Chapter 5

Cauchy's Integral Formulae and Related Theorems

CAUCHY'S INTEGRAL FORMULAE

If $f(z)$ is analytic inside and on a simple closed curve C and a is any point inside C [Fig. 5-1], then

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz \qquad (1)$$

where C is traversed in the positive (counterclockwise) sense.

Also the nth derivative of $f(z)$ at $z=a$ is given by

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz \qquad n = 1,2,3,\ldots \qquad (2)$$

The result (*1*) can be considered a special case of (*2*) with $n=0$ if we define $0! = 1$.

The results (*1*) and (*2*) are called *Cauchy's integral formulae* and are quite remarkable because they show that if a function $f(z)$ is known *on* the simple closed curve C then the values of the function and all its derivatives can be found at all points *inside C*. Thus if a function of a complex variable has a first derivative, i.e. is analytic, in a simply-connected region $\mathcal{R}$, all its higher derivatives exist in $\mathcal{R}$. This is not necessarily true for functions of real variables.

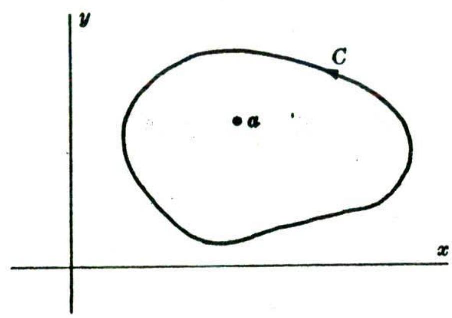

Fig. 5-1

SOME IMPORTANT THEOREMS

The following is a list of some important theorems which are consequences of Cauchy's integral formulae.

1. **Morera's theorem** (converse of Cauchy's theorem).

   If $f(z)$ is continuous in a simply-connected region $\mathcal{R}$ and if $\oint_C f(z)\,dz = 0$ around every simple closed curve C in $\mathcal{R}$, then $f(z)$ is analytic in $\mathcal{R}$.

2. **Cauchy's inequality.**

   If $f(z)$ is analytic inside and on a circle C of radius r and centre at $z=a$, then

   $$|f^{(n)}(a)| \le \frac{M\cdot n!}{r^n} \qquad n = 0,1,2,\ldots \qquad (3)$$

   where M is a constant such that $|f(z)| < M$ on C, i.e. M is an upper bound of $|f(z)|$ on C.

3. Liouville's theorem.

Suppose that for all z in the entire complex plane, (i) $f(z)$ is analytic and (ii) $f(z)$ is bounded, i.e. $|f(z)| < M$ for some constant M. Then $f(z)$ must be a constant.

4. Fundamental theorem of algebra.

Every polynomial equation $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0$ with degree $n \geqq 1$ and $a_n \neq 0$ has at least one root.

From this it follows that $P(z) = 0$ has exactly n roots, due attention being paid to multiplicities of roots.

5. Gauss' mean value theorem.

If $f(z)$ is analytic inside and on a circle C with centre at a and radius r, then $f(a)$ is the mean of the values of $f(z)$ on C, i.e.,

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{i\theta})\,d\theta \tag{4}$$

6. Maximum modulus theorem.

If $f(z)$ is analytic inside and on a simple closed curve C and is not identically equal to a constant, then the maximum value of $|f(z)|$ occurs on C.

7. Minimum modulus theorem.

If $f(z)$ is analytic inside and on a simple closed curve C and $f(z) \neq 0$ inside C, then $|f(z)|$ assumes its minimum value on C.

8. The argument theorem.

Let $f(z)$ be analytic inside and on a simple closed curve C except for a finite number of poles inside C. Then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = N - P \tag{5}$$

where N and P are respectively the number of zeros and poles of $f(z)$ inside C.

For a generalization of this theorem see Problem 90.

9. Rouché's theorem.

If $f(z)$ and $g(z)$ are analytic inside and on a simple closed curve C and if $|g(z)| < |f(z)|$ on C, then $f(z) + g(z)$ and $f(z)$ have the same number of zeros inside C.

10. Poisson's integral formulas for a circle.

Let $f(z)$ be analytic inside and on the circle C defined by $|z| = R$. Then if $z = re^{i\theta}$ is any point inside C, we have

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\,f(Re^{i\phi})}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\,d\phi \tag{6}$$

If $u(r,\theta)$ and $v(r,\theta)$ are the real and imaginary parts of $f(re^{i\theta})$ while $u(R,\phi)$ and $v(R,\phi)$ are the real and imaginary parts of $f(Re^{i\phi})$, then

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\,u(R,\phi)}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\,d\phi \tag{7}$$

$$v(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\,v(R,\phi)}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\,d\phi \tag{8}$$

These results are called *Poisson's integral formulas for a circle.* They express the values of a harmonic function inside a circle in terms of its values on the boundary.

11. Poisson's integral formulae for a half plane

Let $f(z)$ be analytic in the upper half $y \geqq 0$ of the z plane and let $\zeta = \xi + i\eta$ be any point in this upper half plane. Then

$$f(\zeta) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\eta f(x)}{(x-\xi)^2+\eta^2}\,dx \tag{9}$$

In terms of the real and imaginary parts of $f(\zeta)$ this can be written

$$u(\xi,\eta) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\eta\, u(x,0)}{(x-\xi)^2+\eta^2}\,dx \tag{10}$$

$$v(\xi,\eta) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\eta\, v(x,0)}{(x-\xi)^2+\eta^2}\,dx \tag{11}$$

These are called *Poisson's integral formulae for a half plane.* They express the values of a harmonic function in the upper half plane in terms of the values on the x axis [the boundary] of the half plane.

Solved Problems

CAUCHY'S INTEGRAL FORMULAE

1. If $f(z)$ is analytic inside and on the boundary C of a simply-connected region $\mathcal{R}$, prove *Cauchy's integral formula*

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

Method 1.

The function $f(z)/(z-a)$ is analytic inside and on C except at the point $z = a$ (see Fig. 5-2). By Theorem 4, Page 97, we have

$$\oint_C \frac{f(z)}{z-a}\,dz = \oint_\Gamma \frac{f(z)}{z-a}\,dz \tag{1}$$

where we can choose Γ as a circle of radius ϵ with centre at a. Then an equation for Γ is $|z-a| = \epsilon$ or $z - a = \epsilon e^{i\theta}$ where $0 \leqq \theta < 2\pi$. Substituting $z = a + \epsilon e^{i\theta}$, $dz = i\epsilon e^{i\theta}$, the integral on the right of (1) becomes

$$\oint_\Gamma \frac{f(z)}{z-a}\,dz = \int_0^{2\pi} \frac{f(a+\epsilon e^{i\theta})\, i\epsilon e^{i\theta}}{\epsilon e^{i\theta}}\,d\theta$$

$$= i\int_0^{2\pi} f(a+\epsilon e^{i\theta})\,d\theta$$

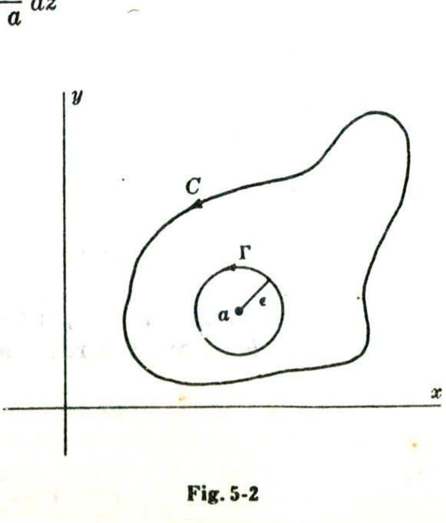

Fig. 5-2

Thus we have from (1),

$$\oint_C \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f(a+\epsilon e^{i\theta})\,d\theta \tag{2}$$

Taking the limit of both sides of (2) and making use of the continuity of $f(z)$, we have

$$\oint_C \frac{f(z)}{z-a}\,dz = \lim_{\epsilon\to 0} i\int_0^{2\pi} f(a+\epsilon e^{i\theta})\,d\theta$$

$$= i\int_0^{2\pi} \lim_{\epsilon\to 0} f(a+\epsilon e^{i\theta})\,d\theta = i\int_0^{2\pi} f(a)\,d\theta = 2\pi i\, f(a) \tag{3}$$

so that we have, as required,

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

Method 2. The right side of equation (1) of Method 1 can be written as

$$\oint_\Gamma \frac{f(z)}{z-a}\,dz = \oint_\Gamma \frac{f(z)-f(a)}{z-a}\,dz + \oint_\Gamma \frac{f(a)}{z-a}\,dz$$

$$= \oint_\Gamma \frac{f(z)-f(a)}{z-a}\,dz + 2\pi i\, f(a)$$

using Problem 21, Chapter 4. The required result will follow if we can show that

$$\oint_\Gamma \frac{f(z)-f(a)}{z-a}\,dz = 0$$

But by Problem 21, Chapter 3,

$$\oint_\Gamma \frac{f(z)-f(a)}{z-a}\,dz = \oint_\Gamma f'(a)\,dz + \oint_\Gamma \eta\,dz = \oint_\Gamma \eta\,dz$$

Then choosing Γ so small that for all points on Γ we have $|\eta| < \delta/2\pi$, we find

$$\left|\oint_\Gamma \eta\,dz\right| < \left(\frac{\delta}{2\pi}\right)(2\pi\epsilon) = \epsilon$$

Thus $\oint_\Gamma \eta\,dz = 0$ and the proof is complete.

2. If $f(z)$ is analytic inside and on the boundary C of a simply-connected region $\mathcal{R}$, prove that

$$f'(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}\,dz$$

From Problem 1 if a and $a+h$ lie in $\mathcal{R}$, we have

$$\frac{f(a+h)-f(a)}{h} = \frac{1}{2\pi i}\oint_C \frac{1}{h}\left\{\frac{1}{z-(a+h)} - \frac{1}{z-a}\right\} f(z)\,dz = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a-h)(z-a)}$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a)^2} + \frac{h}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a-h)(z-a)^2}$$

The result follows on taking the limit as $h\to 0$ if we can show that the last term approaches zero.

To show this we use the fact that if Γ is a circle of radius ϵ and centre a which lies entirely in $\mathcal{R}$ (see Fig. 5-3), then

$$\frac{h}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a-h)(z-a)^2}$$

$$= \frac{h}{2\pi i}\oint_\Gamma \frac{f(z)\,dz}{(z-a-h)(z-a)^2}$$

Choosing h so small in absolute value that $a+h$ lies in Γ and $|h| < \epsilon/2$, we have by Problem 7(c), Chapter 1, and the fact that Γ has equation $|z-a| = \epsilon$,

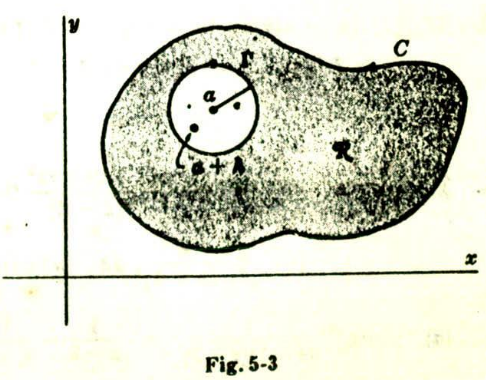

Fig. 5-3

$$|z-a-h| \geq |z-a| - |h| > \epsilon - \epsilon/2 = \epsilon/2$$

Also since $f(z)$ is analytic in $\mathcal{R}$, we can find a positive number M such that $|f(z)| < M$.

Then since the length of Γ is $2\pi\epsilon$, we have

$$\left|\frac{h}{2\pi i}\oint_\Gamma \frac{f(z)\,dz}{(z-a-h)(z-a)^2}\right| \leq \frac{|h|}{2\pi}\frac{M(2\pi\epsilon)}{(\epsilon/2)(\epsilon^2)} = \frac{2|h|M}{\epsilon^2}$$

and it follows that the left side approaches zero as $h \to 0$, thus completing the proof.

It is of interest to observe that the result is equivalent to

$$\frac{d}{da}f(a) = \frac{d}{da}\left\{\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz\right\} = \frac{1}{2\pi i}\oint_C \frac{\partial}{\partial a}\left\{\frac{f(z)}{z-a}\right\}dz$$

which is an extension to contour integrals of *Leibnitz's rule* for differentiating under the integral sign.

3. Prove that under the conditions of Problem 2,

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}dz \qquad n = 0,1,2,3,\ldots$$

The cases where $n=0$ and 1 follow from Problems 1 and 2 respectively provided we define $f^{(0)}(a) = f(a)$ and $0! = 1$.

To establish the case where $n=2$, we use Problem 2 where a and $a+h$ lie in $\mathcal{R}$ to obtain

$$\begin{aligned}\frac{f'(a+h)-f'(a)}{h} &= \frac{1}{2\pi i}\oint_C \frac{1}{h}\left\{\frac{1}{(z-a-h)^2} - \frac{1}{(z-a)^2}\right\}f(z)\,dz \\ &= \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^3}dz + \frac{h}{2\pi i}\oint_C \frac{3(z-a)-2h}{(z-a-h)^2(z-a)^3}f(z)\,dz\end{aligned}$$

The result follows on taking the limit as $h \to 0$ if we can show that the last term approaches zero. The proof is similar to that of Problem 2, for using the fact that the integral around C equals the integral around Γ, we have

$$\left|\frac{h}{2\pi i}\oint_\Gamma \frac{3(z-a)-2h}{(z-a-h)^2(z-a)^3}f(z)\,dz\right| \leq \frac{|h|}{2\pi}\frac{M(2\pi\epsilon)}{(\epsilon/2)^2(\epsilon^3)} = \frac{4|h|M}{\epsilon^4}$$

Since M exists such that $|\{3(z-a)-2h\}f(z)| < M$.

In a similar manner we can establish the result for $n = 3,4,\ldots$ (see Problems 36 and 37).

The result is equivalent to (see last paragraph of Problem 2)

$$\frac{d^n}{da^n}f(a) = \frac{d^n}{da^n}\left\{\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz\right\} = \frac{1}{2\pi i}\oint_C \frac{\partial^n}{\partial a^n}\left\{\frac{f(z)}{z-a}\right\}dz$$

4. If $f(z)$ is analytic in a region $\mathcal{R}$, prove that $f'(z), f''(z), \ldots$ are analytic in $\mathcal{R}$.

This follows from Problems 2 and 3.

5. Evaluate (*a*) $\displaystyle\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}dz$,

(*b*) $\displaystyle\oint_C \frac{e^{2z}}{(z+1)^4}dz$ where C is the circle $|z| = 3$.

(*a*) Since $\dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$, we have

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz \;=\; \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\,dz \;-\; \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\,dz$$

By Cauchy's integral formula with $a = 2$ and $a = 1$ respectively, we have

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\,dz \;=\; 2\pi i\{\sin \pi(2)^2 + \cos \pi(2)^2\} \;=\; 2\pi i$$

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\,dz \;=\; 2\pi i\{\sin \pi(1)^2 + \cos \pi(1)^2\} \;=\; -2\pi i$$

since $z = 1$ and $z = 2$ are inside C and $\sin \pi z^2 + \cos \pi z^2$ is analytic inside C. Then the required integral has the value $2\pi i - (-2\pi i) = 4\pi i$.

(b) Let $f(z) = e^{2z}$ and $a = -1$ in the Cauchy integral formula

$$f^{(n)}(a) \;=\; \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz \tag{1}$$

If $n = 3$, then $f'''(z) = 8e^{2z}$ and $f'''(-1) = 8e^{-2}$. Hence (1) becomes

$$8e^{-2} \;=\; \frac{3!}{2\pi i}\oint_C \frac{e^{2z}}{(z+1)^4}\,dz$$

from which we see that the required integral has the value $8\pi i e^{-2}/3$.

6. Prove Cauchy's integral formula for multiply-connected regions.

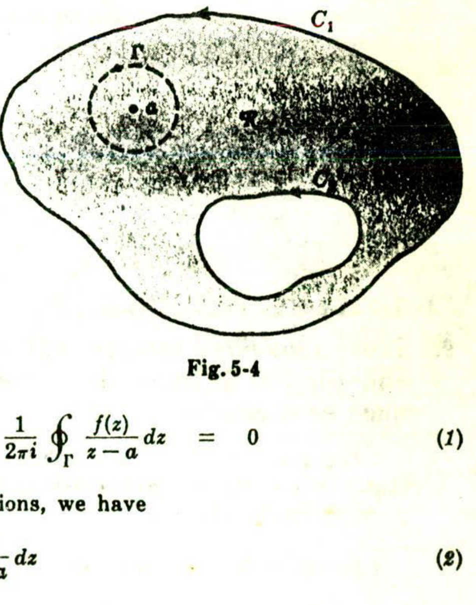

Fig. 5-4

We present a proof for the multiply-connected region $\mathcal{R}$ bounded by the simple closed curves C_1 and C_2 as indicated in Fig. 5-4. Extensions to other multiply-connected regions are easily made (see Problem 40).

Construct a circle Γ having centre at any point a in $\mathcal{R}$ so that Γ lies entirely in $\mathcal{R}$. Let $\mathcal{R}'$ consist of the set of points in $\mathcal{R}$ which are exterior to Γ. Then the function $\dfrac{f(z)}{z-a}$ is analytic inside and on the boundary of $\mathcal{R}'$. Hence by Cauchy's theorem for multiply-connected regions (Problem 16, Chapter 4),

$$\frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z-a}\,dz \;-\; \frac{1}{2\pi i}\oint_{C_2} \frac{f(z)}{z-a}\,dz \;-\; \frac{1}{2\pi i}\oint_{\Gamma} \frac{f(z)}{z-a}\,dz \;=\; 0 \tag{1}$$

But by Cauchy's integral formula for simply-connected regions, we have

$$f(a) \;=\; \frac{1}{2\pi i}\oint_{\Gamma} \frac{f(z)}{z-a}\,dz \tag{2}$$

so that from (1),

$$f(a) \;=\; \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z-a}\,dz \;-\; \frac{1}{2\pi i}\oint_{C_2} \frac{f(z)}{z-a}\,dz \tag{3}$$

Then if C represents the entire boundary of $\mathcal{R}$ (suitably traversed so that an observer moving around C always has $\mathcal{R}$ lying to his left), we can write (3) as

$$f(a) \;=\; \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

In a similar manner we can show that the other Cauchy integral formulae

$$f^{(n)}(a) \;=\; \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz \qquad n = 1, 2, 3, \ldots$$

hold for multiply-connected regions (see Problem 40).

MORERA'S THEOREM

7. Prove *Morera's theorem* (the converse of Cauchy's theorem): If $f(z)$ is continuous in a simply-connected region $\mathcal{R}$ and if

$$\oint_C f(z)\,dz = 0$$

around every simple closed curve C in $\mathcal{R}$, then $f(z)$ is analytic in $\mathcal{R}$.

If $\oint_C f(z)\,dz = 0$ independent of C, it follows by Problem 17, Chapter 4, that $F(z) = \int_a^z f(z)\,dz$ is independent of the path joining a and z, so long as this path is in $\mathcal{R}$.

Then by reasoning identical with that used in Problem 18, Chapter 4, it follows that $F(z)$ is analytic in $\mathcal{R}$ and $F'(z) = f(z)$. However, by Problem 2, it follows that $F'(z)$ is also analytic if $F(z)$ is. Hence $f(z)$ is analytic in $\mathcal{R}$.

CAUCHY'S INEQUALITY

8. If $f(z)$ is analytic inside and on a circle C of radius r and centre at $z=a$, prove *Cauchy's inequality*

$$|f^{(n)}(a)| \leq \frac{M\cdot n!}{r^n} \qquad n = 0,1,2,3,\ldots$$

where M is a constant such that $|f(z)| < M$.

We have by Cauchy's integral formulae,

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz \qquad n = 0,1,2,3,\ldots$$

Then by Problem 3, Chapter 4, since $|z-a| = r$ on C and the length of C is $2\pi r$,

$$|f^{(n)}(a)| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz\right| \leq \frac{n!}{2\pi}\cdot\frac{M}{r^{n+1}}\cdot 2\pi r = \frac{M\cdot n!}{r^n}$$

LIOUVILLE'S THEOREM

9. Prove *Liouville's theorem*: If for all z in the entire complex plane, (i) $f(z)$ is analytic and (ii) $f(z)$ is bounded [i.e. we can find a constant M such that $|f(z)| < M$], then $f(z)$ must be a constant.

Let a and b be any two points in the z plane. Suppose that C is a circle of radius r having centre at a and enclosing point b (see Fig. 5-5).

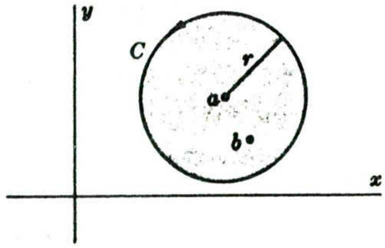

Fig. 5-5

From Cauchy's integral formula, we have

$$f(b) - f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-b}\,dz - \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz = \frac{b-a}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-b)(z-a)}$$

Now we have

$$|z-a| = r, \qquad |z-b| = |z-a+a-b| \geqq |z-a|-|a-b| = r-|a-b| \geqq r/2$$

if we choose r so large that $|a-b| < r/2$. Then since $|f(z)| < M$ and the length of C is $2\pi r$, we have by Problem 3, Chapter 4,

$$|f(b)-f(a)| = \frac{|b-a|}{2\pi}\left|\oint_C \frac{f(z)\,dz}{(z-b)(z-a)}\right| \leqq \frac{|b-a|\,M(2\pi r)}{2\pi(r/2)r} = \frac{2\,|b-a|\,M}{r}$$

Letting $r \to \infty$ we see that $|f(b)-f(a)| = 0$ or $f(b) = f(a)$, which shows that $f(z)$ must be a constant.

Another method. Letting $n = 1$ in Problem 8 and replacing a by z we have,

$$|f'(z)| \leq M/r$$

Letting $r \to \infty$, we deduce that $|f'(z)| = 0$ and so $f'(z) = 0$. Hence $f(z) =$ constant, as required.

FUNDAMENTAL THEOREM OF ALGEBRA

10. Prove the *fundamental theorem of algebra*: Every polynomial equation $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0$, where the degree $n \geq 1$ and $a_n \neq 0$, has at least one root.

If $P(z) = 0$ has no root, then $f(z) = \dfrac{1}{P(z)}$ is analytic for all z. Also $|f(z)| = \dfrac{1}{|P(z)|}$ is bounded (and in fact approaches zero) as $|z| \to \infty$.

Then by Liouville's theorem (Problem 9) it follows that $f(z)$ and thus $P(z)$ must be a constant. Thus we are led to a contradiction and conclude that $P(z) = 0$ must have at least one root or, as is sometimes said, $P(z)$ has at least one *zero*.

11. Prove that every polynomial equation $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0$, where the degree $n \geq 1$ and $a_n \neq 0$, has exactly n roots.

By the fundamental theorem of algebra (Problem 10), $P(z)$ has at least one root. Denote this root by α. Then $P(\alpha) = 0$. Hence

$$\begin{aligned} P(z) - P(\alpha) &= a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n - (a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_n\alpha^n) \\ &= a_1(z-\alpha) + a_2(z^2-\alpha^2) + \cdots + a_n(z^n-\alpha^n) \\ &= (z-\alpha)\,Q(z) \end{aligned}$$

where $Q(z)$ is a polynomial of degree $(n-1)$.

Applying the fundamental theorem of algebra again, we see that $Q(z)$ has at least one zero which we can denote by β [which may equal α] and so $P(z) = (z-\alpha)(z-\beta)\,R(z)$. Continuing in this manner we see that $P(z)$ has exactly n zeros.

GAUSS' MEAN VALUE THEOREM

12. Let $f(z)$ be analytic inside and on a circle C with centre at a. Prove *Gauss' mean value theorem* that the mean of the values of $f(z)$ on C is $f(a)$.

By Cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz \tag{1}$$

If C has radius r, the equation of C is $|z-a| = r$ or $z = a + re^{i\theta}$. Thus (*1*) becomes

$$f(a) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+re^{i\theta})\,ire^{i\theta}}{re^{i\theta}}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})\,d\theta$$

which is the required result.

MAXIMUM MODULUS THEOREM

13. Prove the *maximum modulus theorem*: If $f(z)$ is analytic inside and on a simple closed curve C, then the maximum value of $|f(z)|$ occurs on C, unless $f(z)$ is a constant.

Method 1.

Since $f(z)$ is analytic and hence continuous inside and on C, it follows that $|f(z)|$ does have a maximum value M for at least one value of z inside or on C. Suppose this maximum value is not attained on the boundary of C but is attained at an interior point a, i.e. $|f(a)| = M$. Let C_1 be a circle inside C with centre at a (see Fig. 5-6). If we exclude $f(z)$ from being a constant inside C_1, then there must be a point inside C_1, say b, such that $|f(b)| < M$ or, what is the same thing, $|f(b)| = M - \epsilon$ where $\epsilon > 0$.

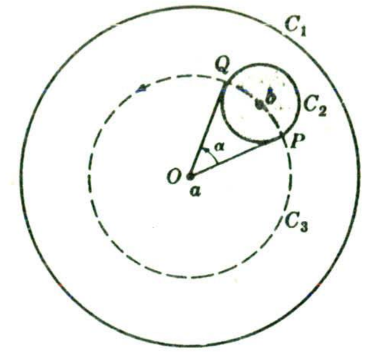

Fig. 5-6

Now by the continuity of $|f(z)|$ at b, we see that for any $\epsilon > 0$ we can find $\delta > 0$ such that

$$\big|\,|f(z)| - |f(b)|\,\big| \;<\; \tfrac{1}{2}\epsilon \quad \text{whenever} \quad |z - b| < \delta \tag{1}$$

i.e.,

$$|f(z)| \;<\; |f(b)| + \tfrac{1}{2}\epsilon \;=\; M - \epsilon + \tfrac{1}{2}\epsilon \;=\; M - \tfrac{1}{2}\epsilon \tag{2}$$

for all points interior to a circle C_2 with centre at b and radius δ, as shown shaded in the figure.

Construct a circle C_3 with centre at a which passes through b (dashed in Fig. 5-6). On part of this circle [namely that part PQ included in C_2] we have from (2), $|f(z)| < M - \frac{1}{2}\epsilon$. On the remaining part of the circle we have $|f(z)| \leqq M$.

If we measure θ counterclockwise from OP and let $\angle POQ = \alpha$, it follows from Problem 12 that if $r = |b - a|$,

$$f(a) \;=\; \frac{1}{2\pi}\int_0^{\alpha} f(a + re^{i\theta})\,d\theta \;+\; \frac{1}{2\pi}\int_{\alpha}^{2\pi} f(a + re^{i\theta})\,d\theta$$

Then

$$\begin{aligned}
|f(a)| &\leqq \frac{1}{2\pi}\int_0^{\alpha} |f(a + re^{i\theta})|\,d\theta \;+\; \frac{1}{2\pi}\int_{\alpha}^{2\pi} |f(a + re^{i\theta})|\,d\theta \\
&\leqq \frac{1}{2\pi}\int_0^{\alpha} (M - \tfrac{1}{2}\epsilon)\,d\theta \;+\; \frac{1}{2\pi}\int_{\alpha}^{2\pi} M\,d\theta \\
&= \frac{\alpha}{2\pi}(M - \tfrac{1}{2}\epsilon) \;+\; \frac{M}{2\pi}(2\pi - \alpha) \\
&= M \;-\; \frac{\alpha\epsilon}{4\pi}
\end{aligned}$$

i.e. $|f(a)| = M \leqq M - \dfrac{\alpha\epsilon}{4\pi}$, an impossible situation. By virtue of this contradiction we conclude that $|f(z)|$ cannot attain its maximum at any interior point of C and so must attain its maximum on C.

Method 2.

From Problem 12, we have

$$|f(a)| \;\leqq\; \frac{1}{2\pi}\int_0^{2\pi} |f(a + re^{i\theta})|\,d\theta \tag{3}$$

Let us suppose that $|f(a)|$ is a maximum so that $|f(a + re^{i\theta})| \leqq |f(a)|$. If $|f(a + re^{i\theta})| < |f(a)|$ for one value of θ then, by continuity of f, it would hold for a finite arc, say $\theta_1 < \theta < \theta_2$. But in such case the mean value of $|f(a + re^{i\theta})|$ is less than $|f(a)|$, which would contradict (3). It follows therefore that in any neighbourhood of a, i.e. for $|z - a| < \delta$, $f(z)$ must be a constant. If $f(z)$ is not a constant, the maximum value of $|f(z)|$ must occur on C.

For another method, see Problem 57.

MINIMUM MODULUS THEOREM

14. Prove the *minimum modulus theorem*: Let $f(z)$ be analytic inside and on a simple closed curve C. Prove that if $f(z) \neq 0$ inside C, then $|f(z)|$ must assume its minimum value on C.

Since $f(z)$ is analytic inside and on C and since $f(z) \neq 0$ inside C, it follows that $1/f(z)$ is analytic inside C. By the maximum modulus theorem it follows that $1/|f(z)|$ cannot assume its maximum value inside C and so $|f(z)|$ cannot assume its minimum value inside C. Then since $|f(z)|$ has a minimum, this minimum must be attained on C.

15. Give an example to show that if $f(z)$ is analytic inside and on a simple closed curve C and $f(z) = 0$ at some point inside C, then $|f(z)|$ need not assume its minimum value on C.

Let $f(z) = z$ for $|z| \leqq 1$, so that C is a circle with centre at the origin and radius one. We have $f(z) = 0$ at $z = 0$. If $z = re^{i\theta}$, then $|f(z)| = r$ and it is clear that the minimum value of $|f(z)|$ does not occur on C but occurs inside C where $r = 0$, i.e. at $z = 0$.

THE ARGUMENT THEOREM

16. Let $f(z)$ be analytic inside and on a simple closed curve C except for a pole $z = \alpha$ of order (multiplicity) p inside C. Suppose also that inside C $f(z)$ has only one zero $z = \beta$ of order (multiplicity) n and no zeros on C. Prove that

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = n - p$$

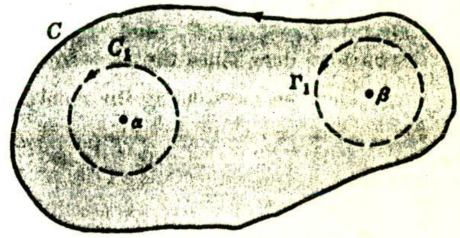

Fig. 5-7

Let C_1 and Γ_1 be non-overlapping circles lying inside C and enclosing $z = \alpha$ and $z = \beta$ respectively. Then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\oint_{C_1} \frac{f'(z)}{f(z)}\,dz + \frac{1}{2\pi i}\oint_{\Gamma_1} \frac{f'(z)}{f(z)}\,dz \tag{1}$$

Since $f(z)$ has a pole of order p at $z = \alpha$, we have

$$f(z) = \frac{F(z)}{(z-\alpha)^p} \tag{2}$$

where $F(z)$ is analytic and different from zero inside and on C_1. Then taking logarithms in (2) and differentiating, we find

$$\frac{f'(z)}{f(z)} = \frac{F'(z)}{F(z)} - \frac{p}{z-\alpha} \tag{3}$$

so that

$$\frac{1}{2\pi i}\oint_{C_1} \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\oint_{C_1} \frac{F'(z)}{F(z)}\,dz - \frac{p}{2\pi i}\oint_{C_1} \frac{dz}{z-\alpha} = 0 - p = -p \tag{4}$$

Since $f(z)$ has a zero of order n at $z = \beta$, we have

$$f(z) = (z-\beta)^n G(z) \tag{5}$$

where $G(z)$ is analytic and different from zero inside and on Γ_1.

Then by logarithmic differentiation, we have

$$\frac{f'(z)}{f(z)} = \frac{n}{z-\beta} + \frac{G'(z)}{G(z)} \tag{6}$$

so that

$$\frac{1}{2\pi i}\oint_{\Gamma_1} \frac{f'(z)}{f(z)}\,dz = \frac{n}{2\pi i}\oint_{\Gamma_1} \frac{dz}{z-\beta} + \frac{1}{2\pi i}\oint \frac{G'(z)}{G(z)}\,dz = n \tag{7}$$

Hence from (1), (4) and (7), we have the required result

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\oint_{C_1} \frac{f'(z)}{f(z)}\,dz + \frac{1}{2\pi i}\oint_{\Gamma_1} \frac{f'(z)}{f(z)}\,dz = n - p$$

17. Let $f(z)$ be analytic inside and on a simple closed curve C except for a finite number of poles inside C. Suppose that $f(z) \neq 0$ on C. If N and P are respectively the number of zeros and poles of $f(z)$ inside C, counting multiplicities, prove that

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = N - P$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_j$ and $\beta_1, \beta_2, \ldots, \beta_k$ be the respective poles and zeros of $f(z)$ lying inside C [Fig. 5-8] and suppose their multiplicities are $p_1, p_2, \ldots, p_j$ and $n_1, n_2, \ldots, n_k$.

Enclose each pole and zero by non-overlapping circles $C_1, C_2, \ldots, C_j$ and $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$. This can always be done since the poles and zeros are isolated.

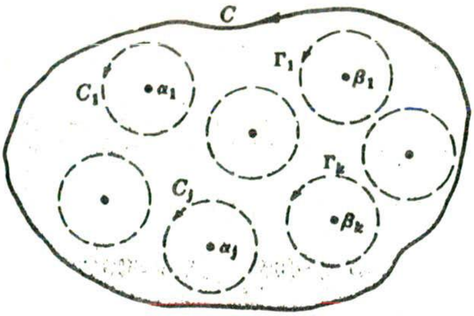

Fig. 5-8

Then we have, using the results of Problem 16,

$$\begin{aligned}\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz &= \sum_{r=1}^{j}\frac{1}{2\pi i}\oint_{\Gamma_r}\frac{f'(z)}{f(z)}\,dz + \sum_{r=1}^{k}\frac{1}{2\pi i}\oint_{C_r}\frac{f'(z)}{f(z)}\,dz \\ &= \sum_{r=1}^{j} n_r - \sum_{r=1}^{k} p_r \\ &= N - P\end{aligned}$$

ROUCHE'S THEOREM

18. Prove *Rouché's theorem:* If $f(z)$ and $g(z)$ are analytic inside and on a simple closed curve C and if $|g(z)| < |f(z)|$ on C, then $f(z) + g(z)$ and $f(z)$ have the same number of zeros inside C.

Let $F(z) = g(z)/f(z)$ so that $g(z) = f(z)F(z)$ or briefly $g = fF$. Then if N_1 and N_2 are the number of zeros inside C of $f + g$ and f respectively, we have by Problem 17, using the fact that these functions have no poles inside C,

$$N_1 = \frac{1}{2\pi i}\oint_C \frac{f'+g'}{f+g}\,dz, \qquad N_2 = \frac{1}{2\pi i}\oint_C \frac{f'}{f}\,dz$$

Then

$$\begin{aligned}N_1 - N_2 &= \frac{1}{2\pi i}\oint_C \frac{f' + f'F + fF'}{f + fF}\,dz - \frac{1}{2\pi i}\oint_C \frac{f'}{f}\,dz \\ &= \frac{1}{2\pi i}\oint_C \frac{f'(1+F) + fF'}{f(1+F)}\,dz - \frac{1}{2\pi i}\oint_C \frac{f'}{f}\,dz \\ &= \frac{1}{2\pi i}\oint_C \left\{\frac{f'}{f} + \frac{F'}{1+F}\right\}dz - \frac{1}{2\pi i}\oint_C \frac{f'}{f}\,dz \\ &= \frac{1}{2\pi i}\oint_C \frac{F'}{1+F}\,dz = \frac{1}{2\pi i}\int_C F'(1 - F + F^2 - F^3 + \cdots)\,dz \\ &= 0\end{aligned}$$

using the given fact that $|F| < 1$ on C so that the series is uniformly convergent on C and term by term integration yields the value zero. Thus $N_1 = N_2$ as required.

19. Use Rouché's theorem (Problem 18) to prove that every polynomial of degree n has exactly n zeros (fundamental theorem of algebra).

Suppose the polynomial to be $a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$, where $a_n \neq 0$. Choose $f(z) = a_nz^n$ and $g(z) = a_0 + a_1z + a_2z^2 + \cdots + a_{n-1}z^{n-1}$.

If C is a circle having centre at the origin and radius $r > 1$, then on C we have

$$\left|\frac{g(z)}{f(z)}\right| = \frac{|a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}|}{|a_n z^n|}$$

$$\leq \frac{|a_0| + |a_1|\,r + |a_2|\,r^2 + \cdots + |a_{n-1}|\,r^{n-1}}{|a_n|\,r^n}$$

$$\leq \frac{|a_0|\,r^{n-1} + |a_1|\,r^{n-1} + |a_2|\,r^{n-1} + \cdots + |a_{n-1}|\,r^{n-1}}{|a_n|\,r^n}$$

$$= \frac{|a_0| + |a_1| + |a_2| + \cdots + |a_{n-1}|}{|a_n|\,r}$$

Then by choosing r large enough we can make $\left|\frac{g(z)}{f(z)}\right| < 1$, i.e. $|g(z)| < |f(z)|$. Hence by Rouché's theorem the given polynomial $f(z) + g(z)$ has the same number of zeros as $f(z) = a_n z^n$. But since this last function has n zeros all located at $z = 0$, $f(z) + g(z)$ also has n zeros and the proof is complete.

20. Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.

Consider the circle C_1: $|z| = 1$. Let $f(z) = 12$, $g(z) = z^7 - 5z^3$. On C_1 we have

$$|g(z)| = |z^7 - 5z^3| \leq |z^7| + |5z^3| \leq 6 < 12 = |f(z)|$$

Hence by Rouché's theorem $f(z) + g(z) = z^7 - 5z^3 + 12$ has the same number of zeros inside $|z| = 1$ as $f(z) = 12$, i.e. there are no zeros inside C_1.

Consider the circle C_2: $|z| = 2$. Let $f(z) = z^7$, $g(z) = 12 - 5z^3$. On C_2 we have

$$|g(z)| = |12 - 5z^3| \leq |12| + |5z^3| \leq 60 < 2^7 = |f(z)|$$

Hence by Rouché's theorem $f(z) + g(z) = z^7 - 5z^3 + 12$ has the same number of zeros inside $|z| = 2$ as $f(z) = z^7$, i.e. all the zeros are inside C_2.

Hence all the roots lie inside $|z| = 2$ but outside $|z| = 1$, as required.

POISSON'S INTEGRAL FORMULAE FOR A CIRCLE

21. (*a*) Let $f(z)$ be analytic inside and on the circle C defined by $|z| = R$, and let $z = re^{i\theta}$ be any point inside C. Prove that

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\, f(Re^{i\phi})\, d\phi$$

(*b*) If $u(r,\theta)$ and $v(r,\theta)$ are the real and imaginary parts of $f(re^{i\theta})$, prove that

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\,u(R,\phi)\,d\phi}{R^2 - 2Rr\cos(\theta - \phi) + r^2}$$

$$v(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\,v(R,\phi)\,d\phi}{R^2 - 2Rr\cos(\theta - \phi) + r^2}$$

The results are called *Poisson's integral formulae for the circle.*

(*a*) Since $z = re^{i\theta}$ is any point inside C, we have by Cauchy's integral formula

$$f(z) = f(re^{i\theta}) = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w - z}\, dw \qquad (1)$$

The *inverse of the point* z with respect to C lies outside C and is given by $R^2/\bar{z}$. Hence by Cauchy's theorem,

$$0 = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w - R^2/\bar{z}}\, dw \qquad (2)$$

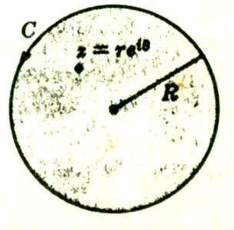

Fig. 5-9

If we subtract (2) from (1), we find

$$f(z) \quad = \quad \frac{1}{2\pi i}\oint_C \left\{\frac{1}{w-z} - \frac{1}{w-R^2/\bar{z}}\right\} f(w)\,dw$$

$$= \quad \frac{1}{2\pi i}\oint_C \frac{z-R^2/\bar{z}}{(w-z)(w-R^2/\bar{z})}\,f(w)\,dw \tag{3}$$

Now let $z = re^{i\theta}$ and $w = Re^{i\phi}$. Then since $\bar{z} = re^{-i\theta}$, (3) yields

$$f(re^{i\theta}) \quad = \quad \frac{1}{2\pi i}\int_0^{2\pi} \frac{\{re^{i\theta} - (R^2/r)e^{i\theta}\}\,f(Re^{i\phi})\,iRe^{i\phi}\,d\phi}{\{Re^{i\phi} - re^{i\theta}\}\{Re^{i\phi} - (R^2/r)e^{i\theta}\}}$$

$$= \quad \frac{1}{2\pi}\int_0^{2\pi} \frac{(r^2-R^2)\,e^{i(\theta+\phi)}\,f(Re^{i\phi})\,d\phi}{(Re^{i\phi}-re^{i\theta})(re^{i\phi}-Re^{i\theta})}$$

$$= \quad \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\,f(Re^{i\phi})\,d\phi}{(Re^{i\phi}-re^{i\theta})(Re^{-i\phi}-re^{-i\theta})}$$

$$= \quad \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\,f(Re^{i\phi})\,d\phi}{R^2 - 2Rr\cos(\theta-\phi) + r^2}$$

(b) Since $f(re^{i\theta}) = u(r,\theta) + i\,v(r,\theta)$ and $f(Re^{i\phi}) = u(R,\phi) + i\,v(R,\phi)$, we have from part (a),

$$u(r,\theta) + i\,v(r,\theta) \quad = \quad \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\{u(R,\phi) + i\,v(R,\phi)\}\,d\phi}{R^2 - 2Rr\cos(\theta-\phi) + r^2}$$

$$= \quad \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\,u(R,\phi)\,d\phi}{R^2 - 2Rr\cos(\theta-\phi) + r^2} \; + \; \frac{i}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\,v(R,\phi)\,d\phi}{R^2 - 2Rr\cos(\theta-\phi) + r^2}$$

Then the required result follows on equating real and imaginary parts.

POISSON'S INTEGRAL FORMULAE FOR A HALF PLANE

22. Derive Poisson's formulae for the half plane [see Page 120].

Let C be the boundary of a semicircle of radius R [see Fig. 5-10] containing ζ as an interior point. Since C encloses ζ but does not enclose $\bar{\zeta}$, we have by Cauchy's integral formula,

$$f(\zeta) \quad = \quad \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-\zeta}\,dz, \qquad 0 \quad = \quad \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-\bar{\zeta}}\,dz$$

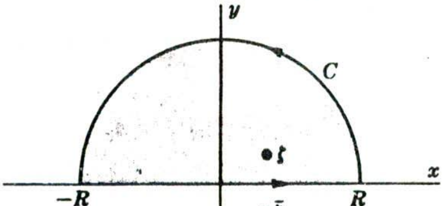

Fig. 5-10

Then by subtraction,

$$f(\zeta) \quad = \quad \frac{1}{2\pi i}\oint_C f(z)\left\{\frac{1}{z-\zeta} - \frac{1}{z-\bar{\zeta}}\right\}dz$$

$$= \quad \frac{1}{2\pi i}\oint_C \frac{(\zeta-\bar{\zeta})\,f(z)\,dz}{(z-\zeta)(z-\bar{\zeta})}$$

Letting $\zeta = \xi + i\eta$, $\bar{\zeta} = \xi - i\eta$, this can be written

$$f(\zeta) \quad = \quad \frac{1}{\pi}\int_{-R}^{R} \frac{\eta\,f(x)\,dx}{(x-\xi)^2+\eta^2} \; + \; \frac{1}{\pi}\int_\Gamma \frac{\eta\,f(z)\,dz}{(z-\zeta)(z-\bar{\zeta})}$$

where Γ is the semicircular arc of C. As $R \to \infty$, this last integral approaches zero [see Problem 76] and we have

$$f(\zeta) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\eta\,f(x)\,dx}{(x-\xi)^2+\eta^2}$$

Writing $f(\zeta) = f(\xi+i\eta) = u(\xi,\eta) + i\,v(\xi,\eta)$, $f(x) = u(x,0) + i\,v(x,0)$, we obtain as required,

$$u(\xi,\eta) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\eta\,u(x,0)\,dx}{(x-\xi)^2+\eta^2}, \qquad v(\xi,\eta) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\eta\,v(x,0)\,dx}{(x-\xi)^2+\eta^2}$$

MISCELLANEOUS PROBLEMS

23. Let $f(z)$ be analytic in a region $\mathcal{R}$ bounded by two concentric circles C_1 and C_2 and on the boundary [Fig. 5-11]. Prove that if z_0 is any point in $\mathcal{R}$, then

$$f(z_0) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i}\oint_{C_2}\frac{f(z)}{z-z_0}\,dz$$

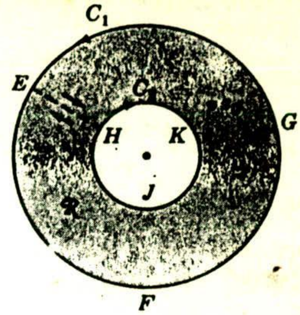

Fig. 5-11

Method 1.

Construct cross-cut EH connecting circles C_1 and C_2. Then $f(z)$ is analytic in the region bounded by $EFGEHKJHE$. Hence by Cauchy's integral formula,

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\oint_{EFGEHKJHE}\frac{f(z)}{z-z_0}\,dz \\ &= \frac{1}{2\pi i}\oint_{EFGE}\frac{f(z)}{z-z_0}\,dz + \frac{1}{2\pi i}\int_{EH}\frac{f(z)}{z-z_0}\,dz + \frac{1}{2\pi i}\oint_{HKJH}\frac{f(z)}{z-z_0}\,dz + \frac{1}{2\pi i}\int_{HE}\frac{f(z)}{z-z_0}\,dz \\ &= \frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i}\oint_{C_2}\frac{f(z)}{z-z_0}\,dz \end{aligned}$$

since the integrals along EH and HE cancel.

Similar results can be established for the derivatives of $f(z)$.

Method 2. The result also follows from equation (*3*) of Problem 6 if we replace the simple closed curves C_1 and C_2 by the circles of Fig. 5-11.

24. Prove that $\displaystyle\int_0^{2\pi}\cos^{2n}\theta\,d\theta = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}2\pi$ where $n = 1, 2, 3, \ldots$.

Let $z = e^{i\theta}$. Then $dz = ie^{i\theta}\,d\theta = iz\,d\theta$ or $d\theta = dz/iz$ and $\cos\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta}) = \frac{1}{2}(z+1/z)$. Hence if C is the unit circle $|z| = 1$, we have

$$\begin{aligned} \int_0^{2\pi}\cos^{2n}\theta\,d\theta &= \oint_C\left\{\frac{1}{2}\left(z+\frac{1}{z}\right)\right\}^{2n}\frac{dz}{iz} \\ &= \frac{1}{2^{2n}i}\oint_C\frac{1}{z}\left\{z^{2n} + \binom{2n}{1}(z^{2n-1})\left(\frac{1}{z}\right) + \cdots + \binom{2n}{k}(z^{2n-k})\left(\frac{1}{z}\right)^k + \cdots + \left(\frac{1}{z}\right)^{2n}\right\}dz \\ &= \frac{1}{2^{2n}i}\oint_C\left\{z^{2n-1} + \binom{2n}{1}z^{2n-3} + \cdots + \binom{2n}{k}z^{2n-2k-1} + \cdots + z^{-2n}\right\}dz \\ &= \frac{1}{2^{2n}i}\cdot 2\pi i\binom{2n}{n} = \frac{1}{2^{2n}}\binom{2n}{n}2\pi \\ &= \frac{1}{2^{2n}}\frac{(2n)!}{n!\,n!}2\pi = \frac{(2n)(2n-1)(2n-2)\cdots(n)(n-1)\cdots 1}{2^{2n}\,n!\,n!}2\pi \\ &= \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}2\pi \end{aligned}$$

25. If $f(z) = u(x,y) + i\,v(x,y)$ is analytic in a region $\mathcal{R}$, prove that u and v are harmonic in $\mathcal{R}$.

In Problem 6, Chapter 3, we proved that u and v are harmonic in $\mathcal{R}$, i.e. satisfy the equation $\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = 0$, under the *assumption* of existence of the second partial derivatives of u and v, i.e. the existence of $f''(z)$.

This assumption is no longer necessary since we have in fact proved in Problem 4 that if $f(z)$ is analytic in $\mathcal{R}$ then *all* the derivatives of $f(z)$ exist.

26. Prove *Schwarz's theorem*: Let $f(z)$ be analytic for $|z| \leqq R$, $f(0) = 0$ and $|f(z)| \leqq M$. Then

$$|f(z)| \leqq \frac{M|z|}{R}$$

The function $f(z)/z$ is analytic in $|z| \leqq R$. Hence on $|z| = R$ we have by the maximum modulus theorem,

$$\left|\frac{f(z)}{z}\right| \leqq \frac{M}{R}$$

However, since this inequality must also hold for points inside $|z| = R$, we have for $|z| \leqq R$, $|f(z)| \leqq M|z|/R$ as required.

27. Let $f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ where x is real. Show that the function (*a*) has a first derivative at all values of x for which $0 \leqq x \leqq 1$ but (*b*) does not have a second derivative in $0 \leqq x \leqq 1$. (*c*) Reconcile these conclusions with the result of Problem 4.

(*a*) The only place where there is any question as to existence of the first derivative is at $x = 0$. But at $x = 0$ the derivative is

$$\lim_{\Delta x \to 0} \frac{f(0+\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(\Delta x)^2 \sin(1/\Delta x) - 0}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \Delta x \sin(1/\Delta x) = 0$$

and so exists.

At all other values of x in $0 \leqq x \leqq 1$, the derivative is given (using elementary differentiation rules) by

$$x^2 \cos(1/x)\{-1/x^2\} + (2x)\sin(1/x) = 2x \sin(1/x) - \cos(1/x)$$

(*b*) From part (*a*), we have

$$f'(x) = \begin{cases} 2x\sin(1/x) - \cos(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

The second derivative exists for all x such that $0 < x \leqq 1$. At $x = 0$ the second derivative is given by

$$\lim_{\Delta x \to 0} \frac{f'(0+\Delta x) - f'(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\,\Delta x \sin(1/\Delta x) - \cos(1/\Delta x) - 0}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \{2\sin(1/\Delta x) - (1/\Delta x)\cos(1/\Delta x)\}$$

which does not exist.

It follows that the second derivative of $f(x)$ does not exist in $0 \leqq x \leqq 1$.

(*c*) According to Problem 4, if $f(z)$ is analytic in a region $\mathcal{R}$ then all higher derivatives exist and are analytic in $\mathcal{R}$. The above results do not conflict with this, since the function $f(z) = z^2 \sin(1/z)$ is not analytic in any region which includes $z = 0$.

28. (*a*) If $F(z)$ is analytic inside and on a simple closed curve C except for a pole of order m at $z = a$ inside C, prove that

$$\frac{1}{2\pi i}\oint_C F(z)\,dz = \lim_{z \to a} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m F(z)\}$$

(*b*) How would you modify the result in (*a*) if more than one pole were inside C?

(a) If $F(z)$ has a pole of order m at $z = a$, then $F(z) = f(z)/(z-a)^m$ where $f(z)$ is analytic inside and on C, and $f(a) \neq 0$. Then by Cauchy's integral formula,

$$\frac{1}{2\pi i}\oint_C F(z)\,dz = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^m}\,dz = \frac{f^{(m-1)}(a)}{(m-1)!}$$

$$= \lim_{z\to a} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m F(z)\}$$

(b) Suppose there are two poles at $z = a_1$ and $z = a_2$ inside C, of orders m_1 and m_2 respectively. Let Γ_1 and Γ_2 be circles inside C having radii ϵ_1 and ϵ_2 and centres at a_1 and a_2 respectively. Then

$$\frac{1}{2\pi i}\oint_C F(z)\,dz = \frac{1}{2\pi i}\oint_{\Gamma_1} F(z)\,dz + \frac{1}{2\pi i}\oint_{\Gamma_2} F(z)\,dz \qquad (1)$$

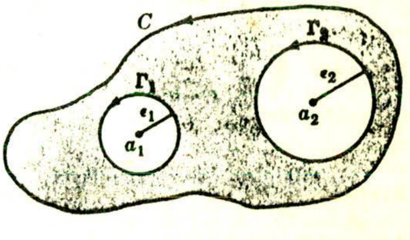

Fig. 5-12

If $F(z)$ has a pole of order m_1 at $z = a_1$, then

$$F(z) = \frac{f_1(z)}{(z-a_1)^{m_1}} \qquad \text{where } f_1(z) \text{ is analytic and } f_1(a_1) \neq 0$$

If $F(z)$ has a pole of order m_2 at $z = a_2$, then

$$F(z) = \frac{f_2(z)}{(z-a_2)^{m_2}} \qquad \text{where } f_2(z) \text{ is analytic and } f_2(a_2) \neq 0$$

Then by (1) and part (a),

$$\frac{1}{2\pi i}\oint_C F(z)\,dz = \frac{1}{2\pi i}\oint_{\Gamma_1} \frac{f_1(z)}{(z-a_1)^{m_1}}\,dz + \frac{1}{2\pi i}\oint_{\Gamma_2} \frac{f_2(z)}{(z-a_2)^{m_2}}\,dz$$

$$= \lim_{z\to a_1} \frac{1}{(m_1-1)!}\frac{d^{m_1-1}}{dz^{m_1-1}}\{(z-a_1)^{m_1} F(z)\}$$

$$+ \lim_{z\to a_2} \frac{1}{(m_2-1)!}\frac{d^{m_2-1}}{dz^{m_2-1}}\{(z-a_2)^{m_2} F(z)\}$$

If the limits on the right are denoted by R_1 and R_2, we can write

$$\oint_C F(z)\,dz = 2\pi i(R_1 + R_2)$$

where R_1 and R_2 are called the *residues* of $F(z)$ at the poles $z = a_1$ and $z = a_2$.

In general if $F(z)$ has a number of poles inside C with residues $R_1, R_2, \ldots$, then $\oint_C F(z)\,dz =$ $2\pi i$ times the sum of the residues. This result is called the *residue theorem.* Applications of this theorem together with generalization to singularities other than poles, are treated in Chap. 7.

29. Evaluate $\oint_C \frac{e^z}{(z^2+\pi^2)^2}\,dz$ where C is the circle $|z| = 4$.

The poles of $\frac{e^z}{(z^2+\pi^2)^2} = \frac{e^z}{(z-\pi i)^2(z+\pi i)^2}$ are at $z = \pm\pi i$ inside C and are both of order two.

Residue at $z = \pi i$ is $\lim_{z\to\pi i} \frac{1}{1!}\frac{d}{dz}\left\{(z-\pi i)^2 \frac{e^z}{(z-\pi i)^2(z+\pi i)^2}\right\} = \frac{\pi+i}{4\pi^3}.$

Residue at $z = -\pi i$ is $\lim_{z\to-\pi i} \frac{1}{1!}\frac{d}{dz}\left\{(z+\pi i)^2 \frac{e^z}{(z-\pi i)^2(z+\pi i)^2}\right\} = \frac{\pi-i}{4\pi^3}.$

Then $\oint_C \frac{e^z}{(z^2+\pi^2)^2}\,dz = 2\pi i(\text{sum of residues}) = 2\pi i\left(\frac{\pi+i}{4\pi^3} + \frac{\pi-i}{4\pi^3}\right) = \frac{i}{\pi}.$

Supplementary Problems

CAUCHY'S INTEGRAL FORMULAE

30. Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}\,dz$ if C is (a) the circle $|z|=3$, (b) the circle $|z|=1$. *Ans.* (a) e^2, (b) 0

31. Evaluate $\oint_C \frac{\sin 3z}{z+\pi/2}\,dz$ if C is the circle $|z|=5$. *Ans.* $2\pi i$

32. Evaluate $\oint_C \frac{e^{3z}}{z-\pi i}\,dz$ if C is (a) the circle $|z-1|=4$, (b) the ellipse $|z-2|+|z+2|=6$.
Ans. (a) $-2\pi i$, (b) 0

33. Evaluate $\frac{1}{2\pi i}\oint_C \frac{\cos \pi z}{z^2-1}\,dz$ around a rectangle with vertices at: (a) $2\pm i, -2\pm i$; (b) $-i, 2-i, 2+i, i$.
Ans. (a) 0, (b) $-\frac{1}{2}$

34. Show that $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2+1}\,dz = \sin t$ if $t>0$ and C is the circle $|z|=3$.

35. Evaluate $\oint_C \frac{e^{iz}}{z^3}\,dz$ where C is the circle $|z|=2$. *Ans.* $-\pi i$

36. Prove that $f'''(a) = \frac{3!}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a)^4}$ if C is a simple closed curve enclosing $z=a$ and $f(z)$ is analytic inside and on C.

37. Prove Cauchy's integral formulae for all positive integral values of n. [*Hint*: Use mathematical induction.]

38. Find the value of (a) $\oint_C \frac{\sin^6 z}{z-\pi/6}\,dz$, (b) $\oint_C \frac{\sin^6 z}{(z-\pi/6)^3}\,dz$ if C is the circle $|z|=1$.
Ans. (a) $\pi i/32$, (b) $21\pi i/16$

39. Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{(z^2+1)^2}\,dz$ if $t>0$ and C is the circle $|z|=3$. *Ans.* $\frac{1}{2}(\sin t - t\cos t)$

40. Prove Cauchy's integral formulae for the multiply-connected region of Fig. 4-26, Page 115.

MORERA'S THEOREM

41. (a) Determine whether $G(z) = \int_1^z \frac{d\zeta}{\zeta}$ is independent of the path joining 1 and z.
(b) Discuss the relationship of your answer to part (a) with Morera's theorem.

42. Does Morera's theorem apply in a multiply-connected region? Justify your answer.

43. (a) If $P(x,y)$ and $Q(x,y)$ are conjugate harmonic functions and C is any simple closed curve, prove that $\oint_C P\,dx + Q\,dy = 0$.
(b) If for all simple closed curves C in a region $\mathcal{R}$, $\oint_C P\,dx + Q\,dy = 0$, is it true that P and Q are conjugate harmonic functions, i.e. is the converse of (a) true? Justify your conclusion.

CAUCHY'S INEQUALITY

44. (a) Use Cauchy's inequality to obtain estimates for the derivatives of $\sin z$ at $z=0$ and (b) determine how good these estimates are.

45. (*a*) Show that if $f(z) = 1/(1-z)$, then $f^{(n)}(z) = n!/(1-z)^{n+1}$.

 (*b*) Use (*a*) to show that the Cauchy inequality is "best possible", i.e. the estimate of growth of the nth derivative cannot be improved for *all* functions.

46. Prove that the equality in Cauchy's inequality (*3*), Page 118, holds if and only if $f(z) = kMz^n/r^n$ where $|k| = 1$.

47. Discuss Cauchy's inequality for the function $f(z) = e^{-1/z^2}$ in the neighbourhood of $z=0$.

LIOUVILLE'S THEOREM

48. The function of a real variable defined by $f(x) = \sin x$ is (*a*) analytic everywhere and (*b*) bounded, i.e. $|\sin x| \leqq 1$ for all x but it is certainly not a constant. Does this contradict Liouville's theorem? Explain.

49. A non-constant function $F(z)$ is such that $F(z+a) = F(z)$, $F(z+bi) = F(z)$ where $a>0$ and $b>0$ are given constants. Prove that $F(z)$ cannot be analytic in the rectangle $0 \leqq x \leqq a$, $0 \leqq y \leqq b$.

FUNDAMENTAL THEOREM OF ALGEBRA

50. (*a*) Carry out the details of proof of the fundamental theorem of algebra to show that the particular function $f(z) = z^4 - z^2 - 2z + 2$ has exactly four zeros. (*b*) Determine the zeros of $f(z)$.
 Ans. (*b*) $1, 1, -1 \pm i$

51. Determine all the roots of the equations (*a*) $z^3 - 3z + 4i = 0$, (*b*) $z^4 + z^2 + 1 = 0$.
 Ans. (*a*) i, $\frac{1}{2}(-i \pm \sqrt{15})$, (*b*) $\frac{1}{2}(-1 \pm \sqrt{3}\,i)$, $\frac{1}{2}(1 \pm \sqrt{3}\,i)$

GAUSS' MEAN VALUE THEOREM

52. Evaluate $\frac{1}{2\pi}\int_0^{2\pi} \sin^2(\pi/6 + 2e^{i\theta})\,d\theta$ *Ans.* 1/4

53. Show that the mean value of any harmonic function over a circle is equal to the value of the function at the centre.

54. Find the mean value of $x^2 - y^2 + 2y$ over the circle $|z - 5 + 2i| = 3$. *Ans.* 5

55. Prove that $\int_0^{\pi} \ln \sin\theta\, d\theta = -\pi \ln 2$. [*Hint.* Consider $f(z) = \ln(1+z)$.]

MAXIMUM MODULUS THEOREM

56. Find the maximum of $|f(z)|$ in $|z| \leqq 1$ for the functions $f(z)$ given by (*a*) $z^2 - 3z + 2$, (*b*) $z^4 + z^2 + 1$, (*c*) $\cos 3z$, (*d*) $(2z+1)/(2z-1)$.

57. (*a*) If $f(z)$ is analytic inside and on the simple closed curve C enclosing $z=a$, prove that

$$\{f(a)\}^n = \frac{1}{2\pi i}\oint_C \frac{\{f(z)\}^n}{z-a}\,dz \qquad n = 0, 1, 2, \ldots$$

 (*b*) Use (*a*) to prove that $|f(a)|^n \leqq M^n/2\pi D$ where D is the minimum distance from a to the curve C and M is the maximum value of $|f(z)|$ on C.

 (*c*) By taking the nth root of both sides of the inequality in (*b*) and letting $n \to \infty$, prove the maximum modulus theorem.

58. Let $U(x,y)$ be harmonic inside and on a simple closed curve C. Prove that the (*a*) maximum and (*b*) minimum values of $U(x,y)$ are attained on C. Are there other restrictions on $U(x,y)$?

59. Verify Problem 58 for the functions (*a*) $x^2 - y^2$ and (*b*) $x^3 - 3xy^2$ if C is the circle $|z| = 1$.

60. Is the maximum modulus theorem valid for multiply-connected regions? Justify your answer.

THE ARGUMENT THEOREM

61. If $f(z) = z^5 - 3iz^2 + 2z - 1 + i$, evaluate $\oint_C \frac{f'(z)}{f(z)}\,dz$ where C encloses all the zeros of $f(z)$.
Ans. $10\pi i$

62. Let $f(z) = \frac{(z^2+1)^2}{(z^2+2z+2)^3}$. Evaluate $\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz$ where C is the circle $|z| = 4$. *Ans.* -2

63. Evaluate $\oint_C \frac{f'(z)}{f(z)}\,dz$ if C is the circle $|z| = \pi$ and (a) $f(z) = \sin \pi z$, (b) $f(z) = \cos \pi z$, (c) $f(z) = \tan \pi z$.
Ans. (a) $14\pi i$, (b) $12\pi i$, (c) $2\pi i$

64. If $f(z) = z^4 - 2z^3 + z^2 - 12z + 20$ and C is the circle $|z| = 5$, evaluate $\oint_C \frac{z\,f'(z)}{f(z)}\,dz$. *Ans.* $4\pi i$

ROUCHE'S THEOREM

65. If $a > e$, prove that the equation $az^n = e^z$ has n roots inside $|z| = 1$.

66. Prove that $ze^z = a$ where $a \neq 0$ is real has infinitely many roots.

67. Prove that $\tan z = az$, $a > 0$ has (a) infinitely many real roots, (b) only two pure imaginary roots if $0 < a < 1$, (c) all real roots if $a \geqq 1$.

68. Prove that $z \tan z = a$, $a > 0$ has infinitely many real roots but no imaginary roots.

POISSON'S INTEGRAL FORMULAE FOR A CIRCLE

69. Show that
$$\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\,d\phi = 2\pi$$
(a) with, (b) without Poisson's integral formula for a circle.

70. Show that (a) $$\int_0^{2\pi} \frac{e^{\cos\phi}\cos(\sin\phi)}{5 - 4\cos(\theta - \phi)}\,d\phi = \frac{2\pi}{3}e^{\cos\theta}\cos(\sin\theta)$$
(b) $$\int_0^{2\pi} \frac{e^{\cos\phi}\sin(\sin\phi)}{5 - 4\cos(\theta - \phi)}\,d\phi = \frac{2\pi}{3}e^{\cos\theta}\sin(\sin\theta)$$

71. (a) Prove that the function $U(r,\theta) = \frac{2}{\pi}\tan^{-1}\left(\frac{2r\sin\theta}{1 - r^2}\right)$, $0 < r < 1$, $0 \leqq \theta < 2\pi$ is harmonic inside the circle $|z| = 1$.
(b) Show that $\lim_{r \to 1-} U(r,\theta) = \begin{cases} 1 & 0 < \theta < \pi \\ -1 & \pi < \theta < 2\pi \end{cases}$.
(c) Can you derive the expression for $U(r,\theta)$ from Poisson's integral formula for a circle?

72. If $f(z)$ is analytic inside and on the circle C defined by $|z| = R$ and if $z = re^{i\theta}$ is any point inside C, show that
$$f'(re^{i\theta}) = \frac{i}{2\pi}\int_0^{2\pi} \frac{R(R^2 - r^2)\,f(Re^{i\phi})\sin(\theta - \phi)}{[R^2 - 2Rr\cos(\theta - \phi) + r^2]^2}\,d\phi$$

73. Verify that the functions u and v of equations (7) and (8), Page 119, satisfy Laplace's equation.

POISSON'S INTEGRAL FORMULAE FOR A HALF PLANE

74. Find a function which is harmonic in the upper half plane $y > 0$ and which on the x axis takes the values -1 if $x < 0$ and 1 if $x > 0$. *Ans.* $1 - (2/\pi)\tan^{-1}(y/x)$

75. Work Problem 74 if the function takes the values -1 if $x < -1$, 0 if $-1 < x < 1$, and 1 if $x > 1$.
Ans. $1 - \frac{1}{\pi}\tan^{-1}\left(\frac{y}{x+1}\right) - \frac{1}{\pi}\tan^{-1}\left(\frac{y}{x-1}\right)$

76. Prove the statement made in Problem 22 that the integral over Γ approaches zero as $R \to \infty$.

77. Prove that under suitable restrictions on $f(x)$,

$$\lim_{\eta \to 0+} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\eta\, f(x)}{(x-\xi)^2 + \eta^2}\, dx \quad = \quad f(\xi)$$

and state these restrictions.

78. Verify that the functions u and v of equations *(10)* and *(11)*, Page 120, satisfy Laplace's equation.

MISCELLANEOUS PROBLEMS

79. Evaluate $\dfrac{1}{2\pi i} \oint_C \dfrac{z^2\, dz}{z^2+4}$ where C is the square with vertices at ± 2, $\pm 2 + 4i$. *Ans.* i

80. Evaluate $\oint_C \dfrac{\cos^2 tz}{z^3}\, dz$ where C is the circle $|z| = 1$ and $t > 0$. *Ans.* $-2\pi i t^2$

81. (*a*) Show that $\oint_C \dfrac{dz}{z+1} = 2\pi i$ if C is the circle $|z| = 2$.

(*b*) Use (*a*) to show that

$$\oint_C \frac{(x+1)\,dx + y\,dy}{(x+1)^2 + y^2} \;=\; 0, \qquad \oint_C \frac{(x+1)\,dy - y\,dx}{(x+1)^2 + y^2} \;=\; 2\pi$$

and verify these results directly.

82. Find all functions $f(z)$ which are analytic everywhere in the entire complex plane and which satisfy the conditions (*a*) $f(2-i) = 4i$ and (*b*) $|f(z)| < e^2$ for all z.

83. If $f(z)$ is analytic inside and on a simple closed curve C, prove that

$$(a) \quad f'(a) \;=\; \frac{1}{2\pi} \int_0^{2\pi} e^{-i\theta} f(a + e^{i\theta})\, d\theta$$

$$(b) \quad \frac{f^{(n)}(a)}{n!} \;=\; \frac{1}{2\pi} \int_0^{2\pi} e^{-ni\theta} f(a + e^{i\theta})\, d\theta$$

84. Prove that $8z^4 - 6z + 5 = 0$ has one root in each quadrant.

85. Show that (*a*) $\displaystyle\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta)\, d\theta = 0$, (*b*) $\displaystyle\int_0^{2\pi} e^{\cos\theta} \sin(\sin\theta)\, d\theta = 2\pi$.

86. Extend the result of Problem 23 so as to obtain formulae for the derivatives of $f(z)$ at any point in $\mathcal{R}$.

87. Prove that $z^3 e^{1-z} = 1$ has exactly two roots inside the circle $|z| = 1$.

88. If $t > 0$ and C is any simple closed curve enclosing $z = -1$, prove that

$$\frac{1}{2\pi i} \oint_C \frac{z e^{zt}}{(z+1)^3}\, dz \;=\; \left(t - \frac{t^2}{2}\right) e^{-t}$$

89. Find all functions $f(z)$ which are analytic in $|z| < 1$ and which satisfy the conditions (*a*) $f(0) = 1$ and (*b*) $|f(z)| \geqq 1$ for $|z| < 1$.

90. Let $f(z)$ and $g(z)$ be analytic inside and on a simple closed curve C except that $f(z)$ has zeros at $a_1, a_2, \ldots, a_m$ and poles at $b_1, b_2, \ldots, b_n$ of orders (multiplicities) $p_1, p_2, \ldots, p_m$ and $q_1, q_2, \ldots, q_n$ respectively. Prove that

$$\frac{1}{2\pi i} \oint_C g(z) \frac{f'(z)}{f(z)}\, dz \;=\; \sum_{k=1}^{m} p_k\, g(a_k) \;-\; \sum_{k=1}^{n} q_k\, g(b_k)$$

91. If $f(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n$ where $a_0 \neq 0, a_1, \ldots, a_n$ are complex constants and C encloses all the zeros of $f(z)$, evaluate (a) $\frac{1}{2\pi i}\oint_C \frac{z f'(z)}{f(z)}\,dz$, (b) $\frac{1}{2\pi i}\oint_C \frac{z^2 f'(z)}{f(z)}\,dz$ and interpret the results. *Ans.* (a) $-a_1/a_0$, (b) $(a_1^2 - 2a_0 a_2)/a_0^2$

92. Find all functions $f(z)$ which are analytic in the region $|z| \leqq 1$ and are such that (a) $f(0) = 3$ and (b) $|f(z)| \leqq 3$ for all z such that $|z| < 1$.

93. Prove that $z^6 + 192z + 640 = 0$ has one root in the first and fourth quadrants and two roots in the second and third quadrants.

94. Prove that the function $xy(x^2 - y^2)$ cannot have an absolute maximum or minimum inside the circle $|z| = 1$.

95. (a) If a function is analytic in a region $\mathcal{R}$, is it bounded in $\mathcal{R}$? (b) In view of your answer to (a), is it necessary to state that $f(z)$ is bounded in Liouville's theorem?

96. Find all functions $f(z)$ which are analytic everywhere, have a zero of order two at $z = 0$, satisfy the condition $|f'(z)| \leqq 6|z|$ for all z, and are such that $f(i) = -2$.

97. Prove that all the roots of $z^5 + z - 16i = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.

98. If U is harmonic inside and on a simple closed curve C, prove that

$$\oint_C \frac{\partial U}{\partial n}\,ds = 0$$

where n is a unit normal to C in the z plane and s is the arc length parameter.

99. A theorem of Cauchy states that all the roots of the equation $z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n = 0$, where $a_1, a_2, \ldots, a_n$ are real, lie inside the circle $|z| = 1 + \max\{a_1, a_2, \ldots, a_n\}$, i.e. $|z| = 1$ plus the maximum of the values $a_1, a_2, \ldots, a_n$. Verify this theorem for the special cases (a) $z^3 - z^2 + z - 1 = 0$, (b) $z^4 + z^2 + 1 = 0$, (c) $z^4 - z^2 - 2z + 2 = 0$, (d) $z^4 + 3z^2 - 6z + 10 = 0$.

100. Prove the theorem of Cauchy stated in Problem 99.

101. Let $P(z)$ be any polynomial. If m is any positive integer and $\omega = e^{2\pi i/m}$, prove that

$$\frac{P(1) + P(\omega) + P(\omega^2) + \cdots + P(\omega^{m-1})}{m} = P(0)$$

and give a geometric interpretation.

102. Is the result of Problem 101 valid for any function $f(z)$? Justify your answer.

103. Prove *Jensen's theorem:* If $f(z)$ is analytic inside and on the circle $|z| = R$ except for zeros at $a_1, a_2, \ldots, a_m$ of multiplicities $p_1, p_2, \ldots, p_m$ and poles at $b_1, b_2, \ldots, b_n$ of multiplicities $q_1, q_2, \ldots, q_n$ respectively, and if $f(0)$ is finite and different from zero, then

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|f(Re^{i\theta})|\,d\theta = \ln|f(0)| + \sum_{k=1}^{m} p_k \ln\left(\frac{R}{|a_k|}\right) - \sum_{k=1}^{n} q_k \ln\left(\frac{R}{|b_k|}\right)$$

[*Hint.* Consider $\oint_C \ln z\,\{f'(z)/f(z)\}\,dz$ where C is the circle $|z| = R$.]

Chapter 6

Infinite Series
Taylor's and Laurent's Series

SEQUENCES OF FUNCTIONS

The ideas of Chapter 2, Pages 40 and 41, for sequences and series of constants are easily extended to sequences and series of functions.

Let $u_1(z), u_2(z), \ldots, u_n(z), \ldots,$ denoted briefly by $\{u_n(z)\}$, be a sequence of functions of z defined and single-valued in some region of the z plane. We call $U(z)$ the *limit* of $u_n(z)$ as $n \to \infty$, and write $\lim_{n\to\infty} u_n(z) = U(z)$, if given any positive number ϵ we can find a number N [depending in general on both ϵ and z] such that

$$|u_n(z) - U(z)| < \epsilon \qquad \text{for all } n > N$$

In such case we say that the sequence *converges* or is *convergent* to $U(z)$.

If a sequence converges for all values of z (points) in a region $\mathcal{R}$, we call $\mathcal{R}$ the *region of convergence* of the sequence. A sequence which is not convergent at some value (point) z is called *divergent* at z.

The theorems on limits given on Page 40 can be extended to sequences of functions.

SERIES OF FUNCTIONS

From the sequence of functions $\{u_n(z)\}$ let us form a new sequence $\{S_n(z)\}$ defined by

$$\begin{aligned} S_1(z) &= u_1(z) \\ S_2(z) &= u_1(z) + u_2(z) \\ &\vdots \\ S_n(z) &= u_1(z) + u_2(z) + \cdots + u_n(z) \end{aligned}$$

where $S_n(z)$, called the nth *partial sum*, is the sum of the first n terms of the sequence $\{u_n(z)\}$.

The sequence $S_1(z), S_2(z), \ldots$ or $\{S_n(z)\}$ is symbolized by

$$u_1(z) + u_2(z) + \cdots = \sum_{n=1}^{\infty} u_n(z) \tag{1}$$

called an *infinite series*. If $\lim_{n\to\infty} S_n(z) = S(z)$, the series is called *convergent* and $S(z)$ is its *sum*; otherwise the series is called *divergent*. We sometimes write $\sum_{n=1}^{\infty} u_n(z)$ as $\Sigma\, u_n(z)$ or $\Sigma\, u_n$ for brevity.

As we have already seen, a necessary condition that the series (*1*) converge is $\lim_{n\to\infty} u_n(z) = 0$, but this is not sufficient. See, for example, Problem 150, Chapter 2, and also Problems 67(*c*), 67(*d*) and 111(*a*).

If a series converges for all values of z (points) in a region $\mathcal{R}$, we call $\mathcal{R}$ the *region of convergence* of the series.

ABSOLUTE CONVERGENCE

A series $\sum_{n=1}^{\infty} u_n(z)$ is called *absolutely convergent* if the series of absolute values, i.e. $\sum_{n=1}^{\infty} |u_n(z)|$, converges.

If $\sum_{n=1}^{\infty} u_n(z)$ converges but $\sum_{n=1}^{\infty} |u_n(z)|$ does not converge, we call $\sum_{n=1}^{\infty} u_n(z)$ *conditionally convergent.*

UNIFORM CONVERGENCE OF SEQUENCES AND SERIES

In the definition of limit of a sequence of functions it was pointed out that the number N depends in general on ϵ and the particular value of z. It may happen, however, that we can find a number N such that $|u_n(z) - U(z)| < \epsilon$ for all $n > N$, where the same number N holds for all z in a region $\mathcal{R}$ [i.e. N depends only on ϵ and not on the particular value of z (point) in the region]. In such case we say that $u_n(z)$ *converges uniformly*, or is *uniformly convergent*, to $U(z)$ for all z in $\mathcal{R}$.

Similarly if the sequence of partial sums $\{S_n(z)\}$ converges uniformly to $S(z)$ in a region, we say that the infinite series *(1) converges uniformly*, or is *uniformly convergent*, to $S(z)$ in the region.

If we call $R_n(z) = u_{n+1}(z) + u_{n+2}(z) + \cdots = S(z) - S_n(z)$ the *remainder* of the infinite series (*1*) after n terms, we can equivalently say that the series is uniformly convergent to $S(z)$ in $\mathcal{R}$ if given any $\epsilon > 0$ we can find a number N such that for all z in $\mathcal{R}$,

$$|R_n(z)| = |S(z) - S_n(z)| < \epsilon \qquad \text{for all } n > N$$

POWER SERIES

A series having the form

$$a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots = \sum_{n=0}^{\infty} a_n(z-a)^n \tag{2}$$

is called a *power series* in $z - a$. We shall sometimes indicate (*2*) briefly by $\Sigma a_n(z-a)^n$.

Clearly the power series (*2*) converges for $z = a$, and this may indeed be the only point for which it converges [see Problem 13(*b*)]. In general, however, the series converges for other points as well. In such case we can show that there exists a positive number R such that (*2*) converges for $|z-a| < R$ and diverges for $|z-a| > R$, while for $|z-a| = R$ it may or may not converge.

Geometrically if Γ is a circle of radius R with centre at $z = a$, then the series (*2*) converges at all points inside Γ and diverges at all points outside Γ, while it may or may not converge on the circle Γ. We can consider the special cases $R = 0$ and $R = \infty$ respectively to be the cases where (*2*) converges only at $z = a$ or converges for all (finite) values of z. Because of this geometrical interpretation, R is often called the *radius of convergence* of (*2*) and the corresponding circle is called the *circle of convergence.*

SOME IMPORTANT THEOREMS

For reference purposes we list here some important theorems involving sequences and series. Many of these will be familiar from their analogs for real variables.

A. General Theorems

Theorem 1. If a sequence has a limit, the limit is unique [i.e. it is the only one].

Theorem 2. Let $u_n = a_n + ib_n$, $n = 1, 2, 3, \ldots$, where a_n and b_n are real. Then a necessary and sufficient condition that $\{u_n\}$ converge is that $\{a_n\}$ and $\{b_n\}$ converge.

Theorem 3. Let $\{a_n\}$ be a real sequence with the property that

$$\text{(i)} \quad a_{n+1} \geqq a_n \quad \text{or} \quad a_{n+1} \leqq a_n, \qquad \text{(ii)} \quad |a_n| < M \ \text{(a constant)}$$

Then $\{a_n\}$ converges.

If the first condition in Property (i) holds the sequence is called *monotonic increasing*, while if the second condition holds it is called *monotonic decreasing*. If Property (ii) holds, the sequence is said to be *bounded*. Thus the theorem states that every bounded monotonic (increasing or decreasing) sequence has a limit.

Theorem 4. A necessary and sufficient condition that $\{u_n\}$ converges is that given any $\epsilon > 0$, we can find a number N such that $|u_p - u_q| < \epsilon$ for all $p > N$, $q > N$.

This result, which has the advantage that the limit itself is not present, is called *Cauchy's convergence criterion.*

Theorem 5. A necessary condition that Σu_n converge is that $\lim_{n\to\infty} u_n = 0$. However, the condition is not sufficient.

Theorem 6. Multiplication of each term of a series by a constant different from zero does not affect the convergence or divergence. Removal (or addition) of a finite number of terms from (or to) a series does not affect the convergence or divergence.

Theorem 7. A necessary and sufficient condition that $\sum_{n=1}^{\infty} (a_n + ib_n)$ converge, where a_n and b_n are real, is that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge.

B. Theorems on Absolute Convergence

Theorem 8. If $\sum_{n=1}^{\infty} |u_n|$ converges, then $\sum_{n=1}^{\infty} u_n$ converges. In words, an absolutely convergent series is convergent.

Theorem 9. The terms of an absolutely convergent series can be rearranged in any order and all such rearranged series converge to the same sum. Also the sum, difference and product of absolutely convergent series is absolutely convergent.

These are not so for conditionally convergent series (see Problem 127).

C. Special Tests for Convergence

Theorem 10. (*Comparison tests.*)

(*a*) If $\Sigma |v_n|$ converges and $|u_n| \leqq |v_n|$, then Σu_n converges absolutely.

(*b*) If $\Sigma |v_n|$ diverges and $|u_n| \geqq |v_n|$, then $\Sigma |u_n|$ diverges but Σu_n may or may not converge.

Theorem 11. (*Ratio test.*)

If $\lim_{n\to\infty} \left|\dfrac{u_{n+1}}{u_n}\right| = L$, then Σu_n converges (absolutely) if $L < 1$ and diverges if $L > 1$. If $L = 1$, the test fails.

Theorem 12. (*n*th *Root test.*)

If $\lim_{n\to\infty} \sqrt[n]{|u_n|} = L$, then Σu_n converges (absolutely) if $L < 1$ and diverges if $L > 1$. If $L = 1$, the test fails.

Theorem 13. (*Integral test.*) If $f(x) \geqq 0$ for $x \geqq a$, then $\Sigma f(n)$ converges or diverges according as $\lim_{M\to\infty} \int_a^M f(x)\,dx$ converges or diverges.

Theorem 14. (*Raabe's test.*)

If $\lim_{n\to\infty} n\left(1 - \left|\dfrac{u_{n+1}}{u_n}\right|\right) = L$, then Σu_n converges (absolutely) if $L > 1$ and diverges or converges conditionally if $L < 1$. If $L = 1$, the test fails.

Theorem 15. (*Gauss' test.*)

If $\left|\frac{u_{n+1}}{u_n}\right| = 1 - \frac{L}{n} + \frac{c_n}{n^2}$ where $|c_n| < M$ for all $n > N$, then Σu_n converges (absolutely) if $L > 1$ and diverges or converges conditionally if $L \leqq 1$.

Theorem 16. (*Alternating series test.*)

If $a_n \geqq 0$, $a_{n+1} \leqq a_n$ for $n = 1, 2, 3, \ldots$ and $\lim_{n\to\infty} a_n = 0$, then $a_1 - a_2 + a_3 - \cdots = \Sigma(-1)^{n-1} a_n$ converges.

D. Theorems on Uniform Convergence

Theorem 17. (*Weierstrass M test.*)

If $|u_n(z)| \leqq M_n$ where M_n is independent of z in a region $\mathcal{R}$ and ΣM_n converges, then $\Sigma u_n(z)$ is uniformly convergent in $\mathcal{R}$.

Theorem 18. The sum of a uniformly convergent series of continuous functions is continuous, i.e. if $u_n(z)$ is continuous in $\mathcal{R}$ and $S(z) = \Sigma u_n(z)$ is uniformly convergent in $\mathcal{R}$, then $S(z)$ is continuous in $\mathcal{R}$.

Theorem 19. If $\{u_n(z)\}$ are continuous in $\mathcal{R}$, $S(z) = \Sigma u_n(z)$ is uniformly convergent in $\mathcal{R}$ and C is a curve in $\mathcal{R}$, then

$$\int_C S(z)\,dz = \int_C u_1(z)\,dz + \int_C u_2(z)\,dz + \cdots$$

or

$$\int_C \{\Sigma u_n(z)\}\,dz = \Sigma \int_C u_n(z)\,dz$$

In words, a uniformly convergent series of continuous functions can be integrated term by term.

Theorem 20. If $u_n'(z) = \frac{d}{dz}u_n(z)$ exists in $\mathcal{R}$, $\Sigma u_n'(z)$ converges uniformly in $\mathcal{R}$ and $\Sigma u_n(z)$ converges in $\mathcal{R}$, then $\frac{d}{dz}\Sigma u_n(z) = \Sigma u_n'(z)$.

Theorem 21. If $\{u_n(z)\}$ are analytic and $\Sigma u_n(z)$ is uniformly convergent in $\mathcal{R}$, then $S(z) = \Sigma u_n(z)$ is analytic in $\mathcal{R}$.

E. Theorems on Power Series

Theorem 22. A power series converges uniformly and absolutely in any region which lies entirely inside its circle of convergence.

Theorem 23.

(*a*) A power series can be differentiated term by term in any region which lies entirely inside its circle of convergence.

(*b*) A power series can be integrated term by term along any curve C which lies entirely inside its circle of convergence.

(*c*) The sum of a power series is continuous in any region which lies entirely inside its circle of convergence.

These follow from Theorems 17, 18, 19 and 21.

Theorem 24. (*Abel's theorem.*)

Let $\Sigma a_n z^n$ have radius of convergence R and suppose that z_0 is a point on the circle of convergence such that $\Sigma a_n z_0^n$ converges. Then $\lim_{z\to z_0} \Sigma a_n z^n = \Sigma a_n z_0^n$ where $z \to z_0$ from within the circle of convergence.

Extensions to other power series are easily made.

Theorem 25. If $\Sigma a_n z^n$ converges to zero for all z such that $|z| < R$ where $R > 0$, then $a_n = 0$. Equivalently, if $\Sigma a_n z^n = \Sigma b_n z^n$ for all z such that $|z| < R$, then $a_n = b_n$.

TAYLOR'S THEOREM

Let $f(z)$ be analytic inside and on a simple closed curve C. Let a and $a+h$ be two points inside C. Then

$$f(a+h) \quad = \quad f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a) + \cdots \qquad (3)$$

or writing $z = a+h,\ h = z-a,$

$$f(z) \quad = \quad f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \cdots \qquad (4)$$

This is called *Taylor's theorem* and the series (3) or (4) is called a *Taylor series* or *expansion* for $f(a+h)$ or $f(z)$.

The region of convergence of the series (4) is given by $|z-a| < R$, where the radius of convergence R is the distance from a to the nearest singularity of the function $f(z)$. On $|z-a| = R$, the series may or may not converge. For $|z-a| > R$, the series diverges.

If the nearest singularity of $f(z)$ is at infinity, the radius of convergence is infinite, i.e. the series converges for all z.

If $a = 0$ in (3) or (4), the resulting series is often called a *Maclaurin series.*

SOME SPECIAL SERIES

The following list shows some special series together with their regions of convergence. In the case of multiple-valued functions, the principal branch is used.

1. $e^z \quad = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots \qquad |z| < \infty$

2. $\sin z \quad = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots (-1)^{n-1}\frac{z^{2n-1}}{(2n-1)!} + \cdots \qquad |z| < \infty$

3. $\cos z \quad = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots (-1)^{n-1}\frac{z^{2n-2}}{(2n-2)!} + \cdots \qquad |z| < \infty$

4. $\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots (-1)^{n-1}\frac{z^n}{n} + \cdots \qquad |z| < 1$

5. $\tan^{-1} z \quad = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots (-1)^{n-1}\frac{z^{2n-1}}{2n-1} + \cdots \qquad |z| < 1$

6. $(1+z)^p \quad = 1 + pz + \frac{p(p-1)}{2!}z^2 + \cdots + \frac{p(p-1)\cdots(p-n+1)}{n!}z^n + \cdots \qquad |z| < 1$

   This is the *binomial theorem* or *formula*. If $(1+z)^p$ is multiple-valued the result is valid for that branch of the function which has the value 1 when $z = 0$.

LAURENT'S THEOREM

Let C_1 and C_2 be concentric circles of radii R_1 and R_2 respectively and centre at a [Fig. 6-1]. Suppose that $f(z)$ is single-valued and analytic on C_1 and C_2 and in the ring-shaped region $\mathcal{R}$ [also called *annulus* or *annular region*] between C_1 and C_2, shown shaded in the figure. Let $a+h$ be any point in $\mathcal{R}$. Then we have

$$f(a+h) \quad = \quad a_0 + a_1h + a_2h^2 + \cdots$$

$$+ \frac{a_{-1}}{h} + \frac{a_{-2}}{h^2} + \frac{a_{-3}}{h^3} + \cdots \qquad (5)$$

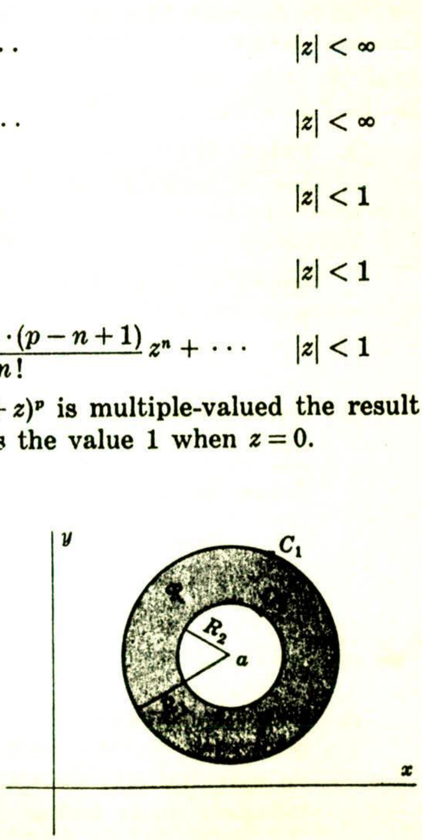

Fig. 6-1

where
$$\left.\begin{aligned} a_n &= \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{(z-a)^{n+1}}\,dz \qquad n = 0, 1, 2, \ldots \\ a_{-n} &= \frac{1}{2\pi i}\oint_{C_2} (z-a)^{n-1} f(z)\,dz \qquad n = 1, 2, 3, \ldots \end{aligned}\right\} \tag{6}$$

C_1 and C_2 being traversed in the positive direction with respect to their interiors.

We can in the above integrations replace C_1 and C_2 by any concentric circle C between C_1 and C_2 [see Problem 100]. Then the coefficients (6) can be written in a single formula,

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz \qquad n = 0, \pm1, \pm2, \ldots \tag{7}$$

With an appropriate change of notation, we can write the above as

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots \tag{8}$$

where
$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta \qquad n = 0, \pm1, \pm2, \ldots \tag{9}$$

This is called *Laurent's theorem* and (5) or (8) with coefficients (6), (7) or (9) is called a *Laurent series* or *expansion*.

The part $a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$ is called the *analytic part* of the Laurent series, while the remainder of the series which consists of inverse powers of $z-a$ is called the *principal part*. If the principal part is zero, the Laurent series reduces to a Taylor series.

CLASSIFICATION OF SINGULARITIES

It is possible to classify the singularities of a function $f(z)$ by examination of its Laurent series. For this purpose we assume that in Fig. 6-1, $R_2 = 0$, so that $f(z)$ is analytic inside and on C_1 except at $z = a$ which is an isolated singularity [see Page 67]. In the following, all singularities are assumed isolated unless otherwise indicated.

1. **Poles.** If $f(z)$ has the form (8) in which the principal part has only a finite number of terms given by
$$\frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots + \frac{a_{-n}}{(z-a)^n}$$
where $a_{-n} \neq 0$, then $z = a$ is called a *pole of order n*. If $n = 1$, it is called a *simple pole*.

   If $f(z)$ has a pole at $z = a$, then $\lim_{z\to a} f(z) = \infty$ [see Problem 32].

2. **Removable singularities.** If a single-valued function $f(z)$ is not defined at $z = a$ but $\lim_{z\to a} f(z)$ exists, then $z = a$ is called a *removable singularity*. In such case we define $f(z)$ at $z = a$ as equal to $\lim_{z\to a} f(z)$.

   Example: If $f(z) = \sin z/z$, then $z = 0$ is a removable singularity since $f(0)$ is not defined but $\lim_{z\to 0} \sin z/z = 1$. We define $f(0) = \lim_{z\to 0} \sin z/z = 1$. Note that in this case
$$\frac{\sin z}{z} = \frac{1}{z}\left\{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right\} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$

3. **Essential singularities.** If $f(z)$ is single-valued, then any singularity which is not a pole or removable singularity is called an *essential singularity*. If $z = a$ is an essential singularity of $f(z)$, the principal part of the Laurent expansion has infinitely many terms.

   Example: Since $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots$, $z = 0$ is an essential singularity.

The following two related theorems are of interest (see Problems 153-155):

Casorati-Weierstrass theorem. In any neighbourhood of an isolated essential singularity a, an otherwise analytic function $f(z)$ comes arbitrarily close to any complex number A. In symbols, given any positive numbers δ and ϵ and any complex number A, there exists a value of z inside the circle $|z-a| = \delta$ for which $|f(z) - A| < \epsilon$.

Picard's theorem. In the neighbourhood of an isolated essential singularity a, an otherwise analytic function $f(z)$ can take on any value whatsoever with perhaps one exception.

4. **Branch points.** A point $z = z_0$ is called a *branch point* of the multiple-valued function $f(z)$ if the branches of $f(z)$ are interchanged when z describes a closed path about z_0 [see Page 37]. Since each of the branches of a multiple-valued function is analytic, all of the theorems for analytic functions, in particular Taylor's theorem, apply.

   Example: The branch of $f(z) = z^{1/2}$ which has the value 1 for $z = 1$, has a Taylor series of the form $a_0 + a_1(z-1) + a_2(z-1)^2 + \cdots$ with radius of convergence $R = 1$ [the distance from $z = 1$ to the nearest singularity, namely the branch point $z = 0$].

5. **Singularities at infinity.** By letting $z = 1/w$ in $f(z)$, we obtain the function $f(1/w) = F(w)$. Then the nature of the singularity at $z = \infty$ [the point at infinity] is defined to be the same as that of $F(w)$ at $w = 0$.

   Example: $f(z) = z^3$ has a pole of order 3 at $z = \infty$, since $F(w) = f(1/w) = 1/w^3$ has a pole of order 3 at $w = 0$. Similarly $f(z) = e^z$ has an essential singularity at $z = \infty$, since $F(w) = f(1/w) = e^{1/w}$ has an essential singularity at $w = 0$.

ENTIRE FUNCTIONS

A function which is analytic everywhere in the finite plane [i.e. everywhere except at ∞] is called an *entire function* or *integral function*. The functions e^z, $\sin z$, $\cos z$ are entire functions.

An entire function can be represented by a Taylor series which has an infinite radius of convergence. Conversely if a power series has an infinite radius of convergence, it represents an entire function.

Note that by Liouville's theorem [Chapter 5, Page 119] a function which is analytic *everywhere including* ∞ must be a constant.

MEROMORPHIC FUNCTIONS

A function which is analytic everywhere in the finite plane except at a finite number of poles is called a *meromorphic function*.

Example: $\dfrac{z}{(z-1)(z+3)^2}$ which is analytic everywhere in the finite plane except at the poles $z = 1$ (simple pole) and $z = -3$ (pole of order two) is a meromorphic function.

LAGRANGE'S EXPANSION

Let z be that root of $z = a + \zeta\,\phi(z)$ which has the value $z = a$ when $\zeta = 0$. Then if $\phi(z)$ is analytic inside and on a circle C containing $z = a$, we have

$$z = a + \sum_{n=1}^{\infty} \frac{\zeta^n}{n!} \frac{d^{n-1}}{da^{n-1}} \{[\phi(a)]^n\} \tag{11}$$

More generally, if $F(z)$ is analytic inside and on C, then

$$F(z) = F(a) + \sum_{n=1}^{\infty} \frac{\zeta^n}{n!} \frac{d^{n-1}}{da^{n-1}} \{F'(a)\,[\phi(a)]^n\} \tag{12}$$

The expansion (*12*) and the special case (*11*) are often referred to as *Lagrange's expansions*.

ANALYTIC CONTINUATION

Suppose that we do not know the precise form of an analytic function $f(z)$ but only know that inside some circle of convergence C_1 with centre at a [Fig. 6-2] $f(z)$ is represented by a Taylor series

$$a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots \qquad (13)$$

Choosing a point b inside C_1, we can find the value of $f(z)$ and its derivatives at b from *(13)* and thus arrive at a new series

$$b_0 + b_1(z-b) + b_2(z-b)^2 + \cdots \qquad (14)$$

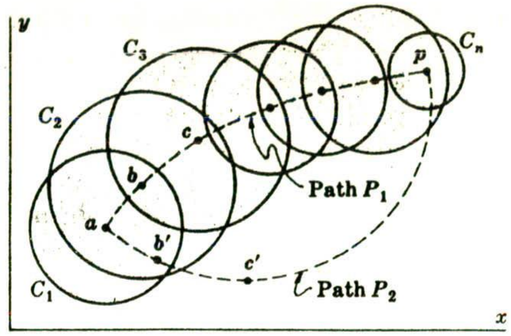

Fig. 6-2

having circle of convergence C_2. If C_2 extends beyond C_1, then the values of $f(z)$ and its derivatives can be obtained in this extended portion and so we have achieved ore information concerning $f(z)$.

We say in this case that $f(z)$ has been *extended analytically* beyond C_1 and call the process *analytic continuation* or *analytic extension.*

The process can of course be repeated indefinitely. Thus choosing point c inside C_2, we arrive at a new series having circle of convergence C_3 which may extend beyond C_1 and C_2, etc.

The collection of all such power series representations, i.e. all possible analytic continuations, is defined as the analytic function $f(z)$ and each power series is sometimes called an *element* of $f(z)$.

In performing analytic continuations we must avoid singularities. For example, there cannot be any singularity in Fig. 6-2 which is both inside C_2 and on the boundary of C_1, since otherwise *(14)* would diverge at this point. In some cases the singularities on a circle of convergence are so numerous that analytic continuation is impossible. In these cases the boundary of the circle is called a *natural boundary* or barrier [see Prob. 30]. The function represented by a series having a natural boundary is called a *lacunary* function.

In going from circle C_1 to circle C_n [Fig. 6-2], we have chosen the path of centres $a, b, c, \ldots, p$ which we represent briefly by *path* P_1. Many other paths are also possible, e.g. $a, b', c', \ldots, p$ represented briefly by *path* P_2. A question arises as to whether one obtains the same series representation valid inside C_n when one chooses different paths. The answer is *yes* so long as the region bounded by paths P_1 and P_2 has no singularity.

For further discussion of analytic continuation, see Chapter 10.

Solved Problems

SEQUENCES AND SERIES OF FUNCTIONS

1. Using the definition, prove that $\lim_{n\to\infty}\left(1+\frac{z}{n}\right) = 1$ for all z.

Given any number $\epsilon > 0$, we must find N such that $|1 + z/n - 1| < \epsilon$ for $n > N$. Then $|z/n| < \epsilon$, i.e. $|z|/n < \epsilon$ if $n > |z|/\epsilon = N$.

2. **(a) Prove that the series $z(1-z) + z^2(1-z) + z^3(1-z) + \cdots$ converges for $|z| < 1$, and (b) find its sum.**

The sum of the first n terms of the series is

$$\begin{aligned} S_n(z) &= z(1-z) + z^2(1-z) + \cdots + z^n(1-z) \\ &= z - z^2 + z^2 - z^3 + \cdots + z^n - z^{n+1} \\ &= z - z^{n+1} \end{aligned}$$

Now $|S_n(z) - z| = |-z^{n+1}| = |z|^{n+1} < \epsilon$ for $(n+1)\ln|z| < \ln \epsilon$, i.e. $n+1 > \dfrac{\ln\epsilon}{\ln|z|}$ or $n > \dfrac{\ln\epsilon}{\ln|z|} - 1$ if $z \neq 0$.

If $z = 0$, $S_n(0) = 0$ and $|S_n(0) - 0| < \epsilon$ for all n.

Hence $\lim\limits_{n\to\infty} S_n(z) = z$, the required sum for all z such that $|z| < 1$.

Another method.

Since $S_n(z) = z - z^{n+1}$, we have [by Problem 41, Chapter 2, in which we showed that $\lim\limits_{n\to\infty} z^n = 0$ if $|z| < 1$]

$$\text{Required sum} = S(z) = \lim_{n\to\infty} S_n(z) = \lim_{n\to\infty} (z - z^{n+1}) = z$$

ABSOLUTE AND UNIFORM CONVERGENCE

3. **(a) Prove that the series in Problem 2 converges uniformly to the sum z for $|z| \leqq \frac{1}{2}$.**

(b) Does the series converge uniformly for $|z| \leqq 1$? Explain.

(a) In Problem 2 we have shown that $|S_n(z) - z| < \epsilon$ for all $n > \dfrac{\ln\epsilon}{\ln|z|} - 1$, i.e. the series converges to the sum z for $|z| < 1$ and thus for $|z| \leqq \frac{1}{2}$.

Now if $|z| \leqq \frac{1}{2}$, the largest value of $\dfrac{\ln\epsilon}{\ln|z|} - 1$ occurs where $|z| = \frac{1}{2}$ and is given by $\dfrac{\ln\epsilon}{\ln(1/2)} - 1 = N$. It follows that $|S_n(z) - z| < \epsilon$ for all $n > N$ where N depends only on ϵ and not on the particular z in $|z| \leqq \frac{1}{2}$. Thus the series converges uniformly to z for $|z| \leqq \frac{1}{2}$.

(b) The same argument given in part (a) serves to show that the series converges uniformly to sum z for $|z| \leqq .9$ or $|z| \leqq .99$ by using $N = \dfrac{\ln\epsilon}{\ln(.9)} - 1$ and $N = \dfrac{\ln\epsilon}{\ln(.99)} - 1$ respectively.

However, it is clear that we cannot extend the argument to $|z| \leqq 1$ since this would require $N = \dfrac{\ln\epsilon}{\ln 1} - 1$ which is infinite, i.e. there is no finite value of N which can be used in this case. Thus the series does not converge uniformly for $|z| \leqq 1$.

4. **(a) Prove that the sequence $\left\{\dfrac{1}{1+nz}\right\}$ is uniformly convergent to zero for all z such that $|z| \geqq 2$. (b) Can the region of uniform convergence in (a) be extended? Explain.**

(a) We have $\left|\dfrac{1}{1+nz} - 0\right| < \epsilon$ when $\dfrac{1}{|1+nz|} < \epsilon$ or $|1+nz| > 1/\epsilon$. Now $|1+nz| \leqq |1| + |nz| = 1 + n|z|$ and $1 + n|z| \geqq |1+nz| > 1/\epsilon$ for $n > \dfrac{1/\epsilon - 1}{|z|}$. Thus the sequence converges to zero for $|z| > 2$.

To determine whether it converges uniformly to zero, note that the largest value of $\dfrac{1/\epsilon - 1}{|z|}$ in $|z| \geqq 2$ occurs for $|z| = 2$ and is given by $\frac{1}{2}\{(1/\epsilon) - 1\} = N$. It follows that $\left|\dfrac{1}{1+nz} - 0\right| < \epsilon$ for all $n > N$ where N depends only on ϵ and not on the particular z in $|z| \geqq 2$. Thus the sequence is uniformly convergent to zero in this region.

(b) If δ is any positive number, the largest value of $\frac{(1/\epsilon)-1}{|z|}$ in $|z| \geqq \delta$ occurs for $|z| = \delta$ and is given by $\frac{(1/\epsilon)-1}{\delta}$. As in part (a), it follows that the sequence converges uniformly to zero for all z such that $|z| \geqq \delta$, i.e. in any region which excludes all points in a neighbourhood of $z = 0$.

Since δ can be chosen arbitrarily close to zero, it follows that the region of (a) can be extended considerably.

5. Show that (a) the sum function in Problem 2 is discontinuous at $z=1$, (b) the limit in Problem 4 is discontinuous at $z=0$.

(a) From Problem 2, $S_n(z) = z - z^{n+1}$, $S(z) = \lim_{n\to\infty} S_n(z)$. If $|z|<1$, $S(z) = \lim_{n\to\infty} S_n(z) = z$. If $z=1$, $S_n(z) = S_n(1) = 0$ and $\lim_{n\to\infty} S_n(1) = 0$. Hence $S(z)$ is discontinuous at $z=1$.

(b) From Problem 4 if we write $u_n(z) = \frac{1}{1+nz}$ and $U(z) = \lim_{n\to\infty} u_n(z)$ we have $U(z) = 0$ if $z \neq 0$ and 1 if $z=0$. Thus $U(z)$ is discontinuous at $z=0$.

These are consequences of the fact [see Problem 16] that if a series of continuous functions is uniformly convergent in a region $\mathcal{R}$, then the sum function must be continuous in $\mathcal{R}$. Hence if the sum function is not continuous, the series cannot be uniformly convergent. A similar result holds for sequences.

6. Prove that the series of Problem 2 is absolutely convergent for $|z|<1$.

Let
$$\begin{aligned} T_n(z) &= |z(1-z)| + |z^2(1-z)| + \cdots + |z^n(1-z)| \\ &= |1-z|\{|z| + |z|^2 + |z|^3 + \cdots + |z|^n\} \\ &= |1-z||z|\left\{\frac{1-|z|^n}{1-|z|}\right\} \end{aligned}$$

If $|z|<1$, then $\lim_{n\to\infty} |z|^n = 0$ and $\lim_{n\to\infty} T_n(z)$ exists so that the series converges absolutely.

Note that the series of absolute values converges in this case to $\frac{|1-z|\,|z|}{1-|z|}$.

SPECIAL CONVERGENCE TESTS

7. If $\Sigma|v_n|$ converges and $|u_n| \leqq |v_n|$, $n = 1,2,3,\ldots,$ prove that $\Sigma|u_n|$ also converges (i.e. establish the comparison test for convergence).

Let $S_n = |u_1| + |u_2| + \cdots + |u_n|$, $T_n = |v_1| + |v_2| + \cdots + |v_n|$.

Since $\Sigma|v_n|$ converges, $\lim_{n\to\infty} T_n$ exists and equals T, say. Also since $|v_n| \geqq 0$, $T_n \leqq T$.

Then $S_n = |u_1| + |u_2| + \cdots + |u_n| \leqq |v_1| + |v_2| + \cdots + |v_n| \leqq T$ or $0 \leqq S_n \leqq T$.

Thus S_n is a bounded monotonic increasing sequence and must have a limit [Theorem 3, Page 141], i.e. $\Sigma|u_n|$ converges.

8. Prove that $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for any constant $p>1$.

We have
$$\frac{1}{1^p} = \frac{1}{1^{p-1}}$$
$$\frac{1}{2^p} + \frac{1}{3^p} \leqq \frac{1}{2^p} + \frac{1}{2^p} = \frac{1}{2^{p-1}}$$
$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} \leqq \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} = \frac{1}{4^{p-1}}$$

etc., where we consider $1,2,4,8,\ldots$ terms of the series. It follows that the sum of any finite number of terms of the given series is less than the geometric series

$$\frac{1}{1^{p-1}} + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \frac{1}{8^{p-1}} + \cdots \quad = \quad \frac{1}{1 - 1/2^{p-1}}$$

which converges for $p > 1$. Thus the given series, sometimes called the *p series*, converges.

By using a method analogous to that used here together with the comparison test for divergence [Theorem 10(*b*), Page 141], we can show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for $p \leqq 1$.

9. Prove that an absolutely convergent series is convergent.

Given that $\Sigma |u_n|$ converges, we must show that Σu_n converges.

Let $S_M = u_1 + u_2 + \cdots + u_M$ and $T_M = |u_1| + |u_2| + \cdots + |u_M|$. Then

$$\begin{aligned} S_M + T_M &= (u_1 + |u_1|) + (u_2 + |u_2|) + \cdots + (u_M + |u_M|) \\ &\leqq 2|u_1| + 2|u_2| + \cdots + 2|u_M| \end{aligned}$$

Since $\Sigma |u_n|$ converges and $u_n + |u_n| \geqq 0$ for $n = 1, 2, 3, \ldots$, it follows that $S_M + T_M$ is a bounded monotonic increasing sequence and so $\lim_{M\to\infty} (S_M + T_M)$ exists.

Also since $\lim_{M\to\infty} T_M$ exists [because by hypothesis the series is absolutely convergent],

$$\lim_{M\to\infty} S_M \quad = \quad \lim_{M\to\infty} (S_M + T_M - T_M) \quad = \quad \lim_{M\to\infty} (S_M + T_M) \; - \; \lim_{M\to\infty} T_M$$

must also exist and the result is proved.

10. Prove that $\sum_{n=1}^{\infty} \frac{z^n}{n(n+1)}$ converges (absolutely) for $|z| \leqq 1$.

If $|z| \leqq 1$, then $\left|\frac{z^n}{n(n+1)}\right| = \frac{|z|^n}{n(n+1)} \leqq \frac{1}{n(n+1)} \leqq \frac{1}{n^2}$.

Taking $u_n = \frac{z^n}{n(n+1)}$, $v_n = \frac{1}{n^2}$ in the comparison test and recognizing that $\Sigma \frac{1}{n^2}$ converges by Problem 8 with $p = 2$, we see that $\Sigma |u_n|$ converges, i.e. Σu_n converges absolutely.

11. Establish the ratio test for convergence.

We must show that if $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = L < 1$, then $\Sigma |u_n|$ converges or, by Problem 9, Σu_n is (absolutely) convergent.

By hypothesis, we can choose an integer N so large that for all $n \geqq N$, $\left|\frac{u_{n+1}}{u_n}\right| \leqq r$ where r is some constant such that $L < r < 1$. Then

$$\begin{aligned} |u_{N+1}| &\leqq r|u_N| \\ |u_{N+2}| &\leqq r|u_{N+1}| < r^2|u_N| \\ |u_{N+3}| &\leqq r|u_{N+2}| < r^3|u_N| \end{aligned}$$

etc. By addition,

$$|u_{N+1}| + |u_{N+2}| + \cdots \leqq |u_N|(r + r^2 + r^3 + \cdots)$$

and so $\Sigma |u_n|$ converges by the comparison test since $0 < r < 1$.

12. Find the region of convergence of the series $\sum_{n=1}^{\infty} \frac{(z+2)^{n-1}}{(n+1)^3 4^n}$.

If $u_n = \frac{(z+2)^{n-1}}{(n+1)^3 4^n}$, then $u_{n+1} = \frac{(z+2)^n}{(n+2)^3 4^{n+1}}$. Hence, excluding $z = -2$ for which the given series converges, we have

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| \quad = \quad \lim_{n\to\infty} \left|\frac{(z+2)}{4}\frac{(n+1)^3}{(n+2)^3}\right| \quad = \quad \frac{|z+2|}{4}$$

Then the series converges (absolutely) for $\dfrac{|z+2|}{4} < 1$, i.e. $|z+2| < 4$. The point $z = -2$ is included in $|z+2| < 4$.

If $\dfrac{|z+2|}{4} = 1$, i.e. $|z+2| = 4$, the ratio test fails. However it is seen that in this case

$$\left|\frac{(z+2)^{n-1}}{(n+1)^3\, 4^n}\right| = \frac{1}{4(n+1)^3} \leq \frac{1}{n^3}$$

and since $\sum \dfrac{1}{n^3}$ converges [p series with $p = 3$], the given series converges (absolutely).

It follows that the given series converges (absolutely) for $|z+2| \leq 4$. Geometrically this is the set of all points inside and on the circle of radius 4 with centre at $z = -2$, called the *circle of convergence* [shown shaded in Fig. 6-3]. The *radius of convergence* is equal to 4.

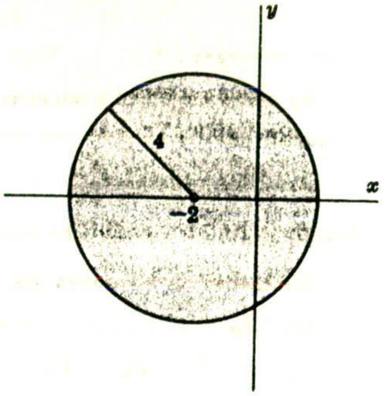

Fig. 6-3

13. Find the region of convergence of the series (*a*) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^{2n-1}}{(2n-1)!}$, (*b*) $\displaystyle\sum_{n=1}^{\infty} n!\, z^n$.

(*a*) If $u_n = \dfrac{(-1)^{n-1} z^{2n-1}}{(2n-1)!}$, then $u_{n+1} = \dfrac{(-1)^n z^{2n+1}}{(2n+1)!}$. Hence, excluding $z = 0$ for which the given series converges, we have

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|-\frac{z^2(2n-1)!}{(2n+1)!}\right| = \lim_{n\to\infty}\frac{(2n-1)!\,|z|^2}{(2n+1)(2n)(2n-1)!}$$

$$= \lim_{n\to\infty}\frac{|z|^2}{(2n+1)(2n)} = 0$$

for all finite z. Thus the series converges (absolutely) for all z, and we say that the series converges for $|z| < \infty$. We can equivalently say that the circle of convergence is infinite or that the radius of convergence is infinite.

(*b*) If $u_n = n!\, z^n$, $u_{n+1} = (n+1)!\, z^{n+1}$. Then excluding $z = 0$ for which the given series converges, we have

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\, z^{n+1}}{n!\, z^n}\right| = \lim_{n\to\infty}(n+1)|z| = \infty$$

Thus the series converges only for $z = 0$.

THEOREMS ON UNIFORM CONVERGENCE

14. Prove the Weierstrass M test, i.e. if in a region $\mathcal{R}$, $|u_n(z)| \leqq M_n$, $n = 1, 2, 3, \ldots$, where M_n are positive constants such that ΣM_n converges, then $\Sigma u_n(z)$ is uniformly (and absolutely) convergent in $\mathcal{R}$.

The remainder of the series $\Sigma u_n(z)$ after n terms is $R_n(z) = u_{n+1}(z) + u_{n+2}(z) + \cdots$. Now

$$|R_n(z)| = |u_{n+1}(z) + u_{n+2}(z) + \cdots| \leqq |u_{n+1}(z)| + |u_{n+2}(z)| + \cdots$$
$$\leqq M_{n+1} + M_{n+2} + \cdots$$

But $M_{n+1} + M_{n+2} + \cdots$ can be made less than ϵ by choosing $n > N$, since ΣM_n converges. Since N is clearly independent of z, we have $|R_n(z)| < \epsilon$ for $n > N$, and the series is uniformly convergent. The absolute convergence follows at once from the comparison test.

15. Test for uniform convergence in the indicated region:

$(a)\ \sum_{n=1}^{\infty} \frac{z^n}{n\sqrt{n+1}},\ |z| \leqq 1;\quad (b)\ \sum_{n=1}^{\infty} \frac{1}{n^2+z^2},\ 1 < |z| < 2;\quad (c)\ \sum_{n=1}^{\infty} \frac{\cos nz}{n^3},\ |z| \leqq 1.$

(a) If $u_n(z) = \frac{z^n}{n\sqrt{n+1}}$, then $|u_n(z)| = \frac{|z|^n}{n\sqrt{n+1}} \leqq \frac{1}{n^{3/2}}$ if $|z| \leqq 1$. Calling $M_n = \frac{1}{n^{3/2}}$, we see that ΣM_n converges (p series with $p = 3/2$). Hence by the Weierstrass M test the given series converges uniformly (and absolutely) for $|z| \leqq 1$.

(b) The given series is $\frac{1}{1^2+z^2} + \frac{1}{2^2+z^2} + \frac{1}{3^2+z^2} + \cdots$. The first two terms can be omitted without affecting the uniform convergence of the series. For $n \geqq 3$ and $1 < |z| < 2$, we have

$$|n^2+z^2| \geqq |n^2| - |z^2| \geqq n^2 - 4 \geqq \tfrac{1}{2}n^2 \quad \text{or} \quad \left|\frac{1}{n^2+z^2}\right| \leqq \frac{2}{n^2}$$

Since $\sum_{n=3}^{\infty} \frac{2}{n^2}$ converges, it follows from the Weierstrass M test (with $M_n = 2/n^2$) that the given series converges uniformly (and absolutely) for $1 < |z| < 2$.

Note that the convergence, and thus uniform convergence, breaks down if $|z| = 1$ or $|z| = 2$ [namely at $z = \pm i$ and $z = \pm 2i$]. Hence the series cannot converge uniformly for $1 \leqq |z| \leqq 2$.

(c) If $z = x + iy$, we have

$$\frac{\cos nz}{n^3} = \frac{e^{inz} + e^{-inz}}{2n^3} = \frac{e^{inx-ny} + e^{-inx+ny}}{2n^3}$$

$$= \frac{e^{-ny}(\cos nx + i\sin nx)}{2n^3} + \frac{e^{ny}(\cos nx - i\sin nx)}{2n^3}$$

The series $\sum_{n=1}^{\infty} \frac{e^{ny}(\cos nx - i\sin nx)}{2n^3}$ and $\sum_{n=1}^{\infty} \frac{e^{-ny}(\cos nx + i\sin nx)}{2n^3}$ cannot converge for $y > 0$ and $y < 0$ respectively [since in these cases the nth term does not approach zero]. Hence the series does not converge for all z such that $|z| \leqq 1$, and so cannot possibly be uniformly convergent in this region.

The series does converge for $y = 0$, i.e. if z is real. In this case $z = x$ and the series becomes $\sum_{n=1}^{\infty} \frac{\cos nx}{n^3}$. Then since $\left|\frac{\cos nx}{n^3}\right| \leqq \frac{1}{n^3}$ and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, it follows from the Weierstrass M test (with $M_n = 1/n^3$) that the given series converges uniformly in any interval on the real axis.

16. Prove Theorem 18, Page 142, i.e. if $u_n(z)$, $n = 1, 2, 3, \ldots$, are continuous in $\mathcal{R}$ and $\sum_{n=1}^{\infty} u_n(z)$ is uniformly convergent to $S(z)$ in $\mathcal{R}$, then $S(z)$ is continuous in $\mathcal{R}$.

If $S_n(z) = u_1(z) + u_2(z) + \cdots + u_n(z)$, and $R_n(z) = u_{n+1}(z) + u_{n+2}(z) + \cdots$ is the remainder after n terms, it is clear that

$$S(z) = S_n(z) + R_n(z) \quad \text{and} \quad S(z+h) = S_n(z+h) + R_n(z+h)$$

and so

$$S(z+h) - S(z) = S_n(z+h) - S_n(z) + R_n(z+h) - R_n(z) \tag{1}$$

where z and $z+h$ are in $\mathcal{R}$.

Since $S_n(z)$ is the sum of a finite number of continuous functions, it must also be continuous. Then given $\epsilon > 0$, we can find δ so that

$$|S_n(z+h) - S_n(z)| < \epsilon/3 \quad \text{whenever} \quad |h| < \delta \tag{2}$$

Since the series, by hypothesis, is uniformly convergent, we can choose N so that for all z in $\mathcal{R}$,

$$|R_n(z)| < \epsilon/3 \quad \text{and} \quad |R_n(z+h)| < \epsilon/3 \quad \text{for } n > N \tag{3}$$

Then from (1), (2) and (3),

$$|S(z+h) - S(z)| \leqq |S_n(z+h) - S_n(z)| + |R_n(z+h)| + |R_n(z)| < \epsilon$$

for $|h| < \delta$ and all z in $\mathcal{R}$, and so the continuity is established.

17. Prove Theorem 19, Page 142, i.e. if $\{u_n(z)\}$, $n = 1, 2, 3, \ldots$, are continuous in $\mathcal{R}$, $S(z) = \sum_{n=1}^{\infty} u_n(z)$ is uniformly convergent in $\mathcal{R}$ and C is a curve in $\mathcal{R}$, then

$$\int_C S(z)\,dz = \int_C \left(\sum_{n=1}^{\infty} u_n(z)\right) dz = \sum_{n=1}^{\infty} \int_C u_n(z)\,dz$$

As in Problem 16, we have $S(z) = S_n(z) + R_n(z)$ and so since these are continuous in $\mathcal{R}$ [by Problem 16] their integrals exist, i.e.,

$$\int_C S(z)\,dz = \int_C S_n(z)\,dz + \int_C R_n(z)\,dz$$
$$= \int_C u_1(z)\,dz + \int_C u_2(z)\,dz + \cdots + \int_C u_n(z)\,dz + \int_C R_n(z)\,dz$$

By hypothesis the series is uniformly convergent, so that given any $\epsilon > 0$ we can find a number N independent of z in $\mathcal{R}$ such that $|R_n(z)| < \epsilon$ when $n > N$. Denoting by L the length of C, we have [using Property 5, Page 93]

$$\left|\int_C R_n(z)\,dz\right| < \epsilon L$$

Then $\left|\int_C S(z)\,dz - \int_C S_n(z)\,dz\right|$ can be made as small as we like by choosing n large enough, and the result is proved.

THEOREMS ON POWER SERIES

18. If a power series $\Sigma a_n z^n$ converges for $z = z_0 \neq 0$, prove that it converges (*a*) absolutely for $|z| < |z_0|$, (*b*) uniformly for $|z| \leqq |z_1|$ where $|z_1| < |z_0|$.

(*a*) Since $\Sigma a_n z_0^n$ converges, $\lim_{n\to\infty} a_n z_0^n = 0$ and so we can make $|a_n z_0^n| < 1$ by choosing n large enough, i.e. $|a_n| < \dfrac{1}{|z_0|^n}$ for $n > N$. Then

$$\sum_{N+1}^{\infty} |a_n z^n| = \sum_{N+1}^{\infty} |a_n|\,|z|^n \leqq \sum_{N+1}^{\infty} \frac{|z|^n}{|z_0|^n} \tag{1}$$

But the last series in (*1*) converges for $|z| < |z_0|$ and so by the comparison test the first series converges, i.e. the given series is absolutely convergent.

(*b*) Let $M_n = \dfrac{|z_1|^n}{|z_0|^n}$. Then ΣM_n converges, since $|z_1| < |z_0|$. As in part (*a*), $|a_n z^n| < M_n$ for $|z| \leqq |z_1|$ so that, by the Weierstrass M test, $\Sigma a_n z^n$ is uniformly convergent.

It follows that a power series is uniformly convergent in any region which lies entirely inside its circle of convergence.

19. Prove that both the power series $\sum_{n=0}^{\infty} a_n z^n$ and the corresponding series of derivatives $\sum_{n=0}^{\infty} n a_n z^{n-1}$ have the same radius of convergence.

Let $R > 0$ be the radius of convergence of $\Sigma a_n z^n$. Let $0 < |z_0| < R$. Then as in Problem 18 we can choose N so that $|a_n| < \dfrac{1}{|z_0|^n}$ for $n > N$.

Thus the terms of the series $\Sigma\,|n a_n z^{n-1}| = \Sigma n\,|a_n|\,|z|^{n-1}$ can for $n > N$ be made less than corresponding terms of the series $\Sigma n \dfrac{|z|^{n-1}}{|z_0|^n}$ which converges, by the ratio test, for $|z| < |z_0| < R$.

Hence $\Sigma n a_n z^{n-1}$ converges absolutely for all points such that $|z| < |z_0|$ (no matter how close $|z_0|$ is to R), i.e. for $|z| < R$.

If however $|z| > R$, $\lim_{n\to\infty} a_n z^n \neq 0$ and thus $\lim_{n\to\infty} n a_n z^{n-1} \neq 0$, so that $\Sigma n a_n z^{n-1}$ does not converge.

Thus R is the radius of convergence of $\Sigma n a_n z^{n-1}$. This is also true if $R = 0$.

Note that the series of derivatives may or may not converge for values of z such that $|z| = R$.

20. Prove that in any region which lies entirely within its circle of convergence, a power series (a) represents a continuous function, say $f(z)$, (b) can be integrated term by term to yield the integral of $f(z)$, (c) can be differentiated term by term to yield the derivative of $f(z)$.

We consider the power series $\Sigma a_n z^n$, although analogous results hold for $\Sigma a_n(z-a)^n$.

(a) This follows from Problem 16 and the fact that each term $a_n z^n$ of the series is continuous.

(b) This follows from Problem 17 and the fact that each term $a_n z^n$ of the series is continuous and thus integrable.

(c) From Problem 19 the derivative of a power series converges within the circle of convergence of the original power series and therefore is uniformly convergent in any region entirely within the circle of convergence. Thus the required result follows from Theorem 20, Page 142.

21. Prove that the series $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ has a finite value at all points inside and on its circle of convergence but that this is not true for the series of derivatives.

By the ratio test the series converges for $|z| < 1$ and diverges for $|z| > 1$. If $|z| = 1$, then $|z^n/n^2| = 1/n^2$ and the series is convergent (absolutely). Thus the series converges for $|z| \leqq 1$ and so has a finite value inside and on its circle of convergence.

The series of derivatives is $\sum_{n=1}^{\infty} \frac{z^{n-1}}{n}$. By the ratio test the series converges for $|z| < 1$. However, the series does not converge for all z such that $|z| = 1$, for example if $z = 1$ the series diverges.

TAYLOR'S THEOREM

22. Prove Taylor's theorem: If $f(z)$ is analytic inside a circle C with centre at a, then for all z inside C,

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \cdots$$

Let z be any point inside C. Construct a circle C_1 with centre at a and enclosing z (see Fig. 6-4). Then by Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{w-z}\,dw \qquad (1)$$

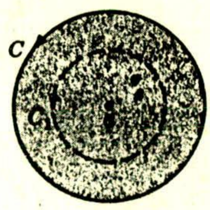

Fig. 6-4

We have

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{(w-a)}\left\{\frac{1}{1-(z-a)/(w-a)}\right\}$$

$$= \frac{1}{(w-a)}\left\{1 + \left(\frac{z-a}{w-a}\right) + \left(\frac{z-a}{w-a}\right)^2 + \cdots + \left(\frac{z-a}{w-a}\right)^{n-1} + \left(\frac{z-a}{w-a}\right)^n \frac{1}{1-(z-a)/(w-a)}\right\}$$

or

$$\frac{1}{w-z} = \frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + \cdots + \frac{(z-a)^{n-1}}{(w-a)^n} + \left(\frac{z-a}{w-a}\right)^n \frac{1}{w-z} \qquad (2)$$

Multiplying both sides of (2) by $f(w)$ and using (1), we have

$$f(z) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-a}\,dw + \frac{z-a}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^2}\,dw + \cdots + \frac{(z-a)^{n-1}}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^n}\,dw + U_n \qquad (3)$$

where

$$U_n = \frac{1}{2\pi i}\oint_{C_1}\left(\frac{z-a}{w-a}\right)^n \frac{f(w)}{w-z}\,dw$$

Using Cauchy's integral formulae

$$f^{(n)}(a) \quad = \quad \frac{n!}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^{n+1}}\,dw \qquad n = 0, 1, 2, 3, \ldots$$

(3) becomes

$$f(z) \quad = \quad f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1} + U_n$$

If we can now show that $\lim_{n\to\infty} U_n = 0$, we will have proved the required result. To do this we note that since w is on C_1,

$$\left|\frac{z-a}{w-a}\right| \quad = \quad \gamma \quad < \quad 1$$

where γ is a constant. Also we have $|f(w)| < M$ where M is a constant, and

$$|w-z| \quad = \quad |(w-a)-(z-a)| \quad \geqq \quad r_1 - |z-a|$$

where r_1 is the radius of C_1. Hence from Property 5, Page 93, we have

$$\begin{aligned} |U_n| \quad &= \quad \frac{1}{2\pi}\left|\oint_{C_1}\left(\frac{z-a}{w-a}\right)^n \frac{f(w)}{w-z}\,dw\right| \\ &\leqq \quad \frac{1}{2\pi}\,\frac{\gamma^n M}{r_1 - |z-a|}\cdot 2\pi r_1 \quad = \quad \frac{\gamma^n M r_1}{r_1 - |z-a|} \end{aligned}$$

and we see that $\lim_{n\to\infty} U_n = 0$, completing the proof.

23. Let $f(z) = \ln(1+z)$, where we consider that branch which has the value zero when $z = 0$. (*a*) Expand $f(z)$ in a Taylor series about $z = 0$. (*b*) Determine the region of convergence for the series in (*a*). (*c*) Expand $\ln\left(\frac{1+z}{1-z}\right)$ in a Taylor series about $z = 0$.

(*a*)

$$\begin{aligned} f(z) \quad &= \quad \ln(1+z) & f(0) \quad &= \quad 0 \\ f'(z) \quad &= \quad \frac{1}{1+z} \quad = \quad (1+z)^{-1} & f'(0) \quad &= \quad 1 \\ f''(z) \quad &= \quad -(1+z)^{-2} & f''(0) \quad &= \quad -1 \\ f'''(z) \quad &= \quad (-1)(-2)(1+z)^{-3} & f'''(0) \quad &= \quad 2! \\ &\ \ \vdots & &\ \ \vdots \\ f^{(n+1)}(z) \quad &= \quad (-1)^n n!\,(1+z)^{-(n+1)} & f^{(n+1)}(0) \quad &= \quad (-1)^n n! \end{aligned}$$

Then

$$\begin{aligned} f(z) \quad = \quad \ln(1+z) \quad &= \quad f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \frac{f'''(0)}{3!}z^3 + \cdots \\ &= \quad z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots \end{aligned}$$

Another method. If $|z| < 1$,

$$\frac{1}{1+z} \quad = \quad 1 - z + z^2 - z^3 + \cdots$$

Then integrating from 0 to z yields

$$\ln(1+z) \quad = \quad z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

(*b*) The nth term is $u_n = \frac{(-1)^{n-1}z^n}{n}$. Using the ratio test,

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| \quad = \quad \lim_{n\to\infty}\left|\frac{nz}{n+1}\right| \quad = \quad |z|$$

and the series converges for $|z| < 1$. The series can be shown to converge for $|z| = 1$ except for $z = -1$.

This result also follows from the fact that the series converges in a circle which extends to the nearest singularity (i.e. $z = -1$) of $f(z)$.

(c) From the result in (a) we have, on replacing z by $-z$,

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

$$\ln(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \cdots$$

both series convergent for $|z| < 1$. By subtraction, we have

$$\ln\left(\frac{1+z}{1-z}\right) = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots\right) = \sum_{n=0}^{\infty} \frac{2z^{2n+1}}{2n+1}$$

which converges for $|z| < 1$. We can also show that this series converges for $|z| = 1$ except for $z = \pm 1$.

24. (a) Expand $f(z) = \sin z$ in a Taylor series about $z = \pi/4$ and (b) determine the region of convergence of this series.

(a) $f(z) = \sin z,\ f'(z) = \cos z,\ f''(z) = -\sin z,\ f'''(z) = -\cos z,\ f^{IV}(z) = \sin z,\ \ldots$

$f(\pi/4) = \sqrt{2}/2,\ f'(\pi/4) = \sqrt{2}/2,\ f''(\pi/4) = -\sqrt{2}/2,\ f'''(\pi/4) = -\sqrt{2}/2,\ f^{IV}(\pi/4) = \sqrt{2}/2,\ \ldots$

Then, since $a = \pi/4$,

$$\begin{aligned} f(z) &= f(a) + f'(a)(z-a) + \frac{f''(a)(z-a)^2}{2!} + \frac{f'''(a)(z-a)^3}{3!} + \cdots \\ &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(z-\pi/4) - \frac{\sqrt{2}}{2\cdot 2!}(z-\pi/4)^2 - \frac{\sqrt{2}}{2\cdot 3!}(z-\pi/4)^3 + \cdots \\ &= \frac{\sqrt{2}}{2}\left\{1 + (z-\pi/4) - \frac{(z-\pi/4)^2}{2!} - \frac{(z-\pi/4)^3}{3!} + \cdots\right\} \end{aligned}$$

Another method.

Let $u = z - \pi/4$ or $z = u + \pi/4$. Then we have,

$$\begin{aligned} \sin z &= \sin(u + \pi/4) = \sin u \cos(\pi/4) + \cos u \sin(\pi/4) \\ &= \frac{\sqrt{2}}{2}(\sin u + \cos u) \\ &= \frac{\sqrt{2}}{2}\left\{\left(u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots\right) + \left(1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \cdots\right)\right\} \\ &= \frac{\sqrt{2}}{2}\left\{1 + u - \frac{u^2}{2!} - \frac{u^3}{3!} + \frac{u^4}{4!} + \cdots\right\} \\ &= \frac{\sqrt{2}}{2}\left\{1 + (z-\pi/4) - \frac{(z-\pi/4)^2}{2!} - \frac{(z-\pi/4)^3}{3!} + \cdots\right\} \end{aligned}$$

(b) Since the singularity of $\sin z$ nearest to $\pi/4$ is at infinity, the series converges for all finite values of z, i.e. $|z| < \infty$. This can also be established by the ratio test.

LAURENT'S THEOREM

25. Prove *Laurent's theorem*: If $f(z)$ is analytic inside and on the boundary of the ring-shaped region $\mathcal{R}$ bounded by two concentric circles C_1 and C_2 with centre at a and respective radii r_1 and $r_2 (r_1 > r_2)$, then for all z in $\mathcal{R}$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-a)^n}$$

where

$$a_n = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^{n+1}}\,dw \qquad n = 0, 1, 2, \ldots$$

$$a_{-n} = \frac{1}{2\pi i}\oint_{C_2} \frac{f(w)}{(w-a)^{-n+1}}\,dw \qquad n = 1, 2, 3, \ldots$$

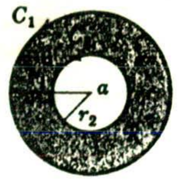

Fig. 6-5

By Cauchy's integral formula [see Problem 23, Page 131] we have

$$f(z) \quad = \quad \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-z}\,dw \;-\; \frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{w-z}\,dw \qquad (1)$$

Consider the first integral in (1). As in Problem 22, equation (2), we have

$$\begin{aligned}\frac{1}{w-z} &= \frac{1}{(w-a)\{1-(z-a)/(w-a)\}}\\ &= \frac{1}{w-a}+\frac{z-a}{(w-a)^2}+\cdots+\frac{(z-a)^{n-1}}{(w-a)^n}+\left(\frac{z-a}{w-a}\right)^n\frac{1}{w-z} \qquad (2)\end{aligned}$$

so that

$$\begin{aligned}\frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-z}\,dw &= \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-a}\,dw+\frac{z-a}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^2}\,dw\\ &\qquad +\cdots+\frac{(z-a)^{n-1}}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^n}\,dw+U_n\\ &= a_0+a_1(z-a)+\cdots+a_{n-1}(z-a)^{n-1}+U_n \qquad (3)\end{aligned}$$

where

$$a_0 = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-a}\,dw,\quad a_1=\frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^2}\,dw,\quad \ldots,\quad a_{n-1}=\frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^n}\,dw$$

and

$$U_n = \frac{1}{2\pi i}\oint_{C_1}\left(\frac{z-a}{w-a}\right)^n\frac{f(w)}{w-z}\,dw$$

Let us now consider the second integral in (1). We have on interchanging w and z in (2),

$$\begin{aligned}-\frac{1}{w-z} &= \frac{1}{(z-a)\{1-(w-a)/(z-a)\}}\\ &= \frac{1}{z-a}+\frac{w-a}{(z-a)^2}+\cdots+\frac{(w-a)^{n-1}}{(z-a)^n}+\left(\frac{w-a}{z-a}\right)^n\frac{1}{z-w}\end{aligned}$$

so that

$$\begin{aligned}-\frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{w-z}\,dw &= \frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{z-a}\,dw+\frac{1}{2\pi i}\oint_{C_2}\frac{w-a}{(z-a)^2}f(w)\,dw\\ &\qquad +\cdots+\frac{1}{2\pi i}\oint_{C_2}\frac{(w-a)^{n-1}}{(z-a)^n}f(w)\,dw+V_n\\ &= \frac{a_{-1}}{z-a}+\frac{a_{-2}}{(z-a)^2}+\cdots+\frac{a_{-n}}{(z-a)^n}+V_n \qquad (4)\end{aligned}$$

where

$$a_{-1}=\frac{1}{2\pi i}\oint_{C_2}f(w)\,dw,\quad a_{-2}=\frac{1}{2\pi i}\oint_{C_2}(w-a)f(w)\,dw,\quad\ldots,\quad a_{-n}=\frac{1}{2\pi i}\oint_{C_2}(w-a)^{n-1}f(w)\,dw$$

and

$$V_n = \frac{1}{2\pi i}\oint_{C_2}\left(\frac{w-a}{z-a}\right)^n\frac{f(w)}{z-w}\,dw$$

From (1), (3) and (4) we have

$$\begin{aligned}f(z) &= \{a_0+a_1(z-a)+\cdots+a_{n-1}(z-a)^{n-1}\}\\ &\qquad+\left\{\frac{a_{-1}}{z-a}+\frac{a_{-2}}{(z-a)^2}+\cdots+\frac{a_{-n}}{(z-a)^n}\right\}+U_n+V_n \qquad (5)\end{aligned}$$

The required result follows if we can show that (a) $\lim_{n\to\infty}U_n = 0$ and (b) $\lim_{n\to\infty}V_n=0$. The proof of (a) follows from Problem 22. To prove (b) we first note that since w is on C_2,

$$\left|\frac{w-a}{z-a}\right| = \kappa < 1$$

where κ is a constant. Also we have $|f(w)| < M$ where M is a constant and

$$|z-w| = |(z-a)-(w-a)| \geqq |z-a|-r_2$$

Hence from Property 5, Page 93, we have

$$|V_n| = \frac{1}{2\pi}\left|\oint_{C_2}\left(\frac{w-a}{z-a}\right)^n \frac{f(w)}{z-w}\,dw\right|$$
$$\leq \frac{1}{2\pi}\frac{\kappa^n M}{|z-a|-r_2}2\pi r_2 = \frac{\kappa^n M r_2}{|z-a|-r_2}$$

Then $\lim_{n\to\infty} V_n = 0$ and the proof is complete.

26. Find Laurent series about the indicated singularity for each of the following functions. Name the singularity in each case and give the region of convergence of each series.

(*a*) $\dfrac{e^{2z}}{(z-1)^3}$; $z=1$. Let $z-1=u$. Then $z=1+u$ and

$$\frac{e^{2z}}{(z-1)^3} = \frac{e^{2+2u}}{u^3} = \frac{e^2}{u^3}\cdot e^{2u} = \frac{e^2}{u^3}\left\{1+2u+\frac{(2u)^2}{2!}+\frac{(2u)^3}{3!}+\frac{(2u)^4}{4!}+\cdots\right\}$$
$$= \frac{e^2}{(z-1)^3}+\frac{2e^2}{(z-1)^2}+\frac{2e^2}{z-1}+\frac{4e^2}{3}+\frac{2e^2}{3}(z-1)+\cdots$$

$z=1$ is a *pole of order 3*, or *triple pole*.

The series converges for all values of $z \neq 1$.

(*b*) $(z-3)\sin\dfrac{1}{z+2}$; $z=-2$. Let $z+2=u$ or $z=u-2$. Then

$$(z-3)\sin\frac{1}{z+2} = (u-5)\sin\frac{1}{u} = (u-5)\left\{\frac{1}{u}-\frac{1}{3!\,u^3}+\frac{1}{5!\,u^5}-\cdots\right\}$$
$$= 1-\frac{5}{u}-\frac{1}{3!\,u^2}+\frac{5}{3!\,u^3}+\frac{1}{5!\,u^4}-\cdots$$
$$= 1-\frac{5}{z+2}-\frac{1}{6(z+2)^2}+\frac{5}{6(z+2)^3}+\frac{1}{120(z+2)^4}-\cdots$$

$z=-2$ is an *essential singularity*.

The series converges for all values of $z \neq -2$.

(*c*) $\dfrac{z-\sin z}{z^3}$; $z=0$.

$$\frac{z-\sin z}{z^3} = \frac{1}{z^3}\left\{z-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\cdots\right)\right\}$$
$$= \frac{1}{z^3}\left\{\frac{z^3}{3!}-\frac{z^5}{5!}+\frac{z^7}{7!}-\cdots\right\} = \frac{1}{3!}-\frac{z^2}{5!}+\frac{z^4}{7!}-\cdots$$

$z=0$ is a *removable singularity*.

The series converges for all values of z.

(*d*) $\dfrac{z}{(z+1)(z+2)}$; $z=-2$. Let $z+2=u$. Then

$$\frac{z}{(z+1)(z+2)} = \frac{u-2}{(u-1)u} = \frac{2-u}{u}\cdot\frac{1}{1-u} = \frac{2-u}{u}(1+u+u^2+u^3+\cdots)$$
$$= \frac{2}{u}+1+u+u^2+\cdots = \frac{2}{z+2}+1+(z+2)+(z+2)^2+\cdots$$

$z=-2$ is a pole of order 1, or simple pole.

The series converges for all values of z such that $0<|z+2|<1$.

(*e*) $\dfrac{1}{z^2(z-3)^2}$; $z=3$. Let $z-3=u$. Then by the binomial theorem,

$$\begin{aligned}
\frac{1}{z^2(z-3)^2} &= \frac{1}{u^2(3+u)^2} = \frac{1}{9u^2(1+u/3)^2} \\
&= \frac{1}{9u^2}\left\{1 + (-2)\left(\frac{u}{3}\right) + \frac{(-2)(-3)}{2!}\left(\frac{u}{3}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(\frac{u}{3}\right)^3 + \cdots\right\} \\
&= \frac{1}{9u^2} - \frac{2}{27u} + \frac{1}{27} - \frac{4}{243}u + \cdots \\
&= \frac{1}{9(z-3)^2} - \frac{2}{27(z-3)} + \frac{1}{27} - \frac{4(z-3)}{243} + \cdots
\end{aligned}$$

$z = 3$ is a *pole of order 2* or *double pole*.

The series converges for all values of z such that $0 < |z-3| < 3$.

27. Expand $f(z) = \dfrac{1}{(z+1)(z+3)}$ in a Laurent series valid for (*a*) $1 < |z| < 3$, (*b*) $|z| > 3$, (*c*) $0 < |z+1| < 2$, (*d*) $|z| < 1$.

(*a*) Resolving into partial fractions, $\dfrac{1}{(z+1)(z+3)} = \dfrac{1}{2}\left(\dfrac{1}{z+1}\right) - \dfrac{1}{2}\left(\dfrac{1}{z+3}\right)$.

If $|z| > 1$,

$$\frac{1}{2(z+1)} = \frac{1}{2z(1+1/z)} = \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) = \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \cdots$$

If $|z| < 3$,

$$\frac{1}{2(z+3)} = \frac{1}{6(1+z/3)} = \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right) = \frac{1}{6} - \frac{z}{18} + \frac{z^2}{54} - \frac{z^3}{162} + \cdots$$

Then the required Laurent expansion valid for both $|z| > 1$ and $|z| < 3$, i.e. $1 < |z| < 3$, is

$$\cdots - \frac{1}{2z^4} + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{6} + \frac{z}{18} - \frac{z^2}{54} + \frac{z^3}{162} - \cdots$$

(*b*) If $|z| > 1$, we have as in part (*a*),

$$\frac{1}{2(z+1)} = \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \cdots$$

If $|z| > 3$,

$$\frac{1}{2(z+3)} = \frac{1}{2z(1+3/z)} = \frac{1}{2z}\left(1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \cdots\right) = \frac{1}{2z} - \frac{3}{2z^2} + \frac{9}{2z^3} - \frac{27}{2z^4} + \cdots$$

Then the required Laurent expansion valid for both $|z| > 1$ and $|z| > 3$, i.e. $|z| > 3$, is by subtraction

$$\frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \frac{40}{z^5} + \cdots$$

(*c*) Let $z + 1 = u$. Then

$$\begin{aligned}
\frac{1}{(z+1)(z+3)} &= \frac{1}{u(u+2)} = \frac{1}{2u(1+u/2)} = \frac{1}{2u}\left(1 - \frac{u}{2} + \frac{u^2}{4} - \frac{u^3}{8} + \cdots\right) \\
&= \frac{1}{2(z+1)} - \frac{1}{4} + \frac{1}{8}(z+1) - \frac{1}{16}(z+1)^2 + \cdots
\end{aligned}$$

valid for $|u| < 2$, $u \neq 0$ or $0 < |z+1| < 2$.

(*d*) If $|z| < 1$,

$$\frac{1}{2(z+1)} = \frac{1}{2(1+z)} = \tfrac{1}{2}(1 - z + z^2 - z^3 + \cdots) = \tfrac{1}{2} - \tfrac{1}{2}z + \tfrac{1}{2}z^2 - \tfrac{1}{2}z^3 + \cdots$$

If $|z| < 3$, we have by part (*a*),

$$\frac{1}{2(z+3)} = \frac{1}{6} - \frac{z}{18} + \frac{z^2}{54} - \frac{z^3}{162} + \cdots$$

Then the required Laurent expansion, valid for both $|z| < 1$ and $|z| < 3$, i.e. $|z| < 1$, is by subtraction

$$\frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 - \frac{40}{81}z^3 + \cdots$$

This is a *Taylor series.*

LAGRANGE'S EXPANSION

28. Prove Lagrange's expansion (*11*) on Page 145.

Let us assume that C is taken so that there is only one simple zero of $z = a + \zeta\phi(z)$ inside C. Then from Problem 90, Page 137, with $g(z) = z$ and $f(z) = z - a - \zeta\phi(z)$, we have

$$\begin{aligned} z &= \frac{1}{2\pi i}\oint_C w\left\{\frac{1-\zeta\phi'(w)}{w-a-\zeta\phi(w)}\right\}dw \\ &= \frac{1}{2\pi i}\oint_C \frac{w}{w-a}\{1-\zeta\phi'(w)\}\left\{\frac{1}{1-\zeta\phi(w)/(w-a)}\right\}dw \\ &= \frac{1}{2\pi i}\oint_C \frac{w}{w-a}\{1-\zeta\phi'(w)\}\left\{\sum_{n=0}^{\infty}\zeta^n\phi^n(w)/(w-a)^n\right\}dw \\ &= \frac{1}{2\pi i}\oint_C \frac{w}{w-a}dw + \sum_{n=1}^{\infty}\frac{\zeta^n}{2\pi i}\oint_C\left\{\frac{w\,\phi^n(w)}{(w-a)^{n+1}} - \frac{w\,\phi^{n-1}(w)\,\phi'(w)}{(w-a)^n}\right\}dw \\ &= a - \sum_{n=1}^{\infty}\frac{\zeta^n}{2\pi i}\oint_C \frac{w}{n}\frac{d}{dw}\left\{\frac{\phi^n(w)}{(w-a)^n}\right\}dw \\ &= a + \sum_{n=1}^{\infty}\frac{\zeta^n}{2\pi i n}\oint_C \frac{\phi^n(w)}{(w-a)^n}dw \\ &= a + \sum_{n=1}^{\infty}\frac{\zeta^n}{n!}\frac{d^{n-1}}{da^{n-1}}[\phi^n(a)] \end{aligned}$$

ANALYTIC CONTINUATION

29. Show that the series (*a*) $\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$ and (*b*) $\sum_{n=0}^{\infty}\frac{(z-i)^n}{(2-i)^{n+1}}$ are analytic continuations of each other.

(*a*) By the ratio test, the series converges for $|z| < 2$ [shaded in Fig. 6-6]. In this circle the series [which is a geometric series with first term $\frac{1}{2}$ and ratio $z/2$] can be summed and represents the function $\frac{1/2}{1-z/2} = \frac{1}{2-z}$.

(*b*) By the ratio test, the series converges for $\left|\frac{z-i}{2-i}\right| < 1$, i.e. $|z-i| < \sqrt{5}$, [see Fig. 6-6]. In this circle the series [which is a geometric series with first term $1/(2-i)$ and ratio $(z-i)/(2-i)$] can be summed and represents the function $\frac{1/(2-i)}{1-(z-i)/(2-i)} = \frac{1}{2-z}$.

Since the power series represent the same function in the regions common to the interiors of the circles $|z| = 2$ and $|z-i| = \sqrt{5}$, it follows that they are analytic continuations of each other.

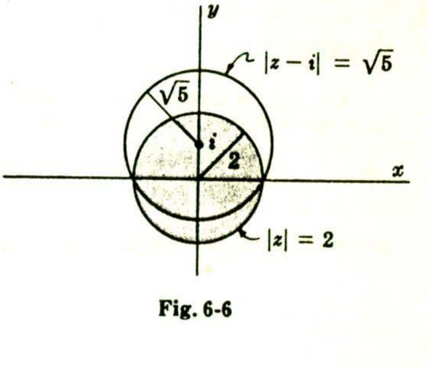

Fig. 6-6

30. Prove that the series $1 + z + z^2 + z^4 + z^8 + \cdots = 1 + \sum_{n=0}^{\infty} z^{2^n}$ cannot be continued analytically beyond $|z| = 1$.

Let $F(z) = 1 + z + z^2 + z^4 + z^8 + \cdots$. Then $F(z) = z + F(z^2)$, $F(z) = z + z^2 + F(z^4)$, $F(z) = z + z^2 + z^4 + F(z^8)$, $\ldots$.

From these it is clear that the values of z given by $z = 1$, $z^2 = 1$, $z^4 = 1$, $z^8 = 1$, ... are all singularities of $F(z)$. These singularities all lie on the circle $|z| = 1$. Given any small arc of this circle, there will be infinitely many such singularities. These represent an impassable barrier and analytic continuation beyond $|z| = 1$ is therefore impossible. The circle $|z| = 1$ constitutes a *natural boundary*.

MISCELLANEOUS PROBLEMS

31. Let $\{f_k(z)\}$, $k = 1, 2, 3, \ldots$ be a sequence of functions analytic in a region $\mathcal{R}$. Suppose that

$$F(z) = \sum_{k=1}^{\infty} f_k(z)$$

is uniformly convergent in $\mathcal{R}$. Prove that $F(z)$ is analytic in $\mathcal{R}$.

Let $S_n(z) = \sum_{k=1}^{n} f_k(z)$. By definition of uniform convergence, given any $\epsilon > 0$ we can find a positive integer N depending on ϵ and not on z such that for all z in $\mathcal{R}$,

$$|F(z) - S_n(z)| < \epsilon \qquad \text{for all } n > N \tag{1}$$

Now suppose that C is any simple closed curve lying entirely in $\mathcal{R}$ and denote its length by L. Then by Problem 16, since $f_k(z)$, $k = 1, 2, 3, \ldots$ are continuous, $F(z)$ is also continuous so that $\oint_C F(z)\,dz$ exists. Also, using (1) we see that for $n > N$,

$$\left| \oint_C F(z)\,dz - \sum_{k=1}^{n} \oint_C f_k(z)\,dz \right| = \left| \oint_C \{F(z) - S_n(z)\}\,dz \right| < \epsilon L$$

Because ϵ can can be made as small as we please, we see that

$$\oint_C F(z)\,dz = \sum_{k=1}^{\infty} \oint_C f_k(z)\,dz$$

But by Cauchy's theorem, $\oint_C f_k(z)\,dz = 0$. Hence

$$\oint_C F(z)\,dz = 0$$

and so by Morera's theorem (Page 118, Chapter 5) $F(z)$ must be analytic.

32. Prove that an analytic function cannot be bounded in the neighbourhood of an isolated singularity.

Let $f(z)$ be analytic inside and on a circle C of radius r, except at the isolated singularity $z = a$ taken to be the centre of C. Then by Laurent's theorem $f(z)$ has a Laurent expansion

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k \tag{1}$$

where the coefficients a_k are given by equation (7), Page 144. In particular,

$$a_{-n} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{-n+1}}\,dz \qquad n = 1, 2, 3, \ldots \tag{2}$$

Now if $|f(z)| < M$ for a constant M, i.e. if $f(z)$ is bounded, then from (2),

$$|a_{-n}| = \frac{1}{2\pi} \left| \oint_C (z-a)^{n-1} f(z)\,dz \right| \leqq \frac{1}{2\pi} r^{n-1} \cdot M \cdot 2\pi r = M r^n$$

Hence since r can be made arbitrarily small, we have $a_{-n} = 0$, $n = 1, 2, 3, \ldots$, i.e. $a_{-1} = a_{-2} = a_{-3} = \cdots = 0$, and the Laurent series reduces to a Taylor series about $z = a$. This shows that $f(z)$ is analytic at $z = a$ so that $z = a$ is not a singularity, contrary to hypothesis. This contradiction shows that $f(z)$ cannot be bounded in the neighbourhood of an isolated singularity.

33. Prove that if $z \neq 0$, then

$$e^{\frac{1}{2}\alpha(z-1/z)} = \sum_{n=-\infty}^{\infty} J_n(\alpha)\, z^n$$

where $$J_n(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - \alpha\sin\theta)\, d\theta \qquad n = 0, 1, 2, \ldots$$

The point $z = 0$ is the only finite singularity of the function $e^{\frac{1}{2}\alpha(z-1/z)}$ and it follows that the function must have a Laurent series expansion of the form

$$e^{\frac{1}{2}\alpha(z-1/z)} = \sum_{n=-\infty}^{\infty} J_n(\alpha)\, z^n \tag{1}$$

which holds for $|z| > 0$. By equation (7), Page 144, the coefficients $J_n(\alpha)$ are given by

$$J_n(\alpha) = \frac{1}{2\pi i}\oint_C \frac{e^{\frac{1}{2}\alpha(z-1/z)}}{z^{n+1}}\, dz \tag{2}$$

where C is any simple closed curve having $z = 0$ inside.

Let us in particular choose C to be a circle of radius 1 having centre at the origin; i.e. the equation of C is $|z| = 1$ or $z = e^{i\theta}$. Then (2) becomes

$$\begin{aligned}
J_n(\alpha) &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{\frac{1}{2}\alpha(e^{i\theta} - e^{-i\theta})}}{e^{i(n+1)\theta}}\, i\, e^{i\theta}\, d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{i\alpha\sin\theta - in\theta}\, d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \cos(\alpha\sin\theta - n\theta)\, d\theta + \frac{i}{2\pi}\int_0^{2\pi} \sin(\alpha\sin\theta - n\theta)\, d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - \alpha\sin\theta)\, d\theta
\end{aligned}$$

using the fact that $I = \int_0^{2\pi} \sin(\alpha\sin\theta - n\theta)\, d\theta = 0$. This last result follows since on letting $\theta = 2\pi - \phi$, we find

$$I = \int_0^{2\pi} \sin(-\alpha\sin\phi - 2\pi n + n\phi)\, d\phi = -\int_0^{2\pi} \sin(\alpha\sin\phi - n\phi)\, d\phi = -I$$

so that $I = -I$ and $I = 0$. The required result is thus established.

The function $J_n(\alpha)$ is called a *Bessel function* of the first kind of order n.

For further discussion of Bessel functions, see Chapter 10.

34. The *Legendre polynomials* $P_n(t)$, $n = 0, 1, 2, 3, \ldots$ are defined by *Rodrigues' formula*

$$P_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}(t^2 - 1)^n$$

(a) Prove that if C is any simple closed curve enclosing the point $z = t$, then

$$P_n(t) = \frac{1}{2\pi i}\cdot\frac{1}{2^n}\oint_C \frac{(z^2-1)^n}{(z-t)^{n+1}}\, dz$$

This is called *Schlaefli's representation* for $P_n(t)$, or *Schlaefli's formula*.

(b) Prove that

$$P_n(t) = \frac{1}{2\pi}\int_0^{2\pi} (t + \sqrt{t^2-1}\cos\theta)^n\, d\theta$$

(a) By Cauchy's integral formulae, if C encloses point t,

$$f^{(n)}(t) = \frac{d^n}{dt^n} f(t) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-t)^{n+1}}\, dz$$

Then taking $f(t) = (t^2-1)^n$ so that $f(z) = (z^2-1)^n$, we have the required result

$$P_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}(t^2-1)^n = \frac{1}{2^n}\cdot\frac{1}{2\pi i}\oint_C \frac{(z^2-1)^n}{(z-t)^{n+1}}\,dz$$

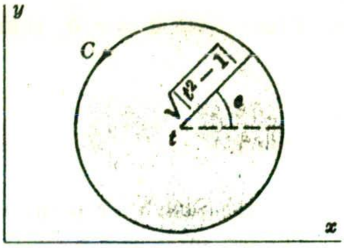

Fig. 6-7

(b) Choose C as a circle with centre at t and radius $\sqrt{|t^2-1|}$ as shown in Fig. 6-7. Then an equation for C is $|z-t| = \sqrt{|t^2-1|}$ or $z = t+\sqrt{t^2-1}\,e^{i\theta}$, $0 \leqq \theta < 2\pi$. Using this in part (a), we have

$$P_n(t) = \frac{1}{2^n}\cdot\frac{1}{2\pi i}\int_0^{2\pi}\frac{\{(t+\sqrt{t^2-1}\,e^{i\theta})^2-1\}^n\sqrt{t^2-1}\,ie^{i\theta}\,d\theta}{(\sqrt{t^2-1}\,e^{i\theta})^{n+1}}$$

$$= \frac{1}{2^n}\cdot\frac{1}{2\pi}\int_0^{2\pi}\frac{\{(t^2-1)+2t\sqrt{t^2-1}\,e^{i\theta}+(t^2-1)e^{2i\theta}\}^n e^{-in\theta}\,d\theta}{(t^2-1)^{n/2}}$$

$$= \frac{1}{2^n}\cdot\frac{1}{2\pi}\int_0^{2\pi}\frac{\{(t^2-1)e^{-i\theta}+2t\sqrt{t^2-1}+(t^2-1)e^{i\theta}\}^n\,d\theta}{(t^2-1)^{n/2}}$$

$$= \frac{1}{2^n}\cdot\frac{1}{2\pi}\int_0^{2\pi}\frac{\{2t\sqrt{t^2-1}+2(t^2-1)\cos\theta\}^n\,d\theta}{(t^2-1)^{n/2}}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(t+\sqrt{t^2-1}\cos\theta)^n\,d\theta$$

For further discussion of Legendre polynomials, see Chapter 10.

Supplementary Problems

SEQUENCES AND SERIES OF FUNCTIONS

35. Using the definition, prove: (a) $\lim_{n\to\infty}\frac{3n-2z}{n+z} = 3$, (b) $\lim_{n\to\infty}\frac{nz}{n^2+z^2} = 0$.

36. If $\lim_{n\to\infty} u_n(z) = U(z)$ and $\lim_{n\to\infty} v_n(z) = V(z)$, prove that (a) $\lim_{n\to\infty}\{u_n(z)\pm v_n(z)\} = U(z)\pm V(z)$, (b) $\lim_{n\to\infty}\{u_n(z)\,v_n(z)\} = U(z)\,V(z)$, (c) $\lim_{n\to\infty} u_n(z)/v_n(z) = U(z)/V(z)$ if $V(z)\neq 0$.

37. (a) Prove that the series $\frac{1}{2}+\frac{z}{2^2}+\frac{z^2}{2^3}+\cdots = \sum_{n=1}^{\infty}\frac{z^{n-1}}{2^n}$ converges for $|z|<2$ and (b) find its sum.

Ans. (a) $S_n(z) = \{1-(z/2)^n\}/(2-z)$ and $\lim_{n\to\infty} S_n(z)$ exists if $|z|<2$, (b) $S(z) = 1/(2-z)$

38. (a) Determine the set of values of z for which the series $\sum_{n=0}^{\infty}(-1)^n(z^n+z^{n+1})$ converges and (b) find its sum. *Ans.* (a) $|z|<1$, (b) 1

39. (a) For what values of z does the series $\sum_{n=1}^{\infty}\frac{1}{(z^2+1)^n}$ converge and (b) what is its sum?

Ans. (a) All z such that $|z^2+1|>1$, (b) $1/z^2$

40. If $\lim_{n\to\infty}|u_n(z)| = 0$, prove that $\lim_{n\to\infty} u_n(z) = 0$. Is the converse true? Justify your answer.

41. Prove that for all finite z, $\lim_{n\to\infty} z^n/n! = 0$.

42. Let $\{a_n\}$, $n = 1, 2, 3, \ldots$ be a sequence of positive numbers having zero as a limit. Suppose that $|u_n(z)| \leqq a_n$ for $n = 1, 2, 3, \ldots$. Prove that $\lim_{n \to \infty} u_n(z) = 0$.

43. Prove that the convergence or divergence of a series is not affected by adding (or removing) a finite number of terms.

44. Let $S_n = z + 2z^2 + 3z^3 + \cdots + nz^n$, $T_n = z + z^2 + z^3 + \cdots + z^n$. (*a*) Show that $S_n = (T_n - nz^{n+1})/(1-z)$. (*b*) Use (*a*) to find the sum of the series $\sum_{n=1}^{\infty} nz^n$ and determine the set of values for which the series converges. *Ans.* (*b*) $z/(1-z)^2$, $|z| < 1$

45. Find the sum of the series $\sum_{n=0}^{\infty} \frac{n+1}{2^n}$. *Ans.* 4

ABSOLUTE AND UNIFORM CONVERGENCE

46. (*a*) Prove that $u_n(z) = 3z + 4z^2/n$, $n = 1, 2, 3, \ldots$, converges uniformly to $3z$ for all z inside or on the circle $|z| = 1$. (*b*) Can the circle of part (*a*) be enlarged? Explain.

47. (*a*) Determine whether the sequence $u_n(z) = nz/(n^2 + z^2)$ [Problem 35(*b*)] converges uniformly to zero for all z inside $|z| = 3$. (*b*) Does the result of (*a*) hold for all finite values of z?

48. Prove that the series $1 + az + a^2z^2 + \cdots$ converges uniformly to $1/(1 - az)$ inside or on the circle $|z| = R$ where $R < 1/|a|$.

49. Investigate the (*a*) absolute and (*b*) uniform convergence of the series

$$\frac{z}{3} + \frac{z(3-z)}{3^2} + \frac{z(3-z)^2}{3^3} + \frac{z(3-z)^3}{3^4} + \cdots$$

Ans. (*a*) Converges absolutely if $|z - 3| < 3$ or $z = 0$. (*b*) Converges uniformly for $|z - 3| \leqq R$ where $0 < R < 3$; does not converge uniformly in any neighbourhood which includes $z = 0$.

50. Investigate the (*a*) absolute and (*b*) uniform convergence of the series in Problem 38.
Ans. (*a*) Converges absolutely if $|z| < 1$. (*b*) Converges uniformly if $|z| \leqq R$ where $R < 1$.

51. Investigate the (*a*) absolute and (*b*) uniform convergence of the series in Problem 39.
Ans. (*a*) Converges absolutely if $|z^2 + 1| > 1$. (*b*) Converges uniformly if $|z^2 + 1| \geqq R$ where $R > 1$.

52. Let $\{a_n\}$ be a sequence of positive constants having limit zero; and suppose that for all z in a region $\mathcal{R}$, $|u_n(z)| \leqq a_n$, $n = 1, 2, 3, \ldots$. Prove that $\lim_{n \to \infty} u_n(z) = 0$ uniformly in $\mathcal{R}$.

53. (*a*) Prove that the sequence $u_n(z) = nze^{-nz^2}$ converges to zero for all finite z such that $\text{Re}\,\{z^2\} > 0$, and represent this region geometrically. (*b*) Discuss the uniform convergence of the sequence in (*a*).
Ans. (*b*) Not uniformly convergent in any region which includes $z = 0$.

54. If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge absolutely, prove that $\sum_{n=0}^{\infty} c_n$, where $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$, converges absolutely.

55. Prove that if each of two series is absolutely and uniformly convergent in $\mathcal{R}$, their product is absolutely and uniformly convergent in $\mathcal{R}$.

SPECIAL CONVERGENCE TESTS

56. Test for convergence:

(*a*) $\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$, (*b*) $\sum_{n=1}^{\infty} \frac{n}{3^n - 1}$, (*c*) $\sum_{n=1}^{\infty} \frac{n+3}{3n^2 - n + 2}$, (*d*) $\sum_{n=1}^{\infty} \frac{(-1)^n}{4n+3}$, (*e*) $\sum_{n=1}^{\infty} \frac{2n-1}{\sqrt{n^3 + n + 2}}$.

Ans. (*a*) conv., (*b*) conv., (*c*) div., (*d*) conv., (*e*) div.

57. Investigate the convergence of:

(a) $\sum_{n=1}^{\infty} \frac{1}{n+|z|}$, (b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+|z|}$, (c) $\sum_{n=1}^{\infty} \frac{1}{n^2+|z|}$, (d) $\sum_{n=1}^{\infty} \frac{1}{n^2+z}$.

Ans. (a) Diverges for all finite z. (b) Converges for all z. (c) Converges for all z. (d) Converges for all z except $z = -n^2$, $n = 1, 2, 3, \ldots$.

58. Investigate the convergence of $\sum_{n=0}^{\infty} \frac{ne^{n\pi i/4}}{e^n - 1}$. *Ans.* Conv.

59. Find the region of convergence of (a) $\sum_{n=0}^{\infty} \frac{(z+i)^n}{(n+1)(n+2)}$, (b) $\sum_{n=1}^{\infty} \frac{1}{n^2 \cdot 3^n}\left(\frac{z+1}{z-1}\right)^n$, (c) $\sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n!}$.

Ans. (a) $|z+i| \leqq 1$, (b) $|(z+1)/(z-1)| \leqq 3$, (c) $|z| < \infty$

60. Investigate the region of absolute convergence of $\sum_{n=1}^{\infty} \frac{n(-1)^n(z-i)^n}{4^n(n^2+1)^{5/2}}$.

Ans. Conv. abs. for $|z - i| \leqq 4$.

61. Find the region of convergence of $\sum_{n=0}^{\infty} \frac{e^{2\pi i n z}}{(n+1)^{3/2}}$.

Ans. Converges if $\text{Im } z \geqq 0$.

62. Prove that the series $\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})$ diverges although the nth term approaches zero.

63. Let N be a positive integer and suppose that for all $n > N$, $|u_n| > 1/(n \ln n)$. Prove that $\sum_{n=1}^{\infty} u_n$ diverges.

64. Establish the validity of the (a) nth root test [Theorem 12], (b) integral test [Theorem 13], on Page 141.

65. Find the interval of convergence of $1 + 2z + z^2 + 2z^3 + z^4 + 2z^5 + \cdots$. *Ans.* $|z| < 1$

66. Prove Raabe's test (Theorem 14) on Page 141.

67. Test for convergence: (a) $\frac{1}{2\ln^2 2} + \frac{1}{3\ln^2 3} + \frac{1}{4\ln^2 4} + \cdots$, (b) $\frac{1}{5} + \frac{1\cdot 4}{5\cdot 8} + \frac{1\cdot 4\cdot 7}{5\cdot 8\cdot 11} + \cdots$, (c) $\frac{2}{5} + \frac{2\cdot 7}{5\cdot 10} + \frac{2\cdot 7\cdot 12}{5\cdot 10\cdot 15} + \cdots$, (d) $\frac{\ln 2}{2} + \frac{\ln 3}{3} + \frac{\ln 4}{4} + \cdots$.

Ans. (a) conv., (b) conv., (c) div., (d) div.

THEOREMS ON UNIFORM CONVERGENCE AND POWER SERIES

68. Determine the regions in which each of the following series is uniformly convergent:

(a) $\sum_{n=1}^{\infty} \frac{z^n}{3^n+1}$, (b) $\sum_{n=1}^{\infty} \frac{(z-i)^{2n}}{n^2}$, (c) $\sum_{n=1}^{\infty} \frac{1}{(n+1)z^n}$, (d) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2+|z|^2}$

Ans. (a) $|z| \leqq R$ where $R < 3$. (b) $|z-i| \leqq 1$. (c) $|z| \geqq R$ where $R > 1$. (d) All z

69. Prove Theorem 20, Page 142.

70. State and prove theorems for sequences analogous to Theorems 18, 19 and 20, Page 142, for series.

71. (a) By differentiating both sides of the identity

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \qquad |z| < 1$$

find the sum of the series $\sum_{n=1}^{\infty} nz^n$ for $|z| < 1$. Justify all steps.

(b) Find the sum of the series $\sum_{n=1}^{\infty} n^2 z^n$ for $|z| < 1$.

Ans. (a) $z/(1-z)^2$ [compare Problem 44], (b) $(1+z)/(1-z)^3$

72. Let z be real and such that $0 \leqq z \leqq 1$, and let $u_n(z) = nze^{-nz^2}$. (a) Find $\lim_{n\to\infty} \int_0^1 u_n(z)\,dz$. (b) Find $\int_0^1 \left\{\lim_{n\to\infty} u_n(z)\right\} dz$. (c) Explain why the answers to (a) and (b) are not equal. [See Problem 53.]

Ans. (a) 1/2, (b) 0

73. Prove Abel's theorem [Theorem 24, Page 142].

74. (a) Prove that $\dfrac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \cdots$ for $|z| < 1$.

(b) If we choose that branch of $f(z) = \tan^{-1} z$ such that $f(0) = 0$, use (a) to prove that

$$\tan^{-1} z = \int_0^z \frac{dz}{1+z^2} = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots$$

(c) Prove that $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots$.

75. Prove Theorem 25, Page 142.

76. (a) Determine $Y(z) = \sum_{n=0}^{\infty} a_n z^n$ such that for all z in $|z| \leqq 1$, $Y'(z) = Y(z)$, $Y(0) = 1$. State all theorems used and verify that the result obtained is a solution.

(b) Is the result obtained in (a) valid outside of $|z| \leqq 1$? Justify your answer.

(c) Show that $Y(z) = e^z$ satisfies the differential equation and conditions in (a).

(d) Can we identify the series in (a) with e^z? Explain.

Ans. (a) $Y(z) = 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \cdots$

77. (a) Use series methods on the differential equation $Y''(z) + Y(z) = 0$, $Y(0) = 0$, $Y'(0) = 1$ to obtain the series expansion

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

(b) How could you obtain a corresponding series for $\cos z$?

TAYLOR'S THEOREM

78. Expand each of the following functions in a Taylor series about the indicated point and determine the region of convergence in each case.

(a) e^{-z}; $z = 0$ (b) $\cos z$; $z = \pi/2$ (c) $1/(1+z)$; $z = 1$ (d) $z^3 - 3z^2 + 4z - 2$; $z = 2$ (e) ze^{2z}; $z = -1$

79. If each of the following functions were expanded into a Taylor series about the indicated points, what would be the region of convergence? Do not perform the expansion.

(a) $\sin z/(z^2+4)$; $z = 0$ (c) $(z+3)/(z-1)(z-4)$; $z = 2$ (e) $e^z/z(z-1)$; $z = 4i$ (g) $\sec \pi z$; $z = 1$

(b) $z/(e^z+1)$; $z = 0$ (d) $e^{-z^2}\sinh(z+2)$; $z = 0$ (f) $\coth 2z$; $z = 0$

Ans. (a) $|z| < 2$, (b) $|z| < \pi$, (c) $|z-2| < 1$, (d) $|z| < \infty$, (e) $|z-4i| < 4$, (f) $|z| < \pi/2$, (g) $|z-1| < 1/2$

80. Verify the expansions 1, 2, 3 for e^z, $\sin z$ and $\cos z$ on Page 143.

81. Show that $\sin z^2 = z^2 - \dfrac{z^6}{3!} + \dfrac{z^{10}}{5!} - \dfrac{z^{14}}{7!} + \cdots$, $|z| < \infty$.

82. Prove that $\tan^{-1} z = z - \dfrac{z^3}{3} + \dfrac{z^5}{5} - \dfrac{z^7}{7} + \cdots$, $|z| < 1$.

83. Show that

(a) $\tan z = z + \dfrac{z^3}{3} + \dfrac{2z^5}{15} + \cdots$, $|z| < \pi/2$

(b) $\sec z = 1 + \dfrac{z^2}{2} + \dfrac{5z^4}{24} + \cdots$, $|z| < \pi/2$

(c) $\csc z = \dfrac{1}{z} + \dfrac{z}{6} + \dfrac{7z^3}{360} + \cdots$, $0 < |z| < \pi$

84. By replacing z by iz in the expansion of Problem 82, obtain the result in Problem 23(c) on Page 155.

85. How would you obtain series for (a) $\tanh z$, (b) $\operatorname{sech} z$, (c) $\operatorname{csch} z$ from the series in Problem 83?

86. Prove the uniqueness of the Taylor series expansion of $f(z)$ about $z = a$.
[*Hint.* Assume $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n = \sum_{n=0}^{\infty} d_n(z-a)^n$ and show that $c_n = d_n$, $n = 0, 1, 2, 3, \ldots$.]

87. Prove the binomial Theorem 6 on Page 143.

88. If we choose that branch of $\sqrt{1+z^3}$ having the value 1 for $z = 0$, show that

$$\frac{1}{\sqrt{1+z^3}} = 1 - \frac{1}{2}z^3 + \frac{1\cdot 3}{2\cdot 4}z^6 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}z^9 + \cdots \qquad |z| < 1$$

89. (a) Choosing that branch of $\sin^{-1} z$ having the value zero for $z = 0$, show that

$$\sin^{-1} z = z + \frac{1}{2}\frac{z^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{z^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{z^7}{7} + \cdots \qquad |z| \le 1$$

(b) Prove that the result in (a) is valid for $z = i$.

90. (a) Expand $f(z) = \ln(3 - iz)$ in powers of $z - 2i$, choosing that branch of the logarithm for which $f(0) = \ln 3$, and (b) determine the region of convergence.

Ans. (a) $\ln 5 - \dfrac{i(z-2i)}{5} + \dfrac{(z-2i)^2}{2\cdot 5^2} + \dfrac{i(z-2i)^3}{3\cdot 5^3} - \dfrac{(z-2i)^4}{4\cdot 5^4} - \cdots$ (b) $|z - 2i| < 5$

LAURENT'S THEOREM

91. Expand $f(z) = 1/(z-3)$ in a Laurent series valid for (a) $|z| < 3$, (b) $|z| > 3$.

Ans. (a) $-\dfrac{1}{3} - \dfrac{1}{9}z - \dfrac{1}{27}z^2 - \dfrac{1}{81}z^3 - \cdots$ (b) $z^{-1} + 3z^{-2} + 9z^{-3} + 27z^{-4} + \cdots$

92. Expand $f(z) = \dfrac{z}{(z-1)(2-z)}$ in a Laurent series valid for:

(a) $|z| < 1$, (b) $1 < |z| < 2$, (c) $|z| > 2$, (d) $|z-1| > 1$, (e) $0 < |z-2| < 1$.

Ans. (a) $-\dfrac{1}{2}z - \dfrac{3}{4}z^2 - \dfrac{7}{8}z^3 - \dfrac{15}{16}z^4 - \cdots$ (b) $\cdots + \dfrac{1}{z^2} + \dfrac{1}{z} + 1 + \dfrac{1}{2}z + \dfrac{1}{4}z^2 + \dfrac{1}{8}z^3 + \cdots$

(c) $-\dfrac{1}{2} - \dfrac{3}{z^2} - \dfrac{7}{z^3} - \dfrac{15}{z^4} - \cdots$ (d) $-(z-1)^{-1} - 2(z-1)^{-2} - 2(z-1)^{-3} - \cdots$

(e) $1 - 2(z-2)^{-1} - (z-2) + (z-2)^2 - (z-2)^3 + (z-2)^4 - \cdots$

93. Expand $f(z) = 1/z(z-2)$ in a Laurent series valid for (a) $0 < |z| < 2$, (b) $|z| > 2$.

94. Find an expansion of $f(z) = z/(z^2+1)$ valid for $|z-3| > 2$.

95. Expand $f(z) = 1/(z-2)^2$ in a Laurent series valid for (a) $|z| < 2$, (b) $|z| > 2$

96. Expand each of the following functions in a Laurent series about $z = 0$, naming the type of singularity in each case.

(a) $(1 - \cos z)/z$, (b) e^{z^2}/z^3, (c) $z^{-1}\cosh z^{-1}$, (d) $z^2 e^{-z^4}$, (e) $z \sinh \sqrt{z}$.

Ans. (a) $\dfrac{z}{2!} - \dfrac{z^3}{4!} + \dfrac{z^5}{6!} - \cdots$; removable singularity

(b) $\dfrac{1}{z^3} + \dfrac{1}{z} + \dfrac{z}{2!} + \dfrac{z^3}{3!} + \dfrac{z^5}{4!} + \dfrac{z^7}{5!} + \cdots$; pole of order 3

(c) $\dfrac{1}{z} - \dfrac{1}{2!\,z^3} + \dfrac{1}{4!\,z^5} - \cdots$; essential singularity

(d) $z^2 - z^6 + \dfrac{z^{10}}{2!} - \dfrac{z^{14}}{3!} + \cdots$; ordinary point

(e) $z^{3/2} + \dfrac{z^{5/2}}{3!} + \dfrac{z^{7/2}}{5!} + \dfrac{z^{9/2}}{7!} + \cdots$; branch point

97. Show that if $\tan z$ is expanded into a Laurent series about $z = \pi/2$, (a) the principal part is $-1/(z - \pi/2)$, (b) the series converges for $0 < |z - \pi/2| < \pi/2$, (c) $z = \pi/2$ is a simple pole.

98. Determine and classify all the singularities of the functions:
(*a*) $1/(2\sin z - 1)^2$, (*b*) $z/(e^{1/z} - 1)$, (*c*) $\cos(z^2 + z^{-2})$, (*d*) $\tan^{-1}(z^2 + 2z + 2)$, (*e*) $z/(e^z - 1)$.

Ans. (*a*) $\pi/6 + 2m\pi$, $(2m+1)\pi - \pi/6$, $m = 0, \pm 1, \pm 2, \ldots$; poles of order 2
(*b*) $i/2m\pi$, $m = \pm 1, \pm 2, \ldots$; simple poles, $z = 0$; essential singularity, $z = \infty$; pole of order 2
(*c*) $z = 0, \infty$; essential singularities (*d*) $z = -1 \pm i$; branch points
(*e*) $z = 2m\pi i$, $m = \pm 1, \pm 2, \ldots$; simple poles, $z = 0$; removable singularity, $z = \infty$; essential singularity

99. (*a*) Expand $f(z) = e^{z/(z-2)}$ in a Laurent series about $z = 2$ and (*b*) determine the region of convergence of this series. (*c*) Classify the singularities of $f(z)$.

Ans. (*a*) $e\left\{1 + 2(z-2)^{-1} + \dfrac{2^2(z-2)^{-2}}{2!} + \dfrac{2^3(z-2)^{-3}}{3!} + \cdots\right\}$ (*b*) $|z-2| > 0$ (*c*) $z = 2$; essential singularity, $z = \infty$; removable singularity

100. Establish the result (*7*), Page 144, for the coefficients in a Laurent series.

101. Prove that the only singularities of a rational function are poles.

102. Prove the converse of Problem 101, i.e. if the only singularities of a function are poles, the function must be rational.

LAGRANGE'S EXPANSION

103. Show that the root of the equation $z = 1 + \zeta z^p$, which is equal to 1 when $\zeta = 0$, is given by

$$z = 1 + \zeta + \frac{2p}{2!}\zeta^2 + \frac{(3p)(3p-1)}{3!}\zeta^3 + \frac{(4p)(4p-1)(4p-2)}{4!}\zeta^4 + \cdots$$

104. Calculate the root in Problem 103 if $p = 1/2$ and $\zeta = 1$, (*a*) by series and (*b*) exactly, and compare the two answers. *Ans.* 2.62 to two decimal accuracy

105. By considering the equation $z = \alpha + \frac{1}{2}\zeta(z^2 - 1)$, show that

$$\frac{1}{\sqrt{1 - 2\alpha\zeta + \zeta^2}} = 1 + \sum_{n=1}^{\infty} \frac{\zeta^n}{2^n\, n!} \frac{d^n}{d\alpha^n}(\alpha^2 - 1)^n$$

106. Show how Lagrange's expansion can be used to solve Kepler's problem of determining that root of $z = a + \zeta \sin z$ for which $z = a$ when $\zeta = 0$.

107. Prove the Lagrange expansion (*12*) on Page 145.

ANALYTIC CONTINUATION

108. (*a*) Prove that $F_2(z) = \dfrac{1}{1+i}\displaystyle\sum_{n=0}^{\infty}\left(\frac{z+i}{1+i}\right)^n$ is an analytic continuation of $F_1(z) = \displaystyle\sum_{n=0}^{\infty} z^n$, showing graphically the regions of convergence of the series.
(*b*) Determine the function represented by all analytic continuations of $F_1(z)$. *Ans.* (*b*) $1/(1-z)$

109. Let $F_1(z) = \displaystyle\sum_{n=0}^{\infty} \frac{z^{n+1}}{3^n}$. (*a*) Find an analytic continuation of $F_1(z)$ which converges for $z = 3 - 4i$.
(*b*) Determine the value of the analytic continuation in (*a*) for $z = 3 - 4i$. *Ans.* (*b*) $-3 - \frac{9}{4}i$

110. Prove that the series

$$z^{1!} + z^{2!} + z^{3!} + \cdots$$

has the natural boundary $|z| = 1$.

MISCELLANEOUS PROBLEMS

111. (*a*) Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if the constant $p \leqq 1$.
(*b*) Prove that if p is complex the series in (*a*) converges if $\text{Re}\{p\} > 1$.
(*c*) Investigate the convergence or divergence of the series in (*a*) if $\text{Re}\{p\} \leqq 1$.

112. Test for convergence or divergence: (a) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+i}$, (b) $\sum_{n=1}^{\infty} \frac{n+\sin^2 n}{ie^n+(2-i)n}$, (c) $\sum_{n=1}^{\infty} n \sin^{-1}(1/n^3)$, (d) $\sum_{n=2}^{\infty} \frac{(i)^n}{n \ln n}$, (e) $\sum_{n=1}^{\infty} \coth^{-1} n$, (f) $\sum_{n=1}^{\infty} ne^{-n^2}$.

Ans. (a) div., (b) conv., (c) conv., (d) conv., (e) div., (f) conv.

113. Euler presented the following argument to show that $\sum_{-\infty}^{\infty} z^n = 0$:

$$\frac{z}{1-z} = z + z^2 + z^3 + \cdots = \sum_1^{\infty} z^n, \qquad \frac{z}{z-1} = \frac{1}{1-1/z} = 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots = \sum_0^{-\infty} z^n$$

Then adding, $\sum_{-\infty}^{\infty} z^n = 0$. Explain the fallacy.

114. Show that for $|z-1| < 1$, $z \ln z = (z-1) + \frac{(z-1)^2}{1 \cdot 2} - \frac{(z-1)^3}{2 \cdot 3} + \frac{(z-1)^4}{3 \cdot 4} - \cdots$.

115. Expand $\sin^3 z$ in a Maclaurin series. *Ans.* $\sum_{n=1}^{\infty} \frac{(3-3^{2n-1})z^{2n-1}}{4(2n-1)!}$

116. Given the series $z^2 + \frac{z^2}{1+z^2} + \frac{z^2}{(1+z^2)^2} + \frac{z^2}{(1+z^2)^3} + \cdots$.

(a) Show that the sum of the first n terms is $S_n(z) = 1 + z^2 - 1/(1+z^2)^{n-1}$.

(b) Show that the sum of the series is $1+z^2$ for $z \neq 0$, and 0 for $z=0$; and hence that $z=0$ is a point of discontinuity.

(c) Show that the series is not uniformly convergent in the region $|z| \leqq \delta$ where $\delta > 0$.

117. If $F(z) = \frac{3z-3}{(2z-1)(z-2)}$, find a Laurent series of $F(z)$ about $z=1$ convergent for $\frac{1}{2} < |z-1| < 1$.

Ans. $\cdots - \frac{1}{8}(z-1)^{-4} + \frac{1}{4}(z-1)^{-3} - \frac{1}{2}(z-1)^{-2} + (z-1)^{-1} - 1 - (z-1) - (z-1)^2 - \cdots$

118. Let $G(z) = (\tan^{-1} z)/z^4$. (a) Expand $G(z)$ in a Laurent series. (b) Determine the region of convergence of the series in (a). (c) Evaluate $\oint_C G(z)\,dz$ where C is a square with vertices at $2 \pm 2i$, $-2 \pm 2i$. *Ans.* (a) $\frac{1}{z^3} - \frac{1}{3z} + \frac{z}{5} - \frac{z^3}{7} + \cdots$ (b) $|z| > 0$ (c) $-1/3$

119. For each of the functions ze^{1/z^2}, $(\sin^2 z)/z$, $1/z(4-z)$ which have singularities at $z=0$: (a) give a Laurent expansion about $z=0$ and determine the region of convergence; (b) state in each case whether $z=0$ is a removable singularity, essential singularity or a pole; (c) evaluate the integral of the function about the circle $|z|=2$.

Ans. (a) $z + z^{-1} + z^{-3}/2! + z^{-5}/3! + \cdots$; $|z| > 0$, $2z - 2z^3/3 + 4z^5/45 - \cdots$; $|z| \geqq 0$, $z^{-1}/4 + 1/16 + z/64 + z^2/256 + \cdots$; $0 < |z| < 4$

(b) essential singularity, removable singularity, pole (c) $2\pi i$, 0, $\pi i/2$

120. (a) Investigate the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$. (b) Does your answer to (a) contradict Problem 8, Page 148? *Ans.* (a) diverges

121. (a) Show that the series $\frac{\sin z}{1^2+1} + \frac{\sin^2 z}{2^2+1} + \frac{\sin^3 z}{3^2+1} + \cdots$, where $z = x+iy$, converges absolutely in the region bounded by $\sin^2 x + \sinh^2 y = 1$. (b) Graph the region of (a).

122. If $|z| > 0$, prove that

$$\cosh(z + 1/z) = c_0 + c_1(z + 1/z) + c_2(z^2 + 1/z^2) + \cdots$$

where

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} \cos n\phi \cosh(2\cos\phi)\,d\phi$$

123. If $f(z)$ has simple zeros at $1-i$ and $1+i$, double poles at $-1+i$ and $-1-i$, but no other finite singularities, prove that the function must be given by

$$f(z) = \kappa \frac{z^2-2z+2}{(z^2+2z+2)^2}$$

where κ is an arbitrary constant.

124. Prove that for all z, $e^z \sin z = \sum_{n=1}^{\infty} \frac{2^{n/2} \sin(n\pi/4)}{n!} z^n$.

125. Show that $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$, justifying all steps. [*Hint.* Use Problem 23.]

126. Investigate the uniform convergence of the series $\sum_{n=1}^{\infty} \frac{z}{[1+(n-1)z][1+nz]}$.

[*Hint.* Resolve the nth term into partial fractions and show that the nth partial sum is $S_n(z) = 1 - \frac{1}{1+nz}$.]

Ans. Not uniformly convergent in any region which includes $z = 0$; uniformly convergent in a region $|z| \geqq \delta$, where δ is any positive number.

127. If $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges to S, prove that the rearranged series $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots = \frac{3}{2}S$. Explain.

[*Hint.* Take 1/2 of the first series and write it as $0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + \cdots$; then add term by term to the first series. Note that $S = \ln 2$, as shown in Problem 125.]

128. Prove that the *hypergeometric series*

$$1 + \frac{a \cdot b}{1 \cdot c} z + \frac{a(a+1)\,b(b+1)}{1 \cdot 2 \cdot c(c+1)} z^2 + \frac{a(a+1)(a+2)\,b(b+1)(b+2)}{1 \cdot 2 \cdot 3 \cdot c(c+1)(c+2)} z^3 + \cdots$$

(*a*) converges absolutely if $|z| < 1$, (*b*) diverges for $|z| > 1$, (*c*) converges absolutely for $z = 1$ if $\text{Re}\{a + b - c\} < 0$, (*d*) satisfies the differential equation $z(1-z)Y'' + \{c - (a+b+1)z\}Y' - abY = 0$.

129. Prove that for $|z| < 1$,

$$(\sin^{-1} z)^2 = z^2 + \frac{2}{3} \cdot \frac{z^4}{2} + \frac{2 \cdot 4}{3 \cdot 5} \cdot \frac{z^6}{3} + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} \cdot \frac{z^8}{4} + \cdots$$

130. Prove that $\sum_{n=1}^{\infty} \frac{1}{n^{1+i}}$ diverges.

131. Show that $\frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} - \frac{1}{4 \cdot 5} + \cdots = 2 \ln 2 - 1$

132. Locate and name all the singularities of $\frac{z^6 + 1}{(z-1)^3 (3z+2)^2} \sin\left(\frac{z^2}{z-3}\right)$.

133. By using only properties of infinite series, prove that

(*a*) $\left\{1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \cdots\right\}\left\{1 + b + \frac{b^2}{2!} + \frac{b^3}{3!} + \cdots\right\} = \left\{1 + (a+b) + \frac{(a+b)^2}{2!} + \cdots\right\}$

(*b*) $\left\{1 - \frac{a^2}{2!} + \frac{a^4}{4!} - \frac{a^6}{6!} + \cdots\right\}^2 + \left\{a - \frac{a^3}{3!} + \frac{a^5}{5!} - \frac{a^7}{7!} + \cdots\right\}^2 = 1$

134. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges for $|z| < R$ and $0 \leqq r < R$, prove that

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$$

135. Use Problem 134 to prove Cauchy's inequality (Page 118), namely

$$|f^{(n)}(0)| \leqq \frac{M \cdot n!}{r^n} \qquad n = 0, 1, 2, \ldots$$

136. If a function has six zeros of order 4, and four poles of orders 3, 4, 7 and 8, but no other singularities in the finite plane, prove that it has a pole of order 2 at $z = \infty$.

137. State whether each of the following functions are entire, meromorphic or neither.

(*a*) $z^2 e^{-z}$, (*b*) $\cot 2z$, (*c*) $(1 - \cos z)/z$, (*d*) $\cosh z^2$, (*e*) $z \sin(1/z)$, (*f*) $z + 1/z$, (*g*) $\sin\sqrt{z}/\sqrt{z}$, (*h*) $\sqrt{\sin z}$.

Ans. (*a*) entire, (*b*) meromorphic, (*c*) entire, (*d*) entire, (*e*) neither, (*f*) meromorphic, (*g*) entire, (*h*) neither

138. If $-\pi < \theta < \pi$, prove that

$$\ln(2 \cos \theta/2) = \cos\theta - \tfrac{1}{2}\cos 2\theta + \tfrac{1}{3}\cos 3\theta - \tfrac{1}{4}\cos 4\theta + \cdots$$

139. (*a*) Expand $1/\ln(1+z)$ in a Laurent series about $z=0$ and (*b*) determine the region of convergence.

Ans. (*a*) $\frac{1}{z}+\frac{z}{2}-\frac{z}{12}+\frac{z^2}{24}+\frac{89z^3}{720}+\cdots$ (*b*) $0<|z|<1$

140. If $S(z) = a_0 + a_1z + a_2z^2 + \cdots$, prove that

$$\frac{S(z)}{1-z} = a_0 + (a_0+a_1)z + (a_0+a_1+a_2)z^2 + \cdots$$

giving restrictions if any.

141. Show that the series

$$\frac{1}{1+|z|} - \frac{1}{2+|z|} + \frac{1}{3+|z|} - \frac{1}{4+|z|} + \cdots$$

(*a*) is not absolutely convergent but (*b*) is uniformly convergent for all values of z.

142. Prove that $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges at all points of $|z| \leqq 1$ except $z=1$.

143. Prove that the solution of $z = a + \zeta e^z$, which has the value a when $\zeta = 0$, is given by

$$z = a + \sum_{n=1}^{\infty} \frac{n^{n-1}e^{na}\zeta^n}{n!}$$

if $|\zeta| < |e^{-(a+1)}|$.

144. Find the sum of the series $1 + \cos\theta + \frac{\cos 2\theta}{2!} + \frac{\cos 3\theta}{3!} + \cdots$. *Ans.* $e^{\cos\theta}\cos(\sin\theta)$

145. Let $F(z)$ be analytic in the finite plane and suppose that $F(z)$ has period 2π, i.e. $F(z+2\pi) = F(z)$. Prove that

$$F(z) = \sum_{n=-\infty}^{\infty} \alpha_n e^{inz} \qquad \text{where} \qquad \alpha_n = \frac{1}{2\pi}\int_0^{2\pi} F(z)\, e^{-inz}\, dz$$

The series is called the *Fourier series* for $F(z)$.

146. Prove that the series

$$\sin\theta + \tfrac{1}{3}\sin 3\theta + \tfrac{1}{5}\sin 5\theta + \cdots$$

is equal to $\pi/4$ if $0<\theta<\pi$, and to $-\pi/4$ if $-\pi<\theta<0$.

147. Prove that $|z|=1$ is a natural boundary for the series $\sum_{n=0}^{\infty} 2^{-n} z^{3^n}$.

148. If $f(z)$ is analytic and not identically zero in the region $0 < |z-z_0| < R$, and if $\lim_{z\to z_0} f(z) = 0$, prove that there exists a positive integer n such that $f(z) = (z-z_0)^n g(z)$ where $g(z)$ is analytic at z_0 and different from zero.

149. If $f(z)$ is analytic in a deleted neighbourhood of z_0 and $\lim_{z\to z_0} |f(z)| = \infty$, prove that $z = z_0$ is a pole of $f(z)$.

150. Explain why Problem 149 does not hold for $f(x) = e^{1/x^2}$ where x is real.

151. (*a*) Show that the function $f(z) = e^{1/z}$ can assume any value except zero. (*b*) Discuss the relationship of the result of (*a*) to the Casorati-Weierstrass theorem and Picard's theorem.

152. (*a*) Determine whether the function $g(z) = z^2 - 3z + 2$ can assume any complex value. (*b*) Is there any relationship of the result in (*a*) to the theorems of Casorati-Weierstrass and Picard? Explain.

153. Prove the Casorati-Weierstrass theorem stated on Page 145. [*Hint.* Use the fact that if $z=a$ is an essential singularity of $f(z)$, then it is also an essential singularity of $1/\{f(z)-A\}$.]

154. (*a*) Prove that along any ray through $z=0$, $|z+e^z| \to \infty$.
(*b*) Does the result in (*a*) contradict the Casorati-Weierstrass theorem?

155. (*a*) Prove that an entire function $f(z)$ can assume any value whatsoever, with perhaps one exception.
(*b*) Illustrate the result of (*a*) by considering $f(z) = e^z$ and stating the exception in this case.
(*c*) What is the relationship of the result to the Casorati-Weierstrass and Picard theorems?

156. Prove that every entire function has a singularity at infinity. What type of singularity must this be? Justify your answer.

157. Prove that:

(*a*) $$\frac{\ln(1+z)}{1+z} = z - (1+\tfrac{1}{2})z^2 + (1+\tfrac{1}{2}+\tfrac{1}{3})z^3 - \cdots, \qquad |z|<1$$

(*b*) $$\{\ln(1+z)\}^2 = z^2 - (1+\tfrac{1}{2})\frac{2z^3}{3} + (1+\tfrac{1}{2}+\tfrac{1}{3})\frac{2z^4}{4} - \cdots, \qquad |z|<1$$

158. Find the sum of the following series if $|a| < 1$:

$$(a)\ \sum_{n=1}^{\infty} na^n \sin n\theta, \qquad (b)\ \sum_{n=1}^{\infty} n^2a^n \sin n\theta$$

159. Show that $$e^{\sin z} = 1 + z + \frac{z^2}{2} - \frac{z^4}{8} - \frac{z^5}{15} + \cdots, \qquad |z| < \infty.$$

160. (*a*) Show that $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges for $|z| \leqq 1$.
(*b*) Show that the function $F(z)$, defined as the collection of all possible analytic continuations of the series in (*a*), has a singular point at $z = 1$.
(*c*) Reconcile the results of (*a*) and (*b*).

161. Let $\displaystyle\sum_{n=1}^{\infty} a_n z^n$ converge inside a circle of convergence of radius R. There is a theorem which states that the function $F(z)$ defined by the collection of all possible continuations of this series, has at least one singular point on the circle of convergence. (*a*) Illustrate the theorem by several examples. (*b*) Can you prove the theorem?

162. Show that

$$u(r,\theta) = \frac{R^2-r^2}{2\pi}\int_0^{2\pi} \frac{U(\phi)\,d\phi}{R^2 - 2rR\cos(\theta-\phi) + r^2}$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \{a_n \cos n\theta + b_n \sin n\theta\}$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi} U(\phi)\cos n\phi\,d\phi, \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} U(\phi)\sin n\phi\,d\phi$$

163. Let $\dfrac{z}{e^z-1} = 1 + B_1 z + \dfrac{B_2z^2}{2!} + \dfrac{B_3z^3}{3!} + \cdots$. (*a*) Show that the numbers B_n, called the *Bernoulli numbers*, satisfy the recursion formula $(B+1)^n = B^n$ where B^k is formally replaced by B_k after expanding. (*b*) Using (*a*) or otherwise, determine $B_1, \ldots, B_6$.
Ans. (*b*) $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$, $B_5 = 0$, $B_6 = \frac{1}{42}$

164. (*a*) Prove that $\dfrac{z}{e^z-1} = \dfrac{z}{2}\left(\coth\dfrac{z}{2} - 1\right)$. (*b*) Use Problem 163 and part (*a*) to show that $B_{2k+1} = 0$ if $k = 1, 2, 3, \ldots$.

165. Derive the series expansions:

(*a*) $$\coth z = \frac{1}{z} + \frac{z}{3} - \frac{z^3}{45} + \cdots + \frac{B_{2n}(2z)^{2n}}{(2n)!\,z} + \cdots, \qquad |z| < \pi$$

(*b*) $$\cot z = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} + \cdots(-1)^n\frac{B_{2n}(2z)^{2n}}{(2n)!\,z} + \cdots, \qquad |z| < \pi$$

(*c*) $$\tan z = z + \frac{z^3}{3} + \frac{2z^5}{15} + \cdots(-1)^{n-1}\frac{2(2^{2n}-1)B_{2n}(2z)^{2n-1}}{(2n)!}, \qquad |z| < \pi/2$$

(*d*) $$\csc z = \frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + \cdots(-1)^{n-1}\frac{2(2^{2n-1}-1)B_{2n}z^{2n-1}}{(2n)!} + \cdots, \qquad |z| < \pi$$

[*Hint.* For (*a*) use Problem 164; for (*b*) replace z by iz in (*a*); for (*c*) use $\tan z = \cot z - 2\cot 2z$; for (*d*) use $\csc z = \cot z + \tan z/2$.]

Chapter 7

The Residue Theorem
Evaluation of Integrals and Series

RESIDUES

Let $f(z)$ be single-valued and analytic inside and on a circle C except at the point $z=a$ chosen as the centre of C. Then, as we have seen in Chapter 6, $f(z)$ has a Laurent series about $z=a$ given by

$$\begin{aligned} f(z) &= \sum_{n=-\infty}^{\infty} a_n(z-a)^n \\ &= a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots \end{aligned} \tag{1}$$

where
$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz \qquad n = 0, \pm1, \pm2, \ldots \tag{2}$$

In the special case $n=-1$, we have from (2)

$$\oint_C f(z)\,dz = 2\pi i\, a_{-1} \tag{3}$$

Formally we can obtain (3) from (1) by integrating term by term and using the results (Problems 21 and 22, Chapter 4)

$$\oint_C \frac{dz}{(z-a)^p} = \begin{cases} 2\pi i & p=1 \\ 0 & p = \text{integer} \neq 1 \end{cases} \tag{4}$$

Because of the fact that (3) involves only the coefficient a_{-1} in (1), we call a_{-1} the *residue* of $f(z)$ at $z=a$.

CALCULATION OF RESIDUES

To obtain the residue of a function $f(z)$ at $z=a$, it may appear from (1) that the Laurent expansion of $f(z)$ about $z=a$ must be obtained. However, in the case where $z=a$ is a pole of order k there is a simple formula for a_{-1} given by

$$a_{-1} = \lim_{z\to a} \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\{(z-a)^k f(z)\} \tag{5}$$

If $k=1$ (simple pole) the result is especially simple and is given by

$$a_{-1} = \lim_{z\to a}(z-a)f(z) \tag{6}$$

which is a special case of (5) with $k=1$ if we define $0!=1$.

Example 1: If $f(z) = \dfrac{z}{(z-1)(z+1)^2}$, then $z=1$ and $z=-1$ are poles of orders one and two respectively. We have, using (6) and (5) with $k=2$,

Residue at $z=1$ is $\displaystyle\lim_{z\to 1}(z-1)\left\{\frac{z}{(z-1)(z+1)^2}\right\} = \frac{1}{4}$

Residue at $z=-1$ is $\displaystyle\lim_{z\to -1}\frac{1}{1!}\frac{d}{dz}\left\{(z+1)^2\left(\frac{z}{(z-1)(z+1)^2}\right)\right\} = -\frac{1}{4}$

If $z=a$ is an essential singularity, the residue can sometimes be found by using known series expansions.

Example 2: If $f(z) = e^{-1/z}$, then $z = 0$ is an essential singularity and from the known expansion for e^u with $u = -1/z$ we find

$$e^{-1/z} = 1 - \frac{1}{z} + \frac{1}{2!\,z^2} - \frac{1}{3!\,z^3} + \cdots$$

from which we see that the residue at $z = 0$ is the coefficient of $1/z$ and equals -1.

THE RESIDUE THEOREM

Let $f(z)$ be single-valued and analytic inside and on a simple closed curve C except at the singularities $a, b, c, \ldots$ inside C which have residues given by $a_{-1}, b_{-1}, c_{-1}, \ldots$ [see Fig. 7-1]. Then the *residue theorem* states that

$$\oint_C f(z)\,dz = 2\pi i(a_{-1} + b_{-1} + c_{-1} + \cdots) \qquad (7)$$

i.e. the integral of $f(z)$ around C is $2\pi i$ times the sum of the residues of $f(z)$ at the singularities enclosed by C. Note that (7) is a generalization of (3). Cauchy's theorem and integral formulae are special cases of this theorem (see Problem 75).

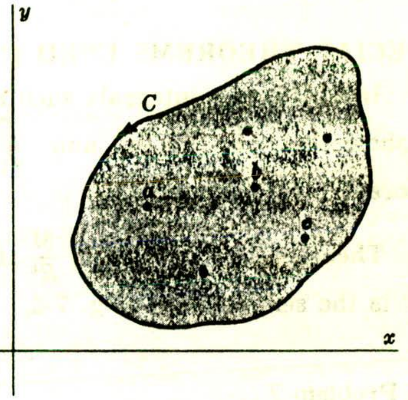

Fig. 7-1

EVALUATION OF DEFINITE INTEGRALS

The evaluation of definite integrals is often achieved by using the residue theorem together with a suitable function $f(z)$ and a suitable closed path or contour C, the choice of which may require great ingenuity. The following types are most common in practice.

1. $\int_{-\infty}^{\infty} F(x)\,dx$, $F(x)$ is a rational function.

   Consider $\oint_C F(z)\,dz$ along a contour C consisting of the line along the x axis from $-R$ to $+R$ and the semicircle Γ above the x axis having this line as diameter [Fig. 7-2]. Then let $R \to \infty$. If $F(x)$ is an even function this can be used to evaluate $\int_0^{\infty} F(x)\,dx$. See Problems 7-10.

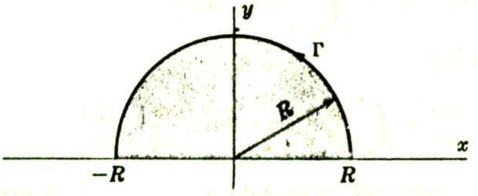

Fig. 7-2

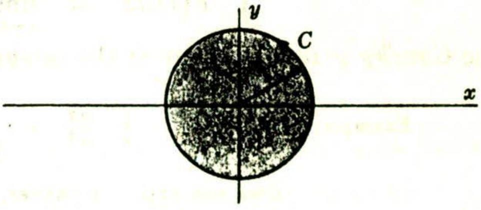

Fig. 7-3

2. $\int_0^{2\pi} G(\sin\theta, \cos\theta)\,d\theta$, $G(\sin\theta, \cos\theta)$ is a rational function of $\sin\theta$ and $\cos\theta$.

   Let $z = e^{i\theta}$. Then $\sin\theta = \dfrac{z - z^{-1}}{2i}$, $\cos\theta = \dfrac{z + z^{-1}}{2}$ and $dz = ie^{i\theta}\,d\theta$ or $d\theta = dz/iz$. The given integral is equivalent to $\oint_C F(z)\,dz$ where C is the unit circle with centre at the origin [Fig. 7-3]. See Problems 11-14.

3. $\displaystyle\int_{-\infty}^{\infty} F(x) \begin{Bmatrix} \cos mx \\ \sin mx \end{Bmatrix} dx$, $F(x)$ is a rational function.

Here we consider $\displaystyle\oint_C F(z)\, e^{imz}\, dz$ where C is the same contour as that in Type 1. See Problems 15-17, and 37.

4. Miscellaneous integrals involving particular contours. See Problems 18-23.

SPECIAL THEOREMS USED IN EVALUATING INTEGRALS

In evaluating integrals such as those of Types 1 and 3 above, it is often necessary to show that $\displaystyle\int_\Gamma F(z)\, dz$ and $\displaystyle\int_\Gamma e^{imz} F(z)\, dz$ approach zero as $R \to \infty$. The following theorems are fundamental.

Theorem 1. If $|F(z)| \leqq \dfrac{M}{R^k}$ for $z = Re^{i\theta}$, where $k > 1$ and M are constants, then if Γ is the semicircle of Fig. 7-2,

$$\lim_{R\to\infty} \int_\Gamma F(z)\, dz \;=\; 0$$

See Problem 7.

Theorem 2. If $|F(z)| \leqq \dfrac{M}{R^k}$ for $z = Re^{i\theta}$, where $k > 0$ and M are constants, then if Γ is the semicircle of Fig. 7-2,

$$\lim_{R\to\infty} \int_\Gamma e^{imz} F(z)\, dz \;=\; 0$$

See Problem 15.

THE CAUCHY PRINCIPAL VALUE OF INTEGRALS

If $F(x)$ is continuous in $a \leqq x \leqq b$ except at a point x_0 such that $a < x_0 < b$, then if ϵ_1 and ϵ_2 are positive we define

$$\int_a^b F(x)\, dx \;=\; \lim_{\substack{\epsilon_1\to 0\\ \epsilon_2\to 0}} \left\{ \int_a^{x_0-\epsilon_1} F(x)\, dx + \int_{x_0+\epsilon_2}^b F(x)\, dx \right\}$$

In some cases the above limit does not exist for $\epsilon_1 \neq \epsilon_2$ but does exist if we take $\epsilon_1 = \epsilon_2 = \epsilon$. In such case we call

$$\int_a^b F(x)\, dx \;=\; \lim_{\epsilon\to 0} \left\{ \int_a^{x_0-\epsilon} F(x)\, dx + \int_{x_0+\epsilon}^b F(x)\, dx \right\}$$

the *Cauchy principal value* of the integral on the left.

Example: $\displaystyle\int_{-1}^1 \frac{dx}{x^3} \;=\; \lim_{\substack{\epsilon_1\to 0\\ \epsilon_2\to 0}} \left\{ \int_{-1}^{-\epsilon_1} \frac{dx}{x^3} + \int_{\epsilon_2}^1 \frac{dx}{x^3} \right\} \;=\; \lim_{\substack{\epsilon_1\to 0\\ \epsilon_2\to 0}} \left\{ \frac{1}{2\epsilon_2^2} - \frac{1}{2\epsilon_1^2} \right\}$

does not exist. However, the Cauchy principal value with $\epsilon_1 = \epsilon_2 = \epsilon$ does exist and equals zero.

DIFFERENTIATION UNDER THE INTEGRAL SIGN. LEIBNITZ'S RULE

A useful method for evaluating integrals employs *Leibnitz's rule* for differentiation under the integral sign. This rule states that

$$\frac{d}{d\alpha} \int_a^b F(x, \alpha)\, dx \;=\; \int_a^b \frac{\partial F}{\partial \alpha}\, dx$$

The rule is valid if a and b are constants, α is a real parameter such that $\alpha_1 \leqq \alpha \leqq \alpha_2$ where α_1 and α_2 are constants, and $F(x, \alpha)$ is continuous and has a continuous partial derivative with respect to α for $a \leqq x \leqq b$, $\alpha_1 \leqq \alpha \leqq \alpha_2$. It can be extended to cases where the limits a and b are infinite or dependent on α.

SUMMATION OF SERIES

The residue theorem can often be used to sum various types of series. The following results are valid under very mild restrictions on $f(z)$ which are generally satisfied whenever the series converge. See Problems 24-32, and 38.

1. $\sum_{-\infty}^{\infty} f(n) \;=\; -\{\text{sum of residues of } \pi \cot \pi z\, f(z) \text{ at all the poles of } f(z)\}$
2. $\sum_{-\infty}^{\infty} (-1)^n f(n) \;=\; -\{\text{sum of residues of } \pi \csc \pi z\, f(z) \text{ at all the poles of } f(z)\}$
3. $\sum_{-\infty}^{\infty} f\left(\frac{2n+1}{2}\right) \;=\; \{\text{sum of residues of } \pi \tan \pi z\, f(z) \text{ at all the poles of } f(z)\}$
4. $\sum_{-\infty}^{\infty} (-1)^n f\left(\frac{2n+1}{2}\right) \;=\; \{\text{sum of residues of } \pi \sec \pi z\, f(z) \text{ at all the poles of } f(z)\}$

MITTAG-LEFFLER'S EXPANSION THEOREM

1. Suppose that the only singularities of $f(z)$ in the finite z plane are the simple poles $a_1, a_2, a_3, \ldots$ arranged in order of increasing absolute value.
2. Let the residues of $f(z)$ at $a_1, a_2, a_3, \ldots$ be $b_1, b_2, b_3, \ldots$.
3. Let C_N be circles of radius R_N which do not pass through any poles and on which $|f(z)| < M$, where M is independent of N and $R_N \to \infty$ as $N \to \infty$.

Then *Mittag-Leffler's expansion theorem* states that

$$f(z) \;=\; f(0) + \sum_{n=1}^{\infty} b_n \left\{ \frac{1}{z - a_n} + \frac{1}{a_n} \right\}$$

SOME SPECIAL EXPANSIONS

1. $\csc z \;=\; \frac{1}{z} - 2z\left(\frac{1}{z^2 - \pi^2} - \frac{1}{z^2 - 4\pi^2} + \frac{1}{z^2 - 9\pi^2} - \cdots\right)$
2. $\sec z \;=\; \pi\left(\frac{1}{(\pi/2)^2 - z^2} - \frac{3}{(3\pi/2)^2 - z^2} + \frac{5}{(5\pi/2)^2 - z^2} - \cdots\right)$
3. $\tan z \;=\; 2z\left(\frac{1}{(\pi/2)^2 - z^2} + \frac{1}{(3\pi/2)^2 - z^2} + \frac{1}{(5\pi/2)^2 - z^2} + \cdots\right)$
4. $\cot z \;=\; \frac{1}{z} + 2z\left(\frac{1}{z^2 - \pi^2} + \frac{1}{z^2 - 4\pi^2} + \frac{1}{z^2 - 9\pi^2} + \cdots\right)$
5. $\operatorname{csch} z \;=\; \frac{1}{z} - 2z\left(\frac{1}{z^2 + \pi^2} - \frac{1}{z^2 + 4\pi^2} + \frac{1}{z^2 + 9\pi^2} - \cdots\right)$
6. $\operatorname{sech} z \;=\; \pi\left(\frac{1}{(\pi/2)^2 + z^2} - \frac{3}{(3\pi/2)^2 + z^2} + \frac{5}{(5\pi/2)^2 + z^2} - \cdots\right)$
7. $\tanh z \;=\; 2z\left(\frac{1}{z^2 + (\pi/2)^2} + \frac{1}{z^2 + (3\pi/2)^2} + \frac{1}{z^2 + (5\pi/2)^2} + \cdots\right)$
8. $\coth z \;=\; \frac{1}{z} + 2z\left(\frac{1}{z^2 + \pi^2} + \frac{1}{z^2 + 4\pi^2} + \frac{1}{z^2 + 9\pi^2} + \cdots\right)$

Solved Problems

RESIDUES AND THE RESIDUE THEOREM

1. Let $f(z)$ be analytic inside and on a simple closed curve C except at point a inside C.

(a) Prove that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n \qquad \text{where} \qquad a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz, \quad n = 0, \pm 1, \pm 2, \ldots$$

i.e. $f(z)$ can be expanded into a converging Laurent series about $z = a$.

(b) Prove that

$$\oint_C f(z)\,dz = 2\pi i\, a_{-1}$$

(a) This follows from Problem 25 of Chapter 6.

(b) If we let $n = -1$ in the result of (a), we find

$$a_{-1} = \frac{1}{2\pi i}\oint_C f(z)\,dz, \qquad \text{i.e.} \qquad \oint_C f(z)\,dz = 2\pi i\, a_{-1}$$

We call a_{-1} the *residue* of $f(z)$ at $z = a$.

2. Prove the *residue theorem*. If $f(z)$ is analytic inside and on a simple closed curve C except at a finite number of points $a, b, c, \ldots$ inside C at which the residues are $a_{-1}, b_{-1}, c_{-1}, \ldots$ respectively, then

$$\oint_C f(z)\,dz = 2\pi i(a_{-1} + b_{-1} + c_{-1} + \cdots)$$

i.e. $2\pi i$ times the sum of the residues at all singularities enclosed by C.

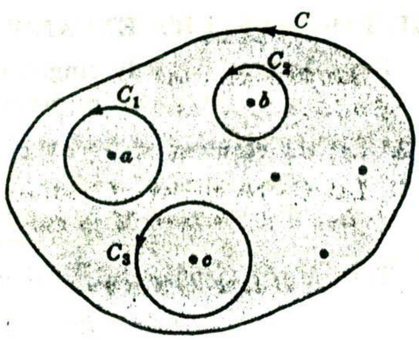

Fig. 7-4

With centres at $a, b, c, \ldots$ respectively construct circles $C_1, C_2, C_3, \ldots$ which lie entirely inside C as shown in Fig. 7-4. This can be done since $a, b, c, \ldots$ are interior points. By Theorem 5, Page 97, we have

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \oint_{C_3} f(z)\,dz + \cdots \qquad (1)$$

But by Problem 1,

$$\oint_{C_1} f(z)\,dz = 2\pi i\, a_{-1}, \qquad \oint_{C_2} f(z)\,dz = 2\pi i\, b_{-1}, \qquad \oint_{C_3} f(z)\,dz = 2\pi i\, c_{-1}, \qquad \ldots \qquad (2)$$

Then from (1) and (2) we have, as required,

$$\oint_C f(z)\,dz = 2\pi i(a_{-1} + b_{-1} + c_{-1} + \cdots) = 2\pi i\,(\text{sum of residues})$$

The proof given here establishes the residue theorem for simply-connected regions containing a finite number of singularities of $f(z)$. It can be extended to regions with infinitely many isolated singularities and to multiply-connected regions (see Problems 96 and 97).

3. Let $f(z)$ be analytic inside and on a simple closed curve C except at a pole a of order m inside C. Prove that the residue of $f(z)$ at a is given by

$$a_{-1} = \lim_{z\to a} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}$$

Method 1. If $f(z)$ has a pole a of order m, then the Laurent series of $f(z)$ is

$$f(z) = \frac{a_{-m}}{(z-a)^m} + \frac{a_{-m+1}}{(z-a)^{m-1}} + \cdots + \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots \qquad (1)$$

Then multiplying both sides by $(z-a)^m$, we have

$$(z-a)^m f(z) \quad = \quad a_{-m} + a_{-m+1}(z-a) + \cdots + a_{-1}(z-a)^{m-1} + a_0(z-a)^m + \cdots \qquad (2)$$

This represents the Taylor series about $z=a$ of the analytic function on the left. Differentiating both sides $m-1$ times with respect to z, we have

$$\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\} \quad = \quad (m-1)!\,a_{-1} + m(m-1)\cdots 2a_0(z-a) + \cdots$$

Thus on letting $z \to a$,

$$\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\} \quad = \quad (m-1)!\,a_{-1}$$

from which the required result follows.

Method 2. The required result also follows directly from Taylor's theorem on noting that the coefficient of $(z-a)^{m-1}$ in the expansion (2) is

$$a_{-1} \quad = \quad \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}\bigg|_{z=a}$$

Method 3. See Problem 28, Chapter 5, Page 132.

4. Find the residues of (a) $f(z) = \dfrac{z^2-2z}{(z+1)^2(z^2+4)}$ and (b) $f(z) = e^z \csc^2 z$ at all its poles in the finite plane.

(a) $f(z)$ has a double pole at $z=-1$ and simple poles at $z=\pm 2i$.

Method 1.

Residue at $z=-1$ is

$$\lim_{z\to -1}\frac{1}{1!}\frac{d}{dz}\left\{(z+1)^2\cdot\frac{z^2-2z}{(z+1)^2(z^2+4)}\right\} \quad = \quad \lim_{z\to -1}\frac{(z^2+4)(2z-2)-(z^2-2z)(2z)}{(z^2+4)^2} \quad = \quad -\frac{14}{25}$$

Residue at $z=2i$ is

$$\lim_{z\to 2i}\left\{(z-2i)\cdot\frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}\right\} \quad = \quad \frac{-4-4i}{(2i+1)^2(4i)} \quad = \quad \frac{7+i}{25}$$

Residue at $z=-2i$ is

$$\lim_{z\to -2i}\left\{(z+2i)\cdot\frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}\right\} \quad = \quad \frac{-4+4i}{(-2i+1)^2(-4i)} \quad = \quad \frac{7-i}{25}$$

Method 2.

Residue at $z=2i$ is

$$\lim_{z\to 2i}\left\{\frac{(z-2i)(z^2-2z)}{(z+1)^2(z^2+4)}\right\} \quad = \quad \left\{\lim_{z\to 2i}\frac{z^2-2z}{(z+1)^2}\right\}\left\{\lim_{z\to 2i}\frac{z-2i}{z^2+4}\right\}$$

$$= \quad \frac{-4-4i}{(2i+1)^2}\cdot\lim_{z\to 2i}\frac{1}{2z} \quad = \quad \frac{-4-4i}{(2i+1)^2}\cdot\frac{1}{4i} \quad = \quad \frac{7+i}{25}$$

using L'Hospital's rule. In a similar manner, or by replacing i by $-i$ in the result, we can obtain the residue at $z=-2i$.

(b) $f(z) = e^z\csc^2 z = \dfrac{e^z}{\sin^2 z}$ has double poles at $z = 0, \pm\pi, \pm 2\pi, \ldots$, i.e. $z = m\pi$ where $m = 0, \pm 1, \pm 2, \ldots$.

Method 1.

Residue at $z = m\pi$ is

$$\lim_{z\to m\pi}\frac{1}{1!}\frac{d}{dz}\left\{(z-m\pi)^2\frac{e^z}{\sin^2 z}\right\}$$

$$= \quad \lim_{z\to m\pi}\frac{e^z[(z-m\pi)^2\sin z + 2(z-m\pi)\sin z - 2(z-m\pi)^2\cos z]}{\sin^3 z}$$

Letting $z - m\pi = u$ or $z = u + m\pi$, this limit can be written

$$\lim_{u\to 0} e^{u+m\pi}\left\{\frac{u^2 \sin u + 2u \sin u - 2u^2 \cos u}{\sin^3 u}\right\}$$

$$= e^{m\pi}\left\{\lim_{u\to 0} \frac{u^2 \sin u + 2u \sin u - 2u^2 \cos u}{\sin^3 u}\right\}$$

The limit in braces can be obtained using L'Hospital's rule. However, it is easier to first note that $\lim_{u\to 0} \frac{u^3}{\sin^3 u} = \lim_{u\to 0}\left(\frac{u}{\sin u}\right)^3 = 1$ and thus write the limit as

$$e^{m\pi} \lim_{u\to 0}\left(\frac{u^2 \sin u + 2u \sin u - 2u^2 \cos u}{u^3} \cdot \frac{u^3}{\sin^3 u}\right)$$

$$= e^{m\pi} \lim_{u\to 0} \frac{u^2 \sin u + 2u \sin u - 2u^2 \cos u}{u^3} = e^{m\pi}$$

using L'Hospital's rule several times. In evaluating this limit we can instead use the series expansions $\sin u = u - u^3/3! + \cdots$, $\cos u = 1 - u^2/2! + \cdots$.

Method 2 (using Laurent's series).

In this method we expand $f(z) = e^z \csc^2 z$ in a Laurent series about $z = m\pi$ and obtain the coefficient of $1/(z - m\pi)$ as the required residue. To make the calculation easier let $z = u + m\pi$. Then the function to be expanded in a Laurent series about $u = 0$ is $e^{m\pi+u}\csc^2(m\pi + u) = e^{m\pi}e^u \csc^2 u$. Using the Maclaurin expansions for e^u and $\sin u$, we find using long division

$$e^{m\pi}e^u \csc^2 u = \frac{e^{m\pi}\left(1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots\right)}{\left(u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots\right)^2} = \frac{e^{m\pi}\left(1 + u + \frac{u^2}{2} + \cdots\right)}{u^2\left(1 - \frac{u^2}{6} + \frac{u^4}{120} - \cdots\right)^2}$$

$$= \frac{e^{m\pi}\left(1 + u + \frac{u^2}{2!} + \cdots\right)}{u^2\left(1 - \frac{u^2}{3} + \frac{2u^4}{45} + \cdots\right)} = e^{m\pi}\left(\frac{1}{u^2} + \frac{1}{u} + \frac{5}{6} + \frac{u}{3} + \cdots\right)$$

and so the residue is $e^{m\pi}$.

5. Find the residue of $F(z) = \dfrac{\cot z \coth z}{z^3}$ at $z = 0$.

We have as in Method 2 of Problem 4(*b*),

$$F(z) = \frac{\cos z \cosh z}{z^3 \sin z \sinh z} = \frac{\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right)\left(1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\right)}{z^3\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)\left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)}$$

$$= \frac{\left(1 - \frac{z^4}{6} + \cdots\right)}{z^5\left(1 - \frac{z^4}{90} + \cdots\right)} = \frac{1}{z^5}\left(1 - \frac{7z^4}{45} + \cdots\right)$$

and so the residue (coefficient of $1/z$) is $-7/45$.

Another method. The result can also be obtained by finding

$$\lim_{z\to 0} \frac{1}{4!}\frac{d^4}{dz^4}\left\{z^5 \frac{\cos z \cosh z}{z^3 \sin z \sinh z}\right\}$$

but this method is much more laborious than that given above.

6. Evaluate $\dfrac{1}{2\pi i}\oint_C \dfrac{e^{zt}}{z^2(z^2 + 2z + 2)}\,dz$ around the circle C with equation $|z| = 3$.

The integrand $\dfrac{e^{zt}}{z^2(z^2 + 2z + 2)}$ has a double pole at $z = 0$ and two simple poles at $z = -1 \pm i$ [roots of $z^2 + 2z + 2 = 0$]. All these poles are inside C.

Residue at $z = 0$ is

$$\lim_{z\to 0} \frac{1}{1!}\frac{d}{dz}\left\{z^2 \frac{e^{zt}}{z^2(z^2+2z+2)}\right\} = \lim_{z\to 0} \frac{(z^2+2z+2)(te^{zt}) - (e^{zt})(2z+2)}{(z^2+2z+2)^2} = \frac{t-1}{2}$$

Residue at $z = -1+i$ is

$$\lim_{z\to -1+i}\left\{[z-(-1+i)]\frac{e^{zt}}{z^2(z^2+2z+2)}\right\} = \lim_{z\to -1+i}\left\{\frac{e^{zt}}{z^2}\right\}\lim_{z\to -1+i}\left\{\frac{z+1-i}{z^2+2z+2}\right\}$$

$$= \frac{e^{(-1+i)t}}{(-1+i)^2}\cdot\frac{1}{2i} = \frac{e^{(-1+i)t}}{4}$$

Residue at $z = -1-i$ is

$$\lim_{z\to -1-i}\left\{[z-(-1-i)]\frac{e^{zt}}{z^2(z^2+2z+2)}\right\} = \frac{e^{(-1-i)t}}{4}$$

Then by the residue theorem

$$\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz = 2\pi i\,(\text{sum of residues})$$

$$= 2\pi i\left\{\frac{t-1}{2} + \frac{e^{(-1+i)t}}{4} + \frac{e^{(-1-i)t}}{4}\right\}$$

$$= 2\pi i\left\{\frac{t-1}{2} + \frac{1}{2}e^{-t}\cos t\right\}$$

i.e.,

$$\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz = \frac{t-1}{2} + \frac{1}{2}e^{-t}\cos t$$

DEFINITE INTEGRALS OF THE TYPE $\int_{-\infty}^{\infty} F(x)\,dx$

7. If $|F(z)| \leqq M/R^k$ for $z = Re^{i\theta}$ where $k > 1$ and M are constants, prove that $\lim_{R\to\infty}\int_\Gamma F(z)\,dz = 0$ where Γ is the semicircular arc of radius R shown in Fig. 7-5.

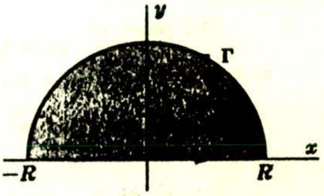

Fig. 7-5

By Property 5, Page 93, we have

$$\left|\int_\Gamma F(z)\,dz\right| \leqq \frac{M}{R^k}\cdot\pi R = \frac{\pi M}{R^{k-1}}$$

since the length of arc $L = \pi R$. Then

$$\lim_{R\to\infty}\left|\int_\Gamma F(z)\,dz\right| = 0 \quad \text{and so} \quad \lim_{R\to\infty}\int_\Gamma F(z)\,dz = 0$$

8. Show that for $z = Re^{i\theta}$, $|f(z)| \leqq \dfrac{M}{R^k}$, $k > 1$ if $f(z) = \dfrac{1}{z^6+1}$.

If $z = Re^{i\theta}$, $|f(z)| = \left|\dfrac{1}{R^6e^{6i\theta}+1}\right| \leqq \dfrac{1}{|R^6e^{6i\theta}|-1} = \dfrac{1}{R^6-1} \leqq \dfrac{2}{R^6}$ if R is large enough (say $R > 2$, for example) so that $M = 2$, $k = 6$.

Note that we have made use of the inequality $|z_1+z_2| \geqq |z_1| - |z_2|$ with $z_1 = R^6e^{6i\theta}$ and $z_2 = 1$.

9. Evaluate $\displaystyle\int_0^\infty \frac{dx}{x^6+1}$.

Consider $\displaystyle\oint_C \frac{dz}{z^6+1}$, where C is the closed contour of Fig. 7-5 consisting of the line from $-R$ to R and the semicircle Γ, traversed in the positive (counterclockwise) sense.

Since $z^6+1 = 0$ when $z = e^{\pi i/6}, e^{3\pi i/6}, e^{5\pi i/6}, e^{7\pi i/6}, e^{9\pi i/6}, e^{11\pi i/6}$, these are simple poles of $1/(z^6+1)$. Only the poles $e^{\pi i/6}$, $e^{3\pi i/6}$ and $e^{5\pi i/6}$ lie within C. Then using L'Hospital's rule,

$$\text{Residue at } e^{\pi i/6} = \lim_{z\to e^{\pi i/6}} \left\{(z-e^{\pi i/6})\frac{1}{z^6+1}\right\} = \lim_{z\to e^{\pi i/6}} \frac{1}{6z^5} = \frac{1}{6}e^{-5\pi i/6}$$

$$\text{Residue at } e^{3\pi i/6} = \lim_{z\to e^{3\pi i/6}} \left\{(z-e^{3\pi i/6})\frac{1}{z^6+1}\right\} = \lim_{z\to e^{3\pi i/6}} \frac{1}{6z^5} = \frac{1}{6}e^{-5\pi i/2}$$

$$\text{Residue at } e^{5\pi i/6} = \lim_{z\to e^{5\pi i/6}} \left\{(z-e^{5\pi i/6})\frac{1}{z^6+1}\right\} = \lim_{z\to e^{5\pi i/6}} \frac{1}{6z^5} = \frac{1}{6}e^{-25\pi i/6}$$

Thus
$$\oint_C \frac{dz}{z^6+1} = 2\pi i\{\tfrac{1}{6}e^{-5\pi i/6} + \tfrac{1}{6}e^{-5\pi i/2} + \tfrac{1}{6}e^{-25\pi i/6}\} = \frac{2\pi}{3}$$

i.e.,
$$\int_{-R}^{R} \frac{dx}{x^6+1} + \int_\Gamma \frac{dz}{z^6+1} = \frac{2\pi}{3} \tag{1}$$

Taking the limit of both sides of (1) as $R\to\infty$ and using Problems 7 and 8, we have

$$\lim_{R\to\infty}\int_{-R}^{R} \frac{dx}{x^6+1} = \int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3} \tag{2}$$

Since $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^6+1} = 2\int_0^\infty \frac{dx}{x^6+1}$, the required integral has the value $\pi/3$.

10. Show that $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)^2(x^2+2x+2)} = \frac{7\pi}{50}$.

The poles of $\dfrac{z^2}{(z^2+1)^2(z^2+2z+2)}$ enclosed by the contour C of Fig. 7-5 are $z=i$ of order 2 and $z=-1+i$ of order 1.

Residue at $z=i$ is $\displaystyle\lim_{z\to i}\frac{d}{dz}\left\{(z-i)^2\frac{z^2}{(z+i)^2(z-i)^2(z^2+2z+2)}\right\} = \frac{9i-12}{100}$.

Residue at $z=-1+i$ is $\displaystyle\lim_{z\to -1+i}(z+1-i)\frac{z^2}{(z^2+1)^2(z+1-i)(z+1+i)} = \frac{3-4i}{25}$.

Then
$$\oint_C \frac{z^2\,dz}{(z^2+1)^2(z^2+2z+2)} = 2\pi i\left\{\frac{9i-12}{100}+\frac{3-4i}{25}\right\} = \frac{7\pi}{50}$$

or
$$\int_{-R}^{R} \frac{x^2\,dx}{(x^2+1)^2(x^2+2x+2)} + \int_\Gamma \frac{z^2\,dz}{(z^2+1)^2(z^2+2z+2)} = \frac{7\pi}{50}$$

Taking the limit as $R\to\infty$ and noting that the second integral approaches zero by Problem 7, we obtain the required result.

DEFINITE INTEGRALS OF THE TYPE $\int_0^{2\pi} G(\sin\theta,\cos\theta)\,d\theta$

11. Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta}$.

Let $z=e^{i\theta}$. Then $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i} = \dfrac{z-z^{-1}}{2i}$, $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2} = \dfrac{z+z^{-1}}{2}$, $dz = iz\,d\theta$ so that

$$\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta} = \oint_C \frac{dz/iz}{3-2(z+z^{-1})/2+(z-z^{-1})/2i} = \oint_C \frac{2\,dz}{(1-2i)z^2+6iz-1-2i}$$

where C is the circle of unit radius with centre at the origin (Fig. 7-6).

The poles of $\dfrac{2}{(1-2i)z^2+6iz-1-2i}$ are the simple poles

$$z = \frac{-6i \pm \sqrt{(6i)^2 - 4(1-2i)(-1-2i)}}{2(1-2i)}$$

$$= \frac{-6i \pm 4i}{2(1-2i)} = 2-i,\ (2-i)/5$$

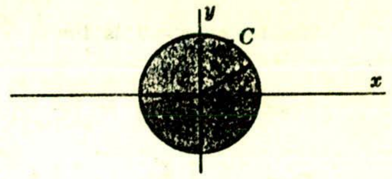

Fig. 7-6

Only $(2-i)/5$ lies inside C.

$$\text{Residue at } (2-i)/5 = \lim_{z\to(2-i)/5} \{z-(2-i)/5\}\left\{\frac{2}{(1-2i)z^2+6iz-1-2i}\right\}$$

$$= \lim_{z\to(2-i)/5} \frac{2}{2(1-2i)z+6i} = \frac{1}{2i} \quad \text{by L'Hospital's rule.}$$

$$\text{Then} \quad \oint_C \frac{2\,dz}{(1-2i)z^2+6iz-1-2i} = 2\pi i\left(\frac{1}{2i}\right) = \pi, \quad \text{the required value.}$$

12. Show that $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ if $a > |b|$.

Let $z = e^{i\theta}$. Then $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i} = \dfrac{z-z^{-1}}{2i}$, $dz = ie^{i\theta}\,d\theta = iz\,d\theta$ so that

$$\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \oint_C \frac{dz/iz}{a+b(z-z^{-1})/2i} = \oint_C \frac{2\,dz}{bz^2+2aiz-b}$$

where C is the circle of unit radius with centre at the origin, as shown in Fig. 7-6.

The poles of $\dfrac{2}{bz^2+2aiz-b}$ are obtained by solving $bz^2+2aiz-b=0$ and are given by

$$z = \frac{-2ai \pm \sqrt{-4a^2+4b^2}}{2b} = \frac{-ai \pm \sqrt{a^2-b^2}\,i}{b}$$

$$= \left\{\frac{-a+\sqrt{a^2-b^2}}{b}\right\}i,\ \left\{\frac{-a-\sqrt{a^2-b^2}}{b}\right\}i$$

Only $\dfrac{-a+\sqrt{a^2-b^2}}{b}i$ lies inside C, since

$$\left|\frac{-a+\sqrt{a^2-b^2}}{b}i\right| = \left|\frac{\sqrt{a^2-b^2}-a}{b}\cdot\frac{\sqrt{a^2-b^2}+a}{\sqrt{a^2-b^2}+a}\right| = \left|\frac{b}{(\sqrt{a^2-b^2}+a)}\right| < 1 \quad \text{if } a > |b|$$

$$\text{Residue at } z_1 = \frac{-a+\sqrt{a^2-b^2}}{b}i = \lim_{z\to z_1}(z-z_1)\frac{2}{bz^2+2aiz-b}$$

$$= \lim_{z\to z_1}\frac{2}{2bz+2ai} = \frac{1}{bz_1+ai} = \frac{1}{\sqrt{a^2-b^2}\,i}$$

by L'Hospital's rule.

$$\text{Then} \quad \oint_C \frac{2\,dz}{bz^2+2aiz-b} = 2\pi i\left(\frac{1}{\sqrt{a^2-b^2}\,i}\right) = \frac{2\pi}{\sqrt{a^2-b^2}}, \quad \text{the required value.}$$

13. Show that $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\,d\theta = \frac{\pi}{12}$.

If $z = e^{i\theta}$, then $\cos\theta = \dfrac{z+z^{-1}}{2}$, $\cos 3\theta = \dfrac{e^{3i\theta}+e^{-3i\theta}}{2} = \dfrac{z^3+z^{-3}}{2}$, $dz = iz\,d\theta$ so that

$$\int_0^{2\pi}\frac{\cos 3\theta}{5-4\cos\theta}\,d\theta = \oint_C \frac{(z^3+z^{-3})/2}{5-4(z+z^{-1})/2}\frac{dz}{iz} = -\frac{1}{2i}\oint_C \frac{z^6+1}{z^3(2z-1)(z-2)}\,dz$$

where C is the contour of Fig. 7-6.

The integrand has a pole of order 3 at $z=0$ and a simple pole $z=\tfrac{1}{2}$ inside C.

Residue at $z=0$ is $\lim_{z\to 0}\frac{1}{2!}\frac{d^2}{dz^2}\left\{z^3\cdot\frac{z^6+1}{z^3(2z-1)(z-2)}\right\} = \frac{21}{8}$.

Residue at $z=\frac{1}{2}$ is $\lim_{z\to 1/2}\left\{\left(z-\frac{1}{2}\right)\cdot\frac{z^6+1}{z^3(2z-1)(z-2)}\right\} = -\frac{65}{24}$.

Then $-\frac{1}{2i}\oint_C \frac{z^6+1}{z^3(2z-1)(z-2)}\,dz = -\frac{1}{2i}(2\pi i)\left\{\frac{21}{8}-\frac{65}{24}\right\} = \frac{\pi}{12}$ as required.

14. Show that $\displaystyle\int_0^{2\pi}\frac{d\theta}{(5-3\sin\theta)^2} = \frac{5\pi}{32}$.

Letting $z=e^{i\theta}$, we have $\sin\theta=(z-z^{-1})/2i$, $dz = ie^{i\theta}\,d\theta = iz\,d\theta$ and so

$$\int_0^{2\pi}\frac{d\theta}{(5-3\sin\theta)^2} = \oint_C\frac{dz/iz}{\{5-3(z-z^{-1})/2i\}^2} = -\frac{4}{i}\oint_C\frac{z\,dz}{(3z^2-10iz-3)^2}$$

where C is the contour of Fig. 7-6.

The integrand has poles of order 2 at $z = \frac{10i\pm\sqrt{-100+36}}{6} = \frac{10i\pm 8i}{6} = 3i,\ i/3$. Only the pole $i/3$ lies inside C.

$$\text{Residue at } z=i/3 = \lim_{z\to i/3}\frac{d}{dz}\left\{(z-i/3)^2\cdot\frac{z}{(3z^2-10iz-3)^2}\right\}$$

$$= \lim_{z\to i/3}\frac{d}{dz}\left\{(z-i/3)^2\cdot\frac{z}{(3z-i)^2(z-3i)^2}\right\} = -\frac{5}{256}.$$

Then $$-\frac{4}{i}\oint_C\frac{z\,dz}{(3z^2-10iz-3)^2} = -\frac{4}{i}(2\pi i)\left(\frac{-5}{256}\right) = \frac{5\pi}{32}$$

Another method.

From Problem 12, we have for $a>|b|$,

$$\int_0^{2\pi}\frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Then by differentiating both sides with respect to a (considering b as constant) using Leibnitz's rule, we have

$$\frac{d}{da}\int_0^{2\pi}\frac{d\theta}{a+b\sin\theta} = \int_0^{2\pi}\frac{\partial}{\partial a}\left(\frac{1}{a+b\sin\theta}\right)d\theta = -\int_0^{2\pi}\frac{d\theta}{(a+b\sin\theta)^2}$$

$$= \frac{d}{da}\left(\frac{2\pi}{\sqrt{a^2-b^2}}\right) = \frac{-2\pi a}{(a^2-b^2)^{3/2}}$$

i.e., $$\int_0^{2\pi}\frac{d\theta}{(a+b\sin\theta)^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}}$$

Letting $a=5$ and $b=-3$, we have

$$\int_0^{2\pi}\frac{d\theta}{(5-3\sin\theta)^2} = \frac{2\pi(5)}{(5^2-3^2)^{3/2}} = \frac{5\pi}{32}$$

DEFINITE INTEGRALS OF THE TYPE $\int_{-\infty}^{\infty}F(x)\begin{Bmatrix}\cos mx\\ \sin mx\end{Bmatrix}dx$

15. If $|F(z)|\leq\frac{M}{R^k}$ for $z=Re^{i\theta}$ where $k>0$ and M are constants, prove that

$$\lim_{R\to\infty}\int_\Gamma e^{imz}F(z)\,dz = 0$$

where Γ is the semicircular arc of Fig. 7-5 and m is a positive constant.

If $z=Re^{i\theta}$, $\int_\Gamma e^{imz}F(z)\,dz = \int_0^\pi e^{imRe^{i\theta}}F(Re^{i\theta})\,iRe^{i\theta}\,d\theta$. Then

$$\left|\int_0^\pi e^{imRe^{i\theta}} F(Re^{i\theta})\, iRe^{i\theta}\, d\theta\right| \leq \int_0^\pi |e^{imRe^{i\theta}} F(Re^{i\theta})\, iRe^{i\theta}|\, d\theta$$

$$= \int_0^\pi |e^{imR\cos\theta - mR\sin\theta} F(Re^{i\theta})\, iRe^{i\theta}|\, d\theta$$

$$= \int_0^\pi e^{-mR\sin\theta} |F(Re^{i\theta})|\, R\, d\theta$$

$$\leq \frac{M}{R^{k-1}} \int_0^\pi e^{-mR\sin\theta}\, d\theta = \frac{2M}{R^{k-1}} \int_0^{\pi/2} e^{-mR\sin\theta}\, d\theta$$

Now $\sin\theta \geqq 2\theta/\pi$ for $0 \leqq \theta \leqq \pi/2$, as can be seen geometrically from Fig. 7-7 or analytically from Prob. 99.

Then the last integral is less than or equal to

$$\frac{2M}{R^{k-1}} \int_0^{\pi/2} e^{-2mR\theta/\pi}\, d\theta = \frac{\pi M}{mR^k}(1 - e^{-mR})$$

As $R \to \infty$ this approaches zero, since m and k are positive, and the required result is proved.

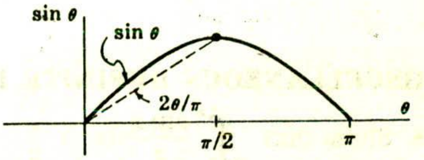

Fig. 7-7

16. Show that $\displaystyle\int_0^\infty \frac{\cos mx}{x^2+1}\, dx = \frac{\pi}{2} e^{-m}, \quad m > 0.$

Consider $\displaystyle\oint_C \frac{e^{imz}}{z^2+1}\, dz$ where C is the contour of Fig. 7-5. The integrand has simple poles at $z = \pm i$, but only $z = i$ lies inside C.

Residue at $z = i$ is $\displaystyle\lim_{z\to i}\left\{(z-i)\frac{e^{imz}}{(z-i)(z+i)}\right\} = \frac{e^{-m}}{2i}$. Then

$$\oint_C \frac{e^{imz}}{z^2+1}\, dz = 2\pi i\left(\frac{e^{-m}}{2i}\right) = \pi e^{-m}$$

or

$$\int_{-R}^{R} \frac{e^{imx}}{x^2+1}\, dx + \int_\Gamma \frac{e^{imz}}{z^2+1}\, dz = \pi e^{-m}$$

i.e.,

$$\int_{-R}^{R} \frac{\cos mx}{x^2+1}\, dx + i\int_{-R}^{R} \frac{\sin mx}{x^2+1}\, dx + \int_\Gamma \frac{e^{imz}}{z^2+1}\, dz = \pi e^{-m}$$

and so

$$2\int_0^R \frac{\cos mx}{x^2+1}\, dx + \int_\Gamma \frac{e^{imz}}{z^2+1}\, dz = \pi e^{-m}$$

Taking the limit as $R \to \infty$ and using Problem 15 to show that the integral around Γ approaches zero, we obtain the required result.

17. Evaluate $\displaystyle\int_{-\infty}^\infty \frac{x \sin \pi x}{x^2+2x+5}\, dx.$

Consider $\displaystyle\oint_C \frac{ze^{i\pi z}}{z^2+2z+5}\, dz$ where C is the contour of Fig. 7-5. The integrand has simple poles at $z = -1 \pm 2i$, but only $z = -1+2i$ lies inside C.

Residue at $z = -1+2i$ is $\displaystyle\lim_{z\to -1+2i}\left\{(z+1-2i)\cdot\frac{ze^{i\pi z}}{z^2+2z+5}\right\} = (-1+2i)\frac{e^{-i\pi-2\pi}}{4i}$. Then

$$\oint_C \frac{ze^{i\pi z}}{z^2+2z+5}\, dz = 2\pi i(-1+2i)\left(\frac{e^{-i\pi-2\pi}}{4i}\right) = \frac{\pi}{2}(1-2i)e^{-2\pi}$$

or

$$\int_{-R}^{R} \frac{xe^{i\pi x}}{x^2+2x+5}\, dx + \int_\Gamma \frac{ze^{i\pi z}}{z^2+2z+5}\, dz = \frac{\pi}{2}(1-2i)e^{-2\pi}$$

i.e.,

$$\int_{-R}^{R} \frac{x\cos \pi x}{x^2+2x+5}\, dx + i\int_{-R}^{R} \frac{x\sin \pi x}{x^2+2x+5}\, dx + \int_\Gamma \frac{ze^{i\pi z}}{z^2+2z+5}\, dz = \frac{\pi}{2}(1-2i)e^{-2\pi}$$

Taking the limit as $R \to \infty$ and using Problem 15 to show that the integral around Γ approaches zero, this becomes

$$\int_{-\infty}^{\infty} \frac{x\cos\pi x}{x^2+2x+5}\,dx + i\int_{-\infty}^{\infty}\frac{x\sin\pi x}{x^2+2x+5}\,dx = \frac{\pi}{2}e^{-2\pi} - i\pi e^{-2\pi}$$

Equating real and imaginary parts,

$$\int_{-\infty}^{\infty} \frac{x\cos\pi x}{x^2+2x+5}\,dx = \frac{\pi}{2}e^{-2\pi}, \qquad \int_{-\infty}^{\infty}\frac{x\sin\pi x}{x^2+2x+5}\,dx = -\pi e^{-2\pi}$$

Thus we have obtained the value of another integral in addition to the required one.

MISCELLANEOUS DEFINITE INTEGRALS

18. Show that $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

The method of Problem 16 leads us to consider the integral of e^{iz}/z around the contour of Fig. 7-5. However, since $z=0$ lies on this path of integration and since we cannot integrate through a singularity, we modify that contour by indenting the path at $z=0$, as shown in Fig. 7-8, which we call contour C' or $ABDEFGHJA$.

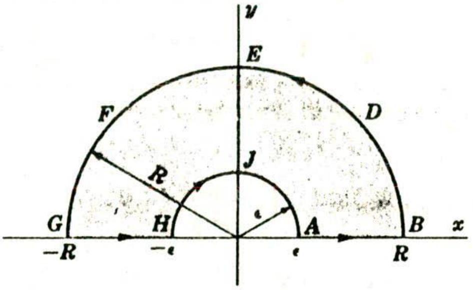

Fig. 7-8

Since $z=0$ is outside C', we have

$$\oint_{C'} \frac{e^{iz}}{z}\,dz = 0$$

or

$$\int_{-R}^{-\epsilon}\frac{e^{ix}}{x}\,dx + \int_{HJA}\frac{e^{iz}}{z}\,dz + \int_{\epsilon}^{R}\frac{e^{ix}}{x}\,dx + \int_{BDEFG}\frac{e^{iz}}{z}\,dz = 0$$

Replacing x by $-x$ in the first integral and combining with the third integral, we find

$$\int_{\epsilon}^{R}\frac{e^{ix}-e^{-ix}}{x}\,dx + \int_{HJA}\frac{e^{iz}}{z}\,dz + \int_{BDEFG}\frac{e^{iz}}{z}\,dz = 0$$

or

$$2i\int_{\epsilon}^{R}\frac{\sin x}{x}\,dx = -\int_{HJA}\frac{e^{iz}}{z}\,dz - \int_{BDEFG}\frac{e^{iz}}{z}\,dz$$

Let $\epsilon \to 0$ and $R\to\infty$. By Problem 15, the second integral on the right approaches zero. Letting $z=\epsilon e^{i\theta}$ in the first integral on the right, we see that it approaches

$$-\lim_{\epsilon\to 0}\int_{\pi}^{0}\frac{e^{i\epsilon e^{i\theta}}}{\epsilon e^{i\theta}}\,i\epsilon e^{i\theta}\,d\theta = -\lim_{\epsilon\to 0}\int_{\pi}^{0} ie^{i\epsilon e^{i\theta}}\,d\theta = \pi i$$

since the limit can be taken under the integral sign.

Then we have

$$\lim_{\substack{R\to\infty\\ \epsilon\to 0}} 2i\int_{\epsilon}^{R}\frac{\sin x}{x}\,dx = \pi i \qquad \text{or} \qquad \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

19. Prove that

$$\int_0^\infty \sin x^2\,dx = \int_0^\infty \cos x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

Let C be the contour indicated in Fig. 7-9, where AB is the arc of a circle with centre at O and radius R. By Cauchy's theorem,

$$\oint_C e^{iz^2}\,dz = 0$$

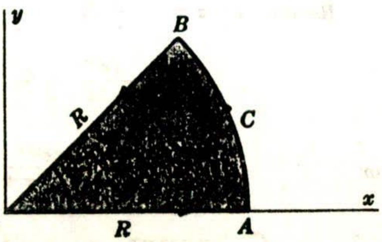

Fig. 7-9

or

$$\int_{OA} e^{iz^2}\,dz + \int_{AB} e^{iz^2}\,dz + \int_{BO} e^{iz^2}\,dz = 0 \tag{1}$$

Now on OA, $z = x$ (from $x = 0$ to $x = R$); on AB, $z = Re^{i\theta}$ (from $\theta = 0$ to $\theta = \pi/4$); on BO, $z = re^{\pi i/4}$ (from $r = R$ to $r = 0$). Hence from (1),

$$\int_0^R e^{ix^2}\,dx + \int_0^{\pi/4} e^{iR^2e^{2i\theta}}\, iRe^{i\theta}\,d\theta + \int_R^0 e^{ir^2e^{\pi i/2}}\, e^{\pi i/4}\,dr = 0 \tag{2}$$

i.e.,

$$\int_0^R (\cos x^2 + i \sin x^2)\,dx = e^{\pi i/4}\int_0^R e^{-r^2}\,dr - \int_0^{\pi/4} e^{iR^2\cos 2\theta - R^2 \sin 2\theta}\, iRe^{i\theta}\,d\theta \tag{3}$$

We consider the limit of (3) as $R \to \infty$. The first integral on the right becomes [see Problem 14, Chapter 10]

$$e^{\pi i/4}\int_0^\infty e^{-r^2}\,dr = \frac{\sqrt{\pi}}{2}e^{\pi i/4} = \frac{1}{2}\sqrt{\frac{\pi}{2}} + \frac{i}{2}\sqrt{\frac{\pi}{2}} \tag{4}$$

The absolute value of the second integral on the right of (3) is

$$\begin{aligned}\left|\int_0^{\pi/4} e^{iR^2\cos 2\theta - R^2\sin 2\theta}\, iRe^{i\theta}\,d\theta\right| &\leqq \int_0^{\pi/4} e^{-R^2 \sin 2\theta}\, R\,d\theta \\ &= \frac{R}{2}\int_0^{\pi/2} e^{-R^2\sin\phi}\,d\phi \\ &\leqq \frac{R}{2}\int_0^{\pi/2} e^{-2R^2\phi/\pi}\,d\phi \\ &= \frac{\pi}{4R}(1 - e^{-R^2})\end{aligned}$$

where we have used the transformation $2\theta = \phi$ and the inequality $\sin\phi \geqq 2\phi/\pi$, $0 \leqq \phi \leqq \pi/2$ (see Problem 15). This shows that as $R \to \infty$ the second integral on the right of (3) approaches zero. Then (3) becomes

$$\int_0^\infty (\cos x^2 + i \sin x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}} + \frac{i}{2}\sqrt{\frac{\pi}{2}}$$

and so equating real and imaginary parts we have, as required,

$$\int_0^\infty \cos x^2\,dx = \int_0^\infty \sin x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

20. Show that $\displaystyle\int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \frac{\pi}{\sin p\pi}$, $0 < p < 1$.

Consider $\displaystyle\oint_C \frac{z^{p-1}}{1+z}\,dz$. Since $z = 0$ is a branch point, choose C as the contour of Fig. 7-10 where the positive real axis is the branch line and where AB and GH are actually coincident with the x axis but are shown separated for visual purposes.

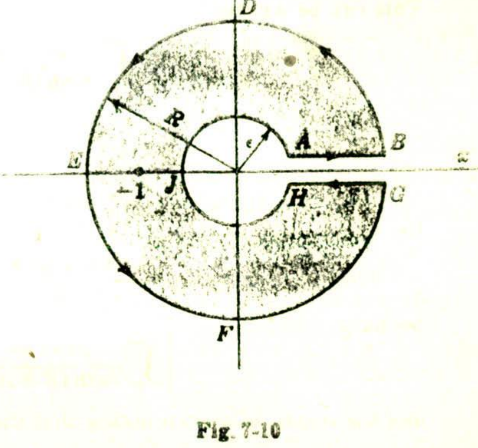

Fig. 7-10

The integrand has the simple pole $z = -1$ inside C.

Residue at $z = -1 = e^{\pi i}$ is

$$\lim_{z\to -1} (z+1)\frac{z^{p-1}}{1+z} = (e^{\pi i})^{p-1} = e^{(p-1)\pi i}$$

Then $\displaystyle\oint_C \frac{z^{p-1}}{1+z}\,dz = 2\pi i e^{(p-1)\pi i}$ or, omitting the integrand,

$$\int_{AB} + \int_{BDEFG} + \int_{GH} + \int_{HJA} = 2\pi i e^{(p-1)\pi i}$$

We thus have

$$\int_{\epsilon}^{R} \frac{x^{p-1}}{1+x}\,dx + \int_{0}^{2\pi} \frac{(Re^{i\theta})^{p-1}\,iRe^{i\theta}\,d\theta}{1+Re^{i\theta}} + \int_{R}^{\epsilon} \frac{(xe^{2\pi i})^{p-1}}{1+xe^{2\pi i}}\,dx$$
$$+ \int_{2\pi}^{0} \frac{(\epsilon e^{i\theta})^{p-1}\,i\epsilon e^{i\theta}\,d\theta}{1+\epsilon e^{i\theta}} = 2\pi i e^{(p-1)\pi i}$$

where we have used $z = xe^{2\pi i}$ for the integral along GH, since the argument of z is increased by 2π in going around the circle $BDEFG$.

Taking the limit as $\epsilon \to 0$ and $R \to \infty$ and noting that the second and fourth integrals approach zero, we find

$$\int_0^\infty \frac{x^{p-1}}{1+x}\,dx + \int_\infty^0 \frac{e^{2\pi i(p-1)}\,x^{p-1}}{1+x}\,dx = 2\pi e^{(p-1)\pi i}$$

or

$$(1 - e^{2\pi i(p-1)}) \int_0^\infty \frac{x^{p-1}}{1+x}\,dx = 2\pi i e^{(p-1)\pi i}$$

so that

$$\int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \frac{2\pi i e^{(p-1)\pi i}}{1-e^{2\pi i(p-1)}} = \frac{2\pi i}{e^{p\pi i} - e^{-p\pi i}} = \frac{\pi}{\sin p\pi}$$

21. Prove that $\displaystyle\int_0^\infty \frac{\cosh ax}{\cosh x}\,dx = \frac{\pi}{2\cos(\pi a/2)}$ where $|a| < 1$.

Consider $\displaystyle\oint_C \frac{e^{az}}{\cosh z}\,dz$ where C is a rectangle having vertices at $-R, R, R+\pi i, -R+\pi i$ (see Fig. 7-11).

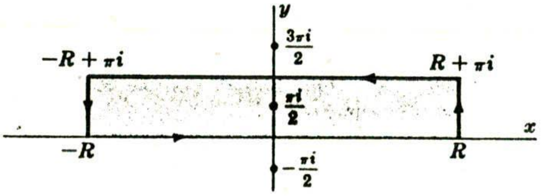

Fig. 7-11

The poles of $e^{az}/\cosh z$ are simple and occur where $\cosh z = 0$, i.e. $z = (n+\frac{1}{2})\pi i$, $n = 0, \pm 1, \pm 2, \ldots$. The only pole enclosed by C is $\pi i/2$.

Residue of $\dfrac{e^{az}}{\cosh z}$ at $z = \pi i/2$ is

$$\lim_{z \to \pi i/2} (z - \pi i/2)\frac{e^{az}}{\cosh z} = \frac{e^{a\pi i/2}}{\sinh(\pi i/2)} = \frac{e^{a\pi i/2}}{i\sin(\pi/2)} = -ie^{a\pi i/2}$$

Then by the residue theorem,

$$\oint_C \frac{e^{az}}{\cosh z}\,dz = 2\pi i(-ie^{a\pi i/2}) = 2\pi e^{a\pi i/2}$$

This can be written

$$\int_{-R}^{R} \frac{e^{ax}}{\cosh x}\,dx + \int_0^\pi \frac{e^{a(R+iy)}}{\cosh(R+iy)}\,i\,dy + \int_R^{-R} \frac{e^{a(x+\pi i)}}{\cosh(x+\pi i)}\,dx$$
$$+ \int_\pi^0 \frac{e^{a(-R+iy)}}{\cosh(-R+iy)}\,i\,dy = 2\pi e^{a\pi i/2} \qquad (1)$$

As $R \to \infty$ the second and fourth integrals on the left approach zero. To show this let us consider the second integral. Since

$$|\cosh(R+iy)| = \left|\frac{e^{R+iy}+e^{-R-iy}}{2}\right| \geqq \tfrac{1}{2}\{|e^{R+iy}| - |e^{-R-iy}|\} = \tfrac{1}{2}(e^R - e^{-R}) \geqq \tfrac{1}{4}e^R$$

we have

$$\left|\int_0^\pi \frac{e^{a(R+iy)}}{\cosh(R+iy)}\,i\,dy\right| \leqq \int_0^\pi \frac{e^{aR}}{\frac{1}{4}e^R}\,dy = 4\pi e^{(a-1)R}$$

and the result follows on noting that the right side approaches zero as $R \to \infty$ since $|a| < 1$. In a similar manner we can show that the fourth integral on the left of (1) approaches zero as $R \to \infty$. Hence (1) becomes

$$\lim_{R\to\infty} \left\{ \int_{-R}^{R} \frac{e^{ax}}{\cosh x}\,dx + e^{a\pi i} \int_{-R}^{R} \frac{e^{ax}}{\cosh x}\,dx \right\} = 2\pi e^{a\pi i/2}$$

since $\cosh(x + \pi i) = -\cosh x$. Thus

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ax}}{\cosh x}\,dx = \int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh x}\,dx = \frac{2\pi e^{a\pi i/2}}{1 + e^{a\pi i}} = \frac{2\pi}{e^{a\pi i/2} + e^{-a\pi i/2}} = \frac{\pi}{\cos(\pi a/2)}$$

Now

$$\int_{-\infty}^{0} \frac{e^{ax}}{\cosh x}\,dx + \int_{0}^{\infty} \frac{e^{ax}}{\cosh x}\,dx = \frac{\pi}{\cos(\pi a/2)}$$

Then replacing x by $-x$ in the first integral, we have

$$\int_{0}^{\infty} \frac{e^{-ax}}{\cosh x}\,dx + \int_{0}^{\infty} \frac{e^{ax}}{\cosh x}\,dx = 2\int_{0}^{\infty} \frac{\cosh ax}{\cosh x}\,dx = \frac{\pi}{\cos(\pi a/2)}$$

from which the required result follows.

22. Prove that $\displaystyle\int_{0}^{\infty} \frac{\ln(x^2+1)}{x^2+1}\,dx = \pi \ln 2$.

Consider $\displaystyle\oint_C \frac{\ln(z+i)}{z^2+1}\,dz$ around the contour C consisting of the real axis from $-R$ to R and the semicircle Γ of radius R (see Fig. 7-12).

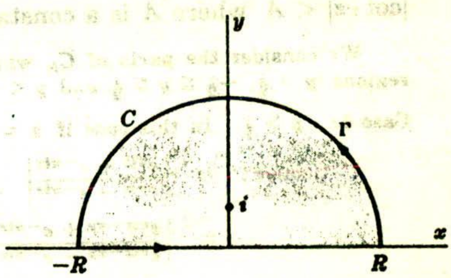

Fig. 7-12

The only pole of $\ln(z+i)/(z^2+1)$ inside C is the simple pole $z = i$, and the residue is

$$\lim_{z\to i} (z - i)\frac{\ln(z+i)}{(z-i)(z+i)} = \frac{\ln(2i)}{2i}$$

Hence by the residue theorem,

$$\oint_C \frac{\ln(z+i)}{z^2+1}\,dz = 2\pi i\left\{\frac{\ln(2i)}{2i}\right\} = \pi \ln(2i) = \pi \ln 2 + \tfrac{1}{2}\pi^2 i \qquad (1)$$

on writing $\ln(2i) = \ln 2 + \ln i = \ln 2 + \ln e^{\pi i/2} = \ln 2 + \pi i/2$ using principal values of the logarithm. The result can be written

$$\int_{-R}^{R} \frac{\ln(x+i)}{x^2+1}\,dx + \int_{\Gamma} \frac{\ln(z+i)}{z^2+1}\,dz = \pi \ln 2 + \tfrac{1}{2}\pi^2 i$$

or

$$\int_{-R}^{0} \frac{\ln(x+i)}{x^2+1}\,dx + \int_{0}^{R} \frac{\ln(x+i)}{x^2+1}\,dx + \int_{\Gamma} \frac{\ln(z+i)}{z^2+1}\,dz = \pi \ln 2 + \tfrac{1}{2}\pi^2 i$$

Replacing x by $-x$ in the first integral, this can be written

$$\int_{0}^{R} \frac{\ln(i-x)}{x^2+1}\,dx + \int_{0}^{R} \frac{\ln(i+x)}{x^2+1}\,dx + \int_{\Gamma} \frac{\ln(z+i)}{z^2+1}\,dz = \pi \ln 2 + \tfrac{1}{2}\pi^2 i$$

or, since $\ln(i-x) + \ln(i+x) = \ln(i^2 - x^2) = \ln(x^2+1) + \pi i$,

$$\int_{0}^{R} \frac{\ln(x^2+1)}{x^2+1}\,dx + \int_{0}^{R} \frac{\pi i}{x^2+1}\,dx + \int_{\Gamma} \frac{\ln(z+i)}{z^2+1}\,dz = \pi \ln 2 + \tfrac{1}{2}\pi^2 i \qquad (2)$$

As $R \to \infty$ we can show that the integral around Γ approaches zero (see Problem 101). Hence on taking real parts we find, as required,

$$\lim_{R\to\infty} \int_{0}^{R} \frac{\ln(x^2+1)}{x^2+1}\,dx = \int_{0}^{\infty} \frac{\ln(x^2+1)}{x^2+1}\,dx = \pi \ln 2$$

23. Prove that $\int_0^{\pi/2} \ln \sin x \, dx = \int_0^{\pi/2} \ln \cos x \, dx = -\frac{1}{2}\pi \ln 2$

Letting $x = \tan\theta$ in the result of Problem 22, we find

$$\int_0^{\pi/2} \frac{\ln(\tan^2\theta + 1)}{\tan^2\theta + 1} \sec^2\theta \, d\theta = -2\int_0^{\pi/2} \ln \cos\theta \, d\theta = \pi \ln 2$$

from which

$$\int_0^{\pi/2} \ln \cos\theta \, d\theta = -\tfrac{1}{2}\pi \ln 2 \qquad (1)$$

which establishes part of the required result. Letting $\theta = \pi/2 - \phi$ in (1), we find

$$\int_0^{\pi/2} \ln \sin\phi \, d\phi = -\tfrac{1}{2}\pi \ln 2$$

SUMMATION OF SERIES

24. Let C_N be a square with vertices at

$$(N+\tfrac{1}{2})(1+i), \quad (N+\tfrac{1}{2})(-1+i),$$
$$(N+\tfrac{1}{2})(-1-i), \quad (N+\tfrac{1}{2})(1-i)$$

as shown in Fig. 7-13. Prove that on C_N, $|\cot \pi z| < A$ where A is a constant.

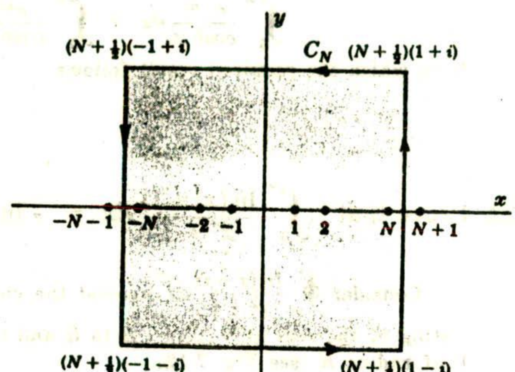

Fig. 7-13

We consider the parts of C_N which lie in the regions $y > \frac{1}{2}$, $-\frac{1}{2} \le y \le \frac{1}{2}$ and $y < -\frac{1}{2}$.

Case 1: $y > \frac{1}{2}$. In this case if $z = x + iy$,

$$|\cot \pi z| = \left|\frac{e^{\pi iz} + e^{-\pi iz}}{e^{\pi iz} - e^{-\pi iz}}\right| = \left|\frac{e^{\pi ix - \pi y} + e^{-\pi ix + \pi y}}{e^{\pi ix - \pi y} - e^{-\pi ix + \pi y}}\right|$$
$$\le \frac{|e^{\pi ix - \pi y}| + |e^{-\pi ix + \pi y}|}{|e^{-\pi ix + \pi y}| - |e^{\pi ix - \pi y}|}$$
$$= \frac{e^{-\pi y} + e^{\pi y}}{e^{\pi y} - e^{-\pi y}} = \frac{1 + e^{-2\pi y}}{1 - e^{-2\pi y}} \le \frac{1 + e^{-\pi}}{1 - e^{-\pi}} = A_1$$

Case 2: $y < -\frac{1}{2}$. Here as in Case 1,

$$|\cot \pi z| \le \frac{|e^{\pi ix - \pi y}| + |e^{-\pi ix + \pi y}|}{|e^{\pi ix - \pi y}| - |e^{-\pi ix + \pi y}|} = \frac{e^{-\pi y} + e^{\pi y}}{e^{-\pi y} - e^{\pi y}} = \frac{1 + e^{2\pi y}}{1 - e^{2\pi y}} \le \frac{1 + e^{-\pi}}{1 - e^{-\pi}} = A_1$$

Case 3: $-\frac{1}{2} \le y \le \frac{1}{2}$. Consider $z = N + \frac{1}{2} + iy$. Then

$$|\cot \pi z| = |\cot \pi(N + \tfrac{1}{2} + iy)| = |\cot(\pi/2 + \pi iy)| = |\tanh \pi y| \le \tanh(\pi/2) = A_2$$

If $z = -N - \frac{1}{2} + iy$, we have similarly

$$|\cot \pi z| = |\cot \pi(-N - \tfrac{1}{2} + iy)| = |\tanh \pi y| \le \tanh(\pi/2) = A_2$$

Thus if we choose A as a number greater than the larger of A_1 and A_2, we have $|\cot \pi z| < A$ on C_N where A is independent of N. It is of interest to note that we actually have $|\cot \pi z| \le A_1 = \coth(\pi/2)$ since $A_2 < A_1$.

25. Let $f(z)$ be such that along the path C_N of Fig. 7-13, $|f(z)| \le \frac{M}{|z|^k}$ where $k > 1$ and M are constants independent of N. Prove that

$$\sum_{-\infty}^{\infty} f(n) = -\{\text{sum of residues of } \pi \cot \pi z \, f(z) \text{ at the poles of } f(z)\}$$

Case 1: $f(z)$ has a finite number of poles.

In this case we can choose N so large that the path C_N of Fig. 7-13 encloses all poles of $f(z)$. The poles of $\cot \pi z$ are simple and occur at $z = 0, \pm 1, \pm 2, \ldots$.

Residue of $\pi \cot \pi z\, f(z)$ at $z = n$, $n = 0, \pm 1, \pm 2, \ldots$, is

$$\lim_{z\to n} (z-n)\pi \cot \pi z\, f(z) = \lim_{z\to n} \pi\left(\frac{z-n}{\sin \pi z}\right)\cos \pi z\, f(z) = f(n)$$

using L'Hospital's rule. We have assumed here that $f(z)$ has no poles at $z = n$, since otherwise the given series diverges.

By the residue theorem,

$$\oint_{C_N} \pi \cot \pi z\, f(z)\, dz = \sum_{n=-N}^{N} f(n) + S \tag{1}$$

where S is the sum of the residues of $\pi \cot \pi z\, f(z)$ at the poles of $f(z)$. By Problem 24 and our assumption on $f(z)$, we have

$$\left|\oint_{C_N} \pi \cot \pi z\, f(z)\, dz\right| \le \frac{\pi A M}{N^k}(8N+4)$$

since the length of path C_N is $8N+4$. Then taking the limit as $N \to \infty$ we see that

$$\lim_{N\to\infty} \oint_{C_N} \pi \cot \pi z\, f(z)\, dz = 0 \tag{2}$$

Thus from (1) we have as required,

$$\sum_{-\infty}^{\infty} f(n) = -S \tag{3}$$

Case 2: $f(z)$ has infinitely many poles.

If $f(z)$ has an infinite number of poles, we can obtain the required result by an appropriate limiting procedure. See Problem 103.

26. Prove that $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a}\coth \pi a$ where $a > 0$.

Let $f(z) = \dfrac{1}{z^2+a^2}$ which has simple poles at $z = \pm ai$.

Residue of $\dfrac{\pi \cot \pi z}{z^2+a^2}$ at $z = ai$ is

$$\lim_{z\to ai} (z-ai)\frac{\pi \cot \pi z}{(z-ai)(z+ai)} = \frac{\pi \cot \pi a i}{2ai} = -\frac{\pi}{2a}\coth \pi a$$

Similarly the residue at $z = -ai$ is $\dfrac{-\pi}{2a}\coth \pi a$, and the sum of the residues is $-\dfrac{\pi}{a}\coth \pi a$. Then by Problem 25,

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = -(\text{sum of residues}) = \frac{\pi}{a}\coth \pi a$$

27. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{2a}\coth \pi a - \frac{1}{2a^2}$ where $a > 0$.

The result of Problem 26 can be written in the form

$$\sum_{n=-\infty}^{-1} \frac{1}{n^2+a^2} + \frac{1}{a^2} + \sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a}\coth \pi a$$

or

$$2\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} + \frac{1}{a^2} = \frac{\pi}{a}\coth \pi a$$

which gives the required result.

28. Prove that $\dfrac{1}{1^2}+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\cdots = \dfrac{\pi^2}{6}$.

We have
$$F(z) = \frac{\pi\cot\pi z}{z^2} = \frac{\pi\cos\pi z}{z^2\sin\pi z} = \frac{\left(1-\frac{\pi^2z^2}{2!}+\frac{\pi^4z^4}{4!}-\cdots\right)}{z^3\left(1-\frac{\pi^2z^2}{3!}+\frac{\pi^4z^4}{5!}-\cdots\right)}$$
$$= \frac{1}{z^3}\left(1-\frac{\pi^2z^2}{2!}+\cdots\right)\left(1+\frac{\pi^2z^2}{3!}+\cdots\right) = \frac{1}{z^3}\left(1-\frac{\pi^2z^2}{3}+\cdots\right)$$
so that the residue at $z=0$ is $-\pi^2/3$.

Then as in Problems 26 and 27,
$$\oint_{C_N}\frac{\pi\cot\pi z}{z^2}\,dz = \sum_{n=-N}^{-1}\frac{1}{n^2}+\sum_{n=1}^{N}\frac{1}{n^2}-\frac{\pi^2}{3} = 2\sum_{n=1}^{N}\frac{1}{n^2}-\frac{\pi^2}{3}$$

Taking the limit as $N\to\infty$ we have, since the left side approaches zero,
$$2\sum_{n=1}^{\infty}\frac{1}{n^2}-\frac{\pi^2}{3} = 0 \qquad\text{or}\qquad \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Another method. Take the limit as $a\to 0$ in the result of Problem 27. Then using L'Hospital's rule,
$$\lim_{a\to 0}\sum_{n=1}^{\infty}\frac{1}{n^2+a^2} = \sum_{n=1}^{\infty}\frac{1}{n^2} = \lim_{a\to 0}\frac{\pi a\coth\pi a-1}{2a^2} = \frac{\pi^2}{6}$$

29. If $f(z)$ satisfies the same conditions given in Problem 25, prove that
$$\sum_{-\infty}^{\infty}(-1)^n f(n) = -\{\text{sum of residues of } \pi\csc\pi z\, f(z) \text{ at the poles of } f(z)\}$$

We proceed in a manner similar to that in Problem 25. The poles of $\csc\pi z$ are simple and occur at $z=0,\pm1,\pm2,\ldots$.

Residue of $\pi\csc\pi z\, f(z)$ at $z=n$, $n=0,\pm1,\pm2,\ldots$, is
$$\lim_{z\to n}(z-n)\pi\csc\pi z\, f(z) = \lim_{z\to n}\pi\left(\frac{z-n}{\sin\pi z}\right)f(z) = (-1)^n f(n)$$

By the residue theorem,
$$\oint_{C_N}\pi\csc\pi z\, f(z)\,dz = \sum_{n=-N}^{N}(-1)^n f(n) + S \tag{1}$$
where S is the sum of the residues of $\pi\csc\pi z\, f(z)$ at the poles of $f(z)$.

Letting $N\to\infty$, the integral on the left of (1) approaches zero (Problem 106) so that, as required, (1) becomes
$$\sum_{-\infty}^{\infty}(-1)^n f(n) = -S \tag{2}$$

30. Prove that $\displaystyle\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{(n+a)^2} = \frac{\pi^2\cos\pi a}{\sin^2\pi a}$ where a is real and different from $0,\pm1,\pm2,\ldots$.

Let $f(z)=\dfrac{1}{(z+a)^2}$ which has a double pole at $z=-a$.

Residue of $\dfrac{\pi\csc\pi z}{(z+a)^2}$ at $z=-a$ is
$$\lim_{z\to -a}\frac{d}{dz}\left\{(z+a)^2\cdot\frac{\pi\csc\pi z}{(z+a)^2}\right\} = -\pi^2\csc\pi a\cot\pi a$$

Then by Problem 29,
$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{(n+a)^2} = -(\text{sum of residues}) = \pi^2\csc\pi a\cot\pi a = \frac{\pi^2\cos\pi a}{\sin^2\pi a}$$

31. Prove that if $a \neq 0, \pm 1, \pm 2, \ldots$, then

$$\frac{a^2+1}{(a^2-1)^2} - \frac{a^2+4}{(a^2-4)^2} + \frac{a^2+9}{(a^2-9)^2} - \cdots = \frac{1}{2a^2} - \frac{\pi^2 \cos \pi a}{2 \sin^2 \pi a}$$

The result of Problem 30 can be written in the form

$$\frac{1}{a^2} - \left\{\frac{1}{(a+1)^2} + \frac{1}{(a-1)^2}\right\} + \left\{\frac{1}{(a+2)^2} + \frac{1}{(a-2)^2}\right\} + \cdots = \frac{\pi^2 \cos \pi a}{\sin^2 \pi a}$$

or

$$\frac{1}{a^2} - \frac{2(a^2+1)}{(a^2-1)^2} + \frac{2(a^2+4)}{(a^2-4)^2} - \frac{2(a^2+9)}{(a^2-9)^2} + \cdots = \frac{\pi^2 \cos \pi a}{\sin^2 \pi a}$$

from which the required result follows. Note that the grouping of terms in the infinite series is permissible since the series is absolutely convergent.

32. Prove that $\dfrac{1}{1^3} - \dfrac{1}{3^3} + \dfrac{1}{5^3} - \dfrac{1}{7^3} + \cdots = \dfrac{\pi^3}{32}$.

We have

$$F(z) = \frac{\pi \sec \pi z}{z^3} = \frac{\pi}{z^3 \cos \pi z} = \frac{\pi}{z^3(1 - \pi^2 z^2/2! + \cdots)}$$

$$= \frac{\pi}{z^3}\left(1 + \frac{\pi^2 z^2}{2} + \cdots\right) = \frac{\pi}{z^3} + \frac{\pi^3}{2z} + \cdots$$

so that the residue at $z = 0$ is $\pi^3/2$.

The residue of $F(z)$ at $z = n + \frac{1}{2}$, $n = 0, \pm 1, \pm 2, \ldots$ [which are the simple poles of $\sec \pi z$], is

$$\lim_{z \to n + 1/2} \{z - (n + \tfrac{1}{2})\} \frac{\pi}{z^3 \cos \pi z} = \frac{\pi}{(n+\frac{1}{2})^3} \lim_{z \to n+1/2} \frac{z - (n+\frac{1}{2})}{\cos \pi z} = \frac{-(-1)^n}{(n+\frac{1}{2})^3}$$

If C_N is a square with vertices at $N(1+i)$, $N(1-i)$, $N(-1+i)$, $N(-1-i)$, then

$$\oint_{C_N} \frac{\pi \sec \pi z}{z^3}\, dz = -\sum_{n=-N}^{N} \frac{(-1)^n}{(n+\frac{1}{2})^3} + \frac{\pi^3}{2} = -8 \sum_{n=-N}^{N} \frac{(-1)^n}{(2n+1)^3} + \frac{\pi^3}{2}$$

and since the integral on the left approaches zero as $N \to \infty$, we have

$$\sum_{-\infty}^{\infty} \frac{(-1)^n}{(2n+1)^3} = 2\left\{\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots\right\} = \frac{\pi^3}{16}$$

from which the required result follows.

MITTAG-LEFFLER'S EXPANSION THEOREM

33. Prove Mittag-Leffler's expansion theorem (see Page 175).

Let $f(z)$ have poles at $z = a_n$, $n = 1, 2, \ldots$, and suppose that $z = \zeta$ is not a pole of $f(z)$. Then the function $\dfrac{f(z)}{z - \zeta}$ has poles at $z = a_n$, $n = 1, 2, 3, \ldots$ and ζ.

Residue of $\dfrac{f(z)}{z-\zeta}$ at $z = a_n$, $n = 1, 2, 3, \ldots$, is $\displaystyle\lim_{z \to a_n} (z - a_n) \frac{f(z)}{z - \zeta} = \frac{b_n}{a_n - \zeta}$.

Residue of $\dfrac{f(z)}{z-\zeta}$ at $z = \zeta$ is $\displaystyle\lim_{z \to \zeta} (z - \zeta) \frac{f(z)}{z - \zeta} = f(\zeta)$.

Then by the residue theorem,

$$\frac{1}{2\pi i} \oint_{C_N} \frac{f(z)}{z - \zeta}\, dz = f(\zeta) + \sum_n \frac{b_n}{a_n - \zeta} \qquad (1)$$

where the last summation is taken over all poles inside circle C_N of radius R_N (Fig. 7-14).

Suppose that $f(z)$ is analytic at $z = 0$. Then putting $\zeta = 0$ in (1), we have

$$\frac{1}{2\pi i} \oint_{C_N} \frac{f(z)}{z}\, dz = f(0) + \sum_n \frac{b_n}{a_n} \qquad (2)$$

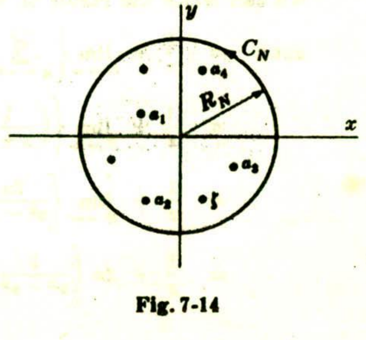

Fig. 7-14

Subtraction of (*2*) from (*1*) yields

$$f(\zeta) - f(0) + \sum_n b_n\left(\frac{1}{a_n - \zeta} - \frac{1}{a_n}\right) = \frac{1}{2\pi i}\oint_{C_N} f(z)\left\{\frac{1}{z-\zeta} - \frac{1}{z}\right\} dz$$

$$= \frac{\zeta}{2\pi i}\oint_{C_N} \frac{f(z)}{z(z-\zeta)}\,dz \qquad (3)$$

Now since $|z-\zeta| \geqq |z| - |\zeta| = R_N - |\zeta|$ for z on C_N, we have, if $|f(z)| \leqq M$,

$$\left|\oint_{C_N} \frac{f(z)}{z(z-\zeta)}\,dz\right| \leqq \frac{M\cdot 2\pi R_N}{R_N(R_N - |\zeta|)}$$

As $N \to \infty$ and therefore $R_N \to \infty$, it follows that the integral on the left approaches zero, i.e.,

$$\lim_{N\to\infty}\oint_{C_N} \frac{f(z)}{z(z-\zeta)}\,dz = 0$$

Hence from (*3*), letting $N \to \infty$, we have as required

$$f(\zeta) = f(0) + \sum_n b_n\left(\frac{1}{\zeta - a_n} + \frac{1}{a_n}\right)$$

the result on Page 175 being obtained on replacing ζ by z.

34. Prove that $\cot z = \dfrac{1}{z} + \sum_n \left(\dfrac{1}{z - n\pi} + \dfrac{1}{n\pi}\right)$ where the summation extends over $n = \pm 1, \pm 2, \ldots$.

Consider the function $f(z) = \cot z - \dfrac{1}{z} = \dfrac{z\cos z - \sin z}{z \sin z}$. Then $f(z)$ has simple poles at $z = n\pi$, $n = \pm 1, \pm 2, \pm 3, \ldots$, and the residue at these poles is

$$\lim_{z\to n\pi}(z - n\pi)\left(\frac{z\cos z - \sin z}{z\sin z}\right) = \lim_{z\to n\pi}\left(\frac{z - n\pi}{\sin z}\right)\lim_{z\to n\pi}\left(\frac{z\cos z - \sin z}{z}\right) = 1$$

At $z = 0$, $f(z)$ has a removable singularity since

$$\lim_{z\to 0}\left(\cot z - \frac{1}{z}\right) = \lim_{z\to 0}\left(\frac{z\cos z - \sin z}{z\sin z}\right) = 0$$

by L'Hospital's rule. Hence we can define $f(0) = 0$.

By Problem 110 it follows that $f(z)$ is bounded on circles C_N having centre at the origin and radius $R_N = (N + \frac{1}{2})\pi$. Hence by Problem 33,

$$\cot z - \frac{1}{z} = \sum_n \left(\frac{1}{z - n\pi} + \frac{1}{n\pi}\right)$$

from which the required result follows.

35. Prove that $\cot z = \dfrac{1}{z} + 2z\left\{\dfrac{1}{z^2 - \pi^2} + \dfrac{1}{z^2 - 4\pi^2} + \cdots\right\}$.

We can write the result of Problem 34 in the form

$$\cot z = \frac{1}{z} + \lim_{N\to\infty}\left\{\sum_{n=-N}^{-1}\left(\frac{1}{z - n\pi} + \frac{1}{n\pi}\right) + \sum_{n=1}^{N}\left(\frac{1}{z - n\pi} + \frac{1}{n\pi}\right)\right\}$$

$$= \frac{1}{z} + \lim_{N\to\infty}\left\{\left(\frac{1}{z+\pi} + \frac{1}{z-\pi}\right) + \left(\frac{1}{z+2\pi} + \frac{1}{z-2\pi}\right) + \cdots + \left(\frac{1}{z+N\pi} + \frac{1}{z-N\pi}\right)\right\}$$

$$= \frac{1}{z} + \lim_{N\to\infty}\left\{\frac{2z}{z^2 - \pi^2} + \frac{2z}{z^2 - 4\pi^2} + \cdots + \frac{2z}{z^2 - N^2\pi^2}\right\}$$

$$= \frac{1}{z} + 2z\left\{\frac{1}{z^2 - \pi^2} + \frac{1}{z^2 - 4\pi^2} + \cdots\right\}$$

MISCELLANEOUS PROBLEMS

36. Evaluate $\dfrac{1}{2\pi i}\displaystyle\int_{a-i\infty}^{a+i\infty} \frac{e^{zt}}{\sqrt{z+1}}\,dz$ where a and t are any positive constants.

The integrand has a branch point at $z=-1$. We shall take as branch line that part of the real axis to the left of $z=-1$. Since we cannot cross this branch line, let us consider

$$\oint_C \frac{e^{zt}}{\sqrt{z+1}}\,dz$$

where C is the contour $ABDEFGHJKA$ shown in Fig. 7-15. In this figure EF and HJ actually lie on the real axis but have been shown separated for visual purposes. Also, FGH is a circle of radius ϵ while BDE and JKA represent arcs of a circle of radius R.

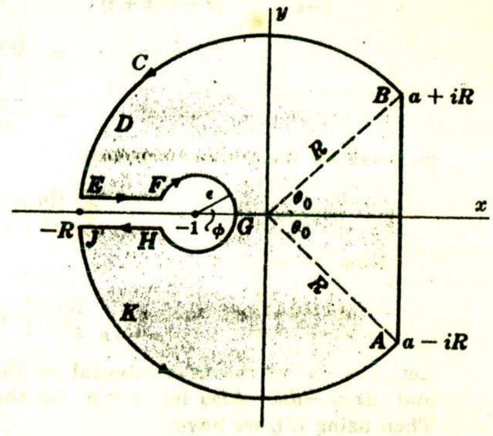

Fig. 7-15

Since $e^{zt}/\sqrt{z+1}$ is analytic inside and on C, we have by Cauchy's theorem

$$\oint_C \frac{e^{zt}}{\sqrt{z+1}}\,dz = 0 \tag{1}$$

Omitting the integrand, this can be written

$$\int_{AB} + \int_{BDE} + \int_{EF} + \int_{FGH} + \int_{HJ} + \int_{JKA} = 0 \tag{2}$$

Now on BDE and JKA, $z = Re^{i\theta}$ where θ goes from θ_0 to π and π to $2\pi-\theta_0$ respectively.

On EF, $z+1 = ue^{\pi i}$, $\sqrt{z+1} = \sqrt{u}\,e^{\pi i/2} = i\sqrt{u}$; whereas on HJ, $z+1 = ue^{-\pi i}$, $\sqrt{z+1} = \sqrt{u}\,e^{-\pi i/2} = -i\sqrt{u}$. In both cases $z=-u-1$, $dz=-du$, where u varies from $R-1$ to ϵ along EF and ϵ to $R-1$ along HJ.

On FGH, $z+1 = \epsilon e^{i\phi}$ where ϕ goes from $-\pi$ to π.

Thus (2) can be written

$$\int_{a-iR}^{a+iR} \frac{e^{zt}}{\sqrt{z+1}}\,dz + \int_{\theta_0}^{\pi} \frac{e^{Re^{i\theta}t}}{\sqrt{Re^{i\theta}+1}}\,iRe^{i\theta}\,d\theta + \int_{R-1}^{\epsilon} \frac{e^{-(u+1)t}(-du)}{i\sqrt{u}}$$

$$+ \int_{\pi}^{-\pi} \frac{e^{(\epsilon e^{i\phi}-1)t}}{\sqrt{\epsilon e^{i\phi}+1}}\,i\epsilon e^{i\phi}\,d\phi + \int_{\epsilon}^{R-1} \frac{e^{-(u+1)t}(-du)}{-i\sqrt{u}}$$

$$+ \int_{\pi}^{2\pi-\theta_0} \frac{e^{Re^{i\theta}t}}{\sqrt{Re^{i\theta}+1}}\,iRe^{i\theta}\,d\theta = 0 \tag{3}$$

Let us now take the limit as $R\to\infty$ and $\epsilon\to 0$. We can show (see Problem 111) that the second, fourth and sixth integrals approach zero. Hence we have

$$\int_{a-i\infty}^{a+i\infty} \frac{e^{zt}}{\sqrt{z+1}}\,dz = \lim_{\substack{\epsilon\to 0\\ R\to\infty}} 2i\int_{\epsilon}^{R-1} \frac{e^{-(u+1)t}}{\sqrt{u}}\,du = 2i\int_0^{\infty} \frac{e^{-(u+1)t}}{\sqrt{u}}\,du$$

or letting $u = v^2$,

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{e^{zt}}{\sqrt{z+1}}\,dz = \frac{1}{\pi}\int_0^{\infty} \frac{e^{-(u+1)t}}{\sqrt{u}}\,du = \frac{2e^{-t}}{\pi}\int_0^{\infty} e^{-v^2t}\,dv = \frac{e^{-t}}{\sqrt{\pi t}}$$

37. Prove that $\displaystyle\int_0^{\infty} \frac{(\ln u)^2}{u^2+1}\,du = \frac{\pi^3}{8}$.

Let C be the closed curve of Fig. 7-16 below where Γ_1 and Γ_2 are semicircles of radii ϵ and R respectively and centre at the origin. Consider

$$\oint_C \frac{(\ln z)^2}{z^2+1}\,dz$$

Since the integrand has a simple pole $z = i$ inside C and since the residue at this pole is

$$\lim_{z\to i} (z-i)\frac{(\ln z)^2}{(z-i)(z+i)} = \frac{(\ln i)^2}{2i} = \frac{(\pi i/2)^2}{2i} = \frac{-\pi^2}{8i}$$

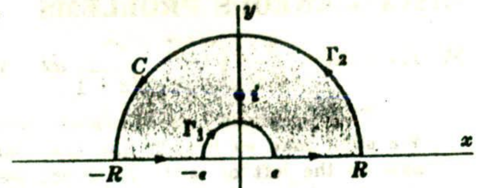

Fig. 7-16

we have by the residue theorem

$$\oint_C \frac{(\ln z)^2}{z^2+1}\,dz = 2\pi i\left(\frac{-\pi^2}{8i}\right) = \frac{-\pi^3}{4} \tag{1}$$

Now

$$\oint_C \frac{(\ln z)^2}{z^2+1}\,dz = \int_{-R}^{-\epsilon}\frac{(\ln z)^2}{z^2+1}\,dz + \int_{\Gamma_1}\frac{(\ln z)^2}{z^2+1}\,dz + \int_{\epsilon}^{R}\frac{(\ln z)^2}{z^2+1}\,dz + \int_{\Gamma_2}\frac{(\ln z)^2}{z^2+1}\,dz \tag{2}$$

Let $z = -u$ in the first integral on the right so that $\ln z = \ln(-u) = \ln u + \ln(-1) = \ln u + \pi i$ and $dz = -du$. Also let $z = u$ (so that $dz = du$ and $\ln z = \ln u$) in the third integral on the right. Then using (1), we have

$$\int_{\epsilon}^{R}\frac{(\ln u+\pi i)^2}{u^2+1}\,du + \int_{\Gamma_1}\frac{(\ln z)^2}{z^2+1}\,dz + \int_{\epsilon}^{R}\frac{(\ln u)^2}{u^2+1}\,du + \int_{\Gamma_2}\frac{(\ln z)^2}{z^2+1}\,dz = \frac{-\pi^3}{4}$$

Now let $\epsilon \to 0$ and $R \to \infty$. Since the integrals around Γ_1 and Γ_2 approach zero, we have

$$\int_0^\infty \frac{(\ln u+\pi i)^2}{u^2+1}\,du + \int_0^\infty \frac{(\ln u)^2}{u^2+1}\,du = \frac{-\pi^3}{4}$$

or

$$2\int_0^\infty \frac{(\ln u)^2}{u^2+1}\,du + 2\pi i\int_0^\infty \frac{\ln u}{u^2+1}\,du - \pi^2\int_0^\infty \frac{du}{u^2+1} = \frac{-\pi^3}{4}$$

Using the fact that $\displaystyle\int_0^\infty \frac{du}{u^2+1} = \tan^{-1}u\Big|_0^\infty = \frac{\pi}{2}$,

$$2\int_0^\infty \frac{(\ln u)^2}{u^2+1}\,du + 2\pi i\int_0^\infty \frac{\ln u}{u^2+1}\,du = \frac{\pi^3}{4}$$

Equating real and imaginary parts, we find

$$\int_0^\infty \frac{(\ln u)^2}{u^2+1}\,du = \frac{\pi^3}{8}, \qquad \int_0^\infty \frac{\ln u}{u^2+1}\,du = 0$$

the second integral being a by-product of the evaluation.

38. Prove that

$$\frac{\coth\pi}{1^3} + \frac{\coth 2\pi}{2^3} + \frac{\coth 3\pi}{3^3} + \cdots = \frac{7\pi^3}{180}$$

Consider

$$\oint_{C_N}\frac{\pi\cot\pi z\coth\pi z}{z^3}\,dz$$

taken around the square C_N shown in Fig. 7-17. The poles of the integrand are located at: $z = 0$ (pole of order 5); $z = \pm1, \pm2, \ldots$ (simple poles); $z = \pm i, \pm 2i, \ldots$ (simple poles).

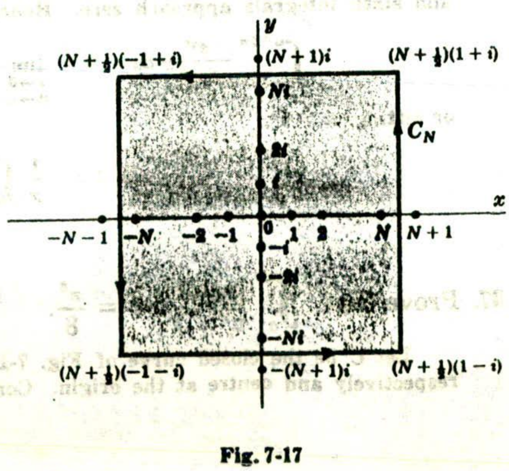

Fig. 7-17

By Problem 5 (replacing z by πz) we see that:

Residue at $z = 0$ is $\dfrac{-7\pi^3}{45}$.

Residue at $z = n$ ($n = \pm1, \pm2, \ldots$) is

$$\lim_{z\to n}\left\{\frac{(z-n)}{\sin \pi z}\cdot\frac{\pi\cos \pi z\coth \pi z}{z^3}\right\} = \frac{\coth n\pi}{n^3}$$

Residue at $z = ni$ $(n = \pm 1, \pm 2, \ldots)$ is

$$\lim_{z\to ni}\left\{\frac{(z-ni)}{\sinh \pi z}\cdot\frac{\pi\cot \pi z\cosh \pi z}{z^3}\right\} = \frac{\coth n\pi}{n^3}$$

Hence by the residue theorem,

$$\oint_{C_N}\frac{\pi\cot \pi z\coth \pi z}{z^3}\,dz = \frac{-7\pi^3}{45} + 4\sum_{n=1}^{N}\frac{\coth n\pi}{n^3}$$

Taking the limit as $N\to\infty$, we find as in Problem 25 that the integral on the left approaches zero and the required result follows.

Supplementary Problems

RESIDUES AND THE RESIDUE THEOREM

39. For each of the following functions determine the poles and the residues at the poles:

(a) $\dfrac{2z+1}{z^2-z-2}$, (b) $\left(\dfrac{z+1}{z-1}\right)^2$, (c) $\dfrac{\sin z}{z^2}$, (d) $\operatorname{sech} z$, (e) $\cot z$.

Ans. (a) $z = -1, 2$; $1/3, 5/3$
(b) $z = 1$; 4
(c) $z = 0$; 1
(d) $z = \frac{1}{2}(2k+1)\pi i$; $(-1)^{k+1}i$ where $k = 0, \pm 1, \pm 2, \ldots$
(e) $z = k\pi i$; 1 where $k = 0, \pm 1, \pm 2, \ldots$

40. Prove that $\displaystyle\oint_C \frac{\cosh z}{z^3}\,dz = \pi i$ if C is the square with vertices at $\pm 2 \pm 2i$.

41. Show that the residue of $(\csc z \operatorname{csch} z)/z^3$ at $z = 0$ is $-1/60$.

42. Evaluate $\displaystyle\oint_C \frac{e^z\,dz}{\cosh z}$ around the circle C defined by $|z| = 5$. *Ans.* $8\pi i$

43. Find the zeros and poles of $f(z) = \dfrac{z^2+4}{z^3+2z^2+2z}$ and determine the residues at the poles.
Ans. Zeros: $z = \pm 2i$ Res; at $z = 0$ is 2 Res; at $z = -1+i$ is $-\frac{1}{2}(1-3i)$ Res; at $z = -1-i$ is $-\frac{1}{2}(1+3i)$

44. Evaluate $\displaystyle\oint_C e^{-1/z}\sin(1/z)\,dz$ where C is the circle $|z| = 1$. *Ans.* $2\pi i$

45. Let C be a square bounded by $x = \pm 2$, $y = \pm 2$. Evaluate $\displaystyle\oint_C \frac{\sinh 3z}{(z-\pi/4)^3}\,dz$. *Ans.* $-9\pi\sqrt{2}/2$

46. Evaluate $\displaystyle\oint_C \frac{2z^2+5}{(z+2)^3(z^2+4)z^2}\,dz$ where C is (a) $|z-2i| = 6$, (b) the square with vertices at $1+i$, $2+i$, $2+2i$, $1+2i$.

47. Evaluate $\displaystyle\oint_C \frac{2+3\sin \pi z}{z(z-1)^2}\,dz$ where C is a square having vertices at $3+3i$, $3-3i$, $-3+3i$, $-3-3i$.
Ans. $-6\pi i$

48. Evaluate $\displaystyle\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z(z^2+1)}\,dz$, $t > 0$ around the square with vertices at $1+i$, $-1+i$, $-1-i$, $1-i$.
Ans. $1-\cos t$

DEFINITE INTEGRALS

49. Prove that $\displaystyle\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$.

50. Evaluate $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)^2}$. *Ans.* $5\pi/288$

51. Evaluate $\displaystyle\int_0^{2\pi} \frac{\sin 3\theta}{5-3\cos\theta}\,d\theta$. *Ans.* 0

52. Evaluate $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5+4\cos\theta}\,d\theta$.

53. Prove that $\displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta}{5-4\cos 2\theta}\,d\theta = \frac{3\pi}{8}$.

54. Prove that if $m>0$, $\displaystyle\int_0^\infty \frac{\cos mx}{(x^2+1)^2}\,dx = \frac{\pi e^{-m}(1+m)}{4}$.

55. (*a*) Find the residue of $\dfrac{e^{iz}}{(z^2+1)^5}$ at $z=i$. (*b*) Evaluate $\displaystyle\int_0^\infty \frac{\cos x}{(x^2+1)^5}\,dx$.

56. If $a^2 > b^2+c^2$, prove that $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta+c\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2-c^2}}$.

57. Prove that $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{(5-3\cos\theta)^4}\,d\theta = \frac{135\pi}{16{,}384}$.

58. Evaluate $\displaystyle\int_0^\infty \frac{dx}{x^4+x^2+1}$. *Ans.* $\pi\sqrt{3}/6$

59. Evaluate $\displaystyle\int_{-\infty}^\infty \frac{dx}{(x^2+4x+5)^2}$. *Ans.* $\pi/2$

60. Prove that $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$.

61. Discuss the validity of the following solution to Problem 19. Let $u = (1+i)x/\sqrt{2}$ in the result $\displaystyle\int_0^\infty e^{-u^2}\,du = \tfrac{1}{2}\sqrt{\pi}$ to obtain $\displaystyle\int_0^\infty e^{-ix^2}\,dx = \tfrac{1}{2}(1-i)\sqrt{\pi/2}$ from which $\displaystyle\int_0^\infty \cos x^2\,dx = \int_0^\infty \sin x^2\,dx = \tfrac{1}{2}\sqrt{\pi/2}$ on equating real and imaginary parts.

62. Show that $\displaystyle\int_0^\infty \frac{\cos 2\pi x}{x^4+x^2+1}\,dx = \frac{-\pi}{2\sqrt{3}}\,e^{-\pi/\sqrt{3}}$.

SUMMATION OF SERIES

63. Prove that $\displaystyle\sum_{n=1}^\infty \frac{1}{(n^2+1)^2} = \frac{\pi}{4}\coth\pi + \frac{\pi^2}{4}\operatorname{csch}^2\pi - \frac{1}{2}$.

64. Prove that (*a*) $\displaystyle\sum_{n=1}^\infty \frac{1}{n^4} = \frac{\pi^4}{90}$, (*b*) $\displaystyle\sum_{n=1}^\infty \frac{1}{n^6} = \frac{\pi^6}{945}$.

65. Prove that $\displaystyle\sum_{n=1}^\infty \frac{(-1)^{n-1}\,n\sin n\theta}{n^2+\alpha^2} = \frac{\pi}{2}\,\frac{\sinh\alpha\theta}{\sinh\alpha\pi}$, $-\pi<\theta<\pi$.

66. Prove that $\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \cdots = \dfrac{\pi^2}{12}$.

67. Prove that $\displaystyle\sum_{n=-\infty}^\infty \frac{1}{n^4+4a^4} = \frac{\pi}{4a^3}\left\{\frac{\sinh 2\pi a + \sin 2\pi a}{\cosh 2\pi a - \cos 2\pi a}\right\}$.

68. Prove that $\displaystyle\sum_{n=-\infty}^\infty \sum_{m=-\infty}^\infty \frac{1}{(m^2+a^2)(n^2+b^2)} = \frac{\pi^2}{ab}\coth\pi a\coth\pi b$.

MITTAG-LEFFLER'S EXPANSION THEOREM

69. Prove that $\csc z = \frac{1}{z} - 2z\left(\frac{1}{z^2-\pi^2} - \frac{1}{z^2-4\pi^2} + \frac{1}{z^2-9\pi^2} - \cdots\right)$.

70. Prove that $\operatorname{sech} z = \pi\left(\frac{1}{(\pi/2)^2+z^2} - \frac{3}{(3\pi/2)^2+z^2} + \frac{5}{(5\pi/2)^2+z^2} - \cdots\right)$.

71. (a) Prove that $\tan z = 2z\left(\frac{1}{(\pi/2)^2-z^2} + \frac{1}{(3\pi/2)^2-z^2} + \frac{1}{(5\pi/2)^2-z^2} + \cdots\right)$.

(b) Use the result in (a) to show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$.

72. Prove the expansions (a) 2, (b) 4, (c) 5, (d) 7, (e) 8 on Page 175.

73. Prove that $\sum_{k=1}^{\infty} \frac{1}{z^2+4k^2\pi^2} = \frac{1}{2z}\left\{\frac{1}{2} - \frac{1}{z} + \frac{1}{e^z-1}\right\}$.

74. Prove that $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \frac{\pi^4}{96}$.

MISCELLANEOUS PROBLEMS

75. Prove that Cauchy's theorem and integral formulae can be obtained as special cases of the residue theorem.

76. Prove that the sum of the residues of the function $\frac{2z^5-4z^2+5}{3z^6-8z+10}$ at all the poles is 2/3.

77. If n is a positive integer, prove that $\int_0^{2\pi} e^{\cos\theta}\cos(n\theta - \sin\theta)\,d\theta = \frac{2\pi}{n!}$.

78. Evaluate $\oint_C z^3 e^{1/z}\,dz$ around the circle C with equation $|z-1| = 4$. *Ans.* 1/24

79. Prove that under suitably stated conditions on the function:

(a) $\int_0^{2\pi} f(e^{i\theta})\,d\theta = 2\pi f(0)$, (b) $\int_0^{2\pi} f(e^{i\theta})\cos\theta\,d\theta = -\pi f'(0)$.

80. Show that (a) $\int_0^{2\pi} \cos(\cos\theta)\cosh(\sin\theta)\,d\theta = 2\pi$

(b) $\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)\cos\theta\,d\theta = \pi$.

81. Prove that $\int_0^{\infty} \frac{\sin ax}{e^{2\pi x}-1}\,dx = \frac{1}{4}\coth\frac{a}{2} - \frac{1}{2a}$.

[*Hint*. Integrate $e^{aiz}/(e^{2\pi z}-1)$ around a rectangle with vertices at $0, R, R+i, i$ and let $R \to \infty$.]

82. Prove that $\int_0^{\infty} \frac{\sin ax}{e^x+1}\,dx = \frac{1}{2a} - \frac{\pi}{2\sinh \pi a}$.

33. If a, p and t are positive constants, prove that $\int_{a-i\infty}^{a+i\infty} \frac{e^{zt}}{z^2+p^2}\,dz = \frac{\sin pt}{p}$.

84. Prove that $\int_0^{\infty} \frac{\ln x}{x^2+a^2}\,dx = \frac{\pi \ln 2}{2a}$.

85. If $-\pi < a < \pi$, prove that $\int_{-\infty}^{\infty} e^{i\lambda x}\frac{\sinh ax}{\sinh \pi x}\,dx = \frac{\sin a}{\cos a + \cosh\lambda}$.

86. Prove that $\displaystyle\int_0^\infty \frac{dx}{(4x^2+\pi^2)\cosh x} = \frac{\ln 2}{2\pi}$.

87. Prove that (a) $\displaystyle\int_0^\infty \frac{\ln x}{x^4+1}\,dx = \frac{-\pi^2\sqrt{2}}{16}$, (b) $\displaystyle\int_0^\infty \frac{(\ln x)^2}{x^4+1}\,dx = \frac{3\pi^3\sqrt{2}}{64}$.

[*Hint.* Consider $\displaystyle\oint_C \frac{(\ln z)^2}{z^4+1}\,dz$ around a semicircle properly indented at $z=0$.]

88. Evaluate $\displaystyle\int_0^\infty \frac{\ln x}{(x^2+1)^2}\,dx$. *Ans.* $\frac{1}{4}\pi\ln 2$

89. Prove that if $|a|<1$ and $b>0$, $\displaystyle\int_0^\infty \frac{\sinh ax}{\sinh x}\cos bx\,dx = \frac{\pi}{2}\left(\frac{\sin a\pi}{\cos a\pi + \cosh b\pi}\right)$.

90. Prove that if $-1<p<1$, $\displaystyle\int_0^\infty \frac{\cos px}{\cosh x}\,dx = \frac{\pi}{2\cosh(p\pi/2)}$.

91. Prove that $\displaystyle\int_0^\infty \frac{\ln(1+x)}{1+x^2}\,dx = \frac{\pi\ln 2}{2}$.

92. If $\alpha>0$ and $-\pi/2<\beta<\pi/2$, prove that

(a) $\displaystyle\int_0^\infty e^{-\alpha x^2\cos\beta}\cos(\alpha x^2\sin\beta)\,dx = \tfrac{1}{2}\sqrt{\pi/\alpha}\cos(\beta/2)$.

(b) $\displaystyle\int_0^\infty e^{-\alpha x^2\cos\beta}\sin(\alpha x^2\sin\beta)\,dx = \tfrac{1}{2}\sqrt{\pi/\alpha}\sin(\beta/2)$.

93. Prove that $\displaystyle\csc^2 z = \sum_{n=-\infty}^{\infty}\frac{1}{(z-n\pi)^2}$.

94. If α and p are real and such that $0<|p|<1$ and $0<|\alpha|<\pi$, prove that

$$\int_0^\infty \frac{x^{-p}\,dx}{x^2+2x\cos\alpha+1} = \left(\frac{\pi}{\sin p\pi}\right)\left(\frac{\sin p\alpha}{\sin\alpha}\right)$$

95. Prove that $\displaystyle\int_0^1 \frac{dx}{\sqrt[3]{x^2-x^3}} = \frac{2\pi}{\sqrt{3}}$. [*Hint.* Consider the contour of Fig. 7-18.]

96. Prove the residue theorem for multiply-connected regions.

97. Find sufficient conditions under which the residue theorem (Problem 2) is valid if C encloses infinitely many isolated singularities.

98. Let C be a circle with equation $|z|=4$. Determine the value of the integral

$$\oint_C z^2\csc\frac{1}{z}\,dz$$

if it exists.

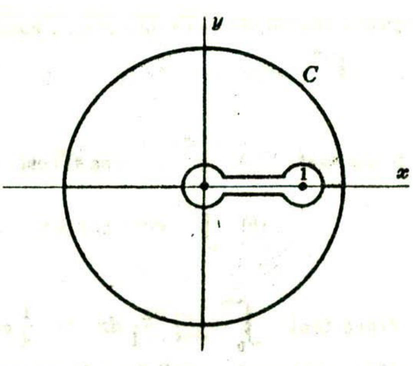

Fig. 7-18

99. Give an analytical proof that $\sin\theta \geqq 2\theta/\pi$ for $0\leqq\theta\leqq\pi/2$.
[*Hint.* Consider the derivative of $(\sin\theta)/\theta$, showing that it is a decreasing function.]

100. Prove that $\displaystyle\int_0^\infty \frac{x}{\sinh\pi x}\,dx = \frac{1}{4}$.

101. Verify that the integral around Γ in equation (2) of Problem 22 goes to zero as $R\to\infty$.

102. (a) If r is real, prove that $\displaystyle\int_0^\pi \ln(1-2r\cos\theta+r^2)\,d\theta = \begin{cases}0 & \text{if } |r|\leqq 1\\ \pi\ln r^2 & \text{if } |r|\geqq 1\end{cases}$.

(b) Use the result in (a) to evaluate $\displaystyle\int_0^{\pi/2}\ln\sin\theta\,d\theta$ (see Problem 23).

103. Complete the proof of Case 2 in Problem 25.

104. If $0 < p < 1$, prove that $\int_0^\infty \frac{x^{-p}}{x-1}\,dx = \pi \cot p\pi$ in the Cauchy principal value sense.

105. Show that $\sum_{n=-\infty}^{\infty} \frac{1}{n^4+n^2+1} = \frac{\pi\sqrt{3}}{3}\tanh\left(\frac{\pi\sqrt{3}}{2}\right)$

106. Verify that as $N \to \infty$ the integral on the left of (1) in Problem 29 goes to zero.

107. Prove that $\frac{1}{1^5} - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \cdots = \frac{5\pi^5}{1536}$.

108. Prove the results given on Page 175 for (a) $\sum_{-\infty}^{\infty} f\left(\frac{2n+1}{2}\right)$ and (b) $\sum_{-\infty}^{\infty} (-1)^n f\left(\frac{2n+1}{2}\right)$.

109. If $-\pi \leqq \theta \leqq \pi$, prove that $\sum_{n=1}^{\infty} \frac{(-1)^n \sin n\theta}{n^3} = \frac{\theta(\pi-\theta)(\pi+\theta)}{12}$.

110. Prove that the function $\cot z - 1/z$ of Problem 34 is bounded on the circles C_N.

111. Show that the second, fourth and sixth integrals in equation (3) of Problem 36 approach zero as $\epsilon \to 0$ and $R \to \infty$.

112. Prove that $\frac{1}{\cosh(\pi/2)} - \frac{1}{3\cosh(3\pi/2)} + \frac{1}{5\cosh(5\pi/2)} - \cdots = \frac{\pi}{8}$.

113. Prove that $\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{e^{zt}}{\sqrt{z}}\,dz = \frac{1}{\sqrt{\pi t}}$ where a and t are any positive constants.

114. Prove that $\sum_{n=1}^{\infty} \frac{\coth n\pi}{n^7} = \frac{19\pi^7}{56{,}700}$.

115. Prove that $\int_0^\infty \frac{dx}{(x^2+1)\cosh \pi x} = \frac{4-\pi}{2}$.

116. Prove that $\frac{1}{1^3 \sinh \pi} - \frac{1}{2^3 \sinh 2\pi} + \frac{1}{3^3 \sinh 3\pi} - \cdots = \frac{\pi^3}{360}$.

117. Prove that if a and t are any positive constants,

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{zt}\cot^{-1} z\,dz = \frac{\sin t}{t}$$