Chapter 8

Conformal Mapping

TRANSFORMATIONS OR MAPPINGS

The set of equations

$$\left.\begin{array}{rcl} u & = & u(x,y) \\ v & = & v(x,y) \end{array}\right\} \qquad (1)$$

defines, in general, a *transformation* or *mapping* which establishes a correspondence between points in the uv and xy planes. The equations (*1*) are called *transformation equations*. If to each point of the uv plane there corresponds one and only one point of the xy plane, and conversely, we speak of a *one to one* transformation or mapping. In such case a set of points in the xy plane [such as a curve or region] is *mapped* into a set of points in the uv plane [curve or region] and conversely. The corresponding sets of points in the two planes are often called *images* of each other.

JACOBIAN OF A TRANSFORMATION

Under the transformation (*1*) a closed region $\mathcal{R}$ of the xy plane is in general mapped into a closed region $\mathcal{R}'$ of the uv plane. Then if ΔA_{xy} and ΔA_{uv} denote respectively the areas of these regions, we can show that if u and v are continuously differentiable,

$$\lim \frac{\Delta A_{uv}}{\Delta A_{xy}} = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| \qquad (2)$$

where lim denotes the limit as ΔA_{xy} (or ΔA_{uv}) approaches zero and where the determinant

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \qquad (3)$$

is called the *Jacobian of the transformation* (*1*).

If we solve (*1*) for x and y in terms of u and v, we obtain the transformation $x = x(u,v)$, $y = y(u,v)$, often called the *inverse transformation* corresponding to (*1*). If x and y are single-valued and continuously differentiable, the Jacobian of this transformation is $\frac{\partial(x,y)}{\partial(u,v)}$ and can be shown equal to the reciprocal of $\frac{\partial(u,v)}{\partial(x,y)}$ [see Problem 7]. Thus if one Jacobian is different from zero in a region, so also is the other.

Conversely we can show that if u and v are continuously differentiable in a region $\mathcal{R}$ and if the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ does not vanish in $\mathcal{R}$, then the transformation (*1*) is one to one.

COMPLEX MAPPING FUNCTIONS

A case of special interest occurs when u and v are real and imaginary parts of an analytic function of a complex variable $z = x+iy$, i.e. $w = u+iv = f(z) = f(x+iy)$.

In such case the Jacobian of the transformation is given by

$$\frac{\partial(u,v)}{\partial(x,y)} = |f'(z)|^2 \tag{4}$$

(see Problem 5). It follows that the transformation is one to one in regions where $f'(z) \neq 0$. Points where $f'(z) = 0$ are called *critical points.*

CONFORMAL MAPPING

Suppose that under transformation (*1*) point (x_0, y_0) of the xy plane is mapped into point (u_0, v_0) of the uv plane [Figs. 8-1 and 8-2] while curves C_1 and C_2 [intersecting at (x_0, y_0)] are mapped respectively into curves C_1' and C_2' [intersecting at (u_0, v_0)]. Then if the transformation is such that the angle at (x_0, y_0) between C_1 and C_2 is equal to the angle at (u_0, v_0) between C_1' and C_2' both in magnitude and sense, the transformation or mapping is said to be *conformal* at (x_0, y_0). A mapping which preserves the magnitudes of angles but not necessarily the sense is called *isogonal.*

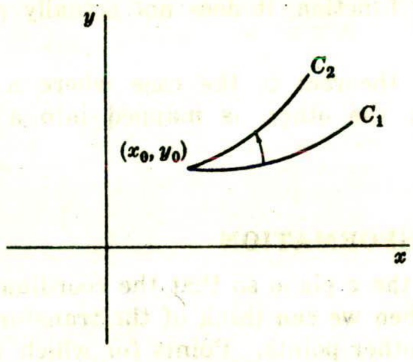


Fig. 8-1

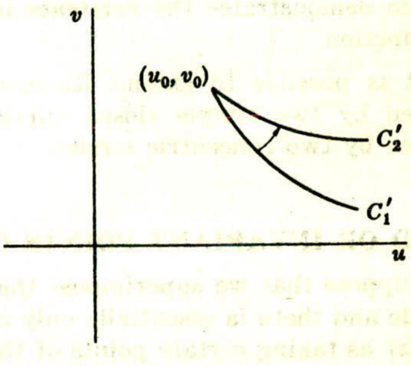


Fig. 8-2

The following theorem is fundamental.

Theorem. If $f(z)$ is analytic and $f'(z) \neq 0$ in a region $\mathcal{R}$, then the mapping $w = f(z)$ is conformal at all points of $\mathcal{R}$.

For conformal mappings or transformations small figures in the neighbourhood of a point z_0 in the z plane map into similar small figures in the w plane and are magnified [or reduced] by an amount given approximately by $|f'(z_0)|^2$, called the *area magnification factor* or simply *magnification factor*. Short distances in the z plane in the neighbourhood of z_0 are magnified [or reduced] in the w plane by an amount given approximately by $|f'(z_0)|$, called the *linear magnification factor*. Large figures in the z plane usually map into figures in the w plane which are far from similar.

RIEMANN'S MAPPING THEOREM

Let C [Fig. 8-3] be a simple closed curve in the z plane forming the boundary of a region $\mathcal{R}$. Let C' [Fig. 8-4] be a circle of radius one and centre at the origin [the *unit circle*] forming the boundary of region $\mathcal{R}'$ in the w plane. The region $\mathcal{R}'$ is sometimes called the *unit disk*. Then *Riemann's mapping theorem* states that there exists a function $w = f(z)$, analytic in $\mathcal{R}$, which maps each point ot $\mathcal{R}$ into a corresponding point of $\mathcal{R}'$ and each point of C into a corresponding point of C', the correspondence being one to one.

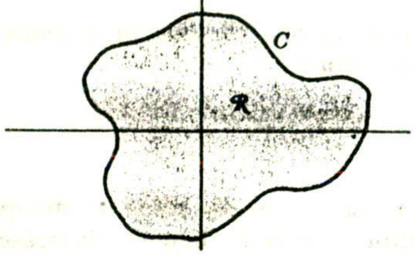


Fig. 8-3

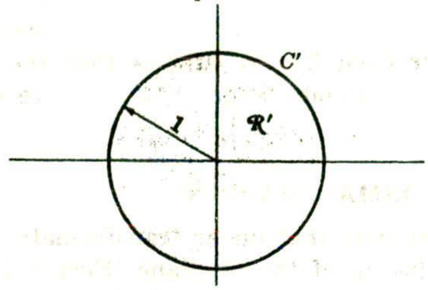


Fig. 8-4

This function $f(z)$ contains three arbitrary real constants which can be determined by making the centre of C' correspond to some given point in $\mathcal{R}$, while some point on C' corresponds to a given point on C. It should be noted that while Riemann's mapping theorem demonstrates the *existence* of a mapping function, it does not actually produce this function.

It is possible to extend Riemann's mapping theorem to the case where a region bounded by two simple closed curves, one inside the other, is mapped into a region bounded by two concentric circles.

FIXED OR INVARIANT POINTS OF A TRANSFORMATION

Suppose that we superimpose the w plane on the z plane so that the coordinate axes coincide and there is essentially only one plane. Then we can think of the transformation $w = f(z)$ as taking certain points of the plane into other points. Points for which $z = f(z)$ will however remain fixed, and for this reason we call them the *fixed* or *invariant points* of the transformation.

Example: The fixed or invariant points of the transformation $w = z^2$ are solutions of $z^2 = z$, i.e. $z = 0, 1$.

SOME GENERAL TRANSFORMATIONS

In the following α, β are given complex constants while a, θ_0 are real constants.

1. **Translation.** $w = z + \beta$

   By this transformation, figures in the z plane are *displaced* or *translated* in the direction of vector β.

2. **Rotation.** $w = e^{i\theta_0} z$

   By this transformation, figures in the z plane are rotated through an angle θ_0. If $\theta_0 > 0$ the rotation is counterclockwise, while if $\theta_0 < 0$ the rotation is clockwise.

3. **Stretching.** $w = az$

   By this transformation, figures in the z plane are stretched (or contracted) in the direction z if $a > 1$ (or $0 < a < 1$). We consider contraction as a special case of stretching.

4. **Inversion.** $w = 1/z$

SUCCESSIVE TRANSFORMATIONS

If $w = f_1(\zeta)$ maps region $\mathcal{R}_\zeta$ of the ζ plane into region $\mathcal{R}_w$ of the w plane while $\zeta = f_2(z)$ maps region $\mathcal{R}_z$ of the z plane into region $\mathcal{R}_\zeta$, then $w = f_1[f_2(z)]$ maps $\mathcal{R}_z$ into $\mathcal{R}_w$. The functions f_1 and f_2 define ***successive transformations*** from one plane to another which are equivalent to a single transformation. These ideas are easily generalized.

THE LINEAR TRANSFORMATION

The transformation

$$w = \alpha z + \beta \tag{5}$$

where α and β are given complex constants, is called a *linear transformation.* Since we can write (*5*) in terms of the successive transformations $w = \zeta + \beta$, $\zeta = e^{i\theta_0}\tau$, $\tau = az$ where $\alpha = ae^{i\theta_0}$, we see that a general linear transformation is a combination of the transformations of translation, rotation and stretching.

THE BILINEAR OR FRACTIONAL TRANSFORMATION

The transformation

$$w = \frac{\alpha z + \beta}{\gamma z + \delta}, \qquad \alpha\delta - \beta\gamma \neq 0 \tag{6}$$

is called a *bilinear* or *fractional transformation.* This transformation can be considered as combinations of the transformations of translation, rotation, stretching and inversion.

The transformation (*6*) has the property that circles in the z plane are mapped into circles in the w plane, where by circles we include circles of infinite radius which are straight lines. See Problems 14 and 15.

The transformation maps any three distinct points of the z plane into three distinct points of the w plane, one of which may be at infinity.

If z_1, z_2, z_3, z_4 are distinct, then the quantity

$$\frac{(z_4 - z_1)(z_2 - z_3)}{(z_2 - z_1)(z_4 - z_3)} \tag{7}$$

is called the *cross ratio* of z_1, z_2, z_3, z_4. This ratio is invariant under the bilinear transformation, and this property can be used in obtaining specific bilinear transformations mapping three points into three other points.

MAPPING OF A HALF PLANE ON TO A CIRCLE

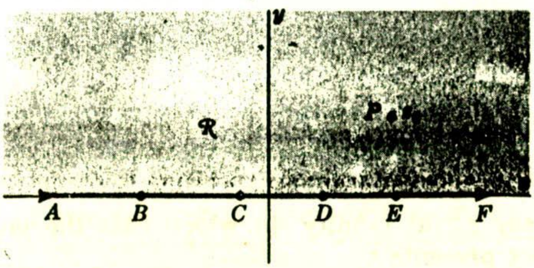


Fig. 8-5

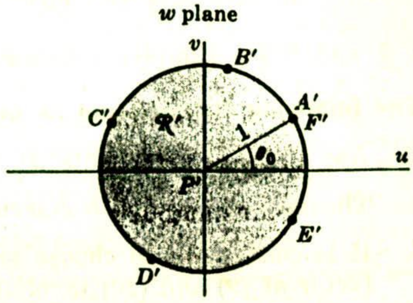


Fig. 8-6

Let z_0 be any point P in the upper half of the z plane denoted by $\mathcal{R}$ in Fig. 8-5 above. Then the transformation

$$w = e^{i\theta_0}\left(\frac{z - z_0}{z - \bar{z}_0}\right) \tag{8}$$

maps this upper half plane in a one to one manner on to the interior $\mathcal{R}'$ of the unit circle $|w| = 1$, and conversely. Each point of the x axis is mapped on to the boundary of the circle. The constant θ_0 can be determined by making one particular point of the x axis correspond to a given point on the circle.

In the above figures we have used the convention that unprimed points such as $A, B, C,$ etc., in the z plane correspond to primed points $A', B', C',$ etc., in the w plane. Also, in the case where points are at infinity we indicate this by an arrow such as at A and F in Fig. 8-5 which correspond respectively to A' and F' (the same point) in Fig. 8-6 above. As point z moves on the boundary of $\mathcal{R}$ [i.e. the real axis] from $-\infty$ (point A) to $+\infty$ (point F), w moves counterclockwise along the unit circle from A' back to A'.

THE SCHWARZ-CHRISTOFFEL TRANSFORMATION

Consider a polygon [Fig. 8-7] in the w plane having vertices at $w_1, w_2, \ldots, w_n$ with corresponding interior angles $\alpha_1, \alpha_2, \ldots, \alpha_n$ respectively. Let the points $w_1, w_2, \ldots, w_n$ map respectively into points $x_1, x_2, \ldots, x_n$ on the real axis of the z plane [Fig. 8-8].

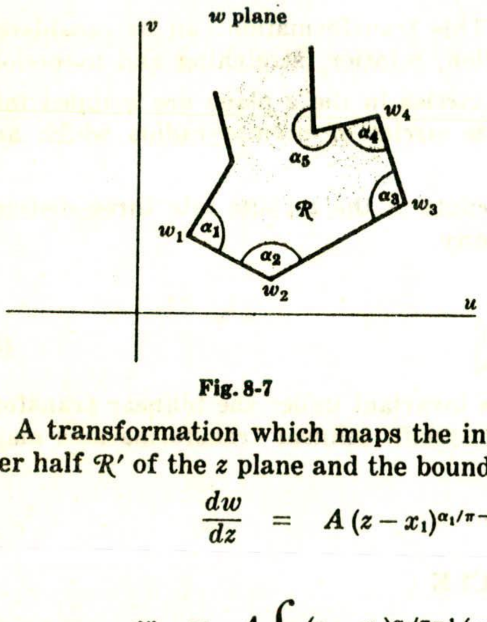


Fig. 8-7

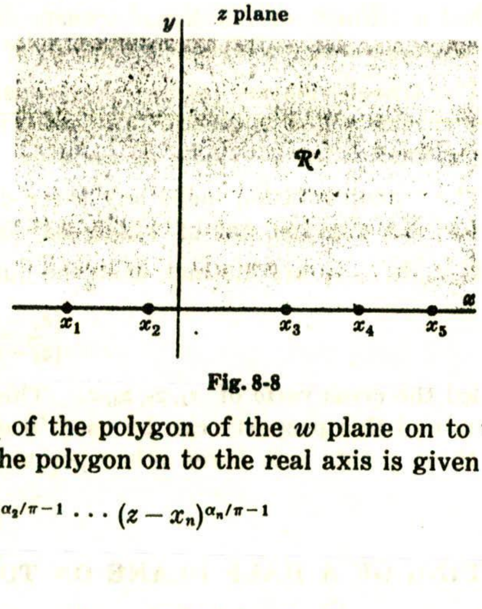


Fig. 8-8

A transformation which maps the interior $\mathcal{R}$ of the polygon of the w plane on to the upper half $\mathcal{R}'$ of the z plane and the boundary of the polygon on to the real axis is given by

$$\frac{dw}{dz} = A\,(z - x_1)^{\alpha_1/\pi - 1}(z - x_2)^{\alpha_2/\pi - 1} \cdots (z - x_n)^{\alpha_n/\pi - 1} \tag{9}$$

or

$$w = A\int (z - x_1)^{\alpha_1/\pi - 1}(z - x_2)^{\alpha_2/\pi - 1} \cdots (z - x_n)^{\alpha_n/\pi - 1}\,dz + B \tag{10}$$

where A and B are complex constants.

The following facts should be noted:

1. Any three of the points $x_1, x_2, \ldots, x_n$ can be chosen at will.
2. The constants A and B determine the size, orientation and position of the polygon.
3. It is convenient to choose one point, say x_n, at infinity in which case the last factor of (9) and (10) involving x_n is not present.
4. Infinite open polygons can be considered as limiting cases of closed polygons.

TRANSFORMATIONS OF BOUNDARIES IN PARAMETRIC FORM

Suppose that in the z plane a curve C [Fig. 8-9], which may or may not be closed, has parametric equations given by

$$x = F(t), \quad y = G(t) \tag{11}$$

where we assume that F and G are continuously differentiable. Then the transformation

$$z = F(w) + i\,G(w) \tag{12}$$

maps curve C on to the real axis C' of the w plane [Fig. 8-10].

z plane

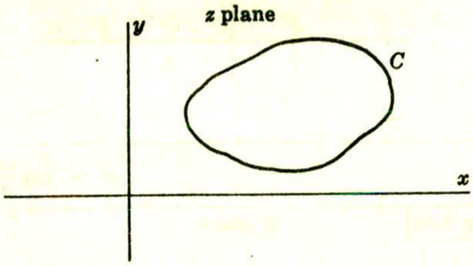


Fig. 8-9

w plane

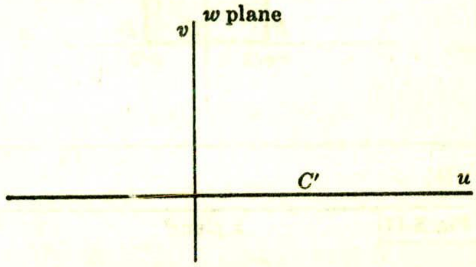


Fig. 8-10

SOME SPECIAL MAPPINGS

For reference purposes we list here some special mappings which are useful in practice. For convenience we have listed separately the mapping functions which map the given region $\mathcal{R}$ of the w or z plane on to the upper half of the z or w plane or the unit circle in the z or w plane, depending on which mapping function is simpler. As we have already seen there exists a transformation [equation (8)] which maps the upper half plane on to the unit circle.

A. Mappings on the Upper Half Plane

A–1 Infinite sector of angle π/m $\qquad w = z^m, \quad m \geqq 1/2$

Fig. 8-11 z plane

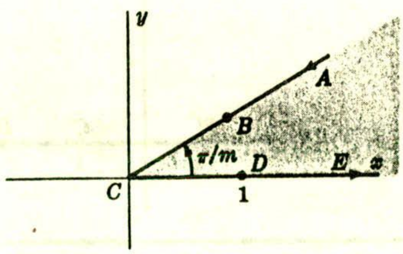
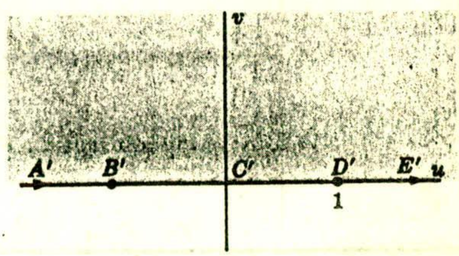


Fig. 8-12 w plane



A–2 Infinite strip of width a $\qquad w = e^{\pi z/a}$

Fig. 8-13 z plane

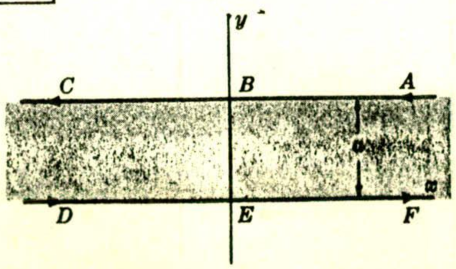
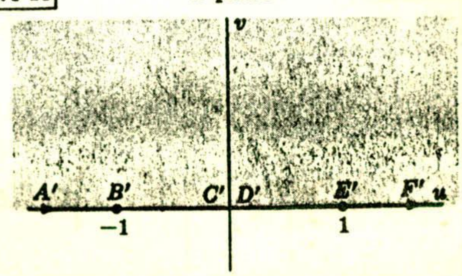


Fig. 8-14 w plane



A–3 **Semi-Infinite strip of width a**

(*a*) $w = \sin \frac{\pi z}{a}$

Fig. 8-15 *z* plane

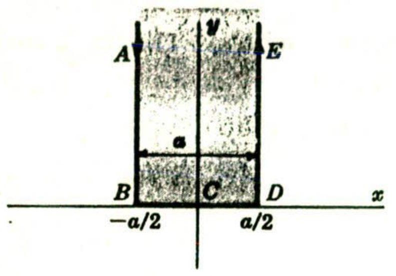
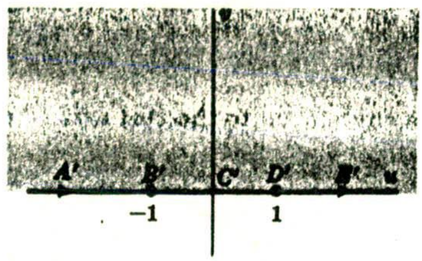


Fig. 8-16 *w* plane



(*b*) $w = \cos \frac{\pi z}{a}$

Fig. 8-17 *z* plane

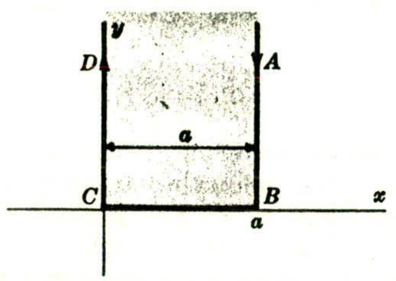
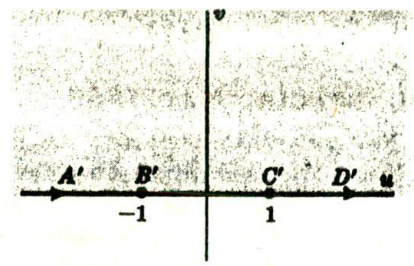


Fig. 8-18 *w* plane



(*c*) $w = \cosh \frac{\pi z}{a}$

Fig. 8-19 *z* plane

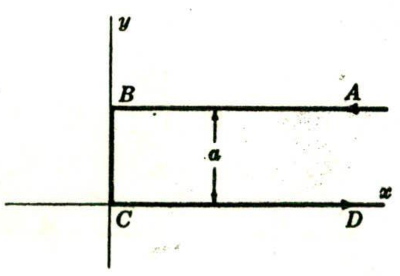
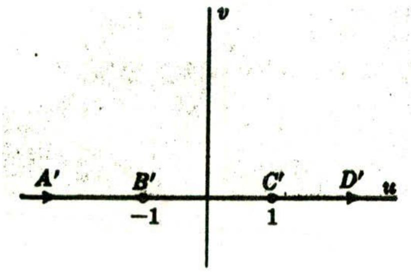


Fig. 8-20 *w* plane



A–4 **Half plane with semicircle removed** $w = \frac{a}{2}\left(z + \frac{1}{z}\right)$

Fig. 8-21 *z* plane

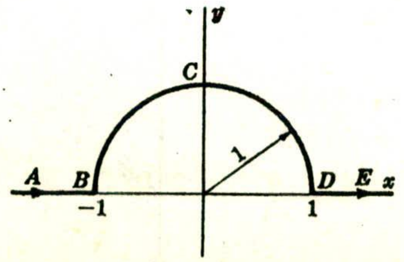
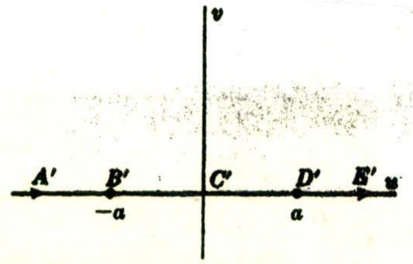


Fig. 8-22 *w* plane



A–5 **Semicircle** $w = \left(\frac{1+z}{1-z}\right)^2$

Fig. 8-23 *z* plane

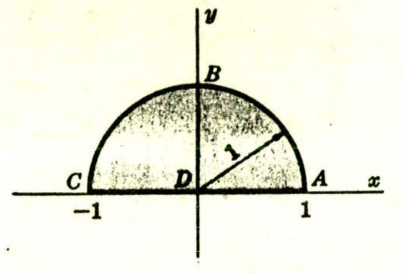
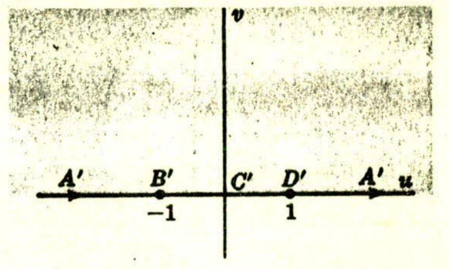


Fig. 8-24 *w* plane



A–6 **Sector of a circle** $w = \left(\frac{1+z^m}{1-z^m}\right)^2, \quad m \ge \frac{1}{2}$

Fig. 8-25 *z* plane

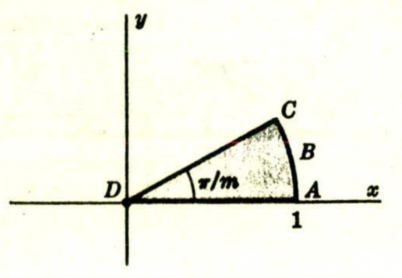
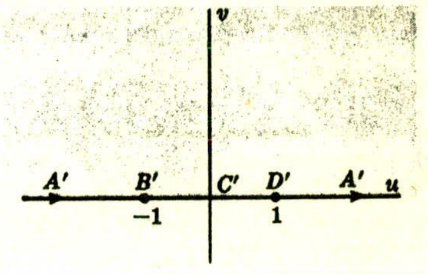


Fig. 8-26 *w* plane



A–7 **Lens-shaped region of angle π/m**
[ABC and CDA are circular arcs.] $w = e^{2mi\cot^{-1}p}\left(\frac{z+1}{z-1}\right)^m, \quad m \ge 2$

Fig. 8-27 *z* plane

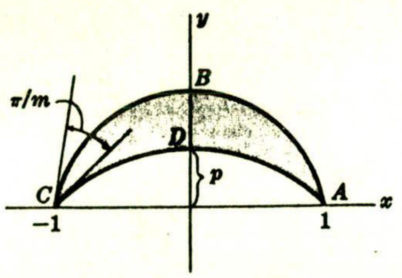
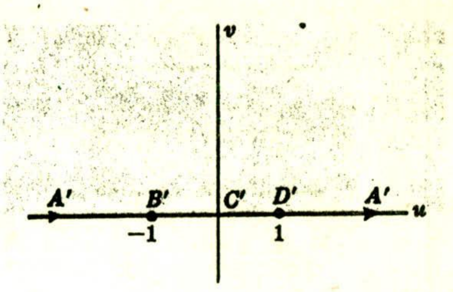


Fig. 8-28 *w* plane



A–8 **Half plane with circle removed** $w = \coth(\pi/z)$

Fig. 8-29 *z* plane

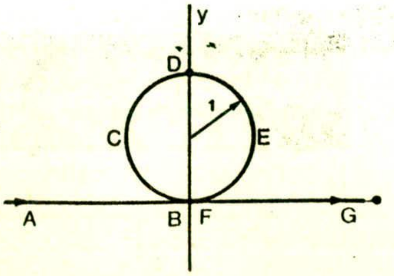


Fig. 8-30 *w* plane

A-9 Exterior of parabola $y^2 = 4p(p - x)$ $\qquad w = i(\sqrt{z} - \sqrt{p})$

Fig. 8-31 *z* plane

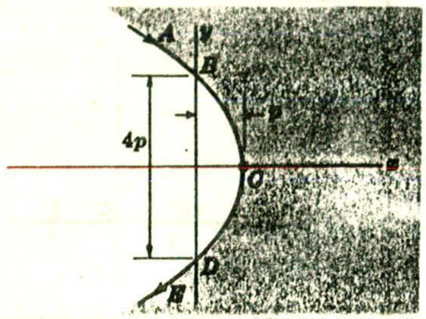
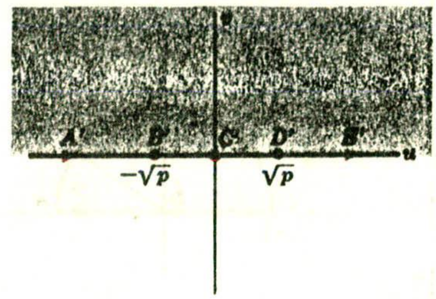


Fig. 8-32 *w* plane



A-10 Interior of the parabola $y^2 = 4p(p - x)$ $\qquad w = e^{\pi i \sqrt{z/p}}$

Fig. 8-33 *z* plane

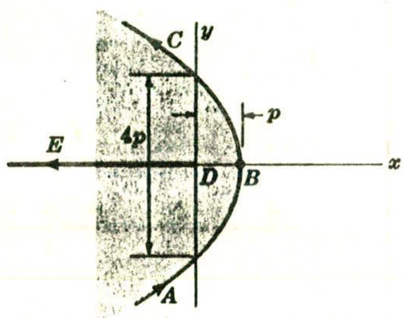
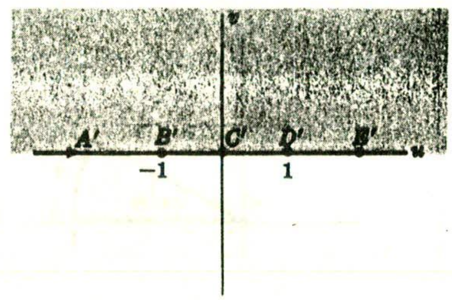


Fig. 8-34 *w* plane



A-11 Plane with two semi-infinite parallel cuts $\qquad w = -\pi i + 2 \ln z - z^2$

Fig. 8-35 *w* plane

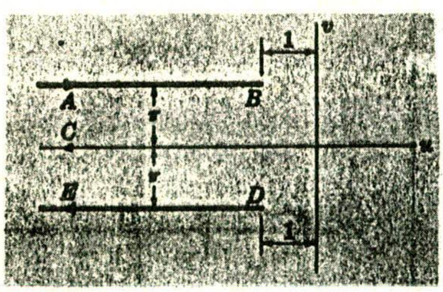
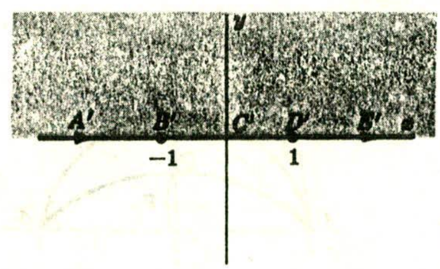


Fig. 8-36 *z* plane



A-12 Channel with right angle bend $\qquad w = \frac{2}{\pi}\{\tanh^{-1} p\sqrt{z} - p \tan^{-1}\sqrt{z}\}$

Fig. 8-37 *w* plane

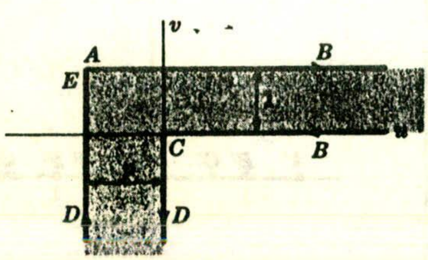
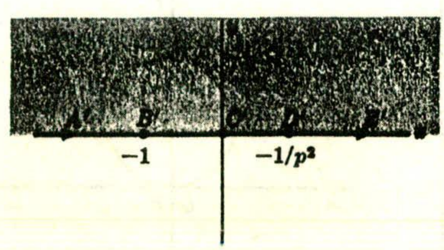


Fig. 8-38 *z* plane



A–13 **Interior of triangle** $w = \int_0^z t^{\alpha/\pi - 1}(1-t)^{\beta/\pi - 1}\,dt$

Fig. 8-39 *w* plane

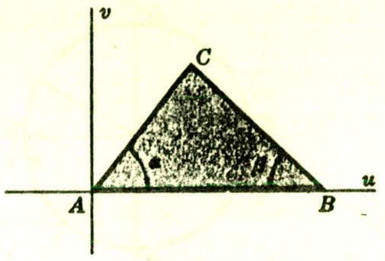
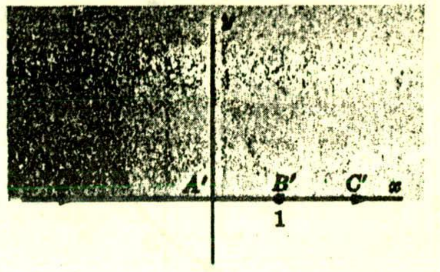


Fig. 8-40 *z* plane



A–14 **Interior of rectangle** $w = \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad 0 < k < 1$

Fig. 8-41 *w* plane

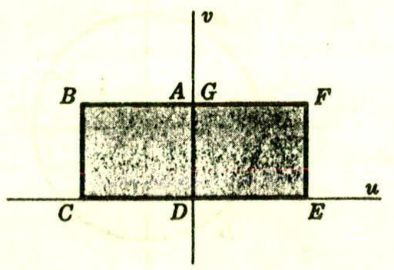
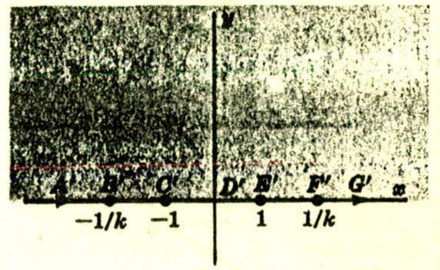


Fig. 8-42 *z* plane



B. Mappings on the Unit Circle

B–1 **Exterior of unit circle** $w = 1/z$

Fig. 8-43 *w* plane

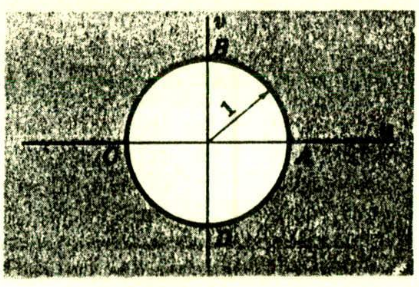
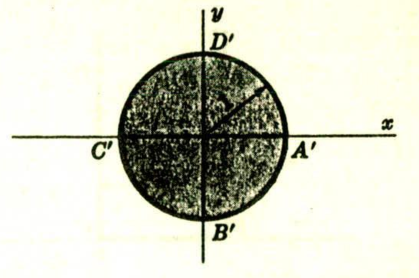


Fig. 8-44 *z* plane



B–2 **Exterior of ellipse** $w = \frac{1}{2}(ze^{-\alpha} + z^{-1}e^{\alpha})$

Fig. 8-45 *w* plane

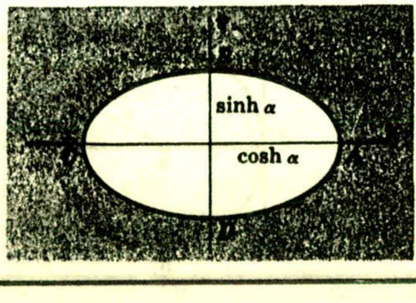
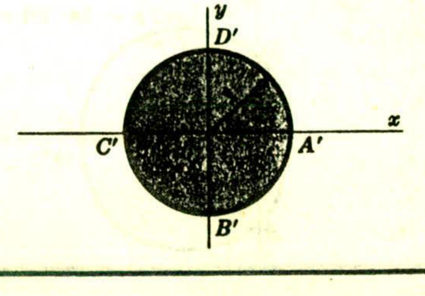


Fig. 8-46 *z* plane



B-3 **Exterior of parabola** $y^2 = 4p(p-x)$ $\qquad w = 2\sqrt{\frac{p}{z}} - 1$

Fig. 8-47 **z plane**

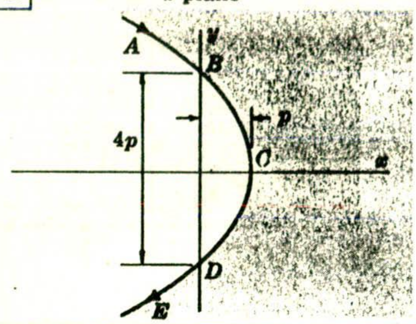
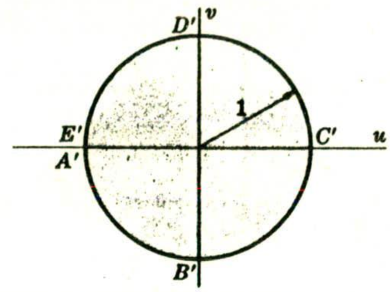


Fig. 8-48 **w plane**



B-4 **Interior of parabola** $y^2 = 4p(p-x)$ $\qquad w = \tan^2 \frac{\pi}{4} \sqrt{\frac{z}{p}}$

Fig. 8-49 **z plane**

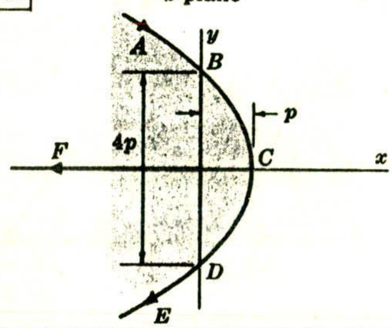
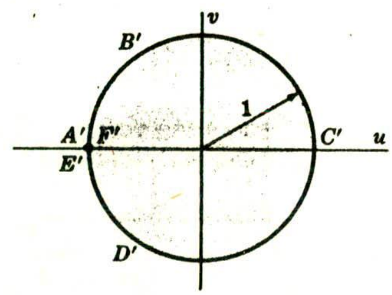


Fig. 8-50 **w plane**



C. Miscellaneous Mappings

C-1 **Semi-infinite strip of width a on to quarter plane** $\qquad w = \sin \frac{\pi z}{2a}$

Fig. 8-51 **z plane**

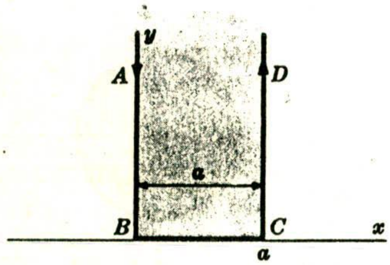
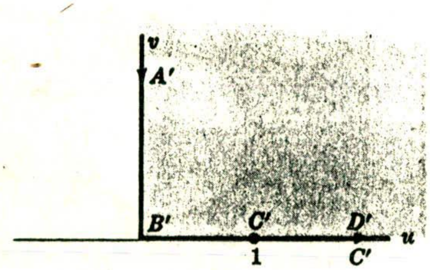


Fig. 8-52 **w plane**



C-2 **Interior of cardioid on to circle** $\qquad w = z^2$

Fig. 8-53 **w plane**

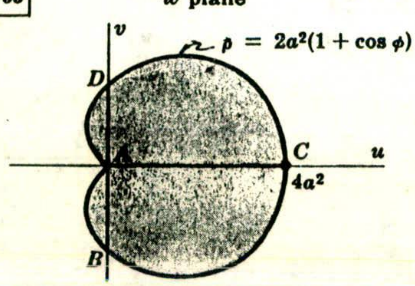
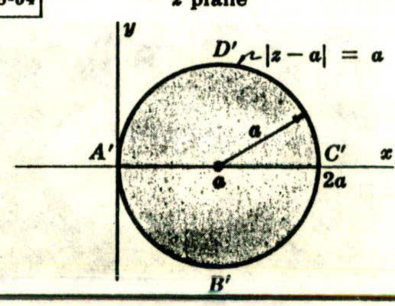


Fig. 8-54 **z plane**



C-3 **Annulus on to rectangle** $w = \ln z$

Fig. 8-55 z plane

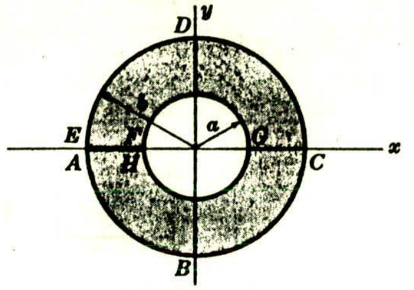
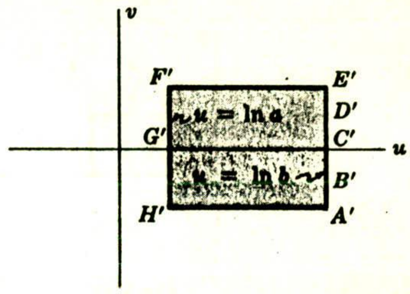


Fig. 8-56 w plane



C-4 **Semi-infinite strip on to infinite strip** $w = \ln \coth (z/2)$

Fig. 8-57 z plane

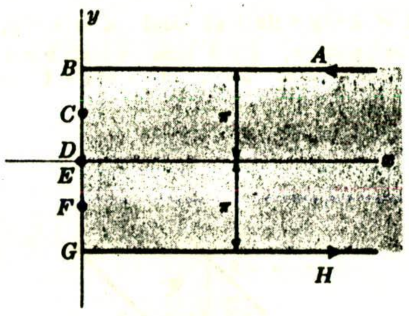
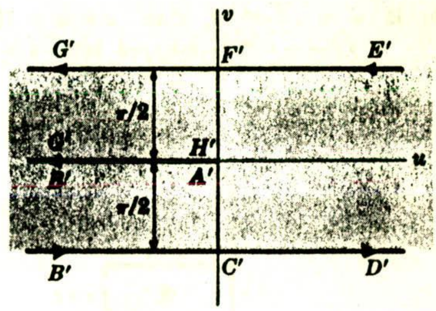


Fig. 8-58 w plane



C-5 **Plane with two semi-infinite cuts on to infinite strip** $w = z + e^z$

Fig. 8-59 w plane

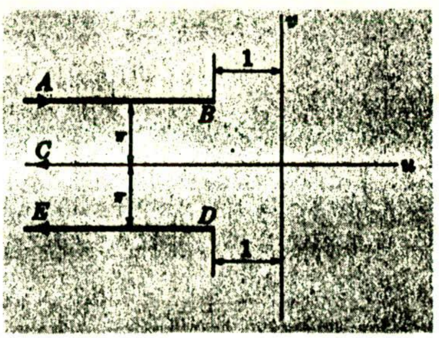
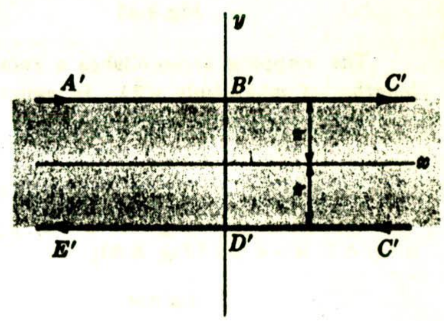


Fig. 8-60 z plane



Solved Problems

TRANSFORMATIONS

1. Let the rectangular region $\mathcal{R}$ [Fig. 8-61 below] in the z plane be bounded by $x = 0$, $y = 0$, $x = 2$, $y = 1$. Determine the region $\mathcal{R}'$ of the w plane into which $\mathcal{R}$ is mapped under the transformations:
   (a) $w = z + (1 - 2i)$, (b) $w = \sqrt{2}\, e^{\pi i/4} z$, (c) $w = \sqrt{2}\, e^{\pi i/4} z + (1 - 2i)$.

   (a) If $w = z + (1 - 2i)$, then $u + iv = x + iy + 1 - 2i = (x + 1) + i(y - 2)$ and $u = x + 1$, $v = y - 2$.

Line $x = 0$ is mapped into $u = 1$; $y = 0$ into $v = -2$; $x = 2$ into $u = 3$; $y = 1$ into $v = -1$ [Fig. 8-62]. Similarly, we can show that each point of $\mathcal{R}$ is mapped into one and only one point of $\mathcal{R}'$ and conversely.

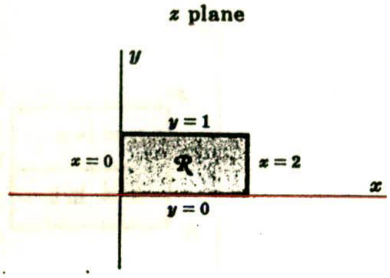


Fig. 8-61

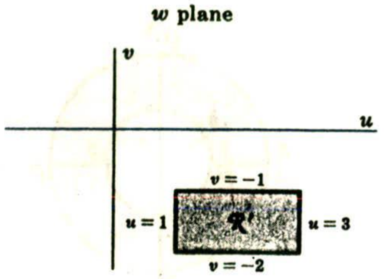


Fig. 8-62

The transformation or mapping accomplishes a *translation* of the rectangle. In general, $w = z + \beta$ accomplishes a translation of any region.

(*b*) If $w = \sqrt{2}\, e^{\pi i/4} z$, then $u + iv = (1+i)(x+iy) = x - y + i(x+y)$ and $u = x - y$, $v = x + y$. Line $x = 0$ is mapped into $u = -y$, $v = y$ or $u = -v$; $y = 0$ into $u = x$, $v = x$ or $u = v$; $x = 2$ into $u = 2 - y$, $v = 2 + y$ or $u + v = 4$; $y = 1$ into $u = x - 1$, $v = x + 1$ or $v - u = 2$ [Fig. 8-64].

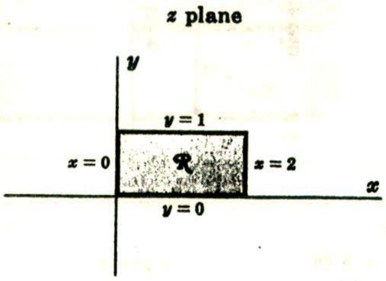


Fig. 8-63

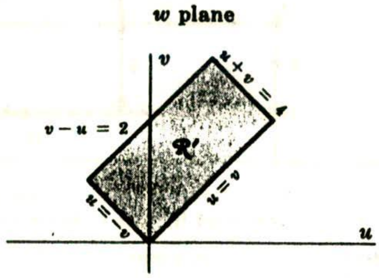


Fig. 8-64

The mapping accomplishes a *rotation* of $\mathcal{R}$ (through angle $\pi/4$ or 45°) and a *stretching* of lengths (of magnitude $\sqrt{2}$). In general the transformation $w = \alpha z$ accomplishes a rotation and stretching of a region.

(*c*) If $w = \sqrt{2}\, e^{\pi i/4} z + (1 - 2i)$, then $u + iv = (1+i)(x+iy) + 1 - 2i$ and $u = x - y + 1$, $v = x + y - 2$. The lines $x = 0$, $y = 0$, $x = 2$, $y = 1$ are mapped respectively into $u + v = -1$, $u - v = 3$, $u + v = 3$, $u - v = 1$ [Fig. 8-66].

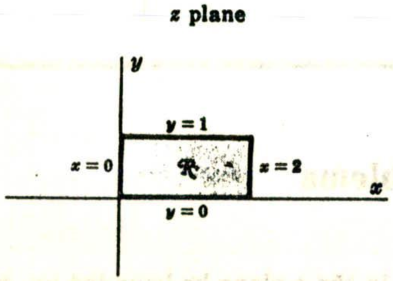


Fig. 8-65

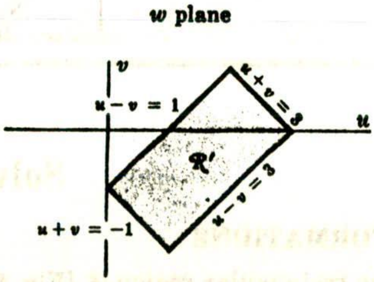


Fig. 8-66

The mapping accomplishes a rotation and stretching as in (*b*) and a subsequent translation. In general the transformation $w = \alpha z + \beta$ accomplishes a rotation, stretching and translation. This can be considered as two successive mappings $w = \alpha z_1$ (rotation and stretching) and $z_1 = z + \beta/\alpha$ (translation).

2. **Determine the region of the w plane into which each of the following is mapped by the transformation $w = z^2$.**

(a) **First quadrant of the z plane.**

Let $z = re^{i\theta}$, $w = \rho e^{i\phi}$. Then if $w = z^2$, $\rho e^{i\phi} = r^2 e^{2i\theta}$ and $\rho = r^2$, $\phi = 2\theta$. Thus points in the z plane at (r, θ) are rotated through angle 2θ. Since all points in the first quadrant [Fig. 8-67] of the z plane occupy the region $0 \leq \theta \leq \pi/2$, they map into $0 \leq \phi \leq \pi$ or the upper half of the w plane [Fig. 8-68].

z plane

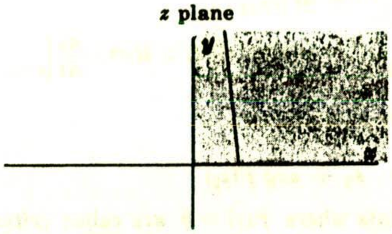


Fig. 8-67

w plane

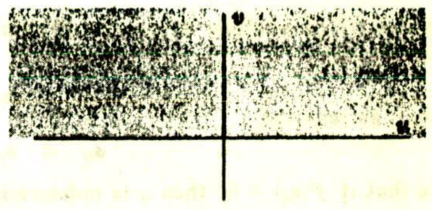


Fig. 8-68

(b) **Region bounded by $x = 1$, $y = 1$ and $x + y = 1$.**

Since $w = z^2$ is equivalent to $u + iv = (x + iy)^2 = x^2 - y^2 + 2ixy$, we see that $u = x^2 - y^2$, $v = 2xy$. Then line $x = 1$ maps into $u = 1 - y^2$, $v = 2y$ or $u = 1 - v^2/4$; line $y = 1$ into $u = x^2 - 1$, $v = 2x$ or $u = v^2/4 - 1$; line $x + y = 1$ or $y = 1 - x$ into $u = x^2 - (1 - x)^2 = 2x - 1$, $v = 2x(1 - x) = 2x - 2x^2$ or $v = \frac{1}{2}(1 - u^2)$ on eliminating x.

The regions appear shaded in Figures 8-69 and 8-70 below where points A, B, C map into A', B', C'. Note that the angles of triangle ABC are equal respectively to the angles of curvilinear triangle $A'B'C'$. This is a consequence of the fact that the mapping is *conformal*.

z plane

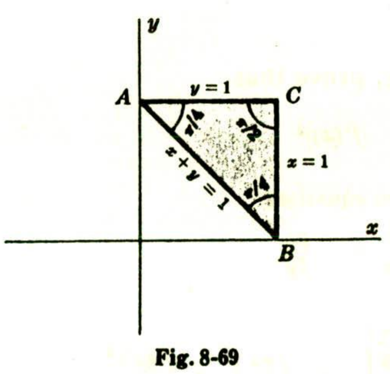


Fig. 8-69

w plane

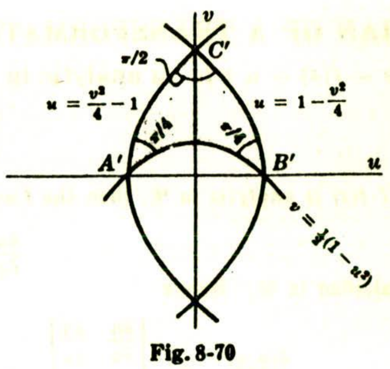


Fig. 8-70

CONFORMAL TRANSFORMATIONS

3. **Consider the transformation $w = f(z)$ where $f(z)$ is analytic at z_0 and $f'(z_0) \neq 0$. Prove that under this transformation the tangent at z_0 to any curve C in the z plane passing through z_0 [Fig. 8-71] is rotated through the angle $\arg f'(z_0)$.**

z plane

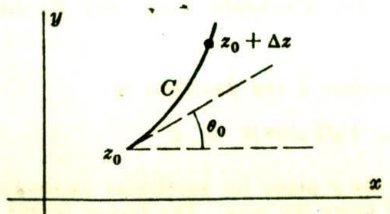


Fig. 8-71

w plane

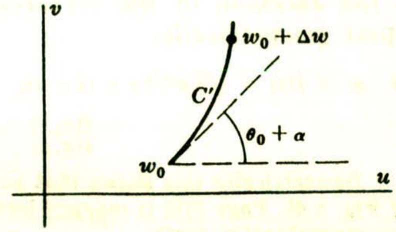


Fig. 8-72

As a point moves from z_0 to $z_0 + \Delta z$ along C, the image point moves along C' in the w plane from w_0 to $w_0 + \Delta w$. If the parameter used to describe the curve is t, then corresponding to the path $z = z(t)$ [or $x = x(t)$, $y = y(t)$] in the z plane, we have the path $w = w(t)$ [or $u = u(t)$, $v = v(t)$] in the w plane.

The derivatives dz/dt and dw/dt represent tangent vectors to corresponding points on C and C'.

Now $\dfrac{dw}{dt} = \dfrac{dw}{dz}\cdot\dfrac{dz}{dt} = f'(z)\dfrac{dz}{dt}$ and, in particular at z_0 and w_0,

$$\left.\frac{dw}{dt}\right|_{w=w_0} = f'(z_0)\left.\frac{dz}{dt}\right|_{z=z_0} \tag{1}$$

provided $f(z)$ is analytic at $z = z_0$. Writing $\left.\dfrac{dw}{dt}\right|_{w=w_0} = \rho_0 e^{i\phi_0}$, $f'(z) = Re^{i\alpha}$, $\left.\dfrac{dz}{dt}\right|_{z=z_0} = r_0 e^{i\theta_0}$, we have from (1)

$$\rho_0 e^{i\phi_0} = Rr_0 e^{i(\theta_0+\alpha)} \tag{2}$$

so that, as required,

$$\phi_0 = \theta_0 + \alpha = \theta_0 + \arg f'(z_0) \tag{3}$$

Note that if $f'(z_0) = 0$, then α is indeterminate. Points where $f'(z) = 0$ are called *critical points*.

4. Prove that the angle between two curves C_1 and C_2 passing through the point z_0 in the z plane [see Figures 8-1 and 8-2, Page 201] is preserved [in magnitude and sense] under the transformation $w = f(z)$, i.e. the mapping is conformal, if $f(z)$ is analytic at z_0 and $f'(z_0) \neq 0$.

By Problem 3 each curve is rotated through the angle $\arg f'(z_0)$. Hence the angle between the curves must be preserved, both in magnitude and sense, in the mapping.

JACOBIAN OF A TRANSFORMATION

5. If $w = f(z) = u + iv$ is analytic in a region $\mathcal{R}$, prove that

$$\frac{\partial(u,v)}{\partial(x,y)} = |f'(z)|^2$$

If $f(z)$ is analytic in $\mathcal{R}$, then the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

are satisfied in $\mathcal{R}$. Hence

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{vmatrix} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

$$= \left|\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right|^2 = |f'(z)|^2$$

using Problem 5, Chapter 3, Page 72.

6. Find the Jacobian of the transformation in (a) Problem 1(c), (b) Problem 2 and interpret geometrically.

(a) If $w = f(z) = \sqrt{2}\, e^{\pi i/4} z + (1 - 2i)$, then by Problem 5 the Jacobian is

$$\frac{\partial(u,v)}{\partial(x,y)} = |f'(z)|^2 = |\sqrt{2}\, e^{\pi i/4}|^2 = 2$$

Geometrically this shows that any region in the z plane [in particular rectangular region $\mathcal{R}$ of Fig. 8-65, Page 212] is mapped into a region of twice the area. The factor $|f'(z)|^2 = 2$ is called the *magnification factor*.

Another method. The transformation is equivalent to $u = x - y,\ v = x + y$ and so

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$$

(*b*) If $w = f(z) = z^2$, then

$$\frac{\partial(u,v)}{\partial(x,y)} = |f'(z)|^2 = |2z|^2 = |2x+2iy|^2 = 4(x^2+y^2)$$

Geometrically, a small region in the z plane having area A and at approximate distance r from the origin would be mapped into a region of the w plane having area $4r^2A$. Thus regions far from the origin would be mapped into regions of greater area than similar regions near the origin.

Note that at the critical point $z=0$ the Jacobian is zero. At this point the transformation is not conformal.

7. Prove that $\dfrac{\partial(u,v)}{\partial(x,y)} \cdot \dfrac{\partial(x,y)}{\partial(u,v)} = 1$.

Corresponding to the transformation (*1*) $u = u(x,y),\ v = v(x,y)$, with Jacobian $\dfrac{\partial(u,v)}{\partial(x,y)}$, we have the inverse transformation (*2*) $x = x(u,v),\ y = y(u,v)$, with Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$.

From (*1*), $$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy, \qquad dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy.$$

From (*2*), $$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv, \qquad dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv.$$

Hence, $$du = \frac{\partial u}{\partial x}\left\{\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right\} + \frac{\partial u}{\partial y}\left\{\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right\}$$
$$= \left\{\frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u}\right\}du + \left\{\frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v}\right\}dv$$

from which $$\frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u} = 1, \qquad \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v} = 0 \tag{3}$$

Similarly we find $$\frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v} = 1, \qquad \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u} = 0 \tag{4}$$

Using (*3*) and (*4*) and the rule for products of determinants (see Problem 94), we have

$$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$
$$= \begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

8. Discuss Problem 7 if u and v are real and imaginary parts of an analytic function $f(z)$.

In this case $\dfrac{\partial(u,v)}{\partial(x,y)} = |f'(z)|^2$ by Problem 5. If the inverse to $w = f(z)$ is $z = g(w)$ assumed single-valued and analytic, then $\dfrac{\partial(x,y)}{\partial(u,v)} = |g'(w)|^2$. The result of Problem 7 is a consequence of the fact that

$$|f'(z)|^2\,|g'(w)|^2 \;=\; \left|\frac{dw}{dz}\right|^2 \cdot \left|\frac{dz}{dw}\right|^2 \;=\; 1$$

since $dw/dz = 1/(dz/dw)$.

BILINEAR OR FRACTIONAL TRANSFORMATIONS

9. Find a bilinear transformation which maps points z_1, z_2, z_3 of the z plane into points w_1, w_2, w_3 of the w plane respectively.

If w_k corresponds to z_k, $k = 1, 2, 3$, we have

$$w - w_k \;=\; \frac{\alpha z + \beta}{\gamma z + \delta} - \frac{\alpha z_k + \beta}{\gamma z_k + \delta} \;=\; \frac{(\alpha\delta - \beta\gamma)(z - z_k)}{(\gamma z + \delta)(\gamma z_k + \delta)}$$

Then

$$w - w_1 = \frac{(\alpha\delta - \beta\gamma)(z - z_1)}{(\gamma z + \delta)(\gamma z_1 + \delta)}, \qquad w - w_3 = \frac{(\alpha\delta - \beta\gamma)(z - z_3)}{(\gamma z + \delta)(\gamma z_3 + \delta)} \tag{1}$$

Replacing w by w_2, and z by z_2,

$$w_2 - w_1 = \frac{(\alpha\delta - \beta\gamma)(z_2 - z_1)}{(\gamma z_2 + \delta)(\gamma z_1 + \delta)}, \qquad w_2 - w_3 = \frac{(\alpha\delta - \beta\gamma)(z_2 - z_3)}{(\gamma z_2 + \delta)(\gamma z_3 + \delta)} \tag{2}$$

By division of (1) and (2), assuming $\alpha\delta - \beta\gamma \neq 0$,

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} \;=\; \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} \tag{3}$$

Solving for w in terms of z gives the required transformation. The right-hand side of (3) is called the *cross ratio* of z_1, z_2, z_3 and z.

10. Find a bilinear transformation which maps points $z = 0, -i, -1$ into $w = i, 1, 0$ respectively.

Method 1. Since $w = \dfrac{\alpha z + \beta}{\gamma z + \delta}$, we have

$$(1)\quad i = \frac{\alpha(0) + \beta}{\gamma(0) + \delta}, \qquad (2)\quad 1 = \frac{\alpha(-i) + \beta}{\gamma(-i) + \delta}, \qquad (3)\quad 0 = \frac{\alpha(-1) + \beta}{\gamma(-1) + \delta}$$

From (3), $\beta = \alpha$. From (1), $\delta = \beta/i = -i\alpha$. From (2), $\gamma = i\alpha$. Then

$$w \;=\; \frac{\alpha z + \alpha}{i\alpha z - i\alpha} \;=\; \frac{1}{i}\left(\frac{z+1}{z-1}\right) \;=\; -i\left(\frac{z+1}{z-1}\right)$$

Method 2. Use Problem 9. Then

$$\frac{(w - i)(1 - 0)}{(w - 0)(1 - i)} \;=\; \frac{(z - 0)(-i + 1)}{(z + 1)(-i - 0)}. \qquad \text{Solving, } w \;=\; -i\left(\frac{z+1}{z-1}\right).$$

11. If z_0 is in the upper half of the z plane, show that the bilinear transformation $w = e^{i\theta_0}\left(\dfrac{z - z_0}{z - \bar{z}_0}\right)$ maps the upper half of the z plane into the interior of the unit circle in the w plane, i.e. $|w| \leqq 1$.

We have

$$|w| \;=\; \left|e^{i\theta_0}\left(\frac{z - z_0}{z - \bar{z}_0}\right)\right| \;=\; \left|\frac{z - z_0}{z - \bar{z}_0}\right|$$

From Fig. 8-73 if z is in the upper half plane, $|z - z_0| \leqq |z - \bar{z}_0|$, the equality holding if and only if z is on the x axis. Hence $|w| \leqq 1$, as required.

The transformation can also be derived directly (see Problem 61).

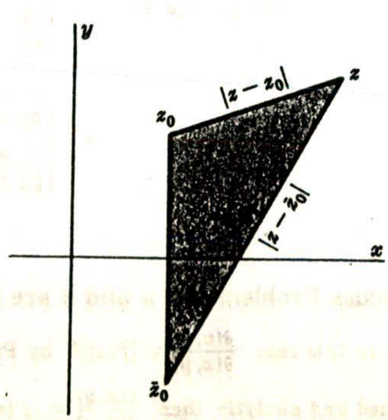


Fig. 8-73

12. Find a bilinear transformation which maps the upper half of the z plane into the unit circle in the w plane in such a way that $z = i$ is mapped into $w = 0$ while the point at infinity is mapped into $w = -1$.

We have $w = 0$ corresponding to $z = i$, and $w = -1$ corresponding to $z = \infty$. Then from $w = e^{i\theta_0}\left(\frac{z - z_0}{z - \bar{z}_0}\right)$ we have $0 = e^{i\theta_0}\left(\frac{i - z_0}{i - \bar{z}_0}\right)$ so that $z_0 = i$. Corresponding to $z = \infty$ we have $w = e^{i\theta_0} = -1$. Hence the required transformation is

$$w = (-1)\left(\frac{z - i}{z + i}\right) = \frac{i - z}{i + z}$$

The situation is described graphically in Figures 8-74 and 8-75.

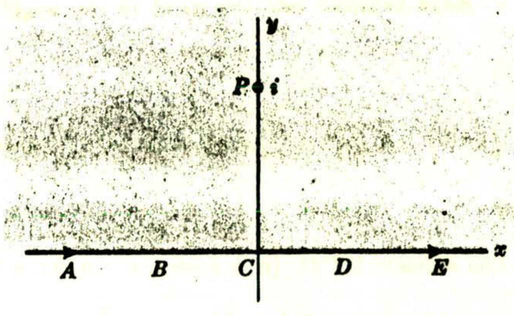


Fig. 8-74

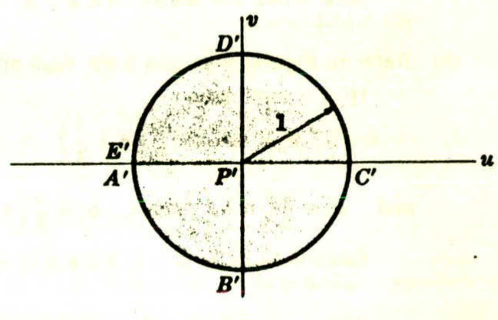


Fig. 8-75

13. Find the fixed or invariant points of the transformation $w = \dfrac{2z - 5}{z + 4}$.

The fixed points are solutions to $z = \dfrac{2z - 5}{z + 4}$ or $z^2 + 2z + 5 = 0$, i.e. $z = -1 \pm 2i$.

14. Prove that the bilinear transformation can be considered as a combination of the transformations of translation, rotation, stretching and inversion.

By division, $w = \dfrac{\alpha z + \beta}{\gamma z + \delta} = \dfrac{\alpha}{\gamma} + \dfrac{\beta\gamma - \alpha\delta}{\gamma(\gamma z + \delta)} = \lambda + \dfrac{\mu}{z + \nu}$ where $\lambda = \alpha/\gamma$, $\mu = (\beta\gamma - \alpha\delta)/\gamma^2$ and $\nu = \delta/\gamma$ are constants. The transformation is equivalent to $\zeta = z + \nu$, $\tau = 1/\zeta$ and $w = \lambda + \mu\tau$ which are combinations of the transformations of translation, rotation, stretching and inversion.

15. Prove that the bilinear transformation transforms circles of the z plane into circles of the w plane, where by circles we include circles of infinite radius which are straight lines.

The general equation of a circle in the z plane is by Problem 44, Chapter 1, $Az\bar{z} + Bz + \bar{B}\bar{z} + C = 0$, where $A > 0$, $C > 0$ and B is complex. If $A = 0$ the circle reduces to a straight line.

Under the transformation of inversion, $w = 1/z$ or $z = 1/w$, this equation becomes $Cw\bar{w} + \bar{B}w + B\bar{w} + A = 0$, a circle in the w plane.

Under the transformation of rotation and stretching, $w = az$ or $z = w/a$, this equation becomes $Aw\bar{w} + (B\bar{a})w + (\bar{B}a)\bar{w} + Ca\bar{a} = 0$, also a circle.

Similarly we can show either analytically or geometrically that under the transformation of translation, circles are transformed into circles.

Since by Problem 14 a bilinear transformation can be considered as a combination of translation, rotation, stretching and inversion, the required result follows.

SPECIAL MAPPING FUNCTIONS

16. Verify the entries (*a*) A-2, Page 205 (*b*) A-4, Page 206 (*c*) B-1, Page 209.

(*a*) Refer to Figures 8-13 and 8-14, Page 205.

If $z = x + iy$, then

$$w = u + iv = e^{\pi z/a} = e^{\pi(x+iy)/a} = e^{\pi x/a}(\cos \pi y/a + i \sin \pi y/a)$$

or $u = e^{\pi x/a} \cos \pi y/a$, $v = e^{\pi x/a} \sin \pi y/a$.

The line $y = 0$ [the real axis in the z plane; DEF in Fig. 8-13] maps into $u = e^{\pi x/a}$, $v = 0$ [the positive real axis in the w plane; $D'E'F'$ in Fig. 8-14]. The origin E [$z = 0$] maps into E' [$w = 1$] while D [$x = -\infty$, $y = 0$] and F [$x = +\infty$, $y = 0$] map into D' [$w = 0$] and F' [$w = \infty$] respectively.

The line $y = a$ [ABC in Fig. 8-13] maps into $u = -e^{\pi x/a}$, $v = 0$ [the negative real axis in the w plane; $A'B'C'$ in Fig. 8-14]. The points A [$x = +\infty$, $y = a$] and C [$x = -\infty$, $y = a$] map into A' [$w = -\infty$] and C' [$w = 0$] respectively.

Any point for which $0 < y < a$, $-\infty < x < \infty$ maps uniquely into one point in the uv plane for which $v > 0$.

(*b*) Refer to Figures 8-21 and 8-22, Page 206.

If $z = re^{i\theta}$, then

$$w = u + iv = \frac{a}{2}\left(z + \frac{1}{z}\right) = \frac{a}{2}\left(re^{i\theta} + \frac{1}{r}e^{-i\theta}\right) = \frac{a}{2}\left(r + \frac{1}{r}\right)\cos\theta + \frac{ia}{2}\left(r - \frac{1}{r}\right)\sin\theta$$

and $u = \dfrac{a}{2}\left(r + \dfrac{1}{r}\right)\cos\theta$, $v = \dfrac{a}{2}\left(r - \dfrac{1}{r}\right)\sin\theta$.

Semicircle BCD [$r = 1$, $0 \leqq \theta \leqq \pi$] maps into line segment $B'C'D'$ [$u = a\cos\theta$, $v = 0$, $0 \leqq \theta \leqq \pi$, i.e. $-a \leqq u \leqq a$].

The line DE [$\theta = 0$, $r > 1$] maps into line $D'E'$ $\left[u = \dfrac{a}{2}\left(r + \dfrac{1}{r}\right),\ v = 0\right]$; line AB [$\theta = \pi$, $r > 1$] maps into line $A'B'$ $\left[u = -\dfrac{a}{2}\left(r + \dfrac{1}{r}\right),\ v = 0\right]$.

Any point of the z plane for which $r \geqq 1$ and $0 < \theta < \pi$ maps uniquely into one point of the uv plane for which $v \geqq 0$.

(*c*) Refer to Figures 8-43 and 8-44, Page 209.

If $z = re^{i\theta}$ and $w = \rho e^{i\phi}$, then $w = 1/z$ becomes $\rho e^{i\phi} = \dfrac{1}{re^{i\theta}} = \dfrac{1}{r}e^{-i\theta}$ from which $\rho = 1/r$, $\phi = -\theta$.

The circle $ABCD$ [$\rho = 1$] in the w plane maps into the circle $A'B'C'D'$ [$r = 1$] of the z plane. Note that if $ABCD$ is described counterclockwise, $A'B'C'D'$ is described clockwise.

Any point exterior to the circle $ABCD$ [$\rho > 1$] is mapped uniquely into a point interior to the circle $A'B'C'D'$ [$r < 1$].

THE SCHWARZ-CHRISTOFFEL TRANSFORMATION

17. Establish the validity of the Schwarz-Christoffel transformation.

We must show that the mapping function obtained from

$$\frac{dw}{dz} = A(z - x_1)^{\alpha_1/\pi - 1}(z - x_2)^{\alpha_2/\pi - 1}\cdots(z - x_n)^{\alpha_n/\pi - 1} \tag{1}$$

maps a given polygon of the w plane [Fig. 8-76 below] into the real axis of the z plane [Fig. 8-77 below].

To show this observe that from (*1*) we have

$$\begin{aligned}\arg dw = \arg dz + \arg A + \left(\frac{\alpha_1}{\pi} - 1\right)\arg(z - x_1) + \left(\frac{\alpha_2}{\pi} - 1\right)\arg(z - x_2) \\ + \cdots + \left(\frac{\alpha_n}{\pi} - 1\right)\arg(z - x_n)\end{aligned} \tag{2}$$

As z moves along the real axis from the left toward x_1, let us assume that w moves along a side of the polygon toward w_1. When z crosses from the left of x_1 to the right of x_1, $\theta_1 = \arg(z - x_1)$ changes from π to 0 while all other terms in (*2*) stay constant. Hence $\arg dw$ decreases by $(\alpha_1/\pi - 1)\arg(z - x_1) = (\alpha_1/\pi - 1)\pi = \alpha_1 - \pi$ or, what is the same thing, increases by $\pi - \alpha_1$ [an increase being in the counterclockwise direction].

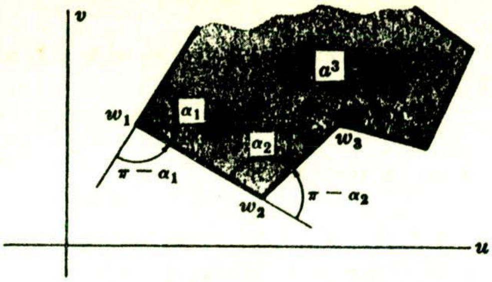


Fig. 8-76

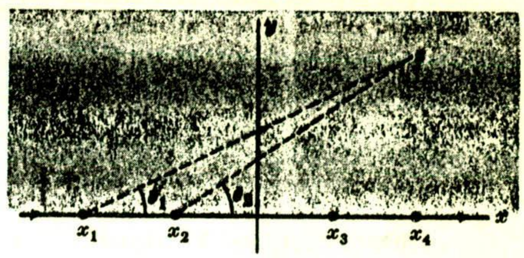


Fig. 8-77

It follows from this that the direction through w_1 turns through the angle $\pi - \alpha_1$, and thus w now moves along the side w_1w_2 of the polygon.

When z moves through x_2, $\theta_1 = \arg(z - x_1)$ and $\theta_2 = \arg(z - x_2)$ change from π to 0 while all other terms stay constant. Hence another turn through angle $\pi - \alpha_2$ in the w plane is made. By continuing the process we see that as z traverses the x axis, w traverses the polygon, and conversely.

We can actually prove that the interior of the polygon (if it is closed) is mapped on to the upper half plane by *(1)* [see Problem 26].

18. Prove that for closed polygons the sum of the exponents $\frac{\alpha_1}{\pi} - 1, \frac{\alpha_2}{\pi} - 1, \ldots, \frac{\alpha_n}{\pi} - 1$ in the Schwarz-Christoffel transformation *(9)* or *(10)*, Page 204, is equal to -2.

The sum of the exterior angles of any closed polygon is 2π. Then

$$(\pi - \alpha_1) + (\pi - \alpha_2) + \cdots + (\pi - \alpha_n) = 2\pi$$

and dividing by $-\pi$, we obtain as required,

$$\left(\frac{\alpha_1}{\pi} - 1\right) + \left(\frac{\alpha_2}{\pi} - 1\right) + \cdots + \left(\frac{\alpha_n}{\pi} - 1\right) = -2$$

19. If in the Schwarz-Christoffel transformation *(9)* or *(10)*, Page 204, one point, say x_n, is chosen at infinity, show that the last factor is not present.

In *(9)*, Page 204, let $A = K/(-x_n)^{\alpha_n/\pi - 1}$ where K is a constant. Then the right side of *(9)* can be written

$$K(z - x_1)^{\alpha_1/\pi - 1}(z - x_2)^{\alpha_2/\pi - 1} \cdots (z - x_{n-1})^{\alpha_{n-1}/\pi - 1}\left(\frac{x_n - z}{x_n}\right)^{\alpha_n/\pi - 1}$$

As $x_n \to \infty$, this last factor approaches 1; this is equivalent to removal of the factor.

20. Determine a function which maps each of the indicated regions in the w plane on to the upper half of the z plane.

(a)

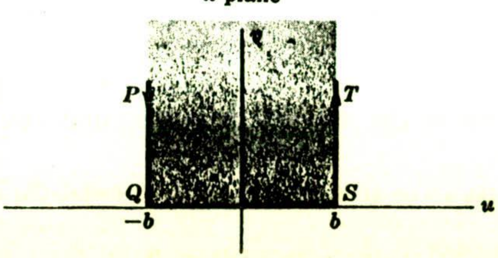


Fig. 8-78

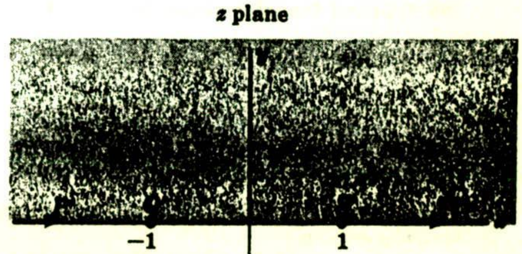


Fig. 8-79

Let points P, Q, S and T [Fig. 8-78 above] map respectively into P', Q', S' and T' [Fig. 8-79 above]. We can consider $PQST$ as a limiting case of a polygon (a triangle) with two vertices at Q and S and the third vertex P or T at infinity.

By the Schwarz-Christoffel transformation, since the angles at Q and S are equal to $\pi/2$, we have

$$\frac{dw}{dz} = A(z+1)^{\frac{\pi/2}{\pi}-1}(z-1)^{\frac{\pi/2}{\pi}-1} = \frac{A}{\sqrt{z^2-1}} = \frac{K}{\sqrt{1-z^2}}$$

Integrating,
$$w = K\int \frac{dz}{\sqrt{1-z^2}} + B = K\sin^{-1}z + B$$

When $z = 1$, $w = b$. Hence (1) $b = K\sin^{-1}(1) + B = K\pi/2 + B$.

When $z = -1$, $w = -b$. Hence, (2) $-b = K\sin^{-1}(-1) + B = -K\pi/2 + B$.

Solving (1) and (2) simultaneously, we find $B = 0$, $K = 2b/\pi$. Then

$$w = \frac{2b}{\pi}\sin^{-1}z \qquad \text{or} \qquad z = \sin\frac{\pi w}{2b}$$

The result is equivalent to entry A-3(a) in the table on Page 206 if we interchange w and z, and let $b = a/2$.

(b)

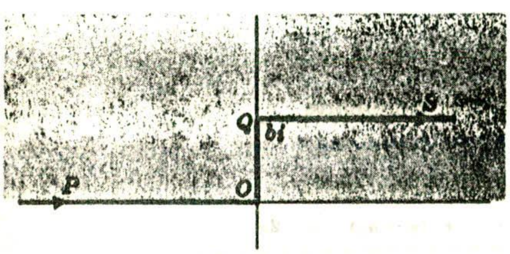


Fig. 8-80

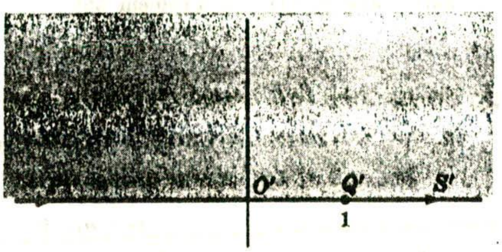


Fig. 8-81

Let points P, O, Q [$w = bi$] and S map into P', O', Q' [$z = 1$] and S' respectively. Note that P, S, P', S' are at infinity (as indicated by the arrows) while O and O' are the origins [$w = 0$ and $z = 0$] of the w and z planes. Since the interior angles at O and Q are $\pi/2$ and $3\pi/2$ respectively, we have by the Schwarz-Christoffel transformation,

$$\frac{dw}{dz} = A(z-0)^{\frac{\pi/2}{\pi}-1}(z-1)^{\frac{3\pi/2}{\pi}-1} = A\sqrt{\frac{z-1}{z}} = K\sqrt{\frac{1-z}{z}}$$

Then
$$w = K\int\sqrt{\frac{1-z}{z}}\,dz$$

To integrate this, let $z = \sin^2\theta$ and obtain

$$w = 2K\int\cos^2\theta\,d\theta = K\int(1+\cos 2\theta)\,d\theta = K(\theta + \tfrac{1}{2}\sin 2\theta) + B$$
$$= K(\theta + \sin\theta\cos\theta) + B = K\left(\sin^{-1}\sqrt{z} + \sqrt{z(1-z)}\right) + B$$

When $z = 0$, $w = 0$ so that $B = 0$. When $z = 1$, $w = bi$ so that $bi = K\pi/2$ or $K = 2bi/\pi$. Then the required transformation is

$$w = \frac{2bi}{\pi}\left(\sin^{-1}\sqrt{z} + \sqrt{z(1-z)}\right)$$

21. Find a transformation which maps a polygon in the w plane on to the unit circle in the ζ plane.

A polygon in the w plane can be mapped on to the x axis of the z plane by the Schwarz-Christoffel transformation

$$w = A\int(z-x_1)^{\alpha_1/\pi-1}(z-x_2)^{\alpha_2/\pi-1}\cdots(z-x_n)^{\alpha_n/\pi-1}\,dz + B \qquad (1)$$

A transformation which maps the upper half of the z plane into the unit circle in the ζ plane is

$$\zeta = \frac{i-z}{i+z} \qquad \text{or} \qquad z = i\left(\frac{1-\zeta}{1+\zeta}\right) \tag{2}$$

on replacing w by ζ and taking $\theta = \pi$, $z_0 = i$ in equation (8), Page 204.

If we let $x_1, x_2, \ldots, x_n$ map into $\zeta_1, \zeta_2, \ldots, \zeta_n$ respectively on the unit circle, then we have for $k = 1, 2, \ldots, n$,

$$z - x_k = i\left(\frac{1-\zeta}{1+\zeta}\right) - i\left(\frac{1-\zeta_k}{1+\zeta_k}\right) = \frac{-2i(\zeta-\zeta_k)}{(1+\zeta)(1+\zeta_k)}$$

Also, $dz = -2i\,d\zeta/(1+\zeta)^2$. Substituting into (1) and simplifying using the fact that the sum of the exponents $\frac{\alpha_1}{\pi}-1, \frac{\alpha_2}{\pi}-1, \ldots, \frac{\alpha_n}{\pi}-1$ is -2, we find the required transformation

$$w = A'\int (\zeta-\zeta_1)^{\alpha_1/\pi-1}(\zeta-\zeta_2)^{\alpha_2/\pi-1}\cdots(\zeta-\zeta_n)^{\alpha_n/\pi-1}\,d\zeta + B$$

where A' is a new arbitrary constant.

TRANSFORMATIONS OF BOUNDARIES IN PARAMETRIC FORM

22. Let C be a curve in the z plane with parametric equations $x = F(t)$, $y = G(t)$. Show that the transformation

$$z = F(w) + iG(w)$$

maps curve C on to the real axis of the w plane.

If $z = x + iy$, $w = u + iv$, the transformation can be written

$$x + iy = F(u+iv) + iG(u+iv)$$

Then $v = 0$ [the real axis of the w plane] corresponds to $x + iy = F(u) + iG(u)$, i.e. $x = F(u)$, $y = G(u)$, which represents the curve C.

23. Find a transformation which maps the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the z plane on to the real axis of the w plane.

A set of parametric equations for the ellipse is given by $x = a\cos t$, $y = b\sin t$ where $a > 0$, $b > 0$. Then by Problem 22 the required transformation is $z = a\cos w + ib\sin w$.

MISCELLANEOUS PROBLEMS

24. Find a function which maps the interior of a triangle in the w plane [Fig. 8-82] on to the upper half of the z plane.

Let vertices P $[w = 0]$ and Q $[w = 1]$ of the triangle map into points P' $[z = 0]$ and Q' $[z = 1]$ on the z plane while the third vertex R maps into R' $[z = \infty]$.

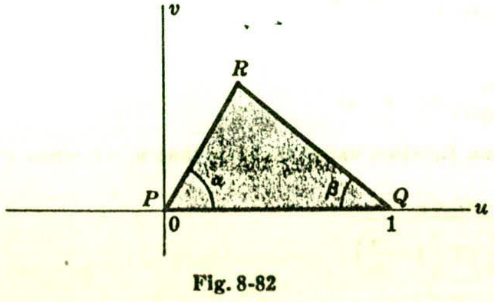


Fig. 8-82

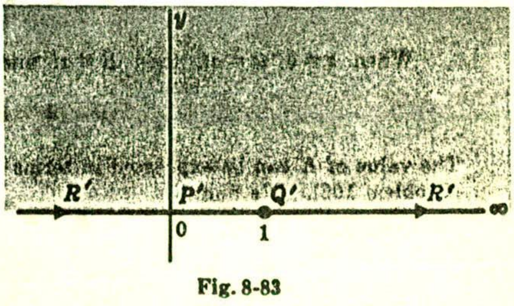


Fig. 8-83

By the Schwarz-Christoffel transformation,

$$\frac{dw}{dz} = A\, z^{\alpha/\pi - 1}(z-1)^{\beta/\pi - 1} = K\, z^{\alpha/\pi - 1}(1-z)^{\beta/\pi - 1}$$

Then by integration,

$$w = K\int_0^z \zeta^{\alpha/\pi - 1}(1-\zeta)^{\beta/\pi - 1}\, d\zeta + B$$

Since $w = 0$ when $z = 0$, we have $B = 0$. Also since $w = 1$ when $z = 1$, we have

$$1 = K\int_0^1 \zeta^{\alpha/\pi - 1}(1-\zeta)^{\beta/\pi - 1}\, d\zeta = \frac{\Gamma(\alpha/\pi)\,\Gamma(\beta/\pi)}{\Gamma\left(\frac{\alpha+\beta}{\pi}\right)}$$

using properties of the beta and gamma functions [Chapter 10]. Hence

$$K = \frac{\Gamma\left(\frac{\alpha+\beta}{\pi}\right)}{\Gamma(\alpha/\pi)\,\Gamma(\beta/\pi)}$$

and the required transformation is

$$w = \frac{\Gamma\left(\frac{\alpha+\beta}{\pi}\right)}{\Gamma(\alpha/\pi)\,\Gamma(\beta/\pi)}\int_0^z \zeta^{\alpha/\pi - 1}(1-\zeta)^{\beta/\pi - 1}\, d\zeta$$

Note that this agrees with entry A-13 on Page 209, since the length of side AB in Fig. 8-39 is

$$\int_0^1 \zeta^{\alpha/\pi - 1}(1-\zeta)^{\beta/\pi - 1}\, d\zeta = \frac{\Gamma(\alpha/\pi)\,\Gamma(\beta/\pi)}{\Gamma\left(\frac{\alpha+\beta}{\pi}\right)}$$

25. (*a*) Find a function which maps the shaded region in the w plane of Fig. 8-84 on to the upper half of the z plane of Fig. 8-85.

(*b*) Discuss the case where $b \to 0$.

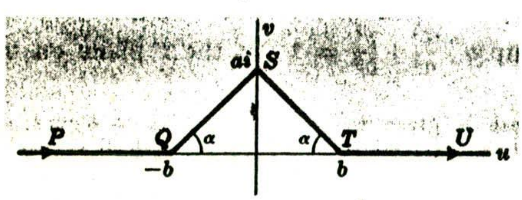


Fig. 8-84

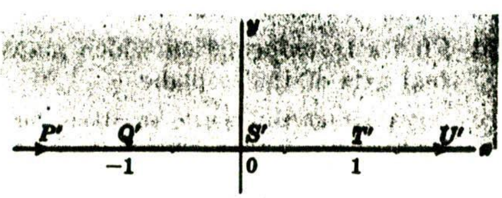


Fig. 8-85

(*a*) The interior angles at Q and T are each $\pi - \alpha$, while the angle at S is $2\pi - (\pi - 2\alpha) = \pi + 2\alpha$. Then by the Schwarz-Christoffel transformation we have

$$\frac{dw}{dz} = A\,(z+1)^{(\pi-\alpha)/\pi - 1}\, z^{(\pi+2\alpha)/\pi - 1}\,(z-1)^{(\pi-\alpha)/\pi - 1}$$

$$= \frac{A\, z^{2\alpha/\pi}}{(z^2-1)^{\alpha/\pi}} = \frac{K\, z^{2\alpha/\pi}}{(1-z^2)^{\alpha/\pi}}$$

Hence by integration

$$w = K\int_0^z \frac{\zeta^{2\alpha/\pi}}{(1-\zeta^2)^{\alpha/\pi}}\, d\zeta + B$$

When $z = 0$, $w = ai$; then $B = ai$ and

$$w = K\int_0^z \frac{\zeta^{2\alpha/\pi}}{(1-\zeta^2)^{\alpha/\pi}}\, d\zeta + ai \qquad (1)$$

The value of K can be expressed in terms of the gamma function using the fact that $w = b$ when $z = 1$ [Problem 102]. We find

$$K = \frac{(b - ai)\sqrt{\pi}}{\Gamma\left(\frac{\alpha}{\pi} + \frac{1}{2}\right)\Gamma\left(1 - \frac{\alpha}{\pi}\right)} \qquad (2)$$

(b) As $b \to 0$, $\alpha \to \pi/2$ and the result in (a) reduces to

$$w = ai - ai \int_0^z \frac{\zeta\, d\zeta}{\sqrt{1-\zeta^2}} = ai\sqrt{1-z^2}$$

$$= a\sqrt{z^2-1}$$

In this case Fig. 8-84 reduces to Fig. 8-86. The result for this case can be found directly from the Schwarz-Christoffel transformation by considering $PQSTU$ as a polygon with interior angles at Q, S and T equal to $\pi/2$, 2π, and $\pi/2$ respectively.

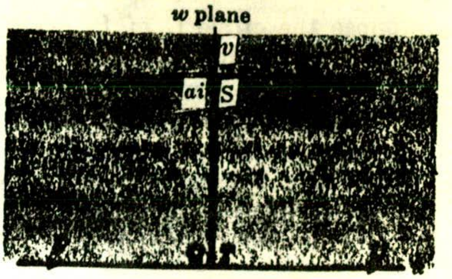


Fig. 8-86

26. Prove that the Schwarz-Christoffel transformation of Problem 17 maps the interior of the polygon on to the upper half plane.

It suffices to prove that the transformation maps the interior on to the unit circle, since we already know [Problem 11] that the unit circle can be mapped on to the upper half plane.

Suppose that the function mapping polygon P in the w plane on to the unit circle C in the z plane is given by $w = f(z)$ where $f(z)$ is analytic inside C.

We must now show that to each point a inside P there corresponds one and only one point, say z_0, such that $f(z_0) = a$.

Now by Cauchy's integral formula, since a is inside P,

$$\frac{1}{2\pi i}\oint_P \frac{dw}{w-a} = 1$$

Then since $w - a = f(z) - a$,

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)-a}\,dz = 1$$

But $f(z) - a$ is analytic inside C. Hence from Problem 17, Chapter 5, we have shown that there is only one zero (say z_0) of $f(z) - a$ inside C, i.e. $f(z_0) = a$, as required.

27. Let C be a circle in the z plane having its centre on the real axis, and suppose further that it passes through $z = 1$ and has $z = -1$ as an interior point. Determine the image of C in the w plane under the transformation $w = f(z) = \frac{1}{2}(z + 1/z)$.

We have $dw/dz = \frac{1}{2}(1 - 1/z^2)$. Since $dw/dz = 0$ at $z = 1$, it follows that $z = 1$ is a critical point. From the Taylor series of $f(z) = \frac{1}{2}(z + 1/z)$ about $z = 1$, we have

$$w - 1 = \tfrac{1}{2}[(z-1)^2 - (z-1)^3 + (z-1)^4 - \cdots]$$

By Problem 100 we see that angles with vertices at $z = 1$ are doubled under the transformation. In particular, since the angle at $z = 1$ exterior to C is π, the angle at $w = 1$ exterior to the image C' is 2π. Hence C' has a sharp tail at $w = 1$ (see Fig. 8-88). Other points of C' can be found directly.

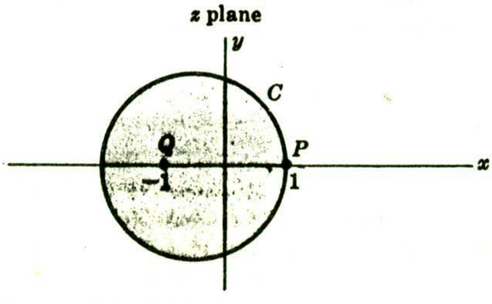


Fig. 8-87

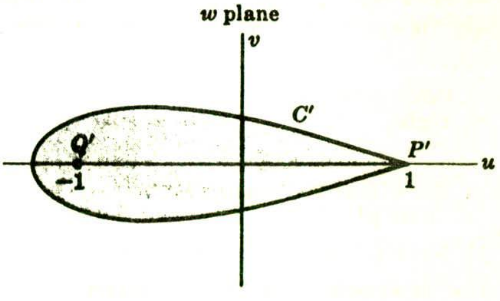


Fig. 8-88

It is of interest to note that in this case C encloses the circle $|z| = 1$ which under the transformation is mapped into the slit from $w = -1$ to $w = 1$. Thus as C approaches $|z| = 1$, C' approaches the straight line joining $w = -1$ to $w = 1$.

28. Suppose the circle C of Problem 27 is moved so that its centre is in the upper half plane but that it still passes through $z=1$ and encloses $z=-1$. Determine the image of C under the transformation $w = \frac{1}{2}(z + 1/z)$.

As in Problem 27, since $z=1$ is a critical point, we will obtain the sharp tail at $w=1$ [Fig. 8-90]. If C does not entirely enclose the circle $|z|=1$ [as shown in Fig. 8-89], the image C' will not entirely enclose the image of $|z|=1$ [which is the slit from $w=-1$ to $w=1$]. Instead, C' will only enclose that portion of the slit which corresponds to the part of $|z|=1$ inside C. The appearance of C' is therefore as shown in Fig. 8-90. By changing C appropriately, other shapes similar to C' can be obtained.

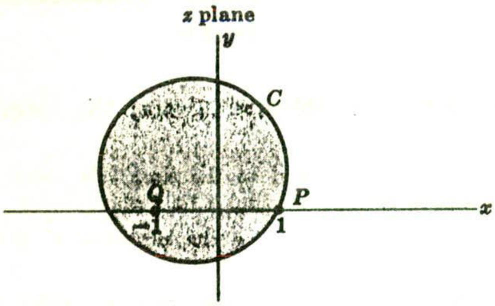


Fig. 8-89

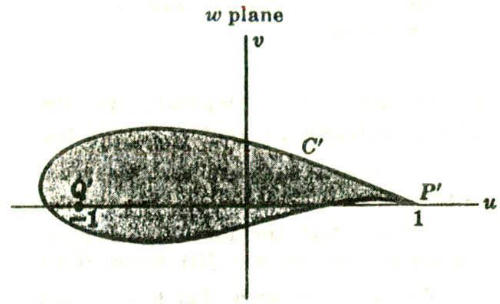


Fig. 8-90

The fact that C' resembles the cross-section of the wing of an airplane, sometimes called an *airfoil*, is important in aerodynamic theory (see Chapter 10) and was first used by *Joukowski*. For this reason shapes such as C' are called *Joukowski airfoils* or *profiles* and $w = \frac{1}{2}(z + 1/z)$ is called a *Joukowski transformation*.

Supplementary Problems

TRANSFORMATIONS

29. Given triangle T in the z plane with vertices at i, $1-i$, $1+i$. Determine the triangle T' into which T is mapped under the transformations (*a*) $w = 3z + 4 - 2i$, (*b*) $w = iz + 2 - i$, (*c*) $w = 5e^{\pi i/3}z - 2 + 4i$. What is the relationship between T and T' in each case?

30. Sketch the region of the w plane into which the interior of triangle T of Problem 29 is mapped under the transformations (*a*) $w = z^2$, (*b*) $w = iz^2 + (2-i)z$, (*c*) $w = z + 1/z$.

31. (*a*) Show that by means of the transformation $w = 1/z$ the circle C given by $|z-3| = 5$ is mapped into the circle $|w + 3/16| = 5/16$. (*b*) Into what region is the interior of C mapped?

32. (*a*) Prove that under the transformation $w = (z-i)/(iz-1)$ the region $\text{Im}\{z\} \geqq 0$ is mapped into the region $|w| \leqq 1$. (*b*) Into what region is $\text{Im}\{z\} \leqq 0$ mapped under the transformation?

33. (*a*) Show that the transformation $w = \frac{1}{2}(ze^{-\alpha} + z^{-1}e^{\alpha})$ where α is real, maps the interior of the circle $|z|=1$ on to the exterior of an ellipse [see entry B-2 in the table on Page 209].

(*b*) Find the lengths of the major and minor axes of the ellipse in (*a*) and construct the ellipse.

Ans. (*b*) $2\cosh\alpha$ and $2\sinh\alpha$ respectively.

34. Determine the equation of the curve in the w plane into which the straight line $x + y = 1$ is mapped under the transformations (*a*) $w = z^2$, (*b*) $w = 1/z$.

Ans. (*a*) $u^2 + 2v = 1$, (*b*) $u^2 + 2uv + 2v^2 = u + v$

35. Show that $w = \left(\frac{1+z}{1-z}\right)^{2/3}$ maps the unit circle on to a wedge-shaped region and illustrate graphically.

36. (a) Show that the transformation $w = 2z - 3i\bar{z} + 5 - 4i$ is equivalent to $u = 2x + 3y + 5$, $v = 2y - 3x - 4$.

(b) Determine the triangle in the uv plane into which triangle T of Problem 29 is mapped under the transformation in (a). Are the triangles similar?

37. Express the transformations (a) $u = 4x^2 - 8y$, $v = 8x - 4y^2$ and (b) $u = x^3 - 3xy^2$, $v = 3x^2y - y^3$ in the form $w = F(z, \bar{z})$. *Ans.* (a) $w = (1+i)(z^2 + \bar{z}^2) + (2 - 2i)z\bar{z} + 8iz$, (b) $w = z^3$

CONFORMAL TRANSFORMATIONS

38. The straight lines $y = 2x$, $x + y = 6$ in the xy plane are mapped on to the w plane by means of the transformation $w = z^2$. (a) Show graphically the images of the straight lines in the w plane. (b) Show analytically that the angle of intersection of the straight lines is the same as the angle of intersection of their images and explain why this is so.

39. Work Problem 38 if the transformation is (a) $w = \frac{1}{z}$, (b) $w = \frac{z-1}{z+1}$.

40. The interior of the square $\mathcal{S}$ with vertices at $1, 2, 1+i, 2+i$ is mapped into a region $\mathcal{S}'$ by means of the transformations (a) $w = 2z + 5 - 3i$, (b) $w = z^2$, (c) $w = \sin \pi z$. In each case sketch the regions and verify directly that the interior angles of $\mathcal{S}'$ are right angles.

41. (a) Sketch the images of the circle $(x-3)^2 + y^2 = 2$ and the line $2x + 3y = 7$ under the transformation $w = 1/z$. (b) Determine whether the images of the circle and line of (a) intersect at the same angles as the circle and line. Explain.

42. Work Problem 41 for the case of the circle $(x-3)^2 + y^2 = 5$ and the line $2x + 3y = 14$.

43. (a) Work Problem 38 if the transformation is $w = 3z - 2i\bar{z}$.

(b) Is your answer to part (b) the same? Explain.

44. Prove that a necessary and sufficient condition for the transformation $w = F(z, \bar{z})$ to be conformal in a region $\mathcal{R}$ is that $\partial F/\partial \bar{z} = 0$ and $\partial F/\partial z \neq 0$ in $\mathcal{R}$ and explain the significance of this.

JACOBIANS

45. (a) For each part of Problem 29, determine the ratio of the areas T and T'. (b) Compare your findings in part (a) with the magnification factor $|dw/dz|^2$ and explain the significance.

46. Find the Jacobian of the transformations (a) $w = 2z^2 - iz + 3 - i$, (b) $u = x^2 - xy + y^2$, $v = x^2 + xy + y^2$. *Ans.* (a) $|4z - i|^2$, (b) $4(x^2 + y^2)$

47. Prove that a polygon in the z plane is mapped into a similar polygon in the w plane by means of the transformation $w = F(z)$ if and only if $F'(z)$ is a constant different from zero.

48. The analytic function $F(z)$ maps the interior $\mathcal{R}$ of a circle C defined by $|z| = 1$ into a region $\mathcal{R}'$ bounded by a simple closed curve C'. Prove that (a) the length of C' is $\oint_C |F'(z)|\,|dz|$, (b) the area of $\mathcal{R}'$ is $\iint_{\mathcal{R}} |F'(z)|^2\,dx\,dy$.

49. Prove the result (2) on Page 200.

50. Find the ratio of areas of the triangles in Problem 36(b) and compare with the magnification factor as obtained from the Jacobian.

51. Let $u = u(x,y)$, $v = v(x,y)$ and $x = x(\xi,\eta)$, $y = y(\xi,\eta)$.

(a) Prove that $\dfrac{\partial(u,v)}{\partial(\xi,\eta)} = \dfrac{\partial(u,v)}{\partial(x,y)} \cdot \dfrac{\partial(x,y)}{\partial(\xi,\eta)}$.

(b) Interpret the result of (a) geometrically.

(c) Generalize the result in (a).

52. Show that if $w = u + iv = F(z)$, $z = x + iy = G(\zeta)$ and $\zeta = \xi + i\eta$, the result in Problem 51(a) is equivalent to the relation

$$\left|\frac{dw}{d\zeta}\right| = \left|\frac{dw}{dz}\right|\left|\frac{dz}{d\zeta}\right|$$

BILINEAR OR FRACTIONAL TRANSFORMATIONS

53. Find a bilinear transformation which maps the points $i, -i, 1$ of the z plane into $0, 1, \infty$ of the w plane respectively. *Ans.* $w = (1-i)(z-i)/2(z-1)$

54. (a) Find a bilinear transformation which maps the vertices $1+i$, $-i$, $2-i$ of a triangle T of the z plane into the points $0, 1, i$ of the w plane.

(b) Sketch the region into which the interior of triangle T is mapped under the transformation obtained in (a).

Ans. (a) $w = (2z - 2 - 2i)/\{(i-1)z - 3 - 5i\}$

55. Prove that the result of (a) two successive bilinear transformations, (b) any number of successive bilinear transformations is also a bilinear transformation.

56. If $a \neq b$ are the two fixed points of the bilinear transformation, show that it can be written in the form

$$\frac{w-a}{w-b} = K\left(\frac{z-a}{z-b}\right)$$

where K is a constant.

57. If $a = b$ in Problem 56, show that the transformation can be written in the form

$$\frac{1}{w-a} = \frac{1}{z-a} + k$$

where k is a constant.

58. Prove that the most general bilinear transformation which maps $|z| = 1$ on to $|w| = 1$ is

$$w = e^{i\theta}\left(\frac{z-p}{\bar{p}z-1}\right)$$

where p is a constant,

59. Show that the transformation of Problem 58 maps $|z| < 1$ on to (a) $|w| < 1$ if $|p| < 1$ and (b) $|w| > 1$ if $|p| > 1$.

60. Discuss Problem 58 if $|p| = 1$.

61. Work Problem 11 directly.

62. (a) If z_1, z_2, z_3, z_4 are any four different points of a circle, prove that the cross ratio is real.

(b) Is the converse of part (a) true? *Ans.* (b) Yes

THE SCHWARZ-CHRISTOFFEL TRANSFORMATION

63. Use the Schwarz-Christoffel transformation to determine a function which maps each of the indicated regions in the w plane on to the upper half of the z plane.

(a)

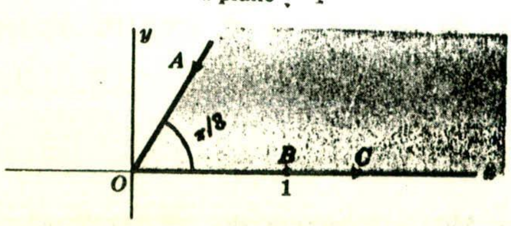


Fig. 8-91

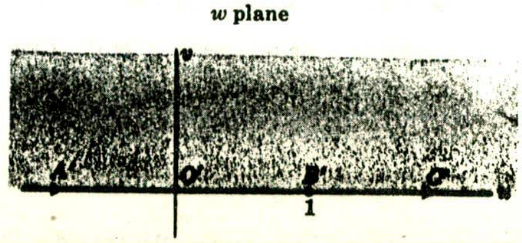


Fig. 8-92

(b) z plane

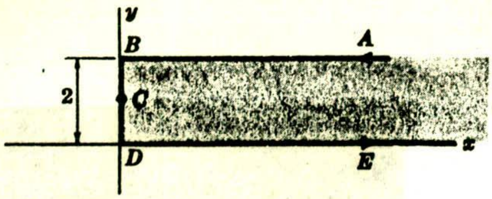


Fig. 8-93

w plane

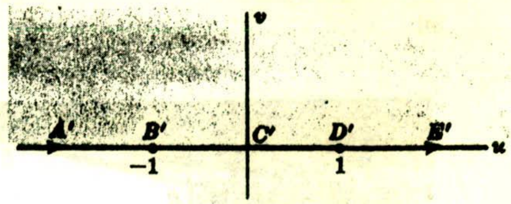


Fig. 8-94

(c) z plane

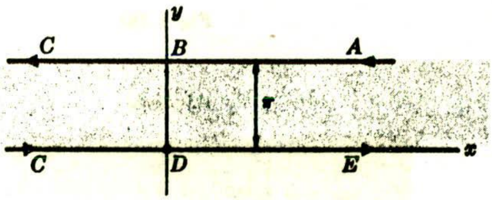


Fig. 8-95

w plane

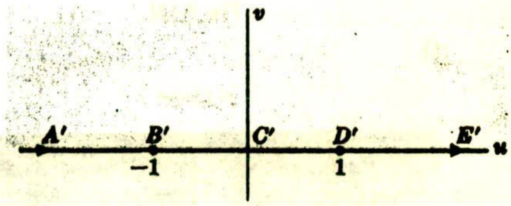


Fig. 8-96

(d) z plane

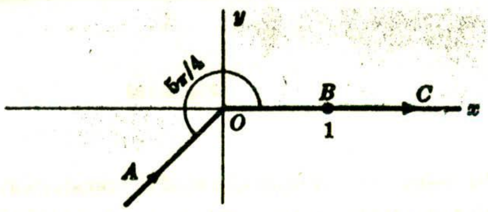


Fig. 8-97

w plane

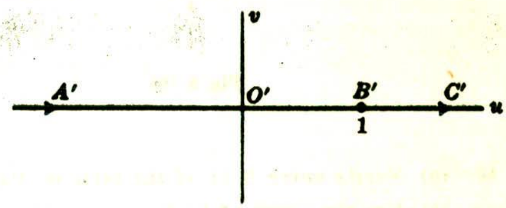


Fig. 8-98

Ans. (a) $w = z^3$, (b) $w = \cosh(\pi z/2)$, (c) $w = e^z$, (d) $w = z^{4/5}$

64. Verify entry A-14 in the table on Page 209 by using the Schwarz-Christoffel transformation.

65. Find a function which maps the infinite shaded region of Fig. 8-99 on to the upper half of the z plane [Fig. 8-100] so that P, Q, R map into P', Q', R' respectively [where P, R, P', R' are at infinity as indicated by the arrows]. *Ans.* $z = (w + \pi - \pi i)^2$

w plane

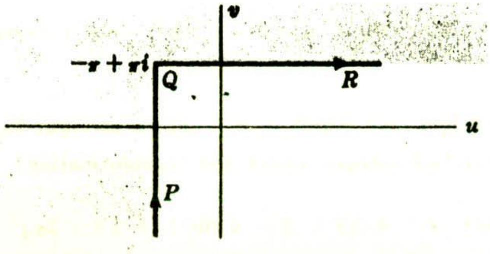


Fig. 8-99

z plane

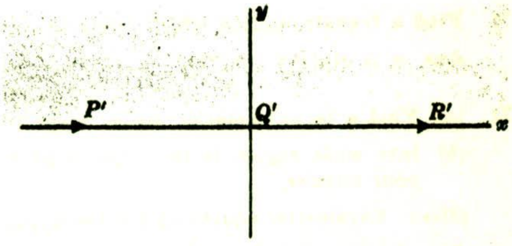


Fig. 8-100

66. Verify entry A-12 in the table on Page 208 by using the Schwarz-Christoffel transformation.

67. **Find a function which maps each of the indicated shaded regions in the w plane on to the upper half of the z plane.**

(*a*)

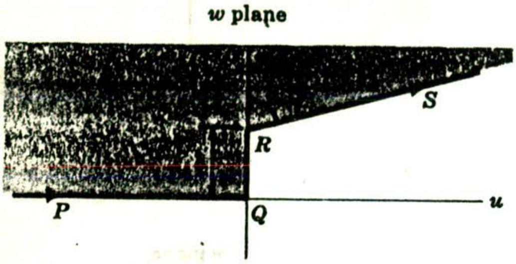


Fig. 8-101

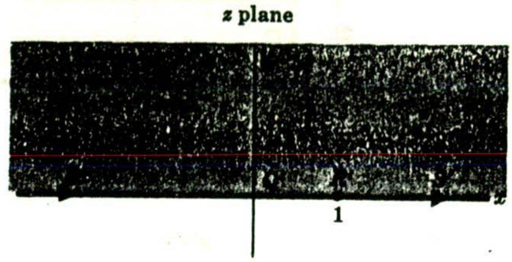


Fig. 8-102

(*b*)

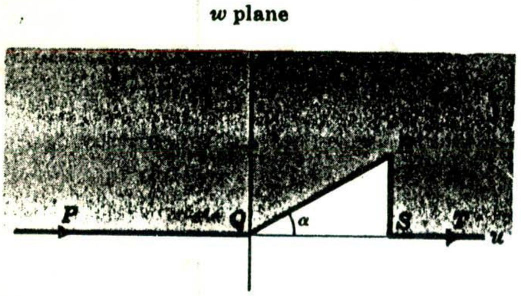


Fig. 8-103

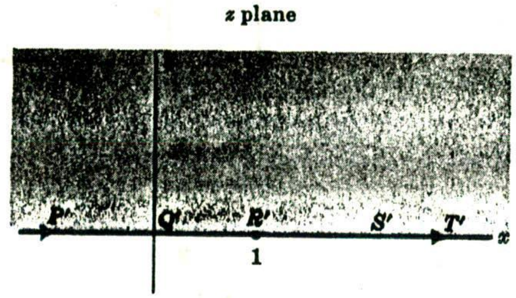


Fig. 8-104

68. (*a*) Verify entry A-11 of the table on Page 208 by using the Schwarz-Christoffel transformation.

(*b*) Use the result of (*a*) together with entry A-2 of the table on Page 205 to arrive at the entry C-5 in the table on Page 211.

TRANSFORMATIONS OF BOUNDARIES IN PARAMETRIC FORM

69. (*a*) Find a transformation which maps the parabola $y^2 = 4p(p-x)$ into a straight line.

(*b*) Discuss the relationship of your answer to entry A-9 in the table on Page 208.

Ans. (*a*) One possibility is $z = p - pw^2 + 2piw = p(1+iw)^2$ obtained by using the parametric equations $x = p(1-t^2)$, $y = 2pt$.

70. Find a transformation which maps the hyperbola $x = a\cosh t$, $y = a\sinh t$ into a straight line.

Ans. $z = a(\cosh w + i\sinh w)$

71. Find a transformation which maps the cycloid $x = a(t - \sin t)$, $y = a(1-\cos t)$ into a straight line.

Ans. $z = a(w + i - ie^{-iw})$

72. (*a*) Find a transformation which maps the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$ into a straight line.

(*b*) Into what region is the interior of the hypocycloid mapped under the transformation? Justify your answer.

[*Hint.* Parametric equations for the hypocycloid are $x = a\cos^3 t$, $y = a\sin^3 t$, $0 \leqq t < 2\pi$.]

Ans. (*a*) $z = a(\cos^3 w + i\sin^3 w)$

73. Two sets of parametric equations for the parabola $y = x^2$ are (*a*) $x = t$, $y = t^2$ and (*b*) $x = \pm e^t$, $y = e^{2t}$. Use these parametric equations to arrive at two possible transformations mapping the parabola into a straight line and determine whether there is any advantage in using one rather than the other.

MISCELLANEOUS PROBLEMS

74. (a) Show that the transformation $w = 1/z$ maps the circle $|z-a| = a$, where $a > 0$, into a straight line. Illustrate graphically showing the region into which the interior of the circle is mapped, as well as various points of the circle.

(b) Show how the result in (a) can be used to derive the transformation for the upper half plane into the unit circle.

75. Prove that the function $w = (z^2/a^2) - 1$ maps one loop of the lemniscate $r^2 = 2a^2 \cos 2\theta$ on to the unit circle.

76. Prove that the function $w = z^2$ maps the circle $|z-a| = a$, $a > 0$, on to the cardioid $\rho = 2a^2(1+\cos\phi)$ [see entry C-2 in the table on Page 210].

77. Show that the Joukowsky transformation $w = z + k^2/z$ can be written as

$$\frac{w-2k}{w+2k} = \left(\frac{z-k}{z+k}\right)^2$$

78. (a) Let $w = F(z)$ be a bilinear transformation. Show that the most general linear transformation for which $F\{F(z)\} = z$ is given by

$$\frac{w-p}{w-q} = k\frac{z-p}{z-q}$$

where $k^2 = 1$.

(b) What is the result in (a) if $F\{F[F(z)]\} = z$?

(c) Generalize the results in (a) and (b).

Ans. (b) Same as (a) with $k^3 = 1$.

79. (a) Determine a transformation which rotates the ellipse $x^2 + xy + y^2 = 5$ so that the major and minor axes are parallel to the coordinate axes. (b) What are the lengths of the major and minor axes?

80. Find a bilinear transformation which maps the circle $|z-1| = 2$ on to the line $x + y = 1$.

81. Verify the transformations (a) A-6, (b) A-7, (c) A-8, in the table on Page 207.

82. Consider the stereographic projection of the complex plane on to a unit sphere tangent to it [see Page 6]. Let an XYZ rectangular coordinate system be constructed so that the Z axis coincides with NS while the X and Y axes coincide with the x and y axes of Fig. 1-6, Page 6. Prove that the point (X, Y, Z) of the sphere corresponding to (x, y) on the plane is such that

$$X = \frac{x}{x^2+y^2+1}, \qquad Y = \frac{y}{x^2+y^2+1}, \qquad Z = \frac{x^2+y^2}{x^2+y^2+1}$$

83. Prove that a mapping by means of stereographic projection is conformal.

84. (a) Prove that by means of a stereographic projection, arc lengths of the sphere are magnified in the ratio $(x^2+y^2+1):1$.

(b) Discuss what happens to regions in the vicinity of the north pole. What effect does this produce on navigational charts?

85. Let $u = u(x,y)$, $v = v(x,y)$ be a transformation of points of the xy plane on to points of the uv plane.

(a) Show that in order that the transformation preserve angles, it is necessary and sufficient that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2, \qquad \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = 0$$

(b) Deduce from (a) that we must have either

$$\text{(i)}\ \ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{or} \qquad \text{(ii)}\ \ \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

Thus conclude that $u + iv$ must be an analytic function of $x + iy$.

86. Find the area of the ellipse $ax^2 + bxy + cy^2 = 1$ where $a > 0$, $c > 0$ and $b^2 < 4ac$.
Ans. $2\pi/\sqrt{4ac - b^2}$

87. A transformation $w = f(z)$ of points in a plane is called *involutory* if $z = f(w)$. In this case a single repetition of the transformation restores each point to its original position. Find conditions on $\alpha, \beta, \gamma, \delta$ in order that the bilinear transformation $w = (\alpha z + \beta)/(\gamma z + \delta)$ be involutory. *Ans.* $\delta = -\alpha$

88. Show that the transformations (*a*) $w = (z+1)/(z-1)$, (*b*) $w = \ln \coth (z/2)$ are involutory.

89. Find a bilinear transformation which maps $|z| \leqq 1$ on to $|w - 1| \leqq 1$ so that the points $1, -i$ correspond to $2, 0$ respectively.

90. Discuss the significance of the vanishing of the Jacobian for a bilinear transformation.

91. Prove that the bilinear transformation $w = (\alpha z + \beta)/(\gamma z + \delta)$ has one fixed point if and only if $(\delta + \alpha)^2 = 4(\alpha\delta - \beta\gamma) \neq 0$.

92. (*a*) Show that the transformation $w = (\alpha z + \bar{\gamma})/(\gamma z + \bar{\alpha})$ where $|\alpha|^2 - |\gamma|^2 = 1$ transforms the unit circle and its interior into itself.

(*b*) Show that if $|\gamma|^2 - |\alpha|^2 = 1$ the interior is mapped into the exterior.

93. Suppose under the transformation $w = F(z, \bar{z})$ any intersecting curves C_1 and C_2 in the z plane map respectively into corresponding intersecting curves C_1' and C_2' in the w plane. Prove that if the transformation is conformal then (*a*) $F(z, \bar{z})$ is a function of z alone, say $f(z)$, and (*b*) $f(z)$ is analytic.

94. (*a*) Prove the multiplication rule for determinants [see Problem 7]:

$$\begin{vmatrix} a_1 & b_1 \\ c_1 & d_1 \end{vmatrix} \begin{vmatrix} a_2 & b_2 \\ c_2 & d_2 \end{vmatrix} = \begin{vmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + c_1c_2 & c_1b_2 + d_1d_2 \end{vmatrix}$$

(*b*) Show how to generalize the result in (*a*) to third order and higher order determinants.

95. Find a function which maps on to each other the shaded regions of Figures 8-105 and 8-106, where QS has length b.

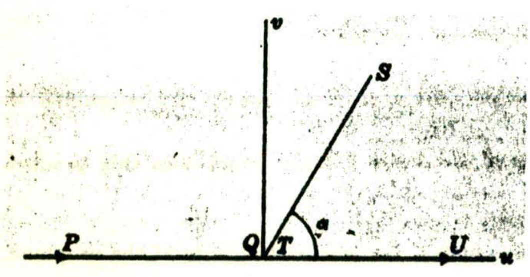


Fig. 8-105

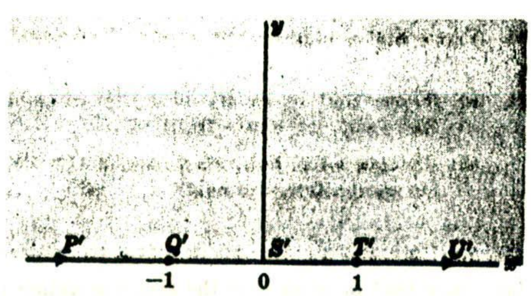


Fig. 8-106

96. (*a*) Show that the function $w = \displaystyle\int_0^z \frac{dt}{(1 - t^6)^{1/3}}$ maps a regular hexagon into the unit circle.

(*b*) What is the length of a side of the hexagon in (*a*)?

Ans. (*b*) $\frac{1}{6}\sqrt[3]{2}\,\Gamma(\frac{1}{3})$

97. Show that the transformation $w = (Az^2 + Bz + C)/(Dz^2 + Ez + F)$ can be considered as a combination of two bilinear transformations separated by a transformation of the type $\tau = \zeta^2$.

98. Find a function which maps a regular polygon of n sides into the unit circle.

99. Verify the entries: (*a*) A-9, Page 208; (*b*) A-10, Page 208; (*c*) B-3, Page 210; (*d*) B-4, Page 210; (*e*) C-3, Page 211; (*f*) C-4, Page 211.

100. Suppose the mapping function $w = f(z)$ has the Taylor series expansion

$$w = f(z) = f(a) + f'(a)(z-a) + \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \cdots$$

Show that if $f^{(k)}(a) = 0$ for $k = 0, 1, \ldots, n-1$ while $f^{(n)}(a) \neq 0$, then angles in the z plane with vertices at $z = a$ are multiplied by n in the w plane.

101. Determine a function which maps the infinite strip $-\pi/4 \leqq x \leqq \pi/4$ on to the interior of the unit circle $|w| \leqq 1$ so that $z = 0$ corresponds to $w = 0$. *Ans.* $w = \tan z$

102. Verify the value of K obtained in equation (*2*) of Problem 25.

103. Find a function which maps the upper half plane on to the interior of a triangle with vertices at $w = 0, 1, i$ corresponding to $z = 0, 1, \infty$ respectively.

Ans. $$w = \frac{\Gamma(3/4)}{\sqrt{\pi}\,\Gamma(1/4)} \int_0^z t^{-1/2}(1-t)^{-3/4}\,dt$$

Chapter 9

Physical Applications of Conformal Mapping

BOUNDARY VALUE PROBLEMS

Many problems of science and engineering when formulated mathematically lead to *partial differential equations* and associated conditions called *boundary conditions*. The problem of determining solutions to a partial differential equation which satisfy the boundary conditions is called a *boundary-value problem*.

It is of fundamental importance, from a mathematical as well as physical viewpoint, that one should not only be able to find such solutions (i.e. that solutions *exist*) but that for any given problem there should be only one solution (i.e. the solution is *unique*).

HARMONIC AND CONJUGATE FUNCTIONS

A function satisfying *Laplace's equation*

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0 \qquad (1)$$

in a region $\mathcal{R}$ is called *harmonic* in $\mathcal{R}$. As we have already seen, if $f(z) = u(x,y) + i\,v(x,y)$ is analytic in $\mathcal{R}$, then u and v are harmonic in $\mathcal{R}$.

Example: If $f(z) = 4z^2 - 3iz = 4(x+iy)^2 - 3i(x+iy) = 4x^2 - 4y^2 + 3y + i(8xy - 3x)$, then $u = 4x^2 - 4y^2 + 3y$, $v = 8xy - 3x$. Since u and v satisfy Laplace's equation, they are harmonic.

The functions u and v are called *conjugate functions*; and given one, the other can be determined within an arbitrary additive constant [see Chapter 3].

DIRICHLET AND NEUMANN PROBLEMS

Let $\mathcal{R}$ [Fig. 9-1] be a simply-connected region bounded by a simple closed curve C. Two types of boundary-value problems are of great importance.

1. **Dirichlet's problem** seeks the determination of a function Φ which satisfies Laplace's equation (*1*) [i.e. is harmonic] in $\mathcal{R}$ and takes prescribed values on the boundary C.

2. **Neumann's problem** seeks the determination of a function Φ which satisfies Laplace's equation (*1*) in $\mathcal{R}$ and whose normal derivative $\partial\Phi/\partial n$ takes prescribed values on the boundary C.

The region $\mathcal{R}$ may be unbounded. For example $\mathcal{R}$ can be the upper half plane with the x axis as the boundary C.

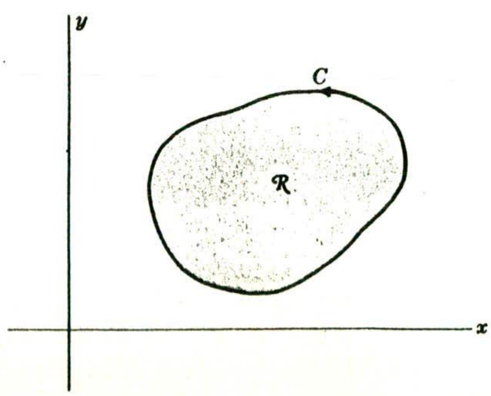


Fig. 9-1

It can be shown that solutions to both the Dirichlet and Neumann problems exist and are unique [the Neumann problem within an arbitrary additive constant] under very mild restrictions on the boundary conditions [see Problems 29 and 80].

It is of interest that a Neumann problem can be stated in terms of an appropriately stated Dirichlet problem (see Problem 79). Hence if we can solve the Dirichlet problem we can (at least theoretically) solve a corresponding Neumann problem.

THE DIRICHLET PROBLEM FOR THE UNIT CIRCLE. POISSON'S FORMULA

Let C be the unit circle $|z| = 1$ and $\mathcal{R}$ be its interior. A function which satisfies Laplace's equation [i.e. is harmonic] at each point (r, θ) in $\mathcal{R}$ and takes on the prescribed value $F(\theta)$ on C [i.e. $\Phi(1, \theta) = F(\theta)$], is given by

$$\Phi(r, \theta) \quad = \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - r^2)\, F(\phi)\, d\phi}{1 - 2r\cos(\theta - \phi) + r^2} \tag{2}$$

This is called *Poisson's formula for a circle* [see Chapter 5, Page 119].

THE DIRICHLET PROBLEM FOR THE HALF PLANE

A function which is harmonic in the half plane $y > 0$ [$\text{Im}\{z\} > 0$] and which takes on the prescribed value $G(x)$ on the x axis [i.e. $\Phi(x, 0) = G(x)$, $-\infty < x < \infty$], is given by

$$\Phi(x, y) \quad = \quad \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y\, G(\eta)\, d\eta}{y^2 + (x - \eta)^2} \tag{3}$$

This is sometimes called *Poisson's formula for the half plane* [see Chapter 5, Page 120].

SOLUTIONS TO DIRICHLET AND NEUMANN PROBLEMS BY CONFORMAL MAPPING

The Dirichlet and Neumann problems can be solved for any simply-connected region $\mathcal{R}$ which can be mapped conformally by an analytic function on to the interior of a unit circle or half plane. [By Riemann's mapping theorem this can always be accomplished, at least in theory.] The basic ideas involved are as follows.

(*a*) Use the mapping function to transform the boundary-value problem for the region $\mathcal{R}$ into a corresponding one for the unit circle or half plane.

(*b*) Solve the problem for the unit circle or half plane.

(*c*) Use the solution in (*b*) to solve the given problem by employing the inverse mapping function.

Important theorems used in this connection are as follows.

Theorem 1. Let $w = f(z)$ be analytic in a region $\mathcal{R}$ of the z plane. Then there exists a unique inverse $z = g(w)$ in $\mathcal{R}$, provided $f'(z) \neq 0$ in $\mathcal{R}$ [thus insuring that the mapping is conformal at each point of $\mathcal{R}$].

Theorem 2. Let $\Phi(x, y)$ be harmonic in $\mathcal{R}$ and suppose that $\mathcal{R}$ is mapped into $\mathcal{R}'$ of the w plane by the mapping function $w = f(z)$ where $f(z)$ is analytic and $f'(z) \neq 0$ so that $x = x(u, v)$, $y = y(u, v)$. Then $\Phi(x, y) = \Phi[x(u, v), y(u, v)] \equiv \Psi(u, v)$ is harmonic in $\mathcal{R}'$. In words, a harmonic function is transformed into a harmonic function under a transformation $w = f(z)$ which is analytic [see Problem 4].

Theorem 3. If $\Phi = a$ [a constant] on the boundary or part of the boundary C of a region in the z plane, then $\Psi = a$ on its image C' in the w plane. Similarly if the normal derivative of Φ is zero, i.e. $\partial\Phi/\partial n = 0$ on C, then the normal derivative of Ψ is zero on C'.

Applications to Fluid Flow

BASIC ASSUMPTIONS

The solution of many important problems in fluid flow, also referred to as *fluid dynamics, hydrodynamics* or *aerodynamics,* is often achieved by complex variable methods under the following assumptions.

1. **The fluid flow is two-dimensional,** i.e. the basic flow pattern and characteristics of the fluid motion in any plane are essentially the same as in any parallel plane. This permits us to confine our attention to just a single plane which we take to be the z plane. Figures constructed in this plane are interpreted as cross-sections of corresponding infinite cylinders perpendicular to the plane. For example, in Fig. 9-7, Page 237, the circle represents an infinite cylindrical obstacle around which the fluid flows. Naturally, an infinite cylinder is nothing more than a *mathematical model* of a physical cylinder which is so long that end effects can be reasonably neglected.

2. **The flow is stationary or steady,** i.e. the velocity of the fluid at any point depends only on the position (x, y) and not on time.

3. **The velocity components are derivable from a potential,** i.e. if V_x and V_y denote the components of velocity of the fluid at (x, y) in the positive x and y directions respectively, there exists a function Φ, called the *velocity potential,* such that

$$V_x = \frac{\partial\Phi}{\partial x}, \qquad V_y = \frac{\partial\Phi}{\partial y} \tag{4}$$

   An equivalent assumption is that if C is any simple closed curve in the z plane and V_t is the tangential component of velocity on C, then

$$\oint_C V_t\,ds \;=\; \oint_C V_x\,dx + V_y\,dy \;=\; 0 \tag{5}$$

   See Problem 48.

   Either of the integrals in (*5*) is called the *circulation* of the fluid along C. When the circulation is zero the flow is called *irrotational* or *circulation free.*

4. **The fluid is incompressible,** i.e. the density, or mass per unit volume of the fluid, is constant. If V_n is the normal component of velocity on C this leads to the conclusion (see Problem 48) that

$$\oint_C V_n\,ds \;=\; \oint_C V_x\,dy - V_y\,dx \;=\; 0 \tag{6}$$

   or

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} \;=\; 0 \tag{7}$$

   which expresses the condition that the quantity of fluid contained inside C is a constant, i.e. the quantity entering C is equal to the quantity leaving C. For this reason equation (*6*), or the equivalent (*7*), is called the *equation of continuity.*

5. **The fluid is non-viscous,** i.e. has no viscosity or internal friction. A moving viscous fluid tends to adhere to the surface of an obstacle placed in its path. If there is no viscosity, the pressure forces on the surface are perpendicular to the surface. A fluid which is non-viscous and incompressible is often called an *ideal fluid.* It must of course be realized that such a fluid is only a mathematical model of a real fluid in which such effects can be safely assumed negligible.

THE COMPLEX POTENTIAL

From (*4*) and (*7*) it is seen that the velocity potential Φ is harmonic, i.e. satisfies Laplace's equation

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0 \tag{8}$$

It follows that there must exist a conjugate harmonic function, say $\Psi(x, y)$, such that

$$\Omega(z) = \Phi(x,y) + i\Psi(x,y) \tag{9}$$

is analytic. By differentiation we have, using (*4*),

$$\frac{d\Omega}{dz} = \Omega'(z) = \frac{\partial\Phi}{\partial x} + i\frac{\partial\Psi}{\partial x} = \frac{\partial\Phi}{\partial x} - i\frac{\partial\Phi}{\partial y} = V_x - iV_y \tag{10}$$

Thus the velocity [sometimes called the *complex velocity*] is given by

$$\mathcal{V} = V_x + iV_y = \overline{d\Omega/dz} = \overline{\Omega'(z)} \tag{11}$$

and has magnitude

$$V = |\mathcal{V}| = \sqrt{V_x^2 + V_y^2} = |\overline{\Omega'(z)}| = |\Omega'(z)| \tag{12}$$

Points at which the velocity is zero, i.e. $\Omega'(z) = 0$, are called *stagnation points*.

The function $\Omega(z)$, of fundamental importance in characterizing a flow, is called the *complex potential.*

EQUIPOTENTIAL LINES AND STREAMLINES

The one parameter families of curves

$$\Phi(x,y) = \alpha, \quad \Psi(x,y) = \beta \tag{13}$$

where α and β are constants, are orthogonal families called respectively the *equipotential lines* and *streamlines* of the flow [although the more appropriate terms *equipotential curves* and *stream curves* are sometimes used]. In steady motion, streamlines represent the actual paths of fluid particles in the flow pattern.

The function Ψ is called the *stream function* while, as already seen, the function Φ is called the *velocity potential function* or briefly the *velocity potential.*

SOURCES AND SINKS

In the above development of theory we assumed that there were no points in the z plane [i.e. lines in the fluid] at which fluid appears or disappears. Such points are called *sources* and *sinks* respectively [also called *line sources* and *line sinks*]. At such points, which are singular points, the equation of continuity (*7*), and hence (*8*), fail to hold. In particular the circulation integral in (*5*) may not be zero around closed curves C which include such points.

No difficulty arises in using the above theory, however, provided we introduce the proper singularities into the complex potential $\Omega(z)$ and note that equations such as (*7*) and (*8*) then hold in any region which excludes these singular points.

SOME SPECIAL FLOWS

Theoretically, any complex potential $\Omega(z)$ can be associated with, or interpreted as, a particular two-dimensional fluid flow. The following are some simple cases arising in practice. [Note that a constant can be added to all complex potentials without affecting the flow pattern.]

1. **Uniform Flow.** The complex potential corresponding to the flow of a fluid at constant speed V_0 in a direction making an angle δ with the positive x direction is (Fig. 9-2 below)

$$\Omega(z) \;=\; V_0\, e^{-i\delta} z \tag{14}$$

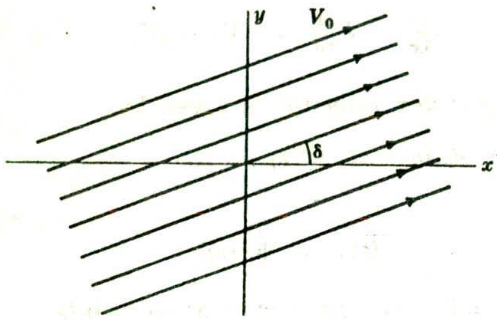


Fig. 9-2

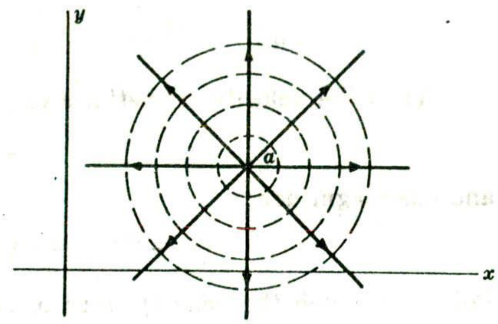


Fig. 9-3

2. **Source at $z = a$.** If fluid is emerging at constant rate from a line source at $z = a$ (Fig. 9-3 above), the complex potential is

$$\Omega(z) \;=\; k \ln(z-a) \tag{15}$$

where $k > 0$ is called the *strength* of the source. The streamlines are shown heavy while the equipotential lines are dashed.

3. **Sink at $z = a$.** In this case the fluid is disappearing at $z = a$ (Fig. 9-4 below) and the complex potential is obtained from that of the source by replacing k by $-k$, giving

$$\Omega(z) \;=\; -k \ln(z-a) \tag{16}$$

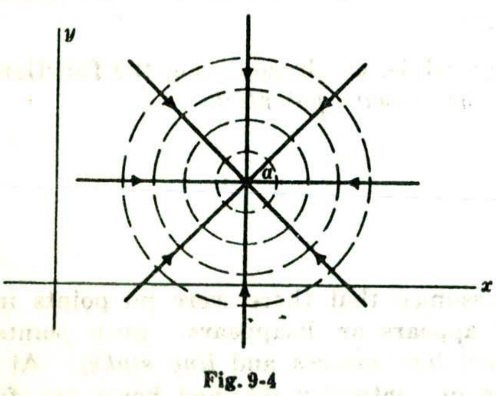


Fig. 9-4

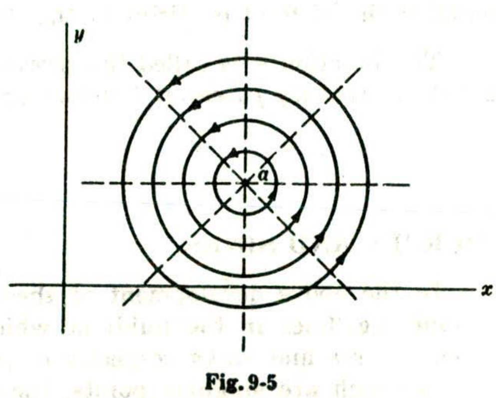


Fig. 9-5

4. **Flow with Circulation.** The flow corresponding to the complex potential

$$\Omega(z) \;=\; -ik \ln(z-a) \tag{17}$$

is as indicated in Fig. 9-5 above. The magnitude of the velocity of fluid at any point is in this case inversely proportional to the distance from a.

The point $z=a$ is called a *vortex* and k is called its *strength*. The circulation [see equation (5)] about any simple closed curve C enclosing $z=a$ is equal in magnitude to $2\pi k$. Note that by changing k to $-k$ in (17) the complex potential corresponding to a "clockwise" vortex is obtained.

5. **Superposition of Flows.** By addition of complex potentials, more complicated flow patterns can be described. An important example is obtained by considering the flow due to a source at $z=-a$ and a sink of equal strength at $z=a$. Then the complex potential is

$$\Omega(z) = k\ln(z+a) - k\ln(z-a) = k\ln\left(\frac{z+a}{z-a}\right) \tag{18}$$

By letting $a\to 0$ and $k\to\infty$ in such a way that $2ka=\mu$ is finite we obtain the complex potential

$$\Omega(z) = \frac{\mu}{z} \tag{19}$$

This is the complex potential due to a *doublet* or *dipole*, i.e. the combination of a source and sink of equal strengths separated by a very small distance. The quantity μ is called the *dipole moment.*

FLOW AROUND OBSTACLES

An important problem in fluid flow is that of determining the flow pattern of a fluid initially moving with uniform velocity V_0 in which an obstacle has been placed.

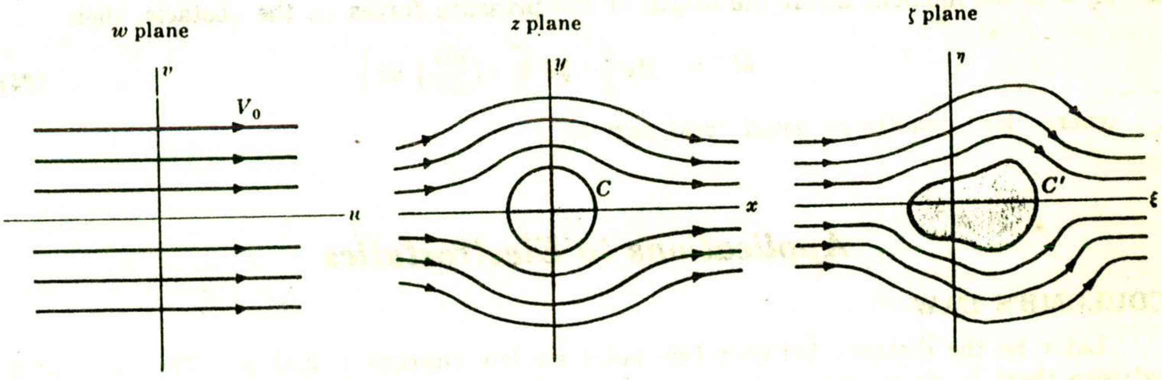


Fig. 9-6 Fig. 9-7 Fig. 9-8

A general principle involved in this type of problem is to design a complex potential having the form

$$\Omega(z) = V_0 z + G(z) \tag{20}$$

(if the flow is in the z plane) where $G(z)$ is such that $\lim_{|z|\to\infty} G'(z) = 0$, which means physically that far from the obstacle the velocity has constant magnitude (in this case V_0). Furthermore, the complex potential must be chosen so that one of the streamlines represents the boundary of the obstacle.

A knowledge of conformal mapping functions is often useful in obtaining complex potentials. For example, the complex potential corresponding to the uniform flow in the w plane of Fig. 9-6 is given by $V_0 w$. By use of the mapping function $w = z + a^2/z$ [see entry A-4, Page 206] the upper half w plane of Fig. 9-6 is transformed into the upper half z plane exterior to circle C, and the complex potential for the flow of Fig. 9-7 is given by

$$\Omega(z) = V_0\left(z + \frac{a^2}{z}\right) \tag{21}$$

Similarly if $z = F(\zeta)$ maps C and its exterior on to C' and its exterior [see Fig. 9-8], then the complex potential for the flow of Fig. 9-8 is obtained by replacing z by $F(\zeta)$ in *(21)*. The complex potential can also be obtained on going directly from the w to the ζ plane by means of a suitable mapping function.

Using the above and introducing other physical phenomena such as circulation, we can describe the flow pattern about variously shaped airfoils and thus describe the motion of an airplane in flight.

BERNOULLI'S THEOREM

If P denotes the pressure in a fluid and V is the speed of the fluid, then *Bernoulli's theorem* states that

$$P + \tfrac{1}{2}\sigma V^2 = K \tag{22}$$

where σ is the fluid density and K is a constant along any streamline.

THEOREMS OF BLASIUS

1. Let X and Y be the net forces, in the positive x and y directions respectively, due to fluid pressure on the surface of an obstacle bounded by a simple closed curve C. Then if Ω is the complex potential for the flow,

$$X - iY = \tfrac{1}{2}i\sigma \oint_C \left(\frac{d\Omega}{dz}\right)^2 dz \tag{23}$$

2. If M is the moment about the origin of the pressure forces on the obstacle, then

$$M = \text{Re}\left\{-\tfrac{1}{2}\sigma \oint_C z\left(\frac{d\Omega}{dz}\right)^2 dz\right\} \tag{24}$$

where "Re" denotes as usual "real part of".

Applications to Electrostatics

COULOMB'S LAW

Let r be the distance between two point electric charges q_1 and q_2. Then the force between them is given in magnitude by *Coulomb's law* which states that

$$F = \frac{q_1 q_2}{\kappa r^2} \tag{25}$$

and is one of repulsion or attraction according as the charges are like (both positive or both negative) or unlike (one positive and the other negative). The constant κ in *(25)*, which is called the *dielectric constant*, depends on the medium; in a vacuum $\kappa = 1$, in other cases $\kappa > 1$. In the following we assume $\kappa = 1$ unless otherwise specified.

ELECTRIC FIELD INTENSITY. ELECTROSTATIC POTENTIAL

Suppose we are given a charge distribution which may be continuous, discrete, or a combination. This charge distribution sets up an electric field. If a unit positive charge (small enough so as not to affect the field appreciably) is placed at any point A not already occupied by charge, the force acting on this charge is called the *electric field intensity* at A and is denoted by $\mathcal{E}$. This force is derivable from a potential Φ which is sometimes called the *electrostatic potential*. In symbols,

$$\mathcal{E} = -\text{grad}\,\Phi = -\nabla\Phi \tag{26}$$

If the charge distribution is two-dimensional, which is our main concern here, then

$$\mathcal{E} \;=\; E_x + iE_y \;=\; -\frac{\partial \Phi}{\partial x} - i\frac{\partial \Phi}{\partial y} \qquad \text{where } E_x = -\frac{\partial \Phi}{\partial x},\; E_y = -\frac{\partial \Phi}{\partial y} \tag{27}$$

In such case if E_t denotes the component of the electric field intensity tangential to any simple closed curve C in the z plane,

$$\oint_C E_t\,ds \;=\; \oint_C E_x\,dx + E_y\,dy \;=\; 0 \tag{28}$$

GAUSS' THEOREM

Let us confine ourselves to charge distributions which can be considered two-dimensional. If C is any simple closed curve in the z plane having a net charge q in its interior (actually an infinite cylinder enclosing a net charge q) and E_n is the normal component of the electric field intensity, then *Gauss' theorem* states that

$$\oint_C E_n\,ds \;=\; 4\pi q \tag{29}$$

If C does not enclose any net charge, this reduces to

$$\oint_C E_n\,ds \;=\; \oint_C E_x\,dy - E_y\,dx \;=\; 0 \tag{30}$$

It follows that in any region not occupied by charge,

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} \;=\; 0 \tag{31}$$

From (27) and (31), we have

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \;=\; 0 \tag{32}$$

i.e. Φ is harmonic at all points not occupied by charge.

THE COMPLEX ELECTROSTATIC POTENTIAL

From the above it is evident that there must exist a harmonic function Ψ conjugate to Φ such that

$$\Omega(z) \;=\; \Phi(x,y) + i\Psi(x,y) \tag{33}$$

is analytic in any region not occupied by charge. We call $\Omega(z)$ the *complex electrostatic potential* or, briefly, *complex potential*. In terms of this, (27) becomes

$$\mathcal{E} \;=\; -\frac{\partial \Phi}{\partial x} - i\frac{\partial \Phi}{\partial y} \;=\; -\frac{\partial \Phi}{\partial x} + i\frac{\partial \Psi}{\partial y} \;=\; -\overline{\frac{d\Omega}{dz}} \;=\; -\overline{\Omega'(z)} \tag{34}$$

and the magnitude of $\mathcal{E}$ is given by $E = |\mathcal{E}| = |-\overline{\Omega'(z)}| = |\Omega'(z)|$.

The curves (cylindrical surfaces in three dimensions)

$$\Phi(x,y) = \alpha, \qquad \Psi(x,y) = \beta \tag{35}$$

are called *equipotential lines* and *flux lines* respectively.

LINE CHARGES

The analogy of the above with fluid flow is quite apparent. The electric field in electrostatic problems corresponds to the velocity field in fluid flow problems, the only difference being a change of sign in the corresponding complex potentials.

The ideas of sources and sinks of fluid flow have corresponding analogues for electrostatics. Thus the complex (electrostatic) potential due to a line charge q per unit length at z_0 (in a vacuum) is given by

$$\Omega(z) = -2q \ln(z - z_0) \tag{36}$$

and represents a source or sink according as $q < 0$ or $q > 0$. Similarly we talk about doublets or dipoles, etc. If the medium is not a vacuum, we replace q in (*36*) by q/κ.

CONDUCTORS

If a solid is perfectly conducting, i.e. is a *perfect conductor*, all charge is located on its surface. Thus if we consider the surface represented by the simple closed curve C in the z plane, the charges are in equilibrium on C and hence C is an equipotential line.

An important problem is the calculation of potential due to a set of charged cylinders. This can be accomplished by use of conformal mapping.

CAPACITANCE

Two conductors having charges of equal magnitude q but of opposite sign, have a difference of potential, say V. The quantity C defined by

$$q = CV \tag{37}$$

depends only on the geometry of the conductors and is called the *capacitance*. The conductors themselves form what is called a *condenser* or *capacitor*.

Applications to Heat Flow

HEAT FLUX

Consider a solid having a temperature distribution which may be varying. We are often interested in the quantity of heat conducted per unit area per unit time across a surface located in the solid. This quantity, sometimes called the *heat flux* across the surface, is given by

$$\mathcal{Q} = -K \operatorname{grad} \Phi \tag{38}$$

where Φ is the temperature and K, assumed to be a constant, is called the *thermal conductivity* and depends on the material of which the solid is made.

THE COMPLEX TEMPERATURE

If we restrict ourselves to problems of two-dimensional type, then

$$\mathcal{Q} = -K\left(\frac{\partial\Phi}{\partial x} + i\frac{\partial\Phi}{\partial y}\right) = Q_x + iQ_y \qquad \text{where } Q_x = -K\frac{\partial\Phi}{\partial x},\ Q_y = -K\frac{\partial\Phi}{\partial y} \tag{39}$$

Let C be any simple closed curve in the z plane (representing the cross-section of a cylinder). If Q_t and Q_n are the tangential and normal components of the heat flux and if *steady state* conditions prevail so that there is no net accumulation of heat inside C, then we have

$$\oint_C Q_n\,ds = \oint_C Q_x\,dy - Q_y\,dx = 0, \qquad \oint_C Q_t\,ds = \oint_C Q_x\,dx + Q_y\,dy = 0 \tag{40}$$

assuming no sources or sinks inside C. The first equation of (40) yields

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = 0 \tag{41}$$

which becomes on using (39),

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

i.e. Φ is harmonic. Introducing the harmonic conjugate function Ψ, we see that

$$\Omega(z) = \Phi(x,y) + i\Psi(x,y) \tag{42}$$

is analytic. The families of curves

$$\Phi(x,y) = \alpha, \quad \Psi(x,y) = \beta \tag{43}$$

are called *isothermal lines* and *flux lines* respectively, while $\Omega(z)$ is called the *complex temperature*.

The analogies with fluid flow and electrostatics are evident and procedures used in these fields can be similarly employed in solving various temperature problems.

Solved Problems

HARMONIC FUNCTIONS

1. Show that the functions (*a*) $x^2 - y^2 + 2y$ and (*b*) $\sin x \cosh y$ are harmonic in any finite region $\mathcal{R}$ of the z plane.

(*a*) If $\Phi = x^2 - y^2 + 2y$, we have $\frac{\partial^2 \Phi}{\partial x^2} = 2$, $\frac{\partial^2 \Phi}{\partial y^2} = -2$. Then $\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$ and Φ is harmonic in $\mathcal{R}$.

(*b*) If $\Phi = \sin x \cosh y$, we have $\frac{\partial^2 \Phi}{\partial x^2} = -\sin x \cosh y$, $\frac{\partial^2 \Phi}{\partial y^2} = \sin x \cosh y$. Then $\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$ and Φ is harmonic in $\mathcal{R}$.

2. Show that the functions of Problem 1 are harmonic in the w plane under the transformation $z = w^3$.

If $z = w^3$, then $x + iy = (u+iv)^3 = u^3 - 3uv^2 + i(3u^2v - v^3)$ and $x = u^3 - 3uv^2$, $y = 3u^2v - v^3$.

(*a*)
$$\begin{aligned}\Phi &= x^2 - y^2 + 2y = (u^3 - 3uv^2)^2 - (3u^2v - v^3)^2 + 2(3u^2v - v^3) \\ &= u^6 - 15u^4v^2 + 15u^2v^4 - v^6 + 6u^2v - 2v^3\end{aligned}$$

Then $\frac{\partial^2 \Phi}{\partial u^2} = 30u^4 - 180u^2v^2 + 30v^4 + 12v$, $\quad \frac{\partial^2 \Phi}{\partial v^2} = -30u^4 + 180u^2v^2 - 30v^4 - 12v$

and $\frac{\partial^2 \Phi}{\partial u^2} + \frac{\partial^2 \Phi}{\partial v^2} = 0$ as required.

(*b*) We must show that $\Phi = \sin(u^3 - 3uv^2)\cosh(3u^2v - v^3)$ satisfies $\frac{\partial^2 \Phi}{\partial u^2} + \frac{\partial^2 \Phi}{\partial v^2} = 0$. This can readily be established by straightforward but tedious differentiation.

This problem illustrates a general result proved in Problem 4.

3. Prove that $\dfrac{\partial^2\Phi}{\partial x^2}+\dfrac{\partial^2\Phi}{\partial y^2} = |f'(z)|^2\left(\dfrac{\partial^2\Phi}{\partial u^2}+\dfrac{\partial^2\Phi}{\partial v^2}\right)$ **where** $w=f(z)$ **is analytic and** $f'(z)\neq 0$.

The function $\Phi(x,y)$ is transformed into a function $\Phi[x(u,v),\,y(u,v)]$ by the transformation. By differentiation we have

$$\frac{\partial\Phi}{\partial x} = \frac{\partial\Phi}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial\Phi}{\partial v}\frac{\partial v}{\partial x}, \qquad \frac{\partial\Phi}{\partial y} = \frac{\partial\Phi}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial\Phi}{\partial v}\frac{\partial v}{\partial y}$$

$$\begin{aligned}
\frac{\partial^2\Phi}{\partial x^2} &= \frac{\partial\Phi}{\partial u}\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial\Phi}{\partial u}\right)+\frac{\partial\Phi}{\partial v}\frac{\partial^2 v}{\partial x^2}+\frac{\partial v}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial\Phi}{\partial v}\right)\\
&= \frac{\partial\Phi}{\partial u}\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\left[\frac{\partial}{\partial u}\left(\frac{\partial\Phi}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial\Phi}{\partial u}\right)\frac{\partial v}{\partial x}\right]\\
&\quad+\frac{\partial\Phi}{\partial v}\frac{\partial^2 v}{\partial x^2}+\frac{\partial v}{\partial x}\left[\frac{\partial}{\partial u}\left(\frac{\partial\Phi}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial\Phi}{\partial v}\right)\frac{\partial v}{\partial x}\right]\\
&= \frac{\partial\Phi}{\partial u}\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\left[\frac{\partial^2\Phi}{\partial u^2}\frac{\partial u}{\partial x}+\frac{\partial^2\Phi}{\partial v\,\partial u}\frac{\partial v}{\partial x}\right]+\frac{\partial\Phi}{\partial v}\frac{\partial^2 v}{\partial x^2}+\frac{\partial v}{\partial x}\left[\frac{\partial^2\Phi}{\partial u\,\partial v}\frac{\partial u}{\partial x}+\frac{\partial^2\Phi}{\partial v^2}\frac{\partial v}{\partial x}\right]
\end{aligned}$$

Similarly,

$$\frac{\partial^2\Phi}{\partial y^2} = \frac{\partial\Phi}{\partial u}\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}\left[\frac{\partial^2\Phi}{\partial u^2}\frac{\partial u}{\partial y}+\frac{\partial^2\Phi}{\partial v\,\partial u}\frac{\partial v}{\partial y}\right]+\frac{\partial\Phi}{\partial v}\frac{\partial^2 v}{\partial y^2}+\frac{\partial v}{\partial y}\left[\frac{\partial^2\Phi}{\partial u\,\partial v}\frac{\partial u}{\partial y}+\frac{\partial^2\Phi}{\partial v^2}\frac{\partial v}{\partial y}\right]$$

Adding,

$$\begin{aligned}
\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2} &= \frac{\partial\Phi}{\partial u}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial\Phi}{\partial v}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\frac{\partial^2\Phi}{\partial u^2}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]\\
&\quad+2\frac{\partial^2\Phi}{\partial u\,\partial v}\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right]+\frac{\partial^2\Phi}{\partial v^2}\left[\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right] \qquad (1)
\end{aligned}$$

Since u and v are harmonic, $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$, $\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}=0$. Also, by the Cauchy-Riemann equations, $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}$, $\dfrac{\partial v}{\partial x}=-\dfrac{\partial u}{\partial y}$. Then

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2 = \left|\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right|^2 = |f'(z)|^2$$

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0$$

Hence (*1*) becomes
$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2} = |f'(z)|^2\left(\frac{\partial^2\Phi}{\partial u^2}+\frac{\partial^2\Phi}{\partial v^2}\right)$$

4. Prove that a harmonic function $\Phi(x,y)$ **remains harmonic under the transformation** $w=f(z)$ **where** $f(z)$ **is analytic and** $f'(z)\neq 0$.

This follows at once from Problem 3, since if $\dfrac{\partial^2\Phi}{\partial x^2}+\dfrac{\partial^2\Phi}{\partial y^2}=0$ and $f'(z)\neq 0$, then $\dfrac{\partial^2\Phi}{\partial u^2}+\dfrac{\partial^2\Phi}{\partial v^2}=0$.

5. If a **is real, show that the real and imaginary parts of** $w=\ln(z-a)$ **are harmonic functions in any region** $\mathcal{R}$ **not containing** $z=a$.

Method 1.

If $\mathcal{R}$ does not contain a, then $w=\ln(z-a)$ is analytic in $\mathcal{R}$. Hence the real and imaginary parts are harmonic in $\mathcal{R}$.

Method 2.

Let $z-a = re^{i\theta}$. Then if principal values are used for θ, $w = u+iv = \ln(z-a) = \ln r + i\theta$ so that $u = \ln r$, $v = \theta$.

In the polar coordinates (r,θ), Laplace's equation is $\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = 0$ and by direct substitution we find that $u = \ln r$ and $v = \theta$ are solutions if $\mathcal{R}$ does not contain $r = 0$, i.e. $z = a$.

Method 3.

If $z-a = re^{i\theta}$, then $x-a = r\cos\theta$, $y = r\sin\theta$ and $r = \sqrt{(x-a)^2+y^2}$, $\theta = \tan^{-1}\{y/(x-a)\}$. Then $w = u+iv = \frac{1}{2}\ln\{(x-a)^2+y^2\} + i\tan^{-1}\{y/(x-a)\}$ and $u = \frac{1}{2}\ln\{(x-a)^2+y^2\}$, $v = \tan^{-1}\{y/(x-a)\}$. Substituting these into Laplace's equation $\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2} = 0$, we find after straightforward differentiation that u and v are solutions if $z \neq a$.

DIRICHLET AND NEUMANN PROBLEMS

6. Find a function harmonic in the upper half of the z plane, $\text{Im}\{z\} > 0$, which takes the prescribed values on the x axis given by $G(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$.

We must solve for $\Phi(x,y)$ the boundary-value problem

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2} = 0, \quad y>0; \qquad \lim_{y\to 0+}\Phi(x,y) = G(x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases}$$

This is a Dirichlet problem for the upper half plane [see Fig. 9-9].

The function $A\theta + B$, where A and B are real constants, is harmonic since it is the imaginary part of $A\ln z + B$.

To determine A and B note that the boundary conditions are $\Phi = 1$ for $x > 0$, i.e. $\theta = 0$ and $\Phi = 0$ for $x < 0$, i.e. $\theta = \pi$. Thus

(1) $1 = A(0) + B$, (2) $0 = A(\pi) + B$

from which $A = -1/\pi$, $B = 1$.

Then the required solution is

$$\Phi = A\theta + B = 1 - \frac{\theta}{\pi} = 1 - \frac{1}{\pi}\tan^{-1}\left(\frac{y}{x}\right)$$

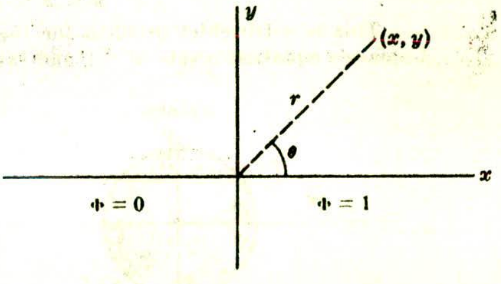


Fig. 9-9

Another method, using Poisson's formula for the half plane.

$$\Phi(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y\,G(\eta)\,d\eta}{y^2+(x-\eta)^2} = \frac{1}{\pi}\int_{-\infty}^{0}\frac{y[0]\,d\eta}{y^2+(x-\eta)^2} + \frac{1}{\pi}\int_{0}^{\infty}\frac{y[1]\,d\eta}{y^2+(x-\eta)^2}$$

$$= \frac{1}{\pi}\tan^{-1}\left(\frac{\eta-x}{y}\right)\Big|_0^{\infty} = \frac{1}{2}+\frac{1}{\pi}\tan^{-1}\left(\frac{x}{y}\right) = 1-\frac{1}{\pi}\tan^{-1}\left(\frac{y}{x}\right)$$

7. Solve the boundary-value problem

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2} = 0, \quad y>0;$$

$$\lim_{y\to 0+}\Phi(x,y) = G(x) = \begin{cases} T_0 & x<-1 \\ T_1 & -1<x<1 \\ T_2 & x>1 \end{cases}$$

where T_0, T_1, T_2 are constants.

This is a Dirichlet problem for the upper half plane [see Fig. 9-10].

The function $A\theta_1 + B\theta_2 + C$ where A, B and C are real constants, is harmonic since it is the imaginary part of $A\ln(z+1) + B\ln(z-1) + C$.

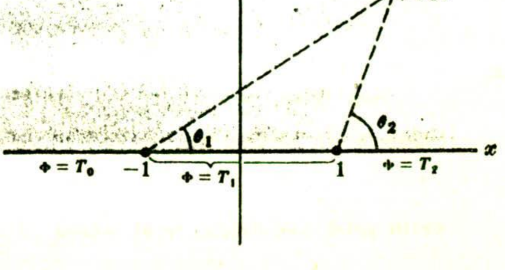


Fig. 9-10

To determine A, B, C note that the boundary conditions are: $\Phi = T_2$ for $x > 1$, i.e. $\theta_1 = \theta_2 = 0$; $\Phi = T_1$ for $-1 < x < 1$, i.e. $\theta_1 = 0$, $\theta_2 = \pi$; $\Phi = T_0$ for $x < -1$, i.e. $\theta_1 = \pi$, $\theta_2 = \pi$. Thus

(1) $T_2 = A(0) + B(0) + C$ (2) $T_1 = A(0) + B(\pi) + C$ (3) $T_0 = A(\pi) + B(\pi) + C$

from which $C = T_2$, $B = (T_1 - T_2)/\pi$, $A = (T_0 - T_1)/\pi$.

Then the required solution is

$$\Phi = A\theta_1 + B\theta_2 + C = \frac{T_0 - T_1}{\pi}\tan^{-1}\left(\frac{y}{x+1}\right) + \frac{T_1 - T_2}{\pi}\tan^{-1}\left(\frac{y}{x-1}\right) + T_2$$

Another method, using Poisson's formula for the half plane.

$$\begin{aligned}\Phi(x,y) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y\,G(\eta)\,d\eta}{y^2+(x-\eta)^2} \\ &= \frac{1}{\pi}\int_{-\infty}^{-1}\frac{y\,T_0\,d\eta}{y^2+(x-\eta)^2} + \frac{1}{\pi}\int_{-1}^{1}\frac{y\,T_1\,d\eta}{y^2+(x-\eta)^2} + \frac{1}{\pi}\int_{1}^{\infty}\frac{y\,T_2\,d\eta}{y^2+(x-\eta)^2} \\ &= \frac{T_0}{\pi}\tan^{-1}\left(\frac{\eta-x}{y}\right)\Big|_{-\infty}^{-1} + \frac{T_1}{\pi}\tan^{-1}\left(\frac{\eta-x}{y}\right)\Big|_{-1}^{1} + \frac{T_2}{\pi}\tan^{-1}\left(\frac{\eta-x}{y}\right)\Big|_{1}^{\infty} \\ &= \frac{T_0-T_1}{\pi}\tan^{-1}\left(\frac{y}{x+1}\right) + \frac{T_1-T_2}{\pi}\tan^{-1}\left(\frac{y}{x-1}\right) + T_2\end{aligned}$$

8. Find a function harmonic inside the unit circle $|z| = 1$ and taking the prescribed values given by $F(\theta) = \begin{cases}1 & 0 < \theta < \pi \\ 0 & \pi < \theta < 2\pi\end{cases}$ on its circumference.

This is a Dirichlet problem for the unit circle [Fig. 9-11] in which we seek a function satisfying Laplace's equation inside $|z| = 1$ and taking the values 0 on arc ABC and 1 on arc CDE.

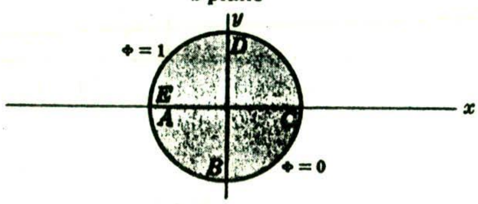


Fig. 9-11

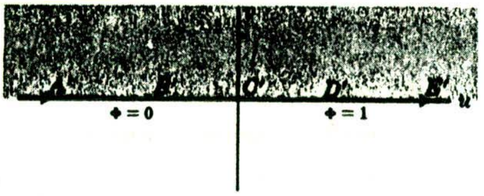


Fig. 9-12

Method 1, using conformal mapping.

We map the interior of the circle $|z| = 1$ on to the upper half of the w plane [Fig. 9-12] by using the mapping function $z = \dfrac{i-w}{i+w}$ or $w = i\left(\dfrac{1-z}{1+z}\right)$ [see Problem 12, Chapter 8, Page 217, and interchange z and w].

Under this transformation, arcs ABC and CDE are mapped on to the negative and positive real axis $A'B'C'$ and $C'D'E'$ respectively of the w plane. Then by Problem 81, the boundary conditions $\Phi = 0$ on arc ABC and $\Phi = 1$ on arc CDE become respectively $\Phi = 0$ on $A'B'C'$ and $\Phi = 1$ on $C'D'E'$.

Thus we have reduced the problem to finding a function Φ harmonic in the upper half w plane and taking the values 0 for $u < 0$ and 1 for $u > 0$. But this problem has already been solved in Problem 6 and the solution (replacing x by u and y by v) is given by

$$\Phi = 1 - \frac{1}{\pi}\tan^{-1}\left(\frac{v}{u}\right) \tag{1}$$

Now from $w = i\left(\dfrac{1-z}{1+z}\right)$, we find $u = \dfrac{2y}{(1+x)^2+y^2}$, $v = \dfrac{1-(x^2+y^2)}{(1+x)^2+y^2}$. Then substituting these in (1), we find the required solution

$$\Phi = 1 - \frac{1}{\pi}\tan^{-1}\left(\frac{2y}{1-[x^2+y^2]}\right) \tag{2}$$

or in polar coordinates (r, θ), where $x = r\cos\theta$, $y = r\sin\theta$,

$$\Phi = 1 - \frac{1}{\pi}\tan^{-1}\left(\frac{2r\sin\theta}{1-r^2}\right) \tag{3}$$

Method 2, using Poisson's formula.

$$\Phi(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{F(\phi)\,d\phi}{1-2r\cos(\theta-\phi)+r^2}$$

$$= \frac{1}{2\pi}\int_0^{\pi} \frac{d\phi}{1-2r\cos(\theta-\phi)+r^2} = 1 - \frac{1}{\pi}\tan^{-1}\left(\frac{2r\sin\theta}{1-r^2}\right)$$

by direct integration [see Problem 69(*b*), Chapter 5, Page 136].

APPLICATIONS TO FLUID FLOW

9. (*a*) Find the complex potential for a fluid moving with constant speed V_0 in a direction making an angle δ with the positive x axis [see Fig. 9-13].

(*b*) Determine the velocity potential and stream function.

(*c*) Determine the equations for the streamlines and equipotential lines.

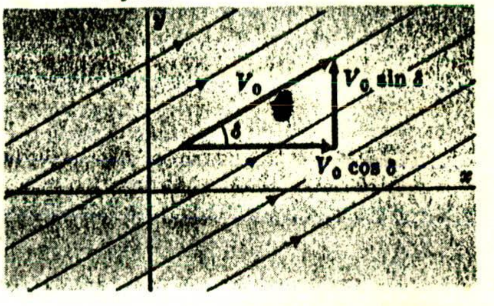


Fig. 9-13

(*a*) The x and y components of velocity are

$$V_x = V_0\cos\delta, \qquad V_y = V_0\sin\delta$$

The complex velocity is

$$\mathcal{V} = V_x + iV_y = V_0\cos\delta + iV_0\sin\delta = V_0 e^{i\delta}$$

The complex potential $\Omega(z)$ is given by

$$\frac{d\Omega}{dz} = \bar{\mathcal{V}} = V_0 e^{-i\delta}$$

Then integrating,

$$\Omega(z) = V_0 e^{-i\delta} z$$

omitting the constant of integration.

(*b*) The velocity potential Φ and stream function Ψ are the real and imaginary parts of the complex potential. Thus

$$\Omega(z) = \Phi + i\Psi = V_0 e^{-i\delta} z = V_0(x\cos\delta + y\sin\delta) + iV_0(y\cos\delta - x\sin\delta)$$

and

$$\Phi = V_0(x\cos\delta + y\sin\delta), \qquad \Psi = V_0(y\cos\delta - x\sin\delta)$$

Another method.

$$(1)\quad \frac{\partial\Phi}{\partial x} = V_x = V_0\cos\delta \qquad\qquad (2)\quad \frac{\partial\Phi}{\partial y} = V_y = V_0\sin\delta$$

Solving for Φ in (*1*), $\Phi = (V_0\cos\delta)x + G(y)$. Substituting in (*2*), $G'(y) = V_0\sin\delta$ and $G(y) = (V_0\sin\delta)y$, omitting the constant of integration. Then

$$\Phi = (V_0\cos\delta)x + (V_0\sin\delta)y$$

From the Cauchy-Riemann equations,

$$(3)\quad \frac{\partial\Psi}{\partial y} = \frac{\partial\Phi}{\partial x} = V_x = V_0\cos\delta \qquad\qquad (4)\quad \frac{\partial\Psi}{\partial x} = -\frac{\partial\Phi}{\partial y} = -V_y = -V_0\sin\delta$$

Solving for Ψ in (*3*), $\Psi = (V_0\cos\delta)y + H(x)$. Substituting in (*4*), $H'(x) = -V_0\sin\delta$ and $H(x) = -(V_0\sin\delta)x$, omitting the constant of integration. Then

$$\Psi = (V_0\cos\delta)y - (V_0\sin\delta)x$$

(*c*) The streamlines are given by $\Psi = V_0(y\cos\delta - x\sin\delta) = \beta$ for different values of β. Physically, under steady-state conditions, a streamline represents the path actually taken by a fluid particle, in this case a straight line path.

The equipotential lines are given by $\Phi = V_0(x\cos\delta + y\sin\delta) = \alpha$ for different values of α. Geometrically they are lines perpendicular to the streamlines; all points on an equipotential line are at equal potential.

10. The complex potential of a fluid flow is given by $\Omega(z) = V_0\left(z + \frac{a^2}{z}\right)$ where V_0 and a are positive constants. (*a*) Obtain equations for the streamlines and equipotential lines, represent them graphically and interpret physically. (*b*) Show that we can interpret the flow as that around a circular obstacle of radius a. (*c*) Find the velocity at any point and determine its value far from the obstacle. (*d*) Find the stagnation points.

(*a*) Let $z = re^{i\theta}$. Then

$$\Omega(z) \quad = \quad \Phi + i\Psi \quad = \quad V_0\left(re^{i\theta} + \frac{a^2}{r}e^{-i\theta}\right) \quad = \quad V_0\left(r + \frac{a^2}{r}\right)\cos\theta \;+\; iV_0\left(r - \frac{a^2}{r}\right)\sin\theta$$

from which $\Phi = V_0\left(r + \frac{a^2}{r}\right)\cos\theta, \qquad \Psi = V_0\left(r - \frac{a^2}{r}\right)\sin\theta$

The streamlines are given by Ψ = constant = β, i.e.,

$$V_0\left(r - \frac{a^2}{r}\right)\sin\theta \quad = \quad \beta$$

These are indicated by the heavy curves of Fig. 9-14 and show the actual paths taken by fluid particles. Note that $\Psi = 0$ corresponds to $r = a$ and $\theta = 0$ or π.

The equipotential lines are given by Φ = constant = α, i.e.,

$$V_0\left(r + \frac{a^2}{r}\right)\cos\theta \quad = \quad \alpha$$

These are indicated by the dashed curves of Fig. 9-14 and are orthogonal to the family of streamlines.

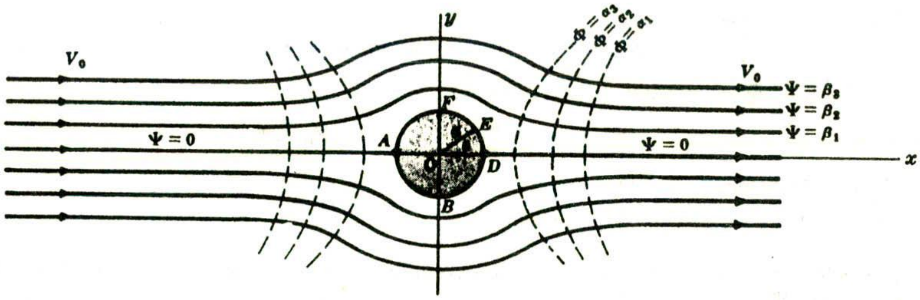


Fig. 9-14

(*b*) The circle $r = a$ represents a streamline; and since there cannot be any flow across a streamline, it can be considered as a circular obstacle of radius a placed in the path of the fluid.

(*c*) We have

$$\Omega'(z) \quad = \quad V_0\left(1 - \frac{a^2}{z^2}\right) \quad = \quad V_0\left(1 - \frac{a^2}{r^2}e^{-2i\theta}\right) \quad = \quad V_0\left(1 - \frac{a^2}{r^2}\cos 2\theta\right) \;+\; i\,\frac{V_0 a^2}{r^2}\sin 2\theta$$

Then the complex velocity is

$$\mathcal{V} \quad = \quad \overline{\Omega'(z)} \quad = \quad V_0\left(1 - \frac{a^2}{r^2}\cos 2\theta\right) \;-\; i\,\frac{V_0 a^2}{r^2}\sin 2\theta \qquad (1)$$

and its magnitude is

$$V \quad = \quad |\mathcal{V}| \quad = \quad \sqrt{\left\{V_0\left(1 - \frac{a^2}{r^2}\cos 2\theta\right)\right\}^2 + \left\{\frac{V_0 a^2}{r^2}\sin 2\theta\right\}^2}$$

$$= \quad V_0\sqrt{1 - \frac{2a^2\cos 2\theta}{r^2} + \frac{a^4}{r^4}} \qquad (2)$$

Far from the obstacle, we see from (*1*) that $\mathcal{V} = V_0$ approximately, i.e. the fluid is travelling in the direction of the positive x axis with constant speed V_0.

(*d*) The stagnation points, i.e. points at which the velocity is zero, are given by

$$\Omega'(z) = 0, \quad \text{i.e.} \quad V_0\left(1 - \frac{a^2}{z^2}\right) = 0 \quad \text{or} \quad z = a \quad \text{and} \quad z = -a$$

The stagnation points are therefore at A and D in Fig. 9-14.

11. Show that under the transformation $w = z + \frac{a^2}{z}$ the fluid flow in the z plane considered in Problem 10 is mapped into a uniform flow with constant velocity V_0 in the w plane.

The complex potential for the flow in the w plane is given by

$$V_0\left(z + \frac{a^2}{z}\right) = V_0 w$$

which represents uniform flow with constant velocity V_0 in the w plane [compare entry A-4 in the table on Page 206].

In general, the transformation $w = \Omega(z)$ maps the fluid flow in the z plane with complex potential $\Omega(z)$ into a uniform flow in the w plane. This is very useful in determining complex potentials of complicated fluid patterns through a knowledge of mapping functions.

12. Fluid emanates at a constant rate from an infinite line source perpendicular to the z plane at $z = 0$ [Fig. 9-15]. (*a*) Show that the speed of the fluid at a distance r from the source is $V = k/r$ where k is a constant. (*b*) Show that the complex potential is $\Omega(z) = k \ln z$. (*c*) What modification should be made in (*b*) if the line source is at $z = a$? (*d*) What modification is made in (*b*) if the source is replaced by a sink in which fluid is disappearing at a constant rate?

(*a*) Consider a portion of the line source of unit length [Fig. 9-16]. If V_r is the radial velocity of the fluid at distance r from the source and σ is the density of the fluid (assumed incompressible so that σ is constant), then:

Mass of fluid per unit time emanating from line source of unit length
= Mass of fluid crossing surface of cylinder of radius r and height 1
= (Surface area)(Radial velocity)(Fluid density)
$= (2\pi r \cdot 1)(V_r)(\sigma) = 2\pi r V_r \sigma$

If this is to be a constant κ, then

$$V_r = \frac{\kappa}{2\pi\sigma r} = \frac{k}{r}$$

where $k = \kappa/2\pi\sigma$ is called the *strength* of the source.

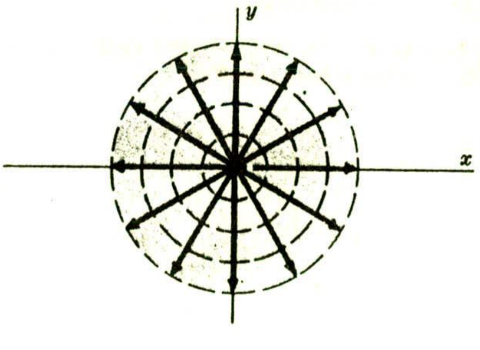


Fig. 9-15

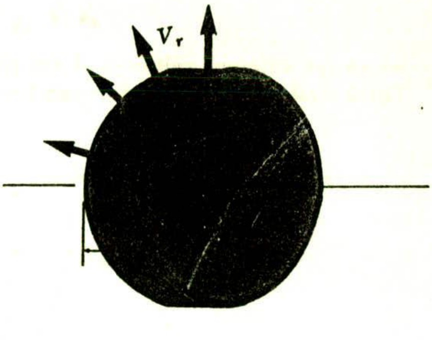


Fig. 9-16

(*b*) Since $V_r = \frac{\partial\Phi}{\partial r} = \frac{k}{r}$, we have on integrating and omitting the constant of integration, $\Phi = k \ln r$. But this is the real part of $\Omega(z) = k \ln z$ which is therefore the required complex potential.

(*c*) If the line source is at $z = a$ instead of $z = 0$, replace z by $z - a$ to obtain the complex potential $\Omega(z) = k \ln (z - a)$.

(*d*) If the source is replaced by a sink, the complex potential is $\Omega(z) = -k \ln z$, the minus sign arising from the fact that the velocity is directed toward $z = 0$.

Similarly, $\Omega(z) = -k \ln (z - a)$ is the complex potential for a sink at $z = a$.

13. (*a*) Find the complex potential due to a source at $z=-a$ and a sink at $z=a$ of equal strengths k. (*b*) Determine the equipotential lines and streamlines and represent graphically. (*c*) Find the speed of the fluid at any point.

(*a*) Complex potential due to source at $z=-a$ of strength k is $k\ln(z+a)$.

Complex potential due to sink at $z=a$ of strength k is $-k\ln(z-a)$.

Then by superposition:

Complex potential due to source at $z=-a$ and sink at $z=a$ of strengths k is

$$\Omega(z) = k\ln(z+a) - k\ln(z-a) = k\ln\left(\frac{z+a}{z-a}\right)$$

(*b*) Let $z+a = r_1e^{i\theta_1}$, $z-a = r_2e^{i\theta_2}$. Then

$$\Omega(z) = \Phi + i\Psi = k\ln\left(\frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}}\right) = k\ln\left(\frac{r_1}{r_2}\right) + ik(\theta_1-\theta_2)$$

so that $\Phi = k\ln(r_1/r_2)$, $\Psi = k(\theta_1-\theta_2)$. The equipotential lines and streamlines are thus given by

$$\Phi = k\ln(r_1/r_2) = \alpha, \qquad \Psi = k(\theta_1-\theta_2) = \beta$$

Using $r_1 = \sqrt{(x+a)^2+y^2}$, $r_2 = \sqrt{(x-a)^2+y^2}$, $\theta_1 = \tan^{-1}\left(\frac{y}{x+a}\right)$, $\theta_2 = \tan^{-1}\left(\frac{y}{x-a}\right)$, the equipotential lines are given by

$$\frac{\sqrt{(x+a)^2+y^2}}{\sqrt{(x-a)^2+y^2}} = e^{\alpha/k}$$

This can be written in the form

$$[x - a\coth(\alpha/k)]^2 + y^2 = a^2\operatorname{csch}^2(\alpha/k)$$

which for different values of α are circles having centres at $a\coth(\alpha/k)$ and radii equal to $a\,|\operatorname{csch}(\alpha/k)|$.

These circles are shown by the dashed curves of Fig. 9-17.

The streamlines are given by

$$\tan^{-1}\left(\frac{y}{x+a}\right) - \tan^{-1}\left(\frac{y}{x-a}\right) = \beta/k$$

or taking the tangent of both sides and simplifying,

$$x^2 + [y + a\cot(\beta/k)]^2 = a^2\csc^2(\beta/k)$$

which for different values of β are circles having centres at $-a\cot(\beta/k)$ and radii $a\,|\csc(\beta/k)|$. These circles, which pass through $(-a,0)$ and $(a,0)$, are shown heavy in Fig. 9-17.

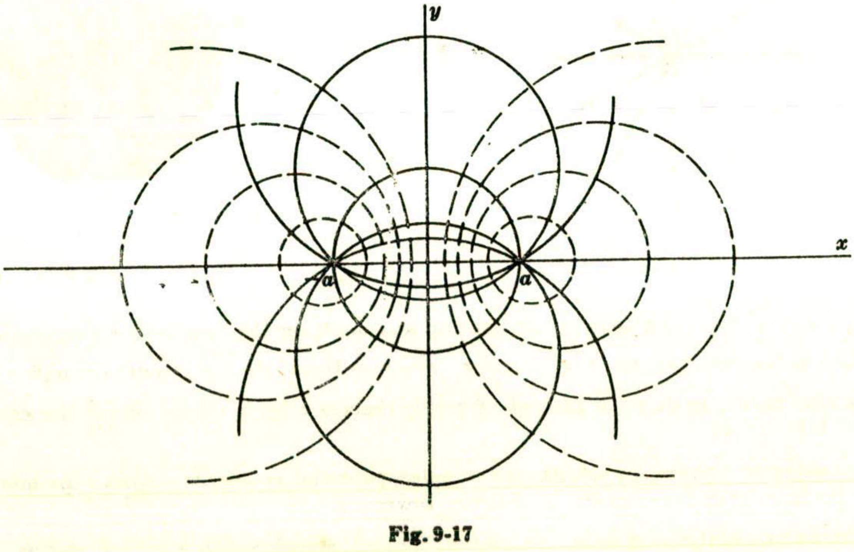


Fig. 9-17

(c) Speed $= |\Omega'(z)| = \left|\frac{k}{z+a} - \frac{k}{z-a}\right| = \frac{2ka}{|z^2-a^2|}$

$$= \frac{2ka}{|a^2 - r^2e^{2i\theta}|} = \frac{2ka}{\sqrt{a^4 - 2a^2r^2\cos 2\theta + r^4}}$$

14. Discuss the motion of a fluid having complex potential $\Omega(z) = ik\ln z$ where $k>0$.

If $z = re^{i\theta}$, then $\Omega(z) = \Phi + i\Psi = ik(\ln r + i\theta) = ik\ln r - k\theta$ or $\Phi = -k\theta$, $\Psi = k\ln r$.

The streamlines are given by

$$\Psi = \text{constant} \quad \text{or} \quad r = \text{constant}$$

which are circles having common centre at $z=0$ [shown heavy in Fig. 9-18].

The equipotential lines, given by θ = constant, are shown dashed in Fig. 9-18.

Since $\Omega'(z) = \frac{ik}{z} = \frac{ik}{r}e^{-i\theta} = \frac{k\sin\theta}{r} + \frac{ik\cos\theta}{r}$, the complex velocity is given by

$$\mathcal{V} = \overline{\Omega'(z)} = \frac{k\sin\theta}{r} - \frac{ik\cos\theta}{r}$$

and shows that the direction of fluid flow is clockwise as indicated in the figure. The speed is given by $V = |\mathcal{V}| = k/r$.

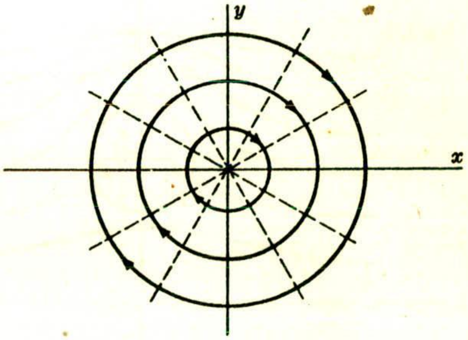


Fig. 9-18

Thus the complex potential describes the flow of a fluid which is rotating around $z=0$. The flow is sometimes referred to as a *vortex flow* and $z=0$ is called a *vortex*.

15. Show that the circulation about the vortex in Problem 14 is given by $\gamma = 2\pi k$.

If curve C encloses $z=0$, the circulation integral is given by

$$\gamma = \oint_C V_t\,ds = \oint_C V_x\,dx + V_y\,dy = \oint_C -\frac{\partial\Phi}{\partial x}dx - \frac{\partial\Phi}{\partial y}dy = \oint_C -d\Phi$$

$$= \int_0^{2\pi} k\,d\theta = 2\pi k$$

In terms of the circulation the complex potential can be written $\Omega(z) = \frac{i\gamma}{2\pi}\ln z$.

16. Discuss the motion of a fluid having complex potential

$$\Omega(z) = V_0\left(z + \frac{a^2}{z}\right) + \frac{i\gamma}{2\pi}\ln z$$

This complex potential has the effect of superimposing a circulation on the flow of Problem 10.

If $z = re^{i\theta}$,

$$\Omega(z) = \Phi + i\Psi = V_0\left(r + \frac{a^2}{r}\right)\cos\theta - \frac{\gamma\theta}{2\pi} + i\left\{V_0\left(r - \frac{a^2}{r}\right)\sin\theta + \frac{\gamma}{2\pi}\ln r\right\}$$

Then the equipotential lines and streamlines are given by

$$V_0\left(r + \frac{a^2}{r}\right)\cos\theta - \frac{\gamma\theta}{2\pi} = \alpha, \qquad V_0\left(r - \frac{a^2}{r}\right)\sin\theta + \frac{\gamma}{2\pi}\ln r = \beta$$

There are in general two stagnation points occurring where $\Omega'(z) = 0$, i.e.

$$V_0\left(1 - \frac{a^2}{z^2}\right) + \frac{i\gamma}{2\pi z} = 0 \quad \text{or} \quad z = \frac{-i\gamma}{4\pi V_0} \pm \sqrt{a^2 - \frac{\gamma^2}{16\pi^2V_0^2}}$$

In case $\gamma = 4\pi aV_0$, there is only one stagnation point.

Since $r = a$ is a streamline corresponding to $\beta = \frac{\gamma}{2\pi}\ln a$, the flow can be considered as one about a circular obstacle as in Problem 10. Far from this obstacle the fluid has velocity V_0 since $\lim_{|z|\to\infty} \Omega'(z) = V_0$.

The flow pattern changes, depending on the magnitude of γ. In Figures 9-19 and 9-20 we have shown two of the many possible ones. Fig. 9-19 corresponds to $\gamma < 4\pi a V_0$; the stagnation points are situated at A and B. Fig. 9-20 corresponds to $\gamma > 4\pi a V_0$ and there is only one stagnation point in the fluid at C.

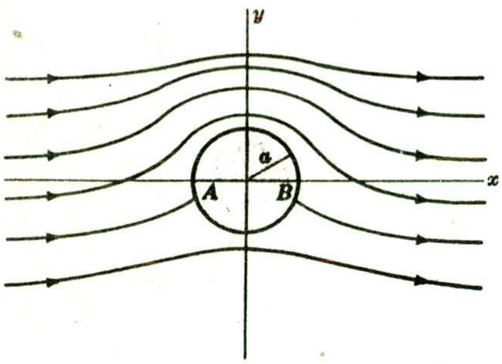


Fig. 9-19

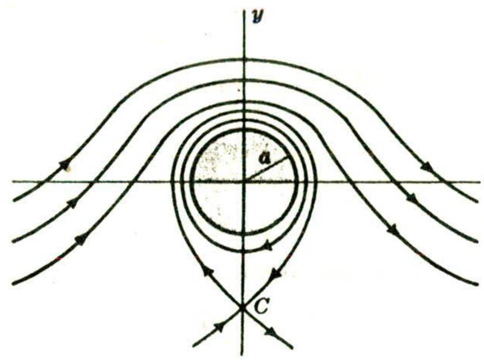


Fig. 9-20

THEOREMS OF BLASIUS

17. Let $\Omega(z)$ be the complex potential describing the flow about a cylindrical obstacle of unit length whose boundary in the z plane is a simple closed curve C. Prove that the net fluid force on the obstacle is given by

$$\bar{F} = X - iY = \tfrac{1}{2} i\sigma \oint_C \left(\frac{d\Omega}{dz}\right)^2 dz$$

where X and Y are the components of force in the positive x and y directions respectively and σ is the fluid density.

The force acting on the element of area ds in Fig. 9-21 is normal to ds and given in magnitude by $P\,ds$ where P is the pressure. On resolving this force into components parallel to the x and y axes, we see that it is given by

$$\begin{aligned} dF &= dX + i\,dY \\ &= -P\,ds \sin\theta + iP\,ds\cos\theta \\ &= iP\,ds\,(\cos\theta + i\sin\theta) \\ &= iP\,ds\,e^{i\theta} \\ &= iP\,dz \end{aligned}$$

using the fact that

$$\begin{aligned} dz &= dx + i\,dy \\ &= ds\cos\theta + i\,ds\sin\theta \\ &= ds\,e^{i\theta} \end{aligned}$$

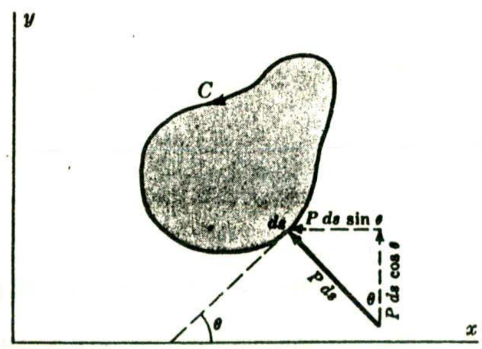


Fig. 9-21

Since C represents a streamline, we have by Bernoulli's theorem, $P + \frac{1}{2}\sigma V^2 = K$ or $P = K - \frac{1}{2}\sigma V^2$, where V is the fluid speed on the streamline. Also by Problem 49 we have, $\frac{d\Omega}{dz} = Ve^{-i\theta}$.

Then, integrating over C, we find

$$F = X + iY = \oint_C iP\,dz = i\oint_C (K - \tfrac{1}{2}\sigma V^2)\,dz$$
$$= -\tfrac{1}{2}i\sigma \oint_C V^2\,dz = -\tfrac{1}{2}i\sigma\oint_C V^2 e^{i\theta}\,ds$$
$$= -\tfrac{1}{2}i\sigma\oint_C (V^2 e^{2i\theta})(e^{-i\theta}\,ds)$$

or

$$\bar{F} = X - iY = \tfrac{1}{2}i\sigma\oint_C (V^2 e^{-2i\theta})(e^{i\theta}\,ds)$$
$$= \tfrac{1}{2}i\sigma\oint_C \left(\frac{d\Omega}{dz}\right)^2 dz$$

18. Let M denote the total moment about the origin of the pressure forces on the obstacle in Problem 17. Prove that

$$M = \text{Re}\left\{-\tfrac{1}{2}\sigma\oint_C z\left(\frac{d\Omega}{dz}\right)^2 dz\right\}$$

We consider counterclockwise moments as positive. The moment about the origin of the force acting on element ds of Fig. 9-21 is

$$dM = (P\,ds\sin\theta)y + (P\,ds\cos\theta)x = P(y\,dy + x\,dx)$$

since $ds\sin\theta = dy$ and $ds\cos\theta = dx$. Then on using Bernoulli's equation, the total moment is

$$M = \oint_C P(y\,dy + x\,dx) = \oint_C (K - \tfrac{1}{2}\sigma V^2)(y\,dy + x\,dx)$$
$$= K\oint_C (y\,dy + x\,dx) - \tfrac{1}{2}\sigma\oint_C V^2(y\,dy + x\,dx)$$
$$= 0 \qquad - \tfrac{1}{2}\sigma\oint_C V^2(x\cos\theta + y\sin\theta)\,ds$$

where we have used the fact that $\oint_C (y\,dy + x\,dx) = 0$ since $y\,dy + x\,dx$ is an exact differential. Hence

$$M = -\tfrac{1}{2}\sigma\oint_C V^2(x\cos\theta + y\sin\theta)\,ds$$
$$= \text{Re}\left\{-\tfrac{1}{2}\sigma\oint_C V^2(x+iy)(\cos\theta - i\sin\theta)\,ds\right\}$$
$$= \text{Re}\left\{-\tfrac{1}{2}\sigma\oint_C V^2 z e^{-i\theta}\,ds\right\} = \text{Re}\left\{-\tfrac{1}{2}\sigma\oint_C z(V^2 e^{-2i\theta})(e^{i\theta}\,ds)\right\}$$
$$= \text{Re}\left\{-\tfrac{1}{2}\sigma\oint_C z\left(\frac{d\Omega}{dz}\right)^2 dz\right\}$$

Sometimes we write this result in the form $M + iN = -\tfrac{1}{2}\sigma\oint_C z\left(\frac{d\Omega}{dz}\right)^2 dz$ where N has no simple physical significance.

19. Find the net force acting on the cylindrical obstacle of Problem 16.

The complex potential for the flow in Problem 16 is

$$\Omega = V_0\left(z + \frac{a^2}{z}\right) + \frac{i\gamma}{2\pi}\ln z$$

where V_0 is the speed of the fluid at distances far from the obstacle and γ is the circulation. By Problem 17 the net force acting on the cylindrical obstacle is given by F, where

$$F = X - iY = \tfrac{1}{2}i\sigma \oint_C \left(\frac{d\Omega}{dz}\right)^2 dz = \tfrac{1}{2}i\sigma \oint_C \left\{V_0\left(1 - \frac{a^2}{z^2}\right) + \frac{i\gamma}{2\pi z}\right\}^2 dz$$

$$= \tfrac{1}{2}i\sigma \oint_C \left\{V_0^2\left(1 - \frac{a^2}{z^2}\right)^2 + \frac{2iV_0\gamma}{2\pi z}\left(1 - \frac{a^2}{z^2}\right) - \frac{\gamma^2}{4\pi^2 z^2}\right\} dz = -\sigma V_0 \gamma$$

Then $X = 0$, $Y = \sigma V_0 \gamma$ and it follows that there is a net force in the positive y direction of magnitude $\sigma V_0 \gamma$. In the case where the cylinder is horizontal and the flow takes place in a vertical plane this force is called the *lift* on the cylinder.

APPLICATIONS TO ELECTROSTATICS

20. (*a*) Find the complex potential due to a line of charge q per unit length perpendicular to the z plane at $z = 0$.

(*b*) What modification should be made in (*a*) if the line is at $z = a$?

(*c*) Discuss the similarity with the complex potential for a line source or sink in fluid flow.

(*a*) The electric field due to a line charge q per unit length is radial and the normal component of the electric vector is constant and equal to E_r while the tangential component is zero (see Fig. 9-22). If C is any cylinder of radius r with axis at $z = 0$, then by Gauss' theorem,

$$\oint_C E_n \, ds = E_r \oint_C ds = E_r \cdot 2\pi r = 4\pi q$$

and

$$E_r = \frac{2q}{r}$$

Since $E_r = -\dfrac{\partial \Phi}{\partial r}$ we have $\Phi = -2q \ln r$, omitting the constant of integration. This is the real part of $\Omega(z) = -2q \ln z$ which is the required complex potential.

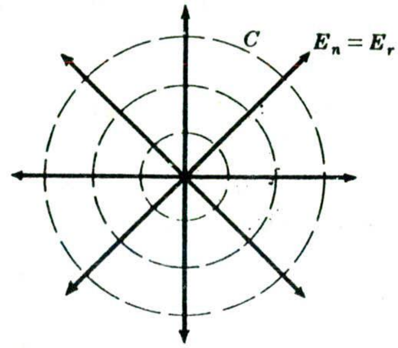


Fig. 9-22

(*b*) If the line of charge is at $z = a$, the complex potential is $\Omega(z) = -2q \ln(z - a)$.

(*c*) The complex potential has the same form as that for a line source of fluid if $k = -2q$ [see Problem 12]. If q is a positive charge, this corresponds to a line sink.

21. (*a*) Find the potential at any point of the region shown in Fig. 9-23 if the potentials on the x axis are given by V_0 for $x > 0$ and $-V_0$ for $x < 0$.

(*b*) Determine the equipotential and flux lines.

(*a*) We must find a function, harmonic in the plane, which takes on the values V_0 for $x > 0$, i.e. $\theta = 0$, and $-V_0$ for $x < 0$, i.e. $\theta = \pi$. As in Problem 6, if A and B are real constants $A\theta + B$ is harmonic. Then $A(0) + B = V_0$, $A(\pi) + B = -V_0$ from which $A = -2V_0/\pi$, $B = V_0$ so that the required potential is

$$V_0\left(1 - \frac{2}{\pi}\theta\right) = V_0\left(1 - \frac{2}{\pi}\tan^{-1}\frac{y}{x}\right)$$

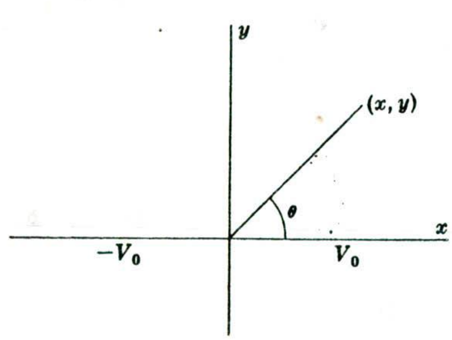


Fig. 9-23

in the upper half plane $y > 0$. The potential in the lower half plane is obtained by symmetry.

(*b*) The equipotential lines are given by $V_0\left(1 - \frac{2}{\pi}\tan^{-1}\frac{y}{x}\right) = \alpha$, i.e. $y = mx$ where m is a constant. These are straight lines passing through the origin.

The flux lines are the orthogonal trajectories of the lines $y = mx$ and are given by $x^2 + y^2 = \beta$. They are circles with centre at the origin.

Another method. A function conjugate to $V_0\left(1 - \frac{2}{\pi}\tan^{-1}\frac{y}{x}\right)$ is $-\frac{2V_0}{\pi}\ln r$. Then the flux lines are given by $r = \sqrt{x^2+y^2} = \text{constant}$, which are circles with centre at the origin.

22. (*a*) Find the potential due to a line charge q per unit length at $z = z_0$ and a line charge $-q$ per unit length at $z = \bar{z}_0$.

(*b*) Show that the potential due to an infinite plane [*ABC* in Fig. 9-25] kept at zero potential (ground potential) and a line charge q per unit length parallel to this plane can be found from the result in (*a*).

(*a*) The complex potential due to the two line charges [Fig. 9-24] is

$$\Omega(z) \;=\; -2q\ln(z-z_0) + 2q\ln(z-\bar{z}_0) \;=\; 2q\ln\left(\frac{z-\bar{z}_0}{z-z_0}\right)$$

Then the required potential is the real part of this, i.e.,

$$\Phi \;=\; 2q\ \text{Re}\left\{\ln\left(\frac{z-\bar{z}_0}{z-z_0}\right)\right\} \tag{1}$$

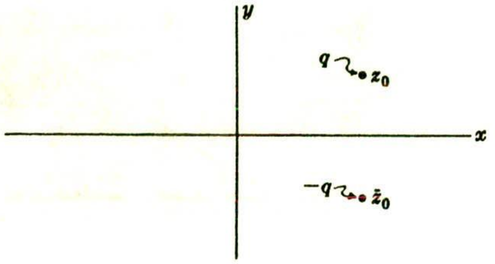


Fig. 9-24

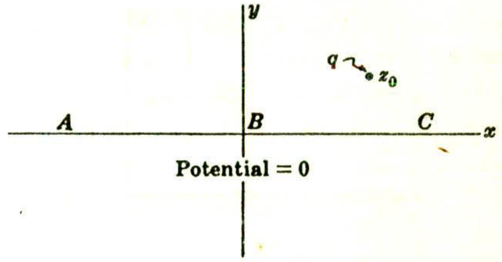


Fig. 9-25

(*b*) To prove this we must show that the potential (*1*) reduces to $\Phi = 0$ on the x axis, i.e. *ABC* in Fig. 9-25 is at potential zero. This follows at once from the fact that on the x axis, $z = x$ so that

$$\Omega \;=\; 2q\ln\left(\frac{x-\bar{z}_0}{x-z_0}\right) \qquad \text{and} \qquad \bar{\Omega} \;=\; 2q\ln\left(\frac{x-z_0}{x-\bar{z}_0}\right) \;=\; -\Omega$$

i.e. $\Phi = \text{Re}\{\Omega\} = 0$ on the x axis.

Thus we can replace the charge $-q$ at $\bar{z}_0$ [Fig. 9-24] by a plane *ABC* at potential zero [Fig. 9-25] and conversely.

23. Two infinite parallel planes, separated by a distance a, are grounded (i.e. are at potential zero). A line charge q per unit length is located between the planes at a distance b from one plane. Determine the potential at any point between the planes.

Let *ABC* and *DEF* in Fig. 9-26 represent the two planes perpendicular to the z plane, and suppose the line charge passes through the imaginary axis at the point $z = bi$.

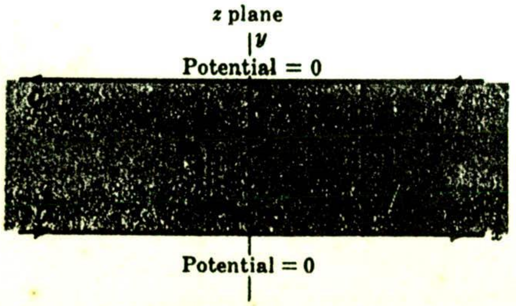


Fig. 9-26

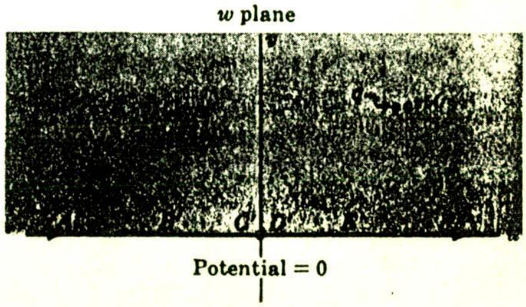


Fig. 9-27

From entry A-2 in the table on Page 205 we see that the transformation $w = e^{\pi z/a}$ maps the shaded region of Fig. 9-26 on to the upper half w plane of Fig. 9-27. The line charge q at $z = bi$ in Fig. 9-26 is mapped into the line charge q at $w = e^{\pi bi/a}$. The boundary $ABCDEF$ of Fig. 9-26 (at potential zero) is mapped into the x axis $A'B'C'D'E'F'$ (at potential zero) where C' and D' are coincident at $w = 0$.

By Problem 22 the potential at any point of the shaded region in Fig. 9-27 above is

$$\Phi \quad = \quad 2q \ \text{Re}\left\{\frac{w - e^{-\pi bi/a}}{w - e^{\pi bi/a}}\right\}$$

Then the potential at any point of the shaded region in Fig. 9-26 is

$$\Phi \quad = \quad 2q \ \text{Re}\left\{\frac{e^{\pi z/a} - e^{-\pi bi/a}}{e^{\pi z/a} - e^{\pi bi/a}}\right\}$$

APPLICATIONS TO HEAT FLOW

24. A semi-infinite slab (shaded in Fig. 9-28) has its boundaries maintained at the indicated temperatures where T is constant. Find the steady-state temperature.

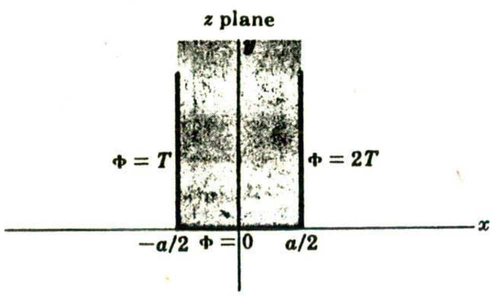


Fig. 9-28

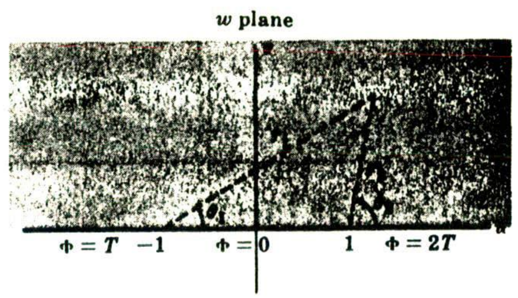


Fig. 9-29

The shaded region of the z plane is mapped into the upper half of the w plane [Fig. 9-29] by the mapping function $w = \sin(\pi z/a)$ which is equivalent to $u = \sin(\pi x/a)\cosh(\pi y/a)$, $v = \cos(\pi x/a)\sinh(\pi y/a)$ [see entry A-3(a) in the table on Page 205].

We must now solve the equivalent problem in the w plane. We use the method of Problem 7 to find that the solution in the w plane is

$$\Phi \quad = \quad \frac{T}{\pi}\tan^{-1}\left(\frac{v}{u+1}\right) - \frac{2T}{\pi}\tan^{-1}\left(\frac{v}{u-1}\right) + 2T$$

and the required solution to the problem in the z plane is therefore

$$\Phi \quad = \quad \frac{T}{\pi}\tan^{-1}\left\{\frac{\cos(\pi x/a)\sinh(\pi y/a)}{\sin(\pi x/a)\cosh(\pi y/a) + 1}\right\} - \frac{2T}{\pi}\tan^{-1}\left\{\frac{\cos(\pi x/a)\sinh(\pi y/a)}{\sin(\pi x/a)\cosh(\pi y/a) - 1}\right\} + 2T$$

25. Find the steady-state temperature at any point of the region shown shaded in Fig. 9-30 if the temperatures are maintained as indicated.

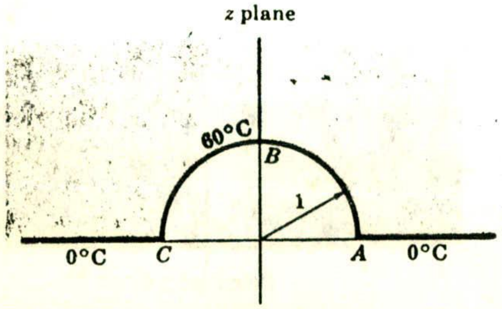


Fig. 9-30

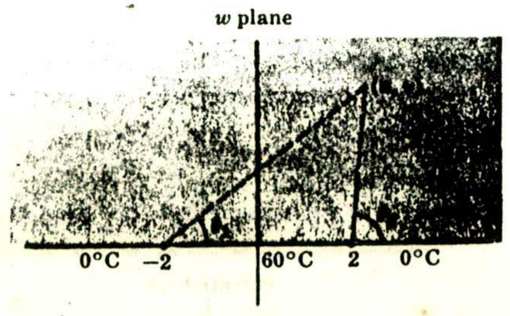


Fig. 9-31

The shaded region of the z plane is mapped on to the upper half of the w plane by means of the mapping function $w = z + \frac{1}{z}$ [entry A-4 in the table on Page 206] which is equivalent to

$$u + iv = x + iy + \frac{1}{x+iy} = x + \frac{x}{x^2+y^2} + i\left(y - \frac{y}{x^2+y^2}\right), \quad \text{i.e.} \quad u = x + \frac{x}{x^2+y^2}, \quad v = y - \frac{y}{x^2+y^2}$$

The solution to the problem in the w plane is, using the method of Problem 7,

$$\frac{60}{\pi}\tan^{-1}\left(\frac{v}{u-2}\right) - \frac{60}{\pi}\tan^{-1}\left(\frac{v}{u+2}\right)$$

Then substituting the values of u and v, the solution to the required problem in the z plane is

$$\frac{60}{\pi}\tan^{-1}\left\{\frac{y(x^2+y^2-1)}{(x^2+y^2+1)x - 2(x^2+y^2)}\right\} - \frac{60}{\pi}\tan^{-1}\left\{\frac{y(x^2+y^2-1)}{(x^2+y^2+1)x + 2(x^2+y^2)}\right\}$$

or, in polar coordinates,

$$\frac{60}{\pi}\tan^{-1}\left\{\frac{(r^2-1)\sin\theta}{(r^2+1)\cos\theta - 2r}\right\} - \frac{60}{\pi}\tan^{-1}\left\{\frac{(r^2-1)\sin\theta}{(r^2+1)\cos\theta + 2r}\right\}$$

MISCELLANEOUS PROBLEMS

26. A region is bounded by two infinitely long concentric cylindrical conductors of radii r_1 and r_2 ($r_2 > r_1$) which are charged to potentials Φ_1 and Φ_2 respectively [see Fig. 9-32]. Find the (*a*) potential and (*b*) electric field vector everywhere in the region.

(*a*) Consider the function $\Omega = A \ln z + B$ where A and B are real constants. If $z = re^{i\theta}$, then

$$\Omega = \Phi + i\Psi = A \ln r + Ai\theta + B$$

or

$$\Phi = A \ln r + B, \qquad \Psi = A\theta$$

Now Φ satisfies Laplace's equation, i.e. is harmonic, everywhere in the region $r_1 < r < r_2$ and reduces to $\Phi = \Phi_1$ and $\Phi = \Phi_2$ on $r = r_1$ and $r = r_2$ provided A and B are chosen so that

$$\Phi_1 = A \ln r_1 + B, \qquad \Phi_2 = A \ln r_2 + B$$

i.e.,

$$A = \frac{\Phi_2 - \Phi_1}{\ln(r_2/r_1)}, \qquad B = \frac{\Phi_1 \ln r_2 - \Phi_2 \ln r_1}{\ln(r_2/r_1)}$$

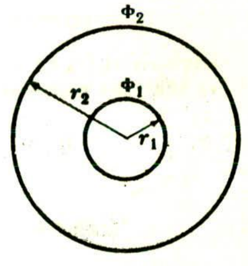


Fig. 9-32

Then the required potential is

$$\Phi = \frac{(\Phi_2 - \Phi_1)}{\ln(r_2/r_1)}\ln r + \frac{\Phi_1 \ln r_2 - \Phi_2 \ln r_1}{\ln(r_2/r_1)}$$

(*b*) Electric field vector

$$= \mathcal{E} = -\text{grad}\,\Phi = -\frac{\partial \Phi}{\partial r} = \frac{\Phi_1 - \Phi_2}{\ln(r_2/r_1)} \cdot \frac{1}{r}$$

Note that the lines of force, or flux lines, are orthogonal to the equipotential lines, and some of these are indicated by the dashed lines of Fig. 9-33.

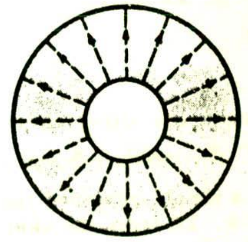


Fig. 9-33

27. Find the capacitance of the condenser formed by the two cylindrical conductors in Problem 26.

If Γ is any simple closed curve containing the inner cylinder and q is the charge on this cylinder, then by Gauss' theorem and the results of Problem 26 we have

$$\oint_\Gamma E_n\, ds = \int_{\theta=0}^{2\pi}\left\{\frac{\Phi_1 - \Phi_2}{\ln(r_2/r_1)} \cdot \frac{1}{r}\right\} r\, d\theta = \frac{2\pi(\Phi_1 - \Phi_2)}{\ln(r_2/r_1)} = 4\pi q$$

Then $q = \dfrac{\Phi_1 - \Phi_2}{2 \ln (r_2/r_1)}$ and so

$$\text{Capacitance } C = \frac{\text{charge}}{\text{difference in potential}} = \frac{q}{\Phi_1 - \Phi_2} = \frac{1}{2 \ln (r_2/r_1)}$$

which depends only on the geometry of the condensers, as it should.

The above result holds if there is a vacuum between the conductors. If there is a medium of dielectric constant κ between the conductors, we must replace q by q/κ and in this case the capacitance is $1/[2\kappa \ln (r_2/r_1)]$.

28. Two circular cylindrical conductors of equal radius R and centres at distance D from each other [Fig. 9-34] are charged to potentials V_0 and $-V_0$ respectively. (*a*) Determine the charge per unit length needed to accomplish this. (*b*) Find an expression for the capacitance.

(*a*) We use the results of Problem 13, since we can replace any of the equipotential curves (surfaces) by circular conductors at the specified potentials. Placing $\alpha = -V_0$ and $\alpha = V_0$ and noting that $k = 2q$, we find that the centres of the circles are at

$$x = -a \coth (V_0/2q) \quad \text{and} \quad x = a \coth (V_0/2q)$$

so that (*1*) $D = 2a \coth (V_0/2q)$

The radius R of the circles is

(*2*) $R = a \operatorname{csch} (V_0/2q)$

Division of (*1*) by (*2*) yields $2 \cosh (V_0/2q) = D/R$ so that the required charge is

$$q = \frac{V_0}{2 \cosh^{-1} (D/2R)}$$

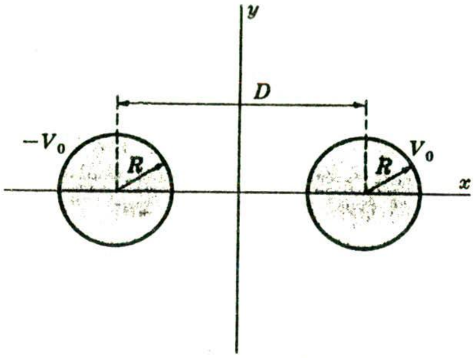


Fig. 9-34

(*b*) $$\text{Capacitance } C = \frac{\text{charge}}{\text{difference in potential}} = \frac{q}{2V_0} = \frac{1}{4 \cosh^{-1} (D/2R)}$$

The result holds for a vacuum. If there is a medium of dielectric constant κ, we must divide the result by κ.

Note that the capacitance depends as usual only on the geometry. The result is fundamental in the theory of transmission line cables.

29. Prove the uniqueness of the solution to Dirichlet's problem.

Dirichlet's problem is the problem of determining a function Φ which satisfies $\dfrac{\partial^2 \Phi}{\partial x^2} + \dfrac{\partial^2 \Phi}{\partial y^2} = 0$ in a simply-connected region $\mathcal{R}$ and which takes on a prescribed value $\Phi = f(x, y)$ on the boundary C of $\mathcal{R}$. To prove the uniqueness, we must show that if such a solution exists it is the only one. To do this suppose that there are two different solutions, say Φ_1 and Φ_2. Then

$$\frac{\partial^2 \Phi_1}{\partial x^2} + \frac{\partial^2 \Phi_1}{\partial y^2} = 0 \text{ in } \mathcal{R} \quad \text{and} \quad \Phi_1 = f(x, y) \text{ on } C \tag{1}$$

$$\frac{\partial^2 \Phi_2}{\partial x^2} + \frac{\partial^2 \Phi_2}{\partial y^2} = 0 \text{ in } \mathcal{R} \quad \text{and} \quad \Phi_2 = f(x, y) \text{ on } C \tag{2}$$

Subtracting and letting $G = \Phi_1 - \Phi_2$, we have

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = 0 \text{ in } \mathcal{R} \quad \text{and} \quad G = 0 \text{ on } C \tag{3}$$

To show that $\Phi_1 = \Phi_2$ identically, we must show that $G = 0$ identically in $\mathcal{R}$.

Let $F = G$ in Problem 31, Chapter 4, Page 112 to obtain

$$\oint_C G\left(\frac{\partial G}{\partial x}dx - \frac{\partial G}{\partial y}dy\right) = -\iint_{\mathcal{R}} \left[G\left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2}\right) + \left(\frac{\partial G}{\partial x}\right)^2 + \left(\frac{\partial G}{\partial y}\right)^2\right] dx\,dy \qquad (4)$$

Suppose that G is not identically equal to a constant in $\mathcal{R}$. From the fact that $G = 0$ on C, and $\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = 0$ identically in $\mathcal{R}$, (4) becomes

$$\iint_{\mathcal{R}} \left[\left(\frac{\partial G}{\partial x}\right)^2 + \left(\frac{\partial G}{\partial y}\right)^2\right] dx\,dy = 0$$

But this contradicts the assumption that G is not identically equal to a constant in $\mathcal{R}$, since in such case

$$\iint_{\mathcal{R}} \left[\left(\frac{\partial G}{\partial x}\right)^2 + \left(\frac{\partial G}{\partial y}\right)^2\right] dx\,dy > 0$$

It follows that G must be constant in $\mathcal{R}$, and by continuity we must have $G = 0$. Thus $\Phi_1 = \Phi_2$ and there is only one solution.

30. An infinite wedge shaped region $ABDE$ of angle $\pi/4$ [shaded in Fig. 9-35] has one of its sides (AB) maintained at constant temperature T_1. The other side BDE has part BD [of unit length] insulated while the remaining part DE is maintained at constant temperature T_2. Find the temperature everywhere in the region.

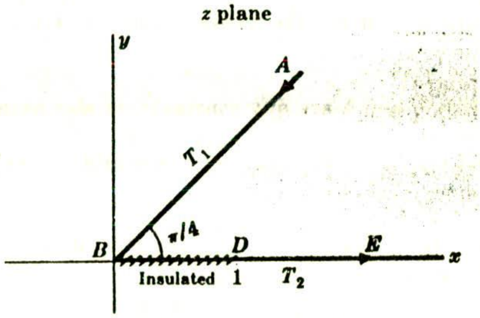


Fig. 9-35

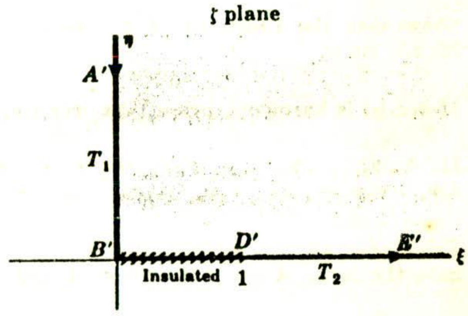


Fig. 9-36

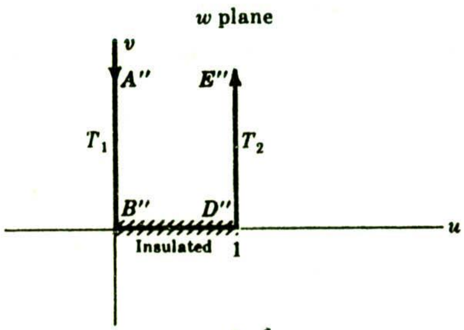


Fig. 9-37

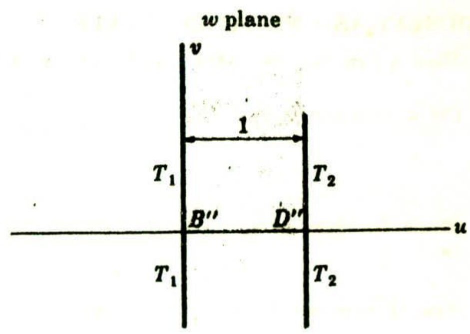


Fig. 9-38

By the transformation $\zeta = z^2$, the shaded region of the z plane [Fig. 9-35] is mapped into the region shaded in Fig. 9-36 with the indicated boundary conditions [see entry A-1 in the table on Page 205].

By the transformation $\zeta = \sin(\pi w/2)$, the shaded region of the ζ plane [Fig. 9-36] is mapped into the region shaded in Fig. 9-37 with the indicated boundary conditions [see entry C-1 in the table on Page 210].

Now the temperature problem represented by Fig. 9-37 with $B''D''$ insulated is equivalent to the temperature problem represented by Fig. 9-38 since, by symmetry, no heat transfer can take place across $B''D''$. But this is the problem of determining the temperature between two parallel planes kept at constant temperatures T_1 and T_2 respectively. In this case the temperature variation is linear and so must be given by $T_1 + (T_2 - T_1)u$.

From $\zeta = z^2$ and $\zeta = \sin(\pi w/2)$ we have on eliminating ζ, $w = \frac{2}{\pi}\sin^{-1} z^2$ or $u = \frac{2}{\pi}\operatorname{Re}\{\sin^{-1} z^2\}$. Then the required temperature is

$$T_1 + \frac{2(T_2 - T_1)}{\pi}\operatorname{Re}\{\sin^{-1} z^2\}$$

In polar coordinates (r, θ) this can be written as [see Problem 95],

$$T_1 + \frac{2(T_2 - T_1)}{\pi}\sin^{-1}\{\tfrac{1}{2}\sqrt{r^4 + 2r^2\cos 2\theta + 1} - \tfrac{1}{2}\sqrt{r^4 - 2r^2\cos 2\theta + 1}\}$$

Supplementary Problems

HARMONIC FUNCTIONS

31. Show that the functions (a) $2xy + y^3 - 3x^2y$, (b) $e^{-x}\sin y$ are harmonic.

32. Show that the functions of Problem 31 remain harmonic under the transformations (a) $z = w^2$, (b) $z = \sin w$.

33. If $\Phi(x, y)$ is harmonic, prove that $\Phi(x + a, y + b)$, where a and b are any constants, is also harmonic.

34. If $\Phi_1, \Phi_2, \ldots, \Phi_n$ are harmonic in a region $\mathcal{R}$ and $c_1, c_2, \ldots, c_n$ are any constants, prove that $c_1\Phi_1 + c_2\Phi_2 + \cdots + c_n\Phi_n$ is harmonic in $\mathcal{R}$.

35. Prove that all the harmonic functions which depend only on the distance r from a fixed point must have the form $A \ln r + B$ where A and B are any constants.

36. If $F(z)$ is analytic and different from zero in a region $\mathcal{R}$, prove that the real and imaginary parts of $\ln F(z)$ are harmonic in $\mathcal{R}$.

DIRICHLET AND NEUMANN PROBLEMS

37. Find a function harmonic in the upper half z plane $\operatorname{Im}\{z\} > 0$ which takes the prescribed values on the x axis given by $G(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$. *Ans.* $1 - (2/\pi)\tan^{-1}(y/x)$

38. Work Problem 37 if $G(x) = \begin{cases} 1 & x < -1 \\ 0 & -1 < x < 1 \\ -1 & x > 1 \end{cases}$.

Ans. $1 - \frac{1}{\pi}\tan^{-1}\left(\frac{y}{x-1}\right) - \frac{1}{\pi}\tan^{-1}\left(\frac{y}{x+1}\right)$

39. Find a function harmonic inside the circle $|z| = 1$ and taking the values $F(\theta) = \begin{cases} T & 0 < \theta < \pi \\ -T & \pi < \theta < 2\pi \end{cases}$ on its circumference. *Ans.* $T\left\{1 - \frac{2}{\pi}\tan^{-1}\left(\frac{2r\sin\theta}{1 - r^2}\right)\right\}$

40. Work Problem 39 if $F(\theta) = \begin{cases} T & 0 < \theta < \pi/2 \\ 0 & \pi/2 < \theta < 3\pi/2 \\ -T & 3\pi/2 < \theta < 2\pi \end{cases}$.

41. Work Problem 39 if $F(\theta) = \begin{cases} \sin\theta & 0 < \theta < \pi \\ 0 & \pi < \theta < 2\pi \end{cases}$.

42. Find a function harmonic inside the circle $|z| = 2$ and taking the values $F(\theta) = \begin{cases} 10 & 0 < \theta < \pi \\ 0 & \pi < \theta < 2\pi \end{cases}$.
Ans. $10\left\{1 - \frac{1}{\pi}\tan^{-1}\left(\frac{4r\sin\theta}{4 - r^2}\right)\right\}$

43. Show by direct substitution that the answers obtained in (*a*) Problem 6, (*b*) Problem 7, (*c*) Problem 8 are actually solutions to the corresponding boundary-value problems.

44. Find a function $\Phi(x, y)$ harmonic in the first quadrant $x > 0$, $y > 0$ which takes on the values $V(x, 0) = -1$, $V(0, y) = 2$. *Ans.* $\frac{3}{\pi}\tan^{-1}\left(\frac{2xy}{x^2 - y^2}\right) - 1$

45. Find a function $\Phi(x, y)$ which is harmonic in the first quadrant $x > 0$, $y > 0$ and which satisfies the boundary conditions $\Phi(x, 0) = e^{-x}$, $\partial\Phi/\partial x\,|_{x=0} = 0$.

APPLICATIONS TO FLUID FLOW

46. Sketch the streamlines and equipotential lines for fluid motion in which the complex potential is given by (*a*) $z^2 + 2z$, (*b*) z^4, (*c*) e^{-z}, (*d*) $\cos z$.

47. Discuss the fluid flow corresponding to the complex potential $\Omega(z) = V_0(z + 1/z^2)$.

48. Verify the statements made before equations (*5*) and (*6*) on Page 234.

49. Derive the relation $d\Omega/dz = Ve^{-i\theta}$, where V and θ are defined as in Problem 17.

50. Referring to Problem 10, (*a*) show that the speed of the fluid at any point E [Fig. 9-14] is given by $2V_0|\sin\theta|$ and (*b*) determine at what points on the cylinder the speed is greatest.

51. (*a*) If P is the pressure at point E of the obstacle in Fig. 9-14 of Problem 10 and P_∞ is the pressure far from the obstacle, show that

$$P - P_\infty = \tfrac{1}{2}\sigma V_0^2(1 - 4\sin^2\theta)$$

(*b*) Show that a vacuum is created at points B and F if the speed of the fluid is equal to or greater than $V_0 = \sqrt{2P_\infty/3\sigma}$. This is often called *cavitation*.

52. Derive equation (*19*), Page 237, by a limiting procedure applied to equation (*18*).

53. Discuss the fluid flow due to three sources of equal strength k located at $z = -a, 0, a$.

54. Discuss the fluid flow due to two sources at $z = \pm a$ and a sink at $z = 0$ if the strengths all have equal magnitude.

55. Prove that under the transformation $w = F(z)$ where $F(z)$ is analytic, a source (or sink) in the z plane at $z = z_0$ is mapped into a source (or sink) of equal strength in the w plane at $w = w_0 = F(z_0)$.

56. Show that the total moment on the cylindrical obstacle of Problem 10 is zero and explain physically.

57. If $\Psi(x, y)$ is the stream function, prove that the mass rate of flow of fluid across an arc C joining points (x_1, y_1) and (x_2, y_2) is $\sigma\{\Psi(x_2, y_2) - \Psi(x_1, y_1)\}$.

58. (*a*) Show that the complex potential due to a source of strength $k > 0$ in a fluid moving with speed V_0 is $\Omega = V_0 z + k\ln z$ and (*b*) discuss the motion.

59. A source and sink of equal strengths m are located at $z = \pm 1$ between the parallel lines $y = \pm 1$. Show that the complex potential for the fluid motion is

$$\Omega = m \ln \left\{ \frac{e^{\pi(z+1)} - 1}{e^{\pi(z-1)} - 1} \right\}$$

60. Given a source of fluid at $z = z_0$ and a wall $x = 0$. Prove that the resulting flow is equivalent to removing the wall and introducing another source of equal strength at $z = -z_0$.

61. Fluid flows between the two branches of the hyperbola $ax^2 - by^2 = 1$, $a > 0$, $b > 0$. Prove that the complex potential for the flow is given by $K \cosh^{-1} \alpha z$ where K is a positive constant and $\alpha = \sqrt{ab/(a+b)}$.

APPLICATIONS TO ELECTROSTATICS

62. Two semi-infinite plane conductors, as indicated in Fig. 9-39 below, are charged to constant potentials Φ_1 and Φ_2 respectively. Find the (*a*) potential Φ and (*b*) electric field $\mathcal{E}$ everywhere in the shaded region between them. *Ans.* (*a*) $\Phi = \Phi_2 + \left(\frac{\Phi_1 - \Phi_2}{\alpha}\right)\theta$ (*b*) $\mathcal{E} = (\Phi_2 - \Phi_1)/\alpha r$

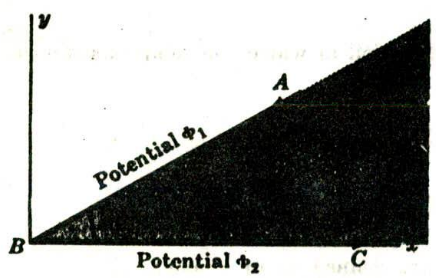


Fig. 9-39

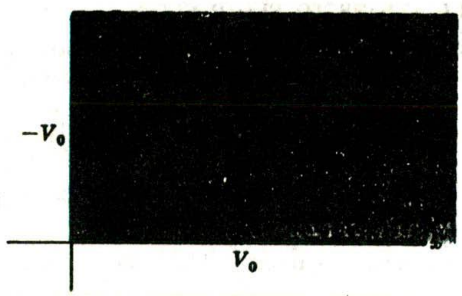


Fig. 9-40

63. Find the (*a*) potential and (*b*) electric field everywhere in the shaded region of Fig. 9-40 above if the potentials on the positive x and y axes are constant and equal to V_0 and $-V_0$ respectively.

Ans. $V_0\left\{1 - \frac{2}{\pi}\tan^{-1}\left(\frac{2xy}{x^2 - y^2}\right)\right\}$

64. An infinite region has in it 3 wires located at $z = -1, 0, 1$ and maintained at constant potentials $-V_0, 2V_0, -V_0$ respectively. Find the (*a*) potential and (*b*) electric field everywhere.
Ans. (*a*) $V_0 \ln \{z(z^2 - 1)\}$

65. Prove that the capacity of a capacitor is invariant under a conformal transformation.

66. The semi-infinite plane conductors AB and BC which intersect at angle α are grounded [Fig. 9-41]. A line charge q per unit length is located at point z_1 in the shaded region at equal distances a from AB and BC. Find the potential. *Ans.* $\text{Im}\left\{-2qi \ln\left(\frac{z^{\pi/\alpha} - z_1^{\pi/\alpha}}{z^{\pi/\alpha} - \bar{z}_1^{\pi/\alpha}}\right)\right\}$

67. Work Problem 66 if q is at a distance a from AB and b from BC.

68. Work Problem 23 if there are two line charges, q per unit length and $-q$ per unit length, located at $z = bi$ and $z = ci$ respectively, where $0 < b < a$, $0 < c < a$ and $b \neq c$.

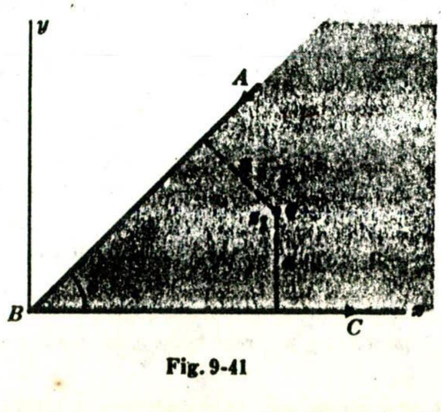


Fig. 9-41

69. An infinitely long circular cylinder has half of its surface charged to constant potential V_0 while the other half is grounded, the two halves being insulated from each other. Find the potential everywhere.

APPLICATIONS TO HEAT FLOW

70. (a) Find the steady-state temperature at any point of the region shown shaded in Fig. 9-42 below and (b) determine the isothermal and flux lines. *Ans.* (a) $60 - (120/\pi)\tan^{-1}(y/x)$

71. Find the steady-state temperature at the point (2, 1) of the region shown shaded in Fig. 9-43 below.

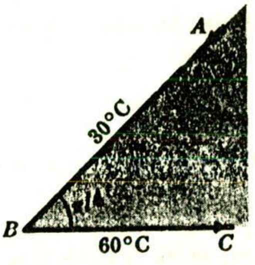


Fig. 9-42

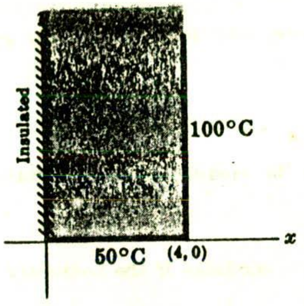


Fig. 9-43

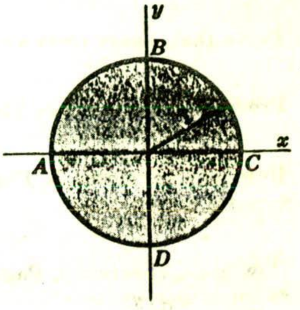


Fig. 9-44

72. The convex portions ABC and ADC of a unit cylinder [Fig. 9-44 above] are maintained at temperatures 40° C and 80° C respectively. (a) Find the steady-state temperature at any point inside. (b) Determine the isothermal and flux lines.

73. Find the steady-state temperature at the point (5, 2) in the shaded region of Fig. 9-45 below if the temperatures are maintained as shown. *Ans.* 45.9° C

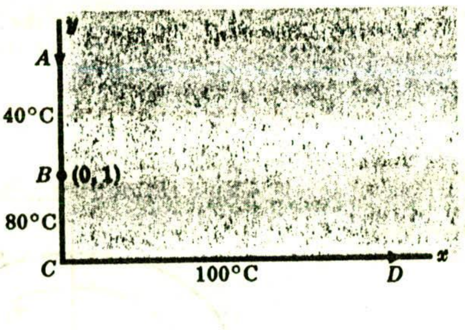


Fig. 9-45

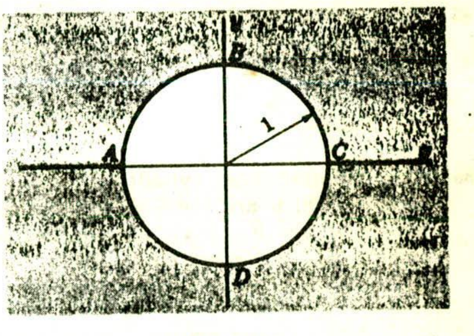


Fig. 9-46

74. An infinite conducting plate has in it a circular hole $ABCD$ of unit radius [Fig. 9-46 above]. Temperatures of 20° C and 80° C are applied to arcs ABC and ADC and maintained indefinitely. Find the steady-state temperature at any point of the plate.

MISCELLANEOUS PROBLEMS

75. If $\Phi(x, y)$ is harmonic, prove that $\Phi(x/r^2, y/r^2)$ where $r = \sqrt{x^2+y^2}$ is also harmonic.

76. Prove that if U and V are continuously differentiable, then

$$(a)\quad \frac{\partial U}{\partial n} = \frac{\partial U}{\partial x}\frac{dx}{ds} + \frac{\partial U}{\partial y}\frac{dy}{ds} \qquad (b)\quad \frac{\partial V}{\partial s} = -\frac{\partial V}{\partial x}\frac{dy}{ds} + \frac{\partial V}{\partial y}\frac{dx}{ds}$$

where n and s denote the outward drawn normal and arc length parameter respectively to a simple closed curve C.

77. If U and V are conjugate harmonic functions, prove that (a) $\dfrac{\partial U}{\partial n} = \dfrac{\partial V}{\partial s}$, (b) $\dfrac{\partial U}{\partial s} = -\dfrac{\partial V}{\partial n}$.

78. Prove that the function $\dfrac{1 - r^2}{1 - 2r\cos\theta + r^2}$ is harmonic in every region which does not include the point $r = 1$, $\theta = 0$.

79. Let it be required to solve the Neumann problem, i.e. to find a function V harmonic in a region $\mathcal{R}$ such that on the boundary C of $\mathcal{R}$, $\partial V/\partial n = G(s)$ where s is the arc length parameter. Let $H(s) = \int_a^s G(s)\,ds$ where a is any point of C, and suppose that $\oint_C G(s)\,ds = 0$. Show that to find V we must find the conjugate harmonic function U which satisfies the condition $U = -H(s)$ on C. This is an equivalent Dirichlet problem. [*Hint*. Use Problem 77.]

80. Prove that, apart from an arbitrary additive constant, the solution to the Neumann problem is unique.

81. Prove Theorem 3, Page 234.

82. How must Theorem 3, Page 234, be modified if the boundary condition $\Phi = a$ on C is replaced by $\Phi = f(x,y)$ on C?

83. How must Theorem 3, Page 234, be modified if the boundary condition $\partial\Phi/\partial n = 0$ on C is replaced by $\partial\Phi/\partial n = g(x,y)$ on C?

84. If a fluid motion is due to some distribution of sources, sinks and doublets and if C is some curve such that no flow takes place across it, then the distribution of sources, sinks and doublets to one side of C is called the *image* of the distribution of sources, sinks and doublets on the other side of C. Prove that the image of a source inside a circle C is a source of equal strength at the inverse point together with a sink of equal strength at the centre of C. [Point P is called the *inverse* of point Q with respect to a circle C with centre at O if OPQ is a straight line and $OP \cdot OQ = a^2$ where a is the radius of C.]

85. A source of strength $k > 0$ is located at point z_0 in a fluid which is contained in the first quadrant where the x and y axes are considered as rigid barriers. Prove that the speed of the fluid at any point is given by

$$k\,|(z-z_0)^{-1} + (z-\bar{z}_0)^{-1} + (z+z_0)^{-1} + (z+\bar{z}_0)^{-1}|$$

86. Two infinitely long cylindrical conductors having cross-sections which are confocal ellipses with foci at $(-c,0)$ and $(c,0)$ [see Fig. 9-47] are charged to constant potentials Φ_1 and Φ_2 respectively. Show that the capacitance per unit length is equal to

$$\frac{2\pi}{\cosh^{-1}(R_2/c) - \cosh^{-1}(R_1/c)}$$

[*Hint*. Use the transformation $z = c\cosh w$.]

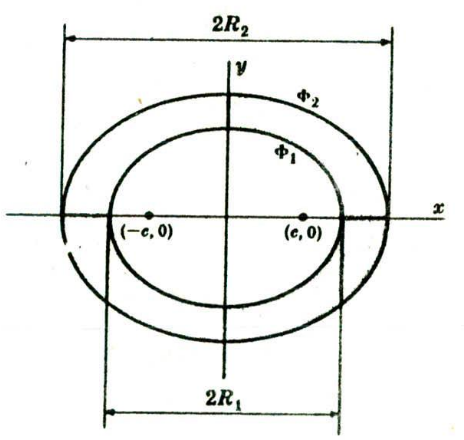


Fig. 9-47

87. In Problem 86 suppose that Φ_1 and Φ_2 represent constant temperatures applied to the elliptic cylinders. Find the steady-state temperature at any point in the conducting region between the cylinders.

88. A circular cylinder obstacle of radius a rests at the bottom of a channel of fluid which at distances far from the obstacle flows with velocity V_0 [see Fig. 9-48].

(*a*) Prove that the complex potential is given by

$$\Omega(z) = \pi a V_0 \coth(\pi a/z)$$

(*b*) Show that the speed at the top of the cylinder is $\frac{1}{4}\pi^2 V_0$ and compare with that for a circular obstacle in the middle of a fluid.

(*c*) Show that the difference in pressure between top and bottom points of the cylinder is $\sigma\pi^4 V_0^2/32$.

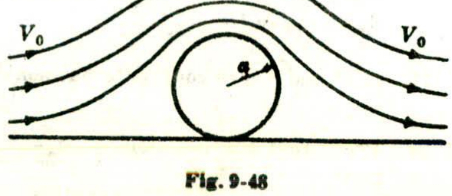


Fig. 9-48

89. (a) Show that the complex potential for fluid flow past the elliptic cylinder of Fig. 9-49 is given by

$$\Omega(z) = V_0\left\{\zeta + \frac{(a+b)^2}{4\zeta}\right\}$$

where $\zeta = \frac{1}{2}(z + \sqrt{z^2 - c^2})$ and $c^2 = a^2 - b^2$.

(b) Prove that the fluid speed at the top and bottom of the cylinder is $V_0(1 + b/a)$. Discuss the case $a = b$. [*Hint.* Express the complex potential in terms of elliptic coordinates (ξ, η) where $z = x + iy = c\cosh(\xi + i\eta) = c\cosh\zeta$.]

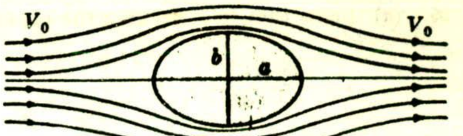


Fig. 9-49

90. Show that if the flow in Problem 89 is in a direction making an angle δ with the positive x axis, the complex potential is given by the result in (a) with $\zeta = \frac{1}{2}(z + \sqrt{z^2 - c^2})e^{i\delta}$.

91. In the *theory of elasticity*, the equation

$$\nabla^4\Phi = \nabla^2(\nabla^2\Phi) = \frac{\partial^4\Phi}{\partial x^4} + 2\frac{\partial^4\Phi}{\partial x^2\,\partial y^2} + \frac{\partial^4\Phi}{\partial y^4} = 0$$

called the *biharmonic equation*, is of fundamental importance. Solutions to this equation are called *biharmonic*. Prove that if $F(z)$ and $G(z)$ are analytic in a region $\mathcal{R}$, then the real part of $\bar{z}F(z) + G(z)$ is biharmonic in $\mathcal{R}$.

92. Show that biharmonic functions (see Problem 91) do not, in general, remain biharmonic under a conformal transformation.

93. (a) Show that $\Omega(z) = K\ln\sinh(\pi z/a)$, $k > 0$, $a > 0$ represents the complex potential due to a row of fluid sources at $z = 0, \pm ai, \pm 2ai, \ldots$.

(b) Show that, apart from additive constants, the potential and stream functions are given by

$$\Phi = K\ln\{\cosh(2\pi x/a) - \cos(2\pi y/a)\}, \qquad \Psi = K\tan^{-1}\left\{\frac{\tan(\pi y/a)}{\tanh(\pi x/a)}\right\}$$

(c) Graph some of the streamlines for the flow.

94. Prove that the complex potential of Problem 93 is the same as that due to a source located halfway between the parallel lines $y = \pm 3a/2$.

95. Verify the statement made at the end of Problem 30 [compare Problem 137, Chapter 2, Page 62].

96. A condenser is formed from an elliptic cylinder, with major and minor axes of lengths $2a$ and $2b$ respectively, together with a flat plate AB of length $2h$ [see Fig. 9-50 below]. Show that the capacitance is equal to $\dfrac{2\pi}{\cosh^{-1}(a/h)}$.

97. A fluid flows with uniform velocity V_0 through a semi-infinite channel of width D and emerges through the opening AB [Fig. 9-51 below]. (a) Find the complex potential for the flow. (b) Determine the streamlines and equipotential lines and obtain graphs of some of these.
[*Hint.* Use entry C-5 in the table on Page 211.]

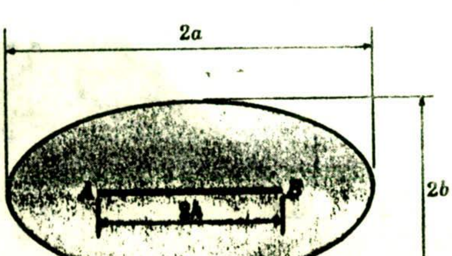


Fig. 9-50

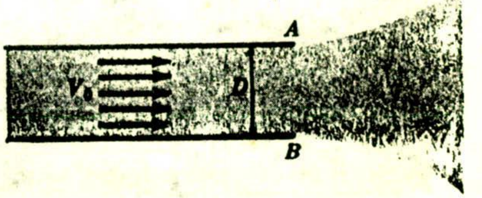


Fig. 9-51

98. Give a potential theory interpretation to Problem 30.

99. (*a*) Show that in a vacuum the capacitance of the parallel cylindrical conductors in Fig. 9-52 is

$$\frac{1}{2\cosh^{-1}\left(\dfrac{D^2 - R_1^2 - R_2^2}{2R_1R_2}\right)}$$

(*b*) Examine the case $R_1 = R_2 = R$ and compare with Problem 28.

100. Show that in a vacuum the capacitance of the two parallel cylindrical conductors in Fig. 9-53 is

$$\frac{1}{2\cosh^{-1}\left(\dfrac{R_1^2 + R_2^2 - D^2}{2R_1R_2}\right)}$$

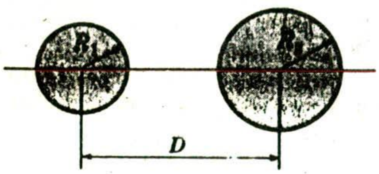


Fig. 9-52

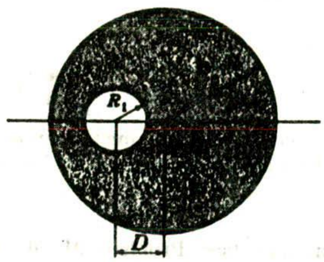


Fig. 9-53

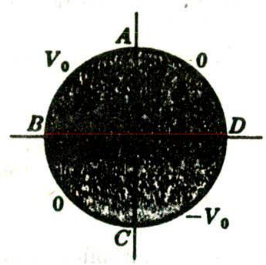


Fig. 9-54

101. Find the potential at any point of the unit cylinder of Fig. 9-54 if AB, BC, CD and DA are kept at potentials $V_0, 0, -V_0$ and 0 respectively.

Ans. $\dfrac{V_0}{\pi}\left(\tan^{-1}\dfrac{2r\sin\theta}{1-r^2} + \tan^{-1}\dfrac{2r\cos\theta}{1-r^2}\right)$

102. The shaded region of Fig. 9-55 represents an infinite conducting half plane in which lines AD, DE and DB are maintained at temperatures $0, T$ and $2T$ respectively, where T is a constant. (*a*) Find the temperature everywhere. (*b*) Give an interpretation involving potential theory.

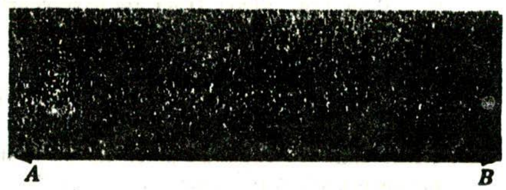


Fig. 9-55

103. Work the preceding problem if (*a*) DE is insulated, (*b*) AB is insulated.

104. In Fig. 9-55 suppose that DE represents an obstacle perpendicular to the base of an infinite channel in which a fluid is flowing from left to right so that far from the obstacle the speed of the fluid is V_0. Find (*a*) the speed and (*b*) the pressure at any point of the fluid.

105. Find the steady-state temperature at the point (3, 2) in the shaded region of Fig. 9-56.

106. An infinite wedge shaped region $ABCD$ of angle $\pi/4$ [shaded in Fig. 9-57] has one of its sides (CD) maintained at 50° C; the other side ABC has the part AB at temperature 25° C while part BC, of unit length, is insulated. Find the steady-state temperature at any point.

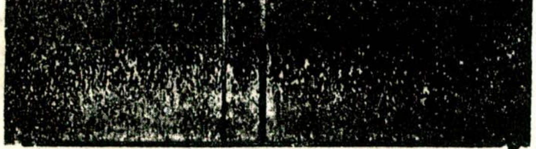


Fig. 9-56

Fig. 9-57

Chapter 10

Special Topics

ANALYTIC CONTINUATION

Let $F_1(z)$ be a function of z which is analytic in a region $\mathcal{R}_1$ [Fig. 10-1]. Suppose that we can find a function $F_2(z)$ which is analytic in a region $\mathcal{R}_2$ and which is such that $F_1(z) = F_2(z)$ in the region common to $\mathcal{R}_1$ and $\mathcal{R}_2$. Then we say that $F_2(z)$ is an *analytic continuation* of $F_1(z)$. This means that there is a function $F(z)$ analytic in the combined regions $\mathcal{R}_1$ and $\mathcal{R}_2$ such that $F(z) = F_1(z)$ in $\mathcal{R}_1$ and $F(z) = F_2(z)$ in $\mathcal{R}_2$. Actually it suffices for $\mathcal{R}_1$ and $\mathcal{R}_2$ to have only a small arc in common, such as LMN in Fig. 10-2.

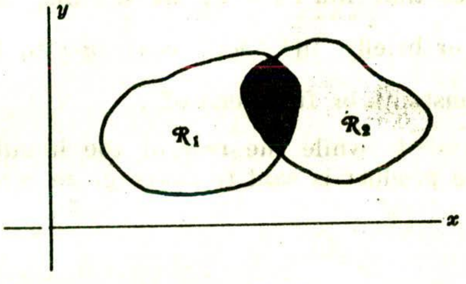


Fig. 10-1

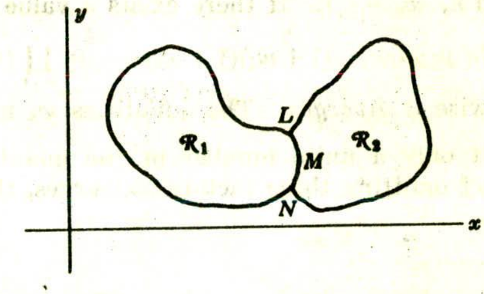


Fig. 10-2

By analytic continuation to regions $\mathcal{R}_3$, $\mathcal{R}_4$, etc., we can extend the original region of definition to other parts of the complex plane. The functions $F_1(z)$, $F_2(z)$, $F_3(z)$, ..., defined in $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \ldots$ respectively, are sometimes called *function elements* or briefly *elements*. It is sometimes impossible to extend a function analytically beyond the boundary of a region. We then call the boundary a *natural boundary*.

If a function $F_1(z)$ defined in $\mathcal{R}_1$ is continued analytically to region $\mathcal{R}_n$ along two different paths [Fig. 10-3], then the two analytic continuations will be identical if there is no singularity between the paths. This is the *uniqueness theorem for analytic continuation*.

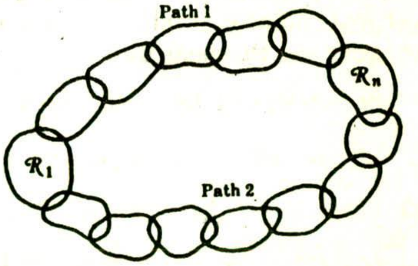


Fig. 10-3

If we do get different results, we can show that there is a singularity (specifically a *branch point*) between the paths. It is in this manner that we arrive at the various branches of multiple-valued functions. In this connection the concept of Riemann surfaces [Chapter 2] proves valuable.

We have already seen how functions represented by power series may be continued analytically (Chapter 6). In this chapter we consider how functions with other representations (such as integrals) may be continued analytically.

SCHWARZ'S REFLECTION PRINCIPLE

Suppose that $F_1(z)$ is analytic in the region $\mathcal{R}_1$ [Fig. 10-4] and that $F_1(z)$ assumes real values on the part LMN of the real axis.

Then *Schwarz's reflection principle* states that the analytic continuation of $F_1(z)$ into region $\mathcal{R}_2$ (considered as a mirror image or reflection of $\mathcal{R}_1$ with LMN as the mirror) is given by

$$F_2(z) = \overline{F_1(\bar{z})} \qquad (1)$$

The result can be extended to cases where LMN is a curve instead of a straight line segment.

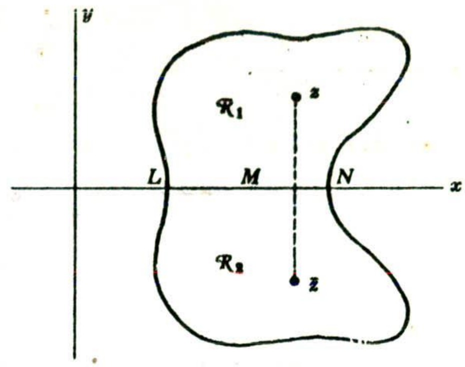


Fig. 10-4

INFINITE PRODUCTS

Let $P_n = (1+w_1)(1+w_2)\cdots(1+w_n)$ be denoted by $\prod_{k=1}^{n}(1+w_k)$ where we suppose that for all k, $w_k \neq -1$. If there exists a value $P \neq 0$ such that $\lim_{n\to\infty} P_n = P$, we say that the *infinite product* $(1+w_1)(1+w_2)\cdots \equiv \prod_{k=1}^{\infty}(1+w_k)$, or briefly $\Pi(1+w_k)$, *converges* to P; otherwise it *diverges*. The quantities w_k may be constants or functions of z.

If only a finite number of the quantities $w_k = -1$ while the rest of the infinite product omitting these factors converges, the infinite product is said to *converge to zero.*

ABSOLUTE, CONDITIONAL AND UNIFORM CONVERGENCE OF INFINITE PRODUCTS

If the infinite product $\Pi(1+|w_k|)$ converges, we say that $\Pi(1+w_k)$ is *absolutely convergent.*

If $\Pi(1+w_k)$ converges but $\Pi(1+|w_k|)$ diverges, we say that $\Pi(1+w_k)$ is *conditionally convergent.*

An important theorem, analogous to one for infinite series, states that an absolutely convergent infinite product is convergent, i.e. if $\Pi(1+|w_k|)$ converges then $\Pi(1+w_k)$ converges (see Problem 65).

The concept of *uniform convergence* of infinite products is easily defined by analogy with infinite series or sequences in general. Thus if $\prod_{k=1}^{n}\{1+w_k(z)\} = P_n(z)$ and $\prod_{k=1}^{\infty}\{1+w_k(z)\} = P(z)$, we say that $P_n(z)$ *converges uniformly* to $P(z)$ in a region $\mathcal{R}$ if, given any $\epsilon > 0$, we can find a number N, depending only on ϵ and not on the particular value of z in $\mathcal{R}$, such that $|P_n(z) - P(z)| < \epsilon$ for all $n > N$.

As in the case of infinite series, certain things can be done with absolutely or uniformly convergent infinite products that cannot necessarily be done for infinite products in general. Thus, for example, we can rearrange factors in an absolutely convergent infinite product without changing the value.

SOME IMPORTANT THEOREMS ON INFINITE PRODUCTS

1. A necessary condition that $\Pi(1+w_k)$ converge is that $\lim_{n\to\infty} w_n = 0$. However, the condition is not sufficient, i.e. even if $\lim_{n\to\infty} w_n = 0$ the infinite product may diverge.
2. If $\Sigma |w_k|$ converges [i.e. if Σw_k converges absolutely], then $\Pi(1+|w_k|)$, and thus $\Pi(1+w_k)$, converges [i.e. $\Pi(1+w_k)$ converges absolutely]. The converse theorem also holds.
3. If an infinite product is absolutely convergent, its factors can be altered without affecting the value of the product.
4. If in a region $\mathcal{R}$, $|w_k(z)| < M_k$, $k = 1,2,3,\ldots$, where M_k are constants such that ΣM_k converges, then $\Pi\{1+w_k(z)\}$ is uniformly (and absolutely) convergent. This is the analogue of the Weierstrass M test for series.
5. If $w_k(z)$, $k=1,2,3,\ldots$, are analytic in a region $\mathcal{R}$ and $\Sigma w_k(z)$ is uniformly convergent in $\mathcal{R}$, then $\Pi\{1+w_k(z)\}$ converges to an analytic function in $\mathcal{R}$.

WEIERSTRASS' THEOREM FOR INFINITE PRODUCTS

Let $f(z)$ be analytic for all z [i.e. $f(z)$ is an *entire function*] and suppose that it has simple zeros at $a_1, a_2, a_3, \ldots$ where $0 < |a_1| < |a_2| < |a_3| < \cdots$ and $\lim_{n\to\infty} |a_n| = \infty$. Then $f(z)$ can be expressed as an infinite product of the form

$$f(z) \quad = \quad f(0)\, e^{f'(0)z/f(0)} \prod_{k=1}^{\infty} \left\{ \left(1 - \frac{z}{a_k}\right) e^{z/a_k} \right\} \tag{2}$$

A generalization of this states that if $f(z)$ has zeros at $a_k \neq 0$, $k=1,2,3,\ldots$, of respective multiplicities or orders μ_k, and if for some integer N, $\sum_{k=1}^{\infty} 1/a_k^N$ is absolutely convergent, then

$$f(z) \quad = \quad f(0)\, e^{G(z)} \prod_{k=1}^{\infty} \left\{ \left(1 - \frac{z}{a_k}\right) e^{\frac{z}{a_k} + \frac{1}{2}\frac{z^2}{a_k^2} + \cdots + \frac{1}{N-1}\frac{z^{N-1}}{a_k^{N-1}}} \right\}^{\mu_k} \tag{3}$$

where $G(z)$ is an entire function. The result is also true if some of the a_k's are poles, in which case their multiplicities are negative.

The results (*2*) and (*3*) are sometimes called *Weierstrass' factor theorems.*

SOME SPECIAL INFINITE PRODUCTS

1. $\sin z \quad = \quad z\left\{1 - \frac{z^2}{\pi^2}\right\}\left\{1 - \frac{z^2}{(2\pi)^2}\right\}\cdots \quad = \quad z\prod_{k=1}^{\infty}\left(1 - \frac{z^2}{k^2\pi^2}\right)$

2. $\cos z \quad = \quad \left\{1 - \frac{z^2}{(\pi/2)^2}\right\}\left\{1 - \frac{z^2}{(3\pi/2)^2}\right\}\cdots \quad = \quad \prod_{k=1}^{\infty}\left(1 - \frac{4z^2}{(2k-1)^2\pi^2}\right)$

3. $\sinh z \quad = \quad z\left\{1 + \frac{z^2}{\pi^2}\right\}\left\{1 + \frac{z^2}{(2\pi)^2}\right\}\cdots \quad = \quad \prod_{k=1}^{\infty}\left(1 + \frac{z^2}{k^2\pi^2}\right)$

4. $\cosh z \quad = \quad \left\{1 + \frac{z^2}{(\pi/2)^2}\right\}\left\{1 + \frac{z^2}{(3\pi/2)^2}\right\}\cdots \quad = \quad \prod_{k=1}^{\infty}\left(1 + \frac{4z^2}{(2k-1)^2\pi^2}\right)$

THE GAMMA FUNCTION

For $\text{Re}\{z\} > 0$, we define the gamma function by

$$\Gamma(z) \quad = \quad \int_0^{\infty} t^{z-1} e^{-t}\, dt \tag{4}$$

Then (see Problem 11) we have the *recursion formula*

$$\Gamma(z+1) \;=\; z\,\Gamma(z) \qquad \text{where } \Gamma(1) = 1 \tag{5}$$

If z is a positive integer n, we see from (5) that

$$\Gamma(n+1) \;=\; n(n-1)\cdots(1) \;=\; n! \tag{6}$$

so that the gamma function is a generalization of the factorial. For this reason the gamma function is also called the *factorial function* and is written as $z!$ rather than $\Gamma(z+1)$, in which case we define $0! = 1$.

From (5) we also see that if z is real and positive, then $\Gamma(z)$ can be determined by knowing the values of $\Gamma(z)$ for $0 < z < 1$. If $z = \frac{1}{2}$, we have [Problem 14]

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi} \tag{7}$$

For $\text{Re}\{z\} \leqq 0$, the definition (4) breaks down since the integral diverges. By analytic continuation, however, we can define $\Gamma(z)$ in the left-hand plane. Essentially this amounts to use of (5) [see Problem 15]. At $z = 0, -1, -2, \ldots,$ $\Gamma(z)$ has simple poles [see Problem 16].

PROPERTIES OF THE GAMMA FUNCTION

The following list shows some important properties of the gamma function. The first two can be taken as definitions from which all other properties can be deduced.

1. $$\Gamma(z+1) \;=\; \lim_{k\to\infty} \frac{1\cdot 2\cdot 3\cdots k}{(z+1)(z+2)\cdots(z+k)}\,k^z \;=\; \lim_{k\to\infty} \Pi(z,k)$$

   where $\Pi(z,k)$ is sometimes called *Gauss' Π function*.

2. $$\frac{1}{\Gamma(z)} \;=\; z e^{\gamma z} \prod_{k=1}^{\infty} \left\{1 + \frac{z}{k}\right\} e^{-z/k}$$

   where $\gamma = \lim_{p\to\infty}\left\{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p} - \ln p\right\} = .5772157\ldots$ is called *Euler's constant*.

3. $$\Gamma(z)\,\Gamma(1-z) \;=\; \frac{\pi}{\sin \pi z}$$

   In particular if $z = \frac{1}{2}$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

4. $$2^{2z-1}\,\Gamma(z)\,\Gamma(z+\tfrac{1}{2}) \;=\; \sqrt{\pi}\,\Gamma(2z)$$

   This is sometimes called the *duplication formula* for the gamma function.

5. If $m = 1, 2, 3, \ldots,$

   $$\Gamma(z)\,\Gamma\left(z+\frac{1}{m}\right)\Gamma\left(z+\frac{2}{m}\right)\cdots\Gamma\left(z+\frac{m-1}{m}\right) \;=\; m^{\frac{1}{2}-mz}(2\pi)^{(m-1)/2}\,\Gamma(mz)$$

   Property 4 is a special case of this with $m = 2$.

6. $$\frac{\Gamma'(z)}{\Gamma(z)} \;=\; -\gamma + \left(\frac{1}{1} - \frac{1}{z}\right) + \left(\frac{1}{2} - \frac{1}{z+1}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{z+n-1}\right) + \cdots$$

7. $$\Gamma'(1) \;=\; \int_0^\infty e^{-t} \ln t\, dt \;=\; -\gamma$$

8. $$\Gamma(z) \;=\; \frac{1}{e^{2\pi i z} - 1} \oint_C t^{z-1} e^{-t}\, dt$$

   where C is the contour in Fig. 10-5. This is an analytic continuation to the left-hand half plane of the gamma function defined in (4).

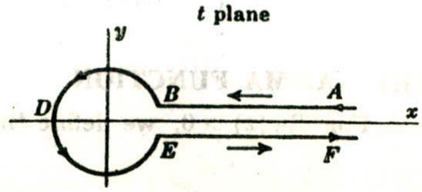


Fig. 10-5

9. Another contour integral using contour C [Fig. 10-5] is given by

$$\Gamma(z) = \frac{i}{2\sin \pi z}\oint_C (-t)^{z-1} e^{-t}\, dt = -\frac{1}{2\pi i}\oint_C (-t)^{-z} e^{-t}\, dt$$

THE BETA FUNCTION

For $\operatorname{Re}\{m\} > 0$, $\operatorname{Re}\{n\} > 0$, we define the *beta function* by

$$B(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\, dt \tag{8}$$

As seen in Problem 18, this is related to the gamma function according to

$$B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \tag{9}$$

Various integrals can be expressed in terms of the beta function and thus in terms of the gamma function. Two interesting results are

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \tfrac{1}{2}B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{2\,\Gamma(m+n)} \tag{10}$$

$$\int_0^\infty \frac{t^{p-1}}{1+t}\, dt = B(p, 1-p) = \Gamma(p)\,\Gamma(1-p) = \frac{\pi}{\sin p\pi} \tag{11}$$

the first holding for $\operatorname{Re}\{m\} > 0$ and $\operatorname{Re}\{n\} > 0$, and the second holding for $0 < \operatorname{Re}\{p\} < 1$.

For $\operatorname{Re}\{m\} \leqq 0$ and $\operatorname{Re}\{n\} \leqq 0$, the definition *(8)* can be extended by use of analytic continuation.

DIFFERENTIAL EQUATIONS

Suppose we are given the *linear differential equation*

$$Y'' + p(z)\,Y' + q(z)\,Y = 0 \tag{12}$$

If $p(z)$ and $q(z)$ are analytic at a point a, then a is called an *ordinary point* of the differential equation. Points at which $p(z)$ or $q(z)$ or both are not analytic are called *singular points* of the differential equation.

Example 1: For $Y'' + zY' + (z^2-4)Y = 0$, every point is an ordinary point.

Example 2: For $(1-z^2)Y'' - 2zY' + 6Y = 0$ or $Y'' - \frac{2z}{1-z^2}Y' + \frac{6}{1-z^2}Y = 0$, $z = \pm 1$ are singular points; all other points are ordinary points.

If $z = a$ is a singular point but $(z-a)\,p(z)$ and $(z-a)^2 q(z)$ are analytic at $z = a$, then $z = a$ is called a *regular singular point*. If $z = a$ is neither an ordinary point or a regular singular point, it is called an *irregular singular point*.

Example 3: In Example 2, $z = 1$ is a regular singular point since $(z-1)\left(-\frac{2z}{1-z^2}\right) = \frac{2z}{z+1}$ and $(z-1)^2\left(\frac{6}{1-z^2}\right) = \frac{6-6z}{z+1}$ are analytic at $z = 1$. Similarly, $z = -1$ is a regular singular point.

Example 4: $z^3Y'' + (1-z)Y' - 2Y = 0$ has $z = 0$ as a singular point. Also, $z\left(\frac{1-z}{z^3}\right) = \frac{1-z}{z^2}$ and $z^2\left(-\frac{2}{z^3}\right) = -\frac{2}{z}$ are not analytic at $z = 0$, so that $z = 0$ is an irregular singular point.

If $Y_1(z)$ and $Y_2(z)$ are two solutions of *(12)* which are not constant multiples of each other, we call the solutions *linearly independent*. In such case, if A and B are any constants the general solution of *(12)* is

$$Y = AY_1 + BY_2 \tag{13}$$

The following theorems are fundamental.

Theorem 1. If $z=a$ is an ordinary point of (*12*), then there exist two linearly independent solutions of (*12*) having the form

$$\sum_{k=0}^{\infty} a_k(z-a)^k \tag{14}$$

where the constants a_k are determined by substitution in (*12*). In doing this it may be necessary to expand $p(z)$ and $q(z)$ in powers of $(z-a)$. In practice it is desirable to replace $(z-a)$ by a new variable.

The solutions (*14*) converge in a circle with centre at a which extends up to the nearest singularity of the differential equation.

Example 5: The equation $(1-z^2)Y'' - 2zY' + 6Y = 0$ [see Example 2] has a solution of the form $\Sigma a_k z^k$ which converges inside the circle $|z| = 1$.

Theorem 2. If $z=a$ is a regular singular point, then there exists at least one solution having the form

$$(z-a)^c \sum_{k=0}^{\infty} a_k(z-a)^k \tag{15}$$

where c is a constant. By substituting into (*12*) and equating the lowest power of $(z-a)$ to zero, a quadratic equation for c (called the *indicial equation*) is obtained. If we call the solutions of this quadratic equation c_1 and c_2, the following situations arise.

1. $c_1 - c_2 \neq$ *an integer*. In this case there are two linearly independent solutions having the form (*15*).
2. $c_1 = c_2$. Here one solution has the form (*15*) while the other linearly independent solution has the form

$$\ln(z-a) \sum_{k=0}^{\infty} b_k(z-a)^{k+c} \tag{16}$$

3. $c_1 - c_2 =$ *an integer* $\neq 0$. In this case there is either one solution of the form (*15*) or two linearly independent solutions having this form. If only one solution of the form (*15*) can be found, the other linearly independent solution has the form (*16*).

All solutions obtained converge in a circle with centre at a which extends up to the nearest singularity of the differential equation.

SOLUTION OF DIFFERENTIAL EQUATIONS BY CONTOUR INTEGRALS

It is often desirable to seek a solution of a linear differential equation in the form

$$Y(z) = \oint_C K(z,t)\, G(t)\, dt \tag{17}$$

where $K(z,t)$ is called the *kernel*. One useful possibility occurs if $K(z,t) = e^{zt}$, in which case

$$Y(z) = \oint_C e^{zt}\, G(t)\, dt \tag{18}$$

Such solutions may occur where the coefficients in the differential equation are rational functions (see Problems 25 and 26).

BESSEL FUNCTIONS

Bessel's differential equation of order n is given by

$$z^2Y'' + zY' + (z^2-n^2)Y = 0 \tag{19}$$

A solution of this equation if $n \geqq 0$ is

$$J_n(z) = \frac{z^n}{2^n \Gamma(n+1)}\left\{1 - \frac{z^2}{2(2n+2)} + \frac{z^4}{2\cdot 4(2n+2)(2n+4)} - \cdots\right\} \tag{20}$$

and is called *Bessel's function of the first kind of order n.*

If n is not an integer, the general solution of (18) is

$$Y = A\,J_n(z) + B\,J_{-n}(z) \tag{21}$$

where A and B are arbitrary constants. However, if n is an integer then $J_{-n}(z) = (-1)^n J_n(z)$ and (20) fails to yield the general solution. The general solution in this case can be found as in Problems 182 and 183.

Bessel functions have many interesting and important properties, among them being the following.

1. $$e^{z(t-1/t)/2} = \sum_{n=-\infty}^{\infty} J_n(z)\,t^n$$

   The left side is often called the *generating function* for the Bessel functions of the first kind for integer values of n.

2. $$z\,J_{n-1}(z) - 2n\,J_n(z) + z\,J_{n+1}(z) = 0$$

   This is called the *recursion formula* for *Bessel functions* [see Problem 27].

3. $$\frac{d}{dz}\{z^n J_n(z)\} = z^n J_{n-1}(z), \qquad \frac{d}{dz}\{z^{-n} J_n(z)\} = -z^{-n} J_{n+1}(z)$$

4. $$J_n(z) = \frac{1}{\pi}\int_0^\pi \cos(n\phi - z\sin\phi)\,d\phi, \qquad n = \text{integer}$$

5. $$J_n(z) = \frac{1}{\pi}\int_0^\pi \cos(n\phi - z\sin\phi)\,d\phi - \frac{\sin n\pi}{\pi}\int_0^\infty e^{-n\phi - z\sinh\phi}\,d\phi$$

6. $$\int_0^z t\,J_n(at)\,J_n(bt)\,dt = \frac{z\{a\,J_n(bz)\,J_n'(az) - b\,J_n(az)\,J_n'(bz)\}}{b^2 - a^2}, \qquad a \neq b$$

7. $$\int_0^z t\,J_n(at)\,J_n(bt)\,dt = \frac{az\,J_n(bz)\,J_{n-1}(az) - bz\,J_n(az)\,J_{n-1}(bz)}{b^2 - a^2}, \qquad a \neq b$$

8. $$\int_0^z t\{J_n(at)\}^2\,dt = \frac{z^2}{2}\left[\{J_n(az)\}^2 - J_{n-1}(az)\,J_{n+1}(az)\right]$$

9. $$J_n(z) = \frac{1}{2\pi i}\oint_C t^{-n-1}\,e^{\frac{1}{2}z(t-1/t)}\,dt, \qquad n = 0, \pm 1, \pm 2, \ldots$$

   where C is any simple closed curve enclosing $t = 0$.

10. $$J_n(z) = \frac{z^n}{1\cdot 3\cdot 5\cdots(2n-1)\pi}\int_{-1}^1 e^{izt}\,(1-t^2)^{n-1/2}\,dt = \frac{z^n}{1\cdot 3\cdot 5\cdots(2n-1)\pi}\int_0^\pi \cos(z\cos\phi)\,\sin^{2n}\phi\,d\phi$$

A second solution to Bessel's differential equation if n is a positive integer, is called *Bessel's function of the second kind of order n* or *Neumann's function* and is given by

$$Y_n(z) = J_n(z)\ln z - \frac{1}{2}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k-n} - \frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)(n+k)!}\left(\frac{z}{2}\right)^{2k+n}\{G(k) + G(n+k)\} \tag{22}$$

where $G(k) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$ and $G(0) = 0$.

If $n = 0$, we have

$$Y_0(z) = J_0(z)\ln z + \frac{z^2}{2^2} - \frac{z^4}{2^2 4^2}(1+\tfrac{1}{2}) + \frac{z^6}{2^2 4^2 6^2}(1+\tfrac{1}{2}+\tfrac{1}{3}) - \cdots \tag{23}$$

In terms of these the general solution of (19) if n is a positive integer can be written

$$Y = A\,J_n(z) + B\,Y_n(z) \tag{24}$$

LEGENDRE FUNCTIONS

Legendre's differential equation of order n is given by

$$(1-z^2)Y'' - 2zY' + n(n+1)Y = 0 \tag{25}$$

The general solution of this equation is

$$\begin{aligned} Y = {} & A\left\{1 - \frac{n(n+1)}{2!}z^2 + \frac{n(n-2)(n+1)(n+3)}{4!}z^4 - \cdots\right\} \\ & + B\left\{z - \frac{(n-1)(n+2)}{3!}z^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}z^5 - \cdots\right\} \end{aligned} \tag{26}$$

If n is not an integer, these series solutions converge for $|z| < 1$. If n is zero or a positive integer, polynomial solutions of degree n are obtained. We call these polynomial solutions *Legendre polynomials* and denote them by $P_n(z)$, $n = 0,1,2,3,\ldots$. By choosing these so that $P_n(1) = 1$, we find that they can be expressed by *Rodrigues' formula*

$$P_n(z) = \frac{1}{2^n n!}\frac{d^n}{dz^n}(z^2-1)^n \tag{27}$$

from which $P_0(z) = 1$, $P_1(z) = z$, $P_2(z) = \frac{1}{2}(3z^2-1)$, $P_3(z) = \frac{1}{2}(5z^3-3z)$, etc.

The following are some properties of Legendre polynomials.

1. $$\frac{1}{\sqrt{1-2zt+t^2}} = \sum_{n=0}^{\infty} P_n(z)\,t^n$$

   This is called the *generating function* for Legendre polynomials.

2. $$P_n(z) = \frac{(2n)!}{2^n(n!)^2}\left\{z^n - \frac{n(n-1)}{2(2n-1)}z^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}z^{n-4} - \cdots\right\}$$

3. $$P_n(z) = \frac{1}{2\pi i}\oint_C \frac{(t^2-1)^n}{2^n(t-z)^{n+1}}\,dt$$

   where C is any simple closed curve enclosing the pole $t = z$.

4. $$\int_{-1}^{1} P_m(z)\,P_n(z)\,dz = \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{2}{2n+1} & \text{if } m = n \end{cases}$$

   [See Problems 30 and 31.]

5. $$P_n(z) = \frac{1}{\pi}\int_0^{\pi} [z + \sqrt{z^2-1}\cos\phi]^n\,d\phi$$

   [See Problem 34, Chapter 6.]

6. $$(n+1)P_{n+1}(z) - (2n+1)z\,P_n(z) + n\,P_{n-1}(z) = 0$$

   This is called the *recursion formula for Legendre polynomials* [see Prob. 32].

7. $$(2n+1)P_n(z) = P'_{n+1}(z) - P'_{n-1}(z)$$

If n is a positive integer or zero, the general solution of Legendre's equation can be written as

$$Y = AP_n(z) + BQ_n(z) \tag{28}$$

where $Q_n(z)$ is an infinite series convergent for $|z|<1$ obtained from (26). If n is not a positive integer, there are two infinite series solutions obtained from (26) which are convergent for $|z|<1$. These solutions to Legendre's equation are called *Legendre functions.* They have properties analogous to those of the Legendre polynomials.

THE HYPERGEOMETRIC FUNCTION

The function defined by

$$F(a,b;c;z) = 1 + \frac{a\cdot b}{1\cdot c}z + \frac{a(a+1)b(b+1)}{1\cdot 2\cdot c(c+1)}z^2 + \cdots \tag{29}$$

is called the *hypergeometric function* and is a solution to *Gauss' differential equation* or the *hypergeometric equation*

$$z(1-z)Y'' + \{c-(a+b+1)z\}Y' - abY = 0 \tag{30}$$

The series (29) is absolutely convergent for $|z|<1$ and divergent for $|z|>1$. For $|z|=1$ it converges absolutely if $\text{Re}\{c-a-b\} > 0$.

If $|z|<1$ and $\text{Re}\{c\} > \text{Re}\{b\} > 0$, we have

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt \tag{31}$$

For $|z|>1$ the function can be defined by analytic continuation.

THE ZETA FUNCTION

The *zeta function,* studied extensively by Riemann in connection with the theory of numbers, is defined for $\text{Re}\{z\} > 1$ by

$$\zeta(z) = \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \cdots = \sum_{k=1}^{\infty}\frac{1}{k^z} \tag{32}$$

It can be extended by analytic continuation to other values of z. This extended definition of $\zeta(z)$ has the interesting property that

$$\zeta(1-z) = 2^{1-z}\pi^{-z}\,\Gamma(z)\cos(\pi z/2)\,\zeta(z) \tag{33}$$

Other interesting properties are as follows.

1. $$\zeta(z) = \frac{1}{\Gamma(z)}\int_0^\infty \frac{t^{z-1}}{e^t+1}\,dt \qquad \text{Re}\{z\} > 0$$

2. The only singularity of $\zeta(z)$ is a simple pole at $z=1$ having residue 1.

3. If B_k, $k = 1,2,3,\ldots$, is the coefficient of z^{2k} in the expansion

$$\tfrac{1}{2}z\cot(\tfrac{1}{2}z) = 1 - \sum_{k=1}^{\infty}\frac{B_k z^{2k}}{(2k)!}$$

then $$\zeta(2k) = \frac{2^{2k-1}\pi^{2k}B_k}{(2k)!} \qquad k = 1,2,3,\ldots$$

We have, for example, $B_1 = 1/6$, $B_2 = 1/30$, $\ldots$, from which $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, $\ldots$. The numbers B_k are called *Bernoulli numbers.* For another definition of the Bernoulli numbers see Problem 163, Page 171.

4. $$\frac{1}{\zeta(z)} = \left(1-\frac{1}{2^z}\right)\left(1-\frac{1}{3^z}\right)\left(1-\frac{1}{5^z}\right)\left(1-\frac{1}{7^z}\right)\cdots = \prod_p \left(1-\frac{1}{p^z}\right)$$

where the product is taken over all positive primes p.

Riemann conjectured that all zeros of $\zeta(z)$ are situated on the line $\text{Re}\{z\} = \frac{1}{2}$, but as yet this has neither been proved nor disproved. It has, however, been shown by Hardy that there are infinitely many zeros which do lie on this line.

ASYMPTOTIC SERIES

A series
$$a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots = \sum_{n=0}^{\infty} \frac{a_n}{z^n} \tag{34}$$
is called an *asymptotic series* for a function $F(z)$ if for any specified positive integer M,
$$\lim_{z\to\infty} z^M \left\{F(z) - \sum_{n=0}^{M} \frac{a_n}{z^n}\right\} = 0 \tag{35}$$
In such case we write
$$F(z) \sim \sum_{n=0}^{\infty} \frac{a_n}{z^n} \tag{36}$$

Asymptotic series, and formulae involving them, are very useful in evaluation of functions for large values of the variable, which might otherwise be difficult. In practice, an asymptotic series may diverge. However, by taking the sum of successive terms of the series, stopping just before the terms begin to increase, we may obtain a good approximation for $F(z)$.

Various operations with asymptotic series are permissible. For example, asymptotic series may be added, multiplied or integrated term by term to yield another asymptotic series. However, differentiation is not always possible. For a given range of values of z an asymptotic series, if it exists, is unique.

THE METHOD OF STEEPEST DESCENTS

Let $I(z)$ be expressible in the form
$$I(z) = \int_C e^{zF(t)}\,dt \tag{37}$$
where C is some path in the t plane. Since $F(t)$ is complex, we can consider z to be real.

The method of steepest descents is a method for finding an asymptotic formula for (37) valid for large z. Where applicable, it consists of the following steps.

1. Determine the points at which $F'(t) = 0$. Such points are called *saddle points*, and for this reason the method is also called the *saddle point method*.

   We shall assume that there is only one saddle point, say t_0. The method can be extended if there is more than one.

2. Assuming $F(t)$ analytic in a neighbourhood of t_0, obtain the Taylor series expansion
$$F(t) \doteq F(t_0) + \frac{F''(t_0)(t-t_0)^2}{2!} + \cdots = F(t_0) - u^2 \tag{38}$$
   Now deform contour C so that it passes through the saddle point t_0, and is such that $\text{Re}\{F(t)\}$ is largest at t_0 while $\text{Im}\{F(t)\}$ can be considered equal to the constant $\text{Im}\{F(t_0)\}$ in the neighbourhood of t_0. With these assumptions, the variable u defined by (38) is real and we obtain to a high degree of approximation
$$I(z) = e^{zF(t_0)} \int_{-\infty}^{\infty} e^{-zu^2}\left(\frac{dt}{du}\right) du \tag{39}$$

where from (38), we can find constants $b_0, b_1, \ldots$ such that

$$\frac{dt}{du} = b_0 + b_1 u + b_2 u^2 + \cdots \tag{40}$$

3. Substitute (40) into (39) and perform the integrations to obtain the required asymptotic expansion

$$I(z) \sim \sqrt{\frac{\pi}{z}}\, e^{zF(t_0)} \left\{ b_0 + \frac{1}{2}\frac{b_2}{z} + \frac{1\cdot 3}{2\cdot 2}\frac{b_4}{z^2} + \frac{1\cdot 3\cdot 5}{2\cdot 2\cdot 2}\frac{b_6}{z^3} + \cdots \right\} \tag{41}$$

For many practical purposes the first term provides enough accuracy and we find

$$I(z) \sim \sqrt{\frac{-2\pi}{zF''(t_0)}}\, e^{zF(t_0)} \tag{42}$$

Methods similar to the above are also known as *Laplace's method* and the *method of stationary phase.*

SPECIAL ASYMPTOTIC EXPANSIONS

1. The Gamma Function

$$\Gamma(z+1) \sim \sqrt{2\pi z}\, z^z e^{-z} \left\{ 1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51{,}840z^3} + \cdots \right\} \tag{43}$$

This is sometimes called *Stirling's asymptotic formula for the gamma function.* It holds for large values of $|z|$ such that $-\pi < \arg z < \pi$.

If n is real and large, we have

$$\Gamma(n+1) = \sqrt{2\pi n}\, n^n e^{-n} e^{\theta/12n} \qquad \text{where } 0 < \theta < 1 \tag{44}$$

In particular, if n is a large positive integer we have

$$n! \sim \sqrt{2\pi n}\, n^n e^{-n} \tag{45}$$

called *Stirling's asymptotic formula for $n!$.*

2. Bessel Functions

$$J_n(z) \sim \sqrt{\frac{2}{\pi z}} \{P(z)\cos(z - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi) + Q(z)\sin(z - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi)\} \tag{46}$$

where

$$\left.\begin{aligned} P(z) &= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k [4n^2-1^2][4n^2-3^2]\cdots[4n^2-(4k-1)^2]}{(2k)!\,2^{6k} z^{2k}} \\ Q(z) &= \sum_{k=1}^{\infty} \frac{(-1)^k [4n^2-1^2][4n^2-3^2]\cdots[4n^2-(4k-3)^2]}{(2k-1)!\,2^{6k-3} z^{2k-1}} \end{aligned}\right\} \tag{47}$$

This holds for large values of $|z|$ such that $-\pi < \arg z < \pi$.

3. The Error Function

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt \sim 1 + \frac{ze^{-z^2}}{\pi}\sum_{k=1}^{\infty}(-1)^k \frac{\Gamma(k-\frac{1}{2})}{z^{2k}} \tag{48}$$

This result holds for large values of $|z|$ such that $-\pi/2 < \arg z < \pi/2$. For $\pi/2 < \arg z < 3\pi/2$ the result holds if we replace z by $-z$ on the right.

4. The Exponential Integral

$$\operatorname{Ei}(z) = \int_z^{\infty} \frac{e^{-t}}{t}\,dt \sim e^{-z}\sum_{k=0}^{\infty}\frac{(-1)^k k!}{z^{k+1}} \tag{49}$$

This result holds for large values of $|z|$ such that $-\pi < \arg z < \pi$.

ELLIPTIC FUNCTIONS

The integral

$$z = \int_0^w \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \qquad |k| < 1 \tag{50}$$

is called an *elliptic integral of the first kind.* The integral exists if w is real and such that $|w| < 1$. By analytic continuation we can extend it to other values of w. If $t = \sin\theta$ and $w = \sin\phi$, the integral (50) assumes an equivalent form

$$z = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} \tag{51}$$

where we often write $\phi = \text{am}\, z$.

If $k = 0$, (50) becomes $z = \sin^{-1} w$ or, equivalently, $w = \sin z$. By analogy, we denote the integral in (50) when $k \neq 0$ by $\text{sn}^{-1}(w; k)$ or briefly $\text{sn}^{-1} w$ when k does not change during a given discussion. Thus

$$z = \text{sn}^{-1} w = \int_0^w \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \tag{52}$$

This leads to the function $w = \text{sn}\, z$ which is called an *elliptic function* or sometimes a *Jacobian elliptic function.*

By analogy with the trigonometric functions, it is convenient to define other elliptic functions

$$\text{cn}\, z = \sqrt{1 - \text{sn}^2 z}, \qquad \text{dn}\, z = \sqrt{1 - k^2 \text{sn}^2 z} \tag{53}$$

Another function which is sometimes used is $\text{tn}\, z = (\text{sn}\, z)/(\text{cn}\, z)$.

The following list shows various properties of these functions.

1. $\text{sn}(0) = 0$, $\text{cn}(0) = 1$, $\text{dn}(0) = 1$, $\text{sn}(-z) = -\text{sn}\, z$, $\text{cn}(-z) = \text{cn}\, z$, $\text{dn}(-z) = \text{dn}\, z$
2. $\dfrac{d}{dz}\text{sn}\, z = \text{cn}\, z\, \text{dn}\, z$, $\dfrac{d}{dz}\text{cn}\, z = -\text{sn}\, z\, \text{dn}\, z$, $\dfrac{d}{dz}\text{dn}\, z = -k^2\, \text{sn}\, z\, \text{cn}\, z$
3. $\text{sn}\, z = \sin(\text{am}\, z)$, $\text{cn}\, z = \cos(\text{am}\, z)$
4. $$\text{sn}(z_1 + z_2) = \frac{\text{sn}\, z_1\, \text{cn}\, z_2\, \text{dn}\, z_2 + \text{cn}\, z_1\, \text{dn}\, z_1\, \text{sn}\, z_2}{1 - k^2\, \text{sn}^2 z_1\, \text{sn}^2 z_2} \tag{54}$$

   $$\text{cn}(z_1 + z_2) = \frac{\text{cn}\, z_1\, \text{cn}\, z_2 - \text{sn}\, z_1\, \text{sn}\, z_2\, \text{dn}\, z_1\, \text{dn}\, z_2}{1 - k^2\, \text{sn}^2 z_1\, \text{sn}^2 z_2} \tag{55}$$

   $$\text{dn}(z_1 + z_2) = \frac{\text{dn}\, z_1\, \text{dn}\, z_2 - k^2\, \text{sn}\, z_1\, \text{sn}\, z_2\, \text{cn}\, z_1\, \text{cn}\, z_2}{1 - k^2\, \text{sn}^2 z_1\, \text{sn}^2 z_2} \tag{56}$$

   These are called *addition formulae* for the elliptic functions.

5. The elliptic functions have two periods, and for this reason they are often called *doubly-periodic functions.* Let us write

   $$K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} \tag{57}$$

   $$K' = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k'^2 \sin^2\theta}} \tag{58}$$

   where k and k', called the *modulus* and *complementary modulus* respectively, are such that $k' = \sqrt{1-k^2}$. Then the periods of $\text{sn}\, z$ are $4K$ and $2iK'$, the periods of $\text{cn}\, z$ are $4K$ and $2K + 2iK'$, and the periods of $\text{dn}\, z$ are $2K$ and $4iK'$. It follows that there exists a periodic set of parallelograms [often called *period parallelograms*] in the complex plane in which the values of an elliptic function repeat. The smallest of these is often referred to as a *unit cell* or briefly a *cell.*

The above ideas can be extended to other elliptic functions. Thus there exist *elliptic integrals of the second* and *third kinds* defined respectively by

$$z = \int_0^w \sqrt{\frac{1-k^2t^2}{1-t^2}}\,dt = \int_0^\phi \sqrt{1-k^2\sin^2\theta}\,d\theta \tag{59}$$

$$z = \int_0^w \frac{dt}{(1+nt^2)\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^\phi \frac{d\theta}{(1+n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} \tag{60}$$

Solved Problems

ANALYTIC CONTINUATION

1. Let $F(z)$ be analytic in a region $\mathcal{R}$ and suppose that $F(z) = 0$ at all points on an arc PQ inside $\mathcal{R}$ [Fig. 10-6]. Prove that $F(z) = 0$ throughout $\mathcal{R}$.

Choose any point, say z_0, on arc PQ. Then in some circle of convergence C with centre at z_0 [this circle extending at least to the boundary of $\mathcal{R}$ where a singularity may exist], $F(z)$ has a Taylor series expansion

$$F(z) = F(z_0) + F'(z_0)(z-z_0) + \tfrac{1}{2}F''(z_0)(z-z_0)^2 + \cdots$$

But by hypothesis $F(z_0) = F'(z_0) = F''(z_0) = \cdots = 0$. Hence $F(z) = 0$ inside C.

By choosing another arc inside C, we can continue the process. In this manner we can show that $F(z) = 0$ throughout $\mathcal{R}$.

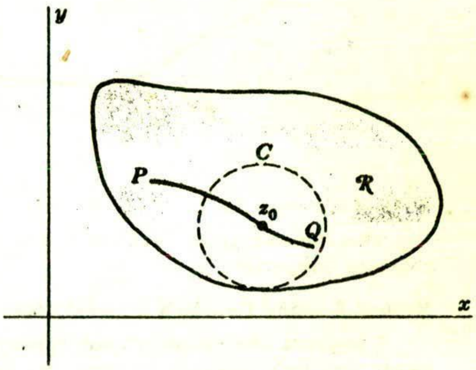


Fig. 10-6

2. Given that the identity $\sin^2 z + \cos^2 z = 1$ holds for real values of z, prove that it also holds for all complex values of z.

Let $F(z) = \sin^2 z + \cos^2 z - 1$ and let $\mathcal{R}$ be a region of the z plane containing a portion of the x axis [Fig. 10-7].

Since $\sin z$ and $\cos z$ are analytic in $\mathcal{R}$, it follows that $F(z)$ is analytic in $\mathcal{R}$. Also $F(z) = 0$ on the x axis. Hence by Problem 1, $F(z) = 0$ identically in $\mathcal{R}$, which shows that $\sin^2 z + \cos^2 z = 1$ for all z in $\mathcal{R}$. Since $\mathcal{R}$ is arbitrary, we obtain the required result.

This method is useful in proving for complex values many of the results true for real values.

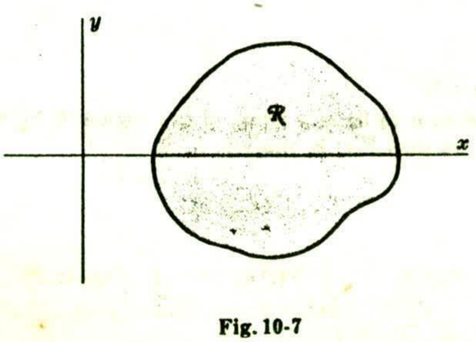


Fig. 10-7

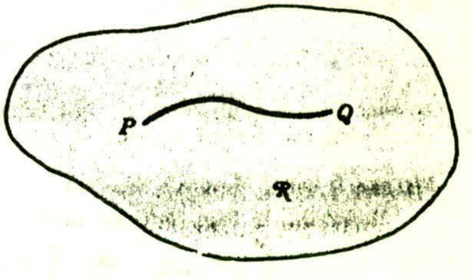


Fig. 10-8

3. Let $F_1(z)$ and $F_2(z)$ be analytic in a region $\mathcal{R}$ [Fig. 10-8] and suppose that on an arc PQ in $\mathcal{R}$, $F_1(z) = F_2(z)$. Prove that $F_1(z) = F_2(z)$ in $\mathcal{R}$.

This follows from Problem 1 by choosing $F(z) = F_1(z) - F_2(z)$.

4. Let $F_1(z)$ be analytic in region $\mathcal{R}_1$ [Fig. 10-9] and on the boundary $JKLM$. Suppose that we can find a function $F_2(z)$ analytic in region $\mathcal{R}_2$ and on the boundary $JKLM$ such that $F_1(z) = F_2(z)$ on $JKLM$. Prove that the function

$$F(z) = \begin{cases} F_1(z) \text{ for } z \text{ in } \mathcal{R}_1 \\ F_2(z) \text{ for } z \text{ in } \mathcal{R}_2 \end{cases}$$

is analytic in the region $\mathcal{R}$ which is composed of $\mathcal{R}_1$ and $\mathcal{R}_2$ [sometimes written $\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2$].

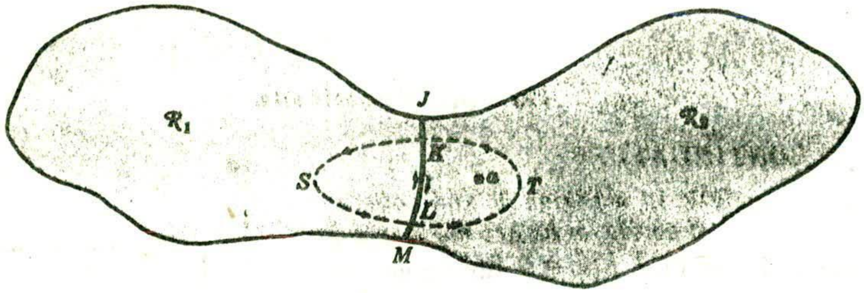


Fig. 10-9

Method 1.

This follows from Problem 3, since there can be only one function $F_2(z)$ in $\mathcal{R}_2$ satisfying the required properties.

Method 2, using Cauchy's integral formulae.

Construct the simple closed curve $SLTKS$ (dashed in Fig. 10-9) and let a be any point inside. From Cauchy's integral formula, we have (since $F_2(z)$ is analytic inside and on $LTKL$ and since $F_2(z) = F(z)$ on LTK)

$$F_2(a) = \frac{1}{2\pi i}\oint_{LTKL} \frac{F_2(z)}{z-a}\,dz = \frac{1}{2\pi i}\int_{LTK} \frac{F(z)}{z-a}\,dz + \frac{1}{2\pi i}\int_{KL} \frac{F(z)}{z-a}\,dz$$

Also we have by Cauchy's theorem (since $F_1(z)/(z-a)$ is analytic inside and on $KSLK$ and since $F_1(z) = F(z)$ on KSL)

$$0 = \frac{1}{2\pi i}\oint_{KSLK} \frac{F_1(z)}{z-a}\,dz = \frac{1}{2\pi i}\int_{KSL} \frac{F(z)}{z-a}\,dz + \frac{1}{2\pi i}\int_{LK} \frac{F(z)}{z-a}\,dz$$

Adding, using the fact that $F(z) = F_1(z) = F_2(z)$ on LK so that the integrals along KL and LK cancel, we have since $F(a) = F_2(a)$

$$F(a) = \frac{1}{2\pi i}\oint_{LTKSL} \frac{F(z)}{z-a}\,dz$$

In a similar manner we find

$$F^{(n)}(a) = \frac{n!}{2\pi i}\oint_{LTKSL} \frac{F(z)}{(z-a)^{n+1}}\,dz$$

so that $F(z)$ is analytic at a. But since we can choose a to be any point in the region $\mathcal{R}$ by suitably modifying the dashed contour of Fig. 10-9, it follows that $F(z)$ is analytic in $\mathcal{R}$.

Method 3, using Morera's theorem.

Referring to Fig. 10-9, we have

$$\oint_{KSLTK} F(z)\,dz = \int_{KSL} F(z)\,dz + \int_{LK} F(z)\,dz + \int_{KL} F(z)\,dz + \int_{LTK} F(z)\,dz$$
$$= \oint_{KSLK} F_1(z)\,dz + \oint_{KLTK} F_2(z)\,dz = 0$$

by Cauchy's theorem. Thus the integral around any simple closed path in $\mathcal{R}$ is zero, and so by Morera's theorem $F(z)$ must be analytic.

The function $F_2(z)$ is called an *analytic continuation* of $F_1(z)$.

5. (a) Prove that the function defined by $F_1(z) = z - z^2 + z^3 - z^4 + \cdots$ is analytic in the region $|z| < 1$. (b) Find a function which represents all possible analytic continuations of $F_1(z)$.

(a) By the ratio test, the series converges for $|z| < 1$. Then the series represents an analytic function in this region.

(b) For $|z| < 1$, the sum of the series is $F_2(z) = z/(1+z)$. But this function is analytic at all points except $z = -1$. Since $F_2(z) = F_1(z)$ inside $|z| = 1$, it is the required function.

6. (a) Prove that the function defined by $F_1(z) = \int_0^\infty t^3 e^{-zt}\,dt$ is analytic at all points z for which $\text{Re}\{z\} > 0$. (b) Find a function which is the analytic continuation of $F_1(z)$ into the left-hand plane $\text{Re}\{z\} < 0$.

(a) On integrating by parts, we have

$$\begin{aligned}
\int_0^\infty t^3 e^{-zt}\,dt &= \lim_{M\to\infty} \int_0^M t^3 e^{-zt}\,dt \\
&= \lim_{M\to\infty} \left\{ (t^3)\left(\frac{e^{-zt}}{-z}\right) - (3t^2)\left(\frac{e^{-zt}}{z^2}\right) + (6t)\left(\frac{e^{-zt}}{-z^3}\right) - (6)\left(\frac{e^{-zt}}{z^4}\right) \right\}\Bigg|_0^M \\
&= \lim_{M\to\infty} \left\{ \frac{6}{z^4} - \frac{M^3 e^{-Mz}}{z} - \frac{3M^2 e^{-Mz}}{z^2} - \frac{6M e^{-Mz}}{z^3} - \frac{6 e^{-Mz}}{z^4} \right\} \\
&= \frac{6}{z^4} \qquad \text{if } \text{Re}\{z\} > 0
\end{aligned}$$

(b) For $\text{Re}\{z\} > 0$, the integral has the value $F_2(z) = 6/z^4$. But this function is analytic at all points except $z = 0$. Since $F_2(z) = F_1(z)$ for $\text{Re}\{z\} > 0$, we see that $F_2(z) = 6/z^4$ must be the required analytic continuation.

SCHWARZ'S REFLECTION PRINCIPLE

7. Prove Schwarz's reflection principle (see Page 266).

Refer to Fig. 10-4, Page 266. On the real axis $[y = 0]$ we have $F_1(z) = F_1(x) = \overline{F_1(x)} = \overline{F_1(\bar{z})}$. Then by Problem 3 we have only to prove that $\overline{F_1(\bar{z})} = F_2(z)$ is analytic in $\mathcal{R}_2$.

Let $F_1(z) = U_1(x,y) + i\,V_1(x,y)$. Since this is analytic in $\mathcal{R}_1$ [i.e. $y > 0$], we have by the Cauchy-Riemann equations,

$$\frac{\partial U_1}{\partial x} = \frac{\partial V_1}{\partial y}, \qquad \frac{\partial V_1}{\partial x} = -\frac{\partial U_1}{\partial y} \tag{1}$$

where these partial derivatives are continuous.

Now $F_1(\bar{z}) = F_1(x - iy) = U_1(x,-y) + i\,V_1(x,-y)$, and so $\overline{F_1(\bar{z})} = U_1(x,-y) - i\,V_1(x,-y)$. If this is to be analytic in $\mathcal{R}_2$ we must have, for $y > 0$,

$$\frac{\partial U_1}{\partial x} = \frac{\partial(-V_1)}{\partial(-y)}, \qquad \frac{\partial(-V_1)}{\partial x} = -\frac{\partial U_1}{\partial(-y)} \tag{2}$$

But these are equivalent to (1), since $\dfrac{\partial(-V_1)}{\partial(-y)} = \dfrac{\partial V_1}{\partial y}$, $\dfrac{\partial(-V_1)}{\partial x} = -\dfrac{\partial V_1}{\partial x}$ and $\dfrac{\partial U_1}{\partial(-y)} = -\dfrac{\partial U_1}{\partial y}$. Hence the required result follows.

INFINITE PRODUCTS

8. Prove that a necessary and sufficient condition for $\prod_{k=1}^\infty (1 + |w_k|)$ to converge is that $\Sigma |w_k|$ converges.

Sufficiency. If $x > 0$, then $1 + x \leqq e^x$ so that

$$P_n = \prod_{k=1}^n (1 + |w_k|) = (1+|w_1|)(1+|w_2|)\cdots(1+|w_n|) \leqq e^{|w_1|}e^{|w_2|}\cdots e^{|w_n|} = e^{|w_1|+|w_2|+\cdots+|w_n|}$$

If $\sum_{k=1}^{\infty} |w_k|$ converges, it follows that P_n is a bounded monotonic increasing sequence and so has a limit, i.e. $\prod_{k=1}^{\infty} (1 + |w_k|)$, converges.

Necessity. If $S_n = \sum_{k=1}^{n} |w_k|$, we have

$$P_n \;=\; (1+|w_1|)(1+|w_2|)\cdots(1+|w_n|) \;\geqq\; 1 + |w_1| + |w_2| + \cdots + |w_n| \;=\; 1 + S_n \;\geqq\; 1$$

If $\lim_{n\to\infty} P_n$ exists, i.e. the infinite product converges, it follows that S_n is a bounded monotonic increasing sequence and so has a limit, i.e. $\sum_{k=1}^{\infty} |w_k|$ converges.

9. Prove that $\prod_{k=1}^{\infty}\left(1 - \frac{z^2}{k^2}\right)$ converges.

Let $w_k = -\frac{z^2}{k^2}$. Then $|w_k| = \frac{|z|^2}{k^2}$ and $\Sigma\,|w_k| = |z|^2\,\Sigma\,\frac{1}{k^2}$ converges. Hence by Problem 8, the infinite product is absolutely convergent and thus convergent.

10. Prove that $\sin z = z\left(1 - \frac{z^2}{\pi^2}\right)\left(1 - \frac{z^2}{4\pi^2}\right)\left(1 - \frac{z^2}{9\pi^2}\right)\cdots = z\prod_{k=1}^{\infty}\left(1 - \frac{z^2}{k^2\pi^2}\right)$.

From Problem 35, Chapter 7, Page 192, we have

$$\int_0^z \left(\cot t - \frac{1}{t}\right) dt \;=\; \ln\left(\frac{\sin t}{t}\right)\Bigg|_0^z \;=\; \ln\left(\frac{\sin z}{z}\right)$$

$$= \int_0^z \left(\frac{2t}{t^2 - \pi^2} + \frac{2t}{t^2 - 4\pi^2} + \cdots\right) dt$$

$$= \sum_{k=1}^{\infty} \ln\left(1 - \frac{z^2}{k^2\pi^2}\right) \;=\; \ln \prod_{k=1}^{\infty}\left(1 - \frac{z^2}{k^2\pi^2}\right)$$

Then $\sin z = z\prod_{k=1}^{\infty}\left(1 - \frac{z^2}{k^2\pi^2}\right)$.

THE GAMMA FUNCTION

11. Prove that $\Gamma(z+1) = z\,\Gamma(z)$ using definition (4), Page 267.

Integrating by parts, we have if $\text{Re}\{z\} > 0$,

$$\Gamma(z+1) \;=\; \int_0^\infty t^z e^{-t}\,dt \;=\; \lim_{M\to\infty} \int_0^M t^z e^{-t}\,dt$$

$$= \lim_{M\to\infty}\left\{(t^z)(-e^{-t})\Big|_0^M - \int_0^M (zt^{z-1})(-e^{-t})\,dt\right\}$$

$$= z\int_0^\infty t^{z-1} e^{-t}\,dt \;=\; z\,\Gamma(z)$$

12. Prove that $\Gamma(m) = 2\int_0^\infty x^{2m-1} e^{-x^2}\,dx, \quad m > 0.$

If $t = x^2$, we have

$$\Gamma(m) \;=\; \int_0^\infty t^{m-1} e^{-t}\,dt \;=\; \int_0^\infty (x^2)^{m-1} e^{-x^2}\,2x\,dx \;=\; 2\int_0^\infty x^{2m-1} e^{-x^2}\,dx$$

The result also holds if $\text{Re}\{m\} > 0$.

13. Prove that $\Gamma(z)\,\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$.

We first prove it for real values of z such that $0 < z < 1$. By analytic continuation we can then extend it to other values of z.

From Problem 12, we have for $0 < m < 1$,

$$\Gamma(m)\,\Gamma(1-m) = \left\{2\int_0^\infty x^{2m-1}e^{-x^2}\,dx\right\}\left\{2\int_0^\infty y^{1-2m}e^{-y^2}\,dy\right\}$$

$$= 4\int_0^\infty\int_0^\infty x^{2m-1}y^{1-2m}e^{-(x^2+y^2)}\,dx\,dy$$

In terms of polar coordinates (r, θ) with $x = r\cos\theta$, $y = r\sin\theta$ this becomes

$$4\int_{\theta=0}^{\pi/2}\int_{r=0}^\infty (\tan^{1-2m}\theta)(re^{-r^2})\,dr\,d\theta = 2\int_0^{\pi/2}\tan^{1-2m}\theta\,d\theta = \frac{\pi}{\sin m\pi}$$

using Problem 20, Page 185, with $x = \tan^2\theta$ and $p = 1-m$.

14. Prove that $\Gamma(\tfrac{1}{2}) = 2\displaystyle\int_0^\infty e^{-u^2}\,du = \sqrt{\pi}$.

From Problem 12, letting $m = \frac{1}{2}$, we have

$$\Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-x^2}\,dx$$

From Problem 13, letting $z = \frac{1}{2}$, we have

$$\{\Gamma(\tfrac{1}{2})\}^2 = \pi \qquad \text{or} \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

since $\Gamma(\frac{1}{2}) > 0$. Thus the required result follows.

Another method. As in Problem 13,

$$\{\Gamma(\tfrac{1}{2})\}^2 = \left\{2\int_0^\infty e^{-x^2}\,dx\right\}\left\{2\int_0^\infty e^{-y^2}\,dy\right\}$$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^\infty e^{-r^2}r\,dr\,d\theta = \pi$$

from which $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

15. By use of analytic continuation, show that $\Gamma(-\tfrac{1}{2}) = -2\sqrt{\pi}$.

If $\operatorname{Re}\{z\} > 0$, $\Gamma(z)$ is defined by (4), Page 267, but this definition cannot be used for $\operatorname{Re}\{z\} \leqq 0$. However, we can use the recursion formula $\Gamma(z+1) = z\,\Gamma(z)$, which holds for $\operatorname{Re}\{z\} > 0$, to extend the definition for $\operatorname{Re}\{z\} \leqq 0$, i.e. it provides an analytic continuation into the left-hand plane.

Substituting $z = -\frac{1}{2}$ in $\Gamma(z+1) = z\,\Gamma(z)$, we find $\Gamma(\frac{1}{2}) = -\frac{1}{2}\Gamma(-\frac{1}{2})$ or $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$ using Problem 14.

16. (*a*) **Prove that** $\Gamma(z) = \dfrac{\Gamma(z+n+1)}{z(z+1)(z+2)\cdots(z+n)}$.

(*b*) **Use (*a*) to show that $\Gamma(z)$ is an analytic function except for simple poles in the left-hand plane at** $z = 0, -1, -2, -3, \ldots$.

(*a*) We have $\Gamma(z+1) = z\,\Gamma(z)$, $\Gamma(z+2) = (z+1)\,\Gamma(z+1) = (z+1)z\,\Gamma(z)$, $\Gamma(z+3) = (z+2)\,\Gamma(z+2) = (z+2)(z+1)z\,\Gamma(z)$ and, in general, $\Gamma(z+n+1) = (z+n)(z+n-1)\cdots(z+2)(z+1)z\,\Gamma(z)$ from which the required result follows.

(*b*) We know that $\Gamma(z)$ is analytic for $\operatorname{Re}\{z\} > 0$, from definition (4), Page 267. Also, it is clear from the result in (*a*) that $\Gamma(z)$ is defined and analytic for $\operatorname{Re}\{z\} \geqq -n$ except for the simple poles at $z = 0, -1, -2, \ldots, -n$. Since this is the case for any positive integer n, the required result follows.

17. Use Weierstrass' factor theorem for infinite products [equation (2), Page 267] to obtain the infinite product for the gamma function [Property 2, Page 268].

Let $f(z) = 1/\Gamma(z+1)$. Then $f(z)$ is analytic everywhere and has simple zeros at $z = -1, -2, -3, \cdots$. By Weierstrass' factor theorem, we find

$$\frac{1}{\Gamma(z+1)} = e^{f'(0)z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-z/k}$$

To determine $f'(0)$, let $z = 1$. Then since $\Gamma(2) = 1$, we have

$$1 = e^{f'(0)} \prod_{k=1}^{\infty} \left(1 + \frac{1}{k}\right) e^{-1/k}$$

$$= e^{f'(0)} \lim_{M \to \infty} \prod_{k=1}^{M} \left(1 + \frac{1}{k}\right) e^{-1/k}$$

Taking logarithms, we see that

$$f'(0) = \lim_{M\to\infty} \left\{\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{M} - \ln\left[\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\cdots\left(1+\frac{1}{M}\right)\right]\right\}$$

$$= \lim_{M\to\infty} \left\{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{M} - \ln M\right\} = \gamma$$

where γ is Euler's constant. Then the required result follows on noting that $\Gamma(z+1) = z\,\Gamma(z)$.

THE BETA FUNCTION

18. Prove that $B(m,n) = B(n,m)$.

Letting $t = 1 - u$,

$$B(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\,dt = \int_0^1 (1-u)^{m-1} u^{n-1}\,du = B(n,m)$$

19. Prove that $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = 2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta$.

Let $t = \sin^2\theta$. Then

$$B(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\,dt = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1}\, 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = 2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta$$

by Problem 18.

20. Prove that $B(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\,dt = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$.

From Problem 12, we have on transforming to polar coordinates,

$$\Gamma(m)\,\Gamma(n) = \left\{2\int_0^\infty x^{2m-1}e^{-x^2}\,dx\right\}\left\{2\int_0^\infty y^{2n-1}e^{-y^2}\,dy\right\}$$

$$= 4\int_0^\infty \int_0^\infty x^{2m-1}y^{2n-1}e^{-(x^2+y^2)}\,dx\,dy$$

$$= 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} (\cos^{2m-1}\theta \sin^{2n-1}\theta)(r^{2m+2n-1}e^{-r^2})\,dr\,d\theta$$

$$= \left\{2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta\right\}\left\{\int_0^\infty r^{2(m+n)-1}e^{-r^2}\,dr\right\}$$

$$= B(m,n)\,\Gamma(m+n)$$

where we have used Problem 19 and Problem 12 with r replacing t and $m+n$ replacing m. From this the required result follows.

21. Evaluate (a) $\int_0^2 \sqrt{x(2-x)}\,dx$, (b) $\int_0^{\pi/2} \sqrt{\tan\theta}\,d\theta$.

(a) Letting $x = 2t$, the integral becomes

$$\int_0^1 \sqrt{4t(1-t)}\,2\,dt = 4\int_0^1 t^{1/2}(1-t)^{1/2}\,dt = 4\,B(3/2,3/2)$$

$$= 4\,\frac{\Gamma(3/2)\,\Gamma(3/2)}{\Gamma(3)} = \frac{4(\frac{1}{2}\sqrt{\pi})(\frac{1}{2}\sqrt{\pi})}{2} = \frac{\pi}{2}$$

(b)
$$\int_0^{\pi/2} \sqrt{\tan\theta}\,d\theta = \int_0^{\pi/2} \sin^{1/2}\theta\,\cos^{-1/2}\theta\,d\theta = \tfrac{1}{2}B(\tfrac{3}{4},\tfrac{1}{4})$$

$$= \tfrac{1}{2}\,\Gamma(\tfrac{3}{4})\,\Gamma(\tfrac{1}{4}) = \frac{1}{2}\,\frac{\pi}{\sin(\pi/4)} = \frac{\pi\sqrt{2}}{2}$$

using Problems 13, 19 and 20.

22. Show that $\int_0^4 y^{3/2}(16-y^2)^{1/2}\,dy = \frac{64}{21}\sqrt{\frac{2}{\pi}}\,\{\Gamma(\frac{1}{4})\}^2$.

Let $y^2 = 16t$, i.e. $y = 4t^{1/2}$, $dy = 2t^{-1/2}\,dt$. Then the integral becomes

$$\int_0^1 \{8t^{3/4}\}\{4(1-t)^{1/2}\}\{2t^{-1/2}\,dt\} = 64\int_0^1 t^{1/4}(1-t)^{1/2}\,dt$$

$$= 64\,B(\tfrac{5}{4},\tfrac{3}{2}) = \frac{64\,\Gamma(\frac{5}{4})\,\Gamma(\frac{3}{2})}{\Gamma(\frac{11}{4})} = \frac{64(\frac{1}{4})\,\Gamma(\frac{1}{4})\,(\frac{1}{2})\,\Gamma(\frac{1}{2})}{\frac{7}{4}\cdot\frac{3}{4}\,\Gamma(\frac{3}{4})}$$

$$= \frac{128\sqrt{\pi}}{21}\,\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} = \frac{128\sqrt{\pi}}{21}\,\frac{\{\Gamma(\frac{1}{4})\}^2}{\Gamma(\frac{1}{4})\,\Gamma(\frac{3}{4})} = \frac{64}{21}\sqrt{\frac{2}{\pi}}\,\{\Gamma(\tfrac{1}{4})\}^2$$

using the fact that $\Gamma(\frac{1}{4})\,\Gamma(\frac{3}{4}) = \pi/[\sin(\pi/4)] = \pi\sqrt{2}$ [Problem 13].

DIFFERENTIAL EQUATIONS

23. Determine the singular points of each of the following differential equations and specify whether they are regular or irregular.

(a) $z^2Y'' + zY' + (z^2 - n^2)Y = 0$ or $Y'' + \frac{1}{z}Y' + \left(\frac{z^2-n^2}{z^2}\right)Y = 0$.

$z = 0$ is a singular point. Since $z(1/z) = 1$ and $z^2\left(\frac{z^2-n^2}{z^2}\right) = z^2 - n^2$ are analytic at $z = 0$, it is a regular singular point.

(b) $(z-1)^4Y'' + 2(z-1)^3Y' + Y = 0$ or $Y'' + \frac{2}{z-1}Y' + \frac{1}{(z-1)^4}Y = 0$.

At the singular point $z = 1$, $(z-1)\left(\frac{2}{z-1}\right) = 2$ is analytic but $(z-1)^2\cdot\frac{1}{(z-1)^4} = \frac{1}{(z-1)^2}$ is not analytic. Then $z = 1$ is an irregular singular point.

(c) $z^2(1-z)Y'' + Y' - Y = 0$ or $Y'' + \frac{1}{z^2(1-z)}Y' - \frac{1}{z^2(1-z)}Y = 0$.

At the singular point $z = 0$, $z\left\{\frac{1}{z^2(1-z)}\right\} = \frac{1}{z(1-z)}$ and $z^2\left\{\frac{-1}{z^2(1-z)}\right\} = \frac{-1}{1-z}$ are not both analytic. Hence $z = 0$ is an irregular singular point.

At the singular point $z = 1$, $(z-1)\cdot\left\{\frac{1}{z^2(1-z)}\right\} = \frac{-1}{z^2}$ and $(z-1)^2\left\{\frac{-1}{z^2(1-z)}\right\} = \frac{z-1}{z^2}$ are both analytic. Hence $z = 1$ is a regular singular point.

24. Find the general solution of Bessel's differential equation

$$z^2Y'' + zY' + (z^2 - n^2)Y = 0 \qquad \text{where } n \neq 0, \pm 1, \pm 2, \ldots$$

The point $z = 0$ is a regular singular point. Hence there is a series solution of the form $Y = \sum_{k=-\infty}^{\infty} a_k z^{k+c}$ where $a_k = 0$ for $k = -1, -2, -3, \ldots$. By differentiation, omitting the summation limits, we have

$$Y' = \Sigma (k+c) a_k z^{k+c-1}, \qquad Y'' = \Sigma (k+c)(k+c-1) a_k z^{k+c-2}$$

Then

$$z^2Y'' = \Sigma (k+c)(k+c-1)\, a_k z^{k+c}$$

$$zY' = \Sigma (k+c)\, a_k z^{k+c}$$

$$(z^2 - n^2)Y = \Sigma a_k z^{k+c+2} - \Sigma n^2 a_k z^{k+c}$$

$$= \Sigma a_{k-2} z^{k+c} - \Sigma n^2 a_k z^{k+c}$$

Adding, $$z^2Y'' + zY' + (z^2 - n^2)Y = \Sigma \{[(k+c)^2 - n^2]\, a_k + a_{k-2}\} z^{k+c} = 0$$

from which we obtain

$$[(k+c)^2 - n^2]\, a_k + a_{k-2} = 0 \tag{1}$$

If $k = 0$, $(c^2 - n^2)a_0 = 0$; and if $a_0 \neq 0$, we obtain the *indicial equation* $c^2 - n^2 = 0$ with roots $c = \pm n$.

Case 1: $c = n$.

From (*1*), $[(k+n)^2 - n^2]\, a_k + a_{k-2} = 0$ or $k(2n+k)\, a_k + a_{k-2} = 0$.

If $k = 1$, $a_1 = 0$. If $k = 2$, $a_2 = -\dfrac{a_0}{2(2n+2)}$. If $k = 3$, $a_3 = 0$. If $k = 4$, $a_4 = -\dfrac{a_2}{4(2n+4)} = \dfrac{a_0}{2 \cdot 4(2n+2)(2n+4)}$, etc. Then

$$Y = \Sigma a_k z^{k+c} = a_0 z^n \left\{1 - \frac{z^2}{2(2n+2)} + \frac{z^4}{2 \cdot 4(2n+2)(2n+4)} - \cdots \right\} \tag{2}$$

Case 2: $c = -n$.

The result obtained is

$$Y = a_0 z^{-n} \left\{1 - \frac{z^2}{2(2-2n)} + \frac{z^4}{2 \cdot 4(2n+2)(2n+4)} - \cdots \right\} \tag{3}$$

which can be obtained formally from Case 1 on replacing n by $-n$.

The general solution if $n \neq 0, \pm 1, \pm 2, \ldots$ is given by

$$Y = Az^n \left\{1 - \frac{z^2}{2(2n+2)} + \frac{z^4}{2 \cdot 4(2n+2)(2n+4)} - \cdots \right\}$$

$$+ Bz^{-n} \left\{1 - \frac{z^2}{2(2-2n)} + \frac{z^4}{2 \cdot 4(2-2n)(4-2n)} - \cdots \right\} \tag{4}$$

If $n = 0, \pm 1, \pm 2, \ldots$ only one solution is obtained. To find the general solution in this case we must proceed as in Problems 175 and 176.

Since the singularity nearest to $z = 0$ is at infinity, the solutions should converge for all z. This is easily shown by the ratio test.

SOLUTION OF DIFFERENTIAL EQUATIONS BY CONTOUR INTEGRALS

25. (*a*) Obtain a solution of the equation $zY'' + (2n+1)Y' + zY = 0$ having the form $Y = \oint_C e^{zt} G(t)\, dt$. (*b*) By letting $Y = z^r U$ and choosing the constant r appropriately, obtain a contour integral solution of $z^2U'' + zU' + (z^2 - n^2)U = 0$.

(*a*) If $Y = \oint_C e^{zt} G(t)\, dt$, we find $Y' = \oint_C te^{zt} G(t)\, dt$, $Y'' = \oint_C t^2 e^{zt} G(t)\, dt$.

Then integrating by parts, assuming that C is chosen so that the functional values at the initial and final points P are equal [and the integrated part is zero], we have

$$zY = \oint_C ze^{zt}\,G(t)\,dt = e^{zt}\,G(t)\Big|_P^P - \oint_C e^{zt}\,G'(t)\,dt = -\oint_C e^{zt}\,G'(t)\,dt$$

$$(2n+1)Y' = \oint_C (2n+1)t\,e^{zt}\,G(t)\,dt$$

$$\begin{aligned} zY'' &= \oint_C zt^2\,e^{zt}\,G(t)\,dt = \oint_C (ze^{zt})\{t^2\,G(t)\}\,dt \\ &= e^{zt}\{t^2\,G(t)\}\Big|_P^P - \oint_C e^{zt}\{t^2\,G(t)\}'\,dt \\ &= -\oint_C e^{zt}\{t^2\,G(t)\}'\,dt \end{aligned}$$

Thus

$$zY'' + (2n+1)Y' + zY = 0 = \oint_C e^{zt}[-G'(t) + (2n+1)t\,G(t) - \{t^2\,G(t)\}']\,dt$$

This is satisfied if we choose $G(t)$ so that the integrand is zero, i.e.

$$-G'(t) + (2n+1)t\,G(t) - \{t^2\,G(t)\}' = 0 \quad \text{or} \quad G'(t) = \frac{(2n-1)t}{t^2+1}\,G(t)$$

Solving, $G(t) = A\,(t^2+1)^{n-1/2}$ where A is any constant. Hence a solution is

$$Y = A\oint_C e^{zt}\,(t^2+1)^{n-1/2}\,dt$$

(*b*) If $Y = z^rU$, then $Y' = z^rU' + rz^{r-1}U$ and $Y'' = z^rU'' + 2rz^{r-1}U' + r(r-1)z^{r-2}U$. Hence

$$\begin{aligned} zY'' + (2n+1)Y' + zY &= z^{r+1}U'' + 2rz^rU' + r(r-1)z^{r-1}U \\ &\quad + (2n+1)z^rU' + (2n+1)rz^{r-1}U + z^{r+1}U \\ &= z^{r+1}U'' + [2rz^r + (2n+1)z^r]U' \\ &\quad + [r(r-1)z^{r-1} + (2n+1)rz^{r-1} + z^{r+1}]U \end{aligned}$$

The given differential equation is thus equivalent to

$$z^2U'' + (2r+2n+1)zU' + [z^2+r^2+2nr]U = 0$$

Letting $r = -n$, this becomes $z^2U'' + zU' + (z^2-n^2)U = 0$.

Hence a contour integral solution is

$$U = z^nY = Az^n\oint_C e^{zt}\,(t^2+1)^{n-1/2}\,dt$$

26. Obtain the general solution of $Y'' - 3Y' + 2Y = 0$ by the method of contour integrals.

Let $Y = \oint_C e^{zt}\,G(t)\,dt$, $Y' = \oint_C te^{zt}\,G(t)\,dt$, $Y'' = \oint_C t^2\,e^{zt}\,G(t)\,dt$. Then

$$Y'' - 3Y' + 2Y = \oint_C e^{zt}\,(t^2-3t+2)\,G(t)\,dt = 0$$

is satisfied if we choose $G(t) = 1/(t^2-3t+2)$. Hence

$$Y = \oint_C \frac{e^{zt}}{t^2-3t+2}\,dt$$

If we choose C so that the simple pole $t=1$ lies inside C while $t=2$ lies outside C, the integral has the value $2\pi ie^z$. If $t=2$ lies inside C while $t=1$ lies outside C, the integral has the value $2\pi ie^{2z}$.

The general solution is given by $Y = Ae^z + Be^{2z}$.

BESSEL FUNCTIONS

27. Prove that $zJ_{n-1}(z) - 2nJ_n(z) + zJ_{n+1}(z) = 0.$

Differentiating with respect to t both sides of the identity

$$e^{\frac{1}{2}z(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(z)\,t^n$$

yields

$$e^{\frac{1}{2}z(t-1/t)}\left\{\frac{z}{2}\left(1+\frac{1}{t^2}\right)\right\} = \sum_{n=-\infty}^{\infty}\frac{z}{2}\left(1+\frac{1}{t^2}\right)J_n(z)\,t^n = \sum_{n=-\infty}^{\infty} n\,J_n(z)\,t^{n-1}$$

i.e.,

$$\sum_{n=-\infty}^{\infty} z\,J_n(z)\,t^n + \sum_{n=-\infty}^{\infty} z\,J_n(z)\,t^{n-2} = \sum_{n=-\infty}^{\infty} 2n\,J_n(z)\,t^{n-1}$$

Equating coefficients of t^n on both sides, we have

$$z\,J_n(z) + z\,J_{n+2}(z) = 2(n+1)\,J_{n+1}(z)$$

and the required result follows on replacing n by $n-1$.

Since we have used the generating function, the above result is established only for integral values of n. The result also holds for non-integral values of n [see Problem 114].

28. Prove $J_n(z) = \dfrac{1}{2\pi i}\oint_C t^{-n-1}\,e^{\frac{1}{2}z(t-1/t)}\,dt,$ where C is a simple closed curve enclosing $t=0$.

We have

$$e^{\frac{1}{2}z(t-1/t)} = \sum_{m=-\infty}^{\infty} J_m(z)\,t^m$$

so that

$$t^{-n-1}\,e^{\frac{1}{2}z(t-1/t)} = \sum_{m=-\infty}^{\infty} t^{m-n-1}\,J_m(z)$$

and

$$\oint_C t^{-n-1}\,e^{\frac{1}{2}z(t-1/t)}\,dt = \sum_{m=-\infty}^{\infty} J_m(z)\oint_C t^{m-n-1}\,dt \qquad (1)$$

Now by Problems 21 and 22, Chapter 4, Page 108, we have

$$\oint_C t^{m-n-1}\,dt = \begin{cases} 2\pi i & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \qquad (2)$$

Thus the series on the right of (1) reduces to $2\pi i\,J_n(z)$, from which the required result follows.

29. Prove that if $a \neq b$,

$$\int_0^z t\,J_n(at)\,J_n(bt)\,dt = \frac{z\{a\,J_n(bz)\,J_n'(az) - b\,J_n(az)\,J_n'(bz)\}}{b^2-a^2}$$

$Y_1 = J_n(at)$ and $Y_2 = J_n(bt)$ satisfy the respective differential equations

$$(1) \quad t^2Y_1'' + t\,Y_1' + (a^2t^2 - n^2)\,Y_1 = 0$$

$$(2) \quad t^2Y_2'' + t\,Y_2' + (b^2t^2 - n^2)\,Y_2 = 0$$

Multiplying (1) by Y_2, (2) by Y_1 and subtracting, we find

$$t^2(Y_2Y_1'' - Y_1Y_2'') + t(Y_2Y_1' - Y_1Y_2') = (b^2-a^2)t^2\,Y_1Y_2$$

This can be written

$$t\frac{d}{dt}(Y_2Y_1' - Y_1Y_2') + (Y_2Y_1' - Y_1Y_2') = (b^2-a^2)t\,Y_1Y_2$$

or

$$\frac{d}{dt}\{t(Y_2Y_1' - Y_1Y_2')\} = (b^2-a^2)t\,Y_1Y_2$$

Integrating with respect to t from 0 to z yields

$$(b^2-a^2)\int_0^z t\,Y_1Y_2\,dt = t(Y_2Y_1' - Y_1Y_2')\Big|_0^z$$

or since $a \neq b$

$$\int_0^z t\,J_n(at)\,J_n(bt)\,dt = \frac{z\{a\,J_n(bz)\,J_n'(az) - b\,J_n(az)\,J_n'(bz)\}}{b^2-a^2}$$

LEGENDRE FUNCTIONS

30. Prove that $\int_{-1}^{1} P_m(z)\, P_n(z)\, dz = 0$ if $m \neq n$.

We have

$$(1) \quad (1-z^2)P_m'' - 2zP_m' + m(m+1)P_m = 0$$

$$(2) \quad (1-z^2)P_n'' - 2zP_n' + n(n+1)P_n = 0$$

Multiplying (1) by P_n, (2) by P_m, and subtracting, we obtain

$$(1-z^2)\{P_n P_m'' - P_m P_n''\} - 2z\{P_n P_m' - P_m P_n'\} = \{n(n+1) - m(m+1)\}P_m P_n$$

which can be written

$$(1-z^2)\frac{d}{dz}\{P_n P_m' - P_m P_n'\} - 2z\{P_n P_m' - P_m P_n'\} = \{n(n+1) - m(m+1)\}P_m P_n$$

or

$$\frac{d}{dz}\{(1-z^2)(P_n P_m' - P_m P_n')\} = \{n(n+1) - m(m+1)\}P_m P_n$$

Integrating from -1 to 1, we have

$$\{n(n+1) - m(m+1)\}\int_{-1}^{1} P_m(z)\, P_n(z)\, dz = (1-z^2)(P_n P_m' - P_m P_n')\Big|_{-1}^{1} = 0$$

from which the required result follows, since $m \neq n$.

The result is often called the *orthogonality principle* for Legendre polynomials and we say that the Legendre polynomials form an *orthogonal set*.

31. Prove that $\int_{-1}^{1} P_m(z)\, P_n(z)\, dz = \frac{2}{2n+1}$ if $m = n$.

Squaring both sides of the identity,

$$\frac{1}{\sqrt{1-2zt+t^2}} = \sum_{n=0}^{\infty} P_n(z)\, t^n$$

we obtain

$$\frac{1}{1-2zt+t^2} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P_m(z)\, P_n(z)\, t^{m+n}$$

Integrating from -1 to 1 and using Problem 30, we find

$$\int_{-1}^{1}\frac{dz}{1-2zt+t^2} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{\int_{-1}^{1} P_m(z)\, P_n(z)\, dz\right\} t^{m+n} = \sum_{n=0}^{\infty}\left\{\int_{-1}^{1}\{P_n(z)\}^2\, dz\right\} t^{2n} \quad (1)$$

But the left side is equal to

$$-\frac{1}{2t}\ln(1-2zt+t^2)\Big|_{-1}^{1} = \frac{1}{t}\ln\left(\frac{1+t}{1-t}\right) = \sum_{n=0}^{\infty}\left\{\frac{2}{2n+1}\right\} t^{2n} \quad (2)$$

using Problem 23(c), Chapter 6, Page 155. Equating coefficients of t^{2n} in the series (1) and (2) yields the required result.

32. Prove that $(n+1)\, P_{n+1}(z) - (2n+1)z\, P_n(z) + n\, P_{n-1}(z) = 0$.

Differentiating with respect to t both sides of the identity

$$\frac{1}{\sqrt{1-2zt+t^2}} = \sum_{n=0}^{\infty} P_n(z)\, t^n$$

we have

$$\frac{z-t}{(1-2zt+t^2)^{3/2}} = \sum_{n=0}^{\infty} n\, P_n(z)\, t^{n-1}$$

Then multiplying by $1-2zt+t^2$, we have

$$(z-t)\sum_{n=0}^{\infty} P_n(z)\,t^n = (1-2zt+t^2)\sum_{n=0}^{\infty} n\,P_n(z)\,t^{n-1}$$

or

$$\sum_{n=0}^{\infty} z\,P_n(z)\,t^n - \sum_{n=0}^{\infty} P_n(z)\,t^{n+1} = \sum_{n=0}^{\infty} n\,P_n(z)\,t^{n-1} - \sum_{n=0}^{\infty} 2nz\,P_n(z)\,t^n + \sum_{n=0}^{\infty} n\,P_n(z)\,t^{n+1}$$

Equating coefficients of t^n on each side, we obtain

$$z\,P_n(z) - P_{n-1}(z) = (n+1)\,P_{n+1}(z) - 2nz\,P_n(z) + (n-1)\,P_{n-1}(z)$$

which yields the required result on simplifying.

THE HYPERGEOMETRIC FUNCTION

33. Show that $F(1/2, 1/2; 3/2; z^2) = \dfrac{\sin^{-1} z}{z}$.

Since $F(a,b;c;z) = 1 + \dfrac{a\cdot b}{1\cdot c}z + \dfrac{a(a+1)\,b(b+1)}{1\cdot 2\cdot c(c+1)}z^2 + \cdots$ we have

$$\begin{aligned} F(1/2,1/2;3/2;z^2) &= 1 + \frac{(1/2)(1/2)}{1\cdot(3/2)}z^2 + \frac{(1/2)(3/2)(1/2)(3/2)}{1\cdot 2\cdot(3/2)(5/2)}z^4 \\ &\quad + \frac{(1/2)(3/2)(5/2)(1/2)(3/2)(5/2)}{1\cdot 2\cdot 3\cdot(3/2)(5/2)(7/2)}z^6 + \cdots \\ &= 1 + \frac{1}{2}\frac{z^2}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{z^4}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{z^6}{7} + \cdots = \frac{\sin^{-1} z}{z} \end{aligned}$$

using Problem 89, Chapter 6, Page 166.

THE ZETA FUNCTION

34. Prove that the zeta function $\zeta(z) = \sum_{k=1}^{\infty}\dfrac{1}{k^z}$ is analytic in the region of the z plane for which $\text{Re}\{z\} \geqq 1+\delta$ where δ is any fixed positive number.

Each term $1/k^z$ of the series is an analytic function. Also, if $x = \text{Re}\{z\} \geqq 1+\delta$ then,

$$\left|\frac{1}{k^z}\right| = \left|\frac{1}{e^{z\ln k}}\right| = \frac{1}{e^{x\ln k}} = \frac{1}{k^x} \leqq \frac{1}{k^{1+\delta}}$$

Since $\Sigma\, 1/k^{1+\delta}$ converges, we see by the Weierstrass M test that $\sum_{k=1}^{\infty}\frac{1}{k^z}$ converges uniformly for $\text{Re}\{z\} \geqq 1+\delta$. Hence by Theorem 21, Page 142, $\zeta(z)$ is analytic in this region.

ASYMPTOTIC EXPANSIONS AND THE METHOD OF STEEPEST DESCENTS

35. (*a*) If $p > 0$, prove that

$$F(z) = \int_z^{\infty}\frac{e^{-t}}{t^p}\,dt = e^{-z}\left\{\frac{1}{z^p} - \frac{p}{z^{p+1}} + \frac{p(p+1)}{z^{p+2}} - \cdots (-1)^n\frac{p(p+1)\cdots(p+n-1)}{z^{p+n}}\right\} + (-1)^{n+1}p(p+1)\cdots(p+n)\int_z^{\infty}\frac{e^{-t}}{t^{p+n+1}}\,dt$$

(*b*) Use (*a*) to prove that

$$F(z) = \int_z^{\infty}\frac{e^{-t}}{t^p}\,dt \sim e^{-z}\left\{\frac{1}{z^p} - \frac{p}{z^{p+1}} + \frac{p(p+1)}{z^{p+2}} - \cdots\right\} = S(z)$$

i.e. the series on the right is an asymptotic expansion of the function on the left.

(*a*) Integrating by parts, we have

$$I_p = \int_z^\infty \frac{e^{-t}}{t^p}\,dt = \lim_{M\to\infty}\int_z^M e^{-t}t^{-p}\,dt$$

$$= \lim_{M\to\infty}\left\{(-e^{-t})(t^{-p})\Big|_z^M - \int_z^M (-e^{-t})(-pt^{-p-1})\,dt\right\}$$

$$= \lim_{M\to\infty}\left\{\frac{e^{-z}}{z^p} - \frac{e^{-M}}{M^p} - p\int_z^M \frac{e^{-t}}{t^{p+1}}\,dt\right\}$$

$$= \frac{e^{-z}}{z^p} - p\int_z^\infty \frac{e^{-t}}{t^{p+1}}\,dt = \frac{e^{-z}}{z^p} - pI_{p+1}$$

Similarly, $I_{p+1} = \dfrac{e^{-z}}{z^{p+1}} - (p+1)I_{p+2}$ so that

$$I_p = \frac{e^{-z}}{z^p} - p\left\{\frac{e^{-z}}{z^{p+1}} - (p+1)I_{p+2}\right\} = \frac{e^{-z}}{z^p} - \frac{pe^{-z}}{z^{p+1}} + p(p+1)I_{p+2}$$

By continuing in this manner, the result follows.

(b) Let $S_n(z) = e^{-z}\left\{\dfrac{1}{z^p} - \dfrac{p}{z^{p+1}} + \dfrac{p(p+1)}{z^{p+2}} - \cdots (-1)^n\dfrac{p(p+1)\cdots(p+n-1)}{z^{p+n}}\right\}$. Then

$$R_n(z) = F(z) - S_n(z) = (-1)^{n+1}p(p+1)\cdots(p+n)\int_z^\infty \frac{e^{-t}}{t^{p+n+1}}\,dt$$

Now for real $z > 0$,

$$|R_n(z)| = p(p+1)\cdots(p+n)\int_z^\infty \frac{e^{-t}}{t^{p+n+1}}\,dt \leqq p(p+1)\cdots(p+n)\int_z^\infty \frac{e^{-t}}{z^{p+n+1}}\,dt$$

$$\leqq \frac{p(p+1)\cdots(p+n)}{z^{p+n+1}}$$

since
$$\int_z^\infty e^{-t}\,dt \leqq \int_0^\infty e^{-t}\,dt = 1$$

Thus
$$\lim_{z\to\infty}|z^n R_n(z)| \leqq \lim_{z\to\infty}\frac{p(p+1)\cdots(p+n)}{z^p} = 0$$

and it follows that $\lim_{z\to\infty} z^n R_n(z) = 0$. Hence the required result is proved for real $z>0$. The result can also be extended to complex values of z.

Note that since $\left|\dfrac{u_{n+1}}{u_n}\right| = \left|\dfrac{p(p+1)\cdots(p+n)/z^{p+n+1}}{p(p+1)\cdots(p+n-1)/z^{p+n}}\right| = \dfrac{p+n}{|z|}$, where u_n is the nth term of the series, we have for all fixed z, $\lim_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| = \infty$ and the series diverges for all z by the ratio test.

36. Show that $\Gamma(z+1) \sim \sqrt{2\pi z}\,z^z e^{-z}\left\{1 + \dfrac{1}{12z} + \dfrac{1}{288z^2} - \dfrac{139}{51{,}840z^3} + \cdots\right\}$.

We have $\Gamma(z+1) = \int_0^\infty \tau^z e^{-\tau}\,d\tau$. By letting $\tau = zt$, this becomes

$$\Gamma(z+1) = z^{z+1}\int_0^\infty t^z e^{-zt}\,dt = z^{z+1}\int_0^\infty e^{z(\ln t - t)}\,dt \qquad (1)$$

which has the form (37), Page 274, where $F(t) = \ln t - t$.

$F'(t) = 0$ when $t = 1$. Letting $t = 1 + w$ we find, using Problem 23, Page 154, or otherwise, the Taylor series

$$F(t) = \ln t - t = \ln(1+w) - (1+w) = \left(w - \frac{w^2}{2} + \frac{w^3}{3} - \frac{w^4}{4} + \cdots\right) - 1 - w$$

$$= -1 - \frac{w^2}{2} + \frac{w^3}{3} - \frac{w^4}{4} + \cdots = -1 - \frac{(t-1)^2}{2} + \frac{(t-1)^3}{3} - \frac{(t-1)^4}{4} + \cdots$$

Hence from (1),
$$\Gamma(z+1) = z^{z+1}e^{-z}\int_0^\infty e^{-z(t-1)^2/2}\,e^{z(t-1)^3/3 - z(t-1)^4/4 + \cdots}\,dt$$
$$= z^{z+1}e^{-z}\int_{-1}^\infty e^{-zw^2/2}\,e^{zw^3/3 - zw^4/4 + \cdots}\,dw \qquad (2)$$

Letting $w = \sqrt{2/z}\,v$, this becomes

$$\Gamma(z+1) = \sqrt{2}\,z^{z+1/2}e^{-z}\int_{-\sqrt{z/2}}^{\infty} e^{-v^2}\,e^{(2/3)\sqrt{2}z^{-1/2}v^3 - z^{-1}v^4 + \cdots}\,dv \qquad (3)$$

For large values of z the lower limit can be replaced by $-\infty$, and on expanding the exponential we have

$$\Gamma(z+1) \sim \sqrt{2}\,z^{z+1/2}e^{-z}\int_{-\infty}^{\infty} e^{-v^2}\,\{1 + (\tfrac{2}{3}\sqrt{2}\,z^{-1/2}v^3 - z^{-1}v^4) + \cdots\}\,dv \qquad (4)$$

or

$$\Gamma(z+1) \sim \sqrt{2\pi z}\,z^z e^{-z}\left\{1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51{,}840z^3} + \cdots\right\} \qquad (5)$$

Although we have proceeded above in a formal manner, the analysis can be justified rigorously.

Another method.

If $F(t) = -1 - \dfrac{(t-1)^2}{2} + \dfrac{(t-1)^3}{3} - \dfrac{(t-1)^4}{4} + \cdots = -1 - u^2$, then

$$u^2 = \frac{(t-1)^2}{2} - \frac{(t-1)^3}{3} + \cdots$$

and by reversion of series or by using the fact that $F(t) = \ln t - t$, we find

$$\frac{dt}{du} = b_0 + b_1u + b_2u^2 + \cdots = \sqrt{2} + \frac{\sqrt{2}}{6}u^2 + \frac{\sqrt{2}}{216}u^4 + \cdots$$

Then from (*41*), Page 275, we find

$$\Gamma(z+1) \sim \sqrt{\frac{\pi}{z}}\,z^{z+1}\,e^{z(\ln 1 - 1)}\left\{\sqrt{2} + \frac{1}{2}\left(\frac{\sqrt{2}}{6}\right)\frac{1}{z} + \frac{1\cdot 3}{2\cdot 2}\left(\frac{\sqrt{2}}{216}\right)\frac{1}{z^2} + \cdots\right\}$$

or

$$\Gamma(z+1) \sim \sqrt{2\pi z}\,z^z e^{-z}\left\{1 + \frac{1}{12z} + \frac{1}{288z^2} + \cdots\right\}$$

Note that since $F''(1) = -1$, we find on using (*42*), Page 275,

$$\Gamma(z+1) \sim \sqrt{2\pi z}\,z^z e^{-z}$$

which is the first term. For many purposes this first term provides sufficient accuracy.

ELLIPTIC FUNCTIONS

37. Prove (*a*) $\dfrac{d}{dz}\,\text{sn}\,z = \text{cn}\,z\,\text{dn}\,z$, (*b*) $\dfrac{d}{dz}\,\text{cn}\,z = -\,\text{sn}\,z\,\text{dn}\,z$.

By definition, if $z = \displaystyle\int_0^w \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$, then $w = \text{sn}\,z$. Hence

(*a*) $$\frac{d}{dz}(\text{sn}\,z) = \frac{dw}{dz} = 1/(dz/dw) = \sqrt{(1-w^2)(1-k^2w^2)} = \text{cn}\,z\,\text{dn}\,z$$

(*b*) $$\frac{d}{dz}(\text{cn}\,z) = \frac{d}{dz}(1-\text{sn}^2 z)^{1/2} = \tfrac{1}{2}(1-\text{sn}^2 z)^{-1/2}\frac{d}{dz}(-\text{sn}^2 z)$$
$$= \tfrac{1}{2}(1-\text{sn}^2 z)^{-1/2}(-2\,\text{sn}\,z)(\text{cn}\,z\,\text{dn}\,z) = -\,\text{sn}\,z\,\text{dn}\,z$$

38. Prove (*a*) $\text{sn}(-z) = -\,\text{sn}\,z$, (*b*) $\text{cn}(-z) = \text{cn}\,z$, (*c*) $\text{dn}(-z) = \text{dn}\,z$.

(*a*) If $z = \displaystyle\int_0^w \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$, then $w = \text{sn}\,z$. Let $t = -\tau$; then

$$z = -\int_0^{-w}\frac{d\tau}{\sqrt{(1-\tau^2)(1-k^2\tau^2)}} \quad\text{or}\quad -z = \int_0^{-w}\frac{d\tau}{\sqrt{(1-\tau^2)(1-k^2\tau^2)}},$$

i.e. $\text{sn}(-z) = -w = -\,\text{sn}\,z$

(*b*) $\text{cn}(-z) = \sqrt{1-\text{sn}^2(-z)} = \sqrt{1-\text{sn}^2 z} = \text{cn}\,z$

(*c*) $\text{dn}(-z) = \sqrt{1-k^2\,\text{sn}^2(-z)} = \sqrt{1-k^2\,\text{sn}^2 z} = \text{dn}\,z$

39. Prove that (a) $\operatorname{sn}(z+2K) = -\operatorname{sn} z$, (b) $\operatorname{cn}(z+2K) = -\operatorname{cn} z$.

We have $z = \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$ so that $\phi = \operatorname{am} z$ and $\sin\phi = \operatorname{sn} z$, $\cos\phi = \operatorname{cn} z$. Now

$$\int_0^{\phi+\pi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{\pi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} + \int_{\pi}^{\phi+\pi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

$$= 2\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} + \int_0^{\phi} \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}}$$

using the transformation $\theta = \pi + \psi$. Hence $\phi + \pi = \operatorname{am}(z+2K)$.

Thus we have

(a) $\operatorname{sn}(z+2K) = \sin\{\operatorname{am}(z+2K)\} = \sin(\phi+\pi) = -\sin\phi = -\operatorname{sn} z$

(b) $\operatorname{cn}(z+2K) = \cos\{\operatorname{am}(z+2K)\} = \cos(\phi+\pi) = -\cos\phi = -\operatorname{cn} z$

40. Prove that (a) $\operatorname{sn}(z+4K) = \operatorname{sn} z$, (b) $\operatorname{cn}(z+4K) = \operatorname{cn} z$, (c) $\operatorname{dn}(z+2K) = \operatorname{dn} z$.

From Problem 39,

(a) $\operatorname{sn}(z+4K) = -\operatorname{sn}(z+2K) = \operatorname{sn} z$

(b) $\operatorname{cn}(z+4K) = -\operatorname{cn}(z+2K) = \operatorname{cn} z$

(c) $\operatorname{dn}(z+2K) = \sqrt{1-k^2\operatorname{sn}^2(z+2K)} = \sqrt{1-k^2\operatorname{sn}^2 z} = \operatorname{dn} z$

Another method. The integrand $\frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}$ has branch points at $t = \pm 1$ and $t = \pm 1/k$ in the t plane [Fig. 10-10]. Consider the integral from 0 to w along two paths C_1 and C_2. We can deform C_2 into the path $ABDEFGHJA + C_1$, where BDE and GHJ are circles of radius ϵ while JAB and EFG, drawn separately for visual purposes, are actually coincident with the x axis.

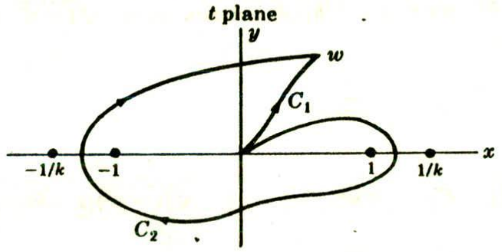


Fig. 10-10

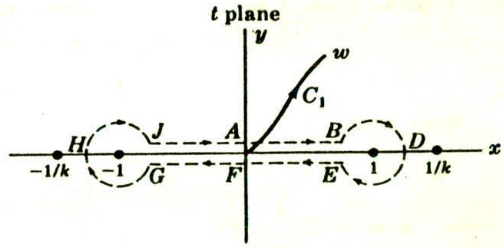


Fig. 10-11

We then have

$$\int_{0\,(C_2)}^{w} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{1-\epsilon} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} + \int_{BDE} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

$$+ \int_{1-\epsilon}^{0} \frac{dx}{-\sqrt{(1-x^2)(1-k^2x^2)}} + \int_0^{-1+\epsilon} \frac{dx}{-\sqrt{(1-x^2)(1-k^2x^2)}}$$

$$+ \int_{GHJ} \frac{dt}{-\sqrt{(1-t^2)(1-k^2t^2)}} + \int_{-1+\epsilon}^{0} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

$$+ \int_{0\,(C_1)}^{w} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

$$= 4\int_0^{1-\epsilon} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} + \int_{0\,(C_1)}^{w} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

$$+ \int_{BDE} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} + \int_{GHJ} \frac{dt}{-\sqrt{(1-t^2)(1-k^2t^2)}}$$

where we have used the fact that in encircling a branch point the sign of the radical is changed

On BDE and GHJ we have $t = 1 - \epsilon e^{i\theta}$ and $t = -1 + \epsilon e^{i\theta}$ respectively. Then the corresponding integrals equal

$$\int_0^{2\pi} \frac{-i\epsilon e^{i\theta}\,d\theta}{\sqrt{(2-\epsilon e^{i\theta})(\epsilon e^{i\theta})\{1-k^2(1-\epsilon e^{i\theta})^2\}}} = -i\sqrt{\epsilon}\int_0^{2\pi} \frac{e^{i\theta/2}\,d\theta}{\sqrt{(2-\epsilon e^{i\theta})\{1-k^2(1-\epsilon e^{i\theta})^2\}}}$$

$$\int_0^{2\pi} \frac{i\epsilon e^{i\theta}\,d\theta}{\sqrt{(\epsilon e^{i\theta})(2-\epsilon e^{i\theta})\{1-k^2(-1+\epsilon e^{i\theta})^2\}}} = i\sqrt{\epsilon}\int_0^{2\pi} \frac{e^{i\theta/2}\,d\theta}{\sqrt{(2-\epsilon e^{i\theta})\{1-k^2(-1+\epsilon e^{i\theta})^2\}}}$$

As $\epsilon \to 0$, these integrals approach zero and we obtain

$$\int_{C_2\,0}^{w} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = 4\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} + \int_{C_1\,0}^{w} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

Now if we write $\quad z = \displaystyle\int_{C_1\,0}^{w} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad$ i.e. $\quad w = \operatorname{sn} z$

then $\quad z + 4K = \displaystyle\int_{C_2\,0}^{w} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad$ i.e. $\quad w = \operatorname{sn}(z+4K)$

and since the value of w is the same in both cases, $\operatorname{sn}(z+4K) = \operatorname{sn} z$.

Similarly we can establish the other results.

41. Prove that (a) $\operatorname{sn}(K+iK') = 1/k$, (b) $\operatorname{cn}(K+iK') = -ik'/k$, (c) $\operatorname{dn}(K+iK') = 0$.

(a) We have $\quad K' = \displaystyle\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}}, \quad$ where $k' = \sqrt{1-k^2}$.

Let $u = 1/\sqrt{1-k'^2t^2}$. When $t=0$, $u=1$; when $t=1$, $u=1/k$. Thus as t varies from 0 to 1, u varies from 1 to $1/k$. By Problem 43, Page 56, with $p = 1/k$, it follows that $\sqrt{1-t^2} = -ik'u/\sqrt{1-k'^2u^2}$. Thus we have by substitution

$$K' = -i\int_1^{1/k} \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}$$

from which

$$K + iK' = \int_0^1 \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}} + \int_1^{1/k} \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}} = \int_0^{1/k} \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}$$

i.e. $\operatorname{sn}(K+iK') = 1/k$.

(b) From Part (a),

$$\operatorname{cn}(K+iK') = \sqrt{1-\operatorname{sn}^2(K+iK')} = \sqrt{1-1/k^2} = -i\sqrt{1-k^2}/k = -ik'/k$$

(c) $\operatorname{dn}(K+iK') = \sqrt{1-k^2\operatorname{sn}^2(K+iK')} = 0$ by Part (a).

42. Prove that (a) $\operatorname{sn}(2K+2iK') = 0$, (b) $\operatorname{cn}(2K+2iK') = 1$, (c) $\operatorname{dn}(2K+2iK') = -1$.

From the addition formulae with $z_1 = z_2 = K+iK'$, we have

(a) $$\operatorname{sn}(2K+2iK') = \frac{2\operatorname{sn}(K+iK')\operatorname{cn}(K+iK')\operatorname{dn}(K+iK')}{1-k^2\operatorname{sn}^4(K+iK')} = 0$$

(b) $$\operatorname{cn}(2K+2iK') = \frac{\operatorname{cn}^2(K+iK') - \operatorname{sn}^2(K+iK')\operatorname{dn}^2(K+iK')}{1-k^2\operatorname{sn}^4(K+iK')} = 1$$

(c) $$\operatorname{dn}(2K+2iK') = \frac{\operatorname{dn}^2(K+iK') - k^2\operatorname{sn}^2(K+iK')\operatorname{cn}^2(K+iK')}{1-k^2\operatorname{sn}^4(K+iK')} = -1$$

43. Prove that (a) $\operatorname{sn}(z+2iK') = \operatorname{sn} z$, (b) $\operatorname{cn}(z+2K+2iK') = \operatorname{cn} z$, (c) $\operatorname{dn}(z+4iK') = \operatorname{dn} z$.

Using Problems 39, 42, 170 and the addition formulae, we have

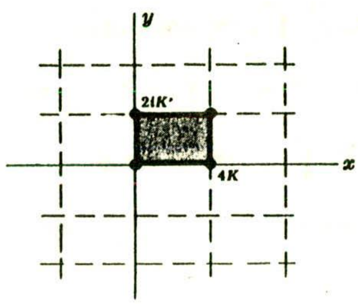
$$\begin{aligned}(a)\quad \operatorname{sn}(z+2iK') &= \operatorname{sn}(z-2K+2K+2iK') \\ &= \frac{\operatorname{sn}(z-2K)\operatorname{cn}(2K+2iK')\operatorname{dn}(2K+2iK') + \operatorname{sn}(2K+2iK')\operatorname{cn}(z-2K)\operatorname{dn}(z-2K)}{1 - k^2\operatorname{sn}^2(z-2K)\operatorname{sn}^2(2K+2iK')} \\ &= \operatorname{sn} z\end{aligned}$$

$$\begin{aligned}(b)\quad \operatorname{cn}(z+2K+2iK') &= \frac{\operatorname{cn} z\operatorname{cn}(2K+2iK') - \operatorname{sn} z\operatorname{sn}(2K+2iK')\operatorname{dn} z\operatorname{dn}(2K+2iK')}{1 - k^2\operatorname{sn}^2 z\operatorname{sn}^2(2K+2iK')} \\ &= \operatorname{cn} z\end{aligned}$$

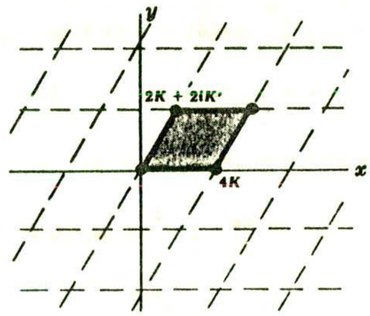
$$\begin{aligned}(c)\quad \operatorname{dn}(z+4iK') &= \operatorname{dn}(z-4K+4K+4iK') \\ &= \frac{\operatorname{dn}(z-4K)\operatorname{dn}(4K+4iK') - k^2\operatorname{sn}(z-4K)\operatorname{sn}(4K+4iK')\operatorname{cn}(z-4K)\operatorname{cn}(4K+4iK')}{1 - k^2\operatorname{sn}^2(z-4K)\operatorname{sn}^2(4K+4iK')} \\ &= \operatorname{dn} z\end{aligned}$$

44. Construct period parallelograms or cells for the functions (*a*) sn z, (*b*) cn z, (*c*) dn z.

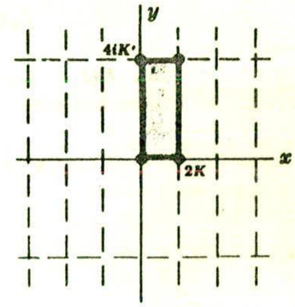
The results are shown in Figures 10-12, 10-13 and 10-14 respectively.



Period Parallelograms
for sn z
Fig. 10-12



Period Parallelograms
for cn z
Fig. 10-13



Period Parallelograms
for dn z
Fig. 10-14

MISCELLANEOUS PROBLEMS

45. Prove that $P_n(z) = F\left(-n, n+1; 1; \frac{1-z}{2}\right)$, $n = 0, 1, 2, 3, \ldots$.

The Legendre polynomials $P_n(z)$ are of degree n and have the value 1 for $z = 1$. Similarly from (*29*), Page 273, it is seen that

$$F\left(-n, n+1; 1; \frac{1-z}{2}\right) = 1 - \frac{n(n+1)}{2}(1-z) + \frac{n(n-1)(n+1)(n+2)}{16}(1-z)^2 + \cdots$$

is a polynomial of degree n having the value 1 for $z = 1$.

The required result follows if we show that P_n and F satisfy the same differential equation. To do this, let $\frac{1-z}{2} = u$, i.e. $z = 1 - 2u$, in Legendre's equation (*25*), Page 272, to obtain

$$u(1-u)\frac{d^2Y}{du^2} + (1-2u)\frac{dY}{du} + n(n+1)Y = 0$$

But this is the hypergeometric equation (*30*), Page 273, with $a = -n$, $b = n+1$, $c = 1$ and $u = (1-z)/2$. Hence the result is proved.

46. Prove that for $m = 1, 2, 3, \ldots$,

$$\Gamma\left(\frac{1}{m}\right)\Gamma\left(\frac{2}{m}\right)\Gamma\left(\frac{3}{m}\right)\cdots\Gamma\left(\frac{m-1}{m}\right) = \frac{(2\pi)^{(m-1)/2}}{\sqrt{m}}$$

We have

$$P = \Gamma\left(\frac{1}{m}\right)\Gamma\left(\frac{2}{m}\right)\cdots\Gamma\left(1-\frac{1}{m}\right) = \Gamma\left(1-\frac{1}{m}\right)\Gamma\left(1-\frac{2}{m}\right)\cdots\Gamma\left(\frac{1}{m}\right)$$

Then multiplying these products term by term and using Problem 13 and Problem 52, Page 25, we find

$$P^2 = \left\{\Gamma\left(\frac{1}{m}\right)\Gamma\left(1-\frac{1}{m}\right)\right\}\left\{\Gamma\left(\frac{2}{m}\right)\Gamma\left(1-\frac{2}{m}\right)\right\}\cdots\left\{\Gamma\left(1-\frac{1}{m}\right)\Gamma\left(\frac{1}{m}\right)\right\}$$

$$= \frac{\pi}{\sin(\pi/m)}\cdot\frac{\pi}{\sin(2\pi/m)}\cdots\frac{\pi}{\sin(m-1)\pi/m}$$

$$= \frac{\pi^{m-1}}{\sin(\pi/m)\sin(2\pi/m)\cdots\sin(m-1)\pi/m} = \frac{\pi^{m-1}}{m/2^{m-1}} = \frac{(2\pi)^{m-1}}{m}$$

or $P = (2\pi)^{(m-1)/2}/\sqrt{m}$, as required.

47. Show that for large positive values of z,

$$J_n(z) \sim \sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{n\pi}{2}-\frac{\pi}{4}\right)$$

By Problem 33, Chapter 6, we have

$$J_n(z) = \frac{1}{\pi}\int_0^\pi \cos(nt - z\sin t)\,dt = \text{Re}\left\{\frac{1}{\pi}\int_0^\pi e^{-int}\,e^{iz\sin t}\,dt\right\}$$

Let $F(t) = i\sin t$. Then $F'(t) = i\cos t = 0$ where $t = \pi/2$. If we let $t = \pi/2 + v$, the integral in braces becomes

$$\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} e^{-in(\pi/2+v)}\,e^{iz\sin(\pi/2+v)}\,dv = \frac{e^{-in\pi/2}}{\pi}\int_{-\pi/2}^{\pi/2} e^{-inv}\,e^{iz\cos v}\,dv$$

$$= \frac{e^{-in\pi/2}}{\pi}\int_{-\pi/2}^{\pi/2} e^{-inv}\,e^{iz(1-v^2/2+v^4/24-\cdots)}\,dv$$

$$= \frac{e^{i(z-n\pi/2)}}{\pi}\int_{-\pi/2}^{\pi/2} e^{-inv}\,e^{-izv^2/2+izv^4/24-\cdots}\,dv$$

Let $v^2 = -2iu^2/z$ or $v = (1-i)u/\sqrt{z}$, i.e. $u = \frac{1}{2}(1+i)\sqrt{z}\,v$. Then the integral can be approximated by

$$\frac{(1-i)\,e^{i(z-n\pi/2)}}{\pi\sqrt{z}}\int_{-\infty}^{\infty} e^{-(1+i)nu/\sqrt{z}}\,e^{-u^2-iu^4/6z-\cdots}\,du$$

or for large positive values of z,

$$\frac{(1-i)\,e^{i(z-n\pi/2)}}{\pi\sqrt{z}}\int_{-\infty}^{\infty} e^{-u^2}\,du = \frac{(1-i)\,e^{i(z-n\pi/2)}}{\sqrt{\pi z}}$$

and the real part is

$$\frac{1}{\sqrt{\pi z}}\left\{\cos\left(z-\frac{n\pi}{2}\right)+\sin\left(z-\frac{n\pi}{2}\right)\right\} = \sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{n\pi}{2}-\frac{\pi}{4}\right)$$

Higher order terms can also be obtained [see Problem 162].

48. If C is the contour of Fig. 10-15, prove that for all values of z

$$\Gamma(z) = \frac{1}{e^{2\pi iz}-1}\oint_C t^{z-1}\,e^{-t}\,dt$$

Referring to Fig. 10-15 below, we see that along AB, $t = x$; along BDE, $t = \epsilon e^{i\theta}$; and along EF, $t = xe^{2\pi i}$. Then

$$\int_{ABDEF} t^{z-1}\,e^{-t}\,dt = \int_R^\epsilon x^{z-1}\,e^{-x}\,dx + \int_0^{2\pi}(\epsilon e^{i\theta})^{z-1}\,e^{-\epsilon e^{i\theta}}\,i\epsilon e^{i\theta}\,d\theta$$

$$+ \int_\epsilon^R x^{z-1}\,e^{2\pi i(z-1)}\,e^{-x}\,dx$$

$$= (e^{2\pi iz}-1)\int_\epsilon^R x^{z-1}\,e^{-x}\,dx + i\int_0^{2\pi}\epsilon^z\,e^{i\theta z}\,e^{-\epsilon e^{i\theta}}\,d\theta$$

Now if $\operatorname{Re}\{z\} > 0$, we have on taking the limit as $\epsilon \to 0$ and $R \to \infty$,

$$\int_C t^{z-1} e^{-t}\, dt = (e^{2\pi i z} - 1) \int_0^\infty x^{z-1} e^{-x}\, dx = (e^{2\pi i z} - 1)\, \Gamma(z)$$

But the functions on both sides are analytic for all z. Hence for all z,

$$\Gamma(z) = \frac{1}{e^{2\pi i z} - 1} \oint_C t^{z-1} e^{-t}\, dt$$

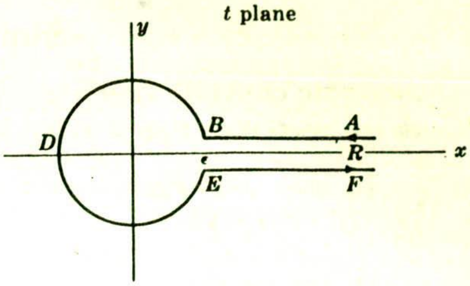


Fig. 10-15

49. Prove that $\operatorname{sn}(z_1 + z_2) = \dfrac{\operatorname{sn} z_1 \operatorname{cn} z_2 \operatorname{dn} z_2 + \operatorname{cn} z_1 \operatorname{sn} z_2 \operatorname{dn} z_1}{1 - k^2 \operatorname{sn}^2 z_1 \operatorname{sn}^2 z_2}$.

Let $z_1 + z_2 = \alpha$, a constant. Then $dz_2/dz_1 = -1$. Let us define $U = \operatorname{sn} z_1$, $V = \operatorname{sn} z_2$. It follows that

$$\frac{dU}{dz_1} = \dot{U} = \operatorname{cn} z_1 \operatorname{dn} z_1, \qquad \frac{dV}{dz_1} = \dot{V} = \frac{dV}{dz_2}\frac{dz_2}{dz_1} = -\operatorname{cn} z_2 \operatorname{dn} z_2$$

where dots denote differentiation with respect to z_1. Then

$$\dot{U}^2 = (1 - U^2)(1 - k^2U^2) \quad \text{and} \quad \dot{V}^2 = (1 - V^2)(1 - k^2V^2)$$

Differentiating and simplifying, we find

$$(1) \quad \ddot{U} = 2k^2U^3 - (1 + k^2)U, \qquad (2) \quad \ddot{V} = 2k^2V^3 - (1 + k^2)V$$

Multiplying (1) by V, (2) by U, and subtracting, we have

$$\ddot{U}V - U\ddot{V} = 2k^2UV(U^2 - V^2) \tag{3}$$

It is easy to verify that

$$\dot{U}^2V^2 - U^2\dot{V}^2 = (1 - k^2U^2V^2)(V^2 - U^2) \tag{4}$$

or

$$\dot{U}V - U\dot{V} = \frac{(1 - k^2U^2V^2)(V^2 - U^2)}{\dot{U}V + U\dot{V}} \tag{5}$$

Dividing equations (3) and (5), we have

$$\frac{\ddot{U}V - U\ddot{V}}{\dot{U}V - U\dot{V}} = \frac{-2k^2UV(\dot{U}V + U\dot{V})}{1 - k^2U^2V^2} \tag{6}$$

But $\ddot{U}V - U\ddot{V} = \dfrac{d}{dz_1}(\dot{U}V - U\dot{V})$ and $-2k^2UV(\dot{U}V + U\dot{V}) = \dfrac{d}{dz_1}(1 - k^2U^2V^2)$, so that (6) becomes

$$\frac{d(\dot{U}V - U\dot{V})}{\dot{U}V - U\dot{V}} = \frac{d(1 - k^2U^2V^2)}{1 - k^2U^2V^2}$$

An integration yields $\dfrac{\dot{U}V - U\dot{V}}{1 - k^2U^2V^2} = c$ (a constant), i.e.,

$$\frac{\operatorname{sn} z_1 \operatorname{cn} z_2 \operatorname{dn} z_2 + \operatorname{cn} z_1 \operatorname{sn} z_2 \operatorname{dn} z_1}{1 - k^2 \operatorname{sn}^2 z_1 \operatorname{sn}^2 z_2} = c$$

is a solution of the differential equation. It is also clear that $z_1 + z_2 = \alpha$ is a solution. The two solutions must be related as follows:

$$\frac{\operatorname{sn} z_1 \operatorname{cn} z_2 \operatorname{dn} z_2 + \operatorname{cn} z_1 \operatorname{sn} z_2 \operatorname{dn} z_1}{1 - k^2 \operatorname{sn}^2 z_1 \operatorname{sn}^2 z_2} = F(z_1 + z_2)$$

Putting $z_2 = 0$, we see that $F(z_1) = \operatorname{sn} z_1$. Then $F(z_1 + z_2) = \operatorname{sn}(z_1 + z_2)$ and the required result follows.

Supplementary Problems

ANALYTIC CONTINUATION

50. (*a*) Show that $F_1(z) = z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \frac{1}{4}z^4 + \cdots$ converges for $|z| < 1$.

(*b*) Show that $F_2(z) = \frac{1}{4}\pi i - \frac{1}{2}\ln 2 + \left(\frac{z-i}{1-i}\right) + \frac{1}{2}\left(\frac{z-i}{1-i}\right)^2 + \frac{1}{3}\left(\frac{z-i}{1-i}\right)^3 + \cdots$ converges for $|z-i| < \sqrt{2}$.

(*c*) Show that $F_1(z)$ and $F_2(z)$ are analytic continuations of each other.

(*d*) Can you find a function which represents all possible analytic continuations of $F_1(z)$? Justify your answer.

Ans. (*d*) $-\ln(1-z)$

51. A function $F(z)$ is represented in $|z-1| < 2$ by the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n (z-1)^{2n}}{2^{2n+1}}$$

Prove that the value of the function at $z = 5$ is 1/16.

52. (*a*) Show that $F_1(z) = \int_0^\infty (1+t)e^{-zt}\,dt$ converges only if $\text{Re}\{z\} > 0$.

(*b*) Find a function which is the analytic continuation of $F_1(z)$ into the left-hand plane.

Ans. (*b*) $(z+1)/z^2$

53. (*a*) Find the region of convergence of $F_1(z) = \int_0^\infty e^{-(z+1)^2 t}\,dt$ and graph this region.

(*b*) Find the value of the analytic continuation of $F_1(z)$ corresponding to $z = 2 - 4i$.

Ans. (*a*) $\text{Re}\{z+1\}^2 > 0$, (*b*) $(-7+24i)/625$

54. (*a*) Prove that $\dfrac{z}{1-z^2} + \dfrac{z^2}{1-z^4} + \dfrac{z^4}{1-z^8} + \cdots = \begin{cases} z/(1-z) & \text{if } |z| < 1 \\ 1/(1-z) & \text{if } |z| > 1 \end{cases}$

(*b*) Discuss these results from the point of view of analytic continuation.

55. Show that the series $\sum_{n=0}^{\infty} z^{3^n}$ cannot be continued analytically beyond the circle $|z| = 1$.

56. If $\sum_{n=1}^{\infty} a_n z^{\beta_n}$ has $|z| = 1$ as a natural barrier, would you expect $\sum_{n=1}^{\infty} (-1)^n a_n z^{\beta_n}$ to have $|z| = 1$ as natural barrier also? Justify your conclusion.

57. Let $\{z_n\}$, $n = 1, 2, 3, \ldots$ be a sequence such that $\lim_{n\to\infty} z_n = a$, and suppose that for all n, $z_n \neq a$. Let $F(z)$ and $G(z)$ be analytic at a and such that $F(z_n) = G(z_n)$, $n = 1, 2, 3, \ldots$.
(*a*) Prove that $F(z) = G(z)$. (*b*) Explain the relationship of the result in (*a*) with analytic continuation.
[*Hint.* Consider the expansion of $F(z) - G(z)$ in a Taylor series about $z = a$.]

SCHWARZ'S REFLECTION PRINCIPLE

58. Work Problem 2 using Schwarz's reflection principle.

59. (*a*) Given that $\sin 2z = 2\sin z\cos z$ holds for all real values of z, prove that it also holds for all complex values of z.

(*b*) Can you use the Schwarz reflection principle to prove that $\tan 2z = (2\tan z)/(1-\tan^2 z)$? Justify your conclusion.

60. Does the Schwarz reflection principle apply if reflection takes place in the imaginary rather than the real axis? Prove your statements.

61. Can you extend the Schwarz reflection principle to apply to reflection in a curve C?

INFINITE PRODUCTS

62. Investigate the convergence of the infinite products

$$(a)\ \prod_{k=1}^{\infty}\left(1+\frac{1}{k^3}\right), \qquad (b)\ \prod_{k=1}^{\infty}\left(1-\frac{1}{\sqrt{k+1}}\right), \qquad (c)\ \prod_{k=1}^{\infty}\left(1+\frac{\cos k\pi}{k^2+1}\right).$$

Ans. (*a*) conv., (*b*) div., (*c*) conv.

63. Prove that a necessary condition for $\prod_{k=1}^{\infty}(1+w_k)$ to converge is that $\lim_{n\to\infty} w_n = 0$.

64. Investigate the convergence of (*a*) $\prod_{k=1}^{\infty}\left(1+\frac{1}{k}\right)$, (*b*) $\prod_{k=1}^{\infty}\left(1+\frac{k}{\sqrt{k^2+1}}\right)$, (*c*) $\prod_{k=1}^{\infty}(1+\cot^{-1}k^2)$.
Ans. (*a*) div., (*b*) div., (*c*) conv.

65. If an infinite product is absolutely convergent, prove that it is convergent.

66. Prove that $\cos z = \prod_{k=1}^{\infty}\left(1-\frac{4z^2}{(2k-1)^2\pi^2}\right)$.

67. Show that $\prod_{k=1}^{\infty}\left(1+\frac{e^{-kz}}{k^2}\right)$ (*a*) converges absolutely and uniformly in the right half plane $\text{Re}\{z\} \geqq 0$ and (*b*) represents an analytic function of z for $\text{Re}\{z\} \geqq 0$.

68. Prove that $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots = \frac{1}{2}$.

69. Prove that $\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots = \frac{1}{2}$.

70. Prove that (*a*) $\sinh z = \prod_{k=1}^{\infty}\left(1+\frac{z^2}{k^2\pi^2}\right)$

(*b*) $\cosh z = \prod_{k=1}^{\infty}\left(1+\frac{4z^2}{(2k-1)^2\pi^2}\right)$.

71. Use infinite products to show that $\sin 2z = 2\sin z\cos z$. Justify all steps.

72. Prove that $\prod_{k=1}^{\infty}\left(1+\frac{1}{k}\sin\frac{z}{k}\right)$ (*a*) converges absolutely and uniformly for all z and (*b*) represents an analytic function.

73. Prove that $\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)e^{-z/k}$ converges.

THE GAMMA FUNCTION

74. Evaluate each of the following by use of the gamma function.

(*a*) $\int_0^{\infty} y^3 e^{-2y}\,dy$ (*c*) $\int_0^{\infty} y^2 e^{-2y^2}\,dy$ (*e*) $\int_0^{\infty} \{ye^{-y^2}\}^{1/4}\,dy$

(*b*) $\int_0^{\infty} u^{3/2} e^{-3u}\,du$ (*d*) $\int_0^{1} \{\ln(1/t)\}^{-1/2}\,dt$

Ans. (*a*) 3/8, (*b*) $\sqrt{3\pi}/36$, (*c*) $\sqrt{2\pi}/16$, (*d*) $\sqrt{\pi}$, (*e*) $\Gamma(5/8)/\sqrt[4]{2}$

75. Prove that $\Gamma(z) = \int_0^1 \{\ln(1/t)\}^{z-1}\,dt$ for $\text{Re}\{z\} > 0$.

76. Show that $\int_1^{\infty} \frac{(x-1)^p}{x^2}\,dx = \Gamma(1+p)\,\Gamma(1-p)$, $-1 < p < 1$.

77. If m, n and a are positive constants, show that

$$\int_0^{\infty} x^m e^{-ax^n}\,dx = \frac{1}{n}a^{-(m+1)/n}\,\Gamma\left(\frac{m+1}{n}\right)$$

78. Show that $\int_0^\infty \frac{e^{-zt}}{\sqrt{t}}\,dt = \sqrt{\frac{\pi}{z}}$ if $\operatorname{Re}\{z\} > 0$.

79. Evaluate $\int_0^1 (x \ln x)^4\,dx$. *Ans.* 24/3125

80. Evaluate (*a*) $\Gamma(-7/2)$, (*b*) $\Gamma(-1/3)$. *Ans.* (*a*) $16\sqrt{\pi}/105$, (*b*) $-3\,\Gamma(2/3)$

81. Show that $\Gamma(-\frac{1}{2} - m) = \frac{(-1)^{m+1}\sqrt{\pi}\,2^{m+1}}{1\cdot 3\cdot 5\cdots(2m+1)}$, $m = 0, 1, 2, \ldots$.

82. Prove that the residue of $\Gamma(z)$ at $z = -m$, $m = 0, 1, 2, 3, \ldots$, is $(-1)^m/m!$ where $0! = 1$ by definition.

83. Use the infinite product representation of the gamma function to prove that

(*a*) $$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

(*b*) $$2^{2z-1}\,\Gamma(z)\,\Gamma(z+\tfrac{1}{2}) = \sqrt{\pi}\,\Gamma(2z)$$

84. Prove that if $y > 0$, $|\Gamma(iy)| = \sqrt{\frac{\pi}{y \sinh \pi y}}$.

85. Discuss Problem 84 if $y < 0$.

86. Prove (*a*) Property 6, (*b*) Property 7, (*c*) Property 9 on Pages 268 and 269.

87. Prove that $\Gamma(\frac{1}{5})\,\Gamma(\frac{4}{5}) = 4\pi^2/\sqrt{5}$.

88. (*a*) By using the infinite product representation of the gamma function, prove that for any positive integer m,

$$\frac{m^{mz}\,\Gamma(z)\,\Gamma(z+1/m)\,\Gamma(z+2/m)\cdots\Gamma(z+[m-1]/m)}{\Gamma(mz)}$$

is a constant independent of z.

(*b*) By letting $z \to 0$ in the result of (*a*), evaluate the constant and thus establish Property 5, Page 268.

THE BETA FUNCTION

89. Evaluate (*a*) $B(3, 5/2)$, (*b*) $B(1/3, 2/3)$. *Ans.* (*a*) 16/315, (*b*) $2\pi/\sqrt{3}$

90. Evaluate each of the following using the beta function.

(*a*) $\int_0^1 t^{-1/3}(1-t)^{2/3}\,dt$, (*b*) $\int_0^1 u^2(1-u^2)^{-1/2}\,du$, (*c*) $\int_0^3 (9-t^2)^{3/2}\,dt$, (*d*) $\int_0^4 \frac{dt}{\sqrt{4t-t^2}}$.

Ans. (*a*) $4\pi/3\sqrt{3}$, (*b*) $\pi/4$, (*c*) $243\pi/16$, (*d*) π

91. Prove that $\frac{B(m+1, n)}{B(m, n+1)} = \frac{m}{n}$.

92. If $a > 0$, prove that $\int_0^a \frac{dy}{\sqrt{a^4 - y^4}} = \frac{\{\Gamma(1/4)\}^2}{4a\sqrt{2\pi}}$.

93. Prove that $\dfrac{B\left(\frac{p+1}{2}, \frac{1}{2}\right)}{B\left(\frac{p+1}{2}, \frac{p+1}{2}\right)} = 2^p$ stating any restrictions on p.

94. Evaluate (*a*) $\int_0^{\pi/2} \sin^6\theta \cos^4\theta\,d\theta$, (*b*) $\int_0^{\pi/2} \sqrt{\tan\theta}\,d\theta$. *Ans.* (*a*) $3\pi/512$, (*b*) $\pi/\sqrt{2}$

95. Prove that $B(m,n) = \frac{1}{2}\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$ where $\text{Re}\{m\} > 0$ and $\text{Re}\{n\} > 0$.

[*Hint.* Let $y = x/(1+x)$.]

96. Prove that $\int_0^\infty \frac{x^3\,dx}{1+x^6} = \frac{\pi}{3\sqrt{3}}$.

97. (*a*) Show that if either m or n (but not both) is a negative integer and if $m+n<0$, then $B(m,n)$ is infinite. (*b*) Investigate $B(m,n)$ when both m and n are negative integers.

DIFFERENTIAL EQUATIONS

98. Determine the singular points of each of the following differential equations and state whether they are regular or irregular.

(*a*) $(1-z^2)Y'' - 2Y' + 6Y = 0$

(*b*) $(2z^4 - z^5)Y'' + zY' + (z^2+1)Y = 0$

(*c*) $z^2(1-z)^2Y'' + (2-z)Y' + 4z^2Y = 0$.

Ans. (*a*) $z = \pm 1$, regular. (*b*) $z = 2$, regular; $z = 0$, irregular. (*c*) $z = 0, 1$, irregular.

99. Solve each of the following differential equations using power series and find the region of convergence. If possible, sum the series and show that the sum satisfies the differential equation.

(*a*) $Y'' + 2Y' + Y = 0$, (*b*) $Y'' + zY = 0$, (*c*) $zY'' + 2Y' + zY = 0$.

Ans. (*a*) $Y = Ae^{-z} + Bze^{-z}$

(*b*) $Y = A\left(1 - \frac{z^3}{3!} + \frac{1\cdot 4}{6!}z^6 - \frac{1\cdot 4\cdot 7}{9!}z^9 + \cdots\right) + B\left(z - \frac{2z^4}{4!} + \frac{2\cdot 5}{7!}z^7 - \frac{2\cdot 5\cdot 8}{10!}z^{10} + \cdots\right)$

(*c*) $Y = \frac{A\sin z + B\cos z}{z}$

100. (*a*) If you solved $(1-z^2)Y'' + 2Y = 0$ by substituting the assumed solution $Y = \Sigma a_n z^n$, what region of convergence would you expect? Explain.

(*b*) Determine whether your expectations in (*a*) are correct by actually finding the series solution.

Ans. (*b*) $Y = A(1-z^2) + B\left(z - \frac{z^3}{1\cdot 3} - \frac{z^5}{3\cdot 5} - \frac{z^7}{5\cdot 7} - \cdots\right)$

101. (*a*) Solve $Y'' + z^2Y = 0$ subject to $Y(0) = 1$, $Y'(0) = -1$ and (*b*) determine the region of convergence.

Ans. (*a*) $Y = 1 - z - \frac{z^4}{3\cdot 4} + \frac{z^5}{4\cdot 5} + \frac{z^8}{3\cdot 4\cdot 7\cdot 8} - \frac{z^9}{4\cdot 5\cdot 8\cdot 9} - \cdots$ (*b*) $|z| < \infty$

102. If $Y = Y_1(z)$ is a solution of $Y'' + p(z)Y' + q(z)Y = 0$, show that the general solution is

$$Y = A\,Y_1(z) + B\,Y_1(z)\int \frac{e^{-\int p(z)\,dz}}{\{Y_1(z)\}^2}\,dz$$

103. (*a*) Solve $zY'' + (1-z)Y' - Y = 0$ and (*b*) determine the region of convergence.

Ans. (*a*) $Y = (A + B\ln z)e^z - B\left\{z + \frac{z^2}{2!}(1 + \tfrac{1}{2}) + \frac{z^3}{3!}(1 + \tfrac{1}{2} + \tfrac{1}{3}) + \cdots\right\}$ (*b*) $|z| > 0$

104. (*a*) Use Problem 102 to show that the solution to the differential equation of Problem 103 can be written as

$$Y = Ae^z + Be^z\int \frac{e^{-z}}{z}\,dz$$

(*b*) Reconcile the result of (*a*) with the series solution obtained in Problem 103.

105. (*a*) Solve $zY'' + Y' - Y = 0$ and (*b*) determine the region of convergence.

Ans. (*a*) $Y = (A + B\ln z)\left\{\frac{z}{(1!)^2} + \frac{z^2}{(2!)^2} + \frac{z^3}{(3!)^2} + \cdots\right\}$

$$- 2B\left\{\frac{z}{(1!)^2} + \frac{z^2}{(2!)^2}(1 + \tfrac{1}{2}) + \frac{z^3}{(3!)^2}(1 + \tfrac{1}{2} + \tfrac{1}{3}) + \cdots\right\}$$

106. Prove that $Y = Ve^{-\frac{1}{2}\int p(z)\,dz}$ transforms the differential equation $Y'' + p(z)Y' + q(z)Y = 0$ into

$$V'' + \{q(z) - \tfrac{1}{2}p'(z) - \tfrac{1}{4}[p(z)]^2\}V = 0$$

107. Use the method of Problem 106 to find the general solution of $zY'' + 2Y' + zY = 0$ [see Prob. 99(c)].

SOLUTION OF DIFFERENTIAL EQUATIONS BY CONTOUR INTEGRALS

108. Use the method of contour integrals to solve each of the following.

(a) $Y'' - Y' - 2Y = 0$, (b) $Y'' + 4Y' + 4Y = 0$, (c) $Y'' + 2Y' + 2Y = 0$.

Ans. (a) $Y = Ae^{2z} + Be^{-z}$, (b) $Y = Ae^{-2z} + Bze^{-2z}$, (c) $Y = e^{-z}(A \sin z + B \cos z)$.

109. Prove that a solution of $zY'' + (a - z)Y' - bY = 0$, where $\text{Re}\{a\} > 0$, $\text{Re}\{b\} > 0$, is given by

$$Y = \int_0^1 e^{zt}\, t^{b-1}\,(1-t)^{a-b-1}\,dt$$

BESSEL FUNCTIONS

110. Prove that $J_{-n}(z) = (-1)^n J_n(z)$ for $n = 0, 1, 2, 3, \ldots$.

111. Prove (a) $\dfrac{d}{dz}\{z^n J_n(z)\} = z^n J_{n-1}(z)$, (b) $\dfrac{d}{dz}\{z^{-n} J_n(z)\} = -z^{-n} J_{n+1}(z)$.

112. Show that (a) $J_0'(z) = -J_1(z)$, (b) $\int z^3 J_2(z)\,dz = z^3 J_3(z) + c$, (c) $\int z^3 J_0(z)\,dz = z^3 J_1(z) - 2z^2 J_2(z) + c$.

113. Show that (a) $J_{1/2}(z) = \sqrt{2/\pi z}\,\sin z$, (b) $J_{-1/2}(z) = \sqrt{2/\pi z}\,\cos z$.

114. Prove the result of Problem 27 for non-integral values of n.

115. Show that $J_{3/2}(z)\sin z - J_{-3/2}\cos z = \sqrt{2/\pi z^3}$.

116. Prove that $J_n'(z) = \frac{1}{2}\{J_{n-1}(z) - J_{n+1}(z)\}$.

117. Prove that (a) $J_n''(z) = \frac{1}{4}\{J_{n-2}(z) - 2\,J_n(z) + J_{n+2}(z)\}$

(b) $J_n'''(z) = \frac{1}{8}\{J_{n-3}(z) - 3\,J_{n-1}(z) + 3\,J_{n+1}(z) - J_{n+3}(z)\}$.

118. Generalize the results in Problems 116 and 117.

119. By direct substitution prove that $J_0(z) = \dfrac{1}{\pi}\displaystyle\int_0^\pi \cos(z \sin\theta)\,d\theta$ satisfies the equation

$$zY'' + Y' + zY = 0$$

120. If $\text{Re}\{z\} > 0$, prove that $\displaystyle\int_0^\infty e^{-zt} J_0(t)\,dt = \frac{1}{\sqrt{z^2+1}}$.

121. Prove that: (a) $\cos(\alpha\cos\theta) = J_0(\alpha) - 2\,J_2(\alpha)\cos 2\theta + 2\,J_4(\alpha)\cos 4\theta + \cdots$

(b) $\sin(\alpha\cos\theta) = 2\,J_1(\alpha)\cos\theta - 2\,J_3(\alpha)\cos 3\theta + 2\,J_5(\alpha)\cos 5\theta - \cdots$.

122. If p is an integer, prove that

$$J_p(x+y) = \sum_{n=-\infty}^{\infty} J_n(x)\,J_{p-n}(y)$$

[*Hint.* Use the generating function.]

123. Establish Property 8, Page 271.

124. If $\operatorname{Re}\{z\} > 0$, prove that $J_n(z) = \dfrac{z^n}{2\pi i}\oint_C e^{\frac{1}{2}(t - z^2/t)}\, t^{-n-1}\, dt$ where C is the contour of Fig. 10-5, Page 268.

125. If $\operatorname{Re}\{z\} > 0$, prove that

$$J_n(z) = \frac{1}{\pi}\int_0^\pi \cos(n\phi - z\sin\phi)\, d\phi - \frac{\sin n\pi}{\pi}\int_0^\infty e^{-n\phi - z\sinh\phi}\, d\phi$$

126. (a) Verify that $Y_0(z)$, given by equation (23) on Page 272, is a solution to Bessel's equation of order zero. (b) Verify that $Y_n(z)$ given by equation (22) on Page 271 is a solution to Bessel's equation of order n.

127. Show that: (a) $z\, Y_{n-1}(z) - 2n\, Y_n(z) + z\, Y_{n+1}(z) = 0$

(b) $\dfrac{d}{dz}\{z^n\, Y_n(z)\} = z^n\, Y_{n-1}(z)$ (c) $\dfrac{d}{dz}\{z^{-n}\, Y_n(z)\} = -z^{-n}\, Y_{n+1}(z)$.

128. Prove that the general solution of

$$V'' + \left\{1 - \frac{(n^2 - 1/4)}{z^2}\right\} V = 0$$

is $V = \sqrt{z}\{A\, J_n(z) + B\, Y_n(z)\}$.

129. Prove that $J_{n+1}(z)\, Y_n(z) - J_n(z)\, Y_{n+1}(z) = 1/z$.

130. Show that the general solution of $V'' + z^{m-2}V = 0$ is

$$V = \sqrt{z}\left\{A\, J_{1/m}\left(\frac{2}{m}z^{m/2}\right) + B\, Y_{1/m}\left(\frac{2}{m}z^{m/2}\right)\right\}$$

131. (a) Show that the general solution to Bessel's equation $z^2 Y'' + z Y' + (z^2 - n^2)Y = 0$ is

$$Y = A\, J_n(z) + B\, J_n(z)\int \frac{dz}{z\, J_n^2(z)}$$

(b) Reconcile this result with that of equation (24), Page 272.

LEGENDRE FUNCTIONS

132. Obtain the Legendre polynomials (a) $P_3(z)$, (b) $P_4(z)$, (c) $P_5(z)$.

Ans. (a) $\frac{1}{2}(5z^3 - 3z)$, (b) $\frac{1}{8}(35z^4 - 30z^2 + 3)$, (c) $\frac{1}{8}(63z^5 - 70z^3 + 15z)$

133. Prove (a) $P'_{n+1}(z) - P'_{n-1}(z) = (2n+1)\, P_n(z)$, (b) $(n+1)\, P_n(z) = P'_{n+1}(z) - z\, P'_n(z)$.

134. Prove that $nP'_{n+1}(z) - (2n+1)zP'_n(z) + (n+1)P'_{n-1}(z) = 0$.

135. Prove that (a) $P_n(-1) = (-1)^n$, (b) $P_{2n+1}(0) = 0$.

136. Prove that $P_{2n}(0) = \dfrac{(-1)^n}{n!}\left(\dfrac{1}{2}\right)\left(\dfrac{3}{2}\right)\left(\dfrac{5}{2}\right)\cdots\left(\dfrac{2n-1}{2}\right) = (-1)^n\dfrac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}$.

137. Verify Property 2, Page 272.

138. If $[n/2]$ denotes the greatest integer $\leqq n/2$, show that

$$P_n(z) = \sum_{k=0}^{[n/2]} \frac{(-1)^k\,(2n-2k)!}{2^n\, k!\,(n-k)!\,(n-2k)!}\, z^{n-2k}$$

139. Prove that the general solution of Legendre's equation $(1 - z^2)Y'' - 2zY' + n(n+1)Y = 0$ for $n = 0, 1, 2, 3, \ldots$ is

$$Y = A\, P_n(z) + B\, Q_n(z) \qquad \text{where} \qquad Q_n(z) = P_n(z)\int_z^\infty \frac{dt}{(t^2 - 1)\{P_n(t)\}^2}$$

140. Use Problem 139 to find the general solution of the differential equation $(1 - z^2)Y'' - 2zY' + 2Y = 0$.

Ans. $Y = Az + B\left\{1 + \frac{1}{2}z\ln\left(\dfrac{z-1}{z+1}\right)\right\}$

THE ZETA FUNCTION

141. If $\operatorname{Re}\{z\} > 0$, prove that

$$\zeta(z) = \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \cdots = \frac{1}{\Gamma(z)}\int_0^\infty \frac{t^{z-1}\,dt}{e^t - 1}$$

142. Prove that $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\cdots = \frac{\pi^2}{6}$ where $2, 3, 5, 7, \ldots$ are the series of prime numbers.

143. Prove that the only singularity of $\zeta(z)$ is a simple pole at $z = 1$ whose residue is equal to 1.

144. Use the analytic continuation of $\zeta(z)$ given by equation (*33*), Page 273, to show that (*a*) $\zeta(-1) = -1/12$, (*b*) $\zeta(-3) = 1/120$.

145. Show that if z is replaced by $1 - z$ in equation (*33*), Page 273, the equation remains the same.

THE HYPERGEOMETRIC FUNCTION

146. Prove that: (*a*) $\ln(1+z) = z\,F(1, 1; 2; -z)$

(*b*) $\dfrac{\tan^{-1} z}{z} = F(1/2, 1; 3/2; -z^2)$.

147. Prove that $\cos 2az = F(a, -a; 1/2; \sin^2 z)$.

148. Prove that $\dfrac{d}{dz}F(a, b; c; z) = \dfrac{ab}{c}F(a+1, b+1; c+1; z)$.

149. If $\operatorname{Re}\{c - a - b\} > 0$ and $c \neq 0, -1, -2, \ldots,$ prove that

$$F(a, b; c; 1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}$$

150. Prove the result (*31*), Page 273.

151. Prove that: (*a*) $F(a, b; c; z) = (1-z)^{c-a-b}\,F(c-a, c-b; c; z)$

(*b*) $F(a, b; c; z) = (1-z)^{-a}\,F(a, c-b; c; z/[z-1])$.

152. Show that for $|z-1| < 1$, the equation $z(1-z)Y'' + \{c - (a+b+1)z\}Y' - abY = 0$ has the solution $F(a, b; a+b-c+1; 1-z)$.

ASYMPTOTIC EXPANSIONS AND THE METHOD OF STEEPEST DESCENTS

153. Prove that

$$\int_p^\infty e^{-zt^2}\,dt = \frac{e^{-zp^2}}{2pz}\left\{1 - \frac{1}{2p^2z} + \frac{1\cdot 3}{(2p^2z)^2} - \cdots (-1)^n\frac{1\cdot 3\cdot 5\cdots(2n-1)}{(2p^2z)^n}\right\}$$

$$\cdot\,(-1)^{n+1}\frac{1\cdot 3\cdot 5\cdots(2n+1)}{(2z)^{n+1}}\int_p^\infty \frac{e^{-zt^2}}{t^{2n+2}}\,dt$$

and thus obtain an asymptotic expansion for the integral on the left.

154. Use Problem 153 to verify the result (*48*) on Page 275.

155. Evaluate $50!$. *Ans.* 3.04×10^{64}

156. Show that for large values of n, $\dfrac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} \sim \dfrac{1}{\sqrt{\pi n}}$.

157. Obtain the asymptotic expansions:

(a) $\displaystyle\int_0^\infty \frac{e^{-zt^2}}{1+t^2}\,dt \sim \frac{1}{2}\sqrt{\frac{\pi}{z}}\left\{1 - \frac{1}{2z} + \frac{1\cdot 3}{(2z)^2} - \frac{1\cdot 3\cdot 5}{(2z)^3} + \cdots\right\}$

(b) $\displaystyle\int_0^\infty \frac{e^{-zt}}{1+t}\,dt \sim \frac{1}{z} - \frac{1!}{z^2} + \frac{2!}{z^3} - \frac{3!}{z^4} + \cdots$

158. Verify the asymptotic expansion (49) on Page 275.

159. Use asymptotic series to evaluate $\displaystyle\int_{10}^\infty \frac{e^{-t}}{t}\,dt$. *Ans.* .915, approx.

160. Under suitable conditions on $F(t)$, prove that

$$\int_0^\infty e^{-zt}F(t)\,dt \sim \frac{F(0)}{z} + \frac{F'(0)}{z^2} + \frac{F''(0)}{z^3} + \cdots$$

161. Perform the steps needed in order to go from (4) to (5) of Problem 36.

162. Prove the asymptotic expansion (46), Page 275, for the Bessel function.

163. If $F(z) \sim \displaystyle\sum_{n=0}^\infty \frac{a_n}{z^n}$ and $G(z) \sim \displaystyle\sum_{n=0}^\infty \frac{b_n}{z^n}$, prove that:

(a) $F(z) + G(z) \sim \displaystyle\sum_{n=0}^\infty \frac{a_n + b_n}{z^n}$

(b) $F(z)\,G(z) \sim \displaystyle\sum_{n=0}^\infty \frac{c_n}{z^n}$ where $c_n = \displaystyle\sum_{k=0}^n a_k b_{n-k}$.

164. If $F(z) \sim \displaystyle\sum_{n=2}^\infty \frac{a_n}{z^n}$, prove that $\displaystyle\int_z^\infty F(z)\,dz \sim \sum_{n=2}^\infty \frac{a_n}{(n-1)\,z^{n-1}}$.

165. Show that for large values of z,

$$\int_0^\infty \frac{dt}{(1+t^2)^z} \sim \frac{\sqrt{\pi}}{2}\left\{\frac{1}{z^{1/2}} + \frac{3}{8z^{3/2}} + \frac{25}{128z^{5/2}} + \cdots\right\}$$

ELLIPTIC FUNCTIONS

166. If $0 < k < 1$, prove that

$$K = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}} = \frac{\pi}{2}\left\{1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \cdots\right\}$$

167. Prove: (a) $\operatorname{sn} 2z = \dfrac{2\operatorname{sn} z\operatorname{cn} z\operatorname{dn} z}{1 - k^2\operatorname{sn}^4 z}$, (b) $\operatorname{cn} 2z = \dfrac{1 - 2\operatorname{sn}^2 z + k^2\operatorname{sn}^4 z}{1 - k^2\operatorname{sn}^4 z}$.

168. If $k = \sqrt{3}/2$, show that (a) $\operatorname{sn}(K/2) = \sqrt{2/3}$, (b) $\operatorname{cn}(K/2) = \sqrt{1/3}$, (c) $\operatorname{dn}(K/2) = \sqrt{1/2}$.

169. Prove that $\dfrac{\operatorname{sn} A + \operatorname{sn} B}{\operatorname{cn} A + \operatorname{cn} B} = \operatorname{tn}\tfrac{1}{2}(A+B)\operatorname{dn}\tfrac{1}{2}(A-B)$.

170. Prove that (a) $\operatorname{sn}(4K + 4iK') = 0$, (b) $\operatorname{cn}(4K + 4iK') = 1$, (c) $\operatorname{dn}(4K + 4iK') = 1$.

171. Prove: (a) $\operatorname{sn} z = z - \frac{1}{6}(1+k^2)z^3 + \frac{1}{120}(1 + 14k + k^4)z^5 + \cdots$

(b) $\operatorname{cn} z = 1 - \frac{1}{2}z^2 + \frac{1}{24}(1 + 4k^2)z^4 + \cdots$

(c) $\operatorname{dn} z = 1 - \frac{1}{2}k^2z^2 + \frac{1}{24}k^2(k^2 + 4)z^4 + \cdots$

172. Prove that $\displaystyle\int_1^\infty \frac{dt}{\sqrt{t^4 - 1}} = \frac{1}{\sqrt{2}}K\left(\frac{1}{\sqrt{2}}\right)$.

173. Use contour integration to prove the results of Problem 40 (b) and (c).

174. (a) Show that $\int_0^{\phi} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = \frac{2}{1+k}\int_0^{\phi_1} \frac{d\phi_1}{\sqrt{1-k^2\sin^2\phi_1}}$ where $k_1 = 2\sqrt{k}/(1+k)$ by using *Landen's transformation*, $\tan\phi = (\sin 2\phi_1)/(k+\cos 2\phi_1)$.

(b) If $0<k<1$, prove that $k<k_1<1$.

(c) Show that by successive applications of Landen's transformation a sequence of moduli k_n, $n=1,2,3,\ldots$ is obtained such that $\lim_{n\to\infty} k_n = 1$. Hence show that if $\Phi = \lim_{n\to\infty}\phi_n$,

$$\int_0^{\phi} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = \sqrt{\frac{k_1k_2k_3\cdots}{k}}\ \ln\tan\left(\frac{\pi}{4}+\frac{\Phi}{2}\right)$$

(d) Explain how the result in (c) can be used in the evaluation of elliptic integrals.

175. Is $\operatorname{tn} z = (\operatorname{sn} z)/(\operatorname{cn} z)$ a doubly periodic function? Explain.

176. Derive the addition formulae for (a) $\operatorname{cn}(z_1+z_2)$, (b) $\operatorname{dn}(z_1+z_2)$ given on Page 276.

MISCELLANEOUS PROBLEMS

177. If $|p|<1$, show that $\int_0^{\pi/2} \tan^p\theta\, d\theta = \frac{1}{2}\pi\sec(p\pi/2)$.

178. If $0<n<2$, show that $\int_0^{\infty} \frac{\sin t}{t^n}\,dt = \frac{\pi\csc(n\pi/2)}{2\,\Gamma(n)}$.

179. If $0<n<1$, show that $\int_0^{\infty} \frac{\cos t}{t^n}\,dt = \frac{\pi\sec(n\pi/2)}{2\,\Gamma(n)}$.

180. Prove that the general solution of $(1-z^2)Y''-4zY'+10Y = 0$ is given by

$$Y = A\,F(5/2,-1;\,1/2;\,z^2) + Bz\,F(3,-1/2;\,3/2;\,z^2)$$

181. Show that: (a) $\int_0^{\infty} \sin t^3\,dt = \frac{1}{6}\Gamma(1/3)$

(b) $\int_0^{\infty} \cos t^3\,dt = \frac{\sqrt{3}}{6}\Gamma(1/3)$.

182. (a) Find a solution of $zY''+Y'+zY = 0$ having the form $(\ln z)\left(\sum_{k=0}^{\infty} a_k z^k\right)$, and thus verify the result (*23*) given on Page 272. (b) What is the general solution?

183. Use the method of Problem 182 to find the general solution of $z^2Y''+zY'+(z^2-n^2)Y = 0$. [See equation (*22*), Page 271.]

184. Show that the general solution of $zU''+(2m+1)U'+zU = 0$ is

$$U = z^{-m}\{A\,J_m(z) + B\,Y_m(z)\}$$

185. (a) Prove that $z^{1/2}J_1(2iz^{1/2})$ is a solution of $zU''-U = 0$. (b) What is the general solution?
Ans. (b) $Y = z^{1/2}\{A\,J_1(2iz^{1/2}) + B\,Y_1(2iz^{1/2})\}$

186. Prove that $\{J_0(z)\}^2 + 2\{J_1(z)\}^2 + 2\{J_2(z)\}^2 + \cdots = 1$.

187. Prove that $e^{z\cos\alpha}J_0(z\sin\alpha) = \sum_{n=0}^{\infty} \frac{P_n(\cos\alpha)}{n!}z^n$.

188. Prove that $\Gamma'(\frac{1}{2}) = -\sqrt{\pi}(\gamma + 2\ln 2)$.

189. (a) Show that $\int_z^{\infty} \frac{e^{-t}}{t}\,dt = -\gamma - \ln z + z - \frac{z^2}{2\cdot 2!} + \frac{z^3}{3\cdot 3!} - \cdots$.

(b) Is the result in (a) suitable for finding the value of $\int_{10}^{\infty} \frac{e^{-t}}{t}\,dt$? Explain. [Compare with Problem 159.]

190. If m is a positive integer, show that $F(\frac{1}{2}, -m; \frac{1}{2}-m; 1) = \dfrac{2 \cdot 4 \cdot 6 \cdots 2m}{1 \cdot 3 \cdot 5 \cdots (2m-1)}$.

191. Prove that $(1+z)\left(1-\frac{z}{2}\right)\left(1+\frac{z}{3}\right)\left(1-\frac{z}{4}\right)\cdots = \dfrac{\sqrt{\pi}}{\Gamma\left(\frac{1+z}{2}\right)\Gamma\left(\frac{2-z}{2}\right)}$.

192. Prove that $\displaystyle\int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = \frac{\pi}{2}F(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2)$.

193. The *associated Legendre functions* are defined by

$$P_n^{(m)}(z) = (1-z^2)^{m/2}\frac{d^m}{dz^m}P_n(z)$$

(*a*) Determine $P_3^{(2)}(z)$.

(*b*) Prove that $P_n^{(m)}(z)$ satisfies the differential equation

$$(1-z^2)Y'' - 2zY' + \left\{n(n+1) - \frac{m^2}{1-z^2}\right\}Y = 0$$

(*c*) Prove that $\displaystyle\int_{-1}^{1} P_n^{(m)}(z)\,P_l^{(m)}(z)\,dz = 0$ if $n \neq l$.

This is called the *orthogonality property for the associated Legendre functions.*

Ans. (*a*) $15z(1-z^2)$

194. Prove that if m, n and r are positive constants,

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+r)^{m+n}}\,dx = \frac{B(m,n)}{r^m(1+r)^{m+n}}$$

[*Hint.* Let $x = (r+1)y/(r+y)$.]

195. Prove that if m, n, a and b are positive constants,

$$\int_0^{\pi/2} \frac{\sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}} = \frac{B(m,n)}{2a^n b^m}$$

[*Hint.* Let $x = \sin^2\theta$ in Problem 194 and choose r appropriately.]

196. Prove that: (*a*) $\dfrac{z}{2} = J_1(z) + 3J_3(z) + 5J_5(z) + \cdots$

(*b*) $\dfrac{z^2}{8} = 1^2J_2(z) + 2^2J_4(z) + 3^2J_6(z) + \cdots$

197. If m is a positive integer, prove that:

(*a*) $P_{2m}(z) = \dfrac{(-1)^m(2m)!}{2^{2m}(m!)^2}F(-m, m+\tfrac{1}{2}; \tfrac{1}{2}; z^2)$

(*b*) $P_{2m+1}(z) = \dfrac{(-1)^m(2m+1)!}{2^{2m}(m!)^2}z\,F(-m, m+\tfrac{3}{2}; \tfrac{3}{2}; z^2)$

198. (*a*) Prove that $1/(\operatorname{sn} z)$ has a simple pole at $z=0$ and (*b*) find the residue at this pole. *Ans.* 1

199. Prove that $\{\Gamma(\tfrac{1}{4})\}^2 = 8\sqrt{\pi}\,\dfrac{4\cdot 6\cdot 8\cdot 10\cdot 12\cdot 14\cdot 16\cdot 18\cdots}{5\cdot 5\cdot 9\cdot 9\cdot 13\cdot 13\cdot 17\cdot 17\cdots}$.

200. If $|z|<1$, prove *Euler's identity*: $(1+z)(1+z^2)(1+z^3)\cdots = \dfrac{1}{(1-z)(1-z^3)(1-z^5)\cdots}$.

201. If $|z|<1$, prove that $(1-z)(1-z^2)(1-z^3)\cdots = 1 + \displaystyle\sum_{n=1}^{\infty}(-1)^n\{z^{n(3n-1)/2} + z^{n(3n+1)/2}\}$.

202. (*a*) Prove that $\dfrac{z}{1+z} + \dfrac{z^2}{(1+z)(1+z^2)} + \dfrac{z^4}{(1+z)(1+z^2)(1+z^4)} + \cdots$ converges for $|z|<1$ and $|z|>1$.

(*b*) Show that in each region the series represents an analytic function, say $F_1(z)$ and $F_2(z)$ respectively.

(*c*) Are $F_1(z)$ and $F_2(z)$ analytic continuations of each other? Is $F_1(z) = F_2(z)$ identically? Justify your answers.

203. (a) Show that the series $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges at all points of the region $|z| \leq 1$.

(b) Show that the function represented by all analytic continuations of the series in (a) has a singularity at $z = 1$ and reconcile this with the result in (a).

204. Let $\Sigma a_n z^n$ have a finite circle of convergence C and let $F(z)$ be the function represented by all analytic continuations of this series. Prove that $F(z)$ has at least one singularity on C.

205. Prove that $\dfrac{\text{cn } 2z + \text{dn } 2z}{1 + \text{cn } 2z} = \text{dn}^2 z.$

206. Prove that a function which is not identically constant cannot have two periods whose ratio is a real irrational number.

207. Prove that a function, not identically constant, cannot have three or more independent periods.

208. (a) If a doubly-periodic function is analytic everywhere in a cell [period parallelogram], prove that it must be a constant. (b) Deduce that a doubly-periodic function, not identically constant, has at least one singularity in a cell.

209. Let $F(z)$ be a doubly-periodic function. (a) Prove that if C is the boundary of its period parallelogram, then $\oint_C F(z)\,dz = 0$. (b) Prove that the number of poles inside a period parallelogram equals the number of zeros, due attention being paid to their multiplicities.

210. Prove that the Jacobian elliptic functions sn z, cn z and dn z (a) have exactly two zeros and two poles in each cell and that (b) each function assumes any given value exactly twice in each cell.

211. Prove that $$\left(1+\frac{1}{1^2}\right)\left(1+\frac{1}{4^2}\right)\left(1+\frac{1}{7^2}\right)\cdots = \frac{\{\Gamma(1/3)\}^2}{\left\{\Gamma\left(\frac{1+i}{3}\right)\right\}^2\left\{\Gamma\left(\frac{1-i}{3}\right)\right\}^2}.$$

212. Prove that $$\int_0^{\pi/2} e^{-z\tan\theta}\,d\theta \sim \frac{1}{z} - \frac{2!}{z^3} + \frac{4!}{z^5} - \frac{6!}{z^7} + \cdots$$

213. Prove that $$P_n(\cos\theta) = 2\left\{\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\right\}\left\{\cos n\theta + \frac{1\cdot 2n}{2\cdot(2n-1)}\cos(n-2)\theta + \frac{1\cdot 3\cdot 2n(2n-2)}{2\cdot 4\cdot(2n-1)(2n-3)}\cos(n-4)\theta + \cdots\right\}$$

[*Hint.* $1 - 2t\cos\theta + t^2 = (1-te^{i\theta})(1-te^{-i\theta})$.]

214. (a) Prove that $\Gamma(z)$ is a meromorphic function and (b) determine the principal part at each of its poles.

215. If $\text{Re}\{n\} > -1/2$, prove that
$$J_n(z) = \frac{z^n}{2^n\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\int_{-1}^{1} e^{izt}(1-t^2)^{n-1/2}\,dt$$
$$= \frac{z^n}{2^n\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\int_0^{\pi} \cos(z\cos\theta)\sin^{2n}\theta\,d\theta$$

216. Prove that $$\int_0^{\infty} t^n J_m(t)\,dt = \frac{2^n\,\Gamma\left(\frac{m+n+1}{2}\right)}{\Gamma\left(\frac{m-n+1}{2}\right)}.$$

217. Prove that $$\int_0^{\pi/2} \cos^p\theta\cos q\theta\,d\theta = \frac{\pi\,\Gamma(p+1)}{2^{p+1}\,\Gamma\left(\frac{2+p+q}{2}\right)\Gamma\left(\frac{2+p-q}{2}\right)}.$$

218. Prove that $$\{\Gamma(\tfrac{1}{4})\}^2 = 4\sqrt{\pi}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}.$$