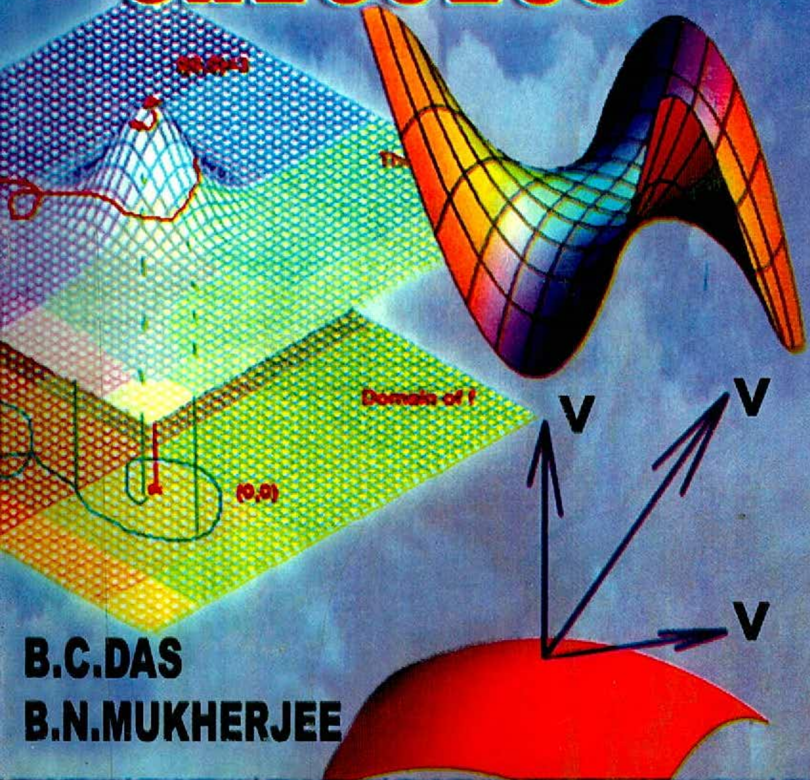
Revised 51st Edition

$\dfrac{dy}{dx} = \dfrac{\text{DIFFERENTIAL}}{\text{CALCULUS}}$



B.C.DAS
B.N.MUKHERJEE

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Revised 51st Edition

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DIFFERENTIAL CALCULUS

B. C. DAS, M.Sc.

PROFESSOR OF MATHEMATICS (RETD.),
PRESIDENCY COLLEGE, CALCUTTA;
EX-LECTURER IN APPLIED MATHEMATICS,
CALCUTTA UNIVERSITY

B. N. MUKHERJEE, M.A.

Premchand Roychand Scholar
PROFESSOR OF MATHEMATICS (RETD.),
SCOTTISH CHURCH COLLEGE, CALCUTTA

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PREFACE TO THE 51st EDITION

This edition of Differential Calculus, is almost the reprint of the previous one with a few rectifications of the printing mistakes.

Some rearrangements and additions-alterations also been done in a few chapters to upgrade and comply with the recent trends of paper setting in different examinations.

We are thankful to Dr. Dinabandhu Chatterjee, M.Sc., D.Phil, of Dinabandhu College, Howrah (Retd.) for revision of this edition.

Our thanks are due to Dr. Purnendu Dhar, M.Sc., Ph. D., for bringing out this edition in time.

Any suggestions for the improvement of this book will be received, with thanks.

Kolkata,
January, 2009 *Copyright Holders.*

PREFACE TO THE 50th EDITION

This edition of Differential Calculus, is thoroughly revised and updated to comply with the latest syllabus of the Indian Universities.

A few chapters like Number System, Function, Maxima and Minima, have been modified.

An attempt has been taken to create interest about the basic need to study Calculus by demonstrating and explaining A Few Real-Life Events or features by help of Calculus.

A lot of interesting Miscellaneous Worked out Examples have been incorporated to most of the chapters for the benefit of the students.

We are thankful to Dr. Dinabandhu Chatterjee, M.Sc., D.Phil, of Dinabandhu College, Howrah (Retd.) or revision of this edition.

We convey thanks to Sri. Santanu Kr. Ganguli, F.C.A. for introducing some new ideas.

We convey thanks to Dr. Purnendu Dhar, M.Sc., Ph. D., for publishing this revised edition in time.

Any suggestions for the improvement of this book will be received, with thanks.

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June, 2006 *Copyright Holders.*

PREFACE TO THE FIRST EDITION

This book is prepared with a view to be used as a textbook for the B. A. and B. Sc. students of the Indian Universities. We have tried to make the expositions of the fundamental principles clear as well as concise without going into unnecessary details; and at the same time an attempt has been made to make the treatment as much rigorous and up-to-date as possible within the scope of the elementary work.

Proofs of certain important fundamental results and theorems which are assumed in the earlier chapters of the book for the convenience of the beginners and average students have been given in the Appendix. A brief account of the theory of infinite series, especially the power series, is given in the Appendix in order to emphasise their peculiarity in the application of the principles of Calculus. Important formulae of this book are given in the beginning for ready reference. A good number of typical examples have been worked out by way of illustrations.

Many varied types of examples have been given for exercise in order that the students might acquire a good grasp of the applications of the principles of Calculus. University questions of recent years have been added at the end to give the students and idea of the standard of the examination. Our thanks are due to the authorities and the staff of the K. P. Basu Printing Works, Calcutta, who, in spite of their various preoccupations, had the kindness to complete the printing so efficiently in a short period of time. Any criticism, correction and suggestion towards the improvement of the book from teachers and students will be thankfully received.

CALCUTTA
June, 1949

B. C. D
B. N. M

LATEST SYLLABUS FOR
DIFFERENTIAL CALCULUS (GENERAL)

Rational numbers. Geometrical representations. Irrational numbers. Real number represented as point on a line - Linear Continuum. Acquaintance with basic properties of real number (No deduction or proof included).

Sequence : Definition of bounds of a sequence and monotone sequence. Limit of a sequence. Statements of limit theorems. Concept of convergence and divergence of monotone sequences - applications of the theorems, in particular, definition of e.

Statements of Cauchy's general principle of convergence and its application.

Infinite series of constant terms : Convergence and Divergence (definitions). Cauchy's principle as applied to infinite series (application only). Series of positive terms : Statements of Comparison test, D.Alembert's Ratio test, Cauchy's nth root test and Raabe's test - Applications. Alternating series: Statements of Leibnitz's test and its application.

Real-valued functions defined on an interval : Limit of a function (Cauchy's definition). Algebra of limits. Continuity of a function at a point and in an interval.

Acquiantance (no proof) with the important properties of continuous functions on closed intervals. Statement of existence of inverse function of a strictly monotone function and its continuity.

Derivative - its geometrical and physical interpretation. Sign of derivative-Monotonic increasing and decreasing functions. Relation between continuity and derivability. Defferential - application in finding approximation.

Successive derivative- Leibnitz's Theorem and its application.

Statement of Rolle's Theorem and its geometrical interpretation. Mean value Theorems of Lagrange's and Cauchy. Statement of Taylor's and Maclaurin's Theorems with Lagrange's and Cauchy's form of remainders. Taylor's and Maclaurin's Infinite series for functions like

e^x, $\sin x, \cos x$, $(1+x)^n$, $\log(1+x)$.[with restrictions wherever necessary]

Indeterminate Forms : L'Hospital's Rule : Statement and problems only.

Application of the principle of Maxima and Minima for a function of single variable in geometrical, physical and other problems.

Functions of two and three variables : Their geometrical representations. Limit and Continuity (definitions only) for functions of two variables. Partial derivatives : Knowledge and use of Chain Rule. Exact differentials (emphasis on solving problems only). Functions of two variables - successive partial derivatives : Statement of Schwarz's Theorem on Commutative property of mixed derivatives. Euler's Theorem on homogeneous function of two and three variables. Maxima and minima of functions of not more than three variables - Lagrange's Method of undetermined multiplier - Problems only. Implicit function in case of function of two variables (existence assumed) and derivative.

Applications of Differential Calculus : Tangents and Normals, Pedal equation and Pedal of a Curve. Rectilinear Asymptotes (Cartesian only), Curvature of plane curves. Envelope of family of straight lines and of curves (Problems only).

Definitions and examples of singular points (viz. Node, Cusp, Isolated point).

University of Calcutta
Senate House
Kolkata 700 073

CONTENTS

CHAPTERS

GREEK ALPHABETS USED IN THE BOOK

α (alpha)	ρ (rho)
β (beta)	σ (sigma)
γ (gamma)	τ (tau)
δ (delta)	ξ (xi)
ϕ (phi)	η (eta)
ψ (psi)	ζ (zeta)
κ (kappa)	ε (epsilon)
λ (lambda)	θ (theta)
μ (mu)	Δ (cap. delta)
ν (nu)	Σ (cap. sigma)
π (pi)	Γ (cap. gamma)

I A Few Real-Life Events Explained By Calculus

Calculus, invented from the visions of master minds, took little time to break through the barrier of theoretical enquiry to practical fields of human activities. The apparently unrelated areas of application of this pattern of the novel methods of calculus enabled it to determine the dimensions of a container in the shelves of the grocer to the angle of elevation of the rainbow in the sky, the rate of drying of a moist piece of cloth kept hung in the air, or to predict the laws governing the motion of planets in the heavens far from our planet.

Let us make a voyage to a few of the areas where calculus reigns.

I. ***To Assess the Dimensions of a can to minimise the cost of material.***

A cylindrical can is to be made to hold 1 litre of oil. To find the dimensions that will minimise of the metal to manufacture the can.

Let us consider a cylindrical can of radius r and height h. The volume of the can is

$$\pi r^2 h = 1000, \text{ for } 1 \text{ litre} = 1000 \text{ c.c}$$

$$\text{or, } h = \frac{1000}{\pi r^2} \qquad \dots \quad \text{(i)}$$

Now, total metal sheet required to make the can

= total surface area of the can (A)

$= 2\pi r^2 + 2\pi r h$

$= 2\pi r^2 + 2\pi r \times \left(\dfrac{1000}{\pi r^2}\right)$

using (i)

$= 2\pi r^2 + \dfrac{2000}{r}$

Area $2(\pi r^2)$ Area $(2\pi r)h$

Now, we are to minimise A.

Calculus teaches that for a maximum or minimum of A, we have $\dfrac{dA}{dr} = 0$

i.e., $= 4\pi r + \dfrac{2000}{r^2} = 0$ or, $r^3 = \dfrac{500}{\pi}$.

Now, $\dfrac{d^2 A}{dr^2} = 4\pi + \dfrac{4000}{r^3} > 0$

indicating that A is minimum when $r^3 = \dfrac{500}{\pi}$ i.e., $r = \sqrt[3]{\dfrac{500}{\pi}}$

and the value of h corresponding to this value of r is given by

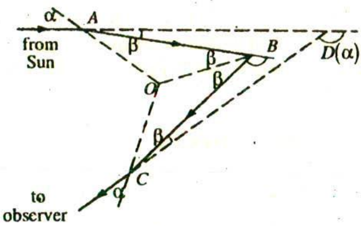
$$h = \frac{1000}{\pi r^2} = 2\sqrt[3]{\frac{500}{\pi}} \approx 2r.$$

Now you have got the logic behind the fact that cylindricaal containers are made so that their heights are approximately double the radius.

II. *The calculus of Rainbow*

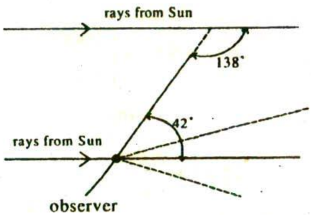
Rainbow in the sky have fascinated mankind from ancient times. Formation of rainbow is a beautiful phenomenon of nature caused by scattering of sunlight by waterdrops floating in air.

The formation of a rainbow can be explained in the following way.



Formation of the primary rainbow

A ray of light meets the spherical surface of the raindrop at A, after refraction it takes up the path along AB inside the drop, at B the ray suffers total internal refraction to move along BC. At C it is refracted again to emerge out of the raindrop.



If α be the angle of incidence of the ray of light at A, then $D(\alpha)$, the amount of clockwise deviation that the ray has undergone during this three-stage process.

Thus, $D(\alpha) = (\alpha - \beta) + (\pi - 2\beta) + (\alpha - \beta) = \pi + 2\alpha - 4\beta.$

A little calculus will show that D is minimum when $\alpha = 59 \cdot 4^\circ$ and $D_{min} = 138^\circ$.

The above figure shows that the angle of elevation from the observer up to the highest point of the rainbow is $(180^\circ - 138^\circ) = 42^\circ$.

III. *Problem of moist piece of cloth.*

A moist piece of cloth in the open air loses its moisture at a rate proportional to the moisture content. If a sheet hung in the wind loses half its moisture during the first hour, when will it lose 99% of moisture, weather conditions remaining unchanged ?

To solve this problem found in our everyday life, let m_0 be the initial moisture content and m be the value of m_0 after t hours.

These facts can be put in the language of Calculus as

$\frac{dm}{dt} = km$ where $k\ (< 0)$ is constant of proportionality.

This, being *a differential equation,* has its solution $m = m_0 e^{kt}$.

Given that $m = \frac{1}{2} m_0$, when $t = 1$.

then $\frac{1}{2} m_0 = m_0 e^{K} \Rightarrow e^{K} = \frac{1}{2}, k = -\log_e^2$.

Now, if the cloth loses 99% of moisture after T hours

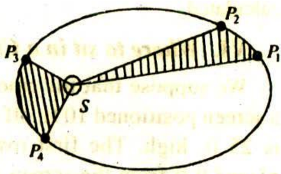
$$\frac{1}{100} m_0 = m_0 e^{KT}$$

$$\Rightarrow e^{KT} = \frac{1}{100} \Rightarrow T = \frac{\log_e 100}{\log_e 2} \text{ hrs.} = 6 \text{ hrs. } 39 \text{ mins.}$$

The techinque of this problem can be used to the population growth of a country, or, the decay of a radioactive substance, the growth of bacteria in a medium and in many other investigations.

IV. *Calculus in planetary motion.*

Kepler formulated three laws of planetary motion, for they fitted the astronomical data. Now you can see how calculus can help to deduce the laws.



Since a planet during its motion round the sun, is acted on by an attraction towards the sun, the force or acceleration along the tangent to the path is zero, which can be represented by an equation of the form $\frac{1}{r} \cdot \frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right) = 0$ where r is the distance of the planet from the sun and θ is the angle traced out by the planet in time t. Integrating the above relation, we have

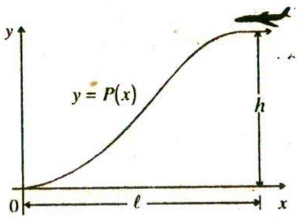
$$r^2 \frac{d\theta}{dt} = \text{constant.}$$

But $\frac{1}{2}r^2\frac{d\theta}{dt}$ is the rate at which the radius vector r sweeps out area around the sun.

Thus we find that the rate of description of sectiorial area by the radius joining sun to the planet is constant, which proves *Kepler's Second. Law* of planetary motion which states that the radius-vector joining any planet to the sun sweeps out equal areas in equal intervals of time.

The other two laws can be similarly deduced.

V. *Where should a pilot of an aircraft start descent*



An approach path for an aircraft, as shown in the figure, satisfies the following conditions :

(i) The cruising altitude is h when the descent starts at a horizontal distance l from the touchdown at the origin.

(ii) The pilot maintains an uniform horizontal velocity v through out.

(iii) The absolute value of the vertical acceleration must not exceed a constant k, which is less than the acceleration due to gravity.

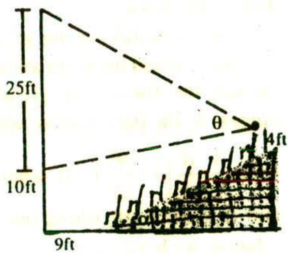
If the altitude y and the horizontal distance from the origin be x, then we may assume

$$y = ax^3 + bx^2 + cx + d.$$

The conditions (i), (ii), (iii) give $\frac{6hv^2}{l^2} = k$.

For a given k, v and h, the path and the horizontal distance l can be calculated.

VI. *Where to sit in a Cinema Hall.*



We suppose that the cinema Hall has a screen positioned 10 ft. off the floor and is 25 ft. high. The first row of seats is placed 9 ft from the screen and the rows are 3 ft. apart. The floor of the seating area is inclined at angle $\alpha = 20°$ above the horizontal and the distance up the incline where an observer sits is x. The cinema hall has 21 rows of seats, so $0 \le x \le 60$.

An observer decides that the best place to sit is in the row where the angle θ, subtended by the screen at the observer's eye is a maximum.

Now, let us translate the problem in mathematical language. Obviously, this is a maximizing problem, and we are to find the value of x which makes θ a maximum.

Now, $2ab\cos\theta = a^2 + b^2 - (25)^2$

i.e., $\cos\theta = \dfrac{a^2 + b^2 - 625}{2ab}$

Where $a^2 = (9 + x\cos\alpha)^2 + (31 - x\sin\alpha)^2$

and $b^2 = (9 + x\cos\alpha)^2 + (x\sin\alpha - 6)^2$.

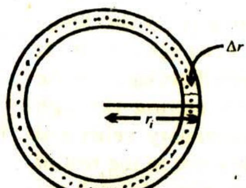
Here numerical computations are complicated, but $\dfrac{d\theta}{dx} = 0$ will give the required value of x.

VII. *Flow of Blood through Veins and arteries.*

The laminar flow of a fluid is govermed by the law :

$$v(r) = \frac{P}{4\eta l}(R^2 - r^2)$$

which gives the velocity v of a fluid that flows along a tube of radius R and length l, at a distance r from the central axis; and P is the pressure difference between the ends of the tube η being the viscosity of the fluid.



Now we use this law to calculate the rate of flow or *flux* (i.e. volume per unit time) *of blood within blood vessels* (i.e. veins or arteries). Let us consider smaller, equally spaced radii $r_1, r_2, \cdots$. We see that the approximate area of the ring with inner radius r_{i-1} and outer radius r_i is

$$2\pi r_i \Delta r, \text{ where } \Delta r = r_i - r_{i-1}$$

When Δr is small enough, v can be assumed to be constant having value $v(r_i)$.

So the volume of blood that flows per unit time can be approximated by the expression

$$(2\pi r_i \Delta r)v(r_i) = 2\pi r_i v(r_i)\Delta r.$$

Thus the total volume of blood flowing across a Cross-section per unit time is

$$\sum_{i=1}^{n} 2\pi r_i v(r_i)\Delta r.$$

Using the concept of Definite Integral as the limit of a sum, the volume of blood that passes a cross-section of a blood vessel per unit time is given by

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi r_i v(r_i)\Delta r$$

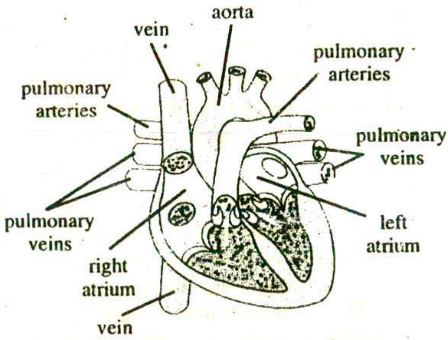
$$\doteq \int_0^R 2\pi r v(r) dr$$

$$= \int_0^R 2\pi r \cdot \frac{P}{4\eta l}\left(R^2 - r^2\right) dr$$

i.e., $F = \dfrac{\pi P R^4}{8\eta l}$.

VIII. *Cardiac Out Put*

The figure below shows the human cardiovascular system. Blood returns from the body through the veins, enters the right atrium of the heart, and is pumped to the lungs for oxygenation. It then flows back into the left atrium through pulmonary veins and then out to the rest of the body through the aorta. The **Cardiac output** of the heart is measured by the volume of blood pumped by the heart per unit time, i.e., the rate of flow of blood into the aorta.



The *dye dilution method* is used to measure the cardiac out put. Dye is injected into the right atrium and flows through the heart into the aorta. A probe inserted into the aorta measures the concentration of dye leaving the heart at equally spaced times over a time interval $[0, T]$ until the dye has cleared. Let $c(t)$ be the concentration of the dye at time t. If the interval $[0, T]$ is divided into a large number of subintervals of equal length Δt, then the amount of dye that flows past the measuring point during the subinterval $t = t_{i-1}$ to $t = t_i$ is approximately concentration $\times$ volume $= c(t_i) \times (F\Delta t)$,

where F is the rate of flow that we are trying to determine.

Thus the total amount of dye $= \sum_{i=1}^{n} c(t_i)F\Delta t$.

Making $n \to \infty$, the amount of dye A is given by

$$A = F\sum_{i=1}^{n} c(t_i)\Delta t = F\int_0^T c(t)dt.$$

so that $F = \dfrac{A}{\int_0^T c(t)dt}$.

Thus the problem reduces to the evaluation of a definite integral.

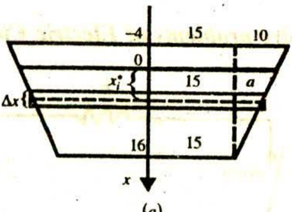
As concrete example, if $A = 5$, $\Delta t = 1$, $T = 10$.

$F = \dfrac{5}{\int_0^1 c(t)dt} = \dfrac{5}{41.87}$, by *numerical integration using Simpson's Rule.*

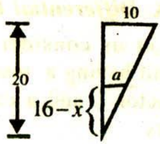
$= 0.12$ litre/Second.

IX. *Application to the Technology of DAM.*

A dam has the shape of a trapezoid as shown in the figure below. The height of the dam is 20 metre and width is 50 m at the top and 30 m at the bottom. We are to determine the force on the dam due to hydrostatic pressure when the water level is 4-m from the top of the dam.



(a)



(b)

We choose a vertical x-axis with origin at the surface of water as shown in the figure

The depth of water is 16m, so we divide the interval [0, 16] into subintervals of equal lengths with end points x_i and $\bar{x} \in [x_{i-1}, x_i]$. The ith horizontal strip of the dam is approximately a rectangle with height Δx and width w_i. Now from similar triangles [See fig. (b) above]

$$\frac{a}{16-\bar{x}_i} = \frac{10}{20} \text{ i.e., } a = 8 - \frac{\bar{x}_i}{2}$$

and so $w_i = 2(15+a) = 46 - \bar{x}_i$

If A_i be the area of the ith strip, then

$A_i = w_i \Delta x = (46 - \bar{x}_i)\Delta x$.

If Δx is small, then the pressure P_i on the ith strip is almost constant, and

$P_i = \rho g d = 1000 g \bar{x}_i$.

The force F_i due to hydrostatic pressure is given by

$F_i = Force \times area = P_i A_i = 1000 g \bar{x}_i (46 - \bar{x}_i)\Delta x$.

Adding the forces and taking the limit as $n \to \infty$.

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} 1000 g \bar{x}_i (46 - \bar{x}_i)\Delta x$$

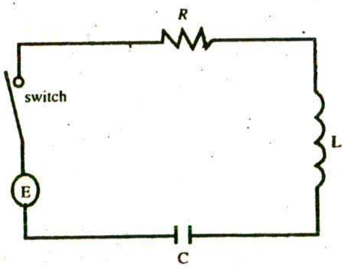
$$= \int_0^{16} 1000 g x (46 - x) dx$$

$$= 1000 \times 9.8 \left[23x^2 - \frac{x^3}{3} \right]_0^{16}$$

$= 4.43 \times 10^7$ Newton.

X. *Differential Equation to solve problems of Electric Circuit.*

Let us consider an electric circuit having a resistance R, an inductor L and a capacitor c is series.



A battery or generator supplies an electro-motive force E.

If the charge on the capacitor at time t be $Q(t)$, then current $= I = \frac{dQ}{dt}$ and the drop of voltage across the resistance, inductor and capacitor are RI, $L\frac{dI}{dt}$ and $\frac{Q}{c}$ respectively.

By Kirchoff's law

$$L\frac{dI}{dt} + RI + \frac{Q}{c} = E(t).$$

Since $I = \frac{dQ}{dt}$, the above equation becomes

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{c}Q = E(t) \qquad \dots \qquad (1)$$

which is a second order linear differential equation with constant coefficients.

Let us move to particular case, where $R = 40\,\Omega$,

$L = 1H, C = 16\times 10^{-4}\,F$ and $E(t) = 100\cos(10t)$ and initial charge and current are both zero.

Then equation (1) becomes

$$\frac{d^2Q}{dt^2} + 40\frac{dQ}{dt} + 625Q = 100\cos(10t).$$

The auxiliarly equation is $m^2 + 40m + 625 = 0$

which gives $m = -20 \pm 15i$.

And the general solution of (1) turns up to be

$$Q(t) = e^{-20t}\left(A\cos 15t + B\sin 15t\right) + \frac{4}{697}\left\{21\cos(10t) + 16\sin(10t)\right\}.$$

The initial conditions $Q = 0, \frac{dQ}{dt} = 0$ at $t = 0$ gives

$$A = -\frac{84}{697}, B = -\frac{464}{2091}$$

So,

$$Q(t) = \frac{4}{697}\left[\frac{e^{-20t}}{3}\left\{-63\cos(15t) - 116\sin(15t) + 21\cos(10t) + 16\sin(10t)\right\}\right]$$

and

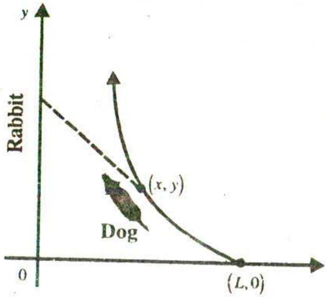
$$I = \frac{dQ}{dt} = \frac{1}{2091}\left[\frac{e^{-20t}}{3}\left\{-1920\cos(15t) + 13060\sin(15t)\right\}\right.$$
$$\left. + 120\left\{16\cos(10t) - 21\sin(10t)\right\}\right]$$

For large values of t, $e^{-20t} \to 0$.

So, $Q(t) = \frac{4}{697}\left\{21\cos(10t) + 16\sin(10t)\right\}$.

XI. *When a dog chases a rabbit.*

A dog sees a rabbit running in a straight line across an open field and gives chase. In a rectangular co-ordinate system (as shown in the figure below), we assume :



(i) The rabbit is at the origin and the dog is at the point $(L, 0)$ at the instant the dog first sees the rabbit.

(ii) The rabbit runs up the y-axis and the dog always runs straight for the rabbit.

(iii) The dog has the same speed as the rabbit.

If the dog moves along the path whose equation is $y = f(x)$, then it can be shown that y satisfies the differential equation

$$x\frac{d^2y}{dx^2} = \left(1+\frac{dy}{dx}\right)^2 \qquad \dots \qquad \dots \qquad (1)$$

With the initial conditions $y = 0, \frac{dy}{dx} = 0$ when $x = 0$

$$y = \frac{x^2 - L^2}{4L} - \frac{1}{2}L\log\left(\frac{x}{L}\right).$$

It will be interesting to find that the dog will never be able to catch the rabbit.

II LIST OF IMPORTANT FORMULÆ

I. Important Limits.

(i) $\underset{x \to 0}{Lt} \dfrac{\sin x}{x} = 1$, where x is in radian measure.

(ii) $\underset{n \to \pm\infty}{Lt} \left(1 + \dfrac{1}{n}\right)^n = e$ or, $\underset{x \to 0}{Lt} (1 + x)^{1/x} = e$.

(iii) $\underset{x \to 0}{Lt} \dfrac{1}{x} \log(1 + x) = 1$.

(iv) $\underset{x \to 0}{Lt} \dfrac{e^x - 1}{x} = 1$

(v) $\underset{x \to 0}{Lt} \dfrac{(1 + x)^n - 1}{x} = n$

(vi) $\underset{x \to \infty}{Lt} \dfrac{x^n}{n!} = 0$.

(vi) $\underset{n \to \infty}{Lt} x^n = 0 \ (-1 < x < 1)$.

II. Standard Derivatives.

$\dfrac{d}{dx}(x^n) = nx^{n-1}$;

$\dfrac{d}{dx}\left(\dfrac{1}{x^n}\right) = -\dfrac{n}{x^{n+1}}$.

$\dfrac{d}{dx}(x) = 1$;

$\dfrac{d}{dx}(\sqrt{x}) = \dfrac{1}{2\sqrt{x}}$.

$\dfrac{d}{dx}(e^x) = e^x$;

$\dfrac{d}{dx}(a^x) = a^x \log_e a$.

$\dfrac{d}{dx}(\log x) = \dfrac{1}{x}$;

$\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x} \log_a e$.

$\dfrac{d}{dx}(\sin x) = \cos x$;

$\dfrac{d}{dx}(\cos x) = -\sin x$.

$\dfrac{d}{dx}(\tan x) = \sec^2 x$;

$\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$.

$\dfrac{d}{dx}(\sec x) = \sec x \tan x$;

$\dfrac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

$\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1 - x^2}}$;

$\dfrac{d}{dx}(\cos^{-1} x) = -\dfrac{1}{\sqrt{1 - x^2}}$.

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}; \qquad \frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}.$$

$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}; \qquad \frac{d}{dx}\left(\operatorname{cosec}^{-1} x\right) = -\frac{1}{x\sqrt{x^2-1}}.$$

$$\frac{d}{dx}\left(\sinh x\right) = \cosh x; \qquad \frac{d}{dx}\left(\cosh x\right) = \sinh x.$$

$$\frac{d}{dx}\left(\tanh x\right) = \operatorname{sech}^2 x; \qquad \frac{d}{dx}\left(\coth x\right) = -\operatorname{cosech}^2 x.$$

$$\frac{d}{dx}\left(\operatorname{sech} x\right) = -\operatorname{sech} x \tanh x; \frac{d}{dx}\left(\operatorname{cosech} x\right) = -\operatorname{cosech} x \coth x.$$

$$\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{1+x^2}}; \qquad \frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}} \ (x > 1)$$

$$\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2} \ (x < 1); \ \frac{d}{dx}\left(\coth^{-1} x\right) = -\frac{1}{x^2-1} \ (x > 1)$$

$$\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = -\frac{1}{x\sqrt{1-x^2}} \ (x \le 1);$$

$$\frac{d}{dx}\left(\operatorname{cosech}^{-1} x\right) = -\frac{1}{x\sqrt{x^2+1}}$$

III. Fundamental theorems on Differentiation.

(i) $\frac{d}{dx}(c) = 0.$ (ii) $\frac{d}{dx}\{c\phi(x)\} = c\phi'(x).$

(iii) $\frac{d}{dx}\{\phi(x) \pm \psi(x)\} = \phi'(x) \pm \psi'(x).$

(iv) $\frac{d}{dx}\{\phi(x) \times \psi(x)\} = \phi(x)\psi'(x) \pm \psi(x)\phi'(x).$

Derivative of the product of two functions
=first function × derivative of the second
+second function × derivative of the first.

(v) $\frac{d}{dx}\{\phi_1(x)\phi_2(x)\ldots\phi_n(x)\}$

$$= \phi_1'(x)\{\phi_2(x)\phi_3(x)\ldots\} + \phi_2'(x)\{\phi_1(x)\phi_3(x)\ldots\} + \ldots + \phi_n'(x)\{\phi_1(x)\phi_2(x)\ldots\}$$

(vi) $\frac{d}{dx}\left\{\frac{\phi(x)}{\psi(x)}\right\}=\frac{\phi'(x)\psi(x)-\psi'(x)\phi(x)}{\{\psi(x)\}^2}$, $\psi(x)\neq 0$.

Derivative of the quotient of two functions

$$=\frac{(\text{Deriv. of Num.})\times\text{Denom.}-(\text{Deriv. of Denom.})\times\text{Num.}}{(\text{Denom.})^2}.$$

(vii) If $y = f(v)$, $v = \phi(x)$,

$$\frac{dy}{dx}=\frac{dy}{dv}\cdot\frac{dv}{dx}.$$

(viii) $\frac{dy}{dx}\times\frac{dx}{dy}=1$, *i.e.*, $\frac{dy}{dx}=1\Big/\frac{dx}{dy}\left(\frac{dx}{dy}\text{ and }\frac{dy}{dx}\neq 0\right)$.

IV. Meaning of the derivatives and differential.

(i) $\frac{dy}{dx}=\tan\psi$, where ψ is the angle which the tangent at any point to the curve $y = f(x)$ makes with the positive direction of the x-axis.

$\frac{dy}{dx}$ = rate of change of y with respect to x.

$dy = f'(x)\,dx$, if $y = f(x)$.

V. The nth derivatives of some special functions.

$D^n(x^n) = n!$.

$D^n(x^m)=\frac{m!}{(m-n)!}\cdot x^{m-n}$ (m being a positive integer $> n$).

$D^n(e^{ax}) = a^n e^{ax}$; $D^n(e^x) = e^x$.

$D^n\left(\frac{1}{x+a}\right)=\frac{(-1)^n n!}{(x+a)^{n+1}}$; $\{D^n\log(x+a)\}=\frac{(-1)^{n-1}(n-1)!}{(x+a)^n}$.

$D^n\{\sin(ax+b)\}=a^n\sin\left(\frac{n\pi}{2}+ax+b\right)$.

$D^n\{\cos(ax+b)\}=a^n\cos\left(\frac{n\pi}{2}+ax+b\right)$.

$D^n(\sin ax)=a^n\sin\left(\frac{n\pi}{2}+ax\right)$; $D^n(\cos ax)=a^n\cos\left(\frac{n\pi}{2}+ax\right)$.

$$D^n \left(e^{ax} \sin bx\right) = \left(a^2 + b^2\right)^{n/2} e^{ax} \sin\left(bx + n \tan^{-1} b/a\right).$$

$$D^n \left(e^{ax} \cos bx\right) = \left(a^2 + b^2\right)^{n/2} e^{ax} \cos\left(bx + n \tan^{-1} b/a\right).$$

$$D^n \left(\frac{1}{x^2 + a^2}\right) = \frac{(-1)^n n!}{a^{n+2}} \sin^{n+1}\theta \sin(n+1)\theta,$$

where $\theta = \tan^{-1}(a/x)$.

$$D^n \left(\tan^{-1} x\right) = (-1)^{n-1}(n-1)!\sin^n \theta \sin n\theta, \text{ where } \theta = \cot^{-1} x.$$

Leibnitz's Theorem

$$(uv)_n = u_n v + {}^n c_1 u_{n-1} v_1 + {}^n c_2 u_{n-2} v_2 + \ldots + u v_n.$$

VI. **(i) Mean Value Theorem.**

$$f(x+h) = f(x) + hf'(x + \theta h),\ 0 < \theta < 1.$$

(ii) Taylor's Series (*finite form*),

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(x) + R.$$

Remainder $R_n = \frac{h^n}{n!} f^n(x + \theta h),\ 0 < \theta < 1$ (*Lagrange's form*)

or, $= \frac{h^n (1-\theta)^{n-1}}{(n-1)!} f^n(x + \theta h)$ (*Cauchy's form*)

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots\ldots$$

$$+ \frac{(x-a)^{n+1}}{(n-1)!} f^{n-1}(a) + \frac{(x-a)^n}{n!} f^n\{a + \theta(x-a)\},$$

$0 < \theta < 1$.

Remainder $R_n = \frac{(x-a)^n}{n!} f^n\{a + \theta(x-a)\}$.

(iii) Maclaurin's Series (*finite form*)

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots$$

$$+ \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{n!} f^n(\theta x),\ 0 < \theta < 1.$$

(iv) Taylor's Series (*extended to infinity*)

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \text{.....} \text{ to } \infty .$$

(v) Maclaurin's Series (*extended to infinity*)

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \text{......} \text{ to } \infty .$$

(vi) Expansions of some well-known functions in series.

$$\left.\begin{aligned}
&(a)\ e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots \\
&(b)\ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots \\
&(c)\ \cos x = x - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots \\
&(d)\ \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots + \frac{x^{2n+1}}{(2n+1)!} + \ldots \\
&(e)\ \cosh x = x + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots + \frac{x^{2n}}{(2n)!} + \ldots
\end{aligned}\right]\ \text{for all values of } x.$$

$$(f)\ \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots + (-1)^{n-1} \frac{x^n}{n} + \ldots$$

for $-1 < x \le 1$.

$$(g)\ (1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots \qquad \text{for } -1 < x < 1.$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots \qquad \text{for } -1 < x < 1.$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots \qquad \text{for } -1 < x < 1$$

$$(1-x)^{-\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \ldots \qquad \text{for } -1 < x < 1.$$

$$(h)\ \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + \ldots$$

for $-1 \le x \le 1$.

$$(i)\ \sin^{-1} x = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1.3}{2.4} \cdot \frac{x^5}{3} + \ldots \qquad \text{for } -1 \le x \le 1.$$

VII. Maxima and Minima.

(i) If $f'(c) = 0$ and $f''(c)$ is *negative*, then

$f(x)$ is a *maximum* for $x = c$

(ii) If $f'(c) = 0$ and $f''(c)$ is *positive*, then

$f(x)$ is a *minimum* for $x = c$.

(iii) If $f'(c) = f''(c) = \dots = f^{r-1}(c) = 0$ and $f^{r}(c) \neq 0$, then

(*a*) if r is *even*, then $f(x)$ is a *maximum* or a *minimum* for $x = c$, according as $f^{r}(c)$ is *negative* or *positive* ;

(*b*) if r is odd, there is neither a maximum nor a minimum for $f(x)$ at $x = c$.

(iv) *Alternative criterion* for maxima and minima :

(*a*) $f(x)$ is a maximum if $f'(x+h)$ changes sign from $+$ to $-$ and

(*b*) $f(x)$ is a minimum if $f'(x+h)$ changes sign from $-$ to $+$,

as h, being numerically infinitely small, changes from $-$ to $+$.

VIII. Indererminate forms.

(i) Form $\dfrac{0}{0}$.

$$\underset{x \to a}{Lt} \frac{\phi(x)}{\psi(x)} = \underset{x \to a}{Lt} \frac{\phi'(x)}{\psi'(x)} = \frac{\phi'(a)}{\psi'(a)}. \qquad (\textit{L' Hospital's Theorem})$$

(ii) Form $\dfrac{\infty}{\infty}$, $\underset{x \to a}{Lt} \dfrac{\phi(x)}{\psi(x)} = \underset{x \to a}{Lt} \dfrac{\phi'(x)}{\psi'(x)}$.

IX. Partial Differentiation.

(i) If $u = f(x, y)$ and if f_{yx}, f_{xy} exist and f_{yx} (or f_{xy}) is continuous, then $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$.

(ii) If $f(x, y)$ be a homogeneous function of degree n in x, y, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y). \qquad [\textit{Euler's Theorem}]$$

Similarly, $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} + z\dfrac{\partial f}{\partial z} = nf(x, y, z)$.

(iii) If $f(x, y) = 0$, $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}$ $\quad (f_y \neq 0)$

(iv) If $u = f(x, y)$, where $x = \phi(t), y = \psi(t)$,

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}.$$

(v) If $u = f(x, y)$, $du = f_x dx + f_y dy$.

Similarly if $u = f(x, y, z)$, $du = f_x dx + f_y dy + f_z dz$.

(vi) If $u = f(x_1, x_2)$, where $x_1 = \phi_1(x, y)$, $x_2 = \phi_2(x, y)$,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial x} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial x}; \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial y} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial y}.$$

Tangent and Normal.

(i) *Equation of the tangent*

$$Y - y = \frac{dy}{dx}(X - x), \text{ or, } (X - x)f_x + (Y - y)f_y = 0.$$

(ii) *Equation of the normal*

$$Y - y = -\frac{dx}{dy}(X - x), \text{ or, } \frac{(X - x)}{f_x} = \frac{(Y - y)}{f_y}.$$

(iii) *Cartesian subtangent* $= y/y_1$; *subnormal* $= yy_1$

Length of tangent $= y\sqrt{1 + y_1^2}/y_1$.

Length of normal $= y\sqrt{1 + y_1^2}$.

(iv) *Arc-differential (Cartesian)*

$$\frac{dx}{ds} = \cos\psi, \quad \frac{dy}{ds} = \sin\psi, \quad \frac{dy}{dx} = \tan\psi.$$

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1; \quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{ds}{dt}\right)^2.$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}; \quad \frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

$$ds^2 = dx^2 + dy^2$$

(v) *Angle between tangent and radius vector* (ϕ)

$$\tan\phi = \frac{rd\theta}{dr}; \quad \sin\phi = \frac{rd\theta}{ds}; \quad \cos\phi = \frac{dr}{ds},$$

$$\psi = \theta + \phi .$$

(vi) *Arc differential* (*Polar*)

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \ ; \quad \frac{ds}{dr} = \sqrt{1 + \left(\frac{r\,d\theta}{dr}\right)^2}$$

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \ ; \quad ds = \sqrt{1 + \left(\frac{r\,d\theta}{dr}\right)^2}\, dr .$$

$$ds^2 = dr^2 + r^2 d\theta^2$$

(vii) *Polar subtangent* $= r^2 \dfrac{d\theta}{dr} = -\dfrac{d\theta}{du}$, where $r = \dfrac{1}{u}$.

Polar subtangent $= \dfrac{dr}{d\theta}$.

Perpendicular from pole on tangent (p)

$$p = r \sin\phi \ ; \quad \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = u^2 + \left(\frac{du}{d\theta}\right)^2 .$$

(viii) *Pedal equations of some well-known curves*

(1) Circle $x^2 + y^2 = a^2$ (centre) $\cdots r = p$.

(2) Circle $x^2 + y^2 = a^2$ (point on the circumference) $\cdots r^2 = 2ap$ ..

(3) Parabola $y^2 = 4ax$ (focus) $\cdots p^2 = ar$.

(4) Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (focus) $\cdots \dfrac{b^2}{p^2} = \dfrac{2a}{r} - 1$.

(5) Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (centre) $\cdots \dfrac{a^2 b^2}{p^2} + r^2 = a^2 + b^2$.

(6) Hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ (focus) $\cdots \dfrac{b^2}{p^2} = \dfrac{2a}{r} + 1$.

(7) Hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ (centre) $\cdots \dfrac{a^2 b^2}{p^2} - r^2 = a^2 - b^2$.

(8) Rect. Hyperbola $x^2 - y^2 = a^2$ (centre) $\cdots p^2 = ar$.

(9) Parabola $r = \dfrac{2a}{1 \pm \cos\theta}$ (focus) $\cdots p^2 = ar$.

(10) Cardioide $r = a(1 \pm \cos\theta)$ (pole) $\cdots r^3 = 2ap^2$.

(11) Lemniscate $\left.\begin{array}{l} r^2 = a^2 \cos 2\theta \\ r^2 = a^2 \sin 2\theta \end{array}\right\}$ $\cdots r^3 = a^2 p$.

(12) $\left.\begin{array}{l} r^n = a^n \cos n\theta \\ r^n = a^n \sin n\theta \end{array}\right\}$ (pole) $\cdots r^{n+1} = a^n p$.

XI. Curvature.

$$\rho = \frac{ds}{d\psi} \; [s = f(\psi)].$$

$$\rho = \frac{\left(1 + y_1^2\right)^{\frac{3}{2}}}{y_2} \; [y = f(x)]; \; y_2 \neq 0.$$

$$\rho = \frac{\left(1 + x_1^2\right)^{\frac{3}{2}}}{x_2} \; [x = f(y)]; \; x_2 \neq 0.$$

$$\rho = \frac{\left(x'^2 + y'^2\right)^{\frac{3}{2}}}{x'y'' - y'x''} \; [x = \phi(t),\; y = \psi(t)].$$

$$\rho = \frac{\left(f_x^2 + f_y^2\right)^{\frac{3}{2}}}{f_{xx} f_y^2 - 2 f_{xy} f_x f_y + f_{yy} f_x^2} \; [f(x, y) = 0].$$

$$\rho = \frac{\left(r^2 + r_1^2\right)^{\frac{3}{2}}}{r^2 + 2r_1^2 - r r_2} \qquad [r = f(\theta)].$$

$$\rho = \frac{\left(u^2 + u_1^2\right)^{\frac{3}{2}}}{u^3 \left(u + u_2\right)} \qquad [u = f(\theta)].$$

$$\rho = \frac{r\,dr}{dp} \quad [p = f(r)]. \qquad \rho = p + \frac{d^2 p}{d\psi^2} \quad [p = f(\psi)].$$

$$\rho = Lt \frac{x^2}{2y} \quad [\text{at the origin, } x\text{-axis } (y = 0) \text{ being tangent}].$$

$$\rho = Lt \frac{y^2}{2x} \quad \left[\text{at the origin, } y\text{-axis } (x=0) \text{ being tangent}\right].$$

$$\rho = \sqrt{a^2+b^2}\,.\,Lt \frac{x^2+y^2}{ax+by} \quad \left[\text{at the origin, } ax+by=0 \text{ being tangent}\right].$$

Chord of curvature through the pole $= 2\rho \sin\phi$.

Chord of curvature $\begin{cases} \text{parallel to } x\text{-axis} = 2\rho \sin\psi. \\ \text{parallel to } y\text{-axis} = 2\rho \cos\psi. \end{cases}$

Centre of curvature

$$\bar{x} = x - \frac{y_1\left(1+y_1^2\right)}{y_2}, \quad \bar{y} = y + \frac{1+y_1^2}{y_2} \quad [y = f(x)]$$

$$\bar{x} = x - \frac{1+x_1^2}{x_2}, \quad \bar{y} = y - \frac{x_1\left(1+x_1^2\right)}{y_2} \quad [x = f(y)]$$

$$\bar{x} = x - \rho \sin\psi, \quad \bar{y} = y + \rho \cos\psi.$$

Radius of curvature of the evolute $= \dfrac{d^2 s}{d\psi^2}$

III THE HISTORICAL DEVELOPMENT OF CALCULUS

The English and German mathematicians, respectively, Isaac Newton and Gottfried Wilhelm Leibniz invented calculus in the 17th century, but isolated results about its fundamental problems had been known for thousands of years. For example, the Egyptians discovered the rule for the volume of a pyramid as well as an approximation of the area of a circle. In ancient Greece Archimedes proved that if c is the circumference and d the diameter of a circle, then $3\frac{1}{7}d < c < 3\frac{10}{71}$. His proof extended the method of inscribed and circumscribed figures developed by the Greek astronomer and mathematician Eudoxus. Archimedes used the same technique for his other results on areas and volumes. Archimedes discovered his results by means of heuristic arguments involving parallel slices of the figures and the law of the lever. Unfortunately, the Method was only rediscovered in the 19th century, so later mathematicians believed that the Greeks deliberately concealed their secret methods. During the late middle ages in Europe, mathematicians studied translations of Archimedes' treatises from Arabic. At the same time, philosophers were studying problems of change and the infinite, such as the addition of infinitely small quantities. Greek thinkers had seen only contradictions there, but medieval thinkers aided mathematics by making the infinite, philosophically respectable.

Isaac Newton

G. W. Leibniz

Archimedes

F. B. Cavalieri

By the early 17th century, mathematicians had developed methods for finding areas and volumes of a great variety of figures. In his Geometry by Indivisibles, the Italian mathematician F. B. Cavalieri (1598-1647), a student of the Italian physicist and astronomer Galileo, expanded on the work of the German astronomer Johannes Kepler on measuring volumes. He used what he called

J. Kepler

"indivisible magnitudes" to investigate areas under the curves. Also, his theorem on the volumes of figures contained between parallel planes (now called Cavalieri's theorem) was known all over Europe. At about the same time, the French mathematician Rene Descartes' La Geometric appeared. In this important work, Descartes showed how to use algebra to describe curves and obtain an algebraic analysis of geometric problems. A codiscoverer of this analytic geometry was the French mathematician Pierre de Fermat, who also discovered a method of finding the greatest or least value of some algebraic expressions-a method close to those now used in differential calculus. About 20 years later, the English mathematician John Wallis (1616-1703) published The Arithmetic of Infinites, in which he extrapolated from patterns that held for finite processes to get formulœ for infinite processes. His colleague at Cambridge University was Newton's teacher, the English mathematician Isaac Barrow (1630-77), who published a book that stated geometrically the inverse relationship between problems of finding tangents and areas, a relationship known today as the fundamental theorem of calculus.

John Wallis

Isaac Barrow

Bernoulli

Eluer

Although many other mathematicians of the time came close to discovering calculus, the real founders were Newton and Leibniz.

Newton's discovery (1665-66) combined infinite sums (infinite series), the binomial (q. v.) theorem for fractional exponents, and the algebraic expression of the inverse relation between tangents and areas into methods we know today as calculus. Newton, however, was reluctant to publish, so Leibniz became recognized as a codiscoverer because he published his discovery of differential calculus in 1684 and of integral calculus in 1686. It was Leibniz, also, who replaced Newton's symbols with those familiar today. In the following years, one problem that led to new results and concepts was that of describing mathematically the motion of a vibrating string. Leibniz's students, the Bernoulli family of Swiss mathematicians (see Bernoulli, Daniel), used calculus to solve this and other problems, such as finding the curve of quickest descent connecting two given points in a vertical plane. In the 18th century, the great Swiss-Russian mathematician Leonhard Euler, who had studied with Johann Bernoulli (1667-1748), wrote his Introduction to the Analysis of Infinites, which summarized known results and also contained much new material, such as a strictly analytic treatment of trigonometric and exponential functions.

A. L. Cauchy

N. I. Lobachevsky

P. G. L. Dirichlet

Despite these advances in technique, calculus remained without logical foundations. Only in 1821 did the French mathematician A. L. Cauchy succeeded in giving a secure foundation to the subject by his theory of limits, a purely arithmetic theory that did not depend on geometric intuition or infinitesimals. Cauchy then showed how this could be used to give a logical account of the ideas of continuity, derivatives, integrals and infinite series. In the next

J. W. R. Dedekind

decade, the Russian mathematician N. I. Lobachevsky and German mathematician P. G. L. Dirichlet (1805-59) both gave the definition of a function as a correspondence between two sets of real numbers, and the logical foundations of calculus were completed by the German mathematician J. W. R. Dedekind (1831-1916) in his theory of real numbers in 1872.

Gauss