DIFFERENTIAL CALCULUS

1 AN IDEA OF NUMBER SYSTEM

1.1 Introduction : Calculus - origin and extension.

Calculus, fundamentally different from Arithmetic, Algebra or Geometry, is essentially concerned with change and motion; calculus deals with quantities that *approach* other quantities. When there is continuous and gradual change, however small the change be, Calculus, with its novel concept of '*limit*' and limiting operations, is the right mathematics to apply.

Though invented initially to meet the mechanical or geometrical needs, today Calculus and its extensions in mathematical analysis are far reaching. Besides being used in theoretical fields of enquiry, Calculus is now used in determining the orbits of artificial satellites and space-craft, in predicting population size, in estimating how fast the price of an agricultural commodity rises, in forecasting weather, in measuring the cardiac output of the heart and in a large variety of other areas.

However, diverse be the area of application of subject, the common theme is the way or manner in which one quantity changes with another when the change in the later is very small or, more properly, with *the rate of change* of one quantity with the other.

In these investigations one has to deal with the relations between pure numbers which represent the magnitude of the quantities. That is why we begin our study of Calculus with a short discussion on number system.

1.2. Numbers.

The earliest concept of numbers originated from counting, and the first set of numbers which was known to men, was the set of *positive integers*. The arithmetical process of subtraction needed an extension to *negative integers*, and *zero* was included as a number. The process of division required a further extension to *rational numbers*, which are defined to be numbers of the form $\frac{m}{n}$ where m and n are integers, ultimately prime to each other, n being positive and not equal to zero. It may be noted that terminating decimals, as also recurring decimals, which are expressible in the form $\frac{m}{n}$ fall under this category.

1.3. Geometrical representation of rational numbers; rational points.

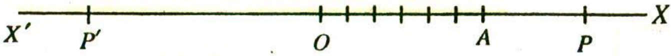


Fig 1.2.1

Take a line $\overleftrightarrow{X'OX}$ extending indefinitely in either direction, for reference, and a suitable point O on it as origin. A suitable length OA on it being chosen as unit, if we divide $\overline{OA}$ into n equal parts, and take a length OP (or OP') equal to m such parts (towards the right of O if m be positive, and towards the left if m be negative), the length OP (or OP'), or the point P (or P', as the case may be) represents the rational number $\frac{m}{n}$. The point P, representing a rational number, is called a *rational point.*

1.4 Properties of rational numbers.

(i) Rational numbers are *well-ordered.* This means that of two unequal rational numbers a and b, either $a > b$ or $a < b$; also if $a > b$ and $b > c$, then $a > c$, etc. In other words, rational numbers are well arranged in respect of their magnitudes, points representing higher numbers always falling to the right of those representing smaller ones, and vice versa, in their geometrical representation.

(ii) Rational numbers are everywhere *dense*; in other words, between any two rational numbers, however close, or within any interval on the axis representing rational numbers, however small, there is an infinite number of rational numbers or points.

This may be easily seen from the fact that, however close the two rational numbers a and b may be, $\frac{1}{2}(a+b)$ is a rational number lying between them. Similarly, between a and $\frac{1}{2}(a+b)$, as also between $\frac{1}{2}(a+b)$ and b, we can insert rational numbers, and so on. Thus there is an infinite number of rational numbers between a and b.

1.5. Irrational numbers.

Whereas all rational numbers are represented by points on the axis, and though in any interval, however small, there is an infinite number of rational points, still the converse, that every point on the axis must represent some rational number, is not true;

e.g., $OP \equiv \sqrt{2}$ *is not rational.*

OA being unity, if $\overline{AB}$ be taken at right angles to $\overline{OA}$ and equal to it, $\overline{OB}$ is joined and on $\overline{OX}$, $\overline{OP}$ be cut off, equal to OB, then OP represents a number equal to $\sqrt{2}$, which is not rational.

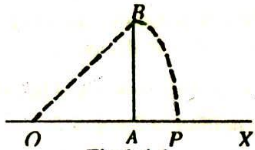


Fig 1.4.1

Proof: For, if $\sqrt{2} = m/n$ where m, n are integers prime to each other, then $m^2 = 2n^2$, showing that m^2 and so m is an even integer (for, the square of an odd integer is evidently odd). Let $m = 2m'$ where m' is an integer. Then we get $n^2 = 2m'^2$ and so n is also an even integer. Thus m and n, which have a common factor 2, cannot be prime to one another, thus leading to a contradiction.

Similarly, equations like $x^3 = 7$, $4x^4 = 13$, etc. cannot be solved in terms of rational numbers alone. Besides radicals, there are other types of number like e, π, . . . (called *transcendental* numbers) which are not rational.

There are, therefore, numbers other than rational numbers, which are called irrational numbers, thus leading to a further extension of numbers.

1.6. Relations of irrational numbers to rational numbers; representation of numbers (rational as well as irrational) as sections of rational numbers.

Consider the number $\sqrt{2}$. There is no rational number whose square is 2. The system of rational numbers, therefore, can be divided into two classes, say L and R, such that all numbers of the L-class have their squares less than 2, and those of the R-class have squares greater than 2. Hence, every number of the R-class > every number of the L-class.

Thus, $1, 1\cdot4, 1\cdot41, 1\cdot414, 1\cdot4142, \ldots$ belong to L-class

and $2, 1\cdot5, 1\cdot42, 1\cdot415, 1\cdot4143, \ldots$ belong to R-class.

The differences of the corresponding numbers of the two classes are, respectively,

$1, 0\cdot1, 0\cdot01, 0\cdot001, 0\cdot0001, \ldots\ldots$

Proceeding in this manner (by expressing $\sqrt{2}$ in a decimal form, which will lead to an endless decimal not recurring, and choosing the rational numbers of the two classes by stopping at any stage) we can find a member of the L-class and a member of the R-class which differ from one another by as little as we please. Our common sense notion, therefore, demands the existence of a number x, and a corresponding point P on the axis, such that P divides the class L from the class R.

But this number x is not rational and belongs to neither of the two classes. Further, x^2 is neither > 2 nor < 2.

For, if $x^2 > 2$, let $x^2 = 2 + \varepsilon$. Then however small ε may be, we can get rational numbers of the R-class whose squares being > 2 will differ from 2 by less than ε. Such rational numbers of the R-class will lie to the left of x, and so the assumption that x is the point dividing the two classes is untenable. Similarly, $x^2 \not< 2$.

$\therefore x^2 = 2$, or, $x = \sqrt{2}$, and being not rational as proved before, it belongs neither to class L nor to class R. The point P is thus only a point of section of the two classes of rational numbers L and R defined before, not belonging to either class, and representing the irrational number $\sqrt{2}$.

This leads to a new idea of defining *numbers as sections of rational numbers,* as follows :

"If by some means or other we divide all rational numbers into two classes L and R, such that each class contains at least one rational number, every rational number belongs to either L or R, and each number belonging to R-class > every number of the L-class, then we obtain a section of rational numbers which defines a number, rational or irrational; the particular mode of division defines a particular number by its section."

Three cases may arise : (i) That L-class has a greatest number, but the R-class has no least; *e.g.*, let all rational numbers > 5 belong to R-class, and the number 5, as also all rational numbers < 5 belong to L-class. The section in this case represents the rational number 5, which belongs here to one of the two classes, namely, the L-class. (ii) The L-class has no greatest number, but the R-class has a least one; *e.g.*, all rational numbers $< -3 \cdot 5$ belong to L-class and $-3 \cdot 5$ with all rational numbers greater than this belongs to R-class. Here the section represents the rational number $-3 \cdot 5$, and the number itself belongs to R-class. (iii)

The L-class has no greatest number and the R-class has no least number; *e.g.*, all rational numbers whose cubes are < 7 belong to L-class, and those whose cubes are > 7 belong to R-class; there is no rational number, as can be shown, whose cube is equal to 7. The section in this case represents the irrational number $\sqrt[3]{7}$ and belongs to none of the classes L and R which consist of rational numbers only.

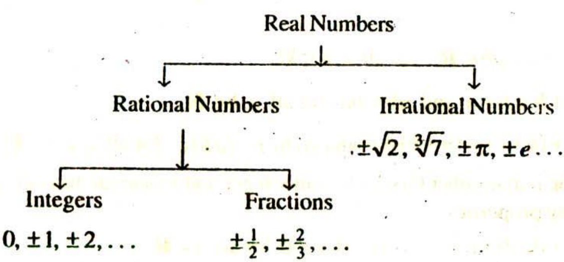
It may be noted that the case in which the L-class has a greatest number and the R-class has a least number simultaneously is not possible, for otherwise, between these two rational numbers there would be an infinite number of rational numbers as proved before, and they would belong to none of the two classes.

This extension of our conception of numbers as sections of rational numbers gives us a more satisfactory basis of defining all numbers in a uniform way. We no longer think of numbers as isolated members, but as an aggregate of rational numbers divided into sections.

1.7. Real numbers.

All kinds of numbers, rational as well as irrational, positive and negative, including zero, constitute what are called real numbers.

The contents and classification of real number system will be understood at a glance from the scheme given below.



It may be thought that just as from rational numbers, by dividing them into two classes by sections, we get, in addition to rational numbers, a new type of numbers, namely irrational numbers; similarly by sections of real numbers again, we may expect a further extension of numbers. But this is not true. In this connection we state the following theorem :

(given in the next page)

Dedekind's theorem (on sections of real numbers) :

If real numbers be divided into two classes L and R in such a way that

(i) every real number belongs to one class or the other,

(ii) each class contains at least one number, and

(iii) any number of the L-class is less than every number of the R-class,

then there exists a real number 'a' which effects this section, i.e., which has the property that all numbers less than 'a' belong to L-class, and all numbers greater than 'a' belong to R-class, the number 'a' itself may belong to either class.

[*For proof, see Hardy's Pure Mathematics.*]

Thus as sections of real numbers we get real numbers alone (unlike that in case of rational numbers), and not any other new type of numbers.

Thus no further extension of numbers is possible; and the aggregate of real numbers is complete. The correspondence (one to one) between *all* the points on the line $\overleftrightarrow{X'OX}$ without exception (called the **linear continuum**) and the system of *all* real numbers, rational and irrational (constituting what is called the **arithmetical continuum**), is now perfect.

1.8 Fundamental Properties of real numbers

Properties involving 'addition' and 'multiplication' of real numbers

(i) If a, b are any two real numbers, then $(a+b)$ and ab are also real numbers, i.e.,

$$(a+b),\ ab \in \mathbb{R}, \text{ for all } a, b \in \mathbb{R}.$$

(ii) $a+b=b+a$ and $ab=ba$, for all $a, b \in \mathbb{R}$.

(iii) $a+(b+c)=(a+b)+c$ and $a(bc)=(ab)c$, for all $a, b, c \in \mathbb{R}$.

(iv) The real number 0 (which is an integer and a rational number) has the following properties :

$$a+0=0+a=a,\ a.0=0.a=0 \text{ for all } a \in \mathbb{R}.$$

Division by zero is meaningless in the set of real numbers.

(v) For every real number a, there exists a real number $-a$, such that $a+(-a)=-a+a=0$.

(vi) For every real number 1 (which is an integer and a rational number) has the following properties :

$$a.1=1.a=a \text{ for all } a \in \mathbb{R}.$$

(vii) For every a $(\neq 0) \in \mathbb{R}$, there exists $\frac{1}{a} \in \mathbb{R}$, where $a.\frac{1}{a} = \frac{1}{a}.a = 1$.

(viii) $a-b \in \mathbb{R}$ for all $a,b \in \mathbb{R}$, where $a-b$ is defined as

$$a-b = a+(-b).$$

(ix) $a.(b+c) = a.b + a.c$, for all $a,b,c \in \mathbb{R}$.

(x) If a is any real number $(\neq 0)$ and b is any real number, then $\frac{b}{a}$ is defined as $\frac{b}{a} = b.\frac{1}{a}$

(xi) $a.(-b) = (-a).b = -(a.b)$ and $(-a).(-b) = ab$, for all $a,b \in \mathbb{R}$.

(xii) $1 \neq 0$.

(xiii) For $a,b \in \mathbb{R}$, $a.b = 0 \Rightarrow$ at least one of a and b must be zero.

1.9 Properties regarding order relation in $\mathbb{R}$.

(i) For any two real numbers a, b one and only one of the statement : "$a > b$", "$a < b$", "$a = b$" must be true.

(ii) $a > b$ and $b > c \Rightarrow a > c$ for all $a,b,c \in \mathbb{R}$.

(iii) $a > b$ and $c > 0 \Rightarrow ac > bc$, for all $a, b, c \in \Re\,\mathbb{R}$.

(iv) $a > b \Rightarrow a+c > b+c$, for all $a, b, c \in \Re\,\mathbb{R}$.

(v) $a > 0$, if and only if $-a < 0$.

(vi) $a > b$, if and only if $a-b > 0$ and $a < b$, if and only if $a-b < 0$, where $a,b \in \mathbb{R}$.

(vii) $a > b$ and $c < 0 \Rightarrow ac < bc$, where $a,b,c \in \mathbb{R}$.

(viii) $a^2 \geq 0$ for all $a \in \mathbb{R}$ ($x \geq y$ means $x > y$, or, $x = y$)

(ix) Between any two distinct numbers, there exist an infinite number of rational numbers as well as an infinite numbers of irrational numbers.

(x) If $a\,(>0)$ and b are two real numbers, then there exists at least one positive integer n such that $n\,a > b$.

Note : 1. For any real number a, one an only one of '$a > 0$', '$a < 0$', '$a = 0$' must be true.

2. A real number a is said to be positive or negtive according as $a > 0$, or, $a < 0$.

3. It should be remembered that the real number 0 (which is also a rational number) is neither positive nor negative.

1.10. Integers

In Art 1.2, it has been discussed that in the process of counting *number* of elements of a finite set (e.g., the number of rooms in a house, the number of trees in a garden, the number of students in a class, etc.) the *natural numbers* denoted by the symbols, 1, 2, 3, . . . we obtained, and the set of natural numbers is denoted by $\mathbb{N}$.

Now for every natural number n, the numbers given by the symbol $-n$ is introduced together with the introduction of the number 'zero' expressed by the symbol 0, where $x+0=0+x=x$ for every $x \in \mathbb{N}. U\{-n:n \in \mathbb{N}\} \cup \{0\}$ and $n+(-n)=-n+n=0$ for every $x \in \mathbb{N}$.

Elements of the set $\mathbb{N} \cup \{-n : n \in \mathbb{N}\} \cup \{0\}$ are called *integers*. The set of all integers is denoted by $\mathbb{Z}$.

Remarks : The integer 0 is neither positive nor negative.

Factors of an Integer

An integer $a(\neq 0)$ is called a *factor* or a *divisor* of an integer b, if b can be expressed as $b = a c$ (or, $c\, a$), for some integer c. In this case c will also be a factor of b if $c \neq 0$.

For example, as $2.3 = (-2)(-3) = 6$, $2, -2, 3, -3$ are factors of 6.

We observe that if a is any non-zero integer, then $1, -1, a, -a$ are factors of a.

Prime Numbers

An integer p (> 1) is called a prime number if p has no factor besides $1, -1, p, -p$. For example, 2, 3, 5, 7, . . . are *prime numbers*. The integer 2 is the least prime number.

Remark : From the definition of prime numbers, it follows that the integers 0, 1 are not prime numbers.

Relatively Prime Numbers

The integer a and b are said to be *relatively prime* or, *co-primes or, prime to each other* if the integer 1 is the only positive integer which is a common factor of a and b. We note that 6 and 29 are relatively prime integers, while, 9 and 24 are not relatively prime to each other.

An Important Property of Prime Numbers

If p be a prime number and a, b are integers where p is a factor of the product ab, then p is a factor of at least a and b.

Even and Odd integers

It can be shown that any integer is either of the form $2m$, or, of the form of $2m+1$, where m is an integer.

An integer of the form $2m$ (where m is an integer) is called an *even integer*, while an integer of the form $(2m+1)$, (where m is an integer) is called an *odd integer*.

Thus $0, \pm 2, \pm 4, \pm 6, \ldots$ are even integers and $\pm 1, \pm 3, \pm 5, \ldots$ are odd integers.

Note : The integer 0 is an even number.

1.11 Intervals in $\mathbb{R}$.

Let $a, b \in \mathbb{R}$ and $a < b$.

(i) The set $\{x : x \in \mathbb{R} \text{ and } a < x < b\}$ is denoted by (a, b) and is called as *open interval.*

(ii) The set $\{x : x \in \mathbb{R} \text{ and } a \le x \le b\}$ is denoted by $[a, b]$ and is called an *closed interval.*

(iii) The sets $\{x : x \in \mathbb{R} \text{ and } a \le x < b\}$ denoted by $[a, b)$ and $\{x : x \in \mathbb{R} \text{ and } a < x \le b\}$ denoted by $(a, b]$ are called *Semiopen* or *Semiclosed intervals.*

The set $\mathbb{R}$ is also regarded as an *interval* and is denoted by $(-\infty, \infty)$ where we write $\mathbb{R} = \{x : -\infty < x < \infty\}$.

It should be noted that the *symbols* $\infty, -\infty$ *do not represent real numbers.*

For any real number a, the set $\{x : x \in \mathbb{R} \text{ and } x \ge a\}$ is denoted by $[a, \infty)$ and this is a *semiopen interval.*

Similarly the sets $\{x : x \in \mathbb{R} \text{ and } x > a\}$, $\{x : x \in \mathbb{R} \text{ and } x \le a\}$, $\{x : x \in \mathbb{R} \text{ and } x < a\}$ are respectively denoted by (a, ∞), $(-\infty, a]$, $(-\infty, a)$ and these sets are also called *intervals.*

1.12. Complex numbers.

In order to fill up the gaps and bring about a uniformity in the theory of equations, as also in all other theories of higher mathematics, it has been found necessary to introduce a class of numbers, called complex numbers. A *complex number* has been defined by modern mathematicians as an *ordered couple of real numbers, i.e.*, a pair of real numbers united symbolically in a particular order for the purpose of technical convenience. Thus a complex number is, strictly speaking, not a single number at all, but a pair of real numbers with a proper order. If the order is reversed, we get a different complex number. A complex number may be expressed in the form $[a, b]$, where a and b are two real numbers. It is also represented, for convenience, in the form $a + ib$, where the symbol i has no meaning by itself; it merely indicates the order in which the real numbers a and b are considered. In defining all ordinary algebraical operations with regard to complex numbers it has been found convenient to associate the symbol i with the property, $i^2 = -1$ in which case all operations consistent with the algebra of real numbers may be applied to the case of complex numbers.

For geometrical representation and further introduction into the algebra of complex numbers see *Chapter VI*, Das and Mukherjees' *Higher Trigonometry.*

1.12 Miscellaneous Worked out Examples

Ex. 1. Prove that $\sqrt{3}$ is not a rational number. *[B. P. 1997]*

Solution : Since, $1 < 3 < 4$, $1 < \sqrt{3} < 2$, which shows that $\sqrt{3}$ cannot be an integer.

Now, if possible, let $\sqrt{3}$ be a rational number.

We assume, $\sqrt{3} = \dfrac{p}{q}$, ... (1)

where $q > 1$ and p and q are positive integers prime to each other.

From (1) $\dfrac{p^2}{q^2} = 3$, *i.e.*, $\dfrac{p^2}{q} = 3q$ ... (2)

Since p and q are positive integers prime to each other, p^2 and q are also positive integers prime to each other. Again, since $q > 1$, $\dfrac{p^2}{q}$ represents a rational number, which is not an integer, but $3q$ represents a positive integer. So, from (2), we get, a positive rational number which is not an integer is equal to a positive integer. But this is not possible. Hence, our initial assumption cannot be true, *i.e.*, $\sqrt{3}$ cannot be a rational number.

Ex. 2. Prove that $\log_2 6$ is an irrational number.

Solution : $\log_2 6 = \log_2(2\times3) = \log_2 2 + \log_2 3 = 1 + \log_2 3$.

$\log_2 6$ will be a rational number, if $\log_2 3$ is rational. If possible. let us assume that $\log_2 3$ is rational and $\log_2 3 = \frac{p}{q}$, when $q \neq 0$ and p, q are positive integers prime to each other.

Now, $\because \log_2 3 = \frac{p}{q}$, $2^{\frac{p}{q}} = 3$ or, $2^p = 3^q$.

Obviously, 2 and 3 are prime to each other, and p and q are also assumed to be integers prime to each other. So, the equation, $2^p = 3^q$ cannot hold.

Therefore, $\log_2 3$ cannot be a rational number.

Since, the sum of a rational number and an irrational number is irrational, $\log_2 6$ cannot be a rational number.

Ex. 3. Prove that $\sqrt{3}+\sqrt{2}$ is an irrational number.

Solution : Let us assume the contrary, *i.e.*, $\sqrt{3}+\sqrt{2}$ is rational.

$$\because \left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right) = 1$$

$$\sqrt{3}-\sqrt{2} = \frac{1}{\sqrt{3}+\sqrt{2}}$$

is also a rational number, since it is the quotient of two rational numbers 1 and $\left(\sqrt{3}+\sqrt{2}\right)$.

Thus, $\sqrt{2} = \frac{1}{2}\left\{\left(\sqrt{3}+\sqrt{2}\right)-\left(\sqrt{3}-\sqrt{2}\right)\right\}$, being the difference of two rational numbers, is rational. Thus we arrive at an absurd conclusion.

Hence, our initial assumption that $\left(\sqrt{3}+\sqrt{2}\right)$ is rational is wrong.

So, $\sqrt{3}+\sqrt{2}$ is an irrational number.

Ex.4. Show that no positive integer m other than a square number has a square root within the aggregate of rational numbers.

Solution : Let m be a positive integer which is not a square number. Then we are to prove that $\sqrt{m}$ cannot be rational.

Since m is not a square number, we have a positive integer n such that

$$n^2 < m < (n+1)^2$$

$$\text{or,} \quad n < \sqrt{m} < n+1$$

whence it follows that $\sqrt{m}$ cannot be an integer.

If possible, let us assume that $\sqrt{m}$ is rational.

Then $\sqrt{m} = \frac{p}{q}$, where $q > 1$ and p and q are integers prime to each other.

$$\because \quad \frac{p}{q} = \sqrt{m}$$

$$\text{or,} \quad \frac{p^2}{q^2} = m$$

$$\text{or,} \quad \frac{p^2}{q} = mq. \qquad \dots \quad (1)$$

p and q being integers prime to each other, p^2 and q are also positive integers prime to each other. Also, since $q > 1$, $\frac{p^2}{q}$ is rational number which is not an integer.

But, m and q being both positive integers, mq is also a positive integer. So, from (1) we get a rational number which is not an integer = a positive integer, which is impossible.

Therefore, our assumption cannot be true, *i.e.*, $\sqrt{m}$ cannot be a rational number.

Ex. 5. (*i*) If r and s are any two rational numbers, prove that $(r+s)$ and $(r \times s)$ are also rational numbers. [*C. P. 2003, B. P. 1992, 93, 95*]

(*ii*) Give examples to show that the sum and product of two irrational numbers may be rational or irrational. [*B. P. 1994*]

Solution : (*i*) If possible, let $r+s=p$, where r and s are rational numbers, while p is an irrational number.

$$\therefore \quad r = p - s$$

since the difference of an irrational number and a rational number cannot be a rational number, so r cannot be a rational number, which contradicts our initial condition that r is rational. Therefore, the sum of two rational numbers cannot be irrational; *i.e.*, $(r+s)$ is a rational number, when r

and s are rational numbers. Next, if possible, let the product of two rational numbers be an irrational number.

i.e., let $r \times s = q$, where r and s are rational and q an irrational number,

$\therefore \; r = \frac{q}{s} \quad (s \neq 0)$

since division of an irrational number by a rational number is not a rational number, $\frac{q}{s}$ is not rational, but r is rational Thus q cannot be an irrational number, it must be a rational number.

$\therefore \; r \times s$ is a rational number.

(ii) We have $\sqrt{2}$ and $\sqrt{3}$ are two irrational numbers. Their sum $\sqrt{3}+\sqrt{2}$ is also an irrational number. [See Ex. 3. above]

Again, product of two irrational numbers $\sqrt{2}$ and $\sqrt{3}$.

$$\sqrt{2}\sqrt{3} = \sqrt{2 \times 3} = \sqrt{6} \cdot$$

If possible, let $\sqrt{6}$ be a rational number.

then $\sqrt{6} = \frac{p}{q}$, where p and q are positive integers, prime to each other and $q > 1$

$$\therefore \; 6 = \frac{p^2}{q^2}, \; i.e., \; \frac{p^2}{q} = 6q .$$

since p and q are prime to each other, p^2 and q have no common factor and so $\frac{p^2}{q}$ cannot be an integer, but $6q$ is obviously an integer.

Thus, a fraction is equal to an integer, which is not possible.

So, product of two irrational numbers $\sqrt{2}$ and $\sqrt{3}$ is also an irrational number.

Next, let us consider two irrational numbers $5+\sqrt{2}$ and $5-\sqrt{2}$.

There sum $(5+\sqrt{2})+(5-\sqrt{2}) = 10$, a rational number;

and their product $(5+\sqrt{2})(5-\sqrt{2}) = 25-2 = 23$.

a rational number.

Hence, it is shown that the sum and product of irrational numbers may be irrational or rational.

Ex. 6. Examine whether $\log_{10}5$ is a rational number. [*B. P. 1999, 2001*]

Solution : If possible, let $\log_{10}5$ be rational and $\log_{10}5 = \frac{p}{q}$, where p and q integers, prime to each other, $q > p$

$\therefore \quad (10)^{\frac{p}{q}} = 5$

or, $10^p = 5^q$

or, $2^p \cdot 5^p = 5^q$,

i.e., $2^p = 5^{q-p}$ ... (1)

since p and $(q-p)$ are both positive integers and 2 and 5 are prime to each other, equation (1) cannot hold.

Hence $\log_{10}5$ cannot be rational, *i.e.*, $\log_{10}5$ is an irrational number.

EXAMPLE - I

1. Define a rational number. Show that $\sqrt{2}$ is not a rational number. [*B.P. 1981, '86, '95, '97, C.P. '98*]
2. Show that $\sqrt{3}$ is not rational. [*B.P. 2002*]
3. Prove that $\log_{10}7$ is not rational.
4. If, $x\sqrt{2} + y\sqrt{3} = 0$, where x and y are both rational, prove that $x = 0 = y$.
5. Define an irrational number. Give examples to show that the sum and product of two irrational numbers may be rational or irrational. [*B.P.1994*]
6. Given that r and s are two rational numbers, prove that $r+s$, $r-s$, rs and r/s $(s \neq 0)$ are rational numbers. [*B.P. '82, '93, '95*]
7. Prove that the sum of or the difference between, a rational number and an irrational number cannot be a rational number.

2 FUNCTIONS

2.1. Introduction.

In higher mathematics and various branches of science very often we have to deal with *changeable quantities* which are *interrelated* to one another, and in many such cases we have occasions to investigate how one of these quantities changes with a gradual change in the other. For example, in a given amount of gas enclosed in a cylinder with a movable piston, and kept at a constant temperature, the volume and pressure are interdependent, and a change in one produces a corresponding change in the other ; or again, for a falling particle, the height from the ground depends on the time, and changes with it; the area of a circle changes with its radius, etc.

In Differential Calculus we deal with the way in which one quantity varies with another when the change in the latter is ultimately very small, or more properly, with the *rate of change* of one quantity with another, as also other allied problems.[1]

In these investigations we shall be dealing with the relations between pure numbers which represent the magnitudes (with proper signs) of the quantities, and not with the concrete quantities themselves, so that the results will be general in nature, applicable to any pair of interdependent quantities under similar mathematical conditions.

In the following discussions we shall be concerned with the system of *real numbers* only, meaning by real numbers, zero, integers, rational and irrational numbers, positive or negative.

2.2. Preliminary Definitions and Notations.

Aggregate or **Set :** A system of real numbers defined in any way whatever is called an 'aggregate' or 'set' of numbers.

Illustration : The aggregate of positive integers; the aggregate of all negative rational numbers; the aggregate of all real numbers positive or negative; the aggregate of all rational numbers from -3 to $+7$; the aggregate of numbers $\frac{1}{1}, \frac{1}{-2}, \frac{1}{3}, \frac{1}{-4}, \frac{1}{5}, \frac{1}{-6}$, etc.

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[1] *While investigating problems of this type, Newton (in England) and Leibnitz (in Germany) were independently led to the investigation of the principles of Calculus, towards the close of the seventeenth century. The principles of Calculus, in some form, were also known to the Hindus in India much earlier.*

Variable : Let x be a symbol used during any mathematical investigation, to which may be assigned any numerical value out of a given set of real numbers. Then x is called a 'variable' or a 'real variable', and the totality of the values of x constitutes what is called the **domain** of x.

Illustration : In the expression $x\,!$, x may be considered a real variable whose domain is the aggregate of positive integers.

Note. Variables are usually denoted by latter letters of the alphabet, such as x, y, z, u, v, w, g, h, etc.

Continuous Variable : If x assumes successively every numerical value of an aggregate of *all* real numbers from a given number 'a' to another given number 'b', then x is called a 'continuous real variable'.

The **domain** or **interval** (as it will be sometimes called) of x in this case is denoted by $[a, b]$ or $a \le x \le b$.

If a *be omitted* from the domain, it is indicated as $a < x \le b$.

In the last case the domain is said to be *open* at the left end, whereas the domain $a \le x \le b$ is said to be *closed*. The interval $a < x < b$ is open at both ends, a and b being both excluded from the domain of possible values of x.

Illustration : In the expression $\sqrt{(5-x)(x+3)}$, x is a continuous real variable whose domain is $-3 \le x \le 5$; again, in $\sqrt{x+2}\,/\sqrt{7-x}$, the real variable x has the interval $-2 \le x < 7$. In $\sin^{-1} x$, the interval of x is $-1 \le x \le 1$.

The domain of the variable x in any expression containing x, as in the above cases, consists of those values of x for which the expression has a definite real value.

The interval $[a, b]$ is very often graphically represented on the x-axis by means of the length bounded by the two points $A\ (x = a)$ and $B\ (x = b)$.

Fig 2.2.1

The *length of the interval* $[a, b]$ is obviously $AB = OB - OA = b - a$.

Constant : A symbol which retains the same numerical value throughout a set of mathematical operations is called a constant.

Note. Constants (other than numerical constants like 2, -3, e, π, etc.) are usually denoted by the earlier letters of the alphabet, such as a, b, α, β, etc.

Absolute Value : By absolute value of a quantity x, as distinguished from its algebraical value, we mean its magnitude or numerical value, taken with a positive sign. It is represented by the notation $|x|$ which is $= x$, 0 or $-x$ according as $x >, =$ or < 0.
From the very definition the following results are apparent, *viz.*,

(i) $|a \pm b| \le |a| + |b|$ or more generally,

$|a \pm b \pm c \pm ......| \le |a| + |b| + |c| + ...$

(ii) $|a \pm b| \ge |a| \sim |b|$, *i.e.*, $\big||a| - |b|\big|$.

Illustration : $|-2| = 2$, $|6| = 6$, $|-2+6| < 2+6$,

$|-2-6| = 2+6$, $|-6-2| > 6 \sim 2$, $|2-6| = 2 \sim 6$, etc.

Note. *Meaning of the symbol* $|x-a| < \delta$.

Since $|x-a| < \delta$, if $x > a$, $x - a < \delta$, *i.e.*, $x < a + \delta$; and if $x < a$, $a - x < \delta$, *i.e.*, $a - \delta < x$. Hence, combining the two, we see that $|x-a| < \delta$ means $a - \delta < x < a + \delta$. Similarly, $|x-a| \le \delta$ means $a - \delta \le x \le a + \delta$. Symbol $0 < |x-a| \le \delta$ means $a - \delta \le x \le a + \delta$, but $x \ne a$.

Thus, $|x| < \delta$ means $-\delta < x < \delta$.

Functions : *By a function of x, defined for a given domain, is understood a quantity which has a single and definite value for every value of x in its domain.* [See note 1.]

In other words, "*If x and y be two real variables so related that, corresponding to every value of x within a defined domain, we get a definite value of y, then y is said to be a function of x defined in its domain.*"

In this case, the variable x, to which we may arbitrarily assign different values in the given domain, is referred to as the **independent variable** (or, *argument*), and y is called the **dependent variable** (or, *function*).

[See *note* 2.]

We shall generally denote functions of x by such symbols as $f(x)$, $\psi(x)$, $F(x)$, $\phi(x)$, etc., where the mathematical forms of these funcitons may or may not be obtainable.

Note 1. When an expression or equation which defines a function gives two or more values of the function for each value of x, we call the function *multiple-valued.* The definition given above refers to a *single-valued* function with which we are mainly concerned in all mathematical investigations. A multiple-valued function, with proper limitations imposed on its value to be used in any particular investigation, can in general be treated as defining two or more different single-valued functions of x; e.g., $y = \sin^{-1} x \left(-1 \le x \le 1\right)$ can be broken up into $y = \sin^{-1} x$,

where (i) $-\frac{1}{2}\pi \le y \le \frac{1}{2}\pi$,

(ii) $\frac{1}{2}\pi \le y \le \frac{3}{2}\pi$,

(iii) $\frac{3}{2}\pi \le y \le \frac{5}{2}\pi$, etc.;

again $y^2 = x$ can be broken up into $y = +\sqrt{x}$ and $y = -\sqrt{x}$, and so on.

More generally (without restricting to single-valued functions only), a funciton of x may be defined as follows :

If two quantities x and y are so related that, corresponding to values of x, there are values of y, then y is said to be a function of x.

Note 2. If y be a function of the variable x, it will generally be open to us also to regard x as a function of y by virtue of the functional relation between x and y, the proper domain of y being taken into account in this case, because it may so happen that the domain in which y is defined is not the domain in which x is defined. For example, $y = \sqrt{x}$ can be written as $x = y^2$, the domain of x in the former relation being $x \ge 0$, and that of y in the latter is the aggregate of all real numbers, positive or negative.

In the latter case, y will be the independent variable, x the dependent one.

Note 3. A function may be undefined (i.e., may not have a definite value) for some particular value or values of x in a given interval. In this connection we may make the following remark :

Division by zero (symbols $\frac{a}{0}$, $\frac{0}{0}$) is undefined.

The quotient of two finite numbers a and b (*viz.*, $\frac{a}{b}$) is defined as the definite finite number x such that $a = bx$. Now, obviously, in the division, zero value of b is excluded; for, if $b = 0$, then $a\,(= bx) = 0$, and x can be any number. Hence, the above definition rules out division by zero. Therefore, forms $\frac{a}{0}$, $\frac{0}{0}$ are undefined.

The following simple illustration shows how *division by zero leads to fallacious results*.

Suppose, $x = y\ (x \neq 0,\ \ y \neq 0)$, $\therefore\ x^2 = xy$.

$\therefore\ x^2 - y^2 = xy - y^2$

or, $(x+y)(x-y) = y(x-y)$.

Hence, dividing out by $x-y$, $x+y=y$, *i.e.*, $2y = y$, or $2=1$.

The fallacy is due to the fact that we have divided by $x-y$ which is equal to zero.

Similarly, the assumption $\frac{0}{0}=1$, on the basis that anything divided by itself is 1, leads to fallacious results, as shown below.

$$3\times\frac{0}{0} = 3\times 1 = 3\ ;\ \text{again,}\ 3\times\frac{0}{0} = \frac{3\times 0}{0} = \frac{0}{0} = 1,\quad \therefore\ 3 = 1.$$

From the above remarks, it will be apparent that

the function $f(x) = \dfrac{x^2-25}{x-5}$ *is not defined* for $x = 5$;

the function $f(x) = \sin\dfrac{1}{x}$ *is not defined* for $x = 0$; etc.

Note 4. If $f(x)$ denotes a certain function of x, then in case $f(x)$ is given by a mathematical expression involving x, then $f(a)$, i.e., the value of the function for $x=a$, may, in general [but not always, as explained in note 3 above, and also in (iv), Art. 2.4], be obtained by putting a for x in the expression for $f(x)$.

Thus, If $f(x) = \sin x$, $\quad f(0) = \sin 0 = 0$;

If $f(x) = x^2 - 5x + 1$, $f(1) = -3$, $f(-1) = 7$;

If $f(x) = x^2$, $f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$; etc.,

whereas, If $f(x) = x\cos\dfrac{1}{x}$, $f(0)$ is undefined.

2.3. Graphical representation of functions.

Let $y = f(x)$ be a real valued function with domain A $(\subseteq \Re)$. Then the *graph* of the function $y = f(x)$ is defined to be the *set* of *all points in the plane with cartesian coordinates* $(x,\ f(x))$, where $y = f(x)$, $x \in$ A.

Taking the straight line $\overleftrightarrow{X'OX}$, with origin O on it as usual, to represent the real variable x, the value of the function, y or $f(x)$, may be represented parallel to the line $\overleftrightarrow{Y'OY}$ drawn at right angles to $\overleftrightarrow{X'OX}$, as in ordinary graphs. Corresponding to every value of x (in the assigned domain) the point is plotted whose ordinate gives the corresponding value of the function. The assemblage of the points, which may or may not form a continuous line, represents the graph of the function.

In drawing the graph it is not necessary to know the exact mathematical relation between x and y (which may or may not be obtainable). It will be sufficient if we know the definite value of y corresponding to every value (at least a large number of values) of x in the defined domain.

The graph at once presents to the eye the way in which the function is related to, and changes with the argument.

2.4. Some remarks on functions.

From the very definition the following points should be clear :

(i) *It is not essential for a function to be expressible by a mathematical form always.* For example, suppose x hour after noon on a certain day, the temperature of a patient is T degree. Now, to each value of x (up to a certain number, depending on our contemplated period of observation), there corresponds a definite value of T. Hence, T is a function of x by definition. But T cannot be expressed analytically by a mathematical expression in terms of x. Nevertheless, we can draw a graph which is the temperature chart of the patient, giving an idea how T changes with the time x. For other examples, see (vii) of the next article.

(ii) In some cases *a function may have different mathematical forms for different ranges of its domain of existence*; for illustration, see (v) of the next article.

(iii) *A function may be undefined for some value or values of the argument,* as has already been remarked and illustrated in note 3, Art. 2.2. Also, every function cannot be defined in every interval; thus, $\sin^{-1} x$ cannot be defined in the interval (2, 3), for $\sin^{-1} 2$ has no meaning, there being no angle whose sine is 2.

(iv) *A function may be defined arbitrarily.* For instance, we may define a function as

$$f(x) = x^2 \quad \text{when} \quad x < 0,$$

$$f(0) = 3,$$

$$f(x) = \frac{1}{2} - x \quad \text{when} \quad x > 0.$$

The function is thus definitely defined for all real values of x.

(v) The functions $\frac{x^2-25}{x-5}$ and $x+5$ are different functions. The former is undefined at $x=5$, and so its domain of existence is the aggregate of all real numbers excepting 5 for the arguement x. The latter exists for all real values of x. Hence, though for other values of x the two functions are equal, there is a point of distinction at $x=5$.

A third function might be defined as $f(x)=\frac{x^2-25}{x-5}$ when $x \neq 5$, and $f(5)=20$. Then the function is again different from either of the first two. It exists for all real values of x including $x=5$, but at its value is different from that of the second function $x+5$.

If we define a fourth function by saying that $f(x)=\frac{x^2-25}{x-5}$ when $x \neq 5$, and $f(5)=10$, then this function is identical with the function $x+5$.

2.5. Examples of functions.

Below is given a number of examples of functions of a variety of types, with their graphs in certain cases which will help to form a clear notion about functions and will further elucidate the remarks of the previous article.

(i) Analytical functions like x^2, $\frac{2x^3+7}{x^2+9}$, etc., or more generally, *polynomials in* x of the type $f(x) \equiv a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$ (where n is a positive integer), or *rational algebraic functions* of the type $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.

The domain of these are generally the set of all real numbers; in the last case the zeroes of the denominator are excluded, for, the function is not defined at these points.

(ii) $f(x) = x$ when $x > 0$
$= 0$ when $x = 0$,
$= -x$ when $x < 0$.

The graph, as shown in Fig 2.5.1, consists of two lines $\overrightarrow{OA}$ and $\overrightarrow{OB}$ which bisect the angles $\angle XOY$ and $\angle YOX'$ respectively.

This is also the graph of the function $f(x)=|x|$.

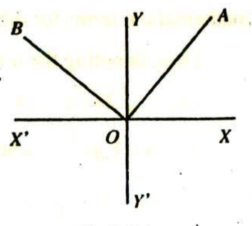

Fig 2.5.1

(iii) $f(x) = \sqrt{x}$.

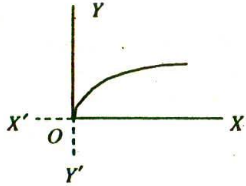

Fig 2.5.2

$f(x)$ is defined for $x = 0$, and all positive values of x; the graph is a continuous curve (parabola) in the first quadrant.

(iv) $f(x) = x\,!$

or, $f(x) =$ sum of the first x terms of $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots$

The functions are defined only for positive integral values of x.

The graph in each case consists of a series of isolated points.

(v) The height y from the ground, at a time x, of a perfectly elastic ball originally dropped from a height h.

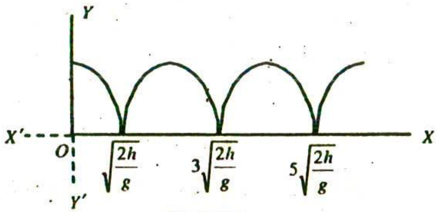

Fig 2.5.3

Here y is defined for all positive values of x, but expressed by different mathematical terms for different ranges of the values of x.

Thus, denoting the time of fall (from start to first impact),

i.e., $\sqrt{2h/g}$ *by* x_1,

$y = h - \frac{1}{2} g x^2$, when $0 \le x \le x_1$ (*i.e.*, before first impact),

$y = (x - x_1)\sqrt{2gh} - \frac{1}{2} g (x - x_1)^2$, when $x_1 < x \le 3x_1$ (*i.e.*, between first and second impacts),

$$y = (x - 3x_1)\sqrt{2gh} - \tfrac{1}{2} g (x - 3x_1)^2, \quad \text{when} \quad 3x_1 < x \le 5x_1$$

(*i.e.*, between second and third impacts), etc.

The graph, as shown, consists of a series of parabolic arcs, on the positive side of the x-axis.

(vi) $y = \dfrac{x^2}{x}$.

For $x \neq 0$, $y = x$; for $x = 0$, y is not known (undefined).

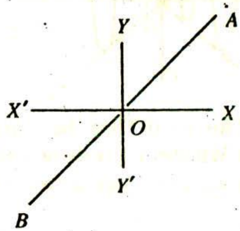

Fig 2.5.4

The graph is that of the straight line $y = x$, with the origin left out.

(vii) $y = [x]$, where $[x]$ denotes the greatest integer not exceeding x.

For $0 \le x < 1$, $y = 0$; $1 \le x < 2$, $y = 1$;

$2 \le x < 3$, $y = 2$;

$-1 \le x < 0$, $y = -1$;

$-2 \le x < -1$, $y = -2$; etc.

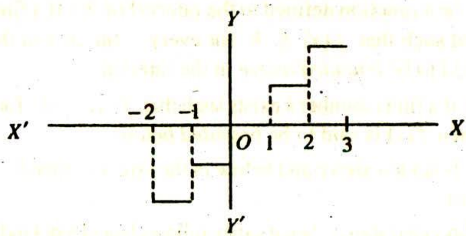

Fig 2.5.5

Thus, the graph consists of parallel segments of lines in which the right-hand end-points are left out.

(viii) $y = x \sin \dfrac{1}{x}$.

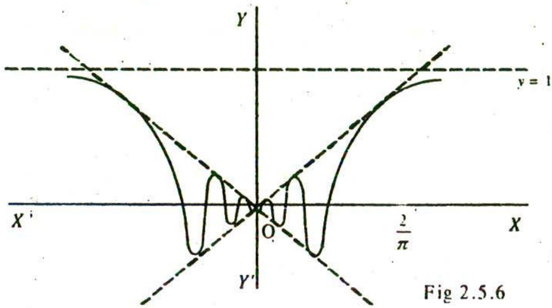

Fig 2.5.6

Here y is not defined for $x = 0$. Thus, the domain of x is the aggregate of all real numbers except 0. Whether x is positive or negative as the numerical value of x is very large, the value of y approaches 1, while always remaining less than 1.

The graph shown in Fig 2.5.6, which is continuous everywhere excepting at $x = 0$, where a point is missing on the graph. Near O, on either side, the graph has an inifinite number of oscillations with gradually diminishing amplitude. The graph is comprised between the lines $y = x$ and $y = -x$.

(ix) Functions like e^x, $\log x$, $\sin x$, $\cos x$, $\sin^{-1} x$, $\cos^{-1} x$, etc., which are not algebraic functions, are called *Transcendental functions.* For graphs of first two, *see* Art. 19.9, and for some others see next page, *i.e.* Fig 2.5.7.

2.6. Bounded functions and their Bounds.

Let $f(x)$ be a function defined in the interval (a, b). If a finite number K can be found such that $f(x) \le K$ for every value of x in the interval, then $f(x)$ is said to be *bounded above* in the interval.

Similarly, if a finite number k exists such that $f(x) \ge k$ for every x in the interval, then $f(x)$ is said to be *bounded below.*

If $f(x)$ is bounded above and below in the interval. then it is said to be simply *bounded.*

If $f(x)$ is bounded above, then it easily follows from Dedekind's Theorem that there exists a definite finite number M such that $M \ge f(x)$ for every value of x in the interval, but ε being any pre-assigned positive quantity, however small, there is at least one value of x in the interval for which $f(x) > M - \varepsilon$. This number M is called the *upper bound* of the function in the interval.

In a similar way, if $f(x)$ be bounded below, then there exists a definite finite number m such that $m \le f(x)$ for every x in the interval, but given any pre-assigned positive number ε, however small, there is at least one value of x for which $f(x) < m + \varepsilon$. This number m is called the *lower bound* of $f(x)$ in the interval.

We know, $\sin x$, $\cos x$ are bounded functions in the interval $[-p, p]$, the upper bounds of both being 1 and their lower bounds being -1.

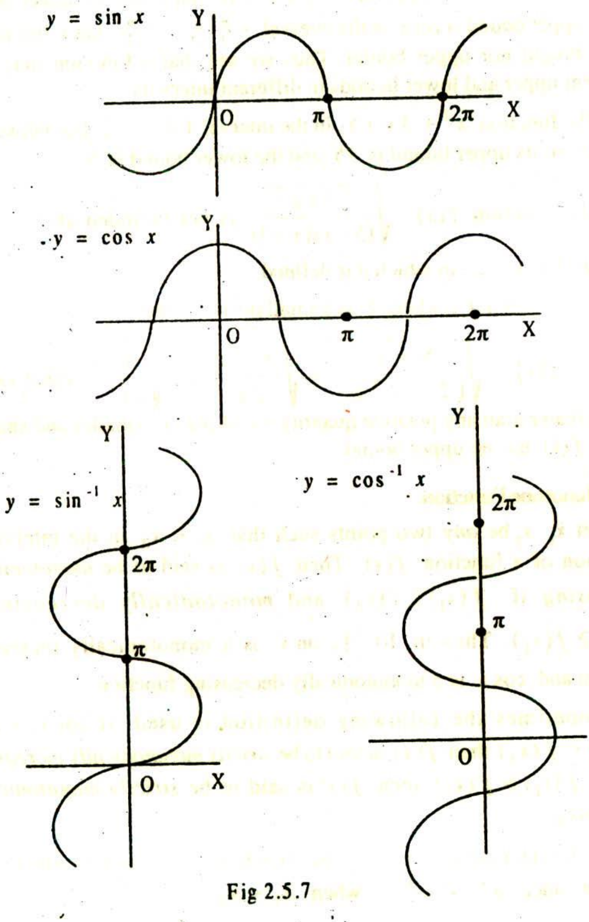

Fig 2.5.7

For $\sin x$, $M=1$; now, taking $\varepsilon = \frac{1}{2}$, we can find at least one value, say $\frac{\pi}{3}$, of x, such that $\sin\frac{\pi}{3} > 1-\frac{1}{2}=\frac{1}{2}$. Other values of x can be obtained from tables, for which $\sin x > \frac{1}{2}$. The function defined in Ex. (v), § 2.5 is a bounded function in the interval $[0, \infty)$, the upper bound being h and the lower bound being zero.

For the interval $0 \le x < \frac{\pi}{2}$, $\tan x$ has the lower bound zero, but no upper bound. For the interval $-\frac{\pi}{2} < x \le 0$, $\tan x$ has no lower bound, but its upper bound is zero. In the interval $-\frac{\pi}{2} \le x < \frac{\pi}{2}$, $\tan x$ has neither lower bound nor upper bound. Thus we see that a function may have different upper and lower bounds in different intervals.

The function x^2+3x+5, in the interval $1 \le x \le 2$ lies between 9 and 15; so its upper bound is 15 and the lower bound is 9.

The function $f(x)=\sqrt{\dfrac{2+x}{(5-x)(x-3)}}$ is not bounded above in the interval $3 < x < 5$, in which it is defined.

Let $x = 3+\varepsilon$, where ε is a small positive number.

$$f(x)=\sqrt{\frac{5+\varepsilon}{(2-\varepsilon)\varepsilon}} > \sqrt{\frac{5+\varepsilon}{2\varepsilon}} > \sqrt{\frac{5}{2\varepsilon}}$$, which can be made greater than any positive quantity by taking ε smaller and smaller. Hence $f(x)$ has no upper bound.

2.7. Monotone Function.

Let x_1, x_2 be *any* two points such that $x_1 < x_2$ in the interval of definition of a function $f(x)$. Then $f(x)$ is said to be *monotonically increasing* if $f(x_1) \le f(x_2)$ and *monotonically decreasing* if $f(x_1) \ge f(x_2)$. Thus in $[0, \frac{\pi}{2}]$, $\sin x$ is a monotonically increasing function and $\cos x$ is a monotonically decreasing function.

Sometimes the following definition is used. If for $x_1 < x_2$, $f(x_1) < f(x_2)$ then $f(x)$ is said to be *strictly monotonically increasing*, and if $f(x_1) > f(x_2)$ then $f(x)$ is said to be *strictly monotonically decreasing*.

In the interval $0 \le x < \infty$, the function e^x is a strictly increasing function, since $e^{x_1} < e^{x_2}$. when $x_1 < x_2$.

In the interval $0 \le x < \infty$, the function $f(x) = \dfrac{3x+5}{2x+1}$ is a strictly decreasing function, for, $f(x_1) > f(x_2)$ when $x_1 < x_2$.

The example (vii) of § 2.5 is an example of a function defined in the interval (0, 3), which is monotonically increasing but not strictly increasing.

2.8. Classification of Functions

(I) Even and Odd Functions

Let $f(x)$ be a function defined in a domain $D(\in R)$ where D is such that $x \in D \Rightarrow -x \in D$. The function $f(x)$ is said to be an *even function* if $f(-x) = f(x)$, for all $x \in D$ and $f(x)$ is called to be an *odd function* if $f(-x) = -f(x)$ for all $x \in D$.

The graph of an even function is symmetrical about the axis of y while the graph of an odd function is symmetrical in opposite quadrants.

Every function can be expressed as the sum of an even an odd funtion.

It should be noted that *inverse* of an even function is not defined.

Examples : $f(x) = x^2$, $f(x) = \cos x$ when $x \in \mathbb{R}$ are even functions, for $f(-x) = (-x)^2 = f(x)$, $F(-x) = \cos(-x) = \cos x = F(x)$.

Again $\phi(x) = x^3$, $\psi(x) = \sin x$, where $x \in \mathbb{R}$ are odd functions, for $\phi(-x) = (-x)^3 = -x^3 = -\phi(x)$, $\psi(-x) = -\sin x = -\psi(x)$.

(II) Periodic functions

A function $f(x)$ defined in a domain D is said to be a *periodic function* of peiod μ if μ be the *least* positive real number such that $f(x+\mu) = f(x)$ for all $x \in D$ [Here, $x + \mu \in D$, for all $x \in D$].

$f(x) = \cos x$, $x \in \mathbb{R}$ periodic function of period 2π, since 2π is the least positive number such that $f(x+2\pi) = \sin(x+2\pi) = \sin x = f(x)$, for all $x \in \mathbb{R}$.

(III) Explicit and Implicit Functions

If $D\ (\subseteq \mathbb{R})$ be the domain of a function f, we can express the function as $y = f(x)$, $x \in D$. ... (1)

If a function can be expressed in the form (1), the function is said to be expressed *explicitly* and we say that the function is *explicit*.

$f(x) = x^3 + 2x^2 + 10x$, $x \in \mathbb{R}$, is an explicit function.

Now, let x, y be two variables where the relation between x and y is expressed by an equation, say, $\phi(x, y) = 0$, then it is called an *implicit* function.

If $x^2 + y^2 = a^2$, then $\phi(x, y) = x^2 + y^2 - a^2 = 0$ is an implicit function.

Here, $y = \pm\sqrt{a^2 - x^2}$, $-a \le x \le a$.

So, we have two explicit functions, viz.,

$$y_1 = \sqrt{a^2 - x^2}, \ -a \le x \le a$$

and $$y_2 = -\sqrt{a^2 - x^2}, \ -a \le x \le a$$

(IV) Parametric Function

Let $x = f(t)$ and $y = \phi(t)$ be two functions of the variable t in the interval $a \le t \le b$.

By eleminating t fom the relations $x = f(t)$, $y = \phi(t)$ we shall have a relation connecting x and y, i.e., y can be regarded as a function of x

Such functions are called parametric functions. If $x = at^2$, $y = 2at$, we can easily see that $y^2 = 4ax$, i.e., $y^2 - 4ax = 0$, which is an implicit function of x and y.

Here, $x = at^2$, $y = 2at$ together constitute a parametric function, t being called *parameter*.

2.9. Composite function : Function of a function

Let $y = f(u)$ be a real valuesd function defined in a domain $D_1 (\subseteq \mathbb{R})$ and $u = g(x)$ be another function with domain D, where $u = g(x) \in D_1$ for all $x \in D$.

Here, $f: D_1 \to \mathbb{R}$, $g: D \to \mathbb{R}$ are two mapping where the range of g i.e., $g(D)$ is a subset of D_1. Then the composite mapping $f \circ g: D \to \mathbb{R}$ can be defined where $(f \circ g)(x) = f\{g(x)\}$, $x \in D$.

This composite mapping is called the *composite funciton* of two real value functins $y = f(u)$, $u \in D_1$ and $u = g(x)$, $u \in D$ where $g(D) \subseteq D_1$ and the composite functions is given by $y = f\{g(x)\}$, $x \in D$ and so the *composite funciton* can be called *a function of function*.

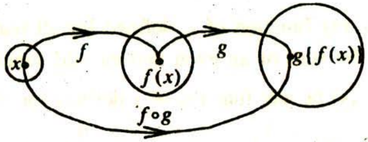

Fig 2.9

2.10. Inverse of Function

Let $f : A \to B$ be a function defined by $y = f(x)$, such that f is *bijective*, i.e., both *one-one* and *onto*. Then there exists as unique function $g : B \to A$, such that $f(x) = y \Leftrightarrow g(y) = x$, for all $x \in A$ an for all $y \in B$. In such case a situation g is said to be the *inverse* of f and we write

$$g = f^{-1} : B \to A$$

If f and g are inverse to each other, $(f \circ g)(x) = (g \circ f)(x) = x$.

i.e., $f\{g(x)\} = g\{f(x)\}$.

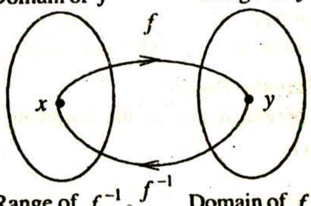

Fig 2.10

2.11 Miscellaneous Worked out Examples.

Ex. 1. Show that $f(x) = \log\left(x + \sqrt{1+x^2}\right)$ is an odd function of x.

Solution : $f(x) + f(-x) = \log\left(x + \sqrt{1+x^2}\right) + \log\left(-x + \sqrt{1+x^2}\right)$

$$= \log\left\{\left(x + \sqrt{1+x^2}\right)\left(-x + \sqrt{1+x^2}\right)\right\}$$

$$= \log\left\{1 + x^2 - x^2\right\} = \log 1 = 0$$

i.e., $f(x) = -f(-x)$

Thus $f(x)$ is an odd function of x.

Ex. 2. Prove that any function of x, defined for all real values of x, can be expressed as the sum of an even and an odd function of x.

Solution : Let, $f(x)$ be any function of x defined for all real values of x.

$\therefore$ We can write, $f(x)=\frac{1}{2}\{f(x)+f(-x)\}+\frac{1}{2}\{f(x)-f(-x)\}$

$$=\phi(x)+\psi(x)\text{, say}$$

Now, as $\phi(x)=\frac{1}{2}\{f(x)+f(-x)\}$,

$$\phi(-x)=\frac{1}{2}\{f(-x)+f(x)\}=\phi(x),$$

So, $\phi(x)$ is an even function of x.

Also, $\psi(x)=\frac{1}{2}\{f(x)-f(-x)\}$

$$\psi(-x)=\frac{1}{2}\{f(-x)-f(x)\}=-\frac{1}{2}\{f(x)-f(-x)\}=-\psi(x),$$

so that $\psi(x)$ is an odd function of x.

Thus any function $f(x)$ of the real variable x can be expressed as the sum of an even function and an odd function of x.

Ex. 3. (*i*) Define a Periodic Function.

(*ii*) Find the period, if any, of the following functions :

(*a*) $\sin(ax)$;

(*b*) $|\cos x|$;

(*c*) $2\cos\frac{1}{3}(x-\pi)$;

(*d*) $\sin^4 x+\cos^4 x$.

Solution :

(*i*) If $f(x)$ be such that, $f(x+k)=f(x)$, for all values of x within the domain of definition of $f(x)$, then $f(x)$ is called a Periodic Function, and k is called its period. Here k is generally taken, if its exists, to be the least number except 0.

(*ii*) (*a*) $f(x)=\sin ax$

$$\because\quad \sin\left\{a\left(x+\frac{2\pi}{a}\right)\right\}=\sin(ax+2\pi)=\sin ax,$$

$\sin ax$ is a periodic function of period $\frac{2\pi}{a}$.

(*b*) Here, $f(x)=|\cos x|=\sqrt{\cos^2 x}=\sqrt{\frac{1}{2}(1+\cos 2x)}$

Since the function $\cos 2x$ is periodic with period

$$\frac{2\pi}{2}=\pi,\ \sqrt{\frac{1}{2}\left(1+\cos^2 x\right)}=|\cos x|$$

is also a periodic function with period π.

(*c*) $f(x)=2\cos\frac{1}{3}(x-\pi)$

$$=2\cos\left(\frac{x}{3}-\frac{\pi}{3}\right)$$

$$=\cos\frac{x}{3}+\sqrt{3}\sin\frac{x}{3}.$$

Obviously, $f(x)$ is a periodic function with period 6π.

(*d*) $f(x)=\sin^4 x+\cos^4 x=\left(\sin^2 x+\cos^2 x\right)^2-2\sin^2 x\cos^2 x$

$$=1-\frac{1}{2}\cdot\sin^2 2x=\frac{3}{4}+\frac{1}{4}\cos 4x.$$

since $\cos 4x$ is a periodic function with period $\frac{2\pi}{4}=\frac{\pi}{2}$, $f(x)$ is a periodic function with period $\frac{\pi}{2}$.

Ex. 4. Find the domain of definition of the following functions :

(*i*) $f(x)=\sqrt{x-1}+\sqrt{5-x}$ [*B. P. 1993*]

(*ii*) $f(x)=\sqrt{(3x-1)(7-x)}$. [*C. P. 1994*]

(*iii*) $f(x)=\sqrt{8+2x-3x^2}$. [*C. P. 1996*]

(*iv*) $f(x)=\log\left(x^2-5x+6\right)$. [*C. P. 1993, 2000, 2006*]

(*v*) $f(x)=\sqrt{\log\frac{4x-x^2}{3}}$. [*B. P. 1995; C. P. 2007*]

(*vi*) $f(x)=\log\sqrt{\frac{5x-x^2}{4}}$. [*C. P. 1997*]

(*vii*) $f(x)=\frac{1}{\sqrt{|x|-x}}$. [*C. P. 2005*]

Solution : (*i*) $f(x)=\sqrt{x-1}+\sqrt{5-x}$

Since $f(x)$ is real, the values of x must be such that both $\sqrt{x-1}$ and $\sqrt{5-x}$ are real quantities, which requires that $(x-1)\geq 0$ and $(5-x)\geq 0$

i.e., $x\geq 1$ and $x\leq 5$.

So, domain of definition of $f(x)$ is $1\leq x\leq 5$ or, [1, 5]

(*ii*) $f(x)=\sqrt{(3x-1)(7-x)}$

In order that $f(x)$ may be defined,

$(3x-1)(7-x)$ must be non-negative.

i.e., $(3x-1)(7-x)\geq 0$

i.e., either, $(3x-1)\geq 0$ and $(7-x)\geq 0$ ... (1)

or, $(3x-1)\leq 0$ and $(7-x)\leq 0$ ... (2)

From (1), $\frac{1}{3}\leq x\leq 7$

Again from (2) $x\geq\frac{1}{3}$ and $7\leq x$

But these two relations cannot hold simultaneously.

So, the domain of definition of $f(x)$ is $\frac{1}{3}\leq x\leq 7$

(*iii*) $f(x)=\sqrt{8+2x-3x^2}$

For $f(x)$ to be defined,

$8+2x-3x^2\geq 0,$

i.e., $(3x+4)(2-x)\geq 0$

i.e., either, $(3x+4)\geq 0$ and $(2-x)\geq 0$ ... (1)

or, $(3x+4)\leq 0$ and $(2-x)\leq 0$ ... (2)

From (1) $-\frac{4}{3}\leq x\leq 2$.

Again, from (2) $x\leq -\frac{4}{3}$ and $x>2$

But these relations cannot hold simultaneously.

So, domain of definition of $f(x)$ is $-\frac{4}{3}\leq x\leq 2$.

(*iv*) $f(x) = \log\left(x^2 - 5x + 6\right)$

$f(x)$ is defined for all real values of x that make $x^2 - 5x + 6 > 0$

or, $(x-2)(x-3) > 0$

This inequality holds for all real values of x, except those that lie between 2 and 3, including $x = 2$ and $x = 3$.

So, domain of definition of $f(x)$ is all real values of x, except $2 \le x \le 3$.

(*v*) $f(x) = \sqrt{\log \dfrac{4x - x^2}{3}}$

$f(x)$ is defined, if $\log \dfrac{4x - x^2}{3} \ge 0$, i.e., $\ge \log 1$

or, $4x - x^2 \ge 3$ or, $x^2 - 4x + 3 \le 0$ or, $(x-1)(x-3) \le 3$

This inequality is satisfied if $0 < x < 4$.

(*vi*) $f(x) = \log\sqrt{\dfrac{5x - x^2}{4}}$

$f(x)$ is defined for those values of x which make

$5x - x^2 = x(5-x) > 0$

Domain of $f(x)$ is $0 < x < 5$.

(*vii*) $f(x) = \dfrac{1}{\sqrt{|x| - x}}$

$f(x)$ is defined, when $|x| - x > 0$

i.e., $|x| > x$

and this inequality is satisfied for all values of $x < 0$.

So, domain of definition of $f(x)$ is $(-\infty, 0)$ or, $-\infty < x < 0$.

Ex. 5. Show that the domain of definition of the function

$$f(x) = \log \frac{1-x}{1+x} \text{ is } -1 < x < 1.$$

Also, show that for $x_1, x_2 \in (-1, 1)$, $f(x_1) + f(x_2) = f\left(\dfrac{x_1 + x_2}{1 + x_1 x_2}\right)$

Solution : $\because \log x$ is defined for positive values of x only,

$f(x)$ is defined only when $\dfrac{1-x}{1+x} > 0$

i.e., when both $(1-x) > 0$ and $(1+x) > 0$

or, when $(1-x) < 0$, and $(1+x) < 0$

These two sets of inequalities are satisfied for $-1 < x < 1$.

Thus the domain of definition of $f(x)$ is $(-1,\ 1)$.

$$\text{Now, } f\left(\frac{x_1+x_2}{1+x_1x_2}\right) = \log\frac{1-\dfrac{x_1+x_2}{1+x_1x_2}}{1+\dfrac{x_1+x_2}{1+x_1x_2}}$$

$$= \log\frac{(1-x_1)(1-x_2)}{(1+x_1)(1+x_2)}$$

$$= \log\frac{1-x_1}{1-x_1} + \log\frac{1-x_2}{1-x_2}$$

$$= f(x_1) + f(x_2).$$

Ex. 6. Find the domain and the range of the function $f(x) = \dfrac{|x|}{x}$.

[*C. P. 1995, 2008*]

Solution : Here, $f(x) = \dfrac{|x|}{x}$.

Obviously, $f(x)$ is defined for all real values of x, except $x = 0$.
Hence the domain of $f(x)$ is $-\infty < x < \infty$, except $x = 0$

Again, $\because\ |x| = x$, when $x > 0$

$= -x$, when $x < 0$,

$\dfrac{|x|}{x} = 1$ when $x > 0$ and $\dfrac{|x|}{x} = -1$ when $x < 0$.

So that range of $f(x)$ is $[-1,\ 1]$.

Ex. 7. If $f(x) = \max.\left(x,\ \dfrac{1}{x}\right)$ for $x > 0$, where max. $(a,\ b)$ denotes the greater of the two real numbers a and b, find the value of $f(c)\cdot f\left(\dfrac{1}{c}\right)$ for $c > 0$.

[*C. P. 1988*]

Solution : $\because\ f(x) = \max.\left(x,\ \dfrac{1}{x}\right)$,

when $c \geq 1$, $f(c) = \max.\left(c, \dfrac{1}{c}\right) = c$

and when $0 < c < 1$, $f(c) = \max.\left(c, \frac{1}{c}\right) = \frac{1}{c}$.

Again, $f\left(\frac{1}{c}\right) = \max.\left(\frac{1}{c}, c\right) = \max.\left(c, \frac{1}{c}\right) = f(c)$

$\therefore$ when $c \geq 1$, $f(c)f\left(\frac{1}{c}\right) = \{f(c)\}^2 = c^2$

and when $0 < c < 1$, $f(c)f\left(\frac{1}{c}\right) = \left(\frac{1}{c}\right)^2 = \frac{1}{c^2}$.

Ex. 8. If $f(x) = \frac{|x|}{x}$ and $c\,(\neq 0)$ be any real number, show that

$$|f(c) - f(-c)| = 2.$$ [*C. P. 1994*]

Solution : We have $|x| = x$, when $x > 0$
$= -x$, when $x < 0$

So, when $c > 0$, $f(c) = 1$ and $f(-c) = -1$

and $|f(c) - f(-c)| = |1 + 1| = 2$ ... (1)

Again, when $c < 0$, $f(c) = -1$ and $f(-c) = 1$

So, $|f(c) - f(-c)| = |-1 - 1| = 2$ ... (2)

Combining (1) and (2), if $c\,(\neq 0)$ be any real number,

$$|f(c) - f(-c)| = 2.$$

Ex. 9. If $f(x) = x + |x|$, find $f(3)$ and $f(-3)$ [*B. P. 1995*]

Solution : $f(3) = 3 + |3| = 3 + 3 = 6$

$f(-3) = 3 + |-3| = 3 + 3 = 6.$

Ex. 10. If $2f\left(\frac{1}{x}\right) - f(x) = 5x$, find the value of $f\left(x + \frac{1}{x}\right)$.

Solution : We have, $2f\left(\frac{1}{x}\right) - f(x) = 5x$ ... (1)

Replacing x by $\frac{1}{x}$,

$$2f(x) - f\left(\frac{1}{x}\right) = \frac{5}{x} \quad (2)$$

From (1) and (2), $f(x)=\frac{5x}{3}+\frac{10}{3x}$

$$\therefore \quad f\left(x+\frac{1}{x}\right)=\frac{5}{3}\left(x+\frac{1}{x}\right)+\frac{10}{3\left(x+\frac{1}{x}\right)}$$

$$=\frac{5\left\{\left(x+\frac{1}{x}\right)^2+2\right\}}{3\left(x+\frac{1}{x}\right)}.$$

Ex. 11. (*i*) The function f satisfies the equation $f(x+y)=f(x)+f(y)$. Show that

(*a*) $f(0)=0$,

(*b*) $f(x)$ is an odd function,

(*c*) if x is an integer and $f(1)=a$, then $f(x)=ax$

(*ii*) Find the natural number a for which

$$\sum_{k=1}^{n} f(a+k)=16\left(2^n-1\right),$$

where the function f satisfies the relation $f(x+y)=f(x)\cdot f(y)$ for all natural x, y and $f(1)=2$.

Solution : Given that $f(x+y)=f(x)+f(y)$ ... (1)

(*a*) Putting $x=y=0$ on both sides of (1),

$f(0)=f(0)+f(0)$, *i.e.*, $f(0)=0$

(*b*) Putting $y=-x$ on both the sides of (1)

$f(0)=f(x)+f(-x)$

i.e., $f(0)=f(x)+f(-x)$, *i.e.*, $f(x)=-f(-x)$ $\because f(0)=0$

$\therefore$ $f(x)$ is an odd function.

(*c*) Again, putting $x=y=1$ on both the sides of (1),

$f(2)=f(1)+f(1)=2f(1)=2a$ $[\because f(1)=a]$

$f(3)=f(2+1)=f(2)+f(1)=2a+a=3a$

If x be a positive integer, then

$f(x) = f(1+x-1) = f(1) + f(x-1) = a + f(x-1)$

$f(x-1) = f(1+x-2) = f(1) + f(x-2) = a + f(x-2)$

Similarly, $f(x-2) = a + f(x-3) = 2a + f(x-4)$

Thus, $f(x) = a + f(x-1) = 2a + f(x-2) = 3a + f(x-3)$

$= 4a + f(x-4)$

... ...

$= (x-1)a + f(1)$

$= (x-1)a + a = ax$

If x be a negative integer, putting $x = -y$, where y is a positive integer, we get

$f(x) = f(-y) = -f(y)$ [$\because$ $f(x)$ is an odd function]

$= -ay = a(-y) = ax$

Thus, when x is any integer, $f(x) = ax$.

(*ii*) Here, $f(x+y) = f(x) \cdot f(y)$ (1)

Putting $x = y = 1$,

$f(2) = f(1) \cdot f(1) = 2 \cdot 2 = 2^2$ $[\because f(1) = 2]$

Putting $x = 2,\ y = 1$

$f(3) = f(2) \cdot f(1) = 2^2 \cdot 2 = 2^3$

Similarly, $f(4) = f(3) \cdot f(1) = 2^3 \cdot 2 = 2^4$

and in general, $f(n) = 2^n$

Now, $\because \sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$

$\because \sum_{k=1}^{n} f(a) \cdot f(k) = 16(2^n - 1)$ [from (1)]

or, $f(a)[f(1) + f(2) + f(3) + \cdots + f(n)] = 16(2^n - 1)$

or, $f(a)[2 + 2^2 + 2^3 + \cdots + \cdots + 2^n] = 16(2^n - 1)$

or, $f(a) \times \dfrac{2(2^n - 1)}{2-1} = 16(2^n - 1)$

or, $f(a) = 8 = 2^3 = f(3)$

$\therefore a = 3$.

Ex. 12. Solve : $4\{x\} = x + [x]$,

where $\{x\}$ and $[x]$ denote the fractional and integral parts of a real number x respectively.

Solution : If x is any real number, $x = [x] + \{x\}$

$\because \; 4\{x\} = x + [x]$

$$4\{x\} = [x] + \{x\} + [x]$$

or, $3\{x\} = 2[x]$ *i.e.,* $\{x\} = \frac{2}{3}[x]$ . . . (1)

Numerical value of $\{x\}$ is less than 1. So, the only integral values of $[x]$ which will satisfy (1) are 1 and (-1).

So, $\{x\} = \frac{2}{3}$, when, $[x] = 1$

and $\{x\} = -\frac{2}{3}$, when, $[x] = -1$

$\because \; x = 1 + \frac{2}{3} = \frac{5}{3}$ or, $x = -1 - \frac{2}{3} = -\frac{5}{3}$

Ex.13. If $f(x) = \cos(\log x)$, then show that

$$f(x)\cdot f(y) - \frac{1}{2}\left\{f\left(\frac{x}{y}\right) + f(xy)\right\} = 0$$

Solution : $\because \; f(x) = \cos(\log x)$

$f(x)f(y) = \cos(\log x)\,\cos(\log y)$ . . . (1)

and $f\left(\frac{x}{y}\right) + f(xy) = \cos\left\{\log\left(\frac{x}{y}\right)\right\} + \cos\{\log(xy)\}$

$$= 2\cos\frac{\log\left(\frac{x}{y}\right) + \log(xy)}{2}\cdot\cos\frac{\log\left(\frac{x}{y}\right) - \log(xy)}{2}$$

$$= 2\cos\frac{\left(\log\frac{x}{y}\cdot xy\right)}{2}\cdot\cos\frac{\log\frac{x}{y}\cdot\frac{1}{xy}}{2}$$

$$= 2\cos\left(\frac{\log x^2}{2}\right)\cdot\cos\left(\frac{1}{2}\log\frac{1}{y^2}\right)$$

$$= 2\cos(\log x)\cos(\log y) \quad [\because \; \cos(-\theta) = \cos\theta]$$

$$\therefore\ f(x)\cdot f(y)-\frac{1}{2}\left\{f\left(\frac{x}{y}\right)+f(xy)\right\}$$

$$=\cos(\log x)\cos(\log y)-\frac{1}{2}\cdot 2\cos(\log x)\cos(\log y)=0\cdot$$

Ex. 14. *If* $f(x+3)=2x^2-3x+1$, *find* $f(x+1)$

Solution : $f(x+1)=f(x-2+3)$

$$=2(x-2)^2-3(x-2)+1$$

$$=2x^2-11x+15.$$

Ex. 15. *Find the range of the following functions :*

(*i*) $f(x)=\dfrac{1}{2-\cos 3x}$;

(*ii*) $\cos(2x-\beta)+\sin(2x-\beta)$;

(*iii*) $\log_2\dfrac{\sin x-\cos x+3\sqrt{2}}{\sqrt{2}}$.

Solution : (*i*) $y=f(x)=\dfrac{1}{2-\cos 3x}$

or, $\dfrac{1}{y}=2-\cos 3x$

or, $\cos 3x=2-\dfrac{1}{y}$

$\because\ -1\le\cos 3x\le 1$,

$-1\le 2-\dfrac{1}{y}\le 1$

or, $-3\le -\dfrac{1}{y}\le -1$

or, $\dfrac{1}{3}\le y\le 1$. $\quad [\,y>0,\ \because\ -1\le\cos 3x\le 1\,]$

Required range is $\left[\dfrac{1}{3},\ 1\right]$.

(*ii*) Let $y=\cos(2x-\beta)+\sin(2x-\beta)$

$$=\sqrt{2}\left[\cos(2x-\beta)\cos\frac{\pi}{4}+\sin(2x-\beta)\sin\frac{\pi}{4}\right]$$

$$= \sqrt{2}\cos\left(2x+\frac{\pi}{4}-\beta\right)$$

$$\because \quad -1 \le \cos\left(2x+\frac{\pi}{4}-\beta\right) \le 1$$

$-1 \le \dfrac{y}{\sqrt{2}} \le 1$, *i.e.* , $-\sqrt{2} \le y \le \sqrt{2}$

range is $\left[-\sqrt{2},\ \sqrt{2}\right]$.

(*iii*) Let , $y = \log_2 \dfrac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}}$

$$\therefore \quad 2^y = \frac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}} = \sin\left(x-\frac{\pi}{4}\right)+3$$

or, $2^y - 3 = \sin\left(x-\dfrac{\pi}{4}\right)$

$$\because \quad -1 \le \sin\left(x-\frac{\pi}{4}\right) \le 1$$

$-1 \le 2^y - 3 \le 1$

or, $2 \le 2^y < 4$

i.e., $2^1 \le 2^y < 2^2$

i.e., $1 \le y \le 2$ range is $[1,\ 2]$.

Ex. 16. Find the domain of $f(x) = \sqrt{4+x} + \sqrt{9-x}$. [*B. P. 2004*]

Solution : $f(x)$ has real value if $4+x \ge 0$ and $9-x \ge 0$.

i.e., $x \ge -4$ and $x \le 9$

i.e., if $x \in [-4, \infty)$ and $x \in (-\infty, 9]$.

Now $[-4, \infty) \cap (-\infty, 9] = [-4, 9]$.

Hence, the domain of $f(x)$ is $[-4, 9]$.

Ex. 17. Find the domain of function $f(x) = \sqrt{\dfrac{1-|x|}{2-|x|}}$.

Solution : $f(x)$ will have real value if

$$\frac{1-|x|}{2-|x|} \geq 0 \qquad \ldots \quad (1)$$

and $2-|x| \neq 0$ ... (2)

The relation (1) holds if

(I) $1 \geq |x|$ and $2 > |x|$, or, if (II) $1-|x| \leq 0$, and $2-|x| < 0$

(I) if $1 \geq x$ and $2 > |x|$

then $-1 \leq x \leq 1$ and $-2 < x < 2$.

i.e., $x \in [-1, 1]$ and $x \in (-2, 2)$

i.e., $x \in [-1, 1] \cap (-2, 2)$

i.e., $x \in [-1, 1]$ ... (3)

(II) If $1-|x| \leq 0$ and $2-|x| < 0$

then $1 \leq |x|$ and $2 < |x|$

i.e., $|x| > 2$

i.e., $x > 2$ or, $x < -2$

i.e., $x \in (-\infty, -2) \cup (2, \infty)$ ... (4)

Now (2) holds if $|x| \neq 2$, i.e., if $x \neq \pm 2$ ... (5)

From (3), (4) and (5) the required domain is

$$[-1, 1] \cup (-\infty, -2) \cup (2, \infty).$$

Ex. 18. Find the domain of the function $f(x) = \log_{(2x-5)}(x^2 - 3x - 10)$.

Solution : $f(x)$ is defined if $x^2 - 3x - 10 > 0$ ... (1)

and $2x - 5 > 0$, $2x - 5 \neq 1$ ... (2)

Relation (1) holds if $(x-5)(x+2) > 0$

i.e., if either, $(x-5) > 0$ and $(x+2) > 0$

or, if $(x-5) < 0$, and $(x+2) < 0$

i.e., if $x > 5$ and $x > -2$, or, if $x < 5$ and $x < -2$

i.e., if $x < 5$ or $x < -2$

So relation (1) is valid if $x \in (5, \infty) \cup (-\infty, -2)$ ... (3)

Again, relation (2) holds if $x > \frac{5}{2}$ and $x \neq 3$

i.e., if $x \in (\frac{5}{2}, \infty) - \{3\}$ ... (4)

Hence the required domain is the common portion of (3) and (4), i.e., $(5, \infty)$.

Ex. 19. Find the range of the function $f(x) = \log_2 \dfrac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}}$.

Solution : Let $y = \log_2 \dfrac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}}$

or, $2^y = (\frac{1}{\sqrt{2}}\sin x - \frac{1}{\sqrt{2}}\cos x) + 3$

or, $2^y - 3 = \sin(x - \frac{\pi}{4})$

Since for all real vaues of x

$$-1 \leq \sin(x - \frac{\pi}{4}) \leq 1, \qquad -1 \leq 2^y - 3 \leq 1$$

or, $2 \leq 2^y \leq 4 \Rightarrow 2^1 \leq 2^y \leq 2^2 \therefore 1 \leq y \leq 2$, since the base is $2 > 1$.

Hence the range is $[1, 2]$.

Ex. 20. Find the period of each of the following functions :

(i) $\cot\frac{x}{2}$, (ii) $3\sin\frac{x}{2} + 4\cos\frac{x}{2}$.

Solution : (i) $\because$ $\cot(\pi + x) = \cot x$, , $\cot x$ is a periodic function of period π.

So, $\cot\frac{x}{2}$ is also a periodic function, the period being $\dfrac{\pi}{\frac{1}{2}} = 2\pi$

(ii) $\sin(2\pi + x) = \sin x$, so $\sin x$ is a periodic function of the period 2π and hence $\sin\frac{x}{2}$ is a periodic function of the period $\dfrac{2\pi}{\frac{1}{2}} = 4\pi$.

Again $\cos\frac{x}{4}$ is a periodic function of period $\dfrac{2\pi}{\frac{1}{4}} = 8\pi$.

Since 8π is a rational multiple of 4π, $3\sin\frac{x}{2} + 4\cos\frac{x}{2}$ is a periodic function, the period being the l.c.m. of 4π and 8π, i.e., 8π.

Ex. 21. Show that $\sin^3 x + \cos^3 x$ is a periodic function. Find the period.

Solution : $\sin^3 x + \cos^3 x = \frac{1}{4}(3\sin x - \sin 3x) + \frac{1}{4}(\cos 3x + 3\cos x)$

$$= \tfrac{1}{4}\{3(\sin x + \cos x) + (\cos 3x - \sin 3x)\}$$

Both $\sin x$ and $\cos x$ are periodic functions, period of each of the them being 2π.

Again $\cos 3x$ and $\sin 3x$ are also periodic functions, period of each of them being $\frac{2\pi}{3}$.

Now l.c.m. of 2π and $\frac{2\pi}{3}$ is 2π.

Hence $f(x)$ is a periodic function of period 2π.

Ex. 22. Find the inverse of the function $f(x) = \log_e(x + \sqrt{x^2 + 1})$.

Solution : Let $y = \log_e(x + \sqrt{x^2 + 1})$.

then $e^y = x + \sqrt{x^2 + 1}$

or, $x^2 + 1 = (e^y - x)^2 = e^{2y} - 2xe^y + x^2$

or, $e^{2y} - 2xe^y = 1$

or, $x = \dfrac{e^{2y} - 1}{2e^y}$

Interchanging x and y, we have $f^{-1}(x) = \dfrac{e^{2x} - 1}{2e^x} = \frac{1}{2}(e^x - e^{-x})$.

EXAMPLES-II

1. If $y = 6$ for every value of x, can y be regarded as a function of x?
2. If y = the number of windows in the house numbered x on a particular road, is y a function of x?
3. Given $f(x) = x^2 - 10x + 3$, find $f(0)$ and $f(-2)$.
4. If $f(x) = \sec x + \cos x$, then show that $f(x) = f(-x)$.
5. If $f(x) = b\dfrac{x - a}{b - a} + a\dfrac{x - b}{a - b}$, then show that
$$f(a) + f(b) = f(a + b).$$
6. If $f(x) = x^2 - 3x + 7$, then show that
$$\{f(x + h) - f(x)\}/h = 2x - 3 + h.$$
7. Show that

(i) $\dfrac{1 - \tan x}{\cos x - \sin x}$ is not defined for $x = \frac{1}{4}\pi$.

(ii) $\sqrt{x^2 - 5x + 6}$ is not defined for $2 < x < 3$.

(iii) $\dfrac{x^2 - 5x + 6}{x^2 - 8x + 12}$ is not defined for $x = 2$.

Also find $f(-5)$ and $f(6)$ in this case.

8. Draw the graphs of the following functions :

(i) $y = 1$ when $x > 0$,
$= 0$ when $x = 0$,
$= -1$ when $x < 0$.

(ii) $y = x^2$ for $x \neq 1$,
$= 2$ for $x = 1$.

(iii) $f(x) = 1$ when x is an integer,
$= 0$ when x is not an integer.

(iv) $y = \cos \dfrac{1}{x}$.

(v) $y = x \cos \dfrac{1}{x}$.

(vi) $y = x - [x]$,

where $[x]$ denotes the greatest integer not greater than x.

(vii) $f(x) = \sqrt{1 - (x-1)^2}$.

(viii) $f(x) = \dfrac{|x|}{x}$.

(ix) $f(x) = 1 - \dfrac{\sin \pi x}{\sin \pi x}$.

(x) $f(x) = \sqrt{x^2}$,

where the positive sign of the square root is to be taken.

(xi) $f(x) = 0$ when $|x| > 1$,
$= 1 + x$ when $-1 \leq x \leq 0$,
$= 1 - x$ when $0 < x \leq 1$.

9. (i) Show that $f(x) = \sec x$, in the interval $0 \leq x < \frac{1}{2}\pi$, has the lower bound 1, and no upper bound.

(ii) Show that $f(x) = 2x^2 + 4x + 6$, in the interval $0 \leq x \leq 1$, has the lower bound 6 and the upper bound 12.

(iii) Show that $f(x)=\left(\dfrac{1-\theta}{1+\theta x}\right)^n$, when $0<\theta<1$, $-1<x<1$ and n a positive integer, is bounded.

10. (i) Show that $f(x)=\dfrac{x}{x+1}$ is monotone ascending for $x>0$.

(ii) Show that

$$f(x)=\frac{1}{x+1}+\frac{1}{x+2}+\dots\dots+\frac{1}{x+n},\quad x>0,$$

is monotone descending.

(iii) Show that $f(x)=\left(1+\dfrac{1}{x}\right)^x$, $x>0$, is monotone ascending.

11. Given the relation $y^2-6y-x+7=0$, which of the following statements is true ?

(i) The equation defines x as a function of y for all values of y.

(ii) The equation defines y as a function of x for all values of x.

12. A taxi company charges one rupee for one kilometre or less from start, and at a rate of (i) 50 paise per kilometre (ii) 50 paise per kilometre or any fraction thereof, for additional distance. Express analytically the fare F (in rupee) as a function of the distance d (in kilometre), and draw the graph of the function.

13. Find the domain of definition of the following functions :

(i) $f(x)=1$, when x is rational.
$\quad\quad = 0$, when x is irrational.

(ii) $f(x)=\log\dfrac{1-x}{1+x}$.

(iii) $f(x)=\dfrac{|x|}{x}$. [C.P. 1995]

(iv) $f(x)=\sqrt{\log\dfrac{5x-x^2}{4}}$. [C.P. 1997]

(v) $f(x)=\dfrac{1}{\sqrt{|x|-x}}$.

(vi) $f(x)=\sqrt{x-1}+\sqrt{5-x}$. [B.P. 1993]

(vii) $f(x)=\log\left(x^2-5x+6\right)$. [C.P. 1993, 2000]

(viii) $f(x) = \sqrt{\dfrac{1 - |x|}{2 - |x|}}$.

(ix) $f(x) = \log_3 \dfrac{1}{\sqrt{x^2 - 1}}$.

14. If the function f satisfies the relation $f(x + y) = f(x) + f(y)$, for all real values of x and y, prove that :

(*i*) $f(0) = 0$;

(*ii*) $f(-x) = -f(x)$;

(iii) $f(x) = ax$, where x is any integer and $f(1) = a$.

15. If $f(x) = 2^{x(x-1)}$ for $1 \le x < \infty$, show that

$$f^{-1}(x) = \frac{1}{2}\left(1 + \sqrt{1 + 4 \log_2 x}\right).$$

ANSWERS

1. Yes. 2. Yes. 3. 3; 27. 7. (iii) $\frac{8}{11}$; does not exist.

11. (i) True. (ii) Not true: true only for values of $x^3 - 2$.

12. (i) $\begin{cases} F = 1, \text{ for } 0 < d \le 1; \\ F = 1 + \frac{1}{2}(d - 1), \text{ for } d > 1 \end{cases}$

(ii) $\begin{cases} F = 1, \text{ for } 0 < d \le 1; \\ F = 1 + \frac{1}{2}m, \text{ for } m < d \le m + 1, \text{ where } m \text{ is a positive integer.} \end{cases}$

13. (i) The set of all rational numbers ;

(ii) $(-1, 1)$; (iii) $-\infty < x < \infty$, except $x = 0$;

(iv) $0 < x < 5$; (v) $-\infty < x < 0$;

(vi) $1 \le x \le 5$; (vii) all x, except $2 \le x \le 3$;

(viii) $-1 \le x \le 1$, $2 < x < \infty$, $-\infty < x < -2$;

(ix) $-1 < x < \infty$.

3 LIMIT

3.1. Introduction.

The idea of '*LIMIT*' forms the most outstanding concept in Calculus and plays an important role in the development of the subject. It is this process of limit, or *limiting operation,* which marks the line of difference of Calculus with Algebra, the latter being based upon the four fundamental operations, *viz.*, addition, subtraction, multiplication and division. The real essence and strength of this subject, an important part of Mathematical Analysis, lies in the concept of limit upon which is built the new and broad structure of Calculus.

3.2. Limit of an Independent Variable.

Suppose, x is a real variable which takes up different values $x = 1 \cdot 9, 1 \cdot 99, 1 \cdot 999, 1 \cdot 9999, 1 \cdot 99999, \ldots$ It is obvious that as the variable x passes through successive values, the difference of x from a real number 2, gradually diminishes and finally *becomes* and *remains* less than *any pre-assigned positive quantity,* however small. We say that x *approaches or tends to* the value 2, remaining always less than 2.

Again, we consider the values $x = 2 \cdot 1, 2 \cdot 01, 2 \cdot 001, 2 \cdot 0001, 2 \cdot 00001, \ldots$ etc. Here also the difference of the successive values of x, from the real number 2, gradually diminishes and ultimately *becomes* and *remains* less than any pre-assigned positive quantity, however small. In this case, we say x *approaches* or, *tends to* 2, remaining always greater than 2.

In either case, $|x - 2| < \varepsilon$ where ε is a pre-assigned positive quantity, however small, we may imagine and we write :

$$\text{limit } x \to 2 \text{ or, } \lim x \to 2 \text{ or, Lt } x \to 2.$$

3.3. Geometrical Idea of the Limit of a Variable.

Suppose, the point A on the real number axis $X'X$ represents the real number $x = 2$, while a point P represents a real variable x. Further, let us suppose that the points $A_1, A_2, A_3, \ldots\ldots$ etc. represent the real numbers $1 \cdot 9, 1 \cdot 99, 1 \cdot 999, \ldots$ etc. respectively and the points $B_1, B_2, B_3, \ldots\ldots$ etc. represents the real numbers $2 \cdot 1, 2 \cdot 01, 2 \cdot 001, \ldots\ldots$ etc.

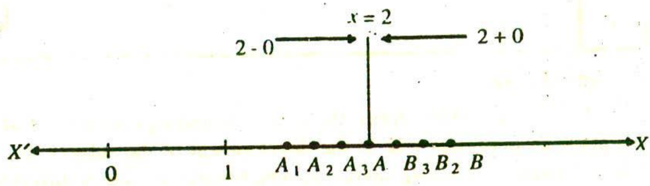

Fig 3.3.1

The variable x gradually approaches towards the real number 2 by assuming the successive values $1 \cdot 9$, $1 \cdot 99$, $1 \cdot 999$, ... etc. and the point P gradually approaches towards A from its *left side* after passing successively through the points $A_1, A_2, A_3, \ldots\ldots$ etc., but it never meets the point A: In this case, we say that the point *P approaches A from the left*, and denote it by the symbol $x \to a - 0$ or, simply by $x \to a-$. Again, let us consider the sequence of numbers $2 \cdot 1, 2 \cdot 01, 2 \cdot 001, \ldots\ldots$ etc. When the variable x gradually approaches towards 2 by assuming the successive values $2 \cdot 1$, $2 \cdot 01$, $2 \cdot 001, \ldots\ldots$ etc., then the point P gradually approaches towards A from the right side after passing successively through the points $B_1, B_2, B_3, \ldots\ldots$ etc., but it never meets the point A. In this case, we say that the point *P approaches A from the right* and denote by the symbol $x \to a + 0$, or, simply by $x \to a+$.

Note 1. If $x \to a - 0$ (or, $x \to a-$), the assumed values of x are always *less than a* and the numerical difference between the assumed value of x and a, i.e., $|x - a|$ is *less than* any pre-assigned positive quantity, however small, but *x is not equal to a* ($x \neq a$).

Note 2. If $x \to a + 0$ (or, $x \to a+$), the assumed values of x are always *greater than a* and the numerical difference between the assumed value of x and a, i.e., $|x - a|$ is *less than* any pre-assigned positive quantity, however small, but *x is not equal to a* ($x \neq a$).

Note 3. The symbols : $x \to a$ is read as "x tends to a",

$x \to a - 0$ (or, $x \to a-$) is read as "x tends to a from the left"

and $x \to a + 0$ (or, $x \to a+$) is read as "x tends to a from the right".

3.4. Idea of Limit of a Function.

Let $y = f(x)$ be a function of a real variable x. A question may arise, what happens to the function $f(x)$ as $x \to a$?

We examine the case by an example. Consider the function defined as follows :

1. $f(x) = \frac{x^2 - 4}{x - 2}$, when $x \neq 2$.

$= 3$, when $x = 2$. ... (1)

Obviously, when $x \neq 2$, $f(x) = x + 2$.

We prepare the following table showing values of x and $f(x)$, where he variable x approaches 2 either from the left or from the right.

x	1·9	1·99	1·999	...	2·1	2·01	2·001	...
$f(x)$	3·9	3·99	3·999	...	4·1	4·01	4·001	...

It is clear from the above table that as x gradually approaches 2, assuming values either less than or greater than 2, and sufficiently close to 2, the values of $f(x)$ gradually approach the number 4, or, in other words, $|f(x) - 4|$, i.e., the numerical difference between the value of $f(x)$ and 4 can be made less than any pre-assigned positive number, however small.

We write $f(x) \to 4$, when $x \to 2$ or, symbolically

$$\underset{x \to 2}{Lt} f(x) = 4$$

or,

$$\underset{x \to 2}{Lt} \frac{x^2 - 4}{x - 2} = 4.$$

It is interesting to note that $\underset{x \to 2}{Lt} \frac{x^2 - 4}{x - 2} = 4$ does *not necessarily imply* that $f(2) = 4$.

In the example cited above, $\underset{x \to 2}{Lt} \frac{x^2 - 4}{x - 2} = 4$, but $f(2) = 3$, i.e., $\underset{x \to 2}{Lt} f(x) \neq f(2)$.

2. If, on the other hand, we define $f(x)$ as

$$f(x) = \frac{x^2 - 4}{x - 2} \text{ when } x \neq 2$$

$= 4$. when $x = 2$ ... (2)

then, $\underset{x \to 2}{Lt} f(x) = 4$ and $f(2) = 4$, i.e., the limiting value is numerically equal to the value of the function at the point in question.

Here, $\underset{x \to 2}{Lt} f(x) = f(2)$.

Further, if $$f(x) = \frac{x^2 - 4}{x - 2}, \quad \ldots \quad (3)$$

$f(x)$ becomes *undefined* at $x = 2$, but as discussed earlier, $\underset{x \to 2}{Lt} f(x) = 4$,

i.e., $\underset{x \to 2}{Lt} f(x)$ exists and has a finite value 4.

Thus, we see that the limiting value of a function at any specific point *is in no way dependent* on the value of the function at that point. The distinction between $\underset{x \to a}{Lt} f(x)$ and $f(a)$ has been discussed in details and explained with illustrations in art. 3.6.

The graph of the function $f(x)$ as defined in (1) is shown in fig. 3.4.1 and the *graphical representation* of the $f(x)$ as defined in (2) is shown in fig. 3.4.2 below.

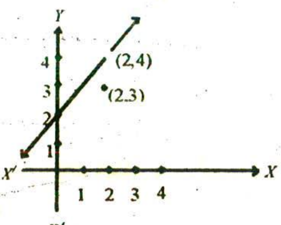

Fig 3.4.1

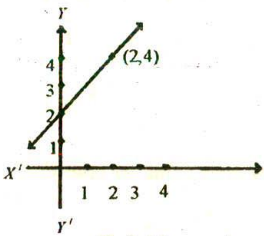

Fig 3.4.2

By the expressions 'the variable x approaches the constant number a' or simply '*x tends to the value a*', we mean that x assumes successive values whose numerical differences from a, *i.e.*, the successive values of $|x - a|$, become gradually less and less, and $|x - a|$ can ultimately be taken to be less than any small quantity we can name or imagine (*i.e.*, less than any pre-assigned positive quantity, however small), and we denote this by the symbol $x \to a$.

Here the successive values of x may be greater than as well as less than a.

If the variable x, remaining always greater than a, approaches a, such that ultimately $x - a$ is less than any pre-assigned positive quantity, however small (but $x \neq a$ actually), then we say that x *approaches or tends to 'a' from the right*, and denote it by the symbol $x \to a + 0$ or simply by $x \to a +$.

Similarly, when x is less than a always, and $a - x$ is ultimately less than any pre-assigned positive quantity, however small, we say that x *tends to 'a' from the left*, and denote it by $x \to a - 0$ or simply by $x \to a -$.

Illustraion : When the successive values of x are $1 \cdot 9, 1 \cdot 99, 1 \cdot 999 \ldots \ldots$, we say $x \to 2 - 0$, and when the successive values of x are $2 \cdot 1, 2 \cdot 01, 2 \cdot 001 \ldots \ldots$, we say $x \to 2 + 0$. If the successive values of x are $2 + 1,\ 2 + \frac{1}{2},\ 2 + \frac{1}{3},\ 2 + \frac{1}{4},\ 2 + \frac{1}{5}, \ldots$, we say $x \to 2+$.

3.5. Limit of a function.

$\underset{x \to a}{Lt} f(x)$: When x approaches a constant quantity a from either side (but $\neq a$) if there exists a definite finite number l towards which $f(x)$ approaches[1], such that the numerical difference of $f(x)$ and l can be made as small as we please (*i.e.*, less than any pre-assigned positive quantity, however small) by taking x sufficiently close to a, then l is defined as the limit of $f(x)$ as x tends to a. This is symbolically written as $\underset{x \to a}{Lt} f(x) = l$.

Mathematically speaking, $\underset{x \to a}{Lt} f(x) = l$, *provided, given any pre-assigned positive quantity* ε, *however small, we can determine another positive quantity* δ *(depending on* ε*) such that* $|f(x) - l| < \varepsilon$ *for all values of x satisfying* $0 < |x - a| \leq \delta$, *i.e. whenever* $a - \delta \leq x \leq a + \delta$, but $x \neq a$.

Ex. (i). $\underset{x \to 3}{\text{Lt}} \dfrac{x^2 - 9}{x - 3} = 6$. For, if $x = 3 + \delta_1$, whether δ_1 be positive or negative, $\dfrac{x^2 - 9}{x - 3} = \dfrac{(x-3)(x+3)}{x-3} = \dfrac{\delta_1(6 + \delta_1)}{\delta_1} = 6 + \delta_1$, and, by taking δ_1 numerically small enough, the difference of $\dfrac{x^2 - 9}{x - 3}$ and 6 can be made

[1] As a particular case $f(x)$ may remain always equal to l when x is sufficiently close to a.

as small as we like. It may be noted here that however small δ_1 may be, since $\delta_1 \neq 0$, we can cancel the factor $x-3$, *i.e.*, δ_1 between the numerator and denominator in this case. Hence, $\underset{x\to 3}{Lt}\dfrac{x^2-9}{x-3}=6$. But when $x=3$, the function $\dfrac{x^2-9}{x-3}$ is non-existent or undefined, for, we cannot cancel the factor $x-3$, which is equal to zero in that case. Thus, writing $f(x)=\dfrac{x^2-9}{x-3}$, $\underset{x\to 3}{Lt} f(x)=6$, whereas $f(3)$ does not exist or is undefined.

Ex. (ii). $\underset{x\to 0}{Lt}\, x\sin\frac{1}{x}=0$. For $\sin\frac{1}{x}$, whatever small value x may have provided it is not exactly equal to zero, is a finite quantity lying between $+1$ and -1, and so by taking x numerically small enough (*i.e.*, sufficiently near to zero), we can make $x\sin\frac{1}{x}$ numerically as small as we like, *i.e.*, $\left|x\sin\frac{1}{x}-0\right|$ is less than any assignable quantity. Hence the limit is zero.

Here also the value of $x\sin\frac{1}{x}$, when x is exactly equal to zero, is non-existent.

Ex. (iii). $\underset{x\to -1}{Lt}\dfrac{x^2-7}{x+3}=-3$. For, writing $x=-1\pm\delta$, we can show that the numerical difference of $\dfrac{x^2-7}{x+3}$ and -3 can be made as small as we like by taking δ small enough.

In this case the value of $\dfrac{x^2-7}{x+3}$, when x is exactly $=-1$, is also available, and that is also equal to -3.

$\underset{x\to a}{Lt} f(x)$ **:** The limit of a function $f(x)$, as x approaches the value a from the right (*i.e.*, from bigger values), is that quantity l_1, (if one such exists), towards which $f(x)$ approaches, and from which the numerical difference of $f(x)$ can be made as small as we please by making x approach a sufficiently closely, all the time keeping it greater than a. It is called the Right-hand limit of $f(x)$ as x tends to a, and is written as

$$\underset{x\to a+0}{Lt} f(x)=l_1.$$

Mathematically, $\underset{x \to a+0}{Lt} f(x) = l_1$, *provided, given any pre-assigned positive quantity* ε, *however small, we can determine a positive quantity* δ, *such that* $|f(x) - l_1| < \varepsilon$ *whenever* $0 < x - a \le \delta$, *i.e.,* $a < x \le a + \delta$.

$\underset{x \to a+0}{Lt} f(x)$ is sometimes denoted by the symbol $f(a+0)$.

Ex. (iv). $\underset{x \to 2+0}{Lt} \left\{ \dfrac{1}{5 + e^{\frac{1}{x-2}}} \right\} = 0$, as can be shown by writing $x = 2 + \delta$ where δ is positive, and then making δ arbitrarily small when the denominator becomes arbitrarily large.

$\underset{x \to a-0}{\textbf{Lt}} \textbf{f}(\textbf{x})$: In a similar way, we may define the **Left-hand limit** of a function $f(x)$ as x tends to a as follows :

$\underset{x \to a-0}{Lt} f(x) = l_2$, *provided, a quantity* l_2 *can be obtained such that, given any pre-assigned positive quantity* ε, *however small, we can determine a positive quantity* δ, *so that* $|f(x) - l_2| < \varepsilon$ *whenever* $0 < a - x \le \delta$, *i.e.,* $a - \delta \le x < a$.

$\underset{x \to a-0}{Lt} f(x)$ is sometimes denoted by $f(a-0)$.

Illustration : $\underset{x \to 2-0}{Lt} \left\{ \dfrac{1}{5 + e^{\frac{1}{x-2}}} \right\} = \dfrac{1}{5}$.

As $x \to 2-0$, $x-2$ is negative, and becomes numerically smaller and smaller, so that $e^{\frac{1}{x-2}}$ approaches zero in this case.

It may be noted that when $\underset{x \to a+0}{Lt} f(x) = \underset{x \to a-0}{Lt} f(x)$, each of these is equal to $\underset{x \to a}{Lt} f(x)$. Conversely, for $\underset{x \to a}{Lt} f(x)$ to exist, each of $\underset{x \to a+0}{Lt} f(x)$ and $\underset{x \to a-0}{Lt} f(x)$ must exist, and must be equal to one another, and this common value is $\underset{x \to a}{Lt} f(x)$.

If $\underset{x \to a+0}{Lt} f(x) \ne \underset{x \to a-0}{Lt} f(x)$, or even one of them does not exist, then $\underset{x \to a}{Lt} f(x)$ does not exist.

Ex. (v). In the above example (iv),

since $\underset{x\to 2+0}{Lt}\left\{\frac{1}{5+e^{\frac{1}{x-2}}}\right\} \neq \underset{x\to 2-0}{Lt}\left\{\frac{1}{5+e^{\frac{1}{x-2}}}\right\}$,

$\underset{x\to 2}{Lt}\left\{\frac{1}{5+e^{\frac{1}{x-2}}}\right\}$ does not exist.

Ex. (vi). Again, consider $\underset{x\to 2}{Lt}\ x^2$.

Here, $\underset{x\to 2+0}{Lt}\ x^2 = 2^2 = \underset{x\to 2-0}{Lt}\ x^2$ [See *Art. 3.13, Ex.I*]

$\therefore \quad \underset{x\to 2}{Lt}\ x^2 = 2^2 = 4$.

Ex. (vii). Consider $f(x) = \sqrt{x-2}$.

Here, $\underset{x\to 2-0}{Lt}\ f(x)$ does not exist, since, for values of $x<2$ (however near to 2), $f(x)$ does not exist.

$\therefore \underset{x\to 2}{Lt}\ \sqrt{x-2}$ does not exist.

3.6. Distinction between $\underset{x\to a}{\text{Lt}}\ f(x)$ and $f(a)$.

The statement $\underset{x\to a}{Lt}\ f(x)$ is a statement about the value of $f(x)$ when x has any value arbitrarily near to a, *except a*. In this case, we do not care to know what happens when x is put equal to a. But $f(a)$ stands for the value of $f(x)$ when x is exactly equal to a, obtained either by the definition of the function at a, or else by substitution of a for x in the expression $f(x)$, when it exists.

Note. Five distinct cases may arise.

(i) $f(a)$ does not exist, but $\underset{x\to a}{Lt}\ f(x)$ exists.

This is illustrated by Ex. (i) of Art. 3.5.

(ii) $f(a)$ exists, but $\underset{x\to a}{Lt}\ f(x)$ does not exist.

Suppose $f(x) = 1$ when $x > 0$,

$= 0$ when $x = 0$,

$= -1$ when $x < 0$.

Here, $\underset{x \to 0+0}{Lt} f(x) = 1$; because when x, remaining greater than 0, becomes arbitrarily near to 0, $f(x)$ always remains equal to 1 and hence $|f(x)-1|$, being $= 0$, is $<$ any pre-assigned positive number ε_1, for any positive value of x less than δ, however small.

Similarly, $\underset{x \to 0-0}{Lt} f(x) = -1$; because when x, remaining less than zero, becomes arbitrarily near to 0, $f(x)$ always remains equal to -1.

Since $\underset{x \to 0+0}{Lt} f(x) \neq \underset{x \to 0-0}{Lt} f(x)$,

$\therefore \underset{x \to 0}{Lt} f(x)$ does not exist; but, by definition, $f(0) = 0$, here.

(iii) $f(a)$ *and* $\underset{x \to a}{Lt} f(x)$ *both exist but are unequal.*

Let $f(x) = 0$ for $x \neq 0$,
$= 1$ for $x = 0$.

As in (ii), it can be easily shown here that

$$\underset{x \to 0+0}{Lt} f(x) = 0 = \underset{x \to 0-0}{Lt} f(x).$$

$\therefore \underset{x \to 0}{Lt} f(x) = 0$. But, by definition, $f(0) = 1$.

(iv) $f(a)$ *and* $\underset{x \to a}{Lt} f(x)$ both exist and are equal.

This is illustrated by Ex. (iii) of Art. 3.5.

(v) *Neither* $f(a)$ *nor* $\underset{x \to a}{Lt} f(x)$ exists.

Let $f(x) = \sin \frac{1}{x}$.

Here, $\underset{x \to 0}{Lt} \sin \frac{1}{x}$ does not exist [*See Ex.* 4, Art. 3.13] and $f(0)$ does not exist, as it would involve division by zero, and is otherwise undefined.

3.7. Symbols $+\infty$ and $-\infty$.

If a variable x, assuming positive values only, increases without limit (*i.e.*, ultimately becomes and remains greater than any pre-assigned positive number, however large), we say that x *tends to infinity*, and write it as $x \to \infty$.

Similarly, if a variable x, assuming negative values only, increases numerically without limit (*i.e.*, $-x$ ultimately becomes and remains greater than any pre-assigned positive number, however large), we say that *x tends to minus infinity* and write it as $x \to -\infty$.

Note. It should be borne in mind that there is no number such as ∞ or $-\infty$ towards which x approaches. The symbols are used only to indicate that the numerical value of x increases without limit.

3.8. Function tending to infinity : $\underset{x \to a}{\text{Lt}} f(x) = \pm\infty$.

As x approaches a either from the right or left, if $f(x)$ tends to infinity with the same sign in both cases, then we say that, as x tends to a, tends to infinity[1], (or loosely, the limit of $f(x)$ is infinite), positive or negative as the case may be; and write it as $\underset{x \to a}{Lt} f(x) = \infty$ or $-\infty$.

If, however, as x approaches a from both sides, $f(x)$ tends to infinity with different signs, we say 'does not possess any limit as x tends to a'.

The formal definitions are as follows :

If, corresponding to any pre-assigned positive quantity N, however large, we can determine a positive quantity d, such that $f(x) > N$ *whenever* $0 < |x - a| \le \delta$, *we say*

$$\underset{x \to a}{Lt} f(x) = \infty.$$

If, in the above circumstances, $-f(x) > N$ *whenever* $0 < |x - a| \le \delta$,

we say $\underset{x \to a}{Lt} f(x) = -\infty$.

Similarly, we may define the cases

$$\underset{x \to a+0}{Lt} f(x) = \infty, \qquad \underset{x \to a-0}{Lt} f(x) = \infty,$$

$$\underset{x \to a+0}{Lt} f(x) = -\infty, \qquad \underset{x \to a-0}{Lt} f(x) = -\infty.$$

Illustration : $\underset{x \to 0+0}{Lt} \dfrac{1}{x^2} = \infty$, $\underset{x \to 0-0}{Lt} \dfrac{1}{x^2} = \infty$, $\therefore \underset{x \to 0}{Lt} \dfrac{1}{x^2} = \infty$,

$\underset{x \to 0+0}{Lt} \dfrac{1}{x} = \infty$, $\underset{x \to 0-0}{Lt} \dfrac{1}{x} = -\infty$, $\therefore \underset{x \to 0}{Lt} \dfrac{1}{x}$. does not exist. In either case, however, $f(0)$ does not exist.

3.9. Limit of a function as the variable tends to infinity : $\underset{x \to \infty}{\text{Lt}} f(x)$.

As x, remaining positive, becomes larger and larger, if there exists a

[1] According to some modern writers, this is described as '$f(x)$ becoming infinitely large', and infinite limit is not recognised as a limit.

definite finite number l towards which $f(x)$ continually approaches, such that the numerical difference of $f(x)$ and l can be made as small as we please by taking x large enough, we say $\underset{x \to \infty}{Lt} f(x) = l$.

Mathematically, $\underset{x \to \infty}{Lt} f(x) = l$, *provided, given any pre-assigned positive quantity* ε, *however small, we can determine a positive quantity M, such that* $| f(x) - l | < \varepsilon$ *for all values of* $x > M$.

Similarly, $\underset{x \to -\infty}{Lt} f(x) = l'$, provided, given any pre-assigned positive quantity ε, however small, we can determine a positive quantity M, such that $| f(x) - l' | < \varepsilon$ for all values of $-x > M$.

In a similar way, we may define the cases

$$\underset{x \to \infty}{Lt} f(x) = \infty, \quad \underset{x \to \infty}{Lt} f(x) = -\infty, \quad \underset{x \to -\infty}{Lt} f(x) = \infty, \text{ etc.}$$

Illustration : $\underset{x \to \infty}{Lt} \frac{1}{x} = 0, \quad \underset{x \to -\infty}{Lt} \frac{1}{x^2} = 0,$

$$\underset{x \to \infty}{Lt} e^{\frac{1}{x}} = 1, \quad \underset{x \to -\infty}{Lt} x^2 = \infty, \text{ etc.}$$

3.10. Fundamental Theorems on Limit.

We give below some fundamental theorems on limit which are of frequent use.

If $\underset{x \to a}{Lt} f(x) = l$, and $\underset{x \to a}{Lt} \phi(x) = l'$, where l and l' are finite quantities, then

(i) $\underset{x \to a}{Lt} \{f(x) \pm \phi(x)\} = l \pm l'$.

(ii) $\underset{x \to a}{Lt} \{f(x) \times \phi(x)\} = l l'$.

(iii) $\underset{x \to a}{Lt} \left\{\frac{f(x)}{\phi(x)}\right\} = \frac{l}{l'}$, provided $l' \neq 0$.

(iv) $\underset{x \to a}{Lt} F\{f(x)\} = F\left\{\underset{x \to a}{Lt} f(x)\right\}$, *i.e.*, $= F(l)$,

where $F(u)$ is a function of u which is continuous[1] for $u = l$.

[1] See next chapter.

(v) If $\phi(x) < f(x) < \psi(x)$ in a certiain neighbourhood of the point 'a' and $\underset{x\to a}{Lt}\ \phi(x) = l$ and $\underset{x\to a}{Lt}\ \psi(x) = l$, then $\underset{x\to a}{Lt}\ f(x)$ exists and is equal to l.

In particular, if $|f(x)| < |g(x)|$, *i.e.*, $f(x)$ lies between $-g(x)$ and $g(x)$, and if $\underset{x\to a}{Lt}\ g(x) = 0$, then $\underset{x\to a}{Lt}\ f(x) = 0$.

(vi) If $\underset{x\to a}{Lt}\ \phi(x) = l_1$ and $\underset{x\to a}{\text{Lt}}\ \psi(x) = l_2$ and if $\phi(x) < \psi(x)$ in a certain neighbourhood of a except a, then $l_1 \le l_2$.

Proof :

(i) Since $\underset{x\to a}{\text{Lt}}\ f(x) = l$, $\underset{x\to a}{\text{Lt}}\ \phi(x) = l'$, we can, when any positive number ε is given, choose positive numbers δ_1, δ_2, such that

$$|f(x) - l| < \tfrac{1}{2}\varepsilon \text{ when } 0 < |x - a| \le \delta_1, \quad \ldots \quad (1)$$

$$|\phi(x) - l'| < \tfrac{1}{2}\varepsilon \text{ when } 0 < |x - a| \le \delta_2, \quad \ldots \quad (2)$$

Let δ be any positive number which is smaller than both δ_1 and δ_2; then the inequalities (1) and (2) both hold good when $0 < |x - a| \le \delta$.

Now $|\{f(x) - l\} + \{\phi(x) - l'\}| \le |f(x) - l| + |\phi(x) - l'|$,

$$\therefore \quad |\{f(x) + \phi(x)\} - \{l + l'\}| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon, \text{ i.e.}, < \varepsilon,$$

when $0 < |x - a| \le \delta$.

$\therefore$ by definition, $l + l'$ is the limit of $\{f(x) + \phi(x)\}$ as $x \to a$. Similarly, it can be shown that $l - l'$ is the limit of $\{f(x) - \phi(x)\}$ as $x \to a$.

Hence, $\underset{x\to a}{\text{Lt}}\ \{f(x) \pm \varphi(x)\} = l \pm l'$.

(ii) We have $f(x)\phi(x) - ll' = \{f(x) - l\}\{\phi(x) - l'\}$

$$+ l'\{f(x) - l\} + l\{\phi(x) - l'\}.$$

$$\therefore \quad |f(x)\phi(x) - ll'| \le |f(x) - l||\phi(x) - l'|$$

$$+ |l'||f(x) - l| + |l||\phi(x) - l'|. \quad \ldots \quad (1)$$

Now, ε being any pre-assigned positive quantity, and choosing any other positive quantity k,

$$k, \frac{\varepsilon}{3|l|}, \frac{\varepsilon}{3|l'|}, \frac{\varepsilon}{3k}$$

are known positive quantities, and

since $\underset{x \to a}{Lt} f(x) = l, \ \underset{x \to a}{Lt} \phi(x) = l'$,

we can choose positive numbers $\delta_1, \delta_2, \delta_3, \delta_4$, such that

$$|f(x) - l| < k \quad \text{when} \quad 0 < |x - a| \le \delta_1,$$

$$|\phi(x) - l'| < \frac{\varepsilon}{3k} \quad \text{when} \quad 0 < |x - a| \le \delta_2,$$

$$|f(x) - l| < \frac{\varepsilon}{3|l'|} \quad \text{when} \quad 0 < |x - a| \le \delta_3,$$

and $|\phi(x) - l'| < \dfrac{\varepsilon}{3|l|}$ when $0 < |x - a| \le \delta_4$.

Hence, if δ be the least of the positive numbers $\delta_1, \delta_2, \delta_3, \delta_4$ all the above four inequalities hold when

$$0 < |x - a| \le \delta,$$

and so from (1),

$$|f(x)\phi(x) - ll'| < k.\frac{\varepsilon}{3k} + |l'|.\frac{\varepsilon}{3|l'|} + |l|.\frac{\varepsilon}{3|l|},$$

i.e., $< \dfrac{\varepsilon}{3} + \dfrac{\varepsilon}{3} + \dfrac{\varepsilon}{3}$, *i.e.*, $< \varepsilon$ when $0 < |x - a| \le \delta$.

$\therefore$ by definition, $\underset{x \to a}{Lt} \{f(x)\varphi(x)\} = ll'$.

(iii) We have $\left|\dfrac{f(x)}{\varphi(x)} - \dfrac{l}{l'}\right| = \left|\dfrac{l'\{f(x) - l\} - l\{\varphi(x) - l'\}}{l'\varphi(x)}\right|$

$$\le \frac{|l'||f(x) - l| + |l||\varphi(x) - l'|}{|l'||\varphi(x)|}. \quad \ldots \quad (1)$$

Now, since $\underset{x \to a}{Lt} \phi(x) = l'$, there exists a positive number δ_1 such

that $|\phi(x) - l'| < \frac{1}{2}|l'|$ when $0 < |x - a| \le \delta_1$, for $l' \ne 0$.

$$\therefore \quad |l'| - |\phi(x)| \le |\phi(x) - l'| < \tfrac{1}{2}|l'|$$

or, $|\phi(x)| > \frac{1}{2}|l'|$ when $0 < |x - a| \le \delta_1$. ... (2)

Also, there exist positive numbers δ_2, δ_3, such that

$$|f(x)-l| < \varepsilon' \quad \text{for} \quad 0 < |x-a| \leq \delta_2 ,$$

$$|\phi(x)-l'| < \varepsilon' \quad \text{for} \quad 0 < |x-a| \leq \delta_3 \qquad \ldots \quad (3)$$

where ε' is any chosen positive number.

If δ be the smallest of δ_1, δ_2, δ_3, then it follows from (1), (2), (3) that, when $0 < |x-a| \leq \delta_1$

$$\left|\frac{f(x)}{\varphi(x)}-\frac{l}{l'}\right| < \frac{\{|l'|+|l|\}\varepsilon}{\frac{1}{2}\{|l'|\}^2}.$$

Now ε being any pre-assinged positive number, if we choose $\varepsilon' = \frac{1}{2}\varepsilon\{l'\}^2/\{|l|+|l'|\}$, we get

$$\left|\frac{f(x)}{\phi(x)}-\frac{l}{l'}\right| < \varepsilon \text{ when } 0 < |x-a| \leq \delta .$$

Hence, $\displaystyle\underset{x \to a}{Lt} \frac{f(x)}{\phi(x)} = \frac{l}{l'}$.

(iv) Let $u = f(x)$; since $F(u)$ is continuous for $u = l$.

$$|F(u)-F(l)| < \varepsilon \quad \text{when} \quad |u-l| \leq \delta_1 ,$$

i.e., when $|f(x)-l| < \delta_1$, $\qquad \ldots \quad (1)$

Again, since $f(x) \to l$ as $x \to a$,

$$|f(x)-l| < \delta_1 \text{ when } 0 < |x-a| \leq \delta . \qquad \ldots \quad (2)$$

Combining (1) and (2),

$$|F\{f(x)\}-F(l)| < \varepsilon \quad \text{when} \quad 0 < |x-a| \leq \delta ,$$

i.e., $\displaystyle\underset{x \to a}{Lt} F\{f(x)\} = F(l)$.

(v) Assume that the inequalities $\varphi(x) < f(x) < \psi(x)$ are satisfied when $0 < |x-a| < \delta_1$.

Since $\underset{x \to a}{Lt}\ \phi(x) = l$, $|\phi(x)-l| < \varepsilon$ when $0 < |x-a| \leq \delta_2$.

i.e., $l - \varepsilon < \phi(x) < l + \varepsilon$ when $0 < |x-a| \leq \delta_2$.

Similarly, $l-\varepsilon<\psi(x)<l+\varepsilon$ when $0<|x-a|\leq\delta_3$.

If δ be the smallest of the numbers δ_1, δ_2, δ_3, then all the above inequalities are satisfied when $0<|x-a|\leq\delta$. Under these conditions, $l-\varepsilon<\phi(x)<f(x)$.

Also $l+\varepsilon>\psi(x)>f(x)$, $\therefore\ l-\varepsilon<f(x)<l+\varepsilon$,

i.e., $|f(x)-l|<\varepsilon$ when $0<|x-a|\leq\delta$.

$\therefore\quad f(x)\to l$ as $x\to a$.

(vi) Let us suppose that the inequality $\phi(x)<\psi(x)$ holds good when $0<|x-a|<\delta_1$, *i.e.*, in the neighbourhood

$$a-\delta_1<x<a+\delta_1,\quad x\neq a, \qquad \ldots \quad (1)$$

If possible, suppose $l_1>l_2$.

Let us choose $\varepsilon=\frac{1}{2}(l_1-l_2)$, a positive number.

Since $\underset{x\to a}{Lt}\ \phi(x)=l_1$,

$\therefore\quad |\phi(x)-l_1|<\varepsilon$ when $0<|x-a|\leq\delta_2$,

i.e., $l_1-\varepsilon<\phi(x)<l_1+\varepsilon$ when $a-\delta_2\leq x\leq a+\delta_2$.

$\therefore\quad l_1-\frac{1}{2}(l_1-l_2)<\phi(x)$ when $a-\delta_2\leq x\leq a+\delta_2$.

i.e., $\frac{1}{2}(l_1+l_2)<\phi(x)$. $\qquad\ldots\quad(2)$

Again, since $\underset{x\to a}{Lt}\ \psi(x)=l_2$,

$\therefore\quad l_2-\varepsilon<\psi(x)<l_2+\varepsilon$ when $a-\delta_3\leq x\leq a+\delta_3$.

$\therefore\quad \psi(x)<l_2+\frac{1}{2}(l_1-l_2)$,

i.e., $\psi(x)<\frac{1}{2}(l_1+l_2)$ when $a-\delta_3\leq x\leq a+\delta_3$. $\ldots\quad(3)$

Let δ be the smallest of the numbers δ_1, δ_2, δ_3, then all the above inequalities (1), (2), (3) hold good in the interval $a-\delta\leq x\leq a+\delta$.

$\therefore$ from (2) and (3), $\phi(x)>\frac{1}{2}(l_1+l_2)>\psi(x)$, *i.e.,* $\phi(x) > \psi(x)$ in $a - \delta \le x \le a + \delta$,

which contradicts our hypothesis that $\phi(x) < \psi(x)$ in that interval. Hence our assumption $l_1 > l_2$ is incorrect.

$\therefore\ l_1 \not> l_2$, i.e., $l_1 \le l_2$.

Note. The first two theorems may be extended to any finite number of functions. In languages, the first three theorems may be stated as follows :

(i) *The limit of the sum or difference of any finite number of functions is equal to the sum or difference of the limits of the functions taken separately.*

(ii) *The limit of the product of a finite number of functions is equal to the product of their limits taken separately.*

(iii) *The limit of the quotient of two functions is equal to the quotient of their limits, provided the limit of the denominator is not zero.*

As examples of (iv), we get

$$\underset{x \to a}{Lt} \log f(x) = \log\left\{\underset{x \to a}{Lt} f(x)\right\} = \log l, \text{ provided } l > 0,$$

$$\underset{x \to a}{Lt}\ e^{f(x)} = e^{\underset{x \to a}{Lt} f(x)} = e^l,$$

$$\underset{x \to a}{Lt} \{f(x)\}^n = \left\{\underset{x \to a}{Lt} f(x)\right\}^n = l^n, \text{ etc.}$$

3.11. Some Important Limits.

(i) $\underset{x \to 0}{\mathrm{Lt}} \dfrac{\sin x}{x} = 1$, where x is expressed in radian measure.

From elementary Trigonometry[1], we know that if x be the radian measure of any positive acute angle, i.e., $0 < x < \frac{1}{2}\pi$, then

$$\sin x < x < \tan x, \text{ or, } \cos x < \frac{\sin x}{x} < 1. \qquad \ldots \qquad (1)$$

$$\therefore\quad 0 < 1 - \frac{\sin x}{x} < 1 - \cos x, \quad i.e., < 2\sin^2\frac{1}{2}x.$$

[1] See Das and Mukherjees' *Intermediate Trigonometry.*

But $\quad 2\sin^2 \tfrac{1}{2}x < 2\left(\tfrac{1}{2}x\right)^2, \quad$ *i.e.*, $< \tfrac{1}{2}x^2$.

Hence, $\quad 0 < 1 - \dfrac{\sin x}{x} < \tfrac{1}{2}x^2.$

Now, since $x^2 \to 0$ as $x \to 0+0$, we get

$$\underset{x \to 0+0}{Lt}\left(1 - \frac{\sin x}{x}\right) = 0, \text{ i.e., } \underset{x \to 0+0}{Lt} \frac{\sin x}{x} = 1.$$

Alternatively, noting that $\cos x \to 1$ as $x \to 0$, we can conclude directly from (1) that $Lt \dfrac{\sin x}{x} = 1$.

When $-\tfrac{1}{2}\pi < x < 0$, putting $x = -z$, we get $0 < z < \tfrac{1}{2}\pi$.

Also, $\dfrac{\sin x}{x} = \dfrac{\sin(-z)}{-z} = \dfrac{\sin z}{z}$.

Hence, $\underset{x \to 0-0}{Lt} \dfrac{\sin x}{x} = \underset{z \to 0+0}{Lt} \dfrac{\sin z}{z} = 1.$

Hence the result.

(ii) (a) $\underset{n \to +\infty}{Lt}\left(1 + \dfrac{1}{n}\right)^n = e$, ($n \to \infty$ through positive integral values).

(b) $\underset{x \to \pm\infty}{Lt}\left(1 + \dfrac{1}{x}\right)^x = e.$

Proof : We have already seen that

$$\underset{n \to \infty}{Lt}\left(1 + \frac{1}{n}\right)^n = e, \text{ (n is a positive integer).}$$ [*See Art. 5.12*]

Now, let x be any large positive number. Then we can get two consecutive positive integers n, $n + 1$, such that

$$n \le x < n + 1. \quad \therefore \quad 1 + \frac{1}{n} \ge 1 + \frac{1}{x} > 1 + \frac{1}{n+1},$$

and each being > 1, and as $n + 1 > x \ge n$,

$$\left(1 + \frac{1}{n}\right)^{n+1} > \left(1 + \frac{1}{x}\right)^x > \left(1 + \frac{1}{n+1}\right)^n, \text{ or}$$

$$\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n > \left(1+\frac{1}{x}\right)^x > \left(1+\frac{1}{n+1}\right)^{n+1} \Big/ \left(1+\frac{1}{n+1}\right).$$

Now when $x \to \infty$, $n \to \infty$ also, and n being a positive integer, both $\left(1+\frac{1}{n}\right)^n$ and $\left(1+\frac{1}{n+1}\right)^{n+1} \to e$, as proved before. Also, $1+\frac{1}{n} \to 1$ and $1+\frac{1}{n+1} \to 1$. Hence the two extremes in the above inequality tend to a common limit e, and so $\left(1+\frac{1}{x}\right)^x \to e$.

Lastly, suppose $x = -p$, where p is a large positive number; then as $p \to \infty$, $x \to -\infty$.

Then $\left(1+\frac{1}{x}\right)^x = \left(1-\frac{1}{p}\right)^{-p} = \left(\frac{p}{p-1}\right)^p = \left(1+\frac{1}{p-1}\right)^p$

$$= \left(1+\frac{1}{q}\right)^{q+1} \qquad (\text{where } q = p-1).$$

Now, if $p \to \infty$, $q \to \infty$ and hence

$$\left(1+\frac{1}{q}\right)^{q+1} = \left(1+\frac{1}{q}\right)^q \cdot \left(1+\frac{1}{q}\right) \to e.$$

Thus, $\left(1+\frac{1}{x}\right)^x \to e$ as $x \to -\infty$.

Hence, we see that $\underset{x \to \pm\infty}{Lt} \left(1+\frac{1}{x}\right)^x = e$,

x being not confined to be integral here.

Cor. $\underset{x \to 0}{\text{Lt}} (1+x)^{1/x} = e$.

Proof:

In the above result, replacing y by $\frac{1}{x}$ as $x \to \pm\infty$, $y \to 0$, and we get

$$\underset{y \to 0}{Lt} (1+y)^{1/y} = e, \text{ or, } \underset{x \to 0}{\text{Lt}} (1+x)^{1/x} = e.$$

(iii) $\underset{x \to 0}{Lt} \frac{1}{x} \log(1+x) = 1.$

Proof : We have

$$\underset{x \to 0}{Lt} \frac{1}{x} \log(1+x) = \underset{x \to 0}{Lt} \log(1+x)^{1/x}$$

$$= \log\left\{\underset{x \to 0}{Lt}(1+x)^{1/x}\right\} \qquad \text{by § 3.8 (iv)]}$$

$$= \log e = 1.$$

(iv) $\underset{x \to 0}{Lt} \frac{e^x - 1}{x} = 1.$

Proof : Put $e^x = 1 + z$.

Then $x = \log(1+z)$, and as $x \to 0$, $z \to 0$.

Thus,
$$\underset{x \to 0}{Lt} \frac{e^x - 1}{x} = \underset{z \to 0}{Lt} \frac{z}{\log(1+z)}$$

$$= \underset{z \to 0}{Lt}\left[1 \Big/ \left\{\frac{1}{z}\log(1+z)\right\}\right]$$

$$= 1 \Big/ \underset{z \to 0}{Lt}\left\{\frac{1}{z}\log(1+z)\right\}$$

$$= \frac{1}{1} = 1. \qquad [\text{by (iii)}]$$

(v) $\underset{x \to a}{Lt} \frac{x^n - a^n}{x - a} = n a^{n-1}$

for all rational values of n, provided a is positive.

CASE I. *When n is a positive integer.*

By actual division, we have

$$\frac{x^n - a^n}{x - a} = x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots\ldots + a^{n-1}.$$

$\therefore$ reqd. limit $= \underset{x \to a}{Lt}\left(x^{n-1} + x^{n-2}a + \ldots\ldots + a^{n-1}\right) = na^{n-1}$,

since the limit of each of the n terms as $x \to a$ as a^{n-1}, and the limit of the sum of a finite number of terms is equal to the sum of their limits (*Art. 3.8*).

CASE II. *When n is a negative integer.*

Suppose $n = -m$, where m is a positive integer and $a \neq 0$.

Then $$\frac{x^n - a^n}{x - a} = \frac{x^{-m} - a^{-m}}{x - a} = -\frac{1}{x^m a^m} \cdot \frac{x^m - a^m}{x - a}.$$

Now, as $x \to a$, the limiting value of $\dfrac{1}{x^m a^m} = \dfrac{1}{a^m a^m} = \dfrac{1}{a^{2m}}$ and

as $x \to a$, the limiting value of $\dfrac{x^m - a^m}{x - a} = m a^{m-1}$, by Case I.

$$\therefore \underset{x \to a}{Lt} \frac{x^n - a^n}{x - a} = -\frac{1}{a^{2m}} m a^{m-1} = -m a^{-m-1} = n a^{n-1}.$$

CASE III. *When n is a rational fraction.*

Suppose $n = p/q$, where q is a positive integer and p any integer, positive or negative.

Let us put $x^{1/q} = y$ and $a^{1/q} = b$

$$\therefore \frac{x^n - a^n}{x - a} = \frac{x^{p/q} - a^{p/q}}{x - a} = \frac{y^p - b^p}{y^q - b^q} = \frac{(y^p - b^p)/(y - b)}{(y^q - b^q)/(y - b)}.$$

Now, as $x \to a$, $x^{1/q} \to a^{1/q}$, $\therefore y \to b$. Again, as $y \to b$, the limiting value of the numerator of the right side $= pb^{p-1}$ (*by Cases I and II*) and that of the denominator $= qb^{q-1}$ (*by Case I*).

$$\therefore \underset{x \to a}{Lt} \frac{x^n - a^n}{x - a} = \frac{pb^{p-1}}{qb^{q-1}} = \frac{p}{q} b^{p-q} = \frac{p}{q} a^{\frac{p}{q}-1} = na^{n-1}.$$

When $n = 0$, the limit is $\underset{x \to a}{Lt} \dfrac{0}{x - a} = 0$.

(vi) $$\underset{x \to 0}{\textbf{Lt}} \frac{(1 + x)^n - 1}{x} = n.$$

Proof : We have

$$\underset{x \to 0}{Lt} \frac{(1 + x)^n - 1}{x} = \underset{x \to 0}{Lt} \left\{ \frac{(1 + x)^n - 1}{\log(1 + x)} \times \frac{\log(1 + x)}{x} \right\}$$

$$= \underset{x \to 0}{Lt} \frac{(1 + x)^n - 1}{\log(1 + x)} \times \underset{x \to 0}{Lt} \frac{\log(1 + x)}{x}.$$

Now, put $(1+x)^n = 1+z$. Then $n\log(1+x) = \log(1+z)$.

Hence, as $x \to 0$, $\log(1+z) \to 0$ and so $z \to 0$.

Thus, $$\underset{x\to 0}{Lt}\frac{(1+x)^n-1}{\log(1+x)} = \underset{z\to 0}{Lt}\frac{nz}{\log(1+z)}$$

$$= \underset{z\to 0}{Lt}\left[n\Big/\left\{\frac{1}{z}\log(1+z)\right\}\right]$$

$$= n\Big/\underset{z\to 0}{Lt}\left\{\frac{1}{z}\log(1+z)\right\}$$

$$= \frac{n}{1} = n \qquad \text{[by (iii)].}$$

Also, $$\underset{x\to 0}{Lt}\frac{\log(1+x)}{x} = 1 \qquad \text{[by (iii)].}$$

Hence, $$\underset{x\to 0}{Lt}\frac{(1+x)^n-1}{x} = n\times 1 = n.$$

This result also follows by replacing x by $x+1$ and a by 1 in (v) above.

3.12. Cauchy's necessary and sufficient condition for the existence of a limit.

The necessary and sufficient condition that the limit $\underset{x\to a}{Lt} f(x)$ *exists and is finite is that, corresponding to any pre-assigned positive number* ε, *however small (but not equal to zero), we can find a positive number* δ *such that* x_1 *and* x_2 *being any two quantities satisfying* $0<|x-a|\le\delta$, $|f(x_1)-f(x_2)|<\varepsilon$.

To prove that the condition is necessary, let $\underset{x\to a}{Lt} f(x)$ exist, and be finite, and $= l$ (say).

Then, given any pre-assigned positive number ε, we can find a positive number δ, such that

$$|f(x)-l| < \tfrac{1}{2}\varepsilon$$

when $0<|x-a|\le\delta$. If now x_1 and x_2 be any two quantities satisfying $0<|x-a|\le\delta$,

then $$\left|f(x_1)-f(x_2)\right| = \left|\{f(x_1)-l\}-\{f(x_2)-l\}\right|$$
$$\leq \left|f(x_1)-l\right| + \left|f(x_2)-l\right|$$
$$< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon, \quad i.e., < \varepsilon.$$

Hence, the condition is necessary.

The proof that the condition is sufficient is beyond the scope of the present treatise.

Note. In some cases even if we may not know the value of a limit beforehand, we can determine by the above test whether a limit exists or not. Illustrations of this are given below.

Ex. 1. *Show that* $\underset{x\to 0}{Lt} \cos\frac{1}{x}$ *does not exist.*

In order that the limit may exist, it must be possible to find a positive number δ such that, x_1 and x_2 satisfying $0 < |x| \leq \delta$,

$$\left|\cos\frac{1}{x_1} - \cos\frac{1}{x_2}\right| < \varepsilon$$

where ε is any pre-assigned positive quantity.

Now, whatever δ we may choose, if we take $x_1 = 1/(2n\pi)$ and $x_2 = 1/\{(2n+1)\pi\}$, by taking n a sufficiently large positive integer, both x_1 and x_2 will satisfy $0 < |x| \leq \delta$.

But in this case,

$$\left|\cos(1/x_1) - \cos(1/x_2)\right| = \left|\cos 2n\pi - \cos(2n+1)\pi\right| = 2,$$

a finite quantity, and is not less than any chosen ε.

Thus, the necessary condition is not satisfied, and so the required limit does not exist.

Here, the right-hand limit as also the left-hand limit are both non-existent.

Ex. 2. *Show that* $\underset{x\to 0}{Lt} \dfrac{1}{2+e^{1/x}}$ *does not exist.*

Here, taking $x_1 = -1/n$, $x_2 = 1/n$, whatever δ we may choose, by taking n a sufficiently large positive integer, we can make x_1 and x_2 both satisfy $0 < |x| \le \delta$. But in this case,

$$\left|\frac{1}{2+e^{1/x_1}} - \frac{1}{2+e^{1/x_2}}\right| = \left|\frac{1}{2+e^{-n}} - \frac{1}{2+e^{n}}\right|$$

$$= \frac{1}{2+e^{-n}} - \frac{1}{2+e^{n}} > \frac{1}{2+e^{-1}} - \frac{1}{2+e},$$

which is a finite quantity and so cannot be less than any chosen ε however small.

Thus, the necessary condition being not satisfied, the limit in question does not exist.

Here, the right-hand limit exists and = 0, and the left-hand limit exists and = $\frac{1}{2}$.

3.13. Illustrative Examples.

Ex. 1. *Find the value of* $\underset{x \to 2}{Lt}\ x^2$.

By taking successive values of x, which always remaining less than 2 tend to 2, *viz.*, $x = 1{\cdot}9, 1{\cdot}99, 1{\cdot}999, \ldots$, we see that x^2 has the values $3{\cdot}61, 3{\cdot}9601, 3{\cdot}996001, \ldots$ which tend to 4, and we can make the difference between 4 and x^2 smaller than any positive number however small by taking x sufficiently near to 2. Hence, the left-hand limit is 4.

Similarly, by taking values of x, which always remaining greater than 2 approach 2, *viz.*, $x = 2{\cdot}1, 2{\cdot}01, 2{\cdot}001, \ldots$, we see that x^2 has the values $4{\cdot}41, 4{\cdot}0401, 4{\cdot}004001$, which continually approach 4. Hence, as before, the right-hand limit is 4.

Hence, the value of the required limit is 4.

Note. Exactly in the same way, we can show that $\underset{x \to a}{Lt}\ x^n = a^n$, where n is an integer or a rational fraction (except when $a = 0$ and n is negative).

Ex. 2. *Show that* (i) $\underset{\theta \to 0}{Lt}\ \sin\theta = 0$;

(ii) $\underset{\theta \to 0}{Lt}\ \cos\theta = 1$;

(iii) $\underset{\theta \to \alpha}{Lt}\ \sin\theta = \sin\alpha$;

(iv) $\underset{\theta \to \alpha}{Lt}\ \cos\theta = \cos\alpha$.

(i) Since, from the definition of sine of a real angle θ in trigonometry, with the help of a figure, it may be easily seen that $|\sin\theta-0|$, *i.e.*, $|\sin\theta|$ can be made less than any positive number ε, however small, by making $|\theta|$ arbitrarily small, it follows that $\underset{\theta\to 0}{Lt}\ \sin\theta=0$.

(ii) $\underset{\theta\to 0}{Lt}(1-\cos\theta)=\underset{\theta\to 0}{Lt}\ 2\sin^2\tfrac{1}{2}\theta=2\times\underset{\theta\to 0}{Lt}\left(\sin\tfrac{1}{2}\theta\times\sin\tfrac{1}{2}\theta\right)$

$$=2\times 0\ [\textit{by (i)}]=0.$$

$\therefore\quad \underset{\theta\to 0}{Lt}\ \cos\theta=1.$

(iii) $\sin\theta-\sin\alpha=2\sin\tfrac{1}{2}(\theta-\alpha)\cos\tfrac{1}{2}(\theta+\alpha).$

As $\theta\to\alpha$, $\tfrac{1}{2}(\theta-\alpha)\to 0$, $\quad\therefore\ \underset{\theta\to\alpha}{Lt}\ \sin\tfrac{1}{2}(\theta-\alpha)=0.$

Also, $\left|\cos\tfrac{1}{2}(\theta+\alpha)\right|\le 1.\quad\therefore\ \underset{\theta\to\alpha}{Lt}(\sin\theta-\sin\alpha)=0,$

i.e., $\underset{\theta\to\alpha}{Lt}\ \sin\theta=\sin\alpha.$

(iv) Since $\cos\theta-\cos\alpha=2\sin\tfrac{1}{2}(\alpha-\theta)\sin\tfrac{1}{2}(\theta+\alpha)$, it follows, as in (iii), that $\underset{\theta\to\alpha}{Lt}(\cos\theta-\cos\alpha)=0$, *i.e.*, $\underset{\theta\to\alpha}{Lt}\ \cos\theta=\cos\alpha$.

Ex. 3. *Apply* (δ, ε) *definition of limit to illustrate that*

$$\underset{x\to 4}{Lt}(2x-2)=6.$$

Let us choose $\varepsilon=0{\cdot}01$.

Then, $|(2x-2)-6|<0{\cdot}01$ if $|2x-8|<0{\cdot}01$, *i.e.*, if $|x-4|<0{\cdot}005$, *i.e.*, $\delta=0{\cdot}005$. Similarly, if $\varepsilon=0{\cdot}001$, $\delta=0{\cdot}0005$; and so on.

Thus, δ depends upon ε, *i.e.*, the nearer $(2x-2)$ is to 6, the nearer x is to 4. We have

$$|(2x-2)-6|<0{\cdot}01 \quad\text{if}\quad 0<|x-4|<0{\cdot}005,$$

$$|(2x-2)-6|<0{\cdot}001 \quad\text{if}\quad 0<|x-4|<0{\cdot}0005,$$

and generally, $|(2x-2)-6|<\varepsilon$ if $0<|x-4|<\tfrac{1}{2}\varepsilon.$

Hence, 6 is the limit of $2x-2$ as $x\to 4$.

Ex. 4. Draw the graph of $\sin(1/x)$ and show that neither the right-hand limit nor the left-hand limit exists as x tends to zero.

When $x=0, \sin\frac{1}{x}$ is meaningless and hence its value is not known.

For all other values of x, $\sin(1/x)$ exists and may take any value from -1 to 1. Thus, the graph is a continuous curve with a break at $x=0$ and is comprised between the lines $y=1$ and $y=-1$.

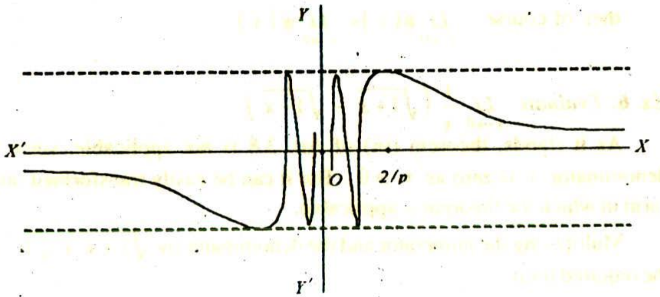

Fig. 3.13.1

As $x \to 0+0$, by passing successively through values $2/n\pi$, where n is a positive integer which can be made as large as we like, $\sin(1/x)$ passes through values 0, -1, 0, 1, etc. taking intermediate values at intermediate points. Now, it is evident that these values are taken more frequently as x comes nearer to 0 and so $\sin(1/x)$ does not approach any fixed value as $x \to 0+0$, but oscillates through all values between -1 and $+1$, *i.e.*, the function has no right-hand limit. Since, when x is negative, $\sin(1/x) = -\sin(1/z)$, where $z=(-x)$ is positive, the function behaves exactly in the same way when $x \to 0-0$. Hence, the left-hand limit also does not exist for the function.

Note. Hence, it follows that $\underset{x\to 0}{Lt} \sin\frac{1}{x}$ also does not exist.

Ex. 5. Give an example to illustrate the following limit-inequality :

If $\underset{x\to a}{Lt}\ \phi(x)=A$ *and* $\underset{x\to a}{Lt}\ \psi(x)=B$ *and if* $\phi(x)<\psi(x)$ in a certain neighbourhood of a except a, then $A \le B$.[1]

Suppose, $\phi(x)=5+x^2$; $\psi(x)=5+3x^2$.

$\therefore \quad \underset{x\to 0}{Lt}\ \phi(x)=5=\underset{x\to 0}{Lt}\ \psi(x).$

[1] For proof of this important theorem see Appendix.

But, $\phi(x) < \psi(x)$ if $x \neq 0$.

Thus, the limits of the two functions are equal, even though $\phi(x) < \psi(x)$ for all values of x on which the limits depend.

If, however, $\phi(x) = 5 + x^2$, $\psi(x) = 7 + 3x^2$,

then of course, $\underset{x \to 0}{Lt}\, \phi(x) < \underset{x \to 0}{Lt}\, \psi(x)$.

Ex. 6. *Evaluate* $\underset{x \to 0}{Lt}\, \frac{1}{x}\left(\sqrt{1+x} - \sqrt{1-x}\right)$.

As it stands, theorem (iii) of Art. 3.8 is not applicable, since the denominator x is zero as $x \to 0$. But it can be easily transformed into a form in which the theorem is applicable.

Multiplying the numerator and the denominator by $\sqrt{1+x} + \sqrt{1-x}$, the required limit

$$= \underset{x \to 0}{Lt}\, \frac{2x}{x\left(\sqrt{1+x} + \sqrt{1-x}\right)}$$

$$= \underset{x \to 0}{Lt}\, \frac{2}{\left(\sqrt{1+x} + \sqrt{1-x}\right)} = \frac{2}{2} = 1,$$

since $\underset{x \to 0}{Lt}\, \sqrt{1+x} = \underset{y \to 1}{Lt}\, \sqrt{y} = 1$ (*putting* $1 + x = y$),

and similarly $\underset{x \to 0}{Lt}\, \sqrt{1-x} = 1$.

Ex. 7. *If* $-1 < x < 1$, *then* $\underset{n \to \infty}{Lt}\, x^n = 0$. (*$n$ is a positive integer*).

Let us first consider the case when $0 < x < 1$.

Put $x = 1 - p$, so that $0 < p < 1$. Since $(1-p)(1+p) = 1 - p^2$, which is less than 1, we have $1 - p < 1/(1+p)$.

$$\therefore\ x^n = (1-p)^n < \frac{1}{(1+p)^n} < \frac{1}{1+np} < \frac{1}{np}.$$

$\therefore$ x^n can be made less than any given positive number ε by taking n large enough $\left(i.e.\text{ taking } n > \frac{1}{\varepsilon p}\right)$; but x^n is positive.

$\therefore$ $Lt\, x^n = 0$ when $n \to \infty$.

Since $(-x)^n = (-1)^n x^n$, the result also holds for $-1 < x < 0$. When $x = 0$, $x^n = 0$ for every positive value of n. Hence $Lt\, x^n = 0$ when $n \to \infty$.

Note. When $x > 1$, putting x for $1+p$ in the inequality $(1+p)^n > 1+np > np$, it can be shown that $x^n > k$, where k is any positive number, however large, for all values of $n > k/p$.

Hence, it follows that, for $x > 1$, $\underset{n\to\infty}{Lt}\, x^n = \infty$.

Ex. 8. Prove that (n being a positive integer)

(i) $\underset{n\to\infty}{Lt}\, nx^n = 0$ when $|x| < 1$.

(ii) $\underset{n\to\infty}{Lt}\, \dfrac{x^n}{n} = 0$ when $|x| \le 1$.

$= \infty$ when $x > 1$.

(iii) $\underset{n\to\infty}{Lt}\, \dfrac{x^n}{n!} = 0$ for all values of x.

(iv) $\underset{n\to\infty}{Lt}\, \dfrac{m(m-1)(m-2)\ldots(m-n+1)}{n!}x^n = 0$ when $|x| < 1$.

3.14 Miscellaneous Worked out Examples

Ex. 1. Evaluate the following :

(*i*) $\lim\limits_{x\to\infty}\left(x-\sqrt{x^2+x}\right)$ [*C. P. 1983*]

(*ii*) $\lim\limits_{x\to\infty}\left\{x-\sqrt{(x-a)(x-b)}\right\}$ [*C. P. 1992, B. P. 1999*]

Solution : (*i*) $\lim\limits_{x\to\infty}\left(x-\sqrt{x^2+x}\right)$

$$= \lim_{x\to\infty}\frac{\left(x-\sqrt{x^2+x}\right)\left(x+\sqrt{x^2+x}\right)}{x+\sqrt{x^2+x}}$$

$$= \lim_{x\to\infty}\frac{-x}{x\left\{1+\sqrt{1+\frac{1}{x}}\right\}}$$

$$= \lim_{x\to\infty}\frac{-1}{1+\sqrt{1+\frac{1}{x}}}$$

$$= -\frac{1}{2}, \quad \therefore \lim_{x\to\infty}\frac{1}{x} = 0.$$

(*ii*) $\lim\limits_{x \to \infty} \left\{ x - \sqrt{(x-a)(x-b)} \right\}$

$$= \lim_{x \to \infty} \frac{\left\{ x - \sqrt{(x-a)(x-b)} \right\}\left\{ x + \sqrt{(x-a)(x-b)} \right\}}{x + \sqrt{(x-a)(x-b)}}$$

$$= \lim_{x \to \infty} \frac{x^2 - \{ x^2 - (a-b)x + ab \}}{\left\{ x + \sqrt{(x-a)(x-b)} \right\}}$$

$$= \lim_{x \to \infty} \frac{x\left\{ -\frac{ab}{x} + (a+b) \right\}}{x\left\{ 1 + \sqrt{\left(1 - \frac{a}{x}\right)\left(1 - \frac{b}{x}\right)} \right\}}$$

$$= \lim_{x \to \infty} \frac{(a+b) - \frac{ab}{x}}{1 + \sqrt{\left(1 - \frac{a}{x}\right)\left(1 - \frac{b}{x}\right)}}$$

$$= \frac{1}{2}(a+b), \quad \because \quad \lim_{x \to \infty} \frac{ab}{x} = 0, \ \lim_{x \to \infty} \frac{a}{x} = 0, \ \lim_{x \to \infty} \frac{b}{x} = 0.$$

Ex. 2. Show that :

(*i*) $\lim\limits_{x \to 0} \dfrac{x^2 \sin\left(\frac{1}{x}\right)}{\sin x} = 0.$ [*B. P. 1989, '91*]

(*ii*) $\lim\limits_{x \to \pi} \dfrac{\sin x}{\pi - x} = 1.$ [*B. P. 1995*]

(*iii*) $\lim\limits_{x \to \frac{\pi}{4}} (\sec 2x - \tan 2x) = 0.$ [*C. P. 1980*]

(*iv*) $\lim\limits_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0.$ [*B. P. 1990*]

(*v*) $\lim\limits_{x \to 1} (1-x) \tan \dfrac{\pi x}{2} = \dfrac{2}{\pi}.$ [*B. P. 1991, '93*]

(*vi*) $\lim\limits_{x \to 0} \dfrac{\sin x - \tan x}{x^3} = -\dfrac{1}{2}.$ [*B. P. 1992*]

Solution : (*i*) $\lim\limits_{x\to 0} \dfrac{x^2 \sin\left(\frac{1}{x}\right)}{\sin x} = \lim\limits_{x\to 0} \left\{ x \cdot \sin\left(\frac{1}{x}\right) \cdot \dfrac{1}{\frac{\sin x}{x}} \right\}$

$$= \lim_{x\to 0} (x) \lim_{x\to 0} \sin\left(\frac{1}{x}\right) . \frac{1}{\lim\limits_{x\to 0} \frac{\sin x}{x}} = 0 .$$

since, $\lim\limits_{x\to 0} x = 0$, $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$, $\left|\sin\left(\frac{1}{x}\right)\right| \le 1$.

(*ii*) $\lim\limits_{x\to \pi} \dfrac{\sin x}{\pi - x} = \lim\limits_{\theta\to 0} \dfrac{\sin(\pi - \theta)}{\theta}$,

where, $\pi - x = \theta$ $\therefore$ $x = \pi - \theta$, and $\theta \to 0$ as $x \to \pi$.

$$= \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 ,$$

(*iii*) $\lim\limits_{x\to \frac{\pi}{4}} (\sec 2x - \tan 2x)$

$$= \lim_{x\to \frac{\pi}{4}} \left(\frac{1 - \sin 2x}{\cos 2x} \right) = \lim_{x\to \frac{\pi}{4}} \left\{ \frac{(\cos x - \sin x)^2}{\cos^2 x - \sin^2 x} \right\}$$

$$= \lim_{x\to \frac{\pi}{4}} \frac{\cos x - \sin x}{\cos x + \sin x} = 0 .$$

(*iv*) $\lim\limits_{x\to 0} x \sin\left(\frac{1}{x}\right)$

$= 0$, since $\lim\limits_{x\to 0} x = 0$ and $\left|\sin\frac{1}{x}\right| \le 1$.

(*v*) $\lim\limits_{x\to 0} (1 - x) \tan\dfrac{\pi x}{2}$

$= \lim\limits_{\theta\to 0} \theta \tan\left\{\frac{\pi}{2}(1 - \theta)\right\}$, where $1 - x = \theta$, i.e., $1 - \theta$ and

$\theta \to 0$ as $x \to 1$

$$= \lim_{\theta\to 0} \theta \tan\left(\frac{\pi}{2} - \frac{\pi\theta}{2}\right),$$

$$= \lim_{\theta \to 0} \theta \cdot \cot \frac{\pi\theta}{2},$$

$$= \lim_{\theta \to 0} \theta \cdot \frac{\cos \frac{\pi\theta}{2}}{\sin \frac{\pi\theta}{2}} = \lim_{\theta \to 0} \cos \frac{\pi\theta}{2} \cdot \lim_{\theta \to 0} \frac{1}{\frac{\sin \frac{\pi\theta}{2}}{\frac{\pi\theta}{2}}} \times \frac{2}{\pi} = \frac{2}{\pi}$$

(*vi*) $$\lim_{x \to 0} \frac{\sin x - \tan x}{x^3} = \lim_{x \to 0} \frac{1}{x^3} \left(\sin x - \frac{\sin x}{\cos x} \right)$$

$$= \lim_{x \to 0} \frac{1}{x^3} \cdot \sin x \left(\frac{\cos x - 1}{\cos x} \right) = \lim_{x \to 0} \left\{ \frac{1}{\cos x} \cdot \frac{\sin x}{x} \cdot \frac{1}{x^2} \left(-2\sin^2 \frac{x}{2} \right) \right\},$$

$$= \lim_{x \to 0} \left(\frac{1}{\cos x} \right) \cdot \lim_{x \to 0} \left(\frac{\sin x}{x} \right) \cdot \lim_{x \to 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 \times \left(-\frac{1}{2} \right)$$

$$= 1 \times 1 \times (1)^2 \times \left(-\frac{1}{2} \right) = -\frac{1}{2}.$$

Ex. 3. Evaluate :

(*i*) $$\lim_{n \to \infty} \frac{2^{n+1} + 3^{n+1}}{2^n + 3^n}.$$

(*ii*) $$\lim_{x \to 1} \frac{x^2 - x \log + \log x - 1}{x - 1}$$

(*iii*) $$\lim_{x \to 0} \frac{1 - \cos^3 x}{\tan^2 x}$$

(*iv*) $$\lim_{h \to 0} \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h}$$

(*v*) $$\lim_{x \to \infty} \frac{\sqrt{x}}{\sqrt{x + \sqrt{x + \sqrt{x}}}}.$$

Solution : (*i*) $$\lim_{n \to \infty} \frac{2^{n+1} + 3^{n+1}}{2^n + 3^n}$$

$$= \lim_{n\to\infty} \frac{3^{n+1}\left\{1+\left(\frac{2}{3}\right)^{n+1}\right\}}{3^n\left\{1+\left(\frac{2}{3}\right)^n\right\}} = 3$$

for, $\lim_{n\to\infty}\left(\frac{2}{3}\right)^{n+1} = 0$ and $\lim_{n\to\infty}\left(\frac{2}{3}\right)^n = 0.$

(*ii*) $\lim_{x\to1} \frac{x^2 - x\log x + \log x - 1}{x-1}$

$$= \lim_{x\to1}\left[\frac{x^2-1}{x-1} \frac{(x-1)\log x}{x-1}\right]$$

$$= \lim_{x\to1}[(x+1) - \log x] \quad \because\ x-1 \neq 0$$

$$= 2 - \lim_{x\to1}\log x = 2, \quad \because \lim_{x\to1}\ \log x = 0.$$

(*iii*) $\lim_{x\to0} \frac{1-\cos^3 x}{\tan^2 x}$

$$= \lim_{x\to0} \frac{(1-\cos x)(1+\cos x+\cos^2 x)}{\sin^2 x}\cdot\cos^2 x$$

$$= \lim_{x\to0} \frac{(1-\cos x)(1+\cos^2 x+\cos x)}{1-\cos^2 x}\cdot\cos^2 x$$

$$= \lim_{x\to0} \frac{(1+\cos^2 x+\cos x)}{1+\cos x}, \quad \because\ 1-\cos x \neq 0$$

$$= \frac{3}{2}, \qquad \therefore\ \lim_{x\to0}\ \cos x = 1.$$

(*iv*) $\lim_{x\to0} \frac{(a+h)^2\sin(a+h) - a^2\sin a}{h}$

$$= \lim_{x\to0}\left\{\frac{a^2\{\sin(a+h)-\sin a\} + 2ha\sin(a+h) + h^2\sin(a+h)}{h}\right\}$$

$$= \lim_{h\to0} \frac{a^2\cdot 2\cos\left(a+\frac{h}{2}\right)\sin\frac{h}{2}}{h}$$

$$+ \lim_{h\to0} 2a\sin(a+h) + \lim_{h\to0} h\sin(a+h)$$

$$= a^2 \cdot \lim_{h\to 0} \cos\left(a+\frac{h}{2}\right) \cdot \lim_{h\to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}} = 2a\sin a + 0$$

$$= a^2 \cdot \cos a \times (1) + 2a\sin a$$

$$= a^2 \cos a + 2a\sin a.$$

(*v*) $\lim_{x\to\infty} \dfrac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}$.

$$= \lim_{x\to\infty} \frac{1}{\sqrt{1+\sqrt{\frac{x}{x^2}+\sqrt{\frac{x}{x^4}}}}}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{1+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x^3}}}}}$$

$$= \lim_{y\to 0} \frac{1}{\sqrt{1+\sqrt{y+\sqrt{y^3}}}},$$ where $y=\frac{1}{x}$

and as $x\to\infty$, $y\to 0$

$$= \frac{1}{1} = 1.$$

Ex. 4. Evaluate :

(*i*) $\lim_{x\to\frac{\pi}{4}} \dfrac{4\sqrt{2}-(\cos x+\sin x)^5}{1-\sin 2x}$.

(*ii*) $\lim_{x\to 0} \dfrac{x\tan 2x - 2x\tan x}{(1-\cos 2x)^2}$.

(*iii*) $\lim_{x\to 0} \left\{\tan\left(\dfrac{\pi}{4}+x\right)\right\}^{\frac{1}{x}}$.

(*iv*) $\lim_{x\to\frac{\pi}{2}} \left(x\tan x - \dfrac{\pi}{2}\sec x\right)$.

(v) $\lim_{x\to 0} \frac{\sqrt{\cos x} - \sqrt[3]{\cos x}}{\sin^2 x}$.

Solution :

(i) $\lim_{x\to\frac{\pi}{4}} \frac{4\sqrt{2}-(\cos x+\sin x)^5}{(1-\sin 2x)} = \lim_{x\to\frac{\pi}{4}} \frac{4\sqrt{2}\left\{1-\frac{1}{4\sqrt{2}}(\cos x+\sin x)^5\right\}}{1-\sin 2x}$

$$= 4\sqrt{2}\cdot \lim_{x\to\frac{\pi}{4}} \frac{\left\{1-\left(\frac{1}{\sqrt{2}}\cos x+\frac{1}{\sqrt{2}}\sin x\right)^5\right\}}{1-\sin 2x}$$

$$= 4\sqrt{2}\cdot \lim_{x\to\frac{\pi}{4}} \frac{\left\{1-\cos^5\left(x-\frac{\pi}{4}\right)\right\}}{1-\sin 2x}$$

$$= 4\sqrt{2}\cdot \lim_{\theta\to 0} \frac{1-\cos^5\theta}{1-\cos 2\theta}, \text{ where, } \theta = x-\frac{\pi}{4} \text{ and } \theta\to 0 \text{ , as } x\to\frac{\pi}{4}.$$

$$= 4\sqrt{2}\cdot \lim_{\theta\to 0} \frac{(1-\cos\theta)(1+\cos\theta+\cos^2\theta+\cos^3\theta+\cos^4\theta)}{2\sin^2\theta}$$

$$= 2\sqrt{2}\cdot \lim_{\theta\to 0} \frac{2\sin^2\frac{\theta}{2}}{\sin^2\theta} \times \lim_{\theta\to 0}(1+\cos\theta+\cos^2\theta+\cos^3\theta+\cos^4\theta)$$

$$= 2\sqrt{2} \lim_{\theta\to 0} \frac{2\theta^2\sin^2\frac{\theta}{2}}{\frac{\theta^2}{4}\times 4} \times \frac{1}{\lim_{\theta\to 0}\frac{\sin^2}{\theta^2}\times\theta^2} \times$$

$$\lim_{\theta\to 0}(1+\cos\theta+\cos^2\theta+\cos^3\theta+\cos^4\theta)$$

$$= 2\sqrt{2} \lim_{\theta\to 0} \frac{1}{2}\left(\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\right)^2 \times \frac{1}{\lim_{\theta\to 0}\left(\sin\frac{\theta}{\theta}\right)^2} \times$$

$$(1+\cos\theta+\cos^2\theta+\cos^3\theta+\cos^4\theta)$$

$$= 2\sqrt{2}\times(1)^2\times\frac{1}{(1)^2}\times(1+1+1+1+1) = 5\sqrt{2}.$$

(ii) $$\lim_{x\to 0}\frac{x\tan 2x-2x\tan x}{(1-\cos 2x)^2}=\lim_{x\to 0}\frac{\dfrac{2x\tan x}{1-\tan^2 x}-2x\tan x}{(2\sin^2 x)^2}$$

$$=\lim_{x\to 0}\frac{2x\tan x\left\{\dfrac{1}{1-\tan^2 x}-1\right\}}{4\sin^4 x}=\frac{1}{2}\lim_{x\to 0}\frac{x\tan^3 x}{x(1-\tan^2 x)}$$

$$=\frac{1}{2}\lim_{x\to 0}\frac{x^4\tan^3 x}{x^3\sin^4 x(1-\tan^2 x)}$$

$$=\frac{1}{2}\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^3\times\frac{1}{\lim x\to 0\left(\dfrac{\sin x}{x}\right)^4}\times\lim_{x\to 0}\frac{1}{1-\tan^2 x}$$

$$=\frac{1}{2}(1)^3\times\frac{1}{(1)^4}\times\frac{1}{1}=\frac{1}{2}$$

since, $\lim_{x\to 0}\left(\dfrac{\tan x}{x}\right)=\lim_{x\to 0}\dfrac{\sin x}{x}\times\lim_{x\to 0}\dfrac{1}{\cos x}=1\times 1=1.$

(iii) $$\lim_{x\to 0}\left\{\tan\left(\frac{\pi}{4}+x\right)\right\}^{\frac{1}{x}}=\lim_{x\to 0}\left\{\frac{1+\tan x}{1+\tan x}\right\}^{\frac{1}{x}}.$$

$$=\frac{\lim_{x\to 0}\left\{(1+\tan x)^{\frac{1}{\tan x}}\right\}^{\left(\frac{\tan x}{x}\right)}}{\lim_{x\to 0}\left\{(1-\tan x)^{-\frac{1}{\tan x}}\right\}^{-\left(\frac{\tan x}{x}\right)}}=\frac{\lim_{x\to 0}\left\{(1+\tan x)^{\frac{1}{\tan x}}\right\}^{\lim_{x\to 0}\left(\frac{\tan x}{x}\right)}}{\lim_{x\to 0}\left\{(1-\tan x)^{-\frac{1}{\tan x}}\right\}^{\lim_{x\to 0}\left(-\frac{\tan x}{x}\right)}}$$

$$=\frac{\lim_{\theta\to 0}\left\{(1+\theta)^{\frac{1}{\theta}}\right\}^{1}}{\lim_{\theta\to 0}\left\{(1+\beta)^{\frac{1}{\beta}}\right\}^{-1}}=\frac{e}{e^{-1}}=e^2$$, where, $\tan x=\theta,\quad -\tan x=-\beta.$

(iv) $\lim_{x\to\frac{\pi}{2}}\left(x\tan x-\frac{\pi}{2}\sec x\right)$

$$=\lim_{x\to\frac{\pi}{2}}\left(\frac{x\sin x-\frac{\pi}{2}}{\cos x}\right)=\lim_{z\to 0}\left\{\frac{\left(\frac{\pi}{2}-z\right)\sin\left(\frac{\pi}{2}-z\right)-\frac{\pi}{2}}{\cos\left(\frac{\pi}{2}-z\right)}\right\}$$

where, $z=\frac{\pi}{2}-x \therefore\ z\to 0$ as $x\to\frac{\pi}{2}$.

$$=\lim_{z\to 0}\frac{1}{\sin z}\left\{\frac{\pi}{2}\cos z-z\cos z-\frac{\pi}{2}\right\}$$

$$=\lim_{z\to 0}\frac{-z\cos z}{\sin z}-\frac{\pi}{2}\cdot\lim_{z\to 0}\frac{1-\cos z}{\sin z}$$

$$=\frac{\lim_{z\to 0}\cos z}{\lim_{z\to 0}\left(\frac{\sin z}{z}\right)}-\frac{\pi}{2}\cdot\lim_{z\to 0}\frac{2\sin^2\frac{z}{2}}{2\sin\frac{z}{2}\cos\frac{z}{2}}$$

$$=-\frac{1}{1}-\frac{\pi}{2}\cdot\lim_{z\to 0}\tan\frac{z}{2},\quad \because\ \sin\frac{z}{2}\neq 0 \text{ as } z\to 0.$$

$$=-1-\frac{\pi}{2}\cdot 0=-1.$$

(v) Let us substitute, $\cos x=t^6$.

Obviously, $t\to 1$ as $x\to 0$ and $\sqrt{\cos x}=t^3$, $\sqrt[3]{\cos x}=t^2$

and $\sin^2 x=1-\cos^2 x=1-t^{12}$

$$\therefore\ \lim_{x\to 0}\frac{\sqrt{\cos x}-\sqrt[3]{\cos^2 x}}{\sin^2 x}$$

$$=\lim_{t\to 1}\frac{t^3-t^2}{1-t^{12}}=\lim_{t\to 1}\frac{t^2(t-1)}{-(t^{12}-1)}$$

$$=(-1)\frac{\lim_{t\to 1}(t^2)}{\lim_{t\to 1}\frac{t^{12}-1}{t-1}}$$

$$=(-1)\cdot\frac{1}{12(1)^{12-1}}.\qquad \because\ \lim_{z\to a}\frac{z^n-a^n}{z-a}=na^{n-1}$$

$$=-\frac{1}{12}.$$

EXAMPLES-III

1. Evaluate the following limits :

(i) $\underset{x\to 1}{Lt}\dfrac{x^2+2x-2}{2x+2}$. (ii) $\underset{x\to 1}{Lt}\dfrac{x^2-3x+2}{x^2-4x+3}$.

(iii) $\underset{x\to 0}{Lt}\dfrac{a-\sqrt{a^2-x^2}}{x^2}$. (iv) $\underset{x\to 0}{Lt}\dfrac{\sqrt{1+2x}-\sqrt{1-3x}}{x}$.

2. Find the value of

$$\underset{x\to 0}{Lt}\frac{a_0x^n+a_1x^{n-1}+\ldots+a_n}{b_0x^n+b_1x^{n-1}+\ldots+b_n} \qquad (b_n\neq 0).$$

3. Do the following limits exist ? If so, find their values :

(i) $\underset{x\to \pi}{Lt}\dfrac{1}{\pi-x}$. (ii) $\underset{x\to \pi}{Lt}\dfrac{\sin x}{\pi-x}$.

4. Find the values of :

(i) $\underset{x\to 0}{Lt}\dfrac{\tan x}{x}$. (ii) $\underset{x\to 0}{Lt}\dfrac{\sin\left(x^2\right)}{x}$.

(iii) $\underset{x\to 0}{Lt}\dfrac{1-\cos x}{x}$. (iv) $\underset{x\to 0}{Lt}\dfrac{1-\cos x}{x^2}$.

(v) $\underset{x\to 0}{Lt}\dfrac{x^2\sin(1/x)}{\sin x}$. (vi) $\underset{x\to 0}{Lt}\dfrac{\operatorname{cosec} x-\cot x}{x}$.

(vii) $\underset{x\to 0}{Lt}\dfrac{\sin x^{\circ}}{x}$. (viii $\underset{x\to 0}{Lt}\dfrac{\sin^{-1}x}{x}$.

(ix) $\underset{x\to 0}{Lt}\dfrac{\tan^{-1}x}{x}$. (x) $\underset{x\to \infty}{Lt}\dfrac{\sin x}{x}$.

(xi) $\underset{x\to \infty}{Lt}\dfrac{\sin x}{x+\cos x}$. (xii) $\underset{x\to \infty}{Lt}\dfrac{x+1}{x^2+1}$.

(xiii) $\underset{x\to 0}{Lt}\left(\dfrac{1}{\sin x}-\dfrac{1}{\tan x}\right)$.

5. A function $f(x)$ is defined as follows :

$$f(x) = x \quad \text{when } x > 0.$$
$$= 0 \quad \text{when } x = 0,$$
$$= -x \quad \text{when } x < 0.$$

Find the value of $\underset{x \to 0}{Lt} f(x)$.

6. A function $\phi(x)$ is defined as follows :

$$\phi(x) = x^2 \quad \text{when } x < 1,$$
$$= 2 \cdot 5 \quad \text{when } x = 1,$$
$$= x^2 + 2 \quad \text{when } x > 1.$$

Does $\underset{x \to 1}{Lt} \phi(x)$ exist ?

7. Do the following limits exist ?

(i) $\underset{x \to 2}{Lt} [x]$, where $[x]$ denotes the integral part of x.

(ii) $\underset{x \to 1}{Lt} \left\{x^2 + \sqrt{x-1}\right\}$.

(iii) $\underset{x \to \frac{1}{2}\pi}{Lt} \dfrac{e^{\tan x} - 1}{e^{\tan x} + 1}$.

8. Given $f(x) = ax^2 + bx + c$, show that

$$\underset{h \to 0}{Lt} \left\{ \frac{f(x+h) - f(x)}{h} \right\} = 2ax + b.$$

9. Given $f(x) = |x|$, show that

$\underset{h \to 0}{Lt} \{f(h) - f(0)\}/h$ does not exist.

10. If $\phi(x) = \left\{(x+2)^2 - 4\right\}/x$, show that

$\underset{x \to 0}{Lt} \phi(x) = 4$, although $\phi(0)$ does not exist.

11. Show that $\underset{x \to 2}{Lt} \dfrac{2x^2 - 8}{x - 2} = 8$.

Appl[illegible] (δ, ε) definition. [illegible] $= 0 \cdot 1$.

12. (i) Is $\underset{x\to a}{Lt}\dfrac{x^2}{x-a}-\underset{x\to a}{Lt}\dfrac{a^2}{x-a}=\underset{x\to a}{Lt}\dfrac{x^2-a^2}{x-a}$?

(ii) Is $\underset{x\to a}{Lt}\left(x^2-a^2\right)\times\underset{x\to a}{Lt}\dfrac{1}{x-a}=\underset{x\to a}{Lt}\left\{\left(x^2-a^2\right)\times\dfrac{1}{x-a}\right\}$?

13. Evaluate

(i) $\underset{n\to\infty}{Lt}\left(\dfrac{1}{n^2}+\dfrac{2}{n^2}+\dfrac{3}{n^2}+\ldots+\dfrac{n}{n^2}\right)$.

(ii) $\underset{n\to\infty}{Lt}\left(\dfrac{1^2+2^2+3^2+\ldots+n^2}{n^3}\right)$.

14. Does $\underset{x\to 0}{Lt} f(x)$ exist, when

(i) $f(x)=\left(2^{1/x}+2^x+1/2^x\right)$?

(ii) $f(x)=\left(\sin\dfrac{1}{x}+x\sin\dfrac{1}{x}+x^2\sin\dfrac{1}{x}\right)$?

15. Evaluate

(i) $\underset{n\to\infty}{Lt}\ x^n$.

(ii) $\underset{n\to\infty}{Lt}\dfrac{x^n}{x^n+1}$. [*C. H. 1957*]

(iii) $\underset{n\to\infty}{Lt}\dfrac{x^n f(x)+g(x)}{x^n+1}$. [*C.H. 1956*]

(iv) $\underset{n\to\infty}{Lt}\dfrac{x^n-1}{x^n+1}$.

16. Find the value of $\underset{n\to\infty}{Lt}\dfrac{2}{\pi}\text{arc}\tan nx$.

17. Evaluate

(i) $\underset{x\to\infty}{Lt}\ \sin n\pi x$.

(ii) $\underset{x\to\infty}{Lt}\dfrac{1}{1+n\sin^2 nx}$. [*C.H. 1957*]

(ii) $\underset{x \to \infty}{Lt} \dfrac{1}{1 + n \sin^2 nx}$. [C.H. 1957]

18. (i) Prove that

$\underset{x \to 0}{Lt} \tan^{-1} \dfrac{a}{x^2} = -\frac{1}{2}\pi, 0$ or $\frac{1}{2}\pi$ according as a is negative, zero or positive.

(ii) Draw the graph of the function $f(x)$ where

$$f(x) = \underset{t \to 0}{Lt} \left(\frac{2x}{\pi} \tan^{-1} \frac{x}{t^2} \right).$$

19. If $f(x) = \underset{n \to \infty}{Lt} \dfrac{1}{1 + x^{2n}}$, show that $f(x) = 1, \frac{1}{2}$ or 0 according as $|x| <, =$ or > 1. [C.H. 1950]

Draw the graph of $f(x)$ in this case.

20. The function $y = f(x)$ is defined as follows :

$f(x) = 0$ when $x^2 > 1$

$f(x) = 1$ when $x^2 < 1$

$f(x) = \frac{1}{2}$ when $x^2 = 1$.

Using the idea of a limit, show that the above function can be represented by

$f(x) = \underset{n \to \infty}{Lt} \dfrac{1}{1 + x^{2n}}$, for all values of x. [C.P. 1949]

ANSWERS

1. (i) $\frac{1}{4}$, (ii) $\frac{1}{2}$, (iii) $\frac{1}{2a}$, (iv) $\frac{5}{2}$,

2. $\frac{a_n}{b_n}$ 3. (i) Does not exist. (ii) 1.

4. (i) 1, (ii) 0, (iii) 0, (iv) $\frac{1}{2}$, (v) 0,

(vi) $\frac{1}{2}$, (vii) $\frac{\pi}{180}$, (viii) 1, (ix) 1, (x) 0,

(xi) 0, (xii) 0, (xiii) 0.

5. 0. 6. Does not exist.

7. (i) Does not exist. (ii) Does not exist (iii) Does not exist

11. 0·05.

12. (i) No, (ii) No.

13. (i) $\frac{1}{2}$, (ii) $\frac{1}{3}$.

14. (i) No, (ii) No.

15. (i) $+\infty$ when $x > 1$; 0 when $-1 < x < 1$; 1 when $x = 1$; no limit exists when $x \le -1$,

(ii) 0 when $-1 < x < 1$; $\frac{1}{2}$ when $x = 1$; 1 when $x < -1$ or > 1; not defined when $x = -1$,

(iii) $f(x)$ when $|x| > 1$; $g(x)$ when $|x| < 1$;

$\frac{1}{2}\{f(x) + g(x)\}$ when $x = 1$; undefined when $x = -1$,

(iv) -1 when $-1 < x < 1$; 0 when $x = 1$; 1 when $|x| > 1$.

16. 1 when $x > 0$; 0 when $x = 0$; -1 when $x < 0$.

17. (i) 0 when x is an integer; no limit exists if x is not an integer

(ii) 0.

4 CONTINUITY

4.1. We have a commonsense idea of what a continuous curve is. For instance, in Art. 2.5, the curves of example (ii), (iii), (v) are continuous, while those of (vi) and (vii) are discontinuous, the curve in (vi) having a point of discontinuity at the origin O. A function $f(x)$ is commonly said to be continuous provided its graph is a continuous curve, and, if there is any discontinuity or break at any point on the curve, the function is said to be discontinuous for the corresponding value of x. The general notions of continuity of a function $f(x)$ for any value of the variable x require that the function should be finite at the point, and for a very small change in x, the change in the value of $f(x)$ should also be small, or in other words, as we approach the particular value of x from either side the function should also approach the corresponding value of $f(x)$, and ultimately coincide with it at the point. If $f(x)$ be non-existent at a point, so that the corresponding point on the graph is missing, or else, if the value of $f(x)$ suddenly jumps as x passes from one side to the other of the particular value, or $f(x)$ becomes infinitely large at a point, then the function is discontinuous there.

We proceed below to give a formal mathematical definition of continuity.

4.2. Continuity.

A function $f(x)$ is said to be continuous for $x = a$, provided $\underset{x \to a}{Lt}\ f(x)$ exists, is finite, and is equal to $f(a)$.

In other words, for $f(x)$ to be continuous at $x = a$,

$$\underset{x \to a+0}{Lt} f(x) = \underset{x \to a-0}{Lt} f(x) = f(a)$$

or briefly, $f(a+0) = f(a-0) = f(a)$

This may also be written in the form $\underset{h \to 0}{Lt}\ f(a+h) = f(a)$.

If $f(x)$ be continuous for every value of x in the interval $[a, b]$, it is said to be *continuous throughout the interval*.

A function which is not continuous at a point is said to have a discontinuity at that point.

Ex (i). $f(x) = x^2$ is continuous for any value a of x,

for, $\underset{x \to a}{Lt}\ x^2 = a^2$,

(ii) $f(x) = \cos\frac{1}{x}$ is discontinuous at $x = 0$, since $\underset{x \to 0}{Lt} \cos\frac{1}{x}$ does not exist [See *Ex.*, 1, § *3.10*]

(iii) $f(x) = \frac{1}{x^2}$ is discontinuous at $x = 0$, since $\underset{x \to 0}{Lt}\left(\frac{1}{x^2}\right)$ is not finite.

(iv) $f(x) = x \sin\frac{1}{x}$ when $x \neq 0$, and $f(0) = 0$, then $f(x)$ is continuous at $x = 0$, for $\underset{x \to 0}{Lt}\ x \sin\frac{1}{x} = 0$.

[See *Ex.* (ii), § *3.6*]

(v) If $f(x) = \frac{1}{5 + e^{1/(x-2)}}$ when $x \neq 2$, and $f(2) = \frac{1}{2}$, then $f(x)$ is discontinuous at $x = 2$, since $\underset{x \to 2+0}{Lt} f(x) \neq \underset{x \to 2-0}{Lt} f(x)$ here, so that $\underset{x \to 2}{Lt} f(x)$ does not exist.

(vi) $f(x) = e^{-(a-x)^{-2}}$ is discontinuous at $x = a$, since though $\underset{x \to a+0}{Lt} f(x) = \underset{x \to a-0}{Lt} f(x) = 0$, *i.e.*, $\underset{x \to a}{Lt} f(x)$ exists and $= 0$, $f(a)$ is undefined.

Corresponding to the analytical definition of limit, we have the following *analytical definition* of continuity of a function at a point :

The function $f(x)$ is continuous at $x = a$ provided $f(a)$ exists and given any pre-assigned positive quantity ε, however small, we can determine a positive quantity δ such that $|f(x) - f(a)| < \varepsilon$ for all values of x satisfying $a - \delta \le x \le a + \delta$.

4.3. Different classes of Discontinuity.

(A) If $f(a+0) \neq f(a-0)$, then $f(x)$ is said to have an *ordinary discontinuity* at $x = a$. In this case, $f(a)$ may or may not exist, or if it exists, it may be equal to one of $f(a+0)$ and $f(a-0)$ or may be equal to neither.

To these is to be added the case where only one of $f(a+0)$ and $f(a-0)$ exists and $f(a)$ exists, but is not equal to that.

Illustration : $f(x)=\left(2+e^{\frac{1}{x}}\right)^{-1}$ has an ordinary discontinuity at $x=0$, for $\underset{x\to 0+0}{Lt} f(x)=0$, and $\underset{x\to 0-0}{Lt} f(x)=\frac{1}{2}$.

Note. Continuity on one side.

In case where $f(x)$ is undefined on one side of a (say, for $x>a$), if $f(a+0)$ exists and is equal to $f(a)$ (which also exists and is finite), we say, as a special case, that $f(x)$ is continuous at $x=a$.

(B) If $f(a+0)=f(a-0)\neq f(a)$, or $f(a)$ is not defined, then $f(x)$ is said to have a *removable discontinuity* at $x=a$.

Illustration : $f(x)=(x^2-a^2)/(x-a)$ has a removable discontinuity at $x=a$. For, $f(a)$ is undefined here, though $\underset{x\to a}{Lt} f(x)$ exists, and $=2a$.

Again, if $f(x)=1$ when $x=a$, and $f(x)=e^{-(x-a)^{-2}}$ when $x\neq a$, $f(x)$ has a removable discontinuity at a, for, $\underset{x\to a}{Lt} f(x)=0$, whereas $f(x)=1$ as defined.

It may be noted that a function which has removable discontinuity at a point can be made continuous there by suitably defining the function at the particular point only.

The two classes of discontinuities (A) and (B) are termed *simple discontinuities.*

(C) If one or both of $f(a+0)$ and $f(a-0)$ tend to $+\infty$ or $-\infty$, then $f(x)$ is said to have an *infinite discontinuity* at *a*. Here, $f(a)$ may or may not exist.

Illustration : $f(x)=e^{\frac{1}{x-a}}$ has an infinite discontinuity at $x=a$, since $f(a-0)\to\infty$, $f(x)=\dfrac{3x^2}{(x-2)^2}$ has an infinite discontinuity at $x=2$.

(D) Any point of discontinuity which is not a point of simple discontinuity, nor an infinite discontinuity, is called a point of *oscillatory discontinuity.* At such a point the funtion may *oscillate finitely* or *oscillate infinitely,* and does not tend to a limit, or tends to to $+\infty$ or $-\infty$.

Illustration : $f(x)=\sin\frac{1}{x}$ oscillate finitely at $x=0$.

$f(x)=\frac{1}{x-a}\sin\frac{1}{x-a}$ oscillates infinitely at $x=a$. $Lt\ x^n\ (x=-1)$ oscillates finitely, and $Lt\ x^n\ (x<-1)$ oscillates infinitely as $n\to\infty$.

4.4. Some properties of continuous functions.

(*i*) *The sum or difference of two continuous funcitons is a continuous function* ;

i.e., if $f(x)$ and $\phi(x)$ are both continuous at $x=a$, then $f(x)\pm\phi(x)$ is continuous at $x=a$.

For in this case, by definition of continuity, $\underset{x\to a}{Lt}\ f(x)$ exists, and

$=\underset{x\to a}{Lt}\ f(a)$, as also $\underset{x\to a}{Lt}\ \phi(x)=\phi(a)$.

Hence, $\underset{x\to a}{Lt}\ \{f(x)\pm\phi(x)\}=\underset{x\to a}{Lt}\ f(x)\pm\underset{x\to a}{Lt}\ \phi(x)$

[See § 3.8(*i*)]

$=f(a)\pm\phi(a)$,

whence, by definition, $f(x)\pm\phi(x)$ is continuous at $x=a$.

Note 1. The result may be extended to the case of any finite number of functions.

Note 2. If $f(x)$ is continuous at $x=a$, and $\phi(x)$ is not, then $f(x)\pm\phi(x)$ is discontinuous at $x=a$, and behaves like $\phi(x)$.

(*ii*) *The product of two continuous functions is continuous function*;

i.e., $f(x)$ and $\phi(x)$ being continuous at $x=a$, $f(x)\times\phi(x)$ is continuous there.

Proof is exactly similar to that in the above case, depending on the corresponding limit theorem [See § 3.8 (*ii*)].

Note. This result may also be extended to any finite number of functions.

(*iii*) *The quotient of two continuous functions is a continuous funciton, provided the denominator is not zero anywhere for the range of values considered* ;

i.e., if $f(x)$ and $\phi(x)$ be both continuous at $x=a$, and $\phi(a)\neq 0$, then $f(x)/\phi(x)$ is continuous there.

Proof depends on the corresponding limit theorem [See § 3.8(*iii*)].

(*iv*) *If f(x) be continuous at* $x = a$, *and* $f(a) \neq 0$, *then in the neighbourhood of* $x = a$, $f(x)$, *has the same sign as that of* $f(a)$, *i.e., we can get a positive quantity* δ *such that* $f(x)$ *preserves the same sign as that of* $f(a)$ *for every value of x in the interval* $a-\delta < x < a+\delta$.

Let $f(x) = \sin x$, $a = \frac{1}{2}\pi$; then $f(a) = 1$ and hence $\neq 0$ and *positive*. Let us take $\delta = \frac{1}{4}\pi$. Then in the interval $\frac{1}{2}\pi - \frac{1}{4}\pi < x < \frac{1}{2}\pi + \frac{1}{4}\pi$ *i.e.*, $\frac{1}{4}\pi < x < \frac{3}{4}\pi$, *f(x) is always positive.*

Since $f(x)$ is continuous at $x = a$, from definition, if ε be any chosen positive number, we can get a positive quantity δ, such that

$$|f(x) - f(a)| < \varepsilon, \text{ i.e., } f(a) - \varepsilon < f(x) < f(a) + \varepsilon \qquad \dots (1)$$

for all values of x satisfying $a-\delta < x < a+\delta$.

As $f(a) \neq 0$ here, if $f(a)$ be positive, choose $\varepsilon = \frac{1}{2} f(a)$ then from (1), $f(x) > f(a) - \varepsilon$, *i.e.*, $> \dfrac{1}{2} f(a)$, and is accordingly positive when $a - \delta < x < a + \delta$.

If $f(a)$ be negative, choose $\varepsilon = -\frac{1}{2} f(a)$, and then we have from (1), $f(x) < f(a) + \varepsilon$ *i.e.*, $< f(a) - \frac{1}{2} f(a)$, *i.e.*, $< \dfrac{1}{2} f(a)$, and is accordingly negative when $a - \delta < x < a + \delta$.

Thus, whatever be the sign of $f(a)$, we can find δ such that $f(x)$ has the same sign as that of $f(a)$ in the range $a-\delta < x < a+\delta$.

(*v*) *If* $f(x)$ *be continuous throughout the interval* $[a, b]$, *and if* $f(a)$ *and* $f(b)$ *be of opposite signs, then there is at least one value, say x, of x within the interval for which* $f(\xi) = 0$.

Let $f(x) = \cos x$, $a = 0$, $b = \pi$. Then $f(a) = 1$, $f(b) = -1$. Now, $\cos x = 0$ if $x = \frac{1}{2}\pi$, which obviously lies in the interval $(0, \pi)$. and so here $\xi = \frac{1}{2}\pi$. Similarly, if we take $a = 0$, $b = 3\pi$, we get another value of ξ, *viz.*, $\frac{3}{2}\pi$, besides $\frac{1}{2}\pi$.

Let $OA = a$, $OB = b$. Bisect the interval AB at C_1. If $f(x)$ be not zero at C_1, it must be opposite in sign to one of $f(a)$ and $f(b)$ which are given to be of opposite signs. Suppose $f(x)$ has opposite signs at C_1 and B. Bisect C_1B at C_2. If $f(x)$ be not zero at C_2, it must have opposite signs at the extremities of one of the intervals C_1C_2 or C_2B. Bisect that particular interval at C_3. Proceeding in this manner n times, unless $f(x)$ is zero at one of these points of bisection, we can get an interval C_{n-1}, C_n (say) within AB, at the opposite extremities of which $f(x)$ will have opposite signs. This interval C_{n-1}, C_n is clearly is clearly $1/2^n$ of the interval AB, *i.e.*, $= (b-a)/2^n$, and taking n large enough, can be made as small as we like.

But $f(x)$ being a continuous function for every value of x within the interval AB, corresponding to any point C_n in it, given by $x = c$ say, if $f(c)$ be not zero, it must be possible [*by (iv) above*] to get a positive quantity δ such that $f(x)$ will retain the same sign, namely that of $f(c)$, in the interval $[c-\delta, c+\delta]$. Now whatever δ we may choose, $= (b-a)/2^n$ can be made less than δ by taking n large enough, and it has been shown that the extremities of the interval C_{n-1}, C_n which falls within $[c-\delta, c+\delta]$, $f(x)$ has got opposite signs. We are thus led to a contradiction if $f(x)$ is not zero anywhere within the interval AB. Hence there must be some point in the interval, given by $x = \xi$ (say), where $f(\xi) = 0$ under the circumstances.

(*vi*) *If* $f(x)$ *is continuous throughout the interval* $[a, b]$ *and if* $f(a) \neq f(b)$, *then* $f(x)$ *assumes every value between* $f(a)$ and $f(b)$ *at least once in the interval.*

Let $f(x) = x^2$, $a = 0, b = 1$; then $f(a) = 0$, $f(b) = 1$.

Let c be any number between 0 and 1. Then $f(x) = x^2 = c$, which evidently lies in $[0, 1]$.

Let $f(x) = \sin x, a = 0, b = \frac{5\pi}{2}$; then $f(a) = 0$ $f(b) = 1$, so $f(a) \neq f(b)$. Let c be any number lying in $[0, 1]$. Then $\sin x = c$, if $x = n\pi + (-1)^n \sin^{-1} c$, $n = 0, \pm 1, \pm 2, \ldots$ Now, for $n = 0, 1, 2$ only, x lies in the interval $\left[0, \frac{5\pi}{2}\right]$. That is, when $x = \sin^{-1} c$, or $\pi - \sin^{-1} c$, or $2\pi + \sin^{-1} c$, we have $f(x) = c$. Thus, $f(x)$ assumes the value c at least once (here 3 times and in the previous example once only).

Let k be any quantity intermediate between $f(a)$ and $f(b)$ which are given to be unequal. Let $\phi(x) = f(x) - k$. Then since $f(x)$ is continuous in the interval $[a, b]$, $\phi(x)$ is also continuous. Also $\phi(a) = f(a) - k$ and $\phi(b) = f(b) - k$ are of opposite signs, since k lies between $f(a)$ and $f(b)$. Hence by (v) above, there is a value $x = \xi$ in the interval, for which $\phi(\xi) = 0$, *i.e.*, $f(\zeta) = k$. In other words, $f(x)$ assumes the value k at some point in the interval.

(*vii*) *A function which is continuous throughout a closed interval is bounded therein.*

The function $f(x) = \sin x$ is continuous in the closed interval $0 \le x \le \pi$, and has the upper bound at $x = \frac{1}{2}\pi$ and lower bound at $x = 0$ or p, hence it is bounded.

Let the function $f(x)$ be continuous throughout the closed interval $[a, b]$. Let us divide all the real numbers in the interval into two classes L, R, putting a number x in L if $f(x)$ is bounded in (a, x), and in R otherwise. Members of L-class exist in this case, since $f(x)$ being continuous at a (to the right), corresponding to any pre-assigned positive number e, we can get a positive number d such that $f(a) - \varepsilon < f(x) < f(a) + \varepsilon$ (and accordingly $f(x)$ is bounded) in the interval $[a, a+\delta]$, so that $a+\delta$ belongs to L-class. If now numbers of R-class also exist in the interval, then by Dedekind's theorem, ther exists a definite number c (say) in the interval, which represents the section. [To include all real numbers in the classification, we put all numbers less than a in L, and all numbers greater than b in R here.]

Now since $f(x)$ is continuous at c, for any given positive *quantity* e, *we can determine a positive number* d *such that* $f(c) - \varepsilon < f(x) < f(c) + \varepsilon$ *within the interval* $(c - \delta, c + \delta)$, i.e., $f(x)$ *is bounded therein. Also* $c - \delta$ *belonging to* L-*class*, $f(x)$ *is bounded in* $[a, c - \delta]$. *Hence,* $f(x)$ *is bounded throughout the interval* $(a, c + \delta)$. *But* $c + \delta$ *belonging to* R-*class,* $f(x)$ *is not bounded in* $[a, c - \delta]$. *This contradiction shows that no number of the* R-*class can exist in the interval* $[a, b]$; *in other words,* $f(x)$ *is bounded throughout the interval* $[a, b]$.

(*viii*) *A continuous function in an interval actually attains its upper and lower bounds, at least once each, in the interval.*

The function $f(x) = \sin x$ is continuous in the interval $0 \le x \le \pi$. Its upper bound 1 is attained at the point $x = \frac{1}{2}\pi$ and the lower bound 0 is attained at the point $x=0$ and $x=$ p. Thus, $f(x)$ attains its upper and lower bounds, at least once each (here the upper bound is attained once, whereas the lower bound is attained twice).

(ix) A function $f(x)$, *continuous in a closed interval* $[a, b]$, *attains every intermidiate value between its upper and lower bounds in the interval, at least once.*

Let $f(x)=x^2$, $a=-1, b=2$, then the upper bound of $f(x)$ is 4 and its lower bound is 0. Let c be any number in $[0, 4]$. Now, if $0 \le c \le 1$, then $f(x) = x^2 = c$, if $x = \pm\sqrt{c}$ which lies in $(-1, 2)$; and if $1 < c \le 4$, then $f(x)=x^2=c$ if $x=\pm\sqrt{c}$, of which only $+\sqrt{c}$ lies in the interval $(-1, 2)$. Thus, $f(x)$ attains the value c at least once.

Let $f(x)$ be continuous in the closed interval $[a, b]$, and let M and m be its upper and lower bounds in the interval. If possible, let there be no point in the interval where $f(x)=M$. Then $M - f(x) > 0$ for all points in the interval. Now, since $f(x)$ is continuous, $M - f(x)$ is also continous, and so $1/\{M - f(x)\}$ is continuous in the interval. Thus $1/\{M - f(x)\}$ is bounded in the interval, *i.e.*, $1/\{M - f(x)\} \le k$, where k is a fixed positive number.

$$\therefore \quad M - f(x) \ge \frac{1}{k}, \text{ or, } f(x) \le M - \frac{1}{k}.$$

This contradicts the assumption that M is the upper bound of $f(x)$ in the interval.

Hence, $f(x)$ must assume the value M at some point in the interval.

Similarly, it may be proved that $f(x)$ assumes the value m also in the interval.

It now follows from (vi) that $f(x)$ assumes every intermediate value between M and m.

4.5. Continuity of some Elementary Functions.

(*i*) *Function* x^n, where n is any rational number.

We know that $\underset{x \to a}{Lt} x^n = a^n$, for all values of n, except when $a=0$ and n is negative [*See Note, Ex. 1, § 3.11*].

Hence, x^n is continuous for all values of x when n is positive, and continuous for all values of x except 0 when n is negative.

When n is negative and $= -m$, say, where m is positive

$$x^n = x^{-m} = 1/x^m,$$

which either does not tend to a limit or $\to \infty$ as $x \to 0$.

(*ii*) *Polynomials.*

Since the polynomial $a_0 x^n + a_1 x^{n-1} + \ldots + a^n$ is the sum of a finite number of positive integral power of x (each multiplied by a constant) each of which is continuous for all values of x, the polynomial itself [*by §4.4(i)*] is continuous for all values of x.

(*iii*) *Rational Algebraic Functions.*

Rational algebraic functions like

$$\frac{a_0 x^n + a_1 x^{n-1} + \ldots + a^n}{b_0 x^n + b_1 x^{n-1} + \ldots + b^n}$$

being the quotient of two polynomials which are continuous for all values of x, are continuous for all values of x except those which make the denominator zero [*by § 4.4(iii)*].

(*iv*) *Trignometric Functions.*

Since the limiting values of $\sin x$ and $\cos x$ when $x \to a$, where a has any value, are $\sin a$ and $\cos a$ [*See Ex. 2, § 3.11*], it follows that $\sin x$ and $\cos x$ are continuous for all values of x.

Since $\tan x = \sin x / \cos x$, $\tan x$ is continuous for all values of x except those which make $\cos x$ zero, *i.e.*, except for $x = (2n+1)\frac{1}{2}\pi$. Similarly, $\sec x$ is continuous for all values of x except for $x = (2n+1)\frac{1}{2}\pi$ and $\cot x$ and $\operatorname{cosec} x$ are continuous for all values of x, except when $x = 0$ or any multiple of π when $\sin x = 0$.

(*v*) *Inverse Circular Functions.*

Inverse circular functions being many valued, we make a convention of defining their domains in such a way as to make them single-valued. Throughout the book we shall suppose (unless otherwise stated) that $\sin^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, $\operatorname{cosec}^{-1} x$ lie between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ (both values inclusive) and $\cos^{-1} x$, $\sec^{-1} x$ lie between 0 and π (both values inclusive), which are the principal values of these inverse functions. It should be noted. however, that and have no existence *outside* the *closed* interval of x, and and have no existence *outside* the *closed* interval $[-1,1]$ of x, and $\operatorname{cosec}^{-1} x$ and $\sec^{-1} x$ have no existence *inside* the *open* interval $(-1,1)$.

All the inverse circular functions are continuous for all values for which they exist ; this follows immediately from the continuity of the corresponding circular functions.

(vi) *Function* e^x.

Corresponding to the positive number ε, however small, we can choose n sufficiently large such that $(1+\varepsilon)^n > e$, since $(1+\varepsilon)^n > 1+ne$, and e is finite.

Thus, $e^{\frac{1}{n}} - 1 < \varepsilon$.

Hence, if $0 < x < 1/n$, $e^x - 1 < e^{\frac{1}{n}} - 1 < \varepsilon$,

and therefore $\underset{x \to 0+}{Lt} \left(e^x - 1\right) = 0$, or , $\underset{x \to 0+}{Lt} e^x = 1$.

If x be negative, putting $x = -y$, $\underset{x \to 0-}{Lt} e^x = \underset{y \to 0+}{Lt} 1/e^y = 1$.

Hence, $\underset{x \to 0}{Lt} e^x = 1$.

$\therefore \quad \underset{x \to c}{Lt} e^{x-c} = 1$, *i.e.*, $\underset{x \to c}{Lt} e^x = e^c$.

$\therefore \quad e^x$ is continuous at any point $x = c$.

(vii) *Function* $\log x$, $x > 0$.

It should be noted that $\log x$ is defined only for values of $x > 0$.

Let $\log x = y$ and $\log(x+h) = y+k$.

Then $e^y = x$ and $e^{y+k} = x+h$;

$\therefore h = e^{y+k} - e^y$.

As e^y is a continuous function of y, $e^{y+k} \to e^y$,

i.e., $h \to 0$ as $k \to$

Thus, $\{\log(x+h) - \log x\} \to 0$ as $k \to 0$,

i.e., as $h \to 0$.

Hence, $\log x$ is continuous.

4.6. Illustrative Examples.

Ex. 1. *A function* $f(x)$ *is defined as follows :*

$f(x) = x$ *when* $x > 0$, $f(0) = 0$, $f(x) = -x$ *when* $x < 0$.

Prove that the function is continuous at $x = 0$.

Here, $\underset{x \to 0+0}{Lt} f(x) = \underset{x \to 0+0}{Lt} x = 0$ and $\underset{x \to 0-0}{Lt} f(x) = \underset{x \to 0-0}{Lt} (-x) = 0$.

Thus $\underset{x\to 0+0}{Lt} f(x) = \underset{x\to 0-0}{Lt} f(x) = f(0) = 0$ here,

Hence, $f(x)$ is continuous at $x = 0$,

For its graph, see figure of § 2.5 (ii).

Ex. 2. *A function $f(x)$ is defined as follows :*

$$f(x) = x\sin\frac{1}{x} \quad \textit{for } x \neq 0$$
$$= 0 \quad \textit{for } x = 0,$$

Show that $f(x)$ is continuous at $x = 0$. [V. P. 1999]

Since $|\sin(1/x)| \leq 1$, by making $|x| < \varepsilon$,

we can make $|x\sin(1/x)| < \varepsilon$,

where ε is any pre-assigned positive quantity, however small.

Hence, $\underset{x\to 0}{Lt}\ x\sin\frac{1}{x} = 0$. Also, $f(0) = 0$, as defined.

Thus, $\underset{x\to 0}{Lt}\ f(x) = f(0)$.

For its graph, see figure of §2.5 (viii).

Note. It should be noted that the function $x\sin(1/x)$ is continuous for all values of x, except for $x = 0$; because when $x = 0$, $x\sin(1/x)$ is undefined. In the above example, the discontinuity of $x\sin(1/x)$ at $x = 0$ has been *removed by definition of $f(0)$.*

Ex. 3. A function $f(x)$ is defined as follows :

$$f(x) = \frac{1}{2} - x \quad \textit{when } 0 < x < \tfrac{1}{2},$$
$$= \frac{1}{2} \quad \textit{when } x = \frac{1}{2},$$
$$= \frac{3}{2} - x \quad \textit{when } \tfrac{1}{2} < x < 1.$$

Show that $f(x)$ is discontinuous at $x = \frac{1}{2}$.

Here, $\underset{x\to \frac{1}{2}-0}{Lt} f(x) = \underset{x\to \frac{1}{2}-0}{Lt}\left(\frac{1}{2} - x\right) = \frac{1}{2} - \frac{1}{2} = 0,$

$\underset{x\to \frac{1}{2}+0}{Lt} f(x) = \underset{x\to \frac{1}{2}+0}{Lt}\left(\frac{3}{2} - x\right) = \frac{3}{2} - \frac{1}{2} = 1.$

Since $\underset{x\to\frac{1}{2}}{Lt} f(x)$ does not exist,

hence $f(x)$ is discontinuous at $x=\frac{1}{2}$.

Ex. 4. A function $f(x)$ is defined in $(0, 3)$ in the following way

$$f(x)=x^2 \quad when \quad 0<x<1,$$
$$=x \quad when \quad 1\le x<2,$$
$$=\frac{1}{4}x^3 \quad when \quad 2\le x<3.$$

Show that $f(x)$ is discontinuous at $x=1$ and $x=2$. [*C. P. 1941*]

when $x=1$, $f(x)=x$. $\therefore f(1)=1$.

$$\underset{x\to1-0}{Lt} f(x)=\underset{x\to1-0}{Lt} x^2=1;$$

also, $\underset{x\to1+0}{Lt} f(x)=\underset{x\to1+0}{Lt} x=1.$

Hence, $\underset{x\to1-0}{Lt} f(x)=\underset{x\to1+0}{Lt} f(x)=f(1)$ here.

$\therefore$ $f(x)$ is continuous at $x=1$.

Similarly, it can be shown (from the definition of the function in the relevant ranges) that $\underset{x\to2-0}{Lt} f(x)=\underset{x\to2+0}{Lt} f(x)=f(2)=2$.

Hence, $f(x)$ is continuous at $x=2$.

Ex. 5. Show that the function $f(x)=|x|+|x-1|+|x-2|$ is continuous at the points $x=0, 1, 2$.

Here, $$f(x)=\begin{cases}-x-(x-1)-(x-2)=-3x+3, & \text{for } x<0\\ x-(x-1)-(x-2)=-x+3, & \text{for } 0\le x<1\\ x+(x-1)-(x-2)=x+1, & \text{for } 1\le x<2\\ x+(x-1)+(x-2)=3x-3, & \text{for } x\ge2\end{cases}$$

Now, $\underset{x\to0-}{Lt} f(x)=\underset{x\to0-}{Lt}(-3x+3)=3$

$\underset{x\to0+}{Lt} f(x)=\underset{x\to0+}{Lt}(-x+3)=3$ and $f(0)=3$.

$\because \underset{x\to 0-}{Lt} f(x)= \underset{x\to 0+}{Lt} f(x)=f(0)$, $f(x)$ is continous at $x=0$

Again, $\underset{x\to 1-}{Lt} f(x)= \underset{x\to 1+}{Lt} f(x)=f(1)=2,$

and $\underset{x\to 2-}{Lt} f(x)= \underset{x\to 2+}{Lt} f(x)=f(2)=3$

$\therefore$ $f(x)$ is continuous at $x=1$ and $x=2$.

Ex. 6. *Show that the function f defined by $f(x)=x-[x]$, where $[x]$ denotes the integral part of x is discontinuous for all integral values of x and continuous for all others.*

We have, $f(x)=\begin{cases} x-(\alpha-1), \text{ for } \alpha-1<x<\alpha \\ 0, \text{ for } x=\alpha \\ x-\alpha, \text{ for } \alpha<x<\alpha+1 \end{cases}$

where α is an integer.

$$\underset{\alpha-}{Lt} f(x)= \underset{x\to\alpha-}{Lt}(x-\alpha+1)=1$$

$$\underset{\alpha+}{Lt} f(x)= \underset{x\to\alpha+}{Lt}(x-\alpha)=0$$

$$\because \underset{x\to\alpha-}{Lt} f(x)\neq \underset{x\to\alpha+}{Lt} f(x)=f(\alpha)$$

$f(x)$ is not continuous at $=\alpha$.

Since α is any integer, $f(x)$ is discontinuous for all integral values of x.

$f(x)$ is obviously continuous for other values of x.

Ex. 7. *Show that the function f defined by*

$$f(x)=\frac{x-1}{1+e^{\frac{1}{x-1}}}, \textit{for } x\neq 1$$

$$=0, \qquad \textit{for } x=1$$

is continuous at $x=1$.

$$e^{\frac{1}{x-1}}\to\infty \text{ as } x\to 1-\Rightarrow \underset{x\to 1-}{Lt} f(x)=0$$

$$e^{\frac{1}{x-1}}\to\infty \text{ as } x\to 1+\Rightarrow \underset{x\to 1+}{Lt} f(x)=0.$$

So, $\underset{x\to 1-}{Lt} f(x)=f(1)=\underset{x\to 1+}{Lt} f(x)=0$.

Hence, $f(x)$ is continuous at $x=1$.

4.7. Some Miscellaneous Worked out Examples

Ex. 1. *Discuss the continuity of the following functions at the points indicated :*

(i) $$\left.\begin{aligned} f(x) &= x \quad \text{when } 0 < x < 1. \\ &= 2 - x \quad \text{when } 1 \le x \le 2. \\ &= x - \tfrac{1}{2}x^2 \text{ when } x > 2 \end{aligned}\right\} \text{at } x = 2$$ [C. P. 1989, '97, 2005]

(ii) $$\left.\begin{aligned} f(x) &= \frac{\tan^2 x}{3x}, \quad x \ne 0 \\ &= \tfrac{2}{3}, \quad x = 0 \end{aligned}\right\} \text{at } x = 0$$ [B. P. 1994]

(iii) $$\left.\begin{aligned} f(x) &= x^2 + x, \quad 0 \le x \\ &= 2, \quad x = 1 \\ &= 2x^3 - x + 1, \quad 1 < x \le 2 \end{aligned}\right\} \text{at } x = 1$$ [C. P. 1992]

(iv) $$\left.\begin{aligned} f(x) &= \frac{x^4 + 4x^3 + 2x}{\sin x}, \quad x \ne 0 \\ &= 0, \quad x = 0 \end{aligned}\right\} \text{at } x = 0$$ [C. P. 1994]

Solution : (i) Here $\lim\limits_{x \to 2-} f(x) = \lim\limits_{x \to 2-} (2 - x) = 0$

$$\lim_{x \to 2+} f(x) = \lim_{x \to 2+} \left(x - \frac{1}{2}x^2\right) = 0$$

and $f(2) = 2 - 2 = 0$

$$\because \lim_{x \to 2-} f(x) = \lim_{x \to 2+} f(x) = f(2),$$

$f(x)$ is continuous at $x = 2$.

(ii) We have,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\tan^2 x}{3x} = \lim_{x \to 0} \left\{ \frac{\sin^2 x}{x^2} \cdot \frac{1}{\cos^2 x} \cdot \frac{x}{3} \right\}$$

$$= \lim_{x \to 0} \left(\frac{\sin x}{x} \right)^2 \cdot \lim_{x \to 0} \left(\frac{1}{\cos^2 x} \right) \cdot \lim_{x \to 0} \left(\frac{1}{3} x \right)$$

$$= (1)^2 \times (1)^2 \times (0) = 0.$$

But $f(0)=\frac{2}{3}$

$\because \lim\limits_{x\to 0} f(x) \neq f(0)$, $f(x)$ is not continuous at $x=0$.

(*iii*) Here, $\lim\limits_{x\to 1-} f(x)=\lim\limits_{x\to 1-}\left(x^2+x\right)=2$

$$\lim_{x\to 1+} f(x)=\lim_{x\to 1+}\left(2x^3-x+1\right)=2$$

and $f(1)=2$

So, $\lim\limits_{x\to 1-} f(x)=\lim\limits_{x\to 1+} f(x)=f(1)$

Hence, $f(x)$ is continuous at $x=1$.

(*iv*) $\because f(x)=\dfrac{x^4+4x^3+2x}{\sin x}$, when $x\neq 0$,

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{x^3+4x^2+2}{\dfrac{\sin x}{x}}=\frac{\lim\limits_{x\to 0}\left(x^3+4x^2+2\right)}{\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)}=\frac{2}{1}=2$$

But $f(0)=0$

Thus $\lim\limits_{x\to 0} f(x)\neq f(0)$

Hence, $f(x)$ is not continuous at $x=0$.

Ex. 2. (*i*) Find $f(0)$, so that $f(x)=x\sin\left(\dfrac{\pi}{x}\right)$, for $x\neq 0$ may be continuous at $x=0$. [*C. P. 1982, '86*]

(*ii*) $f(x)=\dfrac{\sin\left(a^2x^2\right)}{x}$, $x\neq 0$ and $f(0)=k$. Find the value of k for which $f(x)$ is continuous at $x=0$. [*B. P. 1992*]

(*iii*) What should be the value of $f(0)$ so that f defined by $f(x)=\dfrac{x^2-x}{x}$, for $x\neq 0$ be continuous at $x=0$? [*C. P. 1997*]

Solution : (*i*) Here, $f(x)=x\sin\left(\dfrac{\pi}{x}\right)$, $x\neq 0$.

for all real values of x, $\left|\sin\left(\dfrac{\pi}{x}\right)\right|\leq 1$, $\lim\limits_{x\to 0} x\cdot\sin\left(\dfrac{\pi}{x}\right)=0$.

If $f(x)$ is continuous at $x=0$, $\lim_{x\to 0} f(x)=f(0)$

$\therefore f(0)=0, \quad \because \lim_{x\to 0} f(x)=0.$

(*ii*) Here, $f(x)=\dfrac{\sin(a^2x^2)}{x}, \quad x\neq 0$

$$\therefore \lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{\sin(a^2x^2)}{x}$$

$$=\lim_{x\to 0}\left\{\frac{\sin(a^2x^2)}{a^2x^2}\cdot a^2x\right\}$$

$$=a^2\cdot\lim_{x\to 0}\frac{\sin(a^2x^2)}{a^2x^2}\cdot\lim_{x\to 0}x=a^2\times 1\times(0)=0.$$

since $f(x)$ is continuous at $x=0, f(0)=\lim_{x\to 0} f(x)$ *i.e.*, $k=0$.

(*iii*) $f(x)=\dfrac{x^2-x}{x}, \; x\neq 0$

$=x-1, \quad \because x\neq 0$

$\lim_{x\to 0} f(x)=-1.$

So, in order that $f(x)$ may be continuous at $x=0$,

$f(0)=\lim_{x\to 0} f(x)$ *i.e.*, $f(0)=-1$.

Ex. 3. The function $f(x)=\dfrac{\log(1+ax)-\log(1-bx)}{x}$ is not defined at $x=0$. Find the value of $f(0)$ so that $f(x)$ is continuous at $x=0$.

Solution : Here, $\lim_{x\to 0} f(x)=\lim_{x\to 0}\dfrac{\log(1+ax)-\log(1-bx)}{x}$

$$=a\cdot\lim_{x\to 0}\frac{\log(1+ax)}{ax}-(-b)\lim_{x\to 0}\frac{\log(1-bx)}{(-bx)}$$

$$=a\times 1+b\times(1)=a+b.$$

For $f(x)$ to be continuous at $x=0$, $f(0)$ should be defined and the value of $f(0)$ must be equal to the limiting value of $f(x)$ as $x\to 0$. Hence, $f(0)=a+b$.

Ex. 4. Find the values of a and b such that the function

$$f(x) = x+\sqrt{2}a\sin x, \quad 0 \le x \le \frac{\pi}{4}$$
$$= 2x\cot x+b, \quad \frac{\pi}{4} \le x \le \frac{\pi}{2}$$
$$= a\cos 2x - b\sin x, \quad \frac{\pi}{4} \le x \le \pi$$

is continuous for all values of x in the interval $0 \le x \le \pi$.

Solution : The function $f(x)$ will be continuous for all values of x in $0 \le x \le \pi$, if it is continuous at $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$. So, we must have,

$$\lim_{x \to \frac{\pi}{4}-} f(x) = \lim_{x \to \frac{\pi}{4}+} f(x) = f\left(\frac{\pi}{4}\right) \quad \ldots \quad (1)$$

$$\text{and } \lim_{x \to \frac{\pi}{2}+} f(x) = \lim_{x \to \frac{\pi}{2}+} f(x) = f\left(\frac{\pi}{2}\right) \quad \ldots \quad (2)$$

$$\text{Now, } \lim_{x \to \frac{\pi}{4}+} f(x) = \lim_{x \to 4+} f(x+\sqrt{2}a\sin x) = \frac{\pi}{4}+a$$

$$\lim_{x \to \frac{\pi}{4}-} f(x) = \lim_{x \to \frac{\pi}{4}-} (2x\cot x+b) = \frac{\pi}{2}+b$$

$$\text{and } f\left(\frac{\pi}{4}\right) = 2\cdot\frac{\pi}{4}\cdot\cot\frac{\pi}{4}+b = \frac{\pi}{2}+b.$$

$$\text{Also, } \lim_{x \to \frac{\pi}{2}-} f(x) = \lim_{x \to \frac{\pi}{2}-} (2x\cot x+b) = b$$

$$\lim_{x \to \frac{\pi}{2}+} f(x) = \lim_{x \to \frac{\pi}{2}+} (a\cos 2x - b\sin x) = -a-b$$

$$\text{and } f\left(\frac{\pi}{2}\right) = a\cos\left(2\cdot\frac{\pi}{2}\right) - b\sin\frac{\pi}{2} = -a-b$$

From (1) and (2) we have,

$$\frac{\pi}{2}+b = \frac{\pi}{4}+a \text{ or, } a-b = \frac{\pi}{4} \quad \ldots \quad (3)$$

$$b = -a-b, \quad a = -2b \quad \ldots \quad (4)$$

Solving (3) and (4), $a = \frac{\pi}{6}$, $b = -\frac{\pi}{12}$.

Ex. 5. The function f is defined as follows :

$$f(x) = -2\sin x, \quad -\pi \le x \le -\frac{\pi}{2}$$
$$= a\sin x + b, \quad -\frac{\pi}{2} < x \frac{\pi}{2},$$
$$= \cos x, \quad \frac{\pi}{2} \le x \le \pi.$$

If $f(x)$ is continuous in the interval $-\pi \le x \le \pi$ find the values of a and b.

Solution : For continuity of $f(x)$ at $x = -\frac{\pi}{2}$,

$$\lim_{x \to -\frac{\pi}{2}-} f(x) = \lim_{x \to -\frac{\pi}{2}+} f(x) = f\left(-\frac{\pi}{2}\right)$$

i.e., $\lim_{x \to -\frac{\pi}{2}-} (-2\sin x) = \lim_{x \to -\frac{\pi}{2}+} (a\sin x + b) = -2\sin\left(-\frac{\pi}{2}\right)$

i.e., $-a + b = 2$ . . . (1)

And, for continuity of $f(x)$ at $x = \frac{\pi}{2}$,

$$\lim_{x \to \frac{\pi}{2}-} f(x) = \lim_{x \to \frac{\pi}{2}+} f(x) = f\left(\frac{\pi}{2}\right),$$

i.e., $\lim_{x \to \frac{\pi}{2}-} (a\sin x + b) = \lim_{x \to \frac{\pi}{2}+} (\cos x) = \cos\left(\frac{\pi}{2}\right)$

i.e., $a + b = 0$ . . . (2)

Solving (1) and (2), we get, $a = -1, b = 1$.

Ex. 6. Find the points where the function $f(x) = \frac{1}{\log|x|}$ is discontinuous.

Solution : $f(x) = \frac{1}{\log|x|}$

Obviously, the function is not defined for $x = 0$, and hence $x = 0$ s a point of discontinuity of the function.

Again, when $\log|x| = 0$, *i.e.*, $x = \pm 1$, the function is not defined.

Hence, the function has three points of discontinuity, viz., $x = -1$, $x = 0$ and $x = 1$.

Ex. 7. Show that the function $f(x)=[x]+[-x]$, where $[x]$ denotes the greatest integer not exceeding x, has removable discontinuity for integral values of x.

Solution : Let, $x=k$ be any integer.

$[k]=k$ and $[-k]=-k$

$\therefore\ f(k)=k-k=0$

Now, $\lim\limits_{x\to k+} f(x)=\lim\limits_{h\to 0} f(k+h)$

$\lim\limits_{h\to 0}[k+h]+\lim\limits_{h\to 0}[-k-h]=k-(k+1)=-1$

and $\lim\limits_{x\to k-} f(x)=\lim\limits_{h\to 0} f(k-h)$

$\lim\limits_{h\to 0}[k-h]+\lim\limits_{h\to 0}[-k+h]=(k-1)-(k)=-1$

$\therefore\ \lim\limits_{x\to k} f(x)=-1$, but $f(k)=0$

So, f has a discontinuity at $x=k$, where k is any integer. If however, we define $f(k)=-1$, then the function becomes continuous at $x=k$.

Hence the function has a removable discontinuity for integral values of x.

Ex. 8. Let $f(x)$ be a continuous function and $\phi(x)$ be a discontinuous function. Prove that $f(x)+\phi(x)$ is a discontinuous function.

Solution : Let, $\psi(x)=f(x)+\phi(x)$, where $f(x)$ is a continuous and $\phi(x)$ is a discontinuous function.

If possible, let $\psi(x)$ be a continuous function.

Since, $f(x)$ is a continuous function, $\psi(x)-f(x)$ is also a continuous function, *i.e.*, $\phi(x)$ is a continuous function. But this contradicts the given condition.

So, $\psi(x)$ *i.e.*, $f(x)+\phi(x)$ must be a discontinuous function.

Ex. 9. Let, f be a function, such that for all real values of x, y, $f(x+y)=f(x)+f(y)$. If f is continuous at $x=a$, then prove that f is continuous for all real values of x.

Solution : Since $f(x)$ is continuous at $x=a$,

$$\lim_{x\to a+} f(x)=f(a)$$

or, $f(a)=\lim_{h\to 0} f(a+h)=\lim_{h\to 0}[f(a)+f(h)]$,

$$[\because f(x+y)=f(x)+f(y)]$$

$$=\lim_{h\to 0} f(a)+\lim_{h\to 0} f(h)=f(a)+\lim_{h\to 0} f(h)$$

$$\therefore \quad \lim_{h\to 0} f(h)=0 \qquad \ldots \quad (1)$$

Similarly, $f(a)=\lim_{h\to 0} f(a-h)=\lim_{h\to 0} f(a)+\lim_{h\to 0} f(-h)$

$$=f(a)+\lim_{h\to 0} f(-h)$$

$$\therefore \quad \lim_{h\to 0} f(-h)=0 \qquad \ldots \quad (2)$$

Now, let k be any real number.

$$\lim_{h\to 0} f(k+h)=\lim_{h\to 0}[f(k)+f(h)]$$

$$=f(k)+\lim_{h\to 0} f(h)=f(k) \qquad [\text{ from (1) }]$$

and, $\lim_{h\to 0} f(k-h)=\lim_{h\to 0}[f(k)+f(-h)]$

$$=\lim_{h\to 0} f(k)+\lim_{h\to 0} f(-h)$$

$$=f(k) \qquad [\text{ from (2) }]$$

Also, putting $x=0$, $y=0$, we have $f(0)=f(0)+f(0)$,

so that $f(0)=0$

$\therefore \; f(k)=f(k+0)=f(k)+f(0)=f(k)$

Thus $\lim_{h\to 0+} f(k+h)=\lim_{h\to 0-} f(k+h)=f(k)$

$\therefore$ $f(x)$ is continuous at $x=k$.

Since k is arbitrary, $f(x)$ is continuous for all $x\in R$.

Ex. 10. (*i*) Show that $f(x)=\sin x$ is continuous for all real values of x.

[*B.P. 1999*]

(*ii*) Apply $\in-\delta$ definition to show that the following functions are continuous at the indicated points :

(*a*) $\left.\begin{array}{l} f(x) = x\sin\left(\dfrac{1}{x}\right), \quad x \neq 0 \\ f(0) = 0 \end{array}\right\}$ at $x = 0$ [*C. P. 1981, '95*]

(*b*) $\left.\begin{array}{l} f(x) = x^2\cos\left(\dfrac{1}{x}\right) \quad x \neq 0 \\ f(0) = 0 \end{array}\right\}$ at $x-0$ [*B. P. 1996*]

Solution : (*i*) $f(x) = \sin x$ will be continuous at $x = \alpha$, if for any $\in>0$, we can find δ, such that $|\sin x - \sin\alpha| < \in$, whenever, $|x-\alpha| < \delta$.

$$\text{We have, } |\sin x - \sin\alpha| = \left|2\sin\frac{x-\alpha}{2}\cos\frac{x+\alpha}{2}\right|$$

$$= 2\left|\sin\frac{x-\alpha}{2}\right|\cdot\left|\cos\frac{x+\alpha}{2}\right|.$$

Since, $\left|\cos\dfrac{x+\alpha}{2}\right| \leq 1$, for all real values of x

and $\left|\sin\dfrac{x-\alpha}{2}\right| < \left|\dfrac{x-\alpha}{2}\right|$, for $0 < \left|\dfrac{x-\alpha}{2}\right| < \dfrac{\pi}{2}$,

$$|\sin x - \sin\alpha| = 2\left|\sin\frac{x-\alpha}{2}\right|\left|\cos\frac{x+\alpha}{2}\right| < 2\left|\frac{x-\alpha}{2}\right|$$

$= |x-\alpha| < \in$ for $|x-\alpha| < \delta$

The relations are satisfied by taking $\delta = \in$

So, $|\sin x - \sin\alpha| < \in$, whenever, $|x-\alpha| < \in$.

So, $f(x) = \sin x$ is continuous at $x = \alpha$. Since α is any real number, $\sin x$ is continuous for real values of x.

(*ii*) (*a*) $\left.\begin{array}{l} f(x) = x\sin\left(\dfrac{1}{x}\right), \quad x \neq 0 \\ f(0) = 0 \end{array}\right\}$ at $x = 0$

$$\text{Now, } |f(x) - f(0)| = \left|x\sin\left(\frac{1}{x}\right) - 0\right|$$

$$= \left| x \sin \frac{1}{x} \right| = |x| \cdot \left| \sin \frac{1}{x} \right|$$

$$\leq x \text{, since } \left| \sin \frac{1}{x} \right| \leq 1$$

$$< \in \text{ for } |x - 0| < \in.$$

The relations are satisfied, if $\delta = \in$. So, $f(x)$ is continuous at $x = 0$.

(*b*) Here, $f(x) = x^2 \cos\left(\frac{1}{x}\right)$, when $x \neq 0$

$$f(0) = 0.$$

For continuity of $f(x)$ at $x = 0$, we are to find a δ depending upon $\in$, such that

$|f(x) - f(0)| < \in$, for $|x - 0| < \delta$.

i.e., $\left| x^2 \cos \frac{1}{x} - 0 \right| < \in$ for, $|x| < \delta$.

Since $\left| \cos \frac{1}{x} \right| \leq 1$, relations are satisfied, if we take $|x^2| < \in$ for $|x| < \delta$.

So, the relations are satisfied if $\delta = \sqrt{\in}$.

Hence, $f(x)$ is continuous at $x = 0$.

EXAMPLES-IV

1. A function $f(x)$ is defined as follows :
   $f(x) = x^2$ when $x \neq 1$, $f(x) = 2$ when $x = 1$.
   Is continuous at $x = 1$?
2. Are the following functions continuous at the origin ?
   (i) $f(x) = \sin(1/x)$ when $x \neq 0$, $f(0) = 0$.
   (ii) $f(x) = x \cos(1/x)$ when $x \neq 0$, $f(0) = 0$.
   (iii) $f(x) = x \cos(1/x)$ when $x \neq 0$, $f(0) = 1$.
   (iv) $f(x) = \dfrac{\sin \frac{1}{x}}{\frac{1}{x}}$ when $x \neq 0$.
   $= 1$ when $x = 0$.

(iv) $f(x) = \sin x \cos \frac{1}{x}$ when $x \neq 0$.
$= 0$ when $x = 0$.

3. A function $\phi(x)$ is defined as follows :

$$\phi(x) = x^2 \quad \text{when} \quad x < 1,$$
$$= 2.5 \quad \text{when} \quad x = 1,$$
$$= x^2 + 2 \quad \text{when} \quad x > 1.$$

Is $\phi(x)$ continuous at $x = 1$?

4. A function $f(x)$ is defined in the following way ;

$$f(x) = -x \quad \text{when} \quad x \leq 0,$$
$$= x \quad \text{when} \quad 0 < x < 1,$$
$$= 2 - x \quad \text{when} \quad x \geq 1.$$

Show that it is continuous at $x = 0$ and $x = 1$. [*C.P. 1942*]

5. A function $f(x)$ is defined as follows :

$$f(x) = 1,\ 0 \text{ or } -1 \quad \text{according as} \quad x >, = \text{ or } < 0,$$

Show that it is discontinuous at $x = 0$.

6. The function $f(x) = \dfrac{x^2 - 16}{x - 4}$ is undefined at $x = 4$.

What value must be assigned to $f(4)$, if $f(x)$ is to be continuous at $x = 4$?

7. Determine whether the following functions are continuous at $x = 0$.

(i) $f(x) = \left(x^4 + x^3 + 2x^2\right) / \sin x$, $f(0) = 0$.

(ii) $f(x) = \left(x^4 + 4x^3 + 2x\right) / \sin x$, $f(0) = 0$.

8. Find the points of discontinuity of the following functions :

(i) $\dfrac{x^3 + 2x + 5}{x^2 - 8x + 12}$ (ii) $\dfrac{x^3 + 2x + 5}{x^2 - 8x + 16}$

9. A function $f(x)$ is defined as follows :

$$f(x) = 3 + 2x \quad \text{for} \quad -\tfrac{3}{2} \leq x < 0$$
$$= 3 - 2x \quad \text{for} \quad 0 \leq x < \tfrac{3}{2}$$
$$= -3 - 2x \quad \text{for} \quad x \geq \tfrac{3}{2}.$$

Show that $f(x)$ is continuous at $x = 0$ and discontinuous at $x = \frac{3}{2}$.

10. The function $y = f(x)$ is defined as follows : $f(x) = 0$ when, $f(x) = 1$ when $x^2 < 1$, $f(x) = \frac{1}{2}$ when $x^2 = 1$. Draw a diagram of the function and discuss from diagram that, except at points $x = 1$ and $x = -1$, the function is continuous. Discuss also why the function is discontinuous at these two points although it has a value for every value of x.

Examine the continuity of the functions at $x = 0$ (Ex. 11-14)

11. $f(x) = (1 + x)^{1/x}$, when $x \neq 0$
$= 1$, when $x = 0$.

12. $f(x) = (1 + 2x)^{1/x}$, when $x \neq 0$
$= e^2$, when $x = 0$.

13. $f(x) = e^{-1/x^2}$, when $x \neq 0$
$= 1$, when $x = 0$.

14. $f(x) = \dfrac{e^{-1/x}}{1 + e^{1/x}}$, when $x \neq 0$
$= 1$, when $x = 0$.

15. The function f is defined by

$$f(x) = 2x - [x] + \sin\frac{1}{x}, \text{ for } x \neq 0$$
$$= 0, \qquad \text{for } x = 0,$$

where $[x]$ denotes the greatest integer not greater than x.

Examine the continuity of $f(x)$ at $x = 0$ and $x = 2$.

ANSWERS

1. No

2. (i) No. (ii) Yes. (iii) No. (iv) No. (v) Yes

3. No. **6.** 8. **7.** (i) Continuous. (ii) Discontinuous

8. (i) 6, 2. (ii) 4. **11.** Discontinuous. **12.** Continuous.

13. Discontinuous. **14.** Discontinuous. **15.** Discontinuous.

5 SEQUENCE

5.1. Set of Real Numbers.

A set of real numbers is a well-defined collection of objects which are called *members* or *elements* of the set. The term 'well-defined' means that given any real number, it can be determined without ambiguity whether the real number belongs or does not belong to the set.

Examples :

1. The set of natural numbers : $N = \{1, 2, 3, ..., n, ...\}$
2. The set of all integers : $Z = \{0, \pm 1, \pm 2, ..., \pm n, ...\}$
3. The set of all integers between 3.1 and 8.7. $X = \{4, 5, 6, 7, 8\}$

A set is *finite*, if it is empty or contains a finite number of elements, otherwise, a set is *infinite*. The set defined in example 3 above is finite, while the sets in examples 1 and 2 are infinite.

5.2. Greatest and Least Members of a Set.

A number L is the *greatest member* of a set S of real numbers, if

1. L is itself a member of S, and
2. $L \geq x$, where x is any element of the set

Similarly, l is the *least member* of a set S of real number, if

1. l is a member of S, and
2. $l \leq x$, where x is any member of the set

Examples :

1. In the set of natural numbers $\{1, 2, 3, ... , n, ...\}$, 1 is the least member, but it has no greatest member.
2. For the set of numbers $\left\{1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...\right\}$, 1 is the greatest member, but it has no least member.
3. For the set of numbers $\{7, 8, 9, 10, 11, 12\}$, 7 is the least member and 12 is the greatest member.

5.3. Bounds of a Set.

Given a set S of real numbers, if there exists a number G such that $x \leq G$, for *every number* x of S, then we say that the set is *bounded above* and G is an *upper bound*, or a *rough upper bound*.

If G is an upper bound of a set S, then any number greater than G is also an upper bound of S. So, if a set is bounded above, the number of upper bounds is infinite. The least M of all the bounds is called is the *exact upper bound* or, the *least upper bound* or *Supremum*.

Similarly, if there exists a number g such that $x \geq g$, for every number x in S, we say that the set is bounded below and g is called a *lower bound* or, a *rough lower bound.*

The least m of all the lower bounds is called the *exact lower bound* or the *greatest lower bound* or, *infimum* or *the lower bound* of S.

5.4. Neighbourhood of a Point : Points of Accumulation.

(i) Let ξ be a real number and ε be an *arbitrary* positive number. Then the Set of real numbers in the open interval $(\xi - \varepsilon, \xi + \varepsilon)$ is called an ε*-neighbourhood* of ξ. For each separate choice of ε, we may form a separate neighbourhood of ξ.

(ii) **Deleted Neighbourhood** : The set of real numbers in the open interval $(\xi - \varepsilon, \xi + \varepsilon)$ excluding the point ξ itself is called the *deleted* ε*-neighbourhood* of ξ, where ξ is a real number and ε is an arbitrary positive number, however small.

(iii) **Point of Accumulation** : A number ξ, which may or may not belong to a set S of real numbers, is called a *point* of *accumulation* or *cluster point* of S, if every neighbourhood of ξ, however small, contains an infinite number of members of S.

Evidently, a finite set of real numbers cannot have any point of accumulation. Cluster point is also called *Limiting Point.*

Examples :

1. For the set $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\right\}$, $L = 1$, $m = 0$, $M = 1$, limiting point is 0, but l does not exist.

2. For the set $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots\right\}$, $L = \frac{1}{2}$, $m = \frac{1}{2}$, $M = 1$, limiting point is 1, but L does not exist.

5.5. Sequence of Numbers.

A set of real numbers $x_1, x_2, x_3, \dots\dots\dots, x_n, \dots.$ such that corresponding to every positive integer n, there exists a real number x_n of the set, is called a *sequence*. The individual numbers are called *elements* of the sequence. The sequence whose n^{th} element is x_n is briefly denoted by $\{x_n\}$. If the sequence terminates after a finite number of terms it is called a *finite sequence*, otherwise, it is an *infinite sequence*. In what follows, we shall be concerned with infinite sequences only and the word infinite may not be used always.

Examples.

1. $\{2, 4, 6, 8, 10\}$ is a finite sequence
2. $\{n\}$ is the infinite sequence $\{1, 2, 3, \ldots, n, \ldots\}$
3. $\left\{\frac{1}{n}\right\}$ is the infinite sequence $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$
4. $\{x_n = (-1)^{n-1}\}$ is the infinite sequence $\{1, -1, 1, -1, \ldots\}$
5. $\{x_n = 1 + (-1)^n\}$ gives the infinite sequence $\{0, 2, 0, 2, \ldots\}$
6. If $x_n = \sec\left(\frac{1}{2}\pi\sqrt{n}\right)$, then $\{x_n\}$ *does not give a sequence*, for x_n becomes undefined when n is the square of an odd positive integer.

5.6. Bounded Sequence.

Given a sequence $\{x_n\}$, if there exists a number K, such that K is greater than or equal to any member x_n of the sequence, i.e., $K \geq x_n$, where x_n is *any element* of $\{x_n\}$, then the sequence is said to be *bounded above*, K being called the *rough upper bound*. Of all the rough upper bounds, the least one is called the *exact upper bound* of the sequence. If K is the exact upper bound of the sequence $\{x_n\}$, then there exists *at least one member* of $\{x_n\}$, $x_n > K - \varepsilon$ where ε is a preassigned positive number, however small.

Similarly, given a sequence $\{x_n\}$, if there exists a number k, which is less than or equal to any member of the sequence, i.e., if $k \leq \{x_n\}$, for all n, then the sequence $\{x_n\}$ is said to be *bounded below*, k being called the *rough lower bound*. Of all the rough lower bounds, the *greatest one* is called the *exact lower bound* of the sequence. If k is the exact lower bound of the sequence $\{x_n\}$, then there exists *at least one member* of $\{x_n\}$, such that $x_n < k + \varepsilon$.

If a sequence is *bounded both above and below*, it is called a *bounded sequence*.

Examples :

1. The sequence $\left\{1+\frac{1}{n}\right\}$ is *bounded above*, the upper bound being 2.

2. The sequence $\left\{\frac{1}{n}\right\}$ is *bounded below*, the lower bound being 0.

3. The sequence $\left\{2+(-1)^n \cdot \frac{1}{n}\right\}$ is *bounded*, for it is bounded both above and below, the upper and lower bounds being 2·5 and 1 respectively.

4. The sequence $\{0, 3, -2, 5, -4, 7, \ldots\}$, i.e $\{1+(-1)^n n\}$ is *unbounded*, as it has neither upper nor lower bound.

5.7. Monotonic Sequence.

A sequence $\{x_n\}$ is said to be *monotonic*

(a) *increasing* (or, more correctly *non-decreasing*), if $x_n \le x_{n+1}$ for every n ;

(b) *decreasing* (or, *non-increasing*), if $x_n \ge x_{n+1}$ for all n ;

(c) *strictly increasing*, if $x_n < x_{n+1}$ for all n ;

(d) *strictly decreasing*, if $x_n > x_{n+1}$ for all n.

Monotonic sequences are also called *monotone* sequences.

A sequence is said to be *monotonic sequence*, if it is monotonic increasing or monotonic decreasing.

Examples :

1. The sequence $\{x_n\}$, where $x_n = \frac{n}{n+1}$, is strictly increasing.

2. The sequence $\{x_n\}$, where $x_n = \frac{n+1}{n}$, is strictly decreasing.

3. The sequence $\{1, 0, 1, 0, 1, 0, \ldots\}$ is neither increasing nor decreasing.

5.8. Limit of a Sequence.

The idea of *limit* forms the most outstanding concept in Mathematical Analysis and it plays an important role in the discussion of convergence of an infinite sequence.

Let us consider the sequence $\{x_n\}$, where $x_n = \frac{1}{n}$.

If we put $n = 1, 10, 100, 1000, \ldots$ successively, the respective values of x_n are 1, 0·1, 0·01, 0·001. . .

Obviously, as n increases, $\frac{1}{n}$ steadily decreases, but always remains positive. For large value of n, the difference of x_n from 0 is very small and we can make this difference less than any preassigned positive quantity, however, small, by making n sufficiently large. For example, if we like to make this difference less than 0·000001, n should be greater than 10^6. Thus, the value of x_n can be made as near to 0 as we please by taking n sufficiently large. This is expressed as

$$x_n \to 0 \text{ as } n \to \infty \text{ or, } \lim_{n \to \infty} x_n = 0.$$

Therefore, we have the formal definition of limit of an infinite sequence.

Definition : A sequence $\{x_n\}$ is said to have finite limit l, if for any pre-assigned positive quantity ε, however small, there corresponds a positive integer N, such that $|x_n - l| < \varepsilon$, for $n > N$.

This state of affairs is expressed as $\lim_{n \to \infty} x_n = l$

5.9. Convergent Sequence.

An infinite sequence $\{x_n\}$ is said to be *convergent* and has the limit l, if corresponding to any arbitrary small positive number ε, we can find a positive integer N (depending upon ε) such that

$$|x_n - l| < \varepsilon, \text{ for } n \geq N.$$

i.e., $l - \varepsilon < x_n < l + \varepsilon$, when $n \geq N$.

This is expressed by saying that ' x_n tends to the limit l, as n tends to infinity' and expressed as $\lim_{n \to \infty} x_n = l$.

By the symbol $n \to \infty$, it is meant that n takes up successively an endless series of integral values which ultimately become and remain greater than any arbitrarily assigned positive integer.

Example.

Find the limit of the sequence $\left\{\frac{1}{n}\right\}$ as $n \to \infty$.

$$\because \left|\frac{1}{n} - 0\right| < \varepsilon, \text{ when } \frac{1}{n} < \varepsilon,$$

by taking $N = \frac{1}{\varepsilon}$ or, the integral part of $\frac{1}{\varepsilon}$ (when it is a fraction),

$$\left|\frac{1}{n} - 0\right| < \varepsilon, \text{ if } n \geq N.$$

Therefore, $\lim_{n \to \infty} \frac{1}{n} = 0$. Hence, the sequence $\left\{\frac{1}{n}\right\}$ *converges* to 0.

Note: The limit of a sequence may or may not be a term of the sequence.

For example, the convergent sequence $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$ has the limit 0, but no member of the sequence is equal to 0.

Again, let us consider the sequence $\{x_n\}$, where $x_n = \frac{1}{n}$, when n is odd and $x_n = 0$, when n is even, *i.e.*, the sequence $\left\{1, 0, \frac{1}{3}, 0, \frac{1}{5}, \ldots\right\}$.

Obviously, the sequence is convergent and converges to the limit 0 which is equal to an infinite number of terms of the sequence.

5.10. Non-Convergent Sequences.

A sequence $\{x_n\}$ is said to diverge to $+\infty$, if for any number K whatever is assigned, there corresponds a positive integer N, such that $x_n > K$ for all $n > N$. This situation is expressed by $\lim_{n \to \infty} x_n = \infty$.

In this case the sequence $\{x_n\}$ is called *divergent.*

Here ∞ is no real number, it is a symbol to denote a large positive number greater than the greatest number one can imagine.

A sequence $\{x_n\}$ is said to diverge to $-\infty$ if, when any number K whatever is assigned, there always exists a positive integer N such that $x_n < K$ for all $n > N$, and we write $\lim_{n \to \infty} x_n = -\infty$.

Here, K is generally chosen a *negative* number, *large* in *absolute value.*

A sequence which is neither convergent nor divergent is called an *Oscillatory Sequence.*

In an oscillatory sequence $\{x_n\}$, if a constant c exists such that $|x_n| < c$, for all n, then the sequence is said to *oscillate finitely*, otherwise, it is said to *oscillate infinitely.*

Note. A monotonic sequence cannot oscillate.

Examples.

1. The sequence $1^2, 2^2, 3^2, \ldots$, *i.e.*, $\{n^2\}$ diverges to $+\infty$.
2. The sequence $-1, -2, -3, \ldots$, *i.e.*, $\{-n\}$ diverges to $-\infty$.

3. The sequence $-1, 1, -1, \ldots$, *i.e.*, $\{(-1)^n\}$ oscillates finitely between -1 and 1.

4. The sequence $-1, \sqrt{2}, -\sqrt{3}, 2, -\sqrt{5}, \ldots$, *i.e.*, $\{(-1)^n \sqrt{n}\}$ oscillates infinitely.

5.11. A Few Important Theorems.

Theorem I. *A convergent sequence determines its limit uniquely.*

Proof: If possible, let l_1 and l_2 be two *distinct* limits of a convergent sequence $\{x_n\}$.

Since $l_1 \neq l_2$, we can take $|l_1 - l_2| = \delta$, where δ is a non-zero positive number. Now, let us choose a positive number ε, such that $\varepsilon < \delta$.

Since $\{x_n\}$ possesses two distinct limits l_1 and l_2,

$$|x_n - l_1| < \tfrac{1}{2}\varepsilon \text{ for } n > N_1 \text{ and } |x_n - l_2| < \tfrac{1}{2}\varepsilon \text{ and } n > N_2,$$

where N_1 and N_2 depends on the given ε.

Thus, for $n > N = \max(N_1, N_2)$

$$\begin{aligned} |l_1 - l_2| &= |(x_n - l_2) - (x_n - l_1)| \\ &\leq |x_n - l_2| + |x_n - l_1| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ for } n > N. \end{aligned}$$

Thus, $\delta < \varepsilon$ and we arrive at a contradiction. Thus, the assumption that the sequence $\{x_n\}$ has two distinct lim·ts is not true. Hence, the theorem.

Theorem II. *Every convergent sequence is bounded.*

Proof: Let the sequence $\{x_n\}$ be bounded having a unique limit l.

Then $|x_n - l| < \varepsilon$ for all $n \geq N$, N being a positive integer depending upon ε, however small.

$$\text{i.e. } l - \varepsilon < x_n < l + \varepsilon, \text{ when } n \geq N.$$

Let L and M be the least and the greatest of the numbers

$$x_1, x_2, x_3, \ldots, x_{N-1}, l - \varepsilon, l + \varepsilon.$$

Then we have $L \le x_n \le M$ for all values of n.

Thus the sequence $\{x_n\}$ is bounded.

Note. The *converse* of this theorem is not always true. For example, the sequence $\{1+(-1)^n\}$, *i.e.*, $\{0, 2, 0, 2, \dots\}$ is *bounded* but *not convergent*.

Also, the sequence $\left\{1, \frac{1}{2}, 1, \frac{1}{3}, 1, \dots\right\}$ is *bounded* but not convergent.

Theorem III. *A monotonic increasing sequence which is bounded above is convergent and converges to its exact upper bound or supremum.*

Proof : A monotonic increasing sequence $\{x_n\}$ is always bounded below, for $x_n \ge x_1$ for all n. Again, since $\{x_n\}$ is bounded above, if exact upper bound or the supremum of $\{x_n\}$ is M,

(i) $x_n \le M$ or for all n, and

(ii) for any given $\varepsilon\ (>0)$ there exists at least one member of the sequence $\{x_n\}$, say x_N, such that $x_N > M - \varepsilon$.

Since the sequence $\{x_n\}$ is monotonic increasing, $x_n > M - \varepsilon$, for $n \ge N$. Again from (i) $x_n < M$, for all n, whereby $x_n < M + \varepsilon$ for each n.

Thus, $M - \varepsilon < x_n < M + \varepsilon$, for all $n \ge N$, and so $\lim_{n\to\infty} x_n = M$.

Hence, the theorem is established.

Theorem IV. *A monotonic decreasing sequence bounded below is convergent and converges to its exact lower bound.*

The proof is exactly similar to that of Theorem III proved above.

Theorem V. *A monotonic increasing sequence diverges to $+\infty$, if it is not bounded above.*

Proof : Since the sequence $\{x_n\}$ is monotonically increasing, $x_{n+1} \ge x_n$ for all n, and $\{x_n\}$ being not bounded above, there exists at least one member, say, x_m, of the sequence such that $x_m > M$; where M is a large positive number. The sequence being monotonic increasing, $x_{m+1}, x_{m+2}, \dots$ are all greater than M.

Therefore $x_n > M$ for all $n \geq m$.

i.e., $\lim_{n \to \infty} x_n = \infty$.

Thus, the sequence $\{x_n\}$ diverges to $+\infty$.

Theorem VI. *A monotonic decreasing sequence diverges to $-\infty$, if not bounded below.*

The proof is similar to the proof of Theorem V.

5.12. An Important Sequence.

The sequence $\{x_n\}$, where $x_n = \left(1 + \frac{1}{n}\right)^n$ is convergent.

[*C. P. 2004*]

It will be shown that the given sequence is monotonic increasing and bounded above.

Using Binomial expansion,

$$x_n = \left(1 + \frac{1}{n}\right)^n = 1 + n.\frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{1}{n^3} + \ldots$$

$$\ldots + \frac{n(n-1)(n-2)\ldots 2.1}{n!} \cdot \frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots$$

$$\ldots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\left(1 - \frac{n-1}{n}\right) \quad \ldots \quad (1)$$

Replacing n by $(n + 1)$,

$$x_{n+1} = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n+1}\right) + \frac{1}{3!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right) + \ldots$$

$$\ldots + \frac{1}{(n+1)!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)\ldots\left(1 - \frac{n}{n+1}\right) \quad \ldots (2)$$

From (1) and (2), we observe that

(i) the first two terms of x_n and x_{n+1} are equal, each being 1;

(ii) $\because$ $\dfrac{1}{n+1} < \dfrac{1}{n}$, $1 - \dfrac{1}{n+1} > 1 - \dfrac{1}{n}$ and so on ; thus excepting the first two terms, every term of x_{n+1} is greater than the corresponding terms in x_n.

(iii) x_n contains $(n+1)$ terms, while x_{n+1} contains $(n+2)$ terms and all the terms are positive.

Hence $x_{n+1} \geq x_n$ for all n, *i.e.*, the sequence $\{x_n\}$ is monotonic increasing.

Next we note that $x_n \geq 2$ for all n, *i.e.*, $\{x_n\}$ is bounded below and 2 is a lower bound.

Also, $$x_n = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots$$

$$\ldots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\left(1 - \frac{n-1}{n}\right)$$

$$< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}}$$

$$= 1 + \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} = 3 - 2^{1-n} < 3$$

i.e., $x_n < 3$, for all n.

Thus, $2 < x_n < 3$ for all n.

Hence, the sequence $\{x_n\}$ is bounded.

Since the sequence $\{x_n\}$ is increasing and bounded , it is convergent.

Note. Since the sequence is convergent, its limit exists and this limiting value is denoted by 'e',

i.e., $$\lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n = e, \text{ where } 2 < e < 3.$$

This number e is a *transcendental number*.

5.13. Bernoulli's Inequality.

For every positive integer $n \geq 2$, and $1 + p > 0$, $(1 + p)^n > 1 + np$

Proof : The method of mathematical *induction* will be used to establish the result.

We note that, when $n = 2$; $(1 + p)^2 = 1 + 2p + p^2 > 1 + 2p$.

Thus the inequality holds for $n = 2$. Let us assume that the relation holds for any particular value $k\,(\geq 2)$ of n.

$\therefore \quad (1+p)^k > 1+kp$

or, $(1 + p)(1 + p)^k > (1 + p)(1 + kp)$, since $1 + p > 0$,

or, $(1 + p)^{k+1} > 1 + (1 + k)p + kp^2 > 1 + (k + 1)p$,

Thus, we see that the relation is true for $n = k + 1$, if it is assumed to be true for $n = k$.

But, it has been proved to be true for $n = 2$, so it is true for $n = 3$, and as it is true for $n = 3$, it is true for $n = 4$ and so on.

Thus, the inequality holds good for any integral value of $n > 2$.

Note. The above inequality is true for $n > 1$, even if n is not a positive integer.

5.14. Null Sequence.

A sequence is said to be a *null sequence*, if $\lim_{n \to \infty} x_n = 0$, *i.e.*, for any positive number e, however small, there exists a positive integer N, such that $|x_n| < \varepsilon$ for all $n > N$.

Example :

Then sequence $\{x^n\}$ is a null sequence if $|x| < 1$.

If $x = 0$, each number of the sequence is 0 and $x^n \to 0$ as $n \to \infty$.

When $x \neq 0$, $|x|$ is a positive proper fraction, we write

$$|x| = \frac{1}{1+h}, \ (h > 0)$$

Now, $|x^n| = \dfrac{1}{(1+h)^n} < \dfrac{1}{1+nh}$,

$\because \quad (1+h)^n > 1+nh$ [Bernoulli's inequality]

\ $\quad |x|^n < \dfrac{1}{nh} < \varepsilon$ if $n > N$, where N is the integral part of $\dfrac{1}{h\varepsilon}$.

Hence, the sequence $\{x^n\}$ is a null sequence if $|x| < 1$.

5.15. Cauchy Sequences.

A sequence $\{x^n\}$ is called a Cauchy Sequence, if corresponding to any pre-assigned positive number e, however small, there exists a positive integer N, such that for $n \geq N$,

$$|x_{n+p} - x_n| < \varepsilon \text{ for all positive integral values of } p.$$

Example : $\left\{\frac{1}{n}\right\}$ is a Cauchy Sequence.

$$\text{Here, } |x_{n+p} - x_n| = \left|\frac{1}{n+p} - \frac{1}{n}\right| = \frac{1}{n}\left|1 - \frac{1}{1+\frac{p}{n}}\right| < \frac{1}{n} < \varepsilon \text{ for } n > N$$

and taking $N > \frac{1}{\varepsilon}$ or the integral part of $\frac{1}{\varepsilon}$.

Hence, $\left\{\frac{1}{n}\right\}$ is a Cauchy Sequence.

5.16. Cauchy's General Principle of Convergence.

A necessary and sufficient condition for the convergence of the sequence $\{x_n\}$ is that corresponding to any pre-assigned positive number e, however small, there cxists a positive integer N, such that for $n \geq N$, $|x_{n+p} - x_n| < \varepsilon$, for all positive integral values of p.

Condition Necessary : Let the sequence $\{x_n\}$ be convergent, *i.e.*, the sequence has a finite limit, say l. Then, for given e, however small, there exists a positive integer N, such that

$$|x_n - l| < \tfrac{1}{2}\varepsilon \text{ for all } n \geq N.$$

Then it follows that

$$|x_{n+p} - l| < \tfrac{1}{2}\varepsilon \text{ for all } n \geq N \text{ and } p > 0$$

Hence,
$$\begin{aligned} |x_{n+p} - x_n| &= |x_{n+p} - l + l - x_n| \\ &\leq |x_{n+p} - l| + |x_n - l| \\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon \end{aligned}$$

i.e., $|x_{n+p} - x_n| < \varepsilon$, for $n \geq N$ and $p > 0$.

Thus the condition is proved to be *necessary*.

Condition Sufficient : Next let us suppose that $\left| x_{n+p} - x_n \right| < \varepsilon$, for $n \geq N$, p being any positive integer. Then

$x_n - \varepsilon < x_{n+p} < x_n + \varepsilon$, for all positive integral values of p.

$\left\{ x_{n+p} \right\}$ is thus bounded as $p \to \infty$.

Let, L and M be the lower and upper bounds respectively. Then

$$L \leq a_n - \varepsilon \text{ and } M \leq a_n + \varepsilon .$$

Thus, $M - L \leq \left(a_n + \varepsilon \right) - \left(a_n - \varepsilon \right) = 2\varepsilon$

Since eis arbitrarily small, this implies that $M - L = 0$ in the limit.

Therefore, $M - \varepsilon \leq x_{n+p} \leq M + \varepsilon$.

It follows that $x_{n+p} \to M$ for all integral values of p.

Thus, the sequence $\left\{ x_n \right\}$ is convergent and the condition is proved to be *sufficient.*

5.17. Theorems on Limits of Sequences.

Let $\left\{ x_n \right\}$ and $\left\{ y_n \right\}$ be two sequences, such that

$$\lim_{n \to \infty} x_n = A \text{ and } \lim_{n \to \infty} y_n = B ;$$

Then, (i) $\lim_{n \to \infty} \left(x_n + y_n \right) = A + B$,

(ii) $\lim_{n \to \infty} \left(x_n - y_n \right) = A - B$,

(iii) $\lim_{n \to \infty} \left(x_n \cdot y_n \right) = A \cdot B$,

(iv) $\lim_{n \to \infty} \left(\dfrac{x_n}{y_n} \right) = \dfrac{A}{B}$, Provided $B \neq 0$.

Proof : (i) Since $\lim_{n \to \infty} x_n = A$, given any $\varepsilon > 0$, there exists a positive integer N_1, such that $\left| x_n - A \right| < \frac{1}{2} \varepsilon$ for all $n > N_1$

Also, since $\lim_{n \to \infty} y_n = B$, for any $\varepsilon > 0$, there exists a positive integer N_2, such that $\left| y_n - B \right| < \frac{1}{2} \varepsilon$ for all $n > N_2$.

If $N = \max\left\{ N_1, N_2 \right\}$ then for $n > N$

$$\left| x_n - A \right| < \tfrac{1}{2} \varepsilon \text{ and } \left| y_n - B \right| < \tfrac{1}{2} \varepsilon$$

Hence, $\left|(x_n + y_n) - (A + B)\right| \le |x_n - A| + |y_n - B| < \varepsilon$ for all $n > N$.

Therefore, by definition $\lim\limits_{n \to \infty}(x_n + y_n) = A + B$.

(ii) Proof of this part is similar to that of (i)

(iii) We have,

$$|x_n y_n - AB| = |x_n(y_n - B) + B(x_n - A)|$$

$$\le |x_n||y_n - B| + |B||x_n - A|$$

Since $\{x_n\}$ is convergent, it is bounded and there exists a positive number M, such that $\{x_n\} < M$, for all values of n.

Then, $|x_n y_n - AB| < M|y_n - B| + \{|B| + \lambda\}|x_n - A| \ldots$ (1)

where 1 is any positive integer.

Now, let e be any pre-assigned positive number, however small. Then we can find two positive integers N_1 and N_2, such that

$$|x_n - A| < \frac{\varepsilon}{2\{|B| + \lambda\}}, \text{ for } n > N_1 \qquad \ldots \quad (2)$$

and $$|y_n - B| < \frac{\varepsilon}{2M}, \qquad \text{for } n > N_2 \qquad \ldots \quad (3)$$

If N be any positive integer greater than both N_1, N_2, we get from (1), (2) and (3)

$$|x_n y_n - AB| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon, \text{ for all } n > N.$$

Hence, $\lim\limits_{n \to \infty} x_n y_n = AB$

Note. In (1) we have taken $\{|B| + \lambda\}$ instead of $|B|$ in (1); otherwise, the inequality (2) fails if $B = 0$.

(iv) Here,

$$\left|\frac{x_n}{y_n} - \frac{A}{B}\right| = \left|\frac{B.x_n - A.y_n}{B.y_n}\right| = \left|\frac{B(x_n - A) + A(B - y_n)}{B.y_n}\right|$$

$$\le \frac{|B||x_n - A| + |A||B - y_n|}{|B||y_n|} \qquad \ldots \quad (1)$$

Since, $\lim_{n \to \infty} y_n = B \neq 0$, there exists a positive integer N_1, such that

$$|B - y_n| < \tfrac{1}{2}|B|, \text{ for } n \geq N_1$$

or, $$|B| - |y_n| < |B - y_n| < \tfrac{1}{2}|B|$$

or, $$\tfrac{1}{2}|B| < |y_n| \qquad \ldots \quad (2)$$

From (1) and (2)

$$\left|\frac{x_n}{y_n} - \frac{A}{B}\right| < \frac{|B||x_n - A| + |A||B - y_n|}{\frac{1}{2}|B|^2}$$

$$< \frac{2}{|B|}|x_n - A| + \frac{2\{|A| + \lambda\}}{|B|^2}|B - y_n| \qquad \ldots \quad (3)$$

where λ is a positive number.

Let ε be a positive number, however small ; then there exists positive numbers N_2, N_3 such that

$$|x_n - A| < \tfrac{1}{4}\varepsilon|B|, \text{ for every } n \geq N_2 \qquad \ldots \quad (4)$$

$$\text{and } |B - y_n| < \frac{|B|^2}{\{|A| + \lambda\}} \cdot \frac{1}{4}\varepsilon, \text{ for every } n \geq N_3 \qquad \ldots \quad (5)$$

If N be a positive integer, greater than each of N_1, N_2, N_3 then using 3), (4) and (5), we have

$$\left|\frac{x_n}{y_n} - \frac{A}{B}\right| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon, \text{ for every } n \geq N$$

Hence, $\lim_{n \to \infty} \frac{x_n}{y_n} = \frac{A}{B}$.

.18. Illustrative Examples.

Ex. 1. If $x_n = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{n(n+1)}$, then show that $\{x_n\}$ s a bounded monotonic increasing sequence.

[*C.P. 1963, B.P. 1984, 1994*]

Here, $x_n = \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$

$= 1-\frac{1}{n+1} = \frac{n}{n+1}$.

Similarly, $x_{n+1} = \frac{n+1}{n+2}$.

So, $x_{n+1}-x_n = \frac{n+1}{n+2}-\frac{n}{n+1} = \frac{1}{(n+1)(n+2)} > 0$, for all positive integral value of n.

Hence, $\{x_n\}$ is a *monotonic increasing* sequence.

Again, since $\frac{n}{n+1} > 0$, $x_n > 0$.

Also, $\frac{n}{n+1} < 1$, for all positive values of n.

Thus, $0 < x_n < 1$.

Hence, the sequence $\{x_n\}$ is *bounded* also.

Ex. 2. If $x_n = \frac{3n-1}{n+2}$, prove that the sequence $\{x_n\}$ is monotone increasing and bounded. [*V.P. 2000*]

Here, $x_n = \frac{3n-1}{n+2}$ and $x_{n+1} = \frac{3n+2}{n+3}$

$\therefore$ $x_{n+1}-x_n = \frac{7}{(n+2)(n+3)} > 0$, for all positive integral values of n.

Hence, the sequence is *monotone increasing.*

Again, $x_n = \frac{3n-1}{n+2} = \frac{3(n+2)-7}{n+2} = 3-\frac{7}{n+2}$

Since $\frac{7}{n+2}$ is positive for any positive integral value of n, $x_n < 3$.

Also, $x_n \geq \frac{2}{3}$.

i.e., $\frac{2}{3} \leq x_n < 3$.

Hence the sequence is *bounded.*

Ex. 3. Prove that $\lim_{n\to\infty} n^{1/n} = 1$, where n is a positive integer.

[*V. P. 1998*]

For $n > 1$, $\sqrt[n]{n} > 1$,

Let, $\sqrt[n]{n} = 1 + h_n$, where $h_n > 0$,

then $n = (1 + h_n)^n = 1 + n.h_n + \frac{n(n-1)}{2!}h_n^2 + \ldots + h_n^n$

$> \frac{1}{2}n(n-1)h_n^2$, since all the terms are positive.

$\therefore \quad 0 < h_n < \sqrt{2/(n-1)}$

$\therefore \quad \lim_{n\to\infty} h_n = 0$, hence $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Ex. 4. Show that the sequences given by

(i) $x_n = 2 + (-1)^n . 2^{-n}$ and (ii) $x_n = \frac{1}{n}\sin\frac{n\pi}{2}$ are convergent.

Find the limits. [*B.P. 1965*]

(i) We have $|2 - x_n| = 2^{-n}$ and $|2 - x_n| < \varepsilon$, where ε is any positive number, however small,

if $2^{-n} < \varepsilon$, *i.e.*, $2^n > \frac{1}{\varepsilon}$, *i.e.*, if $n > \frac{\log(1/\varepsilon)}{\log 2}$.

Thus, if we choose an integer $N > \frac{\log(1/\varepsilon)}{\log 2}$, $|2 - x_n| < \varepsilon$, for all $n > N$.

Ex. 5. Use Cauchy criterion to show that

(i) the sequence $\{x_n\}$ defined by $x_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + (-1)^{n-1}.\frac{1}{n}$ is convergent;

(ii) the sequence $\{x_n\}$ defined $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}$ is divergent;

(i) Choosing $m > n$,

$$|x_m - x_n| = \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \ldots \frac{1}{m}$$

$$= \frac{1}{n+1} - \left(\frac{1}{n+2} - \frac{1}{n+3}\right) - \left(\frac{1}{n+4} - \frac{1}{n+5}\right) - \ldots$$

$$< \frac{1}{n+1} < \varepsilon, \text{ if } n+1 > \frac{1}{\varepsilon}, \text{ if } n > \frac{1}{\varepsilon} - 1.$$

If now N be so chosen that N is equal to the integral part of $\left(\frac{1}{\varepsilon}-1\right)$, then

$\left| x_m - x_n \right| < \varepsilon$, whenever $n > N$.

Hence, the sequence converges by Cauchy's Criterion.

(ii) If we choose $m = 2n$,

$$\left| x_m - x_n \right| = \left| x_{2n} - x_n \right|$$
$$= \frac{1}{n+1} + \frac{1}{n+2} + \ldots \frac{1}{2n}$$
$$> n.. \frac{1}{2n} = \frac{1}{2}$$

Thus $\left| x_m - x_n \right|$ is *not less than* any pre-assigned positive number.

Hence the sequence does not cnverge by Cauchy's Criterion. Since the sequence is monotonic increasing and does not converge, the sequence *diverges* to $+\infty$.

Ex. 6. (i) Show that the sequence

$$\sqrt{2},\ \sqrt{2+\sqrt{2}},\ \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$$

tends to a definite finite limit and find the limit. [*C.P. 1960*]

(ii) Show that the Sequence $\sqrt{2},\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}} \ldots$ converges to 2.

(i) If the n^{th} term be x_n, then $x_{n+1} = \sqrt{2+x_n}$

or, $x_{n+1}^2 - x_n - 2 = 0$ ... (1)

Again, $x_n = \sqrt{2+x_{n-1}}$

or, $x_n^2 - x_{n-1} - 2 = 0$ ... (2)

From (1) and (2)

$$x_{n+1}^2 - x_n^2 = x_n - x_{n-1}$$

or, $(x_{n+1}+x_n)(x_{n+1}-x_n) = x_n - x_{n-1}$

This implies that $x_{n+1} > x_n$. if $x_n > x_{n-1}$ and $x_{n+1} < x_n$, if $x_n < x_{n-1}$.

So, the sequence is monotonic increasing or decreasing according as $a_2 > a_1$ or $a_2 < a_1$.

Here, obviously $x_2 > x_1$.

Thus, the sequence is monotonic *increasing*.

Again from (1) $x_n^2 - x_n - 2 < 0$, $\because$ $x_n < x_{n+1}$

or, $(x_n - 2)(x_n + 1) < 0$

Therefore, x_n lies between -1 and 2, *i.e.*, $-1 < x < 2$.

Thus the sequence is *bounded*.

Since the sequence is monotonic increasing and bounded, it must tend to a definite limit, say l.

Then, $x_n = l$, $x_{n+1} = l$ as $n \to \infty$

So, from (1) $l^2 - l - 2 = 0$,

$\therefore$ $l = 2, -1$.

Since, all the terms of the sequence are positive, $l = 2$.

(ii) Here, $x_1 = \sqrt{2}$, $x_2 = \sqrt{2\sqrt{2}} = \sqrt{2x_1}$, $x_3 = \sqrt{2x_2}, \ldots,$

$$x_n = \sqrt{2x_{n-1}},\ x_{n+1} = \sqrt{2x_n}$$

$x_{n+1}^2 = 2x_n$ and $x_n^2 = 2x_{n-1}$

$\therefore$ $x_{n+1}^2 - x_n^2 = 2(x_n - x_{n-1})$

or, $(x_{n+1} + x_n)(x_{n+1} - x_n) = 2(x_n - x_{n-1})$

Therefore, if $x_n > x_{n-1}$, then $x_{n+1} > x_n$

Since, $x_2 = \sqrt{2\sqrt{2}}$, $x_1 = \sqrt{2}$, $x_2 > x_1$

Therefore, $x_3 > x_2$, $x_4 > x_3, \ldots$

So, $\{x_n\}$ is monotonic *increasing*.

Again, $x_{n+1}^2 = 2x_n$ and $x_{n+1} > x_n$

$\therefore$ $x_n^2 < 2x_n$ or, $x_n(x_n - 2) < 0$

$\therefore$ $0 < x_n < 2$

Thus, $\{x_n\}$ is *bounded*, and so it is convergent.

Let, $\lim\limits_{n \to \infty} x_n = l$

$\therefore \quad \lim_{n\to\infty} x_{n+1}^2 = \lim_{n\to\infty} 2x_n$

or, $l^2 = 2l$ or, $l(l-2) = 0$

since $x_n > 0,\ l \neq 0,\ \therefore l = 2.$

Hence, $\{x_n\}$ converges to 2.

5.19 Miscellaneous Worked Out Examples

Ex. 1. Prove that the sequence $\{x_n\}$, where $x_n = (-1)^n$ is not convergent. [C. P. '98]

Solution : Let us assume that $\{x_n\}$ is convergent.

Then $x_n \to l$ (a finite quantity).

So, for a finite number $\varepsilon = \frac{1}{2}$, (say) there exists a number N, such that $\left|(-1)^n - l\right| < \frac{1}{2} >$ for $n > N$.

When n is even and $n > N$, $|1-l| < \frac{1}{2}$

i.e., $\frac{1}{2} < l < \frac{3}{2}$

and when n is odd and $n > N$, $|-1-l| < \frac{1}{2}$

i.e., $-\frac{3}{2} < l - \frac{1}{2}$

Thus our assumption leads to a contradiction. Hence, $\{x_n\}$ is not convergent.

Ex. 2. (*i*) Prove that the sequence $\left\{\frac{4n+3}{n+3}\right\}$ is bounded and monotone increasing . [C. P. 1989]

(*ii*) If $x_n = \frac{2n+5}{6n-11}$, find the least integer m, such that $\left|x_n - \frac{1}{3}\right| < \frac{1}{10^3}$ for $n > m$. [C. P. 2000]

Solution : (*i*) $\left|\frac{4n+3}{n+2}\right| \leq 4,$

if $4n+3 \leq 4n+8$, *i.e.,* if $3 < 8$, which is true irrespective of the values of n.

Hence the given sequence is bounded.

Let, $x_n = \dfrac{4n+3}{n+2}$

The sequence $\{x_n\}$ will be monotone increasings,

if $\quad x_n \le x_{n+1}$

i.e., if $\dfrac{4n+3}{n+2} \le \dfrac{4(n+1)+3}{(n+1)+2} = \dfrac{4n+7}{n+3}$.

i.e., if $(4n+3)(n+3) \le (n+2)(4n+7)$

i.e., if $4n^2+15n+9 \le 4n^2+15n+14$

i.e., if $9 \le 14$, which is always true.

Hence $\{x_n\}$ is montone increasing.

(*ii*) $\left|x_n - \dfrac{1}{3}\right| < \dfrac{1}{1000}$ gives $\left|\dfrac{2n+5}{6n-11} - \dfrac{1}{3}\right| < \dfrac{1}{1000}$ for $n > m$

i.e., $\left|\dfrac{26}{3(6n-11)}\right| > \dfrac{1}{1000}$ or, $\dfrac{3(6n-11)}{26} < 1000$

i.e., $6n < \dfrac{26000}{3} + 11$

i.e., $n > 1446\dfrac{5}{18}$, for $n > 1$, and $n > m$

Hence the least integer is 1446.

Ex. 3. Show that if $x_n = \dfrac{3n+1}{n+2}$, then the sequence $\{x_n\}$ is strictly increasing. Is the sequence convergent ? Justify your answer. Also find its limit. [*C. P. 1993, 94*]

Solution : The sequence $\{x_n\}$ is monotone increasing if $x_n < x_{n+1}$

i.e., if $\dfrac{3n+1}{n+2} < \dfrac{3(n+1)+1}{(n+1)+2}$

i.e., if $(3n+1)(n+3) < (n+2)(3n+4)$

i.e., if $3n^2+10n+3 < 3n^2+10n+8$,

which is evidently true for all $n>0$.

Hence the sequence is strictly increasing.

Again, $\therefore \left|\dfrac{3n+1}{n+2}\right| \le 3$. if $3n+1 \le 3n+6$, which is true for all $n > 0$.

Hence the sequence $\{x_n\}$ is bounded.

Thus $\{x_n\}$ is monotone increasing and at the same time bounded above, hence it is convergent.

Also, $\lim\limits_{n\to\infty} \dfrac{3n+1}{n+2} = \lim\limits_{n\to\infty} \dfrac{3+\frac{1}{n}}{1+\frac{2}{n}} = \dfrac{3}{1} = 3.$

Hence, the limit of the sequence is 3.

Ex. 4. (*i*) Examine the convergence of the sequences :

(*a*) $2^2, 4^2, 6^2, \cdots$ [*C. P. 1989*]

(*b*) $1, 2, 2^2, 2^3 \cdots$ [*C. P. 1991*]

(*ii*) Use Cauchy's criterion to prove that $\{x_n\}$ converges, when,

$$x_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots + \frac{1}{n!}$$ [*C. P. 1980*]

(*iii*) Evaluate : $\lim\limits_{n\to\infty} \left\{ \dfrac{1^3}{n^4} + \dfrac{2^3}{n^4} + \dfrac{3^3}{n^4} + \ldots\ldots + \dfrac{n^3}{n^4} \right\}$ [*C. P. 1985*]

Solution : (*i*) (*a*) Here, $x_n = (2n)^2$ and $x_n \to \infty$, as $n \to \infty$

Hence the sequence is not convergent.

(*b*) Here, $x_n = 2^n \to \infty$ as $n \to \infty$

Hence, the sequence is divergent.

We have, $\dfrac{1}{n!} = \dfrac{1}{2.3.4.5....n} < \dfrac{1}{2.2.2.2.....2} = \dfrac{1}{2^{n-1}}$,

$$\therefore |x_m - x_n| = \left| \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \ldots + \frac{1}{m!} \right|, \text{ if } m > n$$

$$< \frac{1}{2^n} + \frac{1}{2^{n+1}} + \ldots\ldots + \frac{1}{2^{m-1}}$$

$$= \frac{1}{2^n} \times \frac{1 - \left(\frac{1}{2}\right)^{m+1-n}}{1 - \frac{1}{2}}$$

$$< \frac{1}{2^n} \cdot 2 = \frac{1}{2^{n-1}}.$$

Thus $| x_m - x_n | \to 0$ as $n \to \infty$.

So, for any $\in > 0$, there exists a positive integer N such that $\frac{1}{2^{n-1}} < \in$ for all $n > N$.

i.e., $| x_m - x_n | < \in$ for all $n > N$ and $m > n$.

Hence, by Cauchy's Principle, $\{x_n\}$ is convergent.

$$(iii)\ \lim_{n\to\infty} \left\{\frac{1^3}{n^4}+\frac{2^3}{n^4}+\frac{3^3}{n^4}+\cdots+\frac{n^3}{n^4}\right\} = \lim_{n\to\infty} \left\{\frac{1^3+2^3+3^3+\cdots+n^3}{n^4}\right\}$$

$$= \lim_{n\to\infty} \frac{n^2(n+1)^2}{4n^4}$$

$$= \frac{1}{4}\cdot \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 = \frac{1}{4}.$$

Ex. 5. (*i*) Show that the sequence $\{x_n\}$, where $x_n = \dfrac{3n+1}{n+1}$ is bounded.

[*C. P. 1997, 2006, 2008*]

(*ii*) Show that the sequence $\{x_n\}$, where $x_n = \dfrac{1}{5+6n}$ is decreasing and bounded.

Correct or justify : $\{x_n\}$ is convergent. [*C. P. 1995*]

(*iii*) If $x_n = (-1)^n$ and $y_n = \frac{1}{n}(x_1 + x_2 + x_3 + \cdots + x_n)$ $(n = 1, 2, 3, \cdots)$, prove that the sequence $\{y_n\}$ converges although the sequence $\{x_n\}$ does not do so. [*C. P. 1985*]

Solution : (*i*) Here, $x_n = \dfrac{3n+1}{n+1} \le 3$

if, $3n+1 \le 3n+3$, *i.e.*, if $1<3$, which is true for all n.

Hence the given sequence is bounded above.

(*ii*) Here, $x_n = \dfrac{1}{5+6n}$, so $x_{n+1} = \dfrac{1}{6n+11}$

$$\therefore\ x_{n+1} - x_n = \frac{1}{11+6n} - \frac{1}{5+6n} = \frac{-6}{(5+6n)(11+6n)} < 0, \text{ for } n > 0$$

$\therefore$ $x_{n+1} - x_n < 0$, *i.e.*, $x_{n+1} < x_n$, so that $x_{n+1} < x_n$ is a decreasing sequence.

Again, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{5+6n} = 0.$

Hence, $\{x_n\}$ is a bounded sequence.

Since the sequence is decreasing and bounded, it must be convergent.

(*iii*) See **Ex.1.**, to prove that $\{x_n\}$, where $x_n = (-1)^n$ is divergent.

$$y_n = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n} = \frac{0}{n} = 0,$$

if n is even and $y_n = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n} = -\frac{1}{n}$, if n is odd.

As $n \to \infty$, $\lim_{n\to\infty} y_n = \lim_{n\to\infty}\left(-\frac{1}{n}\right) = 0$

Hence the sequence $\{y_n\}$ is convergent and it converges to 0.

EXAMPLES-V

1. Define the terms '*bounds*', '*supremum*', '*infimum*' and '*point of accumulation*' in connection with a set.
2. (a) Show by suitable examples that the supremum of a set, if it exists, may or may not be a member of the set.

   (b) Give illustrations, where the infimum of a set is and is not a member of the set.
3. (i) Define the terms 'limits', 'bounds' and 'cluster point' as applied to a sequence.

   (ii) Show that the sequence $\frac{n+1}{n}$ is bounded. [*V. P. 1997*]
4. Prove that a sequence can have at most one limit.
5. What do you mean by a monotonic sequence ?

   State whether a monotonic sequence tends or does not tend to a limit under any circumstances. [*C. P. 1980, 94, 2000*]
6. When does a sequence converge ? Prove that a convergent sequence is always bounded. Comment with reasons on the validity of its converse proposition. [*C. P. 1981, 98, B. P. 1997*]
7. (i) Define a monotone Sequence.

(ii) Prove that the sequence $\{x_n\}$, where $x_n = \left(1+\frac{1}{n}\right)^n$ is
(a) monótone increasing, (b) bounded and (c) converges to a limit e, where $2 < e < 3$. [*C. P. 1985, 88, 98*]

8. (i) Show that the sequence $\{x_n\}$, defined by,

$$x_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}$$

is monotonic increasing and also show that it is bounded.

(ii) Show that the sequence $\{x_n\}$, is monotone decreasing, where

$x_n = \left(1+\frac{1}{n}\right)^{n+1}$ [*C. H. 1955*]

9. (i) Discuss the behaviour of the sequence $\{x^n\}$, where x is any real number.

(ii) Prove that the sequence $\{r^n\}$ is convergent, if $|r| < 1$. [*B. P. 1998*]

(iii) Show that the sequence $\left\{\frac{1}{n}\right\}$ converges to 0.

10. (i) When a sequence is said to be a Cauchy Sequence ?

(ii) State Cauchy's general principle of convergence of a sequence and apply it to show that the sequence $\{x_n\}$, where $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$ is convergent. [*V. P. 2002; C. P. 2006*]

11. (i) Give an example of a sequence which is neither monotone increasing nor monotone decreasing. [*V. P. 2001*]

12. (i) Show that $\lim_{n\to\infty} \sqrt[n]{x} = 1, \quad (x > 1)$. [*B. P. 1996, 1997*]

(ii) Prove that $\lim_{n\to\infty} n \cdot x^n = 0, \quad$ for $|x| < 1$.

(iii) Show that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$.

(iv) Prove that $\lim_{n\to\infty} \frac{x^n}{n} = 0, \quad$ for $|x| < 1$.

(v) Prove that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$. [*B. P. 1996*]

13. What is a Cauchy Sequence? Show that a constant sequence is a Cauchy Sequence.

14. Define Cauchy Sequence and show that $\left\{\frac{n}{n+1}\right\}$ is one such sequence. [*V. P. 1997*]

15. Determine the bounds of the following sequences, if there be any :

(i) $1, -1, 1, -1, 1, \ldots$

(ii) $1, 2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4}$

(iii) $-2, -2^2, -2^3, -2^4, \ldots$

(iv) $\frac{3}{2}, \frac{5}{2}, \frac{5}{3}, \frac{7}{3}, \frac{7}{4}, \frac{9}{4}, \frac{9}{5}, \frac{11}{5}, \ldots$

16. Prove that the sequence $\{x_n\}$ is monotonic increasing and bounded; where

(i) $x_n = \dfrac{3n+1}{n+2}$ [*B. P. 1985, V. P. 1999*]

(ii) $x_n = \dfrac{1}{1.3} + \dfrac{1}{3.5} + \dfrac{1}{5.7} + \ldots + \dfrac{1}{(2n-1)(2n+1)}$

(iii) $x_n = \dfrac{1}{n+1} + \dfrac{1}{n+2} + \dfrac{1}{n+3} + \ldots + \dfrac{1}{2n}$

(iv) $x_n = \dfrac{4n+3}{n+2}$ [*C.P. 1980*]

17. (i) If $x_n = \dfrac{n+1}{2n+1}$, show that the sequence $\{x_n\}$ is strictly monotonic decreasing and hence prove that it is convergent. [*C.P. 1996*]

(ii) Show that the sequence $\left\{\frac{3n}{n+1}\right\}$ is monotonic increasing, bounded above and converges to 3. [*C.P. 1991, 2000*]

(iii) Show that the sequence $\left\{\frac{4n+3}{n+2}\right\}$ is bounded and monotonic increasing. [*C.P. 1989*]

18. (i) Show that the sequence $\{x_n\}$, where

$x_n = (-1)^n \dfrac{2n-1}{n}$ and $x_n = \dfrac{1}{n} + (-1)^n.2$

oscillate finitely.

(ii) Show that the sequence $\{x_n\}$, where $x_n = (-1)^n (n+1)$ oscillates infinitely.

19. (i) Show that the sequence $\left\{\left(\frac{n-1}{n+1}\right)^2\right\}$ is convergent. [*V.P. 1995*]

(ii) Show that the sequence $\{x_n\}$, where $x_n = \frac{4n^2-5}{6n}$ is not convergent. [*V. P. 1996*]

(iii) Show that the sequence $\{x_n\}$, where $x_n = \frac{3n+1}{n+2}$ is strictly increasing. Is the sequence convergent ? Justify your answer. [*C. P. 1993*]

20. Find an integer N, such that

(i) $\left|\frac{1}{n} - 0\right| < 0 \cdot 0001$ for every $n > N$.

(ii) $\left|\frac{1}{\sqrt{n+1}} - 0\right| \leq 0 \cdot 1$ for all $n \geq N$. [*C. P. 1988*]

(iii) $\left|\frac{2n+5}{6n-11} - \frac{1}{3}\right| < \frac{1}{10^3}$ for $n > N$.

21. Show that the sequence $\{x_n\}$ is convergent, where

(i) $x_n = 2 + (-1)^n \cdot \frac{1}{n}$; [*C.P. 1992*]

(ii) $x_n = \frac{2n-1}{n}$;

(iii) $x_n = \frac{1}{\sqrt{n}} \sin \frac{n\pi}{2}$;

(iv) $x_n = \frac{1}{n} \sin \frac{n\pi}{2}$;

(v) $x_n = \frac{2n^2+1}{4n^2+1}$; [*C.P. 1980*]

(vi) $x_n = 1 + \left(\frac{-1}{2}\right)^n$. [*C.P. 1968*]

22. Show that the sequence $\{x_n\}$ is divergent, where

(i) $x_n = \sqrt{n}$; (ii) $x_n = \frac{n^2}{n+1}$; (iii) $x_n = \log\left(\frac{1}{n}\right)$.

23. Consider the behaviour of the following sequences with respect to convergence or divergence :

(i) $\{(-1)^n\}$;

(ii) $\left\{(-1)^n . \dfrac{n^2+1}{n}\right\}$;

(iii) $\{(-1)^n (n)\}$;

(iv) $\left\{\dfrac{n^2}{2n^2+1}\right\}$; [*C.P. 1981*]

24. Show that the sequence $\left\{\dfrac{1}{n^p}\right\}$, where $p > 0$, is a null sequence.

25. (i) (a) Show that the sequence $\sqrt{7}, \sqrt{7+\sqrt{7}}, \sqrt{7+\sqrt{7+\sqrt{7}}}, \ldots$ converges to the positive root of the equation $t^2 - t - 7 = 0$.

(b) Show that the sequence $\sqrt{5}, \sqrt{5+\sqrt{5}}, \sqrt{5+\sqrt{5+\sqrt{5}}}, \ldots$ converges to $\frac{1}{2}\left(1+\sqrt{21}\right)$.

(ii) If $x_{n+1} = \sqrt{K + x_n}$, where x_1 and K are positive, show that the sequence $\{x_n\}$ is increasing or decreasing according as x_1 is less or greater than the positive root of $t^2 - t - K = 0$ and has, in either case, this root as limit.

ANSWERS

15. (i) lower bound -1, upper bound 1;
(ii) no upper bound, lower bound 0;
(iii) upper bound -2, no lower bound;
(iv) upper bound $\frac{5}{2}$, lower bound $\frac{1}{2}$.

19. (iii) yes, convergent.

20. (i) $N = 1000$; (ii) $N = 99$; (iii) 1446.

6 INFINITE SERIES

6.1. Infinite Series.

Let us consider an infinite sequence of numbers, $u_1, u_2, u_3, \ldots, u_n, \ldots$ or $\{u_n\}$. The series derived from the terms of this sequence, viz.,

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots$$

or, $\sum_{n=1}^{\infty} u_n$ or, simply $\sum u_n$ is called an *Infinite Series.*

Now, let us form the sequence of the successive partial sums $\{s_n\}$ of the above infinite series, where $s_1 = u_1$, $s_2 = u_1 + u_2$, $s_3 = u_1 + u_2 + u_3, \ldots, s_n = u_1 + u_2 + u_3 + \ldots + u_n$

6.2. Convergence of Infinite Series.

If the sequence $\{s_n\}$ of the partial sums is convergent, the series $\sum u_n$ is said to be *convergent.*

If the limit of the sequence $\{s_n\}$ be a finite number S, then we say that the series $\sum u_n$ converges to S and that S is the *sum* of the series. It may be noted that S is not a sum in the normal sense of the term, actually it is the *limit* of a *sum.*

Definition : An infinite series $\sum u_n$ is said to converge to S, if corresponding to an arbitrary positive number ε, however small it may be, there exists a positive integer N, (depending upon ε), such that

$$|s_n - S| < \varepsilon, \text{ whenever } n > N.$$

If we denote $(u_{n+1} + u_{n+2} + \ldots)$ by R_n, obviously, for the convergence of $\sum u_n$, $|R_n| < \varepsilon$, for $n > N$.

If s_n does not tend to a definite finite limit, but , $s_n \to +\infty$ or $s_n \to -\infty$, the series $\sum u_n$ is properly *divergent* and diverges to $+\infty$ or to $-\infty$ respectively.

If, however, s_n does not tend to a definite finite limit, or to $+\infty$ or to $-\infty$, but *oscillates* finitely or infinitely, then $\sum u_n$ is said to *oscillatory* or *improperly divergent.*

Divergent or oscillatory series are generally called *non-convergent.*

Examples :

(i) Show that $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\dots$ is convergent, and find its sum.

Here, $s_n = \dfrac{1-\left(\frac{1}{3}\right)^n}{1-\frac{1}{3}} = \dfrac{1}{2}\left(3-\dfrac{1}{3^{n-1}}\right) \to \dfrac{3}{2}$, as $n \to \infty$,

So, the series is *convergent* and its sum is $\frac{3}{2}$.

(ii) Show that the series $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\dots$ is convergent; find its sum.

$$S_n = \sum u_n = \sum_{r=1}^{n} \frac{1}{r(r+1)} = \sum_{1}^{n}\left(\frac{1}{r}-\frac{1}{r+1}\right)$$

$$=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\dots+\left(\frac{1}{n-1}-\frac{1}{n}\right)+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$=1-\frac{1}{n+1} \to 1, \text{ as } n \to \infty.$$

So, the series is *convergent* and its sum is 1.

(iii) Examine the convergence of the series : $1+2+3+\dots+n+\dots$

Here, $s_n = \frac{1}{2}n(n+1) \to \infty$, as $n \to \infty$.

The series is *divergent*, as S_n diverges to $+\infty$.

(iv) Show that the series $\sum_{1}^{\infty}(-1)^{n+1} = 1-1+1-1+\dots$ oscillates finitely.

Since, $S = 1-1+1-1+\dots$

$S_n = 1$, when n is an odd integer.

$= 0$, when n is an even integer.

Hence S_n does not tend to a definite limit, as $n \to \infty$. The series *oscillates finitely.*

(v) The series $1-2+3-4+5-6+\dots = \sum_{1}^{\infty}(-1)^{r+1}.r$ oscillates infinitely, for

$$S_n = \tfrac{1}{2}(n+1), \text{ when } n \text{ is odd,}$$

$$= -\tfrac{1}{2}n, \text{ when } n \text{ is even}$$

and it does not approach a finite limit as $n \to \infty$.

6.3. Convergence of Two Important Series.

I. **Geometric Series.** The infinite geometric series

$$a + ar + ar^2 + ar^3 + \dots \qquad (a > 0) \text{ is}$$

(i) *Convergent* when $|r| < 1$,

(ii) *Divergent* when $r \geq 1$ and

(iii) *Oscillates finitely*, if $r = -1$, *oscillates infinitely* if $r < -1$.

Proof. Obviously, the n^{th} partial sum S_n is given by

$$S_n = a + ar + ar^2 + \dots + ar^{n-1} = \frac{a\left(r^n - 1\right)}{r - 1}, \ (r \neq 1)$$

(i) Now, if $r < 1$, $r^n \to 0$ as $n \to \infty$

$$\therefore \quad \lim_{n \to \infty} S_n = \frac{a}{1 - r}$$

and the series *converges* to $\frac{a}{1-r}$.

(ii) If, on the other hand, $r > 1$, $r^n \to \infty$ as $n \to \infty$.

$$\therefore \quad \lim_{n \to \infty} S_n \to +\infty$$

and the series *diverges* to $+\infty$.

(iii) If $r = 1$,

$$S_n = a + a + a + \dots + a = na$$

and na, *i.e.*, $S \to \infty$ as $n \to \infty$.

So, in this case also, the series *diverges* properly.

(iv) When $r < -1$, $\{r^n\}$ oscillates infinitely and the series is oscillatory or improperly divergent.

(v) If, again, $r = -1$,

$$S_n = a - a + a - a + \dots = a, \text{ when } n \text{ is odd}$$

$$= 0, \text{ when } n \text{ is even}$$

and the series *oscillates finitely* between 0 and a.

From the above discussions we arrive at the conclusion that the geometric series converges if $|r| < 1$ and does not converge if $|r| \geq 1$.

II. The p-series.

The infinite series $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots+\frac{1}{n^p}+\ldots$

(i) is convergent if $p>1$, (ii) is divergent if $p \le 1$.

Proof. (i) We suppose that $p>1$.

Let us consider the partial sums of order. 2^n-1, *i.e.*, $S_1, S_3, S_7, S_{15}, \ldots$

$$S_{2^n-1}=1+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\frac{1}{5^p}+\ldots+\frac{1}{7^p}\right)$$

$$+\left(\frac{1}{8^p}+\frac{1}{9^p}+\ldots+\frac{1}{15^p}\right)+\ldots$$

$$+\left(\frac{1}{(2^{n-1})^p}+\ldots+\frac{1}{(2^n-1)^p}\right)$$

$$\le 1+\left(\frac{1}{2^p}+\frac{1}{2^p}\right)+\left(\frac{1}{4^p}+\frac{1}{4^p}+\ldots+\frac{1}{4^p}\right)$$

$$+\left(\frac{1}{8^p}+\frac{1}{8^p}+\ldots+\frac{1}{8^p}\right)$$

$$+\ldots+\left(\frac{1}{(2^{n-1})^p}+\frac{1}{(2^{n-1})^p}+\ldots+\frac{1}{(2^{n-1})^p}\right)$$

$$=1+2\cdot\frac{1}{2^p}+4\cdot\frac{1}{4^p}+8\cdot\frac{1}{8^p}+\ldots+2^{n-1}\cdot\frac{1}{(2^{n-1})^p}$$

$$=1+\frac{1}{2^{p-1}}+\frac{1}{(2^{p-1})^2}+\frac{1}{(2^{p-1})^3}+\ldots+\frac{1}{(2^{p-1})^{n-1}}$$

$$=\frac{1-r^n}{1-r}, \qquad \text{where } r=\frac{1}{2^{p-1}}.$$

$$\therefore \quad S_{2^n-1}<\frac{1}{1-\frac{1}{2^{p-1}}}=\text{constant.}$$

All the terms of the series being positive, the partial sums are monotonic *increasing*.

And for any positive integer m, there exists another positive integer n, such that $2^{n-1} > m$,

$$S_m < S_{2^n-1} < \frac{1}{1-\frac{1}{2^{p-1}}} = \text{Constant.}$$

Thus, the partial sums are *bounded.*

The sequence of partial sums, being monotonic increasing and bounded, must converge.

Hence, the series is convergent if $p > 1$.

(ii) Next, we suppose that $p \le 1$.

Since $p \le 1$, $n^p \le n$.

$$\text{Then, } S_{2^n} = 1+\frac{1}{2^p}+\left(\frac{1}{3^p}+\frac{1}{4^p}\right)+\left(\frac{1}{5^p}+\ldots+\frac{1}{8^p}\right)+\ldots$$
$$\ldots+\left(\frac{1}{\left(2^{n-1}+1\right)^p}+\ldots+\frac{1}{\left(2^n\right)^p}\right)$$

$$\ge 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\ldots+\frac{1}{8}\right)+\ldots$$
$$\ldots+\left(\frac{1}{2^n}+\frac{1}{2^n}+\ldots+\frac{1}{2^n}\right)$$

$$= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots+\frac{1}{2} = 1+\frac{1}{2}n.$$

Therefore, given any number $G>0$, however large $S_{2^n} > G$, whenever $n > 2(G-1)$.

Further, the partial sums are strictly monotonic increasing and not bounded.

Hence, the given series is divergent, when $p \ge 1$.

6.4. Conditions of Convergence.

I Cauchy's General Principle

The necessary and sufficient condition for the convergence of an infinite series $\sum u_n$ *is that corresponding to any arbitrarily chosen positive number* ε, *however small, a positive integer N can be found that for all* $n \ge N$,

$$\left| S_{n+p} - S_n \right| < \left| u_{n+1} + u_{n+2} + \cdots + u_{n+p} \right| < \varepsilon,$$

for every integral value of p.

Let, $S_n = u_1 + u_2 + u_3 + \ldots + u_n$

Then $\sum u_n$ is convergent, if and only if, $\{S_n\}$ is convergent. Now by Cauchy's general principle of convergence of a sequence, $\{S_n\}$ is convergent, if and only if,

$\left| s_{n+p} - s_n \right| < \varepsilon$ for $n \geq N$ and for every positive integral value of p.

Thus, we have $\sum u_n$ is convergent, if and only if

$$\left| s_{n+p} - s_n \right| < \left| u_{n+1} + u_{n+2} + \cdots + u_{n+p} \right| < \varepsilon,$$

when $n \geq N$ for every positive integral value of p.

II. Pringsheim's Theorem.

If the terms of the series $\sum u_n$ of positive terms steadily decrease, then it is necessary for its convergence that $\lim\limits_{n \to \infty} n \,.\, u_n = 0$.

Let the series $\sum u_n$ be convergent. Then for a given ε, however small, there can be found a positive integer N, such that, for all values of $n \geq N$, we have

$$u_{m+1} + u_{m+2} + u_{m+3} + \cdots + u_N < \tfrac{1}{2}\varepsilon$$

Since the terms of the series steadily decrease, each of the terms u_{m+1}, $u_{m+2}, \ldots, u_n$ is greater than or equal to u_n.

Hence, $(n - N)u_n < \frac{1}{2}\varepsilon$, when $n \geq N$.

Since, $\lim u_n = 0$, we can choose $\mu > N$, such that

$$N u_n < \tfrac{1}{2}\varepsilon, \text{ when } n \geq \mu..$$

Thus, $n u_n < \varepsilon$, when $n \geq \mu$. .

This gives $\lim n u_n = 0$.

Note : This condition is *necessary* but *not sufficient*. If we consider the series $\sum u_n$, where $u_n = \dfrac{1}{n \log n}$, $\lim n u_n = 0$, but $\sum u_n$ is divergent.

6.5. Tests of Convergence of Series of Non-negative terms

In order to ascertain the convergence of infinite series it is not always convenient to find the limiting value of S_n as $n \to \infty$. So, a number of methods and rules have been developed for testing the convergence of infinite series. In this section important rules and methods will be discussed.

I *A necessary condition for convergence of* $\sum u_n$ *is that* $\lim u_n = 0$.

Proof: Since the terms of $\sum u_n$ are all positive, it follows from Cauchy's general principle of Convergence (art. 6.4).

$$|s_n - s_{n-1}| < \varepsilon, \text{ i.e., } |u_n| < \varepsilon$$

Hence $\lim u_n = 0$ is a *necessary condition* for convergence of $\sum u_n$.

Note : This condition is *not sufficient*, for example the series $\sum \frac{1}{n}$ is divergent, although $\lim u_n = 0$.

II *If a series* $\sum u_n$ *of positive and decreasing terms be convergent, then* $\lim (n u_n) = 0$.

This theorem as *Pringsheim's* theorem, has already been discussed in art. 6.4.

III Comparision Test

Statement :

Let $\sum u_n$ and $\sum v_n$ be two series with non-negative terms, and suppose that there exists an integer N such that $u_n \le v_n$ for $n > N$.

Then (*i*) $\sum u_n$ converges if $\sum v_n$ converges and (*ii*) $\sum v_n$ diverges if $\sum u_n$ diverges.

Proof: (*i*) We denote the n^{th} partial sums of $\sum u_n$ and $\sum v_n$ is S, S' respectively.

Since $\sum v_n$ converges, given ε (>0) we can find N_1 (depending on ε), such that for all $p \ge 1$, and $n > N_1$.

$$|S'_{n+p} - S'_n| < \varepsilon.$$

Let $N_2 = \max(N, N_1)$. Then for $n > N_2$ we have

$$|S_{n+p} - S_n| < |S'_{n+p} - S'_n| < \varepsilon$$

and since this inequality holds for all positive integral values of p, it follows then $\sum u_n$ is convergent.

(ii) If $n > N$, we have

$$S'_n - S'_N \geq S_n - S_N$$

or, $S'_n \geq S_n - S_N + S'_N$

If now, $n \to \infty$, $S'_n \to \infty$, since $S_n \to \infty$.

Hence the theorem.

Another form of comparison test

If $\sum u_n$ and $\sum v_n$ be two series of positive terms and if $0 < \lim_{n\to\infty} \frac{u_n}{v_n} < \infty$, then either both of them converge or both diverge.

IV. D'Alembert's Ratio Test

Statement :

If $\sum u_n > 0$, and if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \rho$, then

(i) $\sum u_n$ converges if $\rho < 1$,

(ii) $\sum u_n$ diverges if $\rho > 1$. If however,, $\rho = 1$ this test is inconclusive.

Proof : (i) If $\rho < 1$, for any given ε $(< 1 - \rho)$, we can find a positive integer N (depending upon ε) such that for $n \geq N$,

$$\frac{u_{n+1}}{u_n} < \rho + \varepsilon, \text{ or, } u_{n+1} < (\rho + \varepsilon)u_n.$$

In particular,

$$u_{N+1} < (\rho + \varepsilon)u_N$$

$$u_{N+2} < (\rho + \varepsilon)u_{N+1} < (\rho + \varepsilon)^2 u_N$$

$$u_{N+m} < (\rho + \varepsilon)u_{N+m-1} < \ldots < (\rho + \varepsilon)^m u_N$$

Since $0 < (\rho + \varepsilon) < 1$, the series $\sum_{m=1}^{\infty} (\rho + \varepsilon)^m u_N$ converges, being a geometric series with common ratio <1, and hence $\sum_{N+1}^{\infty} u_n$ or $\sum_{1}^{\infty} u_n$ is convergent by the comparison test.

(ii) If $\rho > 1$, for any given $\varepsilon \ (< \rho - 1)$, we can find a positive integer N, such that for $n \geq N$

$$\frac{u_{n+1}}{u_n} > \rho - \varepsilon \text{ or, } u_{n+1} > (\rho - \varepsilon) u_n .$$

Thus, as in (i), we have for $m \geq 1$,

$$u_{N+m} > (\rho - \varepsilon)^m u_N$$

By comparing with the geometric series with common ratio

$$(\rho - \varepsilon) > 1,$$

we can conclude that the series $\sum u_n$ is divergent.

V. Cauchy's Root Test.

Statement :

If $u_n > 0$ and if $\lim_{n \to \infty} (u_n)^{1/n} = \rho$, then

(i) $\sum u_n$ converges if $\rho < 1$,

(ii) diverges if $\rho > 1$,

When $\rho = 1$, the test is inconclusive.

Proof: (i) If $\rho < 1$, given any $\varepsilon \ (< 1 - \rho)$,

we can find a positive integer N, depending upon ε, such that for $n \geq N$,

$$(u_n)^{1/n} < \rho + \varepsilon$$

or, $u_n < (\rho + \varepsilon)^n$

Comparing with the geometric series with common ratio $(\rho + \varepsilon)$, where $0 < (\rho + \varepsilon) < 1$, we conclude that the $\sum u_n$ is convergent.

(ii) Next, let us suppose that $\rho > 1$.

Given any $\varepsilon \ (< \rho - 1)$, we can find an infinity of n, say $N_1, N_2, N_3, \ldots$, such that for these values of n, $(u_n)^{1/n} > \rho - \varepsilon$

or, $u_n > (\rho - \varepsilon)^n$.

Since $(p - \varepsilon) > 0$, u_n cannot tend to zero, so that $\sum u_n$ is divergent.

VI. Raabe's Test

Statement :

Let $\sum u_n$ be a series of positive terms satisfying $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=\rho$, then

(i) $\sum u_n$ converges if $\rho>1$,

(ii) $\sum u_n$ diverges if $\rho<1$,

This test gives no information regarding convergence, if $\rho = 1$.

VII. Logarithmic Test

Statement :

The series $\sum u_n$ of positive terms is convergent or divergent according as

$$\lim_{n\to\infty}\left\{n\log\frac{u_n}{u_{n+1}}-1\right\}>1 \text{ or } <1.$$

This test fails, if the limit is 1.

VIII. Gauss's Test

Statement :

If for a series $\sum u_n$ of positive terms $\frac{u_n}{u_{n+1}}$ be expressed in powers of $1/n$, so that

$$\frac{u_n}{u_{n+1}}=1+\frac{\mu}{n}+O\left(\frac{1}{n^2}\right),$$

then $\sum u_n$ is convergent if $\mu > 1$, and divergent if $\mu \le 1$.

Note. The notation $O\left(1/n^2\right)$ denotes such a function $f(n)$ that for every $n \ge n_0$ (a definite positive integer), $|f(n)| < k\frac{1}{n^2}$, where k is a finite quantity independent of n.

6.6. Mixed Series : Absolute and Conditional Convergence.

The series $\sum u_n$ which contains both positive and negative terms is said to be *absolutely convergent*, if the series $\sum |u_n|$ be convergent.

For example, the series $1-\frac{1}{2}+\frac{1}{2^2}-\frac{1}{2^3}+\ldots$ is absolutely convergent.

If the series $\sum u_n$ be convergent, but the series $\sum |u_n|$ be divergent, then the series $\sum u_n$ is said to be *non-absolutely convergent*.

The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ is non-absolutely convergent.

A convergent series which remains unaffected by rearrangement of its terms is said to be *unconditionally convergent*, while the series which is affected by rearrangement of its terms is called *conditionally convergent*.

$1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \ldots$ is an example of unconditionally convergent series, and

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ is a conditionally convergent series.

6.7. Alternating Series.

A series in which the signs of consecutive terms are alternatively positive and negative is called an *alternating series..*

THEOREM I. *Leibnitz's test for Alternating Series.*

An alternating series $u_1 - u_2 + u_3 - u_4 + \ldots$ is convergent if $\{u_n\}$ be a sequence of positive terms decreasing monotonically to zero.

THEOREM II. An infinite series in which the terms are alternately positive and negative is convergent if each term be numerically less than the preceeding term and $\lim u_n = 0$.

Note: 1. When we say $\sum u_n$ is absolutely convergent we are to test the convergence of $\sum |u_n|$ and not that of $\sum u_n$.

2. To determine the absolute convergence of series we are to use the test developed for positive series.

3. If the terms of an absolutely convergent series be rearranged, the series remains absolutely convergent and its sum also remains unaltered.

6.8. Power Series.

A series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots + a_n(x - x_0)^n + \ldots$$

or, $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$

where the coefficient $a_0, a_1, a_2, \ldots, a_n, \ldots$ are independent of x, is called a *power series* in x.

The simplest and the most important power series is the geometric series : $1 + x + x^2 + \ldots + x^n + \ldots$.

From the discussions of art. 6.3, it is obvious that the above geometric series is convergent (and also absolutely convergent) for $|x| < 1$, *i.e.*, $-1 < x < 1$, diverges to $+\infty$ for $x \geq 1$, and oscillatory when $x < -1$.

The above example and other similar situations reveal that as the variable x, in a power series, changes, the terms also change and the series may change from a convergent to a non-convergent one. This leads to the conclusion : a power series is convergent either for all values of x, or for a certain range of values of x, or for no value of x except zero. It is, therefore, important to ascertain the value or values of the variable x for which a power series is convergent. That is why comes the idea of '*interval of convergence*'.

6.9. Interval of Convergence.

Definition : The *interval of Convergence* of a power series in x is the collection of values of x in an interval such that the series converges for every value of x in this interval, but does not converge for values of x outside the interval.

If a power series $\sum a_n x^n$ converges absolutely for all values of x, inside the interval of convergence $-r < x < r$, and diverges for $|x| > r$, then r is called the *radius of convergence of the power series.*

6.10. Determination of Interval of Convergence.

D'Alembert's Ratio test and Cauchy's Root test will be useful for the determination of interval of convergence of Power Series.

For the power series $\sum a_n x^n$.

(i) if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \to \lambda$, a finite quantity other than zero,

(ii) if $\lim_{n\to\infty} \left\{ |a_n| \right\}^{1/n} = \lambda$ where $\lambda \neq 1$,

then the interval of convergence of the series is $(-r, r)$, where $r = \frac{1}{\lambda}$.

Note : The interval of convergence as determined by the above tests is *open*. The series may be or may not be convergent at the end points $x = \pm r$. To determine the *complete interval of convergence* one should first find the values of x fot which the series is *absolutely convergent* and then test the end-points.

6.11. Properties of Power Series.

Here we state (without proof) certain properties of infinite Power Series, which are often used in obtaining new series.

I. Within its interval convergence, a power series represents a *continous sum function* and has not more than one power series representation in a given interval.

II. Two power series converging in the same interval of convergence can be added and subtracted term by term; thus if

$$f(x) = \sum a_n x^n \text{ and } \phi(x) = \sum b_n x^n,$$

$$f(x) \pm \phi(x) \mp \sum a_n x^n \pm \sum b_n x^n = \sum (a_n \pm b_n) x^n.$$

III. Two power series converging in the same interval of convergence can be multiplied term by term.

Thus, if $f(x) = \sum a_n x^n$ and $\phi(x) = \sum b_n x^n$,

$$f(x).\phi(x) = \sum a_x x^n . \sum b_n x^n$$

$$= \sum (a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0) x^n.$$

This product series is absolutely convergent in the same common interval of convergence.

IV. The quotient of two power series $\sum a_n x^n$ and $\sum b_n x^n$ $(b_0 \neq 0)$ is another power series $\sum c_n x^n$, provided x remains within a sufficiently small interval in which the denominator does not vanish and both numerator and denominator are convergent series.

Thus, $$\frac{a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n}{b_0 + b_1 x + b_2 x^2 + \ldots + b_n x^n} = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n$$

Since, $\sum a_n x^n = \sum b_n x^n . \sum c_n x^n$, $a_0 = b_0 c_0$, $a_1 = c_0 b_1 + c_1 b_0$, *etc.* whence $c_0, c_1, c_2, \ldots$ can be calculated.

V. Limits, term by term, are permissible in case of power series within its interval of convergence.

Thus, if $f(x) = \sum a_n x^n$, $\lim_{x \to a} f(x) = \lim_{x \to a} \sum a_n x^n$.

VI. A power series can be differentiated or integrated, term by term, over any closed interval lying entirely within its interval of convergence, as many times as one wishes.

If $f(x) = \sum a_n x^n$,

$$f'(x_1) = \sum n a_n x_1^{n-1} , \; f''(x_1) = \sum n(n-1) a_1 x^{n-2} , \text{etc.}$$

and $\int_{x_1}^{x_2} f(x)dx = \int_{x_1}^{x_2} \sum a_n x^n dx = \sum \frac{a_n}{n+1}\left(x_2^{n+1} - x_1^{n+1}\right)$,

provided x_1 and x_2 are both interior points of the interval of convergence.

VII. If two power series $\sum a_n x^n$ and $\sum b_n x^n$ both converge in the same interval and both represent the same function $f(x)$, then they are identical, *i.e.*, $a_n = b_n$ for all values of n.

VIII. If $y = f(x) = \sum a_n x^n$ and $F(y) = \sum b_n x^n$,

then $F\{f(x)\} = b_0 + b_1\left\{\sum a_n x^n\right\} + b_2\left\{\sum a_n x^n\right\} + \ldots$

$$= b_0 + b_1\left(a_0 + a_1 x + a_2 x^2 + \ldots\right) + b_2\left(a_0 + a_1 x + a_2 x^2 + \ldots\right) + \ldots$$

$$= c_0 + c_1 x + c_2 x^2 + \ldots$$

for every value of x for which $\sum \left| a_n x^n \right|$ converges and has a sum less than the radius of convergence of $\sum b_n y^n$.

6.12. Illustrative Examples.

Ex. 1. Prove that the series $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots$ is convergent. [*V. P. 1997*]

Here, $u_n = \frac{1}{(2n-1)(2n+1)}$

$$S_n = \sum_{n=1}^{n} u_n = \sum_{n=1}^{n} \frac{1}{(2n-1)(2n+1)} = \sum_{n=1}^{n} \frac{1}{2}\left\{\frac{1}{2n-1} - \frac{1}{2n+1}\right\}$$

$$= \frac{1}{2}\left\{1 - \frac{1}{3}\right\} + \frac{1}{2}\left\{\frac{1}{3} - \frac{1}{5}\right\} + \frac{1}{2}\left\{\frac{1}{2n-1} - \frac{1}{2n+1}\right\}$$

$$= \frac{1}{2}\left\{1 - \frac{1}{2n+1}\right\} \to \frac{1}{2} \text{ as } n \to \infty .$$

Hence, the series is convergent and its sum is $\frac{1}{2}$.

Ex. 2. Examine the convergence of the series :

$$1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{(n-1)!}+\ldots$$

Here, $S_n = 1+1+\frac{1}{2}+\frac{1}{2.3}+\frac{1}{2.3.4}+\ldots+\frac{1}{2.3.4\ldots(n-1)}$

$$< 1+1+\frac{1}{2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{(n-2).(n-1)} \quad (n>4)$$

$$= 1+1+\frac{1}{2}+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{n-2}-\frac{1}{n-1}\right)$$

$$= 1+1+\frac{1}{2}+\left(\frac{1}{2}-\frac{1}{n-1}\right)$$

$$= 3-\frac{1}{n-1} < 3.$$

$\{S_n\}$ is monotonic *increasing* sequence and *bounded above.*

Hence, the series is convergent.

Ex. 3. Show that the series $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\ldots+\frac{n}{2^n}$ is convergent.

[*V. P. 2001*]

Here, $u_n = \frac{n}{2^n}$ and $u_{n+1} = \frac{n+1}{2^{n+1}}$

$$\therefore \quad \frac{u_{n+1}}{u_n} = \frac{n+1}{2^{n+1}} \div \frac{n}{2^n} = \frac{n+1}{2n} = \frac{1}{2}\left(1+\frac{1}{n}\right) \to \frac{1}{2} < 1$$

Hence, the series is convergent.

Ex. 4. Test the convergence of the following series :

(i) $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n-1}}\right)$, (ii) $\sum_{n=2}^{\infty}\frac{1}{\log n}$ [*C. P. 1998*]

(i) Here, $u_n = \frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n-1}} = \frac{\sqrt{n-1}+\sqrt{n}}{\sqrt{n}.\sqrt{n-1}} = \frac{\sqrt{1-\frac{1}{n}}+1}{\sqrt{n}.\sqrt{1-\frac{1}{n}}}$.

Let us introduce a comparison series $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ where

$v_n = \frac{1}{\sqrt{n}}$.

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sqrt{1-\frac{1}{n}}+1}{\sqrt{n}\,.\sqrt{1-\frac{1}{n}}} \cdot \sqrt{n} = \lim_{n\to\infty} \frac{\sqrt{1-\frac{1}{n}}+1}{\sqrt{1-\frac{1}{n}}} = 2$$

But, $\sum v_n = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is *divergent*, for in the p-series, here $p = \frac{1}{2} < 1$.

Hence, the series u_n is also divergent.

(ii) $\because \quad \log n < n$ for all $n > 1$,

$$\frac{1}{\log n} > \frac{1}{n} \text{ for all } n > 2.$$

Let us compare the given series with the *divergent* series

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} + \ldots \qquad \ldots (1)$$

Each term of the given series *exceeds* the corresponding term of the divergent series. Hence, the given series is also divergent.

Ex. 5. Examine the convergence of :

$$\frac{1^2}{2} + \frac{2^2}{2^2} + \frac{3^2}{2^3} + \frac{4^2}{2^4} + \ldots + \frac{n}{2^n}$$

[*C.P. 1993, 2003, 2008 B.P. 1998*]

Here, $u_n = \frac{n^2}{2^n}$ and $u_{n+1} = \frac{(n+1)^2}{2^{n+1}}$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{2^{n+1}} \div \frac{n^2}{2^n} = \frac{1}{2}\left(1+\frac{1}{n}\right)^2 \to \frac{1}{2} \text{ as } n \to \infty$$

Hence, the given series is convergent.

Ex. 6. Examine the convergence of :

$$\frac{1}{3} + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{7}\right)^3 + \left(\frac{4}{9}\right)^4 + \ldots$$

[*C.P. 1986, '92, 2007*]

Here, $u_n = \left(\frac{n}{2n+1}\right)^n$.

$$(u_n)^{1/n} = \frac{n}{2n+1}$$

$$\therefore \quad \lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} = \frac{1}{2} < 1.$$

Hence, by Cauchy's Root test, the given series is convergent.

Ex. 7. Examine the convergence of

$$2+\frac{3}{8}+\frac{4}{27}+\frac{5}{64}+\ldots+\frac{(n+1)}{n^3}+\ldots \quad [\textit{C.P. 1997, 2003}]$$

Here, $u_n = \dfrac{n+1}{n^3} = \dfrac{1+\frac{1}{n}}{n^2}$.

Let us introduce another series $\sum v_n$, where $v_n = \dfrac{1}{n^2}$.

Evidently, $\sum v_n$ is convergent, for here $p = 2 > 1$.

Now, $\lim\limits_{n\to\infty}\dfrac{u_n}{v_n} = \lim\limits_{n\to\infty}\left(\dfrac{1+\frac{1}{n}}{n^2}\times n^2\right) = \lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right) = 1$

Since, $\sum v_n$ is convergent, the given series $\sum u_n$ is also convergent.

Ex. 8. Examine whether the series

$$2x+\frac{3x^2}{8}+\frac{4x^3}{27}+\ldots+\frac{(n+1)x^n}{n^3}+\ldots$$

is convergent or divergent $(x > 0)$.

Since $x > 0$, each term of the given series $\sum u_n$ is positive and

$$\frac{u_{n+1}}{u_n} = \frac{(n+2)}{(n+1)^3}\cdot x^{n+1}\times\frac{n^3}{(n+1)}\cdot\frac{1}{x^n}$$

$$= \frac{n^3(n+2)}{(n+1)^4}\cdot x = \frac{1+\frac{2}{n}}{\left(1+\frac{1}{n}\right)^4}\cdot x \to x \quad \text{as } n\to\infty$$

Therefore, by D'Alembert's Ratio test, the series is convergent, if $x < 1$, and divergent if $x > 1$.

For $x = 1$, this test fails.

When $x = 1$, $u_n = \dfrac{n+1}{n^3}$.

Let us take another series $\sum v_n$, where $v_n = \dfrac{1}{n^2}$.

$\sum v_n$ being a 'p-series' with $p = 2 > 1$, is known to be convergent.

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{n+1}{n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1, \text{ is a finite quantity.}$$

Therefore, $\sum u_n$ is also convergent.

Hence, the given series is convergent if $x \le 1$, and divergent if $x > 1$.

Ex. 9. Examine whether the series is convergent or divergent :

$$x+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1.3}{2.4}\cdot\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7}+\ldots \qquad [\textit{C.P. 1969}]$$

Denoting the given series by $u_0+u_1+u_2+\ldots$, we have

$$u_n=\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}\cdot\frac{x^{2n+1}}{2n+1}$$

$$u_{n+1}=\frac{1.3.5\ldots(2n-1)(2n+1)}{2.4.6\ldots 2n(2n+2)}\cdot\frac{x^{2n+3}}{2n+3}$$

$$\frac{u_n}{u_{n+1}}=\frac{(2n+2)(2n+3)}{(2n+1)(2n+1)}\cdot\frac{1}{x^2}\to\frac{1}{x^2}, \text{ as } n\to\infty.$$

Thus the given series is convergent, if $\frac{1}{x^2}>1$, *i.e.*, if $x^2<1$, *i.e.*, if $-1<x<1$ and it is divergent if $\frac{1}{x^2}<1$, *i.e.*, if $x>1$ or $x<-1$.

When $x=\pm 1$, $\frac{u_n}{u_{n+1}}=\frac{(2n+2)(2n+3)}{(2n+1)^2}$.

$$\therefore \quad \lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\}=\lim_{n\to\infty}\frac{n(6n+5)}{(2n+1)^2}=\frac{3}{2}>1.$$

So, by Raabe's test, the series is convergent for $x=\pm 1$. Hence the given series is convergent, if $-1\le x\le 1$ and is divergent if $x>1$ or $x<-1$.

Ex. 10. Examine the convergence of $\sum_{n=1}^{\infty}\frac{n^n}{n!}$.

[*C.P. 1996, V.P. 2000, B.P. 2001*]

Here, $u_n=\frac{n^n}{n!}$ and $u_{n+1}=\frac{(n+1)^{n+1}}{(n+1)!}$.

$$\frac{u_n}{u_{n+1}}=\frac{n^n}{n!}\times\frac{(n+1)!}{(n+1)^{n+1}}=\frac{n^n}{(n+1)^n}=\frac{1}{\left(1+\frac{1}{n}\right)^n}.$$

$$\therefore \quad \lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\frac{1}{e}<1, \qquad \because\ 2<e<3.$$

$$\therefore \quad \sum_{n=1}^{\infty}\frac{n^n}{n!} \text{ is divergent.}$$

Ex. 11. Examine the convergence of

(i) $1+\frac{3}{2!}+\frac{5}{3!}+\frac{7}{4!}+\ldots$ [*B.P. 1998*]

(ii) $x^2+\frac{2^2}{3.4}x^4+\frac{2^2.4^2}{3.4.5.6}x^6+\frac{2^2.4^2.6^2}{3.4.5.6.7.8}x^8+\ldots\ (x>0)$

[*C. P. 1965, N. B. P. 1981, V. P. 1988*]

(i) Here, $u_n=\frac{(2n-1)}{n!}$, $\quad u_{n+1}=\frac{(2n+1)}{(n+1)!}$

$$\therefore \quad \frac{u_n}{u_{n+1}}=\frac{(n+1)(2n-1)}{(2n+1)}\to\infty, \text{ as } n\to\infty.$$

$\because \quad \lim_{n\to\infty}\frac{u_n}{u_{n+1}}>1$, $\sum u_n$ is convergent.

(ii) Denoting the first term by u_0,

$$u_n=\frac{2^2.4^2.6^2\ldots(2n)^2}{3.4.5\ldots(2n+1)(2n+2)}x^{2n+2}$$

and
$$u_{n+1}=\frac{2^2.4^2.6^2\ldots(2n)^2(2n+2)^2}{3.4.5\ldots(2n+3)(2n+4)}x^{2n+4}$$

so,
$$\frac{u_n}{u_{n+1}}=\frac{(2n+3)(2n+4)}{(2n+2)^2}\cdot\frac{1}{x^2}$$

$$\therefore \quad \lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{\left(1+\frac{3}{2n}\right)\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)^2}\cdot\frac{1}{x^2}=\frac{1}{x^2}.$$

Therefore, the series is convergent $\frac{1}{x^2}>1$, *i.e.*, if $x^2<1$, *i.e.*, if $-1<x<1$ and it is divergent if $\frac{1}{x^2}<1$, *i.e.*, if $x>1$ or $x<-1$.

The ratio test fails when $x=\pm 1$ and we apply Raabe's test.

We have, $n\left(\frac{u_n}{u_{n+1}}-1\right)=n\left\{\frac{(2n+3)(2n+4)}{(2n+2)^2}-1\right\}=\frac{6n^2+8n}{(2n+2)^2}$

$$\therefore \quad \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim_{n\to\infty}\frac{6+\frac{8}{n}}{\left(2+\frac{2}{n}\right)^2}=\frac{3}{2}>1.$$

Hence, the given series is convergent, if $x=\pm 1$.

Thus, the given series is convergent, if $-1 \leq x \leq 1$ and divergent, if $x > 1$ or $x < -1$.

Ex. 12. Determine the region of convergence of the series

(i) $1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \ldots$ [*C.P. 1998*]

(ii) $\frac{x-3}{3} + \frac{1}{2} \cdot \frac{(x-3)^2}{3^2} + \frac{1}{3} \cdot \frac{(x-3)^3}{3^3} + \ldots$

(i) We have, by the Ratio test,

$$\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{x}{n+1} \right|$$

$$\therefore \quad \lim_{x \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = 0 < 1, \text{ for all values of } x.$$

Hence, the series is convergent for every value of x and the region of convergence is $-\infty < x < \infty$.

(ii) Here, $\lim_{x \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{x \to \infty} \left| \frac{(x-3)^{n+1}}{(n+1)3^{n+1}} \div \frac{(x-3)^n}{n \cdot 3^n} \right|$

$$= \left| \frac{x-3}{3} \right| \cdot \lim_{x \to \infty} \frac{n}{n+1} = \left| \frac{x-3}{3} \right|$$

Thus, the series is convergent if $\left| \frac{x-3}{3} \right| < 1$, *i.e.*, $-1 < \frac{x-3}{3} < 1$, *i.e.*, $0 < x < 6$.

The series is also convergent for $x - 3 = -3$, *i.e.*, $x = 0$, but not for $x - 3 = 3$, *i.e.*, $x = 6$.

Hence, the interval of convergence $0 \leq x < 6$.

6.12 Miscellaneous Worked Out Examples

Ex. 1. Examine the series for convergence.

$$\frac{1^2}{2} + \frac{2^2}{2^2} + \frac{3^2}{2^3} + \frac{4^2}{2^4} + \ldots.$$ [*C. P. 1993*]

Solution : Here, $u_n = \frac{n^2}{2^n}$ and $u_{n+1} = \frac{(n+1)^2}{2^{n+1}}$

So, $\frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \frac{1}{2}\left(1 + \frac{1}{n^2}\right)$

$$\therefore \lim_{n\to\infty} \frac{u_n+1}{u_n} = \frac{1}{2} < 1$$

Hence the series $\sum u_n$ where $u_n = \frac{n^2}{2^n}$ is convergent.

Ex. 2. Test the convergence of the series :

$$\frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \ldots\ldots$$ [*C. P. 1994, 2003*]

Solution : Here, $u_n = \frac{x^n}{n^2}$ and $u_{n+1} = \frac{x^{n+1}}{(n+1)^2}$.

$$\therefore \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}.n^2}{x^n.(n+1)^2}\right|$$

$$= \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^2}|x|$$

$$= |x|.$$

Hence $\sum u_n$ converges when $|x| < 1$ and diverges when $|x| > 1$.
When $x = 1$, the ratio test fails

At $x = 1$, $\Sigma u_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots$
which is a *'p'* series with $p = 2 > 1$.

At $x = -1, \Sigma u_n = -\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots..\right)$.

since the series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots.$, being a *p*-series with $p = 2 > 1$, is convergent, by Leibnitz's Test, Σu_n is also convergent.
Hence, the given series is convergent when $|x| \le 1$ and divergent when $|x| > 1$.

Ex. 3. Apply Raab'es Test to examine the convergence of :

(*i*) $1 + \frac{1}{2.3} + \frac{1.3}{2.4.5} + \frac{1.3.5}{2.4.6.7} + \ldots.$ [*C. P. 1990, 93, 94, 2004*]

(ii) Σu_n where $u_n = \dfrac{3.6.9.....3n}{7.10.13....(3n+4)}$ [C. P. 1989]

Solution : (i) Denoting the given series by

$$u_0 + u_1 + u_2 + ...$$

we have $u_n = \dfrac{1.3.5.....(2n-1)}{2.4.6......2n} \cdot \dfrac{1}{2n+1}$

and $u_{n+1} = \dfrac{1.3.5.....(2n+1)}{2.4.6......2n(2n+2)} \cdot \dfrac{1}{2n+3}$.

so that $\dfrac{u_n}{u_{n+1}} = \dfrac{(2n+2)(2n+3)}{(2n+1)^2}$

$$\therefore \lim_{n\to\infty} n\left\{\frac{u_n}{u_{n+1}} - 1\right\} = \lim_{n\to\infty} \frac{n(6n+5)}{2(2n+1)} = \frac{3}{2} > 1.$$

Hence the series is convergent.

(ii) Here, $u_n = \dfrac{3.6.9.....3n}{7.10.13.....(3n+4)}$

$$u_{n+1} = \frac{3.6.9......3n(3n+3)}{7.10.13........(3n+4)(3n+7)}$$

$$\frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3}.$$

By Raabe's Test, $\lim\limits_{n\to\infty} n\left\{\dfrac{u_n}{u_{n+1}} - 1\right\} = \lim\limits_{n\to\infty} \dfrac{4n}{3n+3} = \dfrac{4}{3} > 1$

Hence the series is convergent.

Ex. 4. Use Root-Test to examine the convergence of the following series :

(i) $\dfrac{1}{3} + \left(\dfrac{2}{5}\right)^2 + \left(\dfrac{3}{7}\right)^3 + \cdots$ [C. P. 1986, '92, 2007]

(ii) $\sum u_n$, where $u_n = \left\{\left(\dfrac{n+1}{n}\right)^{n+1} - \left(\dfrac{n+1}{n}\right)\right\}^{-n}$ [C. P. 1990, 2000]

Solution : (*i*) Here, $u_n = \left(\frac{n}{2n+1}\right)^n$, so that

$$(u_n)^{\frac{1}{n}} = \frac{n}{2n+1} = \frac{1}{2+\frac{1}{n}}$$

and $\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \frac{1}{2} < 1.$

Hence the series is convergent by Cauchy's Root-test.

(*ii*) Obviously, here

$$(u_n)^{\frac{1}{n}} = \frac{1}{\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)} = \frac{1}{\left(1+\frac{1}{n}\right)\left\{\left(1+\frac{1}{n}\right)^n - 1\right\}}$$

$$\therefore \lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)\left\{\left(1+\frac{1}{n}\right)^n - 1\right\}} = \frac{1}{e-1} < 1$$

Hence the given series is convergent.

Ex. 5. Examine the convergence of the following series :

(*i*) $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots \qquad (x>0)$ [*B. P. 1997*]

(*ii*) $x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot3}{2\cdot4}\cdot\frac{x^5}{5} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{x^7}{7} + \cdots$ [*C. P. 1994*]

(*iii*) $\frac{a}{b} + \frac{a(a+1)}{b(b+1)} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)} + \cdots \cdots$

where a, b are non-negative integers and $b \neq 0$. [*C. P. 1997*]

(*iv*) $1 + \frac{1+\alpha}{1+\beta} + \frac{(1+\alpha)(2+\alpha)}{(1+\beta)(2+\beta)} + \frac{(1+\alpha)(2+\alpha)(3+\alpha)}{(1+\beta)(2+\beta)(3+\beta)} + \cdots \cdots,$

[*C. P. 1965*]

where $\beta \neq \alpha$.

Solution : (*i*) Here, $u_n = \frac{x^n}{n}$ and $u_{n+1} = \frac{x^{n+1}}{n+1}$

$$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{1}{x} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right) \cdot \frac{1}{x} = \frac{1}{x}$$

Hence the series is convergent if $x<1$, and divergent if $x>1$

If $x=1$, the Ratio-Test is inconclusive.

when $x=1$, the series is $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\cdots$

and it is divergent, by the p-test $\because$ $p=1$ here.

Hence the series is convergent for $0<x<1$ and divergent for $x \geq 1$.

(*ii*) Denoting the given series by $u_0+u_1+u_2+u_3+\cdots\cdots$,

$$\text{we have, } u_n = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots} \cdot \frac{x^{2n+1}}{(2n+1)}$$

$$\text{and, } u_{n+1} = \frac{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}{2\cdot 4\cdot 6\cdots 2n(2n+2)} \cdot \frac{x^{2n+3}}{(2n+3)}$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(2n+2)(2n+3)}{(2n+1)(2n+1)} \cdot \frac{1}{x^2} = \frac{1}{x^2}$$

Thus the given series is convergent if $\frac{1}{x^2}>1$

i.e., if $x^2<1$, *i.e*, if $-1<x<1$ and it is divergent

if $\frac{1}{x^2}<1$, *i.e.*, if $x>1$ or, $x<-1$.

when $x=\pm 1$, the above test fails and we proceed to Raabe's Test

$$\lim_{n\to\infty} n\left\{\frac{u_n}{u_{n+1}}-1\right\} = \lim_{n\to\infty} \frac{n(6n+5)}{(2n+1)^2} = \frac{3}{2}>1$$

So, the series is convergent for $x=\pm 1$.

Hence, the given series is convergent if $-1 \leq x \leq 1$ and is divergent if $x>1$, or, $x<-1$.

(*iii*) Here, $\dfrac{u_n}{u_{n+1}} = \dfrac{b+n}{a+n}$

and $\lim\limits_{n\to\infty} \dfrac{u_n}{u_{n+1}} = 1$

So *D'* Alembert's Ratio Test fails. We proceed to Raabe's Test.

$$\lim_{n\to\infty} n\left\{\frac{u_n}{u_{n+1}}-1\right\} = \lim_{n\to\infty} n\left\{\frac{b+n}{a+n}-1\right\}$$

$$= \lim_{n\to\infty} \frac{n(b-a)}{a+n} = b-a$$

Hence the series is convergent if $(b-a)>1$, divergent if $(b-a)<1$, and when $b=a$, the series becomes

$1+1+1+\cdots\ \cdots$ which is obviously divergent.

(*iv*) Here, $u_n = \dfrac{(1+\alpha)(2+\alpha)\cdots(n+\alpha)}{(1+\beta)(2+\beta)\cdots(n+\beta)}$

and $u_{n+1} = \dfrac{(1+\alpha)(2+\alpha)\cdots(n+\alpha)(n+1+\alpha)}{(1+\beta)(2+\beta)\cdots(n+\beta)(n+1+\beta)}$

that $\dfrac{u_n}{u_{n+1}} = \dfrac{1+n+\beta}{1+n+\alpha}$

$\therefore \quad \lim\limits_{n\to\infty} \dfrac{u_n}{u_{n+1}} = 1$

We proceed to Raabe's Test

$$\lim_{n\to\infty} n\left\{\frac{u_n}{u_{n+1}}-1\right\} = \lim_{n\to\infty} n\left\{\frac{\beta-\alpha}{1+n+\alpha}\right\}$$

$$\lim_{n\to\infty} \frac{n(\beta-\alpha)}{n+1+\alpha} = \lim_{n\to\infty} \frac{\beta-\alpha}{1+\dfrac{1+\alpha}{n}} = \beta-\alpha .$$

Hence the given series if convergent if $(\beta-\alpha)>1$ and

divergent if $(\beta-\alpha)<1$.

EXAMPLES-VI

1. (i) When does an infinite series is said to converge ?

   (ii) Give an example each of (a) a convergent series, (b) a divergent series and (c) an oscillatory series.

   (iii) Show that the addition or removal of a finite number of terms at the beginning of an infinite series will not affect the convergence or divergence of the series.

   (iv) Prove that multiplication of each term of an infinite series by a constant term, different from zero, does not affect the convergence or divergence of the series.

2. (i) Show that the p-series $\sum \frac{1}{n^p}$ converges only when $p > 1$.

   Also show that the harmonic series $\sum \frac{1}{n}$ is divergent.

   [*B. P. 1996,1999*]

   (ii) Show that the series $\frac{1}{\sqrt{1^3}} + \frac{1}{\sqrt{2^3}} + \frac{1}{\sqrt{3^3}} + \ldots$ converges.

   [*C. P. 1987*]

3. (i) State and prove 'comparison test' for convergence or divergence of a series of positive terms.

   (ii) Use comparison test to prove the convergence of

   $$\frac{1}{2^2} + \frac{\sqrt{2}}{3^2} + \frac{\sqrt{3}}{4^2} + \ldots$$ [*C.P. 1988*]

4. (i) State and prove D'Alembert's ratio test for convergence or divergence of a series of positive terms.

   [*C.P. 1982, '87; B.P. 1999*]

   (ii) Hence, show that $1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \ldots$ converges.

   [*C.P. 1992*]

5. (i) State and establish Cauchy's Root test for convergence or divergence of an infinite series. [*C.P. 1980*]

   (ii) Use Cauchy's Root test to examine the convergence of

   $$\frac{1}{3} + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{7}\right)^3 + \left(\frac{4}{9}\right)^4 + \ldots$$ [*C.P. 1986, '92*]

6. State Raabe's test for convergence or divergence of a series of positive terms. [*C.P. 1994*]

7. For the following series, compute the partial sum S_n, and then obtain their sum :

(i) $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$;

(ii) $\sum_{n=1}^{\infty}\frac{1}{(n+3)(n+4)}$. [*C.P. 1999*, 2006]

(iii) $\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}$;

(iv) $\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}$.

8. Use comparison test to examine the convergence or divergence of the series :

(i) $\frac{1}{1.2}+\frac{1}{3.4}+\ldots+\frac{1}{(2n-1)2n}$ [*C.P. 1985*]

(ii) $2+\frac{3}{8}+\frac{4}{27}+\ldots+\frac{n+1}{n^3}+\ldots$ [*C.P. 1997*]

(iii) $\frac{1}{\log 2}+\frac{1}{\log 3}+\frac{1}{\log 4}+\ldots$ [*C.P. 1998*]

(iv) $\frac{1}{2\log 2}+\frac{1}{3\log 3}+\ldots+\frac{1}{n\log n}+\ldots$

(v) $\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}$ [*C.P. 1994*]

(vi) $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n-1}}\right)$ [*C.P. 1998*]

9. Apply D'Alembert's ratio test to examine the convergence of :

(i) $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\ldots+\frac{n}{2^n}+\ldots$ [*C. P. 2006*]

(ii) $\frac{1^2}{2}+\frac{2^2}{2^2}+\frac{3^2}{2^3}+\ldots+\frac{n^2}{2^n}+\ldots$

(iii) $\frac{4}{1!}+\frac{4^2}{2!}+\frac{4^2}{3!}+\ldots+\frac{4^2}{n!}+\ldots$

10. Use Cauchy's root test to investigate the convergence of the series :

(i) $\frac{1+2}{2.1}+\left(\frac{2+2}{2.2}\right)^2+\left(\frac{3+2}{2.3}\right)^3+\ldots+\left(\frac{n+2}{2n}\right)^n+\ldots$

(ii) $\frac{1}{2}+\frac{1}{3^2}+\frac{1}{4^3}+\ldots+\frac{1}{(n+1)^n}+\ldots$

(iii) $1+\frac{2^2}{2^2}+\frac{2^2}{3^3}+\frac{2^4}{4^4}+\ldots+\frac{2^n}{n^n}+\ldots$

(iv) $\sum_{n=1}^{\infty}\left(\frac{n+\sqrt{n}}{2}\right)^n \div \left(n^{n+1}\right)$ [*C.P. 1981*]

11. Examine the following series for convergence or divergence :

(i) $\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}+\ldots$ [*C.P. 1981*]

(ii) $\frac{1.2}{3}+\frac{2.3}{5}+\frac{3.4}{7}+\frac{4.5}{9}+\ldots$

(iii) $\frac{1^2}{2}+\frac{2^2}{2^2}+\frac{3^2}{2^3}+\frac{4^2}{2^4}+\ldots$ [*C.P. 1993*]

(iv) $\frac{1+2}{2^3}+\frac{1+2+3}{3^3}+\frac{1+2+3+4}{4^3}+\ldots$ [*C.P. 1991, 2004*]

(v) $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\ldots$ [*C.P. 1993*]

(vi) $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\ldots+\frac{n}{2^n}+\ldots$ [*C.P. 1996*]

(vii) $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\ldots$ [*C.P. 1986*]

(viii) $\frac{1}{1+1}+\frac{1}{1+2}+\frac{1}{1+2^2}+\frac{1}{1+2^3}+\ldots$ [*C.P. 1991*]

(ix) $\frac{1}{2}+\frac{1.3}{2.4}+\frac{1.3.5}{2.4.6}+\ldots$

(x) $\frac{1.2}{3^2.4^2}+\frac{3.4}{5^2.6^2}+\frac{5.6}{7^2.8^2}+\ldots$

(xi) $1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+\ldots+\frac{n^2}{n!}$ [*C.P. 1991*]

12. Test the convergence or divergence of the of the series $\sum u_n$, where

(i) $u_n = \sqrt{n^3+1} - \sqrt{n^3}$;

(ii) $u_n = \sin\left(\frac{1}{n}\right)$;

(iii) $u_n = \frac{\sqrt{n}}{n^3+1}$;

(iv) $u_n = \frac{x^n}{n!} \quad (x > 0)$; [*C.P. 1987*]

(v) $u_n = \frac{n!}{n^n}$;

(vi) $u_n = \frac{n+1}{n^3} \cdot x^n$;

(vii) $u_n = \frac{2^n}{(n+1)^n}$;

(viii) $u_n = \frac{n^n}{(n+1)^{n+1}}$;

(ix) $u_n = \frac{2.4.6\ldots(2n+2)}{1.3.5\ldots(2n-1)}$;

(x) $u_n = \frac{(\sqrt{2}-1)^n}{n!}$.

13. (i) Show that the series

$$\frac{1}{1+1^2} + \frac{1}{1+2^2} + \frac{1}{1+3^2} + \cdots \text{ converges.}$$

(ii) Show that the series

$$1 + \frac{1}{2.1^2+1} + \frac{1}{2.2^2+1} + \frac{1}{2.3^2+1} + \cdots \text{ is convergent.}$$

[*B.P. 1984, '94*]

14. Apply Cauchy's general principle of convergence to test the following series :

(i) $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$

(ii) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+(-1)^{n-1}\cdot\frac{1}{n}+\ldots$

15. Test the convergence of $\sum_{n=0}^{\infty}\frac{1}{n!}$ by Cauchy's criterion.

16. Show that the following series are divergent :

(i) $\sum_{n=1}^{\infty}\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}$;

(ii) $\sum_{n=1}^{\infty}\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}\cdot\frac{4n+3}{2n+2}$;

(iii) $\sum_{n=1}^{\infty}\frac{n^n}{n!}$;

(iv) $\sum\left\{\sqrt{n+1}-\sqrt{n}\right\}$.

17. Prove that the hypergeometric series

$$1+\frac{\alpha.\beta}{1.\gamma}+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}+\ldots$$

(where α,β,γ are real and none of them is zero or negative integer) converges if $\gamma>(\alpha+\beta)$ and diverges if $\gamma\le(\alpha+\beta)$. [*C.P. 1989*]

18. Discuss the convergence or divergence of the hypergeometric series

$$1+\frac{\alpha.\beta}{1.\gamma}\cdot x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}\cdot x^2+\ldots$$

(where are real and none of them is zero or negative integer) for $|x|<1$, $|x|>1$ and for $x=1$.

19. Show that the series $\sum_{n=1}^{\infty}\frac{n^2-1}{n^2+1}\cdot x^n$ converges for $|x|<1$.

[*C.P. 1999*]

20. Find the interval of convergence of the following series :

(i) $1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$ [*C.P. 1987*]

(ii) $\frac{x}{1^2}+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\ldots$ [*C.P. 1994*]

(iii) $x+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1.3}{2.4}\cdot\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7}+\ldots \quad (x>0)$

(iv) $1-(x-1)+\frac{(x-1)^2}{2}-\frac{(x-1)^3}{3}+\ldots$

(v) $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$

(vi) $1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots$

(vii) $x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots$

ANSWERS

1. (ii) (a) $\sum\frac{1}{n^2}$; (b) $\sum\frac{1}{n}$; (c) $\sum(-1)^n$.

5. (ii) Convergent.

7. (i) 2; (ii) $\frac{1}{4}$; (iii) $\frac{1}{2}$; (iv) $\frac{1}{4}$.

8. (i) Convergent; (ii) Convergent; (iii) Divergent; (iv) Divergent; (v) Convergent; (vi) Divergent.

9. (i) Convergent; (ii) Convergent; (iii) Convergent.

10. (i) Convergent; (ii) Convergent; (iii) Convergent; (iv) Convergent;

11. (i) Convergent; (ii) Divergent; (iii) Convergent; (iv) Divergent; (v) Divergent; (vi) Convergent; (vii) Convergent; (viii) Convergent; (ix) Divergent; (x) Convergent; (xi) Convergent.

12. (i) Convergent; (ii) Divergent; (iii) Convergent; (iv) Convergent; (v) Convergent; (vi) Convergent when $x \le 1$, Divergent when $x > 1$; (vii) Convergent; (viii) Divergent; (ix) Divergent; (x) Convergent.

14. (i) Divergent; (ii) Convergent.

20. (i) $-\infty < x < +\infty$; (ii) $|x| \le 1$; (iii) $-1 \le x \le 1$; (iv) $0 < x \le 2$; (v) $-1 < x \le 1$; (vi) $-\infty < x < +\infty$; (vii) $-\infty < x < +\infty$.

7 DIFFERENTIATION

7.1. Increment.

The increment of a variable in changing from one value to another is the *difference* obtained by subtracting the first value from the second. An increment of x is denoted by Δx (read as delta x) or h. Evidently, increment may be positive or negative according as the variable in changing increases or decreases.

If, in $y = f(x)$, the independent variable x takes an increment Δx (or h), then Δy (or k) denotes the corresponding increment of y, *i.e.*, of $f(x)$, and we have

$$y + \Delta y = f(x + \Delta x), \ i.e., \ \Delta y = f(x + \Delta x) - f(x)$$

or, $$y + k = f(x + h), \ i.e., \ k = f(x + h) - f(x)$$

Illustration : Let $y = x^2$

Suppose.	x increases from 2 to 2·1, i.e., $\Delta x = 0{\cdot}1$;
then	y increases from 4 to 4·41, i.e., $\Delta y = 0{\cdot}41$.
Suppose.	x decreases from 2 to 1·9, i.e., $\Delta x = -0{\cdot}1$;
then	y decreases from 4 to 3·61, i.e., $\Delta y = -0{\cdot}39$.

Increments are always reckoned from the arbitrarily fixed initial value of the independent variable x.

If y decreases as x increases, or the reverse, then Δx and Δy will have opposite signs.

From a fixed initial value 2 of x, if x increases successively to 2·1, 2·01, 2·001, etc. then although the corresponding increments Δx ($= 0{\cdot}1, 0{\cdot}01, 0{\cdot}001, \ldots$) and Δy ($= 0{\cdot}41, 0{\cdot}401, 0{\cdot}004001, \ldots$) are getting smaller, their ratio, i.e., $\frac{\Delta y}{\Delta x}$, being 4·1, 4·01, 4·001, . . . , is approaching a definite number 4, thus, illustrating the fact that the ratio can be brought as near to 4 as we please by making Δx approach zero. Thus, the ratio of the increments $\frac{\Delta y}{\Delta x}$ has a definite finite limit 4 as $\Delta x \to 0$, and, consequently, $\Delta y \to 0$.

7.2. Differential Coefficient (or Derivative).

Let $y = f(x)$ be a finite and single-valued function defined in any interval of x and assume x to have any particular value in the interval. Let Δx (or h) be the increment of x, and let $\Delta y = f(x+\Delta x) - f(x)$ be the corresponding increment of y. If the ratio $\frac{\Delta y}{\Delta x}$ of these increments tends to a definite finite limit as Δx tends to zero, then this limit is called the *differential coefficient* (or *derivative*) of $f(x)$ (or y) for the particular value of x, and is denoted by $f'(x)$, $\frac{d}{dx}\{f(x)\}$, $\frac{dy}{dx}$, $D\{f(x)\}$.

Thus, symbolically, *the differential coefficient of $y=f(x)$ with respect to x (for any particular value of x) is*

$$f'(x) \text{ or, } \frac{dy}{dx} = \underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{Lt} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$f'(x) \text{ or, } \frac{dy}{dx} = \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h}, \text{ provided this limit exists.}$$

If, as $\Delta x \to 0$, $\Delta y/\Delta x \to +\infty$ or $-\infty$, then also we say that the derivative exists, and $\to +\infty$ or $-\infty$.

Note 1. The process of finding the differential coefficient is called *differentiation*, and we are said to *differentiate* $f(x)$ and sometimes to *differentiate* $f(x)$ *with respect to* x, to emphasise that x is the independent variable.

Note 2. $\frac{dy}{dx}$ stands here for the symbol $\frac{d}{dx}(y)$, a limiting process, and hence must not be regarded as a fraction dy divided by dx, although, for convenience of printing, It may sometimes be written as $\frac{dy}{dx}$.

Note 3. The differential coefficient of $f(x)$, for any particular value a of x, is often denoted by $f'(a)$. Thus, from definition,

$$f'(a) = \underset{h \to 0}{Lt} \frac{f(a+h) - f(a)}{h}, \text{ provided this limit exists.}$$

Note 4. If $f'(a)$ is finite, $f(x)$ must be continuous at $x = a$.

$$f'(a) = \underset{h \to 0}{Lt} \frac{f(a+h) - f(a)}{h},$$

We can write $f(a+h)-f(a)=\dfrac{f(a+h)-f(a)}{h}\times h$

$$\therefore \quad \underset{h\to 0}{Lt}\{f(a+h)-f(a)\} = \underset{h\to 0}{Lt}\left\{\frac{f(a+h)-f(a)}{h}\times h\right\}$$

$$= \underset{h\to 0}{Lt}\frac{f(a+h)-f(a)}{h}\times \underset{h\to 0}{Lt}\, h$$

$$= f'(a)\times 0$$

$$= 0\text{, since } f'(a) \text{ is finite.}$$

$$\therefore \quad \underset{h\to 0}{Lt}\, f(a+h) = f(a).$$

$\therefore$ from the definition of continuity, it follows that $f(x)$ is continuous at $x=a$.

Hence, *for the differential coefficient of $f(x)$ to exist finitely for any value of x, the function $f(x)$ must be continuous at the point .*

The converse, however, is not always true, i.e., if a function be continuous at any point, it is not necessarily true that a finite derivative of the function for that value of x should exist. For illustration see Ex. 4 , § 7.5.

Again, a function $f(x)$, though *discontinuous* at a point, may have an *infinite derivative* át a point. [See Ex. 7(ii), Examples VII(A).]

Note 5. The right-hand limit $\underset{h\to 0+0}{Lt}\dfrac{f(x+h)-f(x)}{h}$ for any particular value of x, when it exists, is called the *right-hand derivative of $f(x)$* at that point and is denoted by $Rf'(x)$. Similarly, the left-hand limit $\underset{h\to 0-0}{Lt}\dfrac{f(x+h)-f(x)}{h}$ or, $\underset{h\to 0+0}{Lt}\dfrac{f(x-h)-f(x)}{-h}$, when it exists is called the *left-hand derivative of $f(x)$* at x, denoted by $Lf'(x)$. When these two derivatives both exist and are equal, it is then only that the derivative of $f(x)$ exists at x. When, however, the left-hand and right-hand derivatives of $f(x)$ at x are unequal, or one or both are non-existent then $f(x)$ is said to have no proper derivative at x.

Thus, though $f'(x)$ may not exist at a point, one or both of the right-hand and left-hand derivatives may exist (the two being unequal in the latter case).

For illustration, see § 7.5, Ex. 4.

7.3. Differential coefficients in some standard cases.

(i) Differential coefficient of x^n.

Let $f(x) = x^n$.

Then from definition, $f'(x) = \underset{h \to 0}{Lt} \dfrac{(x+h)^n - x^n}{h}$.

Now, writing X for $x + h$, so that $h = X - x$, and noting that when $h \to 0$, $X \to x$, we get

$$f'(x) = \underset{X \to x}{Lt} \frac{X^n - x^n}{X - x} = nx^{n-1}, \text{ for all rational values of } n.$$

[*See* § 3.9 (v)]

Thus, $\dfrac{d}{dx}\left(x^n\right) = nx^{n-1}$, for all rational values of n.

Otherwise :

$$f'(x) = \underset{h \to 0}{Lt} \frac{(x+h)^n - x^n}{h} = \underset{h \to 0}{Lt}\, x^{n-1} . \frac{(1+h/x)^n - 1}{h/x}$$

[supposing $x \neq 0$]

$$= x^{n-1} . \frac{(1+z)^n - 1}{z} \qquad [\text{ putting } z = h/x\ ]$$

$$= nx^{n-1} \qquad [\textit{ See } \S\, 3.9\, (vi)\ ]$$

The result can also be derived for any rational value of n [$\neq 0$] from the well-known *inequality*.[1]

$$nX^{n-1}(X - x) \gtrless X^n - x^n \gtrless nx^{n-1}(X - x)$$

[upper sign if $n > 1$ or > 0 and lower if $0 < n < 1$]

Whence $nX^{n-1}(X - x) \gtrless \dfrac{X^n - x^n}{X - x} \gtrless nx^{n-1}$

Now putting $X = x + h$, and letting $h \to 0$, we get

$$\underset{h \to 0}{Lt} \frac{(x+h)^n - x^n}{h} = nx^{n-1} \qquad [\textit{ See } \S\, 3.8\, (v)\ ],$$

since both extremes tend to the same limit nx^{n-1}.

When n is a positive integer, the result can also be proved as follows:

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[1] See any text book on Higher Algebra (*e.g.*, See § 10, Chapter XIV, *Barnard & Child*).

$$f'(x) = \underset{h \to 0}{Lt} \frac{(x+h)^n - x^n}{h}$$

$$= \underset{h \to 0}{Lt} \frac{\left\{x^n + nx^{n-1}h + \frac{1}{2}n(n-1)x^{n-1}h^2 + \ldots + h^n\right\} - x^n}{h}$$

(By Binomial Theorem)

$$= \underset{h \to 0}{Lt} \frac{\left\{nx^{n-1}h + \frac{1}{2}n(n-1)x^{n-1}h^2 + \ldots + h^n\right\}}{h}$$

$$= nx^{n-1}.$$

When n is not a positive integer, for an alternative proof, see Ex. 1, § 7.13, See also Ex. 2, § 7.13 for the case when n has any real value, not necessarily rational.

Cor. $\frac{d}{dx}(x) = 1, \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}, \frac{d}{dx}\left(\frac{1}{x^n}\right) = -\frac{n}{x^{n+1}}.$

Note. It is to be noted that in the above formula we tacitly assume those values of x as do not make x^n or x^{n-1} meaningless ; *e.g., if n be a fraction of even denominator, zero and negative values of x are excluded and if $n - 1$ be negative, zero value for x is excluded.*

Following the definition it may be seen in particular, that if $f(x) = x^n$, then $f'(0) = 0$, when $n > 1$, $f'(0) = 1$, when $n = 1$, and $f'(0)$ is non-existent if $n < 1$.

(ii) Differential coefficient of e^x.

Let $f(x) = e^x$. Then from definition,

$$f'(x) = \underset{h \to 0}{Lt} \frac{e^{x+h} - e^x}{h} = \underset{h \to 0}{Lt}\, e^x \cdot \frac{e^h - 1}{h} = e^x$$

since $= \underset{h \to 0}{Lt} \left(e^{h-1}\right)/h = 1$ [See § 3.9 (*iv*).]

Thus, $\frac{d}{dx}(e^x) = e^x$.

(iii) Differential coefficient of a^x.

Let $f(x) = a^x$

Then $$f'(x)= \underset{h\to 0}{Lt}\ \frac{a^{x+h}-a^{x}}{h}=a^{x}\cdot\frac{a^{h}-1}{h}$$

Now, $$\underset{h\to 0}{Lt}\ \frac{a^{h}-1}{h}=\underset{h\to 0}{Lt}\ \frac{e^{h\log a}-1}{h\log a}.\log a$$

$$=\underset{h'\to 0}{Lt}\ \frac{a^{h'}-1}{h'}.\log a\ \left[\text{where } h'=h\log a\right]$$

$$=\log a,\quad \text{since}\quad \underset{h'\to 0}{Lt}\ \frac{a^{h'}-1}{h'}=1.$$

[*See* § 3.9 (iv)].

$\therefore\ f'(x)=a^{x}\log a$.

Thus, $$\frac{d}{dx}\left(a^{x}\right)=a^{x}\log_{e}a.$$

(iv) Differential coefficient of **log x**.

Let $f'(x)=\log x$

Then $$f'(x)=\underset{h\to 0}{Lt}\ \frac{\log(x+h)-\log x}{h}$$

$$=\underset{h\to 0}{Lt}\ \frac{1}{h}\cdot\log\frac{x+h}{x}$$

$$=\underset{h\to 0}{Lt}\ \frac{1}{x}\cdot\frac{x}{h}\log\left(1+\frac{h}{x}\right)$$

$$=\frac{1}{x}\ \underset{z\to 0}{Lt}\ \frac{1}{z}\log(1+z)\left[\text{where}\quad z=\frac{h}{x}\right]$$

$$=\frac{1}{x}.$$ [*See* § 3.9 (iii)]

Thus, $$\frac{d}{dx}(\log x)=\frac{1}{x}.$$

Cor. Proceeding exactly as above it can be easily shown that

$$\frac{d}{dx}\left(\log_{a}x\right)=\frac{1}{x}\log_{a}e.$$

(v) Differential coefficient of sin x.

Let $f(x) = \sin x$.

Then
$$f'(x) = \underset{h\to 0}{Lt} \frac{\sin(x+h) - \sin x}{h}$$
$$= \underset{h\to 0}{Lt} \frac{2\sin\frac{1}{2}h\cos\left(x+\frac{1}{2}h\right)}{h}$$
$$= \underset{h\to 0}{Lt} \left\{\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}\cdot\cos\left(x+\frac{1}{2}h\right)\right\} = \cos x$$

because as $h \to 0$, $\cos x$ being a continuous function of x,
$$\cos\left(x+\tfrac{1}{2}h\right) \to \cos x;$$
also, by § *3.9 (i)*,
$$\underset{h\to 0}{Lt} \left\{\sin\tfrac{1}{2}h \,/\, \left(\tfrac{1}{2}h\right)\right\} = 1.$$

Thus,
$$\frac{d}{dx}(\sin x) = \cos x.$$

(vi) Differential coefficient of cos x.

Let $f(x) = \cos x$
$$f'(x) = \underset{h\to 0}{Lt} \frac{\cos(x+h) - \cos x}{h}$$
$$= \underset{h\to 0}{Lt} \frac{-2\sin\frac{1}{2}h\,\sin\left(x+\frac{1}{2}h\right)}{h}$$
$$= \underset{h\to 0}{Lt} \left\{-\sin\left(x+\frac{1}{2}h\right)\cdot\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}\right\}$$
$$= -\sin x \quad [\textit{as in } (v)]$$

Thus,
$$\frac{d}{dx}(\cos x) = -\sin x.$$

Note. It should be noted that in finding the above differential coefficients of sin x and cos x, we tacitly assume that x is in radian measure, because we make use of the limit $\sin\frac{1}{2}h \,/\, \left(\frac{1}{2}h\right) = 1$ as $h \to 0$, which is true when h is in radian. Hence, the above results require modification when x is given in any other measure.

(vii) Differential coefficient of tan x.

Let $f(x) = \tan x$

Then $f'(x) = \underset{h\to 0}{Lt} \dfrac{\tan(x+h) - \tan x}{h}$

$$= \underset{h\to 0}{Lt} \frac{1}{h}\left\{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}\right\}$$

$$= \underset{h\to 0}{Lt} \left\{\frac{\sin(x+h-x)}{h\cos(x+h)\cos x}\right\}$$

$$= \underset{h\to 0}{Lt} \left\{\frac{\sin h}{h} \cdot \frac{1}{\cos(x+h)\cos x}\right\}$$

$$= \frac{1}{\cos^2 x}, \quad \left[x \neq \tfrac{1}{2}(2n+1)\pi\right]$$

$\because \quad \underset{h\to 0}{Lt} \dfrac{\sin h}{h} = 1$, and $\underset{h\to 0}{Lt} \cos(x+h) = \cos x$

Thus, $\dfrac{d}{dx}(\tan x) = \sec^2 x. \quad \left[x \neq \tfrac{1}{2}(2n+1)\pi\right]$

(viii) Exactly in a similar way, we can get

$$\frac{d}{dx}(\cot x) = -\mathrm{cosec}^2 x. \quad [x \neq n\pi]$$

(ix) Differential coefficient of sec x.

$$\frac{d}{dx}(\sec x) = \underset{h\to 0}{Lt} \frac{\sec(x+h) - \sec x}{h}$$

$$= \underset{h\to 0}{Lt} \frac{1}{h}\left\{\frac{1}{\cos(x+h)} - \frac{1}{\cos x}\right\}$$

$$= \underset{h\to 0}{Lt} \left\{\frac{\cos x - \cos(x+h)}{h\cos(x+h)\cos x}\right\}$$

$$= \underset{h\to 0}{Lt} \frac{2\sin\frac{1}{2}h\sin\left(x+\frac{1}{2}h\right)}{h\cos(x+h)\cos x}$$

$$= \underset{h\to 0}{Lt} \left\{\frac{\sin\frac{1}{2}h}{\frac{1}{2}h} \sin\left(x+\frac{1}{2}h\right)\frac{1}{\cos(x+h)\cos x}\right\}$$

Then $\frac{d}{dx}(\sec x) = 1 \cdot \sin x \cdot \frac{1}{\cos^2 x} = \tan x \sec x,$

since $\underset{h \to 0}{Lt} \frac{\sin \frac{1}{2} h}{\frac{1}{2} h} = 1, \quad \underset{h \to 0}{Lt} \sin\left(x + \frac{1}{2} h\right) = \sin x$

and $\underset{h \to 0}{Lt} \cos(x + h) = \cos x.$

Thus, $\frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{\sec x}) = \mathbf{\sec x \tan x}. \qquad \left[x \neq \frac{1}{2}(2n+1)\pi\right]$

(x) Proceeding exactly in a similar way, we get

$$\frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{cosec\ x}) = -\mathbf{cosec\ x\ cot\ x}. \quad [x \neq n\pi]$$

Note. For an alternative method of differentiating tan x, cot x, sec x and cosec x from a knowledge of the derivatives of sin x and cos x, see § 7·4, Theorem V.

7.4. Fundamental Theorems on Differentiation.

In the following theorems we assume that $\phi(x)$ and $\psi(x)$ are continuous, and $\phi'(x)$ and $\psi'(x)$ exist.

Theorem I. *The differential coefficient of a constant is zero.*

i.e., $\frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{c}) = \mathbf{0}$, where c is a constant

Let $f(x) = c$ for every value of x.

Then
$$f'(x) = \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h}$$
$$= \underset{h \to 0}{Lt} \frac{c - c}{h} = \underset{h \to 0}{Lt} \frac{0}{h} = 0.$$

Theorem II. *The differential coefficient of the product of a constant and a function is the product of the constant and the differential coefficient of the function,*

i.e., $\frac{\mathbf{d}}{\mathbf{dx}}\{\mathbf{c}\varphi(\mathbf{x})\} = \mathbf{c}\frac{\mathbf{d}}{\mathbf{dx}}\varphi(\mathbf{x})$, where c is a constant.

For, $\frac{d}{dx}\{c\phi(x)\} = \underset{h \to 0}{Lt} \frac{c\phi(x+h) - c\phi(x)}{h}$

$$= c \cdot \underset{h \to 0}{Lt} \frac{\phi(x+h) - \phi(x)}{h} = c\phi'(x).$$

Theorem III. *The differential coefficient of the sum or difference of two functions is the sum or difference of their derivatives,*

i.e., $\frac{d}{dx}\{\varphi(x) \pm \psi(x)\} = \varphi'(x) \pm \psi'(x)$.

Let $\quad f(x) = \phi(x) + \psi(x)$

Then $\quad f(x+h) = \phi(x+h) + \psi(x+h)$

Now $f'(x) = \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h}$

$$= \underset{h \to 0}{Lt} \frac{\{\phi(x+h) + \psi(x+h)\} - \{\phi(x) + \psi(x)\}}{h}$$

$$= \underset{h \to 0}{Lt} \left\{\frac{\phi(x+h) - \phi(x)}{h} + \frac{\psi(x+h) - \psi(x)}{h}\right\}$$

$$= \underset{h \to 0}{Lt} \frac{\phi(x+h) - \phi(x)}{h} + \underset{h \to 0}{Lt} \frac{\psi(x+h) - \psi(x)}{h}$$

$$= \phi'(x) + \psi'(x)$$

Similarly, if $f(x) = \phi(x) - \psi(x)$ then $f'(x) = \phi'(x) - \psi'(x)$

Note. The above result can be easily generalized to the case of the *sum or difference of any finite number of functions.*

Illustration :

If $f(x) = e^x - 4\sin x + x^2 + 5$, then $f'(x) = e^x - 4\cos x + 2x$.

Theorem IV. *The differential coefficient of the product of two functions = first function × derivative of the second + second function × derivative of the first.*

i.e., $\frac{d}{dx}\{\varphi(x) \times \psi(x)\} = \varphi(x).\psi'(x) + \psi(x).\varphi'(x)$.

Let $\quad f(x) = \phi(x) \times \psi(x)$

Then $f(x+h) = \phi(x+h) \times \psi(x+h)$

$\therefore \quad f'(x) = \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h}$

$$= \underset{h \to 0}{Lt} \frac{\phi(x+h)\psi(x+h) - \phi(x)\psi(x)}{h}.$$

$$= \underset{h \to 0}{Lt} \frac{\phi(x+h)\psi(x+h) - \phi(x+h)\psi(x) + \phi(x+h)\psi(x) - \phi(x)\psi(x)}{h}$$

$$= \underset{h \to 0}{Lt} \left\{ \phi(x+h) \cdot \frac{\psi(x+h) - \psi(x)}{h} + \psi(x) \cdot \frac{\phi(x+h) - \phi(x)}{h} \right\}$$

$$= \phi(x).\psi'(x) + \psi(x).\phi'(x),$$

by the limit theorems, and the definitions of $\psi'(x)$ and $\phi'(x)$, noting also that $\underset{h \to 0}{Lt}\, \phi(x+h) = \phi(x)$, since $\phi(x)$ is continuous for $\phi'(x)$ to exist.

Note. This result, by repeated application, may also be easily generalized for the product of a finite number of functions in the form

$$\frac{d}{dx}\{\varphi_1(x).\, \varphi_2(x)\,\varphi_3(x) \ldots \varphi_n(x)\}$$
$$= \varphi_1'(x).\{\varphi_2(x)\varphi_3(x) \ldots\} + \varphi_2'(x) \cdot \{\varphi_1(x)\varphi_3(x) \cdots\}$$
$$+ \cdots\cdots + \varphi_n'(x) \cdot \{\varphi_1(x)\varphi_2(x) \cdots\}.$$

Illustration : If $f(x) = e^x \sin x$, then $f(x) = e^x . \cos x + \sin x . e^x$

If $f(x) = x^3 \tan x \log x$, then

$$f'(x) = 3x^2 . \tan x \log x + \sec^2 x . x^3 \log x + \frac{1}{x} \cdot x^3 \tan x$$
$$= x^2 \left(3 \tan x \log x + x \sec^2 x \log x + \tan x\right)$$

Theorem V. *The differential coefficient of the quotient of two functions*

$$= \frac{(\textit{Diff. Coeff. of num}) \times \textit{denom} - (\textit{Diff. Coeff. of denom}) \times \textit{num}}{\textit{Square of denom}}$$

i.e., $\frac{d}{dx}\left\{\frac{\varphi(x)}{\psi(x)}\right\} = \frac{\varphi'(x).\psi(x) - \psi'(x).\varphi(x)}{\{\psi(x)\}^2}$.

provided $\psi(x) \neq 0$

Let $\quad f(x) = \frac{\phi(x)}{\psi(x)}$

Then $f'(x) = \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h}$

$$= \underset{h \to 0}{Lt} \frac{1}{h} \left\{ \frac{\phi(x+h)}{\psi(x+h)} - \frac{\phi(x)}{\psi(x)} \right\}$$

$$= \underset{h\to 0}{Lt}\ \frac{1}{h}\cdot\frac{\phi(x+h)\,\psi(x)-\psi(x+h)\,\phi(x)}{\psi(x+h)\,\psi(x)}$$

$$= \underset{h\to 0}{Lt}\ \frac{\psi(x)\{\phi(x+h)-\phi(x)\}-\phi(x)\{\psi(x+h)-\psi(x)\}}{h\psi(x+h)\,\psi(x)}$$

$$= \underset{h\to 0}{Lt}\ \frac{1}{\psi(x+h)\,\psi(x)}\cdot\left\{\psi(x)\cdot\frac{\phi(x+h)-\phi(x)}{h}-\phi(x)\frac{\psi(x+h)-\psi(x)}{h}\right\}$$

$$= \frac{1}{\{\psi(x)\}^2}\{\psi(x).\phi'(x)-\phi(x).\psi'(x)\},$$

by the limit theorems, and the definitions of $\phi'(x)$ and $\psi'(x)$, and noting also that $\underset{h\to 0}{Lt}\ \psi(x+h)=\psi(x)$ since $\psi(x)$ is continuous for $\psi'(x)$ to exist.

Illustration : (1) If $f(x)=\dfrac{\sin x}{x^2}$, then $f'(x)=\dfrac{x^2\cdot\cos x-2x\sin x}{x^4}$,

(2) If $f(x)=\cot x=\dfrac{\cos x}{\sin x}$,

then $f'(x)=\dfrac{(-\sin x).\sin x-\cos x.\cos x}{\sin^2 x}=-\dfrac{1}{\sin^2 x}$

$=-\text{cosec}^2\, x.$

(3) If $f(x)=\text{cosec}\, x=\dfrac{1}{\sin x}$,

then $f'(x)=\dfrac{0.\sin x-\cos x.1}{\sin^2 x}=-\dfrac{\cos x}{\sin^2 x}=-\text{cosec}\, x\cot x.$

7.5. Illustrative Examples.

Ex. 1. Find, from the first principle, the derivative of $\sqrt{x}$ $(x>0)$

Let $f(x)=\sqrt{x}$.

$\therefore\ f(x)=\underset{h\to 0}{Lt}\ \dfrac{\sqrt{(x+h)}-\sqrt{(x)}}{h}$ [*by definition*]

$$= \underset{h\to 0}{Lt}\ \frac{(x+h)-x}{h\left\{\sqrt{(x+h)}+\sqrt{x}\right\}}=\underset{h\to 0}{Lt}\ \frac{1}{\sqrt{(x+h)}+\sqrt{x}}=\frac{1}{2\sqrt{x}}.$$

Ex. 2. Find, from the first principles, the differential coefficient of $\tan^{-1} x$.

Let $\tan^{-1} x = y$ and $\tan^{-1}(x+h) = y+k$.

Then, as $h \to 0,\ k \to 0$ Also, $x = \tan y,\ x+h = \tan(y+k)$.

$h = (x+h) - x = \tan(y+k) - \tan y$.

$$\frac{d}{dx}\tan^{-1} x = \underset{h\to 0}{Lt}\ \frac{\tan^{-1}(x+h) - \tan^{-1} x}{h}$$

$$= \underset{k\to 0}{Lt}\ \frac{k}{\tan(y+k) - \tan y}$$

$$= \underset{k\to 0}{Lt}\ \frac{k}{\sin k}.\cos(y+k)\cos y$$

$$= \cos^2 y = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

Note. In a similar way, we can work out the derivatives of other inverse circular functions from first principle. These have, however, been worked out by a different method in § 7.8.

Ex. 3. *Find, from definition, the differential coefficient of* $\log \cos x$.

Let us put $\cos x = u$, $\cos(x+h) = u+k$

$\therefore\quad k = \cos(x+h) - \cos x$ and so, when $h \to 0,\ k \to 0$

$$\therefore \frac{d}{dx}(\log \cos x) = \underset{h\to 0}{Lt}\ \frac{\log\cos(x+h) - \log\cos x}{h}$$

$$= \underset{k\to 0}{Lt}\ \frac{\log(u+k) - \log u}{k}.\frac{k}{h}$$

$$= \underset{k\to 0}{Lt}\ \frac{\log(1+k/u)}{k/u}.\frac{1}{u}.\frac{k}{h}$$

As $k \to 0,\ k/u \to 0\ \therefore$ limit of 1st factor $= 1$. [See § *3.9 (iii)*]

$$\text{Again, } \frac{k}{h} = \underset{h\to 0}{Lt}\ \frac{\cos(x+h) - \cos x}{h} = -\frac{\sin\left(x+\frac{1}{2}h\right).\sin\frac{1}{2}h}{\frac{1}{2}h}$$

as $k \to 0$, *i. e.*, $h \to 0$, $k/u \to -\sin x$ Also, $u = \cos x$.

$$\frac{d}{dx}(\log \cos x) = \frac{-\sin x}{\cos x} = -\tan x.$$

Note. Differentiation from 'first principle' or 'definition' means that we are to find out the derivative without assuming any of the rules of differentiation, or the derivative of any standard function, but we are permitted to use fundamental rules of limiting operations (§ *3.8*) and the standard limit results (§ *3.9*).

Ex. 4. *A function is defined in the following way :*

$f(x)=|x|$, *i.e.*, $f(x)=x$, 0, or, $-x$, *according as* $x >, =,$ *or*, <0 ; *show that* $f'(0)$ *does not exist.* [*V. P. 2000*]

$$f'(0)=\underset{h\to 0}{Lt}\frac{f(0+h)-f(0)}{h}=\underset{h\to 0}{Lt}\frac{f(h)}{h},$$

Now. $$=\underset{h\to 0+0}{Lt}\frac{f(h)}{h}=\underset{h\to 0+0}{Lt}\frac{h}{h}=1.$$

and $$=\underset{h\to 0-0}{Lt}\frac{f(h)}{h}=\underset{h\to 0-0}{Lt}\frac{-h}{h}=-1.$$

Since the right-hand derivative is not equal to the left-hand derivative, the derivative at $x=0$ does not exist.

Ex. 5. *A function is defined in the following way :*

$$f(x)=x\sin\frac{1}{x} \text{ for } x\neq 0,\ f(0)=0.$$

Show that $f'(0)$ *does not exist.*

$$f'(0)=\underset{h\to 0}{Lt}\frac{f(0+h)-f(0)}{h}=\underset{h\to 0}{Lt}\frac{h\sin(1/h)}{h},$$

$$=\underset{h\to 0}{Lt}\sin\frac{1}{h},$$ which does not exist. [See § *3.11*, Ex. 4]

$\therefore$ $f'(0)$ does not exist.

Note. In both the Examples 4 and 5, $f(x)$ is continuous at $x=0$ (See § *4.6*, *Ex. 1 and Ex. 2*) but $f(x)$ does not possess derivative at $x=0$.

Ex. 6. *If* $f(x)=x^2\sin\left(\frac{1}{x}\right)$ *when* $x\neq 0$, *and* $f(0)=0$, *find* $f'(0)$.

$$f'(0) = \underset{h\to 0}{Lt} \frac{f(0+h)-f(0)}{h} = \underset{h\to 0}{Lt} \frac{1}{h}\left(h^2 \sin\frac{1}{h} - 0\right)$$

$$= \underset{h\to 0}{Lt} \left(h \sin\frac{1}{h} - 0\right) = 0.$$

[$\because$ when h is not exactly zero, $\sin\frac{1}{h}$ is finite, not exceeding 1 numerically.]

Ex. 7. *Find, from first principles, the derivative of* x^x $(x > 0)$.

Let $f(x) = x^x = e^{x \log x}$.

$$\therefore \quad f'(x) = \underset{h\to 0}{Lt} \frac{e^{(x+h)\log(x+h)} - e^{x\log x}}{h}$$

$$= \underset{h\to 0}{Lt}\, e^{x\log x} \cdot \frac{e^{(x+h)\log(x+h) - x\log x} - 1}{h}$$

$$= e^{x\log x} \cdot \underset{h\to 0}{Lt} \frac{e^z - 1}{z} \cdot \frac{z}{h},$$

where $z = (x+h)\log(x+h) - x\log x$

and hence $z \to 0$, as $h \to 0$,

$$\therefore \; f'(x) = x^x \cdot \underset{z\to 0}{Lt} \frac{e^z - 1}{z} \cdot \underset{h\to 0}{Lt} \frac{z}{h} = x^x \underset{h\to 0}{Lt} \frac{z}{h}, \quad \because \underset{z\to 0}{Lt} \frac{e^z - 1}{z} = 1$$

Now $$\underset{h\to 0}{Lt} \frac{z}{h} = \underset{h\to 0}{Lt} \frac{x\{\log(x+h) - \log x\} + h\log(x+h)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{x}{h} \log\left(1 + \frac{h}{x}\right) + \underset{h\to 0}{Lt} \log(x+h)$$

$$= \underset{h\to 0}{Lt} \frac{1}{k} \log(1+k) + \log x = 1 + \log x.$$

where k being $h/x \to 0$, as $h \to 0$.

$$\therefore \quad f'(x) = x^x(1 + \log x).$$

EXAMPLES-VII(A)

Find, from *first principles*, the derivatives of (*Ex. 1 - 5*):

1. (i) $x^3 + 2x$. (ii) $x^4 + 6$. (iv) $1/x\ (x \neq 0)$
(iv) $1/\sqrt{x}\ (x > 0)$. (v) $\sqrt[3]{x}$. (vi) $\sqrt{x^2 + a^2}$.
(vii) $x + \sqrt{x^2 + 1}$.

2. (i) $e^{\sqrt{x}}$. (ii) $e^{\sin x}$. (iii) 2^{x^2}. (iv) e^x/x.

3. (i) $\log_{10} x$. [*C. P.1941*] (ii) $x \log x$.
(iii) $\log \sin (x/a)$. [*C. P.1930*] (iv) $\log \sec x$.

4. (i) $a \sin (x/a)$. [*C. P.1937*] (ii) $\sin^2 x$.
(iii) $\sin x^2$. (iv) $\sin^{-1} x$. (v) $\sqrt{\tan x}$.
(vi) $(\sin x)/x$. (vii) $x^2 \tan x$.

5. (i) $e^{\cos x}$ at $x = 0$. (ii) $\log \cos x$ at $x = 0$.

6. (i) $f(x) = x^2 \cos (1/x)$ for $x \neq 0$; $f(0) = 0$.
Find $f'(0)$.
(ii) $f(x) = x$ for $0 \leq x \leq \frac{1}{2}$; $f(x) = 1 - x$ for $\frac{1}{2} < x \leq 1$.
Does $f'\left(\frac{1}{2}\right)$ exist ?

7. (i) $f(x) = 3 + 2x$ for $-\frac{3}{2} < x \leq 0$,
$= 3 - 2x$ for $0 < x < \frac{3}{2}$.
Show that $f(x)$ is continuous at $x = 0$ but does not exist.
[*C. P. 1943*]

(ii) $f(x) = 0$ when $0 \leq x < \frac{1}{2}$; $f\left(\frac{1}{2}\right) = 1$, $f(x) = 2$ when $\frac{1}{2} < x \leq 1$

Prove that although $f(x)$ is discontinuous at $x = \frac{1}{2}$, $f'\left(\frac{1}{2}\right)$ exists and its value is infinite.

8.
$$
\begin{aligned}
f(x) &= 1 && \text{for } x < 0, \\
&= 1 + \sin x && \text{for } 0 \leq x < \tfrac{1}{2}\pi, \\
&= 2 + \left(x - \tfrac{1}{2}\pi\right)^2 && \text{for } \tfrac{1}{2}\pi \leq x ;
\end{aligned}
$$

show that $f'(x)$ exists at $x = \frac{1}{2}\pi$ but does not exist at $x = 0$.

9. $$f(x) = 5x - 4 \quad \text{for} \quad 0 < x \le 1,$$
$$= 4x^2 - 3x \quad \text{for} \quad 1 < x < 2,$$
$$= 3x + 4 \quad \text{for} \quad x \ge 2.$$

Discuss the continuity of $f(x)$ for $x = 1$ and 2, and the existence of $f'(x)$ for these values.

10. (i) $$f(x) = x \quad \text{for} \quad 0 < x < 1,$$
$$= 2 - x \quad \text{for} \quad 1 \le x \le 2,$$
$$= x - \tfrac{1}{2}x^2 \quad \text{for} \quad x > 2.$$

Is $f(x)$ continuous at $x = 1$ and 2? Does $f'(x)$ exist for these values ?

(ii) $$\phi(x) = \tfrac{1}{2}(b^2 - a^2), \quad \text{for} \quad 0 \le x \le a$$
$$= \tfrac{1}{2}b^2 - \tfrac{1}{6}x^2 - \tfrac{1}{3}(a^3/x), \quad \text{for} \quad a < x \le b,$$
$$= \tfrac{1}{3}(b^3 - a^3)/x, \quad \text{for} \quad x > b$$

Show that $\phi'(x)$ is continuous for every positive value of x.

[C. P. 1944]

Find the differential coefficients of the following with respect to x (*Ex. 11-13*) :

11. (i) $3x^5 + 7x^4 - 2x^2 - x + 6$. (ii) $(x^2 - 3)^3$.

(iii) $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!}$. (iv) $(x + 2)(x + 1)^2$.

(v) $(3x^6 + 4x^2 - 2)/x^3$. (v) $(1 + x)^3/x$.

(vii) $6x^{-2} - 3x^{-1} + 4$. (viii) $4x^{-\frac{3}{4}} - 6x^{\frac{1}{2}} + 2$.

(ix) $2x^4 + 5x^2 - 4 - \dfrac{1}{x} + \dfrac{2}{x^2} + \dfrac{3}{x^3}$.

(x) $\sqrt{x} + 2\sqrt{(x^2)} + 3\sqrt{(x^3)} + 4\sqrt{(x^4)} + 5\sqrt{(x^5)}$.

(xi) $x\sqrt{x} + x^2\sqrt{x} + \dfrac{x^2}{\sqrt{x}} - \sqrt{x} + \dfrac{1}{\sqrt{x}}$.

(xii) $\dfrac{5x^3}{\sqrt[5]{x^2}} - \dfrac{3x}{\sqrt[3]{x^4}} + \dfrac{7x}{\sqrt[7]{x^2}} + 12\dfrac{\sqrt[4]{x}}{\sqrt[3]{x}}$.

(xiii) $2\sin x - \frac{1}{2}\log x - \frac{1}{5}e^x - 6\tan x - 7\operatorname{cosec} x$.

(xiv) $\log_a x + \log x^a + e^{\log x} + \log e^x + e^{1+x}$.

12. (i) $x^n e^x$. (ii) $x^2 \log x$. (iii) $x^2 \log x^2$.

(iv) $e^x \sin x$. (v) $2^x \sin x$. (vi) $10^x x^{10}$.

(vii) $\cos^2 x$. (viii) $\sec x \tan x$. (ix) $\left(x^2 + 1\right)\sin x$.

(x) $(3x - 7)(3 - 7x)$. (xi) $\left(x^2 + 7\right)\left(x^3 + 10\right)$.

(xii) $(\sin x + \sec x + \tan x)(\operatorname{cosec} x + \cos x + \cot x)$.

(xiii) $\operatorname{cosec}^3 x$. (xiv) $x \tan x \log x$.

(xv) $\sqrt{x}\,.\,e^x \sec x$. (xvi) $(1 + x)(1 + 2x)(1 + 3x)$.

(xvii) $x(1 - x)\left(1 - x^2\right)$. (xviii) $x \sec x \log\left(x e^x\right)$.

(xix) $x.\cot x.\log\left(x^x\right).e^x$.

(xx) [illegible] $\log x \times 10^x \times \sqrt{x}$.

13. (i) $\dfrac{\sin x}{\cos x}$. (ii) $\dfrac{1}{\cos x}$. (iii) $\dfrac{\sin x}{x}$

(iv) $\dfrac{x^4}{\sin x}$. [*C. P. 1940*] (v) $\dfrac{\cot x}{e^x}$.

(vi) $\dfrac{x^n}{\log x}$. (vii) $\dfrac{x}{e^x - 1}$. (viii) $\dfrac{1 + x}{1 - x}$.

(ix) $\dfrac{1 + x^2}{1 - x^2}$. (x) $\dfrac{1 + \sqrt{x}}{1 - \sqrt{x}}$. (xi) $\dfrac{1 + \sin x}{1 - \sin x}$.

(xii) $\dfrac{1 - \cos x}{1 + \cos x}$. (xiii) $\dfrac{\sin x + \cos x}{\sqrt{1 + \sin 2x}}$;

(xiv) $\dfrac{\cos x - \cos 2x}{1 - \cos x}$. (xv) $x \cdot \dfrac{e^x + e^{3x}}{e^x + e^{-x}}$.

(xvi) $\frac{\tan x}{x} \cdot \log\left(\frac{e^x}{x^x}\right)$.

(xvii) $\frac{\sin x + \cos x}{\sin x - \cos x}$.

(xviii) $\frac{\cot x + \operatorname{cosec} x}{\cot x - \operatorname{cosec} x}$.

(xix) $\frac{1 + x + x^2}{1 - x + x^2}$.

(xx) $\frac{x^3 - 2 + x^{-3}}{x - 2 + x^{-1}}$.

(xxi) $\frac{\tan x}{x} \cdot e^x \cdot \log x$.

(xxii) $\frac{\sin x - \cos x}{\sin x + \cos x} \cdot x^2 e^x$.

14. If $y = \sqrt{2x} - \sqrt{\frac{2}{x}} + \frac{x+4}{4-x}$, find $\frac{dy}{dx}$ for $x = 2$.

15. If $f(x) = \frac{x^3 - 8x^2 + 13x - 6}{x^2 - 11x + 10}$, find the values of x for which $f'(x) = 0$.

Is there any value of x for which $f'(x)$ is non-existent ?

16. From the relation

$$1 + x + x^2 + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

deduce the sum of the series $1 + 2x + 3x^2 + \ldots + nx^{n-1}$
and hence, show that

$$1 + 2x + 3x^2 + \ldots \text{ to } \infty = (1 - x)^{-2}; \quad 0 < |x| < 1.$$

17. If
$$f(x) = 1 + x \quad \text{for } x < 0$$
$$= 1 \quad \text{for } 0 \le x \le 1$$
$$= 2x^2 + 4x + 5 \quad \text{for } x > 1,$$

find $f'(x)$ for all values of x for which it exists.

Does $\underset{x \to 0}{Lt} f'(x)$ exist ?

18. (i) If $f(x) = -\frac{1}{2}x^2$ for $x \le 0$

and $f(x) = x^n \sin(1/x)$ for $x > 0$,

find whether $f'(0)$ exists for $n = 1$ and 2.

(ii) If $f(x) = [x]$ where $[x]$ denotes the greatest integer not exceeding x, find $f'(x)$ and draw its graph.

ANSWERS

1. (i) $3x^2 + 2$. (ii) $4x^3$. (iii) $-1/x^2$. (iv) $-\frac{1}{2}x^{-\frac{3}{2}}$.

(v) $\frac{1}{3}x^{-\frac{2}{3}}$ (vi) $\frac{x}{\sqrt{(x^2+a^2)}}$. (vii) $\frac{\{x+\sqrt{(x^2+1)}\}}{\sqrt{(x^2+1)}}$.

2. (i) $e^{\sqrt{x}}/2\sqrt{x}$. (ii) $e^{\sin x}.\cos x$. (iii) $2^{x^2}.\log_e 2.2x$

(iv) $(xe^x - e^x)/x^2$

3. (i) $x^{-1}.\log_{10} e$. (ii) $1 + \log x$. (iii) $a^{-1}\cot(x/a)$. (iv) $\tan x$

4. (i) $\cos(x/a)$ (ii) $\sin 2x$. (iii) $2x\cos x^2$.

(iv) $1/\sqrt{(1-x^2)}$ (v) $\frac{\sec^2 x}{2\sqrt{(\tan x)}}$. (vi) $\frac{x\cos x - \sin x}{x^2}$.

(vii) $2x\tan x + x^2\sec^2 x$.

5. (i) 0. (ii) 0.

6. (i) 0. (ii) No.

9. Continuous for $x = 1$ and 2, but $f'(x)$ exists for $x = 1$ and does not exist for $x = 2$.

10. (i) Continuous at $x = 1$ and 2; $f'(x)$ does not exist for x = 1, but exists at $x = 2$.

11. (i) $15x^4 + 28x^3 - 4x - 1$. (ii) $6x^5 - 36x^3 + 54x$. (iii) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$.

(iv) $3x^2 + 8x + 5$ (v) $9x^2 - 4x^{-2} - 6x^{-4}$

(vi) $-x^{-2} + 3 + 2x$. (vii) $-12x^{-3} + 3x^{-2}$.

(viii) $-3x^{-\frac{7}{4}} + 3x^{-\frac{1}{2}}$. (ix) $8x^3 + 10x + x^{-2} - 4x^{-3} - 9x^{-4}$.

(x) $\frac{1}{2\sqrt{x}} + 2 + \frac{9}{2}x^{\frac{1}{2}} + 8x + \frac{25}{2}x^{\frac{3}{2}}$.

(xi) $\frac{3}{2}x^{\frac{1}{2}} + \frac{5}{2}x^{\frac{3}{2}} + \frac{3}{2}x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$.

(xii) $13x^{\frac{8}{5}} + x^{-\frac{4}{3}} + 5x^{-\frac{2}{7}} - x^{-\frac{13}{12}}$.

(xiii) $2\cos x - \frac{1}{5}x^{-1} - \frac{1}{2}e^{x} - 6\sec^2 x + 7\operatorname{cosec} x \cot x$.

(xiv) $x^{-1}\log_a e + ax^{-1} + 2 + e^{1+x}$.

(xv) $e^x \sec x(1+2x+2x\tan x)/2\sqrt{x}$

(xvi) $18x^2 + 22x + 6$ (xvii) $4x^3 - 3x^2 - 2x + 1$.

(xviii) $\sec x\{1 + x + (1 + x\tan x)(x + \log x)\}$.

(xix) $e^x\left\{x\cot x(1 + 2\log x + x\log x) - x^2\operatorname{cosec}^2 x\log x\right\}$.

(xx) $10^x \cot x\left[\left(-2\sqrt{x}\operatorname{cosec} 2x + \sqrt{x}\log 10 + \frac{1}{2}x^{-\frac{1}{2}}\right)\log x + x^{-\frac{1}{2}}\right]$.

13. (i) $\sec^2 x$. (ii) $\sec x \tan x$. (iii) $(x\cos x - \sin x)/x^2$.

(iv) $x^2(4\sin x - x\cos x)/\sin^2 x$. (v) $-e^{-x}\left(\operatorname{cosec}^2 x + \cot x\right)$

(vi) $x^{n-1}(n\log x - 1)/(\log x)^2$. (vii) $\dfrac{e^x(1-x)-1}{\left(e^x-1\right)^2}$.

(viii) $\dfrac{2}{(1-x)^2}$. (ix) $\dfrac{4x}{\left(1-x^2\right)^2}$ (x) $\dfrac{1}{\sqrt{x}\left(1-\sqrt{x}\right)^2}$

(xi) $\dfrac{2\cos x}{(1-\sin x)^2}$ (xii) $\dfrac{2\sin x}{(1+\cos x)^2}$ (xiii) 0.

(xiv) $-2\sin x$. (xv) $e^{2x}(1+2x)$.

(xvi) $-x^{-1}\tan x + (1-\log x)\sec^2 x$. (xvii) $\dfrac{-2}{(\sin x - \cos x)^2}$.

(xviii) $\dfrac{2\operatorname{cosec} x}{(\cot x - \operatorname{cosec} x)^2}$. (xix) $\dfrac{2\left(1-x^2\right)}{\left(1-x+x^2\right)^2}$

(xx) $2\left(x + 1 - x^{-2} - x^{-3}\right)$.

(xxi) $\dfrac{e^x}{x^2}\left[\left\{x\sec^2 x + (x-1)\tan x\right\}\log x + \tan x\right]$.

(xxii) $e^x\left\{\dfrac{2x^2 - \left(x^2+2x\right)\cos 2x}{(\sin x + \cos x)^2}\right\}$.

14. $2\frac{3}{4}$. **15.** 4, 16 ; non-existent at 1, 10.

16. $\dfrac{1-(n+1)x^{n}+nx^{n+1}}{(1-x)^2}$.

17. 1 if $x<0$, 0 if $0<x<1$, $4(x+1)$ if $x>1$; No.

18. (i) No, Yes.

(ii) $f'(x)=0$ for all values of x except zero and integral values, for which it does not exist.

7.6. Differentiation of a Function of a Function.

Let $y=f(v)$, where $v=\phi(x)$, and $f(v)$ and $\phi(x)$ are continuous. Thus y is also a continuous function of x.*

Let $f'(v)$ and $\phi'(x)$ exist, and be finite.

Assume $v+\Delta v=\phi(x+\Delta x)$ and $y+\Delta y=f(v+\Delta v)$.

It is evident that when $\Delta x\to 0$, $\Delta v\to 0$, and as $\Delta v\to 0$, $\Delta y\to 0$.

Now, $\dfrac{\Delta y}{\Delta x}=\dfrac{\Delta y}{\Delta v}\cdot\dfrac{\Delta v}{\Delta x}$. $\quad[\Delta v\neq 0]$

$\therefore \quad \underset{\Delta x\to 0}{Lt}\dfrac{\Delta y}{\Delta x}=\underset{\Delta v\to 0}{Lt}\dfrac{\Delta y}{\Delta v}\cdot\underset{\Delta x\to 0}{Lt}\dfrac{\Delta v}{\Delta x}$. $\quad[\Delta v\neq 0]$

i.e., $$\frac{dy}{dx}=\frac{dy}{dv}\cdot\frac{dv}{dx}.$$

If $\Delta v=0$, then $\Delta y=0$. [Otherwise $\dfrac{dy}{dv}$, *i.e.*, $f'(v)$ would not be finite.] $\quad\therefore\ \Delta y/\Delta x=0 \quad [\because\ \Delta x\neq 0]$

Hence, $\dfrac{dy}{dx}=\underset{\Delta x\to 0}{Lt}\dfrac{\Delta y}{\Delta x}=0$. Similarly, $\dfrac{dv}{dx}=0$.

Hence, the above relation is true in this case also.

Illustration : Suppose $\quad y=\sin x^2$;

then we can write $\quad y=\sin v$, $\quad$ where $v=x^2$.

$\therefore \quad \dfrac{dy}{dx}=\dfrac{dy}{dv}\cdot\dfrac{dv}{dx}=\cos v.2x=2x\cos x^2$.

The above rule can easily be generalized.

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* Proof depends on the corresponding limit theorem, see § 3.8 (iv).

Thus, if $y = f(v)$, where $v = \phi(w)$, and $w = \psi(x)$.

then $\dfrac{dy}{dx} = \dfrac{dy}{dv} \cdot \dfrac{dv}{dw} \cdot \dfrac{dw}{dx}$; and so on.

7.7. $\dfrac{\mathbf{dy}}{\mathbf{dx}} \times \dfrac{\mathbf{dx}}{\mathbf{dy}} = 1$, *i.e.,* $\dfrac{\mathbf{dy}}{\mathbf{dx}} = 1 \Big/ \dfrac{\mathbf{dx}}{\mathbf{dy}}$, **provided neither derivative is zero.**

Suppose $y = f(x)$, where $f(x)$ is continuous. From this, in most cases, we can treat x as a function of y.

Let $y + \Delta y = f(x + \Delta x)$.

It is evident that when $\Delta y \to 0$, $\Delta x \to 0$.

Now, $\dfrac{\Delta y}{\Delta x} \times \dfrac{\Delta x}{\Delta y} = 1$. $\quad \therefore \dfrac{\Delta y}{\Delta x} = 1 \Big/ \dfrac{\Delta x}{\Delta y}$.

$$\therefore \quad \underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x} = \underset{\Delta y \to 0}{Lt} \left(1 \Big/ \frac{\Delta x}{\Delta y} \right),$$

i.e., $\dfrac{dy}{dx} = 1 \Big/ \dfrac{dx}{dy}$, or, $\dfrac{dy}{dx} \times \dfrac{dx}{dy} = 1$.

7.8. Differential Coefficients of Inverse Circular Functions.

(i) Let $y = \sin^{-1} x$. $\quad [|x| \le 1]$* $\quad \therefore \quad x = \sin y$.

$$\therefore \quad \frac{dx}{dy} = \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}.$$

$$\therefore \quad \text{for } x \neq 1, \text{ or, } -1, \quad \frac{dy}{dx} = 1 \Big/ \frac{dx}{dy} = \frac{1}{\sqrt{1 - x^2}}.$$

Thus, $\dfrac{\mathbf{d}}{\mathbf{dx}} \left(\sin^{-1} \mathbf{x} \right) = \dfrac{\mathbf{1}}{\sqrt{\mathbf{1} - \mathbf{x}^2}}$, $\quad [-1 < x < 1]$

(ii) Let $y = \cos^{-1} x$. $\quad [|x| \le 1]$* $\quad \therefore \quad x = \cos y$.

$$\therefore \quad \frac{dx}{dy} = -\sin y = -\sqrt{1 - \cos^2 y} = -\sqrt{1 - x^2}.$$

$$\therefore \text{ for } x \neq 1, \text{ or, } -1, \quad \frac{dy}{dx} = 1 \Big/ \frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}.$$

---

* The domain for which y exists.

Thus, $\frac{d}{dx}\left(\cos^{-1}x\right) = -\frac{1}{\sqrt{1-x^2}}$, $[-1 < x < 1]$

Note. This also follows immediately from the relation

$$\cos^{-1}x = \tfrac{1}{2}\pi - \sin^{-1}x.$$

(iii) Let $y = \tan^{-1}x$. $\therefore$ $x = \tan y$.

$\therefore$ $\frac{dx}{dy} = \sec^2 y = 1 + \tan^2 y = 1 + x^2$.

$\therefore$ $\frac{dy}{dx} = 1\Big/\frac{dx}{dy} = \frac{1}{1+x^2}$.

Thus, $\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}$.

(iv) Let $y = \cot^{-1}x$. $\therefore$ $x = \cot y$.

$\therefore$ $\frac{dx}{dy} = -\operatorname{cosec}^2 y = -\left(1 + \cot^2 y\right) = -\left(1 + x^2\right)$.

$\therefore$ $\frac{dy}{dx} = 1\Big/\frac{dx}{dy} = -\frac{1}{1+x^2}$.

Thus, $\frac{d}{dx}\left(\cot^{-1}x\right) = -\frac{1}{1+x^2}$.

Note. This also follows immediately from the relation

$$\cot^{-1}x = \tfrac{1}{2}\pi - \tan^{-1}x.$$

(v) Let $y = \sec^{-1}x$. $[|x| \geq 1]$* $\therefore$ $x = \sec y$.

$\therefore$ $\frac{dx}{dy} = \sec y \tan y = \sec y\sqrt{\sec^2 y - 1} = x\sqrt{x^2 - 1}$.

$\therefore$ for $x \neq 1$, or, -1, $\frac{dy}{dx} = 1\Big/\frac{dx}{dy} = \frac{1}{x\sqrt{x^2-1}}$.

Thus, $\frac{d}{dx}\left(\sec^{-1}x\right) = \frac{1}{x\sqrt{x^2-1}}$, $[|x| > 1]$

(vi) Let $y = \operatorname{cosec}^{-1}x$. $[|x| \geq 1]$* $\therefore$ $x = \operatorname{cosec} y$.

* For which y exists.

$$\therefore \quad \frac{dx}{dy} = -\operatorname{cosec} y \cot y = -\operatorname{cosec} y \sqrt{\operatorname{cosec}^2 y - 1} = -x\sqrt{x^2 - 1}$$

$$\therefore \quad \text{for } x \neq 1, \text{ or, } -1, \quad \frac{dy}{dx} = 1 \Big/ \frac{dx}{dy} = -\frac{1}{x\sqrt{x^2 - 1}}.$$

Thus, $$\frac{d}{dx}\left(\operatorname{cosec}^{-1} x\right) = -\frac{1}{x\sqrt{x^2 - 1}}, \quad [|x| > 1].$$

Note. This also follows immediately from the relation

$$\operatorname{cosec}^{-1} x = \tfrac{1}{2}\pi - \sec^{-1} x.$$

7.9. Derivatives of Hyperbolic Functions*.

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x.$$

$$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x.$$

$$\frac{d}{dx}(\tanh x) = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$$

$$= \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x.$$

Similarly, $$\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$$

$$\frac{d}{dx}(\operatorname{sech} x) = \frac{d}{dx}\left(\frac{1}{\cosh x}\right) = \frac{0 \times \cosh x - \sinh x}{\cosh^2 x}$$

$$= -\frac{\sinh x}{\cosh x}. = -\operatorname{sech} x \tanh x$$

Similarly, $$\frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x$$

Let $y = \sinh^{-1} x$. $\quad \therefore \quad x = \sinh y$.

$$\therefore \quad \frac{dx}{dy} = \cosh y = \sqrt{1 + \sinh^2 x} = \sqrt{1 + x^2}.$$

* For the definitions and properties of Hyperbolic Functions, see Authors' *Higher Trigonometry, Chapter XII.*

Since $\quad \dfrac{dy}{dx} = 1\Big/\dfrac{dx}{dy} = \dfrac{1}{\sqrt{1+x^2}}.$

Thus, $\quad \dfrac{d}{dx}\left(\sinh^{-1} x\right) = \dfrac{1}{\sqrt{1+x^2}}.$

Similarly, $\dfrac{d}{dx}\left(\cosh^{-1} x\right) = \dfrac{1}{\sqrt{x^2-1}}, \qquad (x>1)$

$$\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2}, \qquad (x<1)$$

$$\frac{d}{dx}\left(\coth^{-1} x\right) = -\frac{1}{x^2-1}, \qquad (x>1)$$

$$\frac{d}{dx}\left(\operatorname{cosech}^{-1} x\right) = -\frac{1}{x\sqrt{x^2+1}},$$

$$\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = -\frac{1}{x\sqrt{1-x^2}} \quad (x<1)$$

The derivatives of inverse hyperbolic functions can also be obtained by differentiating their values, *viz.*,

$\sinh^{-1} x = \log\left(x+\sqrt{x^2+1}\right)$; $\qquad \cosh^{-1} x = \log\left(x+\sqrt{x^2-1}\right)$

$\tanh^{-1} x = \dfrac{1}{2}\log\dfrac{1+x}{1-x}$; $\qquad \coth^{-1} x = \dfrac{1}{2}\log\dfrac{x+1}{x-1}$;

$\operatorname{cosech}^{-1} x = \log\dfrac{1+\sqrt{1+x^2}}{x}$; $\qquad \operatorname{sech}^{-1} x = \log\dfrac{1+\sqrt{1-x^2}}{x}$

7.10. Logarithmic Differentiation.

If we have a function raised to a power which is also a funtion, or if we have the product of a number of functions, to differentiate such expressions it would be convenient first to take logarithm of the expression and then differentiate. Such a process is called the *logarithmic differentiation.*

(i) Let $y = \{f(x)\}^{\phi(x)}$; to find $\dfrac{dy}{dx}$.

Here, $\log y = \phi(x).\log f(x)$.

Differentiating both sides with respect to x,

$$\frac{1}{y}\frac{dy}{dx}=\phi(x)\cdot\frac{1}{f(x)}f'(x)+\phi'(x).\log f(x).$$

$$\therefore\ \frac{dy}{dx}=\{f(x)\}^{\phi(x)}\left\{\phi(x).\frac{f'(x)}{f(x)}+\phi'(x).\log f(x)\right\}$$

(ii) Let $y=f_1(x)\times f_2(x).....f_n(x)$; to find $\dfrac{dy}{dx}$.

Here, $\log y=\log f_1(x)+\log f_2(x)+.....+\log f_n(x)$.

Differentiate each side with respect to x.

$$\frac{1}{y}\frac{dy}{dx}=\frac{f_1'(x)}{f_1(x)}+\frac{f_2'(x)}{f_2(x)}+.....+\frac{f_n'(x)}{f_n(x)}.$$

Now, multiplying left-side by y and right-side by $f_1(x).f_2(x).....f_n(x)$

$$\frac{dy}{dx}=f_1'(x).f_2(x).f_3(x)...f_n(x)$$
$$+f_2'(x).f_1(x).f_3(x)...f_n(x)+.....$$

Hence, *the differential coefficient of the product of a finite number of functions is found by multiplying the differential coefficient of each function taken separately by the product of all the remaining functions and adding up the results thus formed, as already obtained otherwise.*

[See § 7.4, Theorem IV, Note.]

7.11. Implicit Functions.

In many cases it may be inconvienient or even impossible to solve a given equation of the form $f(x, y)=0$ for y in terms of x. However, the equation may define y as a function of x. In such cases, y is said to be an implicit function of x. If y be a differentiable function of x, then $\dfrac{dy}{dx}$ may be obtained as follows :

Differentiate each term of the equation with respect to x, regarding y as an unknown function of x having a derivative $\dfrac{dy}{dx}$, and then solve the resulting equation for $\dfrac{dy}{dx}$.

Illustration : Find $\frac{dy}{dx}$, if $x^3 - xy^2 + 3y^2 + 2 = 0$.

Differentiating each term with respect to x.

$$3x^2 + \left(-x.2y\frac{dy}{dx} - y^2.1\right) + 6y\frac{dy}{dx} = 0.$$

$$\therefore\ (6y - 2xy)\frac{dy}{dx} = y^2 - 3x^2, \qquad \therefore\ \frac{dy}{dx} = \frac{y^2 - 3x^2}{6y - 2xy}.$$

7.12. Parametric Equations.

Sometimes in the equation of a curve, x and y are expressed in terms of a third variable known as a *parameter*.

In such cases, to find $\frac{dy}{dx}$ it is not essential to eliminate the parameter and express y in terms of x. We may proceed as follows :

Let $x = \phi(t),\ y = \psi(t)$.

Then x may be regarded as a function of t and also y is a function of t.

$$\text{Now, } \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} \qquad \left(\frac{dx}{dt} \neq 0\right).$$

[By § 7.6 and 7.7]

For illustration see § *7.13*, *Ex.* 6.

7.13. Illustrative Examples.

Ex. 1. (*a*) *Show that* $\frac{d}{dx}x^n = nx^{n-1}$ *when* n *is a positive integer, by the product rule.* [Note, Theorem. IV, Art. 7.4]

Let $f(x) = x^n = x.x.\ldots.x$ (n factors)

$\therefore\ f'(x) = x.x\ldots$to $(n-1)$ factors $+ x.x\ldots$to $(n-1)$ factors

$+ x.x\ldots$to $(n-1)$ factors $+\ldots\ldots$to n terms

$= nx^{n-1}$.

(*b*) *Assuming that* $\frac{d}{dx}x^n = nx^{n-1}$ *when* n *is a positive integer, show that the same result is true when* n *is a negative integer, or a rational fraction, positive or negative.*

When n is a negative integer, suppose $n = -m$, where m is a positive integer.

$$\therefore \quad \frac{d}{dx}x^n = \frac{d}{dx}x^{-m}$$

$$= \frac{d}{dx}\frac{1}{x^m} = \frac{0 \times x^m - mx^{m-1} \times 1}{x^{2m}} = -mx^{-m-1} = nx^{n-1}.$$

Next, let us suppose n is a rational fraction, positive or negative and let $n = p/q$, where q is a positive integer and p any integer positive or negative.

Then $y = x^n = x^{p/q}$. Let $z = x^{1/q}$; then $x = z^q$ and $y = z^p$.

Thus, $\frac{dy}{dx} = \frac{dy}{dz} \Big/ \frac{dx}{dz} = \frac{p}{q} z^{p-q} = nx^{(p-q)/q} = nx^{p/q-1} = nx^{n-1}$.

Ex. 2. *Assuming that* $\frac{d}{dx}\{e^{\phi(x)}\} = e^{\phi(x)}\phi'(x)$ *for all real values of* x, *deduce that* $\frac{d}{dx}(x^n) = nx^{n-1}$ *for all real values of* x.

Let $\quad y = x^n = e^{n \log x}$,

Then $\frac{dy}{dx} = \frac{d}{dx}\left(e^{n\log x}\right) = e^{n\log x}.\frac{d}{dx}(n\log x) = x^n.n.\frac{1}{x} = nx^{n-1}$.

Ex. 3. *Find the differential coefficient of* $\sin^2(\log \sec x)$.

Let $y = \{\sin(\log \sec x)\}^2$

$= u^2$, where $u = \sin(\log \sec x) = \sin v$, where $v = \log \sec x$

$= \log w$, where $w = \sec x$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dw}\cdot\frac{dw}{dx} = 2u.\cos v\frac{1}{w}.\sec x \tan x$$

$$= 2\sin(\log \sec x)\cos(\log \sec x).\tan x$$

$$= \sin(2\log \sec x).\tan x.$$

Ex. 4. *Differentiate* $(\sec x)^{\tan x}$.

Let $y = (\sec x)^{\tan x}$. $\therefore$ $\log y = \tan x . \log \sec x$.

Differentiating both sides with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = \tan x . \frac{1}{\sec x} . \sec x \tan x + \sec^2 x \log \sec x$$

$$= \tan^2 x + \sec^2 x \log \sec x$$

$$\therefore \frac{dy}{dx} = (\sec x)^{\tan x}\left(\tan^2 x + \sec^2 x \log \sec x\right)$$

Note. Writing the given function as $e^{\tan x . \log \sec x}$, we may proceed to differenciate it.

Ex. 5. *Find* $\frac{dy}{dx}$, *if* $y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)}}$.

Taking logarithm of both sides,

$$\log y = \tfrac{1}{2}\{\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4)\}.$$

Differentiating both sides with respect to x.

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{(x-1)} + \frac{1}{(x-2)} - \frac{1}{(x-3)} - \frac{1}{(x-4)}\right)$$

$$= -\frac{2x^2 - 10x + 11}{(x-1)(x-2)(x-3)(x-4)}.$$

$$\therefore \quad \frac{dy}{dx} = -\frac{2x^2 - 10x + 11}{(x-1)^{\frac{1}{2}}(x-2)^{\frac{1}{2}}(x-3)^{\frac{3}{2}}(x-4)^{\frac{3}{2}}}.$$

Ex. 6. Find $\frac{dy}{dx}$, if $x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$.

$$\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = -\frac{a\sin\theta}{a(1-\cos\theta)} = -\frac{2\sin\frac{1}{2}\theta \cos\frac{1}{2}\theta}{2\sin^2\frac{1}{2}\theta} = -\cot\tfrac{1}{2}\theta.$$

Ex. 7. *Find* $\frac{dy}{dx}$, *if* $y = \tan^{-1}\frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{\sqrt{1+\sin x} + \sqrt{1-\sin x}}$.

On rationalising the denominator, $y = \tan^{-1}\frac{1-\cos x}{\sin x} = \tan^{-1}\frac{2\sin^2\frac{1}{2}x}{2\sin\frac{1}{2}x \cos\frac{1}{2}x}$

$$= \tan^{-1}\tan\tfrac{1}{2}x = \tfrac{1}{2}x. \qquad \therefore \quad \frac{dy}{dx} = \tfrac{1}{2}.$$

Note. Sometimes an algebraical or trigonometrical transformation as shown in this example considerably shortens the work. The next example also illustrates the same method.

Ex. 8. *If* $y = \tan^{-1} \dfrac{\sqrt{1+x^2}-1}{x}$, *find* $\dfrac{dy}{dx}$.

Putting $x = \tan\theta$, $\dfrac{\sqrt{1+x^2}-1}{x} = \dfrac{\sec\theta - 1}{\tan\theta} = \dfrac{1-\cos\theta}{\sin\theta}$

$$= \frac{2\sin^2 \frac{1}{2}\theta}{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta} = \tan\tfrac{1}{2}\theta$$

Hence, $y = \tan^{-1}\tan\frac{1}{2}\theta = \frac{1}{2}\theta = \frac{1}{2}\tan^{-1} x$.

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{1+x^2}.$$

Ex. 9. *Differentiate* $\sin x$ *with respect to* x^2.

Let $y = \sin x, \; z = x^2$.

$$\therefore \quad \frac{dy}{dz} = \frac{dy}{dx}\cdot\frac{dx}{dz} = \frac{dy}{dx}\bigg/\frac{dz}{dx} = \frac{\cos x}{2x}.$$

Note. This is an example of the differential coefficient of a function of x with respect to another function of x.

Ex. 10. *If* $\sin y = x\sin(a+y)$, *prove that*

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}.$$

From the given relation, we have

$$x = \frac{\sin y}{\sin(a+y)}. \qquad \ldots \quad (1)$$

Hence x is a funciton of y.

$\therefore$ differentiating both sides of (1) with respect to y,

$$\frac{dx}{dy} = \frac{\sin(a+y)\cos y - \sin y\cos(a+y)}{\sin^2(a+y)}$$

$$= \frac{\sin\{(a+y)-y\}}{\sin^2(a+y)} = \frac{\sin a}{\sin^2(a+y)}.$$

Since, by Art. 7.7, $\frac{dy}{dx} = 1 \Big/ \frac{dx}{dy}$, the required result follows.

Ex. 11. *Find the derivative of* $\Delta(x)$, *where*

$$\Delta(x) = \begin{vmatrix} f_1(x) & \phi_1(x) & \psi_1(x) \\ f_2(x) & \phi_2(x) & \psi_2(x) \\ f_3(x) & \phi_3(x) & \psi_3(x) \end{vmatrix}$$

and $f_1(x)$, $f_2(x)$, $f_3(x)$, $\phi_1(x)$, *etc. are different functions of* x.

***First Method* :**

$$\Delta(x+h) - \Delta(x) = \begin{vmatrix} f_1(x+h) & \phi_1(x+h) & \psi_1(x+h) \\ f_2(x+h) & \phi_2(x+h) & \psi_2(x+h) \\ f_3(x+h) & \phi_3(x+h) & \psi_3(x+h) \end{vmatrix}$$

$$- \begin{vmatrix} f_1(x) & \phi_1(x) & \psi_1(x) \\ f_2(x) & \phi_2(x) & \psi_2(x) \\ f_3(x) & \phi_3(x) & \psi_3(x) \end{vmatrix}$$

$$= \begin{vmatrix} f_1(x+h) - f_1(x) & \phi_1(x+h) - \phi_1(x) & \psi_1(x+h) - \psi_1(x) \\ f_2(x+h) & \phi_2(x+h) & \psi_2(x+h) \\ f_3(x+h) & \phi_3(x+h) & \psi_3(x+h) \end{vmatrix}$$

$$+ \begin{vmatrix} f_1(x) & \phi_1(x) & \psi_1(x) \\ f_2(x+h) - f_2(x) & \phi_2(x+h) - \phi_2(x) & \psi_2(x+h) - \psi_2(x) \\ f_3(x) & \phi_3(x) & \psi_3(x) \end{vmatrix}$$

$$+ \begin{vmatrix} f_1(x) & \phi_1(x) & \psi_1(x) \\ f_2(x) & \phi_2(x) & \psi_2(x) \\ f_3(x+h) - f_3(x) & \phi_3(x+h) - \phi_3(x) & \psi_3(x+h) - \psi_3(x) \end{vmatrix}$$

... **(1)**

[The right-side on simplification can be easily shown to be equal to the left-side.]

Dividing (1) throughout by h and letting $h \to 0$, we get

$$\Delta'(x) = \begin{vmatrix} f_1'(x) & \phi_1'(x) & \psi_1'(x) \\ f_2(x) & \phi_2(x) & \psi_2(x) \\ f_3(x) & \phi_3(x) & \psi_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & \phi_1(x) & \psi_1(x) \\ f_2'(x) & \phi_2'(x) & \psi_2'(x) \\ f_3(x) & \phi_3(x) & \psi_3(x) \end{vmatrix}$$

$$+ \begin{vmatrix} f_1(x) & \phi_1(x) & \psi_1(x) \\ f_2(x) & \phi_2(x) & \psi_2(x) \\ f_3'(x) & \phi_3'(x) & \psi_3'(x) \end{vmatrix}.$$

Second Method :

Clearly $\Delta(x) = \sum f_1(x)\{\phi_2(x)\psi_3(x) - \phi_3(x)\psi_2(x)\}$.

$$\Delta'(x) = \sum f_1'(x)\{\phi_2(x)\psi_3(x) - \phi_3(x)\psi_2(x)\}$$

$$+ \sum f_1(x)\{\phi_2'(x)\psi_3(x) - \phi_3'(x)\psi_2(x)\}$$

$$+ \sum f_1(x)\{\phi_2(x)\psi_3'(x) - \phi_3(x)\psi_2'(x)\}.$$

$$\therefore \Delta'(x) = \begin{vmatrix} f_1'(x) & \phi_1(x) & \psi_1(x) \\ f_2'(x) & \phi_2(x) & \psi_2(x) \\ f_3'(x) & \phi_3(x) & \psi_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & \phi_1'(x) & \psi_1(x) \\ f_2(x) & \phi_2'(x) & \psi_2(x) \\ f_3(x) & \phi_3'(x) & \psi_3(x) \end{vmatrix}$$

$$+ \begin{vmatrix} f_1(x) & \phi_1(x) & \psi_1'(x) \\ f_2(x) & \phi_2(x) & \psi_2'(x) \\ f_3(x) & \phi_3(x) & \psi_3'(x) \end{vmatrix}.$$

Thus, the derivative of a third order determinant $\Delta(x)$ is equal to the sum of three determinants, each obtained by differentiating one column of $\Delta(x)$ leaving the other columns unaltered. Similarly, $\Delta'(x)$ is the sum of three determinants, each obtained by defferentiating one row of $\Delta(x)$ leaving the other rows unaltered.

The similar result is true whatever be the order of the determinant.

EXAMPLES - VII (B)

Find the differential coefficient of [Ex. 1 – 7]

1. (i) $\{\phi(x)\}^n$. (ii) $(x^2+5)^7$. (iii) $\sqrt{(x^2+a^2)}$.

(iv) $1/(ax+b)$. (v) $(e^x)^3$. (vi) $\sqrt{\log x}$.

(vii) $\sin^n x$. (viii) $\tan^5 x$. (ix) $\sec^3 x$.

(x) $\left(\sin^{-1} x\right)^3$. (xi) $\left(\tan^{-1} x\right)^2$. (xii) $\sqrt{\phi(x)}$.

2. (i) $e^{\phi(x)}$ (ii) e^{ax}. (iii) e^{ax^2+bx+c}.

(iv) e^{x^4}. (v) $e^{\tan x}$. (vi) $e^{\sin^{-1} x}$.

(vii) $e^{\sqrt{(x+1)}} - e^{\sqrt{(x-1)}}$.

3. (i) $a^{\phi(x)}$. (ii) 7^{x^2+2x}. (iii) $10^{\log x^x}$.

4. (i) $\log \phi(x)$. (ii) $\log \sin x$. (iii) $\log \cos x$.

(iv) $\log(x+a)$. (v) $\log(ax+b)$. (vi) $\log \sqrt{x}$.

(vii) $\log\left(ax^2+bx+c\right)$. (viii) $\log(\log x)$.

(ix) $10^{\log \sin x}$. (x) $\log \tan^{-1} x$.

(xi) $\log(\sec x + \tan x)$. (xii) $\log_x a$.

(xiii) $\log_x \sin x$. (xiv) $\log_x (a+x)$.

(xv) $\log_{\sin x} x$ (xvi) $\log_{\sin x}(\sec x)$.

(xvii) $\log \tan\left(\frac{1}{4}\pi + \frac{1}{2}x\right)$. (xviii) $\log\left(x + \sqrt{x^2 \pm a^2}\right)$.

(xix) $\log\left(\sqrt{x-a} + \sqrt{x-b}\right)$. (xx) $\log_{10}\left(2x + \sqrt{4x^2+1}\right)$.

(xxi) $\frac{1}{2}\log\frac{1+x}{1-x}$.

5. (i) $\sin \phi(x)$. (ii) $\cos \phi(x)$. (iii) $\tan \phi(x)$.

(iv) $\operatorname{cosec} \phi(x)$. (v) $\sec \phi(x)$. (vi) $\cot \phi(x)$.

(vii) $\sin ax$. (viii) $\cos(ax+b)$. (ix) $\cos^2 x$.

(x) $\tan mx$. (xi) $\operatorname{cosec}^3 x$. (xii) $\sin 2x \cos x$.

(xiii) $\cos 2x \cos 3x$. (xiv) $\sin x^0$ (degree). (xv) $e^{ax} \sin bx$.

(xvi) $e^{ax} \cos(bx+c)$. (xvii) $\tan 3x + \cot 4x$.

(xviii) $\sin x \sin 2x \sin 3x$. (xix) $a \tan^2 x + b \cot^2 x$.

(xx) $\sin^m x \cos^n x$. (xxi) $\sin^m x / \cos^n x$.

(xxii) $\cot x \coth x$. (xxiii) $\tanh x - \frac{1}{3}\tanh^3 x$.

(xxiv) $\log \tanh x$.

6. (i) $\sin^{-1}\phi(x)$. (ii) $\tan^{-1}\phi(x)$

(iii) $\sec^{-1}\phi(x)$ (iv) $\sin^{-1} x^2$.

(v) $\tan^{-1}(\sqrt{x})$. (vi) $\tan^{-1}(x/a)$.

(vii) $\sin^{-1}(x/a)$. (viii) $\sec^{-1} x^3$.

(ix) $\cos^{-1}\sqrt{(ax+b)}$. (x) $\cot^{-1}(e^x)$.

(xi) $\sec^{-1}(\tan x)$. (xii) $\tan^{-1}(\sec x)$.

(xiii) $\tan^{-1}(1 + x + x^2)$ (xiv) $\cos^{-1}(8x^4 - 8x^2 + 1)$.

(xv) $\sin^{-1}(3x - 4x^3)$.

(xvi) $\sec(\tan^{-1} x)$. [*C. P. 1940*]

(xvii) $\tan(\sin^{-1} x)$. (xviii) $\tan^{-1}(\tanh \frac{1}{2}x)$.

(xix) $\cot^{-1}(\operatorname{cosec} x + \cot x)$.

(xx) $\tan^{-1}(\sec x + \tan x)$. (xxi) $\cot^{-1}\left(\sqrt{1+x^2} - x\right)$.

(xxii) $\cot^{-1}\dfrac{1+x}{1-x}$. (xxiii) $\cos^{-1}\dfrac{1-x^2}{1+x^2}$.

(xxiv) $\tan^{-1}\dfrac{a+bx}{b-ax}$. (xxv) $\sin^{-1}\dfrac{2x}{1+x^2}$.

(xxvi) $\sec^{-1}\dfrac{x^2+1}{x^2-1}$. (xxvii) $\tan^{-1}\dfrac{2x}{1-x^2}$.

(xxviii) $\tan^{-1}\dfrac{1}{\sqrt{(x^2-1)}}$. [*C. P. 1943*]

(xxix) $\tan^{-1}\dfrac{x}{\sqrt{(1-x^2)}}$. [*C. P. 1938*]

(xxx) $\tan^{-1}\sqrt{\left(\dfrac{x-a}{b-x}\right)}$. (xxxi) $\tan^{-1}\dfrac{3x-x^3}{1-3x^2}$

(xxxii) $\operatorname{sech}^{-1}x-\operatorname{cosech}^{-1}x$. (xxxiii) $\tanh^{-1}\left(\tan\frac{1}{2}x\right)$.

(xxxiv) $\tanh^{-1}\left\{\left(x^2-1\right)/\left(x^2+1\right)\right\}$.

7. (i) $\cos\left\{\sqrt{\left(1+x^2\right)}\right\}$. (ii) $e^{\sqrt{\cot x}}$. [*C. P. 1943, 1948*]

(iii) $e^{\operatorname{cosec}^2\sqrt{x}}$. (iv) $e^{\left(\sin^{-1}x\right)^2}$. (v) $3^{\sqrt{\left(1+x+x^2\right)}}$.

(vi) $\log\tan\frac{1}{2}x$. (vii) $\sqrt{(\log\sin x)}$. (viii) $(\log\sin x)^2$.

(ix) $\cos\left\{2\sin^{-1}(\cos x)\right\}$. (x) $\sin^2\left(\log x^2\right)$.

(xi) $\log\sec(ax+b)^3$. [*C. P. 1941*]

(xii) $\log\left\{2x+4+\sqrt{\left(4x^2+16x-12\right)}\right\}$.

(xiii) $\sqrt{(1+\log x\log\sin x)}$. [*C. P. 1944*]

(xiv) $\tan\log\sin\left(e^{x^2}\right)$.

(xv) $A\left(x+\sqrt{x^2-1}\right)^n+B\left(x-\sqrt{x^2-1}\right)^n$.

8. Find the differential coefficients of :

(i) x^x. (ii) $(1+x)^x$. (iii) $x^{\log x}$. (iv) x^{1+x+x^2}.

(v) a^{a^x}. (vi) e^{e^x}. (vii) e^{x^x}. (viii) x^{e^x}.

(ix) $(\sin x)^{\tan x}$. (x) $x^{\cos^{-1}x}$. [C. P. 1944]

(xi) $(\sin x)^{\log x}$. [C. P. 1943]

(xii) x^{x^x}. [*C. P. 1937*]

(xiii) $(\sin x)^{\cos x}+(\cos x)^{\sin x}$.

(xiv) $(\tan x)^{\cot x}+(\cot x)^{\tan x}$.

9. Find the differential coefficients of :

(i) $(1-x)(1-2x)(1-3x)(1-4x)$.

(ii) $\sqrt[3]{x(x+1)(x+2)}$. (iii) $\sqrt{\left(\frac{1+x}{1-x}\right)}$.

(iv) $\left(\frac{a^2-x^2}{a^2+x^2}\right)^{\frac{1}{2}}$. (v) $\log\left\{e^x\left(\frac{x-1}{x+1}\right)^{\frac{3}{2}}\right\}$.

(vi) $x^3\sqrt{\frac{x^2+4}{x^2+3}}$. [C. P. 1941]

(vii) $\frac{x^3\sqrt{x^2-12}}{\sqrt[3]{20-3x}}$ for $x=4$.

(viii) $\left(\frac{x}{1+\sqrt{1-x^2}}\right)^n$. [C. P. 1935]

(ix) $\frac{(4x+1)^{\frac{1}{4}}}{(2x+3)^{\frac{1}{2}}(5x-1)^{\frac{1}{5}}}$.

(x) $\left(x^{\frac{b+c}{c-a}}\right)^{\frac{1}{a-b}}\times\left(x^{\frac{c+a}{a-b}}\right)^{\frac{1}{b-c}}\times\left(x^{\frac{a+b}{b-c}}\right)^{\frac{1}{c-a}}$

10. Find $\frac{dy}{dx}$ in the following cases :

(i) $3x^4-x^2y+2y^3=0$. [C. P. 1941]

(ii) $x^4+x^2y^2+y^4=0$. [C. P. 1939]

(iii) $x^3+y^3+4x^2y-25=0$.

(iv) $x^3+y^3=3axy$. (v) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

(vi) $ax^2+2hxy+by^2+2gx+2fy+c=0$

(vii) $x=y\log(xy)$. (viii) $x^py^q=(x+y)^{p+q}$.

(ix) $y=x^y$. [C. P. 1940]

(x) $x^y=y^x$. [C. P. 1945, V. P. 2002]

(xi) $x^yy^x=1$. [C. P. 1943, V. P. 2000]

(xii) $(\cos x)^y=(\sin y)^x$. [C. P. 2007]

(xiii) $e^{xy} - 4xy = 2$.

(xiv) $\log(xy) = x^2 + y^2$. [C. P. 1943]

11. Find $\dfrac{dy}{dx}$ when

(i) $x = a\cos\phi,\ y = b\sin\phi$.

(ii) $x = a\cos^3\theta,\ y = b\sin^3\theta$.

(iii) $x = at^2,\ y = 2at$.

(iv) $x = \sin^2\theta,\ y = \tan\theta$. [C. P. 1943]

(v) $x = a\sec^2\theta,\ y = a\tan^3\theta$. [C. P. 1942]

(vi) $x = a\left(\cos t + \log\tan\frac{1}{2}t\right),\ y = a\sin t$.

(vii) $x = a(\cos t + t\sin t),\ y = a(\sin t - t\cos t)$.

(viii) $x = a(2\cos t + \cos 2t),\ y = a(2\sin t - \sin 2t)$

(ix) $x = 2a\sin^2 t\cos 2t,\ y = 2a\sin^2 t\sin 2t$.

(x) $x = 3at/\left(1+t^3\right),\ y = 3at^2/\left(1+t^3\right)$. [C. P. 1941]

(xi) $\tan y = \dfrac{2t}{1-t^2},\ \sin x = \dfrac{2t}{1+t^2}$. [C. P. 1944]

12. If $y = e^{\sin^{-1}x}$ and $z = e^{-\cos^{-1}x}$, then show that $\dfrac{dy}{dz}$ is independent of x.

13. Differentiate the left-side functions with respect to the right-side ones:

(i) x^5 w.r.t. x^2. (ii) $\sec x$ w.r.t. $\tan x$.

(iii) $\log_{10} x$ w.r.t. x^3. (iv) $\tan^{-1}x$ w.r.t. x^2.

(v) $\cos^{-1}\dfrac{1-x^2}{1+x^2}$ w.r.t. $\tan^{-1}\dfrac{2x}{1-x^2}$.

(vi) $\tan^{-1}\dfrac{\sqrt{(1+x^2)}-1}{x}$ w.r.t. $\tan^{-1}x$.

(vii) $x^{\sin^{-1}x}$ w.r.t. $\sin^{-1}x$. [C. P. 1938]

14. Find the differential coefficients of :

(i) $\dfrac{1}{\sqrt{(1+x)}} + \dfrac{1}{\sqrt{(1-x)}}$. (ii) $\dfrac{1}{\sqrt{(x+a)} + \sqrt{(x+b)}}$.

(iii) $\sqrt{\left\{\dfrac{1+x+x^2}{1-x+x^2}\right\}}$. (iv) $\log \dfrac{x^2+x+1}{x^2-x+1}$.

(v) $\log \sqrt{\left\{\dfrac{1+\sin x}{1-\sin x}\right\}}$. (vi) $\log \sqrt{\left\{\dfrac{1-\cos x}{1+\cos x}\right\}}$

(vii) $\tan^{-1} \sqrt{\left(\dfrac{1-x}{1+x}\right)}$

(viii) $\tan^{-1} \dfrac{\cos x}{1+\sin x}$ [*C. P. 1942, 1944, V. P. 1998*]

(ix) $\tan^{-1} \sqrt{\left(\dfrac{1-\cos x}{1+\cos x}\right)}$. (x) $\tan^{-1} \dfrac{\cos x - \sin x}{\cos x + \sin x}$.

(xi) $\sin^{-1} x + \sin^{-1} \sqrt{(1-x^2)}$.

(xii) $\sin^{-1} \left\{2ax\sqrt{(1-a^2x^2)}\right\}$.

(xiii) $\tan^{-1} \dfrac{\sqrt{(1+x^2)}-\sqrt{(1-x^2)}}{\sqrt{(1+x^2)}+\sqrt{(1-x^2)}}$.

(xiv) $\sin \left\{2\tan^{-1} \sqrt{\left(\dfrac{1-x}{1+x}\right)}\right\}$.

(xv) $\dfrac{\sqrt{(1+x)}+\sqrt{(1-x)}}{\sqrt{(1+x)}-\sqrt{(1-x)}}$

(xvi) $\dfrac{\sqrt{(a^2+x^2)}+\sqrt{(a^2-x^2)}}{\sqrt{(a^2+x^2)}-\sqrt{(a^2-x^2)}}$.

(xvii) $\log \left(\dfrac{1+x}{1-x}\right)^{\frac{1}{4}} - \frac{1}{2}\tan^{-1} x$.

(xviii) $\log \sqrt{\left(\dfrac{a\cos x - b\sin x}{a\cos x + b\sin x}\right)}$.

(xix) $\log \dfrac{a + b \tan x}{a - b \tan x}$. [C. P. 1942]

(xx) $\tan^{-1}\left\{\sqrt{\left(\dfrac{a-b}{a+b}\right)} \tan \dfrac{x}{2}\right\}$.

(xxi) $\sin^{-1} \dfrac{a + b \cos x}{b + a \cos x}$. (xxii) $\cos^{-1} \dfrac{3 + 5 \cos x}{5 + 3 \cos x}$.

(xxiii) $\log \sqrt{\left\{\dfrac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\right\}}$.

(xxiv) $\log \sqrt{\left\{\dfrac{\sqrt{(1+x)}+\sqrt{(1-x)}}{\sqrt{(1+x)}-\sqrt{(1-x)}}\right\}}$. (xxv) $x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x}}}$.

Find $\dfrac{dy}{dx}$ in the following cases (Ex. 15—23):

15. $y = \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{6}x^2 + \frac{1}{6}\log\left(1 + x^2\right)$.

16. $y = \frac{1}{2}x\sqrt{\left(a^2 - x^2\right)} + \frac{1}{2}a^2 \sin^{-1} \dfrac{x}{a}$.

17. $y = \log\left(x + \sqrt{x^2 - a^2}\right) + \sec^{-1} \dfrac{x}{a}$.

18. $y = x\sqrt{\left(x^2 + a^2\right)} + a^2 \log\left(x + \sqrt{x^2 + a^2}\right)$.

19. $x = \sqrt{\left(a^2 - y^2\right)} + \dfrac{a}{2} \log \dfrac{a - \sqrt{\left(a^2 - y^2\right)}}{a + \sqrt{\left(a^2 - y^2\right)}}$.

20. $y = \log \dfrac{1 + x}{1 - x} + \dfrac{1}{2} \log \dfrac{1 + x + x^2}{1 - x + x^2} + \sqrt{3} \tan^{-1} \dfrac{x\sqrt{3}}{1 - x^2}$.

21. $y = \dfrac{1}{4\sqrt{2}} \log \dfrac{1 + x\sqrt{2} + x^2}{1 - x\sqrt{2} + x^2} + \dfrac{1}{2\sqrt{2}} \tan^{-1} \dfrac{x\sqrt{2}}{1 - x^2}$.

22. $y=\frac{1}{4}x\left(x^2+1\right)^{\frac{3}{2}}-\frac{1}{8}x\left(x^2+1\right)^{\frac{1}{2}}-\frac{1}{8}\log\left(x+\sqrt{x^2+1}\right)$

23. $$y=\frac{1}{1+x^{n-m}+x^{p-m}}+\frac{1}{1+x^{m-n}+x^{p-n}}+\frac{1}{1+x^{m-p}+x^{n-p}}.$$

24. If $f(x)=\left(\dfrac{a+x}{b+x}\right)^{a+b+2x}$, show that

$$f'(0)=\left(2\log\frac{a}{b}+\frac{b^2-a^2}{ab}\right)\cdot\left(\frac{a}{b}\right)^{a+b}.$$ [*C. P. 1946*]

25. If $f(x)=\log\dfrac{\sqrt{(a+bx)}-\sqrt{(a-bx)}}{\sqrt{(a+bx)}+\sqrt{(a-bx)}}$,

find for what values of x, $\dfrac{1}{f'(x)}=0$.

26. If $$\sin x\sin\left(\frac{\pi}{n}+x\right)\sin\left(\frac{2\pi}{n}+x\right)\ldots\sin\left(\frac{n-1}{n}\pi+x\right)=\frac{\sin nx}{2^{n-1}}.$$

then show that

$$\cot x+\cot\left(\frac{\pi}{n}+x\right)+\cot\left(\frac{2\pi}{n}+x\right)+\ldots+\cot\left(\frac{n-1}{n}\pi+x\right)=n\cot nx.$$ [*C. P. 1945*]

27. (i) From the relation (when n is odd)

$$2^{n-1}\cos\theta\cos\left(\theta+\frac{2\pi}{n}\right)\cos\left(\theta+\frac{4\pi}{n}\right)\ldots\cos\left(\theta+\frac{2(n-1)\pi}{n}\right)=\cos n\theta.$$

deduce that

$$\tan\theta + \tan\left(\theta + \frac{2\pi}{n}\right) + \tan\left(\theta + \frac{4\pi}{n}\right) + \ldots$$

$$+ \tan\left(\theta + \frac{2(n-1)\pi}{n}\right) = n\tan n\theta .$$

(ii) From the identity

$$\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\ldots\ldots\cos\frac{\theta}{2^n} = \frac{\sin\theta}{2^n\sin\left(\theta/2^n\right)},$$

show that

$$\frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{2^2}\tan\frac{\theta}{2^2} + \frac{1}{2^3}\tan\frac{\theta}{2^3} + \ldots\ldots + \frac{1}{2^n}\tan\frac{\theta}{2^n}$$

$$= \frac{1}{2^n}\cot\frac{\theta}{2^n} - \cot\theta$$

28. Find $f'(x)$ in the following cases and determine if it is continuous for $x=0$.

(i) $f(x) = 0$ or $x^2\cos(1/x)$ according a x is or is not zero.

(ii) $f(x) = 0$ or $x^3\cos(1/x)$ according a x is or is not zero.

29. If $(1+x)^n = c_0 + c_1 x + c_2 x^2 + \ldots. + c_n x^n$, then prove that

(i) $c_1 + 2c_2 + 3c_3 + \ldots. + nc_n = n.2^{n-1}$.

(ii) $c_0 + 2c_1 + 3c_2 + \ldots. + (n+1)c_n = (n+2).2^{n-1}$.

30. If $y = 1 + \frac{a_1}{x-a_1} + \frac{a_2 x}{(x-a_1)(x-a_2)}$

$$+ \frac{a_3 x}{(x-a_1)(x-a_2)(x-a_3)},$$

then show that

$$\frac{dy}{dx} = \frac{y}{x}\left(\frac{a_1}{a_1-x} + \frac{a_2}{a_2-x} + \frac{a_3}{a_3-x}\right)$$

31. If $\Delta(x)=\begin{vmatrix}(x-a)^4 & (x-a)^3 & 1\\(x-b)^4 & (x-b)^3 & 1\\(x-c)^4 & (x-c)^3 & 1\end{vmatrix}$, show that

$$\Delta'(x)=3\begin{vmatrix}(x-a)^4 & (x-a)^2 & 1\\(x-b)^4 & (x-b)^2 & 1\\(x-c)^4 & (x-c)^2 & 1\end{vmatrix}.$$

32. If $\Delta(x)=\begin{vmatrix}\sin x & \cos x & \sin x\\ \cos x & -\sin x & \cos x\\ x & 1 & 1\end{vmatrix}$, then show that $\Delta'(x)=1$.

32. If $f(x)=\begin{vmatrix}1 & 1 & 1 & 1\\1 & x & 1 & 1\\1 & 1 & x & 1\\1 & 1 & 1 & x\end{vmatrix}$, prove that $f'(x)=3(x-1)^2$.

34. If the determinant of the 4th order

$$\begin{vmatrix}x & a & a & a\\a & x & a & a\\a & a & x & a\\a & a & a & x\end{vmatrix}$$

be denoted by Δ_4, show that $\Delta'_4=4\Delta_3$.

ANSWERS

1. (i) $n\{\phi(x)\}^{n-1}\phi'(x)$. (ii) $14x(x^2+5)^6$. (iii) $x/\sqrt{(x^2+a^2)}$.

(iv) $-a/(ax+b)^2$. (v) $3(e^x)^3$. (vi) $1/(2x\sqrt{\log x})$.

(vii) $n\sin^{n-1}x\cos x$. (viii) $5\tan^4 x\sec^2 x$. (ix) $3\sec^4 x\tan x$.

(x) $\dfrac{3(\sin^{-1}x)^2}{\sqrt{1-x^2}}$. (xi) $\dfrac{2\tan^{-1}x}{1+x^2}$. (xii) $\dfrac{\phi'(x)}{2\sqrt{\phi(x)}}$.

2. (i) $e^{\phi(x)}.\phi'(x)$ (ii) ae^{ax}. (iii) $(2ax+b).e^{ax^2+bx+c}$.

(iv) $4x^3.e^{x^4}$. (v) $\sec^2 x.e^{\tan x}$. (vi) $e^{\sin^{-1}x}.\frac{1}{\sqrt{1-x^2}}$.

(vii) $\frac{e^{\sqrt{(x+1)}}}{2\sqrt{(x+1)}}-\frac{e^{\sqrt{(x-1)}}}{2\sqrt{(x-1)}}$.

3. (i) $\log_e a.a^{\phi(x)}.\phi'(x)$. (ii) $2(x+1).\log 7.7^{x^2+2x}$.

(iii) $\log_e 10.10^{\log x^x}.(1+\log x)$.

4. (i) $\phi'(x)/\phi(x)$. (ii) $\cot x$. (iii) $-\tan x$. (iv) $1/(x+a)$.

(v) $a/(ax+b)$. (vi) $1/(2x)$. (vii) $(2ax+b)/(ax^2+bx+c)$.

(viii) $1/(x\log x)$. (ix) $10^{\log\sin x}.\log_e 10.\cot x$.

(x) $1/\{(1+x^2)\tan^{-1}x\}$. (xi) $\sec x$.

(xii) $-\log a/\{x(\log x)^2\}$. (xiii) $\frac{x\cot x\log x-\log\sin x}{x(\log x)^2}$.

(xiv) $\frac{x\log x-(a+x)\log(a+x)}{x(a+x)(\log x)^2}$. (xv) $\frac{\log\sin x-x\cot x\log x}{x(\log\sin x)^2}$

(xvi) $\frac{\tan x\log\sin x+\cot x\log\cos x}{(\log\sin x)^2}$. (xvii) $\sec x$.

(xviii) $\frac{1}{\sqrt{x^2\pm a^2}}$. (xix) $\frac{1}{2\sqrt{(x-a)(x-b)}}$.

(xx) $\frac{2\log_{10}e}{\sqrt{4x^2+1}}$. (xxi) $\frac{1}{1-x^2}$.

5. (i) $\cos\phi(x).\phi'(x)$. (ii) $-\sin\phi(x).\phi'(x)$.

(iii) $\sec^2\phi(x).\phi'(x)$. (iv) $-\text{cosec}\,\phi(x)\cot\phi(x).\phi'(x)$.

(v) $\sec\phi(x)\tan\phi(x).\phi'(x)$. (vi) $-\text{cosec}^2\phi(x).\phi'(x)$.

(vii) $a\cos ax$. (viii) $-a\sin(ax+b)$.

(ix) $-\sin 2x$. (x) $m\sec^2 mx$.

(xi) $-3\,\text{cosec}^3 x\cot x$. (xii) $\frac{1}{2}(3\cos 3x+\cos x)$.

(xiii) $-\frac{1}{2}(5\sin 5x+\sin x)$. (xiv) $\frac{\pi}{180}\cos x^0$ (degree).

(xv) $e^{ax}(a\sin bx+b\cos bx)$.

(xvi) $e^{ax}\{a\cos(bx+c)-b\sin(bx+c)\}$.

(xvii) $3\sec^2 3x-4\operatorname{cosec}^2 4x$. (xviii) $\frac{1}{2}(\cos 2x+2\cos 4x-3\cos 6x)$.

(xix) $2\left(a\tan x\sec^2 x-b\cot x\operatorname{cosec}^2 x\right)$.

(xx) $\sin^{m-1}x\cos^{n-1}x\left(m\cos^2 x-n\sin^2 x\right)$.

(xxi) $\dfrac{\sin^{m-1}x}{\cos^{n+1}x}\left(m\cos^2 x+n\sin^2 x\right)$.

(xxii) $-\operatorname{cosec} x\coth x-\cot x\operatorname{cosech}^2 x$.

(xxiii) $\operatorname{sech}^4 x$. (xxiv) $2\operatorname{cosech} 2x$.

6. (i) $\dfrac{\phi'(x)}{\sqrt{1-\{\phi(x)\}^2}}$. (ii) $\dfrac{\phi'(x)}{1+\{\phi(x)\}^2}$ (iii) $\dfrac{\phi'(x)}{\phi(x).\sqrt{\{\phi(x)\}^2-1}}$

(iv) $\dfrac{2x}{\sqrt{1-x^4}}$. (v) $\dfrac{1}{2(1+x)\sqrt{x}}$. (vi) $\dfrac{a}{a^2+x^2}$.

(vii) $\dfrac{1}{\sqrt{a^2-x^2}}$. (viii) $\dfrac{3}{x\sqrt{x^6-1}}$.

(ix) $-\dfrac{1}{\sqrt{1-(ax+b)}}\cdot\dfrac{a}{2\sqrt{(ax+b)}}$. (x) $-\dfrac{e^x}{1+e^{2x}}$.

(xi) $\dfrac{1}{\sin x\sqrt{\sin^2 x-\cos^2 x}}$. (xii) $\dfrac{\sin x}{1+\cos^2 x}$.

(xiii) $\dfrac{1+2x}{1+\left(1+x+x^2\right)^2}$ (xiv) $-\dfrac{4}{\sqrt{1-x^2}}$. (xv) $\dfrac{3}{\sqrt{1-x^2}}$.

(xvi) $\dfrac{x}{\sqrt{1+x^2}}$. (xvii) $\left(1-x^2\right)^{-\frac{3}{2}}$. (xviii) $\frac{1}{2}\operatorname{sech} x$.

(xix) $\frac{1}{2}$. (xx) $\frac{1}{2}$. (xxi) $\dfrac{1}{2}\cdot\dfrac{1}{1+x^2}$.

(xxii) $-\dfrac{1}{1+x^2}$. (xxiii) $\dfrac{2}{1+x^2}$. (xxiv) $\dfrac{1}{1+x^2}$.

(xxv) $\dfrac{2}{1+x^2}$. (xxvi) $-\dfrac{2}{1+x^2}$. (xxvii) $\dfrac{2}{1+x^2}$.

(xxviii) $-\dfrac{1}{x\sqrt{\left(x^2-1\right)}}$. (xxix) $\dfrac{1}{\sqrt{1-x^2}}$. (xxx) $\dfrac{1}{\sqrt{(x-a)(b-x)}}$.

(xxxi) $\frac{3}{1+x^2}$ (xxxii) $-\frac{1}{x}\left[\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}\right]$.

(xxxiii) $\frac{1}{2}\sec x$. (xxxiv) $1/x$.

7. (i) $\frac{-x\sin\left(\sqrt{(1+x^2)}\right)}{\sqrt{(1+x^2)}}$. (ii) $\frac{-e^{\sqrt{\cot x}}}{2\sin^2 x\sqrt{\cot x}}$.

(iii) $-e^{\operatorname{cosec}^2\sqrt{x}}\frac{\cos\sqrt{x}}{\sqrt{x}\sin^3\sqrt{x}}$. (iv) $\frac{2\sin^{-1}x}{\sqrt{1-x^2}}e^{(\sin^{-1}x)^2}$

(v) $\frac{(1+2x)\log 3}{2\sqrt{1+x+x^2}}\cdot 3^{\sqrt{(1+x+x^2)}}$. (vi) $\operatorname{cosec} x$.

(vii) $\frac{1}{2}(\log\sin x)^{-\frac{1}{2}}\cot x$. (viii) $2\cot x\log\sin x$.

(ix) $2\sin 2x$. (x) $2x^{-1}\sin(4\log x)$.

(xi) $3a(ax+b)^2\tan(ax+b)^3$. (xii) $\frac{1}{\sqrt{x^2+4x-3}}$.

(xiii) $\frac{\log\sin x+x\cot x\log x}{2x\sqrt{(1+\log x\log\sin x)}}$.

(xiv) $\sec^2\left(\log\sin e^{x^2}\right)\cot\left(e^{x^2}\right)e^{x^2}.2x$.

(xv) $\frac{n}{\sqrt{x^2-1}}\left\{A\left(x+\sqrt{x^2-1}\right)^n+B\left(x-\sqrt{x^2-1}\right)^n\right\}$.

8. (i) $x^x(\log x+1)$. (ii) $(1+x)^x\left\{\log(1+x)+\frac{x}{1+x}\right\}$.

(iii) $2\log x.x^{(\log x-1)}$. (iv) $x^{1+x+x^2}.\{(2x+1)\log x+x^{-1}+1+x\}$.

(v) $a^{a^x}.a^x(\log a)^2$. (vi) $e^{e^x}.e^x$. (vii) $e^{x^x}.x^x(\log x+1)$.

(viii) $x^{e^x}.e^x\left(x^{-1}+\log x\right)$. (ix) $(\sin x)^{\tan x}.\left(\sec^2 x\log\sin x+1\right)$.

(x) $x^{\cos^{-1}x}.\left\{-\frac{\log x}{\sqrt{(1-x^2)}}+\frac{\cos^{-1}x}{x}\right\}$.

(xi) $(\sin x)^{\log x}\left\{x^{-1}\log\sin x+\log x\cot x\right\}$.

(xii) $x^{x^x}.x^x\{\log x(\log x+1)+1/x\}$.

(xiii) $(\sin x)^{\cos x}\{\cos x \cot x - \sin x \log \sin x\}$
$+ (\cos x)^{\sin x}\{\cos x \log \cos x - \sin x \tan x\}$.

(xiv) $(\tan x)^{\cot x}\left\{\operatorname{cosec}^2 x(1 - \log \tan x)\right\}$
$+ (\cot x)^{\tan x}\left\{\sec^2 x(\log \cot x - 1)\right\}$.

9. (i) $96x^3 - 150x^2 + 70x - 10$. (ii) $\dfrac{3x^2+6x+2}{3\{x(x+1)(x+2)\}^{\frac{2}{3}}}$.

(iii) $\dfrac{1}{(1-x)\sqrt{1-x^2}}$. (iv) $\dfrac{-2a^2x}{(a^2+x^2)\sqrt{a^4-x^4}}$.

(v) $\dfrac{x^2+2}{x^2-1}$. (vi) $\dfrac{x^2(3x^4+20x^2+36)}{(x^2+4)^{\frac{1}{2}}(x^2+3)^{\frac{3}{2}}}$.

(vii) 120.

(viii) $\dfrac{ny}{x\sqrt{(1-x^2)}}$, y being the given function.

(ix) $\dfrac{-(5+2x+18x^2)}{(4x+1)^{\frac{3}{4}}(2x+3)^{\frac{3}{2}}(5x-1)^{\frac{6}{5}}}$. (x) 0.

10. (i) $\dfrac{2x(6x^2-y)}{x^2-6y^2}$. (ii) $-\dfrac{x(2x^2+y^2)}{y(x^2+2y^2)}$. (iii) $-\dfrac{3x^2+8xy}{3y^2+4x^2}$.

(iv) $\dfrac{x^2-ay}{ax-y^2}$. (v) $-\left(\dfrac{y}{x}\right)^{\frac{1}{3}}$. (vi) $-\dfrac{ax+hy+g}{hx+by+f}$.

(vii) $\dfrac{y(x-y)}{x(y-x)}$. (viii) $\dfrac{y}{x}$. (ix) $\dfrac{y^2}{x(1-y\log x)}$.

(x) $\dfrac{y(x\log y - y)}{x(y\log x - x)}$. (xi) $-\dfrac{y^2(1-\log x)}{x^2(1-\log y)}$. (xii) $\dfrac{y\tan x + \log \sin y}{\log \cos x - x\cot y}$.

(xiii) $-\dfrac{y}{x}$. (xiv) $\dfrac{y(2x^2-1)}{x(1-2y^2)}$.

11. (i) $-(b/a)\cot\phi$. (ii) $-\tan\theta$. (iii) $1/t$.

(iv) $\frac{1}{2}\sec^3\theta \operatorname{cosec}\theta$. (v) $\frac{3}{2}\tan\theta$. (vi) $\tan t$.

(vii) $\tan t$. (viii) $-\tan\frac{1}{2}t$. (ix) $\tan 3t$.

(x) $t\left(2-t^3\right)\left(1-2t^3\right)^{-1}$. (xi) 1.

13. (i) $\frac{5}{2}x^3$. (ii) $\sin x$. (iii) $\frac{1}{3}x^{-3}\log_{10}e$.

(iv) $1/\left\{2x\left(1+x^2\right)\right\}$. (v) 1. (vi) $\frac{1}{2}$.

(vii) $x^{\sin^{-1}x}\left(\log x+\sin^{-1}x\cdot\frac{\sqrt{1-x^2}}{x}\right)$.

14. (i) $\frac{1}{2}\left\{(1-x)^{-\frac{3}{2}}-(1+x)^{-\frac{3}{2}}\right\}$

(ii) $\frac{1}{2(a-b)}\left(\frac{1}{\sqrt{(x+a)}}-\frac{1}{\sqrt{(x+b)}}\right)$.

(iii) $\frac{1-x^2}{\left(1+x+x^2\right)^{\frac{1}{2}}\left(1-x+x^2\right)^{\frac{3}{2}}}$. (iv) $\frac{2\left(1-x^2\right)}{1+x^2+x^4}$.

(v) $\sec x$. (vi) $\operatorname{cosec} x$. (vii) $\frac{-1}{2\sqrt{1-x^2}}$.

(viii) $-\frac{1}{2}$. (ix) $\frac{1}{2}$. (x) -1.

(xi) 0. (xii) $\frac{2a}{\sqrt{\left(1-a^2x^2\right)}}$.

(xiii) $\frac{x}{\sqrt{1-x^4}}$. (xiv) $\frac{-x}{\sqrt{1-x^2}}$. (xv) $-\frac{1}{x^2\sqrt{1-x^2}}-\frac{1}{x^2}$

(xvi) $-\frac{2a^2}{x^3}\left(1+\frac{a^2}{\sqrt{a^4-x^4}}\right)$. (xvii) $\frac{x^2}{1-x^4}$.

(xviii) $\frac{-ab}{a^2\cos^2x-b^2\sin^2x}$. (xix) $\frac{2ab}{a^2\cos^2x-b^2\sin^2x}$.

(xx) $\frac{\sqrt{a^2-b^2}}{2(a+b\cos x)}$. (xxi) $\frac{-\sqrt{b^2-a^2}}{b+a\cos x}$. (xxii) $\frac{4}{5+3\cos x}$.

(xxiii) $\frac{1}{\sqrt{1+x^2}}$. (xxiv) $\frac{-1}{2x\sqrt{1-x^2}}$. (xxv) $1-\frac{x^4+x^2+2}{\left(x^3+2x\right)^2}$.

15. $x^2\tan^{-1}x$. **16.** $\sqrt{\left(a^2-x^2\right)}$. **17.** $\frac{1}{x}\sqrt{\frac{x+a}{x-a}}$.

18. $2\sqrt{\left(a^2+x^2\right)}$. **19.** $\frac{y}{\sqrt{\left(a^2-y^2\right)}}$. **20.** $\frac{6}{1-x^6}$.

21. $\dfrac{1}{1+x^4}$. **22.** $x^2\sqrt{x^2+1}$ **23.** 0. 25. $0. \pm (a/b)$

28. (i) $f'(x)=0$ or $\sin(1/x)+2x\cos(1/x)$ according as x is or is not zero ; $f'(x)$ is discontinuous for $x=0$.

(ii) $f'(x)=0$ or $3x^2\cos(1/x)+x\sin(1/x)$ according as x is or is not zero; $f'(x)$ is continuous for $x=0$.

7.14. Significance of derivative and its sign.

A very important aspect of a derivative, following from its definition, is as a **rate-measurer.** This will be clear from the following examples.

Let s denote the length of the path covered by a moving particle in any time. Clearly, as the particle moves continuously, s has a definite value for every value of t, and, accordingly, by definition, s is a function of t. If $s+\Delta s$ be the value of s corresponding to the value $t+\Delta t$ of t, then Δs denotes the distance moved over by the particle in time Δt, the ratio $\dfrac{\Delta s}{\Delta t}$ in the limit, when Δt becomes infinitely small, represents the rate at which the particle is describing its path per unit of time at the moment. But on the other hand, $\underset{\Delta t \to 0}{Lt}\dfrac{\Delta s}{\Delta t}$ is, by definition (since Δs and denote corresponding changes in s and t), the derivative of s with respect to t. Thus, the derivative $\dfrac{ds}{dt}$ represents the rate of change of s with respect to t, *i.e.*, the speed of the moving particle.

More generally, if y be a function of the variable x, changing continuously with x, then Δy being the change in y corresponding to a change Δx of x, the derivative $\dfrac{dy}{dx}=\underset{\Delta x \to 0}{Lt}\dfrac{\Delta y}{\Delta x}$ represents the *rate of change of y with respect to x.*

Thus, v being the velocity of a moving particle at time t, $\dfrac{dv}{dt}$, represents the time-rate of change of velocity, *i.e.*, its accleration; again, if V be the volume of a quantity of gas enclosed in a flexible vessel at a constant temperature when its pressure is p, which we can change at pleasure, $\dfrac{dV}{dp}$ represents the rate of change of volume with pressure and so on.

Next we may note, that y changing with x, *if y increases when x is increased and diminishes when x is diminished,* the corresponding changes and Δy and Δx are of the same sign, and accordingly, the ratio $\dfrac{\Delta y}{\Delta x}$ is positive. Hence, $\dfrac{dy}{dx}$ (when it exists and is 0) is *positive.*

Similarly, *if y decreases when x is increased, or increases when x is diminished $\dfrac{dy}{dx}$ is negetive.*

Conversely, a positive sign of $\dfrac{dy}{dx}$ at a point c indicates that in the neighbourhood of the point y increases or decreases with x, *i.e.,* both y and x increase or decrease together. On the other hand, a negative sign of $\dfrac{dy}{dx}$ means that y decreases when x increases and vice versa near the point.

A formal proof of the above result is given below.

A theorem on the sign of $f'(x)$.

If $f'(a) > 0$, prove that $f(x) < f(a)$ for all values of $x < a$ but sufficiently near to a, and $f(x) > f(a)$ for all values of $x > a$ but sufficiently near to a.

Since $f'(a) > 0$

$$\therefore \quad \underset{h \to 0+0}{Lt} \frac{f(a+h) - f(a)}{h} > 0, \text{ and}$$

$$\underset{h \to 0+0}{Lt} \frac{f(a-h) - f(a)}{-h} > 0.$$

$\therefore$ for all sufficiently small values of h, we have

$$f(a-h) < f(a) < f(a+h).$$

In other words, there exists some *neighbourhood* $(a-\delta, a+\delta)$ of a in which

$$f(x) > f(a) \text{ for } x > a, \ f(x) < f(a) \text{ for } x < a,$$

i.e., $f(x) > f(a)$ for $x > a$ but sufficiently near to a,

and, $f(x) < f(a)$ for $x < a$ but sufficiently near to a.

When the function $y = f(x)$ is represented graphically, a **geometrical interpretation** of the derivative $\frac{dy}{dx}$ corresponding to any value of x may be given as follows :

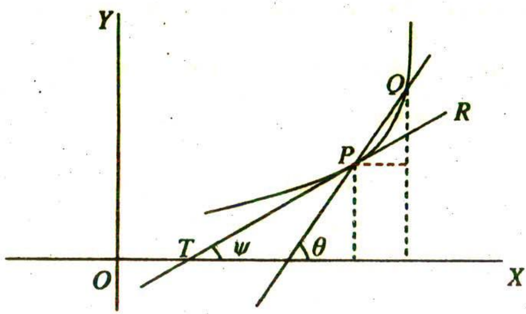

Fig 7.14.1

Let P be a point (x, y) on the curve, and Q a neighbouring point $(x + \Delta x,\ y + \Delta y)$ which may be taken on either side of P, so that Δx may have any sign. The equation to the line $\overline{PQ}$ is (X, Y denoting current co-ordinates)

$$Y - y = \frac{y + \Delta y - y}{x + \Delta x - x}(X - x) = \frac{\Delta y}{\Delta x}(X - x).$$

If θ be the inclination of this line $\overline{PQ}$ to the x-axis, the slope of the line, *i.e.*, $\tan\theta = \frac{\Delta y}{\Delta x}$ ... (1)

Now, let Q approach P along the curve indefinitely closely, so that $\Delta x \to 0$. If the straight line $\overline{PQ}$ tends to a definite limiting position $\overline{TPR}$ as Q approaches P from either side, then $\overline{TPR}$ is called the tangent to the curve at P. In this, if ψ be the inclination of $\overline{TPR}$ to the x-axis, then as $\overline{PQ}$ tends to $\overline{TPR}$, $\theta \to \psi$. Also, as $\Delta x \to 0$, $\underset{\Delta x \to 0}{Lt} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$ from definition. Thus, (1) leads to

$$\tan\psi = \frac{dy}{dx}.$$

Hence, *the derivative* $\frac{dy}{dx}$ *for any value of* x, *when it exists, is the trigonometrical tangent of the inclination (otherwise known as slope or gradient) of the tangent line at the corresponding point P on the curve* $y = f(x)$.

Also, if $\frac{dy}{dx} (= \tan \psi)$ be positive, ψ is acute (as at P in the figure below), and at that point y increases with x. If $\frac{dy}{dx}$, *i.e.*, $\tan \psi$ be negative, ψ is negative (as at Q), or is obtuse (as at R), and y diminishes when x increases, or vice versa.

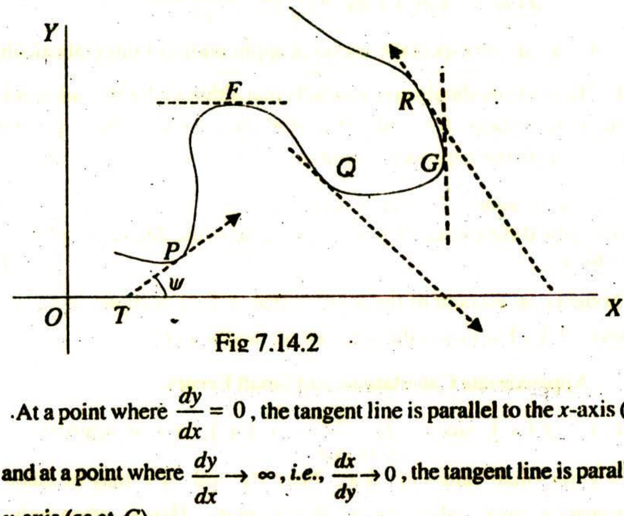

Fig 7.14.2

At a point where $\frac{dy}{dx} = 0$, the tangent line is parallel to the x-axis (as at F), and at a point where $\frac{dy}{dx} \to \infty$, *i.e.*, $\frac{dx}{dy} \to 0$, the tangent line is parallel to the y-axis (as at G).

7.15. Differentials.

If $f'(x)$ is the derivative of $f(x)$, and Δx is an increment of x, then the *differential* of $f(x)$, denoted by the symbol $df(x)$, *is defined* by the relation.

$$df(x) = f'(x)\,\Delta x. \qquad \dots \qquad (1)$$

If $f(x) = x$, then $f'(x) = 1$, and (1) reduces to $dx = \Delta x$. Thus, when x is the independent variable the differential of $x\,(= dx)$ is identical with Δx. Hence, if $y = f(x)$, then the relation (1) becomes

$$\mathbf{dy = f'(x)\, dx} \quad \ldots \quad (2)$$

i.e., the differential of a function is equal to its derivative multiplied by the differential of the independent variable.

Thus, if $y = \tan x$, $dy = \sec^2 x\, dx$.

From the definition of the differential of a function, the following formulæ for finding differentials are obvious :

$$d(c) = 0, \text{ where } c \text{ is a constant ;}$$

$$d(u + v - w) = du + dv - dw\,;$$

$$d(uv) = u\, dv + v\, du;\; d\left(\frac{u}{v}\right) = \frac{v\, du - u dv}{v^2}.$$

Differentials are especially useful in applications of integral calculus.

Note 1. The students should note carefully that although for the independent variable x, increment Δx and differential dx are equal, this is generally not the case with the dependent variable y, *i.e.*, $\Delta y \neq dy$ generally.

Note 2. The relation (2) can be written as $dy / dx = f'(x)$; thus, the quotient of the differentials of y and x is equal to the derivative of y with respect to x.

Probably on account of the position that $f'(x)$ occupies in equation (2) above, $f'(x)$ is called the *differential coefficient.*

7.16. Approximate Calculations and Small Errors.

If $y = f(x)$, since $\underset{\Delta x \to 0}{Lt} \dfrac{\Delta y}{\Delta x} = f'(x)$, Δy is *approximately* $= f'(x)$ for small values of Δx. Thus dy and Δy may be taken as approximately equal, when Δx (= dx) is small. Hence, when only an *approximate value* of the change of a function is desired, it is usually convenient to calulate the value of the corresponding differential and use this value.

Small errors arising in the value of a function due to an assumed small error in the independent variable may also be calculated on the same principle.

As an illustration, let us consider the following case :

The radius of a sphere is found by measurement to be 7 cm; If an error of .01 cm is made in measuring the radius, find the error made in calculating the surface-area of the sphere.

If S cm^2 be the surface-area of the sphere of radius r cm,

$$S = 4\pi r^2.$$

$$\therefore \quad dS = 8\pi r\, dr.$$

Here, $r = 7$ and $dr = 0{\cdot}01$.

$\therefore$ approximate error in the calculation of the surface-area

$$= dS = 8 \times \tfrac{22}{7} \times 7 \times {\cdot}01 = 1{\cdot}76 \text{ cm}^2$$

Note 1. The *actual error is* $4\pi\left\{(7{\cdot}01)^2 - 7^2\right\}\text{cm}^2$ which is very nearly equal to $1{\cdot}76$ cm^2.

Note 2. If dx is the error in x, then the ratio (i) $\dfrac{dx}{x}$ and (ii) $100 \cdot \dfrac{dx}{x}$ are called respectively the *relative error* (*i.e., error per unit* x) and the *percentage error.* They may be easily obtained by logarithmic defferentiation.

7.17. Illustrative Examples.

Ex. 1. *If the area of a circle increases at a uniform rate, show that the rate of increase of the perimeter varies inversely as the radius.* [*C. P. 1930*]

At any time t, let A be the area, P the perimeter and r the radius of the circle.

Then $A = \pi r^2$; $\quad P = 2\pi r$. $\quad \therefore \quad P^2 = 4\pi A$.

$\therefore$ differentiating both sides with respect to t, we have

$$2P\frac{dP}{dt} = 4\pi\frac{dA}{dt}, \qquad i.\,e., \; \frac{dP}{dt} = \frac{2\pi}{P}\cdot\frac{dA}{dt} = \frac{1}{r}\cdot\frac{dA}{dt}.$$

Since $\dfrac{dA}{dt}$ is constant, $\quad \therefore \; \dfrac{dP}{dt} \propto \dfrac{1}{r}$.

Ex. 2. *A ladder* AB, *2.5 m long, leans against a vertical wall. If the lower end A, which is at a distance of ·7 m from the bottom of the wall, is being moved away on the ground from the wall at the rate of ·2 m per second, find how fast is the top B descending on the wall.*

Let the distances of A and B from O, the bottom of the wall, at time t be x and y. Then the velocities of A and B are $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$; hence, as given here $\dfrac{dx}{dt} = {\cdot}2$.

Now, $x^2 + y^2 = 2{\cdot}5^2$.

Differentiating with respect to t, $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$.

$\therefore \quad \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$.

When $x = {\cdot}7$, $y^2 = 2{\cdot}5^2 - {\cdot}7^2 = 5.76 = 2.4^2$. $\quad y = 2{\cdot}4$.

Hence, when $x = {\cdot}7$, $y = 2{\cdot}4$, $\frac{dx}{dt} = {\cdot}2$;

from (1), $\frac{dy}{dt} = -\frac{{\cdot}7}{2{\cdot}4} \times {\cdot}2 = -\frac{7}{12}$ m per second $= -5\frac{5}{6}$ cm /sec.

$\therefore$ the end B is moving at the rate of $5\frac{5}{6}$ cm per second *towards O* $\left(\because dy/dt \text{ is negative}\right)$, *i. e.,* B is descending at that rate.

Ex. 3. *The adiabatic law for the expansion of air is $PV^{1.4} = k$, where k is a constant. If at a given time the volume is observed to be 20 cm³ and the presure is 50 dyne per square centimetre, at what rate is the presure changing, if the volume is decreasing at the rate of 2 cm³ per second?*

$$PV^{1.4} = k$$

Taking logarithm of both sides and differentiating with respect to t.

$$\frac{1}{P}\cdot\frac{dP}{dt} + 1{\cdot}4\frac{1}{V}\cdot\frac{dV}{dt} = 0$$

Since $V = 20$, $P = 50$ and $\frac{dV}{dt} = -2$

$$\therefore \frac{1}{50} \times \frac{dP}{dt} + 1{\cdot}4 \times \frac{1}{20}\cdot(-2) = 0 \quad \therefore \frac{dP}{dt} = 1{\cdot}4 \times 50 = 7$$

Thus the pressure is changing at 7 dyne per second.

Ex. 4. *If* $y = 2x - \tan^{-1} x - \log\left(x + \sqrt{1+x^2}\right)$, *show that y continually increases as x changes from zero to positive infinity.* [*C. P. 1942*]

Here, $$\frac{dy}{dx} = 2 - \frac{1}{1+x^2} - \frac{1}{x+\sqrt{1+x^2}}\cdot\left(1 + \frac{2x}{2\sqrt{1+x^2}}\right)$$

$$= 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}}$$

Since $\left(1+x^2\right)$ and $\sqrt{1+x^2}$ are each greater than 1.

$\therefore \dfrac{1}{1+x^2}$ and $\dfrac{1}{\sqrt{1+x^2}}$ are each less than 1.

$\therefore \dfrac{dy}{dx}$ is positive ; also $y=0$, when $x=0$.

$\therefore$ for positive value of x, y must be positive and continually increases as x increases from 0 to ∞.

Ex. 5. *Find approximately the values of* $\tan 46^0$ *,given* $1^0 = \cdot 01745$ *radian.*

Let $y = \tan x$. $\therefore dy = \sec^2 x\, dx$

Thus, taking $x = 45^0 \left(=\frac{1}{4}\pi\right)$, $dx = 1^0 \times \cdot 01745 = \cdot 01745$, we have $dy = 2 \times \cdot 01745 = \cdot 03490$

Hence, for an increment of 1^0 in the angle, the increament in the value of $\tan 45^\circ$ is $\cdot 03490$.

$\therefore \tan 46^0 = \tan 45^0 + \cdot 03490 \doteq 1 \cdot 03490$ (approximately)

Ex. 6. *If in a triangle the side c and the angle C remain constant while the remaining elements are changed slightly, show that*

$$\frac{da}{\cos A} + \frac{db}{\cos B} = 0$$

In any triangle, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Since c and C are constants. $\therefore c / \sin C = \text{constant} = K$ suppose.

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = K$$

$\therefore a = K \sin A$, and $\therefore da = K \cos A\ dA$

Also, $b = K \sin B$,and $\therefore db = K \cos B\ dB$

$$\therefore \frac{da}{\cos A} + \frac{db}{\cos B} = K.(dA + dB) = K.d(A+B)$$

$$= K.d(\pi - C)$$

$$= K \times 0 = 0 \qquad (\because C \text{ is a constant})$$

7.18 Miscellaneous Worked Out Examples

Ex. 1. (i) A function $f(x)$ is defined in $0 \le x \le 2$ by

$$f(x) = x^2 + x + 1, \quad 0 \le x \le 1,$$
$$= 2x + 1, \quad 1 \le x \le 2.$$

Examine the continuity and derivability of $f(x)$ at $x=1$.

(ii) $f(x)$ is defined in $[0,2]$ by

$$f(x) = x^2 + x, \quad \text{for } 0 \le x < 1,$$
$$= 2, \quad \text{for } x = 1,$$
$$= 2x^3 - x + 1 \quad \text{for } 1 < x \le 2.$$

Examine the continuity and differentiability of $f(x)$ at $x=1$.

[*C. P. 1992, B. P. '95*]

Solution : (*i*) For *continuity* at $x=1$, we have

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} \left(x^2 + x + 1\right) = 3$$

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} (2x + 1) = 3$$

and $f(1) = 2 \cdot 1 + 1 = 3$

Thus, $\lim_{x \to 1-} f(x) = \lim_{x \to 1+} f(x) = f(1)$. So, $f(x)$ is continuous at $x=1$.

For *derivability* at $x=1$,

$$Lf'(1) = \lim_{h \to 0-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0-} \frac{\left\{(1+h)^2 + (1+h) + 1\right\} - 3}{h}$$

$$= \lim_{h \to 0-} \frac{h^2 + 3h}{h} = \lim_{h \to 0-} (h+3) = 3 \quad [\because h \ne 0]$$

$$Rf'(1) = \lim_{h \to 0+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0+} \frac{\left\{2(1+h) + 1\right\} - 3}{h}$$

$$= \lim_{h \to 0+} \frac{2h}{h} = 2 \quad [\because h \ne 0]$$

Thus, $Lf'(1) \ne Rf'(1)$

Hence $f'(1)$ does not exist, *i.e.*, $f(x)$ is *not derivable* at $x = 1$.

(ii) For *continuity* at $x=1$,

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1-}\left(x^2+x\right) = 2$$

$$\lim_{x\to 1+} f(x) = \lim_{x\to 1+}\left(2x^3-x+1\right) = 2$$

and $f(1)=2$

Hence, $f(x)$ is continuous at $x=1$

For *differentiability* at $x=1$,

$$\lim_{x\to 1-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1-} \frac{\left(x^2+x\right)-2}{x-1}$$

$$= \lim_{x\to 1-} \frac{(x-1)(x+2)}{(x-1)}$$

$$= \lim_{x\to 1-} (x+2), \because x-1\neq 0$$

$$=3$$

i.e., $Lf'(1)=3$

$$\lim_{x\to 1+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1+} \frac{\left(2x^3-x+1\right)-2}{x-1}$$

$$= \lim_{x\to 1+} \frac{\left(2x^2+2x+1\right)(x-1)}{(x-1)}$$

$$= \lim_{x\to 1+} \left(2x^2+2x+1\right), \qquad \because x-1\neq 0$$

$$=5,$$

i.e., $Rf'(1)=5$

since $Lf'(1)\neq Rf'(1)$,

$f(x)$ is *not differentiable* at $x=1$.

Ex. 2. (*i*) Show that the function $f(x)$ defined by

$$f(x) = 3+2x \quad \text{for} \quad -\frac{3}{2}<x\le 0$$

$$= 3-2x \quad \text{for} \quad 0<x\le\frac{3}{2}$$

is continuous but not differentiable at $x=0$. [*B. P. 1999, 2006*] (*ii*) If $f(x) = x$, $\quad 0 < x < 1$

$= 2 - x$, $\quad 1 \le x \le 2$

$= x - \frac{1}{2}x^2$, $\quad x > 2$,

show that $f(x)$ is continuous at $x = 1$ and at $x = 2$, and that $f'(2)$ exists, but $f'(1)$ does not. [*C. P. 1989, ,93, '97*]

Solution : (*i*) For *continuity* of $f(x)$ at $x = 0$,

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} (3 + 2x) = 3$$

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} (3 - 2x) = 3$$

and $f(0) = 3 + 2 \times 0 = 3$

Thus, $\lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x) = f(0)$

Hence, $f(x)$ is continuous at $x = 0$.

For differentiability of $f(x)$ at $x = 0$,

$$Lf'(0) = \lim_{h\to 0-} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0-} \frac{\{3 + 2(0+h)\} - 3}{h}$$

$$= \lim_{h\to 0-} \frac{2h}{h} = 2 \quad (\because h \ne 0)$$

$$Rf'(0) = \lim_{h\to 0+} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0+} \frac{\{3 - 2(0+h)\} - 3}{h}$$

$$= \lim_{h\to 0+} \frac{-2h}{h} = -2 \quad (\because h \ne 0)$$

Since $Lf'(0) \ne Rf'(0)$

$f(x)$ is not differentiable at $x = 0$.

Hence, $f(x)$ is *continuous* but *not differentiable* at $x = 0$.

(*ii*) *Continuity* at $x = 1$ and at $x = 0$.

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1-} (x) = 1$$

$$\lim_{x\to 1+} f(x) = \lim_{x\to 1+} (2-x) = 1$$

and $f(1) = 2-1 = 1$.

$\because \lim_{x\to 1-} f(x) = \lim_{x\to 1+} f(x) = f(1)$, $f(x)$ is *continuous* at $x = 1$.

Again, $\lim_{x\to 2-} f(x) = \lim_{x\to 2-} (2-x) = 0$

$$\lim_{x\to 2+} f(x) = \lim_{x\to 2+} \left(x - \frac{1}{2}x^2\right) = 0$$

and $f(2) = 2-2 = 0$.

$\because \lim_{x\to 2-} f(x) = \lim_{x\to 2+} f(x) = f(2)$, $f(x)$ is *continuous* at $x = 2$.

Differentiability at $x = 1$ and at $x = 2$.

$$Lf'(1) = \lim_{x\to 1-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1-} \frac{x-1}{x-1} = 1 \qquad \because x \neq 1$$

$$Rf'(1) = \lim_{x\to 1+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1+} \frac{(2-x)-1}{x-1}$$

$$= \lim_{x\to 1+} \frac{-(x-1)}{x-1} = -1, \qquad \because x \neq 1$$

$\because$ $Lf'(1) \neq Rf'(1)$, $f'(1)$ *does not exist.*

Again, $Lf'(2) = \lim_{x\to 2-} \dfrac{f(x)-f(2)}{x-2}$

$$= \lim_{x\to 2-} \frac{(2-x)-0}{x-2} = -1, \qquad \because x = 2 \neq 0$$

$$Rf'(2) = \lim_{x\to 2+} \frac{f(x)-f(2)}{x-2}$$

$$= \lim_{x\to 2+} \frac{\left(x - \frac{1}{2}x^2\right) - 0}{x-2}$$

$$= \lim_{x\to 2+} \frac{-x(x-2)}{2(x-2)} = -1, \qquad \because x-2 \neq 0$$

$\because$ $Lf'(2) = Rf'(2)$, $f'(2)$ *exists* and $f'(2) = -1$.

Ex. 3. (*i*) Show that the function

$$f(x) = x^2 \sin\left(\frac{1}{x}\right), \qquad x \neq 0$$
$$= 0, \qquad x = 0$$

is both continuous and derivable at $x = 0$.

[*C. P. 1987, '96, 2000*]

(*ii*) Show that the function

$$f(x) = x \cos\left(\frac{1}{x}\right), \qquad x \neq 0$$
$$= 0, \qquad x = 0$$

is continuous at $x = 0$, but has no derivative there.

[*C. P. 1981, 93*]

Solution : (*i*) For continuity at $x = 0$,

we are to find a δ, depending upon $\in$, such that

$$|f(x) - f(0)| < \in \qquad \text{for} \qquad |x - 0| < \delta.$$

$$\text{or, } \left|x^2 \sin\frac{1}{x}\right| < \in \qquad \text{for} \qquad |x| < \delta,$$

Since, $\left|\sin\frac{1}{x}\right| \leq 1$, the above relations will hold, if we take $|x^2| < \in$

for $|x| < \delta$, *i.e.*, $\delta = \sqrt{\in}$.

So, $f(x)$ is *continuous* at $x = 0$.

For *derivability* at $x = 0$,

We have, $$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{x^2 \sin\left(\frac{1}{x}\right)}{x}$$

$$= \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) \qquad \because x \neq 0$$

$$= \lim_{x \to 0} (x) \times \lim_{x \to 0} \left|\sin\left(\frac{1}{x}\right)\right|$$

$$= 0, \qquad \because \left|\sin\left(\frac{1}{x}\right)\right| \leq 1, \text{ and } \lim_{x \to 0} x = 0.$$

Hence $f'(0)$ exists and $f'(0)=0$.

Thus $f(x)$ is both *continuous* and *derivable* at $x=0$.

(*ii*) $\because$ $\left|\cos\left(\frac{1}{x}\right)\right| \le 1$, by making $|x| < \in$,

We can make $\left|\cos\left(\frac{1}{x}\right)\right| \le \in$,

where $\in$ is any pre-assigned positive quantity, however small.

So, $\lim_{x\to 0}\left(x\cos\frac{1}{x}\right)=0$, Also, $f(0)=0$ by definition.

Thus, $\lim_{x\to 0} f(x)=f(0)$

Hence, $f(x)$ is continuous at $x=0$.

For *derivability*, at $x=0$, we have

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{h\cos\left(\frac{1}{h}\right)}{h}$$

$$=\lim_{h\to 0}\cos\left(\frac{1}{h}\right), \quad \because \quad h\ne 0$$

which does not exist.

Hence, $f'(0)$ does not exist.

So, $f(x)$ is *continuous* at $x=0$, but has *no derivative* there.

Ex. 4. If $f(x)=2|x|+|x-2|$ find $f'(1)$. [*C. P. 1992, 2000, '02*]

Solution : We have

$$|x| = x \quad \text{for} \quad x>0$$
$$=0 \quad \text{for} \quad x=0 \qquad \ldots (1)$$
$$=-x \quad \text{for} \quad x<0$$

and

$$|x-2| = x-2, \quad \text{when } x>2$$
$$=0, \quad \text{when } x=2 \qquad \ldots (2)$$
$$=2-x, \quad \text{when } x<2.$$

To find $f'(1)$, we are concerned with values of x in the neighbourhood of $x=1$.

From (1) and (2), $f(x) = 2x + (2 - x)$

$$= x + 2 \qquad \text{for } 0 < x < 2$$

Now, $f'(1) = \lim\limits_{x \to 1} \dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x \to 1} \dfrac{(x+2) - 3}{x - 1}$

$$= \lim_{x \to 1} \frac{x-1}{x-1} = 1, \qquad \because \; x - 1 \neq 0$$

Ex. 5. Let $f(x) = x^2$, when x is rational

$= 0$ when x is irrational

show that $f'(0) = 0$. [*C. P. 1996, 2001*]

Solution : $f'(0) = \lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$

$$= \lim_{h \to 0} \frac{h^2 - 0}{h} \qquad \text{(when } h \text{ is rational)}$$

$$= \lim_{h \to 0} h = 0$$

Again, $f'(0) = \lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$

$$= \lim_{h \to 0} \frac{0 - 0}{h} \qquad \text{(when } h \text{ is irrational)}$$

$$= 0.$$

$\therefore \quad f'(0) = 0.$

Ex. 6. If $f(x)$ be an even function and $f'(0)$ exists, show that

$f'(0) = 0$. [*C. P. 1983*]

Solution : $\because$ $f(x)$ is an even function, $f(x) = f(-x)$.

or, $f'(x) = -f'(-x)$ . . . (1)

since $f'(0)$ exists, putting $x = 0$ in (1)

$$f'(0) = -f'(0) \text{ i.e., } f'(0) = 0.$$

Ex. 7. Show that the derivative of a differentiable odd function is an even function.

Solution : Let $f(x)$ be a differentiable odd fuction.

Then, $f(x) = -f(-x)$

Since, $f(x)$ is differentiable,

$$f'(x) = -\{-f'(-x)\} = f'(-x),$$

so that $f'(x)$ is an even function.

Ex. 8. Find, from definition, the derivatives of :

(*i*) $\sin\left(\sqrt{x}\right)$, $(x>0)$ [*C. P. 1985*]

(*ii*) $\sin(\log x)$, $(x>0)$ (*iii*) $x^2 \cos x$.

Solution : Let $f(x) = \sin\left(\sqrt{x}\right)$ $(x > 0)$

$$\therefore \quad f'(x) = \lim_{h \to 0} \frac{\sin\sqrt{x+h} - \sin\sqrt{x}}{h}$$

$$= \lim_{h \to 0} \left\{ \frac{\sin\sqrt{x+h} - \sin\sqrt{x}}{\sqrt{x+h} - \sqrt{x}} \times \frac{\sqrt{x+h} - \sqrt{x}}{h} \right\} \quad \ldots \quad (1)$$

Now, $\displaystyle\lim_{h \to 0} \frac{\sin\sqrt{x+h} - \sin\sqrt{x}}{\sqrt{x+h} - \sqrt{x}}$

$$= \lim_{k \to 0} \frac{\sin(y+k) - \sin y}{k}, \qquad \text{where } \sqrt{x+h} = y+k$$

$$= \lim_{k \to 0} \frac{2\cos\left(y + \frac{k}{2}\right) \cdot \sin\frac{k}{2}}{k} \qquad \sqrt{x} = y \text{ as } h \to 0,\ k \to 0$$

$$= \lim_{k \to 0} \cos\left(y + \frac{k}{2}\right) \times \lim_{k \to 0} \frac{\sin\frac{k}{2}}{\frac{k}{2}}$$

$$= \cos y \times 1 = \cos y = \cos\sqrt{x}$$

and $$\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{\left(\sqrt{x+h} - \sqrt{x}\right)\left(\sqrt{x+h} + \sqrt{x}\right)}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{x+h-x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}, \qquad \because h \neq 0$$

$$= \frac{1}{2\sqrt{x}}.$$

Using these results in (1)

$$f'(x) = \cos\sqrt{x} \times \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}} \cdot \cos\sqrt{x}.$$

(*ii*) Let $f(x) = \sin(\log x)$, $(x > 0)$

From definition, $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\sin\{\log(x+h)\} - \sin\{\log(x)\}}{h} \qquad \ldots \quad (1)$$

Let, $\log x = u$

If the increment of u be k corresponding to an increment h of x, $u + k = \log(x+h)$.

i.e., $k = \log(x+h) - u = \log(x+h) - \log x$

Obviously, $k \to 0$, as $h \to 0$, and $h \to 0$, when $k \to 0$.

From (1) we have

$$f'(x) = \lim_{h \to 0} \frac{\sin(u+k) - \sin u}{h}$$

$$= \lim_{k \to 0} \frac{\sin(u+k) - \sin u}{k} \times \lim_{h \to 0} \frac{k}{h}$$

$$= \lim_{k \to 0} \frac{2\cos\left(u + \frac{k}{2}\right)\sin\frac{k}{2}}{k} \times \lim_{h \to 0} \frac{\log(x+h) - \log x}{h}$$

$$= \lim_{k \to 0} \cos\left(u + \frac{k}{2}\right) \times \lim_{k \to 0} \frac{\sin\frac{k}{2}}{\frac{k}{2}} \times \lim_{h \to 0} \frac{\log\left(\frac{x+h}{x}\right)}{h}$$

$$= \cos u \times (1) \times \lim_{h \to 0} \frac{\log\left(1 + \frac{h}{x}\right)}{\frac{h}{x} \times x}$$

$$= \cos u \times \frac{1}{x} \cdot \log_{t \to 0} \frac{\log(1+t)}{t}, \quad \text{where } \frac{h}{x} = t \text{ and } t \to 0 \text{ as } h \to 0$$

$$= \frac{\cos(\log x)}{x} \times 1 = \frac{1}{x} \cdot \cos(\log x).$$

(*iii*) Let, $f(x) = x^2 \cos x$

From definition,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 \cos(x+h) - x^2 \cos x}{h}$$

$$= \lim_{h\to 0} \frac{x^2\{\cos(x+h) - \cos x\} + 2xh\cos(x+h) + h^2\cos(x+h)}{h}$$

$$= x^2 \times \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} + 2x \times \lim_{h\to 0} \cos(x+h) + \lim_{h\to 0} \{h\cos(x+h)\}$$

$$= x^2 \times \lim_{h\to 0} \frac{2\sin\left(x+\frac{h}{2}\right)\sin\left(-\frac{h}{2}\right)}{h} + 2x\cos x + 0$$

$$= -x^2 \times \lim_{h\to 0} \sin\left(x+\frac{h}{2}\right) \times \lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} + 2x\cos x$$

$$= -x^2 \sin x \times 1 + 2x\cos x = 2x\cos x - x^2 \sin x$$

Ex. 9. (*i*) If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, show that

$$\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}.$$ [*B. P. 1991*]

(*ii*) If $\sqrt{1-x^{2n}} + \sqrt{1-y^{2n}} = a^n(x^n - y^n)$, show that

$$\frac{dy}{dx} = \left(\frac{x}{y}\right)^{n-1} \cdot \sqrt{\frac{1-y^{2n}}{1-x^{2n}}}.$$

Solution : (*i*) Let us assume, $x = \sin\theta$ and $y = \sin\phi$

$$\therefore \frac{dx}{d\theta} = \cos\theta, \quad \frac{dy}{d\phi} = \cos\phi$$

$$\because \sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$$

$$\cos\theta + \cos\phi = a\left(\sin\theta - \sin\phi\right)$$

or, $2\cos\frac{1}{2}(\theta+\phi)\cos\frac{1}{2}(\theta-\phi) = 2a\cos\frac{1}{2}(\theta+\phi)\sin\frac{1}{2}(\theta-\phi)$

or, $\cot\frac{1}{2}(\theta-\phi) = a, \qquad \because \quad \cos\frac{1}{2}(\theta+\phi) \neq 0$

or, $\theta - \phi = 2\cot^{-1} a = \text{constant}$

or, $\frac{d}{d\theta}(\theta-\phi) = 0$, or, $1 - \frac{d\phi}{d\theta} = 0$ *i.e.*, $\frac{d\phi}{d\theta} = 1$

Now, $\frac{dy}{dx} = \frac{dy}{d\phi}\cdot\frac{d\phi}{d\theta}\cdot\frac{d\theta}{dx} = \frac{\cos\phi}{\cos\theta} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

(*ii*) Here, $\sqrt{1-x^{2n}} + \sqrt{1-y^{2n}} = a^n\left(x^n - y^n\right)$ . . . . (1)

Let us substitute : $x^n = \sin\theta$ and $y^n = \sin\phi$

Then, $nx^{n-1}\frac{dx}{d\theta} = \cos\theta$, *i.e.*, $\frac{dx}{d\theta} = \frac{\cos\theta}{nx^{n-1}}$ . . . (2)

and $ny^{n-1}\frac{dy}{d\phi} = \cos\phi$, *i.e.*, $\frac{dy}{d\varphi} = \frac{\cos\phi}{ny^{n-1}}$ . . . (3)

From (1), $\sqrt{1-\sin^2\theta} + \sqrt{1-\sin^2\phi} = a^n(\sin\theta - \sin\phi)$

or, $\cos\theta + \cos\phi = a^n(\sin\theta - \sin\phi)$

or, $2\cos\frac{1}{2}(\theta+\phi)\cos\frac{1}{2}(\theta-\phi) = 2\cdot a^n\cdot\cos\frac{1}{2}(\theta+\phi)\sin\frac{1}{2}(\theta-\phi)$

or, $\cot\frac{1}{2}(\theta-\phi) = a^n$

i.e., $\theta - \phi = 2\cot^{-1}\left(a^n\right) = \text{constant}$

or, $1 - \frac{d\phi}{d\theta} = 0$, *i.e.*, $\frac{d\phi}{d\theta} = 1$

Now, $\frac{dy}{dx} = \frac{dy}{d\phi}\cdot\frac{d\phi}{d\theta}\cdot\frac{d\theta}{dx} = \frac{\cos\phi}{ny^{n-1}}\times 1\times\frac{nx^{n-1}}{\cos\theta}$ [From (2), (3)]

$$= \left(\frac{x}{y}\right)^{n-1}\cdot\frac{\cos\phi}{\cos\theta} = \left(\frac{x}{y}\right)^{n-1}\cdot\frac{\sqrt{1-y^{2n}}}{\sqrt{1-y^{2n}}}.$$

Ex. 10. (*i*) If $y=\tan^{-1}\dfrac{2x}{1+15x^2}$, prove that

$$\frac{dy}{dx}=\frac{5}{1+25x^2}-\frac{3}{1+9x^2}$$

(*ii*) If the sum of first n terms of a G.P. with common ratio r is S_n, prove that

$$(r-1)\frac{dS_n}{dr}=(n-1)S_n-n\cdot S_{n-1}.$$

Solution : (*i*) $y=\tan^{-1}\dfrac{2x}{1+15x^2}=\tan^{-1}\dfrac{5x-3x}{1+5x\cdot 3x}$

$$=\tan^{-1}(5x)-\tan^{-1}(3x)$$

$$\frac{dy}{dx}=\frac{5}{1+(5x)^2}-\frac{3}{1+(3x)^2}=\frac{5}{1+25x^2}-\frac{3}{1+9x^2}$$

(*ii*) Let the first term of the G. P. be *a*.

Then $S_n=\dfrac{a\left(r^n-1\right)}{r-1}$

or, $(r-1)S_n=a\left(r^n-1\right)$

Differentiating both the sides *w. r. t.* r,

$$(r-1)\frac{dS_n}{dr}+S_n=na\,r^{n-1}$$

Now, the nth term of the G. P. $=a\,r^{n-1}=S_n-S_{n-1}$

$$\therefore\ (r-1)\frac{dS_n}{dr}+S_n=n\left(S_n-S_{n-1}\right)$$

i.e., $(r-1)\dfrac{dS_n}{dr}=(n-1)S_n-n\cdot S_{n-1}$

Ex. 11. (*i*) If $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x\cdots \text{to } \infty}}}$, show that

$$\frac{dy}{dx}=\frac{\cos x}{2y-1}$$

(*ii*) Show that $\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}+\ldots \text{ to } \infty=\frac{1}{1-x}$

$(0<x<1)$

(*iii*) If $y=x+\cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{x+\text{ to }\infty}}}$

show that $\frac{dy}{dx}=\cfrac{1}{2-\cfrac{x}{x+\cfrac{1}{x+\cfrac{1}{x+\text{to }\infty}}}}$

Solution : (*i*) Here, $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x\cdots\text{to }\infty}}}$

$$=\sqrt{\sin x+y}$$

or, $y^2=\sin x+y$

Differentiating both the sides *w.r.t.* x,

$$2y\frac{dy}{dx}=\cos x+\frac{dy}{dx} \qquad \therefore \frac{dy}{dx}=\frac{\cos x}{2y-1}$$

(*ii*) We have, $(1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)\cdots(1+x^n)$

$$=(1-x^2)(1+x^2)(1+x^4)(1+x^8)\cdots(1+x^n)$$

$$=(1-x^4)(1+x^4)(1+x^8)\cdots(1+x^n)$$

$$=(1-x^n)(1+x^n)=1-x^{2n}$$

$\because\ 0<x<1,\ \lim\limits_{n\to\infty} x^{2n}=0$

$$\therefore \lim_{n\to\infty}\{(1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)\cdots(1+x^n)\}$$

$$=\lim_{n\to\infty}(1-x^{2n})=1$$

$$\therefore \log(1-x)+\log(1+x)+\log(1+x^2)+\log(1+x^4)+\log(1+x^8)+\cdots\text{to }\infty=\log 1=0$$

Differentiating both the sides *w.r.t.* x, we get

$$\frac{-1}{1-x}+\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\cdots to \propto=0,$$

whence, $\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\cdots to \propto=\frac{1}{1-x}$.

(*iii*) Here, $y=x+\frac{1}{y}$

or, $y^2=xy+1$

Differentiating both the sides *w.r.t.* x,

$$2y\frac{dy}{dx}=x\frac{dy}{dx}+y$$

$$\therefore\quad \frac{dy}{dx}=\frac{y}{2y-x}=\frac{1}{2-\frac{x}{y}}=\frac{1}{2-\cfrac{x}{x+\cfrac{1}{x+\cfrac{1}{x+\cdots to \propto}}}}$$

Ex. 12. (*i*) If $f(a)=2,\ f'(a)=1,\ g(a)=-1,\ g'(a)=2$, then find the value of

$$\lim_{x\to a}\frac{g(x)f(a)-g(a)f(x)}{x-a}.$$

(*ii*) Show that the function $f(x)=x|x|$, is differentiable at $x=0$.

Solution : (*i*) $\lim_{x\to a}\frac{g(x)f(a)-g(a)f(x)}{x-a}$

$$=\lim_{x\to a}\frac{f(a)\{g(x)-g(a)\}-g(a)\{f(x)-f(a)\}}{x-a}$$

$$=\lim_{x\to a} f(a)\cdot\frac{\{g(x)-g(a)\}}{(x-a)}-\lim_{x\to a} g(a)\frac{\{f(x)-f(a)\}}{(x-a)}$$

$$=f(a)\cdot g'(a)-g(a)\cdot f'(a)$$

$$=2\times2-(-1)\times1=5.$$

(*ii*) Here, $f(x) = x^2, \quad x > 0$

$= 0, \quad x = 0$

$= -x^2, \quad x < 0$

$$Lf'(0) = \lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0-} \frac{-x^2}{x} = 0$$

$$Rf'(0) = \lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{x^2}{x} = 0$$

$\because$ $Lf'(0) = Rf'(0)$, $f(x)$ is differentiable at $x = 0$, and $f'(0) = 0$.

Ex. 13. (*i*) If g be the inverse of the function f and $f'(x) = \dfrac{1}{1+x^n}$, then prove that $g'(x) = 1 + \{g(x)\}^n$.

(*ii*) If $f(x+y) = f(x) \cdot f(y)$ for all real values of x, y, $f(x) \neq 0$ for any real value of x, and $f'(0)$ is defined and $f'(0) = 2$, prove that for all values of x, $f'(x) = 2f(x)$. Hence find $f(x)$.

Solution : (*i*) $\because$ g is the inverse of f,

$g(x) = f^{-1}(x)$, *i.e.*, $f\{g(x)\} = x$

Differentiating both the sides *w.r.t.* x,

$f'\{g(x)\}g'(x) = 1$.

or, $g'(x) = \dfrac{1}{f'\{g(x)\}} = \dfrac{1}{\dfrac{1}{1+\{g(x)\}^n}}$, $\because$ $f'(x) = \dfrac{1}{1+x^n}$

i.e., $g'(x) = 1 + \{g(x)\}^n$.

(*ii*) $\because$ $f(x+y) = f(x) \cdot f(y)$ ... (1)

for all real values of x and y, putting $x = y = 0$ in (1),

$f(0) = f(0) \cdot f(0)$

i.e., $f(0) = 1$, $\because$ $f(0) \neq 0$

Now, $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{f(x) \cdot f(h) - f(x)}{h}$ [from (1)]

$$= f(x) \times \lim_{h \to 0} \frac{f(h) - f(0)}{h}, \quad \because f(0) = 1$$

$$= f(x) \cdot f'(0)$$

$$= 2f(x), \quad \because \quad f'(0) = 2 \text{, given.}$$

$\therefore \quad f'(x) = 2f(x)$, for all real values of x.

Again, $\dfrac{f'(x)}{f(x)} = 2$

On integration $\log\{f(x)\} = 2x + \log A$, A being constant of integration.

Putting $x = 0$, $\log\{f(0)\} = 0 + \log A$

or, $\log A = \log 1 = 0$, $\quad \therefore A = 1$

Hence, $f(x) = e^{2x}$.

EXAMPLES - VII(C)

1. A point moves on the parabola $3y = x^2$ in such a way that when $x = 3$ the abscissa is increasing at the rate of 3cm per second. At what rate is the ordinate increasing at that point ?
2. A toy spherical ballon being inflated, the radius is increasing at the rate of $\frac{1}{11}$ cm per second. At what rate would the volume be increasing at the instant when $r = 7$ cm ?
3. A circular plate of metal expands by heat so that its radius increases at the rate of $0{\cdot}25$ cm per second. Find the rate at which the surface-area is increasing when the radius is 7 cm.
4. The candle-power C of an incandescent lamp and its voltage V are connected by the equation $C = \dfrac{5V^6}{10^{11}}$.

   Find the rate at which the candle-power increases with the voltage when V=200.
5. If Q units be the heat required to raise the temperature of 1 gram of water from $0°C$ to $t°C$, then it is known that
$$Q = t + 10^{-5}.2t^2 + 10^{-7}.3t^3.$$
   Find the specific heat at $50°C$ the specific heat being the rate of increase of heat per unit degree rise of temperature.

6. A man $1 \cdot 5$ m tall walks away from a lamp-post $4 \cdot 5$ m high at the rate of 4 km per hour.

   (i) How fast is the farther end of his shadow moving on the pavement ?

   (ii) How fast is his shadow lengthening ?

7. If a particle moves according to the law $x \propto t^2$, where x is the distance (measured from a fixed point) travelled in time t, show that the velocity will be proportional to time and the rate of change of velocity will be constant.

8. Water is poured into an inverted conical vessel of which the radius of the base is 2 m and height 4 m, at the rate of 77 litre per minute. At what rate is the water-level rising at the instant when the depth is 70 cm?

9. If the side of an equilateral triangle increases at the rate of $\sqrt{3}$ cm per second and its area at the rate of 12 cm^2 per second, find the side of the triangle.

10. If in the rectilinear motion of a particle $s = ut + \frac{1}{2} f t^2$, when u and f are constants, prove that the velocity at time t is $u + ft$ and the acceleration is f.

11. A man is walking at the rate of 5 km per hour towards the foot of a building 16 m high. At what rate is he approaching the top when he is 12 m from the foot of the building ?

12. A circular ink-blot grows at the rate of 2 cm^2 per second. Find the rate at which the radius is increasing after $2\frac{6}{11}$ second.

13. The volume of a right circular cone remains constant. If the radius of the base is increasing at the rate of 3 cm per second, how fast is the altitude changing when the altitude is 8 cm and radius 6 cm ?

14. Sand is being poured on the ground and forms a pile which has always the shape of a right circular cone whose height is equal to the radius of the base. If sand is falling at the rate of $1 \cdot 54 \text{ m}^3/\text{s}$, how fast is the height of the pile increasing when the height is $0 \cdot 7$ m ?

15. The marginal cost of a commodity being the rate of change in the cost for change in the output, if $f(x) = ax \cdot \dfrac{x+b}{x+c} + d \quad (b > c)$ be the total cost of an output x, show that the marginal cost falls continuously as the output increases.

16. (i) An aeroplane is flying horizontally at a height of $\frac{2}{3}$ km with a velocity of 15 km an hour. Find the rate at which it is receding from a fixed point on the ground which it passed over 2 minute ago.

(ii) A kite is 300 m high and there are 500 m of cord out. If the wind moves the kite horizontally at the rate of 5 km per hour directly away from the person who is flying it, how fast is the cord being paid ?

17. If $\phi(x)=(x-1)e^{x}+1$, show that $\phi(x)$ is positive for all positive values of x.

18. If $f(x)=\cos x+\cos^{2} x+x\sin x$, show that $f(x)$ continually diminishes as x increases from 0 to $\frac{1}{2}\pi$.

19. Show that, for $0<\theta<\frac{1}{2}\pi$

(i) $\dfrac{\sin\theta}{\theta}$ continually diminishes as θ continually increases.

(ii) $\dfrac{4\sin\theta}{2+\cos\theta}-\theta$ increases with θ

20. (i) Prove that, If $0<x<\frac{1}{2}\pi$

(a) $1-\dfrac{1}{2!}x^{2}<\cos x<1-\dfrac{1}{2!}x^{2}+\dfrac{1}{4!}x^{4}$

(b) $1-\dfrac{1}{3!}x^{3}<\sin x<x-\dfrac{1}{3!}x^{3}+\dfrac{1}{5!}x^{5}$

(ii) Show that, if $x>0$,

(a) $x>\log(1+x)>x-\frac{1}{2}x^{2}$ (b) $\frac{1}{2}x^{2}+2x+3>(3-x)e^{x}$

21. Given $y=\dfrac{a\sin x+b\cos x}{c\sin x+d\cos x}$, prove that

(i) If $a=1$, $b=2$, $c=3$, $d=4$, then y decreases for all values of x;

(ii) If $a=2$, $b=1$, $c=3$, $d=4$, then y increases for all values of x.

22. Find the range of values of x for which each of the following functions:

(i) $x^{3}-3x^{2}-24x+30$ (ii) $x^{3}-9x^{2}+24x-16$

(iii) $2x^{3}-9x^{2}+12x-3$ (iv) $x^{4}-4x^{3}+4x^{2}+40$

decreases as x increases.

23. Show that the function $x^{3}-3x^{2}+6x-8$ increases with x.

24. Find the approximate values of the following by the method of differentials :

(i) $\log_e 10\cdot1$, given $\log_e 10 = 2\cdot303$.

(ii) $\log_{10} 10\cdot1$, given $\log_{10} e = 0\cdot4343$

(iii) $\sqrt{6\cdot33}$, given $\sqrt{6\cdot25} = 2\cdot5$

(iv) $\sec^2 46^0$, given $1^0 = 0\cdot0175$ radian

(v) $\sin 62^0$, given $\sin 60^0 = 0\cdot86603$

25. What is the approximate change in $\sin\theta$ per minute change in θ when $\theta = 60°$? (given $1' = 0\cdot00029$ radian).

26. Find the approximately the values of :

(i) $x^3 + 4x^2 + 2x + 2$ when $x = 2\cdot00012$

(ii) $x^4 + 4x^2 + 1$ when $x = 1\cdot997$

27. Find approximately the difference in areas of two circles of radii 7 cm and 7·01 cm.

28. What error in the common logarithm of a number will be produced by an error of 1% in the number ? [$\log_{10} e = 0\cdot4343$]

29. Find the relative error (*i.e.*, error per unit area) in calculating the area of a triangle two of whose sides are 5 cm and 6 cm, when the included angle is taken as 45^0 instead of $45^0 2'$.

30. Show that the relative error in computing the volume of a sphere, due to an error in measuring the radius, is approximately equal to three times the relative error in the radius.

31. The angle of elevation of the top of a tower as observed from a distance of 43 m from the foot of the tower is found to be 60^0 ; if the angle of elevation was really $60^0 1'$, obtain approximately the error in the calculated height. [$1' = 0\cdot00029$ radian]

32. The pressure p and the volume v of a gas are connected by the relation $pv^{1\cdot4} = k$, where k is constant. If there be an increase of 0·7 % in the pressure, show that there is a decrease of 0·5 per cent in the volume.

33. An electric current C as measured by a galvanometer is given by the relation $C \propto \tan\theta$. Find the percentage error in the current corresponding to an error 0·7 per cent in the measurement of θ, when $\theta = 45^0$.

34. The time T of a complete oscillation of a simple pendulum of length l is given by the relation $T = 2\pi\sqrt{\frac{l}{g}}$, where g is a constant. Find approximately the percentage error in the calculated value of T correspoinding to an error of 1 per cent in the value of l.

35. (i) In a triangle if the sides and angles receive small variations, but a and B are constants, show that

$$\tan A \; db = b \, dC.$$

(ii) In a triangle if the sides a, b be constants and the base angles A and B vary, show that

$$\frac{dA}{\sqrt{a^2 - b^2 \sin^2 A}} = \frac{dB}{\sqrt{b^2 - a^2 \sin^2 B}}$$

36. If a triangle ABC inscribed in a fixed circle be slightly varied in such a way as to have its vertices always on the circle, show that

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0$$

ANSWERS

1. 6 cm per second.
2. 56 cm^3/s.
3. 11 cm^2 per second.
4. 96.
5. 1·00425.
6. (i) 6 km per hour. (ii) 2 km per hour.
8. 20 cm per second.
9. 8 cm.
11. 3 km per hour.
12. 0·25 cm per second.
13. Decreasing 8 cm per second.
14. 1 m per second.
16. (i) 9 km per hour. (ii) 4 km per hour.
22. (i) $x > 4$ or < -2. (ii) $x > 4$ or < 2. (iii) $1 < x < 2$. (iv) $x < 0$ and $1 < x < 2$.
24. (i) 2·313. (ii) 1·0043. (iii) 2·516. (iv) 2·07. (v) 0·8835.
25. 0·00015.
26. (i) 30·0036. (ii) 32·856.
27. 0·44 cm^2.
28. 0·0043.
29. 0·00058.
31. 0·05 m *i. e.*, 5 cm.
33. 1·1.
34. 0·5.