, construct graphically:

(a) $2z_1 + z_3$ (c) $z_1 + (z_2 + z_3)$ (e) $\frac{1}{3}z_2 - \frac{3}{4}z_1 + \frac{2}{3}z_3$

(b) $(z_1 + z_2) + z_3$ (d) $3z_1 - 2z_2 + 5z_3$

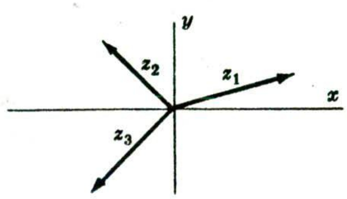

Fig. 1-40

63. If $z_1 = 4 - 3i$ and $z_2 = -1 + 2i$, obtain graphically and analytically (a) $|z_1 + z_2|$, (b) $|z_1 - z_2|$, (c) $\bar{z}_1 - \bar{z}_2$, (d) $|2\bar{z}_1 - 3\bar{z}_2 - 2|$.
Ans. (a) $\sqrt{10}$, (b) $5\sqrt{2}$, (c) $5 + 5i$, (d) 15

64. The position vectors of points A, B and C of triangle ABC are given by $z_1 = 1 + 2i$, $z_2 = 4 - 2i$ and $z_3 = 1 - 6i$ respectively. Prove that ABC is an isosceles triangle and find the lengths of the sides.
Ans. 5, 5, 8

65. Let z_1, z_2, z_3, z_4 be the position vectors of the vertices for quadrilateral $ABCD$. Prove that $ABCD$ is a parallelogram if and only if $z_1 - z_2 - z_3 + z_4 = 0$.

66. If the diagonals of a quadrilateral bisect each other, prove that the quadrilateral is a parallelogram.

67. Prove that the medians of a triangle meet in a point.

68. Let $ABCD$ be a quadrilateral and E, F, G, H the midpoints of the sides. Prove that $EFGH$ is a parallelogram.

69. In parallelogram $ABCD$, point E bisects side AD. Prove that the point where BE meets AC trisects AC.

70. The position vectors of points A and B are $2 + i$ and $3 - 2i$ respectively. (a) Find an equation for line AB. (b) Find an equation for the line perpendicular to AB at its midpoint.

Ans. (a) $z - (2 + i) = t(1 - 3i)$ or $x = 2 + t,\ y = 1 - 3t$ or $3x + y = 7$

(b) $z - (5/2 - i/2) = t(3 + i)$ or $x = 3t + 5/2,\ y = t - 1/2$ or $x - 3y = 4$

71. Describe and graph the locus represented by each of the following: (a) $|z - i| = 2$, (b) $|z + 2i| + |z - 2i| = 6$, (c) $|z - 3| - |z + 3| = 4$, (d) $z(\bar{z} + 2) = 3$, (e) $\text{Im}\{z^2\} = 4$.
Ans. (a) circle, (b) ellipse, (c) hyperbola, (d) circle, (e) hyperbola

72. Find an equation for (a) a circle of radius 2 with centre at $(-3, 4)$, (b) an ellipse with foci at $(0, 2)$ and $(0, -2)$ whose major axis has length 10.
Ans. (a) $|z + 3 - 4i| = 2$ or $(x + 3)^2 + (y - 4)^2 = 4$, (b) $|z + 2i| + |z - 2i| = 10$

73. Describe graphically the region represented by each of the following:
(a) $1 < |z+i| \leqq 2$, (b) $\text{Re}\{z^2\} > 1$, (c) $|z+3i| > 4$, (d) $|z+2-3i| + |z-2+3i| < 10$.

74. Show that the ellipse $|z+3| + |z-3| = 10$ can be expressed in rectangular form as $x^2/25 + y^2/16 = 1$ [see Problem 13(b)].

AXIOMATIC FOUNDATIONS OF COMPLEX NUMBERS

75. Use the definition of a complex number as an ordered pair of real numbers to prove that if the product of two complex numbers is zero then at least one of the numbers must be zero.

76. Prove the commutative laws with respect to (a) addition, (b) multiplication.

77. Prove the associative laws with respect to (a) addition, (b) multiplication.

78. (a) Find real numbers x and y such that $(c,d)\cdot(x,y) = (a,b)$ where $(c,d) \neq (0,0)$.
(b) How is (x,y) related to the result for division of complex numbers given on Page 2?

79. Prove that

$$(\cos\theta_1, \sin\theta_1)(\cos\theta_2, \sin\theta_2)\cdots(\cos\theta_n, \sin\theta_n) = (\cos[\theta_1+\theta_2+\cdots+\theta_n], \sin[\theta_1+\theta_2+\cdots+\theta_n])$$

80. (a) How would you define $(a,b)^{1/n}$ where n is a positive integer?
(b) Determine $(a,b)^{1/2}$ in terms of a and b.

POLAR FORM OF COMPLEX NUMBERS

81. Express each of the following complex numbers in polar form.
(a) $2-2i$, (b) $-1+\sqrt{3}\,i$, (c) $2\sqrt{2}+2\sqrt{2}\,i$, (d) $-i$, (e) -4, (f) $-2\sqrt{3}-2i$, (g) $\sqrt{2}\,i$, (h) $\sqrt{3}/2-3i/2$.
Ans. (a) $2\sqrt{2}$ cis 315° or $2\sqrt{2}\,e^{7\pi i/4}$, (b) 2 cis 120° or $2e^{2\pi i/3}$, (c) 4 cis 45° or $4e^{\pi i/4}$, (d) cis 270° or $e^{3\pi i/2}$, (e) 4 cis 180° or $4e^{\pi i}$, (f) 4 cis 210° or $4e^{7\pi i/6}$, (g) $\sqrt{2}$ cis 90° or $\sqrt{2}\,e^{\pi i/2}$, (h) $\sqrt{3}$ cis 300° or $\sqrt{3}\,e^{5\pi i/3}$.

82. Show that $2+i = \sqrt{5}\,e^{i\tan^{-1}(1/2)}$.

83. Express in polar form: (a) $-3-4i$, (b) $1-2i$.
Ans. (a) $5\,e^{i(\pi+\tan^{-1}4/3)}$, (b) $\sqrt{5}\,e^{-i\tan^{-1}2}$

84. Graph each of the following and express in rectangular form.
(a) $6(\cos 135° + i\sin 135°)$, (b) 12 cis 90°, (c) 4 cis 315°, (d) $2e^{5\pi i/4}$, (e) $5e^{7\pi i/6}$, (f) $3e^{-2\pi i/3}$.
Ans. (a) $-3\sqrt{2}+3\sqrt{2}\,i$, (b) $12i$, (c) $2\sqrt{2}-2\sqrt{2}\,i$, (d) $-\sqrt{2}-\sqrt{2}\,i$, (e) $-5\sqrt{3}/2-(5/2)i$, (f) $-3\sqrt{3}/2-(3/2)i$

85. An airplane travels 150km southeast, 100km due west, 225km 30° north of east, and then 323km northeast. Determine (a) analytically and (b) graphically how far and in what direction it is from its starting point. *Ans.* 375km, 23° north of east (approx.)

86. Three forces as shown in Fig. 1-41 act in a plane on an object placed at O. Determine (a) graphically and (b) analytically what force is needed to prevent the object from moving. [This force is sometimes called the *equilibrant.*]

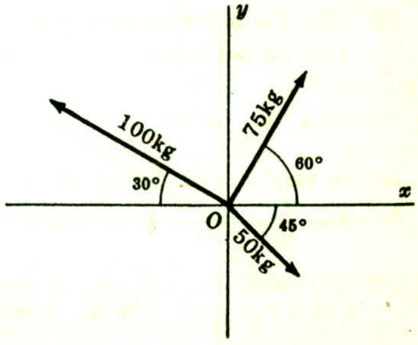

Fig. 1-41

87. Prove that on the circle $z = Re^{i\theta}$, $|e^{iz}| = e^{-R\sin\theta}$.

88. (a) Prove that $r_1e^{i\theta_1} + r_2e^{i\theta_2} = r_3e^{i\theta_3}$ where

$$r_3 = \sqrt{r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1-\theta_2)}$$

$$\theta_3 = \tan^{-1}\left(\frac{r_1\sin\theta_1 + r_2\sin\theta_2}{r_1\cos\theta_1 + r_2\cos\theta_2}\right)$$

(b) Generalize the result in (a).

DE MOIVRE'S THEOREM

89. Evaluate each of the following:

(a) $(5 \text{ cis } 20^\circ)(3 \text{ cis } 40^\circ)$ (c) $\dfrac{(8 \text{ cis } 40^\circ)^3}{(2 \text{ cis } 60^\circ)^4}$ (d) $\dfrac{(3e^{\pi i/6})(2e^{-5\pi i/4})(6e^{5\pi i/3})}{(4e^{2\pi i/3})^2}$ (e) $\left(\dfrac{\sqrt{3}-i}{\sqrt{3}+i}\right)^4\left(\dfrac{1+i}{1-i}\right)^5$

(b) $(2 \text{ cis } 50^\circ)^6$

Ans. (a) $15/2 + (15\sqrt{3}/2)i$, (b) $32 - 32\sqrt{3}\,i$, (c) $-16 - 16\sqrt{3}\,i$, (d) $3\sqrt{3}/2 - (3\sqrt{3}/2)i$, (e) $-\sqrt{3}/2 - (1/2)i$

90. Prove that (a) $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$, (b) $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$.

91. Prove that the solutions of $z^4 - 3z^2 + 1 = 0$ are given by $z = 2\cos 36^\circ, 2\cos 72^\circ, 2\cos 216^\circ, 2\cos 252^\circ$.

92. Show that (a) $\cos 36^\circ = (\sqrt{5}+1)/4$, (b) $\cos 72^\circ = (\sqrt{5}-1)/4$. [*Hint:* Use Problem 91.]

93. Prove that (a) $\dfrac{\sin 4\theta}{\sin \theta} = 8\cos^3\theta - 4 = 2\cos 3\theta + 6\cos\theta - 4$

(b) $\cos 4\theta = 8\sin^4\theta - 8\sin^2\theta + 1$

94. Prove De Moivre's theorem for (a) negative integers, (b) rational numbers.

ROOTS OF COMPLEX NUMBERS

95. Find each of the indicated roots and locate them graphically.

(a) $(2\sqrt{3} - 2i)^{1/2}$, (b) $(-4 + 4i)^{1/5}$, (c) $(2 + 2\sqrt{3}\,i)^{1/3}$, (d) $(-16i)^{1/4}$, (e) $(64)^{1/6}$, (f) $(i)^{2/3}$.

Ans. (a) $2 \text{ cis } 165^\circ$, $2 \text{ cis } 345^\circ$. (b) $\sqrt{2} \text{ cis } 27^\circ$, $\sqrt{2} \text{ cis } 99^\circ$, $\sqrt{2} \text{ cis } 171^\circ$, $\sqrt{2} \text{ cis } 243^\circ$, $\sqrt{2} \text{ cis } 315^\circ$. (c) $\sqrt[3]{4} \text{ cis } 20^\circ$, $\sqrt[3]{4} \text{ cis } 140^\circ$, $\sqrt[3]{4} \text{ cis } 260^\circ$. (d) $2 \text{ cis } 67.5^\circ$, $2 \text{ cis } 157.5^\circ$, $2 \text{ cis } 247.5^\circ$, $2 \text{ cis } 337.5^\circ$. (e) $2 \text{ cis } 0^\circ$, $2 \text{ cis } 60^\circ$, $2 \text{ cis } 120^\circ$, $2 \text{ cis } 180^\circ$, $2 \text{ cis } 240^\circ$, $2 \text{ cis } 300^\circ$. (f) $\text{cis } 60^\circ$, $\text{cis } 180^\circ$, $\text{cis } 300^\circ$.

96. Find all the indicated roots and locate them in the complex plane.

(a) cube roots of 8, (b) square roots of $4\sqrt{2} + 4\sqrt{2}\,i$, (c) fifth roots of $-16 + 16\sqrt{3}\,i$, (d) sixth roots of $-27i$.

Ans. (a) $2 \text{ cis } 0^\circ$, $2 \text{ cis } 120^\circ$, $2 \text{ cis } 240^\circ$. (b) $\sqrt{8} \text{ cis } 22.5^\circ$, $\sqrt{8} \text{ cis } 202.5^\circ$. (c) $2 \text{ cis } 48^\circ$, $2 \text{ cis } 120^\circ$, $2 \text{ cis } 192^\circ$, $2 \text{ cis } 264^\circ$, $2 \text{ cis } 336^\circ$. (d) $\sqrt{3} \text{ cis } 45^\circ$, $\sqrt{3} \text{ cis } 105^\circ$, $\sqrt{3} \text{ cis } 165^\circ$, $\sqrt{3} \text{ cis } 225^\circ$, $\sqrt{3} \text{ cis } 285^\circ$, $\sqrt{3} \text{ cis } 345^\circ$.

97. Solve the equations (a) $z^4 + 81 = 0$, (b) $z^6 + 1 = \sqrt{3}\,i$.

Ans. (a) $3 \text{ cis } 45^\circ$, $3 \text{ cis } 135^\circ$, $3 \text{ cis } 225^\circ$, $3 \text{ cis } 315^\circ$

(b) $\sqrt[6]{2} \text{ cis } 40^\circ$, $\sqrt[6]{2} \text{ cis } 100^\circ$, $\sqrt[6]{2} \text{ cis } 160^\circ$, $\sqrt[6]{2} \text{ cis } 220^\circ$, $\sqrt[6]{2} \text{ cis } 280^\circ$, $\sqrt[6]{2} \text{ cis } 340^\circ$

98. Find the square roots of (a) $5 - 12i$, (b) $8 + 4\sqrt{5}\,i$.

Ans. (a) $3 - 2i, -3 + 2i$. (b) $\sqrt{10} + \sqrt{2}\,i, -\sqrt{10} - \sqrt{2}\,i$

99. Find the cube roots of $-11 - 2i$. *Ans.* $1 + 2i$, $\frac{1}{2} - \sqrt{3} + (1 + \frac{1}{2}\sqrt{3})i$, $-\frac{1}{2} - \sqrt{3} + (\frac{1}{2}\sqrt{3} - 1)i$

POLYNOMIAL EQUATIONS

100. Solve the following equations, obtaining all roots: (a) $5z^2 + 2z + 10 = 0$, (b) $z^2 + (i-2)z + (3-i) = 0$.

Ans. (a) $(-1 \pm 7i)/5$, (b) $1 + i, 1 - 2i$

101. Solve $z^5 - 2z^4 - z^3 + 6z - 4 = 0$. *Ans.* $1, 1, 2, -1 \pm i$

102. (a) Find all the roots of $z^4 + z^2 + 1 = 0$ and (b) locate them in the complex plane.

Ans. $\frac{1}{2}(1 \pm i\sqrt{3})$, $\frac{1}{2}(-1 \pm i\sqrt{3})$

103. Prove that the sum of the roots of $a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n = 0$ where $a_0 \neq 0$ taken r at a time is $(-1)^r a_r/a_0$ where $0 < r < n$.

104. Find two numbers whose sum is 4 and whose product is 8. *Ans.* $2 + 2i$, $2 - 2i$

THE nth ROOTS OF UNITY

105. Find all the (a) fourth roots, (b) seventh roots of unity and exhibit them graphically.
Ans. (a) $e^{2\pi ik/4} = e^{\pi ik/2}$, $k = 0, 1, 2, 3$ (b) $e^{2\pi ik/7}$, $k = 0, 1, \ldots, 6$

106. (a) Prove that $1 + \cos 72° + \cos 144° + \cos 216° + \cos 288° = 0$.
(b) Give a graphical interpretation of the result in (a).

107. Prove that $\cos 36° + \cos 72° + \cos 108° + \cos 144° = 0$ and interpret graphically.

108. Prove that the sum of the products of all the nth roots of unity taken $2, 3, 4, \ldots, (n-1)$ at a time is zero.

109. Find all roots of $(1+z)^5 = (1-z)^5$.
Ans. 0, $(\omega - 1)/(\omega + 1)$, $(\omega^2 - 1)/(\omega^2 + 1)$, $(\omega^3 - 1)/(\omega^3 + 1)$, $(\omega^4 - 1)/(\omega^4 + 1)$, where $\omega = e^{2\pi i/5}$

THE DOT AND CROSS PRODUCT

110. If $z_1 = 2 + 5i$ and $z_2 = 3 - i$, find (a) $z_1 \circ z_2$, (b) $z_1 \times z_2$, (c) $z_2 \circ z_1$, (d) $z_2 \times z_1$, (e) $|z_1 \circ z_2|$, (f) $|z_2 \circ z_1|$, (g) $|z_1 \times z_2|$, (h) $|z_2 \times z_1|$.
Ans. (a) 1, (b) −17, (c) 1, (d) 17, (e) 1, (f) 1, (g) 17, (h) 17

111. Prove that (a) $z_1 \circ z_2 = z_2 \circ z_1$, (b) $z_1 \times z_2 = -z_2 \times z_1$.

112. If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, prove that (a) $z_1 \circ z_2 = r_1 r_2 \cos(\theta_2 - \theta_1)$, (b) $z_1 \times z_2 = r_1 r_2 \sin(\theta_2 - \theta_1)$.

113. Prove that (a) $z_1 \circ (z_2 + z_3) = z_1 \circ z_2 + z_1 \circ z_3$, (b) $z_1 \times (z_2 + z_3) = z_1 \times z_2 + z_1 \times z_3$.

114. Find the area of a triangle having vertices at $-4 - i$, $1 + 2i$, $4 - 3i$. *Ans.* 17

115. Find the area of a quadrilateral having vertices at $(2, -1)$, $(4, 3)$, $(-1, 2)$ and $(-3, -2)$. *Ans.* 18

CONJUGATE COORDINATES

116. Describe each of the following loci expressed in terms of conjugate coordinates $z, \bar{z}$.
(a) $z\bar{z} = 16$, (b) $z\bar{z} - 2z - 2\bar{z} + 8 = 0$, (c) $z + \bar{z} = 4$, (d) $\bar{z} = z + 6i$.
Ans. (a) $x^2 + y^2 = 16$, (b) $x^2 + y^2 - 4x + 8 = 0$, (c) $x = 2$, (d) $y = -3$

117. Write each of the following equations in terms of conjugate coordinates.
(a) $(x-3)^2 + y^2 = 9$, (b) $2x - 3y = 5$, (c) $4x^2 + 16y^2 = 25$.
Ans. (a) $(z-3)(\bar{z}-3) = 9$, (b) $(2i-3)z + (2i+3)\bar{z} = 10i$, (c) $3(z^2 + \bar{z}^2) - 10z\bar{z} + 25 = 0$

POINT SETS

118. Let S be the set of all points $a + bi$, where a and b are rational numbers, which lie inside the square shown shaded in Fig. 1-42. (a) Is S bounded? (b) What are the limit points of S, if any? (c) Is S closed? (d) What are its interior and boundary points? (e) Is S open? (f) Is S connected? (g) Is S an open region or domain? (h) What is the closure of S? (i) What is the complement of S? (j) Is S countable? (k) Is S compact? (l) Is the closure of S compact?

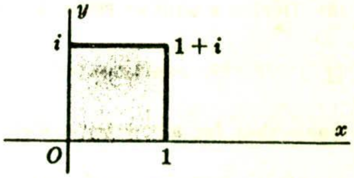

Fig. 1-42

Ans. (a) Yes. (b) Every point inside or on the boundary of the square is a limit point. (c) No. (d) All points of the square are boundary points; there are no interior points. (e) No. (f) No. (g) No. (h) The closure of S is the set of all points inside and on the boundary of the square. (i) The complement of S is the set of all points which are not equal to $a + bi$ when a and b [where $0 < a < 1$, $0 < b < 1$] are rational. (j) Yes. (k) No. (l) Yes.

119. Answer Problem 118 if S is the set of all points inside the square.
Ans. (a) Yes. (b) Every point inside or on the square is a limit point. (c) No. (d) Every point inside is an interior point, while every point on the boundary is a boundary point. (e) Yes. (f) Yes. (g) Yes. (h) The closure of S is the set of all points inside and on the boundary of the square. (i) The complement of S is the set of all points exterior to the square or on its boundary. (j) No. (k) No. (l) Yes.

120. Answer Problem 118 if S is the set of all points inside or on the square.
Ans. (a) Yes. (b) Every point of S is a limit point. (c) Yes. (d) Every point inside the square is an interior point, while every point on the boundary is a boundary point. (e) No. (f) Yes. (g) No. (h) S itself. (i) All points exterior to the square. (j) No. (k) Yes. (l) Yes.

121. Given the point sets $A = \{1, i, -i\}$, $B = \{2, 1, -i\}$, $C = \{i, -i, 1+i\}$, $D = \{0, -i, 1\}$. Find: (a) $A + (B + C)$ or $A \cup (B \cup C)$, (b) $AC + BD$ or $(A \cap C) \cup (B \cap D)$, (c) $(A + C)(B + D)$ or $(A \cup C) \cap (B \cup D)$.
Ans. (a) $\{2, 1, -i, i, 1+i\}$, (b) $\{1, i, -i\}$, (c) $\{1, -i\}$

122. If A, B, C and D are any point sets, prove that (a) $A + B = B + A$, (b) $AB = BA$, (c) $A + (B + C) = (A + B) + C$, (d) $A(BC) = (AB)C$, (e) $A(B + C) = AB + AC$. Give equivalent results using the notations $\cap$ and $\cup$. Discuss how these can be used to define an algebra of sets.

123. If A, B and C are the point sets defined by $|z + i| < 3$, $|z| < 5$, $|z + 1| < 4$, represent graphically each of the following:
(a) $A \cap B \cap C$, (b) $A \cup B \cup C$, (c) $A \cap B \cup C$, (d) $C(A + B)$, (d) $(A \cup B) \cap (B \cup C)$, (e) $AB + BC + CA$, (f) $A\tilde{B} + B\tilde{C} + C\tilde{A}$.

124. Prove that the complement of a set S is open or closed according as S is closed or open.

125. If $S_1, S_2, \ldots, S_n$ are open sets, prove that $S_1 + S_2 + \cdots + S_n$ is open.

126. If a limit point of a set does not belong to the set, prove that it must be a boundary point of the set.

MISCELLANEOUS PROBLEMS

127. Let $ABCD$ be a parallelogram. Prove that $(AC)^2 + (BD)^2 = (AB)^2 + (BC)^2 + (CD)^2 + (DA)^2$.

128. Explain the fallacy: $-1 = \sqrt{-1}\,\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$. Hence $1 = -1$.

129. (a) Show that the equation $z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4 = 0$ where a_1, a_2, a_3, a_4 are real constants different from zero, has a pure imaginary root if $a_3^2 + a_1^2 a_4 = a_1 a_2 a_3$.
(b) Is the converse of (a) true?

130. (a) Prove that
$$\cos^n \phi = \frac{1}{2^{n-1}}\left\{\cos n\phi + n\cos(n-2)\phi + \frac{n(n-1)}{2!}\cos(n-4)\phi + \cdots + R_n\right\}$$
where
$$R_n = \begin{cases} \cos\phi & \text{if } n \text{ is odd} \\ \dfrac{n!}{[(n/2)!]^2} & \text{if } n \text{ is even.} \end{cases}$$
(b) Derive a similar result for $\sin^n \phi$.

131. If $z = 6e^{\pi i/3}$, evaluate $|e^{iz}|$. *Ans.* $e^{-3\sqrt{3}}$

132. Show that for any real numbers p and m, $e^{2mi\cot^{-1}p}\left\{\dfrac{pi+1}{pi-1}\right\}^m = 1$.

133. If $P(z)$ is any polynomial in z with real coefficients, prove that $\overline{P(z)} = P(\bar{z})$.

134. If z_1, z_2 and z_3 are collinear, prove that there exist real constants α, β, γ, not all zero, such that $\alpha z_1 + \beta z_2 + \gamma z_3 = 0$ where $\alpha + \beta + \gamma = 0$.

135. Given the complex number z, represent geometrically (a) $\bar{z}$, (b) $-z$, (c) $1/z$, (d) z^2.

136. Given any two complex numbers z_1 and z_2 not equal to zero, show how to represent graphically using only ruler and compass (a) $z_1 z_2$, (b) z_1/z_2, (c) $z_1^2 + z_2^2$, (d) $z_1^{1/2}$, (e) $z_2^{3/4}$.

137. Prove that an equation for a line passing through the points z_1 and z_2 is given by
$$\arg\{(z - z_1)/(z_2 - z_1)\} = 0$$

138. If $z = x + iy$, prove that $|x| + |y| \leq \sqrt{2}\,|x + iy|$.

139. Is the converse to Problem 51 true? Justify your answer.

140. Find an equation for the circle passing through the points $1-i$, $2i$, $1+i$.
Ans. $|z+1| = \sqrt{5}$ or $(x+1)^2 + y^2 = 5$

141. Show that the locus of z such that $|z-a|\,|z+a| = a^2$, $a > 0$ is a *lemniscate* as shown in Fig. 1-43.

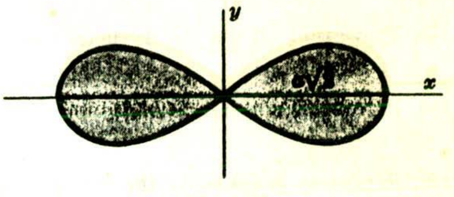

Fig. 1-43

142. Let $p_n = a_n^2 + b_n^2$, $n = 1, 2, 3, \ldots$ where a_n and b_n are positive integers. Prove that for every positive integer M we can always find positive integers A and B such that $p_1 p_2 \cdots p_M = A^2 + B^2$. [Example: If $5 = 2^2 + 1^2$ and $25 = 3^2 + 4^2$, then $5 \cdot 25 = 2^2 + 11^2$.]

143. Prove that

(a) $\cos\theta + \cos(\theta+\alpha) + \cdots + \cos(\theta+n\alpha) = \dfrac{\sin\frac{1}{2}(n+1)\alpha}{\sin\frac{1}{2}\alpha}\cos(\theta+\frac{1}{2}n\alpha)$

(b) $\sin\theta + \sin(\theta+\alpha) + \cdots + \sin(\theta+n\alpha) = \dfrac{\sin\frac{1}{2}(n+1)\alpha}{\sin\frac{1}{2}\alpha}\sin(\theta+\frac{1}{2}n\alpha)$

144. Prove that (a) $\operatorname{Re}\{z\} > 0$ and (b) $|z-1| < |z+1|$ are equivalent statements.

145. A wheel of radius 1·2 metres [Fig. 1-44] is rotating counterclockwise about an axis through its centre at 30 revolutions per minute. (a) Show that the position and velocity of any point P on the wheel are given respectively by $4e^{i\pi t}$ and $4\pi i e^{i\pi t}$, where t is the time in seconds measured from the instant when P was on the positive x axis. (b) Find the position and velocity when $t = 2/3$ and $t = 15/4$.

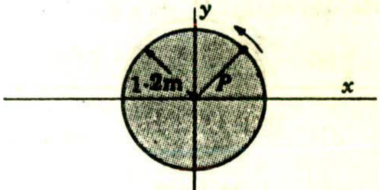

Fig. 1-44

146. Prove that for any integer $m > 1$,

$$(z+a)^{2m} - (z-a)^{2m} = 4maz \prod_{k=1}^{m-1} \{z^2 + a^2\cot^2(k\pi/2m)\}$$

where $\prod_{k=1}^{m-1}$ denotes the product of all the factors indicated from $k = 1$ to $m-1$.

147. If points P_1 and P_2, represented by z_1 and z_2 respectively, are such that $|z_1 + z_2| = |z_1 - z_2|$, prove that (a) z_1/z_2 is a pure imaginary number, (b) $\angle P_1OP_2 = 90°$.

148. Prove that for any integer $m > 1$,

$$\cot\frac{\pi}{2m}\cot\frac{2\pi}{2m}\cot\frac{3\pi}{2m}\cdots\cot\frac{(m-1)\pi}{2m} = 1$$

149. Prove and generalize: (a) $\csc^2(\pi/7) + \csc^2(2\pi/7) + \csc^2(4\pi/7) = 2$
(b) $\tan^2(\pi/16) + \tan^2(3\pi/16) + \tan^2(5\pi/16) + \tan^2(7\pi/16) = 28$

150. If masses m_1, m_2, m_3 are located at points z_1, z_2, z_3 respectively, prove that the centre of mass is given by

$$\hat{z} = \frac{m_1z_1 + m_2z_2 + m_3z_3}{m_1 + m_2 + m_3}$$

Generalize to n masses.

151. Find that point on the line joining points z_1 and z_2 which divides it in the ratio $p:q$.
Ans. $(qz_1 + pz_2)/(q+p)$

152. Show that an equation for a circle passing through 3 points z_1, z_2, z_3 is given by

$$\left(\frac{z-z_1}{z-z_2}\right)\Big/\left(\frac{z_3-z_1}{z_3-z_2}\right) = \left(\frac{\bar{z}-\bar{z}_1}{\bar{z}-\bar{z}_2}\right)\Big/\left(\frac{\bar{z}_3-\bar{z}_1}{\bar{z}_3-\bar{z}_2}\right)$$

153. Prove that the medians of a triangle with vertices at z_1, z_2, z_3 intersect in the point $\frac{1}{3}(z_1 + z_2 + z_3)$.

154. Prove that the rational numbers between 0 and 1 are countable.
[*Hint.* Arrange the numbers as $0, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \ldots$.]

155. Prove that all the real rational numbers are countable.

156. Prove that the irrational numbers between 0 and 1 are not countable.

157. Represent graphically the set of values of z for which (*a*) $|z| > |z-1|$, (*b*) $|z+2| > 1 + |z-2|$.

158. Show that (*a*) $\sqrt[3]{2} + \sqrt{3}$ and (*b*) $2 - \sqrt{2}\,i$ are algebraic numbers.

159. Prove that $\sqrt{2} + \sqrt{3}$ is an irrational number.

160. Let $ABCD\cdots PQ$ represent a regular polygon of n sides inscribed in a circle of unit radius. Prove that the product of the lengths of the diagonals $AC, AD, \ldots, AP$ is $\frac{1}{4}n\csc^2(\pi/n)$.

161. Prove that if $\sin\theta \neq 0$,

(*a*) $$\frac{\sin n\theta}{\sin\theta} = 2^{n-1}\prod_{k=1}^{n-1}\{\cos\theta - \cos(k\pi/n)\}$$

(*b*) $$\frac{\sin(2n+1)\theta}{\sin\theta} = (2n+1)\prod_{k=1}^{n}\left\{1 - \frac{\sin^2\theta}{\sin^2 k\pi/(2n+1)}\right\}.$$

162. Prove $$\cos 2n\theta = (-1)^n\prod_{k=1}^{n}\left\{1 - \frac{\cos^2\theta}{\cos^2(2k-1)\pi/4n}\right\}.$$

163. If the product of two complex numbers z_1 and z_2 is real and different from zero, prove that there exists a real number p such that $z_1 = p\bar{z}_2$.

164. If z is any point on the circle $|z-1| = 1$, prove that $\arg(z-1) = 2\arg z = \frac{2}{3}\arg(z^2 - z)$ and give a geometrical interpretation.

165. Prove that under suitable restrictions (*a*) $z^m z^n = z^{m+n}$, (*b*) $(z^m)^n = z^{mn}$.

166. Prove (*a*) $\operatorname{Re}\{z_1 z_2\} = \operatorname{Re}\{z_1\}\operatorname{Re}\{z_2\} - \operatorname{Im}\{z_1\}\operatorname{Im}\{z_2\}$
(*b*) $\operatorname{Im}\{z_1 z_2\} = \operatorname{Re}\{z_1\}\operatorname{Im}\{z_2\} + \operatorname{Im}\{z_1\}\operatorname{Re}\{z_2\}$.

167. Find the area of the polygon with vertices at $2+3i, 3+i, -2-4i, -4-i, -1+2i$. *Ans.* 47/2

168. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be any complex numbers. Prove *Schwarz's inequality*,

$$\left|\sum_{k=1}^{n} a_k b_k\right|^2 \leqq \left(\sum_{k=1}^{n}|a_k|^2\right)\left(\sum_{k=1}^{n}|b_k|^2\right)$$

Chapter 2

Functions, Limits and Continuity

VARIABLES AND FUNCTIONS

A symbol, such as z, which can stand for any one of a set of complex numbers is called a *complex variable.*

If to each value which a complex variable z can assume there corresponds one or more values of a complex variable w, we say that w is a *function* of z and write $w = f(z)$ or $w = G(z)$, etc. The variable z is sometimes called an *independent variable,* while w is called a *dependent variable.* The *value of a function* at $z = a$ is often written $f(a)$. Thus if $f(z) = z^2$, then $f(2i) = (2i)^2 = -4$.

SINGLE-AND MULTIPLE-VALUED FUNCTIONS

If only one value of w corresponds to each value of z, we say that w is a *single-valued* function of z or that $f(z)$ is single-valued. If more than one value of w corresponds to each value of z, we say that w is a *multiple-valued* or *many-valued* function of z.

A multiple-valued function can be considered as a collection of single-valued functions, each member of which is called a *branch* of the function. It is customary to consider one particular member as a *principal branch* of the multiple-valued function and the value of the function corresponding to this branch as the *principal value.*

Example 1: If $w = z^2$, then to each value of z there is only one value of w. Hence $w = f(z) = z^2$ is a single-valued function of z.

Example 2: If $w = z^{1/2}$, then to each value of z there are two values of w. Hence $w = f(z) = z^{1/2}$ is a multiple-valued (in this case two-valued) function of z.

Whenever we speak of *function* we shall, unless otherwise stated, assume *single-valued function.*

INVERSE FUNCTIONS

If $w = f(z)$, then we can also consider z as a function of w, written $z = g(w) = f^{-1}(w)$. The function f^{-1} is often called the *inverse* function corresponding to f. Thus $w = f(z)$ and $w = f^{-1}(z)$ are *inverse functions* of each other.

TRANSFORMATIONS

If $w = u + iv$ (where u and v are real) is a single-valued function of $z = x + iy$ (where x and y are real), we can write $u + iv = f(x + iy)$. By equating real and imaginary parts this is seen to be equivalent to

$$u = u(x,y), \qquad v = v(x,y) \tag{1}$$

Thus given a point (x,y) in the z plane, such as P in Fig. 2-1 below, there corresponds a point (u,v) in the w plane, say P' in Fig. 2-2 below. The set of equations (*1*) [or the equivalent, $w = f(z)$] is called a *transformation.* We say that point P is *mapped* or *transformed* into point P' by means of the transformation and call P' the *image* of P.

Example: If $w = z^2$, then $u + iv = (x+iy)^2 = x^2 - y^2 + 2ixy$ and the transformation is $u = x^2 - y^2$, $v = 2xy$. The image of a point $(1,2)$ in the z plane is the point $(-3,4)$ in the w plane.

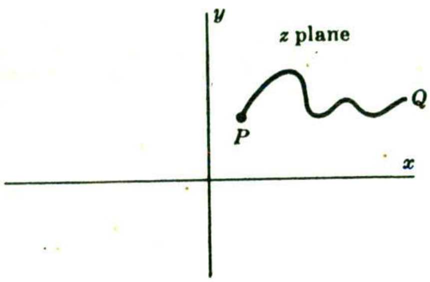

Fig. 2-1

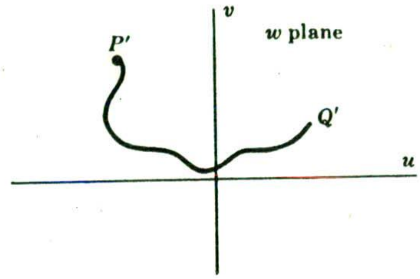

Fig. 2-2

In general, under a transformation, a set of points such as those on curve PQ of Fig. 2-1 is mapped into a corresponding set of points, called the *image*, such as those on curve $P'Q'$ in Fig. 2-2. The particular characteristics of the image depend of course on the type of function $f(z)$, which is sometimes called a *mapping function.* If $f(z)$ is multiple-valued, a point (or curve) in the z plane is mapped in general into more than one point (or curve) in the w plane.

CURVILINEAR COORDINATES

Given the transformation $w = f(z)$ or, equivalently, $u = u(x,y)$, $v = v(x,y)$, we call (x,y) the rectangular coordinates corresponding to a point P in the z plane and (u,v) the *curvilinear coordinates* of P.

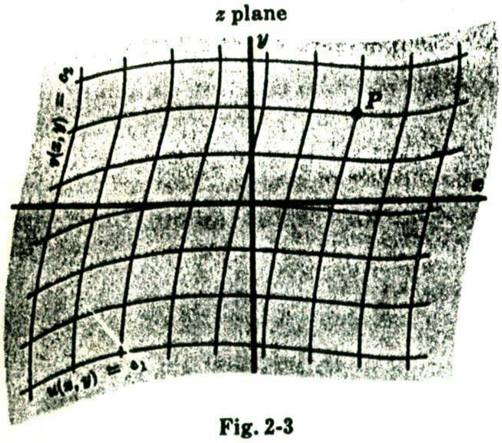

Fig. 2-3

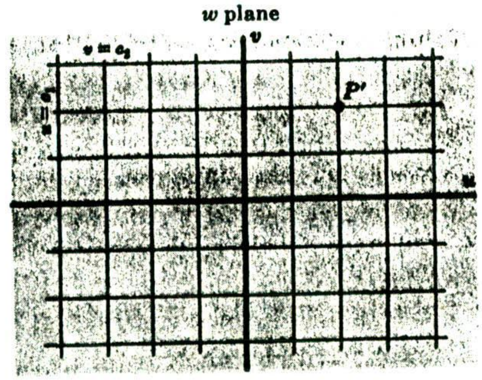

Fig. 2-4

The curves $u(x,y) = c_1$, $v(x,y) = c_2$, where c_1 and c_2 are constants, are called *coordinate curves* [see Fig. 2-3] and each pair of these curves intersects in a point. These curves map into mutually orthogonal lines in the w plane [see Fig. 2-4].

THE ELEMENTARY FUNCTIONS

1. Polynomial Functions are defined by

$$w = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = P(z) \tag{2}$$

where $a_0 \neq 0, a_1, \ldots, a_n$ are complex constants and n is a positive integer called the *degree* of the polynomial $P(z)$.

The transformation $w = az + b$ is called a *linear transformation.*

2. **Rational Algebraic Functions** are defined by

$$w = \frac{P(z)}{Q(z)} \tag{3}$$

where $P(z)$ and $Q(z)$ are polynomials. We sometimes call (3) a *rational transformation.* The special case $w = \dfrac{az+b}{cz+d}$ where $ad - bc \neq 0$ is often called a *bilinear* or *fractional linear transformation.*

3. **Exponential Functions** are defined by

$$w = e^z = e^{x+iy} = e^x(\cos y + i \sin y) \tag{4}$$

where $e = 2.71828\ldots$ is the *natural base of logarithms.* If a is real and positive, we define

$$a^z = e^{z \ln a} \tag{5}$$

where $\ln a$ is the *natural logarithm of a.* This reduces to (4) if $a = e$.

Complex exponential functions have properties similar to those of real exponential functions. For example, $e^{z_1} \cdot e^{z_2} = e^{z_1+z_2}$, $e^{z_1}/e^{z_2} = e^{z_1-z_2}$.

4. **Trigonometric Functions.** We define the trigonometric or circular functions $\sin z$, $\cos z$, etc., in terms of exponential functions as follows.

$$\sin z = \frac{e^{iz}-e^{-iz}}{2i} \qquad \cos z = \frac{e^{iz}+e^{-iz}}{2}$$

$$\sec z = \frac{1}{\cos z} = \frac{2}{e^{iz}+e^{-iz}} \qquad \csc z = \frac{1}{\sin z} = \frac{2i}{e^{iz}-e^{-iz}}$$

$$\tan z = \frac{\sin z}{\cos z} = \frac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})} \qquad \cot z = \frac{\cos z}{\sin z} = \frac{i(e^{iz}+e^{-iz})}{e^{iz}-e^{-iz}}$$

Many of the properties familiar in the case of real trigonometric functions also hold for the complex trigonometric functions. For example, we have

$$\sin^2 z + \cos^2 z = 1 \qquad 1 + \tan^2 z = \sec^2 z \qquad 1 + \cot^2 z = \csc^2 z$$

$$\sin(-z) = -\sin z \qquad \cos(-z) = \cos z \qquad \tan(-z) = -\tan z$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$

$$\tan(z_1 \pm z_2) = \frac{\tan z_1 \pm \tan z_2}{1 \mp \tan z_1 \tan z_2}$$

5. **Hyperbolic Functions** are defined as follows:

$$\sinh z = \frac{e^z - e^{-z}}{2} \qquad \cosh z = \frac{e^z + e^{-z}}{2}$$

$$\operatorname{sech} z = \frac{1}{\cosh z} = \frac{2}{e^z + e^{-z}} \qquad \operatorname{csch} z = \frac{1}{\sinh z} = \frac{2}{e^z - e^{-z}}$$

$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{e^z - e^{-z}}{e^z + e^{-z}} \qquad \coth z = \frac{\cosh z}{\sinh z} = \frac{e^z + e^{-z}}{e^z - e^{-z}}$$

The following properties hold:

$$\cosh^2 z - \sinh^2 z = 1 \qquad 1 - \tanh^2 z = \operatorname{sech}^2 z \qquad \coth^2 z - 1 = \operatorname{csch}^2 z$$

$$\sinh(-z) = -\sinh z \qquad \cosh(-z) = \cosh z \qquad \tanh(-z) = -\tanh z$$

$$\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$$

$$\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$$

$$\tanh(z_1 \pm z_2) = \frac{\tanh z_1 \pm \tanh z_2}{1 \pm \tanh z_1 \tanh z_2}$$

The following relations exist between the trigonometric or circular functions and the hyperbolic functions:

$$\sin iz = i\sinh z \qquad \cos iz = \cosh z \qquad \tan iz = i\tanh z$$
$$\sinh iz = i\sin z \qquad \cosh iz = \cos z \qquad \tanh iz = i\tan z$$

6. **Logarithmic Functions.** If $z = e^w$, then we write $w = \ln z$, called the *natural logarithm* of z. Thus the natural logarithmic function is the inverse of the exponential function and can be defined by

$$w \;=\; \ln z \;=\; \ln r + i(\theta + 2k\pi) \qquad k = 0, \pm 1, \pm 2, \ldots$$

where $z = re^{i\theta} = re^{i(\theta+2k\pi)}$. Note that $\ln z$ is a multiple-valued (in this case infinitely-many-valued) function. The *principal-value* or *principal branch* of $\ln z$ is sometimes defined as $\ln r + i\theta$ where $0 \leqq \theta < 2\pi$. However, any other interval of length 2π can be used, e.g. $-\pi < \theta \leqq \pi$, etc.

The logarithmic function can be defined for real bases other than e. Thus if $z = a^w$, then $w = \log_a z$ where $a > 0$ and $a \neq 0, 1$. In this case $z = e^{w \ln a}$ and so $w = (\ln z)/(\ln a)$.

7. **Inverse Trigonometric Functions.** If $z = \sin w$, then $w = \sin^{-1} z$ is called the *inverse sine of z* or *arc sine of z*. Similarly we define other inverse trigonometric or circular functions $\cos^{-1} z$, $\tan^{-1} z$, etc. These functions, which are multiple-valued, can be expressed in terms of natural logarithms as follows. In all cases we omit an additive constant $2k\pi i$, $k = 0, \pm 1, \pm 2, \ldots$, in the logarithm.

$$\sin^{-1} z = \frac{1}{i}\ln\left(iz + \sqrt{1-z^2}\right) \qquad \csc^{-1} z = \frac{1}{i}\ln\left(\frac{i+\sqrt{z^2-1}}{z}\right)$$

$$\cos^{-1} z = \frac{1}{i}\ln\left(z + \sqrt{z^2-1}\right) \qquad \sec^{-1} z = \frac{1}{i}\ln\left(\frac{1+\sqrt{1-z^2}}{z}\right)$$

$$\tan^{-1} z = \frac{1}{2i}\ln\left(\frac{1+iz}{1-iz}\right) \qquad \cot^{-1} z = \frac{1}{2i}\ln\left(\frac{z+i}{z-i}\right)$$

8. **Inverse Hyperbolic Functions.** If $z = \sinh w$ then $w = \sinh^{-1} z$ is called the *inverse hyperbolic sine of z*. Similarly we define other inverse hyperbolic functions $\cosh^{-1} z$, $\tanh^{-1} z$, etc. These functions, which are multiple-valued, can be expressed in terms of natural logarithms as follows. In all cases we omit an additive constant $2k\pi i$, $k = 0, \pm 1, \pm 2, \ldots$, in the logarithm.

$$\sinh^{-1} z = \ln\left(z + \sqrt{z^2+1}\right) \qquad \operatorname{csch}^{-1} z = \ln\left(\frac{1+\sqrt{z^2+1}}{z}\right)$$

$$\cosh^{-1} z = \ln\left(z + \sqrt{z^2-1}\right) \qquad \operatorname{sech}^{-1} z = \ln\left(\frac{1+\sqrt{1-z^2}}{z}\right)$$

$$\tanh^{-1} z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) \qquad \coth^{-1} z = \frac{1}{2}\ln\left(\frac{z+1}{z-1}\right)$$

9. **The Function** z^α, where α may be complex, is defined as $e^{\alpha \ln z}$. Similarly if $f(z)$ and $g(z)$ are two given functions of z, we can define $f(z)^{g(z)} = e^{g(z)\ln f(z)}$. In general such functions are multiple-valued.

10. **Algebraic and Transcendental Functions.** If w is a solution of the polynomial equation

$$P_0(z)w^n + P_1(z)w^{n-1} + \cdots + P_{n-1}(z)w + P_n(z) \;=\; 0 \tag{6}$$

where $P_0 \neq 0, P_1(z), \ldots, P_n(z)$ are polynomials in z and n is a positive integer, then $w = f(z)$ is called an *algebraic function* of z.

Example: $w = z^{1/2}$ is a solution of the equation $w^2 - z = 0$ and so is an algebraic function of z.

Any function which cannot be expressed as a solution of (6) is called a *transcendental function.* The logarithmic, trigonometric and hyperbolic functions and their corresponding inverses are examples of transcendental functions.

The functions considered in 1-9 above, together with functions derived from them by a finite number of operations involving addition, subtraction, multiplication, division and roots are called *elementary functions.*

BRANCH POINTS AND BRANCH LINES

Suppose that we are given the function $w = z^{1/2}$. Suppose further that we allow z to make a complete circuit (counterclockwise) around the origin starting from point A [Fig. 2-5]. We have $z = re^{i\theta}$, $w = \sqrt{r}\,e^{i\theta/2}$ so that at A, $\theta = \theta_1$ and $w = \sqrt{r}\,e^{i\theta_1/2}$. After a complete circuit back to A, $\theta = \theta_1 + 2\pi$ and $w = \sqrt{r}\,e^{i(\theta_1+2\pi)/2} = -\sqrt{r}\,e^{i\theta_1/2}$. Thus we have not achieved the same value of w with which we started. However, by making a second complete circuit back to A, i.e. $\theta = \theta_1 + 4\pi$, $w = \sqrt{r}\,e^{i(\theta_1+4\pi)/2} = \sqrt{r}\,e^{i\theta_1/2}$ and we then do obtain the same value of w with which we started.

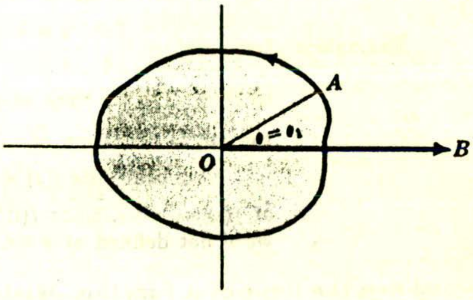

Fig. 2-5

We can describe the above by stating that if $0 \leqq \theta < 2\pi$ we are on one branch of the multiple-valued function $z^{1/2}$, while if $2\pi \leqq \theta < 4\pi$ we are on the other branch of the function.

It is clear that each branch of the function is single-valued. In order to keep the function single-valued, we set up an artificial barrier such as OB where B is at infinity [although any other line from O can be used] which we agree not to cross. This barrier [drawn heavy in the figure] is called a *branch line* or *branch cut,* and point O is called a *branch point.* It should be noted that a circuit around any point other than $z = 0$ does not lead to different values; thus $z = 0$ is the only finite branch point.

RIEMANN SURFACES

There is another way to achieve the purpose of the branch line described above. To see this we imagine that the z plane consists of two sheets superimposed on each other. We now cut the sheets along OB and imagine that the lower edge of the bottom sheet is joined to the upper edge of the top sheet. Then starting in the bottom sheet and making one complete circuit about O we arrive in the top sheet. We must now imagine the other cut edges joined together so that by continuing the circuit we go from the top sheet back to the bottom sheet.

The collection of two sheets is called a *Riemann surface* corresponding to the function $z^{1/2}$. Each sheet corresponds to a branch of the function and on each sheet the function is single-valued.

The concept of Riemann surfaces has the advantage in that the various values of multiple-valued functions are obtained in a continuous fashion.

The ideas are easily extended. For example, for the function $z^{1/3}$ the Riemann surface has 3 sheets; for $\ln z$ the Riemann surface has infinitely many sheets.

LIMITS

Let $f(z)$ be defined and single-valued in a neighbourhood of $z=z_0$ with the possible exception of $z=z_0$ itself (i.e. in a deleted neighbourhood δ of z_0). We say that the number l

is the *limit* of $f(z)$ as z approaches z_0 and write $\lim_{z \to z_0} f(z) = l$ if for any positive number ϵ (however small) we can find some positive number δ (usually depending on ϵ) such that $|f(z) - l| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

In such case we also say that $f(z)$ approaches l as z approaches z_0 and write $f(z) \to l$ as $z \to z_0$. The limit must be independent of the manner in which z approaches z_0.

Geometrically, if z_0 is a point in the complex plane, then $\lim_{z \to z_0} f(z) = l$ if the difference in absolute value between $f(z)$ and l can be made as small as we wish by choosing points z sufficiently close to z_0 (excluding $z = z_0$ itself).

Example: Let $f(z) = \begin{cases} z^2 & z \neq i \\ 0 & z = i \end{cases}$. Then as z gets closer to i (i.e. z approaches i), $f(z)$ gets closer to $i^2 = -1$. We thus *suspect* that $\lim_{z \to i} f(z) = -1$. To *prove* this we must see whether the above definition of limit is satisfied. For this proof see Problem 23.

Note that $\lim_{z \to i} f(z) \neq f(i)$, i.e. the limit of $f(z)$ as $z \to i$ is not the same as the value of $f(z)$ at $z = i$, since $f(i) = 0$ by definition. The limit would in fact be -1 even if $f(z)$ were not defined at $z = i$.

When the limit of a function exists it is unique, i.e. it is the only one (see Problem 26).

If $f(z)$ is multiple-valued, the limit as $z \to z_0$ may depend on the particular branch.

THEOREMS ON LIMITS

If $\lim_{z \to z_0} f(z) = A$ and $\lim_{z \to z_0} g(z) = B$, then

1. $\lim_{z \to z_0} \{f(z) + g(z)\} = \lim_{z \to z_0} f(z) + \lim_{z \to z_0} g(z) = A + B$

2. $\lim_{z \to z_0} \{f(z) - g(z)\} = \lim_{z \to z_0} f(z) - \lim_{z \to z_0} g(z) = A - B$

3. $\lim_{z \to z_0} \{f(z)\, g(z)\} = \left\{\lim_{z \to z_0} f(z)\right\}\left\{\lim_{z \to z_0} g(z)\right\} = AB$

4. $\lim_{z \to z_0} \dfrac{f(z)}{g(z)} = \dfrac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} g(z)} = \dfrac{A}{B}$ if $B \neq 0$

INFINITY

By means of the transformation $w = 1/z$ the point $z = 0$ (i.e. the origin) is mapped into $w = \infty$, called the *point at infinity in the w plane.* Similarly we denote by $z = \infty$ *the point at infinity in the z plane.* To consider the behaviour of $f(z)$ at $z=\infty$, it suffices to let $z=1/w$ and examine the behaviour of $f(1/w)$ at $w=0$.

We say that $\lim_{z \to \infty} f(z) = l$ or $f(z)$ approaches l as z approaches infinity, if for any $\epsilon > 0$ we can find $M > 0$ such that $|f(z) - l| < \epsilon$ whenever $|z| > M$.

We say that $\lim_{z \to z_0} f(z) = \infty$ or $f(z)$ approaches infinity as z approaches z_0, if for any $N > 0$ we can find $\delta > 0$ such that $|f(z)| > N$ whenever $0 < |z - z_0| < \delta$.

CONTINUITY

Let $f(z)$ be defined and single-valued in a neighbourhood of $z=z_0$ as well as at $z=z_0$ (i.e. in a δ neighbourhood of z_0). The function $f(z)$ is said to be *continuous* at $z=z_0$ if $\lim_{z \to z_0} f(z) = f(z_0)$. Note that this implies three conditions which must be met in order that $f(z)$ be continuous at $z = z_0$:

1. $\lim_{z \to z_0} f(z) = l$ must exist
2. $f(z_0)$ must exist, i.e. $f(z)$ is defined at z_0
3. $l = f(z_0)$

Equivalently, if $f(z)$ is continuous at z_0 we can write this in the suggestive form $\lim_{z \to z_0} f(z) = f\left(\lim_{z \to z_0} z\right)$.

Example 1: If $f(z) = \begin{cases} z^2 & z \neq i \\ 0 & z = i \end{cases}$ then from the Example on Page 38, $\lim_{z \to i} f(z) = -1$. But $f(i) = 0$. Hence $\lim_{z \to i} f(z) \neq f(i)$ and the function is not continuous at $z = i$.

Example 2: If $f(z) = z^2$ for all z, then $\lim_{z \to i} f(z) = f(i) = -1$ and $f(z)$ is continuous at $z = i$.

Points in the z plane where $f(z)$ fails to be continuous are called *discontinuities* of $f(z)$, and $f(z)$ is said to be *discontinuous* at these points. If $\lim_{z \to z_0} f(z)$ exists but is not equal to $f(z_0)$, we call z_0 a *removable discontinuity* since by redefining $f(z_0)$ to be the same as $\lim_{z \to z_0} f(z)$ the function becomes continuous.

Alternative to the above definition of continuity, we can define $f(z)$ as continuous at $z = z_0$ if for any $\epsilon > 0$ we can find $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$. Note that this is simply the definition of limit with $l = f(z_0)$ and removal of the restriction that $z \neq z_0$.

To examine the continuity of $f(z)$ at $z = \infty$, we place $z = 1/w$ and examine the continuity of $f(1/w)$ at $w = 0$.

CONTINUITY IN A REGION

A function $f(z)$ is said to be *continuous in a region* if it is continuous at all points of the region.

THEOREMS ON CONTINUITY

Theorem 1. If $f(z)$ and $g(z)$ are continuous at $z = z_0$, so also are the functions $f(z) + g(z)$, $f(z) - g(z)$, $f(z)\,g(z)$ and $\dfrac{f(z)}{g(z)}$, the last only if $g(z_0) \neq 0$. Similar results hold for continuity in a region.

Theorem 2. Among the functions continuous in every finite region are (*a*) all polynomials, (*b*) e^z, (*c*) $\sin z$ and $\cos z$.

Theorem 3. If $w = f(z)$ is continuous at $z = z_0$ and $z = g(\zeta)$ is continuous at $\zeta = \zeta_0$ and if $\zeta_0 = f(z_0)$, then the function $w = g[f(z)]$, called a *function of a function* or *composite function*, is continuous at $z = z_0$. This is sometimes briefly stated as: A continuous function of a continuous function is continuous.

Theorem 4. If $f(z)$ is continuous in a closed region, it is bounded in the region; i.e. there exists a constant M such that $|f(z)| < M$ for all points z of the region.

Theorem 5. If $f(z)$ is continuous in a region, then the real and imaginary parts of $f(z)$ are also continuous in the region.

UNIFORM CONTINUITY

Let $f(z)$ be continuous in a region. Then by definition at each point z_0 of the region and for any $\epsilon > 0$, we can find $\delta > 0$ (which will in general depend on both ϵ and the particular point z_0) such that $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$. If we can find δ depending on ϵ but not on the particular point z_0, we say that $f(z)$ is *uniformly continuous* in the region.

Alternatively, $f(z)$ is uniformly continuous in a region if for any $\epsilon > 0$ we can find $\delta > 0$ such that $|f(z_1) - f(z_2)| < \epsilon$ whenever $|z_1 - z_2| < \delta$ where z_1 and z_2 are any two points of the region.

Theorem. If $f(z)$ is continuous in a *closed* region, it is uniformly continuous there.

SEQUENCES

A function of a positive integral variable, designated by $f(n)$ or u_n, where $n = 1, 2, 3, \ldots$, is called a *sequence*. Thus a sequence is a set of numbers $u_1, u_2, u_3, \ldots$ in a definite order of arrangement and formed according to a definite rule. Each number in the sequence is called a *term* and u_n is called the nth *term*. The sequence $u_1, u_2, u_3, \ldots$ is also designated briefly by $\{u_n\}$. The sequence is called *finite* or *infinite* according as there are a finite number of terms or not. Unless otherwise specified, we shall consider infinite sequences only.

Example 1: The set of numbers $i, i^2, i^3, \ldots, i^{100}$ is a finite sequence; the nth term is given by $u_n = i^n$, $n = 1, 2, \ldots, 100$.

Example 2: The set of numbers $1 + i, \frac{(1+i)^2}{2!}, \frac{(1+i)^3}{3!}, \ldots$ is an infinite sequence; the nth term is given by $u_n = (1+i)^n/n!$, $n = 1, 2, 3, \ldots$.

LIMIT OF A SEQUENCE

A number l is called the *limit* of an infinite sequence $u_1, u_2, u_3, \ldots$ if for any positive number ϵ we can find a positive number N depending on ϵ such that $|u_n - l| < \epsilon$ for all $n > N$. In such case we write $\lim_{n \to \infty} u_n = l$. If the limit of a sequence exists, the sequence is called *convergent*; otherwise it is called *divergent*. A sequence can converge to only one limit, i.e. if a limit exists it is unique.

A more intuitive but unrigorous way of expressing this concept of limit is to say that a sequence $u_1, u_2, u_3, \ldots$ has a limit l if the successive terms get "closer and closer" to l. This is often used to provide a "guess" as to the value of the limit, after which the definition is applied to see if the guess is really correct.

THEOREMS ON LIMITS OF SEQUENCES

If $\lim_{n \to \infty} a_n = A$ and $\lim_{n \to \infty} b_n = B$, then

1. $\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n = A + B$

2. $\lim_{n \to \infty} (a_n - b_n) = \lim_{n \to \infty} a_n - \lim_{n \to \infty} b_n = A - B$

3. $\lim_{n \to \infty} (a_n b_n) = \left(\lim_{n \to \infty} a_n\right)\left(\lim_{n \to \infty} b_n\right) = AB$

4. $\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n} = \frac{A}{B}$ if $B \neq 0$

Further discussion of sequences is given in Chapter 6.

INFINITE SERIES

Let $u_1, u_2, u_3, \ldots$ be a given sequence.

Form a new sequence $S_1, S_2, S_3, \ldots$ defined by

$$S_1 = u_1, \quad S_2 = u_1 + u_2, \quad S_3 = u_1 + u_2 + u_3, \quad \cdots, \quad S_n = u_1 + u_2 + \cdots + u_n$$

where S_n, called the nth *partial sum*, is the sum of the first n terms of the sequence $\{u_n\}$.

The sequence $S_1, S_2, S_3, \ldots$ is symbolized by

$$u_1 + u_2 + u_3 + \cdots = \sum_{n=1}^{\infty} u_n$$

which is called an infinite series. If $\lim_{n\to\infty} S_n = S$ exists, the series is called *convergent* and S is its *sum*; otherwise the series is called *divergent*. A necessary condition that a series converges is $\lim_{n\to\infty} u_n = 0$; however, this is not sufficient (see Problems 40 and 150).

Further discussion of infinite series is given in Chapter 6.

Solved Problems

FUNCTIONS AND TRANSFORMATIONS

1. Let $w = f(z) = z^2$. Find the values of w which correspond to (*a*) $z = -2+i$ and (*b*) $z = 1-3i$, and show how the correspondence can be represented graphically.

(*a*) $w = f(-2+i) = (-2+i)^2 = 4 - 4i + i^2 = 3 - 4i$

(*b*) $w = f(1-3i) = (1-3i)^2 = 1 - 6i + 9i^2 = -8 - 6i$

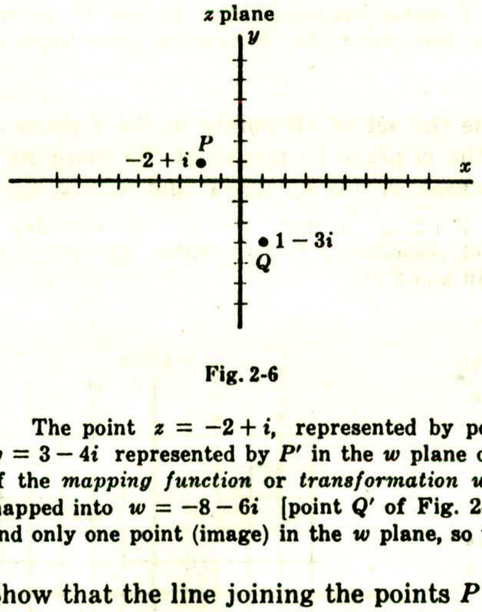

Fig. 2-6

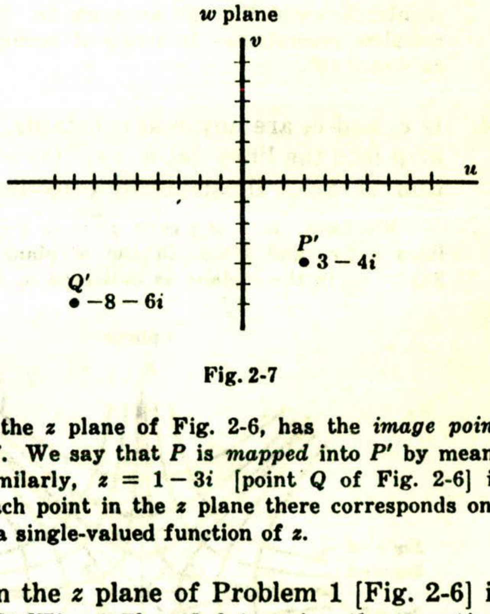

Fig. 2-7

The point $z = -2+i$, represented by point P in the z plane of Fig. 2-6, has the *image point* $w = 3-4i$ represented by P' in the w plane of Fig. 2-7. We say that P is *mapped* into P' by means of the *mapping function* or *transformation* $w = z^2$. Similarly, $z = 1-3i$ [point Q of Fig. 2-6] is mapped into $w = -8-6i$ [point Q' of Fig. 2-7]. To each point in the z plane there corresponds one and only one point (image) in the w plane, so that w is a single-valued function of z.

2. Show that the line joining the points P and Q in the z plane of Problem 1 [Fig. 2-6] is mapped by $w = z^2$ into a curve joining points $P'Q'$ [Fig. 2-7] and determine the equation of this curve.

Points P and Q have coordinates $(-2,1)$ and $(1,-3)$. Then the parametric equations of the line joining these points are given by

$$\frac{x-(-2)}{1-(-2)} = \frac{y-1}{-3-1} = t \qquad \text{or} \qquad x = 3t-2,\ y = 1-4t$$

The equation of the line PQ can be represented by $z = 3t-2+i(1-4t)$. The curve in the w plane into which this line is mapped has the equation

$$\begin{aligned} w = z^2 &= \{3t-2+i(1-4t)\}^2 = (3t-2)^2 - (1-4t)^2 + 2(3t-2)(1-4t)i \\ &= 3 - 4t - 7t^2 + (-4+22t-24t^2)i \end{aligned}$$

Then since $w = u + iv$, the parametric equations of the image curve are given by

$$u = 3 - 4t - 7t^2, \quad v = -4 + 22t - 24t^2$$

By assigning various values to the parameter t, this curve may be graphed.

3. A point P moves in a counterclockwise direction around a circle in the z plane having centre at the origin and radius 1. If the mapping function is $w = z^3$, show that when P makes one complete revolution the image P' of P in the w plane makes three complete revolutions in a counterclockwise direction on a circle having centre at the origin and radius 1.

Let $z = re^{i\theta}$. Then on the circle $|z| = 1$ [Fig. 2-8], $r = 1$ and $z = e^{i\theta}$. Hence $w = z^3 = (e^{i\theta})^3 = e^{3i\theta}$. Letting (ρ, ϕ) denote polar coordinates in the w plane, we have $w = \rho e^{i\phi} = e^{3i\theta}$ so that $\rho = 1$, $\phi = 3\theta$.

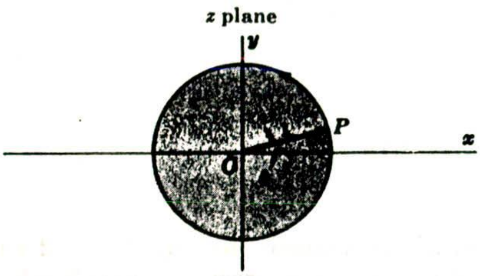

Fig. 2-8

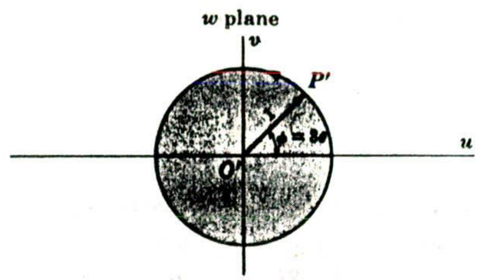

Fig. 2-9

Since $\rho = 1$, it follows that the image point P' moves on a circle in the w plane of radius 1 and centre at the origin [Fig. 2-9]. Also, when P moves counterclockwise through an angle θ, P' moves counterclockwise through an angle 3θ. Thus when P makes one complete revolution, P' makes three complete revolutions. In terms of vectors it means that vector $O'P'$ is rotating three times as fast as vector OP.

4. If c_1 and c_2 are any real constants, determine the set of all points in the z plane which map into the lines (a) $u = c_1$, (b) $v = c_2$ in the w plane by means of the mapping function $w = z^2$. Illustrate by considering the cases $c_1 = 2, 4, -2, -4$ and $c_2 = 2, 4, -2, -4$.

We have $w = u + iv = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$ so that $u = x^2 - y^2$, $v = 2xy$. Then lines $u = c_1$ and $v = c_2$ in the w plane correspond respectively to hyperbolae $x^2 - y^2 = c_1$ and $2xy = c_2$ in the z plane as indicated in Figures 2-10 and 2-11.

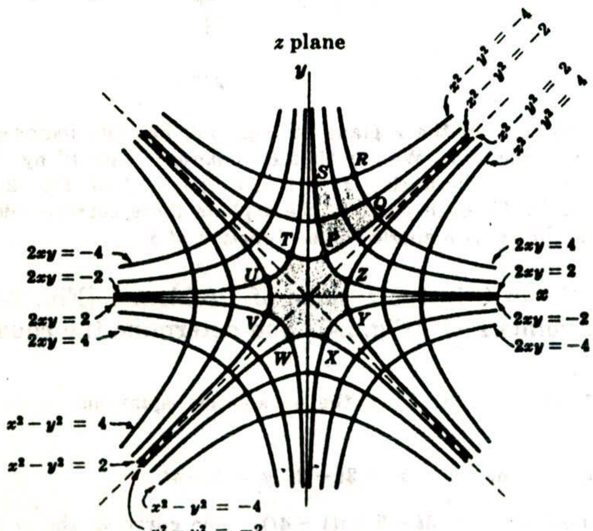

Fig. 2-10

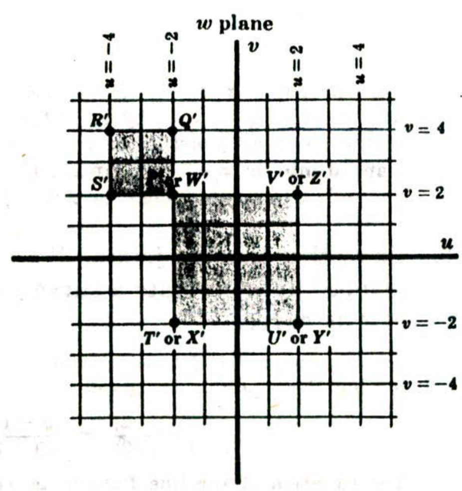

Fig. 2-11

5. Referring to Problem 4, determine: (a) the image of the region in the first quadrant bounded by $x^2 - y^2 = -2$, $xy = 1$, $x^2 - y^2 = -4$ and $xy = 2$; (b) the image of the region in the z plane bounded by all the branches of $x^2 - y^2 = 2$, $xy = 1$, $x^2 - y^2 = -2$ and $xy = -1$; (c) the curvilinear coordinates of that point in the xy plane whose rectangular coordinates are $(2, -1)$.

(a) The region in the z plane is indicated by the shaded portion $PQRS$ of Fig. 2-10. This region maps into the required image region $P'Q'R'S'$ shown shaded in Fig. 2-11. It should be noted that curve $PQRSP$ is traversed in a counterclockwise direction and the image curve $P'Q'R'S'P'$ is also traversed in a counterclockwise direction.

(b) The region in the z plane is indicated by the shaded portion $PTUVWXYZ$ of Fig. 2-10. This region maps into the required image region $P'T'U'V'$ shown shaded in Fig. 2-11.

It is of interest to note that when the boundary of the region $PTUVWXYZ$ is traversed only once, the boundary of the image region $P'T'U'V'$ is traversed twice. This is due to the fact that the eight points P and W, T and X, U and Y, V and Z of the z plane map into the four points P' or W', T' or X', U' or Y', V' or Z' respectively.

However, when the boundary of region $PQRS$ is traversed only once, the boundary of the image region is also traversed only once. The difference is due to the fact that in traversing the curve $PTUVWXYZP$ we are encircling the origin $z=0$, whereas when we are traversing the curve $PQRSP$ we are not encircling the origin.

(c) $u = x^2 - y^2 = (2)^2 - (-1)^2 = 3$, $v = 2xy = 2(2)(-1) = -4$. Then the curvilinear coordinates are $u = 3$, $v = -4$.

MULTIPLE-VALUED FUNCTIONS

6. Let $w^5 = z$ and suppose that corresponding to the particular value $z = z_1$ we have $w = w_1$. (a) If we start at the point z_1 in the z plane [see Fig. 2-12] and make one complete circuit counterclockwise around the origin, show that the value of w on returning to z_1 is $w_1 e^{2\pi i/5}$. (b) What are the values of w on returning to z_1, after 2, 3, ... complete circuits around the origin? (c) Discuss parts (a) and (b) if the paths do not enclose the origin.

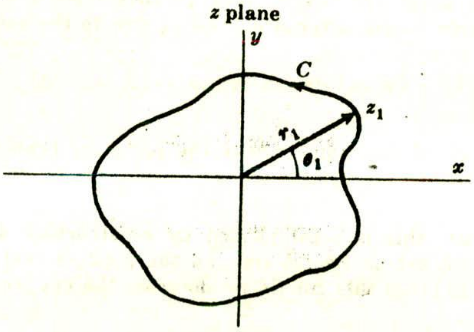

Fig. 2-12

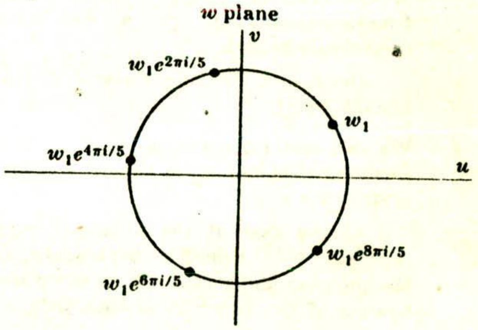

Fig. 2-13

(a) We have $z = re^{i\theta}$, so that $w = z^{1/5} = r^{1/5}e^{i\theta/5}$. If $r = r_1$ and $\theta = \theta_1$, then $w_1 = r_1^{1/5}e^{i\theta_1/5}$.

As θ increases from θ_1 to $\theta_1 + 2\pi$, which is what happens when one complete circuit counterclockwise around the origin is made, we find

$$w = r_1^{1/5}e^{i(\theta_1+2\pi)/5} = r_1^{1/5}e^{i\theta_1/5}e^{2\pi i/5} = w_1 e^{2\pi i/5}$$

(b) After 2 complete circuits around the origin, we find

$$w = r_1^{1/5}e^{i(\theta_1+4\pi)/5} = r_1^{1/5}e^{i\theta_1/5}e^{4\pi i/5} = w_1 e^{4\pi i/5}$$

Similarly after 3 and 4 complete circuits around the origin, we find

$$w = w_1 e^{6\pi i/5} \quad \text{and} \quad w = w_1 e^{8\pi i/5}$$

After 5 complete circuits the value of w is $w_1 e^{10\pi i/5} = w_1$, so that the original value of w is obtained after 5 revolutions about the origin. Thereafter the cycle is repeated [see Fig. 2-13].

Another method. Since $w^5 = z$, we have $\arg z = 5 \arg w$ from which

$$\text{Change in } \arg w = \tfrac{1}{5}(\text{Change in } \arg z)$$

Then if $\arg z$ increases by $2\pi, 4\pi, 6\pi, 8\pi, 10\pi, \ldots$, $\arg w$ increases by $2\pi/5, 4\pi/5, 6\pi/5, 8\pi/5, 2\pi, \ldots$ leading to the same results obtained in (a) and (b).

(c) If the path does not enclose the origin then the increase in $\arg z$ is zero and so the increase in $\arg w$ is also zero. In this case the value of w is w_1, regardless of the number of circuits made.

7. (a) In the preceding problem explain why we can consider w as a collection of five single-valued functions of z.

(b) Explain geometrically the relationship between these single-valued functions.

(c) Show geometrically how we can restrict ourselves to a particular single-valued function.

(a) Since $w^5 = z = re^{i\theta} = re^{i(\theta+2k\pi)}$ where k is an integer, we have

$$w = r^{1/5}e^{i(\theta+2k\pi)/5} = r^{1/5}\{\cos(\theta+2k\pi)/5 + i\sin(\theta+2k\pi)/5\}$$

and so w is a five-valued function of z, the five values being given by $k = 0,1,2,3,4$.

Equivalently, we can consider w as a collection of five single-valued functions, called *branches* of the multiple-valued function, by properly restricting θ. Thus, for example, we can write

$$w = r^{1/5}(\cos\theta/5 + i\sin\theta/5)$$

where we take the five possible intervals for θ given by $0 \leqq \theta < 2\pi$, $2\pi \leqq \theta < 4\pi$, $\ldots$, $8\pi \leqq \theta < 10\pi$, all other such intervals producing repetitions of these.

The first interval, $0 \leqq \theta < 2\pi$, is sometimes called the *principal range* of θ and corresponds to the *principal branch* of the multiple-valued function.

Other intervals for θ of length 2π can also be taken; for example, $-\pi \leqq \theta < \pi$, $\pi \leqq \theta < 3\pi$, etc., the first of these being taken as the principal range.

(b) We start with the (principal) branch

$$w = r^{1/5}(\cos\theta/5 + i\sin\theta/5) \qquad \text{where } 0 \leqq \theta < 2\pi$$

After one complete circuit about the origin in the z plane, θ increases by 2π to give another branch of the function. After another complete circuit about the origin, still another branch of the function is obtained until all five branches have been found, after which we return to the original (principal) branch.

Because different values of $f(z)$ are obtained by successively encircling $z = 0$, we call $z = 0$ a *branch point*.

(c) We can restrict ourselves to a particular single-valued function, usually the principal branch, by insuring that not more than one complete circuit about the branch point is made, i.e. by suitably restricting θ.

In the case of the principal range $0 \leqq \theta < 2\pi$, this is accomplished by constructing a cut, indicated by OA in Fig. 2-14 below, called a *branch cut* or *branch line*, on the positive real axis, the purpose being that we do not allow ourselves to cross this cut (if we do cross the cut, another branch of the function is obtained).

If another interval for θ is chosen, the branch line or cut is taken to be some other line in the z plane emanating from the branch point.

For some purposes, as we shall see later, it is useful to consider the curve of Fig. 2-15 of which Fig. 2-14 is a limiting case.

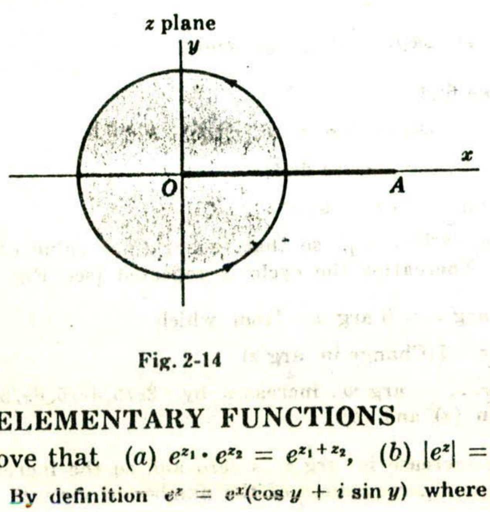

Fig. 2-14

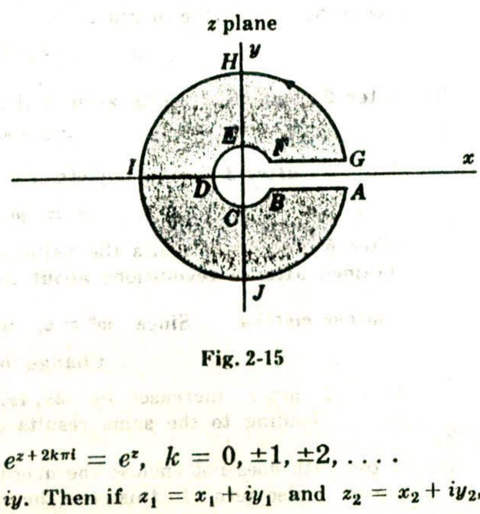

Fig. 2-15

THE ELEMENTARY FUNCTIONS

8. Prove that (a) $e^{z_1} \cdot e^{z_2} = e^{z_1+z_2}$, (b) $|e^z| = e^x$, (c) $e^{z+2k\pi i} = e^z$, $k = 0, \pm1, \pm2, \ldots$.

(a) By definition $e^z = e^x(\cos y + i\sin y)$ where $z = x + iy$. Then if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$,

$$\begin{aligned} e^{z_1} \cdot e^{z_2} &= e^{x_1}(\cos y_1 + i \sin y_1) \cdot e^{x_2}(\cos y_2 + i \sin y_2) \\ &= e^{x_1} \cdot e^{x_2}(\cos y_1 + i \sin y_1)(\cos y_2 + i \sin y_2) \\ &= e^{x_1+x_2}\{\cos(y_1+y_2) + i \sin(y_1+y_2)\} = e^{z_1+z_2} \end{aligned}$$

(*b*) $|e^z| = |e^x(\cos y + i \sin y)| = |e^x|\,|\cos y + i \sin y| = e^x \cdot 1 = e^x$

(*c*) By part (*a*), $e^{z+2k\pi i} = e^z e^{2k\pi i} = e^z(\cos 2k\pi + i \sin 2k\pi) = e^z$

This shows that the function e^z has *period* $2k\pi i$. In particular, it has period $2\pi i$.

9. Prove:

(*a*) $\sin^2 z + \cos^2 z = 1$

(*b*) $e^{iz} = \cos z + i \sin z,\ e^{-iz} = \cos z - i \sin z$

(*c*) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

(*d*) $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

By definition, $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$, $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$. Then

$$\text{(a)}\quad \sin^2 z + \cos^2 z = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 + \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 = -\left(\frac{e^{2iz} - 2 + e^{-2iz}}{4}\right) + \left(\frac{e^{2iz} + 2 + e^{-2iz}}{4}\right) = 1$$

(*b*) (*1*) $e^{iz} - e^{-iz} = 2i \sin z$, (*2*) $e^{iz} + e^{-iz} = 2 \cos z$

Adding (*1*) and (*2*): $2e^{iz} = 2\cos z + 2i \sin z$ and $e^{iz} = \cos z + i \sin z$

Subtracting (*1*) from (*2*): $2e^{-iz} = 2\cos z - 2i \sin z$ and $e^{-iz} = \cos z - i \sin z$

$$\begin{aligned} \text{(c)}\quad \sin(z_1 + z_2) &= \frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{2i} = \frac{e^{iz_1} \cdot e^{iz_2} - e^{-iz_1} \cdot e^{-iz_2}}{2i} \\ &= \frac{(\cos z_1 + i \sin z_1)(\cos z_2 + i \sin z_2) - (\cos z_1 - i \sin z_1)(\cos z_2 - i \sin z_2)}{2i} \\ &= \sin z_1 \cos z_2 + \cos z_1 \sin z_2 \end{aligned}$$

$$\begin{aligned} \text{(d)}\quad \cos(z_1 + z_2) &= \frac{e^{i(z_1+z_2)} + e^{-i(z_1+z_2)}}{2} = \frac{e^{iz_1} \cdot e^{iz_2} + e^{-iz_1} \cdot e^{-iz_2}}{2} \\ &= \frac{(\cos z_1 + i \sin z_1)(\cos z_2 + i \sin z_2) + (\cos z_1 - i \sin z_1)(\cos z_2 - i \sin z_2)}{2} \\ &= \cos z_1 \cos z_2 - \sin z_1 \sin z_2 \end{aligned}$$

10. Prove that the zeros of (*a*) $\sin z$ and (*b*) $\cos z$ are all real and find them.

(*a*) If $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i} = 0$, then $e^{iz} = e^{-iz}$ or $e^{2iz} = 1 = e^{2k\pi i}$, $k = 0, \pm 1, \pm 2, \ldots$.

Hence $2iz = 2k\pi i$ and $z = k\pi$, i.e. $z = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ are the zeros.

(*b*) If $\cos z = \dfrac{e^{iz} + e^{-iz}}{2} = 0$, then $e^{iz} = -e^{-iz}$ or $e^{2iz} = -1 = e^{(2k+1)\pi i}$, $k = 0, \pm 1, \pm 2, \ldots$.

Hence $2iz = (2k+1)\pi i$ and $z = (k + \frac{1}{2})\pi$, i.e. $z = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$ are the zeros.

11. Prove that (*a*) $\sin(-z) = -\sin z$, (*b*) $\cos(-z) = \cos z$, (*c*) $\tan(-z) = -\tan z$.

$$\text{(a)}\quad \sin(-z) = \frac{e^{i(-z)} - e^{-i(-z)}}{2i} = \frac{e^{-iz} - e^{iz}}{2i} = -\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = -\sin z$$

$$\text{(b)}\quad \cos(-z) = \frac{e^{i(-z)} + e^{-i(-z)}}{2} = \frac{e^{-iz} + e^{iz}}{2} = \frac{e^{iz} + e^{-iz}}{2} = \cos z$$

$$\text{(c)}\quad \tan(-z) = \frac{\sin(-z)}{\cos(-z)} = \frac{-\sin z}{\cos z} = -\tan z, \quad \text{using (a) and (b).}$$

Functions of z having the property that $f(-z) = -f(z)$ are called *odd functions*, while those for which $f(-z) = f(z)$ are called *even functions*. Thus $\sin z$ and $\tan z$ are odd functions, while $\cos z$ is an even function.

12. Prove: (*a*) $1 - \tanh^2 z = \operatorname{sech}^2 z$ (*c*) $\cos iz = \cosh z$
(*b*) $\sin iz = i \sinh z$ (*d*) $\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$

(*a*) By definition, $\cosh z = \dfrac{e^z + e^{-z}}{2}$, $\sinh z = \dfrac{e^z - e^{-z}}{2}$. Then

$$\cosh^2 z - \sinh^2 z = \left(\frac{e^z + e^{-z}}{2}\right)^2 - \left(\frac{e^z - e^{-z}}{2}\right)^2 = \frac{e^{2z} + 2 + e^{-2z}}{4} - \frac{e^{2z} - 2 + e^{-2z}}{4} = 1$$

Dividing by $\cosh^2 z$, $\dfrac{\cosh^2 z - \sinh^2 z}{\cosh^2 z} = \dfrac{1}{\cosh^2 z}$ or $1 - \tanh^2 z = \operatorname{sech}^2 z$.

(*b*) $$\sin iz = \frac{e^{i(iz)} - e^{-i(iz)}}{2i} = \frac{e^{-z} - e^{z}}{2i} = i\left(\frac{e^z - e^{-z}}{2}\right) = i \sinh z$$

(*c*) $$\cos iz = \frac{e^{i(iz)} + e^{-i(iz)}}{2} = \frac{e^{-z} + e^{z}}{2} = \frac{e^z + e^{-z}}{2} = \cosh z$$

(*d*) From Problem 9(*c*) and parts (*b*) and (*c*), we have

$$\sin(x+iy) = \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i \cos x \sinh y$$

13. (*a*) If $z = e^w$ where $z = r(\cos\theta + i\sin\theta)$ and $w = u + iv$, show that $u = \ln r$ and $v = \theta + 2k\pi$, $k = 0, \pm1, \pm2, \ldots$ so that $w = \ln z = \ln r + i(\theta + 2k\pi)$. (*b*) Determine the values of $\ln(1-i)$. What is the principal value?

(*a*) Since $z = r(\cos\theta + i\sin\theta) = e^w = e^{u+iv} = e^u(\cos v + i \sin v)$, we have on equating real and imaginary parts,

(*1*) $e^u \cos v = r\cos\theta$ (*2*) $e^u \sin v = r \sin\theta$

Squaring (*1*) and (*2*) and adding, we find $e^{2u} = r^2$ or $e^u = r$ and $u = \ln r$. Then from (*1*) and (*2*), $r\cos v = r\cos\theta$, $r \sin v = r\sin\theta$ from which $v = \theta + 2k\pi$. Hence $w = u + iv = \ln r + i(\theta + 2k\pi)$.

If $z = e^w$, we say that $w = \ln z$. We thus see that $\ln z = \ln r + i(\theta + 2k\pi)$. An equivalent way of saying the same thing is to write $\ln z = \ln r + i\theta$ where θ can assume infinitely many values which differ by 2π.

Note that *formally* $\ln z = \ln(re^{i\theta}) = \ln r + i\theta$ using laws of real logarithms familiar from elementary mathematics.

(*b*) Since $1 - i = \sqrt{2}\, e^{7\pi i/4 + 2k\pi i}$, we have $\ln(1-i) = \ln\sqrt{2} + \left(\dfrac{7\pi i}{4} + 2k\pi i\right) = \dfrac{1}{2}\ln 2 + \dfrac{7\pi i}{4} + 2k\pi i$.

The principal value is $\dfrac{1}{2}\ln 2 + \dfrac{7\pi i}{4}$ obtained by letting $k = 0$.

14. Prove that $f(z) = \ln z$ has a branch point at $z = 0$.

We have $\ln z = \ln r + i\theta$. Suppose that we start at some point $z_1 \neq 0$ in the complex plane for which $r = r_1$, $\theta = \theta_1$ so that $\ln z_1 = \ln r_1 + i\theta_1$ [see Fig. 2-16]. Then after making one complete circuit about the origin in the positive or counterclockwise direction, we find on returning to z_1 that $r = r_1$, $\theta = \theta_1 + 2\pi$ so that $\ln z_1 = \ln r_1 + i(\theta_1 + 2\pi)$. Thus we are on another branch of the function, and so $z = 0$ is a branch point.

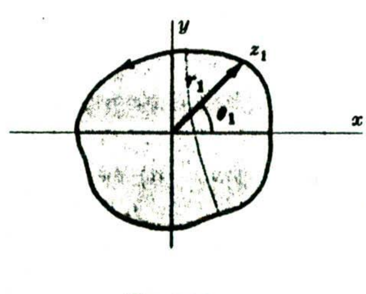

Fig. 2-16

Further complete circuits about the origin lead to other branches and (unlike the case of functions such as $z^{1/2}$ or $z^{1/5}$) we *never* return to the same branch.

It follows that $\ln z$ is an infinitely many-valued function of z with infinitely many branches. That particular branch of $\ln z$ which is real when z is real and positive is called the *principal branch*. To obtain this branch we require that $\theta = 0$ when $z > 0$. To accomplish this we can take $\ln z = \ln r + i\theta$ where θ is chosen so that $0 \leq \theta < 2\pi$ or $-\pi \leq \theta < \pi$, etc.

As a generalization we note that $\ln(z - a)$ has a branch point at $z = a$.

15. Consider the transformation $w = \ln z$. Show that (*a*) circles with centre at the origin in the z plane are mapped into lines parallel to the v axis in the w plane, (*b*) lines or *rays* emanating from the origin in the z plane are mapped into lines parallel to the u axis in the w plane, (*c*) the z plane is mapped into a strip of width 2π in the w plane. Illustrate the results graphically.

We have $w = u + iv = \ln z = \ln r + i\theta$ so that $u = \ln r$, $v = \theta$.

Choose the principal branch as $w = \ln r + i\theta$ where $0 \leqq \theta < 2\pi$.

(*a*) Circles with centre at the origin and radius a have the equation $|z| = r = a$. These are mapped into lines in the w plane whose equations are $u = \ln a$. In Figures 2-17 and 2-18 the circles and lines corresponding to $a = 1/2, 1, 3/2, 2$ are indicated.

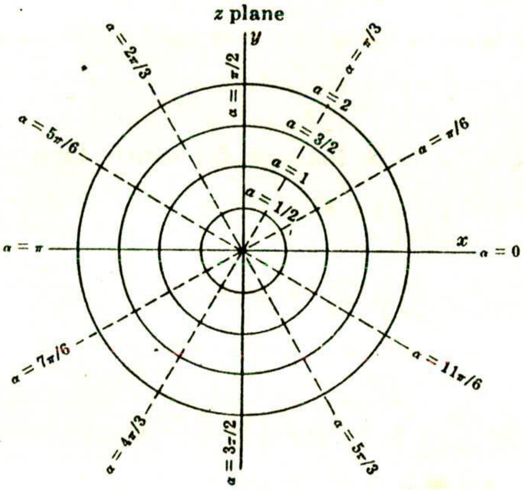

Fig. 2-17

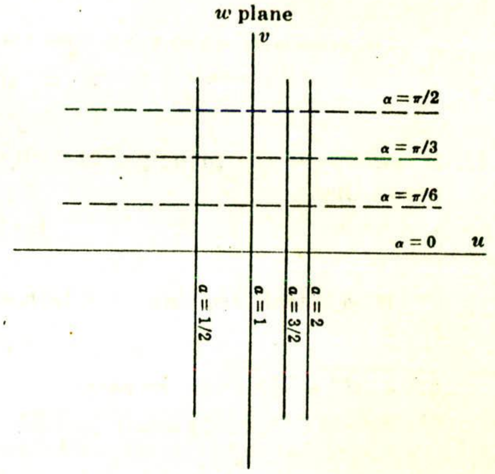

Fig. 2-18

(*b*) Lines or rays emanating from the origin in the z plane (dashed in Fig. 2-17) have the equation $\theta = \alpha$. These are mapped into lines in the w plane (dashed in Fig. 2-18) whose equations are $v = \alpha$. We have shown the corresponding lines for $\alpha = 0, \pi/6, \pi/3$ and $\pi/2$.

(*c*) Corresponding to any given point P in the z plane defined by $z \neq 0$ and having polar coordinates (r, θ) where $0 \leqq \theta < 2\pi$, $r > 0$, there is a point P' in the strip of width 2π shown shaded in Fig. 2-20. Thus the z plane is mapped into this strip. The point $z = 0$ is mapped into a point of this strip sometimes called the *point at infinity*.

If θ is such that $2\pi \leqq \theta < 4\pi$, the z plane is mapped into the strip $2\pi \leqq v < 4\pi$ of Fig. 2-20. Similarly, we obtain the other strips shown in Fig. 2-20.

It follows that given any point $z \neq 0$ in the z plane, there are infinitely many image points in the w plane corresponding to it.

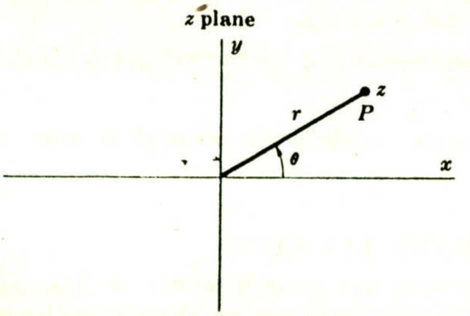

Fig. 2-19

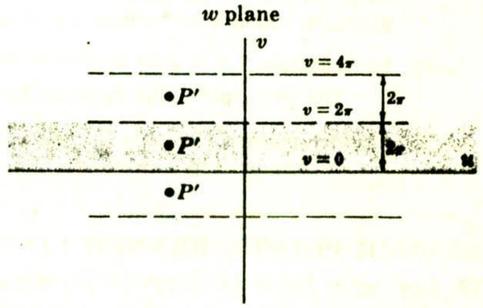

Fig. 2-20

It should be noted that if we had taken θ such that $-\pi \leqq \theta < \pi$, $\pi \leqq \theta < 3\pi$, etc., the strips of Fig. 2-20 would be shifted vertically a distance π.

16. If we choose the principal branch of $\sin^{-1} z$ to be that one for which $\sin^{-1} 0 = 0$, prove that

$$\sin^{-1} z = \frac{1}{i} \ln (iz + \sqrt{1-z^2})$$

If $w = \sin^{-1} z$, then $z = \sin w = \dfrac{e^{iw} - e^{-iw}}{2i}$ from which

$$e^{iw} - 2iz - e^{-iw} = 0 \qquad \text{or} \qquad e^{2iw} - 2ize^{iw} - 1 = 0$$

Solving,
$$e^{iw} = \frac{2iz \pm \sqrt{4-4z^2}}{2} = iz \pm \sqrt{1-z^2} = iz + \sqrt{1-z^2}$$

since $\pm\sqrt{1-z^2}$ is implied by $\sqrt{1-z^2}$. Now $e^{iw} = e^{i(w-2k\pi)}$, $k = 0, \pm 1, \pm 2, \ldots$ so that

$$e^{i(w-2k\pi)} = iz + \sqrt{1-z^2} \qquad \text{or} \qquad w = 2k\pi + \frac{1}{i} \ln (iz + \sqrt{1-z^2})$$

The branch for which $w = 0$ when $z = 0$ is obtained by taking $k = 0$ from which we find, as required,

$$w = \sin^{-1} z = \frac{1}{i} \ln (iz + \sqrt{1-z^2})$$

17. If we choose the principal branch of $\tanh^{-1} z$ to be that one for which $\tanh^{-1} 0 = 0$, prove that

$$\tanh^{-1} z = \frac{1}{2} \ln \left(\frac{1+z}{1-z}\right)$$

If $w = \tanh^{-1} z$, then $z = \tanh w = \dfrac{\sinh w}{\cosh w} = \dfrac{e^{w} - e^{-w}}{e^{w} + e^{-w}}$ from which

$$(1-z)e^{w} = (1+z)e^{-w} \qquad \text{or} \qquad e^{2w} = (1+z)/(1-z)$$

Since $e^{2w} = e^{2(w-k\pi i)}$, we have

$$e^{2(w-k\pi i)} = \frac{1+z}{1-z} \qquad \text{or} \qquad w = k\pi i + \frac{1}{2} \ln \left(\frac{1+z}{1-z}\right)$$

The principal branch is the one for which $k = 0$ and leads to the required result.

18. (*a*) If $z = re^{i\theta}$, prove that $z^i = e^{-(\theta + 2k\pi)} \{\cos (\ln r) + i \sin (\ln r)\}$ where $k = 0, \pm 1, \pm 2, \ldots$.

(*b*) If z is a point on the unit circle with centre at the origin, prove that z^i represents infinitely many real numbers and determine the principal value.

(*c*) Find the principal value of i^i.

(*a*) By definition,
$$z^i = e^{i \ln z} = e^{i\{\ln r + i(\theta + 2k\pi)\}}$$
$$= e^{i \ln r - (\theta + 2k\pi)} = e^{-(\theta + 2k\pi)} \{\cos (\ln r) + i \sin (\ln r)\}$$

The principal branch of the many-valued function $f(z) = z^i$ is obtained by taking $k = 0$ and is given by $e^{-\theta} \{\cos (\ln r) + i \sin (\ln r)\}$ where we can choose θ such that $0 \leqq \theta < 2\pi$.

(*b*) If z is any point on the unit circle with centre at the origin, then $|z| = r = 1$. Hence by part (*a*), since $\ln r = 0$, we have $z^i = e^{-(\theta + 2k\pi)}$ which represents infinitely many real numbers. The principal value is $e^{-\theta}$ where we choose θ such that $0 \leqq \theta < 2\pi$.

(*c*) By definition, $i^i = e^{i \ln i} = e^{i\{i(\pi/2 + 2k\pi)\}} = e^{-(\pi/2 + 2k\pi)}$ since $i = e^{i(\pi/2 + 2k\pi)}$ and $\ln i = i(\pi/2 + 2k\pi)$.

The principal value is given by $e^{-\pi/2}$.

Another method. By part (*b*), since $z = i$ lies on the unit circle with centre at the origin and since $\theta = \pi/2$, the principal value is $e^{-\pi/2}$.

BRANCH POINTS, BRANCH LINES, RIEMANN SURFACES

19. Let $w = f(z) = (z^2 + 1)^{1/2}$. (*a*) Show that $z = \pm i$ are branch points of $f(z)$. (*b*) Show that a complete circuit around both branch points produces no change in the branches of $f(z)$.

(*a*) We have $w = (z^2 + 1)^{1/2} = \{(z-i)(z+i)\}^{1/2}$. Then $\arg w = \frac{1}{2} \arg (z-i) + \frac{1}{2} \arg (z+i)$ so that

$$\text{Change in } \arg w = \tfrac{1}{2}\{\text{Change in } \arg (z-i)\} + \tfrac{1}{2}\{\text{Change in } \arg (z+i)\}$$

Let C [Fig. 2-21] be a closed curve enclosing the point i but not the point $-i$. Then as point z goes once counterclockwise around C,

$$\text{Change in } \arg(z-i) = 2\pi, \qquad \text{Change in } \arg(z+i) = 0$$

so that

$$\text{Change in } \arg w = \pi$$

Hence w does not return to its original value, i.e. a change in branches has occurred. Since a complete circuit about $z = i$ alters the branches of the function, $z = i$ is a branch point. Similarly if C is a closed curve enclosing the point $-i$ but not i, we can show that $z = -i$ is a branch point.

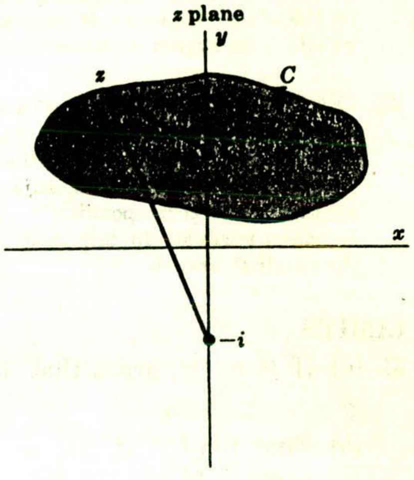

Fig. 2-21

Another method.

Let $z - i = r_1 e^{i\theta_1}$, $z + i = r_2 e^{i\theta_2}$. Then

$$w = \{r_1 r_2 e^{i(\theta_1+\theta_2)}\}^{1/2} = \sqrt{r_1 r_2}\, e^{i\theta_1/2} e^{i\theta_2/2}$$

Suppose we start with a particular value of z corresponding to $\theta_1 = \alpha_1$ and $\theta_2 = \alpha_2$. Then $w = \sqrt{r_1 r_2}\, e^{i\alpha_1/2} e^{i\alpha_2/2}$. As z goes once counterclockwise around i, θ_1 increases to $\alpha_1 + 2\pi$ while θ_2 remains the same, i.e. $\theta_2 = \alpha_2$. Hence

$$\begin{aligned} w &= \sqrt{r_1 r_2}\, e^{i(\alpha_1+2\pi)/2} e^{i\alpha_2/2} \\ &= -\sqrt{r_1 r_2}\, e^{i\alpha_1/2} e^{i\alpha_2/2} \end{aligned}$$

showing that we do not obtain the original value of w, i.e. a change of branches has occurred, showing that $z = i$ is a branch point.

(b) If C encloses both branch points $z = \pm i$ as in Fig. 2-22, then as point z goes counterclockwise around C,

$$\text{Change in } \arg(z-i) = 2\pi$$
$$\text{Change in } \arg(z+i) = 2\pi$$

so that

$$\text{Change in } \arg w = 2\pi$$

Hence a complete circuit around both branch points produces no change in the branches.

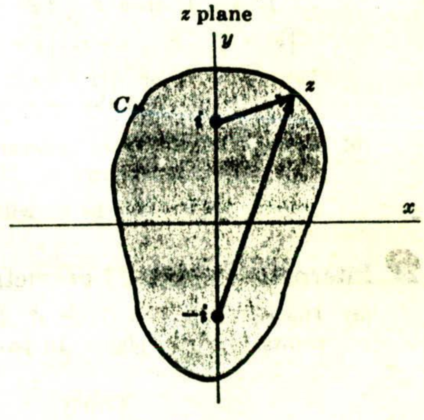

Fig. 2-22

Another method.

In this case, referring to the second method of part (a), θ_1 increases from α_1 to $\alpha_1 + 2\pi$ while θ_2 increases from α_2 to $\alpha_2 + 2\pi$. Thus

$$w = \sqrt{r_1 r_2}\, e^{i(\alpha_1+2\pi)/2} e^{i(\alpha_2+2\pi)/2} = \sqrt{r_1 r_2}\, e^{i\alpha_1/2} e^{i\alpha_2/2}$$

and no change in branch is observed.

20. Determine branch lines for the function of Problem 19.

The branch lines can be taken as those indicated heavy in either of Figures 2-23, 2-24. In both cases, by not crossing these heavy lines we insure the single-valuedness of the function.

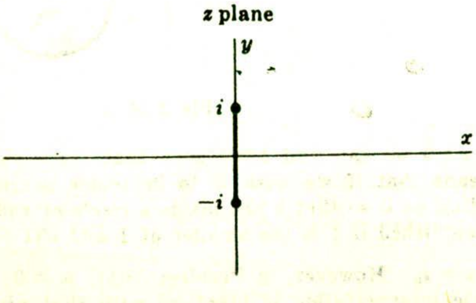

Fig. 2-23

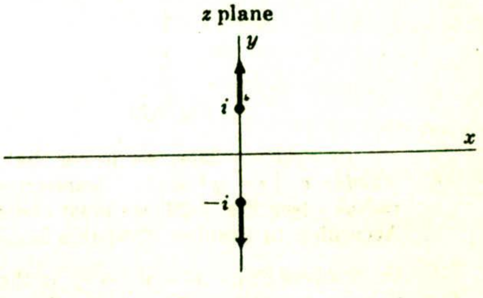

Fig. 2-24

21. Discuss the Riemann surface for the function of Problem 19.

We can have different Riemann surfaces corresponding to Fig. 2-23 or 2-24 of Problem 20. Referring to Fig. 2-23, for example, we imagine that the z plane consists of two sheets superimposed on each other and cut along the branch line. Opposite edges of the cut are then joined, forming the Riemann surface. On making one complete circuit around $z = i$, we start on one branch and wind up on the other. However, if we make one circuit about both $z = i$ and $z = -i$, we do not change branches at all. This agrees with the results of Problem 19.

22. Discuss the Riemann surface for the function $f(z) = \ln z$ [see Problem 14].

In this case we imagine the z plane to consist of infinitely many sheets superimposed on each other and cut along a branch line emanating from the origin $z = 0$. We then connect each cut edge to the opposite cut edge of an adjacent sheet. Then every time we make a circuit about $z = 0$ we are on another sheet corresponding to a different branch of the function. The collection of sheets is the Riemann surface. In this case, unlike Problems 6 and 7, successive circuits never bring us back to the original branch.

LIMITS

23. (*a*) If $f(z) = z^2$, prove that $\lim_{z \to z_0} f(z) = z_0^2$.

(*b*) Find $\lim_{z \to z_0} f(z)$ if $f(z) = \begin{cases} z^2 & z \neq z_0 \\ 0 & z = z_0 \end{cases}$.

(*a*) We must show that given any $\epsilon > 0$ we can find δ (depending in general on ϵ) such that $|z^2 - z_0^2| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

If $\delta \leqq 1$, then $0 < |z - z_0| < \delta$ implies that

$$|z^2 - z_0^2| \;=\; |z - z_0|\,|z + z_0| \;<\; \delta\,|z - z_0 + 2z_0| \;<\; \delta\{|z - z_0| + |2z_0|\} \;<\; \delta(1 + 2|z_0|)$$

Take δ as 1 or $\epsilon/(1 + 2|z_0|)$, whichever is smaller. Then we have $|z^2 - z_0^2| < \epsilon$ whenever $|z - z_0| < \delta$, and the required result is proved.

(*b*) There is no difference between this problem and that in part (*a*), since in both cases we exclude $z = z_0$ from consideration. Hence $\lim_{z \to z_0} f(z) = z_0^2$. Note that the limit of $f(z)$ as $z \to z_0$ has nothing whatsoever to do with the value of $f(z)$ *at* z_0.

24. Interpret Problem 23 geometrically.

(*a*) The equation $w = f(z) = z^2$ defines a transformation or mapping of points of the z plane into points of the w plane. In particular let us suppose that point z_0 is mapped into $w_0 = z_0^2$.

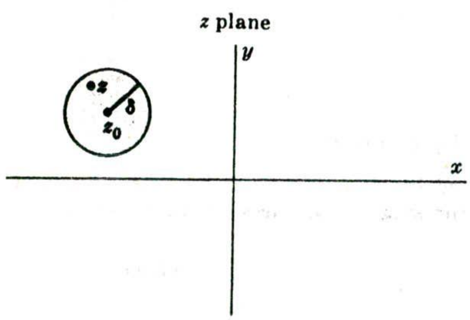

Fig. 2-25

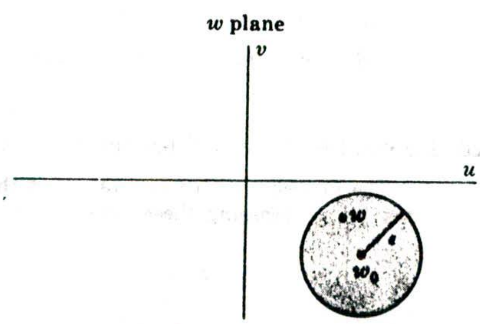

Fig. 2-26

In Problem 23(*a*) we prove that given any $\epsilon > 0$ we can find $\delta > 0$ such that $|w - w_0| < \epsilon$ whenever $|z - z_0| < \delta$. Geometrically this means that if we wish w to be inside a circle of radius ϵ [see Fig. 2-26] we must choose δ (depending on ϵ) so that z lies inside a circle of radius δ. According to Problem 23(*a*) this is certainly accomplished if δ is the smaller of 1 and $\epsilon/(1 + 2|z_0|)$.

(*b*) In Problem 23(*a*), $w = w_0 = z_0^2$ is the image of $z = z_0$. However, in Problem 23(*b*), $w = 0$ is the image of $z = z_0$. Except for this, the geometric interpretation is identical with that given in part (*a*).

25. Prove that $\lim_{z \to i} \frac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i} = 4 + 4i$.

We must show that for any $\epsilon > 0$ we can find $\delta > 0$ such that

$$\left| \frac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i} - (4 + 4i) \right| < \epsilon \quad \text{when} \quad 0 < |z - i| < \delta.$$

Since $z \neq i$, we can write

$$\frac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i} = \frac{[3z^3 - (2 - 3i)z^2 + (5 - 2i)z + 5i][z - i]}{z - i} = 3z^3 - (2 - 3i)z^2 + (5 - 2i)z + 5i$$

on cancelling the common factor $z - i \neq 0$.

Then we must show that for any $\epsilon > 0$, we can find $\delta > 0$ such that

$$|3z^3 - (2 - 3i)z^2 + (5 - 2i)z - 4 + i| < \epsilon \quad \text{when} \quad 0 < |z - i| < \delta$$

If $\delta \leqq 1$, then $0 < |z - i| < \delta$ implies

$$\begin{aligned} |3z^3 - (2 - 3i)z^2 + (5 - 2i)z - 4 + i| &= |z - i|\,|3z^2 + (6i - 2)z - 1 - 4i| \\ &= |z - i|\,|3(z - i + i)^2 + (6i - 2)(z - i + i) - 1 - 4i| \\ &= |z - i|\,|3(z - i)^2 + (12i - 2)(z - i) - 10 - 6i| \\ &< \delta\{3|z - i|^2 + |12i - 2|\,|z - i| + |-10 - 6i|\} \\ &< \delta(3 + 13 + 12) = 28\delta \end{aligned}$$

Taking δ as the smaller of 1 and $\epsilon/28$, the required result follows.

THEOREMS ON LIMITS

26. If $\lim_{z \to z_0} f(z)$ exists, prove that it must be unique.

We must show that if $\lim_{z \to z_0} f(z) = l_1$ and $\lim_{z \to z_0} f(z) = l_2$, then $l_1 = l_2$.

By hypothesis, given any $\epsilon > 0$, we can find $\delta > 0$ such that

$$|f(z) - l_1| < \epsilon/2 \quad \text{when} \quad 0 < |z - z_0| < \delta$$
$$|f(z) - l_2| < \epsilon/2 \quad \text{when} \quad 0 < |z - z_0| < \delta$$

Then

$$|l_1 - l_2| = |l_1 - f(z) + f(z) - l_2| \leqq |l_1 - f(z)| + |f(z) - l_2| < \epsilon/2 + \epsilon/2 = \epsilon$$

i.e. $|l_1 - l_2|$ is less than any positive number ϵ (however small) and so must be zero. Thus $l_1 = l_2$.

27. If $\lim_{z \to z_0} g(z) = B \neq 0$, prove that there exists $\delta > 0$ such that

$$|g(z)| > \tfrac{1}{2}|B| \quad \text{for} \quad 0 < |z - z_0| < \delta$$

Since $\lim_{z \to z_0} g(z) = B$, we can find δ such that $|g(z) - B| < \frac{1}{2}|B|$ for $0 < |z - z_0| < \delta$.

Writing $B = B - g(z) + g(z)$, we have

$$|B| \leqq |B - g(z)| + |g(z)| < \tfrac{1}{2}|B| + |g(z)|$$

i.e.

$$|B| < \tfrac{1}{2}|B| + |g(z)| \quad \text{from which} \quad |g(z)| > \tfrac{1}{2}|B|$$

28. Given $\lim_{z \to z_0} f(z) = A$ and $\lim_{z \to z_0} g(z) = B$, prove that (a) $\lim_{z \to z_0} [f(z) + g(z)] = A + B$, (b) $\lim_{z \to z_0} f(z)\,g(z) = AB$, (c) $\lim_{z \to z_0} \frac{1}{g(z)} = \frac{1}{B}$ if $B \neq 0$, (d) $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{A}{B}$ if $B \neq 0$.

(a) We must show that for any $\epsilon > 0$ we can find $\delta > 0$ such that

$$|[f(z) + g(z)] - (A + B)| < \epsilon \quad \text{when} \quad 0 < |z - z_0| < \delta$$

We have

$$|[f(z) + g(z)] - (A + B)| = |[f(z) - A] + [g(z) - B]| \leqq |f(z) - A| + |g(z) - B| \tag{1}$$

By hypothesis, given $\epsilon > 0$ we can find $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|f(z) - A| < \epsilon/2 \quad \text{when} \quad 0 < |z - z_0| < \delta_1 \tag{2}$$
$$|g(z) - B| < \epsilon/2 \quad \text{when} \quad 0 < |z - z_0| < \delta_2 \tag{3}$$

Then from (1), (2) and (3),

$$|[f(z)+g(z)]-(A+B)| < \epsilon/2+\epsilon/2 = \epsilon \quad \text{when} \quad 0<|z-z_0|<\delta$$

where δ is chosen as the smaller of δ_1 and δ_2.

(b) We have

$$\begin{aligned} |f(z)\,g(z) - AB| &= |f(z)\{g(z)-B\} + B\{f(z)-A\}| \qquad (4)\\ &\leqq |f(z)|\,|g(z)-B| + |B|\,|f(z)-A| \\ &\leqq |f(z)|\,|g(z)-B| + (|B|+1)\,|f(z)-A| \end{aligned}$$

Since $\lim_{z\to z_0} f(z) = A$, we can find δ_1 such that $|f(z)-A| < 1$ for $0<|z-z_0|<\delta_1$. Hence by inequalities 4, Page 2,

$$|f(z)-A| \geqq |f(z)|-|A|, \quad \text{i.e.} \quad 1 \geqq |f(z)|-|A| \quad \text{or} \quad |f(z)| \leqq |A|+1$$

i.e. $|f(z)| < P$ where P is a positive constant.

Since $\lim_{z\to z_0} g(z) = B$, given $\epsilon>0$ we can find $\delta_2>0$ such that $|g(z)-B| < \epsilon/2P$ for $0<|z-z_0|<\delta_2$.

Since $\lim_{z\to z_0} f(z) = A$, given $\epsilon>0$ we can find $\delta_3>0$ such that $|f(z)-A| < \dfrac{\epsilon}{2(|B|+1)}$ for $0<|z-z_0|<\delta_3$.

Using these in (4), we have

$$|f(z)\,g(z)-AB| < P\cdot\frac{\epsilon}{2P} + (|B|+1)\cdot\frac{\epsilon}{2(|B|+1)} = \epsilon$$

for $0<|z-z_0|<\delta$ where δ is the smaller of $\delta_1, \delta_2, \delta_3$, and the proof is complete.

(c) We must show that for any $\epsilon>0$ we can find $\delta>0$ such that

$$\left|\frac{1}{g(z)}-\frac{1}{B}\right| = \frac{|g(z)-B|}{|B|\,|g(z)|} < \epsilon \quad \text{when} \quad 0<|z-z_0|<\delta \qquad (5)$$

By hypothesis, given any $\epsilon>0$ we can find $\delta_1>0$ such that

$$|g(z)-B| < \tfrac{1}{2}|B|^2\epsilon \quad \text{when} \quad 0<|z-z_0|<\delta_1$$

By Problem 27, since $\lim_{z\to z_0} g(z) = B \neq 0$, we can find $\delta_2>0$ such that

$$|g(z)| > \tfrac{1}{2}|B| \quad \text{when} \quad 0<|z-z_0|<\delta_2$$

Then if δ is the smaller of δ_1 and δ_2, we can write

$$\left|\frac{1}{g(z)}-\frac{1}{B}\right| = \frac{|g(z)-B|}{|B|\,|g(z)|} < \frac{\frac{1}{2}|B|^2\epsilon}{|B|\cdot\frac{1}{2}|B|} = \epsilon \quad \text{whenever} \quad 0<|z-z_0|<\delta$$

and the required result is proved.

(d) From parts (b) and (c),

$$\lim_{z\to z_0}\frac{f(z)}{g(z)} = \lim_{z\to z_0}\left\{f(z)\cdot\frac{1}{g(z)}\right\} = \lim_{z\to z_0} f(z)\cdot\lim_{z\to z_0}\frac{1}{g(z)} = A\cdot\frac{1}{B} = \frac{A}{B}$$

This can also be proved directly [see Problem 145].

Note. In the proof of (a) we have used the results $|f(z)-A| < \epsilon/2$ and $|g(z)-B| < \epsilon/2$, so that the final result would come out to be $|f(z)+g(z)-(A+B)| < \epsilon$. Of course the proof would be *just as valid* if we had used 2ϵ [or any other positive multiple of ϵ] in place of ϵ. Similar remarks hold for the proofs of (b), (c) and (d).

29. Evaluate each of the following using theorems on limits.

(a)
$$\begin{aligned} \lim_{z\to 1+i}(z^2-5z+10) &= \lim_{z\to 1+i} z^2 + \lim_{z\to 1+i}(-5z) + \lim_{z\to 1+i} 10 \\ &= \left(\lim_{z\to 1+i} z\right)\left(\lim_{z\to 1+i} z\right) + \left(\lim_{z\to 1+i} -5\right)\left(\lim_{z\to 1+i} z\right) + \lim_{z\to 1+i} 10 \\ &= (1+i)(1+i) - 5(1+i) + 10 = 5-3i \end{aligned}$$

In practice the intermediate steps are omitted.

(b)
$$\lim_{z\to -2i}\frac{(2z+3)(z-1)}{z^2-2z+4} = \frac{\lim_{z\to -2i}(2z+3)\,\lim_{z\to -2i}(z-1)}{\lim_{z\to -2i}(z^2-2z+4)} = \frac{(3-4i)(-2i-1)}{4i} = -\frac{1}{2}+\frac{11}{4}i$$

(c) $\lim_{z \to 2e^{\pi i/3}} \dfrac{z^3+8}{z^4+4z^2+16}$

In this case the limits of the numerator and denominator are each zero and the theorems on limits fail to apply.

However, by obtaining the factors of the polynomials, we see that

$$\lim_{z \to 2e^{\pi i/3}} \frac{z^3+8}{z^4+4z^2+16} = \lim_{z \to 2e^{\pi i/3}} \frac{(z+2)(z-2e^{\pi i/3})(z-2e^{5\pi i/3})}{(z-2e^{\pi i/3})(z-2e^{2\pi i/3})(z-2e^{4\pi i/3})(z-2e^{5\pi i/3})}$$

$$= \lim_{z \to 2e^{\pi i/3}} \frac{(z+2)}{(z-2e^{2\pi i/3})(z-2e^{4\pi i/3})} = \frac{e^{\pi i/3}+1}{2(e^{\pi i/3}-e^{2\pi i/3})(e^{\pi i/3}-e^{4\pi i/3})}$$

$$= \frac{3}{8} - \frac{\sqrt{3}}{8}i$$

Another method. Since $z^6 - 64 = (z^2-4)(z^4+4z^2+16)$, the problem is equivalent to finding

$$\lim_{z \to 2e^{\pi i/3}} \frac{(z^2-4)(z^3+8)}{z^6-64} = \lim_{z \to 2e^{\pi i/3}} \frac{z^2-4}{z^3-8} = \frac{e^{2\pi i/3}-1}{2(e^{\pi i}-1)} = \frac{3}{8} - \frac{\sqrt{3}}{8}i$$

30. Prove that $\lim_{z \to 0} \dfrac{\bar{z}}{z}$ does not exist.

If the limit is to exist it must be independent of the manner in which z approaches the point 0.

Let $z \to 0$ along the x axis. Then $y = 0$, and $z = x + iy = x$ and $\bar{z} = x - iy = x$, so that the required limit is

$$\lim_{x \to 0} \frac{x}{x} = 1$$

Let $z \to 0$ along the y axis. Then $x = 0$, and $z = x + iy = iy$ and $\bar{z} = x - iy = -iy$, so that the required limit is

$$\lim_{y \to 0} \frac{-iy}{iy} = -1$$

Since the two approaches do not give the same answer, the limit does not exist.

CONTINUITY

31. (a) Prove that $f(z) = z^2$ is continuous at $z = z_0$.

(b) Prove that $f(z) = \begin{cases} z^2 & z \neq z_0 \\ 0 & z = z_0 \end{cases}$, where $z_0 \neq 0$, is discontinuous at $z = z_0$.

(a) By Problem 23(a), $\lim_{z \to z_0} f(z) = f(z_0) = z_0^2$ and so $f(z)$ is continuous at $z = z_0$.

Another method. We must show that given any $\epsilon > 0$, we can find $\delta > 0$ (depending on ϵ) such that $|f(z) - f(z_0)| = |z^2 - z_0^2| < \epsilon$ when $|z - z_0| < \delta$. The proof patterns that given in Problem 23(a).

(b) By Problem 23(b), $\lim_{z \to z_0} f(z) = z_0^2$, but $f(z_0) = 0$. Hence $\lim_{z \to z_0} f(z) \neq f(z_0)$ and so $f(z)$ is discontinuous at $z = z_0$ if $z_0 \neq 0$.

If $z_0 = 0$, then $f(z) = 0$; and since $\lim_{z \to z_0} f(z) = 0 = f(0)$, we see that the function is continuous.

32. Is the function $f(z) = \dfrac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i}$ continuous at $z = i$?

$f(i)$ does not exist, i.e. $f(z)$ is not defined at $z = i$. Thus $f(z)$ is not continuous at $z = i$.

By redefining $f(z)$ so that $f(i) = \lim_{z \to i} f(z) = 4 + 4i$ (see Problem 25), it becomes continuous at $z = i$. In such case we call $z = i$ a *removable discontinuity*.

33. Prove that if $f(z)$ and $g(z)$ are continuous at $z = z_0$, so also are

$$(a)\ f(z) + g(z), \qquad (b)\ f(z)\,g(z), \qquad (c)\ \frac{f(z)}{g(z)} \text{ if } g(z_0) \neq 0$$

These results follow at once from Problem 28 by taking $A = f(z_0)$, $B = g(z_0)$ and rewriting $0 < |z - z_0| < \delta$ as $|z - z_0| < \delta$, i.e. *including* $z = z_0$.

34. Prove that $f(z) = z^2$ is continuous in the region $|z| \leqq 1$.

Let z_0 be any point in the region $|z| \leqq 1$. By Problem 23(*a*), $f(z)$ is continuous at z_0. Thus $f(z)$ is continuous in the region since it is continuous at any point of the region.

35. For what values of z are each of the following functions continuous?

(*a*) $f(z) = \dfrac{z}{z^2+1} = \dfrac{z}{(z-i)(z+i)}$. Since the denominator is zero when $z = \pm i$, the function is continuous everywhere except $z = \pm i$.

(*b*) $f(z) = \csc z = \dfrac{1}{\sin z}$. By Problem 10(*a*), $\sin z = 0$ for $z = 0, \pm\pi, \pm 2\pi, \ldots$. Hence $f(z)$ is continuous everywhere except at these points.

UNIFORM CONTINUITY

36. Prove that $f(z) = z^2$ is uniformly continuous in the region $|z| < 1$.

We must show that given any $\epsilon > 0$, we can find $\delta > 0$ such that $|z^2 - z_0^2| < \epsilon$ when $|z - z_0| < \delta$, where δ depends *only* on ϵ and not on the particular point z_0 of the region.

If z and z_0 are any points in $|z| < 1$, then

$$|z^2 - z_0^2| = |z + z_0|\,|z - z_0| \leqq \{|z| + |z_0|\}\,|z - z_0| < 2\,|z - z_0|$$

Thus if $|z - z_0| < \delta$, it follows that $|z^2 - z_0^2| < 2\delta$. Choosing $\delta = \epsilon/2$, we see that $|z^2 - z_0^2| < \epsilon$ when $|z - z_0| < \delta$, where δ depends only on ϵ and not on z_0. Hence $f(z) = z^2$ is uniformly continuous in the region.

37. Prove that $f(z) = 1/z$ is not uniformly continuous in the region $|z| < 1$.

Method 1.

Suppose that $f(z)$ is uniformly continuous in the region. Then for any $\epsilon > 0$ we should be able to find δ, say between 0 and 1, such that $|f(z) - f(z_0)| < \epsilon$ when $|z - z_0| < \delta$ for all z and z_0 in the region.

Let $z = \delta$ and $z_0 = \dfrac{\delta}{1+\epsilon}$. Then $|z - z_0| = \left|\delta - \dfrac{\delta}{1+\epsilon}\right| = \dfrac{\epsilon}{1+\epsilon}\delta < \delta$.

However, $\left|\dfrac{1}{z} - \dfrac{1}{z_0}\right| = \left|\dfrac{1}{\delta} - \dfrac{1+\epsilon}{\delta}\right| = \dfrac{\epsilon}{\delta} > \epsilon$ (since $0 < \delta < 1$).

Thus we have a contradiction, and it follows that $f(z) = 1/z$ cannot be uniformly continuous in the region.

Method 2.

Let z_0 and $z_0 + \zeta$ be any two points of the region such that $|z_0 + \zeta - z_0| = |\zeta| = \delta$. Then

$$|f(z_0) - f(z_0 + \zeta)| = \left|\frac{1}{z_0} - \frac{1}{z_0 + \zeta}\right| = \frac{|\zeta|}{|z_0|\,|z_0 + \zeta|} = \frac{\delta}{|z_0|\,|z_0 + \zeta|}$$

can be made larger than any positive number by choosing z_0 sufficiently close to 0. Hence the function cannot be uniformly continuous in the region.

SEQUENCES AND SERIES

38. Investigate the convergence of the sequences

(*a*) $u_n = \dfrac{i^n}{n}$, $n = 1, 2, 3, \ldots$, (*b*) $u_n = \dfrac{(1+i)^n}{n}$.

(*a*) The first few terms of the sequence are $i, \dfrac{i^2}{2}, \dfrac{i^3}{3}, \dfrac{i^4}{4}, \dfrac{i^5}{5}$, etc., or $i, -\dfrac{1}{2}, \dfrac{-i}{3}, \dfrac{1}{4}, \dfrac{i}{5}, \ldots$. On plotting the corresponding points in the z plane, we suspect that the limit is zero. To prove this we must show that

$$|u_n - l| = |i^n/n - 0| < \epsilon \quad \text{when} \quad n > N \tag{1}$$

Now $|i^n/n - 0| = |i^n/n| = |i|^n/n = 1/n < \epsilon$ when $n > 1/\epsilon$

Let us choose $N = 1/\epsilon$. Then we see that (*1*) is true, and so the sequence converges to zero.

(b) Consider $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(1+i)^{n+1}/(n+1)}{(1+i)^n/n}\right| = \frac{n}{n+1}|1+i| = \frac{n\sqrt{2}}{n+1}$.

For all $n \geqq 10$ (for example), we have $\frac{n\sqrt{2}}{n+1} > \frac{6}{5} = 1.2$. Thus $|u_{n+1}| > 1.2|u_n|$ for $n > 10$, i.e. $|u_{11}| > 1.2\,|u_{10}|$, $|u_{12}| > 1.2\,|u_{11}| > (1.2)^2\,|u_{10}|$, and in general $|u_n| > (1.2)^{n-10}\,|u_{10}|$. It follows that $|u_n|$ can be made larger than any preassigned positive number (no matter how large) and thus the limit of $|u_n|$ cannot exist, and consequently the limit of u_n cannot exist. Thus the sequence diverges.

39. If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, prove that $\lim_{n\to\infty} (a_n + b_n) = A + B$.

By definition, given ϵ we can find N such that

$$|a_n - A| < \epsilon/2, \quad |b_n - B| < \epsilon/2 \quad \text{for } n > N$$

Then for $n > N$,

$$|(a_n + b_n) - (A + B)| = |(a_n - A) + (b_n - B)| \leqq |a_n - A| + |b_n - B| < \epsilon$$

which proves the result.

It is seen that this parallels the proof for limits of functions [Problem 28].

40. Prove that if a series $u_1 + u_2 + u_3 + \cdots$ is to converge, we must have $\lim_{n\to\infty} u_n = 0$.

If S_n is the sum of the first n terms of the series, then $S_{n+1} = S_n + u_n$. Hence if $\lim_{n\to\infty} S_n$ exists and equals S, we have $\lim_{n\to\infty} S_{n+1} = \lim_{n\to\infty} S_n + \lim_{n\to\infty} u_n$ or $S = S + \lim_{n\to\infty} u_n$, i.e. $\lim_{n\to\infty} u_n = 0$.

Conversely, however, if $\lim_{n\to\infty} u_n = 0$ the series may or may not converge. See Problem 150.

41. Prove that $1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}$ if $|z| < 1$.

Let $\quad S_n = 1 + z + z^2 + \cdots + z^{n-1}$

Then $\quad zS_n = z + z^2 + \cdots + z^{n-1} + z^n$

Subtracting, $\quad (1-z)S_n = 1 - z^n \quad$ or $\quad S_n = \frac{1-z^n}{1-z}$

If $|z| < 1$, then we suspect that $\lim_{n\to\infty} z^n = 0$. To prove this we must show that given any $\epsilon > 0$ we can find N such that $|z^n - 0| < \epsilon$ for all $n > N$. The result is certainly true if $z = 0$; hence we can consider $z \neq 0$.

Now $|z^n| = |z|^n < \epsilon$ when $n \ln|z| < \ln\epsilon$ or $n > (\ln\epsilon)/(\ln|z|) = N$ [since if $|z| < 1$, $\ln|z|$ is negative]. We have therefore found the required N, and $\lim_{n\to\infty} z^n = 0$.

Thus $\quad 1 + z + z^2 + \cdots = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-z^n}{1-z} = \frac{1-0}{1-z} = \frac{1}{1-z}$.

The series

$$a + az + az^2 + \cdots = \frac{a}{1-z}$$

is called a *geometric series* with first term equal to a and ratio z, and its sum is $a/(1-z)$ provided $|z| < 1$.

MISCELLANEOUS PROBLEMS

42. Let $w = (z^2+1)^{1/2}$. (a) If $w = 1$ when $z = 0$, and z describes the curve C_1 shown in Fig. 2-27 below, find the value of w when $z = 1$. (b) If z describes the curve C_2 shown in Fig. 2-28 below, is the value of w when $z = 1$ the same as that obtained in (a)?

(a) The branch points of $w = f(z) = (z^2+1)^{1/2} = \{(z-i)(z+i)\}^{1/2}$ are at $z = \pm i$ by Problem 19.

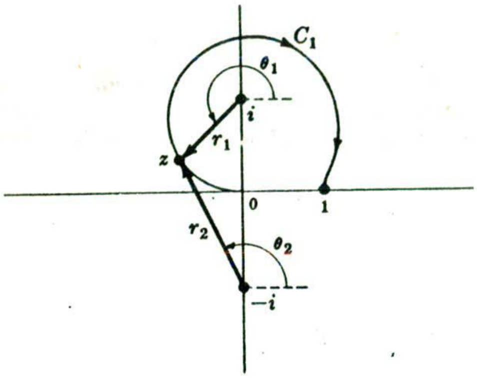

Fig. 2-27

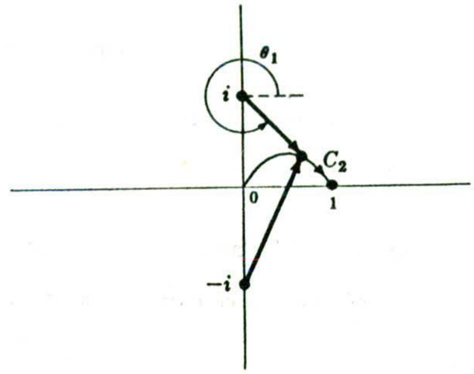

Fig. 2-28

Let (1) $z - i = r_1 e^{i\theta_1}$, (2) $z + i = r_2 e^{i\theta_2}$. Then since θ_1 and θ_2 are determined only within integer multiples of $2\pi i$, we can write

$$w = \sqrt{r_1 r_2}\, e^{i(\theta_1+\theta_2)/2}\, e^{2k\pi i/2} = \sqrt{r_1 r_2}\, e^{i(\theta_1+\theta_2)/2}\, e^{k\pi i} \tag{3}$$

Referring to Fig. 2-27 [or by using the equations (1) and (2)] we see that when z is at 0, $r_1 = 1$, $\theta_1 = 3\pi/2$ and $r_2 = 1$, $\theta_2 = \pi/2$. Since $w = 1$ at $z = 0$, we have from (3), $1 = e^{(k+1)\pi i}$ and we choose $k = -1$ [or $1, -3, \ldots$]. Then

$$w = -\sqrt{r_1 r_2}\, e^{i(\theta_1+\theta_2)/2}$$

As z traverses C_1 from 0 to 1, r_1 changes from 1 to $\sqrt{2}$, θ_1 changes from $3\pi/2$ to $-\pi/4$, r_2 changes from 1 to $\sqrt{2}$, θ_2 changes from $\pi/2$ to $\pi/4$. Then

$$w = -\sqrt{(\sqrt{2})(\sqrt{2})}\, e^{i(-\pi/4+\pi/4)/2} = -\sqrt{2}$$

(b) As in part (a), $w = -\sqrt{r_1 r_2}\, e^{i(\theta_1+\theta_2)/2}$. Referring to Fig. 2-28 we see that as z traverses C_2, r_1 changes from 1 to $\sqrt{2}$, θ_1 changes from $3\pi/2$ to $7\pi/4$, r_2 changes from 1 to $\sqrt{2}$ and θ_2 changes from $\pi/2$ to $\pi/4$. Then

$$w = -\sqrt{(\sqrt{2})(\sqrt{2})}\, e^{i(7\pi/4+\pi/4)/2} = \sqrt{2}$$

which is not the same as the value obtained in (a).

43. Let $\sqrt{1-z^2} = 1$ for $z = 0$. Show that as z varies from 0 to $p > 1$ along the real axis, $\sqrt{1-z^2}$ varies from 1 to $-i\sqrt{p^2-1}$.

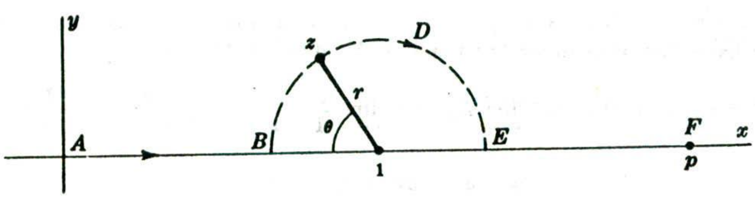

Fig. 2-29

Consider the case where z travels along path $ABDEF$, where BDE is a semi-circle as shown in Fig. 2-29. From this figure, we have

$$1 - z = 1 - x - iy = r\cos\theta - ir\sin\theta$$

so that $\sqrt{1-z^2} = \sqrt{(1-z)(1+z)} = \sqrt{r}\,(\cos\theta/2 - i\sin\theta/2)\,\sqrt{2 - r\cos\theta + ir\sin\theta}$

Along AB: $z = x$, $r = 1 - x$, $\theta = 0$ and $\sqrt{1-z^2} = \sqrt{1-x}\sqrt{1+x} = \sqrt{1-x^2}$.

Along EF: $z = x$, $r = x - 1$, $\theta = \pi$ and $\sqrt{1-z^2} = -i\sqrt{x-1}\sqrt{x+1} = -i\sqrt{x^2-1}$.

Hence as z varies from 0 [where $x = 0$] to p [where $x = p$], $\sqrt{1-z^2}$ varies from 1 to $-i\sqrt{p^2-1}$.

44. Find a mapping function which maps the points $z = 0, \pm i, \pm 2i, \pm 3i, \ldots$ of the z plane into the point $w = 1$ of the w plane [see Figures 2-30 and 2-31].

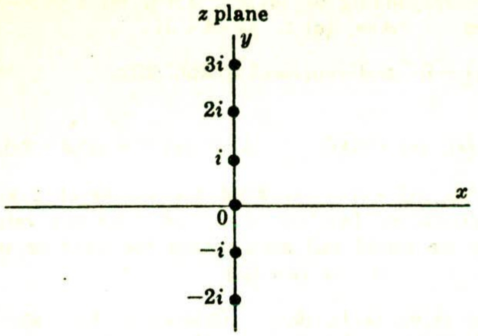

Fig. 2-30

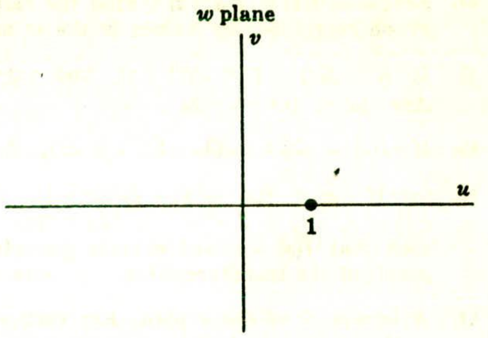

Fig. 2-31

Since the points in the z plane are equally spaced, we are led, because of Problem 15, to consider a logarithmic function of the type $z = \ln w$.

Now if $w = 1 = e^{2k\pi i}$, $k = 0, \pm 1, \pm 2, \ldots$, then $z = \ln w = 2k\pi i$ so that the point $w = 1$ is mapped into the points $0, \pm 2\pi i, \pm 4\pi i, \ldots$.

If, however, we consider $z = (\ln w)/2\pi$, the point $w = 1$ is mapped into $z = 0, \pm i, \pm 2i, \ldots$ as required. Conversely, by means of this mapping function the points $z = 0, \pm i, \pm 2i, \ldots$ are mapped into the point $w = 1$.

Then a suitable mapping function is $z = (\ln w)/2\pi$ or $w = e^{2\pi z}$.

45. If $\lim_{n\to\infty} z_n = l$, prove that $\lim_{n\to\infty} \operatorname{Re}\{z_n\} = \operatorname{Re}\{l\}$ and $\lim_{n\to\infty} \operatorname{Im}\{z_n\} = \operatorname{Im}\{l\}$.

Let $z_n = x_n + iy_n$ and $l = l_1 + il_2$, where x_n, y_n and l_1, l_2 are the real and imaginary parts of z_n and l respectively.

By hypothesis, given any $\epsilon > 0$ we can find N such that $|z_n - l| < \epsilon$ for $n > N$, i.e.,

$$|x_n + iy_n - (l_1 + il_2)| < \epsilon \quad \text{for} \quad n > N$$

or

$$\sqrt{(x_n - l_1)^2 + (y_n - l_2)^2} < \epsilon \quad \text{for} \quad n > N$$

From this it necessarily follows that

$$|x_n - l_1| < \epsilon \quad \text{and} \quad |y_n - l_2| < \epsilon \quad \text{for} \quad n > N$$

i.e. $\lim_{n\to\infty} x_n = l_1$ and $\lim_{n\to\infty} y_n = l_2$, as required.

46. Prove that if $|a| < 1$,

$$(a) \quad 1 + a\cos\theta + a^2\cos 2\theta + a^3\cos 3\theta + \cdots = \frac{1 - a\cos\theta}{1 - 2a\cos\theta + a^2}$$

$$(b) \quad a\sin\theta + a^2\sin 2\theta + a^3\sin 3\theta + \cdots = \frac{a\sin\theta}{1 - 2a\cos\theta + a^2}$$

Let $z = ae^{i\theta}$ in Problem 41. We can do this since $|z| = |a| < 1$. Then

$$1 + ae^{i\theta} + a^2e^{2i\theta} + a^3e^{3i\theta} + \cdots = \frac{1}{1 - ae^{i\theta}}$$

or

$$(1 + a\cos\theta + a^2\cos 2\theta + \cdots) + i(a\sin\theta + a^2\sin 2\theta + \cdots) = \frac{1}{1 - ae^{i\theta}} \cdot \frac{1 - ae^{-i\theta}}{1 - ae^{-i\theta}}$$

$$= \frac{1 - a\cos\theta + ia\sin\theta}{1 - 2a\cos\theta + a^2}$$

The required results follow on equating real and imaginary parts.

Supplementary Problems

FUNCTIONS AND TRANSFORMATIONS

47. Let $w = f(z) = z(2-z)$. Find the values of w corresponding to (a) $z = 1+i$, (b) $z = 2-2i$ and graph corresponding values in the w and z planes. *Ans.* (a) 2, (b) $4+4i$

48. If $w = f(z) = (1+z)/(1-z)$, find (a) $f(i)$, (b) $f(1-i)$ and represent graphically.
Ans. (a) i, (b) $-1-2i$

49. If $f(z) = (2z+1)/(3z-2)$, $z \neq 2/3$, find (a) $f(1/z)$, (b) $f\{f(z)\}$. *Ans.* (a) $(2+z)/(3-2z)$, (b) z

50. (a) If $w = f(z) = (z+2)/(2z-1)$, find $f(0)$, $f(i)$, $f(1+i)$. (b) Find the values of z such that $f(z) = i$, $f(z) = 2-3i$. (c) Show that z is a single-valued function of w. (d) Find the values of z such that $f(z) = z$ and explain geometrically why we would call such values the *fixed* or *invariant points* of the transformation. *Ans.* (a) -2, $-i$, $1-i$, (b) $-i$, $(2+i)/3$

51. A square S in the z plane has vertices at $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$. Determine the region in the w plane into which S is mapped under the transformations (a) $w = z^2$, (b) $w = 1/(z+1)$.

52. Discuss Problem 51 if the square has vertices at $(1,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$.

53. Separate each of the following into real and imaginary parts, i.e. find $u(x,y)$ and $v(x,y)$ such that $f(z) = u+iv$: (a) $f(z) = 2z^2 - 3iz$, (b) $f(z) = z + 1/z$, (c) $f(z) = (1-z)/(1+z)$, (d) $f(z) = z^{1/2}$.

Ans. (a) $u = 2x^2 - 2y^2 + 3y$, $v = 4xy - 3x$

(b) $u = x + x/(x^2+y^2)$, $v = y - y/(x^2+y^2)$

(c) $u = \dfrac{1-x^2-y^2}{(1+x)^2+y^2}$, $v = \dfrac{-2y}{(1+x)^2+y^2}$

(d) $u = r^{1/2}\cos\theta/2$, $v = r^{1/2}\sin\theta/2$ where $x = r\cos\theta$, $y = r\sin\theta$

54. If $f(z) = 1/z = u+iv$, construct several members of the families $u(x,y) = \alpha$, $v(x,y) = \beta$ where α and β are constants, showing that they are families of circles.

MULTIPLE-VALUED FUNCTIONS

55. Let $w^3 = z$ and suppose that corresponding to $z = 1$ we have $w = 1$. (a) If we start at $z = 1$ in the z plane and make one complete circuit counterclockwise around the origin, find the value of w on returning to $z = 1$ for the first time. (b) What are the values of w on returning to $z = 1$ after $2, 3, 4, \ldots$ complete circuits about the origin? Discuss (a) and (b) if the paths do not enclose the origin.
Ans. (a) $e^{2\pi i/3}$, (b) $e^{4\pi i/3}, 1, e^{2\pi i/3}$

56. Let $w = (1-z^2)^{1/2}$ and suppose that corresponding to $z = 0$ we have $w = 1$. (a) If we start at $z = 0$ in the z plane and make one complete circuit counterclockwise so as to include $z = 1$ but not to include $z = -1$, find the value of w on returning to $z = 0$ for the first time. (b) What are the values of w if the circuit in (a) is repeated over and over again? (c) Work parts (a) and (b) if the circuit includes $z = -1$ but does not include $z = 1$. (d) Work parts (a) and (b) if the circuit includes both $z = 1$ and $z = -1$. (e) Work parts (a) and (b) if the circuit excludes both $z = 1$ and $z = -1$. (f) Explain why $z = 1$ and $z = -1$ are branch points. (g) What lines can be taken as branch lines?

57. Find branch points and construct branch lines for the functions (a) $f(z) = \{z/(1-z)\}^{1/2}$, (b) $f(z) = (z^2-4)^{1/3}$, (c) $f(z) = \ln(z - z^2)$.

THE ELEMENTARY FUNCTIONS

58. Prove that (a) $e^{z_1}/e^{z_2} = e^{z_1 - z_2}$, (b) $|e^{iz}| = e^{-y}$.

59. Prove that there cannot be any finite values of z such that $e^z = 0$.

60. Prove that 2π is a period of e^{iz}. Are there any other periods?

61. Find all values of z for which (a) $e^{3z} = 1$, (b) $e^{4z} = i$.
Ans. (a) $2k\pi i/3$, (b) $\frac{1}{8}\pi i + \frac{1}{2}k\pi i$, where $k = 0, \pm 1, \pm 2, \ldots$.

62. Prove (a) $\sin 2z = 2\sin z\cos z$, (b) $\cos 2z = \cos^2 z - \sin^2 z$, (c) $\sin^2(z/2) = \frac{1}{2}(1-\cos z)$, (d) $\cos^2(z/2) = \frac{1}{2}(1+\cos z)$.

63. Prove (a) $1 + \tan^2 z = \sec^2 z$, (b) $1 + \cot^2 z = \csc^2 z$.

64. If $\cos z = 2$, find (a) $\cos 2z$, (b) $\cos 3z$. *Ans.* (a) 7, (b) 26

65. Prove that all the roots of (a) $\sin z = a$, (b) $\cos z = a$, where $-1 \leqq a \leqq 1$, are real.

66. Prove that if $|\sin z| \leq 1$ for all z, then z must be real.

67. Show that (a) $\overline{\sin z} = \sin \bar{z}$, (b) $\overline{\cos z} = \cos \bar{z}$, (c) $\overline{\tan z} = \tan \bar{z}$.

68. For each of the following functions find $u(x,y)$ and $v(x,y)$ such that $f(z) = u + iv$, i.e. separate into real and imaginary parts: (a) $f(z) = e^{3iz}$, (b) $f(z) = \cos z$, (c) $f(z) = \sin 2z$, (d) $f(z) = z^2 e^{2z}$.
Ans. (a) $u = e^{-3y} \cos 3x$, $v = e^{-3y} \sin 3x$, (b) $u = \cos x \cosh y$, $v = -\sin x \sinh y$, (c) $u = \sin 2x \cosh 2y$, $v = \cos 2x \sinh 2y$, (d) $u = e^{2x}\{(x^2 - y^2) \cos 2y - 2xy \sin 2y\}$, $v = e^{2x}\{2xy \cos 2y + (x^2 - y^2) \sin 2y\}$

69. Prove that (a) $\sinh(-z) = -\sinh z$, (b) $\cosh(-z) = \cosh z$, (c) $\tanh(-z) = -\tanh z$.

70. Prove that (a) $\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$, (b) $\cosh 2z = \cosh^2 z + \sinh^2 z$, (c) $1 - \tanh^2 z = \operatorname{sech}^2 z$.

71. Prove that (a) $\sinh^2(z/2) = \frac{1}{2}(\cosh z - 1)$, (b) $\cosh^2(z/2) = \frac{1}{2}(\cosh z + 1)$.

72. Find $u(x,y)$ and $v(x,y)$ such that (a) $\sinh 2z = u + iv$, (b) $z \cosh z = u + iv$.
Ans. (a) $u = \sinh 2x \cos 2y$, $v = \cosh 2x \sin 2y$
(b) $u = x \cosh x \cos y - y \sinh x \sin y$, $v = y \cosh x \cos y + x \sinh x \sin y$

73. Find the value of (a) $4 \sinh(\pi i/3)$, (b) $\cosh(2k+1)\pi i/2$, $k = 0, \pm 1, \pm 2, \ldots$, (c) $\coth 3\pi i/4$.
Ans. (a) $2i\sqrt{3}$, (b) 0, (c) i

74. (a) Show that $\ln\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \left(\frac{4\pi}{3} + 2k\pi\right)i$, $k = 0, \pm 1, \pm 2, \ldots$. (b) What is the principal value?
Ans. (b) $4\pi i/3$

75. Obtain all the values of (a) $\ln(-4)$, (b) $\ln(3i)$, (c) $\ln(\sqrt{3} - i)$ and find the principal value in each case.
Ans. (a) $2 \ln 2 + (\pi + 2k\pi)i$, $2 \ln 2 + \pi i$. (b) $\ln 3 + (\pi/2 + 2k\pi)i$, $\ln 3 + \pi i/2$. (c) $\ln 2 + (11\pi/6 + 2k\pi)i$, $\ln 2 + 11\pi i/6$

76 Show that $\ln(z-1) = \frac{1}{2} \ln\{(x-1)^2 + y^2\} + i \tan^{-1} y/(x-1)$, giving restrictions if any.

77. Prove that (a) $\cos^{-1} z = \frac{1}{i} \ln(z + \sqrt{z^2 - 1})$, (b) $\cot^{-1} z = \frac{1}{2i} \ln\left(\frac{z+i}{z-i}\right)$ indicating any restrictions.

78. Prove that (a) $\sinh^{-1} z = \ln(z + \sqrt{z^2 + 1})$, (b) $\coth^{-1} z = \frac{1}{2} \ln\left(\frac{z+1}{z-1}\right)$.

79. Find all the values of (a) $\sin^{-1} 2$, (b) $\cos^{-1} i$.
Ans. (a) $\pm i \ln(2 + \sqrt{3}) + \pi/2 + 2k\pi$ (b) $-i \ln(\sqrt{2} + 1) + \pi/2 + 2k\pi$, $-i \ln(\sqrt{2} - 1) + 3\pi/2 + 2k\pi$

80. Find all the values of (a) $\cosh^{-1} i$, (b) $\sinh^{-1}\{\ln(-1)\}$.
Ans. (a) $\ln(\sqrt{2} + 1) + \pi i/2 + 2k\pi i$, $\ln(\sqrt{2} - 1) + 3\pi i/2 + 2k\pi i$
(b) $\ln[(2k+1)\pi + \sqrt{(2k+1)^2\pi^2 - 1}] + \pi i/2 + 2m\pi i$,
$\ln[\sqrt{(2k+1)^2\pi^2 - 1} - (2k+1)\pi] + 3\pi i/2 + 2m\pi i$, $k, m = 0, \pm 1, \pm 2, \ldots$

81. Determine all the values of (a) $(1+i)^i$, (b) $1^{\sqrt{2}}$.
Ans. (a) $e^{-\pi/4 + 2k\pi}\{\cos(\frac{1}{2} \ln 2) + i \sin(\frac{1}{2} \ln 2)\}$, (b) $\cos(2\sqrt{2}\, k\pi) + i \sin(2\sqrt{2}\, k\pi)$

82. Find (a) $\operatorname{Re}\{(1-i)^{1+i}\}$, (b) $|(-i)^{-i}|$.
Ans. (a) $e^{\frac{1}{2} \ln 2 - 7\pi/4 - 2k\pi} \cos(7\pi/4 + \frac{1}{2} \ln 2)$, (b) $e^{3\pi/2 + 2k\pi}$

83. Find the real and imaginary parts of z^z where $z = x + iy$.

84. Show that (a) $f(z) = (z^2 - 1)^{1/3}$, (b) $f(z) = z^{1/2} + z^{1/3}$ are algebraic functions of z.

BRANCH POINTS, BRANCH LINES AND RIEMANN SURFACES

85. Prove that $z = \pm i$ are branch points of $(z^2 + 1)^{1/3}$.

86. Construct a Riemann surface for the functions (a) $z^{1/3}$, (b) $z^{1/2}(z-1)^{1/2}$, (c) $\left(\frac{z+2}{z-2}\right)^{1/3}$.

87. Show that the Riemann surface for the function $z^{1/2} + z^{1/3}$ has 6 sheets.

88. Construct Riemann surfaces for the functions (a) $\ln(z+2)$, (b) $\sin^{-1} z$, (c) $\tan^{-1} z$.

LIMITS

89. (a) If $f(z) = z^2 + 2z$, prove that $\lim_{z \to i} f(z) = 2i - 1$.

(b) If $f(z) = \begin{cases} z^2 + 2z & z \neq i \\ 3 + 2i & z = i \end{cases}$, find $\lim_{z \to i} f(z)$ and justify your answer.

90. Prove that $\lim_{z \to 1+i} \dfrac{z^2 - z + 1 - i}{z^2 - 2z + 2} = 1 - \tfrac{1}{2}i$.

91. Guess at a possible value for (a) $\lim_{z \to 2+i} \dfrac{1-z}{1+z}$, (b) $\lim_{z \to 2+i} \dfrac{z^2 - 2iz}{z^2 + 4}$ and investigate the correctness of your guess.

92. If $\lim_{z \to z_0} f(z) = A$ and $\lim_{z \to z_0} g(z) = B$, prove that (a) $\lim_{z \to z_0} \{2f(z) - 3i\,g(z)\} = 2A - 3iB$, (b) $\lim_{z \to z_0} \{p\,f(z) + q\,g(z)\} = pA + qB$ where p and q are any constants.

93. If $\lim_{z \to z_0} f(z) = A$, prove that (a) $\lim_{z \to z_0} \{f(z)\}^2 = A^2$, (b) $\lim_{z \to z_0} \{f(z)\}^3 = A^3$. Can you make a similar statement for $\lim_{z \to z_0} \{f(z)\}^n$? What restrictions, if any, must be imposed?

94. Evaluate using theorems on limits. In each case state precisely which theorems are used.

(a) $\lim_{z \to 2i} (iz^4 + 3z^2 - 10i)$

(b) $\lim_{z \to e^{\pi i/4}} \dfrac{z^2}{z^4 + z + 1}$

(c) $\lim_{z \to i/2} \dfrac{(2z-3)(4z+i)}{(iz-1)^2}$

(d) $\lim_{z \to i} \dfrac{z^2 + 1}{z^6 + 1}$

(e) $\lim_{z \to 1+i} \left\{\dfrac{z - 1 - i}{z^2 - 2z + 2}\right\}^2$

Ans. (a) $-12 + 6i$, (b) $\sqrt{2}\,(1+i)/2$, (c) $-4/3 - 4i$, (d) $1/3$, (e) $-1/4$

95. Find $\lim_{z \to e^{\pi i/3}} (z - e^{\pi i/3}) \left(\dfrac{z}{z^3 + 1}\right)$. *Ans.* $1/6 - i\sqrt{3}/6$

96. Prove that if $f(z) = 3z^2 + 2z$, then $\lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0} = 6z_0 + 2$.

97. If $f(z) = \dfrac{2z - 1}{3z + 2}$, prove that $\lim_{h \to 0} \dfrac{f(z_0 + h) - f(z_0)}{h} = \dfrac{7}{(3z_0 + 2)^2}$ provided $z_0 \neq -2/3$.

98. If we restrict ourselves to that branch of $f(z) = \sqrt{z^2 + 3}$ for which $f(0) = \sqrt{3}$, prove that

$$\lim_{z \to 1} \frac{\sqrt{z^2 + 3} - 2}{z - 1} = \frac{1}{2}$$

99. Explain exactly what is meant by the statements (a) $\lim_{z \to i} 1/(z - i)^2 = \infty$, (b) $\lim_{z \to \infty} \dfrac{2z^4 + 1}{z^4 + 1} = 2$.

100. Show that (a) $\lim_{z \to \pi/2} (\sin z)/z = 2/\pi$, (b) $\lim_{z \to \pi i/2} z^2 \cosh 4z/3 = \pi^2/8$.

101. Show that if we restrict ourselves to that branch of $f(z) = \tanh^{-1} z$ such that $f(0) = 0$, then $\lim_{z \to -i} f(z) = 3\pi i/4$.

CONTINUITY

102. Let $f(z) = \dfrac{z^2 + 4}{z - 2i}$ if $z \neq 2i$, while $f(2i) = 3 + 4i$. (a) Prove that $\lim_{z \to i} f(z)$ exists and determine its value. (b) Is $f(z)$ continuous at $z = 2i$? Explain. (c) Is $f(z)$ continuous at points $z \neq 2i$? Explain.

103. Answer Problem 102 if $f(2i)$ is redefined as equal to $4i$ and explain why any differences should occur.

104. Prove that $f(z) = z/(z^4 + 1)$ is continuous at all points inside and on the unit circle $|z| = 1$ except at four points, and determine these points. *Ans.* $e^{(2k+1)\pi i/4}$, $k = 0, 1, 2, 3$

105. If $f(z)$ and $g(z)$ are continuous at $z = z_0$, prove that $3f(z) - 4i\,g(z)$ is also continuous at $z = z_0$.

106. If $f(z)$ is continuous at $z = z_0$, prove that (a) $\{f(z)\}^2$ and (b) $\{f(z)\}^3$ are also continuous at $z = z_0$. Can you extend the result to $\{f(z)\}^n$ where n is any positive integer?

107. Find all points of discontinuity for the following functions.

(a) $f(z) = \dfrac{2z-3}{z^2+2z+2}$, (b) $f(z) = \dfrac{3z^2+4}{z^4-16}$, (c) $f(z) = \cot z$, (d) $f(z) = \dfrac{1}{z} - \sec z$, (e) $f(z) = \dfrac{\tanh z}{z^2+1}$.

Ans. (a) $-1 \pm i$ (b) $\pm 2, \pm 2i$ (c) $k\pi$, $k = 0, \pm 1, \pm 2, \ldots$ (d) 0, $(k+\frac{1}{2})\pi$, $k = 0, \pm 1, \pm 2, \ldots$ (e) $\pm i$, $(k+\frac{1}{2})\pi i$, $k = 0, \pm 1, \pm 2, \ldots$

108. Prove that $f(z) = z^2 - 2z + 3$ is continuous everywhere in the finite plane.

109. Prove that $f(z) = \dfrac{z^2+1}{z^3+9}$ is (a) continuous and (b) bounded in the region $|z| \leqq 2$.

110. Prove that if $f(z)$ is continuous in a closed region, it is bounded in the region.

111. Prove that $f(z) = 1/z$ is continuous for all z such that $|z| > 0$, but that it is not bounded.

112. Prove that a polynomial is continuous everywhere in the finite plane.

113. Show that $f(z) = \dfrac{z^2+1}{z^2-3z+2}$ is continuous for all z outside $|z| = 2$.

UNIFORM CONTINUITY

114. Prove that $f(z) = 3z - 2$ is uniformly continuous in the region $|z| \leqq 10$.

115. Prove that $f(z) = 1/z^2$ (a) is not uniformly continuous in the region $|z| \leqq 1$ but (b) is uniformly continuous in the region $\frac{1}{2} \leqq |z| \leqq 1$.

116. Prove that if $f(z)$ is continuous in a closed region $\mathcal{R}$ it is uniformly continuous in $\mathcal{R}$.

SEQUENCES AND SERIES

117. Prove that (a) $\lim\limits_{n\to\infty} \dfrac{n^2 i^n}{n^3+1} = 0$, (b) $\lim\limits_{n\to\infty} \left(\dfrac{n}{n+3i} - \dfrac{in}{n+1}\right) = 1 - i$.

118. Prove that for any complex number z, $\lim\limits_{n\to\infty} (1 + 3z/n^2) = 1$.

119. Prove that $\lim\limits_{n\to\infty} n\left(\dfrac{1+i}{2}\right)^n = 0$.

120. Prove that $\lim\limits_{n\to\infty} ni^n$ does not exist.

121. If $\lim\limits_{n\to\infty} |u_n| = 0$, prove that $\lim\limits_{n\to\infty} u_n = 0$. Is the converse true? Justify your conclusion.

122. If $\lim\limits_{n\to\infty} a_n = A$ and $\lim\limits_{n\to\infty} b_n = B$, prove that (a) $\lim\limits_{n\to\infty} (a_n + b_n) = A + B$, (b) $\lim\limits_{n\to\infty} (a_n - b_n) = A - B$, (c) $\lim\limits_{n\to\infty} a_n b_n = AB$, (d) $\lim\limits_{n\to\infty} a_n/b_n = A/B$ if $B \neq 0$.

123. Use theorems on limits to evaluate each of the following.

(a) $\lim\limits_{n\to\infty} \dfrac{in^2 - in + 1 - 3i}{(2n+4i-3)(n-i)}$ (c) $\lim\limits_{n\to\infty} \sqrt{n+2i} - \sqrt{n+i}$

(b) $\lim\limits_{n\to\infty} \left| \dfrac{(n^2+3i)(n-i)}{in^3 - 3n + 4 - i} \right|$ (d) $\lim\limits_{n\to\infty} \sqrt{n}\{\sqrt{n+2i} - \sqrt{n+i}\}$

Ans. (a) $\frac{1}{2}i$, (b) 1, (c) 0, (d) $\frac{1}{2}i$

124. If $\lim\limits_{n\to\infty} u_n = l$, prove that $\lim\limits_{n\to\infty} \dfrac{u_1 + u_2 + \cdots + u_n}{n} = l$.

125. Prove that the series $1 + i/3 + (i/3)^2 + \cdots = \sum\limits_{n=1}^{\infty} (i/3)^{n-1}$ converges and find its sum.
Ans. $(9+3i)/10$

126. Prove that the series $i - 2i + 3i - 4i + \cdots$ diverges.

127. If the series $\sum\limits_{n=1}^{\infty} a_n$ converges to A, and $\sum\limits_{n=1}^{\infty} b_n$ converges to B, prove that $\sum\limits_{n=1}^{\infty} (a_n + ib_n)$ converges to $A + iB$. Is the converse true?

128. Investigate the convergence of $\sum\limits_{n=1}^{\infty} \dfrac{\omega^n}{5^{n/2}}$ where $\omega = \sqrt{3} + i$. *Ans.* conv.

MISCELLANEOUS PROBLEMS

129. Let $w = \{(4-z)(z^2+4)\}^{1/2}$. If $w=4$ when $z=0$, show that if z describes the curve C of Fig. 2-32, then the value of w at $z=6$ is $-4i\sqrt{5}$.

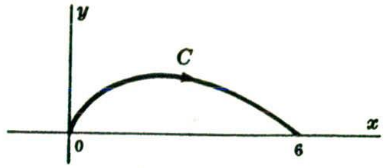

Fig. 2-32

130. Prove that a necessary and sufficient condition for $f(z) = u(x,y) + i\,v(x,y)$ to be continuous at $z = z_0 = x_0 + iy_0$ is that $u(x,y)$ and $v(x,y)$ be continuous at (x_0, y_0).

131. Prove that the equation $\tan z = z$ has only real roots.

132. A student remarked that 1 raised to any power is equal to 1. Was he correct? Explain.

133. Show that $\dfrac{\sin\theta}{2} + \dfrac{\sin 2\theta}{2^2} + \dfrac{\sin 3\theta}{2^3} + \cdots = \dfrac{2\sin\theta}{5 - 4\cos\theta}$.

134. Show that the relation $|f(x+iy)| = |f(x) + f(iy)|$ is satisfied by $f(z) = \sin z$. Can you find any other functions for which it is true?

135. Prove that $\displaystyle\lim_{z\to\infty} \frac{z^3 - 3z + 2}{z^4 + z^2 - 3z + 5} = 0$.

136. Prove that $|\csc z| \leqq 2e/(e^2 - 1)$ if $|y| \geqq 1$.

137. Show that $\text{Re}\,\{\sin^{-1} z\} = \frac{1}{2}\{\sqrt{x^2+y^2+2x+1} - \sqrt{x^2+y^2-2x+1}\}$.

138. If $f(z)$ is continuous in a bounded closed region $\mathcal{R}$, prove that (*a*) there exists a positive number M such that for all z in $\mathcal{R}$, $|f(z)| \leqq M$, (*b*) $|f(z)|$ has a *least upper bound* μ in $\mathcal{R}$ and there exists at least one value z_0 in $\mathcal{R}$ such that $|f(z_0)| = \mu$.

139. Show that $|\tanh \pi(1+i)/4| = 1$.

140. Prove that all the values of $(1-i)^{\sqrt{2}\,i}$ lie on a straight line.

141. Evaluate (*a*) $\cosh \pi i/2$, (*b*) $\tanh^{-1}\infty$. *Ans.* (*a*) 0, (*b*) $(2k+1)\pi i/2$, $k = 0, \pm1, \pm2, \ldots$.

142. If $\tan z = u + iv$, show that

$$u = \frac{\sin 2x}{\cos 2x + \cosh 2y}, \qquad v = \frac{\sinh 2y}{\cos 2x + \cosh 2y}$$

143. Evaluate to 3 decimal place accuracy: (*a*) e^{3-2i}, (*b*) $\sin(5-4i)$.

144. Prove $\text{Re}\left\{\dfrac{1 + i\tan(\theta/2)}{1 - i\tan(\theta/2)}\right\} = \cos\theta$, indicating any exceptional values.

145. If $\displaystyle\lim_{z\to z_0} f(z) = A$ and $\displaystyle\lim_{z\to z_0} g(z) = B \neq 0$, prove that $\displaystyle\lim_{z\to z_0} f(z)/g(z) = A/B$ without first proving that $\displaystyle\lim_{z\to z_0} 1/g(z) = 1/B$.

146. Let $f(z) = \begin{cases} 1 & \text{if } |z| \text{ is rational} \\ 0 & \text{if } |z| \text{ is irrational} \end{cases}$. Prove that $f(z)$ is discontinuous at all values of z.

147. Prove that if $f(z) = u(x,y) + i\,v(x,y)$ is continuous in a region, then (*a*) $\text{Re}\,\{f(z)\} = u(x,y)$ and (*b*) $\text{Im}\,\{f(z)\} = v(x,y)$ are continuous in the region.

148. Prove that all the roots of $z\tan z = k$, where $k > 0$, are real.

149. Prove that if the limit of a sequence exists it must be unique.

150. (*a*) Prove that $\displaystyle\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = 0$.

(*b*) Prove that the series $\displaystyle\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})$ diverges, thus showing that a series whose nth term approaches zero need not converge.

151. If $z_{n+1} = \frac{1}{2}(z_n + 1/z_n)$, $n = 0, 1, 2, \ldots$ and $-\pi/2 < \arg z_0 < \pi/2$, prove that $\displaystyle\lim_{n\to\infty} z_n = 1$.

Chapter 3

Complex Differentiation and The Cauchy-Riemann Equations

DERIVATIVES

If $f(z)$ is single-valued in some region $\mathcal{R}$ of the z plane, the *derivative* of $f(z)$ is defined as

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \tag{1}$$

provided that the limit exists independent of the manner in which $\Delta z \to 0$. In such case we say that $f(z)$ is *differentiable* at z. In the definition (*1*) we sometimes use h instead of Δz. Although differentiability implies continuity, the reverse is not true (see Problem 4).

ANALYTIC FUNCTIONS

If the derivative $f'(z)$ exists at all points z of a region $\mathcal{R}$, then $f(z)$ is said to be *analytic in* $\mathcal{R}$ and is referred to as an *analytic function in* $\mathcal{R}$ or a function *analytic in* $\mathcal{R}$. The terms *regular* and *holomorphic* are sometimes used as synonyms for analytic.

A function $f(z)$ is said to be *analytic at a point* z_0 if there exists a neighbourhood $|z - z_0| < \delta$ at all points of which $f'(z)$ exists.

CAUCHY-RIEMANN EQUATIONS

A necessary condition that $w = f(z) = u(x,y) + iv(x,y)$ be analytic in a region $\mathcal{R}$ is that, in $\mathcal{R}$, u and v satisfy the *Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{2}$$

If the partial derivatives in (*2*) are continuous in $\mathcal{R}$, then the Cauchy-Riemann equations are sufficient conditions that $f(z)$ be analytic in $\mathcal{R}$. See Problem 5.

The functions $u(x,y)$ and $v(x,y)$ are sometimes called *conjugate functions*. Given one we can find the other (within an arbitrary additive constant) so that $u + iv = f(z)$ is analytic (see Problems 7 and 8).

HARMONIC FUNCTIONS

If the second partial derivatives of u and v with respect to x and y exist and are continuous in a region $\mathcal{R}$, then we find from (*2*) that (see Problem 6)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \tag{3}$$

It follows that under these conditions the real and imaginary parts of an analytic function satisfy *Laplace's equation* denoted by

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0 \quad \text{or} \quad \nabla^2 \Psi = 0 \quad \text{where} \quad \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{4}$$

The operator ∇^2 is often called the *Laplacian*.

Functions such as $u(x,y)$ and $v(x,y)$ which satisfy Laplace's equation in a region $\mathcal{R}$ are called *harmonic functions* and are said to be *harmonic in* $\mathcal{R}$.

GEOMETRIC INTERPRETATION OF THE DERIVATIVE

Let z_0 [Fig. 3-1] be a point P in the z plane and let w_0 [Fig. 3-2] be its image P' in the w plane under the transformation $w = f(z)$. Since we suppose that $f(z)$ is single-valued, the point z_0 maps into only one point w_0.

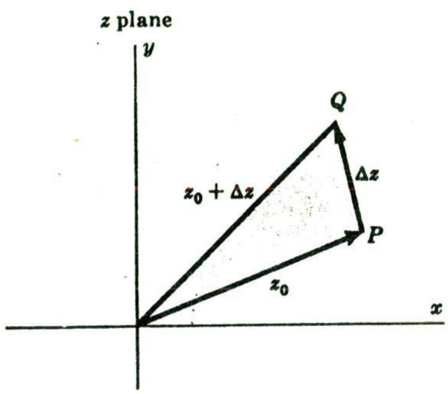

Fig. 3-1

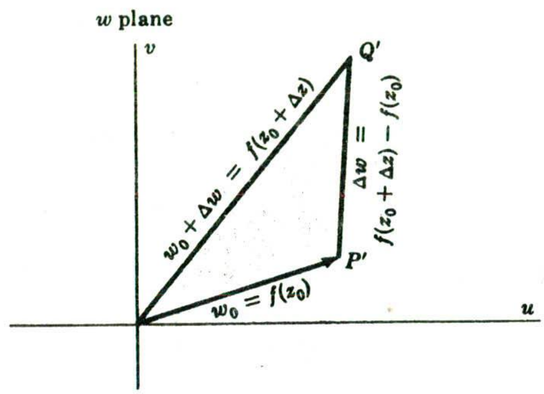

Fig. 3-2

If we give z_0 an increment Δz we obtain the point Q of Fig. 3-1. This point has image Q' in the w plane. Thus from Fig. 3-2 we see that $P'Q'$ represents the complex number $\Delta w = f(z_0 + \Delta z) - f(z_0)$. It follows that the derivative at z_0 (if it exists) is given by

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{Q \to P} \frac{Q'P'}{QP} \tag{5}$$

i.e. the limit of the ratio $Q'P'$ to QP as point Q approaches point P. The above interpretation clearly holds when z_0 is replaced by any point z.

DIFFERENTIALS

Let $\Delta z = dz$ be an increment given to z. Then

$$\Delta w = f(z + \Delta z) - f(z) \tag{6}$$

is called the increment in $w = f(z)$. If $f(z)$ is continuous and has a continuous first derivative in a region, then

$$\Delta w = f'(z)\,\Delta z + \epsilon\,\Delta z = f'(z)\,dz + \epsilon\,dz \tag{7}$$

where $\epsilon \to 0$ as $\Delta z \to 0$. The expression

$$dw = f'(z)\,dz \tag{8}$$

is called the *differential of w or $f(z)$*, or the *principal part* of Δw. Note that $\Delta w \neq dw$ in general. We call dz the *differential of z*.

Because of the definitions (*1*) and (*8*), we often write

$$\frac{dw}{dz} = f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \tag{9}$$

It is emphasized that dz and dw are not the limits of Δz and Δw as $\Delta z \to 0$, since these limits are zero whereas dz and dw are not necessarily zero. Instead, given dz we determine dw from (8), i.e. dw is a dependent variable determined from the independent variable dz for a given z.

It is useful to think of d/dz as being an *operator* which when operating on $w = f(z)$ leads to $dw/dz = f'(z)$.

RULES FOR DIFFERENTIATION

If $f(z)$, $g(z)$ and $h(z)$ are analytic functions of z, the following differentiation rules (identical with those of elementary calculus) are valid.

1. $$\frac{d}{dz}\{f(z)+g(z)\} = \frac{d}{dz}f(z) + \frac{d}{dz}g(z) = f'(z) + g'(z)$$

2. $$\frac{d}{dz}\{f(z)-g(z)\} = \frac{d}{dz}f(z) - \frac{d}{dz}g(z) = f'(z) - g'(z)$$

3. $$\frac{d}{dz}\{c\,f(z)\} = c\frac{d}{dz}f(z) = cf'(z) \qquad \text{where } c \text{ is any constant}$$

4. $$\frac{d}{dz}\{f(z)\,g(z)\} = f(z)\frac{d}{dz}g(z) + g(z)\frac{d}{dz}f(z) = f(z)\,g'(z) + g(z)\,f'(z)$$

5. $$\frac{d}{dz}\left\{\frac{f(z)}{g(z)}\right\} = \frac{g(z)\frac{d}{dz}f(z) - f(z)\frac{d}{dz}g(z)}{[g(z)]^2} = \frac{g(z)\,f'(z) - f(z)\,g'(z)}{[g(z)]^2} \qquad \text{if } g(z) \neq 0$$

6. If $w = f(\zeta)$ where $\zeta = g(z)$ then
$$\frac{dw}{dz} = \frac{dw}{d\zeta}\cdot\frac{d\zeta}{dz} = f'(\zeta)\frac{d\zeta}{dz} = f'\{g(z)\}\,g'(z) \tag{10}$$
Similarly, if $w = f(\zeta)$ where $\zeta = g(\eta)$ and $\eta = h(z)$, then
$$\frac{dw}{dz} = \frac{dw}{d\zeta}\cdot\frac{d\zeta}{d\eta}\cdot\frac{d\eta}{dz} \tag{11}$$
The results (10) and (11) are often called *chain rules* for differentiation of composite functions.

7. If $w = f(z)$, then $z = f^{-1}(w)$; and dw/dz and dz/dw are related by
$$\frac{dw}{dz} = \frac{1}{dz/dw} \tag{12}$$

8. If $z = f(t)$ and $w = g(t)$ where t is a parameter, then
$$\frac{dw}{dz} = \frac{dw/dt}{dz/dt} = \frac{g'(t)}{f'(t)} \tag{13}$$

Similar rules can be formulated for differentials. For example,
$$d\{f(z)+g(z)\} = df(z) + dg(z) = f'(z)\,dz + g'(z)\,dz = \{f'(z) + g'(z)\}\,dz$$
$$d\{f(z)\,g(z)\} = f(z)\,dg(z) + g(z)\,df(z) = \{f(z)\,g'(z) + g(z)\,f'(z)\}\,dz$$

DERIVATIVES OF ELEMENTARY FUNCTIONS

In the following we assume that the functions are defined as in Chapter 2. In the cases where functions have branches, i.e. are multi-valued, the branch of the function on the right is chosen so as to correspond to the branch of the function on the left. Note that the results are identical with those of elementary calculus.

1. $\frac{d}{dz}(c) = 0$
2. $\frac{d}{dz}z^n = nz^{n-1}$
3. $\frac{d}{dz}e^z = e^z$
4. $\frac{d}{dz}a^z = a^z \ln a$
5. $\frac{d}{dz}\sin z = \cos z$
6. $\frac{d}{dz}\cos z = -\sin z$
7. $\frac{d}{dz}\tan z = \sec^2 z$
8. $\frac{d}{dz}\cot z = -\csc^2 z$
9. $\frac{d}{dz}\sec z = \sec z \tan z$
10. $\frac{d}{dz}\csc z = -\csc z \cot z$
11. $\frac{d}{dz}\log_e z = \frac{d}{dz}\ln z = \frac{1}{z}$
12. $\frac{d}{dz}\log_a z = \frac{\log_a e}{z}$
13. $\frac{d}{dz}\sin^{-1} z = \frac{1}{\sqrt{1-z^2}}$
14. $\frac{d}{dz}\cos^{-1} z = \frac{-1}{\sqrt{1-z^2}}$
15. $\frac{d}{dz}\tan^{-1} z = \frac{1}{1+z^2}$
16. $\frac{d}{dz}\cot^{-1} z = \frac{-1}{1+z^2}$
17. $\frac{d}{dz}\sec^{-1} z = \frac{1}{z\sqrt{z^2-1}}$
18. $\frac{d}{dz}\csc^{-1} z = \frac{-1}{z\sqrt{z^2-1}}$
19. $\frac{d}{dz}\sinh z = \cosh z$
20. $\frac{d}{dz}\cosh z = \sinh z$
21. $\frac{d}{dz}\tanh z = \operatorname{sech}^2 z$
22. $\frac{d}{dz}\coth z = -\operatorname{csch}^2 z$
23. $\frac{d}{dz}\operatorname{sech} z = -\operatorname{sech} z \tanh z$
24. $\frac{d}{dz}\operatorname{csch} z = -\operatorname{csch} z \coth z$
25. $\frac{d}{dz}\sinh^{-1} z = \frac{1}{\sqrt{1+z^2}}$
26. $\frac{d}{dz}\cosh^{-1} z = \frac{1}{\sqrt{z^2-1}}$
27. $\frac{d}{dz}\tanh^{-1} z = \frac{1}{1-z^2}$
28. $\frac{d}{dz}\coth^{-1} z = \frac{1}{1-z^2}$
29. $\frac{d}{dz}\operatorname{sech}^{-1} z = \frac{-1}{z\sqrt{1-z^2}}$
30. $\frac{d}{dz}\operatorname{csch}^{-1} z = \frac{-1}{z\sqrt{z^2+1}}$

HIGHER ORDER DERIVATIVES

If $w = f(z)$ is analytic in a region, its derivative is given by $f'(z)$, w' or dw/dz. If $f'(z)$ is also analytic in the region, its derivative is denoted by $f''(z)$, w'' or $\left(\frac{d}{dz}\right)\left(\frac{dw}{dz}\right) = \frac{d^2w}{dz^2}$. Similarly the nth derivative of $f(z)$, if it exists, is denoted by $f^{(n)}(z)$, $w^{(n)}$ or $\frac{d^nw}{dz^n}$ where n is called the *order* of the derivative. Thus derivatives of first, second, third, ... orders are given by $f'(z), f''(z), f'''(z), \ldots$. Computations of these higher order derivatives follow by repeated application of the above differentiation rules.

One of the most remarkable theorems valid for functions of a complex variable and not necessarily valid for functions of a real variable is the following

Theorem. If $f(z)$ is analytic in a region $\mathcal{R}$, so also are $f'(z), f''(z), \ldots$ analytic in $\mathcal{R}$, i.e. all higher derivatives exist in $\mathcal{R}$.

This important theorem is proved in Chapter 5.

L'HOSPITAL'S RULE

Let $f(z)$ and $g(z)$ be analytic in a region containing the point z_0 and suppose that $f(z_0) = g(z_0) = 0$ but $g'(z_0) \neq 0$. Then *L'Hospital's rule* states that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)} \tag{14}$$

In case $f'(z_0) = g'(z_0) = 0$, the rule may be extended. See Problems 21-24.

We sometimes say that the left side of (*14*) has the "indeterminate form" 0/0, although such terminology is somewhat misleading since there is usually nothing indeterminate involved. Limits represented by so-called indeterminate forms ∞/∞, $0 \cdot \infty$, ∞^0, 0^0, 1^∞ and $\infty - \infty$ can often be evaluated by appropriate modifications of L'Hospital's rule.

SINGULAR POINTS

A point at which $f(z)$ fails to be analytic is called a *singular point* or *singularity* of $f(z)$. Various types of singularities exist.

1. **Isolated Singularities.** The point $z = z_0$ is called an *isolated singularity* or *isolated singular point* of $f(z)$ if we can find $\delta > 0$ such that the circle $|z - z_0| = \delta$ encloses no singular point other than z_0 (i.e. there exists a deleted δ neighbourhood of z_0 containing no singularity). If no such δ can be found, we call z_0 a *non-isolated singularity*.

   If z_0 is not a singular point and we can find $\delta > 0$ such that $|z - z_0| = \delta$ encloses no singular point, then we call z_0 an *ordinary point* of $f(z)$.

2. **Poles.** If we can find a positive integer n such that $\lim_{z \to z_0} (z - z_0)^n f(z) = A \neq 0$, then $z = z_0$ is called a *pole of order n*. If $n = 1$, z_0 is called a *simple pole*.

   Example 1: $f(z) = \dfrac{1}{(z-2)^3}$ has a pole of order 3 at $z = 2$.

   Example 2: $f(z) = \dfrac{3z - 2}{(z-1)^2(z+1)(z-4)}$ has a pole of order 2 at $z = 1$, and simple poles at $z = -1$ and $z = 4$.

   If $g(z) = (z - z_0)^n f(z)$, where $f(z_0) \neq 0$ and n is a positive integer, then $z = z_0$ is called a *zero of order n* of $g(z)$. If $n = 1$, z_0 is called a *simple zero*. In such case z_0 is a pole of order n of the function $1/g(z)$.

3. **Branch Points** of multiple-valued functions, already considered in Chapter 2, are singular points.

   Example 1: $f(z) = (z-3)^{1/2}$ has a branch point at $z = 3$.

   Example 2: $f(z) = \ln(z^2 + z - 2)$ has branch points where $z^2 + z - 2 = 0$, i.e. at $z = 1$ and $z = -2$.

4. **Removable Singularities.** The singular point z_0 is called a *removable singularity* of $f(z)$ if $\lim_{z \to z_0} f(z)$ exists.

   Example: The singular point $z = 0$ is a removable singularity of $f(z) = \dfrac{\sin z}{z}$ since $\lim_{z \to 0} \dfrac{\sin z}{z} = 1$.

5. **Essential Singularities.** A singularity which is not a pole, branch point or removable singularity is called an *essential singularity*.

   Example: $f(z) = e^{1/(z-2)}$ has an essential singularity at $z = 2$.

   If a function is single-valued and has a singularity, then the singularity is either a pole or an essential singularity. For this reason a pole is sometimes called a *non-essential singularity*. Equivalently, $z = z_0$ is an essential singularity if we cannot find any positive integer n such that $\lim_{z \to z_0} (z - z_0)^n f(z) = A \neq 0$.

6. **Singularities at Infinity.** The type of singularity of $f(z)$ at $z=\infty$ [the point at infinity; see Pages 6 and 38] is the same as that of $f(1/w)$ at $w=0$.

Example: The function $f(z)=z^3$ has a pole of order 3 at $z=\infty$, since $f(1/w)=1/w^3$ has a pole of order 3 at $w=0$.

For methods of classifying singularities using infinite series, see Chapter 6.

ORTHOGONAL FAMILIES

If $w = f(z) = u(x,y) + i\,v(x,y)$ is analytic, then the one-parameter families of curves

$$u(x,y) = \alpha, \qquad v(x,y) = \beta \tag{15}$$

where α and β are constants, are *orthogonal*, i.e. each member of one family [shown heavy in Fig. 3-3] is perpendicular to each member of the other family [shown dashed in Fig. 3-3] at the point of intersection. The corresponding image curves in the w plane consisting of lines parallel to the u and v axes also form orthogonal families [see Fig. 3-4].

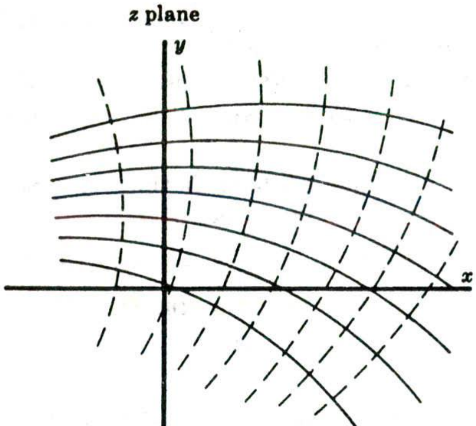

Fig. 3-3

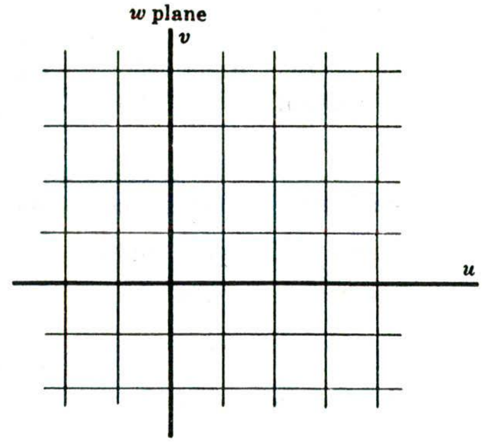

Fig. 3-4

In view of this, one might conjecture that when the mapping function $f(z)$ is analytic the angle between any two intersecting curves C_1 and C_2 in the z plane would equal (both in magnitude and sense) the angle between corresponding intersecting image curves C_1' and C_2' in the w plane. This conjecture is in fact correct and leads to the subject of *conformal mapping* which is of such great importance in both theory and application that two chapters (8 and 9) will be devoted to it.

CURVES

If $\phi(t)$ and $\psi(t)$ are real functions of the real variable t assumed continuous in $t_1 \leqq t \leqq t_2$, the parametric equations

$$z = x + iy = \phi(t) + i\psi(t) = z(t), \qquad t_1 \leqq t \leqq t_2 \tag{16}$$

define a *continuous curve* or *arc* in the z plane joining points $a = z(t_1)$ and $b = z(t_2)$ [see Fig. 3-5 below].

If $t_1 \neq t_2$ while $z(t_1) = z(t_2)$, i.e. $a = b$, the endpoints coincide and the curve is said to be *closed*. A closed curve which does not intersect itself anywhere is called a *simple closed curve*. For example the curve of Fig. 3-6 is a simple closed curve while that of Fig. 3-7 is not.

If $\phi(t)$ and $\psi(t)$ [and thus $z(t)$] have continuous derivatives in $t_1 \leqq t \leqq t_2$, the curve is often called a *smooth curve* or *arc*. A curve which is composed of a finite number of

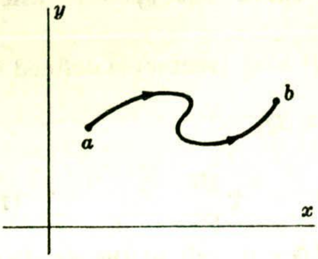

Fig. 3-5

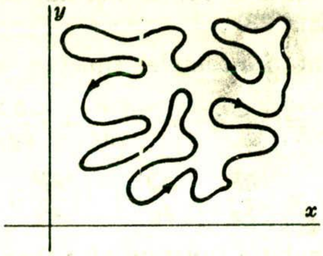

Fig. 3-6

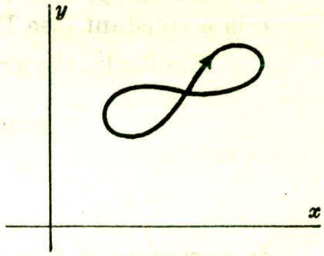

Fig. 3-7

smooth arcs is called a *piecewise* or *sectionally smooth* curve or sometimes a *contour*. For example, the boundary of a square is a piecewise smooth curve or contour.

Unless otherwise specified, whenever we refer to a curve or simple closed curve we shall assume it to be piecewise smooth.

APPLICATIONS TO GEOMETRY AND MECHANICS

We can consider $z(t)$ as a position vector whose terminal point describes a curve C in a definite *sense* or *direction* as t varies from t_1 to t_2. If $z(t)$ and $z(t+\Delta t)$ represent position vectors of points P and Q respectively, then

$$\frac{\Delta z}{\Delta t} = \frac{z(t+\Delta t) - z(t)}{\Delta t}$$

is a vector in the direction of Δz [Fig. 3-8]. If $\lim\limits_{\Delta t \to 0} \dfrac{\Delta z}{\Delta t} = \dfrac{dz}{dt}$ exists, the limit is a vector in the direction of the *tangent* to C at point P and is given by

$$\frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt}$$

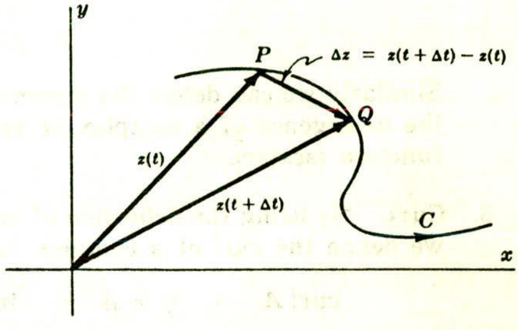

Fig. 3-8

If t is the time, dz/dt represents the *velocity* with which the terminal point describes the curve. Similarly, d^2z/dt^2 represents its *acceleration* along the curve.

COMPLEX DIFFERENTIAL OPERATORS

Let us define the operators ∇ (*del*) and $\bar{\nabla}$ (*del bar*) by

$$\nabla \equiv \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} = 2\frac{\partial}{\partial \bar{z}}, \qquad \bar{\nabla} \equiv \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} = 2\frac{\partial}{\partial z} \tag{17}$$

where the equivalence in terms of the conjugate coordinates z and $\bar{z}$ (Page 7) follows from Problem 32.

GRADIENT, DIVERGENCE, CURL AND LAPLACIAN

The operator ∇ enables us to define the following operations. In all cases we consider $F(x,y)$ as a real continuously differentiable function of x and y (scalar), while $A(x,y) = P(x,y) + iQ(x,y)$ is a complex continuously differentiable function of x and y (vector).

In terms of conjugate coordinates, $F(x,y) = F\left(\dfrac{z+\bar{z}}{2}, \dfrac{z-\bar{z}}{2i}\right) = G(z,\bar{z})$ and $A(x,y) = B(z,\bar{z})$.

1. **Gradient.** We define the *gradient* of a real function F (scalar) by

$$\text{grad } F = \nabla F = \frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y} = 2\frac{\partial G}{\partial \bar{z}} \tag{18}$$

Geometrically, this represents a vector normal to the curve $F(x,y) = c$ where c is a constant (see Problem 33).

Similarly, the gradient of a complex function $A = P + iQ$ (vector) is defined by

$$\text{grad } A = \nabla A = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(P + iQ)$$

$$= \frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} + i\left(\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) = 2\frac{\partial B}{\partial \bar{z}} \qquad (19)$$

In particular if B is an analytic function of z then $\partial B/\partial \bar{z} = 0$ and so the gradient is zero, i.e. $\dfrac{\partial P}{\partial x} = \dfrac{\partial Q}{\partial y}, \dfrac{\partial P}{\partial y} = -\dfrac{\partial Q}{\partial x}$, which shows that the Cauchy-Riemann equations are satisfied in this case.

2. **Divergence.** By using the definition of dot product of two complex numbers (Page 6) extended to the case of operators, we define the *divergence* of a complex function (vector) by

$$\text{div } A = \nabla \circ A = \text{Re}\{\bar{\nabla} A\} = \text{Re}\left\{\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(P + iQ)\right\}$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 2\,\text{Re}\left\{\frac{\partial B}{\partial z}\right\} \qquad (20)$$

Similarly we can define the divergence of a real function. It should be noted that the divergence of a complex or real function (vector or scalar) is always a real function (scalar).

3. **Curl.** By using the definition of cross product of two complex numbers (Page 6), we define the *curl* of a complex function by

$$\text{curl } A = \nabla \times A = \text{Im}\{\bar{\nabla} A\} = \text{Im}\left\{\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(P + iQ)\right\}$$

$$= \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2\,\text{Im}\left\{\frac{\partial B}{\partial z}\right\} \qquad (21)$$

Similarly we can define the curl of a real function.

4. **Laplacian.** The *Laplacian operator* is defined as the dot or scalar product of ∇ with itself, i.e.,

$$\nabla \circ \nabla \equiv \nabla^2 \equiv \text{Re}\{\bar{\nabla}\nabla\} = \text{Re}\left\{\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right\}$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\,\partial \bar{z}} \qquad (22)$$

Note that if A is analytic, $\nabla^2 A = 0$ so that $\nabla^2 P = 0$ and $\nabla^2 Q = 0$, i.e. P and Q are harmonic.

SOME IDENTITIES INVOLVING GRADIENT, DIVERGENCE AND CURL

The following identities hold if A_1, A_2 and A are differentiable functions.

1. $\text{grad}\,(A_1 + A_2) = \text{grad}\,A_1 + \text{grad}\,A_2$
2. $\text{div}\,(A_1 + A_2) = \text{div}\,A_1 + \text{div}\,A_2$
3. $\text{curl}\,(A_1 + A_2) = \text{curl}\,A_1 + \text{curl}\,A_2$
4. $\text{grad}\,(A_1 A_2) = (A_1)(\text{grad}\,A_2) + (\text{grad}\,A_1)(A_2)$
5. $\text{curl grad}\,A = 0$ if A is real or, more generally, if $\text{Im}\{A\}$ is harmonic.
6. $\text{div grad}\,A = 0$ if A is imaginary or, more generally, if $\text{Re}\{A\}$ is harmonic.

Solved Problems

DERIVATIVES

1. Using the definition, find the derivative of $w = f(z) = z^3 - 2z$ at the point where (a) $z = z_0$, (b) $z = -1$.

(a) By definition, the derivative at $z = z_0$ is

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)^3 - 2(z_0 + \Delta z) - \{z_0^3 - 2z_0\}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z_0^3 + 3z_0^2\,\Delta z + 3z_0(\Delta z)^2 + (\Delta z)^3 - 2z_0 - 2\Delta z - z_0^3 + 2z_0}{\Delta z}$$

$$= \lim_{\Delta z \to 0} 3z_0^2 + 3z_0\,\Delta z + (\Delta z)^2 - 2 = 3z_0^2 - 2$$

In general, $f'(z) = 3z^2 - 2$ for all z.

(b) From (a), or directly, we find that if $z_0 = -1$ then $f'(-1) = 3(-1)^2 - 2 = 1$.

2. Show that $\frac{d}{dz}\bar{z}$ does not exist anywhere, i.e. $f(z) = \bar{z}$ is non-analytic anywhere.

By definition, $$\frac{d}{dz}f(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

if this limit exists independent of the manner in which $\Delta z = \Delta x + i\,\Delta y$ approaches zero.

Then $$\frac{d}{dz}\bar{z} = \lim_{\Delta z \to 0} \frac{\overline{z + \Delta z} - \bar{z}}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\overline{x + iy + \Delta x + i\,\Delta y} - \overline{x + iy}}{\Delta x + i\,\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{x - iy + \Delta x - i\,\Delta y - (x - iy)}{\Delta x + i\,\Delta y} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\,\Delta y}{\Delta x + i\,\Delta y}$$

If $\Delta y = 0$, the required limit is $\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$.

If $\Delta x = 0$, the required limit is $\lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1$.

Then since the limit depends on the manner in which $\Delta z \to 0$, the derivative does not exist, i.e. $f(z) = \bar{z}$ is *non-analytic* anywhere.

3. If $w = f(z) = \frac{1+z}{1-z}$, find (a) $\frac{dw}{dz}$ and (b) determine where $f(z)$ is non-analytic.

(a) ***Method 1***, using the definition.

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\dfrac{1 + (z + \Delta z)}{1 - (z + \Delta z)} - \dfrac{1+z}{1-z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{2}{(1 - z - \Delta z)(1 - z)} = \frac{2}{(1-z)^2}$$

independent of the manner in which $\Delta z \to 0$, provided $z \neq 1$.

Method 2, using differentiation rules.

By the quotient rule [see Problem 10(c)] we have if $z \neq 1$,

$$\frac{d}{dz}\left(\frac{1+z}{1-z}\right) = \frac{(1-z)\frac{d}{dz}(1+z) - (1+z)\frac{d}{dz}(1-z)}{(1-z)^2} = \frac{(1-z)(1) - (1+z)(-1)}{(1-z)^2} = \frac{2}{(1-z)^2}$$

(b) The function $f(z)$ is analytic for all finite values of z except $z = 1$ where the derivative does not exist and the function is non-analytic. The point $z = 1$ is a *singular point* of $f(z)$.

4. (a) If $f(z)$ is analytic at z_0, prove that it must be continuous at z_0.

(b) Give an example to show that the converse of (a) is not necessarily true.

(a) Since $f(z_0+h) - f(z_0) = \dfrac{f(z_0+h)-f(z_0)}{h} \cdot h$ where $h = \Delta z \neq 0$, we have

$$\lim_{h\to 0} f(z_0+h) - f(z_0) = \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} \cdot \lim_{h\to 0} h = f'(z_0)\cdot 0 = 0$$

because $f'(z_0)$ exists by hypothesis. Thus

$$\lim_{h\to 0} f(z_0+h) - f(z_0) = 0 \qquad \text{or} \qquad \lim_{h\to 0} f(z_0+h) = f(z_0)$$

showing that $f(z)$ is continuous at z_0.

(b) The function $f(z) = \bar{z}$ is continuous at z_0. However, by Problem 2, $f(z)$ is not analytic anywhere. This shows that a function which is continuous need not have a derivative, i.e. need not be analytic.

CAUCHY-RIEMANN EQUATIONS

5. Prove that a (a) necessary and (b) sufficient condition that $w = f(z) = u(x,y) + i\,v(x,y)$ be analytic in a region $\mathcal{R}$ is that the Cauchy-Riemann equations $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y},\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ are satisfied in $\mathcal{R}$ where it is supposed that these partial derivatives are continuous in $\mathcal{R}$.

(a) *Necessity.* In order for $f(z)$ to be analytic, the limit

$$\lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$
$$= f'(z) = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\{u(x+\Delta x, y+\Delta y) + i\,v(x+\Delta x, y+\Delta y)\} - \{u(x,y) + i\,v(x,y)\}}{\Delta x + i\,\Delta y} \tag{1}$$

must exist independent of the manner in which Δz (or Δx and Δy) approaches zero. We consider two possible approaches.

Case 1. $\Delta y = 0$, $\Delta x \to 0$. In this case (*1*) becomes

$$\lim_{\Delta x\to 0}\left\{\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i\left[\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}\right]\right\} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

provided the partial derivatives exist.

Case 2. $\Delta x = 0$, $\Delta y \to 0$. In this case (*1*) becomes

$$\lim_{\Delta y\to 0}\left\{\frac{u(x, y+\Delta y) - u(x,y)}{i\,\Delta y} + \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}\right\} = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Now $f(z)$ cannot possibly be analytic unless these two limits are identical. Thus a necessary condition that $f(z)$ be analytic is

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad \text{or} \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\ \ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

(b) *Sufficiency.* Since $\partial u/\partial x$ and $\partial u/\partial y$ are supposed continuous, we have

$$\begin{aligned}\Delta u &= u(x+\Delta x, y+\Delta y) - u(x,y)\\ &= \{u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y)\} + \{u(x, y+\Delta y) - u(x,y)\}\\ &= \left(\frac{\partial u}{\partial x} + \epsilon_1\right)\Delta x + \left(\frac{\partial u}{\partial y} + \eta_1\right)\Delta y = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \epsilon_1\,\Delta x + \eta_1\,\Delta y\end{aligned}$$

where $\epsilon_1 \to 0$ and $\eta_1 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$.

Similarly, since $\partial v/\partial x$ and $\partial v/\partial y$ are supposed continuous, we have

$$\Delta v = \left(\frac{\partial v}{\partial x} + \epsilon_2\right)\Delta x + \left(\frac{\partial v}{\partial y} + \eta_2\right)\Delta y = \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \epsilon_2\,\Delta x + \eta_2\,\Delta y$$

where $\epsilon_2 \to 0$ and $\eta_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$. Then

$$\Delta w = \Delta u + i\,\Delta v = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y + \epsilon\,\Delta x + \eta\,\Delta y \tag{2}$$

where $\epsilon = \epsilon_1 + i\epsilon_2 \to 0$ and $\eta = \eta_1 + i\eta_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$.

By the Cauchy-Riemann equations, (*2*) can be written

$$\Delta w = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\Delta y + \epsilon\,\Delta x + \eta\,\Delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\Delta x + i\,\Delta y) + \epsilon\,\Delta x + \eta\,\Delta y$$

Then on dividing by $\Delta z = \Delta x + i\,\Delta y$ and taking the limit as $\Delta z \to 0$, we see that

$$\frac{dw}{dz} = f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

so that the derivative exists and is unique, i.e. $f(z)$ is analytic in $\mathcal{R}$.

6. If $f(z) = u + iv$ is analytic in a region $\mathcal{R}$, prove that u and v are harmonic in $\mathcal{R}$ if they have continuous second partial derivatives in $\mathcal{R}$.

If $f(z)$ is analytic in $\mathcal{R}$ then the Cauchy-Riemann equations (1) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and (2) $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$ are satisfied in $\mathcal{R}$. Assuming u and v have continuous second partial derivatives, we can differentiate both sides of (1) with respect to x and (2) with respect to y to obtain (3) $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 v}{\partial x\,\partial y}$ and (4) $\dfrac{\partial^2 v}{\partial y\,\partial x} = -\dfrac{\partial^2 u}{\partial y^2}$ from which $\dfrac{\partial^2 u}{\partial x^2} = -\dfrac{\partial^2 u}{\partial y^2}$ or $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, i.e. u is harmonic.

Similarly, by differentiating both sides of (1) with respect to y and (2) with respect to x, we find $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$ and v is harmonic.

It will be shown later (Chapter 5) that if $f(z)$ is analytic in $\mathcal{R}$, all its derivatives exist and are continuous in $\mathcal{R}$. Hence the above assumptions will not be necessary.

7. (a) Prove that $u = e^{-x}(x \sin y - y \cos y)$ is harmonic.
(b) Find v such that $f(z) = u + iv$ is analytic.

(a) $$\frac{\partial u}{\partial x} = (e^{-x})(\sin y) + (-e^{-x})(x \sin y - y \cos y) = e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y) = -2e^{-x}\sin y + xe^{-x}\sin y - ye^{-x}\cos y \qquad (1)$$

$$\frac{\partial u}{\partial y} = e^{-x}(x\cos y + y\sin y - \cos y) = xe^{-x}\cos y + ye^{-x}\sin y - e^{-x}\cos y$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(xe^{-x}\cos y + ye^{-x}\sin y - e^{-x}\cos y) = -xe^{-x}\sin y + 2e^{-x}\sin y + ye^{-x}\cos y \qquad (2)$$

Adding (1) and (2) yields $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ and u is harmonic.

(b) From the Cauchy-Riemann equations,

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y \qquad (3)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^{-x}\cos y - xe^{-x}\cos y - ye^{-x}\sin y \qquad (4)$$

Integrate (3) with respect to y, keeping x constant. Then

$$v = -e^{-x}\cos y + xe^{-x}\cos y + e^{-x}(y\sin y + \cos y) + F(x)$$
$$= ye^{-x}\sin y + xe^{-x}\cos y + F(x) \qquad (5)$$

where $F(x)$ is an arbitrary real function of x.

Substitute (5) into (4) and obtain

$$-ye^{-x}\sin y - xe^{-x}\cos y + e^{-x}\cos y + F'(x) = -ye^{-x}\sin y - xe^{-x}\cos y - ye^{-x}\sin y$$

or $F'(x) = 0$ and $F(x) = c$, a constant. Then from (5),

$$v = e^{-x}(y\sin y + x\cos y) + c$$

For another method, see Problem 40.

8. Find $f(z)$ in Problem 7.

Method 1.

We have $f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$.

Putting $y = 0$, $f(x) = u(x,0) + i\,v(x,0)$.

Replacing x by z, $f(z) = u(z,0) + i\,v(z,0)$.

Then from Problem 7, $u(z,0) = 0$, $v(z,0) = ze^{-z}$ and so $f(z) = u(z,0) + i\,v(z,0) = i\,ze^{-z}$, apart from an arbitrary additive constant.

Method 2.

Apart from an arbitrary additive constant, we have from the results of Problem 7,

$$\begin{aligned} f(z) &= u + iv = e^{-x}(x\sin y - y\cos y) + ie^{-x}(y\sin y + x\cos y) \\ &= e^{-x}\left\{x\left(\frac{e^{iy}-e^{-iy}}{2i}\right) - y\left(\frac{e^{iy}+e^{-iy}}{2}\right)\right\} + ie^{-x}\left\{y\left(\frac{e^{iy}-e^{-iy}}{2i}\right) + x\left(\frac{e^{iy}+e^{-iy}}{2}\right)\right\} \\ &= i(x+iy)\,e^{-(x+iy)} = i\,ze^{-z} \end{aligned}$$

Method 3.

We have $x = \dfrac{z+\bar{z}}{2}$, $y = \dfrac{z-\bar{z}}{2i}$. Then substituting into $u(x,y) + i\,v(x,y)$, we find after much tedious labour that $\bar{z}$ disappears and we are left with the result $i\,ze^{-z}$.

In general method 1 is preferable over methods 2 and 3 when both u and v are known. If only u (or v) is known another procedure is given in Problem 101.

DIFFERENTIALS

9. If $w = f(z) = z^3 - 2z^2$, find (a) Δw, (b) dw, (c) $\Delta w - dw$.

(a)
$$\begin{aligned} \Delta w = f(z+\Delta z) - f(z) &= \{(z+\Delta z)^3 - 2(z+\Delta z)^2\} - \{z^3 - 2z^2\} \\ &= z^3 + 3z^2\Delta z + 3z(\Delta z)^2 + (\Delta z)^3 - 2z^2 - 4z\,\Delta z - 2(\Delta z)^2 - z^3 + 2z^2 \\ &= (3z^2 - 4z)\,\Delta z + (3z-2)(\Delta z)^2 + (\Delta z)^3 \end{aligned}$$

(b) dw = principal part of $\Delta w = (3z^2 - 4z)\Delta z = (3z^2 - 4z)dz$, since by definition $\Delta z = dz$.

Note that $f'(z) = 3z^2 - 4z$ and $dw = (3z^2 - 4z)dz$, i.e. $dw/dz = 3z^2 - 4z$.

(c) From (a) and (b), $\Delta w - dw = (3z-2)(\Delta z)^2 + (\Delta z)^3 = \epsilon\,\Delta z$ where $\epsilon = (3z-2)\Delta z + (\Delta z)^2$.

Note that $\epsilon \to 0$ as $\Delta z \to 0$, i.e. $\dfrac{\Delta w - dw}{\Delta z} \to 0$ as $\Delta z \to 0$. It follows that $\Delta w - dw$ is an infinitesimal of higher order than Δz.

DIFFERENTIATION RULES. DERIVATIVES OF ELEMENTARY FUNCTIONS

10. Prove the following assuming that $f(z)$ and $g(z)$ are analytic in a region $\mathcal{R}$.

(a) $$\frac{d}{dz}\{f(z) + g(z)\} = \frac{d}{dz}f(z) + \frac{d}{dz}g(z)$$

(b) $$\frac{d}{dz}\{f(z)\,g(z)\} = f(z)\frac{d}{dz}g(z) + g(z)\frac{d}{dz}f(z)$$

(c) $$\frac{d}{dz}\left\{\frac{f(z)}{g(z)}\right\} = \frac{g(z)\dfrac{d}{dz}f(z) - f(z)\dfrac{d}{dz}g(z)}{[g(z)]^2} \quad \text{if } g(z) \neq 0$$

(a)
$$\begin{aligned} \frac{d}{dz}\{f(z)+g(z)\} &= \lim_{\Delta z \to 0} \frac{f(z+\Delta z) + g(z+\Delta z) - \{f(z)+g(z)\}}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} + \lim_{\Delta z \to 0} \frac{g(z+\Delta z) - g(z)}{\Delta z} = \frac{d}{dz}f(z) + \frac{d}{dz}g(z) \end{aligned}$$

(b)
$$\begin{aligned}\frac{d}{dz}\{f(z)\,g(z)\} &= \lim_{\Delta z \to 0} \frac{f(z+\Delta z)\,g(z+\Delta z) - f(z)\,g(z)}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{f(z+\Delta z)\{g(z+\Delta z) - g(z)\} + g(z)\{f(z+\Delta z) - f(z)\}}{\Delta z} \\ &= \lim_{\Delta z \to 0} f(z+\Delta z)\left\{\frac{g(z+\Delta z) - g(z)}{\Delta z}\right\} + \lim_{\Delta z \to 0} g(z)\left\{\frac{f(z+\Delta z) - f(z)}{\Delta z}\right\} \\ &= f(z)\frac{d}{dz}g(z) + g(z)\frac{d}{dz}f(z)\end{aligned}$$

Note that we have used the fact that $\lim_{\Delta z \to 0} f(z+\Delta z) = f(z)$ which follows since $f(z)$ is analytic and thus continuous (see Problem 4).

Another method.

Let $U = f(z)$, $V = g(z)$. Then $\Delta U = f(z+\Delta z) - f(z)$ and $\Delta V = g(z+\Delta z) - g(z)$, i.e. $f(z+\Delta z) = U + \Delta U$, $g(z+\Delta z) = V + \Delta V$. Thus

$$\begin{aligned}\frac{d}{dz}UV &= \lim_{\Delta z \to 0} \frac{(U+\Delta U)(V+\Delta V) - UV}{\Delta z} = \lim_{\Delta z \to 0} \frac{U\,\Delta V + V\,\Delta U + \Delta U\,\Delta V}{\Delta z} \\ &= \lim_{\Delta z \to 0}\left(U\frac{\Delta V}{\Delta z} + V\frac{\Delta U}{\Delta z} + \frac{\Delta U}{\Delta z}\Delta V\right) = U\frac{dV}{dz} + V\frac{dU}{dz}\end{aligned}$$

where it is noted that $\Delta V \to 0$ as $\Delta z \to 0$, since V is supposed analytic and thus continuous.

A similar procedure can be used to prove (a).

(c) We use the second method in (b). Then

$$\begin{aligned}\frac{d}{dz}\left(\frac{U}{V}\right) &= \lim_{\Delta z \to 0} \frac{1}{\Delta z}\left\{\frac{U+\Delta U}{V+\Delta V} - \frac{U}{V}\right\} = \lim_{\Delta z \to 0} \frac{V\,\Delta U - U\,\Delta V}{\Delta z(V+\Delta V)V} \\ &= \lim_{\Delta z \to 0} \frac{1}{(V+\Delta V)V}\left\{V\frac{\Delta U}{\Delta z} - U\frac{\Delta V}{\Delta z}\right\} = \frac{V(dU/dz) - U(dV/dz)}{V^2}\end{aligned}$$

The first method of (b) can also be used.

11. Prove that (a) $\dfrac{d}{dz}e^z = e^z$, (b) $\dfrac{d}{dz}e^{az} = ae^{az}$ where a is any constant.

(a) By definition, $w = e^z = e^{x+iy} = e^x(\cos y + i \sin y) = u + iv$ or $u = e^x \cos y$, $v = e^x \sin y$.

Since $\dfrac{\partial u}{\partial x} = e^x \cos y = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial v}{\partial x} = e^x \sin y = -\dfrac{\partial u}{\partial y}$, the Cauchy-Riemann equations are satisfied. Then by Problem 5 the required derivative exists and is equal to

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = e^x \cos y + ie^x \sin y = e^z$$

(b) Let $w = e^\zeta$ where $\zeta = az$. Then by part (a) and Problem 39,

$$\frac{d}{dz}e^{az} = \frac{d}{dz}e^\zeta = \frac{d}{d\zeta}e^\zeta \cdot \frac{d\zeta}{dz} = e^\zeta \cdot a = ae^{az}$$

We can also proceed as in part (a).

12. Prove that (a) $\dfrac{d}{dz}\sin z = \cos z$, (b) $\dfrac{d}{dz}\cos z = -\sin z$, (c) $\dfrac{d}{dz}\tan z = \sec^2 z$.

(a) We have $w = \sin z = \sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$. Then

$$u = \sin x \cosh y, \qquad v = \cos x \sinh y$$

Now $\dfrac{\partial u}{\partial x} = \cos x \cosh y = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial v}{\partial x} = -\sin x \sinh y = -\dfrac{\partial u}{\partial y}$ so that the Cauchy-Riemann equations are satisfied. Hence by Problem 5 the required derivative is equal to

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \cos x \cosh y - i \sin x \sinh y = \cos(x+iy) = \cos z$$

Another method.

Since $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$, we have, using Problem 11(*b*),

$$\frac{d}{dz}\sin z = \frac{d}{dz}\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = \frac{1}{2i}\frac{d}{dz}e^{iz} - \frac{1}{2i}\frac{d}{dz}e^{-iz} = \frac{1}{2}e^{iz} + \frac{1}{2}e^{-iz} = \cos z$$

(*b*)
$$\frac{d}{dz}\cos z = \frac{d}{dz}\left(\frac{e^{iz} + e^{-iz}}{2}\right) = \frac{1}{2}\frac{d}{dz}e^{iz} + \frac{1}{2}\frac{d}{dz}e^{-iz}$$
$$= \frac{i}{2}e^{iz} - \frac{i}{2}e^{-iz} = -\frac{e^{iz} - e^{-iz}}{2i} = -\sin z$$

The first method of part (*a*) can also be used.

(*c*) By the quotient rule of Problem 10(*c*) we have

$$\frac{d}{dz}\tan z = \frac{d}{dz}\left(\frac{\sin z}{\cos z}\right) = \frac{\cos z \frac{d}{dz}\sin z - \sin z \frac{d}{dz}\cos z}{\cos^2 z}$$
$$= \frac{(\cos z)(\cos z) - (\sin z)(-\sin z)}{\cos^2 z} = \frac{\cos^2 z + \sin^2 z}{\cos^2 z} = \frac{1}{\cos^2 z} = \sec^2 z$$

13. Prove that $\dfrac{d}{dz}z^{1/2} = \dfrac{1}{2z^{1/2}}$, realizing that $z^{1/2}$ is a multiple-valued function.

A function must be single-valued in order to have a derivative. Thus since $z^{1/2}$ is multiple-valued (in this case two-valued) we must restrict ourselves to one branch of this function at a time.

Case 1.

Let us first consider that branch of $w = z^{1/2}$ for which $w = 1$ where $z = 1$. In this case, $w^2 = z$ so that

$$\frac{dz}{dw} = 2w \quad \text{and so} \quad \frac{dw}{dz} = \frac{1}{2w} \quad \text{or} \quad \frac{d}{dz}z^{1/2} = \frac{1}{2z^{1/2}}$$

Case 2.

Next we consider that branch of $w = z^{1/2}$ for which $w = -1$ where $z = 1$. In this case too, we have $w^2 = z$ so that

$$\frac{dz}{dw} = 2w \quad \text{and} \quad \frac{dw}{dz} = \frac{1}{2w} \quad \text{or} \quad \frac{d}{dz}z^{1/2} = \frac{1}{2z^{1/2}}$$

In both cases we have $\dfrac{d}{dz}z^{1/2} = \dfrac{1}{2z^{1/2}}$. Note that the derivative does not exist at the branch point $z = 0$. In general a function does not have a derivative, i.e. is not analytic, at a branch point. Thus branch points are singular points.

14. Prove that $\dfrac{d}{dz}\ln z = \dfrac{1}{z}$.

Let $w = \ln z$. Then $z = e^w$ and $dz/dw = e^w = z$. Hence

$$\frac{d}{dz}\ln z = \frac{dw}{dz} = \frac{1}{dz/dw} = \frac{1}{z}$$

Note that the result is valid regardless of the particular branch of $\ln z$. Also observe that the derivative does not exist at the branch point $z = 0$, illustrating further the remark at the end of Problem 13.

15. Prove that $\dfrac{d}{dz}\ln f(z) = \dfrac{f'(z)}{f(z)}$.

Let $w = \ln \zeta$ where $\zeta = f(z)$. Then

$$\frac{dw}{dz} = \frac{dw}{d\zeta}\cdot\frac{d\zeta}{dz} = \frac{1}{\zeta}\cdot\frac{d\zeta}{dz} = \frac{f'(z)}{f(z)}$$

16. Prove that (*a*) $\dfrac{d}{dz}\sin^{-1} z = \dfrac{1}{\sqrt{1 - z^2}}$, (*b*) $\dfrac{d}{dz}\tanh^{-1} z = \dfrac{1}{1 - z^2}$.

(*a*) If we consider the principal branch of $\sin^{-1} z$, we have by Problem 22 of Chapter 2 and by Problem 15,

$$\frac{d}{dz}\sin^{-1} z = \frac{d}{dz}\left\{\frac{1}{i}\ln(iz + \sqrt{1-z^2})\right\}$$

$$= \frac{1}{i}\frac{d}{dz}(iz + \sqrt{1-z^2})\Big/(iz + \sqrt{1-z^2})$$

$$= \frac{1}{i}\{i + \tfrac{1}{2}(1-z^2)^{-1/2}(-2z)\}\Big/(iz + \sqrt{1-z^2})$$

$$= \left(1 + \frac{iz}{\sqrt{1-z^2}}\right)\Big/(iz + \sqrt{1-z^2}) = \frac{1}{\sqrt{1-z^2}}$$

The result is also true if we consider other branches.

(b) We have, on considering the principal branch,

$$\tanh^{-1} z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) = \frac{1}{2}\ln(1+z) - \frac{1}{2}\ln(1-z)$$

Then

$$\frac{d}{dz}\tanh^{-1} z = \frac{1}{2}\frac{d}{dz}\ln(1+z) - \frac{1}{2}\frac{d}{dz}\ln(1-z) = \frac{1}{2}\left(\frac{1}{1+z}\right) + \frac{1}{2}\left(\frac{1}{1-z}\right) = \frac{1}{1-z^2}$$

Note that in both parts (a) and (b) the derivatives do not exist at the branch points $z = \pm 1$.

17. Using rules of differentiation, find the derivatives of each of the following:

(a) $\cos^2(2z+3i)$, (b) $z\tan^{-1}(\ln z)$, (c) $\{\tanh^{-1}(iz+2)\}^{-1}$, (d) $(z-3i)^{4z+2}$.

(a) Let $\eta = 2z + 3i$, $\zeta = \cos\eta$, $w = \zeta^2$ from which $w = \cos^2(2z+3i)$. Then using the chain rule, we have

$$\frac{dw}{dz} = \frac{dw}{d\zeta}\cdot\frac{d\zeta}{d\eta}\cdot\frac{d\eta}{dz} = (2\zeta)(-\sin\eta)(2)$$

$$= (2\cos\eta)(-\sin\eta)(2) = -4\cos(2z+3i)\sin(2z+3i)$$

Another method.

$$\frac{d}{dz}\{\cos(2z+3i)\}^2 = 2\{\cos(2z+3i)\}\left\{\frac{d}{dz}\cos(2z+3i)\right\}$$

$$= 2\{\cos(2z+3i)\}\{-\sin(2z+3i)\}\left\{\frac{d}{dz}(2z+3i)\right\}$$

$$= -4\cos(2z+3i)\sin(2z+3i)$$

(b)

$$\frac{d}{dz}\{(z)[\tan^{-1}(\ln z)]\} = z\frac{d}{dz}[\tan^{-1}(\ln z)] + [\tan^{-1}(\ln z)]\frac{d}{dz}(z)$$

$$= z\left\{\frac{1}{1+(\ln z)^2}\right\}\frac{d}{dz}(\ln z) + \tan^{-1}(\ln z)$$

$$= \frac{1}{1+(\ln z)^2} + \tan^{-1}(\ln z)$$

(c)

$$\frac{d}{dz}\{\tanh^{-1}(iz+2)\}^{-1} = -1\{\tanh^{-1}(iz+2)\}^{-2}\frac{d}{dz}\{\tanh^{-1}(iz+2)\}$$

$$= -\{\tanh^{-1}(iz+2)\}^{-2}\left\{\frac{1}{1-(iz+2)^2}\right\}\frac{d}{dz}(iz+2)$$

$$= \frac{-i\{\tanh^{-1}(iz+2)\}^{-2}}{1-(iz+2)^2}$$

(d)

$$\frac{d}{dz}\{(z-3i)^{4z+2}\} = \frac{d}{dz}\{e^{(4z+2)\ln(z-3i)}\} = e^{(4z+2)\ln(z-3i)}\frac{d}{dz}\{(4z+2)\ln(z-3i)\}$$

$$= e^{(4z+2)\ln(z-3i)}\left\{(4z+2)\frac{d}{dz}[\ln(z-3i)] + \ln(z-3i)\frac{d}{dz}(4z+2)\right\}$$

$$= e^{(4z+2)\ln(z-3i)}\left\{\frac{4z+2}{z-3i} + 4\ln(z-3i)\right\}$$

$$= (z-3i)^{4z+1}(4z+2) + 4(z-3i)^{4z+2}\ln(z-3i)$$

18. If $w^3 - 3z^2w + 4\ln z = 0$, find dw/dz.

Differentiating with respect to z, considering w as an implicit function of z, we have

$$\frac{d}{dz}(w^3) - 3\frac{d}{dz}(z^2w) + 4\frac{d}{dz}(\ln z) = 0 \quad \text{or} \quad 3w^2\frac{dw}{dz} - 3z^2\frac{dw}{dz} - 6zw + \frac{4}{z} = 0$$

Then solving for dw/dz, we obtain $\dfrac{dw}{dz} = \dfrac{6zw - 4/z}{3w^2 - 3z^2}$.

19. If $w = \sin^{-1}(t-3)$ and $z = \cos(\ln t)$, find dw/dz.

$$\frac{dw}{dz} = \frac{dw/dt}{dz/dt} = \frac{1/\sqrt{1-(t-3)^2}}{-\sin(\ln t)[1/t]} = -\frac{t}{\sin(\ln t)\sqrt{1-(t-3)^2}}$$

20. In Problem 18, find d^2w/dz^2.

$$\frac{d^2w}{dz^2} = \frac{d}{dz}\left(\frac{dw}{dz}\right) = \frac{d}{dz}\left(\frac{6zw - 4/z}{3w^2 - 3z^2}\right)$$

$$= \frac{(3w^2 - 3z^2)(6z\,dw/dz + 6w + 4/z^2) - (6zw - 4/z)(6w\,dw/dz - 6z)}{(3w^2 - 3z^2)^2}$$

The required result follows on substituting the value of dw/dz from Problem 18 and simplifying.

L'HOSPITAL'S RULE

21. Prove that if $f(z)$ is analytic in a region $\mathcal{R}$ including the point z_0, then

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \eta(z - z_0)$$

where $\eta \to 0$ as $z \to z_0$.

Let $\dfrac{f(z) - f(z_0)}{z - z_0} - f'(z_0) = \eta$ so that

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \eta(z - z_0)$$

Then since $f(z)$ is analytic at z_0 we have as required

$$\lim_{z \to z_0} \eta = \lim_{z \to z_0}\left\{\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right\} = f'(z_0) - f'(z_0) = 0$$

22. Prove that if $f(z)$ and $g(z)$ are analytic at z_0, and $f(z_0) = g(z_0) = 0$ but $g'(z_0) \neq 0$, then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$

By Problem 21 we have, using the fact that $f(z_0) = g(z_0) = 0$,

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \eta_1(z - z_0) = f'(z_0)(z - z_0) + \eta_1(z - z_0)$$
$$g(z) = g(z_0) + g'(z_0)(z - z_0) + \eta_2(z - z_0) = g'(z_0)(z - z_0) + \eta_2(z - z_0)$$

where $\lim_{z \to z_0} \eta_1 = \lim_{z \to z_0} \eta_2 = 0$. Then, as required,

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{\{f'(z_0) + \eta_1\}(z - z_0)}{\{g'(z_0) + \eta_2\}(z - z_0)} = \frac{f'(z_0)}{g'(z_0)}$$

Another method.

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \Big/ \frac{g(z) - g(z_0)}{z - z_0}$$

$$= \left(\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}\right) \Big/ \left(\lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0}\right) = \frac{f'(z_0)}{g'(z_0)}$$

23. Evaluate (a) $\displaystyle\lim_{z \to i} \frac{z^{10} + 1}{z^6 + 1}$, (b) $\displaystyle\lim_{z \to 0} \frac{1 - \cos z}{z^2}$, (c) $\displaystyle\lim_{z \to 0} \frac{1 - \cos z}{\sin z^2}$.

(a) If $f(z) = z^{10} + 1$ and $g(z) = z^6 + 1$, then $f(i) = g(i) = 0$. Also, $f(z)$ and $g(z)$ are analytic at $z = i$. Hence by L'Hospital's rule,

$$\lim_{z\to i}\frac{z^{10}+1}{z^6+1} = \lim_{z\to i}\frac{10z^9}{6z^5} = \lim_{z\to i}\frac{5}{3}z^4 = \frac{5}{3}$$

(*b*) If $f(z) = 1-\cos z$ and $g(z)=z^2$, then $f(0)=g(0)=0$. Also, $f(z)$ and $g(z)$ are analytic at $z=0$. Hence by L'Hospital's rule,

$$\lim_{z\to 0}\frac{1-\cos z}{z^2} = \lim_{z\to 0}\frac{\sin z}{2z}$$

Since $f_1(z) = \sin z$ and $g_1(z) = 2z$ are analytic and equal to zero when $z=0$, we can apply L'Hospital's rule again to obtain the required limit.

$$\lim_{z\to 0}\frac{\sin z}{2z} = \lim_{z\to 0}\frac{\cos z}{2} = \frac{1}{2}$$

(*c*) ***Method 1.*** By repeated application of L'Hospital's rule, we have

$$\lim_{z\to 0}\frac{1-\cos z}{\sin z^2} = \lim_{z\to 0}\frac{\sin z}{2z\cos z^2} = \lim_{z\to 0}\frac{\cos z}{2\cos z^2 - 4z^2\sin z^2} = \frac{1}{2}$$

Method 2. Since $\lim\limits_{z\to 0}\frac{\sin z}{z} = 1$, we have by one application of L'Hospital's rule,

$$\lim_{z\to 0}\frac{1-\cos z}{\sin z^2} = \lim_{z\to 0}\frac{\sin z}{2z\cos z^2} = \lim_{z\to 0}\left(\frac{\sin z}{z}\right)\left(\frac{1}{2\cos z^2}\right)$$

$$= \lim_{z\to 0}\left(\frac{\sin z}{z}\right)\lim_{z\to 0}\left(\frac{1}{2\cos z^2}\right) = (1)\left(\frac{1}{2}\right) = \frac{1}{2}$$

Method 3. Since $\lim\limits_{z\to 0}\frac{\sin z^2}{z^2} = 1$ or, equivalently, $\lim\limits_{z\to 0}\frac{z^2}{\sin z^2} = 1$, we can write

$$\lim_{z\to 0}\frac{1-\cos z}{\sin z^2} = \lim_{z\to 0}\left(\frac{1-\cos z}{z^2}\right)\left(\frac{z^2}{\sin z^2}\right) = \lim_{z\to 0}\frac{1-\cos z}{z^2} = \frac{1}{2}$$

using part (*b*).

24. Evaluate $\lim\limits_{z\to 0}(\cos z)^{1/z^2}$.

Let $w = (\cos z)^{1/z^2}$. Then $\ln w = \frac{\ln\cos z}{z^2}$ where we consider the principal branch of the logarithm. By L'Hospital's rule,

$$\lim_{z\to 0}\ln w = \lim_{z\to 0}\frac{\ln\cos z}{z^2} = \lim_{z\to 0}\frac{(-\sin z)/\cos z}{2z}$$

$$= \lim_{z\to 0}\left(\frac{\sin z}{z}\right)\left(-\frac{1}{2\cos z}\right) = (1)\left(-\frac{1}{2}\right) = -\frac{1}{2}$$

But since the logarithm is a continuous function, we have

$$\lim_{z\to 0}\ln w = \ln\left(\lim_{z\to 0} w\right) = -\frac{1}{2}$$

or $\lim\limits_{z\to 0} w = e^{-1/2}$ which is the required value.

Note that since $\lim\limits_{z\to 0}\cos z = 1$ and $\lim\limits_{z\to 0} 1/z^2 = \infty$, the required limit has the "indeterminate form" 1^∞.

SINGULAR POINTS

25. For each of the following functions locate and name the singularities in the finite z plane and determine whether they are isolated singularities or not.

(*a*) $f(z) = \dfrac{z}{(z^2+4)^2} = \dfrac{z}{\{(z+2i)(z-2i)\}^2} = \dfrac{z}{(z+2i)^2(z-2i)^2}$.

Since $\lim\limits_{z\to 2i}(z-2i)^2 f(z) = \lim\limits_{z\to 2i}\frac{z}{(z+2i)^2} = \frac{1}{8i} \neq 0$, $z=2i$ is a pole of order 2. Similarly $z=-2i$ is a pole of order 2.

Since we can find δ such that no singularity other than $z=2i$ lies inside the circle $|z-2i| = \delta$ (e.g. choose $\delta=1$), it follows that $z=2i$ is an isolated singularity. Similarly $z=-2i$ is an isolated singularity.

(*b*) $f(z) = \sec(1/z)$.

Since $\sec(1/z) = \dfrac{1}{\cos(1/z)}$, the singularities occur where $\cos(1/z) = 0$, i.e. $1/z = (2n+1)\pi/2$ or $z = 2/(2n+1)\pi$, where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$. Also, since $f(z)$ is not defined at $z = 0$, it follows that $z = 0$ is also a singularity.

Now by L'Hospital's rule,

$$\lim_{z \to 2/(2n+1)\pi} \left\{ z - \frac{2}{(2n+1)\pi} \right\} f(z) = \lim_{z \to 2/(2n+1)\pi} \frac{z - 2/(2n+1)\pi}{\cos(1/z)}$$

$$= \lim_{z \to 2/(2n+1)\pi} \frac{1}{-\sin(1/z)\{-1/z^2\}}$$

$$= \frac{\{2/(2n+1)\pi\}^2}{\sin(2n+1)\pi/2} = \frac{4(-1)^n}{(2n+1)^2\pi^2} \neq 0$$

Thus the singularities $z = 2/(2n+1)/\pi$, $n = 0, \pm 1, \pm 2, \ldots$ are *poles of order one*, i.e. *simple poles*. Note that these poles are located on the real axis at $z = \pm 2/\pi, \pm 2/3\pi, \pm 2/5\pi, \ldots$ and that there are infinitely many in a finite interval which includes 0 (see Fig. 3-9).

Since we can surround each of these by a circle of radius δ which contains no other singularity, it follows that they are isolated singularities. It should be noted that the δ required is smaller the closer the singularity is to the origin.

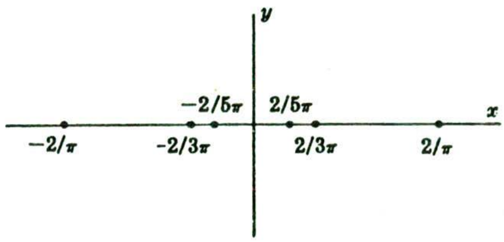

Fig. 3-9

Since we cannot find any positive integer n such that $\lim_{z \to 0} (z-0)^n f(z) = A \neq 0$, it follows that $z = 0$ is an *essential singularity*. Also since every circle of radius δ with centre at $z = 0$ contains singular points other than $z = 0$, no matter how small we take δ, we see that $z = 0$ is a *non-isolated singularity*.

(*c*) $f(z) = \dfrac{\ln(z-2)}{(z^2+2z+2)^4}$.

The point $z = 2$ is a *branch point* and is an *isolated singularity*. Also since $z^2 + 2z + 2 = 0$ where $z = -1 \pm i$, it follows that $z^2 + 2z + 2 = (z+1+i)(z+1-i)$ and that $z = -1 \pm i$ are *poles of order 4* which are *isolated singularities*.

(*d*) $f(z) = \dfrac{\sin\sqrt{z}}{\sqrt{z}}$.

At first sight it appears as if $z = 0$ is a branch point. To test this let $z = re^{i\theta} = re^{i(\theta+2\pi)}$ where $0 \leqq \theta < 2\pi$.

If $z = re^{i\theta}$, we have

$$f(z) = \frac{\sin(\sqrt{r}\, e^{i\theta/2})}{\sqrt{r}\, e^{i\theta/2}}$$

If $z = re^{i(\theta+2\pi)}$, we have

$$f(z) = \frac{\sin(\sqrt{r}\, e^{i\theta/2} e^{\pi i})}{\sqrt{r}\, e^{i\theta/2} e^{\pi i}} = \frac{\sin(-\sqrt{r}\, e^{i\theta/2})}{-\sqrt{r}\, e^{i\theta/2}} = \frac{\sin(\sqrt{r}\, e^{i\theta/2})}{\sqrt{r}\, e^{i\theta/2}}$$

Thus there is actually only one branch to the function, and so $z = 0$ cannot be a branch point. Since $\lim_{z \to 0} \dfrac{\sin\sqrt{z}}{\sqrt{z}} = 1$, it follows in fact that $z = 0$ is a *removable singularity*.

26. (*a*) Locate and name all the singularities of $f(z) = \dfrac{z^8 + z^4 + 2}{(z-1)^3(3z+2)^2}$.

(*b*) Determine where $f(z)$ is analytic.

(*a*) The singularities in the finite z plane are located at $z = 1$ and $z = -2/3$; $z = 1$ is a *pole of order 3* and $z = -2/3$ is a *pole of order 2*.

To determine whether there is a singularity at $z = \infty$ (the point at infinity), let $z = 1/w$. Then

$$f(1/w) = \frac{(1/w)^8 + (1/w)^4 + 2}{(1/w - 1)^3 (3/w + 2)^2} = \frac{1 + w^4 + 2w^8}{w^3(1-w)^3(3+2w)^2}$$

Thus since $w=0$ is a pole of order 3 for the function $f(1/w)$, it follows that $z=\infty$ is a pole of order 3 for the function $f(z)$.

Then the given function has three singularities: a pole of order 3 at $z=1$, a pole of order 2 at $z=-2/3$, and a pole of order 3 at $z=\infty$.

(b) From (a) it follows that $f(z)$ is analytic everywhere in the finite z plane except at the points $z=1$ and $-2/3$.

ORTHOGONAL FAMILIES

27. Let $u(x,y)=\alpha$ and $v(x,y)=\beta$, where u and v are the real and imaginary parts of an analytic function $f(z)$ and α and β are any constants, represent two families of curves. Prove that the families are orthogonal (i.e. each member of one family is perpendicular to each member of the other family at their point of intersection).

Consider any two members of the respective families, say $u(x,y)=\alpha_1$ and $v(x,y)=\beta_1$ where α_1 and β_1 are particular constants [Fig. 3-10].

Differentiating $u(x,y)=\alpha_1$ with respect to x yields

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{dy}{dx} = 0$$

Then the slope of $u(x,y)=\alpha_1$ is

$$\frac{dy}{dx} = -\frac{\partial u}{\partial x}\Big/\frac{\partial u}{\partial y}$$

Similarly the slope of $v(x,y)=\beta_1$ is

$$\frac{dy}{dx} = -\frac{\partial v}{\partial x}\Big/\frac{\partial v}{\partial y}$$

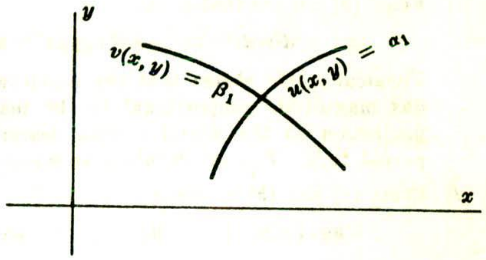

Fig. 3-10

The product of the slopes is, using the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\Big/\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = -\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}\Big/\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = -1$$

Thus the curves are orthogonal.

28. Find the orthogonal trajectories of the family of curves in the xy plane defined by $e^{-x}(x\sin y - y\cos y)=\alpha$ where α is a real constant.

By Problems 7 and 27, it follows that $e^{-x}(y\sin y + x\cos y) = \beta$, where β is a real constant, is the required equation of the orthogonal trajectories.

APPLICATIONS TO GEOMETRY AND MECHANICS

29. An ellipse C has the equation $z=a\cos\omega t+bi\sin\omega t$ where a,b,ω are positive constants, $a>b$, and t is a real variable. (a) Graph the ellipse and show that as t increases from $t=0$ the ellipse is traversed in a counterclockwise direction. (b) Find a unit tangent vector to C at any point.

(a) As t increases from 0 to $\pi/2\omega$, $\pi/2\omega$ to π/ω, π/ω to $3\pi/2\omega$ and $3\pi/2\omega$ to $2\pi/\omega$, point z on C moves from A to B, B to D, D to E and E to A respectively, i.e. it moves in a counterclockwise direction as shown in Fig. 3-11.

(b) A tangent vector to C at any point t is

$$\frac{dz}{dt} = -a\omega\sin\omega t + b\omega i\cos\omega t$$

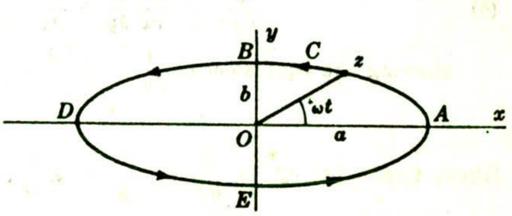

Fig. 3-11

Then a unit tangent vector to C at any point t is

$$\frac{dz/dt}{|dz/dt|} = \frac{-a\omega\sin\omega t + b\omega i\cos\omega t}{|-a\omega\sin\omega t + b\omega i\cos\omega t|} = \frac{-a\sin\omega t + bi\cos\omega t}{\sqrt{a^2\sin^2\omega t + b^2\cos^2\omega t}}$$

30. In Problem 29 suppose that z is the position vector of a particle moving on C and that t is the time.

(a) Determine the velocity and speed of the particle at any time.

(b) Determine the acceleration both in magnitude and direction at any time.

(c) Prove that $d^2z/dt^2 = -\omega^2 z$ and give a physical interpretation.

(d) Determine where the velocity and acceleration have the greatest and least magnitudes.

(a) Velocity $= dz/dt = -a\omega \sin \omega t + b\omega i \cos \omega t$

$$\text{Speed} = \text{magnitude of velocity} = |dz/dt| = \omega\sqrt{a^2 \sin^2 \omega t + b^2 \cos^2 \omega t}$$

(b) Acceleration $= d^2z/dt^2 = -a\omega^2 \cos \omega t - b\omega^2 i \sin \omega t$

$$\text{Magnitude of acceleration} = |d^2z/dt^2| = \omega^2\sqrt{a^2 \cos^2 \omega t + b^2 \sin^2 \omega t}$$

(c) From (b) we see that

$$d^2z/dt^2 = -a\omega^2 \cos \omega t - b\omega^2 i \sin \omega t = -\omega^2(a \cos \omega t + bi \sin \omega t) = -\omega^2 z$$

Physically this states that the acceleration at any time is always directed toward point O and has magnitude proportional to the instantaneous distance from O. As the particle moves, its projection on the x and y axes describes what is sometimes called *simple harmonic motion* of *period* $2\pi/\omega$. The acceleration is sometimes known as the *centripetal acceleration.*

(d) From (a) and (b) we have

$$\text{Magnitude of velocity} = \omega\sqrt{a^2 \sin^2 \omega t + b^2(1 - \sin^2 \omega t)} = \omega\sqrt{(a^2 - b^2) \sin^2 \omega t + b^2}$$

$$\text{Magnitude of acceleration} = \omega^2\sqrt{a^2 \cos^2 \omega t + b^2(1 - \cos^2 \omega t)} = \omega^2\sqrt{(a^2 - b^2) \cos^2 \omega t + b^2}$$

Then the velocity has the greatest magnitude [given by ωa] where $\sin \omega t = \pm 1$, i.e. at points B and E [Fig. 3-11], and the least magnitude [given by ωb] where $\sin \omega t = 0$, i.e. at points A and D.

Similarly the acceleration has the greatest magnitude [given by $\omega^2 a$] where $\cos \omega t = \pm 1$, i.e. at points A and D, and the least magnitude [given by $\omega^2 b$] where $\cos \omega t = 0$, i.e. at points B and E.

Theoretically the planets of our solar system move in elliptical paths with the sun at one focus. In practice there is some deviation from an exact elliptical path.

GRADIENT, DIVERGENCE, CURL AND LAPLACIAN

31. Prove the equivalence of the operators (a) $\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$, (b) $\frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)$ where $z = x + iy$, $\bar{z} = x - iy$.

If F is any continuously differentiable function, then

(a)
$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial F}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial x} = \frac{\partial F}{\partial z} + \frac{\partial F}{\partial \bar{z}}$$

showing the equivalence $\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$.

(b)
$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial F}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial y} = \frac{\partial F}{\partial z}(i) + \frac{\partial F}{\partial \bar{z}}(-i) = i\left(\frac{\partial F}{\partial z} - \frac{\partial F}{\partial \bar{z}}\right)$$

showing the equivalence $\frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)$.

32. Show that (a) $\nabla \equiv \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} = 2\frac{\partial}{\partial \bar{z}}$, (b) $\bar{\nabla} \equiv \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} = 2\frac{\partial}{\partial z}$.

From the equivalences established in Problem 31, we have

(a)
$$\nabla \equiv \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} + i^2\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right) = 2\frac{\partial}{\partial \bar{z}}$$

(b)
$$\bar{\nabla} \equiv \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} - i^2\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right) = 2\frac{\partial}{\partial z}$$

33. If $F(x,y) = c$ [where c is a constant and F is continuously differentiable] is a curve in the xy plane, show that grad $F = \nabla F = \dfrac{\partial F}{\partial x} + i\dfrac{\partial F}{\partial y}$, is a vector normal to the curve.

We have $dF = \dfrac{\partial F}{\partial x}dx + \dfrac{\partial F}{\partial y}dy = 0$. In terms of dot product [see Page 6] this can be written

$$\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right) \circ (dx + i\,dy) = 0$$

But $dx + i\,dy$ is a vector tangent to C. Hence $\nabla F = \dfrac{\partial F}{\partial x}$ [illegible] must be perpendicular to C.

34. Show that $\dfrac{\partial P}{\partial x} - \dfrac{\partial Q}{\partial y} + i\left(\dfrac{\partial Q}{\partial x} + \dfrac{\partial P}{\partial y}\right) = 2\dfrac{\partial B}{\partial \bar{z}}$ where $B(z,\bar{z}) = P(x,y) + i\,Q(x,y)$.

From Problem 32, $\nabla B = 2\dfrac{\partial B}{\partial \bar{z}}$. Hence

$$\nabla B = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(P + iQ) = \frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} + i\left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right) = 2\frac{\partial B}{\partial \bar{z}}$$

35. Let C be the curve in the xy plane defined by $3x^2y - 2y^3 = 5x^4y^2 - 6x^2$. Find a unit vector normal to C at $(1,-1)$.

Let $F(x,y) = 3x^2y - 2y^3 - 5x^4y^2 + 6x^2 = 0$. By Problem 33, a vector normal to C is

$$\nabla F = \frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y} = (6xy - 20x^3y^2 + 12x) + i(3x^2 - 6y^2 - 10x^4y) = -14 + 7i \quad \text{at } (1,-1)$$

Then a unit vector normal to C at $(1,-1)$ is $\dfrac{-14+7i}{|-14+7i|} = \dfrac{-2+i}{\sqrt{5}}$. Another such unit vector is $\dfrac{2-i}{\sqrt{5}}$.

36. If $A(x,y) = 2xy - ix^2y^3$, find (*a*) grad A, (*b*) div A, (*c*) curl A, (*d*) Laplacian of A.

(*a*)
$$\begin{aligned}\text{grad } A &= \nabla A = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(2xy - ix^2y^3) = \frac{\partial}{\partial x}(2xy - ix^2y^3) + i\frac{\partial}{\partial y}(2xy - ix^2y^3) \\ &= 2y - 2ixy^3 + i(2x - 3ix^2y^2) = 2y + 3x^2y^2 + i(2x - 2xy^3)\end{aligned}$$

(*b*)
$$\begin{aligned}\text{div } A &= \nabla \circ A = \text{Re}\{\bar{\nabla}A\} = \text{Re}\left\{\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(2xy - ix^2y^3)\right\} \\ &= \frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^2y^3) = 2y - 3x^2y^2\end{aligned}$$

(*c*)
$$\begin{aligned}\text{curl } A &= \nabla \times A = \text{Im}\{\bar{\nabla}A\} = \text{Im}\left\{\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(2xy - ix^2y^3)\right\} \\ &= \frac{\partial}{\partial x}(-x^2y^3) - \frac{\partial}{\partial y}(2xy) = -2xy^3 - 2x\end{aligned}$$

(*d*)
$$\begin{aligned}\text{Laplacian } A &= \nabla^2 A = \text{Re}\{\bar{\nabla}\nabla A\} = \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} = \frac{\partial^2}{\partial x^2}(2xy - ix^2y^3) + \frac{\partial^2}{\partial y^2}(2xy - ix^2y^3) \\ &= \frac{\partial}{\partial x}(2y - 2ixy^3) + \frac{\partial}{\partial y}(2x - 3ix^2y^2) = -2iy^3 - 6ix^2y\end{aligned}$$

MISCELLANEOUS PROBLEMS

37. Prove that in polar form the Cauchy-Riemann equations can be written

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

We have $x = r\cos\theta$, $y = r\sin\theta$ or $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}(y/x)$. Then

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial r}\left(\frac{x}{\sqrt{x^2+y^2}}\right) + \frac{\partial u}{\partial \theta}\left(\frac{-y}{x^2+y^2}\right) = \frac{\partial u}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta \qquad (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{\partial u}{\partial r}\left(\frac{y}{\sqrt{x^2+y^2}}\right) + \frac{\partial u}{\partial \theta}\left(\frac{x}{x^2+y^2}\right) = \frac{\partial u}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial u}{\partial \theta}\cos\theta \qquad (2)$$

Similarly,

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial v}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial v}{\partial \theta}\sin\theta \tag{3}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{\partial v}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial v}{\partial \theta}\cos\theta \tag{4}$$

From the Cauchy-Riemann equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ we have, using *(1)* and *(4)*,

$$\left(\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta}\right)\cos\theta - \left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\sin\theta = 0 \tag{5}$$

From the Cauchy-Riemann equation $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ we have, using *(2)* and *(3)*,

$$\left(\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta}\right)\sin\theta + \left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\cos\theta = 0 \tag{6}$$

Multiplying *(5)* by $\cos\theta$, *(6)* by $\sin\theta$ and adding yields $\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta} = 0$ or $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$.

Multiplying *(5)* by $-\sin\theta$, *(6)* by $\cos\theta$ and adding yields $\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} = 0$ or $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

38. Prove that the real and imaginary parts of an analytic function of a complex variable when expressed in polar form satisfy the equation [Laplace's equation in polar form]

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r}\frac{\partial \Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Psi}{\partial \theta^2} = 0$$

From Problem 37, *(1)* $\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$, *(2)* $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

To eliminate v differentiate *(1)* partially with respect to r and *(2)* with respect to θ. Then

$$\text{(3)}\quad \frac{\partial^2 v}{\partial r\,\partial \theta} = \frac{\partial}{\partial r}\left(\frac{\partial v}{\partial \theta}\right) = \frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r}$$

$$\text{(4)}\quad \frac{\partial^2 v}{\partial \theta\,\partial r} = \frac{\partial}{\partial \theta}\left(\frac{\partial v}{\partial r}\right) = \frac{\partial}{\partial \theta}\left(-\frac{1}{r}\frac{\partial u}{\partial \theta}\right) = -\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2}$$

But $\frac{\partial^2 v}{\partial r\,\partial \theta} = \frac{\partial^2 v}{\partial \theta\,\partial r}$ assuming the second partial derivatives are continuous. Hence from *(3)* and *(4)*,

$$r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} = -\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} \quad\text{or}\quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Similarly by elimination of u we find $\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$ so that the required result is proved.

39. If $w = f(\zeta)$ where $\zeta = g(z)$, prove that $\frac{dw}{dz} = \frac{dw}{d\zeta}\cdot\frac{d\zeta}{dz}$ assuming f and g are analytic in a region $\mathcal{R}$.

Let z be given an increment $\Delta z \neq 0$ so that $z + \Delta z$ is in $\mathcal{R}$. Then as a consequence ζ and w take on increments $\Delta\zeta$ and Δw respectively, where

$$\Delta w = f(\zeta + \Delta\zeta) - f(\zeta), \qquad \Delta\zeta = g(z + \Delta z) - g(z) \tag{1}$$

Note that as $\Delta z \to 0$, $\Delta w \to 0$ and $\Delta\zeta \to 0$.

If $\Delta\zeta \neq 0$, let us write $\epsilon = \frac{\Delta w}{\Delta\zeta} - \frac{dw}{d\zeta}$ so that $\epsilon \to 0$ as $\Delta\zeta \to 0$ and

$$\Delta w = \frac{dw}{d\zeta}\Delta\zeta + \epsilon\,\Delta\zeta \tag{2}$$

If $\Delta\zeta = 0$ for values of Δz, then *(1)* shows that $\Delta w = 0$ for these values of Δz. For such cases, we define $\epsilon = 0$.

It follows that in both cases, $\Delta\zeta \neq 0$ or $\Delta\zeta = 0$, (2) holds. Then dividing (2) by $\Delta z \neq 0$ and taking the limit as $\Delta z \to 0$, we have

$$\begin{aligned}\frac{dw}{dz} &= \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0}\left(\frac{dw}{d\zeta}\frac{\Delta\zeta}{\Delta z} + \epsilon\frac{\Delta w}{\Delta z}\right) \\ &= \frac{dw}{d\zeta}\cdot\lim_{\Delta z\to 0}\frac{\Delta\zeta}{\Delta z} + \lim_{\Delta z\to 0}\epsilon\cdot\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z} \\ &= \frac{dw}{d\zeta}\cdot\frac{d\zeta}{dz} + 0\cdot\frac{d\zeta}{dz} = \frac{dw}{d\zeta}\cdot\frac{d\zeta}{dz}\end{aligned}$$

40. (a) If $u_1(x,y) = \partial u/\partial x$ and $u_2(x,y) = \partial u/\partial y$, prove that $f'(z) = u_1(z,0) - i\,u_2(z,0)$.
(b) Show how the result in (a) can be used to solve Problems 7 and 8.

(a) From Problem 5, we have $f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = u_1(x,y) - i\,u_2(x,y)$.

Putting $y = 0$, this becomes $f'(x) = u_1(x,0) - i\,u_2(x,0)$.

Then replacing x by z, we have as required $f'(z) = u_1(z,0) - i\,u_2(z,0)$.

(b) Since we are given $u = e^{-x}(x\sin y - y\cos y)$, we have

$$u_1(x,y) = \frac{\partial u}{\partial x} = e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y$$

$$u_2(x,y) = \frac{\partial u}{\partial y} = xe^{-x}\cos y + ye^{-x}\sin y - e^{-x}\cos y$$

so that from part (a),

$$f'(z) = u_1(z,0) - i\,u_2(z,0) = 0 - i(ze^{-z} - e^{-z}) = -i(ze^{-z} - e^{-z})$$

Integrating with respect to z we have, apart from a constant, $f(z) = ize^{-z}$. By separating this into real and imaginary parts, $v = e^{-x}(y\sin y + x\cos y)$ apart from a constant.

41. Prove that $\text{curl grad } A = 0$ if A is real or, more generally, if $\text{Im } A$ is harmonic.

If $A = P + Qi$, $\text{grad } A = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(P + iQ) = \frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} + i\left(\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right)$. Then

$$\begin{aligned}\text{curl grad } A &= \text{Im}\left[\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left\{\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} + i\left(\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right)\right\}\right] \\ &= \text{Im}\left[\frac{\partial^2 P}{\partial x^2} - \frac{\partial^2 Q}{\partial x\,\partial y} + i\left(\frac{\partial^2 P}{\partial x\,\partial y} + \frac{\partial^2 Q}{\partial x^2}\right) - i\left(\frac{\partial^2 P}{\partial y\,\partial x} - \frac{\partial^2 Q}{\partial y^2}\right) + \left(\frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 Q}{\partial y\,\partial x}\right)\right] \\ &= \frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2}\end{aligned}$$

Hence if $Q = 0$, i.e. A is real, or if Q is harmonic, $\text{curl grad } A = 0$.

42. Solve the partial differential equation $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = x^2 - y^2$.

Let $z = x + iy$, $\bar z = x - iy$ so that $x = \frac{z + \bar z}{2}$, $y = \frac{z - \bar z}{2i}$. Then

$$x^2 - y^2 = \tfrac{1}{2}(z^2 + \bar z^2) \quad\text{and}\quad \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \nabla^2 U = 4\frac{\partial^2 U}{\partial z\,\partial \bar z}$$

Thus the given partial differential equation becomes $4\frac{\partial^2 U}{\partial z\,\partial\bar z} = \frac{1}{2}(z^2 + \bar z^2)$ or

$$\frac{\partial}{\partial z}\left(\frac{\partial U}{\partial \bar z}\right) = \frac{1}{8}(z^2 + \bar z^2) \qquad (1)$$

Integrating (1) with respect to z (treating $\bar{z}$ as constant),

$$\frac{\partial U}{\partial \bar{z}} = \frac{z^3}{24} + \frac{z\bar{z}^2}{8} + F_1(\bar{z}) \quad (2)$$

where $F_1(\bar{z})$ is an arbitrary function of $\bar{z}$. Integrating (2) with respect to $\bar{z}$,

$$U = \frac{z^3\bar{z}}{24} + \frac{z\bar{z}^3}{24} + F(\bar{z}) + G(z) \quad (3)$$

where $F(\bar{z})$ is the function obtained by integrating $F_1(\bar{z})$, and $G(z)$ is an arbitrary function of z. Replacing z and $\bar{z}$ by $x+iy$ and $x-iy$ respectively, we obtain

$$U = \tfrac{1}{12}(x^4 - y^4) + F(x-iy) + G(x+iy)$$

Supplementary Problems

DERIVATIVES

43. Using the definition, find the derivative of each function at the indicated points.
(a) $f(z) = 3z^2 + 4iz - 5 + i$; $z = 2$. (b) $f(z) = \dfrac{2z-i}{z+2i}$; $z = -i$. (c) $f(z) = 3z^{-2}$; $z = 1+i$.
Ans. (a) $12 + 4i$ (b) $-5i$ (c) $3/2 + 3i/2$

44. Prove that $\dfrac{d}{dz}(z^2\bar{z})$ does not exist anywhere.

45. Determine whether $|z|^2$ has a derivative anywhere.

46. For each of the following functions determine the singular points, i.e. points at which the function is not analytic. Determine the derivatives at all other points. (a) $\dfrac{z}{z+i}$, (b) $\dfrac{3z-2}{z^2+2z+5}$.
Ans. (a) $-i$, $i/(z+i)^2$; (b) $-1 \pm 2i$, $(19 + 4z - 3z^2)/(z^2+2z+5)^2$

CAUCHY-RIEMANN EQUATIONS

47. Verify that the real and imaginary parts of the following functions satisfy the Cauchy-Riemann equations and thus deduce the analyticity of each function:
(a) $f(z) = z^2 + 5iz + 3 - i$, (b) $f(z) = ze^{-z}$, (c) $f(z) = \sin 2z$.

48. Show that the function $x^2 + iy^3$ is not analytic anywhere. Reconcile this with the fact that the Cauchy-Riemann equations are satisfied at $x = 0$, $y = 0$.

49. Prove that if $w = f(z) = u + iv$ is analytic in a region $\mathcal{R}$, then $\dfrac{dw}{dz} = \dfrac{\partial w}{\partial x} = -i\dfrac{\partial w}{\partial y}$.

50. (a) Prove that the function $u = 2x(1-y)$ is harmonic. (b) Find a function v such that $f(z) = u + iv$ is analytic [i.e. find the conjugate function of u]. (c) Express $f(z)$ in terms of z.
Ans. (b) $2y + x^2 - y^2$, (c) $iz^2 + 2z$

51. Answer Problem 50 for the function $u = x^2 - y^2 - 2xy - 2x + 3y$. *Ans.* (b) $x^2 - y^2 + 2xy - 3x - 2y$

52. Verify that the Cauchy-Riemann equations are satisfied for the functions (a) e^{z^2}, (b) $\cos 2z$, (c) $\sinh 4z$.

53. Determine which of the following functions u are harmonic. For each harmonic function find the conjugate harmonic function v and express $u + iv$ as an analytic function of z.
(a) $3x^2y + 2x^2 - y^3 - 2y^2$, (b) $2xy + 3xy^2 - 2y^3$, (c) $xe^x \cos y - ye^x \sin y$, (d) $e^{-2xy}\sin(x^2 - y^2)$.
Ans. (a) $v = 4xy - x^3 + 3xy^2 + c$, $f(z) = 2z^2 - iz^3 + ic$ (c) $ye^x \cos y + xe^x \sin y + c$, $ze^z + ic$
(b) Not harmonic (d) $-e^{-2xy}\cos(x^2 - y^2) + c$, $-ie^{iz^2} + ic$

54. (a) Prove that $\psi = \ln[(x-1)^2+(y-2)^2]$ is harmonic in every region which does not include the point (1, 2). (b) Find a function ϕ such that $\phi + i\psi$ is analytic. (c) Express $\phi + i\psi$ as a function of z.
Ans. (b) $-2\tan^{-1}\{(y-2)/(x-1)\}$ (c) $2i\ln(z-1-2i)$

55. If $\text{Im}\{f'(z)\} = 6x(2y-1)$ and $f(0) = 3-2i$, $f(1) = 6-5i$, find $f(1+i)$. *Ans.* $6+3i$

DIFFERENTIALS

56. If $w = iz^2 - 4z + 3i$, find (a) Δw, (b) dw, (c) $\Delta w - dw$ at the point $z = 2i$.
Ans. (a) $-8\,\Delta z + i(\Delta z)^2 = -8\,dz + i(dz)^2$, (b) $-8\,dz$, (c) $i(dz)^2$

57. Find (a) Δw and (b) dw if $w = (2z+1)^3$, $z = -i$, $\Delta z = 1+i$. *Ans.* (a) $38-2i$, (b) $6-42i$

58. If $w = 3iz^2 + 2z + 1 - 3i$, find (a) Δw, (b) dw, (c) $\Delta w/\Delta z$, (d) dw/dz where $z = i$.
Ans. (a) $-4\Delta z + 3i(\Delta z)^2$, (b) $-4\,dz$, (c) $-4 + 3i\,\Delta z$, (d) -4

59. (a) If $w = \sin z$, show that $\dfrac{\Delta w}{\Delta z} = (\cos z)\left(\dfrac{\sin\Delta z}{\Delta z}\right) - 2\sin z\left\{\dfrac{\sin^2(\Delta z/2)}{\Delta z}\right\}$.

(b) Assuming $\lim\limits_{\Delta z\to 0}\dfrac{\sin\Delta z}{\Delta z} = 1$, prove that $\dfrac{dw}{dz} = \cos z$.

(c) Show that $dw = (\cos z)\,dz$.

60. (a) If $w = \ln z$, show that if $\Delta z/z = \zeta$, $\dfrac{\Delta w}{\Delta z} = \dfrac{1}{z}\ln\{(1+\zeta)^{1/\zeta}\}$.

(b) Assuming $\lim\limits_{\zeta\to 0}(1+\zeta)^{1/\zeta} = e = 2.71828\ldots$, prove that $\dfrac{dw}{dz} = \dfrac{1}{z}$.

(c) Show that $d(\ln z) = dz/z$.

61. Prove that (a) $d\{f(z)\,g(z)\} = \{f(z)\,g'(z) + g(z)\,f'(z)\}\,dz$
(b) $d\{f(z)/g(z)\} = \{g(z)\,f'(z) - f(z)\,g'(z)\}\,dz/\{g(z)\}^2$
giving restrictions on $f(z)$ and $g(z)$.

DIFFERENTIATION RULES. DERIVATIVES OF ELEMENTARY FUNCTIONS.

62. Prove that if $f(z)$ and $g(z)$ are analytic in a region $\mathcal{R}$, then
(a) $\dfrac{d}{dz}\{2i\,f(z) - (1+i)\,g(z)\} = 2i\,f'(z) - (1+i)\,g'(z)$, (b) $\dfrac{d}{dz}\{f(z)\}^2 = 2\,f(z)\,f'(z)$, (c) $\dfrac{d}{dz}\{f(z)\}^{-1} = -\{f(z)\}^{-2}\,f'(z)$.

63. Using differentiation rules, find the derivatives of each of the following functions: (a) $(1+4i)z^2 - 3z - 2$, (b) $(2z+3i)(z-i)$, (c) $(2z-i)/(z+2i)$, (d) $(2iz+1)^2$, (e) $(iz-1)^{-3}$.
Ans. (a) $(2+8i)z - 3$, (b) $4z+i$, (c) $5i/(z+2i)^2$, (d) $4i - 8z$, (e) $-3i(iz-1)^{-4}$

64. Find the derivatives of each of the following at the indicated points:
(a) $(z+2i)(i-z)/(2z-1)$, $z = i$. (b) $\{z + (z^2+1)^2\}^2$, $z = 1+i$.
Ans. (a) $-6/5 + 3i/5$, (b) $-108 - 78i$

65. Prove that (a) $\dfrac{d}{dz}\sec z = \sec z\tan z$, (b) $\dfrac{d}{dz}\cot z = -\csc^2 z$.

66. Prove that (a) $\dfrac{d}{dz}(z^2+1)^{1/2} = \dfrac{z}{(z^2+1)^{1/2}}$, (b) $\dfrac{d}{dz}\ln(z^2+2z+2) = \dfrac{2z+2}{z^2+2z+2}$ indicating restrictions if any.

67. Find the derivatives of each of the following, indicating restrictions if any.
(*a*) $3\sin^2(z/2)$, (*b*) $\tan^3(z^2-3z+4i)$, (*c*) $\ln(\sec z+\tan z)$, (*d*) $\csc\{(z^2+1)^{1/2}\}$, (*e*) $(z^2-1)\cos(z+2i)$.

Ans. (*a*) $3\sin(z/2)\cos(z/2)$
(*b*) $3(2z-3)\tan^2(z^2-3z+4i)\sec^2(z^2-3z+4i)$
(*c*) $\sec z$
(*d*) $\dfrac{-z\csc\{(z^2+1)^{1/2}\}\cot\{(z^2+1)^{1/2}\}}{(z^2+1)^{1/2}}$
(*e*) $(1-z^2)\sin(z+2i)+2z\cos(z+2i)$

68. Prove that (*a*) $\dfrac{d}{dz}(1+z^2)^{3/2}=3z(1+z^2)^{1/2}$, (*b*) $\dfrac{d}{dz}(z+2\sqrt{z})^{1/3}=\dfrac{1}{3}z^{-1/2}(z+2\sqrt{z})^{-2/3}(\sqrt{z}+1)$.

69. Prove that (*a*) $\dfrac{d}{dz}(\tan^{-1}z)=\dfrac{1}{z^2+1}$, (*b*) $\dfrac{d}{dz}(\sec^{-1}z)=\dfrac{1}{z\sqrt{z^2-1}}$.

70. Prove that (*a*) $\dfrac{d}{dz}\sinh^{-1}z=\dfrac{1}{\sqrt{1+z^2}}$, (*b*) $\dfrac{d}{dz}\operatorname{csch}^{-1}z=\dfrac{-1}{z\sqrt{z^2+1}}$.

71. Find the derivatives of each of the following:
(*a*) $\{\sin^{-1}(2z-1)\}^2$ (*c*) $\cos^{-1}(\sin z-\cos z)$ (*e*) $\coth^{-1}(z\csc 2z)$
(*b*) $\ln\{\cot^{-1}z^2\}$ (*d*) $\tan^{-1}(z+3i)^{-1/2}$ (*f*) $\ln(z-\frac{3}{2}+\sqrt{z^2-3z+2i})$

Ans. (*a*) $2\sin^{-1}(2z-1)/(z-z^2)^{1/2}$ (*d*) $-1/2(z+1+3i)(z+3i)^{1/2}$
(*b*) $-2z/(1+z^4)\cot^{-1}z^2$ (*e*) $(\csc 2z)(1-2z\cot 2z)/(1-z^2\csc^2 2z)$
(*c*) $-(\sin z+\cos z)/(\sin 2z)^{1/2}$ (*f*) $1/\sqrt{z^2-3z+2i}$

72. If $w=\cos^{-1}(z-1)$, $z=\sinh(3\zeta+2i)$ and $\zeta=\sqrt{t}$, find dw/dt.
Ans. $-3[\cosh(3\zeta+2i)]/2(2z-z^2)^{1/2}t^{1/2}$

73. If $w=t\sec(t-3i)$ and $z=\sin^{-1}(2t-1)$, find dw/dz.
Ans. $\sec(t-3i)\{1+t\tan(t-3i)\}(t-t^2)^{1/2}$

74. If $w^2-2w+\sin 2z=0$, find (*a*) dw/dz, (*b*) d^2w/dz^2.
Ans. (*a*) $(\cos 2z)/(1-w)$, (*b*) $\{\cos^2 2z-2(1-w)^2\sin 2z\}/(1-w)^3$

75. Find d^2w/dz^2 at $\zeta=0$ if $w=\cos\zeta$, $z=\tan(\zeta+\pi i)$. *Ans.* $-\cosh^4\pi$

76. Find (*a*) $\dfrac{d}{dz}\{z^{\ln z}\}$, (*b*) $\dfrac{d}{dz}\{[\sin(iz-2)]^{\tan^{-1}(z+3i)}\}$
Ans. (*a*) $2z^{\ln z-1}\ln z$
(*b*) $\{[\sin(iz-2)]^{\tan^{-1}(z+3i)}\}\{i\tan^{-1}(z+3i)\cot(iz-2)+[\ln\sin(iz-2)]/[z^2+6iz-8]\}$

77. Find the second derivatives of each of the following:
(*a*) $3\sin^2(2z-1+i)$, (*b*) $\ln\tan z^2$, (*c*) $\sinh(z+1)^2$, (*d*) $\cos^{-1}(\ln z)$, (*e*) $\operatorname{sech}^{-1}\sqrt{1+z}$.

Ans. (*a*) $24\cos(4z-2+2i)$
(*b*) $4\csc 2z^2-16z^2\csc 2z^2\cot 2z^2$
(*c*) $2\cosh(z+1)^2+4(z+1)^2\sinh(z+1)^2$
(*d*) $(1-\ln z-\ln^2 z)/z^2(1-\ln^2 z)^{3/2}$
(*e*) $-i(1+3z)/4(1+z)^2z^{3/2}$

L'HOSPITAL'S RULE

78. Evaluate (*a*) $\displaystyle\lim_{z\to 2i}\frac{z^2+4}{2z^2+(3-4i)z-6i}$, (*b*) $\displaystyle\lim_{z\to e^{\pi i/3}}(z-e^{\pi i/3})\left(\frac{z}{z^3+1}\right)$, (*c*) $\displaystyle\lim_{z\to i}\frac{z^2-2iz-1}{z^4+2z^2+1}$.
Ans. (*a*) $(16+12i)/25$, (*b*) $(1-i\sqrt{3})/6$, (*c*) $-1/4$

79. Evaluate (*a*) $\displaystyle\lim_{z\to 0}\frac{z-\sin z}{z^3}$, (*b*) $\displaystyle\lim_{z\to m\pi i}(z-m\pi i)\left(\frac{e^z}{\sin z}\right)$. *Ans.* (*a*) $1/6$, (*b*) $e^{m\pi i}/(\cosh m\pi)$

80. Find $\displaystyle\lim_{z\to i}\frac{\tan^{-1}(z^2+1)^2}{\sin^2(z^2+1)}$ where the branch of the inverse tangent is chosen such that $\tan^{-1}0=0$.
Ans. 1

81. Evaluate $\displaystyle\lim_{z\to 0}\left(\frac{\sin z}{z}\right)^{1/z^2}$. *Ans.* $e^{-1/6}$

SINGULAR POINTS

82. For each of the following functions locate and name the singularities in the finite z plane.

(a) $\dfrac{z^2-3z}{z^2+2z+2}$, (b) $\dfrac{\ln(z+3i)}{z^2}$, (c) $\sin^{-1}(1/z)$, (d) $\sqrt{z(z^2+1)}$, (e) $\dfrac{\cos z}{(z+i)^3}$

Ans. (a) $z = -1 \pm i$; simple poles

(b) $z = -3i$; branch point, $z = 0$; pole of order 2

(c) $z = 0$; essential singularity

(d) $z = 0, \pm i$; branch points

(e) $z = -i$; pole of order 3

83. Show that $f(z) = \dfrac{(z+3i)^5}{(z^2-2z+5)^2}$ has double poles at $z = 1 \pm 2i$ and a simple pole at infinity.

84. Show that e^{z^3} has an essential singularity at infinity.

85. Locate and name all the singularities of each of the following functions.
(a) $(z+3)/(z^2-1)$, (b) $\csc(1/z^2)$, (c) $(z^2+1)/z^{3/2}$.

Ans. (a) $z = \pm 1$; simple poles, $z = \infty$; simple pole. (b) $z = 1/\sqrt{m\pi}$, $m = \pm 1, \pm 2, \pm 3, \ldots$; simple poles, $z = 0$; essential singularity, $z = \infty$; pole of order 2. (c) $z = 0$; branch point, $z = \infty$; branch point.

ORTHOGONAL FAMILIES

86. Find the orthogonal trajectories of the following families of curves:
(a) $x^3y - xy^3 = \alpha$, (b) $e^{-x}\cos y + xy = \alpha$.

Ans. (a) $x^4 - 6x^2y^2 + y^4 = \beta$, (b) $2e^{-x}\sin y + x^2 - y^2 = \beta$

87. Find the orthogonal trajectories of the family of curves $r^2\cos 2\theta = \alpha$. *Ans.* $r^2 \sin 2\theta = \beta$

88. By separating $f(z) = z + 1/z$ into real and imaginary parts, show that the families $(r^2+1)\cos\theta = \alpha r$ and $(r^2-1)\sin\theta = \beta r$ are orthogonal trajectories and verify this by another method.

89. If n is any real constant, prove that $r^n = \alpha \sec n\theta$ and $r^n = \beta \csc n\theta$ are orthogonal trajectories.

APPLICATIONS TO GEOMETRY AND MECHANICS

90. A particle moves along a curve $z = e^{-t}(2\sin t + i\cos t)$.

(a) Find a unit tangent vector to the curve at the point where $t = \pi/4$.

(b) Determine the magnitudes of velocity and acceleration of the particle at $t = 0$ and $\pi/2$.

Ans. (a) $\pm i$. (b) Velocity: $\sqrt{5}$, $\sqrt{5}\,e^{-\pi/2}$. Acceleration: 4, $2e^{-\pi/2}$

91. A particle moves along the curve $z = ae^{i\omega t}$. (a) Show that its speed is always constant and equal to ωa. (b) Show that the magnitude of its acceleration is always constant and equal to $\omega^2 a$. (c) Show that the acceleration is always directed toward $z = 0$. (d) Explain the relationship of this problem to the problem of a stone being twirled at the end of a string in a horizontal plane.

92. The position at time t of a particle moving in the z plane is given by $z = 3te^{-4it}$. Find the magnitudes of (a) the velocity, (b) the acceleration of the particle at $t = 0$ and $t = \pi$.

Ans. (a) 3, $3\sqrt{1+16\pi^2}$. (b) 24, $24\sqrt{1+4\pi^2}$

93. A particle P moves along the line $x + y = 2$ in the z plane with a uniform speed of $3\sqrt{2}$ ft/sec from the point $z = -5+7i$ to $z = 10-8i$. If $w = 2z^2 - 3$ and P' is the image of P in the w plane, find the magnitudes of (a) the velocity and (b) the acceleration of P' after 3 seconds.

Ans. (a) $24\sqrt{10}$, (b) 72

GRADIENT, DIVERGENCE, CURL AND LAPLACIAN

94. If $F = x^2y - xy^2$, find (a) ∇F, (b) $\nabla^2 F$. *Ans.* (a) $(2xy - y^2) + i(x^2 - 2xy)$, (b) $2y - 2x$

95. Let $B = 3z^2 + 4\bar{z}$. Find (a) grad B, (b) div B, (c) curl B, (d) Laplacian B.
Ans. (a) 8, (b) $12x$, (c) $12y$, (d) 0

96. Let C be the curve in the xy plane defined by $x^2 - xy + y^2 = 7$. Find a unit vector normal to C at (a) the point $(-1, 2)$, (b) any point.
Ans. (a) $(-4 + 5i)/\sqrt{41}$, (b) $\{2x - y + i(2y - x)\}/\sqrt{5x^2 - 8xy + 5y^2}$

97. Find an equation for the line normal to the curve $x^2y = 2xy + 6$ at the point $(3, 2)$.
Ans. $x = 8t + 3$, $y = 3t + 2$

98. Show that $\nabla^2 |f(z)|^2 = 4\,|f'(z)|^2$. Illustrate by choosing $f(z) = z^2 + iz$.

99. Prove $\nabla^2\{FG\} = F\,\nabla^2 G + G\,\nabla^2 F + 2\,\nabla F \circ \nabla G$

100. Prove div grad $A = 0$ if A is imaginary or, more generally, if Re $\{A\}$ is harmonic.

MISCELLANEOUS PROBLEMS

101. If $f(z) = u(x, y) + i\,v(x, y)$, prove that:
(a) $f(z) = 2u(z/2, -iz/2) + \text{constant}$, (b) $f(z) = 2i\,v(z/2, -iz/2) + \text{constant}$.

102. Use Problem 101 to find $f(z)$ if (a) $u(x, y) = x^4 - 6x^2y^2 + y^4$, (b) $v(x, y) = \sinh x \cos y$.

103. If V is the instantaneous speed of a particle moving along any plane curve C, prove that the normal component of the acceleration at any point of C is given by V^2/R where R is the radius of curvature at the point.

104. Find an analytic function $f(z)$ such that Re $\{f'(z)\} = 3x^2 - 4y - 3y^2$ and $f(1 + i) = 0$.
Ans. $z^3 + 2iz^2 + 6 - 2i$

105. Show that the family of curves

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1$$

with $-a^2 < \lambda < -b^2$ is orthogonal to the family with $\lambda > -b^2 > -a^2$.

106. Prove that the equation $F(x, y) = \text{constant}$ can be expressed as $u(x, y) = \text{constant}$ where u is harmonic if and only if $\dfrac{\partial^2 F/\partial x^2 + \partial^2 F/\partial y^2}{(\partial F/\partial x)^2 + (\partial F/\partial y)^2}$ is a function of F.

107. Illustrate the result in Problem 106 by considering $(y + 2)/(x - 1) = \text{constant}$.

108. If $f'(z) = 0$ in a region $\mathcal{R}$, prove that $f(z)$ must be a constant in $\mathcal{R}$.

109. If $w = f(z)$ is analytic and expressed in polar coordinates (r, θ), prove that

$$\frac{dw}{dz} = e^{-i\theta}\frac{\partial w}{\partial r}$$

110. If u and v are conjugate harmonic functions, prove that

$$dv = \frac{\partial u}{\partial x}\,dy - \frac{\partial u}{\partial y}\,dx$$

111. If u and v are harmonic in a region $\mathcal{R}$, prove that

$$\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

is analytic in $\mathcal{R}$.

112. Prove that $f(z) = |z|^4$ is differentiable but not analytic at $z = 0$.

113. Prove that $\psi = \ln |f(z)|$ is harmonic in a region $\mathcal{R}$ if $f(z)$ is analytic in $\mathcal{R}$ and $f(z)f'(z) \neq 0$ in $\mathcal{R}$.

114. Express the Cauchy-Riemann equations in terms of the curvilinear coordinates (ξ, η) where $x = e^{\xi} \cosh \eta$, $y = e^{\xi} \sinh \eta$.

115. Show that a solution of the differential equation

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E_0 \cos \omega t$$

where L, R, C, E_0 and ω are constants, is given by

$$Q = \operatorname{Re}\left\{\frac{E_0 e^{i\omega t}}{i\omega[R + i(\omega L - 1/\omega C)]}\right\}$$

The equation arises in the *theory of alternating currents* of electricity.

[*Hint.* Rewrite the right-hand side as $E_0 e^{i\omega t}$ and then assume a solution of the form $Ae^{i\omega t}$ where A is to be determined.]

116. Show that $\nabla^2 \{f(z)\}^n = n^2 |f(z)|^{n-2} |f'(z)|^2$, stating restrictions on $f(z)$.

117. Solve the partial differential equation $\dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} = \dfrac{8}{x^2 + y^2}$.

Ans. $U = \frac{1}{2}\{\ln (x^2 + y^2)\}^2 + 2\{\tan^{-1} (y/x)\}^2 + F(x + iy) + G(x - iy)$

118. Prove that $\nabla^4 U = \nabla^2(\nabla^2 U) = \dfrac{\partial^4 U}{\partial x^4} + 2\dfrac{\partial^4 U}{\partial x^2 \partial y^2} + \dfrac{\partial^4 U}{\partial y^4} = 16\dfrac{\partial^4 U}{\partial z^2 \partial \bar{z}^2}$.

119. Solve the partial differential equation $\dfrac{\partial^4 U}{\partial x^4} + 2\dfrac{\partial^4 U}{\partial x^2 \partial y^2} + \dfrac{\partial^4 U}{\partial y^4} = 36(x^2 + y^2)$.

Ans. $U = \frac{1}{16}(x^2 + y^2)^3 + (x + iy) F_1(x - iy) + G_1(x - iy) + (x - iy) F_2(x + iy) + G_2(x + iy)$

Chapter 4

Complex Integration and Cauchy's Theorem

COMPLEX LINE INTEGRALS

Let $f(z)$ be continuous at all points of a curve C [Fig. 4-1] which we shall assume has a finite length, i.e. C is a *rectifiable curve*.

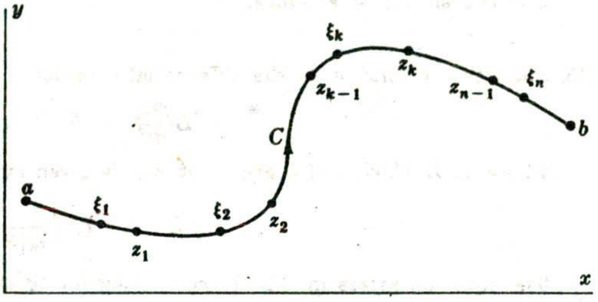

Fig. 4-1

Subdivide C into n parts by means of points $z_1, z_2, \ldots, z_{n-1}$, chosen arbitrarily, and call $a = z_0$, $b = z_n$. On each arc joining z_{k-1} to z_k [where k goes from 1 to n] choose a point ξ_k. Form the sum

$$S_n \;=\; f(\xi_1)(z_1 - a) \;+\; f(\xi_2)(z_2 - z_1) \;+\; \cdots \;+\; f(\xi_n)(b - z_{n-1}) \tag{1}$$

On writing $z_k - z_{k-1} = \Delta z_k$, this becomes

$$S_n \;=\; \sum_{k=1}^{n} f(\xi_k)(z_k - z_{k-1}) \;=\; \sum_{k=1}^{n} f(\xi_k)\,\Delta z_k \tag{2}$$

Let the number of subdivisions n increase in such a way that the largest of the chord lengths $|\Delta z_k|$ approaches zero. Then the sum S_n approaches a limit which does not depend on the mode of subdivision and we denote this limit by

$$\int_a^b f(z)\,dz \quad \text{or} \quad \int_C f(z)\,dz \tag{3}$$

called the *complex line integral* or briefly *line integral* of $f(z)$ along curve C, or the *definite integral* of $f(z)$ from a to b along curve C. In such case $f(z)$ is said to be *integrable* along C. Note that if $f(z)$ is analytic at all points of a region $\mathcal{R}$ and if C is a curve lying in $\mathcal{R}$, then $f(z)$ is certainly integrable along C.

REAL LINE INTEGRALS

If $P(x,y)$ and $Q(x,y)$ are real functions of x and y continuous at all points of curve C, the *real line integral* of $P\,dx + Q\,dy$ along curve C can be defined in a manner similar to that given above and is denoted by

$$\int_C [P(x,y)\,dx \;+\; Q(x,y)\,dy] \quad \text{or} \quad \int_C P\,dx \;+\; Q\,dy \tag{4}$$

the second notation being used for brevity. If C is smooth and has parametric equations $x = \phi(t)$, $y = \psi(t)$ where $t_1 \leqq t \leqq t_2$, the value of (4) is given by

$$\int_{t_1}^{t_2} [P\{\phi(t), \psi(t)\}\phi'(t)\,dt \;+\; Q\{\phi(t), \psi(t)\}\psi'(t)\,dt]$$

Suitable modifications can be made if C is piecewise smooth (see Problem 1).

CONNECTION BETWEEN REAL AND COMPLEX LINE INTEGRALS

If $f(z) = u(x, y) + i v(x, y) = u + iv$ the complex line integral (3) can be expressed in terms of real line integrals as

$$\begin{aligned}\int_C f(z)\,dz &= \int_C (u + iv)(dx + i\,dy) \\ &= \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy \qquad (5)\end{aligned}$$

For this reason (5) is sometimes taken as a definition of a complex line integral.

PROPERTIES OF INTEGRALS

If $f(z)$ and $g(z)$ are integrable along C, then

1. $\int_C \{f(z) + g(z)\}\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$

2. $\int_C A f(z)\,dz = A\int_C f(z)\,dz$ where A = any constant

3. $\int_a^b f(z)\,dz = -\int_b^a f(z)\,dz$

4. $\int_a^b f(z)\,dz = \int_a^m f(z)\,dz + \int_m^b f(z)\,dz$ where points a, b, m are on C.

5. $\left|\int_C f(z)\,dz\right| \leqq ML$

where $|f(z)| \leqq M$, i.e. M is an *upper bound* of $|f(z)|$ on C, and L is the *length* of C.

There are various other ways in which the above properties can be described. For example if T, U and V are successive points on a curve, property 3 can be written $\int_{TUV} f(z)\,dz = -\int_{VUT} f(z)\,dz$.

Similarly if C, C_1 and C_2 represent curves from a to b, a to m and m to b respectively, it is natural for us to consider $C = C_1 + C_2$ and to write property 4 as

$$\int_{C_1+C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

CHANGE OF VARIABLES

Let $z = g(\zeta)$ be a continuous function of a complex variable $\zeta = u + iv$. Suppose that curve C in the z plane corresponds to curve C' in the ζ plane and that the derivative $g'(\zeta)$ is continuous on C' Then

$$\int_C f(z)\,dz = \int_{C'} f\{g(\zeta)\}\,g'(\zeta)\,d\zeta \qquad (6)$$

These conditions are certainly satisfied if g is analytic in a region containing curve C'.

SIMPLY- AND MULTIPLY-CONNECTED REGIONS

A region $\mathcal{R}$ is called *simply-connected* if any simple closed curve [Page 68] which lies in $\mathcal{R}$ can be shrunk to a point without leaving $\mathcal{R}$. A region $\mathcal{R}$ which is not simply-connected is called *multiply-connected.*

For example, suppose $\mathcal{R}$ is the region defined by $|z| < 2$ shown shaded in Fig. 4-2. If Γ is any simple closed curve lying in $\mathcal{R}$ [i.e. whose points are in $\mathcal{R}$], we see that it can be shrunk to a point which lies in $\mathcal{R}$, and thus does not leave $\mathcal{R}$, so that $\mathcal{R}$ is simply-connected. On the other hand if $\mathcal{R}$ is the region defined by $1 < |z| < 2$, shown shaded in Fig. 4-3, then there is a simple closed curve Γ lying in $\mathcal{R}$ which cannot possibly be shrunk to a point without leaving $\mathcal{R}$, so that $\mathcal{R}$ is multiply-connected.

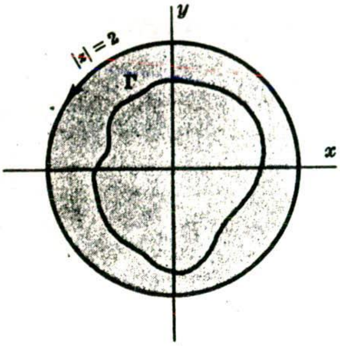

Fig. 4-2

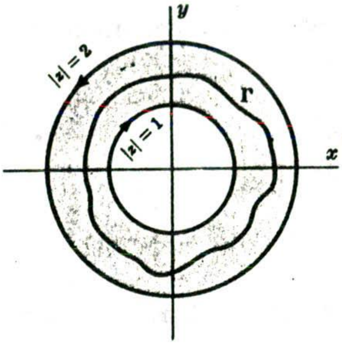

Fig. 4-3

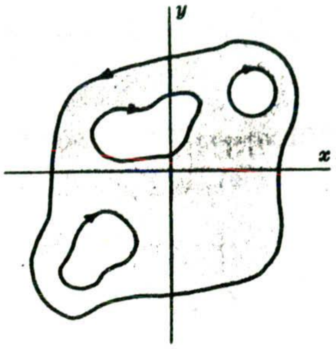

Fig. 4-4

Intuitively, a simply-connected region is one which does not have any "holes" in it, while a multiply-connected region is one which does. Thus the multiply-connected regions of Figures 4-3 and 4-4 have respectively one and three holes in them.

JORDAN CURVE THEOREM

Any continuous, closed curve which does not intersect itself and which may or may not have a finite length is called a *Jordan curve* [see Problem 30]. An important theorem which, although very difficult to prove, seems intuitively obvious is the following.

Jordan Curve Theorem. A Jordan curve divides the plane into two regions having the curve as common boundary. That region which is bounded [i.e. is such that all points of it satisfy $|z| < M$, where M is some positive constant] is called the *interior* or *inside* of the curve, while the other region is called the *exterior* or *outside* of the curve.

It follows from this that the region inside a simple closed curve is a simply-connected region whose boundary is the simple closed curve.

CONVENTION REGARDING TRAVERSAL OF A CLOSED PATH

The boundary C of a region is said to be traversed in the *positive sense* or *direction* if an observer travelling in this direction [and perpendicular to the plane] has the region to the left. This convention leads to the directions indicated by the arrows in Figures 4-2, 4-3 and 4-4. We use the special symbol

$$\oint_C f(z)\,dz$$

to denote integration of $f(z)$ around the boundary C in the positive sense. Note that in the case of a circle [Fig. 4-2] the positive direction is the *counterclockwise direction.* The integral around C is often called a *contour integral.*

GREEN'S THEOREM IN THE PLANE

Let $P(x,y)$ and $Q(x,y)$ be continuous and have continuous partial derivatives in a region $\mathcal{R}$ and on its boundary C. *Green's theorem* states that

$$\oint_C P\,dx + Q\,dy = \iint_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \tag{7}$$

The theorem is valid for both simply- and multiply-connected regions.

COMPLEX FORM OF GREEN'S THEOREM

Let $F(z,\bar{z})$ be continuous and have continuous partial derivatives in a region $\mathcal{R}$ and on its boundary C, where $z = x+iy$, $\bar{z} = x-iy$ are complex conjugate coordinates [see Page 7]. Then Green's theorem can be written in the complex form

$$\oint_C F(z,\bar{z})\,dz = 2i \iint_{\mathcal{R}} \frac{\partial F}{\partial \bar{z}}\,dA \tag{8}$$

where dA represents the element of area $dx\,dy$.

For a generalization of (*8*), see Problem 56.

CAUCHY'S THEOREM. THE CAUCHY-GOURSAT THEOREM

Let $f(z)$ be analytic in a region $\mathcal{R}$ and on its boundary C. Then

$$\oint_C f(z)\,dz = 0 \tag{9}$$

This fundamental theorem, often called *Cauchy's integral theorem* or briefly *Cauchy's theorem*, is valid for both simply- and multiply-connected regions. It was first proved by use of Green's theorem with the added restriction that $f'(z)$ be continuous in $\mathcal{R}$ [see Problem 11]. However, *Goursat* gave a proof which removed this restriction. For this reason the theorem is sometimes called the *Cauchy-Goursat theorem* [see Problems 13-16] when one desires to emphasize the removal of this restriction.

MORERA'S THEOREM

Let $f(z)$ be continuous in a simply-connected region $\mathcal{R}$ and suppose that

$$\oint_C f(z)\,dz = 0 \tag{10}$$

around *every* simple closed curve C in $\mathcal{R}$. Then $f(z)$ is analytic in $\mathcal{R}$.

This theorem, due to *Morera*, is often called the *converse of Cauchy's theorem*. It can be extended to multiply-connected regions. For a proof which assumes that $f'(z)$ is continuous in $\mathcal{R}$, see Problem 22. For a proof which eliminates this restriction, see Problem 7, Chapter 5.

INDEFINITE INTEGRALS

If $f(z)$ and $F(z)$ are analytic in a region $\mathcal{R}$ and such that $F'(z) = f(z)$, then $F(z)$ is called an *indefinite integral* or *anti-derivative* of $f(z)$ denoted by

$$F(z) = \int f(z)\,dz \tag{11}$$

Since the derivative of any constant is zero, it follows that any two indefinite integrals can differ by a constant. For this reason an arbitrary constant c is often added to the right of (*11*).

Example: Since $\frac{d}{dz}(3z^2 - 4\sin z) = 6z - 4\cos z$, we can write

$$\int (6z - 4\cos z)\,dz = 3z^2 - 4\sin z + c$$

INTEGRALS OF SPECIAL FUNCTIONS

Using results on Page 66 [or by direct differentiation], we can arrive at the following results (omitting a constant of integration).

1. $\int z^n\,dz = \dfrac{z^{n+1}}{n+1} \quad n \neq -1$
2. $\int \dfrac{dz}{z} = \ln z$
3. $\int e^z\,dz = e^z$
4. $\int a^z\,dz = \dfrac{a^z}{\ln a}$
5. $\int \sin z\,dz = -\cos z$
6. $\int \cos z\,dz = \sin z$
7. $\int \tan z\,dz = \ln \sec z = -\ln \cos z$
8. $\int \cot z\,dz = \ln \sin z$
9. $\int \sec z\,dz = \ln(\sec z + \tan z) = \ln \tan(z/2 + \pi/4)$
10. $\int \csc z\,dz = \ln(\csc z - \cot z) = \ln \tan(z/2)$
11. $\int \sec^2 z\,dz = \tan z$
12. $\int \csc^2 z\,dz = -\cot z$
13. $\int \sec z \tan z\,dz = \sec z$
14. $\int \csc z \cot z\,dz = -\csc z$
15. $\int \sinh z\,dz = \cosh z$
16. $\int \cosh z\,dz = \sinh z$
17. $\int \tanh z\,dz = \ln \cosh z$
18. $\int \coth z\,dz = \ln \sinh z$
19. $\int \operatorname{sech} z\,dz = \tan^{-1}(\sinh z)$
20. $\int \operatorname{csch} z\,dz = -\coth^{-1}(\cosh z)$
21. $\int \operatorname{sech}^2 z\,dz = \tanh z$
22. $\int \operatorname{csch}^2 z\,dz = -\coth z$
23. $\int \operatorname{sech} z \tanh z\,dz = -\operatorname{sech} z$
24. $\int \operatorname{csch} z \coth z\,dz = -\operatorname{csch} z$
25. $\int \dfrac{dz}{\sqrt{z^2 \pm a^2}} = \ln(z + \sqrt{z^2 \pm a^2})$
26. $\int \dfrac{dz}{z^2 + a^2} = \dfrac{1}{a}\tan^{-1}\dfrac{z}{a}$ or $-\dfrac{1}{a}\cot^{-1}\dfrac{z}{a}$
27. $\int \dfrac{dz}{z^2 - a^2} = \dfrac{1}{2a}\ln\left(\dfrac{z-a}{z+a}\right)$
28. $\int \dfrac{dz}{\sqrt{a^2 - z^2}} = \sin^{-1}\dfrac{z}{a}$ or $-\cos^{-1}\dfrac{z}{a}$
29. $\int \dfrac{dz}{z\sqrt{a^2 \pm z^2}} = \dfrac{1}{a}\ln\left(\dfrac{z}{a + \sqrt{a^2 \pm z^2}}\right)$
30. $\int \dfrac{dz}{z\sqrt{z^2 - a^2}} = \dfrac{1}{a}\cos^{-1}\dfrac{a}{z}$ or $\dfrac{1}{a}\sec^{-1}\dfrac{z}{a}$
31. $\int \sqrt{z^2 \pm a^2}\,dz = \dfrac{z}{2}\sqrt{z^2 \pm a^2} \pm \dfrac{a^2}{2}\ln(z + \sqrt{z^2 \pm a^2})$
32. $\int \sqrt{a^2 - z^2}\,dz = \dfrac{z}{2}\sqrt{a^2 - z^2} + \dfrac{a^2}{2}\sin^{-1}\dfrac{z}{a}$
33. $\int e^{az}\sin bz\,dz = \dfrac{e^{az}(a\sin bz - b\cos bz)}{a^2 + b^2}$
34. $\int e^{az}\cos bz\,dz = \dfrac{e^{az}(a\cos bz + b\sin bz)}{a^2 + b^2}$

SOME CONSEQUENCES OF CAUCHY'S THEOREM

Let $f(z)$ be analytic in a simply-connected region $\mathcal{R}$. Then the following theorems hold.

Theorem 1. If a and z are any two points in $\mathcal{R}$, then

$$\int_a^z f(z)\,dz$$

is *independent of the path* in $\mathcal{R}$ joining a and z.

Theorem 2. If a and z are any two points in $\mathcal{R}$ and

$$G(z) = \int_a^z f(z)\,dz \tag{12}$$

then $G(z)$ is analytic in $\mathcal{R}$ and $G'(z) = f(z)$.

Occasionally, confusion may arise because the variable of integration z in (12) is the same as the upper limit of integration. Since a definite integral depends only on the curve and limits of integration, any symbol can be used for the variable of integration, and for this reason we call it a *dummy variable* or *dummy symbol*. Thus (12) can be equivalently written

$$G(z) = \int_a^z f(\zeta)\,d\zeta \tag{13}$$

Theorem 3. If a and b are any two points in $\mathcal{R}$ and $F'(z) = f(z)$, then

$$\int_a^b f(z)\,dz = F(b) - F(a) \tag{14}$$

This can also be written in the form, familiar from elementary calculus,

$$\int_a^b F'(z)\,dz = F(z)\Big|_a^b = F(b) - F(a) \tag{15}$$

Example: $\displaystyle\int_{3i}^{1-i} 4z\,dz = 2z^2\Big|_{3i}^{1-i} = 2(1-i)^2 - 2(3i)^2 = 18 - 4i$

Theorem 4. Let $f(z)$ be analytic in a region bounded by two simple closed curves C and C_1 [where C_1 lies inside C as in Fig. 4-5 below] and on these curves. Then

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz \tag{16}$$

where C and C_1 are both traversed in the positive sense relative to their interiors [counter-clockwise in Fig. 4-5].

The result shows that if we wish to integrate $f(z)$ along curve C we can equivalently replace C by any curve C_1 so long as $f(z)$ is analytic in the region between C and C_1.

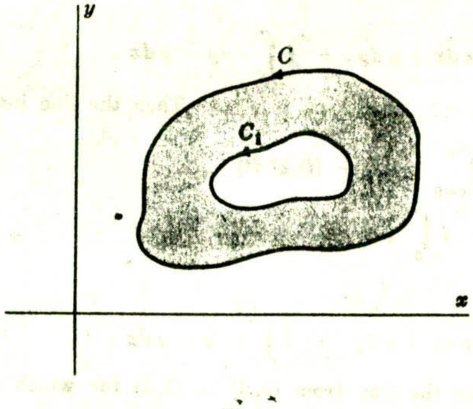

Fig. 4-5

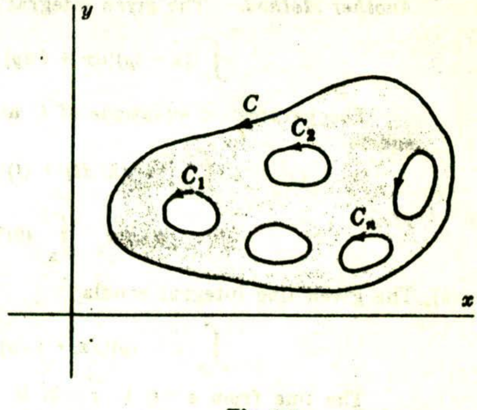

Fig. 4-6

Theorem 5. Let $f(z)$ be analytic in a region bounded by the non-overlapping simple closed curves $C, C_1, C_2, C_3, \ldots, C_n$ [where $C_1, C_2, \ldots, C_n$ are inside C as in Fig. 4-6 above] and on these curves. Then

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots + \oint_{C_n} f(z)\,dz \tag{17}$$

This is a generalization of Theorem 4.

Solved Problems

LINE INTEGRALS

1. Evaluate $\int_{(0,3)}^{(2,4)} (2y + x^2)\,dx + (3x - y)\,dy$ along: (a) the parabola $x = 2t$, $y = t^2 + 3$; (b) straight lines from (0,3) to (2,3) and then from (2,3) to (2,4); (c) a straight line from (0,3) to (2,4).

(a) The points (0,3) and (2,4) on the parabola correspond to $t = 0$ and $t = 1$ respectively. Then the given integral equals

$$\int_{t=0}^{1} \{2(t^2+3) + (2t)^2\}\,2\,dt + \{3(2t) - (t^2+3)\}\,2t\,dt = \int_0^1 (24t^2 + 12 - 2t^3 - 6t)\,dt = 33/2$$

(b) Along the straight line from (0,3) to (2,3), $y = 3$, $dy = 0$ and the line integral equals

$$\int_{x=0}^{2} (6 + x^2)\,dx + (3x - 3)0 = \int_{x=0}^{2} (6 + x^2)\,dx = 44/3$$

Along the straight line from (2,3) to (2,4), $x = 2$, $dx = 0$ and the line integral equals

$$\int_{y=3}^{4} (2y + 4)0 + (6 - y)\,dy = \int_{y=3}^{4} (6 - y)\,dy = 5/2$$

Then the required value $= 44/3 + 5/2 = 103/6$.

(c) An equation for the line joining (0,3) and (2,4) is $2y - x = 6$. Solving for x, we have $x = 2y - 6$. Then the line integral equals

$$\int_{y=3}^{4} \{2y + (2y-6)^2\}\,2\,dy + \{3(2y-6) - y\}\,dy = \int_3^4 (8y^2 - 39y + 54)\,dy = 97/6$$

The result can also be obtained by using $y = \frac{1}{2}(x + 6)$.

2. Evaluate $\int_C \bar{z}\,dz$ from $z = 0$ to $z = 4 + 2i$ along the curve C given by (a) $z = t^2 + it$, (b) the line from $z = 0$ to $z = 2i$ and then the line from $z = 2i$ to $z = 4 + 2i$.

(a) The points $z = 0$ and $z = 4 + 2i$ on C correspond to $t = 0$ and $t = 2$ respectively. Then the line integral equals

$$\int_{t=0}^{2} (\overline{t^2 + it})\,d(t^2 + it) = \int_0^2 (t^2 - it)(2t + i)\,dt = \int_0^2 (2t^3 - it^2 + t)\,dt = 10 - 8i/3$$

Another Method. The given integral equals

$$\int_C (x - iy)(dx + i\,dy) = \int_C x\,dx + y\,dy + i\int_C x\,dy - y\,dx$$

The parametric equations of C are $x = t^2$, $y = t$ from $t = 0$ to $t = 2$. Then the line integral equals

$$\int_{t=0}^{2} (t^2)(2t\,dt) + (t)(dt) + i\int_{t=0}^{2} (t^2)(dt) - (t)(2t\,dt)$$
$$= \int_0^2 (2t^3 + t)\,dt + i\int_0^2 (-t^2)\,dt = 10 - 8i/3$$

(b) The given line integral equals

$$\int_C (x - iy)(dx + i\,dy) = \int_C x\,dx + y\,dy + i\int_C x\,dy - y\,dx$$

The line from $z = 0$ to $z = 2i$ is the same as the line from (0,0) to (0,2) for which $x = 0$, $dx = 0$ and the line integral equals

$$\int_{y=0}^{2} (0)(0) + y\,dy + i\int_{y=0}^{2} (0)(dy) - y(0) = \int_{y=0}^{2} y\,dy = 2$$

The line from $z = 2i$ to $z = 4 + 2i$ is the same as the line from (0,2) to (4,2) for which $y = 2$, $dy = 0$ and the line integral equals

$$\int_{x=0}^{4} x\,dx + 2\cdot 0 + i\int_{x=0}^{4} x\cdot 0 - 2\,dx = \int_0^4 x\,dx + i\int_0^4 -2\,dx = 8 - 8i$$

Then the required value $= 2 + (8 - 8i) = 10 - 8i$.

3. Prove that if $f(z)$ is integrable along a curve C having finite length L and if there exists a positive number M such that $|f(z)| \leqq M$ on C, then

$$\left|\int_C f(z)\,dz\right| \leqq ML$$

By definition we have on using the notation of Page 92,

$$\int_C f(z)\,dz = \lim_{n\to\infty}\sum_{k=1}^{n} f(\xi_k)\,\Delta z_k \tag{1}$$

Now

$$\left.\begin{aligned}\left|\sum_{k=1}^{n} f(\xi_k)\,\Delta z_k\right| &\leqq \sum_{k=1}^{n}|f(\xi_k)|\,|\Delta z_k|\\ &\leqq M\sum_{k=1}^{n}|\Delta z_k|\\ &\leqq ML\end{aligned}\right\} \tag{2}$$

where we have used the facts that $|f(z)| \leqq M$ for all points z on C and that $\sum_{k=1}^{n}|\Delta z_k|$ represents the sum of all the chord lengths joining points z_{k-1} and z_k, where $k = 1, 2, \ldots, n$, and that this sum is not greater than the length of C.

Taking the limit of both sides of (2), using (1), the required result follows.

It is possible to show, more generally, that

$$\left|\int_C f(z)\,dz\right| \leqq \int_C |f(z)|\,|dz|$$

GREEN'S THEOREM IN THE PLANE

4. Prove Green's theorem in the plane if C is a simple closed curve which has the property that any straight line parallel to the coordinate axes cuts C in at most two points.

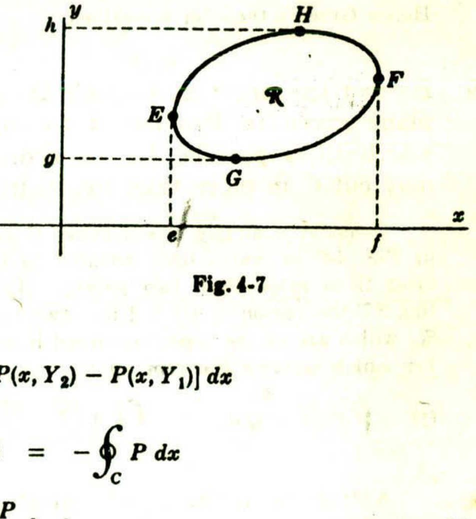

Fig. 4-7

Let the equations of the curves EGF and EHF (see Fig. 4-7) be $y = Y_1(x)$ and $y = Y_2(x)$ respectively. If $\mathcal{R}$ is the region bounded by C, we have

$$\begin{aligned}\iint_{\mathcal{R}} \frac{\partial P}{\partial y}\,dx\,dy &= \int_{x=e}^{f}\left[\int_{y=Y_1(x)}^{Y_2(x)}\frac{\partial P}{\partial y}\,dy\right]dx\\ &= \int_{x=e}^{f} P(x,y)\Big|_{y=Y_1(x)}^{Y_2(x)}\,dx = \int_e^f [P(x,Y_2) - P(x,Y_1)]\,dx\\ &= -\int_e^f P(x,Y_1)\,dx - \int_f^e P(x,Y_2)\,dx = -\oint_C P\,dx\end{aligned}$$

Then

$$\oint_C P\,dx = -\iint_{\mathcal{R}}\frac{\partial P}{\partial y}\,dx\,dy \tag{1}$$

Similarly let the equations of curves GEH and GFH be $x = X_1(y)$ and $x = X_2(y)$ respectively. Then

$$\begin{aligned}\iint_{\mathcal{R}}\frac{\partial Q}{\partial x}\,dx\,dy &= \int_{y=g}^{h}\left[\int_{x=X_1(y)}^{X_2(y)}\frac{\partial Q}{\partial x}\,dx\right]dy = \int_g^h [Q(X_2,y) - Q(X_1,y)]\,dy\\ &= \int_h^g Q(X_1,y)\,dy + \int_g^h Q(X_2,y)\,dy = \oint_C Q\,dy\end{aligned}$$

Then

$$\oint_C Q\,dy = \iint_{\mathcal{R}}\frac{\partial Q}{\partial x}\,dx\,dy \tag{2}$$

Adding (1) and (2), $\displaystyle\oint_C P\,dx + Q\,dy = \iint_{\mathcal{R}}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$

5. Verify Green's theorem in the plane for

$$\oint_C (2xy - x^2)\,dx + (x + y^2)\,dy$$

where C is the closed curve of the region bounded by $y = x^2$ and $y^2 = x$.

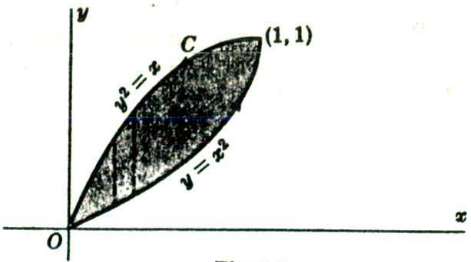

Fig. 4-8

The plane curves $y = x^2$ and $y^2 = x$ intersect at $(0,0)$ and $(1,1)$. The positive direction in traversing C is as shown in Fig. 4-8.

Along $y = x^2$, the line integral equals

$$\int_{x=0}^{1} \{(2x)(x^2) - x^2\}\,dx + \{x + (x^2)^2\}\,d(x^2) \quad = \quad \int_0^1 (2x^3 + x^2 + 2x^5)\,dx \quad = \quad 7/6$$

Along $y^2 = x$, the line integral equals

$$\int_{y=1}^{0} \{2(y^2)(y) - (y^2)^2\}\,d(y^2) + \{y^2 + y^2\}\,dy \quad = \quad \int_1^0 (4y^4 - 2y^5 + 2y^2)\,dy \quad = \quad -17/15$$

Then the required integral $= 7/6 - 17/15 = 1/30$.

$$\iint_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \quad = \quad \iint_{\mathcal{R}} \left\{\frac{\partial}{\partial x}(x + y^2) - \frac{\partial}{\partial y}(2xy - x^2)\right\} dx\,dy$$

$$= \quad \iint_{\mathcal{R}} (1 - 2x)\,dx\,dy \quad = \quad \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} (1 - 2x)\,dy\,dx$$

$$= \quad \int_{x=0}^{1} (y - 2xy)\Big|_{y=x^2}^{\sqrt{x}}\,dx \quad = \quad \int_0^1 (x^{1/2} - 2x^{3/2} - x^2 + 2x^3)\,dx \quad = \quad 1/30$$

Hence Green's theorem is verified.

6. Extend the proof of Green's theorem in the plane given in Problem 4 to curves C for which lines parallel to the coordinate axes may cut C in more than two points.

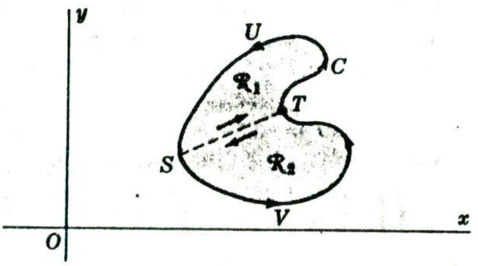

Fig. 4-9

Consider a simple closed curve C such as shown in Fig. 4-9 in which lines parallel to the axes may meet C in more than two points. By constructing line ST the region is divided into two regions $\mathcal{R}_1$ and $\mathcal{R}_2$ which are of the type considered in Problem 4 and for which Green's theorem applies, i.e.,

$$(1)\quad \int_{STUS} P\,dx + Q\,dy \;=\; \iint_{\mathcal{R}_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy, \qquad (2)\quad \int_{SVTS} P\,dx + Q\,dy \;=\; \iint_{\mathcal{R}_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Adding the left-hand sides of (1) and (2), we have, omitting the integrand $P\,dx + Q\,dy$ in each case,

$$\int_{STUS} + \int_{SVTS} \;=\; \int_{ST} + \int_{TUS} + \int_{SVT} + \int_{TS} \;=\; \int_{TUS} + \int_{SVT} \;=\; \int_{TUSVT}$$

using the fact that $\displaystyle\int_{ST} = -\int_{TS}$.

Adding the right-hand sides of (1) and (2), omitting the integrand,

$$\iint_{\mathcal{R}_1} + \iint_{\mathcal{R}_2} \;=\; \iint_{\mathcal{R}}$$

Then

$$\int_{TUSVT} P\,dx + Q\,dy \;=\; \iint_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

and the theorem is proved. We have proved Green's theorem for the simply-connected region of Fig. 4-9 bounded by the simple closed curve C. For more complicated regions it may be necessary to construct more lines, such as ST, to establish the theorem.

Green's theorem is also true for multiply-connected regions, as shown in Problem 7.

7. Show that Green's theorem in the plane is also valid for a multiply-connected region $\mathcal{R}$ such as shown shaded in Fig. 4-10.

The boundary of $\mathcal{R}$, which consists of the exterior boundary $AHJKLA$ and the interior boundary $DEFGD$, is to be traversed in the positive direction so that a person travelling in this direction always has the region on his left. It is seen that the positive directions are as indicated in the figure.

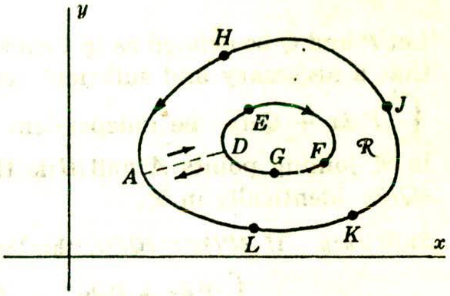

Fig. 4-10

In order to establish the theorem construct a line, such as AD, called a *cross-cut*, connecting the exterior and interior boundaries. The region bounded by $ADEFGDALKJHA$ is simply-connected, and so Green's theorem is valid. Then

$$\oint_{ADEFGDALKJHA} P\,dx + Q\,dy \quad = \quad \iint_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

But the integral on the left, leaving out the integrand, is equal to

$$\int_{AD} + \int_{DEFGD} + \int_{DA} + \int_{ALKJHA} \quad = \quad \int_{DEFGD} + \int_{ALKJHA}$$

since $\int_{AD} = -\int_{DA}$. Thus if C_1 is the curve $ALKJHA$, C_2 is the curve $DEFGD$ and C is the boundary of $\mathcal{R}$ consisting of C_1 and C_2 (traversed in the positive directions with respect to $\mathcal{R}$), then $\int_{C_1} + \int_{C_2} = \oint_C$ and so

$$\oint_C P\,dx + Q\,dy \quad = \quad \iint_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

8. Let $P(x, y)$ and $Q(x, y)$ be continuous and have continuous first partial derivatives at each point of a simply-connected region $\mathcal{R}$. Prove that a necessary and sufficient condition that $\oint_C P\,dx + Q\,dy = 0$ around every closed path C in $\mathcal{R}$ is that $\partial P/\partial y = \partial Q/\partial x$ identically in $\mathcal{R}$.

Sufficiency. Suppose $\partial P/\partial y = \partial Q/\partial x$. Then by Green's theorem,

$$\oint_C P\,dx + Q\,dy \quad = \quad \iint_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \quad = \quad 0$$

where $\mathcal{R}$ is the region bounded by C.

Necessity.

Suppose $\oint_C P\,dx + Q\,dy = 0$ around every closed path C in $\mathcal{R}$ and that $\partial P/\partial y \neq \partial Q/\partial x$ at some point of $\mathcal{R}$. In particular suppose $\partial P/\partial y - \partial Q/\partial x > 0$ at the point (x_0, y_0).

By hypothesis $\partial P/\partial y$ and $\partial Q/\partial x$ are continuous in $\mathcal{R}$ so that there must be some region τ containing (x_0, y_0) as an interior point for which $\partial P/\partial y - \partial Q/\partial x > 0$. If Γ is the boundary of τ, then by Green's theorem

$$\oint_\Gamma P\,dx + Q\,dy \quad = \quad \iint_\tau \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \quad > \quad 0$$

contradicting the hypothesis that $\oint_C P\,dx + Q\,dy = 0$ for *all* closed curves in $\mathcal{R}$. Thus $\partial Q/\partial x - \partial P/\partial y$ cannot be positive.

Similarly we can show that $\partial Q/\partial x - \partial P/\partial y$ cannot be negative and it follows that it must be identically zero, i.e. $\partial P/\partial y = \partial Q/\partial x$ identically in $\mathcal{R}$.

The results can be extended to multiply-connected regions.

9. Let P and Q be defined as in Problem 8. Prove that a necessary and sufficient condition that $\int_A^B P\,dx + Q\,dy$ be independent of the path in $\mathcal{R}$ joining points A and B is that $\partial P/\partial y = \partial Q/\partial x$ identically in $\mathcal{R}$.

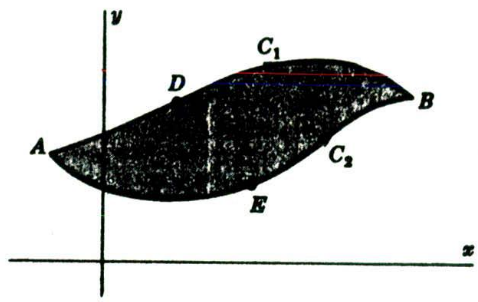

Fig. 4-11

Sufficiency. If $\partial P/\partial y = \partial Q/\partial x$, then by Problem 8

$$\int_{ADBEA} P\,dx + Q\,dy \quad = \quad 0$$

[see Fig. 4-11]. From this, omitting for brevity the integrand $P\,dx + Q\,dy$, we have

$$\int_{ADB} + \int_{BEA} \;=\; 0, \qquad \int_{ADB} = -\int_{BEA} = \int_{AEB} \qquad \text{and so} \qquad \int_{C_1} = \int_{C_2}$$

i.e. the integral is independent of the path.

Necessity.

If the integral is independent of the path, then for all paths C_1 and C_2 in $\mathcal{R}$ we have

$$\int_{C_1} = \int_{C_2}, \qquad \int_{ADB} = \int_{AEB} \qquad \text{and} \qquad \int_{ADBEA} = 0$$

From this it follows that the line integral around any closed path in $\mathcal{R}$ is zero, and hence by Problem 8 that $\partial P/\partial y = \partial Q/\partial x$.

The results can be extended to multiply-connected regions.

COMPLEX FORM OF GREEN'S THEOREM

10. If $B(z, \bar{z})$ is continuous and has continuous partial derivatives in a region $\mathcal{R}$ and on its boundary C, where $z = x + iy$ and $\bar{z} = x - iy$, prove that Green's theorem can be written in complex form as

$$\oint_C B(z, \bar{z})\,dz \quad = \quad 2i \iint_{\mathcal{R}} \frac{\partial B}{\partial \bar{z}}\,dx\,dy$$

Let $B(z, \bar{z}) = P(x, y) + iQ(x, y)$. Then using Green's theorem, we have

$$\begin{aligned}
\oint_C B(z,\bar{z})\,dz &= \oint_C (P + iQ)(dx + i\,dy) \quad = \quad \oint_C P\,dx - Q\,dy \;+\; i\oint_C Q\,dx + P\,dy \\
&= -\iint_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right) dx\,dy \;+\; i\iint_{\mathcal{R}} \left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right) dx\,dy \\
&= i\iint_{\mathcal{R}} \left[\left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right) + i\left(\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right)\right] dx\,dy \\
&= 2i\iint_{\mathcal{R}} \frac{\partial B}{\partial \bar{z}}\,dx\,dy
\end{aligned}$$

from Problem 34, Page 83. The result can also be written in terms of curl B [see Page 70].

CAUCHY'S THEOREM AND THE CAUCHY-GOURSAT THEOREM

11. Prove Cauchy's theorem $\oint_C f(z)\,dz = 0$ if $f(z)$ is analytic with derivative $f'(z)$ which is continuous at all points inside and on a simple closed curve C.

Since $f(z) = u + iv$ is analytic and has a continuous derivative

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

it follows that the partial derivatives $(1)\ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $(2)\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ are continuous inside and on C. Thus Green's theorem can be applied and we have

$$\oint_C f(z)\,dz = \oint_C (u+iv)(dx+i\,dy) = \oint_C u\,dx - v\,dy + i\oint_C v\,dx + u\,dy$$

$$= \iint_{\mathcal{R}} \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_{\mathcal{R}} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy = 0$$

using the Cauchy-Riemann equations (*1*) and (*2*).

By using the fact that Green's theorem is applicable to multiply-connected regions, we can extend the result to multiply-connected regions under the given conditions on $f(z)$.

The *Cauchy-Goursat theorem* [see Problems 13-16] removes the restriction that $f'(z)$ be continuous.

Another method.

The result can be obtained from the complex form of Green's theorem [Problem 10] by noting that if $B(z,\bar{z}) = f(z)$ is independent of $\bar{z}$, then $\partial B/\partial \bar{z} = 0$ and so $\oint_C f(z)\,dz = 0$.

12. Prove (*a*) $\oint_C dz = 0$, (*b*) $\oint_C z\,dz = 0$, (*c*) $\oint_C (z-z_0)\,dz = 0$ where C is any simple closed curve and z_0 is a constant.

These follow at once from Cauchy's theorem since the functions 1, z and $z - z_0$ are analytic inside C and have continuous derivatives.

The results can also be established directly from the definition of an integral (see Problem 90).

13. Prove the *Cauchy-Goursat* theorem for the case of a triangle.

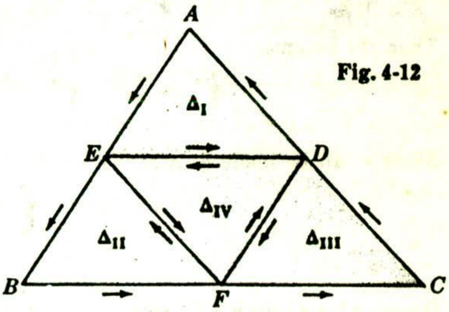

Fig. 4-12

Consider any triangle in the z plane such as ABC, denoted briefly by Δ, in Fig. 4-12. Join the midpoints D, E and F of sides AB, AC and BC respectively to form four triangles indicated briefly by Δ_{I}, Δ_{II}, Δ_{III} and Δ_{IV}.

If $f(z)$ is analytic inside and on triangle ABC we have, omitting the integrand on the right,

$$\oint_{ABCA} f(z)\,dz = \int_{DAE} + \int_{EBF} + \int_{FCD}$$

$$= \left\{\int_{DAE} + \int_{ED}\right\} + \left\{\int_{EBF} + \int_{FE}\right\} + \left\{\int_{FCD} + \int_{DF}\right\} + \left\{\int_{DE} + \int_{EF} + \int_{FD}\right\}$$

$$= \int_{DAED} + \int_{EBFE} + \int_{FCDF} + \int_{DEFD}$$

$$= \oint_{\Delta_{\text{I}}} f(z)\,dz + \oint_{\Delta_{\text{II}}} f(z)\,dz + \oint_{\Delta_{\text{III}}} f(z)\,dz + \oint_{\Delta_{\text{IV}}} f(z)\,dz$$

where in the second line we have made use of the fact that

$$\int_{ED} = -\int_{DE}, \quad \int_{FE} = -\int_{EF}, \quad \int_{DF} = -\int_{FD}$$

Then

$$\left|\oint_{\Delta} f(z)\,dz\right| \leqq \left|\oint_{\Delta_{\text{I}}} f(z)\,dz\right| + \left|\oint_{\Delta_{\text{II}}} f(z)\,dz\right| + \left|\oint_{\Delta_{\text{III}}} f(z)\,dz\right| + \left|\oint_{\Delta_{\text{IV}}} f(z)\,dz\right| \qquad (1)$$

Let Δ_1 be the triangle corresponding to that term on the right of *(1)* having largest value (if there are two or more such terms then Δ_1 is any of the associated triangles). Then

$$\left|\oint_{\Delta} f(z)\,dz\right| \leqq 4\left|\oint_{\Delta_1} f(z)\,dz\right| \qquad (2)$$

By joining midpoints of the sides of triangle Δ_1, we obtain similarly a triangle Δ_2 such that

$$\left|\oint_{\Delta_1} f(z)\,dz\right| \leqq 4\left|\oint_{\Delta_2} f(z)\,dz\right| \qquad (3)$$

so that

$$\left|\oint_{\Delta} f(z)\,dz\right| \leqq 4^2\left|\oint_{\Delta_2} f(z)\,dz\right| \qquad (4)$$

After n steps we obtain a triangle Δ_n such that

$$\left|\oint_{\Delta} f(z)\,dz\right| \leqq 4^n\left|\oint_{\Delta_n} f(z)\,dz\right| \qquad (5)$$

Now $\Delta, \Delta_1, \Delta_2, \Delta_3, \ldots$ is a sequence of triangles each of which is contained in the preceding (i.e. a sequence of *nested triangles*) and there exists a point z_0 which lies in every triangle of the sequence.

Since z_0 lies inside or on the boundary of Δ, it follows that $f(z)$ is analytic at z_0. Then by Problem 21, Page 78,

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \eta(z - z_0) \qquad (6)$$

where for any $\epsilon > 0$ we can find δ such that $|\eta| < \epsilon$ whenever $|z - z_0| < \delta$.

Thus by integration of both sides of *(6)* and using Problem 12,

$$\oint_{\Delta_n} f(z)\,dz = \oint_{\Delta_n} \eta(z - z_0)\,dz \qquad (7)$$

Now if P is the perimeter of Δ, then the perimeter of Δ_n is $P_n = P/2^n$. If z is any point on Δ_n, then as seen from Fig. 4-13 we must have $|z - z_0| < P/2^n < \delta$. Hence from *(7)* and Property 5, Page 93 we have

$$\left|\oint_{\Delta_n} f(z)\,dz\right| = \left|\oint_{\Delta_n} \eta(z - z_0)\,dz\right| \leqq \epsilon \cdot \frac{P}{2^n} \cdot \frac{P}{2^n} = \frac{\epsilon P^2}{4^n}$$

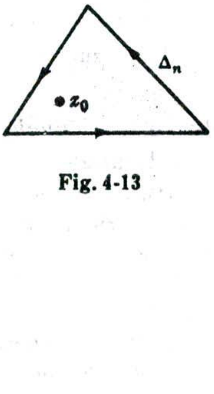

Fig. 4-13

Then *(5)* becomes

$$\left|\oint_{\Delta} f(z)\,dz\right| \leqq 4^n \cdot \frac{\epsilon P^2}{4^n} = \epsilon P^2$$

Since ϵ can be made arbitrarily small it follows that, as required,

$$\oint_{\Delta} f(z)\,dz = 0$$

14. Prove the Cauchy-Goursat theorem for any closed polygon.

Consider for example a closed polygon $ABCDEFA$ such as indicated in Fig. 4-14. By constructing the lines BF, CF and DF the polygon is subdivided into triangles. Then by Cauchy's theorem for triangles [Problem 13] and the fact that the integrals along BF and FB, CF and FC, DF and FD cancel, we find as required

$$\int_{ABCDEFA} f(z)\,dz = \int_{ABFA} f(z)\,dz + \int_{BCFB} f(z)\,dz + \int_{CDFC} f(z)\,dz + \int_{DEFD} f(z)\,dz = 0$$

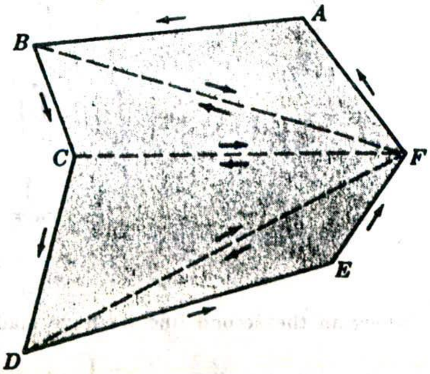

Fig. 4-14

where we suppose that $f(z)$ is analytic inside and on the polygon.

It should be noted that we have proved the result for simple polygons whose sides do not cross. A proof can also be given for any polygon which intersects itself (see Problem 66).

15. Prove the Cauchy-Goursat theorem for any simple closed curve.

Let us assume that C is contained in a region $\mathcal{R}$ in which $f(z)$ is analytic.

Choose n points of subdivision $z_1, z_2, \ldots, z_n$ on curve C [Fig. 4-15] where for convenience of notation we consider $z_0 = z_n$. Construct polygon P by joining these points.

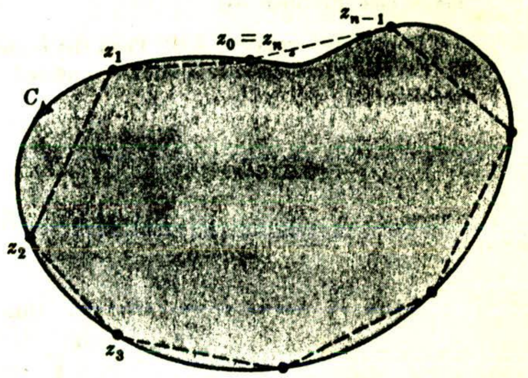

Fig. 4-15

Let us define the sum

$$S_n = \sum_{k=1}^{n} f(z_k)\,\Delta z_k$$

where $\Delta z_k = z_k - z_{k-1}$. Since

$$\lim S_n = \oint_C f(z)\,dz$$

[where the limit on the left means that $n \to \infty$ in such a way that the largest of $|\Delta z_k| \to 0$], it follows that given any $\epsilon > 0$ we can choose N so that for $n > N$

$$\left| \oint_C f(z)\,dz - S_n \right| < \frac{\epsilon}{2} \tag{1}$$

Consider now the integral along polygon P. Since this is zero by Problem 14, we have

$$\begin{aligned}\oint_P f(z)\,dz = 0 &= \int_{z_0}^{z_1} f(z)\,dz + \int_{z_1}^{z_2} f(z)\,dz + \cdots + \int_{z_{n-1}}^{z_n} f(z)\,dz \\ &= \int_{z_0}^{z_1} \{f(z) - f(z_1) + f(z_1)\}\,dz + \cdots + \int_{z_{n-1}}^{z_n} \{f(z) - f(z_n) + f(z_n)\}\,dz \\ &= \int_{z_0}^{z_1} \{f(z) - f(z_1)\}\,dz + \cdots + \int_{z_{n-1}}^{z_n} \{f(z) - f(z_n)\}\,dz + S_n\end{aligned}$$

so that

$$S_n = \int_{z_0}^{z_1} \{f(z_1) - f(z)\}\,dz + \cdots + \int_{z_{n-1}}^{z_n} \{f(z_n) - f(z)\}\,dz \tag{2}$$

Let us now choose N so large that on the lines joining z_0 and z_1, z_1 and $z_2, \ldots, z_{n-1}$ and z_n,

$$|f(z_1) - f(z)| < \frac{\epsilon}{2L}, \quad |f(z_2) - f(z)| < \frac{\epsilon}{2L}, \quad \ldots, \quad |f(z_n) - f(z)| < \frac{\epsilon}{2L} \tag{3}$$

where L is the length of C. Then from (2) and (3) we have

$$|S_n| \leq \left| \int_{z_0}^{z_1} \{f(z_1) - f(z)\}\,dz \right| + \left| \int_{z_1}^{z_2} \{f(z_2) - f(z)\}\,dz \right| + \cdots + \left| \int_{z_{n-1}}^{z_n} \{f(z_n) - f(z)\}\,dz \right|$$

or

$$|S_n| \leq \frac{\epsilon}{2L}\{|z_1 - z_0| + |z_2 - z_1| + \cdots + |z_n - z_{n-1}|\} = \frac{\epsilon}{2} \tag{4}$$

From

$$\oint_C f(z)\,dz = \oint_C f(z)\,dz - S_n + S_n$$

we have, using (1) and (4),

$$\left| \oint_C f(z)\,dz \right| \leq \left| \oint_C f(z)\,dz - S_n \right| + |S_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus since ϵ is arbitrary, it follows that $\oint_C f(z)\,dz = 0$ as required.

16. Prove the Cauchy-Goursat theorem for multiply-connected regions.

We shall present a proof for the multiply-connected region $\mathcal{R}$ bounded by the simple closed curves C_1 and C_2 as indicated in Fig. 4-16. Extensions to other multiply-connected regions are easily made (see Problem 67).

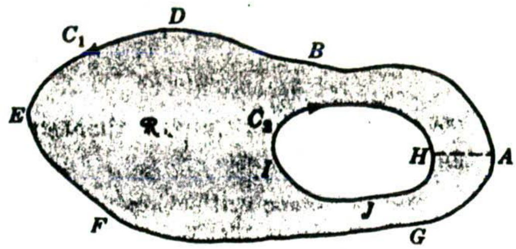

Fig. 4-16

Construct cross-cut AH. Then the region bounded by $ABDEFGAHJIHA$ is simply-connected so that by Problem 15,

$$\oint_{ABDEFGAHJIHA} f(z)\,dz = 0$$

Hence

$$\int_{ABDEFGA} f(z)\,dz + \int_{AH} f(z)\,dz + \int_{HJIH} f(z)\,dz + \int_{HA} f(z)\,dz = 0$$

Since $\int_{AH} f(z)\,dz = -\int_{HA} f(z)\,dz$, this becomes

$$\int_{ABDEFGA} f(z)\,dz + \int_{HJIH} f(z)\,dz = 0$$

This however amounts to saying that

$$\oint_C f(z)\,dz = 0$$

where C is the complete boundary of $\mathcal{R}$ (consisting of $ABDEFGA$ and $HJIH$) traversed in the sense that an observer walking on the boundary always has the region $\mathcal{R}$ on his left.

CONSEQUENCES OF CAUCHY'S THEOREM

17. If $f(z)$ is analytic in a simply-connected region $\mathcal{R}$, prove that $\int_a^b f(z)\,dz$ is independent of the path in $\mathcal{R}$ joining any two points a and b in $\mathcal{R}$.

By Cauchy's theorem,

$$\int_{ADBEA} f(z)\,dz = 0$$

or

$$\int_{ADB} f(z)\,dz + \int_{BEA} f(z)\,dz = 0$$

Hence

$$\int_{ADB} f(z)\,dz = -\int_{BEA} f(z)\,dz = \int_{AEB} f(z)\,dz$$

Thus

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz = \int_a^b f(z)\,dz$$

which yields the required result.

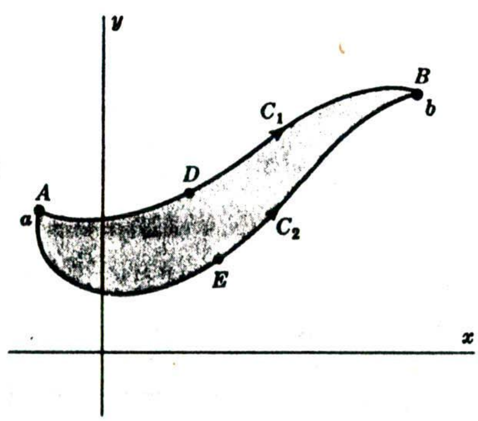

Fig. 4-17

18. Let $f(z)$ be analytic in a simply-connected region $\mathcal{R}$ and let a and z be points in $\mathcal{R}$. Prove that (a) $F(z) = \int_a^z f(u)\,du$ is analytic in $\mathcal{R}$ and (b) $F'(z) = f(z)$.

We have

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\left\{\int_a^{z+\Delta z} f(u)\,du - \int_a^z f(u)\,du\right\} - f(z)$$

$$= \frac{1}{\Delta z}\int_z^{z+\Delta z} [f(u) - f(z)]\,du \qquad (1)$$

By Cauchy's theorem, the last integral is independent of the path joining z and $z + \Delta z$ so long as the path is in $\mathcal{R}$. In particular we can choose as path the straight line segment joining z and $z + \Delta z$ (see Fig. 4-18) provided we choose $|\Delta z|$ small enough so that this path lies in $\mathcal{R}$.

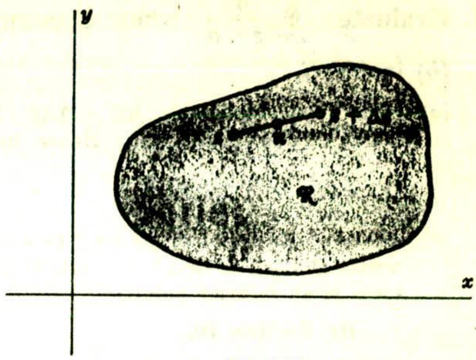

Fig. 4-18

Now by the continuity of $f(z)$ we have for all points u on this straight line path $|f(u) - f(z)| < \epsilon$ whenever $|u - z| < \delta$, which will certainly be true if $|\Delta z| < \delta$.

Furthermore, we have

$$\left| \int_z^{z+\Delta z} [f(u) - f(z)]\,du \right| \quad < \quad \epsilon|\Delta z| \qquad (2)$$

so that from (1)

$$\left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| \quad = \quad \frac{1}{|\Delta z|} \left| \int_z^{z+\Delta z} [f(u) - f(z)]\,du \right| \quad < \quad \epsilon$$

for $|\Delta z| < \delta$. This, however, amounts to saying that $\lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} = f(z)$, i.e. $F(z)$ is analytic and $F'(z) = f(z)$.

19. A function $F(z)$ such that $F'(z) = f(z)$ is called an *indefinite integral* of $f(z)$ and is denoted by $\int f(z)\,dz$. Show that (a) $\int \sin z\,dz = -\cos z + c$, (b) $\int \frac{dz}{z} = \ln z + c$ where c is an arbitrary constant.

(a) Since $\frac{d}{dz}(-\cos z + c) = \sin z$, we have $\int \sin z\,dz = -\cos z + c$.

(b) Since $\frac{d}{dz}(\ln z + c) = \frac{1}{z}$, we have $\int \frac{dz}{z} = \ln z + c$.

20. Let $f(z)$ be analytic in a region $\mathcal{R}$ bounded by two simple closed curves C_1 and C_2 [shaded in Fig. 4-19] and also on C_1 and C_2. Prove that $\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$, where C_1 and C_2 are both traversed in the positive sense relative to their interiors [counterclockwise in Fig. 4-19].

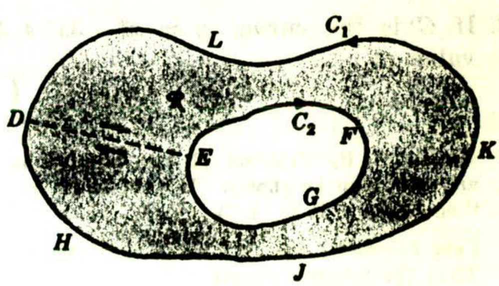

Fig. 4-19

Construct cross-cut DE. Then since $f(z)$ is analytic in the region $\mathcal{R}$, we have by Cauchy's theorem

$$\int_{DEFGEDHJKLD} f(z)\,dz \quad = \quad 0$$

or

$$\int_{DE} f(z)\,dz + \int_{EFGE} f(z)\,dz + \int_{ED} f(z)\,dz + \int_{DHJKLD} f(z)\,dz \quad = \quad 0$$

Hence since $\int_{DE} f(z)\,dz = -\int_{ED} f(z)\,dz$,

$$\int_{DHJKLD} f(z)\,dz \quad = \quad -\int_{EFGE} f(z)\,dz \quad = \quad \int_{EGFE} f(z)\,dz \qquad \text{or} \qquad \oint_{C_1} f(z)\,dz \quad = \quad \oint_{C_2} f(z)\,dz$$

21. Evaluate $\oint_C \frac{dz}{z-a}$ where C is any simple closed curve C and $z = a$ is (a) outside C, (b) inside C.

(a) If a is outside C, then $f(z) = 1/(z-a)$ is analytic everywhere inside and on C. Hence by Cauchy's theorem, $\oint_C \frac{dz}{z-a} = 0$.

(b) Suppose a is inside C and let Γ be a circle of radius ϵ with centre at $z = a$ so that Γ is inside C [this can be done since $z = a$ is an interior point].

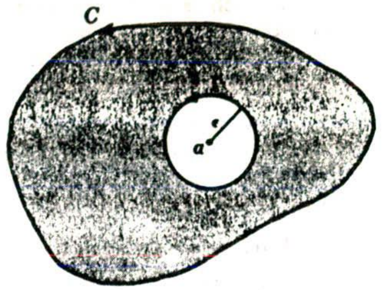

Fig. 4-20

By Problem 20,

$$\oint_C \frac{dz}{z-a} = \oint_\Gamma \frac{dz}{z-a} \qquad (1)$$

Now on Γ, $|z-a| = \epsilon$ or $z - a = \epsilon e^{i\theta}$, i.e. $z = a + \epsilon e^{i\theta}$, $0 \leqq \theta < 2\pi$. Thus since $dz = i\epsilon e^{i\theta}\,d\theta$, the right side of (1) becomes

$$\int_{\theta=0}^{2\pi} \frac{i\epsilon\, e^{i\theta}\,d\theta}{\epsilon e^{i\theta}} = i\int_0^{2\pi} d\theta = 2\pi i$$

which is the required value.

22. Evaluate $\oint_C \frac{dz}{(z-a)^n}$, $n = 2, 3, 4, \ldots$ where $z = a$ is inside the simple closed curve C.

As in Problem 21,

$$\oint_C \frac{dz}{(z-a)^n} = \oint_\Gamma \frac{dz}{(z-a)^n}$$
$$= \int_0^{2\pi} \frac{i\epsilon\, e^{i\theta}\,d\theta}{\epsilon^n e^{in\theta}} = \frac{i}{\epsilon^{n-1}} \int_0^{2\pi} e^{(1-n)i\theta}\,d\theta$$
$$= \frac{i}{\epsilon^{n-1}} \frac{e^{(1-n)i\theta}}{(1-n)i}\bigg|_0^{2\pi} = \frac{1}{(1-n)\,\epsilon^{n-1}}\left[e^{2(1-n)\pi i} - 1\right] = 0$$

where $n \neq 1$.

23. If C is the curve $y = x^3 - 3x^2 + 4x - 1$ joining points $(1,1)$ and $(2,3)$, find the value of

$$\int_C (12z^2 - 4iz)\,dz$$

Method 1. By Problem 17, the integral is independent of the path joining $(1,1)$ and $(2,3)$. Hence any path can be chosen. In particular let us choose the straight line paths from $(1,1)$ to $(2,1)$ and then from $(2,1)$ to $(2,3)$.

Case 1. Along the path from $(1,1)$ to $(2,1)$, $y = 1$, $dy = 0$ so that $z = x + iy = x + i$, $dz = dx$. Then the integral equals

$$\int_{x=1}^{2} \{12(x+i)^2 - 4i(x+i)\}\,dx = \{4(x+i)^3 - 2i(x+i)^2\}\Big|_1^2 = 20 + 30i$$

Case 2. Along the path from $(2,1)$ to $(2,3)$, $x = 2$, $dx = 0$ so that $z = x + iy = 2 + iy$, $dz = i\,dy$. Then the integral equals

$$\int_{y=1}^{3} \{12(2+iy)^2 - 4i(2+iy)\}i\,dy = \{4(2+iy)^3 - 2i(2+iy)^2\}\Big|_1^3 = -176 + 8i$$

Then adding, the required value $= (20 + 30i) + (-176 + 8i) = -156 + 38i$.

Method 2. The given integral equals

$$\int_{1+i}^{2+3i} (12z^2 - 4iz)\,dz = (4z^3 - 2iz^2)\Big|_{1+i}^{2+3i} = -156 + 38i$$

It is clear that Method 2 is easier.

INTEGRALS OF SPECIAL FUNCTIONS

24. Determine (*a*) $\int \sin 3z \cos 3z \, dz$, (*b*) $\int \cot(2z+5)\, dz$.

(*a*) ***Method 1.*** Let $\sin 3z = u$. Then $du = 3\cos 3z\, dz$ or $\cos 3z\, dz = du/3$. Then

$$\int \sin 3z \cos 3z\, dz = \int u \frac{du}{3} = \frac{1}{3}\int u\, du = \frac{1}{3}\frac{u^2}{2} + c$$

$$= \frac{1}{6}u^2 + c = \frac{1}{6}\sin^2 3z + c$$

Method 2.

$$\int \sin 3z \cos 3z\, dz = \frac{1}{3}\int \sin 3z\, d(\sin 3z) = \frac{1}{6}\sin^2 3z + c$$

Method 3. Let $\cos 3z = u$. Then $du = -3\sin 3z\, dz$ or $\sin 3z\, dz = -du/3$. Then

$$\int \sin 3z \cos 3z\, dz = -\frac{1}{3}\int u\, du = -\frac{1}{6}u^2 + c_1 = -\frac{1}{6}\cos^2 3z + c_1$$

Note that the results of Methods 1 and 3 differ by a constant.

(*b*) ***Method 1.***

$$\int \cot(2z+5)\, dz = \int \frac{\cos(2z+5)}{\sin(2z+5)}\, dz$$

Let $u = \sin(2z+5)$. Then $du = 2\cos(2z+5)\, dz$ and $\cos(2z+5)\, dz = du/2$. Thus

$$\int \frac{\cos(2z+5)\, dz}{\sin(2z+5)} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln u + c = \frac{1}{2}\ln \sin(2z+5) + c$$

Method 2.

$$\int \cot(2z+5)\, dz = \int \frac{\cos(2z+5)}{\sin(2z+5)}\, dz = \frac{1}{2}\int \frac{d\{\sin(2z+5)\}}{\sin(2z+5)}$$

$$= \frac{1}{2}\ln \sin(2z+5) + c$$

25. (*a*) Prove that $\int F(z)\, G'(z)\, dz = F(z)\, G(z) - \int F'(z)\, G(z)\, dz$.

(*b*) Find $\int z e^{2z}\, dz$ and $\int_0^1 z e^{2z}\, dz$.

(*c*) Find $\int z^2 \sin 4z\, dz$ and $\int_0^{2\pi} z^2 \sin 4z\, dz$.

(*d*) Evaluate $\int_C (z+2)e^{iz}\, dz$ along the parabola C defined by $\pi^2 y = x^2$ from $(0,0)$ to $(\pi, 1)$.

(*a*) We have

$$d\{F(z)\, G(z)\} = F(z)\, G'(z)\, dz + F'(z)\, G(z)\, dz$$

Integrating both sides yields

$$\int d\{F(z)\, G(z)\} = F(z)\, G(z) = \int F(z)\, G'(z)\, dz + \int F'(z)\, G(z)\, dz$$

Then

$$\int F(z)\, G'(z)\, dz = F(z)\, G(z) - \int F'(z)\, G(z)\, dz$$

The method is often called *integration by parts*.

(*b*) Let $F(z) = z$, $G'(z) = e^{2z}$. Then $F'(z) = 1$ and $G(z) = \frac{1}{2}e^{2z}$, omitting the constant of integration. Thus by part (*a*),

$$\int ze^{2z}\, dz = \int F(z)\, G'(z)\, dz = F(z)\, G(z) - \int F'(z)\, G(z)\, dz$$

$$= (z)(\tfrac{1}{2}e^{2z}) - \int 1 \cdot \tfrac{1}{2}e^{2z}\, dz = \tfrac{1}{2}ze^{2z} - \tfrac{1}{4}e^{2z} + c$$

Hence $$\int_0^1 ze^{2z}\,dz = (\tfrac{1}{2}ze^{2z} - \tfrac{1}{4}e^{2z} + c)\Big|_0^1 = \tfrac{1}{2}e^2 - \tfrac{1}{4}e^2 + \tfrac{1}{4} = \tfrac{1}{4}(e^2+1)$$

(c) Integrating by parts choosing $F(z) = z^2$, $G'(z) = \sin 4z$ we have

$$\int z^2 \sin 4z\,dz = (z^2)(-\tfrac{1}{4}\cos 4z) - \int (2z)(-\tfrac{1}{4}\cos 4z)\,dz$$
$$= -\frac{1}{4}z^2 \cos 4z + \frac{1}{2}\int z\cos 4z\,dz$$

Integrating this last integral by parts, this time choosing $F(z) = z$ and $G'(z) = \cos 4z$, we find

$$\int z\cos 4z\,dz = (z)(\tfrac{1}{4}\sin 4z) - \int (1)(\tfrac{1}{4}\sin 4z)\,dz = \tfrac{1}{4}z\sin 4z + \tfrac{1}{16}\cos 4z$$

Hence $$\int z^2\sin 4z\,dz = -\tfrac{1}{4}z^2\cos 4z + \tfrac{1}{8}z\sin 4z + \tfrac{1}{32}\cos 4z + c$$

and $$\int_0^{2\pi} z^2 \sin 4z\,dz = -\pi^2 + \tfrac{1}{32} - \tfrac{1}{32} = -\pi^2$$

The double integration by parts can be indicated in a suggestive manner by writing

$$\int z^2\sin 4z\,dz = (z^2)(-\tfrac{1}{4}\cos 4z) - (2z)(-\tfrac{1}{16}\sin 4z) + (2)(\tfrac{1}{64}\cos 4z) + c$$
$$= -\tfrac{1}{4}z^2\cos 4z + \tfrac{1}{8}z\sin 4z + \tfrac{1}{32}\cos 4z$$

where the first parentheses in each term [after the first] is obtained by differentiating z^2 successively, the second parentheses is obtained by integrating $\sin 4z$ successively, and the terms alternate in sign.

(d) The points $(0,0)$ and $(\pi,1)$ correspond to $z=0$ and $z = \pi + i$. Since $(z+2)e^{iz}$ is analytic, we see by Problem 17 that the integral is independent of the path and is equal to

$$\int_0^{\pi+i} (z+2)\,e^{iz}\,dz = \left\{(z+2)\left(\frac{e^{iz}}{i}\right) - (1)(-e^{iz})\right\}\Bigg|_0^{\pi+i}$$
$$= (\pi+i+2)\left(\frac{e^{i(\pi+i)}}{i}\right) + e^{i(\pi+i)} - \frac{2}{i} - 1$$
$$= -2e^{-1} - 1 + i(2 + \pi e^{-1} + 2e^{-1})$$

26. Show that $\displaystyle\int \frac{dz}{z^2+a^2} = \frac{1}{a}\tan^{-1}\frac{z}{a} + c_1 = \frac{1}{2ai}\ln\left(\frac{z-ai}{z+ai}\right) + c_2.$

Let $z = a\tan u$. Then

$$\int \frac{dz}{z^2+a^2} = \int \frac{a\sec^2 u\,du}{a^2(\tan^2 u + 1)} = \frac{1}{a}\int du = \frac{1}{a}\tan^{-1}\frac{z}{a} + c_1$$

Also, $$\frac{1}{z^2+a^2} = \frac{1}{(z-ai)(z+ai)} = \frac{1}{2ai}\left(\frac{1}{z-ai} - \frac{1}{z+ai}\right)$$

and so $$\int \frac{dz}{z^2+a^2} = \frac{1}{2ai}\int\frac{dz}{z-ai} - \frac{1}{2ai}\int\frac{dz}{z+ai}$$
$$= \frac{1}{2ai}\ln(z-ai) - \frac{1}{2ai}\ln(z+ai) + c_2 = \frac{1}{2ai}\ln\left(\frac{z-ai}{z+ai}\right) + c_2$$

MISCELLANEOUS PROBLEMS

27. Prove Morera's theorem [Page 95] under the assumption that $f(z)$ has a continuous derivative in $\mathcal{R}$.

If $f(z)$ has a continuous derivative in $\mathcal{R}$, then we can apply Green's theorem to obtain

$$\oint_C f(z)\,dz = \oint_C u\,dx - v\,dy + i\oint_C v\,dx + u\,dy$$
$$= \iint_{\mathcal{R}} \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy + i\iint_{\mathcal{R}}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dx\,dy$$

Then if $\oint_C f(z)\,dz = 0$ around every closed path C in $\mathcal{R}$, we must have

$$\oint_C u\,dx - v\,dy = 0, \qquad \oint_C v\,dx + u\,dy = 0$$

around every closed path C in $\mathcal{R}$. Hence from Problem 8, the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

are satisfied and thus [since these partial derivatives are continuous] it follows [Problem 5, Chapter 3] that $u + iv = f(z)$ is analytic.

28. A force field is given by $F = 3z + 5$. Find the work done in moving an object in this force field along the parabola $z = t^2 + it$ from $z = 0$ to $z = 4 + 2i$.

$$\text{Total work done} = \int_C F \circ dz = \text{Re}\int_C \bar{F}\,dz = \text{Re}\left\{\int_C (3\bar{z} + 5)\,dz\right\}$$

$$= \text{Re}\left\{3\int_C \bar{z}\,dz + 5\int_C dz\right\} = \text{Re}\,\{3(10 - \tfrac{8}{3}i) + 5(4 + 2i)\} = 50$$

using the result of Problem 2.

29. Find (a) $\int e^{ax}\sin bx\,dx$, (b) $\int e^{ax}\cos bx\,dx$.

Omitting the constant of integration, we have

$$\int e^{(a+ib)x}\,dx = \frac{e^{(a+ib)x}}{a + ib}$$

which can be written

$$\int e^{ax}(\cos bx + i\sin bx)\,dx = \frac{e^{ax}(\cos bx + i\sin bx)}{a + ib} = \frac{e^{ax}(\cos bx + i\sin bx)(a - ib)}{a^2 + b^2}$$

Then equating real and imaginary parts,

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2 + b^2}$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2 + b^2}$$

30. Give an example of a continuous, closed, non-intersecting curve which lies in a bounded region $\mathcal{R}$ but which has an infinite length.

Consider equilateral triangle ABC [Fig. 4-21] with sides of unit length. By trisecting each side, construct equilateral triangles DEF, GHJ and KLM. Then omitting sides DF, GJ and KM, we obtain the closed non-intersecting curve $ADEFBGHJCKLMA$ of Fig. 4-22.

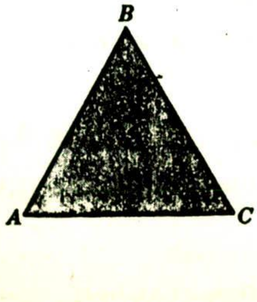

Fig. 4-21

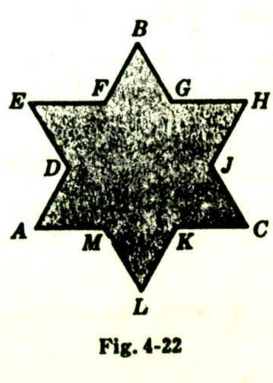

Fig. 4-22

The process can now be continued by trisecting sides DE, EF, FB, BG, GH, etc., and constructing equilateral triangles as before. By repeating the process indefinitely [see Fig. 4-23] we obtain a continuous closed non-intersecting curve which is the boundary of a region with finite area equal to

$$\tfrac{1}{4}\sqrt{3} + (3)(\tfrac{1}{3})^2\frac{\sqrt{3}}{4} + (9)(\tfrac{1}{9})^2\frac{\sqrt{3}}{4} + (27)(\tfrac{1}{27})^2\frac{\sqrt{3}}{4} + \cdots$$

$$= \frac{\sqrt{3}}{4}(1 + \tfrac{1}{3} + \tfrac{1}{9} + \cdots) = \frac{\sqrt{3}}{4}\frac{1}{1-1/3} = \frac{3\sqrt{3}}{8}$$

or 1.5 times the area of triangle ABC, and which has infinite length (see Problem 91).

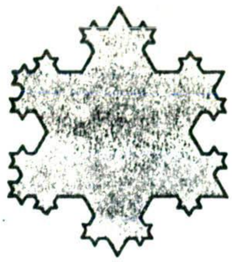

Fig. 4-23

31. Let $F(x,y)$ and $G(x,y)$ be continuous and have continuous first and second partial derivatives in a simply-connected region $\mathcal{R}$ bounded by a simple closed curve C. Prove that

$$\oint_C F\left(\frac{\partial G}{\partial y}dx - \frac{\partial G}{\partial x}dy\right) = -\iint_{\mathcal{R}}\left[F\left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2}\right) + \left(\frac{\partial F}{\partial x}\frac{\partial G}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial G}{\partial y}\right)\right]dx\,dy$$

Let $P = F\dfrac{\partial G}{\partial y}$, $Q = -F\dfrac{\partial G}{\partial x}$ in Green's theorem

$$\oint_C P\,dx + Q\,dy = \iint_{\mathcal{R}}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

Then as required

$$\oint_C F\left(\frac{\partial G}{\partial y}dx - \frac{\partial G}{\partial x}dy\right) = \iint_{\mathcal{R}}\left(\frac{\partial}{\partial x}\left\{-F\frac{\partial G}{\partial x}\right\} - \frac{\partial}{\partial y}\left\{F\frac{\partial G}{\partial y}\right\}\right)dx\,dy$$

$$= -\iint_{\mathcal{R}}\left[F\left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2}\right) + \left(\frac{\partial F}{\partial x}\frac{\partial G}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial G}{\partial y}\right)\right]dx\,dy$$

Supplementary Problems

LINE INTEGRALS

32. Evaluate $\displaystyle\int_{(0,1)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dy$ along (*a*) the curve $y = x^2+1$, (*b*) the straight line joining $(0,1)$ and $(2,5)$, (*c*) the straight lines from $(0,1)$ to $(0,5)$ and then from $(0,5)$ to $(2,5)$, (*d*) the straight lines from $(0,1)$ to $(2,1)$ and then from $(2,1)$ to $(2,5)$.
Ans. (*a*) 88/3, (*b*) 32, (*c*) 40, (*d*) 24

33. (*a*) Evaluate $\displaystyle\oint_C (x+2y)\,dx + (y-2x)\,dy$ around the ellipse C defined by $x = 4\cos\theta$, $y = 3\sin\theta$, $0 \leq \theta < 2\pi$ if C is described in a counterclockwise direction. (*b*) What is the answer to (*a*) if C is described in a clockwise direction? *Ans.* (*a*) -48π, (*b*) 48π

34. Evaluate $\displaystyle\int_C (x^2 - iy^2)\,dz$ along (*a*) the parabola $y = 2x^2$ from $(1,1)$ to $(2,8)$, (*b*) the straight lines from $(1,1)$ to $(1,8)$ and then from $(1,8)$ to $(2,8)$, (*c*) the straight line from $(1,1)$ to $(2,8)$.
Ans. (*a*) $\frac{511}{3} - \frac{49}{5}i$, (*b*) $\frac{518}{3} - 57i$, (*c*) $\frac{518}{3} - 8i$

35. Evaluate $\displaystyle\oint_C |z|^2\,dz$ around the square with vertices at $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$. *Ans.* $-1+i$

36. Evaluate $\int_C (z^2 + 3z)\,dz$ along (*a*) the circle $|z| = 2$ from $(2,0)$ to $(0,2)$ in a counterclockwise direction, (*b*) the straight line from $(2,0)$ to $(0,2)$, (*c*) the straight lines from $(2,0)$ to $(2,2)$ and then from $(2,2)$ to $(0,2)$. *Ans.* $-\frac{44}{3} - \frac{8}{3}i$ for all cases

37. If $f(z)$ and $g(z)$ are integrable, prove that

$$(a)\quad \int_a^b f(z)\,dz \;=\; -\int_b^a f(z)\,dz$$

$$(b)\quad \int_C \{2f(z) - 3i\,g(z)\}\,dz \;=\; 2\int_C f(z)\,dz - 3i\int_C g(z)\,dz.$$

38. Evaluate $\int_i^{2-i} (3xy + iy^2)\,dz$ (*a*) along the straight line joining $z = i$ and $z = 2 - i$, (*b*) along the curve $x = 2t - 2$, $y = 1 + t - t^2$. *Ans.* (*a*) $-\frac{4}{3} + \frac{8}{3}i$, (*b*) $-\frac{1}{3} + \frac{79}{30}i$

39. Evaluate $\oint_C \bar{z}^2\,dz$ around the circles (*a*) $|z| = 1$, (*b*) $|z - 1| = 1$. *Ans.* (*a*) 0, (*b*) $4\pi i$

40. Evaluate $\oint_C (5z^4 - z^3 + 2)\,dz$ around (*a*) the circle $|z| = 1$, (*b*) the square with vertices at $(0,0)$, $(1,0)$, $(1,1)$ and $(1,0)$, (*c*) the curve consisting of the parabolae $y = x^2$ from $(0,0)$ to $(1,1)$ and $y^2 = x$ from $(1,1)$ to $(0,0)$. *Ans.* 0 in all cases

41. Evaluate $\int_C (z^2 + 1)^2\,dz$ along the arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ from the point where $\theta = 0$ to the point where $\theta = 2\pi$. *Ans.* $(96\pi^5a^5 + 80\pi^3a^3 + 30\pi a)/15$

42. Evaluate $\int_C \bar{z}^2\,dz + z^2\,d\bar{z}$ along the curve C defined by $z^2 + 2z\bar{z} + \bar{z}^2 = (2 - 2i)z + (2 + 2i)\bar{z}$ from the point $z = 1$ to $z = 2 + 2i$. *Ans.* 248/15

43. Evaluate $\oint_C \frac{dz}{z - 2}$ around (*a*) the circle $|z - 2| = 4$, (*b*) the circle $|z - 1| = 5$, (*c*) the square with vertices at $2 \pm 2i$, $-2 \pm 2i$. *Ans.* $2\pi i$ in all cases

44. Evaluate $\oint_C (x^2 + iy^2)\,ds$ around the circle $|z| = 2$ where s is the arc length. *Ans.* $8\pi(1 + i)$

GREEN'S THEOREM IN THE PLANE

45. Verify Green's theorem in the plane for $\oint_C (x^2 - 2xy)\,dx + (y^2 - x^3y)\,dy$ where C is a square with vertices at $(0,0)$, $(2,0)$, $(2,2)$, $(0,2)$. *Ans.* common value $= -8$

46. Evaluate $\oint_C (5x + 6y - 3)\,dx + (3x - 4y + 2)\,dy$ around a triangle in the xy plane with vertices at $(0,0)$, $(4,0)$ and $(4,3)$. *Ans.* -18

47. Let C be any simple closed curve bounding a region having area A. Prove that

$$A \;=\; \frac{1}{2}\oint_C x\,dy - y\,dx$$

48. Use the result of Problem 47 to find the area bounded by the ellipse $x = a\cos\theta$, $y = b\sin\theta$, $0 \leqq \theta < 2\pi$. *Ans.* πab

49. Find the area bounded by the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$ shown shaded in Fig. 4-24. [*Hint.* Parametric equations are $x = a\cos^3\theta$, $y = a\sin^3\theta$, $0 \leqq \theta < 2\pi$.] *Ans.* $3\pi a^2/8$

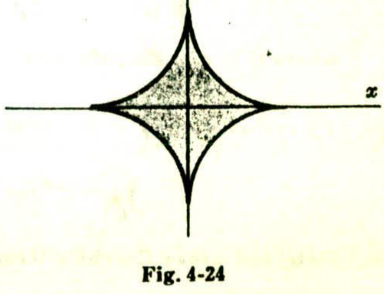

Fig. 4-24

50. Verify Green's theorem in the plane for $\oint_C x^2y\,dx + (y^3 - xy^2)\,dy$ where C is the boundary of the region enclosed by the circles $x^2+y^2 = 4$, $x^2+y^2 = 16$. *Ans.* common value $= 120\pi$

51. (*a*) Prove that $\oint_C (y^2\cos x - 2e^y)\,dx + (2y\sin x - 2xe^y)\,dy = 0$ around any simple closed curve C. (*b*) Evaluate the integral in (*a*) along the parabola $y = x^2$ from $(0,0)$ to (π, π^2). *Ans.* (*b*) $-2\pi e^{\pi^2}$

52. (*a*) Show that $\int_{(2,1)}^{(3,2)} (2xy^3 - 2y^2 - 6y)\,dx + (3x^2y^2 - 4xy - 6x)\,dy$ is independent of the path joining points $(2,1)$ and $(3,2)$. (*b*) Evaluate the integral in (*a*). *Ans.* (*b*) 24

COMPLEX FORM OF GREEN'S THEOREM

53. If C is a simple closed curve enclosing a region of area A, prove that $A = \dfrac{1}{2i}\oint_C \bar{z}\,dz$.

54. Evaluate $\oint_C \bar{z}\,dz$ around (*a*) the circle $|z-2| = 3$, (*b*) the square with vertices at $z = 0, 2, 2i$ and $2+2i$, (*c*) the ellipse $|z-3| + |z+3| = 10$. *Ans.* (*a*) $18\pi i$, (*b*) $8i$, (*c*) $40\pi i$

55. Evaluate $\oint_C (8\bar{z} + 3z)\,dz$ around the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$. *Ans.* $6\pi i a^2$

56. Let $P(z,\bar{z})$ and $Q(z,\bar{z})$ be continuous and have continuous partial derivatives in a region $\mathcal{R}$ and on its boundary C. Prove that

$$\oint_C P(z,\bar{z})\,dz + Q(z,\bar{z})\,d\bar{z} = 2i\iint_{\mathcal{R}} \left(\frac{\partial P}{\partial \bar{z}} - \frac{\partial Q}{\partial z}\right) dA$$

57. Show that the area in Problem 53 can be written in the form $A = \dfrac{1}{4i}\oint_C \bar{z}\,dz - z\,d\bar{z}$.

58. Show that the centroid of the region of Problem 53 is given in conjugate coordinates by $(\hat{z}, \hat{\bar{z}})$ where

$$\hat{z} = -\frac{1}{4Ai}\oint_C z^2\,d\bar{z}, \qquad \hat{\bar{z}} = \frac{1}{4Ai}\oint_C \bar{z}^2\,dz$$

59. Find the centroid of the region bounded above by $|z| = a > 0$ and below by $\operatorname{Im} z = 0$.
Ans. $\hat{z} = 2ai/\pi$, $\hat{\bar{z}} = -2ai/\pi$

CAUCHY'S THEOREM AND THE CAUCHY-GOURSAT THEOREM

60. Verify Cauchy's theorem for the functions (*a*) $3z^2 + iz - 4$, (*b*) $5\sin 2z$, (*c*) $3\cosh(z+2)$ if C is the square with vertices at $1 \pm i$, $-1 \pm i$.

61. Verify Cauchy's theorem for the function $z^3 - iz^2 - 5z + 2i$ if C is (*a*) the circle $|z| = 1$, (*b*) the circle $|z-1| = 2$, (*c*) the ellipse $|z-3i| + |z+3i| = 20$.

62. If C is the circle $|z-2| = 5$, determine whether $\oint_C \dfrac{dz}{z-3} = 0$. (*b*) Does your answer to (*a*) contradict Cauchy's theorem?

63. Explain clearly the relationship between the observations

$$\oint_C (x^2 - y^2 + 2y)\,dx + (2x - 2xy)\,dy = 0 \quad \text{and} \quad \oint_C (z^2 - 2iz)\,dz = 0$$

where C is any simple closed curve.

64. By evaluating $\oint_C e^z\,dz$ around the circle $|z| = 1$, show that

$$\int_0^{2\pi} e^{\cos\theta}\cos(\theta + \sin\theta)\,d\theta = \int_0^{2\pi} e^{\cos\theta}\sin(\theta + \sin\theta)\,d\theta = 0$$

65. State and prove Cauchy's theorem for multiply-connected regions.

66. Prove the Cauchy-Goursat theorem for a polygon, such as *ABCDEFGA* shown in Fig. 4-25, which may intersect itself.

67. Prove the Cauchy-Goursat theorem for the multiply-connected region $\mathcal{R}$ shown shaded in Fig. 4-26.

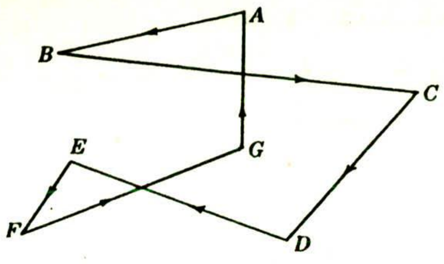

Fig. 4-25

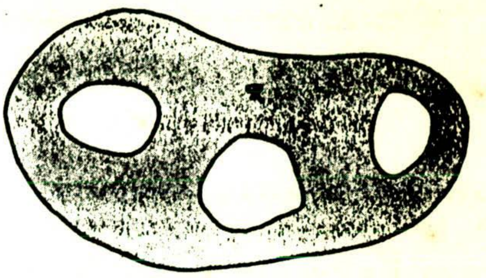

Fig. 4-26

68. (*a*) Prove the Cauchy-Goursat theorem for a rectangle and (*b*) show how the result of (*a*) can be used to prove the theorem for any simple closed curve C.

69. Let P and Q be continuous and have continuous first partial derivatives in a region $\mathcal{R}$. Let C be any simple closed curve in $\mathcal{R}$ and suppose that for any such curve

$$\oint_C P\,dx + Q\,dy \quad = \quad 0$$

(*a*) Prove that there exists an analytic function $f(z)$ such that $\text{Re}\,\{f(z)\,dz\} = P\,dx + Q\,dy$ is an exact differential.

(*b*) Determine p and q in terms of P and Q such that $\text{Im}\,\{f(z)\,dz\} = p\,dx + q\,dy$ and verify that $\oint_C p\,dx + q\,dy = 0$.

(*c*) Discuss the connection between (*a*) and (*b*) and Cauchy's theorem.

70. Illustrate the results of Problem 69 if $P = 2x + y - 2xy$, $Q = x - 2y - x^2 + y^2$ by finding p, q and $f(z)$. *Ans.* One possibility is $p = x^2 - y^2 + 2y - x$, $q = 2x + y - 2xy$, $f(z) = iz^2 + (2 - i)z$.

71. Let P and Q be continuous and have continuous partial derivatives in a region $\mathcal{R}$. Suppose that for any simple closed curve C in $\mathcal{R}$ we have $\oint_C P\,dx + Q\,dy = 0$. (*a*) Prove that $\oint_C Q\,dx - P\,dy = 0$. (*b*) Discuss the relationship of (*a*) with Cauchy's theorem.

CONSEQUENCES OF CAUCHY'S THEOREM

72. Show directly that $\int_{3+4i}^{4-3i} (6z^2 + 8iz)\,dz$ has the same value along the following paths C joining the points $3 + 4i$ and $4 - 3i$: (*a*) a straight line, (*b*) the straight lines from $3 + 4i$ to $4 + 4i$ and then from $4 + 4i$ to $4 - 3i$, (*c*) the circle $|z| = 5$. Determine this value. *Ans.* $238 - 266i$

73. Show that $\int_C e^{-2z}\,dz$ is independent of the path C joining the points $1 - \pi i$ and $2 + 3\pi i$ and determine its value. *Ans.* $\frac{1}{2}e^{-2}(1 - e^{-2})$

74. Given $G(z) = \int_{\pi - \pi i}^{z} \cos 3\zeta\,d\zeta$. (*a*) Prove that $G(z)$ is independent of the path joining $\pi - \pi i$ and the arbitrary point z. (*b*) Determine $G(\pi i)$. (*c*) Prove that $G'(z) = \cos 3z$. *Ans.* (*b*) 0

75. Given $G(z) = \int_{1+i}^{z} \sin \zeta^2\,d\zeta$. (*a*) Prove that $G(z)$ is an analytic function of z. (*b*) Prove that $G'(z) = \sin z^2$.

76. State and prove a theorem corresponding to (*a*) Problem 17, (*b*) Problem 18, (*c*) Problem 20 for the real line integral $\int_C P\,dx + Q\,dy$.

77. Prove Theorem 5, Page 97 for the region of Fig. 4-26.

78. (a) If C is the circle $|z| = R$, show that

$$\lim_{R\to\infty} \oint_C \frac{z^2+2z-5}{(z^2+4)(z^2+2z+2)}\,dz = 0$$

(b) Use the result of (a) to deduce that if C_1 is the circle $|z-2| = 5$, then

$$\oint_{C_1} \frac{z^2+2z-5}{(z^2+4)(z^2+2z+2)}\,dz = 0$$

(c) Is the result in (b) true if C_1 is the circle $|z+1| = 2$? Explain.

INTEGRALS OF SPECIAL FUNCTIONS

79. Find each of the following integrals:

(a) $\int e^{-2z}\,dz$, (b) $\int z \sin z^2\,dz$, (c) $\int \frac{z^2+1}{z^3+3z+2}\,dz$, (d) $\int \sin^4 2z \cos 2z\,dz$

(e) $\int z^2 \tanh(4z^3)\,dz$

Ans. (a) $-\frac{1}{2}e^{-2z} + c$ (b) $-\frac{1}{2}\cos z^2 + c$ (c) $\frac{1}{3}\ln(z^3+3z+2) + c$ (d) $\frac{1}{10}\sin^5 2z + c$ (e) $\frac{1}{12}\ln\cosh(4z^3) + c$

80. Find each of the following integrals:

(a) $\int z \cos 2z\,dz$, (b) $\int z^2 e^{-z}\,dz$, (c) $\int z \ln z\,dz$, (d) $\int z^3 \sinh z\,dz$.

Ans. (a) $\frac{1}{2}z \sin 2z + \frac{1}{4}\cos 2z + c$ (c) $\frac{1}{2}z^2 \ln z - \frac{1}{4} + c$

(b) $-e^{-z}(z^2+2z+2) + c$ (d) $(z^3+6z)\cosh z - 3(z^2+2)\sinh z + c$

81. Evaluate each of the following:

(a) $\int_{\pi i}^{2\pi i} e^{3z}\,dz$, (b) $\int_0^{\pi i} \sinh 5z\,dz$, (c) $\int_0^{\pi+i} z \cos 2z\,dz$.

Ans. (a) 2/3, (b) $-2/5$, (c) $\frac{1}{4}\cosh 2 - \frac{1}{2}\sinh 2 + \frac{1}{2}\pi i \sinh 2$

82. Show that $\int_0^{\pi/2} \sin^2 z\,dz = \int_0^{\pi/2} \cos^2 z\,dz = \pi/4$.

83. Show that $\int \frac{dz}{z^2-a^2} = \frac{1}{2a}\ln\left(\frac{z-a}{z+a}\right) + c_1 = \frac{1}{a}\coth^{-1}\frac{z}{a} + c_2$.

84. Show that if we restrict ourselves to the same branch of the square root,

$$\int z\sqrt{2z+5}\,dz = \frac{1}{20}(2z+5)^{5/2} - \frac{5}{6}(2z+5)^{3/2} + c$$

85. Evaluate $\int \sqrt{1+\sqrt{z+1}}\,dz$, stating conditions under which your result is valid.

Ans. $\frac{4}{5}(1+\sqrt{z+1})^{5/2} - \frac{4}{3}(1+\sqrt{z+1})^{3/2} + c$

MISCELLANEOUS PROBLEMS

86. Use the definition of an integral to prove that along any arbitrary path joining points a and b,

(a) $\int_a^b dz = b - a$, (b) $\int_a^b z\,dz = \frac{1}{2}(b^2 - a^2)$.

87. Prove the theorem concerning change of variables on Page 93.
[*Hint.* Express each side as two real line integrals and use the Cauchy-Riemann equations.]

88. Let $u(x, y)$ be harmonic and have continuous derivatives, of order two at least, in a region $\mathcal{R}$.

(a) Show that

$$v(x,y) = \int_{(a,b)}^{(x,y)} -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

is independent of the path in $\mathcal{R}$ joining (a, b) to (x, y).

(b) Prove that $u + iv$ is an analytic function of $z = x + iy$ in $\mathcal{R}$.

(c) Prove that v is harmonic in $\mathcal{R}$.

89. Work Problem 88 for the special cases (a) $u = 3x^2y + 2x^2 - y^3 - 2y^2$, (b) $u = xe^x \cos y - ye^x \sin y$. [See Problems 53(a) and (c), Page 86.]

90. Using the definition of an integral, verify directly that

$$(a)\ \oint_C dz = 0, \qquad (b)\ \oint_C z\,dz = 0, \qquad (c)\ \oint_C (z - z_0)\,dz = 0$$

where C is a simple closed curve and z_0 is any constant.

91. Find the length of the closed curve of Problem 30 after n steps and verify that as $n \to \infty$ the length of the curve becomes infinite.

92. Evaluate $\int_C \frac{dz}{z^2+4}$ along the line $x + y = 1$ in the direction of increasing x. *Ans.* $\pi/2$

93. Show that $\int_0^\infty xe^{-x} \sin x\,dx = \frac{1}{2}$.

94. Evaluate $\int_{-2-2\sqrt{3}i}^{-2+2\sqrt{3}i} z^{1/2}\,dz$ along a straight line path if we choose that branch of $z^{1/2}$ such that $z^{1/2} = 1$ for $z = 1$. *Ans.* 32/3

95. Does Cauchy's theorem hold for the function $f(z) = z^{1/2}$ where C is the circle $|z| = 1$? Explain.

96. Does Cauchy's theorem hold for a curve, such as $EFGHFJE$ in Fig. 4-27, which intersects itself? Justify your answer.

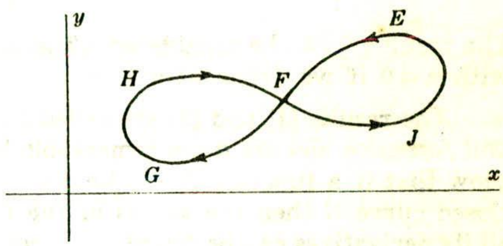

Fig. 4-27

97. If n is the direction of the outward drawn normal to a simple closed curve C, s is the arc length parameter and U is any continuously differentiable function, prove that

$$\frac{\partial U}{\partial n} = \frac{\partial U}{\partial x}\frac{dx}{ds} + \frac{\partial U}{\partial y}\frac{dy}{ds}$$

98. Prove *Green's first identity*,

$$\iint_{\mathcal{R}} U\,\nabla^2 V\,dx\,dy + \iint_{\mathcal{R}} \left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial V}{\partial y}\right) dx\,dy = \oint_C U \frac{\partial V}{\partial n}\,ds$$

where $\mathcal{R}$ is the region bounded by the simple closed curve C, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, while n and s are as in Problem 97.

99. Use Problem 98 to prove *Green's second identity*

$$\iint_{\mathcal{R}} (U\,\nabla^2 V - V\,\nabla^2 U)\,dA = \oint_C \left(U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n}\right) ds$$

where dA is an element of area of $\mathcal{R}$.

100. Write the result of Problem 31 in terms of the operator ∇.

101. Evaluate $\oint_C \frac{dz}{\sqrt{z^2+2z+2}}$ around the unit circle $|z| = 1$ starting with $z = 1$, assuming the integrand positive for this value.

102. If n is a positive integer, show that

$$\int_0^{2\pi} e^{\sin n\theta} \cos(\theta - \cos n\theta)\,d\theta = \int_0^{2\pi} e^{\sin n\theta} \sin(\theta - \cos n\theta)\,d\theta = 0$$