8 SUCCESSIVE DIFFERENTIATION

8.1. Definitions and Notations.

We have seen that the derivative of a function of x, say $f(x)$, is in general a function of x. This new function (*i.e.*, the derivative) may have a derivative, which is called the *second derivative* (*or second differential coefficient*) of $f(x)$, the original derivative being called the *first derivative* (*or first differential coefficient*). Similarly, the derivative of the second derivative is called the *third derivative*; and so on for the *n-th derivative*.

Thus, if $y = x^3$, $\dfrac{dy}{dx} = 3x^2$

Again, $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx}\left(3x^2\right) = 6x$

Since $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ is denoted by $\dfrac{d^2y}{dx^2}$. Therefore, $\dfrac{d^2y}{dx^2}$ (i.e., the second derivative of y with respect to x) in this case is $6x$.

Again, $\dfrac{d}{dx}\left(\dfrac{d^2y}{dx^2}\right) = \dfrac{d}{dx}(6x) = 6$

Since $\dfrac{d}{dx}\left(\dfrac{d^2y}{dx^2}\right)$ is denoted by $\dfrac{d^3y}{dx^3}$

$\therefore$ $\dfrac{d^3y}{dx^3}$ (i.e., the third derivative of y with respect to x) is 6 here.

Similarly, the n-th derivative of y with respect to x is generally denoted by $\dfrac{d^ny}{dx^n}$.

If $y = f(x)$, the successive derivatives are also denoted by

$$
\begin{array}{lllll}
 & y_1, & y_2, & y_3, \ldots\ldots\ldots & y_n \\
\text{or,} & y', & y'' & y''', \ldots\ldots\ldots & y^{(n)} \\
\text{or,} & \dot{y}, & \ddot{y}, & \dddot{y} \ldots\ldots\ldots & \\
\text{or,} & f'(x), & f''(x), & f'''(x) \ldots\ldots\ldots & f^{(n)}(x) \\
\text{or,} & Df(x) & D^2f(x), & D^3f(x) \ldots\ldots & D^nf(x)
\end{array}
$$

D standing for the symbol $\dfrac{d}{dx}$.

8.2. The n-th derivatives* of some special functions.

(i) $y = x^n$, where n is a positive integer.

$y_1 = nx^{n-1}$;

$y_2 = n(n-1)x^{n-2}$;

$y_3 = n(n-1)(n-2)x^{n-3}$ and proceeding in a similar manner,

$y_r = n(n-1)(n-2)\ldots\{n-(r-1)\}x^{n-r}$ $(r<n)$

$y_n = n(n-1)(n-2)\ldots 3.2.1 = n!$,

i.e., $D^n(x^n) = n!$

Cor. Since $y_n = n!$, which is a constant, y_{n+1}, y_{n+2}, . . . etc. all zeroes in this case.

(ii) $y = (ax+b)^m$, where m is any number.

$y_1 = ma(ax+b)^{m-1}$;

$y_2 = m(m-1)a^2(ax+b)^{m-2}$;

$y_3 = m(m-1)(m-2)a^3(ax+b)^{m-3}$; and proceeding similarly,

$y_n = m(m-1)(m-2)\ldots\ldots(m-n+1)a^n(ax+b)^{m-n}$,

$D^n(ax+b)^m = m(m-1)(m-2)(m-n+1)a^n(ax+b)^{m-n}$.

If m be a positive integer greater than n,

Since $m(m-1)(m-2)\ldots\ldots(m-n+1) = \dfrac{m!}{(m-n)!}$,

$$\therefore\ D^n(ax+b)^m = \frac{m!}{(m-n)!}a^n(ax+b)^{m-n},$$

m being a positive integer greater than n.

Note. If m be a positive integer less than n, $D^n(ax+b)^m = 0$

When $m=n$, $D^n(ax+b)^n = a^n.n!$.

(iii) $y = e^{ax}$

$\therefore\ y_1 = ae^{ax}$; $y_2 = a^2e^{ax}$; $y_3 = a^3e^{ax}$; .... $y_n = a^ne^{ax}$.

$$\therefore\ D^n\left(e^{ax}\right) = a^ne^{ax}.$$

Cor. (i) $D^n(e^x) = e^x$

Cor. (ii) $y = a^x = e^{x\log_e a}$, $\therefore\ D^n(a^x) = (\log_e a)^n a^x$.

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* Strictly speaking, in these cases, the n-th derivatives are to be established generally by the method of Induction.

(iv) $y = \dfrac{1}{x+a}$

$\therefore\ y_1 = -1.(x+a)^{-2}$

$y_2 = (-1)(-2)(x+a)^{-3} = (-1)^2 .2!.(x+a)^{-3}$

Similarly, $y_3 = (-1)^3 .3!.(x+a)^{-4}$ etc.

$$\therefore\quad D^n\left(\frac{1}{x+a}\right) = \frac{(-1)^n n!}{(x+a)^{n+1}}.$$

Cor. Proceeding as above, $D^n\left\{\dfrac{1}{(ax+b)^m}\right\} = \dfrac{(-1)^n a^n (m+n-1)!}{(m-1)!\,(ax+b)^{m+n}}$

(v) $y = \log(x+a)$

$\therefore\ y_1 = \dfrac{1}{x+a}$. Hence, as in (iv) above

$$D^n\{\log(x+a)\} = \frac{(-1)^{n-1}(n-1)!}{(x+a)^n}.$$

Cor. $D^n\{\log(ax+b)\} = \dfrac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}$.

(vi) $y = \sin(ax+b)$

$y_1 = a\cos(ax+b) = a\sin\left(\frac{1}{2}\pi + ax + b\right)$

$y_2 = a^2\cos\left(\frac{1}{2}\pi + ax + b\right) = a^2\sin\left(2.\frac{1}{2}\pi + ax + b\right)$

$y_3 = a^3\cos\left(2.\frac{1}{2}\pi + ax + b\right) = a^3\sin\left(3.\frac{1}{2}\pi + ax + b\right)$ etc

$$\therefore\quad D^n\{\sin(ax+b)\} = a^n \sin\left(\frac{n}{2}\pi + ax + b\right)$$

Similarly, $D^n\{\cos(ax+b)\} = a^n\cos\left(\dfrac{n}{2}\pi + ax + b\right)$

As particular cases when $b = 0$,

$$D^n\{\sin ax\} = a^n \sin\left(\frac{n}{2}\pi + ax\right);$$

$$D^n\{\cos ax\} = a^n \cos\left(\frac{n}{2}\pi + ax\right).$$

8.3. The n-th derivatives of rational algebraic functions.

The nth derivative of a fraction whose numerator and denominator are both rational integral algebraic functions may be conveniently obtained by resolving the fraction into *partial fractions*. This is shown in Ex. 4, Art. 8.4. The rules for decomposing a fraction into partial fractions are given in the Appendix.

Even when the denominator of a given algebraic fraction cannot be broken up into *real* linear factors, the above method of decomposition can be used by resolving the denominator into *imaginary* linear factors. In this case, *DeMoivre's theorem* is conveniently applied to put the final result in the real form. This is illustrated in Ex. 5, Art. 8.4.

8.4. Illustrative Examples.

Ex. 1. *If* $y = \sin^3 x$, *find* y_n.

$$\sin 3x = 3 \sin x - 4 \sin^3 x .$$

$$\therefore \quad y = \sin^3 x = \frac{1}{4}(3 \sin x - \sin 3x)$$

$$\therefore \quad y_n = \frac{1}{4}\left\{3 \sin\left(\frac{1}{2} n\pi + x\right) - 3^n \sin\left(\frac{1}{2} n\pi + 3x\right)\right\}$$

Ex. 2. *If* $y = \sin 3x . \cos 2x$, *find* y_n.

$$y = \frac{1}{2} \cdot 2 \sin 3x \cos 2x = \frac{1}{2}(\sin 5x + \sin x)$$

$$\therefore \ y_n = \frac{1}{2}\left\{5^n \sin\left(\frac{1}{2} n\pi + 5x\right) + \sin\left(\frac{1}{2} n\pi + x\right)\right\}.$$

Ex. 3. *If* $y = e^{ax} \sin bx$, *find* y_n.

$$y_1 = e^{ax} . a \sin bx + e^{ax} . \cos bx . b$$
$$= e^{ax} . (a \sin bx + b \cos bx) .$$

Let $a = r \cos \phi$, $b = r \sin \phi$, so that

$$r = \left(a^2 + b^2\right)^{\frac{1}{2}}, \ \phi = \tan^{-1} \frac{b}{a}$$

$$\therefore \ y_1 = r e^{ax} \sin (bx + \phi).$$

Similarly, $y_2 = r e^{ax} \{ a \sin (bx + \phi) + b \cos (bx + \phi)\}$

$$= r^2 e^{ax} \sin (bx + 2\phi), \text{ as before.}$$

In a similar way $y_3 = r^3 e^{ax} \sin(bx + 3\phi)$, etc. and generally

$$y_n = r^n e^{ax} \sin(bx + n\phi),$$

i.e., $D^n(e^{ax} \sin bx) = \left(a^2 + b^2\right)^{\frac{1}{2}n} e^{ax} \sin\left(bx + n \tan^{-1} \dfrac{b}{a}\right)$,

Note. Similarly,

$$D^n(e^{ax} \cos bx) = \left(a^2 + b^2\right)^{\frac{1}{2}n} e^{ax} \cos\left(bx + n \tan^{-1} \frac{b}{a}\right),$$

Again, if $y = e^{ax} \sin(bx + c)$,

$$y_n = \left(a^2 + b^2\right)^{\frac{n}{2}} e^{ax} \sin\left(bx + c + n \tan^{-1} \frac{b}{a}\right)$$

and if $y = e^{ax} \cos(bx + c)$,

$$y_n = \left(a^2 + b^2\right)^{\frac{n}{2}} e^{ax} \cos\left(bx + c + n \tan^{-1} \frac{b}{a}\right)$$

Ex. 4. *If* $y = \dfrac{x^2 + x - 1}{x^3 + x^2 - 6x}$, *find* y_n.

$$x^3 + x^2 - 6x = x(x^2 + x - 6) = x(x+3)(x-2)$$

Let $\dfrac{x^2 + x - 1}{x^3 + x^2 - 6x} = \dfrac{A}{x} + \dfrac{B}{x+3} + \dfrac{C}{x-2}$.

Multiplying both sides by $x(x+3)(x-2)$, we get

$$x^2 + x - 1 = A(x+3)(x-2) + Bx(x-2) + Cx(x+3).$$

Putting $x = 0, -3, 2$ successively on both sides, we get

$$A = \frac{1}{6}, \quad B = \frac{1}{3}, \quad C = \frac{1}{2}.$$

$$\therefore \quad y = \frac{1}{6} \cdot \frac{1}{x} + \frac{1}{3} \cdot \frac{1}{x+3} + \frac{1}{2} \cdot \frac{1}{x-2}.$$

$$\therefore \quad y_n = (-1)^n n! \left\{ \frac{1}{6} \cdot \frac{1}{x^{n+1}} + \frac{1}{3} \cdot \frac{1}{(x+3)^{n+1}} + \frac{1}{2} \cdot \frac{1}{(x-2)^{n+1}} \right\}.$$

Ex. 5. *If* $y = \dfrac{1}{x^2 + a^2}$ *find* y_n.

$$y = \frac{1}{(x + ia)(x - ia)} = \frac{1}{2ia}\left(\frac{1}{x - ia} - \frac{1}{x + ia}\right)$$

$$\therefore \; y_n = \frac{(-1)^n n!}{2ia}\left\{\frac{1}{(x-ia)^{n+1}} - \frac{1}{(x+ia)^{n+1}}\right\}$$

$$= \frac{(-1)^n n!}{2ia}\left\{(x-ia)^{-(n+1)} - (x+ia)^{-(n+1)}\right\}.$$

Put $x = r\cos\theta, \qquad a = r\sin\theta$

So that $r = (x^2+a^2)^{\frac{1}{2}}, \qquad \theta = \tan^{-1} a/x.$

Now, $(x-ia)^{-(n+1)} = r^{-(n+1)}(\cos\theta - i\sin\theta)^{-(n+1)}$

$$= r^{-(n+1)}\{\cos(n+1)\theta + i\sin(n+1)\theta\}$$

$(x+ia)^{-(n+1)} = r^{-(n+1)}(\cos\theta + i\sin\theta)^{-(n+1)}$

$$= r^{-(n+1)}\{\cos(n+1)\theta - i\sin(n+1)\theta\}$$

$$\therefore \quad y_n = \frac{(-1)^n n!}{2ia}\cdot r^{-(n+1)}.2i\sin(n+1)\theta.$$

Since $r = \dfrac{a}{\sin\theta}$, $r^{-(n+1)} = \dfrac{a^{-(n+1)}}{\sin^{-(n+1)}\theta} = \dfrac{\sin^{n+1}\theta}{a^{n+1}}$

$$\therefore D^n\left(\frac{1}{x^2+a^2}\right) = \frac{(-1)^n.n!}{a^{n+2}}\sin^{n+1}\theta\,\sin(n+1)\theta,$$

where $\theta = \tan^{-1}\dfrac{a}{x} = \cot^{-1}\dfrac{x}{a}$.

Note. If $y = \dfrac{1}{(x+b)^2+a^2}$, $y_n = \dfrac{(-1)^n n!}{a^{n+2}}\sin^{n+1}\theta\sin(n+1)\theta$,

where $\theta = \tan^{-1}\{a/(b+x)\} = \cot^{-1}\{(b+x)/a\}$.

Cor. If $y = \tan^{-1}x$, $y_1 = \dfrac{1}{1+x^2}$ hence

$$D^n(\tan^{-1}x) = (-1)^{n-1}(n-1)!\sin^n\theta\,\sin n\theta,$$

where $\theta = \tan^{-1}\dfrac{1}{x} = \cot^{-1}x$.

EXAMPLES - VIII (A)

1. Find y_n in the following cases :

(i) $y=(a+bx)^m$. (ii) $y=1/(ax+b)^m$.

(iii) $y=1/(a-x)$. (iv) $y=\log(ax+b)^p$.

(v) $y=\log\{(a-x)/(a+x)\}$. (vi) $y=\sqrt{x}$.

(vii) $y=1/\sqrt{x}$. (viii) $y=(2-3x)^n$.

(ix) $y=\log(ax+x^2)$ (x) $y=10^{3-2x}$.

(xi) $y=x/(a+bx)$. (xii) $y=(a-x)/(a+x)$.

(xiii) $y=x^n/(x-1)$. (xiv) $y=\sin^2 x$.

(xv) $y=\cos 2x \cos x$. (xvi) $y=\cos^3 x$.

(xvii) $y=\sin^2 x \cos^2 x$. (xviii) $y=\sin x \sin 2x \sin 3x$.

(xix) $y=e^x \cos x$. (xx) $y=e^x \sin x \sin 2x$.

(xxi) $y=e^{3x}\sin 4x$. (xxii) $y=e^x \sin^2 x$.

2. Find y_3, if

(i) $y=x^2 \log x$. (ii) $y=e^{\sin x}$.

(iii) $y=e^{1/x}$. (iv) $y=\sin^{-1} x$.

3. Find the n-th derivatives of the following functions :

(i) $\dfrac{1}{x^2-a^2}$. (ii) $\dfrac{1}{x^2+16}$. [*C.P.1993*] (iii) $\tan^{-1}\dfrac{x}{a}$.

(iv) $\dfrac{x}{x^2+a^2}$. (v) $\dfrac{1}{x^4-a^4}$. (vi) $\dfrac{1}{x^2+x+1}$.

(vii) $\dfrac{1}{(x^2+a^2)(x^2+b^2)}$. (viii) $\dfrac{1}{4x^2+4x+5}$.

(ix) $\dfrac{x^4}{(x-1)(x-2)}$. (x) $\dfrac{x^2+1}{(x-1)(x-2)(x-3)}$. [*C. P. 1993*]

(xi) $\dfrac{x^2}{(x+1)^2(x+2)}$. (xii) $\dfrac{x^2}{(x-a)(x-b)}$

(xiii) $\tan^{-1}\dfrac{1+x}{1-x}$. (xiv) $\sin^{-1}\dfrac{2x}{1+x^2}$.

(xv) $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}$. (xvi) $\cot^{-1}\dfrac{x}{a}$.

4. If $y = x^{2n}$, where n is a positive integer, show that

$$y_n = 2^n \{1.3.5\ldots(2n-1)\} x^n.$$

5. If $u = \sin ax + \cos ax$, show that

$$u_n = a^n \{1 + (-1)^n \sin 2ax\}^{1/2}.$$ [B. P. 1993]

6. If $ax^2 + 2hxy + by^2 = 1$. show that

$$\frac{d^2y}{dx^2} = \frac{h^2 - ab}{(hx + by)^3}.$$

7. Find y_2, if

(i) $\sin x + \cos y = 1$.

(ii) $y = \tan(x + y)$.

(iii) $x^3 + y^3 - 3axy = 0$.

8. (a) Find $\frac{d^2y}{dx^2}$ in the following cases :

(i) If $x = a\cos\theta$, $y = b\sin\theta$.

(ii) If $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$. [C. P. 1988]

(b) If $x = \cos t$ and $y = \log t$, then prove that at $t = \frac{\pi}{2}$

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0.$$ [J.E.E. 1985]

9. If $x = f(t)$, $y = \phi(t)$, then

$$\frac{d^2y}{dx^2} = \frac{x_1y_2 - y_1x_2}{x_1^3}$$ [V. P. 1998]

where suffixes denote differentiations with respect to t.

10. If $x\sin\theta + y\cos\theta = a$ and $x\cos\theta - y\sin\theta = b$ then prove that

$$\frac{d^px}{d\theta^p}\cdot\frac{d^qy}{d\theta^q} - \frac{d^qx}{d\theta^q}\cdot\frac{d^py}{d\theta^p}$$ is a constant.

11. Show that $\frac{d^n}{dx^n}\left(\frac{1}{x^2+1}\right) = \frac{(-1)^n n!}{(x^2+1)^{n+1}} \times$

$$\left\{(n+1)x^n - \binom{n+1}{3}x^{n-2} + \binom{n+1}{5}x^{n-4} - \ldots\right\}.$$

12. If $y = \sin mx$, show that

$$\begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix} = 0,$$

where suffixs of y denote the order of differentiations of y with respect to x.

ANSWERS

1. (i) $(-1)^n m(m-1)(m-2)\ldots(m-n+1)b^n(a-bx)^{m-n}$

(ii) $\dfrac{(-1)^n m(m-1)(m-2)\ldots(m-n+1)a^n}{(a+bx)^{m+n}}$

(iii) $\dfrac{n!}{(a-x)^{n+1}}$. (iv) $\dfrac{(-1)^{n-1}p.a^n(n-1)!}{(ax+b)^n}$

(v) $(n-1)!\left\{\dfrac{-1}{(a-x)^n}+\dfrac{(-1)^n}{(a+x)^n}\right\}$. (vi) $(-1)^{n-1}\cdot\dfrac{1.3.5\ldots(2n-3)}{2^n x^{n-\frac{1}{2}}}$.

(vii) $(-1)^n\cdot\dfrac{1.3.5\ldots(2n-1)}{2^n x^{n+\frac{1}{2}}}$. (viii) $(-1)^n.3^n.n!$

(ix) $(-1)^{n-1}(n-1)!\left\{\dfrac{1}{x^n}+\dfrac{1}{(a+x)^n}\right\}$. (x) $10^{3-2x}.(-2)^n.(\log_e 10)^n$

(xi) $\dfrac{(-1)^{n+1}ab^{n-1}.n!}{(a+bx)^{n+1}}$. (xii) $\dfrac{2a(-1)^n n!}{(a+x)^{n+1}}$.

(xiii) $\dfrac{(-1)^n n!}{(x-1)^{n+1}}$. (xiv) $-2^{n-1}\cos\left(\dfrac{1}{2}n\pi+2x\right)$.

(xv) $\dfrac{1}{2}\left\{3^n\cos\left(\dfrac{1}{2}n\pi+3x\right)+\cos\left(\dfrac{1}{2}n\pi+x\right)\right\}$.

(xvi) $\dfrac{1}{4}\left\{3\cos\left(\dfrac{1}{2}n\pi+x\right)+3^n\cos\left(\dfrac{1}{2}n\pi+3x\right)\right\}$.

(xvii) $-2^{2n-2}\cos\left(\dfrac{1}{2}n\pi+4x\right)$.

(xviii) $\dfrac{1}{4}\left\{4^n\sin\left(\dfrac{1}{2}n\pi+4x\right)+2^n\sin\left(\dfrac{1}{2}n\pi+2x\right)-6^n.\sin\left(\dfrac{1}{2}n\pi+6x\right)\right\}$

(xix) $2^{\frac{1}{2}n} e^x \cos\left(x + \frac{1}{4} n\pi\right)$.

(xx) $\frac{1}{2} e^x \left\{2^{\frac{1}{2}n} \cos\left(x + \frac{1}{4} n\pi\right) - 10^{\frac{1}{2}n} \cos\left(3x + n \tan^{-1} 3\right)\right\}$.

(xxi) $5^n e^{3x} \sin\left(4x + n \tan^{-1} \frac{4}{3}\right)$.

(xxii) $\frac{1}{2} e^x \left\{1 - 5^{\frac{1}{2}n} \cos\left(2x + n \tan^{-1} 2\right)\right\}$.

2. (i) $2/x$ (ii) $-e^{\sin x} \cos x . \sin x . (\sin x + 3)$

(iii) $-\dfrac{\left(1 + 6x + 6x^2\right)e^{\frac{1}{x}}}{x^6}$. (iv) $\dfrac{\left(1 + 2x^2\right)}{\left(1 - x^2\right)^{5/2}}$.

3. (i) $\dfrac{(-1)^n n!}{2a} \left\{\dfrac{1}{(x-a)^{n+1}} - \dfrac{1}{(x+a)^{n+1}}\right\}$.

(ii) $\dfrac{(-1)^n n! \sin^{n+1}\theta \ \sin(n+1)\theta}{4^{n+2}}$, where $\theta = \tan^{-1} \dfrac{4}{x}$.

(iii) $\dfrac{(-1)^{n-1}(n-1)! \ \sin^n \theta \sin n\theta}{a^n}$, where $\theta = \tan^{-1} \dfrac{a}{x}$.

(iv) $\dfrac{(-1)^n n!}{a^{n+1}} \sin^{n+1}\theta \ \cos(n+1)\theta$, where $\theta = \tan^{-1} \dfrac{a}{x}$.

(v) $\dfrac{(-1)^n n!}{4a^3} \left\{\dfrac{1}{(x-a)^{n+1}} - \dfrac{1}{(x+a)^{n+1}} - \dfrac{2}{a^{n+1}} \sin^{n+1}\theta \ \sin(n+1)\theta\right\}$

where $x = a \cot\theta$.

(vi) $\dfrac{(-1)^n . 2^{n+2} n!}{\left(\sqrt{3}\right)^{n+2}} \sin^{n+1}\theta \ \sin(n+1)\theta$, where $\theta = \tan^{-1}\left\{\sqrt{3}/(2x+1)\right\}$.

(vii) $\dfrac{(-1)^n n!}{a^2 - b^2} \left\{\dfrac{\sin^{n+1}\theta \ \sin(n+1)\theta}{b^{n+2}} - \dfrac{\sin^{n+1}\phi \ \sin(n+1)\phi}{a^{n+2}}\right\}$

(viii) $(-1)^n n! \frac{1}{4} \sin^{n+1}\theta \ \sin(n+1)\theta$, where $\cot\theta = x + \frac{1}{2}$

(ix) $(-1)^n n! \left\{\dfrac{16}{(x-2)^{n+1}} - \dfrac{1}{(x-1)^{n+1}}\right\}$, when $n > 2$.

(x) $(-1)^n n!\left\{\dfrac{16}{(x-1)^{n+1}}-\dfrac{5}{(x-2)^{n+1}}+\dfrac{5}{(x-3)^{n+1}}\right\}$.

(xi) $(-1)^n n!\left\{\dfrac{n+1}{(x+1)^{n+2}}-\dfrac{3}{(x+1)^{n+1}}+\dfrac{4}{(x+2)^{n+1}}\right\}$.

(xii) $\dfrac{(-1)^n n!}{(a-b)}\left\{\dfrac{a^2}{(x-a)^{n+1}}-\dfrac{b^2}{(x+a)^{n+1}}\right\}$.

(xiii) $(-1)^{n-1}(n-1)!\sin^n\theta \; \sin n\theta$, where $\cot\theta = x$.

(xiv) $2(-1)^{n-1}(n-1)!\sin^n\theta \; \sin n\theta$, where $\cot\theta = x$.

(xv) $\dfrac{1}{2}(-1)^{n-1}(n-1)!\sin^n\theta \; \sin n\theta$, where $\cot\theta = x$.

(xvi) $\dfrac{(-1)^n\,(n-1)!\,\sin^n\theta\;\sin n\theta}{a^n}$, where $\theta = \cot^{-1}\dfrac{x}{a}$.

7. (i) $-\dfrac{\sin^2 x+\cos y}{\sin^3 y}$ (ii) $-\dfrac{2\left(1+y^2\right)}{y^5}$. (iii) $-\dfrac{2a^3xy}{\left(y^2-ax\right)^3}$.

8. (a) (i) $-\dfrac{b}{a^2}\operatorname{cosec}^3\theta$ (ii) $\dfrac{1}{4a}\sec^4\dfrac{\theta}{2}$

8.5. Leibnitz's Theorem*. (*n-th derivative of the product of two functions*)

If u and v are two functions of x, each possessing derivatives upto nth order, then the nth derivative of their product, i.e.,

$$(uv)_n = u_n v + {}^nc_1 u_{n-1} v_1 + {}^nc_2 u_{n-2} v_2 + \ldots + {}^nc_r u_{n-r} v_r + \ldots + u v_n,$$

where the suffixes of u and v denote the order of differentiations of u and v with respect to x.

Let $y = uv$.

By actual differentiation, we have

$$y_1 = u_1 v + u v_1,$$

$$y_2 = u_2 v + 2u_1 v_1 + u v_2 = u_2 v + {}^2c_1 u_1 v_1 + u v_2,$$

$$y_3 = u_3 v + 3u_2 v_1 + 3u_1 v_2 + u v_3,$$

$$= u_3 v + {}^3c_1 u_2 v_1 + {}^3c_2 u_1 v_2 + u v_3,$$

The theorem is thus seen to be true when $n = 2$ and 3.

* Leibnitz (1646-1716) was a German mathematician, who invented Calculus in Germany, as Newton did in England.

Let us *assume*, therefore, that

$$y_n = u_n v + {}^n c_1 u_{n-1} v_1 + {}^n c_2 u_{n-2} v_2 + \dots + {}^n c_r u_{n-r} v_r + \dots + u v_n,$$

where n has any particular value.

$\therefore$ differentiating,

$$y_{n+1} = u_{n+1} v + \left({}^n c_1 + 1\right) u_n v_1 + \left({}^n c_2 + {}^n c_1\right) u_{n-1} v_2 + \dots$$
$$+ \left({}^n c_r + {}^n c_{r-1}\right) u_{n-r+1} v_r + \dots + u v_{n+1},$$

Since ${}^n c_r + {}^n c_{r-1} = {}^{n+1} c_r$ and ${}^n c_1 + 1 = {}^{n+1} c_1$

$$\therefore \quad y_{n+1} = u_{n+1} v + {}^{n+1} c_1\, u_n v_1 + {}^{n+1} c_2\, u_{n-1} v_2 + \dots$$
$$+ {}^{n+1} c_r\, u_{n-r+1} v_r + \dots + u v_{n+1}.$$

Thus, if the theorem holds for n differentiations, it also holds for $n+1$. But it is proved to hold for 2 and 3 differentiations; hence it holds for four, and so on, and thus the therem is true for every positive integral value of n.

8.6. Important results of symbolic operation.

If $F(D)$ be any rational integral algebraic function of D or $\frac{d}{dx}$ (the symbolic operator), i.e., if

$$F(D) = A_n D^n + A_{n-1} D^{n-1} + \dots\dots + A_1 D + A$$
$$= \sum A_r D^r, \text{ where } A_r \text{ is independent of } D, \text{ then}$$

(i) $F(D) e^{ax} = F(a) e^{ax}$.

(ii) $F(D)\, e^{ax} V = e^{ax} F(D+a)\, V$, *V being a function of x.*

(iii) $F(D^2) \begin{cases} \sin\,(ax+b) \\ \cos\,(ax+b) \end{cases} = F\left(-a^2\right) \begin{cases} \sin\,(ax+b) \\ \cos\,(ax+b) \end{cases}$

Proof :

(i) Since $\quad D^r e^{ax} = a^r e^{ax},$

$$\therefore\ F(D) e^{ax} V = \sum A_r D^r e^{ax}$$
$$= \sum A_r a^r e^{ax}$$
$$= \left(\sum A_r D^r\right) e^{ax}$$
$$= F(a) e^{ax}$$

(ii) Let $y = e^{ax}V$. Since $D^r e^{ax} = a^r e^{ax}$,

$\therefore$ by Leibnitz's Theorem, we have

$$y_n = e^{ax}\left(a^n V + {}^n c_1 a^{n-1} DV + {}^n c_2 a^{n-2} D^2 V + \ldots + D^n V\right)$$

which by analogy with the Binomial theorem may be written as

$$D^n\left(e^{ax}V\right) = e^{ax}(D+a)^n V,$$

$$\begin{aligned} F(D)\, e^{ax}V &= \left(\sum A_r D^r\right) e^{ax} V \\ &= \sum A_r D^r e^{ax} V \\ &= e^{ax} \sum A_r (D+a)^r V \\ &= e^{ax} F(D+a) V. \end{aligned}$$

(iii) We have $D \sin(ax+b) = a\cos(ax+b)$, and so on

$$D^2 \sin(ax+b) = \left(-a^2\right)\sin(ax+b);$$

$$D^{2r} \sin(ax+b) = \left(-a^2\right)^r \sin(ax+b).$$

Hence, as in (i) and (ii), it follows that

$$F\left(D^2\right)\sin(ax+b) = F\left(-a^2\right)\sin(ax+b)$$

Similarly, $F\left(D^2\right)\cos(ax+b) = F\left(-a^2\right)\cos(ax+b)$

8.7 Illustrative Examples.

Ex. 1. *If* $y = e^{ax}x^3$, *find* y_n.

Let $u = e^{ax}$, $v = x^3$. Now, $u_n = a^n e^{ax}$.

$\therefore$ by Leibnitz's Theorem,

$$y_n = a^n e^{ax} x^3 + n.a^{n-1} e^{ax}.3x^2 + \frac{n(n-1)}{2!}\cdot a^{n-1} e^{ax}.6x$$

$$+\frac{n(n-1)(n-2)}{3!}\cdot a^{n-2} e^{ax}.6$$

$$= e^{ax} a^{n-3}\left\{a^3x^3 + 3na^2x^2 + 3n(n-1)ax + n(n-1)(n-2)\right\}$$

Ex. 2. *If* $y = a\cos(\log x) + b\sin(\log x)$, *show that* $x^2 y_2 + xy_1 + y = 0$.

Differentiating,

$$y_1 = -a\sin(\log x).\frac{1}{x} + b\cos(\log x).\frac{1}{x};$$

$$\therefore\ xy_1 = -a\sin(\log x) + b\cos(\log x).$$

Differentiating again,

$$xy_2 + y_1 = -a\cos(\log x).\frac{1}{x} - b\sin(\log x).\frac{1}{x},$$

$$\therefore\ x^2y_2 + xy_1 = -(a\cos\log x + b\sin\log x) = -y,$$

$$\therefore\ x^2y_2 + xy_1 + y = 0.$$

Note. This is called the differential equation formed from the above equation.

Ex. 3. *Differentiate n times the equation*

$$(1+x^2)\,y_2 + (2x-1)y_1 = 0.$$

By Leibnitz's Theorem,

$$\frac{d^n}{dx^n}\{y_2(1+x^2)\} = y_{n+2}(1+x^2) + n.y_{n+1}.2x + \frac{n(n-1)}{2!}y_n\,2$$

$$\frac{d^n}{dx^n}\{y_2(2x-1)\} = y_{n+1}(2x-1) + n.y_n.2$$

adding,

$$(1+x^2)y_{n+2} + \{2(n+1)x-1\}y_{n+1} + n(n+1)\,y_n = 0.$$

Ex. 4. *Find the value of* y_n *for* $x=0$, *when* $y = e^{a\sin^{-1}x}$. [*C. P. 2004*]

From the value of y, when $x=0$, $y=1$

Here $y_1 = e^{a\sin^{-1}x}.a\dfrac{1}{\sqrt{1-x^2}}$ ... (1)

$$= ay\frac{1}{\sqrt{1-x^2}}.$$

$$\therefore\quad y_1^2\left(1-x^2\right) = a^2y^2.$$

Differentiating, $2y_1y_2\left(1-x^2\right) + y_1^2(-2x) = 2a^2yy_1$,

or, $\left(1-x^2\right)y_2 - xy_1 - a^2y = 0.$ ... (2)

Differentiating this n times by Leibnitz's Theorem as in Ex. 3, we easily get $\left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} - \left(n^2+a^2\right)y_n = 0.$

Putting $x=0$, $(y_{n+2})_0 = \left(n^2+a^2\right)(y_n)_0.$ ... (3)

Replacing n by $n-2$, we get, similarly

$$(y_n)_0 = \{(n-2)^2 + a^2\}(y_{n-2})_0$$

$$= \{(n-2)^2 + a^2\}\{(n-4)^2 + a^2\}(y_{n-4})_0$$

Also from (1) and (2), $(y_1)_0 = a,\quad (y_2)_0 = a^2$.

Thus $(y_n)_0 = \{(n-2)^2 + a^2\}\{(n-4)^2 + a^2\}\ldots$

$$(4^2 + a^2)(2^2 + a^2)a^2 \text{ , if } n \text{ is even}$$

and $= \{(n-2)^2 + a^2\}\{(n-4)^2 + a^2\}\ldots$

$$(3^2 + a^2)(1^2 + a^2)a^2, \text{ if } n \text{ is odd.}$$

Note. The value of y_n for $x = 0$ is shortly denoted by $(y_n)_0$.

8.12 Miscellaneous Worked Out Examples

Ex. 1. (*i*) If $F(x) = f(x)\,\phi(x)$ and $f'(x)\,\phi'(x) = k$, (k is a constant), then show that

$$\frac{F''}{F} = \frac{f''}{f} + \frac{\phi''}{\phi} + \frac{2k}{f\phi}, \qquad (F(x) \neq 0)$$

(*ii*) If $x = \sin t$, $y = \sin kt$ where k is a constant, show that

$$\left(1 - x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + k^2 y = 0.$$

Solution : (*i*) $\because$ $F(x) = f(x)\,\phi(x)$,

$$F'(x) = f'(x)\,\phi(x) + f(x)\,\phi'(x) \qquad \ldots \quad (1)$$

Differentiating (1) w.r.t. x,

$$F''(x) = f''(x)\,\phi(x) + f'(x)\,\phi'(x) + f'(x)\,\phi'(x) + f'(x)\,\phi''(x)$$

$$= f''(x)\,\phi(x) + f(x)\,\phi''(x) + 2k$$

$\because$ $F(x) = f(x)\,\phi(x) \neq 0$, dividing the left-hand side by $F(x)$ and the right-hand side by $f(x)\ \phi(x)$, we get

$$\frac{F''(x)}{F(x)} = \frac{f''(x)}{f(x)} + \frac{\phi''(x)}{\phi(x)} + \frac{2k}{f(x)\,\phi(x)}$$

i.e., $\dfrac{F''}{F} = \dfrac{f''}{f} + \dfrac{\phi''}{\phi} + \dfrac{2k}{f\,\phi}$.

(*ii*) $\because\ x = \sin t,\ \dfrac{dx}{dt} = \cos t$

and $y = \sin kt,\ \dfrac{dy}{dt} = k\cos kt$

$$\therefore\ \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{k\cos kt}{\cos t}$$

and
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{k\cos kt}{\cos t}\right)\cdot\frac{dt}{dx}$$

$$= \frac{\cos t\left(-k^2\sin kt\right) - k\cos kt(-\sin t)}{\cos^2 t}\cdot\frac{1}{\cos t}$$

$$= \frac{\left\{-k^2\cos t\sin kt + k\sin t\cdot\cos kt\right\}}{\left(1-\sin^2 t\right)\cos t}$$

$$= \frac{1}{1-x^2}\cdot\left\{-k^2\sin kt + \sin t\left(\frac{k\cos kt}{\cos t}\right)\right\}$$

$$= \frac{1}{1-x^2}\cdot\left\{-k^2y + x\frac{dy}{dx}\right\}$$

or, $\left(1-x^2\right)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + k^2y = 0$

Ex. 2. If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$, prove that $\left(x^2-1\right)y_2 + xy_1 - m^2y = 0$,

where, $y_1 = \dfrac{dy}{dx}$, $\quad y_2 = \dfrac{d^2y}{dx^2}$.

Solution : $\because\ y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$, $\therefore\ a^2 - 2ax + 1 = 0$, where $a = y^{\frac{1}{m}}$

$$\therefore\ a = \frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1}$$

or, $y^{\frac{1}{m}} = x \pm \sqrt{x^2-1}$

Taking logarithm of both the sides,

$$\frac{1}{m}\cdot\log y = \log\left(x \pm \sqrt{x^2-1}\right)$$

Differentiaing both the sides w. r. t. x,

$$\frac{1}{m}\cdot\frac{1}{y}\cdot y_1 = \frac{1}{\left(x \pm \sqrt{x^2-1}\right)} \times \left\{1 \pm \frac{x}{\sqrt{x^2-1}}\right\} \text{ where } y_1 = \frac{dy}{dx}$$

or, $\dfrac{y_1}{my} = \pm\dfrac{1}{\sqrt{x^2-1}}$

or, $\left(x^2-1\right)y_1^2 = m^2 y^2$

Differentiating again w.r.t. x,

$$\left(x^2-1\right)2y_1y_2 + 2x\cdot y_1^2 = m^2\cdot 2y\cdot y_1, \text{ where } y_2 = \frac{d^2y}{dx^2}$$

or, $\left(x^2-1\right)y_2 + xy_1 - m^2y = 0, \qquad 2y_1 \neq 0.$

Ex. 3. (*i*) If $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$ verify that

$$\frac{d^2y}{dx^2} = \frac{1}{4a}\cdot\sec^4\frac{\theta}{2}.$$ [*C. P. 1988*]

(*ii*) If $x = 2\cos\theta - \cos 2\theta$, $y = 2\sin\theta + \sin 2\theta$ find $\dfrac{d^2y}{dx^2}$ at $\theta = \dfrac{\pi}{2}$.

[*C. P. 1990*]

Solution : (*i*) $\because x = a(\theta + \sin\theta)$, $\dfrac{dx}{d\theta} = a(1+\cos\theta)$

and $y = a(1-\cos\theta)$, $\dfrac{dy}{d\theta} = a\sin\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\tan\frac{\theta}{2}\right) = \frac{d}{d\theta}\left(\tan\frac{\theta}{2}\right)\cdot\frac{d\theta}{dx}$$

$$= \frac{1}{2}\sec^2\frac{\theta}{2}\cdot\frac{1}{a(1+\cos\theta)}$$

$$= \frac{1}{4a}\cdot\sec^4\frac{\theta}{2}.$$

(ii) $\because x = 2\cos\theta - \cos 2\theta, \quad \dfrac{dx}{d\theta} = 2\sin 2\theta - 2\sin\theta$

and $\because y = 2\sin\theta + \sin 2\theta, \quad \dfrac{dy}{d\theta} = 2\cos\theta + 2\cos 2\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{2(\cos 2\theta + \cos\theta)}{2(\sin 2\theta - \sin\theta)} = \frac{2\cos\frac{3\theta}{2}\cdot\cos\frac{\theta}{2}}{2\cos\frac{3\theta}{2}\cdot\sin\frac{\theta}{2}}$$

or, $\dfrac{dy}{dx} = \cot\dfrac{\theta}{2}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\cot\frac{\theta}{2}\right) = \frac{d}{d\theta}\left(\cot\frac{\theta}{2}\right)\cdot\frac{d\theta}{dx}$$

$$= \frac{-\operatorname{cosec}^2\frac{\theta}{2}}{2\cdot 2(\sin 2\theta - \sin\theta)}$$

at $\theta = \dfrac{\pi}{2}, \quad \dfrac{d^2y}{dx^2} = \dfrac{-\operatorname{cosec}^2\frac{\pi}{4}}{4\left(\sin\pi - \sin\frac{\pi}{2}\right)} = \dfrac{1}{2}$.

Ex. 4. (*i*) If $\sin x = \dfrac{2t}{1+t^2}$, $\cot y = \dfrac{1-t^2}{2t}$, find the value of $\dfrac{d^2x}{dy^2}$.

[*C. P. 1987*]

(*ii*) If $y = \dfrac{x}{x+1}$, show that $y_5(0) = 5!$ [*C. P. 1991, 2003*]

(*iii*) If $y = 2\cos x\,(\sin x - \cos x)$, show that $(y_{10})_0 = 2^{10}$.

[*C. P. 1992, 2000, 2002, 2005, 2007*]

Solution : (*i*) $\because \sin x = \dfrac{2t}{1+t^2}$,

$$x = \sin^{-1}\frac{2t}{1+t^2} = 2\tan^{-1}(t) \qquad \ldots \quad (1)$$

$$\because \cot y = \frac{1-t^2}{2t}, \quad \tan y = \frac{2t}{1-t^2}$$

or, $y = \tan^{-1}\dfrac{2t}{1-t^2} = 2\tan^{-1}(t) \qquad \ldots \quad (2)$

From (1) and (2) $x = y$

$$\therefore \quad \frac{dx}{dy} = 1 \text{ and } \frac{d^2x}{dy^2} = 0.$$

(ii) $y = \dfrac{x}{x+1} = \dfrac{x+1-1}{x+1} = 1-(x+1)^{-1}$

$$y_1 = (-1)(-1)(x+1)^{-2}$$

$$y_2 = (-1)(-1)(-2)(x+1)^{-3} = (-1)^3 2!(x+1)^{-3}$$

$$y_3 = (-1)^4 \, 3!(x+1)^{-4}$$

Proceeding in this way, we have

$$y_5 = (-1)^6 \, 5!(x+1)^{-6}$$

$$\therefore \quad (y_5)_0 = (-1)^6 \, 5!(1)^{-6} = 5!$$

(iii) $y = 2\cos x(\sin x - \cos x)$

$$= 2\sin x\cos x - 2\cos^2 x$$

$$= \sin 2x - (1+\cos 2x)$$

$$= \sin 2x - \cos 2x - 1$$

$$y_{10} = 2^{10}\sin\left\{10\cdot\frac{\pi}{2} + 2x\right\} - 2^{10}\cos\left\{10\cdot\frac{\pi}{2} + 2x\right\}$$

[vide art. 5·2 (vi)]

If $y = \sin ax$, $y_n = a^n \cdot \sin\left(n\dfrac{\pi}{2} + x\right)$, and

if $y = \cos ax$, $y_n = a^n \cdot \cos\left(n\dfrac{\pi}{2} + x\right)$

$$\therefore (y_{10})_0 = 2^{10}\cdot\sin 5\pi - 2^{10}\cos 5\pi = 2^{10}\cdot 0 - 2^{10}\times(-1) = 2^{10}.$$

Ex. 5. If $y = \dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$, $|x| < 1$, show that

(i) $\left(1-x^2\right)y_2 - 3x\,y_1 - y = 0$

(ii) $\left(1-x^2\right)y_{n+2} - (2n+3)xy_{n+1} - (n+1)^2 y_n = 0$ [C. P. 1993]

Solution : $\because \quad y = \dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$

$$\left(1-x^2\right)y^2 = \left(\sin^{-1} x\right)^2$$

Differentiating w.r.t. x.

$$\left(1-x^2\right)2yy_1 - 2xy^2 = \frac{2\sin^{-1} x}{\sqrt{1-x^2}} = 2y$$

or, $\left(1-x^2\right)y_1 - xy = 1 \qquad (\because \ 2y \neq 0)$

Differentiating again w. r. t. x.,

$$\left(1-x^2\right)y_2 - 2xy_1 - y - xy_1 = 0$$

or, $\left(1-x^2\right)y_2 - 3xy_1 - y = 0 \qquad \ldots \quad (1)$

Differentiating (1) n times by Leibnitz's theorem,

$$y_{n+2}\left(1-x^2\right) + n \cdot y_{n+1}(-2x) + \frac{n(n-1)}{1 \cdot 2} \cdot y_n \quad (-2)$$

$$-3\left\{y_{n+1} \cdot x + n \cdot y_n \cdot 1\right\} - y_n = 0$$

or, $\left(1-x^2\right)y_{n+2} - (2n+3)xy_{n+1} - (n+1)^2 y_n = 0.$

Ex. 6. If $y = \cos\left(10\cos^{-1} x\right)$, show that $\left(1-x^2\right)y_{12} = 21xy_{11}$

[*C. P. 1983*]

Solution : $y = \cos\left(10\cos^{-1} x\right) \qquad \ldots \quad (1)$

$$y_1 = -\sin\left(10\cos^{-1} x\right)\frac{(-10)}{\sqrt{1-x^2}} = \frac{10\sin\left(10\cos^{-1} x\right)}{\sqrt{1-x^2}}$$

$$\left(1-x^2\right)y_1^2 = 100\sin^2\left(10\cos^{-1} x\right)$$

$$= 100\left\{1-\cos^2\left(10\cos^{-1} x\right)\right\}$$

$$= 100\left(1-y^2\right) \qquad [\text{ from } (1).]$$

Differentiating again *w.r.t.* x

$$\left(1-x^2\right)2y_1y_2 - 2x \cdot y_1^2 = -2 \cdot 100y \cdot y_1$$

or, $\left(1-x^2\right)y_2 = xy_1 - 100y \qquad (\because \ 2y_1 \neq 0) \qquad \ldots \quad (2)$

Differentiating (2) 10 times with the help of Leibnitz's theorem,

$$\left(1-x^2\right)y_{12}+10\cdot y_{11}(-2x)+\frac{10\cdot 9}{1\cdot 2}\cdot y_{10}(-2)$$
$$=xy_{11}+10\cdot y_{10}\cdot(1)-100y_{10}$$

or, $\left(1-x^2\right)y_{12}=21\cdot x\cdot y_{11}\cdot$

Ex. 7. (*i*) If $f(x)=x^n$, prove that

$$f(1)+\frac{f'(1)}{1!}+\frac{f''(1)}{2!}+\frac{f'''(1)}{3!}+\cdots+\frac{f^n(1)}{n!}=2^n$$

(*ii*) If $f(x)=\tan x$ and n is a positive integer, prove with the help of Leibnitz's theorem that

$$f^n(0)-{}^nc_2 f^{n-2}(0)+{}^nc_4(0)-\cdots=\sin\left(n\cdot\frac{\pi}{2}\right)$$ [*C. P. 1992*]

Solution : (*i*) $\because\ f(x)=x^n$

$$f'(x)=nx^{n-1},\quad f''(x)=n(n-1)x^{n-2}$$

$$f^r(x)=n(n-1)(n-2)\cdots(n-r+1)\,x^{n-r},\ \cdots,\ f^n(x)=n!$$

$$\therefore\ f(1)+\frac{f'(1)}{1!}+\frac{f''(1)}{2!}+\frac{f'''(1)}{3!}+\cdots+\frac{f^n(1)}{n!}$$

$$=1+\frac{n}{1!}+\frac{n(n-1)}{2!}+\frac{n(n-1)(n-2)}{3!}+\cdots+\frac{n!}{n!}$$

$$=1+{}^nC_1+{}^nC_2+{}^nC_3+\cdots+{}^nC_n=\left(1+1\right)^n=2^n$$

(*ii*) $\because\ f(x)=\tan x=\dfrac{\sin x}{\cos x}$

$$f(x)\cdot\cos x=\sin x \qquad \ldots\quad (1)$$

Applying Leibnitz's theorem to differentiate both the sides n times *w.r.t.* x, we get

$$f^n(x)\cos x+{}^nC_1 f^{n-1}(x)(-\sin x)+{}^nC_2 f^{n-2}(x)(-\cos x)$$
$$+{}^nC_3 f^{n-3}(x)(\sin x)+{}^nC_4 f^{n-4}(x)(\cos x)+\cdots=\sin\left(n\frac{\pi}{2}+x\right)$$

Putting $x=0$ on both the sides,

$$f^n(0)-{}^nC_2 f^{n-2}(0)+{}^nC_4 f^{n-4}(0)+\cdots\ \cdots=\sin\left(\frac{n\pi}{2}\right)$$

Ex. 8. If $y=\cos(m\sin^{-1}x)$, show that

(*i*) $(1-x^2)y_2-xy_1+m^2y=0$

(*ii*) $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}+(m^2-n^2)y=0$

Also, find the value of y_n when $x=0$. [*B. P. 1999*]

Solution : Given, $y=\cos(m\sin^{-1}x)$ ... (1)

Differentiating *w.r.t.* x,

$$\frac{dy}{dx}=y_1=-m\sin(m\sin^{-1}x)\frac{1}{\sqrt{1-x^2}}$$

or, $(1-x^2)y_1^2=m^2\sin^2(m\sin^{-1}x)=m^2\{1-\cos^2(m\sin^{-1}x)\}$

or, $(1-x^2)y_1^2=m^2(1-y^2)$ [from (1)] ... (2)

Differentiating again *w.r.t.* x

$$(1-x^2)2y_1\cdot y_2+y_1^2(-2x)=-m^2\cdot 2y\cdot y_1$$

or, $(1-x^2)y_2-xy_1+m^2y=0,\quad (\because 2y_1\neq 0)$ ... (3)

Differentiating (3) n times by Leibnitz's theorem, we get

$$(1-x^2)y_{n+2}+{}^nC_1y_{n+1}(-2x)+{}^nC_2y_n(-2)-y_{n+1}\cdot x-{}^nC_1y_n(1)+m^2y_n=0$$

or, $(1-x^2)y_{n+2}-2n\,xy_{n+1}-n(n-1)y_n-xy_{n+1}-ny_n+m^2y_n=0$

or, $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}+(m^2-n^2)y_n=0$ ... (4)

Last part : From (1), (2), (3), we have $y=1,\ y_1=0,\ y_2=-m^2$, when $x=0$.

putting n = 1, 2, 3 successively in (4), we get

$$y_3=-m^2y_1=-m^2\times 0=0$$

$$y_4=(2^2-m^2)y_2=-m^2(2^2-m^2)$$

$$y_5=(3^2-m^2)y_3=0$$

$$y_6=(4^2-m^2)y_4=-m^2(2^2-m^2)(4^2-m^2)$$

Thus, $y_n=0$, when n is odd and,

$y_n=-m^2(2^2-m^2)(4^2-m^2)\cdots\{(n-2)^2-m^2\}$, when n is even.

Ex. 9. If $y=e^{\cos^{-1}x}$, show that an equation connecting y_n, y_{n+1} and y_{n+2} is given by $\left(1-x^2\right)y_{n+2}-(2n+1)xy_{n+1}-\left(n^2+1\right)y_n=0$

[*C. P. 1980, B. P. 1996*]

Solution : $$y=e^{\cos^{-1}x} \quad \ldots \quad (1)$$

$$\therefore \; y_1=e^{\cos^{-1}x}\times\frac{-1}{\sqrt{1-x^2}}=-\frac{y}{\sqrt{1-x^2}} \text{ , using (1)}$$

or, $\left(1-x^2\right)y_1^2-y^2=0 \quad \ldots \quad (2)$

Differentiating (2) again *w.r.t.* x,

$$\left(1-x^2\right)2y_1y_2+y_1^2(-2x)-2y\cdot y_1=0$$

or, $\left(1-x^2\right)y_2-xy_1-y=0 \quad \ldots \quad (3)$

Differentiating (3) n times with the help of Leibnitz's theorem, we have,

$$\left(1-x^2\right)y_{n+2}+{}^nC_1y_{n+1}(-2x)+{}^nC_2y_n(-2)-xy_{n+1}$$
$$-{}^nC_1y_n(1)-y_n=0$$

or, $\left(1-x^2\right)y_{n+2}-(2n+1)xy_{n+1}-\left(n^2+1\right)y_n=0$.

Ex. 10. If $x=\sin\theta$, $y=\sin p\theta$, then prove that

(*i*) $\left(1-x^2\right)y_2-xy_1+p^2y=0$

(*ii*) $\left(1-x^2\right)y_{n+2}-(2n+1)xy_{n+1}+\left(p^2-n^2\right)y_n=0$

where y_n denotes the *nth* order derivative of y with respect to x.

[*C. P. 2002*]

Solution : Here, $x=\sin\theta$ and $y=\sin p\theta$

Proceeding exactly as in **Ex. 1.** (*ii*), we have

$$\left(1-x^2\right)y_2-xy_1+p^2y=0 \quad \ldots \quad (1)$$

Differentiating (1) n times with the help of Leibnitz's theorem, we get,

$$\left(1-x^2\right)y_{n+2}+n\cdot y_{n+1}(-2x)+\frac{n(n-1)}{1\cdot 2}\cdot y_n(-2)-xy_{n+1}$$
$$-n\cdot 1\cdot y_n+p^2y_n=0$$

or, $\left(1-x^2\right)y_{n+2}-(2n+1)xy_{n+1}+\left(p^2-n^2\right)y_n=0$.

EXAMPLES - VIII (B)

1. Find y_n in the following cases :

(i) $y=x^2 e^{ax}$. (ii) $y=x^3\sin x$. (iii) $y=x^3\log x$.

(iv) $y=x^2\tan^{-1}x$. (v) $y=e^{ax}\cos bx$. (vi) $y=\log\left(ax+x^2\right)$

(vii) $y=x^n(1-x)^n$. (viii) $y=x^n/(1+x)$.

2. If $y=A\sin mx+B\cos mx$, prove that $y_2+m^2y=0$.

3. If $y=Ae^{mx}+Be^{-mx}$, prove that $y_2-m^2y=0$.

4. If $y=e^{ax}\sin bx$, show that $y_2-2ay_1+\left(a^2+b^2\right)y=0$.

5. If $y=\log\left(x+\sqrt{a^2+x^2}\right)$, show that $\left(a^2+x^2\right)y_2+xy_1=0$.

6. If $y=\log\left(x+\sqrt{1+x^2}\right)^m$, then prove that

$$\left(1+x^2\right)y_2+xy_1-m^2y=0.$$

7. If $y=\tan^{-1}x$, then prove that

(i) $\left(1+x^2\right)y_1=1$, and

(ii) $\left(1+x^2\right)y_{n+1}+2nxy_n+n(n-1)y_{n-1}=0$.

Find also the value of $(y_n)_0$.

8. If $y=\sin^{-1}x$, then show that

(i) $\left(1-x^2\right)y_2-xy_1=0$,

(ii) $\left(1-x^2\right)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n=0$. [*C. P. 1997*]

Find also the value of $(y_n)_0$.

9. If $y=\left(\sin^{-1}x\right)^2$, then show that

(i) $\left(1-x^2\right)y_2-xy_1-2=0$, [*C. P. 1988, 95, 96*]

(ii) $\left(1-x^2\right)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n=0$.

10. If $\log y=\tan^{-1}x$, then prove that

(i) $\left(1+x^2\right)y_2+(2x-1)y_1=0$,

(ii) $\left(1+x^2\right)y_{n+2}+(2nx+2x-1)y_{n+1}+n(n+1)y_n=0$.

11. If $y = a\cos(\log x) + b\sin(\log x)$, then prove that.

$$x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0.$$

[*C.P. 1989, 96, 2007 B.P. 1990, 97, V.P. 1999*]

12. If $y = (x^2-1)^n$, then show that

$$(x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0.$$

13. If $y = e^{a\sin^{-1}x}$, then prove that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0.$$

[*C.P. 1985, 94, 98, 2004*]

14. If $y = \sin(m\sin^{-1}x)$, then show that

(i) $(1-x^2)y_2 - xy_1 + m^2y = 0,$ [*C.P. 1990, 2002, 2008*]

(ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0.$

15. If $y = (ax^2+bx+c)/(1-x)$, show that $(1-x)y_3 = 3y_2$.

16. If $y = e^{-x}\cos x$, prove that $y_4 + 4y = 0.$ [*B. P. 1998, 2001*]

17. If $y = x^{n-1}\log x$, show that $y_n = \dfrac{(n-1)!}{x}$.

[*C.P.1985, 2000, B. P. 1991*]

18. Show that $D^r(e^{ax}x^n) = a^{r-n}x^{n-r}D^n(e^{ax}x^r)$

19. If u, v, w be functions of x and if suffixes denote differentiations with respect to x, prove that

$$\frac{d}{dx}\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_4 & v_4 & w_4 \end{vmatrix}.$$

20. By forming in two different ways the nth derivatives of x^{2n}, show that

$$1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{1^2.2^2} + \frac{n^2(n-1)^2(n-2)^2}{1^2.2^2.3^2} + \ldots = \frac{(2n)!}{(n!)^2}$$

[Equate the nth derivative of the product $x^n.x^n$ to that of x^{2n}.]

21. Prove that $\dfrac{d^n}{dx^n}\left(\dfrac{\sin x}{x}\right)=\left\{P\sin\left(x+\dfrac{n\pi}{2}\right)+Q\cos\left(x+\dfrac{n\pi}{2}\right)\right\}/x^{n+1}$

where $P=x^n-n(n-1)x^{n-2}+n(n-1)(n-2)(n-3)x^{n-4}-\ldots\ldots$

and $Q=nx^{n-1}-n(n-1)(n-2)x^{n-3}+\ldots\ldots$

22. Prove that

$$\frac{d^n}{dx^n}\left(\frac{\cos x}{x}\right)=\left\{P\cos\left(x+\frac{n\pi}{2}\right)-Q\sin\left(x+\frac{n\pi}{2}\right)\right\}/x^{n+1}$$

where P and Q have the same values as in Ex. 21.

23. Prove that

$$\frac{d^n}{dx^n}\left(x^n\sin x\right)=n!\left(P\sin x+Q\cos x\right),$$

where $P=1-\binom{n}{2}\dfrac{x^2}{2!}+\binom{n}{4}\dfrac{x^4}{4!}-\ldots\ldots\ldots\ldots$

and $Q=\binom{n}{1}x-\binom{n}{3}\dfrac{x^3}{3!}+\binom{n}{5}\dfrac{x^5}{5!}-\ldots\ldots$

24. Show that

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right)=(-1)^n\frac{n!}{x^{n+1}}\left(\log x-1-\frac{1}{2}-\frac{1}{3}-\ldots-\frac{1}{n}\right)$$

25. Show that

$$\frac{d^n}{dx^n}\left(e^{-x}x^{n+a}\right)=n!e^{-x}x^a\sum_{r=0}^{n}\binom{n+a}{r}\frac{(-x)^{n-r}}{(n-r)!},\ a>-1.$$

26. If $f(x)=\tan x$, prove that

$$f^n(0)-{}^nc_2f^{n-2}(0)+{}^nc_4f^{n-4}(0)-\ldots=\sin\frac{1}{2}n\pi$$

27. Show that the n^{th} differential coefficient of $\dfrac{1}{1+x+x^2+x^3}$ is

$$\frac{1}{2}(-1)^n n!\sin^{n+1}\theta\left\{\sin(n+1)\theta-\cos(n+1)\theta+(\sin\theta+\cos\theta)^{-n-1}\right\}.$$

where $\theta=\cot^{-1}x$.

ANSWERS

1. (i) $e^{ax}.a^{n-2}\left\{a^2x^2 + 2nax + n(n-1)\right\}$

(ii) $x^3 \sin\left(\frac{1}{2}n\pi + x\right) + 3nx^2 \sin\left\{\frac{1}{2}(n-1)\pi + x\right\} + 3n(n-1)x$

$\times \sin\left\{\frac{1}{2}(n-2)\pi + x\right\} + n(n-1)(n-2)\sin\left\{\frac{1}{2}(n-3)\pi + x\right\}.$

(iii) $(-1)^n.6(n-4)!\frac{1}{x^{n-3}}.$

(iv) $(-1)^n(n-3)!\sin^{n-2}\theta\,\{(n-1)(n-2)\sin n\theta\cos^2\theta$

$-2n(n-2)\sin(n-1)\theta\cos\theta + n(n-1)\sin(n-2)\theta\}$ where $\cot\theta = x.$

(v) $e^{ax}\left\{a^n \cos bx + {}^nc_1 a^{n-1}b\cos\left(bx + \frac{1}{2}\pi\right)\right.$

$\left. + {}^nc_2 a^{n-2}b^2\cos\left(bx + 2.\frac{1}{2}\pi\right) + \ldots + b^n\cos\left(bx + n.\frac{1}{2}\pi\right)\right\}$

(vi) $(-1)^{n-1}(n-1)!\left[\frac{1}{x^n} + \frac{1}{(x+a)^n}\right].$

(vii) $n!\left\{(1-x)^n - \left({}^nc_1\right)^2.(1-x)^{n-1}.x + \left({}^nc_2\right)^2.(1-x)^{n-2}.x^2 - \ldots\right\}$

(viii) $n!/(1+x)^{n+1}.$

7. 0, or $(-1)^{\frac{1}{2}(n-1)}(n-1)!$ according as n is even or odd.

8. 0, or $\{1.3.5\ldots(n-2)\}^2$ according as n is even or odd.

9 EXPANSION OF FUNCTIONS

9.1. Rolle's Theorem.

If (*i*) $f(x)$ *is continuous in the closed interval* $a \le x \le b$,

(*ii*) $f'(x)$ *exists in the open interval* $a < x < b$,

and (*iii*) $f(a) = f(b)$,

then there exists at least one value of x (*say* ξ) ***between a and b*** [*i e.,* $a < \xi < b$], *such that* $f'(\xi) = 0$

Since $f(a) = f(b)$, if $f(x)$ be constant throughout the interval $[a, b]$, being equal to $f(a)$ or $f(b)$, then evidently $f'(x) = 0$ at every point in the interval.

If $f(x)$ be not constant throughout, then it must have values either greater than or less than $f(a)$ or both, in the interval. Suppose $f(x)$ has values greater than $f(a)$. Now, since $f(x)$ is continuous in the interval, it must be bounded and M being its upper bound [which is $> f(a)$ in this case], there must be a value ξ of x in the interval $a < x < b$ for which $f(\xi) = M$

$\therefore$ $f(\xi + h) - f(\xi) \le 0$ *for positive as well as negative values of h.*

$\therefore$ $\dfrac{f(\xi + h) - f(\xi)}{h} \le 0$ if h be positive, and ≥ 0 if h be negative.

$$\text{Hence } \underset{h \to 0+}{Lt} \frac{f(\xi+h) - f(\xi)}{h} \le 0, \text{ and } \underset{h \to 0-}{Lt} \frac{f(\xi+h) - f(\xi)}{h} \ge 0,$$

provided the limits exist.

Now, since $f'(x)$ exists for every value of x in $a < x < b$, $f'(\xi)$ also exists, and so the above two limits must both exist and be equal, and the only equal value they can have is zero. Hence $f'(\xi) = 0$

If $f(x)$ has values less than $f(a)$ in the interval, we can similarly show that $f'(\xi) = 0$, where $f(\xi) = m$, the lower bound of $f(x)$ in the interval.

If $f(x)$ has values both greater than and less than $f(a)$, then there must be an upper bound M which is greater than $f(a)$ and a lower bound m which is less than $f(a)$ and for values of x for which either $f(x) = M$ or $f(x) = m$. $f'(x)$ will be zero.

9.2 Geometrical interpretation of Rolle's Theorem

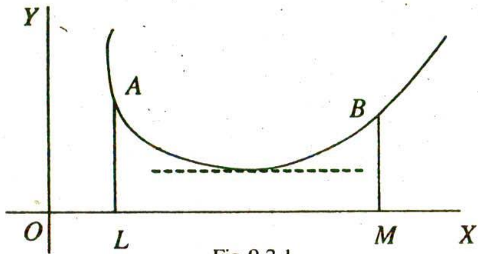


Fig 9.2.1

Let L, M be the points on the number axis $\overrightarrow{OX}$ representing the real numbers a, b respectively. We draw the graph of the function $y = f(x)$ and let A, B be the points on it corresponding to L, M respectively, that is, $LA = f(a)$ and $MB = f(b)$.

From the condition (i) of Rolle's theorem, we say that the graph is a continuous curve between the points A and B; the condition (ii) says that the curve has tangents at every point between A and B and the third condition implies that $LA = MB$.

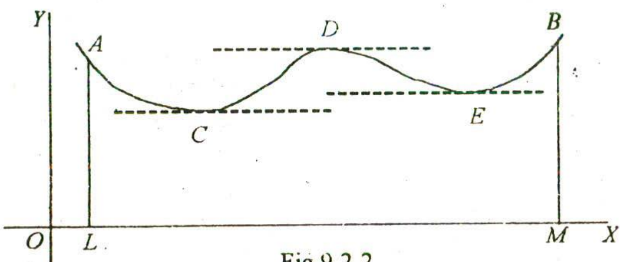


Fig 9.2.2

Now, $f'(\xi)$ is the gradient of the tangent to the curve at $x = \xi$. By Rolle's theorem $f'(x)$ vanishes at least once between $x = a$ and $x = b$. Geometrically we say that we get at least one point C on the graph between A and B such that the tangent at C is parallel to $\overrightarrow{OX}$.

Note. From the above graph, it is clear that there are more points D and E like C.

9.3. Mean Value Theorem. [*Lagrange's form*]

If (i) $f(x)$ is continuous in the closed interval $a \le x \le b$, and (ii) $f'(x)$ exists in the open interval $a < x < b$, then there is at least one value of x (say ξ) between a and b [i.e., $a < \xi < b$], such that

$$\mathbf{f(b) - f(a) = (b - a) f'(\xi)}.$$

Consider the function $\psi(x)$ defined in (a, b) by

$$\psi(x) = f(b) - f(x) - \frac{b - x}{b - a}\{f(b) - f(a)\}.$$

Here $\psi(x)$ is continuous in $a \le x \le b$, since $f(x)$ and $b - x$ are so,

$\therefore$ $\psi'(x) = -f'(x) + \dfrac{f(b) - f(a)}{b - a}$ exists in $a < x < b$, since $f'(x)$ exists in $a < x < b$.

Also, $\psi(a) = 0$, $\psi(b) = 0$, $\therefore$ $\psi(a) = \psi(b)$,

Hence, by Rolle's Theorem, $\psi'(x)$ vanishes for at least one value of x (say ξ) between a and b, *i.e.*, $\psi'(\xi) = 0$,

i.e.,
$$0 = -f'(\xi) + \frac{f(b) - f(a)}{b - a},$$

Whence $f(b) - f(a) = (b - a) f'(\xi)$, $[a < \xi < b]$.

Cor. Since ξ lies between a and b, ξ can be written as $a + \theta(b - a)$ where $0 < \theta < 1$, Putting $b = a + h$ we get another form of the Mean Value Theorem

$$f(a + h) = f(a) + h f'(a + \theta h), \quad \text{where } 0 < \theta < 1,$$

or, $$\mathbf{f(x + h) = f(x) + h f'(x + \theta h)}, \quad \text{where } 0 < \theta < 1.$$

Note. The value of θ usually depends upon both x and h, but there are cases where it is not so dependent. [See. *Ex. 2 and 13, Examples IX (A)*]. Also θ may have more than one value in a given range in some cases. [See *Ex. 12, Examples IX (A)*].

9.4. Geometrical interpretation of Mean Value Theorem.

Let ACB be the graph of $f(x)$ in the interval $[a, b]$ and let a, ξ, b be the abscissæ of the points A, C, B on the curve $y = f(x)$, such that the relation $f(b) - f(a) = (b-a) f'(\xi)$ is satisfied.

Draw $\overline{AL}$, $\overline{BM}$ perpendiculars on $\overline{OX}$ and $\overline{AB}$, perpendicular on $\overline{BM}$. Then $AL = f(a)$, $BM = f(b)$. Let $\overline{CT}$, be the tangent at C.

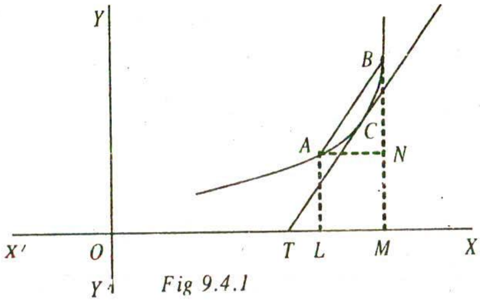


Fig 9.4.1

$$\text{Then, } \frac{f(b) - f(a)}{b-a} = \frac{BM - AL}{LM} = \frac{BN}{AN} = \tan BAN$$

Since $f'(\xi) = \tan CTX$ (*as explained in* § 7.14), it follows from the Mean Value Theorem that $\tan BAN = \tan CTX$, *i.e.*, $m\angle BAN = m\angle CTX$ *i.e.*, $\overline{AB}$ is parallel to $\overline{CT}$.

Hence, we have the following geometrical interpretation of the Mean Value Theorem :

If the graph ACB of $f(x)$ is a continuous curve having everywhere a tangent, then there must be at least on point C intermediate between A and B at which the tangent is parallel to the chord $\overline{AB}$.

9.5. Taylor's Series in finite form. (Generalized Mean Value Theorem)

If $f(x)$ possesses differential coefficients of the first $(n-1)$ orders for every value of x in the closed interval $a \le x \le b$ and the nth derivative of exists in the open interval $a < x < b$ [i.e., if $f^{n-1}(x)$ is continuous in $a \le x \le b$ and $f^{n}(x)$ exists in $a < x < b$], then

∴

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \ldots\ldots$$

$$+ \frac{(b-a)^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{(b-a)^n}{n!}f^n(\xi),$$

where $a < \xi < b$ $\ldots$ (A)

and If $b = a + h$, *So that* $b - a = h$, *then*

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots\ldots$$

$$+ \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{n!}f^n(a+\theta h),$$

where $0 < \theta < 1$ $\ldots$ (B)

or writing x *for* a,

$$\mathbf{f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \ldots.}$$

$$\mathbf{+ \frac{h^{n-1}}{(n-1)!}f^{n-1}(x) + \frac{h^n}{n!}f^n(x+\theta h),}$$

where $0 < \theta < 1$ $\ldots$ (C)

Consider the function $\psi(x)$ defined in $[a, b]$ by

$$\psi(x) = \phi(x) - \frac{(b-x)^n}{(b-a)^n}\phi(a), \qquad \ldots (1)$$

where, $\phi(x) = f(b) - f(x) - (b-x)f'(x) - \frac{(b-x)^2}{2!}f''(x) - \ldots$

$$-\frac{(b-x)^{n-1}}{(n-1)!}f^{n-1}(x) \qquad \ldots (2)$$

Then, evidently $\psi(a) = 0$ and $\psi(b) = 0$ [since $\phi(b) = 0$ is identically],

Now,

$$\phi'(x) = -f'(x) + \{f'(x) - (b-x)f''(x)\} + \left\{(b-x)f''(x) - \frac{(b-x)^2}{2!}f'''(x)\right\} + \ldots$$

$$+ \left\{\frac{(b-x)^{n-2}}{(n-2)!}f^{n-1}(x) - \frac{(b-x)^{n-1}}{(n-1)!}f^n(x)\right\}$$

$$= -\frac{(b-x)^{n-1}}{(n-1)!}f^n(x).$$

Hence, from (1),

$$\psi'(x) = -\frac{(b-x)^{n-1}}{(n-1)!} f^n(x) + \frac{n(b-x)^{n-1}}{(b-a)^n}\phi(a), \quad \ldots \quad (3)$$

Since $\psi(a) = \psi(b)$, and $\psi'(x)$ exists in (a, b), by Rolle's Theorem, $\psi'(\xi) = 0$, where $a < \xi < b$.

$\therefore$ Substituting ξ for x in (3), and cancelling the common factor $(b-\xi)^{n-1}$, we get ultimately

$$\phi(a) = -\frac{(b-a)^n}{n!} f^n(\xi), \text{ and since, from (2)}$$

$$\phi(a) = f(b) - f(a) - (b-a)f'(a) - \ldots - \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a), \text{ the}$$

required result in the form (A) follows by transposition.

Since $a < \xi < b$, we can write $\xi = a + (b-a)\theta$,

i.e., $\xi = a + h\theta$, where $0 < \theta < 1$, and $b - a = h$, and hence the form (B) follows and writing x for a in the form (B), the form (C) can be obtained.

Note 1. The series (A), (B) or (C) is called Taylor's series with the remainder in Lagrange's form, the remainder (after n terms) being $\frac{(b-a)^n}{n!} f^n(\xi)$, or $\frac{h^n}{n!} f^n(a+\theta h)$, or $\frac{h^n}{n!} f^n(x+\theta h)$, $0 < \theta < 1$, which is generally denoted by R_n.

Note 2. Putting $n = 1$ in Taylor's series, we get

$$f(a+h) = f(a) + hf'(a+\theta h), \qquad 0 < \theta < 1,$$

which is the Mean Value Theorem.

So, Taylor's theorem is sometimes called *Mean Value Theorem of the nth order.*

Putting $n = 2$ in Taylor's series, we get

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a+\theta h), \quad 0<\theta<1,$$

which is often called the Mean Value Theorem of the second order and so on.

Note 3. Yet another form of Taylor's series which is found some times useful is obtained by putting x for b in (A). Thus,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a)$$
$$+ \frac{(x-a)^n}{n!}f^n\{a+\theta(x-a)\},\ 0<\theta<1$$

and the function $f(x)$ *is said to be expanded about or in the neighbourhood of* $x = a$.

9.6. Maclaurin's series in finite form.

Putting $x = 0,\ h = x$ in Taylor's series in finite form (C), we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{x^n}{n!}f^n(\theta x),\ 0<\theta<1$$

the corresponding form *of the remainder* R_n *being*

$$\frac{x^n}{n!}f^n(\theta x)$$

The above is known as Maclaurin's series for $f(x)$, and $f(x)$ is said to be *expanded in the neighbourhood of* $x = 0$.

Note. Putting $n = 1, 2$, we get Maclaurin's series of the first and second orders, *viz.*,

$$f(x) = f(0) + xf'(\theta x) \text{ and } f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(\theta x),\ 0<\theta<1.$$

9.7. Cauchy's series in finite form.

In Art. 9.5, if we take $\psi(x) = \phi(x) - \frac{b-x}{b-a}\phi(a)$, the other conditions remaining the same, and carry out the investigation as in that Art., we get

$$\psi'(x) = -\frac{(b-x)^{n-1}}{(n-1)!}f^n(x) + \frac{1}{b-a}\phi(a) \qquad \ldots \quad (4)$$

Since $\psi(a) = \psi(b)$ and $\psi'(x)$ exists in (a, b) by Rolle's theorem, we have $\psi'(\xi) = 0,\ a<\xi<b$.

$\therefore$ Substituting ξ for x in (4), we get

$$\phi(a) = \frac{(b-a)(b-\xi)^{n-1}}{(n-1)!}f^n(\xi) \qquad \ldots \quad (5)$$

Writing $\xi = a + (b-a)\theta$, where $0<\theta<1$, we have $b - \xi = b - a - b\theta + a\theta = (1 - \theta)(b - a)$.

$\therefore \quad (b - \xi)^{n-1} = (1 - \theta)^{n-1}(b - a)^{n-1} = (1 - \theta)^{n-1} h^{n-1}$, since $b - a = h$.

$\therefore$ from (5), we get

$$\phi(a) = \frac{h^n (1 - \theta)^{n-1}}{(n-1)!} f^n(a + \theta h).$$

Now replacing a by x, the required expression for the remainder R_n would come out as

$$\mathbf{R_n} = \frac{\mathbf{h^n (1 - \theta)^{n-1}}}{\mathbf{(n-1)!}} \mathbf{f^n(x + \theta h)}, \quad 0 < \theta < 1.$$

This is known as *Cauchy's form of remainder* in Taylor's expansion. The corresponding form in Maclaurin's expansion is

$$\mathbf{R_n} = \frac{\mathbf{x^n (1 - \theta)^{n-1}}}{\mathbf{(n-1)!}} \mathbf{f^n(\theta x)}, \quad 0 < \theta < 1.$$

This form of remainder is sometimes more useful than that of Lagrange's form. It should be noted that the value of θ in the two forms of the remainder for the same function need not be the same.

9.8 Illustrative Examples.

Ex. 1. (*i*) *If $f'(x) = 0$ for all values of x in an interval, then $f(x)$ is constant in that interval.*

(*ii*) *If $\phi'(x) = \psi'(x)$ in an interval, then $\phi(x)$ and $\psi(x)$ differ by a constant in that interval.*

(i) Suppose, $f'(x) = 0$ at every point in (a, b).

Let us take any two points x_1, x_2 in $[a, b]$, such that $x_2 > x_1$,

By Mean Value Theorem,

$f(x_2) - f(x_1) = (x_2 - x_1) f'(c)$, where $x_1 < c < x_2$

$\qquad = 0$, since $f'(c) = 0$, by hypothesis.

$\therefore \quad f(x_2) = f(x_1)$.

Since x_1, x_2 are any two points in $[a, b]$, it follows that $f(x)$ must be constant throughout $[a, b]$.

(*ii*) *Let* $f(x) = \phi(x) - \psi(x)$.

$\therefore \quad f'(x) = \phi'(x) - \psi'(x) = 0$, everywhere in (a, b).

$\therefore \quad f(x) = \text{constant} = k$, say, by (i).

$\therefore \quad \phi(x) - \psi(x) = k$.

Note : The result (ii) is fundamental in the theory of integration.

Ex. 2. *If* $f(h) = f(0) + hf'(0) + \dfrac{h^2}{2!} f''(\theta h), 0 < \theta < 1$, *find* θ, *when* $h = 1$ *and* $f(x) = (1-x)^{\frac{5}{2}}$ [*C.P. 1944*]

We have $f(h) = (1-h)^{\frac{5}{2}}$, since $f(x) = (1-x)^{\frac{5}{2}}$.

$\therefore \quad f'(h) = -\dfrac{5}{2}(1-h)^{\frac{3}{2}}\ ;\ f''(h) = \dfrac{15}{4}(1-h)^{\frac{1}{2}}$

$\therefore \quad f(0) = 1, f'(0) = -\dfrac{5}{2}$

from the given relation

$$(1-h)^{\frac{5}{2}} = 1 - \frac{5}{2}h + \frac{h^2}{2!} \cdot \frac{15}{4}(1-\theta h)^{\frac{1}{2}}.$$

putting $h = 1, 0 = 1 - \dfrac{5}{2} + \dfrac{15}{4}(1-\theta h)^{\frac{1}{2}}$, whence $(1-\theta)^{\frac{1}{2}} = \dfrac{4}{5}$,

$\therefore \quad 1 - \theta = \dfrac{16}{25} \qquad \therefore \ \theta = \dfrac{9}{25}$,

Ex. 3. *Prove that the Lagrange's remainder after* **n** *terms in the expansion of* e^{ax} *in powers of* x *is*

$$\frac{\left(a^2+b^2\right)^{\frac{1}{2}n}}{n!} x^n e^{a\theta x} \cos\left(b\theta x + n\tan^{-1}\frac{b}{a}\right), 0 < \theta < 1. \text{[C.P. 1942]}$$

Lagrange's remainder after n terms in the expansion of $f(x)$ is

$$\frac{x^n}{n!} f^n(\theta x), 0 < \theta < 1. \qquad \textit{(by Art. 9.6.)} \qquad \text{......(1)}$$

Here, since $f(x) = e^{ax} \cos bx$,

$$f^n(x) = \left(a^2+b^2\right)^{\frac{1}{2}n} e^{ax} \cos\left(bx + n\tan^{-1}\frac{b}{a}\right) \qquad ... \qquad (2)$$

[See *Ex. 3 § 8.4*]

$\therefore$ writing θx for x in (2), we get $f^n(\theta x)$, and substituting this value of $f^n(\theta x)$ in (1), the required remainder is obtained.

Ex. 4. *Prove that the Cauchy's remainder after n terms in the expansion of* $(1+x)^m$ *(m being a negative integer or fraction) in powers of x is*

$$\frac{m(m-1)\cdots(m-n+1)}{(n-1)!}x^n(1+\theta x)^{m-1}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1},\ 0<\theta<1.$$

Cauchy's remainder after *n* terms in the expansion of $f(x)$ is

$$\frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^n(\theta x),\quad 0<\theta<1 \text{ (by Art. 9.7)} \qquad \ldots(1)$$

Here $f(x)=(1+x)^m$,

$\therefore\ f^n(x)=m(m-1)(m-2)\cdots\cdots(m-n+1)(1+x)^{m-n}$,

So, the expression (1) is equivalent to

$$\frac{m(m-1)(m-2)\cdots(m-n+1)}{(n-1)!}x^n(1+\theta)^{n-1}(1+\theta x)^{m-n}$$

which is the required remainder.

Ex. 5. *Show that the Cauchy's remainder after n terms in the expansion of* $\log(1+x)$ *in powers of x is*

$$(-1)^{n-1}\frac{x^n}{1+\theta x}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1},\ 0<\theta<1.$$

Here $f(x)=\log(1+x)$, $\therefore\ f^n(x)=\dfrac{(-1)^{n-1}(n-1)!}{(1+x)^n}$,

Hence, $\dfrac{x^n(1-\theta)^{n-1}}{(n-1)!}f^n(\theta x)=(-1)^{n-1}x^n(1-\theta)^{n-1}\cdot\dfrac{1}{(1+\theta x)^n}$.

which is the required remainder in Cauchy's form.

Ex. 6. *If (i)* $f'(x)$ *exists in* $a\le x\le b$, *(ii)* $f'(a)=\alpha$, $f'(b)=\beta$, $\alpha\ne\beta$, *and (iii)* γ *lies between* α *and* β, *then there exists a value* ξ *of x between a and b such that* $f'(\xi)=\gamma$. **[Darboux's Theorem]**

Suppose, $\alpha<\gamma<\beta$ and let $\phi(x)=f(x)-\gamma(x-a)$,

$\therefore\ \phi'(x)=f'(x)-\gamma$

Since $\phi'(x)$ exists in (a, b), $\phi(x)$ is continuous in $[a, b]$ and therefore attains its lower bound at some point ξ in the interval. [§ 4.4 (*viii*)]

Now, this point cannot be a or b, since $\phi'(a)=f'(a)-\gamma=\alpha-\gamma$ which is negative and $\phi'(b)=f'(b)-\gamma=\beta-\gamma$ and which is positive. Hence, the point ξ is between a and b, and $\phi'(\xi)=0$. $\therefore f'(\xi)-\gamma=0$ $\therefore f'(\xi)=\gamma$ for $a<\xi<b$

Ex. 7. (*a*) *If* (*i*) $\phi(x)$ *and* $\psi(x)$ *are both continuous in* $a\le x\le b$

(*ii*) $\phi'(x)$ *and* $\psi'(x)$ *exists in* $a<x<b$,

and (*iii*) $\psi'(x)\ne 0$ *anywhere in* $a<x<b$,

then there is a value ξ *of* x *between* a *and* b *for which*

$$\frac{\phi(b)-\phi(a)}{\psi(b)-\psi(a)}=\frac{\phi'(\xi)}{\psi'(\xi)}.\qquad \text{[Cauchy's Mean Value Theorem]}$$

(*b*) *If further* $\phi(a)=\psi(a)=0$ *and* $\psi'(x)\ne 0$ *in the neighbourhood of a,*

$$\text{then } \underset{x\to a}{Lt}\frac{\phi(a)}{\psi(a)}=\underset{x\to a}{Lt}\frac{\phi'(x)}{\psi'(x)}, \text{ if the latter limit exists.}$$

[L'Hospital's Theorem]

(*a*) Consider the function $f(x)$ defined by the equation

$$f(x)=\phi(b)-\phi(x)-\frac{\phi(b)-\phi(a)}{\psi(b)-\psi(a)}\{\psi(b)-\psi(x)\},$$

Now, $f(a)=f(b)$, since each $=0$ identically.

Also, $$f'(x)=-\phi'(x)+\frac{\phi(b)-\phi(a)}{\psi(b)-\psi(a)}\psi'(x)$$

Since $f(x)$ satisfies all the conditions of Rolle's theorem

$\therefore$ $f'(\xi)=0$, for some ξ, where $a<\xi<b$, whence the required result follows.

Note 1. The condition $\psi'(x)\ne 0$ anywhere in (*a*, *b*) ensures that $\psi(a)\ne\psi(b)$; for, if $\psi(a)=\psi(b)$, $\psi(x)$ then satisfying all the conditions of Rolle's theorem, $\psi'(x)$ would vanish at some point x in (*a*, *b*).

Note 2. Putting $b-a=h$, we get

$$\frac{\phi(a+h)-\phi(a)}{\psi(a+h)-\psi(a)}=\frac{\phi'(a+\theta h)}{\psi'(a+\theta h)},\ 0<\theta<1,$$

or, writing x for a, $\dfrac{\phi(x+h)-\phi(x)}{\psi(x+h)-\psi(x)}=\dfrac{\phi'(x+\theta h)}{\psi'(x+\theta h)}$, $0<\theta<1$.

Note 3. Mean value Theorem can be deduced from this theorem by putting $\psi(x)=x$

(*b*) We have, for $a<x<b$,

$$\frac{\phi(x)}{\psi(x)}=\frac{\phi(x)-\phi(a)}{\psi(x)-\psi(a)}, \text{ since } \phi(a)=\psi(a)=0 \text{ here.}$$

$$=\frac{\phi'(\xi)}{\psi'(\xi)},\ a<\xi<x, \text{ by Cauchy's theorem.}$$

Taking limits, and noting that $\xi\to a$ as $x\to a$,

We get $\displaystyle \underset{x\to a+0}{Lt}\frac{\phi(x)}{\psi(x)}=\underset{\xi\to a+0}{Lt}\frac{\phi'(\xi)}{\psi'(\xi)}=\underset{x\to a+0}{Lt}\frac{\phi'(x)}{\psi'(x)}$

Again, when $b_1<x<a_1$ [assuming b_1 sufficiently close to a such that $\phi'(x)$ and $\psi'(x)$ exist at every point in the interval, and $\psi'(x)\neq 0$ in it], we may similarly write

$\dfrac{\phi(x)}{\psi(x)}=\dfrac{\phi(a)-\phi(x)}{\psi(a)-\psi(x)}=\dfrac{\phi'(\xi_1)}{\psi'(\xi_1)}$, where $x<\xi_1<a_1$ by Cauchy's Theorem, and making $x\to a$ we get

$$\underset{x\to a-0}{Lt}\frac{\phi(x)}{\psi(x)}=\underset{\xi_1\to a-0}{Lt}\frac{\phi'(\xi_1)}{\psi'(\xi_1)}=\underset{x\to a-0}{Lt}\frac{\phi'(x)}{\psi'(x)}$$

Combining the two cases, L'Hospital's Theorem follows.

EXAMPLES - IX(A)

1. Find the value of ξ in the Mean Value Theorem

$$f(b)-f(a)=(b-a)f'(\xi)$$

(i) if $f(x)=x^2$, $a=1$, $b=2$,

(ii) if $f(x)=\sqrt{x}$, $a=4$, $b=9$, [*C. P. 2006*]

(iii) if $f(x)=x(x-1)(x-2)$, $a=0$, $b=\frac{1}{2}$, [*C.P. 1987, B.P. 1997*]

(iv) if $f(x)=Ax^2+Bx+C$ in $[a,b]$.

2. In the Mean Value Theorem

$$f(x+h)=f(x)+hf'(x+\theta h),$$

if $f(x)=Ax^2+Bx+C$, where $A \neq 0$, show that $\theta=\frac{1}{2}$

Give a geometrical interpretation of the result.

3. In the Mean Value Theorem

$$f(a+h)=f(a)+hf'(a+\theta h),$$

if $a=1$, $h=3$ and $f(x)=\sqrt{x}$, find θ.

4. (i) In the Mean Value Theorem

$$f(h)=f(0)+hf'(\theta h), \quad 0<\theta<1,$$

show that the limiting value of θ *as* $h \to 0+$ is $\frac{1}{2}$ or $\frac{1}{\sqrt{3}}$ according as $f(x)$ is $\cos x$ or $\sin x$. [*C. P. 1994, 2005*]

(ii) In the Mean Value Theorem

$$f(x+h)=f(x)+hf'(x+\theta h), \quad 0<\theta<1,$$

show that the limiting value of θ *as* $h \to 0+$ is $\frac{1}{2}$ whether $f(x)$ is $\sin x$ or $\cos x$. [*C. P. 1994, 2008*]

5. If $f(h)=f(0)+hf'(0)+\frac{h^2}{2!}f''(\theta h)$, $0<\theta<1$, find θ, when $h=7$ and $f(x)=\frac{1}{1+x}$.

6. From the relation

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(\theta x), \quad 0<\theta<1,$$

show that $\log(1+x)>x-\frac{1}{2}x^2$, if $x>0$,

and $\cos x > x-\frac{1}{2}x^2$, if $0<x<\frac{1}{2}\pi$.

7. Show that $\sin x > x-\frac{1}{6}x^3$, if $0<x<\frac{1}{2}\pi$. [*V. P. 1995*]

8. If $f(x)=\tan x$, then $f(0)=0$ and $f(\pi)=0$.
Is Rolle's theorem applicable to $f(x)$ in $(0, \pi)$?

[*C.P. 1982, '86, '96, 2004*]

9. Is the Mean Value Theorem applicable to the functions (i) and (ii) in the intervals $[-1, 1]$ and $[5, 7]$ respectively ?

(i) $f(x) = x \cos \frac{1}{x}$ for $x \neq 0$

$= 0$ for $x = 0$

(ii) $f(x) = 4 - (6 - x)^{\frac{2}{3}}$. [C. P. 2005]

10. If $f'(x)$ exists and > 0 everywhere in the interval (a, b), then show that $f(x)$ is an increasing function in $[a, b]$ and $f'(x) < 0$ everywhere in (a, b), then show that $f(x)$ is decreasing function in (a, b).

11. Show that $2x^3 + 2x^2 - 10x + 6$ is positive if $x > 1$.

12. (i) In the Mean Value Theorem

$$f(a+h) - f(a) = h f'(a + \theta h), \quad 0 < \theta < 1,$$

If $f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x$, and $a = 0$, $h = 3$,

show that θ has got two values and find them.

(ii) In the Mean Value Theorem

$$f(b) - f(a) = (b - a) f'(\xi), \quad a < \xi < b,$$

find ξ, if $f(x) = x^3 - 3x - 1$, $a = -\frac{11}{7}$, $b = \frac{13}{7}$ and give a geometrical interpretation of the result.

13. In the Mean Value Theorem

$$f(x+h) = f(x) + h f'(x + \theta h),$$

(i) find θ where, (a) $f(x) = \frac{1}{x}$, (b) $f(x) = e^x$,

(c) $f(x) = \log x$.

(ii) if $f(x) = a + bx + c\,m^x$

then show that θ is independent of x.

14. Show that

$$(x+h)^{\frac{3}{2}} = x^{\frac{3}{2}} + \frac{3}{2} x^{\frac{1}{2}} h + \frac{3}{2} \cdot \frac{1}{2} \cdot \frac{h^2}{2!} \frac{1}{\sqrt{x + \theta h}}, \quad 0 < \theta < 1$$

Find θ, when $x = 0$.

15. Expand in a finite series in powers of h, and find the remainder in each case :

(i) $\log(x + h)$, (ii) $\sin(x + h)$, (iii) $(x + h)^m$.

16. (i) Apply Taylor's Theorem to obtain the Binomial expansion of $(a+h)^n$, where n is a positive integer.

(ii) If $f(x)$ is a polynomial of degree r, then show that

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)$$

$$+\ldots\ldots+\frac{h^r}{r!}f^r(a).$$

(iii) Expand $5x^2+7x+3$ in powers of $(x-2)$.

17. Expand the following functions in a finite series in powers of x, with the remainder in Lagrange's form in each case :

(i) e^x, (ii) a^x, (iii) $\sin x$,

(iv) $\cos x$, (v) $\log(1+x)$, (vi) $\log(1-x)$,

(vii) $(1+x)^m$, (viii) $\tan^{-1} x$, (ix) $e^x \cos x$,

(x) $e^{ax}\sin bx$.

18. Find the value of θ in the Lagrange's form of remainder R_n for the expansion $\dfrac{1}{1-x}$ in powers of x.

19. Expand the following functions in the neighbourhood of $x=0$ to three terms plus remainder (in Lagrange's form) :

(i) $\sin^2 x$, (ii) $\cos^3 x$, (iii) e^{-x^2}.

20. Expand the following functions in a finite series in powers of x, with the remainder in Cauchy's form in each case :

(i) e^x, (ii) $\cos x$, (iii) $(1-x)^{-1}$.

21. (i) Prove that $\underset{h\to 0}{Lt}\dfrac{f(a+h)-f(a-h)}{2h}=f'(a)$,

provided $f'(x)$ is continuous.

(ii) Prove that $\underset{h\to 0}{Lt}\dfrac{f(a+h)-2f(a)+f(a-h)}{h^2}=f''(a)$,

provided $f''(x)$ is continuous.

22. (i) If $f''(x)$ is continuous in the interval $[a, a+h]$ and $f''(x) \neq 0$, prove that $\underset{h \to 0}{Lt}\ \theta = \frac{1}{2}$, where θ is given by

$$f(a+h) = f(a) + h f'(a+\theta h), \quad 0 < \theta < 1,$$

(ii) Show that the limit when $h \to a$ of θ which occurs in Lagrange's form of remainder $\frac{h^n}{n!} f^n(x+\theta h)$ in the expansion of $f(x+h)$ is $\frac{1}{n+1}$ provided $f^{n+1}(x)$ is continuous and $\neq 0$.

23. In Cauchy's Mean Value Theorem,

(i) if $\phi(x) = \sin x$ and $\psi(x) = \cos x$, or

(ii) if $\phi(x) = e^x$ and $\psi(x) = e^{-x}$, or

(iii) if $\phi(x) = x^2 + x + 1$ and $\psi(x) = 2x^2 + 3x + 4$, then show that θ is independent of both x and h, and is equal to $\frac{1}{2}$.

24. If $f(x) = x^2$, $\phi(x) = x$, then find a value of ξ in terms of a and b in Cauchy's Mean Value Theorem.

25. If $f(x)$ and $\phi(x)$ are continuous in $a \le x \le b$ and differentiable in $a<x<b$ such that $f'(x)$ and $\phi'(x)$ never vanish for the same value of x, then show that

$$\frac{f(\xi) - f(a)}{\phi(b) - \phi(\xi)} = \frac{f'(\xi)}{\phi'(\xi)}, \text{ where } a < \xi < b.$$

26. If $\psi''(x) \neq 0$ for $a < x < b$, then prove that

$$\frac{\phi(b) - \phi(a) - (b-a)\phi'(a)}{\psi(b) - \psi(a) - (b-a)\psi'(a)} = \frac{\phi''(\xi)}{\psi''(\xi)},$$

where $a < \xi < b$

27. If $f(x)$ and $g(x)$ are differentiable in the interval (a, b) then prove that there is a number ξ, $a < \xi < b$, such that

$$\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b-a)\begin{vmatrix} f(a) & f'(\xi) \\ g(a) & g'(\xi) \end{vmatrix}.$$

28. (i) If $f(x), \phi(x), \psi(x)$ are continuous in $a \leq x \leq b$ and differentiable in $a < x < b$, then show that

$$\begin{vmatrix} f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \\ f'(\xi) & \phi'(\xi) & \psi'(\xi) \end{vmatrix} = 0$$

(ii) If $F(x), G(x), H(x)$ are continuous in $a \leq x \leq b$ and diffentiable in $a < x < b$, then prove that

$$\begin{vmatrix} 1 & F(b)-F(a) & F'(\xi) \\ 1 & G(b)-G(a) & G'(\xi) \\ 1 & H(b)-H(a) & H'(\xi) \end{vmatrix} = 0.$$

29. If $f(x) = \begin{vmatrix} \sin x & \sin\alpha & \sin\beta \\ \cos x & \cos\alpha & \cos\beta \\ \tan x & \tan\alpha & \tan\beta \end{vmatrix}$, $0 < \alpha < \beta < \frac{1}{2}\pi$,

show that $f'(\xi) = 0$, where $\alpha < \xi < \beta$. [*C. H. 1955*]

30. Deduce Taylor's Theorem from Cauchy's Mean Value Theorem. [*C. H. 1961*]

[*Assume* $\phi(x) = f(b) - f(x) - (b-x)f'(x) - \ldots$

$$- \frac{(b-x)^{n-1}}{(n-1)!} f^{n-1}(x) \text{ and } \psi(x) = (b-x)^n.]$$

31. If $f(a) = f(c) = f(b) \Rightarrow 0$ where $a < c < b$, and if $f'(x)$ satisfies the conditions of Rolle's Theorem in $[a, b]$, prove that there exists at least one number ξ such that $f''(\xi) = 0$, where $a < \xi < b$.

32. Given that

$$\phi(x) = \frac{(x-b)(x-c)}{(a-b)(a-c)} f(a) + \frac{(x-c)(x-a)}{(b-c)(b-a)} f(b) + \frac{(x-a)(x-b)}{(c-a)(c-b)} f(c) f(x)$$

where $a < c < b$ and $f''(x)$ exists at all points in (a, b). Prove, by considering the function $\phi(x)$, that there exists a number ξ, $a < \xi < b$, such that

$$\frac{f(a)}{(a-b)(a-c)} + \frac{f(b)}{(b-c)(b-a)} + \frac{f(c)}{(c-a)(c-b)} = \frac{1}{2} f''(\xi)$$

33. Given that

$$\phi(x)=\begin{vmatrix} f(x) & x^2 & x & 1 \\ f(b) & b^2 & b & 1 \\ f(a) & a^2 & a & 1 \\ f'(a) & 2a & 1 & 0 \end{vmatrix}$$

and $f''(x)$ exists at all points in (a, b), deduce

$$f(b)=f(a)+(b-a)f'(a)+\tfrac{1}{2}(b-a)f''(\xi), \quad a<\xi<b.$$

34. If $f''(x)$ exists at all points in (a, b) and

$$\frac{f(c)-f(a)}{c-a}=\frac{f(b)-f(c)}{b-c}$$

where $a<c<b$, then show that there is a number such that $a<\xi<b$ and $f''(\xi)=0$.

35. Given that

$$\phi(x)=\begin{vmatrix} f(a) & f(b) & f(x) \\ g(a) & g(b) & g(x) \\ h(a) & h(b) & h(x) \end{vmatrix}$$

and $F(x)=\phi(x)-\dfrac{(x-a)(x-b)}{(c-a)(c-b)}\phi(c)$

where $a<c<b$ and $f''(x)$, $g''(x)$, $h''(x)$ exists throughout the interval (a, b), show that, by considering the function $F(x)$,

$$\phi(c)=\tfrac{1}{2}(c-a)(c-b)\phi''(\xi), \quad a<\xi<b$$

36. If $f''(x)$ exists at all points in (a, b) and if $f(a)=f(b)=0$ and if $f(c)>0$ where $a<c<b$, prove that there is at least one value ξ such that $f''(\xi)<0$, $a<\xi<b$.

ANSWERS

1. (i) 1·5. (ii) 6·25. (iii) $1-\sqrt{\frac{7}{12}}$. (iv) $\frac{1}{2}(a+b)$.

2. The tangent at the middle point of a parabolic arc is parallel to the chord of the arc.

3. $\frac{5}{12}$. **5.** $\frac{2}{7}$. **8.** No. **9.** (i) No. (ii) No.

12. (i) $\theta=\frac{1}{6}\left(3\pm\sqrt{3}\right)$.

(ii) $\xi = \pm 1$; the tangents at these two points are parallel to the line joining the points $\{a, f(a)\}$ and $\{b, f(b)\}$ which is parallel to the x-axis in this case.

13. (i) (*a*) $\dfrac{\sqrt{x^2 + xh} - x}{h}$ (*b*) $\dfrac{1}{h}\log\dfrac{e^h - 1}{h}$ (*c*) $\dfrac{1}{\log(1 + h/x)} - \dfrac{x}{h}$ **14.** $\frac{9}{64}$.

15. (i) $\log x + \dfrac{h}{x} - \dfrac{h^2}{2x^2} + \text{.........} + (-1)^{n-1}\dfrac{h^n}{n!(x + \theta h)^n}$.

(ii) $\sin x + h\cos x - \dfrac{h^2}{2!}\sin x - \dfrac{h^3}{3!}\cos x + \ldots + \dfrac{h^n}{n!}\sin\left(x + \theta h + \dfrac{n\pi}{2}\right)$.

(iii) $x^m + m x^{m-1}h + \dfrac{m(m-1)}{2!}x^{m-2}h^2 + \text{.........}$

$$\ldots + \frac{m(m-1)(m-2)\ldots(m-n+1)}{n!}h^n(x + \theta h)^{m-n}$$

16. (i) $a^n + {}^nc_1 a^{n-1}h + {}^nc_2 a^{n-2}h^2 + \ldots + {}^nc_r a^{n-r}h^r + \ldots + h^n$.

(ii) $37 + 27(x - 2) + 5(x - 2)^2$.

In the following series, in every case, $0 < \theta < 1$.

17. (i) $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots + \dfrac{x^n}{n!}e^{\theta x}$

(ii) $1 + x\log a + \dfrac{x^2}{2!}(\log a) + \dfrac{x^3}{3!}(\log a)^2 + \ldots + \dfrac{x^n}{n!}(\log a)^n a^{\theta x}$.

(iii) $x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots + \dfrac{x^n}{n!}\sin\left(\dfrac{n\pi}{2} + \theta x\right)$.

(iv) $1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \ldots + \dfrac{x^n}{n!}\cos\left(\dfrac{n\pi}{2} + \theta x\right)$.

(v) $x - \dfrac{x^2}{2} + \dfrac{x^3}{4} - \ldots + \dfrac{(-1)^{n-1}}{n}\dfrac{x^n}{(1 + \theta x)^n}$.

(vi) $-x - \dfrac{x^2}{2} - \ldots + \dfrac{1}{n}\dfrac{x^n}{(1 - \theta x)^n}$.

(vii) $1 + mx + \dfrac{m(m-1)}{2!}x^2 + \text{.........}$

$$+ \frac{m(m-1)(m-2)\ldots(m-n+1)}{n!}x^n(x + \theta h)^{m-n}.$$

(viii) $x-\frac{x^2}{3}+\frac{x^3}{5}-\ldots+\frac{(-1)^{n-1}x^n}{n}\sin^n\left(\cot^{-1}\theta x\right)\sin n\left(\cot^{-1}\theta x\right).$

(ix) $1+\frac{x}{1!}\sqrt{2}\cos\left(1\cdot\frac{\pi}{4}\right)+\frac{x^2}{2!}\left(\sqrt{2}\right)^2\cos\left(2\cdot\frac{\pi}{4}\right)+\ldots\ldots\ldots\ldots$

(x) $\frac{x}{1!}r\sin\phi+\frac{x^2}{2!}+r^2\sin 2\phi+\ldots\ldots+\frac{x^n}{n!}r^n e^{a\theta x}\sin\left(b\theta x+n\phi\right).$

where $r=\sqrt{a^2+b^2}$ and $\phi=\tan^{-1}\frac{b}{a}$.

18. $\theta=\frac{1-(1-x)^{1/(n+1)}}{x}.$

19. (i) $x^2-\frac{x^4}{3}+\frac{2x^6}{45}-\frac{4}{315}x^7\sin 2\theta x.$

(ii) $1-\frac{3}{2}x^2-\frac{7}{8}x^4-\frac{1}{160}x^5(81\sin 3\theta x+\sin\theta x).$

(iii) $1-x^2+\frac{1}{2}x^4-\frac{1}{15}x^5e^{-\theta^2x^2}\left(4\theta^5x^5-20\theta^3x^3+15\theta x\right)$.

20. (i) $1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n(1-\theta)^{n-1}}{(n-1)!}e^{\theta x}$

(ii) $1-\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots+\frac{x^n(1-\theta)^{n-1}}{(n-1)!}\cos\left(\frac{\pi x}{2}+\theta x\right)$

(iii) $1+x+x^2+\ldots+\frac{nx^n(1-\theta)^{n-1}}{(1-\theta x)^{n+1}}$

24. $\frac{1}{2}(b+a)$

9.9. Expansion of functions in infinite power series. Taylor's series (extended to infinity).

If $f(x)$, $f'(x)$, $f''(x)$, . . ., $f^n(x)$ exist *finitely* however large n may be in any interval $[x-\delta,\ x+\delta]$ enclosing the point x and if in addition R_n *tends to zero as* n *tends to infinity,* then Taylor's series extended to infinity is valid, and we have

$$\mathbf{f}(\mathbf{x}+\mathbf{h})=\mathbf{f}(\mathbf{x})+\mathbf{h}\mathbf{f}'(\mathbf{x})+\frac{\mathbf{h}^2}{2!}\mathbf{f}''(\mathbf{x})+\ldots\text{ to }\infty,\quad [\,|h|<\delta\,]$$

Denoting the first n terms of the expansion of by S_n and the remainder by R_n we have, by Art. 9.5.

$$f(x+h) = S_n + R_n, \quad i.e., \; f(x+h) - R_n = S_n,$$

Now, let $n \to \infty$; then If $R_n \to 0$, we have

$$f(x+h) = \underset{n\to\infty}{Lt} \; S_n = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \ldots \text{ to } \infty$$

Again, since $f(x+h) - S_n = R_n$, if $f(x+h) = \underset{n\to\infty}{Lt} \; S_n$,

then $\underset{n\to\infty}{Lt} \; R_n = 0$.

Thus, $\underset{n\to\infty}{Lt} \; R_n = 0$ is both *necessary* and *sufficient* condition that $f(x+h)$ can be expanded in an infinite series.

Cor. Another form of Taylor's series which is found often useful is obtained by putting $x-a$ for h in the form (B), Art. 9.5.

Thus, $f(x) = f(a) + (x-a)f'(a) + \dfrac{(x-a)^2}{2!} f''(a) + \cdots$

Maclaurin's series (extended to infinity).

If $f(x)$, $f'(x)$, $f''(x)$, . . ., $f^n(x)$ exist *finitely* however large n may be in any interval $(-\delta, \delta)$ and R_n *tends to zero as* n *tends to infinity,* then Maclaurin's series extended to infinity is valid, and we have

$$\mathbf{f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \cdots}, \text{ where } |x| < \delta.$$

Illustration.

Ex. *Expand the following functions in powers of x in infinite series stating in each case the conditions under which the expansion is valid*:

(i) $\sin x$, (ii) $\cos x$, (iii) e^x, (iv) $\log(1+x)$,

(v) $(1+x)^m$.

(i) Let $f(x) = \sin x$

$\therefore \; f^n(x) = \sin\left(\frac{1}{2}n\pi + x\right)$, so that $f(x)$ possesses derivatives of every order for every value of x. Also, $f^n(0) = \sin\frac{1}{2}n\pi$ which is 0 or ± 1 according as n is even or odd.

$$\therefore \; R_n = \frac{x^n}{n!} f^n(\theta x) = \frac{x^n}{n!} \sin\left(\theta x + \frac{n\pi}{2}\right)$$

$$\therefore \quad |R_n| = \left|\frac{x^n}{n!}\right| \left|\sin\left(\theta x + \frac{n\pi}{2}\right)\right| \le \left|\frac{x^n}{n!}\right|,$$

since $\left|\sin\left(\theta x + \frac{1}{2} n\pi\right)\right| \le 1$.

$R_n \to 0$ as $n \to \infty$, since $\dfrac{x^n}{n!} \to 0$ as $n \to \infty$ for all values of x.

[*Ex. 8, § 3.11*]

Thus the conditions for Maclaurin's infinite expansion are satisfied.

$$\therefore \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots \text{to } \infty, \text{ for all values of } x.$$

(ii) Let $f(x) = \cos x$.

$\therefore \quad f^n(x) = \cos\left(\frac{1}{2} n\pi + x\right), \ \therefore \ f^n(0) = \cos\left(n . \frac{1}{2}\pi\right)$, which is 0 or ± 1 according as n is odd or even.

$$\therefore \quad R_n = \frac{x^n}{n!} f^n(\theta x) = \frac{x^n}{n!} \cos\left(\theta x + \frac{n\pi}{2}\right).$$

Now proceeding as in the case of $\sin x$, we can show that

$R_n \to 0$ as $n \to \infty$, for all values of x.

$$\therefore \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots \text{to } \infty, \text{ for all values of } x.$$

(iii) Let $f(x) = e^x$.

$\therefore \quad f^n(x) = e^x, \ \therefore \ f^n(0) = 1$, thus $f^n(0)$ exists and is finite, however large n may be.

$$\therefore \quad R_n = \frac{x^n}{n!} f^n(\theta x) = \frac{x^n}{n!} e^{\theta x}$$

Now, since $e^{\theta x} < e^{|x|}$ (a finite quantity for a given x)

and $\dfrac{x^n}{n!} \to 0$ as $n \to \infty, R_n \to 0$ as $n \to \infty$.

$$\therefore \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots \text{to } \infty, \text{ for all values of } x.$$

(iv) Let $f(x) = \log(1+x)$.

$$f^n(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$$

which exists for every value of n for $x > -1$.

$$f^n(0) = (-1)^{n-1}(n-1)!$$

If R_n denotes Lagrange's form of remainder, we have

$$R_n = \frac{x^n}{n!} f^n(\theta x) = (-1)^{n-1} \frac{1}{n}\left(\frac{x}{1+\theta x}\right)^n$$

(i) Let $0 \le x \le 1$, so that $\left(\frac{x}{1+\theta x}\right)^n \to 0$ as $n \to \infty$, since $\frac{x}{1+\theta x}$ is positive and less than 1.

Also, $\frac{1}{n} \to 0$ as $n \to \infty$, $\therefore R_n \to 0$ as $n \to \infty$.

(ii) Let $-1 < x < 0$; in this case $\frac{x}{1+\theta x}$ may not be numerically less than unity and hence $\left(\frac{x}{1+\theta x}\right)^n$ may not tend to 0 as $n \to \infty$.

Thus, we fail to draw any definite conclusion from Lagrange's form of remainder. Using Cauchy's form of remainder, we have

$$R_n = \frac{x^n(1-\theta)^{n-1}}{(n-1)!} f^n(\theta x) = (-1)^{n-1} \cdot \frac{x^n}{1+\theta x}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}.$$

Now, $\frac{1-\theta}{1+\theta x}$ is positive and less than 1; hence $\left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \to 0$ as $n \to \infty$.

Also, $x^n \to 0$, as $n \to \infty$, since $-1 < x < 0$; $\frac{1}{1+\theta x}$ is bounded $R_n \to 0$ as $n \to \infty$,

$$\therefore \quad \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots \quad \text{is valid for } -1 < x \le 1.$$

Proceeding similarly we can show that

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots\ldots \quad \text{is valid for } -1 \le x < 1.$$

(v) Let $f(x) = (1+x)^m$, where m *is any real number.*

[*Binomial expansion*]

(i) When m is a *positive integer*, $f^n(x) = 0$, when $n > m$, for every value of x. Hence the expansion stops after the $(m+1)^{\text{th}}$ term and the binomial expansion, being a finite series, is *valid for all values of* x.

(ii) When m is *a negative integer* or a *fraction*,

$$f^n(x) = m(m-1)\ldots(m-n+1)(1+x)^{m-n}, \text{ for } x > -1$$

Hence Cauchy's form of remainder R_n is

$$R_n = \frac{m(m-1)\ldots(m-n+1)}{(n-1)!} x^n (1+\theta x)^{m-1} \left(\frac{1-\theta}{1+\theta x}\right)^{n-1}$$

Let $-1 < x < 1$, *i.e.*, $|x| < 1$; also, $0 < \theta < 1$.

$\therefore \quad 0 < 1 - \theta < 1 + \theta x$.

$\therefore \quad 0 < \dfrac{1-\theta}{1+\theta x} < 1$. $\qquad \therefore \quad 0 < \left(\dfrac{1-\theta}{1+\theta x}\right)^{n-1} < 1$.

(n being a positive integer > 1)

$\therefore \quad \left(\dfrac{1-\theta}{1+\theta x}\right)^{n-1} \to 0$ as $n \to \infty$. Also, $(1+\theta x)^{m-1}$ is finite whether $(m-1)$ is positive or negative.

Again, if $|x| < 1$, $\dfrac{m(m-1)\ldots(m-n+1)}{(n-1)!} x^n \to 0$.

[*Art. 3.11, Ex 8(iv)*]

$\therefore \quad |x| < 1,\ R_n \to 0$ as $n \to \infty$.

$\therefore$ for $|x| < 1$, Maclaurin's infinite expansion for $(1+x)^m$ is valid, m being a negative integer or a fraction.

9.10. Determination of the coefficients in the expansion of f(x) and f (x+h). (*Alternative Method*).

(i) Assuming that $f(x)$ admits of expansion in a convergent power series in x for all values of x within a certiain range, and that the expansion can be differentiated term by term any number of times within this range, we can easily get the coefficient of different powers of x as follows :

Let $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots\ldots \qquad \cdots \quad (1)$

where $a_0, a_1, a_2, \ldots\ldots\ldots$ are constants.

We have by successive differentiations,

$$f'(x)=a_1+2a_2x+3a_3x^2+4a_4x^3+\dots\dots\dots \quad \cdots (2)$$

$$f''(x)=2.1a_2+3.2a_3x+4.3a_4x^2+\dots\dots\dots \quad \cdots (3)$$

$$f'''(x)=3.2.1a_3+4.3.2a_4x+\dots\dots\dots \quad \cdots (4)$$

etc. etc. etc.

Putting $x=0$ in (1), (2), (3), (4),........, we get

$$f(0)=a_0,\ f'(0)=a_1,\ f''(0)=2!a_2,\ f'''(0)=3!a_3\dots$$

Hence, $f(x)=f(0)+xf'(0)+\dfrac{x^2}{2!}f''(0)+\dfrac{x^3}{3!}f'''(0)+\dots\dots$

(ii) Let $f(x+h)$ be a function of h (x being independent of h), and let us assume that it can be expanded in powers of h, and that the expansion can be differentiated with respect to h term by term any number of times within a certain range of values of h. We can easily obtain the coefficients of various powers of h as follows:

Let $f(x+h)=a_0+a_1h+a_2h^2+a_3h^3+\dots\dots\dots \quad \cdots (1)$

where $a_0, a_1, a_2, \dots\dots\dots$ are functions of x, and independent of h.

Since $\dfrac{d}{dh}f(x+h)=\dfrac{d}{dz}f(z)\dfrac{dz}{dh}$ [where $z=x+h$]

$$=f'(z)=f'(x+h),$$

differentiating (1) successively with respect to h, we get

$$f'(x+h)=a_1+2a_2h+3a_3h^2+4a_4h^3+\dots\dots\dots \quad \cdots (2)$$

$$f''(x+h)=2.1a_2+3.2a_3h+4.3a_4h^2+\dots\dots\dots \quad \cdots (3)$$

$$f'''(x+h)=3.2.1a_3+4.3.2a_4h+\dots\dots\dots \quad \cdots (4)$$

etc. etc. etc.

Putting $h=0$ in (1), (2), (3), (4),........, we get

$$f(x)=a_0,\ f'(x)=a_1,\ f''(x)=2!a_2,\ f'''(x)=3!a_3,\ \dots$$

$$\therefore \quad f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\dots$$

Note. Although the *forms* of the series obtained above for $f(x+h)$ and $f(x)$ are identical with the Taylor's and Maclaurin's infinite series for the expansions of these two functions, the above method of proof if used for establishing these two series, is considered as *defective* in as much as it does not enable us to determine exactly the value of x for which the infinite series obtained from each of the functions converges to the value of the function. In fact, Taylor's and Maclaurin's expansions in infinite series do not converge to the functions from which they are developed unless $R_n \to 0$ as $n \to \infty$, even though the function might possess finite differential coefficients of all order and the infinite series may be convergent ; *e.g.*, $f(x) = e^{-1/x^2}$ $(x \neq 0)$, $f(0) = 0$.

Here, $f^r(0) = 0$ for every value of r. But Maclaurin's infinite series for this function, though convergent for all values of x, is not equal to $f(x)$.

9.11. Other methods of Expansion.

The use of Maclaurin's (as also of Taylor's) theorem in expanding a given function in infinite power series is limited in applications because of the unwieldy form of the remainder (*i.e.*, of the *n*th derivative of the function) in many cases. So we employ other methods for expansion. Now, in this connection it should be noted that the operations of algebra like addition, subtration, multiplication, division and operations of calculus like term by term limit and term by term differentiation, though applicable to the sum of a finite number of functions, are not applicable without further examination to the case when the number of terms is infinite, and hence to the infinite power series $\Sigma a_n x^n$. If a power series in x converges (*i.e.*, has a finite sum) for values of x lying within a certain range (called the interval of convergence)[1], then for values of x within that range, operations of algebra and calculus referred to above are applicable, as in the case of polynomials, and the series obtained by such operations would represent the function for which it stands only for those values of x which lie within the interval of convergence. We illustrate below some of these methods.

9.12. Illustrative Examples.

A. Algebraical Method.

Ex. 1. *Expand* $\tan x$ *in powers of* x *as far as* x^5.

$$\text{Since } \tan x = \frac{\sin x}{\cos x} = \frac{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots}{1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \ldots},$$

we may, by actual division, show that

[1] See Chapter 6

$$\tan x = x + \tfrac{1}{2}x^3 + \tfrac{2}{15}x^5 + \ldots$$

Ex. 2. *Expand* $\dfrac{\log(1+x)}{1+x}$ *in powers of* x *as far as* x^4.

Multiplying the two series

$$\log(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \ldots \ (-1 < x < 1)$$

and $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \ldots \quad (-1 < x < 1)$

and collecting together the coefficients of like powers of x, we have

$\dfrac{\log(1+x)}{1+x} = x - \left(1 + \tfrac{1}{2}\right)x^2 + \left(1 + \tfrac{1}{2} + \tfrac{1}{3}\right)x^3 - \left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4}\right)x^4 + \ldots$ for $|x| < 1$ (the common interval of convergence).

B. Method of Undetermined Coefficients.

Ex. 3. *Expand* $\log(1+x)$ *in ascending powers of* x. [*V. P. 1998*]

Let $\log(1+x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots \qquad \ldots (1)$

$\therefore$ differentiating with respect to x,

$$\frac{1}{1+x} = a_1 + 2a_2x + 3a_3x^2 + \cdots \qquad \ldots (2)$$

$$\therefore \ (1+x)(a_1 + 2a_2x + 3a_3x^2 + \ldots) = 1 \qquad \ldots (3)$$

Equating coefficients of x^n on both sides,

$$na_n + (n+1)a_{n+1} = 0 \qquad \ldots (4)$$

Putting $x = 0$ in (1) and (2), $a_0 = \log 1 = 0$, $a_1 = 1$.

Putting $n = 1, 2, 3, \ldots$ in (4), we get $a_2 = -\tfrac{1}{2}$, $a_3 = \tfrac{1}{3}$, $a_4 = -\tfrac{1}{4}$, etc.

$$\therefore \ \log(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \ldots. \qquad \ldots (5)$$

Alternatively

For $|x| < 1$, $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \ldots$

Hence, comparing coefficients of like powers of x on both sides of (2), we obtain $a_1 = 1$, $a_2 = -\tfrac{1}{2}$, $a_3 = \tfrac{1}{3}$, etc.

Note. We shall have now to find for which values of x the series is convergent, and hence represents the function.

It can be shown that the series is convergent for $-1 < x < 1$.

Ex. 4. *Show that*

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \ldots \qquad [\textit{C. P. 1947}]$$

Let $y = \sin^{-1} x = a_o + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n + \ldots\ldots \quad \ldots \quad (1)$

Since $y = \sin^{-1} x$, $\therefore$ differentiating, $y_1 = \dfrac{1}{\sqrt{1-x^2}} \quad \ldots \quad (2)$

$$\therefore \quad y_1 = \left(1-x\right)^{-\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \frac{1.3}{2.4}x^4 + \frac{1.3.5}{2.4.6}x^6 + \ldots \quad \ldots \quad (3)$$

for $-1 < x < 1$ by Binomial expansion.

Also, $y_1 = a_1 + 2a_2 x + 3a_3 x^2 + \ldots\ldots + na_n x^{n-1} + \ldots\ldots \quad \ldots \quad (4)$

Hence, comparing the coefficients of (3) and (4), we get

$$a_1 = 1,\ a_2 = 0,\ a_3 = \frac{1}{2.3},\ a_4 = 0,\ a_5 = \frac{1.3}{2.4.5}, \text{ etc.}$$

Also, putting $x = 0$ in (1), $a_0 = \sin^{-1} 0 = 0$.
Hence, the result.

C. Method of formation of Differential equation.

Ex. 5. *Expand* $(\sin^{-1} x)^2$ *in a series of ascending powers of* x.

Let $y = (\sin^{-1} x)^2$. $\quad \ldots \quad (1)$

Differentiating, $y_1 = \dfrac{2\sin^{-1} x}{\sqrt{1-x^2}}$. $\quad \ldots \quad (2)$

or, $y_1^2\left(1-x^2\right) = 4\left(\sin^{-1} x\right)^2 = 4y.$

Differentiating again, and dividing by $2y_1 \neq 0$,

$$\left(1-x^2\right)y_2 - xy_1 - 2 = 0. \quad \ldots \quad (3)$$

Differentiating this n times by Leibnitz's theorem, we get

$$\left(1-x^2\right)y_{n+2} - 2nxy_{n+1} - n\left(n-1\right)y_n - xy_{n+1} - ny_n = 0,$$

or, $\left(1-x^2\right)y_{n+2} - \left(2n+1\right)xy_{n+1} - n^2 y_n = 0 \quad \ldots \quad (4)$

From (1), (2) and (3), we get $y_0 = 0$, $(y_1)_0 = 0$, $(y_2)_0 = 2$, and from (4), putting $x = 0$, we get

$$\left(y_{n+2}\right)_0 = n^2\left(y_n\right)_0 \quad \ldots \quad (5)$$

$\therefore$ putting $n = 1, 3, 5, \ldots\ldots$ in (5), we get

$$\left(y_3\right)_0 = \left(y_5\right)_0 = \left(y_7\right)_0 = \ldots\ldots = 0$$

and putting $n = 2, 4, 6, \ldots\ldots$ in (5), we have

$$(y_4)_0 = 2^2 (y_2)_0 = 2^2 .2,$$

$$(y_6)_0 = 4^2 (y_4)_0 = 4^2 .2^2 .2.$$

Similarly, $(y_8)_0 = 6^2 .4^2 .2^2 .2$, etc.

Assuming that a Maclaurin's series exists for this function, the coefficients are the values of $y, y_1, y_2, \ldots\ldots, y_n, \ldots\ldots$ when $x = 0$.

Hence,

$$\left(\sin^{-1} x\right)^2 = \frac{1}{2!} \cdot 2x^2 + \frac{2^2}{4!} \cdot 2x^4 + \frac{2^2.4^2}{6!} \cdot 2x^6 + \frac{2^2.4^2.6^2}{8!} \cdot 2x^8 + \ldots$$

Note. It can be shown that this series converges for $x^2 \leq 1$.

D. Differentiation of known series.

Ex. 6. *Assuming expansion of* $\sin x$, *prove that*

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots$$

From the series $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots$

which converges for all values of x, we get the required result by differentiation.

Ex. 7. *Show that* $\sec^2 x = 1 + x^2 + \frac{2}{3} x^4 + \ldots\ldots$

Since $\tan x = x + \frac{1}{3} x^3 + \frac{2}{15} x^5 + \ldots\ldots$

we get the required relult by differentiation.

9.13 Miscellaneous Worked Out Examples

Ex. 1. (*i*) Is Rolle's theorem applicable to $f(x) = \dfrac{1}{2 - x^2}$ in $[-1, 1]$? Justify your answer. [*C. P. 1989*]

(*ii*) Does $f(x) = \cos\left(\dfrac{1}{x}\right)$ satisfy Rolle's Theorem in the interval $-1 \leq x \leq 1$? [*C. P. 1993*]

(*iii*) Is Rolle's Theorem applicable to $f(x) = 1 - x^{\frac{2}{3}}$ in $-1 \leq x \leq 1$? Justify your answer. [*C. P. 1990*]

Solution : (*i*) $f(x)=\dfrac{1}{2-x^2}$ is *continuous* in $-1\le x\le 1$.

$$f'(x)=(-1)\left(2-x^2\right)^{-2}(-2x)=\frac{2x}{\left(2-x^2\right)^2}$$, which exists in $1<x<1$.

And $f(1)=\dfrac{1}{2-1}=1,\quad f(-1)=\dfrac{1}{2-1}=1$

Thus $f(-1)=f(1)$

So, $f(x)$ satisfies all the three conditions of Rolle's Theorem. Hence; Rolle's Theorem is applicable to $f(x)$ in $[-1, 1]$.

(*ii*) $f(x)=\cos\left(\dfrac{1}{x}\right)$

$f(0)=\cos\left(\dfrac{1}{0}\right)$, is undefined

So, $f(x)$ is *not continuous* at $x=0$ which is a point in the interval $-1\le x\le 1$.

Again, $f'(x)=\dfrac{1}{x^2}\sin\left(\dfrac{1}{x}\right)$, which *does not exist* at $x=0$.

So, $f(x)$ does not satisfy the first two conditions of Rolle's Theorem.

Rolle's Theorem is not applicable to $f(x)=\cos\left(\dfrac{1}{x}\right)$ in $[-1,\ 1]$.

(*iii*) Here, $f(x)=1-x^{\frac{2}{3}}$

If $x\ne 0,\quad f'(x)=-\dfrac{2}{3}x^{-\frac{1}{3}}=-\dfrac{2}{3x^{\frac{1}{3}}}$

$f'(0)$ *does not exist.*

$f(x)$ is not derivable at all points in the open interval $-1<x<1$.

Hence, Rolle's Theorem is not applicable to $f(x)=1-x^{\frac{2}{3}}$ in the interval $-1\le x\le 1$.

Ex. 2. **Explain whether Rolle's Theorem is applicable to the function $f(x)=|x|$ in any interval containing the origin.**

[*C. P. 1980, '95, B. P. '95*]

Solution : Here, $f(x) = x, \quad x > 0$
$= 0, \quad x = 0$
$= -x, \quad x < 0$

Let us consider an interval $-a \le x \le a$, where $a > 0$. Obviously, this interval contains the origin.

Here, $f(a) = f(-a) = a$

And since, $\lim_{h \to 0+} f(h) = 0$, $\lim_{h \to 0-} f(h) = 0$

and $f(0) = 0$, $f(x)$ is continuous at $x = 0$ and so at all points in $-a \le x \le a$.

Now, $Lf'(0) = \lim_{h \to 0-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0+} \frac{-h}{h} = -1$

and $Rf'(0) = \lim_{h \to 0+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0+} \frac{h}{h} = +1$

$\because$ $Lf'(0) \ne Rf'(0)$, $f(x)$ is *not derivable* at $x = 0$.

So, it is not derivable at all points in $-a < x < a$.

Second condition of Rolle's Theorem is not satisfied.

Hence, Rolle's Theorem is not applicable to $f(x) = |x|$, in $[-a, a]$, *i.e.*, in any interval containing the origin.

Ex. 3. (*i*) Test the applicability of Rolle's Theorem for the function $f(x) = (x-a)^m \cdot (x-b)^n$ in $a \le x \le b$ where *m*, *n* are positive integers.

(*ii*) If $f(x) = (x-a)^m (x-b)^n$, where *m* and *n* are positive integers, show that '*c*' in Rolle's Theorem divides the segment $a \le x \le b$ in the ratio *m* : *n*. [*C. P. 1998*]

Solution : (*i*) $f(x) = (x-a)^m \cdot (x-b)^n$

since *m* and *n* are positive integers, $f(x)$ is a polynomial of degree $(m+n)$, which is continuous at every point, so it is continuous in $a \le x \le b$.

Also, $f'(x) = m(x-a)^{m-1}(x-b)^n + n(x-a)^m (x-b)^{n-1}$

$$= (x-a)^{m-1} \cdot (x-b)^{n-1} \{m(x-b) + n(x-a)\} \quad \ldots (1)$$

which exists in $a < x < b$.

$f(a) = 0 = f(b)$

So $f(x)$ satisfies all the conditions of Rolle's Theorem. Rolle's theorem is applicable to $f(x)$ in $a \le x \le b$.

(*ii*) Since $f(x)$ satisfies all the conditions of Rolle's Theorem, there is at least one point 'c' in $a < x < b$, such that $f'(c) = 0$.

From (1), $(c-a)^{m-1}(c-b)^{n-1}\{m(c-b)+n(c-a)\} = 0$

$\because a < c < b,\ c-a \neq 0, \quad c-b \neq 0$

$\therefore\ m(c-b)+n(c-a) = 0$ or, $c = \dfrac{mb+na}{m+n}$,

so that the point $x = c$ divides the segment $a \le x \le b$ internally in the ratio $m : n$.

Ex. 4. Verify Rolle's Theorem for the function $f(x) = x^2 - 5x + 6$ in $1 \le x \le 4$. [*C. P. 1991*]

Solution : $f(x) = x^2 - 5x + 6$

$\because f(x)$ is a polynomial function, it is continuous at every point, so it is continuous is $1 \le x \le 4$.

$f'(x) = 2x - 5$, exists in $1 < x < 4$.

$f(1) = 2 = f(4)$

Thus $f(x)$ satisfies all the three conditions of Rolle's theorem. Then there exists at least one point $x = c$ in $1 < x < 4$, such that $f'(c) = 0$.

$\therefore\ 2c - 5 = 0$, or, $c = \dfrac{5}{2}$, which lies is $1 < x < 4$.

Thus Rolle's theorem is verified.

Ex. 5. Discuss the applicability of the Mean value value theorem $f(b) - f(a) = (b-a)f'(\xi)$, $a < \xi < b$.

Find ξ, if the theorem is applicable.

(*i*) $f(x) = x(x-1)(x-3)$, $0 \le x \le 4$ [*C. P. 1992, B. P. 2002*]

(*ii*) $f(x) = |x|$, $-1 \le x \le 1$ [*B. P. 1995*]

(*iii*) $f(x) = |x|$, $0 \le x \le 1$ [*C. P. 1988*]

(*iv*) $f(x) = x(x-1)(x-2)$, $a = 0$, $b = \dfrac{1}{2}$ [*C. P. 1987, B. P. 1997*]

Solution : (*i*) $f(x) = x(x-1)(x-3)$

$$= x^3 - 4x^2 + 3x$$

$$f'(x) = 3x^2 - 8x + 3.$$

$f(x)$ being a polynomial function of x is continuous in $0 \le x \le 4$.

$f'(x)$ also being a polynomial function of x, exists in $0<x<4$.

So, $f(x)$ satisfies the conditions of Lagrange's Mean value theorem in $0 \le x \le 4$.

There exists at least one point ξ in $0 < x < 4$, such that

$$f(b)-f(a)=(b-a)f'(\xi)$$

i.e., $f(4)-f(0)=(4-0)f'(\xi)$

or, $12-0=4\left(3\xi^2-8\xi+3\right)$

or, $3\xi^2-8\xi=0$. $\therefore$ $\xi=0, \frac{8}{3}$.

$\because$ $0<\xi<4$, $\xi=\frac{8}{3}$ (the value $\xi=0$ is rejected)

(*ii*) $f(x)=|x|$, $-1 \le x \le 1$

Here $f(x)=|x|$ is continuous in $-1 \le x \le 1$, but $f'(x)$ does not exist at $x=0$, in the interval [see Ex. 2]. So, Mean Value theorem is not applicable for the function $f(x)=|x|$ in the interval $-1 \le x \le 1$

(*iii*) $f(x)=|x|$, $0 \le x \le 1$

$f(x)=|x|$ is continuous in the interval $-0 \le x \le 1$ and $f'(x)$ exists at all points in $0< x <1$. $f'(x)=1$. Thus $f(x)=|x|$ satisfies the conditions of Mean Value therorem in $0 \le x \le 1$.

So, there exists at least one point ξ in $0<\xi<1$,

such that $f(1)-f(0)=(1-0)\,f'(\xi)$ . . . (1)

i.e., $1-0=1\cdot 1$

so the relation (1) is satisfied *identically*.

Hence, ξ is any number in the open interval $0<x<1$.

(*iv*) $f(x)=x(x-1)(x-2)$, $a=0$, $b=\frac{1}{2}$

Here, $f(x)=x^3-3x^2+2x$

$$f'(x)=3x^2-6x+2$$

$f(x)$ being a polynomial function is continuous is $0 \le x \le \frac{1}{2}$ and

$f'(x)$ also being a polynomial function exists in $0<x<\frac{1}{2}$.

$$f\left(\frac{1}{2}\right)=\frac{3}{8}, \quad f(0)=0$$

$$f\left(\frac{1}{2}\right)-f(0)=\left(\frac{1}{2}-0\right)f'(\xi) \text{ gives}$$

$$\frac{3}{8}=\frac{1}{2}\left(3\xi^2-6\xi+2\right)$$

or, $12\xi^2-24\xi+5=0$

$$\therefore\ \xi=\frac{6\pm\sqrt{21}}{6}$$

$$\because\ 0<\xi<\frac{1}{2}, \qquad \therefore\ \xi=\frac{6-\sqrt{21}}{6}$$

Ex. 6. (*i*) For what range of values of x, $2x^3-9x^2+12x-3$ decreases as x increases? [*C. P. 1986, B. P. 1989*]

(*ii*) Show that x^3+x^2-5x+3 is monotone increasing when $x>1$. [*C. P. 1992*]

(*iii*) Show that $-2x^3+15x^2-36x+6$ is strictly increasing in $2<x<3$. [*C. P. 1993*]

(*iv*) Show that $2x^3-12x^2+24x+6$ is increasing on the real line. [*C. P. 1995*]

(*v*) Separate the intervals in which $2x^3-15x^2+36x+1$ is increasing or decreasing. [*B. P. 1994*]

Solution : (*i*) Let, $f(x)=2x^3-9x^2+12x-3$

$$f'(x)=6x^2-18x+12$$
$$=6(x-1)(x-2)$$

$f(x)$ decreases as x increases if $f'(x)<0$

Obviously, $f'(x)<0$, for $1<x<2$

Hence the required range of values of x is $1<x<2$

(*ii*) Here, $f(x)=x^3+x^2-5x+3$

$$\therefore\ f'(x)=3x^2+2x-5=3(x-1)\left(x+\frac{5}{3}\right)$$

Here, $f'(1)=0$ and $f'(x)>0$, when $x>1$.

Hence, $f(x)$ is monotone increasing when $x > 1$.

(*iii*) Here, $f(x) = -2x^3 + 15x^2 - 36x + 6$

$\therefore \quad f'(x) = -6x^2 + 30x - 36 = -6(x-2)(x-3)$

$f'(x) > 0$ for all values of x satisfying $2 < x < 3$.

Hence, $f(x)$ is strictly increasing in $2 < x < 3$

(*iv*) Here, $f(x) = 2x^3 - 12x^2 + 24x + 6$

$\therefore \quad f'(x) = 6x^2 - 24x + 24 = 6(x-2)^2$

For all real values of x, $f'(x) \geq 0$.

Hence, $f(x)$ is increasing on the real number line.

(*v*) Here, $f(x) = 2x^3 - 15x^2 + 36x + 1$

$\therefore \quad f'(x) = 6x^2 - 30x + 36 = 6(x-2)(x-3)$

Obviously, $f'(x) > 0$ if $x > 3$ or $x < 2$

and, $f'(x) < 0$ if $2 < x < 3$

So, $f(x)$ is a decreasing function when $2 < x < 3$ and it is an increasing function in $(-\infty, 2)$ and $(3, \infty)$.

Ex. 7. (*i*) Show that $\dfrac{x}{1+x} < \log(1+x) < x$, if $x > 0$.

[*C. P. 1987, '89,, B. P. 2000*]

Solution : Let $\phi(x) = \log(1+x) - \dfrac{x}{1+x}$

Here $\phi(0) = 0$

and, $\phi'(x) = \dfrac{1}{1+x} - \dfrac{1+x-x}{(1+x)^2} = \dfrac{x}{(1+x)^2} > 0$, for $x > 0$

$\therefore \quad \phi(x) > 0$ for $x > 0$, and consequently

$$\frac{x}{1+x} < \log(1+x) \text{ for } x > 0 \qquad \ldots \quad (1)$$

Now let, $\psi(x) = x - \log(1+x)$

Here, $\psi(0) = 0$

and $\psi'(x) = 1 - \dfrac{1}{1+x} = \dfrac{x}{1+x} > 0$ for $x > 0$

$\therefore \quad \psi(x) > 0$, when $x > 0$

i.e., $x - \log(1+x) > 0$, when $x > 0$

i.e., $\log(1+x) < x$, for $x > 0$ . . . (2)

From (1) and (2), $\dfrac{x}{1+x} < \log(1+x) < x$, if $x > 0$.

Ex. 8. Show that $\left(\dfrac{\sin x}{x}\right)$ decreases steadily in $0 < x < \dfrac{\pi}{2}$.

[*C. P. 1983, B. P. 1993*]

Solution : Let, $f(x) = \dfrac{\sin x}{x}$

$$\therefore \quad f'(x) = \frac{x\cos x - \sin x}{x^2} \qquad \ldots \quad (1)$$

Let $F(x) = x\cos x - \sin x$

then $F'(x) = \cos x - x\sin x - \cos x$

$$= -x\sin x < 0, \text{ in } 0 < x < \frac{\pi}{2}$$

$\because$ $F(0) = 0$ and $F(x)$ is strictly decreasing function in $0 < x < \dfrac{\pi}{2}$.

$\therefore$ $F(x) < 0$ in $0 < x < \dfrac{\pi}{2}$.

i.e., $x\cos x - \sin x < 0$, in $0 < x < \dfrac{\pi}{2}$

So, from (1) $f(x) < 0$ in $0 < x < \dfrac{\pi}{2}$.

Hence, $\dfrac{\sin x}{x}$ decreases steadily in $0 < x < \dfrac{\pi}{2}$.

Ex. 9. Show that $\dfrac{2}{\pi} < \dfrac{\sin x}{x} < 1$, for $0 < x < \dfrac{\pi}{2}$.

Solution : Let us define a function ϕ, such that

$$\phi(x) = \frac{\sin x}{x}, \quad \text{when } x \neq 0$$
$$= 1 \quad \text{when } x = 0$$

Obviously, $\phi(x)$ is continuous in $0 \le x \le \dfrac{\pi}{2}$ and derivable in

$0 < x < \dfrac{\pi}{2}$ and $\phi'(x) = \dfrac{x\cos x - \sin x}{x^2}$.

Let us consider another function $\psi(x)$, such that

$$\psi(x) = x\cos x - \sin x \text{ defined in } \left[0, \frac{\pi}{2}\right]$$

$$\psi'(x) = -x\sin x < 0 \text{ for } 0 < x < \frac{\pi}{2}$$

$\therefore$ $\psi(x)$ is strictly decreasing in $0 \le x \le \frac{\pi}{2}$.

$\therefore$ $\psi(x) < \psi(0) = 0$ for all x, $0 \le x \le \frac{\pi}{2}$.

$\therefore$ $\phi'(x) < 0$, for $0 \le x \le \frac{\pi}{2}$

So $\phi(x)$ is strictly decreasing in $0 \le x \le \frac{\pi}{2}$.

$\therefore$ $\phi(0) > \phi(x) > \phi\left(\frac{\pi}{2}\right)$ for $0 < x < \frac{\pi}{2}$.

i.e., $1 > \frac{\sin x}{x} > \frac{1}{\frac{\pi}{2}}$

i.e., $\frac{2}{\pi} < \frac{\sin x}{x} < 1$ for $0 < x < \frac{\pi}{2}$.

Ex. 10. Show that $\frac{\tan x}{x} > \frac{x}{\sin x}$, for $0 < x < \frac{\pi}{2}$.

[*C. P. 1983, B. P. 1993*]

Solution : Here, we shall have to show that

$$\frac{\tan x \sin x - x^2}{x \sin x} > 0 \text{ for } 0 < x < \frac{\pi}{2}$$

$\because$ $x \sin x > 0$, when $0 < x < \frac{\pi}{2}$,

it will be enough to show that

$$\tan x \sin x - x^2 > 0, \text{ for } 0 < x < \frac{\pi}{2}$$

Let $F(x) = \tan x \sin x - x^2, \quad 0 < x < \frac{\pi}{2}$

then $F'(x) = \sec^2 x \sin x + \tan x \cos x - 2x$

$$= \sin x \sec^2 x + \sin x - 2x$$

and $F''(x) = \cos x \sec^2 x + \sin x \cdot 2\sec x \sec x \tan x + \cos x - 2$

$$= \sec x + \cos x - 2 + 2\sin x \tan x \sec^2 x$$

$$= \left(\sqrt{\sec x} - \sqrt{\cos x}\right)^2 + 2\sin x \tan x \sec^2 x$$

$$> 0, \text{ for } 0 < x < \frac{\pi}{2}.$$

So, $F'(x)$ is strictly increasing in the interval $0 \le x \le \frac{\pi}{2}$.

i.e., $F(x) > 0$, for $0 < x < \frac{\pi}{2}$

Therefore, $F(x)$ is strictly increasing in $0 \le x \le \frac{\pi}{2}$ and also $F(0) = 0$.

So, $F(x) > 0$, for $0 < x < \frac{\pi}{2}$.

i.e., $\tan x \sin x - x^2 > 0$, for $0 < x < \frac{\pi}{2}$

or, $\dfrac{\tan x \sin x - x^2}{x \sin x} > 0$, in $0 < x < \frac{\pi}{2}$, ($\because$ $x \sin x > 0$ in $0 < x < \frac{\pi}{2}$).

i.e., $\dfrac{\tan x}{x} > \dfrac{x}{\sin x}$, when $0 < x < \frac{\pi}{2}$.

EXAMPLE - IX (B)

1. Expand in infinite series in powers of h :

   (i) e^{x+h}. (ii) $\cos(x+h)$. (iii) $e^h \sin(x+h)$.

2. Expand the following functions in powers of x in infinite series, stating in each case the condition under which the expansion is valid :

   (i) a^x. (ii) $\sinh x$. (iii) $\cosh x$.

   (iv) $\tan^{-1} x$. (v) $\cot^{-1} x$. (vi) $e^{ax} \sin bx$.

   (vii) $e^{ax} \cos bx$. (viii) $e^x \sin x$. (ix) $e^x \cos x$.

   (x) $\dfrac{1}{1+x}$. (xi) $\dfrac{1}{1+x^2}$.

3. Show that

$$\tan^{-1}(x+h) = \tan^{-1} x + (h\sin\theta).\sin\theta - \tfrac{1}{2}(h\sin\theta)^2 \sin 2\theta$$
$$+ \tfrac{1}{3}(h\sin\theta)^3 \sin 3\theta - \ldots, \text{ where } \theta = \cot^{-1} x.$$

4. Find approximately the value of $\sin 60^0 34' 23''$ to 4 places of decimals from the expansion of $\sin(x+h)$ in a series of ascending powers of h by putting

$$x = \tfrac{1}{3}\pi\left(=60^0\right) \text{ and } h = \tfrac{1}{100} \text{ of a radian}\left(=34'23'' \text{nearly}\right).$$

5. Show that

(i) $\log x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \dots$

is true for $0 < x \le 2$.

(ii) $\dfrac{1}{x} = \dfrac{1}{2} - \dfrac{1}{2^2}(x-2) + \dfrac{1}{2^3}(x-2)^2 - \dots$

is true for $0 < x < 4$.

6. Expand e^x. in powers of $(x-1)$.

Verify the following series (*Ex. 7 – 19*):

7. $\sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \dots$

8. $\log(1+x)^{1+x} = x + \dfrac{x^2}{2} - \dfrac{x^3}{6} + \dots$

9. $\log\left(1 - x + x^2\right) = -x + \dfrac{x^2}{2} + \dfrac{2}{3}x^3 + \dfrac{x^4}{4} - \dots$

10. $e^x \sin x = x + x^2 + \dfrac{x^3}{3} + \dfrac{x^5}{30} - \dots$

11. $e^x \log(1+x) = x + \dfrac{x^2}{2!} + \dfrac{2x^3}{3!} + \dfrac{9x^5}{5!} - \dots$

12. $\dfrac{x}{e^x - 1} = 1 - \dfrac{1}{2}x + \dfrac{1}{12}x^2 - \dfrac{1}{720}x^4 + \dots$

13. $x \cot x = 1 - \frac{1}{3}x^2 - \frac{1}{45}x^4 - \dots$

14. $\log(1 + \sin x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{6} - \dots$

15. $\log \sec x = \dfrac{x^2}{2!} + \dfrac{2x^4}{4!} + \dfrac{16x^6}{6!} + \dots$

16. $e^{\sin x} = 1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4 - \dots$

17. $e^{\tan^{-1} x} = 1 + x + \dfrac{x^2}{2} - \dfrac{x^3}{6} - \dots$ [C.P. 1940]

18. $\log\left(x + \sqrt{x^2+1} \right) = x - \dfrac{1}{2}\cdot\dfrac{x^3}{3} + \dfrac{1.3}{2.4}\cdot\dfrac{x^5}{5} - \dots$

19. $(1+x)^{\frac{1}{x}} = e\left(1 - \frac{1}{2}x + \frac{11}{24}x^2 - \frac{7}{16}x^3 + \dots\right)$

20. (i) By differentiation the identity

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots, \quad |x| < 1.$$

show that

$$(1-x)^{-3} = 1 + 3x + \frac{3.4}{1.2}x^2 + \frac{3.4.5}{1.2.3}x^3 + \dots$$

(ii) Differentiating the expansion for $\log(1 + \sin x)$, obtain the expansion for $\sec x - \tan x$.

21. (i) Show that $\sqrt{x}$ and $x^{\frac{5}{2}}$ cannot be expanded in Maclaurin's infinite series.

(ii) Given $f(x) = x^{\frac{3}{2}}$, show that for this function the expansion of $f(x+h)$ fails when $x = 0$, but that there exists a proper fraction θ such that

$$f(x+h) = f(x) + hf'(x) + \tfrac{1}{2}h^2 f''(x + \theta h)$$

holds when $x = 0$. Find q. [C. P. 1949]

22. If $y = (1+x)^n = a_0 + a_1x + a_2x^2 + \dots$, show that

$$(1+x)y_1 = ny$$

and hence obtain the expansion of $(1+x)^n$.

23. If $y = e^{a\sin^{-1}x} = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$,

prove that

(i) $(1-x^2)y_2 = xy_1 + a^2y$,

(ii) $(n+1)(n+2)a_{n+2} = (n^2 + a^2)a_n$,

and hence obtain the expansion of $e^{a\sin^{-1}x}$.

Deduce, from the expansion of $e^{a\sin^{-1}x}$, the expansion of $\sin^{-1}x$.

[C. P. 1945]

24. If $y = e^{m\tan^{-1}x} = a_0 + a_1x + a_2x^2 + \dots$, show that

(i) $(1+x^2)y_1 = my$,

(ii) $(n+1)a_{n+1} + (n-1)a_{n-1} = ma_n$,

and hence obtain the expansion of $e^{m\tan^{-1}x}$.

25. If $y = \sin(m\sin^{-1}x)$, show that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$$

and hence obtain the expansion of $\sin(m\sin^{-1}x)$. [C. P. 1938]

26. If $y = e^{ax}\cos bx$, prove that

$$y_2 - 2ay_1 + (a^2+b^2)y = 0,$$

and hence obtain the expansion of $e^{ax}\cos bx$.

Deduce the expansion of e^{ax} and $\cos bx$. [C. P. 1937]

ANSWERS

1. (i) $e^x\left(1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \dots\right)$

(ii) $\cos x - h\sin x - \frac{h^2}{2!}\cos x + \frac{h^3}{3!}\sin x + \dots$

(iii) $\sin x + \sqrt{2}\,h\sin\left(x + \frac{\pi}{4}\right) + \frac{(\sqrt{2}\,h)^2}{2!}\sin\left(x + \frac{2\pi}{4}\right) + \dots$

2. $\left.\begin{array}{l} \text{(i) } 1 + x\log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \dots \\ \text{(ii) } x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \\ \text{(iii) } 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \end{array}\right\}$ for all values of x

$\left.\begin{array}{l} \text{(iv) } x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \\ \text{(v) } \frac{\pi}{2} - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \dots\right) \end{array}\right\}$ for $-1 \le x \le 1$

(vi) $\frac{x}{1!} r\sin\phi + \frac{x^2}{2!} r^2 \sin 2\phi + \ldots$

(vii) $1 + \frac{x}{1!} r\cos\phi + \frac{x}{2!} r^2 \cos 2\phi + \ldots$

where $r = \sqrt{a^2 + b^2}$ and $\phi = \tan^{-1}\frac{b}{a}$.

(viii) $x\sqrt{2}\sin\frac{\pi}{4} + \frac{x^2}{2!}\left(\sqrt{2}\right)^2 \sin\frac{2\pi}{4} + \ldots + \frac{x^n}{n!} 2^{\frac{n}{2}} \sin\frac{n\pi}{4} + \ldots$

(ix) $1 + x\sqrt{2}\cos\frac{\pi}{4} + \frac{x^2}{2!}\left(\sqrt{2}\right)^2 \cos\frac{2\pi}{4} + \ldots + \frac{x^n}{n!} 2^{\frac{n}{2}} \cos\frac{n\pi}{4} + \ldots$

Ex. (vi) – (ix) are valid for all values of x.

(x) $1 - x + x^2 - x^3 + \ldots \qquad |x| < 1.$

(xi) $1 - x^2 + x^4 - x^6 + \ldots \qquad -1 < x < 1.$

4. 0·8710.

6. $e\left\{1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \ldots\right\}$

20. (ii) $1 - x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \ldots$

21. (ii) $\frac{9}{64}$.

22. $1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$

23. $1 + ax + \frac{a^2x^2}{2!} + \frac{a\left(a^2 + 1^2\right)}{3!}x^3 + \frac{a^2\left(a^2 + 2^2\right)}{4!}x^4$

$$+ \frac{a\left(a^2 + 1^2\right)\left(a^2 + 3^2\right)}{5!}x^5 + \ldots$$

$$\sin^{-1} x = x + \frac{x^3}{3!} + \frac{1^2.3^2}{5!}x^5 + \ldots$$

24. $1 + mx + \frac{m^2}{2!}x^2 + \frac{m\left(m^2 - 2\right)}{3!}x^3 + \frac{m^2\left(m^2 - 8\right)}{4!}x^4 + \ldots$

25. $mx - \frac{m\left(m^2 - 1^2\right)}{3!}x^3 + \frac{m\left(m^2 - 1^2\right)\left(m^2 - 3^2\right)}{5!}x^5 - \ldots$

26. $1 + ax + \frac{a^2 - b^2}{2!}x^2 + \frac{a\left(a^2 - 3b^2\right)}{3!}x^3 + \ldots$

$$e^{ax} = 1 + ax + \frac{a^2x^2}{2!} + \ldots$$

$$\cos bx = 1 - \frac{b^2x^2}{2!} + \frac{b^4x^4}{4!} + \ldots$$

10 MAXIMA AND MINIMA (*Functions of a Single Varaible*)

10.1. By the maximum value of a function $f(x)$ in Calculus we do not necessarily mean the absolutely greatest value attainable by the function. A function $f(x)$ is said to be maximum for a value c of x, provided $f(c)$ is greater than every other value assumed by $f(x)$ in the immediate neighbourhood of $x=c$. Similarly, a minimum value of $f(x)$ is defined to be the value which is less than other value in the immediate neighbourhood. A formal definition is as follows :

A function $f(x)$ is said to have a maximum value for $x=c$, provided we can get a positive quantity δ such that for all values of x in the interval

$$c-\delta < x < c+\delta\,,\ (x \neq c)\ \ f(c) > f(x);$$

i. e., if $f(c+h)-f(c)<0$, for $|h|$ sufficiently small.

Similarly, *the function $f(x)$ has a minimum value for $x=d$, provided we can get an interval $d-\delta' < x < d+\delta'$ within which $f(d)<f(x)\ (x \neq d)$;*

i. e., if $f(d+h)-f(d)>0$, for $|h|$ sufficiently small.

Thus, in the Fig.10.1.1 which represents graphically the function $f(x)$ (a continuous function here), the function has maximum value at P_1, as also at P_2, P_3, P_4, etc. and has minimum values at Q_1, Q_2, Q_3, Q_4, etc. At P_1 for instance, corresponding to $x=OC_1$ $(=c_1$ say), the value of the function, namely, the ordinate P_1C_1 is not necessarily bigger than the value Q_2D_2 at $x=OD_2$, but we can get a range, say $L_1C_1L_2$ in the neighbourhood of C_1 on either side of it, (i.e., we can find a $\delta=L_1C_1=C_1L_2$, say) such that for every value of x within $L_1C_1L_2$ (execept at C_1), the value of the function (represented by the corresponding ordinate) is less than P_1C_1 (the value at C_1).

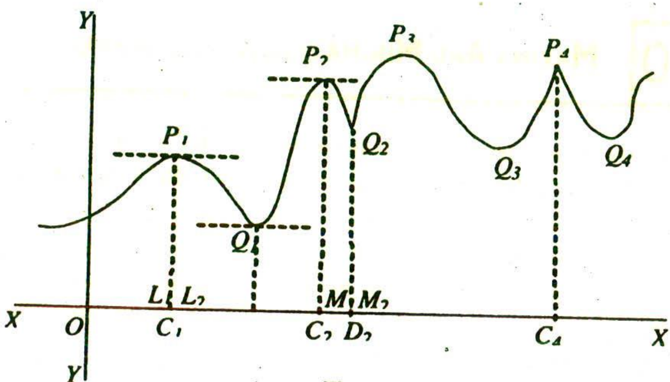


Fig 10.1.1

Hence, by definition, the function is maximum at $x = OC_1$. Similarly, we can find out an interval $M_1D_2M_2$ ($M_1D_2 = D_2M_2 = \delta'$, say) in the neighbourhood of D_2 within which for every other values of x the function is greater than that at D_2. Hence, the function at D_2 (represented by Q_2D_2) is a minimum.

From the figure the following features regarding maxima and minima of a continuous function will be apparent :

(i) that the function may have several maxima and minima in an interval;

(ii) that a maximum value of the function at some poinit may be less than a minimum value of it at another point ($C_1P_1 < D_2Q_2$);

(iii) maximum and minimum values of the function occur alternately, *i.e.*,between any two consecutive maximum values there is a minimum value, and *vice versa*.

10.2. A necessary condition for maximum and minimum.

If $f(x)$ be a maximum, or a minimum at $x = c$, and if $f'(c)$ exists, then $f'(c) = 0$.

By definition, $f(x)$ is a maximum at $x = c$, provided we can find a positive number δ such that

$$f(c+h) - f(c) < 0 \text{ whenever } -\delta < h < \delta, \ (h \neq 0).$$

$\therefore\ \dfrac{f(c+h)-f(c)}{h} < 0$ if h be positive and sufficiently small, >0 if h be negative and numerically sufficiently small.

Thus, $\underset{h\to 0+}{Lt}\ \dfrac{f(c+h)-f(c)}{h} \le 0$, [See *Ex. 5*, § 3·11]

and, similarly, $\underset{h\to 0-}{Lt}\ \dfrac{f(c+h)-f(c)}{h} \ge 0.$

Now, if $f'(c)$ exists, the above two limits, which represent the right-hand and left hand derivatives respectively of $f(x)$ at $x = c$, must be equal. Hence, the only common value of the limit is zero. Thus, $f'(c) = 0$.

Exactly similar is the proof when $f(c)$ is a minimum.

Note. *In case* $f'(c)$ *does not exist,* $f(c)$ *may be a maximum or a minimum*, as is apparent form the figure for points Q_2 and P_4. At the former point $f(x)$ is a minimum, and at the latter it is a maximum. $f'(x)$ is, however, not zero at these points, for, $f'(x)$ does not exist at all at these points.

10.3. Determination of Maxima and Minima.

(A) *If c be a point in the interval in which the function* $f(x)$ *is defined*, and if $\mathbf{f'(c) = 0}$ *and* $\mathbf{f''(c) \neq 0}$, *then* $\mathbf{f(c)}$ is

(i) **a maximum** *if* $\mathbf{f''(c)}$ **is negative** *and*

(ii) **a minimum** *if* **is positive.**

***Proof*:** Suppose $f'(c) = 0$, and $f''(c)$ exists, and $\neq 0$.

By the Mean Value Theorem[1],

$$f(c+h) - f(c) = h f'(c+\theta h), \text{ where } 0 < \theta < 1,$$

$$= \theta h^2 \cdot \frac{f'(c+\theta h) - f'(c)}{\theta h}.$$

Since $0 < \theta < 1$, $\theta h \to 0$ as $h \to 0$, and writing $\theta h = k$, the coefficient of θh^2 on the right side $\to \underset{k\to 0}{Lt}\ \dfrac{f'(c+k) - f'(k)}{k} = f''(c)$.

Accordingly, since θh^2 is positive, $f(c+h) - f(c)$ has the same sign as that of $f''(c)$ when $|h|$ is sufficiently small.

---

[1] Since $f''(x)$ exists, $f'(x)$ also exists in the neighbourhood of c.

$\therefore$ If $f''(c)$ is positive, $f(c+h)-f(c)$ is positive, whether h is positive or negative, provided $|h|$ is small. Hence $f(c)$ is a *minimum*, by definition.

Similarly, If $f''(c)$ is negative, $f(c+h)-f(c)$ is negative, whether h is positive or negative, when $|h|$ is small.

Hence $f(c)$ is a *maximum.*

(B) *Let* c *be an interior point of the interval of definition of the function* $f(x)$, *and let*

$$f'(c)=f''(c)=\cdots=f^{n-1}(c)=0, \text{ and } f^{n}(c)\neq 0;$$

then (i) *If* **n be even, f(c)** *is a* **maximum** *or a* **minimum** *according as* $f^n(c)$ *is* **negative** *or* **positive,**

and (ii) *if* **n be odd, f(c)** *is* **neither a minimum nor a maximum.**

***Proof*:** By the Mean Value Theorem of Higher order, here

$$f(c+h)-f(c)=\frac{h^{n-1}}{(n-1)!}f^{n-1}(c+\theta h),\quad 0<\theta<1,$$

$$=\frac{\theta h^{n}}{(n-1)!}\cdot\frac{f^{n-1}(c+\theta h)-f^{n-1}(c)}{\theta h}.$$

Since $0<\theta<1$, as $h\to 0, \theta h\to 0$, and

the coefficient of $\theta h^n/(n-1)!$, on the right side $\to f^n(c)$

Now, suppose n is *even*; then $\theta h^n/(n-1)!$ is positive.

$\therefore$ $f(c+h)-f(c)$ has the same sign as of $f^n(c)$, whether h is positive or negative, provided $|h|$ is sufficiently small. Hence, *if* $f^n(c)$ *be positive*, $f(c+h)-f(c)$ *is positive* for either sign of h, when $|h|$ is small, and so $f(c)$ *is a minimum*. Similarly, *if* $f^n(c)$ *is negative*, $f(c)$ is a *maximum.*

Next, suppose n *is odd* ; then $\theta h^n/(n-1)!$ is positive or negative according as h is positive or negative. Hence, $f(c+h)-f(c)$ changes in sign with the change of h whatever the sign of $f^n(c)$ may be, and so $f(c)$ cannot be either a maximum or a minimum at $x=c$.

Hence, to determine maxima and minima of $f(x)$, we proceed with the following **working rule :**

Equate $f'(x)$ to zero, and let the roots be $c_1, c_2, c_3, \ldots$. Now work out the value of $f''(c_1)$. If it is negative, then $x = c_1$ makes a maximum. If be positive, then is a minimum of. Similarly test the sign of for the other values of x for which is zero, and determine whether is a maximum or a minimum at these points.

If, in any case above, $f''(c_n) = 0$, use criterion (B).

Note 1. The above criterion for determiniming maxima and minima of $f(x)$ falls at a point where $f'(x)$ is non-existent, even though $f(x)$ may be continuous.

In such a case we should bear in mind that if $f(x)$ be a maximum at a point, immediately to the left of it the value of $f(x)$ is less, and gradually increases towards the value at the point and so $f'(x)$ [which represents the rate of increase of $f(x)$] is positive. Immediately to the right, the value of $f(x)$ is again less, and so $f(x)$ decreases with x increasing and, therefore, $f'(x)$ is negative to the right. Thus changes sign from positive on the left to negative towards the right of the point. [See point P_4, *in the figure of Art. 10.1*]

Similarly, if $f(x)$ be a minimum at any point, $f(x)$ is larger on the left, and diminishes to the value at the point, and again becomes larger on the right, *i.e.*, $f(x)$ increases to the right. Thus $f'(x)$ changes sign here, being negative on the left and positive on the right of the point.

Thus, we have the following **alternative criterion** for maxima and minima : *At a point where $f(x)$ is a maximum or a minimum,* **$f'(x)$ changes sign, from positive on the left to negative on the right if $f(x)$ be a maximum, and from negative on the left to positive on the right if $f(x)$ be a minimum.**

If $f'(x)$ exists at such a point, its change of sign from one side to another takes place through the zero value of $f'(x)$, so that $f'(x) = 0$ at the point. If $f'(x)$ be non-existent at the point, the left-hand and right-hand derivatives are of opposite signs at the point.

Even in the case where the successive derivates exist, instead of proceeding to calculate their values at a point to apply the usual criteria for maxima and minima of $f(x)$ at the point, we may apply effectively in many cases this simple criterion of changing of sign of $f'(x+h)$ as h is changed from negaitve to positive values, being numerically small. [For *illustration see Ex. 4, § 10.5*]

Note 2. At points where $f(x)$ is maximum or a minimum, $f'(x)=0$ when it exists, and accordingly, at these points the tangent line to the graph of $f(x)$ will be parallel to the x-axis (as at P_1, Q_1, P_2, Q_2, P_3, Q_3, etc. in figure of § *10.1*). At points where $f(x)$ is a maximum or a minimum, but $f'(x)$ does not exist (*e.g.*, at Q_2 and P_4), the tangent line to the curve changes its direction abruptly while passing through the point. A special case is where the tangent is parallel to the y-axis, the change in the sign of $f'(x)$ taking place through an infinite value.

Note 3. A maximum or minimum is often called an '*extremum*' (extremal) or '*turning value*'.

The values of x for which $f'(x)$ or $1/f'(x)=0$ are often called '*critical values*' or critical points of $f(x)$.

Note 4. The use of the following principles greatly simplifies the solution of problems on maxima and minima.

(i) Since $f(x)$ and $\log f(x)$ increase and decrease together, $\log f(x)$ is maximum or minimum for any value of x for which $f(x)$ is maximum or minimum.

(ii) When $f(x)$ increases, since $1/f(x)$ decreases, any value of x which renders $f(x)$ a maximum or minimum renders its reciprocal $1/f(x)$ a minimum or a maximum.

(iii) Any value of x which renders $f(x)$ positive and a maximum or a minimum renders $\{f(x)\}^n$ a maximum or a minimum, n being a positive integer.

For examples on maxima and minima of functions of two variables connected by a relation, see Ex. 7 and Ex. 12 of Art. 10.5.

10.4. Elementary methods (*Algebraical and Trigonometrical*).

Certain types of problems on maxima and minima can be solved very simply by elementary algebra or trigonometry[1]. The discussion of the maxima and minima of the quadratic functions or the quotient of two quadratic functions will be found in any text-book on algebra.[2]

In solving simpler problems of maxima and minima of functions of more than one variable, the following elementary results are of great use:

(i) $xy = \{\frac{1}{2}(x+y)\}^2 - \{\frac{1}{2}(x-y)\}^2$.

(ii) $(x+y)^2 = 4xy + (x-y)^2$.

(iii) $x^2 + y^2 = \frac{1}{2}(x+y)^2 + \frac{1}{2}(x-y)^2$.

When *the sum of two positive quantities is given,* it follows from

(i) *that their product is greatest,* and from

(iii) *that the sum of their squares is least, when they are equal.* When *the product of two quantities is given,* from

(ii) *their sum is least when they are equal.*

The above theorems may easily be extended to the cases of more than two quantities.

Thus, *when the sum of any number of positive quantities is given, their product is greatest when they are all equal,* and so on.

For illustrative examples see Art. 10.5, Ex. 9 to 11.

Note. In algebraical or trigonometrical example, by maximum or minimum value of a function we usually mean the greatest or the least value attainable by the function out of all its possible values. In Calculus, however, as has already been remarked, a maximum or a minimum value indicates a local (or relative) maximum or minimum.

10.5 An absolute or Global maximum and an absolute or Global minimum

A real valued function $f(x)$ defined in $[a, b]$ is said to have an *absolute maximum* (or, *a global maximum*) at a point $c \in [a, b]$ if $f(x) \le f(c)$, for all $x \in [a, b]$ and $f(x)$ is said to have an *absolute minumum* (or, *global minimum*) at $x = c \in [a, b]$, if $f(x) \ge f(c)$ for all $x \in [a, b]$, $f(c)$ being called the *absolute maximum* or the *absolute minimum value of* $f(x)$ in $[a, b]$.

To find the Absolute maximum or the Absolute minimum of values of a continuous function defined in a closed interval $[a, b]$, the points $\lambda_1, \lambda_2, \ldots, \lambda_n$ in $[a, b]$ are determined where $f'(\lambda_r) = 0, r = 1, 2, \ldots n$.

[1] See Das & Mukherjee's *Higher Trigonometry, Chap. XV, Sec. B.*

[2] See Ganguly & Mukherjee's *Intermediate Algebra Chap. VI, Art. 6.12.*

Then the Absolute or Global maximum of $f(a)$ in $[a, b]$ is given by $G = \max\{f(a), f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n), f(b)\}$ and the Absolute or global minimum of $f(a)$ in $[a, b]$ is given by $L = \min\{f(a), f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n), f(b)\}$.

Remarks 1. *Any absolute maximum* (or, *minimum* of $f(x)$ is also a *local maximum* (or, minimum) but *the converse is not true.*

2. A function $f(x)$ in $[a, b]$ can have *no local maximum* (or, minimum) *or may have one or more points of local maxima* (or minima).

Example 1. *Find the global minimum if the* $f(x) = x^3 - 6x^2 + 9x + 1$ *in* $[0, 1]$.

Solution. Here, $f(x) = x^3 - 6x^2 + 9x + 1$

$$f'(x) = 3x^2 - 12x + 9$$

For maxima or minima of $f(x)$, $f(x) = 0$

or, $3(x^2 - 4x + 3) = 0$

or, $(x-1)(x-3) = 0 \quad \therefore\ x = 1, 3$

But $3 \notin [0, 1]$

Only critical point is $x = 1$.

Global maximum value $= \max\{f(0), f(1), f(2)\}$.

$= \max\{1, 5, 3\}$

Global minimum $= \min\{f(0), f(1), f(2)\} = \min\{1, 5, 3\} = 1$.

Hence the global maximum and global minimum of $f(x)$ in $[0, 1]$ are 5 and 1 respectively.

Example 2. *Discuss the absolute maximum and absolute minimumof* $f(x) = \tan^{-1} x - \frac{1}{2}\log x$ *in* $[\frac{1}{\sqrt{3}}, \sqrt{3}]$.

Solution. $f(x) = \tan^{-1} x - \frac{1}{2}\log x$

$$f'(x) = \frac{1}{1+x^2} - \frac{1}{2}.\frac{1}{x} = -\frac{(x-1)^2}{2x(1+x^2)}$$

$f'(x) = 0$ gives $x = 1$.

Absolute maximum $= \max\left\{f(\frac{1}{\sqrt{3}}), f(1), f(\sqrt{3})\right\}$

$$= \max\left\{\frac{\pi}{6} + \frac{1}{4}\log 3, \frac{\pi}{4}, \frac{\pi}{3} - \frac{1}{4}\log 3\right\} = \frac{\pi}{6} + \frac{1}{4}\log 3$$

$$\text{Absolute minimum} = \min\left\{f\left(\tfrac{1}{\sqrt{3}}\right),\ f(1),\ f(\sqrt{3})\right\}$$

$$= \min\left\{\frac{\pi}{6}+\frac{1}{4}\log 3,\ \frac{\pi}{4},\ \frac{\pi}{3}-\frac{1}{4}\log 3\right\}$$

$$= \frac{\pi}{6}-\frac{1}{4}\log 3$$

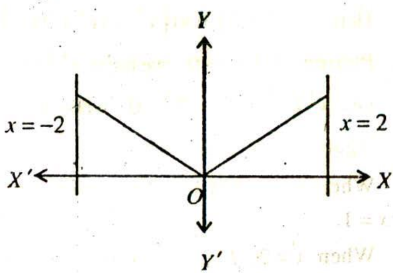
Note : The Absolute max or absolute min does not correspond to $x = 1$, where $f'(x) = 0$.

Example 3. *Examine the function* $f(x) = |x|,\ -2 \le x \le 2$. *for absolutemaximum and absolute minimum.*

Solution. Here, $|x| \ge 0$ for all $x \in [-2, 2]$ and $f(x) = 0$, when $x = 0$.

So, $f(x) \ge f(0)$ for all $x \in [-2, 2]$

Hence $f(x)$ has absolute minimum at $x = 0$ and the absolute minimum value of $f(x)$ is 0.



Also, $|x| \le 0$ for all $x \in [-2, 2]$ and $f(2) = 2,\ f(-2) = 2$.

So $f(x)$ has absolute minima at $x = -2, 2$ and the absolute maximum value of $f(x)$ is 2.

Note : Here $f'(x)$ does not exist at $x = 0$ and $f'(x) \neq 0$ at $x = \pm 2$.

10.6. Illustrative Examples.

Ex. 1. *Find for what values of x, the following expression is maximum and minimum respectively :*

$$2x^3 - 21x^2 + 36x - 20$$

Find also the maximum and minimum values of the expression

[*C.P. 1936*]

Let $f(x) = 2x^3 - 21x^2 + 36x - 20$.

$\therefore\ f'(x) = 6x^2 - 42x + 36$, which exists for all values of x.

Now, when $f(x)$ is a maximum or a minimum, $f'(x) = 0$.

$\therefore$ we should have $6x^2 - 42x + 36 = 0$, *i.e.*, $x^2 - 7x + 6 = 0$, or, $(x-1)(x-6)=0$; $\therefore$ $x=1$ or 6.

Again, $f''(x)=12x-42=6(2x-7)$.

Now, when $x=1$, $f''(x)=-30$, which is *negative*,

when $x=6$, $f''(x)=30$, which is *positive*.

Hence, the given expression is *maximum* for $x=1$, and *minimum* for $x=6$.

The maximum and minimum values of the given expression are respectively $f(1)$, *i.e.*, -3, and $f(6)$, *i.e.*, -128.

Ex. 2. *Investigate for what values of* x,

$$f(x)=5x^6-18x^5+15x^4-10$$

is a maximum or minimum.

Here, $f'(x)=30(x^5-3x^4+2x^3)$,

Putting $f'(x)=0$, we have $x^3(x^2-3x+2)=0$,

i.e., $x^3(x-1)(x-2)=0$, whence $x=0,1$ or 2.

Again, $f''(x)=30(5x^4-12x^3+6x^2)$.

When $x=1$, $f''(x)$ is *negative*, and hence $f(x)$ is a *maximum* for $x=1$.

When $x=2$, $f''(x)$ is *positive*, and hence $f(x)$ is a *minimum* for $x=2$.

When $x=0$, $f''(x)=0$; so the test fails, and we have to examine higher order derivatives.

$$f'''(x)=120(5x^3-9x^2+3x), \quad \therefore f'''(0)=0,$$

$$f^{iv}(x)=360(5x^2-6x+1), \quad \therefore f^{iv}(0) \text{ is } positive,$$

Since *even order* derivative is positive for $x=0$,

$\therefore$ for $x=0$, $f(x)$ is a *minimum*.

Ex. 3. *Show that* $f(x)=x^3-6x^2+24x+4$ *has neither a maximum nor a minimum.*

Here, $f'(x)=3(x^2-4x+8)=3\{(x-2)^2+4\}$

which is always positive and can never be zero.

$\therefore$ $f(x)$ has neither a maximum nor a minimum.

Ex. 4. *Examine* $f(x)=x^3-9x^2+24x-12$ *for maximum or minimum values.*

Here, $f'(x)=3(x^2-6x+8)=3(x-2)(x-4)$.

Putting $f'(x)=0$, we find $x=2$ or 4.

Now, $f'(2-h)=3(-h)(-2-h)=+\text{ve}$,

And $f'(2+h)=3(h)(h-2)=-\text{ve}$, since h is positive and small.

$\therefore$ by § 10.3. Note 1, for $x=2$, $f(x)$ has a *maximum* value, and this is $f(2)=8$.

Again, $f'(4-h)=3.(2-h)(-h)=-\text{ve}$, since h is positive and small,

$$f'(4+h)=3.(2+h)(h)=+\text{ve},$$

$\therefore$ by § 10.3, Note 1, for $x=4, f(x)$ has a *minimum* value, and this is $f(4)=4$.

Note. In this case we could have easily applied rule of Art, 10.3.

Ex. 5. *Find the maxima and minima of*

$$1+2\sin x+3\cos^2 x, \qquad \left(0\le x\le \tfrac{1}{2}\pi\right)$$

Let $f(x)=1+2\sin x+3\cos^2 x$.

Then $f'(x)=2\cos x-6\cos x\sin x$.

$\therefore$ $f'(x)=0$ when $2\cos x(1-3\sin x)=0$, *i.e.*, when $\cos x=0$, and also when $\sin x=\frac{1}{3}$.

$$f''(x)=-2\sin x-6\left(\cos^2 x-\sin^2 x\right).$$

When $\cos x=0$, $x=\frac{1}{2}\pi$, $\therefore \sin x=1$, $\therefore f''(x)=-2+6=4$ (+*ve*).

$\therefore$ for $\cos x=0, f(x)$ is a *minimum*, and the minimum value is 3.

When $\sin x=\frac{1}{3}$,

$$f''(x)=-2\sin x-6\left(1-2\sin^2 x\right)=-\tfrac{2}{3}-6\left(1-\tfrac{2}{9}\right) \textit{ (negative)}$$

Therefore, for $\sin x=\frac{1}{3}, f(x)$ is a *maximum* and the maximum value is $1+2.\frac{1}{3}+3.\left(1-\frac{1}{9}\right)=4\frac{1}{3}$.

Ex. 6. *Examine whether* $x^{\frac{1}{x}}$ *possesses a maximum or a minimum and determine the same.* [*C. P.* 1941, '45]

Let $y=x^{\frac{1}{x}}$, $\therefore$ $\log y=\dfrac{1}{x}\log x$.

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2} - \frac{1}{x^2}\log x = \frac{1}{x^2}(1 - \log x), \qquad \ldots \quad (1)$$

$\therefore$ when $\dfrac{dy}{dx} = 0$, $1 - \log x = 0$, $\therefore \log x = 1 = \log e$, $\therefore x = e$.

Again, differentiating (1) with respect to x,

$$-\frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 + \frac{1}{y}\frac{d^2y}{dx^2} = \frac{x^2.(-1/x) - (1 - \log x)2x}{x^4} = \frac{-3 + 2\log x}{x^3}$$

$\therefore$ when $x = e$, $\dfrac{d^2y}{dx^2} = e^{\frac{1}{e}}.\dfrac{-3+2}{e^3} = -\dfrac{e^{\frac{1}{e}}}{e^3}$, which is *negative*.

$$\left(\because \text{ for } x = e, \ \frac{dy}{dx} = 0\right)$$

for $x = e$, the function is *maximum*, and the maximum value is $e^{\frac{1}{e}}$.

Ex. 7. ***Find the maximum and minimum values of u where***

$$u = \frac{4}{x} + \frac{36}{y} \ \textit{ and } \ x + y = 2. \qquad \text{[C. P. 2006]}$$

Eliminating y between the two given relations

$$u = \frac{4}{x} + \frac{36}{2-x}, \ \therefore \frac{du}{dx} = -\frac{4}{x^2} + \frac{36}{(2-x)^2} = \frac{16(2x^2 + x - 1)}{x^2(2-x)^2}$$

$\dfrac{du}{dx} = 0$ gives $x = \frac{1}{2}$ or -1. Also $\dfrac{d^2u}{dx^2} = \dfrac{8}{x^3} + \dfrac{72}{(2-x)^3}$

When $x = \frac{1}{2}$, $\dfrac{d^2u}{dx^2} = \dfrac{8}{\left(\frac{1}{2}\right)^3} + \dfrac{72}{\left(\frac{3}{2}\right)^3}$ which is *positive.*

$\therefore$ for $x = \frac{1}{2}$, u is a *minimum.*

$\therefore$ *minimum* value of $u = \dfrac{4}{\frac{1}{2}} + \dfrac{36}{2 - \frac{1}{2}} = 32$.

When $x = -1$, $\dfrac{d^2u}{dx^2} = -8 + \dfrac{72}{27}$ which is negetive.

$\therefore$ for $x = -1$, u is a *maximum.*

$\therefore$ *maximum value of* $u = \dfrac{4}{-1} + \dfrac{36}{2+1} = 8$.

Ex. 8. *Examine the function* $f(x)=4-3(x-2)^{\frac{2}{3}}$ *for maxima and minima at* $x=2$.

Here, $f'(x)=-\dfrac{2}{(x-2)^{\frac{1}{3}}}$.

For $x=2$, $f'(x)$ does not exist, the left-hand derivative being $+\infty$ and the right-hand derivative $-\infty$; but $1/f'(x)$ is zero for $x=2$. So, the test of Art. 10.2 fails. Let us apply the criterion of § 10.3, Note 1. Now, $f'(2-h)$ is *positive* and $f'(2+h)$ is *negative.*

Hence, the function has a *maximum* value for $x=2$, and the maximum value is $f(2)$, *i.e.*, 4.

Ex. 9. *A conical tent of given capacity has to be constructed. Find the ratio of the height to the radius of the base for the minimum amount of canvas required for the tent.*

Let r be the radius of the base, h the height, V the volume and S the surface-area of the conical tent.

Then, $V=\frac{1}{3}\pi r^2 h$ ... (1)

and $S=\pi r\sqrt{r^2+h^2}$ ... (2)

Here, V is given as constant.

$$\therefore \quad S^2=\pi^2 r^2\left(r^2+h^2\right)=\pi^2 r^2\left(r^2+\frac{9V^2}{\pi^2 r^4}\right) \quad [\textit{from (1)}]$$

$$=\pi^2 r^4+9V^2.\frac{1}{r^2}.$$

Now, if S is a maximum or a minimum, S^2 is so and hence for maximum or minimum of S, $\dfrac{d}{dr}\left(S^2\right)=0$, *i.e.*, $\dfrac{d}{dr}\left(\pi^2 r^4+9V^2.\dfrac{1}{r^2}\right)=0$,

i.e., $4\pi^2 r^3-18V^2.\dfrac{1}{r^3}=0$. $\therefore\ r^6=\dfrac{9V^2}{2\pi^2}$. $\therefore\ r=\left(\dfrac{3V}{\pi\sqrt{2}}\right)^{\frac{1}{3}}$.

Now, $\dfrac{d^2}{dr^2}\left(S^2\right)=12\pi^2 r^2+54V^2.\dfrac{1}{r^4}$, which is *positive* for

$r=\left(\dfrac{3V}{\pi\sqrt{2}}\right)^{\frac{1}{3}}$.

$\therefore$ for *minimum* amount of canvas,

$$r = \left(\frac{3V}{\pi\sqrt{2}}\right)^{\frac{1}{3}}, \text{ i.e., } r^6 = \frac{9V^2}{2\pi^2} = \frac{9.\frac{1}{9}\pi^2 r^4 h^2}{2\pi^2} \; [\textit{from}\,(1)] = \frac{r^4}{2}.h^2,$$

i.e., $r^2 = \frac{1}{2}h^2$, $\therefore$ $r^2 : h^2 = 1 : 2$ or $r : h = 1 : \sqrt{2}$.

Ex. 10. *Show that for a given perimeter, the area of a triangle is maximum when it is equilateral.*

The area Δ of a triangle $ABC = \sqrt{s(s-a)(s-b)(s-c)}$.

The area Δ of a triangle $ABC = \sqrt{s(s-a)(s-b)(s-c)}$.

Let $s - a = x$, $s - b = y$, $s - c = z$,

$\therefore x + y + z = 3s - (a + b + c) = 3s - 2s = s = \text{const.}$

Now, $\Delta = \sqrt{sxyz}$. Since s is a constant, Δ will be maximum when xyz will be maximum subject to the condition $x + y + z = \text{const.}$, *i.e.*, when $x = y = z$, [See § 10.4]

i.e., when $s - a = s - b = s - c$, *i.e.*, $a = b = c$.

Ex. 11. *Show that the maximum triangle which can be inscribed in a circle is equilateral.*

Area Δ of a triangle ABC inscribed in a circle of radius R

$$= \tfrac{1}{2} bc \sin A = \tfrac{1}{2} \cdot 2R \sin B . 2R \sin C . \sin A$$

$$= 2R^2 \sin A \sin B \sin C$$

$$= R^2 \{\cos(A - B) - \cos(A + B)\} \sin C.$$

Let us suppose C remains constant, while A and B vary. Since R is constant, the above expression will be maximum when $A = B$.

Hence, so long as any two of the angles A, B, C are unequal, the expression $2R^2 \sin A \sin B \sin C$ is not a maximum, that is, it is maximum when $A = B = C$.

Thus, Δ will be maximum when $A = B = C$.

Ex. 12. *Find the maximum and minimum values of*

$$a \sin x + b \cos x.$$

Let $a = r\cos\theta$, $b = r\sin\theta$,

so that $r^2 = a^2 + b^2$ and $\tan\theta = b/a$.

Thus $a\sin x+b\cos x = r(\sin x\cos\theta+\cos x\sin\theta)=r\sin(x+\theta)$

$$=\sqrt{a^2+b^2}\sin\left(x+\tan^{-1}b/a\right)$$

Since the greatest and least values of sine of an angle are 1 and -1, the required maximum and minimum values of the given expression are

$\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$.

Ex. 13. *Assuming Fermat's theorem that a ray of light in passing from a point A in one medium to a point B in another medium takes the path for which the time of description is a minimum, prove the law of refraction.*

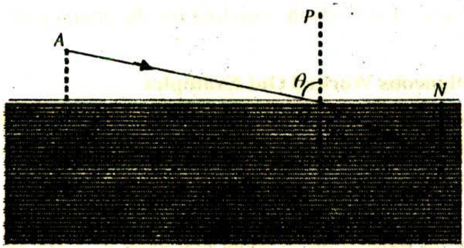


Fig 10.5.1

Let AOB be a possible path of the ray of light, O being the point where it meets the surface of separation $\overline{MON}$ of the two media and let $\overline{POQ}$ be the normal to the common surface $\overline{MN}$ at O, and $\overline{AM}$, $\overline{BN}$ the perpendiculars from A and B on $\overline{MN}$.

Let $m\angle AOP=\theta$, $m\angle BOQ=\phi$, and let v and v' be the velocities of light in the two media. If $AM = a$, $BN = b$, then $AO=a\sec\theta$, $BO=b\sec\phi$. The time taken by the ray of light to travel the path AOB is

$$T=\frac{a\sec\theta}{v}+\frac{b\sec\phi}{v'} \quad \ldots \quad (1)$$

and by Fermat's theorem this is to be a minimum.

Again, since A and B are fixed points,

$$a\tan\theta+b\tan\phi=MO+ON=MN=\text{constant} \quad \ldots \quad (2)$$

so that θ and ϕ are not independent, and we can, thus consider ϕ as a function of θ, which is then the only independent variable.

For T to be minimum, $\dfrac{dT}{d\theta}=0$, giving

$$\frac{a}{v}\sec\theta\tan\theta+\frac{b}{v'}\sec\phi\tan\phi\frac{d\phi}{d\theta}=0,$$

Also, from (2),

$$a\sec^2\theta+b\sec^2\phi\frac{d\phi}{d\theta}=0.$$

From these two, eliminating $\dfrac{d\phi}{d\theta}$, we easily get

$$\frac{\sin\theta}{v}=\frac{\sin\phi}{v'}, \text{ or, } \frac{\sin\theta}{\sin\phi}=\frac{v}{v'}=\mu \text{ (say)}$$

which is the law of refraction, satisfied for the actual path of the ray of light.

10.7 Miscellaneous Worked Out Examples

Ex. 1. (*i*) What do you mean by the maximum or minimum value of a function $f(x)$ at $x=c$? If $f'(x)$ exists, what will be the value of $f'(c)$? Is it neccessary as well as sufficient condition?

[*C. P. 1987, '96, ; B. P. '96, 98*]

(*ii*) Cite an illustration to show that even if $f'(c)$ does not exist, $f'(x)$ may have a maximum or minimum at $x=c$ [*C. P. 1987*]

Solution : (*i*) A function $f(x)$ is said to have a *maximum* (or, a local maximum) at $x=c$, if $f(c)$ is greatest of all the values, *i.e.*, $f(x)\le f(c)$ in some *suitably small neighbourhood* of c.

Analytically, this means

$$f(c+h)-f(c)\le 0, \text{ for } |h| \text{ sufficiently small.}$$

Similarly, $f(x)$ is said to have a *minimum* (or, a local minimum) at $x=c$, if $f(c)$ is smallest of all the values, *i.e.*, $f(x)\ge f(c)$ in some *suitably small neighbourhood* of c, *i.e.*,

$$f(c+h)-f(c)\ge 0, \text{ for } |h| \text{ sufficiently small.}$$

Second part : If $f(x)$ be a maximum or a minimum at $x=c$, and if $f'(c)$ exists, then $f'(c)=0$.

Proof : By definition, $f(x)$ is a maximum at $x=c$, provided we can find a positive number δ, such that

$$f(c+h)-f(c)<0, \text{ whenever } -\delta < h < \delta \ (h \neq 0)$$

$\therefore \quad \dfrac{f(c+h)-f(c)}{h}<0$, if h is positive and sufficiently small and

$\dfrac{f(c+h)-f(c)}{h}>0$ if h is negative and numerically sufficiently small.

Thus $\quad \displaystyle\lim_{h\to 0+} \frac{f(c+h)-f(c)}{h} \le 0$

and $\quad \displaystyle\lim_{h\to 0-} \frac{f(c+h)-f(c)}{h} \ge 0$

Now, if $f'(c)$ exists, the above two limiting values which represent the right-hand and left-hand derivatives respectively of $f(c)$ at $x=c$, must be equal.

Hence, the only common value of the limit is zero, *i.e.*, $f'(c)=0$

Similarly, when $f(c)$ is a minimum, $f'(c)=0$.

Third Part : $f'(c)=0$, if it exists, is a *necessary, but not sufficient* condition that $f(x)$ may have a maximum or minimum value at $x=c$.

As an example, let us consider the function $f(x)=x^5$.

Obviously $f'(0)=0$, but whenever $x>0$, $f(x)>f(0)$ and whenever $x<0$, $f(x)<f(0)$. Hence, $f(x)$ has neither a maximum nor a minimum value at $x=0$

(*ii*) Let us consider the function $\phi(x)=|x|$

Obviously, $\phi(x)$ is minimum at $x=0$, but $\phi'(0)$ does not exist.

Ex. 2. (*i*) Show that the maximum value of xy subject to the condition $3x+4y=5$ is $\dfrac{25}{48}$. [*C. P. 1991, B. P. 1996*]

(*ii*) Show that the function $f(x)=x^3-3x^2+6x+3$ does not possess any maximum or minimum value. [*C. P. 1994*]

Solution : $\quad \because \quad 3x+4y=5, \quad y=\dfrac{1}{4}(5-3x).$

Let $u=xy=\dfrac{1}{4}x(5-3x)=\dfrac{1}{4}\left(5x-3x^2\right).$

$$\frac{du}{dx} = \frac{1}{4}(5-6x)$$

and $\dfrac{d^2u}{dx^2} = -\dfrac{6}{4} = -\dfrac{3}{2}$.

For a maximum or minimum of $u = xy$,

$\dfrac{du}{dx} = 0$, which gives $x = \dfrac{5}{6}$. Also, at $x = \dfrac{5}{6}$, $\dfrac{d^2u}{dx^2} < 0$

$u = xy$ is a maximum at $x = \dfrac{5}{6}$ and the maximum value of

$$u = xy = \frac{1}{4}\left\{5 \times \frac{5}{6} - 3\left(\frac{5}{6}\right)^2\right\} = \frac{25}{48}.$$

(*ii*) $f(x) = x^3 - 3x^2 + 6x + 3$

$f'(x) = 3\left(x^2 - 2x + 2\right)$

$f'(x) = 0$ gives $x^2 - 2x + 2 = 0$, or, $x = 1 \pm i$.

Thus $f'(x)$ does not vanish for any real value of x.

Hence $f(x)$ has neither a maximum nor a minimum.

Ex. 3. When does the function $\sin 3x - 3\sin x$ attain its maximum or minimum values in $(0,\ 2\pi)$? [*C. P. 1981*]

Solution : Here, $f(x) = \sin 3x - 3\sin x$

$f'(x) = 3\cos 3x - 3\cos x$

$f''(x) = -9\sin 3x + 3\sin x$

For, maximum or minimum values of $f(x)$,

$f'(x) = 0$, which gives $3(\cos 3x - \cos x) = 0$

or, $4\cos x\left(\cos^2 x - 1\right) = 0$

$\therefore\ \cos x = 0,\ 1,\ -1$

$\therefore\ x = \dfrac{\pi}{2},\ \dfrac{3\pi}{2} \qquad (\because\ 0 < x < 2\pi\ )$

At $x=\dfrac{\pi}{2}$, $\quad f''(x)=-9\sin\dfrac{3\pi}{2}+3\sin\dfrac{\pi}{2}>0$

At $x=\dfrac{3\pi}{2}$, $\quad f''(x)=-9\sin\dfrac{9\pi}{2}+3\sin\dfrac{3\pi}{2}<0$

$\therefore$ $f(x)$ is maximum at $x=\dfrac{3\pi}{2}$ and minimum at $x=\dfrac{\pi}{2}$.

Ex. 4. (*i*) Show that of all rectangles of given area, the square has the smallest perimeter. [*C. P. 1984, 2008*]

(*ii*) Show that, of all rectangles of given perimeter, square has the largest area.

Solution : Let x be the length and y be the breadth of a rectangle.

Its area $=xy=k$, say, where k is constant.

$\therefore \quad y=\dfrac{k}{x}$

If S be the perimeter of the rectangle, $S=2(x+y)=2\left(x+\dfrac{k}{x}\right)$

$\dfrac{dS}{dx}=2\left(1-\dfrac{k}{x^2}\right)$ and $\dfrac{d^2S}{dx^2}=\dfrac{4k}{x^3}$.

For maximum or minimum of S, $\dfrac{dS}{dx}=0$, which gives

$2\left(1-\dfrac{k}{x^2}\right)=0$, *i.e.*, $x=\sqrt{k}$

when $x=\sqrt{k}$, $\dfrac{d^2S}{dx^2}=\dfrac{2k}{\left(\sqrt{k}\right)^3}=2\sqrt{k}>0$

So, S is minimum when $x=\sqrt{k}$, and when $x=\sqrt{k}$, $y=\sqrt{k}$, $(\because xy=k)$,

i.e. $x=y$.

Hence the perimeter of the rectangle is smallest when the rectangle is a square.

(*ii*) Let, x be the length and y be the breadth of a rectangle.

Perimeter of the rectangle $=2(x+y)=2k$, say, where k is a constant.

$\therefore \quad y=k-x$

Area A of the rectangle is given by

$$A = xy = x(k-x) = kx - x^2$$

$$\frac{dA}{dx} = k - 2x \quad \text{and} \quad \frac{d^2A}{dx^2} = -2$$

For maximum or minimum of A, $\frac{dA}{dx} = 0$, which gives $k - 2x = 0$.

i.e., $x = \frac{1}{2}k$.

for, $x = \frac{1}{2}k$, $\quad \frac{d^2A}{dx^2} = -2 < 0$

$\therefore$ A is maximum when $x = \frac{1}{2}k$.

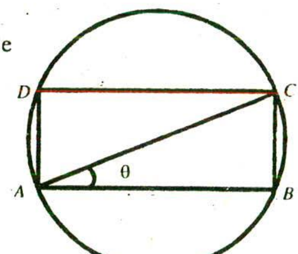
when $x = \frac{1}{2}k$, $y = \frac{1}{2}k$; thus $x = y$.

Hence, the rectangle of given perimeter has largest area when it is a square.

Ex. 5. (*i*) Show that the rectangle inscribed in a circle has maximum area when it is a square.

(*ii*) Find the largest rectangle that can be inscribed within the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. [C. P. 1993]

Solution : Let $ABCD$ be the rectangle inscribed within the circle of radius a.



$\because$ $ABC = \frac{\pi}{2}$,

AC is a diameter of the circle.

Let $\angle CAB = \theta$, then

$AB = AC\cos\theta = 2a\cos\theta$ and

$BC = 2a\sin\theta$.

Area S of the rectangle $ABCD$ is given by

$$S = AB \times BC = 4a^2 \sin\theta\cos\theta = 2a^2 \sin 2\theta$$

$$\frac{dS}{d\theta} = 4a^2\cos 2\theta, \quad \frac{d^2S}{d\theta^2} = -8a^2 \sin 2\theta$$

For, extremum of S, $\frac{dS}{d\theta} = 0$, *i.e.*, $4a^2\cos 2\theta = 0$

$\therefore\ 2\theta = \dfrac{\pi}{2}$, $\left[\because\ 0 \le \theta \le \dfrac{\pi}{2}\right]$ *i.e.*, $\theta = \dfrac{\pi}{4}$.

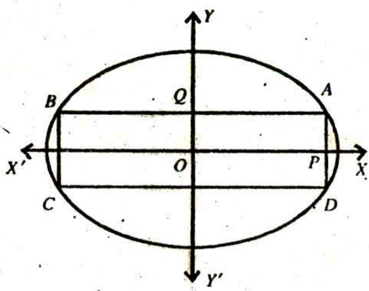
For, $\theta = \dfrac{\pi}{4}$, $\dfrac{d^2S}{d\theta^2} = -8a^2 \sin\dfrac{\pi}{2} = -8a^2 < 0$

$\therefore$ S is maximum when $\theta = \dfrac{\pi}{4}$.

Then, $AB = 2a\cos\dfrac{\pi}{4} = \sqrt{2}a$ and $BC = 2a\sin\dfrac{\pi}{4} = \sqrt{2}a$.

$\because$ $AB = BC$, the rectangle inscribed in the circle with largest area is a square.

(*ii*) Let $ABCD$ be the rectangle inscribed in the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. The length and breadth of the rectangle are parallel to the axes of the ellipse which are coordinate axes also.



Let coordinate of A be $(a\cos\theta, b\sin\theta)$.

S = area of the rectangle $ABCD = 4 \times$ area of the rectangle $OPAQ$

$= 4 \cdot OP \cdot PA = 4a\cos\theta \cdot b\sin\theta = 2ab\sin 2\theta$

$\dfrac{dS}{d\theta} = 4ab\cos 2\theta$; $\dfrac{d^2S}{d\theta^2} = -8ab\sin 2\theta$

For extremum of S, $\dfrac{dS}{d\theta} = 0$, which gives $\cos 2\theta = 0$

$\therefore\ 2\theta = \dfrac{\pi}{2}$, *i.e.*, $\theta = \dfrac{\pi}{4}$.

For, $\theta = \dfrac{\pi}{4}$, $\dfrac{d^2S}{d\theta^2} = -8ab\sin\dfrac{\pi}{2} = -8ab < 0$

$\therefore$ S is maximum when $\theta = \dfrac{\pi}{4}$.

In that case, length of the rectangle $= 2OP = 2a\cos\dfrac{\pi}{4} = \sqrt{2}a$

and breadth of the rectangle $= 2PA = 2b\sin\frac{\pi}{4} = \sqrt{2}b$

and area of the largest rectangle $= 2ab$.

Ex. 6. (*i*) Show that the maximum value of $x^2\log\left(\frac{1}{x}\right)$ is $\frac{1}{2e}$.

[*C. P. 1989*]

(*ii*) Show that $x^{\frac{1}{x}}$ $(x > 0)$ is a maximum at $x = e$. Deduce that $e^{\pi} > \pi^{e}$.

[*C. P. 1992*]

Solution : (*i*) Let, $f(x) = x^2\log\left(\frac{1}{x}\right) = -x^2\log x$.

$\therefore \; f'(x) = -2x\log x - x^2\cdot\frac{1}{x} = -2x\log x - x$

and $f''(x) = -2\log x - 2x\cdot\frac{1}{x} - 1 = -2\log x - 3$.

For, extremum of $f(x)$, $f'(x) = 0$, which gives $-x(2\log x + 1) = 0$,

i.e., $\log x = -\frac{1}{2}$, $\therefore x = e^{-\frac{1}{2}}$.

At $x = e^{-\frac{1}{2}}$, $f''(x) = -2\left(-\frac{1}{2}\right) - 3 = -2 < 0$

$\therefore \; f(x)$ is maximum for $x = e^{-\frac{1}{2}}$

Maximum value of $f(x) = f\left(e^{-\frac{1}{2}}\right) = -e^{-1}\cdot\log\left(e^{-\frac{1}{2}}\right) = \frac{1}{2e}$.

(*ii*) Let $f(x) = x^{\frac{1}{x}}$

$\therefore \; \log f(x) = \frac{1}{x}\cdot\log x$

Differentiating, $\frac{1}{f(x)}\cdot f'(x) = \frac{1}{x}\cdot\frac{1}{x} - \frac{1}{x^2}\log x = \frac{1}{x^2}(1 - \log x)$

$f'(x) = f(x)\left\{\frac{1}{x^2}(1 - \log x)\right\}$

$$f''(x) = f'(x)\left\{\frac{1}{x^2}(1-\log x)\right\} + f(x)\left\{-\frac{3}{x^3}+\frac{2}{x^3}\log x\right\}$$

For, extremum of $f(x)$, $f'(x)=0$, which gives

$$\frac{1}{x^2}(1-\log x)=0 \text{ or, } \log x = 1 = \log e, \quad \therefore x = e$$

$$f''(e) = f'(e)\left\{\frac{1}{e^2}(1-\log e)\right\} + f(e)\left\{-\frac{3}{e^3}+\frac{2}{e^3}\cdot\log e\right\}$$

$$= 0 + e^{\frac{1}{e}}\left\{-\frac{1}{e^3}\right\} < 0$$

$\therefore$ $f(x)$ is maximum at $x=e$.

and maximum value of $f(x)$ is $e^{\frac{1}{e}}$.

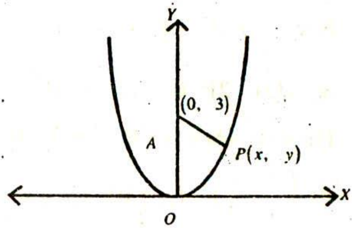
Since $f(x)$ is maximum for $x=e$.

$f(e) > f(\pi)$

or, $e^{\frac{1}{e}} > \pi^{\frac{1}{\pi}}$

i.e., $e^{\pi} > \pi^{e}$. [e, $\pi > 0$]

Ex. 7. Find the point on the parabola $2y = x^2$, which is nearest to the point (0, 3). [*C. P. 1990, 1997*]



Solution : Let $P(x, y)$ be any point on the parabola $y = \frac{1}{2}x^2$ and $A(0, 3)$ is the fixed point

$$AP^2 = (x-0)^2 + (y-3)^2 = x^2 + \left(\frac{x^2}{2}-3\right)^2 = f(x) \text{ (say)}$$

$$f'(x) = 2x + 2x\left(\frac{x^2}{2}-3\right) = x^3 - 4x$$

$$f''(x) = 3x^2 - 4.$$

when AP is minimum, $AP^2 = f(x)$ is also minimum.

For maximum or minimum of $f(x)$, $f'(x)=0$,
which gives, $x\left(x^2-4\right)=0$
i.e., $x=0,\ 2,\ -2$ at $x=0$, $f''(x)=-4<0$
at $x=\pm 2$, $f''(x)=3(4)-4=8>0$
$\therefore$ $f(x)$ is maximum when $x=0$ and minimum when $x=\pm 2$.
$\therefore$ AP is minimum when $x=\pm 2$. When $x=\pm 2$, $y=2$.

Hence, the points on the given parabola, nearest to the point $(0,\ 3)$ are $(2,\ 2)$ and $(-2,\ 2)$.

Ex. 8. Prove that the function $f(x,\ y)=x^3+3x^2+4xy+y^2$ attains a minimum at the point $\left(\frac{2}{3},\ -\frac{4}{3}\right)$. [*C. P. 1990*]

Solution : $f(x,\ y)=x^3+3x^2+4xy+y^2$

$f_x=4y+6x+3x^2$, $f_{xx}=6+6x$, $f_y=6x+2y$, $f_{yy}=2$, $f_{xy}=4$.

For, extremum of $f(x,\ y)$, $f_x=0$, $f_y=0$, which give

$$4y+6x+3x^2=0 \qquad \ldots \qquad \ldots \qquad (1)$$

$$4x+2y=0 \qquad i.e., \qquad y=-2x \qquad \ldots \qquad \ldots \qquad (2)$$

Now, from (1), $-8x+6x+3x^2=0$

or, $x(3x-2)=0$, $\therefore$ $x=0,\ \frac{2}{3}$

From (1) when $x=0$, $y=0$, and when $x=\frac{2}{3}$, $y=-\frac{4}{3}$

At $\left(\frac{2}{3},-\frac{4}{3}\right)$, $f_{xx}=10$, $f_{yy}=2$, $f_{xy}=4$

So, $f_{xx}\times f_{yy}-\left(f_{xy}\right)^2=10\times 2-(4)^2=4>0$

Thus $f(x,\ y)$ has an extremum at $\left(\frac{2}{3},\ -\frac{4}{3}\right)$

Also, $f_{xx}=10>0$

hence, $f(x,\ y)$ has a minimum at $\left(\frac{2}{3},\ -\frac{4}{3}\right)$.

Ex. 9. Find the extreme value of $f(x, y) = 2x^2 - xy + 2y^2 - 20x$.

[C. P. 2000]

Solution : $f(x, y) = 2x^2 - xy + 2y^2 - 20x$

$f_x = 4x - y - 20$, $f_y = -x + 4y$, $f_{xx} = 4$, $f_{yy} = 4$, $f_{xy} = -1$

For extremum of $f(x, y)$, $f_x = 0$, $f_y = 0$

i.e., $4x - y - 20 = 0$, $-x + 4y = 0$

which gives $x = \frac{16}{3}$, $y = \frac{4}{3}$

Again at $\left(\frac{16}{3}, \frac{4}{3}\right)$,

$$f_{xx} \cdot f_{yy} - (f_{xy})^2 = 16 - 1 = 15 > 0$$

and $f_{xx} = 4 > 0$

Hence, $f(x, y)$ has a minimum value at $\left(\frac{16}{3}, \frac{4}{3}\right)$ and

$$f_{\min} = 2 \times \frac{256}{9} - \frac{64}{9} + \frac{32}{9} - 20 \times \frac{16}{3} = -\frac{160}{3}.$$

Ex. 10. (*i*) A wire of length l is to be cut into two pieces, one being bent to form a square and the other to form a circle. How should the wire be cut if the sum of the areas enclosed by the two pieces to be a minimum?

(*ii*) A wire of length 20 metre is bent so as to form a circular sector of maximum area. Find the radius of the circular sector.

[C. P. 1983, 96]

Solution : (*i*) Let the wire be cut in two pieces in the ratio $1 : \lambda$. Then lengths of the pieces will be $\frac{l}{1+\lambda}$ and $\frac{l\lambda}{1+\lambda}$ respectively.

If the length of each side of the square be x and the radius of the circle r, then

$$4x = \frac{l}{1+\lambda}, \text{ i.e., } x = \frac{l}{4(1+\lambda)}.$$

$$\text{and } 2\pi r = \frac{l\lambda}{1+\lambda}, \text{ i.e., } r = \frac{l\lambda}{2\pi(1+\lambda)}.$$

Let A be the sum of the areas of the square and the circle.

$$A = \frac{l^2}{16(1+\lambda)^2} + \frac{\pi l^2 \lambda^2}{4\pi^2 (1+\lambda)^2} = \frac{l^2(\pi + 4\lambda^2)}{16\pi(1+\lambda)^2}$$

$$\frac{dA}{d\lambda} = \frac{l^2}{8\pi} \cdot \frac{(4\lambda - \pi)}{(1+\lambda)^3} \qquad \ldots \quad \ldots \quad (1)$$

and $$\frac{d^2 A}{d\lambda^2} = \frac{l^2}{8\pi} \cdot \frac{(4l + 3\pi - 8\lambda)}{(1+\lambda)^4} \qquad \ldots \quad \ldots \quad (2)$$

For maximum or minimum value of A, $\frac{dA}{d\lambda} = 0$,

which gives from (1) $\lambda = \frac{\pi}{4}$.

Also, from (2), when $\lambda = \frac{\pi}{4}$,

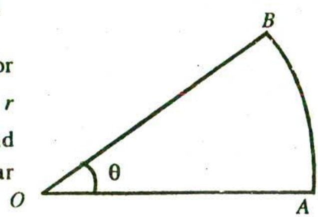
$$\frac{d^2 A}{d\lambda^2} = \frac{l^2}{8\pi} \cdot \frac{(4l + 3\pi - 2\pi)}{(1+\lambda)^4} = \frac{l^2}{8\pi} \cdot \frac{(4l+\pi)}{(1+\lambda)^4} > 0$$

Hence, A will be minimum, when $\lambda = \frac{\pi}{4}$.

In that case, the lengths of the pieces of the wire will be $\dfrac{l}{1+\frac{\pi}{4}}$ and $\dfrac{l \times \frac{\pi}{4}}{1+\frac{\pi}{4}}$ respectively,

i.e., $\frac{4l}{4+\pi}$ and $\frac{\pi l}{4+\pi}$ respectively.

(*ii*) Let, OAB be the circular sector formed by the wire of length 20 metre, r metre be the radius of the circle and $\angle AOB = \theta$, where θ is in circular measure;



then $2r + s = 20$, *i.e.*, $s = 2(10 - r)$ $\quad \ldots \quad (1)$

Area of the sector $AOB = S = \frac{1}{2}r^2\theta$

$$= \frac{1}{2}r^2 \times \frac{s}{r} \quad [\because s = r\theta]$$

$$= \frac{1}{2}r \cdot 2(10-r)$$

or, $S = 10r - r^2$

$\frac{dS}{dr} = 10 - 2r$ and $\frac{d^2S}{dr^2} = -2$

For an extremum of S, $\frac{dS}{dr} = 0$, which gives $r = 5$ and $\frac{d^2S}{dr^2} = -2 < 0$

Hence, S is maximum when $r = 5$ metre, *i.e.*, radius of the circular sector is 5 metre.

EXAMPLES - X

1. Find for which values of x the following functions are maximum and minimum:

   (i) $x^3 - 9x^2 + 15x - 3$. (ii) $4x^3 - 15x^2 + 12x - 2$.

   (iii) $\frac{x^2 - 7x + 6}{x - 10}$. [*C. P. 1939*]

   (iv) $\frac{x^2 + x + 1}{x^2 - x + 1}$. (v) $x^4 - 8x^3 + 22x^2 - 24x + 5$.

2. Find the maximum and minimum values of (iii), (iv) and (v) of Ex. 1.

3. (i) Show that the maximum value of $x + \frac{1}{x}$ is less than its minimum value. [*B.P. 1990, V.P. 2000*]

   (ii) Show that the minimum value of $\frac{(2x-1)(x-8)}{(x-1)(x-4)}$ is greater than its maximum value.

4. Show that $x^3 - 6x^2 + 12x - 3$ is neither a maximum nor a minimum when $x = 2$.

5. Show that the following function possess neither a maximum nor a minimum:

(i) $x^3 - 3x^2 + 6x + 3$. (ii) $x^3 - 3x^2 + 9x - 1$.

(iii) $\sin(x+a)/\sin(x+b)$. (iv) $(ax+b)/(cx+d)$.

6. Show that $x^5 - 5x^4 + 5x^3 - 1$ is neither a maximum nor a minimum when $x = 3$; neither when $x = 0$.

7. Examine for maxima and minima of the following functions :

(i) $\sin x$. (ii) $\cos x$. (iii) x^5.

(iv) x^6. (v) $\frac{1}{5}x^5 - \frac{1}{4}x^4$. (vi) $e^x . \sin x$

8. Test the following functions for maxima and minima at $x = 0$:

(i) $\sin x - x + \dfrac{x^3}{3!} - \dfrac{x^5}{5!}$. (ii) $\cos x - 1 + \dfrac{x^2}{2!} - \dfrac{x^4}{4!}$.

9. Show that

(i) $\sqrt{3}\sin x + 3\cos x$ is a maximum for $x = \frac{1}{6}\pi$.

(ii) $\sin x(1 + \cos x)$ is a maximum for $x = \frac{1}{3}\pi$. [*C. P. 1942, '47*]

(iii) $\sin^3 x \cos x$ is maximum when $x = 1/3\,\pi$.

(iv) $x\sin x + 4\cos x$ is maximum for $x = 0$.

(v) $\sec x + \log\cos^2 x$ is a maximum for $x = 0$ and a minimum for $x = \frac{1}{3}\pi$.

(vi) $\dfrac{2\theta - \sin 2\theta}{\theta^2}$ $(\theta > 0)$ is maximum when $\theta = \frac{1}{2}\pi$.

10. If y is defined as a function of x by the equations

$$y = a(1 - \cos\theta),\ x = a(\theta - \sin\theta),$$

show that y is a maximum when $\theta = \pi$.

11. Show that

(i) the maximum value of $(1/x)^x$ is $e^{1/e}$.

[*C.P. 1990, B.P. 1995, V.P. 2002*]

(ii) the minimum value of $x/\log x$ is e.

(iii) the minimum value of $4e^{2x} + 9e^{-2x}$ is 12. [*C.P. 1994*]

12. (i) Show that $4^x - 8x\log_e 2$ is minimum when $x = 1$.

(ii) Show that $12(\log x + 1) + x^2 - 10x + 3$ is a maximum when $x = 2$ and a minimum when $x = 3$.

(iii) Show that $x^2 \log(1/x)$ is a maximum for $x = 1/\sqrt{e}$.

13. If $f'(x) = (x - a)^{2m}(x - b)^{2n+1}$, when m and n are positive integers, show that $x = a$ gives neither a maximum nor a minimum value of $f(x)$, but $x = b$ gives a minimum.

14. Find the maxima and minima, if any, of $\dfrac{x^4}{(x-1)(x-3)^3}$.

15. If $y = \dfrac{ax + b}{(x - 1)(x - 4)}$ has a turning value at $(2, -1)$, find a and b and show that the turning value is a maximum.

16. Prove that $\sum (x - a_1)^2$ is a minimum when x is the arithmetic mean of $a_1, a_2, a_3, \ldots, a_n$.

17. (i) Given $x/2 + y/3 = 1$, find the maximum value of xy and minimum value of $x^2 + y^2$.

(ii) Given $xy = 4$, find the maximum and minimum values of $4x + 9y$.

[*V.P. 2001*]

18. (i) If $f(x) = 1 - \sqrt{x^2}$, when the square root is to be taken positive, show that $x = 0$ gives a maximum for $f(x)$.

(ii) If $f(x) = a + (x - b)^{\frac{2}{3}} + (x - b)^{\frac{4}{3}}$, show that $f(x)$ is minimum for $x = b$.

(iii) Show that $(x - a)^{\frac{1}{3}}(2x - a)^{\frac{2}{3}}$ is a maximum for $x = \frac{1}{2}a$, a minimum for $x = \frac{5}{6}a$ and neither for $x = a$. [$a > 0$]

(iv) If $f(x) = |x|$, show that $f(0)$ is a minimum although $f'(0)$ does not exist.

19. Show that

(i) the largest rectangle with a given perimeter a square;

(ii) the maximum rectangle inscribable in a circle is a square.

[C..P. 1936]

20. Find the point on the parabola $2y = x^2$ which is nearest to the point $(0, 3)$.

21. P is any point on the curve $y = f(x)$ and C is a fixed point not on the curve. If the length PC is either a maximum or a minimum, show that the line $\overline{PC}$ is perpendicular to the tangent at P.

22. Find the length of the perpendicular from the point $(0, 2)$ upon the line $3x + 4y + 2 = 0$, showing that it is the shortest distance of the point from the line. Find also the foot of the perpendicular.

23. A cylindrical tin can, closed at both ends and of a given capacity, has to be constructed. Show that the amount of tin required will be a minimum when the height is equal to the diameter.

24. By the Post Office regulations, the combined length and girth of a parcel must not exceed 3 metre. Find the volume of the biggest cylindrical (right circular) packet that can be sent by the parcel post.

25. A line drawn through the point $P(1, 8)$ cuts the positive sides of the axes $\overrightarrow{OX}$ and $\overrightarrow{OY}$ at A and B. Find the intercepts of this line on the axes so that

(i) the area of the triangle OAB is a minimum;

(ii) the length of the line $\overline{AB}$ is minimum.

Find also in the above cases the area of the triangle and the length of the line respectively.

26. P is a point on an ellipse whose centre is C, and N is the foot of the perpendicular from C upon the tangent to the ellipse at P; find the maximum value of $\overline{PN}$. [C..P. 1945]

27. The height of a particle projected with velocity u at an angle α with the horizontal is $u \sin \alpha\, t - \frac{1}{2} gt^2$ at any time t. Find the greatest height attained and the time of reaching it.

28. The total waste per mile in an electric conductor is given by $W = C^2 R + \dfrac{1}{R} K^2$, where C is the current, R the resistance, and K a constant. What resistance will make the waste a minimum if the current C is kept constant.?

29. The force F exerted by a circular electric current of radius a on a magnet whose axis coincides with the axis of the coil is given by

$$F \propto x(a^2 + x^2)^{-\frac{5}{2}},$$

where x is the distance of the magnet from the centre of the circle. Show that F is greatest when $x = \frac{1}{2}a$.

30. Assuming that the intensity of light at a point on an illuminated surface varies directly as the sine of the angle at which the ray of light strickes the surface, and inversely as the square of the distance of the source from the point, find how high should a light be placed directly over the centre of a circular field of radius $15\sqrt{2}$ m in order to have a maximum illumination on the boundary.

31. (i) Find the altitude of the right cone of maximum volume that can be inscribed in a sphere of radius a.

(ii) Find the altitude of the right circular cylinder of maximum volume that can be inscribed in a given right circular cone of height h.

32. (i) For a given curved surface of a right circular cone when the volume is maximum, show that the semi-vertical angle is $\sin^{-1}\frac{1}{\sqrt{3}}$.

(ii) For a given volume of a right cone show that, when curved surface is minimum, the semi-vertical angle is $\sin^{-1}\frac{1}{\sqrt{3}}$.

33. An open tank of a given volume consists of a square base with vertical sides. Show that the expense of lining the tank with lead will be least if the height of the tank is half the width.

34. If $\overline{POP'}$ and $\overline{QOQ'}$ be any tow conjugate diameters of an ellipse, and from P and Q are drawn two perpendiculars to the major axis cutting it at M and N respectively, show that $PM + QN$ is a maximum when $\overline{POP'}$ and $\overline{QOQ'}$ are equi-conjugate diameters.

35. A window is in the form of rectangle surmounted by a semi-circle. If the total perimeter be 10 m, find the dimensions so that the greatest possible amount of light may be admitted.

36. A particle is moving in a straight line. Its distance x cm from a fixed point O at any time t second is given by the relation

$$x = t^4 - 10t^3 + 24t^2 + 36t + 12.$$

When is it moving most slowly ?

37. In enclosing a rectangular lawn that has one side along a neighbour's plot, a person has to pay for the fence for the three sides on his own ground and for half of that along the dividing line. What dimensions would give him the least cost if the lawn is to contain 4800 m² ?

38. A gardener having 120 m of fencing wishes to enclose a rectangular plot of land and also to erect a fence across the land parallel to two of the sides. What is the maximum area he can enclose ?

39. A shot is fired with a velocity u at a vertical wall whose distance from the point of projection is x. find the greatest height above the level of the point of projection at which the bullet can hit the wall.

40. From the fixed point A on the circumference of a circle of radius c the perpendicular $\overline{AY}$ is let fall on the tangent at P. Show that the maximum area of the triangle APY is $\frac{3}{8}c^2\sqrt{3}$. [*C..P. 1930*]

41. The intensity of light varies inversely as the square of the distance from the source. If two lights are 15 m apart and one light is 8 times as strong as the other, where should an object be placed between the lights to have the least illumination ?

42. The boundary wall of a house is 2·7 m high, and is at a distance 80 cm from the house. Show that a ladder, one end of which rests on the ground outside the wall and which passes over the wall, must at least be $1{\cdot}3\sqrt{13}$ m long in order to reach the house.

43. A man in a boat $\sqrt{3}$ km from the bank wishes to reach a village that is 11 km distant along the bank from the point nearest to him. He can walk 8 km per hour and row 4 km per hour. Where should he land in order to reach the village in the least time ? Find also the time.

44. If for a steamer the consumption of coal varies as the cube of its speed, show that the most economical rate of steaming against a current will be a speed equal to $1\frac{1}{2}$ times that of the current.

45. For a train the cost of fuel varies as the square of its speed (in km per hour), and the cost is Rs. 24 per hour when the speed is 12 km/h. If other expenses total Rs. 96 per hour, find the most economical speed and the cost for a journey of 100 kilometre.

46. Assuming Fermat's law, that a ray of light in passing from a point A to a point B in the same medium after meeting a reflecting surface takes the path for which the time is minimum, prove the law of reflection.

47. Assuming the law of refraction, if a ray of light passes through a prism in a plane perpendicular to its edge, prove that the deviation in its direction is minimum when the angle of incidence is equal to the angle of emergence.

ANSWERS

1. (i) $x = 1$ (max.), $x = 5$ (min.), (ii) $x = \frac{1}{2}$ (max.), $x = 2$ (min.),
(iii) $x = 4$ (max.), $x = 16$ (min.), (iv) $x = 1$ (max.), $x = -1$ (min.),
(v) $x = 1$ (max.), $x = 2$ (min.), $x = 3$ (min.),

2. Max. value = 1, min. value = 25 for (iii),
Max. value = 3, min. value = $\frac{1}{3}$ for (iv),
Max. value = -3, and min. value = -4 in both cases for (v),

7. (i) $x = (2n + \frac{1}{2})\pi$ (max.), $x = (2n - \frac{1}{2})\pi$ (min.),
(ii) $x = 2n\pi$ (max.), $x = (2n + 1)\pi$ (min.),
(iii) Neither max. nor min. (iv) $x = 0$ gives minimum
(v) $x = 0$ (max.), $x = 1$ (min.),
(vi) $x = 2n\pi + \frac{3}{4}\pi$ (max.), $x = 2n\pi - \frac{1}{4}\pi$ (min.),

8. (i) Neither max. nor min. (ii) Max. for $x = 0$.

14. Min. for $x = 0$, max. for $x = \frac{6}{5}$. **15.** $a = 1$, $b = 0$.

17. (i) $\frac{3}{2}, \frac{36}{13}$, (ii) Max. value = -24 ; min. value = 24.

20. $(\pm 2, 2)$, **22.** 2 units ; $\left(-\frac{6}{5}, \frac{2}{5}\right)$. **24.** $\frac{7}{22}\,\text{m}^3$.

25. (i) 2, 16, (ii) 5, 10.Area fo the triangle in (i) = 16 sq. units ; length of the line in (ii) = $5\sqrt{5}$ units.

26. $a - b$. **27.** $(u^2 \sin^2 \alpha)/(2g)$; $(u \sin \alpha)/g$.

28. $\frac{K}{C}$ units. **30.** 15 m.

31. (i) $\frac{4}{3}a$ (ii) $\frac{1}{3}h$,

35. Height of the rectangle = radius of the semi-circle.

36. At the end of 4 second. **37.** 80 m × 60 m

38. 600 m^2, **39.** $(u^4 - g^2x^2)/(2u^2g)$.

41. 10 m from the stronger light.

43. 1 km from the point nearest to him ; $1\frac{3}{4}$ hour.

45. 24 km/h ; Rs. 800.

11 INDETERMINATE FORMS (*Evaluation of certain limits*)

11.1. The limit of $\phi(x)/\psi(x)$ as $x \to a$ is, in general, equal to the quotient of the limiting values of the numerator and denominator [see *Rule (iii) of Art. 3.7*], but when these two limits are both zero that rule is no longer applicable since the quotient takes the form $\frac{0}{0}$ which is meaningless. We shall consider in the present chapter how to obtain the limiting values of the quotient in such cases, and also the limiting values of the quotient in such cases, and also the limiting values in other cases of meaningless forms, apparently arising out of the indiscriminate use of the rules of Art. 3.7. The name 'indeterminate forms', as applied to these cases, is rather misleading and vague.

11.2. Form $\frac{0}{0}$ (*L' Hospital's theorem*)

If $\phi(x)$, $\psi(x)$ as also their derivatives $\phi'(x)$, $\psi'(x)$ are continuous at $x = a$, and if $\phi(a) = \psi(a) = 0$ [i.e., $\underset{x\to a}{Lt}\, \phi(x) = \underset{x\to a}{Lt}\, \psi(x) = 0$], then

$$\underset{x\to a}{\mathrm{Lt}} \frac{\phi(x)}{\psi(x)} = \underset{x\to a}{\mathrm{Lt}} \frac{\phi'(x)}{\psi'(x)} = \frac{\phi'(a)}{\psi'(a)}$$

provided $\psi'(a) \neq 0$.

Since $\phi(a) = 0$ and $\psi(a) = 0$, we have

$$\phi(x) = \phi(x) - \phi(a) \text{ and } \psi(x) = \psi(x) - \psi(a).$$

Now, by the Mean Value Theorem,

$$\phi(x) - \phi(a) = (x-a)\,\phi'\{a + \theta_1(x-a)\},\ 0 < \theta_1 < 1,$$

$$\psi(x) - \psi(a) = (x-a)\,\psi'\{a + \theta_2(x-a)\},\ 0 < \theta_2 < 1.$$

$$\therefore\ \frac{\phi(x)}{\psi(x)} = \frac{\phi(x)-\phi(a)}{\psi(x)-\psi(a)} = \frac{\phi'\{a+\theta_1(x-a)\}}{\psi'\{a+\theta_2(x-a)\}},$$

$$\therefore\ \underset{x\to a}{Lt} \frac{\phi(x)}{\psi(x)} = \frac{\phi'(a)}{\psi'(a)} = \underset{x\to a}{Lt} \frac{\phi'(x)}{\psi'(x)},$$

provided $\psi'(a) \neq 0$.

Generalization :

In case $\phi'(a)$ and $\psi'(a)$ are both zero, applying the above theorem again, we get

$$\underset{x \to a}{Lt} \frac{\phi'(x)}{\psi'(x)} = \underset{x \to a}{Lt} \frac{\phi''(x)}{\psi''(x)} = \frac{\phi''(a)}{\psi''(a)},$$

provided $\phi''(a)$ and $\psi''(a)$ are continuous at $x = a$, and $\psi''(a) \neq 0$. If, however, $\phi''(a) = \psi''(a) = 0$, then we again apply the above theorem and obtain the limiting value as $\phi'''(a)/\psi'''(a)$, and so on.

[*For* illustration, *see* Ex. 1, Art. 11.8]

Note 1. *The above result can also be established by* Cauchy's Mean Value Theorem. [*See* Ex. 7(a), Art. 9.7]

Note 2. *In the theorem of this article if x tends to ∞ instead of a, then the substitution 1/t for* x *would reduce it to the above form when* t *tends to zero.*

11.3. Form $\frac{\infty}{\infty}$.

If $\underset{x \to a}{Lt} \phi(x) = \infty$ *and* $\underset{x \to a}{Lt} \psi(x) = \infty$, *and if* $\underset{x \to a}{Lt} \frac{\phi'(x)}{\psi'(x)}$ *exists then* $\underset{x \to a}{Lt} \frac{\phi(x)}{\psi(x)}$ *will also exist, and its value is equal to the former limit.*

Let $\underset{x \to a}{Lt} \frac{\phi'(x)}{\psi'(x)} = l$. Then we can determine a positive number δ, such that in the interval $a - \delta < x < a + \delta$ [$x \neq a$], $\frac{\phi'(x)}{\psi'(x)}$ is as near to l as we please. Also, since the limit exists, it follows that for x sufficiently close to a [but $\neq a$] $\phi'(a)$ and $\psi'(a)$ must both exist, and $\psi'(a) \neq 0$ there.

Now first consider the interval $a < x \leq a + \delta$, and x_0 be any particular value therein, and take another value x such that $a < x < x_0$.

Then, by Cauchy's Mean Value Theorem [see § 9.7, *Ex.* 7(*a*)],

$$\frac{\phi(x_0) - \phi(x)}{\psi(x_0) - \psi(x)} = \frac{\phi'(\xi)}{\psi'(\xi)},$$

where $x < \xi < x_0$ and so $a < \xi < a + \delta$

Hence, $$\frac{\phi(x)\left\{\dfrac{\phi(x_0)}{\phi(x)}-1\right\}}{\psi(x)\left\{\dfrac{\psi(x_0)}{\psi(x)}-1\right\}}=\frac{\phi'(\xi)}{\psi'(\xi)},$$

$$\frac{\phi(x)}{\psi(x)}=\frac{\dfrac{\psi(x_0)}{\psi(x)}-1}{\dfrac{\phi(x_0)}{\phi(x)}-1}\cdot\frac{\phi'(\xi)}{\psi'(\xi)} \qquad \ldots \qquad (1)$$

Now keeping x_0 fixed, if we make $x \to a$, $\psi(x) \to \infty$ and $\phi(x) \to \infty$, *and so* $\left\{\dfrac{\phi(x_0)}{\phi(x)}-1\right\}\Big/\left\{\dfrac{\psi(x_0)}{\psi(x)}-1\right\} \to \dfrac{0-1}{0-1}$, *i.e.*, $\to 1$.

Also, $\dfrac{\phi'(\xi)}{\psi'(\xi)}$ is as near to l as we like, by a proper choice of δ.

Hence, from (1), $\dfrac{\phi(x)}{\psi(x)}$ is arbitrarily close to l, as $x \to a+0$.

Thus, $\displaystyle \mathop{Lt}_{x\to a+0}\frac{\phi(x)}{\psi(x)}=l$.

Similarly, considering the interval $a-\delta < x < a$, and proceeding exactly as before, we get $\displaystyle \mathop{Lt}_{x\to a-0}\frac{\phi(x)}{\psi(x)}=l$.

Hence, $\displaystyle \mathop{Lt}_{x\to a}\frac{\phi(x)}{\psi(x)}=\mathop{Lt}_{x\to a}\frac{\phi'(x)}{\psi'(x)}$,

We can also prove a modified form of the above theorem as follows :

If $\displaystyle \mathop{Lt}_{x\to a}\phi(x)$ *and* $\displaystyle \mathop{Lt}_{x\to a}\psi(x)$ *are both infinite, then*

$$\mathop{\mathbf{Lt}}_{\mathbf{x\to a}}\frac{\phi(\mathbf{x})}{\psi(\mathbf{x})}\ \text{(when it exists)}=\mathop{\mathbf{Lt}}_{\mathbf{x\to a}}\frac{\phi'(x)}{\psi'(x)}.$$

$$\mathop{Lt}_{x\to a}\frac{\phi(x)}{\psi(x)}=\mathop{Lt}_{x\to a}\frac{1/\psi(x)}{1/\phi(x)}=\mathop{Lt}_{x\to a}\frac{f(x)}{g(x)},\ \text{say,}$$

[where $f(x)=1/\psi(x)$ and $g(x)=1/\phi(x)$]

which, being of the form $\dfrac{0}{0}$, [see *Art, 11.2*, above]

$$= \underset{x \to a}{Lt} \frac{f'(x)}{g'(x)} = \underset{x \to a}{Lt} \frac{-\psi'(x)/\{\psi(x)\}^2}{-\phi'(x)/\{\phi(x)\}^2} = \underset{x \to a}{Lt}\left[\frac{\psi'(x)}{\phi'(x)}\left\{\frac{\psi(x)}{\phi(x)}\right\}^2\right]$$

$$\therefore \quad \underset{x \to a}{Lt} \frac{\phi(x)}{\psi(x)} = \underset{x \to a}{Lt} \frac{\psi'(x)}{\phi'(x)} \cdot \left\{\underset{x \to a}{Lt} \frac{\psi(x)}{\phi(x)}\right\}^2 \qquad \ldots \quad (1)$$

$$\text{Now, Let } \underset{x \to a}{Lt} \frac{\phi(x)}{\psi(x)} = l \qquad \ldots \quad (2)$$

Three cases arise :

Case I. l is neither zero, nor infinitely large.

Dividing both sides of (1) by l^2, we obtain.

$$\frac{1}{l} = \underset{x \to a}{Lt} \frac{\psi'(x)}{\phi'(x)}; \ \therefore l, \text{ i.e., } \underset{x \to a}{Lt} \frac{\phi(x)}{\psi(x)} = \underset{x \to a}{Lt} \frac{\phi'(x)}{\psi'(x)}.$$

Case II. $l = 0$.

Adding 1 to each side of (2),

$$l + 1 = \underset{x \to a}{Lt} \frac{\phi(x) + \psi(x)}{\psi(x)} = \underset{x \to a}{Lt} \frac{\phi'(x) + \psi'(x)}{\psi'(x)}$$

[by case (I)]

$$= \underset{x \to a}{Lt} \frac{\phi'(x)}{\psi'(x)} + 1.$$

$$\therefore \quad l, \text{ i.e., } \underset{x \to a}{Lt} \frac{\phi(x)}{\psi(x)} = \underset{x \to a}{Lt} \frac{\phi'(x)}{\psi'(x)}.$$

Case III. When l is infinitely large,

$$\underset{x \to a}{Lt} \frac{1}{\{\phi(x)/\psi(x)\}} = \underset{x \to a}{Lt} \frac{\psi(x)}{\phi(x)} = \underset{x \to a}{Lt} \frac{\psi'(x)}{\phi'(x)}$$

$$\therefore \quad \underset{x \to a}{Lt} \frac{\phi(x)}{\psi(x)} = \underset{x \to a}{Lt} \frac{\phi'(x)}{\psi'(x)}.$$

Hence, the theorem is proved in all cases.

For illustration *see* Ex. 3, Art. 11.08.

Note 1. *Theorem is evidently true also when one or both the limits tend to* $-\infty$.

Note 2. *By substituting* $x = 1/t$, *it can be shown that the theorem is also true when* x *tends to* ∞ *instead of* a.

11.4. Form $0\times\infty$.

Such forms arise when we want to find the limiting value of $\phi(x).\psi(x)$ as $x\to a$, where $\phi(x)\to 0$ and $\psi(x)\to\infty$ as $x\to a$.

We can write

$$\phi(x).\psi(x)=\frac{\phi(x)}{1/\psi(x)},\ \text{or,}\ \frac{\psi(x)}{1/\phi(x)}$$

when being of the forms 0/0 and ∞/∞, as , can be evaluated by the methods of Arts. 11.2 and 11.3. [See *Illustrative Ex. 4, Art. 11.8*]

11.5. Form $\infty-\infty$.

Such forms arise when we want to find the limiting value of $\phi(x)-\psi(x)$ as $x\to a$ where $\phi(x)\to\infty$ and $\psi(x)\to\infty$ as .

We can write

$$\phi(x)-\psi(x)=\frac{1/\psi(x)-1/\phi(x)}{1/\{\phi(x)\psi(x)\}}$$

which being of the form 0/0 can be evaluated by the method of Art. 11.2.

[*See Illustrative Example 4, Art. 11.8*]

11.6. Form $0^0, \infty^0, 1^{\pm\infty}$.

These forms occur when we want to evaluate the limits of functions of the form $\{\phi(x)\}^{\psi(x)}$ as $x\to a$,

When (i) both $\phi(x)$ and $\psi(x)\to 0$ as $x\to a$;

(ii) $\phi(x)\to\infty$ and $\psi(x)\to 0$ as $x\to a$;

(iii) $\phi(x)\to 1$ and $\psi(x)\to\pm\infty$ as $x\to a$,

If $\phi(x)>0$, let $y=\{\phi(x)\}^{\psi(x)}$, $\therefore$ $\log y=\psi(x)\log\phi(x)$

$\therefore$ *Lt* log y reduces to the form discussed in Art. 11.4, and, hence, can be evaluated.

Since *Lt* $\log y=\log$ *Lt* y the required limit *Lt* y can be obtained

[See *Illustrative Example 6, Art. 11.8.*]

11.7 Use of power series.

In evaluating limits of certain expressions, it is sometimes found convenient to use the expansions of known functions in the expression in power series in a finite form, and then to take the limit.

[See *Illustrative Example 7, Art. 11.8.*]

11.8. Illustrative Examples.

Ex. 1. *If* $\phi(a), \phi'(a), \phi''(a), \ldots, \phi^{n-1}(a)$ *and* $\psi(a), \psi'(a), \psi''(a), \ldots, \psi^{n-1}(a)$ *are all zero, and* $\psi^n(a) \neq 0$, *then*

$$\underset{x \to a}{Lt} \frac{\phi(x)}{\psi(x)} = \underset{x \to a}{Lt} \frac{\phi^n(x)}{\psi^n(x)}$$

Put $x = a + h$ so that, when $x \to a$, $h \to 0$.

Now, by Taylor's theorem

$$\phi(a+h) = \phi(a) + h\phi'(a) + \frac{h^2}{2!}\phi''(a) + \ldots$$

$$+ \frac{h^{n-1}}{(n-1)!}\phi^{n-1}(a) + \frac{h^n}{n!}\phi^n(a + \theta_1 h)$$

$$= \frac{h^n}{n!}\phi^n(a + \theta_1 h), \text{ where } 0 < \theta_1 < 1,$$

Similarly, $\psi(a+h) = \dfrac{h^n}{n!}\psi^n(a + \theta_2 h)$ where $0 < \theta_2 < 1$,

$$\therefore \quad \underset{x \to a}{Lt} \frac{\phi(x)}{\psi(x)} = \underset{h \to 0}{Lt} \frac{\phi(a+h)}{\psi(a+h)} = \underset{h \to 0}{Lt} \frac{h^n \phi^n(a+\theta_1 h)}{h^n \psi^n(a+\theta_2 h)}$$

$$= \underset{x \to a}{Lt} \frac{\phi^n(x)}{\psi^n(x)} = \frac{\phi^n(a)}{\psi^n(a)},$$

provided $\phi^n(x)$ and $\psi^n(x)$ are continuous at $x = a$.

Ex. 2. *Evaluate* $\underset{x \to 0}{Lt} \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$. [*C.P. 2001*]

The required limit, as it stands, being of the form 0/0, [see § *11.2*]

$$= \underset{x \to 0}{Lt} \frac{e^x + e^{-x} - 2}{1 - \cos x} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \underset{x \to 0}{Lt} \frac{e^x - e^{-x}}{\sin x} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \underset{x \to 0}{Lt} \frac{e^x + e^{-x}}{\cos x} = 2,$$

since $\underset{x \to 0}{Lt}(e^x + e^{-x}) = 1 + 1 = 2$, and $\underset{x \to 0}{Lt} \cos x = 1$.

Ex. 3. *Evaluate* $\underset{x\to\infty}{Lt}\dfrac{x^4}{e^x}$.

The given limit, as it stands, being of the form $\dfrac{\infty}{\infty}$, can be written [by § *11.3*] as

$$= \underset{x\to\infty}{Lt}\frac{4x^3}{e^x}\left(\text{form}\frac{\infty}{\infty}\right) = \underset{x\to\infty}{Lt}\frac{12x^2}{e^x}\left(\text{form}\frac{\infty}{\infty}\right)$$

$$= \underset{x\to\infty}{Lt}\frac{24x}{e^x}\left(\text{form}\frac{\infty}{\infty}\right) = \underset{x\to\infty}{Lt}\frac{24}{e^x} = 0.$$

Ex. 4. *Evaluate* $\underset{x\to\frac{1}{2}\pi}{Lt}(1-\sin x)\tan x$.

The given limit, as it stands, being of the form $0\times\infty$, can be written as

$$= \underset{x\to\frac{1}{2}\pi}{Lt}\frac{1-\sin x}{\cot x}\quad\left[\text{form}\frac{0}{0}\right]$$

$$= \underset{x\to\frac{1}{2}\pi}{Lt}\frac{-\cos x}{-\operatorname{cosec}^2 x} = 0,$$

since $\cos x = 0$ and $\operatorname{cosec} x = 1$ as $x\to\frac{1}{2}\pi$.

Ex. 5. *Evaluate* $\underset{x\to1}{Lt}\left(\dfrac{1}{x^2-1}-\dfrac{2}{x^4-1}\right)$.

The given limit, as it stands, being of the form $\infty-\infty$, can be written as

$$= \underset{x\to1}{Lt}\frac{x^2-1}{x^4-1}\quad\left[\text{form}\frac{0}{0}\right]$$

$$= \underset{x\to1}{Lt}\frac{1}{x^2+1} = \frac{1}{2}.$$

Ex. 6. *Evaluate* $\underset{x\to0}{Lt}(\cos x)^{\cot^2 x}$. [*Patna 1933, V.P. 1997*]

The given limit, as it stands, is of the form 1^∞.

Let $y = (\cos x)^{\cot^2 x}$.

$$\therefore\quad \log y = \cot^2 x\log\cos x = \frac{\log\cos x}{\tan^2 x}.$$

Now, $$\underset{x\to0}{Lt}\log y = \underset{x\to0}{Lt}\frac{\log\cos x}{\tan^2 x}\quad\left[\text{form}\frac{0}{0}\right]$$

$$= \underset{x \to 0}{Lt} \frac{-(\sin x / \cos x)}{2 \tan x \sec^2 x} = \underset{x \to 0}{Lt} \left(-\tfrac{1}{2} \cos^2 x \right)$$

$$= -\tfrac{1}{2} \left(\because \underset{x \to 0}{Lt} \cos^2 x = 1 \right).$$

Since $\underset{x \to 0}{Lt} \log y = \log \underset{x \to 0}{Lt} y$, $\therefore$ $\log \underset{x \to 0}{Lt} y = -\frac{1}{2}$.

$\therefore$ $\underset{x \to 0}{Lt} y = e^{-\frac{1}{2}}$. $\therefore$ the required limit $= e^{-\frac{1}{2}}$.

Ex. 7. *Show that* $\underset{x \to 0}{Lt} \dfrac{x - \sin x}{x^3} = \dfrac{1}{6}$. [*C.P. 1932, 1995*]

Writing down the expansion of $\sin x$ in a finite power series, we have

$$x - \sin x = x - \left\{ x - \frac{x^3}{3!} + \frac{x^5}{5!} \sin\left(\frac{5\pi}{2} + \theta x \right) \right\}, \quad 0 < \theta < 1$$

$$= \frac{x^3}{3!} - \frac{x^5}{5!} \sin\left(\frac{5\pi}{2} + \theta x \right)$$

$$= x^3 \left\{ \frac{1}{3!} - \frac{x^2}{5!} \sin\left(\frac{5\pi}{2} + \theta x \right) \right\},$$

$$\therefore \quad \frac{x - \sin x}{x^3} = \frac{1}{3!} - \frac{x^2}{5!} \sin\left(\frac{5\pi}{2} + \theta x \right).$$

$$\therefore \quad \underset{x \to 0}{Lt} \frac{x - \sin x}{x^3} = \underset{x \to 0}{Lt} \left\{ \frac{1}{3!} - \frac{x^2}{5!} \sin\left(\frac{5\pi}{2} + \theta x \right) \right\} = \frac{1}{3!} = \frac{1}{6};$$

since $\dfrac{x^2}{5!} \sin\left(\dfrac{5\pi}{2} + \theta x \right) \to 0$ as $x \to 0$, $\left| \sin\left(\dfrac{5\pi}{2} + \theta x \right) \right|$ being ≤ 1.

Note. This being of the form 0/0 can also be obtained by the method of Art. 112.

Ex. 8. *Evaluate* $\underset{x \to 0}{Lt} \dfrac{\sqrt{a^2 + ax + x^2} - \sqrt{a^2 - ax + x^2}}{\sqrt{a + x} - \sqrt{a - x}}$

Multiplying both the numerator and denominator by

$$\left(\sqrt{a^2 + ax + x^2} + \sqrt{a^2 - ax + x^2} \right)\left(\sqrt{a + x} + \sqrt{a - x} \right)$$

and simplying, the required limit

$$= \underset{x\to 0}{Lt} \frac{2ax\left(\sqrt{a+x}+\sqrt{a-x}\right)}{2x\left(\sqrt{a^2+ax+x^2}+\sqrt{a^2-ax+x^2}\right)}$$

$$= \underset{x\to 0}{Lt} \frac{a\left(\sqrt{a+x}+\sqrt{a-x}\right)}{\left(\sqrt{a^2+ax+x^2}+\sqrt{a^2-ax+x^2}\right)}$$

Now, the limit of the numerator $= a.2\sqrt{a}$ and that of the denominator $= 2a$. Therefore the required limit $= \sqrt{a}$.

Note. An algebraical or trigonometrical transformation often enables us to obtain the limiting values without using calculus, as shown above, which case belongs to the form 0/0.

Ex. 9. *If* $\underset{x\to 0}{Lt} \frac{\sin 2x + a\sin x}{x^3}$ *be finite, find the value of 'a' and the limit.* [*C.P. 1931, 1994, 2000, 2006*]

The given limit, being of the form 0/0,

$$= \underset{x\to 0}{Lt} \frac{2\cos 2x + a\cos x}{3x^2} \quad (\textit{by § 11.2.})$$

When $x \to 0$, the denominator $3x^2 = 0$; hence, in order that the limiting value of the expression may be finite, the numerator $(2\cos 2x + a\cos x)$ should be zero, as $x \to 0$. $\therefore 2 + a = 0$, *i.e.,* $a = -2$.

When $a = -2$, the given limit becomes

$$= \underset{x\to 0}{Lt} \frac{\sin 2x - 2\sin x}{x^3} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \underset{x\to 0}{Lt} \frac{2\cos 2x - 2\cos x}{3x^2} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \underset{x\to 0}{Lt} \frac{-4\sin 2x + 2\sin x}{6x} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \underset{x\to 0}{Lt} \frac{-8\cos 2x + 2\cos x}{6} = -\frac{6}{6} = -1.$$

Ex.10. *Evaluate* $\underset{x\to 0}{Lt} \left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$ [*C. P. 1947, 1994, 1997, V.P. 1999*]

Let $$u = \underset{x\to 0}{Lt}\left(\frac{\tan x}{x}\right)^{\frac{1}{x}}.$$

$$\therefore \quad \log u = \frac{1}{x}\log\left(\frac{\tan x}{x}\right) = \log\left(\frac{\tan x}{x}\right)\Big/ x.$$

Since $\underset{x\to 0}{Lt}\dfrac{\tan x}{x} = 1$ $\underset{x\to 0}{Lt}\log u$ is of the form $\dfrac{0}{0}$.

$$\therefore \quad \underset{x\to 0}{Lt}\log u = \underset{x\to 0}{Lt}\log\left(\frac{\tan x}{x}\right)\Big/ x$$

$$= \underset{x\to 0}{Lt}\left(\frac{x}{\tan x}\cdot\frac{x\sec^2 x - \tan x}{x^2}\right)\Big/ 1 \qquad (by\ \S\ 11.2.)$$

$$= \underset{x\to 0}{Lt}\frac{2x - \sin 2x}{x\sin 2x} = \underset{x\to 0}{Lt}\frac{2 - 2\cos 2x}{\sin 2x + 2x\cos 2x}$$

$$= \underset{x\to 0}{Lt}\frac{4\sin 2x}{2\cos 2x - 4x\sin 2x} = 0.$$

Since $\underset{x\to 0}{Lt}\log u = \log \underset{x\to 0}{Lt}u$, $\quad \therefore \log \underset{x\to 0}{Lt}u = 0$.

$\therefore \quad \underset{x\to 0}{Lt}u = e^0 = 1$, *i.e.*, the required limit $= 1$.

Otherwise : Writing the finite form of the expansion of $\tan x$ by Maclaurin's theorem,

$$\tan x = x + \tfrac{1}{3}x^3\alpha \text{ where } \alpha = \sec\theta x\left(1 + 2\tan^2\theta x\right), \quad 0<\theta<1.$$

$$\log u = \frac{1}{x}\log\left(\frac{\tan x}{x}\right) = \frac{1}{x}\log\frac{x + \frac{1}{3}x^2\alpha}{x} = \frac{1}{x}\log\left(1 + \tfrac{1}{3}x^2\alpha\right)$$

$$= \frac{1}{\frac{1}{3}x^2\alpha}\log\left(1 + \tfrac{1}{3}x^2\alpha\right)\tfrac{1}{3}x\alpha = \frac{1}{v}\log(1+v)\cdot\tfrac{1}{3}x\alpha,$$

where $v = \frac{1}{3}x^2\alpha$.

When $x\to 0, v\to 0$, also $\underset{v\to 0}{Lt}\dfrac{1}{v}\log(1+v) = 1$.

Hence, $\underset{x\to 0}{Lt}\log u = \underset{v\to 0}{Lt}\dfrac{1}{v}\log(1+v)\cdot \underset{x\to 0}{Lt}\left(\frac{1}{3}xa\right)$,

$\therefore \quad \underset{x\to 0}{Lt}\log u = 0$. Hence, etc.

Ex.11. *Evaluate* $\underset{x\to 0}{Lt}\dfrac{\left(e^{x}-1\right)\tan^{2}x}{x^{3}}$

$$\text{Given limit} = \underset{x\to 0}{Lt}\left\{\frac{e^{x}-1}{x}\cdot\left(\frac{\tan x}{x}\right)^{2}\right\}$$

$$= \underset{x\to 0}{Lt}\frac{e^{x}-1}{x}\times\left(\underset{x\to 0}{Lt}\frac{\tan x}{x}\right)^{2}.$$

Now, $\underset{x\to 0}{Lt}\dfrac{e^{x}-1}{x}\left(\text{being of the form }\dfrac{0}{0}\right)=\underset{x\to 0}{Lt}\dfrac{e^{x}}{1}=e^{0}=1,$

Also, $\underset{x\to 0}{Lt}\dfrac{\tan x}{x}=1,\quad\therefore\quad$ the required limit $=1\times 1^{2}=1.$

Note. Such forms are sometimes called *Compound Indeterminate forms.* In evaluating limits of such forms, the use of the theorems on limit (*Art. 3.8*) is of great help.

11.9 Miscellaneous Worked Out Examples

Ex. 1. Evaluate : $\lim\limits_{x\to 1}\left\{\dfrac{x}{x-1}-\dfrac{1}{\log x}\right\}$ [*C. P. 1981*]

Solution : $\lim\limits_{x\to 1}\left\{\dfrac{x}{x-1}-\dfrac{1}{\log x}\right\}$

$$=\lim_{x\to 1}\left\{\frac{x\log x-x+1}{(x-1)\log x}\right\}\qquad\left[\text{ Form }\frac{0}{0}\right]$$

$$=\lim_{x\to 1}\left\{\frac{\log x+x\cdot\frac{1}{x}-1}{\log x+(x-1)\frac{1}{x}}\right\}\qquad[\text{ By L'Hospitals Rule }]$$

$$=\lim_{x\to 1}\left\{\frac{x\log x}{x\log x+x-1}\right\}\qquad\left[\text{ Form }\frac{0}{0}\right]$$

$$= \lim_{x\to 1} \frac{1+\log x}{2+\log x} = \frac{\lim\limits_{x\to 1}(1+\log x)}{\lim\limits_{x\to 1}(2+\log x)} = \frac{1+0}{2+0} = \frac{1}{2}.$$

Ex. 2. Evaluate : $\lim\limits_{x\to 0}\left\{\frac{1}{x} - \frac{2}{x\left(e^x+1\right)}\right\}$. [*C. P. 1982*]

Solution : $\lim\limits_{x\to 0}\left\{\frac{1}{x} - \frac{2}{x\left(e^x+1\right)}\right\}$

$$= \lim_{x\to 0}\left\{\frac{e^x-1}{x\left(e^x+1\right)}\right\} \qquad \left[\text{ Form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{e^x}{1+(x+1)e^x}$$

$$= \frac{\lim\limits_{x\to 0} e^x}{\lim\limits_{x\to 0}\left\{1+(x+1)e^x\right\}} = \frac{1}{1+1} = \frac{1}{2}.$$

Ex. 3. Evaluate : $\lim\limits_{x\to 0} \frac{\sin x - x}{x^3}$ [*C. P. 1995*]

Solution : $\lim\limits_{x\to 0} \frac{\sin x - x}{x^3}$ $\left[\text{ Form } \frac{0}{0}\right]$

$$= \lim_{x\to 0} \frac{\cos x - 1}{3x^2}$$

$$= \lim_{x\to 0} \frac{-\sin x}{6x} \qquad \left[\text{ Form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{-\cos x}{6} = -\frac{1}{6}.$$

Ex. 4. Find the value of

(*i*) $\lim\limits_{x\to 0} (\cos x)^{\frac{1}{x^2}}$. [*C. P. 1989, 98*] (*ii*) $\lim_{x\to 1}\left(x^{\frac{1}{1-x}}\right)$. [*C. P. 1995*]

(*iii*) $\lim_{x\to\infty}(1+x)^{\frac{1}{x}}$. [*C. P. 1996*]

(*iv*) $\lim_{x\to 0}\left(x^{2\sin x}\right)$. [*C. P. 1996*]

(*v*) $\lim_{x\to 0}(1+\sin x)^{\cot x}$. [*C. P. 1993*]

Solution : (*i*) Let $y=(\cos x)^{\frac{1}{x^2}}$

$\therefore \quad \log y=\frac{1}{x^2}\cdot\log\cos x=\frac{\log\cos x}{x^2}$ [Base of logarithm is e]

$\therefore \quad \lim_{x\to 0}\log y=\lim_{x\to 0}\left\{\frac{\log\cos x}{x^2}\right\}$ [Form $\frac{0}{0}$]

$=\lim_{x\to 0}\frac{-\tan x}{2x}$ [Form $\frac{0}{0}$]

$=\lim_{x\to 0}\frac{-\sec^2 x}{2}=-\frac{1}{2}$.

or, $\log\left\{\lim_{x\to 0} y\right\}=-\frac{1}{2}$

$\therefore \lim_{x\to 0} y=e^{-\frac{1}{2}}$

(*ii*) Let $y=x^{\left(\frac{1}{1-x}\right)}$

or, $\log y=\frac{1}{1-x}\cdot\log x=\frac{\log x}{1-x}$

$\therefore \lim_{x\to 1}\{\log y\}=\lim_{x\to 1}\frac{\log x}{1-x}$ [Form $\frac{0}{0}$]

$=\lim_{x\to 1}\left(-\frac{1}{x}\right)=-1$

$\therefore \quad \log\left\{\lim_{x\to 1} y\right\} = -1$

$\therefore \quad \lim_{x\to 1} \; y = e^{-1}$

$\therefore \quad \lim_{x\to 1} \; x^{\left(\frac{1}{1-x}\right)} = e^{-1}$

(*iii*) $\lim_{x\to\infty} (1+x)^{\frac{1}{x}}$

Let , $z = \dfrac{1}{x}$ then $z \to 0$ as $x \to \infty$

Let $y = (1+x)^{\frac{1}{x}}$

then $\log y = z \cdot \log\left(1 + \dfrac{1}{z}\right)$

Thus $\lim_{z\to 0} \{\log y\} = \lim_{z\to 0}\left\{z\cdot\log\left(1+\dfrac{1}{z}\right)\right\}$

$$= \lim_{z\to 0} \frac{\log\left(1+\dfrac{1}{z}\right)}{\dfrac{1}{z}} \qquad \left[\text{ Form } \frac{\infty}{\infty} \right]$$

$$= \lim_{z\to 0} \frac{\left(\dfrac{1}{1+\dfrac{1}{z}} \times \left(-\dfrac{1}{z^2}\right)\right)}{\left(-\dfrac{1}{z^2}\right)}$$

$$= \lim_{z\to 0} \frac{z}{1+z} \qquad [\text{ as } z \to 0, \; z \neq 0 \;]$$

$$= 0$$

$\therefore \; \log\left\{\lim_{x\to\infty} \; y\right\} = 0$

i.e., $\lim_{x\to\infty} y = e^0 = 1$

Thus $\lim_{x\to\infty} (1+x)^{\frac{1}{x}} = 1$.

(*iv*) Let, $y = x^{2\sin x}$

or, $\log y = 2\sin x \cdot \log x$ [Base of logarithm is e]

or, $\lim\limits_{x\to 0} \{\log y\} = \lim\limits_{x\to 0} \dfrac{2\log x}{\operatorname{cosec} x}$ [Form $\frac{0}{0}$]

$$= \lim_{x\to 0} \frac{\frac{2}{x}}{-\operatorname{cosec} x \cot x}$$

$$= \lim_{x\to 0} \frac{2\sin^2 x}{-x\cos x} \qquad [\text{ Form } \tfrac{0}{0}]$$

$$= -2 \lim_{x\to 0} \frac{2\sin x\cos x}{\cos x - x\sin x}$$

$$= -2 \frac{\lim\limits_{x\to 0} \sin 2x}{\lim\limits_{x\to 0} (\cos x - x\sin x)} = -2 \times \frac{0}{1} = 0.$$

or, $\log\left\{\lim\limits_{x\to 0} y\right\} = 0$

i.e., $\lim\limits_{x\to 0} y = e^0 = 1$

i.e., $\lim\limits_{x\to 0} x^{2\sin x} = 1$

(*v*) Let $y = (1+\sin x)^{\cot x}$

or, $\log y = \cot x \log(1+\sin x)$ [Base of logarithm is e]

$$= \frac{\log(1+\sin x)}{\tan x}$$

$\therefore \lim\limits_{x\to 0} \log y = \lim\limits_{x\to 0} \dfrac{\log(1+\sin x)}{\tan x}$ [Form $\frac{0}{0}$]

$$= \lim_{x\to 0} \frac{\frac{\cos x}{1+\sin x}}{\sec^2 x}$$

$$= \lim_{x\to 0} \frac{\cos^3 x}{1+\sin x} = 1$$

$$\therefore \quad \log\left\{\lim_{x\to 0} y\right\} = 1$$

or, $\lim\limits_{x\to 0} y = e^1 = e$

$$\therefore \quad \lim_{x\to 0}(1+\sin x)^{\cot x} = e$$

Ex. 5. Evaluate :

(*i*) $\lim\limits_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$ [*B. P. 1995, C. P. 1994, '97*]

(*ii*) $\lim\limits_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$ [*C. P. 1990, '98*]

Solution : (*i*) Let, $y = \left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$

$$\therefore \quad \lim_{x\to 0}\{\log y\} = \lim_{x\to 0}\left\{\frac{1}{x}\log\left(\frac{\tan x}{x}\right)\right\}$$

$$= \lim_{x\to 0}\frac{\log\left(\frac{\tan x}{x}\right)}{x}$$ [Form $\frac{0}{0}$ $x \to 0$, since as $x \to 0$ $\frac{\tan x}{x} \to 1$, *i.e.*, $\log\left(\frac{\tan x}{x}\right) \to 0$]

$$= \lim_{x\to 0}\frac{\frac{\sec^2 x}{\tan x} - \frac{1}{x}}{1}$$

$$= \lim_{x\to 0}\left(\frac{1}{\sin x\cos x} - \frac{1}{x}\right)$$

$$s = \lim_{x\to 0}\frac{2x - \sin 2x}{x\sin 2x}$$ [Form $\frac{0}{0}$]

$$= 2 \cdot \lim_{x\to 0} \frac{1-\cos 2x}{\sin 2x + 2x\cos 2x} \qquad \left[\text{ Form } \frac{0}{0}\right]$$

$$= 2 \lim_{x\to 0} \frac{2\sin 2x}{2\cos 2x + 2\cos 2x - 4x\sin 2x} \qquad \left[\text{ Form } \frac{0}{0}\right]$$

$$= 4 \cdot \frac{\lim\limits_{x\to 0} \sin 2x}{\lim\limits_{x\to 0}(4\cos 2x - 4x\sin 2x)} = 4 \times \frac{0}{4} = 0$$

or, $\log\left\{\lim\limits_{x\to 0} y\right\} = 0$

or, $\lim\limits_{x\to 0} y = e^0 = 1$

$\therefore \quad \lim\limits_{x\to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x}} = 1$

(*ii*) Let, $y = \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$

or, $\log y = \frac{1}{x^2} \cdot \log \frac{\sin x}{x} = \frac{\log \frac{\sin x}{x}}{x^2}$

$\because \ \lim\limits_{x\to 0} \left(\frac{\sin x}{x}\right) = 1, \ \lim\limits_{x\to 0} \log\left(\frac{\sin x}{x}\right) = 0$

$$\therefore \ \lim_{x\to 0} (\log y) = \lim_{x\to 0} \frac{\log \frac{\sin x}{x}}{x^2} \qquad \left[\text{ Form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{\frac{x}{\sin x} \times \frac{x\cos x - \sin x}{x^2}}{2x}$$

$$= \lim_{x\to 0} \frac{x\cos x - \sin x}{2x^2 \sin x} \qquad \left[\text{ Form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{\cos x - x\sin x - \cos x}{4x\sin x + 2x^2 \cos x}$$

$$= \lim_{x\to 0} \frac{-\sin x}{2x\cos x + 4\sin x} \qquad \left[\text{ Form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{-\cos x}{2\cos x - 2x\sin x + 4\cos x} = -\frac{1}{6}$$

$$\therefore \log\left\{\lim_{x\to 0} y\right\} = -\frac{1}{6}$$

or, $\lim_{x\to 0} y = e^{-\frac{1}{6}}$

$$\therefore \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} = e^{-\frac{1}{6}}.$$

Ex. 6. Evaluate :

(*i*) $\lim_{x\to 1} \frac{\log(1-x)}{\cot(\pi x)}$ [*C. P. 1993*]

(*ii*) $\lim_{x\to 0} (\cos mx)^{n/x^2}$ [*C. P. 2002*]

Solution : (*i*) $\lim_{x\to 1} \frac{\log(1-x)}{\cot(\pi x)}$ $\left[\text{ Form } \frac{\infty}{\infty}\right]$

$$= \lim_{x\to 1} \frac{-\frac{1}{(1-x)}}{-\pi\,\mathrm{cosec}^2(\pi x)}$$

$$= \lim_{x\to 1} \frac{\sin^2(\pi x)}{\pi(1-x)} \qquad \left[\text{ Form } \frac{0}{0}\right]$$

$$= \lim_{x\to 1} \frac{\{2\sin(\pi x)\cos(\pi x)\}\pi}{\pi\times(-1)} = \lim_{x\to 1}(-2\sin 2\pi x) = 0.$$

(*ii*) $\lim_{x\to 0} (\cos mx)^{n/x^2}$

Let $y = (\cos mx)^{\frac{n}{x^2}}$

then $\log y = \dfrac{n}{x^2} \log(\cos mx)$

$$\lim_{x\to 0}\{\log y\} = n \cdot \lim_{x\to 0} \frac{\log(\cos mx)}{x^2} \qquad \left[\text{ Form } \frac{0}{0}\right]$$

$$= -\frac{mn}{2} \cdot \lim_{x\to 0} \frac{\tan mx}{x} \qquad \left[\text{ Form } \frac{0}{0}\right]$$

$$= -\frac{m^2 n}{2} \cdot \lim_{x\to 0} \frac{\sec^2 mx}{1} = -\frac{1}{2}m^2 n$$

$$\therefore \ \log\left\{\lim_{x\to 0} y\right\} = -\frac{1}{2}m^2 n$$

or, $\displaystyle\lim_{x\to 0} y = e^{-\frac{1}{2}m^2 n}$

$$\therefore \ \lim_{x\to 0} (\cos mx)^{\frac{n}{x^2}} = e^{-\frac{1}{2}m^2 n}.$$

Ex. 7. (*i*) Find *a*, *b* such that $\displaystyle\lim_{x\to 0} \frac{x(1+a\cos x) - b\sin x}{x^3} = 1$

[*C. P. 1990*]

(*ii*) If $\displaystyle\lim_{x\to 0} \frac{\sin 2x + a\sin x}{x^3}$ is finite, find *a* and the value of the limit.

[*C. P. 1994, 2000*]

Solution : (*i*) Here, $\displaystyle\lim_{x\to 0} \frac{x(1+a\cos x) - b\sin x}{x^3}$ $\left[\text{ Form } \dfrac{0}{0}\right]$

$$= \lim_{x\to 0} \frac{1(1+a\cos x) - ax\sin x - b\cos x}{3x^2}$$

$$= \lim_{x\to 0} \frac{1+(a-b)\cos x - ax\sin x}{3x^2} \qquad \dots \quad \dots \quad (1)$$

For (1) to be of the form $\dfrac{0}{0}$, $1+(a-b)=0$

i.e., $b = 1+a$ ... ... (2)

So, the given expression $= \displaystyle\lim_{x\to 0} \frac{1-\cos x - ax\sin x}{3x^2}$ $\left[\text{ Form } \dfrac{0}{0}\right]$

$$= \lim_{x \to 0} \frac{\sin x - a\sin x - ax\cos x}{6x} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{\cos x - a\cos x - a\cos x + ax\sin x}{6}$$

$$= \lim_{x \to 0} \frac{(1-2a)\cos x + ax\sin x}{6} = \frac{1-2a}{6} = 1, \text{ given}$$

$\therefore\ 1 - 2a = 6,\quad$ *i.e.*, $a = -\frac{5}{2}$

From (2), $b = 1 - \frac{5}{2} = -\frac{3}{2}$

Thus, $a = -\frac{5}{2},\ b = -\frac{3}{2}$.

(*ii*) $\lim_{x \to 0} \frac{\sin 2x + a\sin x}{x^3} \qquad \left[\text{Form } \frac{0}{0}\right]$

$$= \lim_{x \to 0} \frac{2\cos 2x + a\cos x}{3x^2}$$

For this limit to be finite, the form should be $\frac{0}{0}$,

i.e., $2 + a = 0$, or, $a = -2$

$$\therefore\ \lim_{x \to 0} \frac{\sin 2x + a\sin x}{x^3} = \lim_{x \to 0} \frac{2\cos 2x - 2\cos x}{3x^2} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{-4\sin 2x + 2\sin x}{6x} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{-8\cos 2x + 2\cos x}{6}$$

$$= \frac{-8+2}{6} = -1.$$

Hence, $a = -2$ and the value of the limit is -1.

EXAMPLES - XI

Evaluate the following limits [*Ex. 1 – 9*]

1. (i) $\underset{x \to 0}{Lt} \dfrac{x - \sin x \cos x}{x^3}$. (ii) $\underset{x \to 0}{Lt} \dfrac{\tan x - x}{x - \sin x}$. [*C. P. 2007*]

(iii) $\underset{x \to 0}{Lt} \dfrac{e^{2x} - 1}{\log(1 + x)}$. (iv) $\underset{x \to a}{Lt} \dfrac{x^n - a^n}{x - a}$.

(v) $\underset{x \to 2}{Lt} \dfrac{x^3 - 2x^2 + 2x - 4}{x^2 - 5x + 6}$. (vi) $\underset{x \to 0}{Lt} \dfrac{e^x + e^{-x} - 2\cos x}{x \sin x}$. [*V.P. 1996*]

(vii) $\underset{x \to a}{Lt} \dfrac{a^x - b^x}{x}$. (viii) $\underset{x \to 0}{Lt} \dfrac{e^x + \sin x - 1}{\log(1 + x)}$.

(ix) $\underset{x \to 0}{Lt} \dfrac{x - \sin^{-1} x}{\sin^3 x}$. (x) $\underset{x \to 0}{Lt} \dfrac{e^x + e^{\sin x}}{x - \sin x}$. [*C. P. 2004*]

(xi) $\underset{x \to 0}{Lt} \dfrac{\tan nx - n \tan x}{n \sin x - \sin nx}$. (xii) $\underset{x \to 2}{Lt} \dfrac{\sqrt{2x} - \sqrt{6 - x}}{3x - 2\sqrt{19 - 5x}}$.

(xiii) $\underset{x \to 0}{Lt} \dfrac{2 \sin x - \sin 2x}{\tan^3 x}$. (xiv) $\underset{x \to 0}{Lt} \dfrac{\sin \log(1 + x)}{\log(1 + \sin x)}$.

(xv) $\underset{x \to 0}{Lt} \left\{ \left(\dfrac{\sin 2x + 2\sin^2 x - 2\sin x}{\cos x - \cos^2 x} \right)^2 + \dfrac{1 - \cos x}{\cos x \sin^2 x} \right\}$.

(xvi) $\underset{x \to \frac{1}{2}\pi}{Lt} \left\{ \dfrac{\left(\frac{1}{2}\pi - x\right) \log \sin x}{e^{\cos x} - 1 + \log\left(1 + x + \frac{1}{2}\pi\right)} + \dfrac{\cos x}{x - \frac{1}{2}\pi} \right\}$.

2. (i) $\underset{x \to \frac{1}{2}\pi}{Lt} \dfrac{\tan 5x}{\tan x}$. (ii) $\underset{x \to 0}{Lt} \dfrac{\log x^2}{\log \cot^2 x}$.

(iii) $\underset{x \to \infty}{Lt} \dfrac{x^n}{e^x}$ (*n being positive*).

(iv) $\underset{x \to \frac{1}{2}}{Lt} \dfrac{\tan 3\pi x}{\sec \pi x}$. (v) $\underset{x \to 0+}{Lt} \dfrac{\log x - \cot \frac{1}{2}\pi x}{\cot \pi x}$.

(vi) $\underset{x \to 0}{Lt} \log_{\tan^2 x} \tan^2 2x$. (vii) $\underset{x \to \infty}{Lt} \dfrac{x^2 + 3x}{1 - 5x^2}$.

3. (i) $\underset{x \to 0}{Lt}\, x^2 \log x^2$. (ii) $\underset{x \to 1}{Lt} \operatorname{cosec}(\pi x) \log x$.

(iii) $\underset{x \to 0}{Lt}\, x \log \sin^2 x$. (iv) $\underset{x \to \frac{1}{2}\pi}{Lt} \sec x \left(x \sin x - \frac{1}{2}\pi \right)$.

(v) $\underset{x \to 0}{Lt} \sin x \,.\, \log x^2$. (vi) $\underset{x \to \frac{1}{2}\pi}{Lt} \sec 5x \cos 7x$.

(vii) $\underset{x \to 0+}{Lt}\, x^m (\log x)^n$, *m and n being positive*.

4. (i) $\underset{x \to \frac{1}{2}\pi}{Lt} (\sec x - \tan x)$. (ii) $\underset{x \to 0}{Lt} \left(x^{-1} - \cot x \right)$.

(iii) $\underset{x \to 0}{Lt} \left(\dfrac{1}{x^2} - \dfrac{1}{\sin^2 x} \right)$. *[C.P. 1996, B.P. 1998]*

(iv) $\underset{x \to 1}{Lt} \left(\dfrac{x}{x-1} - \dfrac{1}{\log x} \right)$. (v) $\underset{x \to 2}{Lt} \left(\dfrac{4}{x^2 - 4} - \dfrac{1}{x-2} \right)$.

(vi) $\underset{x \to 0}{Lt} \left\{ \dfrac{1}{x} - \dfrac{1}{x^2} \log(1 + x) \right\}$.

(vii) $\underset{x \to \infty}{Lt} \left(x - \sqrt{x^2 - 9} \right)$. (viii) $\underset{x \to \infty}{Lt} \left(\sqrt{x^2 + 2x} - x \right)$.

. (i) $\underset{x \to 0}{Lt}\, x^{2x}$. (ii) $\underset{x \to 0}{Lt}\, x^{2 \sin x}$. *[C.P. 1996]*

(iii) $\underset{x \to 0}{Lt} (\sin x)^{2 \tan x}$. (iv) $\underset{x \to 0}{Lt} (\cos x)^{\cot^2 x}$.

[C.P. 1989, 1997]

(v) $\underset{x \to \frac{1}{2}\pi}{Lt} (\sin x)^{\tan x}$. *[C. P. 2005]* (vi) $\underset{x \to 1}{Lt}\, x^{\frac{1}{1-x}}$.

(vii) $\underset{x \to 0}{Lt} \left(\cot^2 x \right)^{\sin x}$. (viii) $\underset{x \to 0}{Lt} \left(1/x^2 \right)^{\tan x}$.

(ix) $\underset{x \to 1-}{Lt} \left(1 - x^2\right)^{1/\log(1-x)}$. (x) $\underset{x \to e}{Lt} \left(\log x\right)^{1/(1-\log x)}$.

(xi) $\underset{x \to \infty}{Lt} \left(1 + \dfrac{1}{x^2}\right)^{x}$. (xii) $\underset{x \to 0}{Lt} \left(\dfrac{\sin x}{x}\right)^{\frac{1}{x}}$.

[C.P. 1985, B.P. 1996, V.P. 1995]

(xiii) $\underset{x \to 0}{Lt} \left(\dfrac{\tan x}{x}\right)^{\frac{1}{x^2}}$. (xiv) $\underset{x \to 0}{Lt} \left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^2}}$.

[C.P. 1990, 1998, V.P. 1998]

6. $\underset{x \to \infty}{Lt} \dfrac{a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n}{b_0 x^m + b_1 x^{m-1} + b_2 x^{m-2} + \ldots + b_m}$.

$(a_0 \neq 0,\ b_0 \neq 0)$, according as $n >=$ or $< m$ [n and m being positive integers].

7. $\underset{x \to \infty}{Lt} \left(a_0 x^m + a_1 x^{m-1} + a_2 x^{m-2} + \ldots + a_m\right)^{\frac{1}{x}}$, m being a positive integer $(a_0 \neq 0)$.

8. $\underset{x \to \infty}{Lt} 2^x \sin \dfrac{a}{2^x}\ (a \neq 0)$. [C. P. 1946]

9. (i) $\underset{x \to \infty}{Lt} \dfrac{x + \cos x}{x + 1}$. (ii) $\underset{x \to 1}{Lt} \dfrac{x^{\frac{3}{2}} - 1 + (x-1)^{\frac{3}{2}}}{(x^2 - 1)^{\frac{3}{2}} - x + 1}$.

10. If $\underset{x \to 0}{Lt} \dfrac{a \sin x - \sin 2x}{\tan^3 x}$ is finite, find the value of a, and the limit.

[C.P. 1997]

11. Adjust the constants a and b in order that

$$\underset{\theta \to 0}{Lt} \frac{\theta(1 + a\cos\theta) - b\sin\theta}{\theta^3} = 1.$$

12. Determine the values of a, b, c so that

(i) $\dfrac{ae^x - b\cos x + ce^{-x}}{x \sin x} \to 2$, as $x \to 0$.

(ii) $\dfrac{(a + b\cos x)x - c\sin x}{x^5} \to 1$, as $x \to 0$.

(iii) $\dfrac{a\sin x - bx + cx^2 + x^3}{2x^2\log(1+x) - 2x^3 + x^4}$ may tend to a finite limit as $x \to 0$, and determine this limit.

Evaluate the following [*Ex. 13 -- 19*]

13. (i) $\underset{x \to 0}{Lt} \dfrac{xe^x - \log(1+x)}{x^2}$. [*C. P. 2007*]

(ii) $\underset{x \to 0}{Lt} \left\{\dfrac{1}{x} - \dfrac{1}{x^2}\log(1+x)\right\}$. [*C.P. 1989, 1991*]

(iii) $\underset{x \to 0}{Lt} \dfrac{x\cos x - \log(1+x)}{x^2}$.

(iv) $\underset{x \to 0}{Lt} \dfrac{\tan x . \tan^{-1} x - x^2}{x^6}$.

14. $\underset{x \to 0}{Lt} \dfrac{e^x - e^{-x} + 2\sin x - 4x}{x^5}$.

15. $\underset{x \to 0}{Lt} \dfrac{\cos x - \log(1+x) + \sin x - 1}{e^x - (1+x)}$.

16. $\underset{x \to \frac{1}{2}\pi}{Lt} \left[\sqrt{\left\{\dfrac{2 + \cos 2x - \sin x}{x\sin 2x + x\cos x}\right\}} - \left(\dfrac{\pi - 2x}{2\sin 2x}\right)^2\right]$.

17. $\underset{x \to a-0}{Lt} \left[\sqrt{(a^2 - x^2)} . \cot\left\{\dfrac{\pi}{2}\sqrt{\dfrac{a-x}{a+x}}\right\}\right]$.

18. $\underset{x \to 2}{Lt} \dfrac{x^2 - 4}{\sqrt{(x+2)} - \sqrt{(3x-2)}}$.

19. $\underset{x \to a+0}{Lt} \dfrac{\sqrt{x} - \sqrt{a} + \sqrt{(x-a)}}{\sqrt{(x^2 - a^2)}}$.

Show that [*Ex. 20 — 26*]

20. $\underset{x\to\infty}{Lt}\, a^x \sin\dfrac{b}{a^x} = 0$ or b according as $0 < a < 1$, or $a > 1$.

21. $\underset{x\to\infty}{Lt}\left\{ x - x^2 \log\left(1 + \dfrac{1}{x} \right)\right\} = \dfrac{1}{2}$.

22. $\underset{x\to 0}{Lt}\, \dfrac{\log\left(1 + x + x^2\right) + \log\left(1 - x + x^2\right)}{\sec x - \cos x} = 1$.

23. $\underset{x\to 0}{Lt}\, \dfrac{e^x - 1}{x^4 \sin x}\left(\dfrac{3\sin x - \sin 3x}{\cos x - \cos 3x}\right)^4 = 1$.

24. $\underset{x\to 0}{Lt}\, \dfrac{\log_{\sin^2 x}(\cos x)}{\log_{\sin^2 \frac{1}{2}x}\left(\cos \frac{1}{2}x\right)} = 4$.

25. $\underset{x\to 0}{Lt}\, \dfrac{(1+x)^{\frac{1}{x}} - e}{x} = -\frac{1}{2}e$.

26. $\left(\underset{x\to\infty}{Lt}\, \dfrac{a_1^{1/x} + a_2^{1/x} + \ldots + a_n^{1/x}}{n} \right)^{nx} = a_1 a_2 \ldots a_n$.

27. Evaluate $\underset{a\to b}{Lt}\, \dfrac{a^x \sin bx - b^x \sin ax}{\tan bx - \tan ax}$.

28. If $\phi(x) = x^2 \sin(1/x)$ and $\psi(x) = \tan x$, show that,

although $\underset{x\to 0}{Lt}\, \dfrac{\phi'(x)}{\psi'(x)}$ does not exist, $\underset{x\to 0}{Lt}\, \dfrac{\phi(x)}{\psi(x)}$ exists and $= 0$.

ANSWERS

1. (i) $\frac{2}{3}$. (ii) 2. (iii) 2. (iv) na^{n-1}. (v) -6.

(vi) 2. (vii) $\log(a/b)$. (viii) 2. (ix) $-\frac{1}{6}$. (x) 1. (xi) 2.

(xii) $\frac{9}{56}$. (xiii) 1. (xiv) 1. (xv) $16\frac{1}{2}$. (xvi) $\frac{1}{2}$.

2. (i) $\frac{1}{5}$. (ii) -1. (iii) 0. (iv) $\frac{1}{3}$ (v) -2 (vi) 1. (vii) $-\frac{1}{5}$

3. (i) 0. (ii) $(-1/\pi)$. (iii) 0. (iv) -1 (v) 0 (vi) $-\frac{7}{5}$. (vii) 0

4. (i) 0. (ii) 0. (iii) $-\frac{1}{3}$. (iv) $\frac{1}{2}$. (v) $-\frac{1}{4}$. (vi). (vii) 0. (viii) 1.

5. (i) 1. (ii) 1. (iii) 1. (iv) $e^{-\frac{1}{2}}$. (v) 1

(vi) $1/e$. (vii) 1. (viii) 1. (ix) e.

(x) $1/e$.. (xi) 1. (xii) 1 (xiii) $e^{\frac{1}{3}}$. (xiv) $e^{-\frac{1}{6}}$.

6. $+\infty$ or $-\infty$ (corresponding to a_0/b_0 being positive or negative) a_0/b_0, 0 according as $n >=, < m$

7. 1. **8.** a. **9.** (i) 1. (ii) $-\dfrac{3}{2}$. **10.** $a=2$; limit $=1$.

11. $a=-\frac{5}{2}$, $b=-\frac{3}{2}$.

12. (i) $a=1,\ b=2,\ c=1$. (ii) $a=120,\ b=60,\ c=180$.

(iii) $a=6,\ b=6,\ c=0$; limit $=\frac{3}{40}$.

13. (i) $\frac{3}{2}$. (ii) $\frac{1}{2}$. (iii) $\frac{1}{2}$. (iv) $\frac{2}{9}$. **14.** $\frac{1}{30}$. **15.** 0.

16. $-\frac{1}{4}$. **17.** $4a/\pi$. **18.** -8. **19.** $1/\sqrt{2a}$.

27. $b^{x-1}\left(b\cos bx-\sin bx\right)\cos^2 bx$.

12 PARTIAL DIFFERENTIATION
(*Functions of two or more variables*)

12.1. Definition.

If three variables u, x, y are so related that for every pair of values of x and y within the defined domain, say, $a \le x \le b$ and $c \le y \le d$, u has a single definite value, u is said to be *a function of the two independe.. variables x and y*, and this is denoted by $u = f(x, y)$.

More generally (*i.e.*, without restricting to single-valued functions only), if the three variables u, x, y are so related that u is determined when x and y are known, u *is said to be a function of the two independent variables x and y.*

Illustration : Since the area of a triangle is determined when its base and altitude are given, the area of a triangle is a function of its base and altitude.

Similarly, the volume of a gas is a function of its pressure and temperature. In a similar way, a function of three or more independent variables can be defined.

Thus, the volume of a paralleloplped is a fuction of three variables, its length, breadth and height.

Note 1. If to each pair of volues of x and y, u has a single definite value, u is called *a single-valued function* (to which the definition refers and with which we are mainly concerned in all mathematical investigations), and if to each set of values of x and y, u has more than one definite value, u is called a *multiple-valued function*. A multiple-valued function with proper limitations imposed on its value can, in general, be treated as defining two or more single-valued functions.

Note 2. *Geometrical representation of* $z = f(x, y)$.

When a single-valued function $z = f(x, y)$ is given, for each pair of values of x and y, there corresponds a point Q in the plane OXY, and if a perpendicular $\overline{QP}$ is then erected of length equal to the value of z obtained from the given relation, the points like P describe what is called a *surface* in three-dimensional space. Thus to a functional relation between three-variables x, y, z, therefore, corresponds a surface referred to axes $\overrightarrow{OX}$, $\overrightarrow{OY}$, $\overrightarrow{OZ}$ in space.

Note 3. Continuity

The function $f(x, y)$ is said to be continuous at the point (a, b), if corresponding to a pre-assigned positive number ε, however small, there exists a positive number δ such that

$$|f(x,y)-f(a,b)|<\varepsilon,$$

whenever $0 \le |x-a| \le \delta$ and $0 \le |y-b| \le \delta$

12.2. Partial Derivatives.

The result of differentiating $u = f(x, y)$, with respect to x, treating y as a constant, is called *the partial derivative of u with respect to x*, and is denoted by one of the symbols $\frac{\partial u}{\partial x}$, $\frac{\partial f}{\partial x}$, $f_x(x, y)$ [or briefly, f_x], u_x, etc.

Analytically, $\frac{\partial f}{\partial x} = \underset{\Delta x \to 0}{Lt} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$,
when this limit exists.

The partial derivative of $u = f(x, y)$ with respect to y is similarly defined and is denoted by $\frac{\partial u}{\partial y}$, $\frac{\partial f}{\partial y}$, $f_y(x, y)$ [or briefly, f_y], u_y, etc.

Thus, $\frac{\partial f}{\partial y} = \underset{\Delta y \to 0}{Lt} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$,
provided this limit exists.

If $u = f(x, y, z)$, then the partial derivative of u with respect to x is the derivative of u with respect to x, when both y and z are regarded as constants.

Thus, $\frac{\partial f}{\partial x} = \underset{\Delta x \to 0}{Lt} \frac{f(x+\Delta x, y, z) - f(x, y, z)}{\Delta x}$

Similarly for $\frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$.

Illustrations :

Let $u = x^2 + xy + y^2$; then

$$\frac{\partial u}{\partial x} = 2x + y; \quad \frac{\partial u}{\partial y} = 2y + x.$$

Let $u = yz + zx + xy$; then

$$\frac{\partial u}{\partial x} = y + z; \quad \frac{\partial u}{\partial y} = z + x; \quad \frac{\partial u}{\partial z} = x + y.$$

Note. The curl ∂ is generally used to denote the symbol of partial derivative, in order to distinguish it from the symbol d of ordinary derivative.

12.3. Successive Partial Derivatives.

Since each of the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ is, in general, a function of x and y, each may possess partial derivatives with respect to these two independent variables, and these are called the second order partial derivatives of u. The usual *notatiions* for these *second order partial derivatives are*

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right), \text{ i.e., } \frac{\partial^2 u}{\partial x^2} \text{ or } f_{xx}, \text{ etc.}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right), \text{ i.e., } \frac{\partial^2 u}{\partial y^2} \text{ or } f_{yy}, \text{ etc.}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right), \text{ i.e., } \frac{\partial^2 u}{\partial x \partial y} \text{ or } f_{xy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right), \text{ i.e., } \frac{\partial^2 u}{\partial y \partial x} \text{ or } f_{yx}, \text{ etc.}$$

Although *for most of the functions that occur in applications we have*

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

i.e., the partial derivative has the same value whether we differentiate partially first with respect to x and then with respect to y or the reverse, *it must not be supposed that the above relation holds good for all functions*; because the equality implies that the two limiting operations involved therein should be commutative, which may not be true always. Ex. 3, Art. 12.4 will elucidate the point. We can prove, in particular, that if the funtions $\frac{\partial^2 u}{\partial y \partial x}$ and $\frac{\partial^2 u}{\partial x \partial y}$ both exist for a particular set of values of x, y, and one of them is continuous there, the equality will hold good.

Proof of the equality $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

If f_x, f_y, f_{xy}, f_{yx} *all exist, and* f_{yx} (or f_{xy}) *is continuous then* $f_{xy} = f_{yx}$.

Proof: Let $\phi(x) = f(x, y+k) - f(x, y)$. $\quad \ldots \quad (1)$

Now applying Mean Value Theorem to $\phi(x)$, we get

$$\phi(x+h) - \phi(x) = h\phi_x(x+\theta h), \quad 0 < \theta < 1,$$

$$= h\{f_x(x+\theta h,\ y+k) - f_x(x+\theta h,\ y)\}$$

$$= h\{F(y+k) - F(y)\}, \text{ say}$$

$$[\text{ where } F(y) = f_x(x+\theta h,\ y)]$$

$$= h\{kF_y(y+\theta' k)\}, \quad 0<\theta'<1,$$

by Mean Value Theorem

$$= hk\{f_{yx}(x+\theta h,\ y+\theta' k)\} \quad \ldots \quad (2)$$

Again from (1), $\phi(x+h) = f(x+h,\ y+k) - f(x+h,\ y)$

$$\therefore \quad \phi(x+h) - \phi(x) = f(x+h,\ y+k) - f(x+h,\ y) - f(x,\ y+k) - f(x, y). \quad \ldots \quad (3)$$

Now, $f_y(x, y) = \underset{k \to 0}{Lt} \dfrac{f(x, y+k) - f(x, y)}{k}$

and $\quad f_{xy}(x, y) = \underset{h \to 0}{Lt} \dfrac{f_y(x+h, y) - f_y(x, y)}{h}$

$$= \underset{h \to 0}{Lt}\ \underset{k \to 0}{Lt} \frac{f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x, y)}{hk}$$

$$= \underset{h \to 0}{Lt}\ \underset{k \to 0}{Lt} \frac{\phi(x+h) - \phi(x)}{hk} \qquad [\textit{from } (3)]$$

$$= \underset{h \to 0}{Lt}\ \underset{k \to 0}{Lt} f_{yx}(x+\theta h, y+\theta' k) \qquad [\textit{from } (2)]$$

$= f_{yx}(x, y)$, since f_{yx} is continuous.

Illustration : *If* $f(x, y) = xy\dfrac{x^2 - y^2}{x^2 + y^2}$, *when* $x \neq 0, \textit{or } y \neq 0$,

$= 0,$ *when* $x = 0,\ y = 0$,

show that at the point $(0, 0)$, $\dfrac{\partial^2 f}{\partial x \partial y} \neq \dfrac{\partial^2 f}{\partial y \partial x}$, [*C. P. 2004, 2007*]

i.e., $f_{xy}(0,0) \neq f_{yx}(0,0)$.

When $x \neq 0$, or $y \neq 0$,

$$f_x(x, y) = y\left\{\frac{x^2 - y^2}{x^2 + y^2} + x\frac{(x^2 + y^2)2x - (x^2 - y^2)2x}{(x^2 + y^2)^2}\right\}$$

$$= y\left\{\frac{x^2 - y^2}{x^2 + y^2} + \frac{4x^2y^2}{(x^2 + y^2)^2}\right\} \quad \ldots \quad (1)$$

Similarly, $$f_y(x, y) = x\left\{\frac{x^2 - y^2}{x^2 + y^2} - \frac{4x^2y^2}{(x^2 + y^2)^2}\right\} \quad \ldots \quad (2)$$

$$f_x(0,0) = \underset{h \to 0}{Lt}\frac{f(h,0) - f(0,0)}{h} = 0.$$

Similarly, $f_y(0,0) = 0$.

From (1) and (2), we see that

$$f_x(0, y) = -y(y \neq 0),\ f_y(x,0) = x(x \neq 0).$$

Again, $$f_{xy}(0, 0) = \underset{h \to 0}{Lt}\frac{f_y(h, 0) - f_y(0, 0)}{h} = \underset{h \to 0}{Lt}\frac{h}{h} = 1,$$

$$f_{yx}(0,0) = \underset{k \to 0}{Lt}\frac{f_x(0,k) - f_x(0,0)}{k} = \underset{k \to 0}{Lt}\frac{-k}{k} = -1$$

$$\therefore \quad f_{xy}(0,0) \neq f_{yx}(0,0)$$

We have similar definitions and notations for partial derivatives of order higher than two.

If, $z = f(x, y)$, the partial derivatives of z are very often denoted by the following notations :

$$\frac{\partial z}{\partial x} = p,\ \frac{\partial z}{\partial y} = q,\ \frac{\partial^2 z}{\partial x^2} = r,\ \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = s,\ \frac{\partial^2 z}{\partial y^2} = t.$$ [1]

Illustrations :

$$u = x^3 + x^2y^2 + y^3$$

$$\frac{\partial u}{\partial x} = 3x^2 + 2xy^2\ ;\ \frac{\partial^2 u}{\partial x^2} = 6x + 2y^2\ ;\ \frac{\partial^2 u}{\partial x \partial y} = 4xy.$$

[1] These notation was first introduced by Monge.

$$\frac{\partial u}{\partial y} = 3y^2 + 2x^2 y; \quad \frac{\partial^2 u}{\partial y^2} = 6y + 2x^2; \quad \frac{\partial^2 u}{\partial y \partial x} = 4xy.$$

12.4. Illustrative Examples.

Ex. 1. *If* $x = r\cos\theta$, $y = r\sin\theta$,

so that $r = \sqrt{(x^2 + y^2)}$; $\theta = \tan^{-1}(y/x)$.

show that

$$\frac{\partial x}{\partial r} \neq 1 \Big/ \frac{\partial r}{\partial x} \quad \textit{and} \quad \frac{\partial x}{\partial \theta} \neq 1 \Big/ \frac{\partial \theta}{\partial x}.$$

Here, $$\frac{\partial x}{\partial r} = \cos\theta \; ; \; \frac{\partial r}{\partial x} = \frac{x}{\sqrt{(x^2 + y^2)}} = \frac{r\cos\theta}{r} = \cos\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \; ; \; \frac{\partial \theta}{\partial x} = -\frac{y}{x^2 + y^2} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}$$

Hence, the required results follow,

Note. If y is a function of a *single variable* x, then we have seen that, under certain circumstances (see § 7.7), $\frac{dy}{dx} \neq 1 \Big/ \frac{dx}{dy}$. *A similar property is not true*, as seen above, *when* y *is a function of more than one variable.*

Ex. 2. *If* $u = f\left(\frac{y}{x}\right)$, *show that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

$u = f(z)$, say, where $z = y/x$.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial x} = f'(z) \cdot \frac{\partial z}{\partial x} = -\frac{y}{x^2} f'(z)$$

Similarly, $$\frac{\partial u}{\partial y} = f'(z) \frac{\partial z}{\partial y} = \frac{1}{x} f'(z).$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{y}{x} f'(z) + \frac{y}{x} f'(z) = 0.$$

Ex.3. *Show that*

$$\underset{x\to 0}{Lt} \; \underset{y\to 0}{Lt} \frac{x-y}{x+y} \neq \underset{y\to 0}{Lt} \; \underset{x\to 0}{Lt} \frac{x-y}{x+y}$$

Left side $= \underset{x\to 0}{Lt} \dfrac{x}{x} = \underset{x\to 0}{Lt} 1 = 1.$

Right side $= \underset{y \to 0}{Lt} \dfrac{-y}{y} = \underset{y \to 0}{Lt} (-1) = -1$.

Hence, the result.

EXAMPLE - XII (A)

1. Find f_x, f_y for the following functions $f(x, y)$:

   (i) $ax^2 + 2hxy + by^2$. (ii) $\tan^{-1}(y/x)$.

   (iii) $1/\sqrt{(x^2 + y^2)}$. (iv) $\log(x^2 + y^2)$.

   (v) $x^2/a^2 + y^2/b^2 = 1$.

2. Find $f_{xx}, f_{xy}, f_{yx}, f_{yy}$ for the following functions $f(x, y)$:

   (i) $x^3 + 3x^2y + 3xy^2 + y^3$. (ii) $e^{x^2+xy+y^2}$.

   (iii) $x \cos y + y \cos x$. (iv) $\log(x^2y + xy^2)$.

3. (i) If $V = x^2 + y^2 + z^2$, show that $xV_x + yV_y + zV_z = 2V$.

   (ii) If $u = x^2y + y^2z + z^2x$, show that

   $$u_x + u_y + u_z = (x + y + z)^2.$$

   (iii) If $u = f(xyz)$, show that

   $$xu_x = yu_y = zu_z.$$

   (iv) If $u = \dfrac{y}{z} + \dfrac{z}{x} + \dfrac{x}{y}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$.

4. (i) If $u = \sin^{-1}\dfrac{x}{y} + \tan^{-1}\dfrac{y}{x}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$.

   [*V.P. 1995, '97*]

   (ii) If $U = \dfrac{x + y}{1 - xy}$ and $V = \dfrac{x(1-y^2) + y(1-x^2)}{(1+x^2)(1+y^2)}$.

   prove that $U_x V_y = U_y V_x$.

5. (*a*) Show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, if

   (i) $u = \log(x^2 + y^2)$. [*C.P. 1990*]

   (ii) $u = \tan^{-1}(y/x)$. [*C.P. 1998*]

   (ii) $u = e^x(x \cos y - y \sin y)$.

(*b*) If $V = z\tan^{-1}\dfrac{y}{x}$, then $\dfrac{\partial^2 V}{\partial x^2}+\dfrac{\partial^2 V}{\partial y^2}+\dfrac{\partial^2 V}{\partial z^2}=0$.

[*B.P. 1998*]

6. (i) If $f(x, y, z) = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$,

show that $f_x + f_y + f_z = 0$.

(ii) If $u = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x & y & z & w \\ x^2 & y^2 & z^2 & w^2 \\ x^3 & y^3 & z^3 & w^3 \end{vmatrix}$,

show that $u_x + u_y + u_z + u_w = 0$.

7. If $V = ax^2 + 2hxy + by^2$, then show that

$$V_x^2 V_{yy} - 2V_x V_y V_{xy} + V_y^2 V_{xx} = 8\left(ab - h^2\right)V.$$

8. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, then show that

(i) $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}=\dfrac{3}{x+y+z}$. [*B.P. 1989, '91, '97*]

(ii) $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=\dfrac{3}{(x+y+z)^2}$.

[*V. P. '97, C. P. '46, '85, 2007, B.P. 2001*]

9. If $V = \sqrt{\left(x^2 + y^2 + z^2\right)}$, show that $V_{xx} + V_{yy} + V_{zz} = 2/V$.

10. If $V = 1\Big/\sqrt{\left(x^2 + y^2 + z^2\right)}$, show that $V_{xx} + V_{yy} + V_{zz} = 0$.

[*C.P. 1976*]

11. If $u = e^{xyz}$, prove that

$$\frac{\partial^3 u}{\partial x\,\partial y\,\partial z} = \left(1 + 3xyz + x^2y^2z^2\right)e^{xyz}.$$ [*C. P. 1947*]

12. (i) If $V = (ax + by)^2 - (x^2 + y^2)$, where $a^2 + b^2 = 2$,

then show that $V_{xx} + V_{yy} = 0$.

(ii) If $u = 3(ax+by+cz)-(x^2+y^2+z^2)$ and $a^2+b^2+c^2=1$, find the value of $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$.

[*C. P. 1934*]

(iii) If $u = ax^2+by^2+cz^2+2fyz+2gzx+2hxy$ and $\sum \frac{\partial^2 u}{\partial x^2}=0$, show that $a+b+c=0$.

13. Show that, if $u(x, y, z)$ satisfies the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0, \text{ then}$$

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ satisfy it, and also

(ii) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}$ satisfies it.

14. If $u=\log\left(x^2+y^2+z^2\right)$, prove that

$$x\frac{\partial^2 u}{\partial y\,\partial z}=y\frac{\partial^2 u}{\partial z\,\partial x}=z\frac{\partial^2 u}{\partial x\,\partial y}$$

15. If $u=\log r$ and $r^2=x^2+y^2+z^2$, prove that

$$r^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=1.$$

[*C.P. 1975, 2008*]

16. If $y=f(x+ct)+\phi(x-ct)$, show that

$$\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2}.$$

[*V.P. 2001*]

17. If $u=f\left(ax^2+2hxy+by^2\right)$, $v=\phi\left(ax^2+2hxy+by^2\right)$ show that $\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right)=\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right)$.

[*C.H.1934*]

18. If (i) $U=x+y+z$, $V=x^2+y^2+z^2$,

$$W=x^3+y^3+z^3-3xyz,$$

(ii) $U=x+y+z$, $V=x^2+y^2+z^2$, $W=yz+zx+xy$, show that in each case $\begin{vmatrix} U_x & U_y & U_z \\ V_x & V_y & V_z \\ W_x & W_y & W_z \end{vmatrix} = 0$.

19. If α, β, γ be the roots of the cubic $x^3 + px^2 + qx + r = 0$, show that

$$\begin{vmatrix} \dfrac{\partial p}{\partial \alpha} & \dfrac{\partial q}{\partial \alpha} & \dfrac{\partial r}{\partial \alpha} \\ \dfrac{\partial p}{\partial \beta} & \dfrac{\partial q}{\partial \beta} & \dfrac{\partial r}{\partial \beta} \\ \dfrac{\partial p}{\partial \gamma} & \dfrac{\partial q}{\partial \gamma} & \dfrac{\partial r}{\partial \gamma} \end{vmatrix}$$

vanishes when any two of the three roots are equal.

20. Show that

$$\frac{\partial^3}{\partial x\, \partial y\, \partial z} \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ \phi_1(y) & \phi_2(y) & \phi_3(y) \\ \psi_1(z) & \psi_2(z) & \psi_3(z) \end{vmatrix} = \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ \phi_1'(y) & \phi_2'(y) & \phi_3'(y) \\ \psi_1'(z) & \psi_2'(z) & \psi_3'(z) \end{vmatrix}$$

where dashes denote differentiations with respect to the variables concerned.

ANSWERS

1. (i) $2(ax + hy)$; $2(hx + by)$. (ii) $-\dfrac{y}{x^2 + y^2}, \dfrac{x}{x^2 + y^2}$.

(iii) $-\dfrac{x}{\left(x^2 + y^2\right)^{\frac{3}{2}}}, -\dfrac{y}{\left(x^2 + y^2\right)^{\frac{3}{2}}}$. (iv) $\dfrac{2x}{x^2 + y^2}, \dfrac{2y}{x^2 + y^2}$.

(v) $\dfrac{2x}{a^2}, \dfrac{2y}{b^2}$.

2. (i) $6(x + y), 6(x + y), 6(x + y), 6(x + y)$.

(ii) $z\left\{(2x + y)^2 + 2\right\}, z\left\{(2x + y)(x + 2y) + 1\right\}$,

$z\left\{(2x + y)(x + 2y) + 1\right\}, z\left\{(x + 2y)^2 + 2\right\}$, where $z = e^{x^2 + xy + y^2}$.

(iii) $-y \cos x, -(\sin x + \sin y), -(\sin x + \sin y), -x \cos y$.

(iv) $-\left\{\dfrac{1}{x^2} + \dfrac{1}{(x + y)^2}\right\}, -\dfrac{1}{(x + y)^2}, -\dfrac{1}{(x + y)^2}, -\left\{\dfrac{1}{y^2} + \dfrac{1}{(x + y)^2}\right\}$.

12. (ii) 0.

12.5. Homogeneous Functions.

A function $f(x, y)$ is s to be *homogeneous* of degree n in the variables x and y, if it can be pressed in the form $x^n \phi\left(\frac{y}{x}\right)$, or in the form $y^n \phi\left(\frac{x}{y}\right)$.

If V be a homogeneous function of degree n in x, y, z, then each of $\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}$ is a homogeneous function of degree (n – 1).

Since $ax^2 + 2hxy + by^2 = x^2\left\{a + 2h\frac{y}{x} + b\left(\frac{y}{x}\right)^2\right\} = x^2 \phi\left(\frac{y}{x}\right)$

$ax^2 + 2hxy + by^2$ is a homogeneous function of degree 2 in x, y.

Similarly, y/x, $x\tan^{-1}(y/x)$, $x^2 \log(y/x)$ are homogeneous function of degree 0, 1 and 2 respectively.

Note 1. An *alternative test* for a function $f(x, y)$ to be homogeneous of degree n is that $f(tx, ty) = t^n f(x, y)$ for all values of t, where t is independent of x and y.

Note 2. The test that a *rational integral algebraic function of* x *and* y should be homogeneous of degree n is that the sum of the indices of x and y in every term must be n.

Note 3. Similarly, a function $f(x, y, z)$ is said to be homogeneous of degree n in the variables x, y, z, if it can be put in the form $x^n f\left(\frac{y}{x}, \frac{z}{x}\right)$, or if $f(tx, ty, tz) = t^n f(x, y, z)$; and so on, for any number of variables.

Thus, $f(x, y, z) = \sqrt{x} + \sqrt{y} + \sqrt{z}$ is a homogeneous function of degree $\frac{1}{2}$, since

$$f(tx, ty, tz) = \sqrt{tx} + \sqrt{ty} + \sqrt{tz} = t^{\frac{1}{2}} f(x, y, z).$$

12.6. Euler's Theorem on Homogeneous Functions.

If $f(x, y)$ *be a homogeneous function of* x *and* y *of degree* n then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y).$$

Since $f(x, y)$ is a homogeneous function of degree n,

let $f(x, y) = x^n \phi(y/x)$

$$= x^n \phi(v), \text{ where } v = y/x.$$

$$\therefore \quad \frac{\partial f}{\partial x} = nx^{n-1}\phi(v) + x^n \phi'(v)\frac{\partial v}{\partial x}$$

$$= nx^{n-1}\phi(v) + x^n \phi'(v) \cdot \frac{-y}{x^2}.$$

$$\frac{\partial f}{\partial y} = x^n \phi'(v)\frac{\partial v}{\partial y} = x^n \phi'(v) \cdot \frac{1}{x}.$$

$$\therefore \quad x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nx^n\phi(v) = nf(x, y).$$

12.7. Differentiation of Implicit Functions.

Let the equation $f(x, y) = 0$ ... (1)

define y as a differentiable function of x, and let f_x and f_y be continuous.

Then, we can find $\frac{dy}{dx}$ in terms of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ as follows :

We have $f(x + \Delta x, y + \Delta y) = 0$.

$$\therefore \quad f(x + \Delta x, y + \Delta y) - f(x, y) = 0 \quad \ldots \quad (2)$$

Now, by the Mean Value Theorem, [see § 9.2]

$$f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)$$

$$= \Delta x \frac{\partial}{\partial x} f(x + \theta_1 \Delta x, y + \Delta y) \quad [0 < \theta_1 < 1].$$

$$f(x, y + \Delta y) - f(x, y)$$

$$= \Delta y \frac{\partial}{\partial y} f(x, y + \theta_2 \Delta y) \quad [0 < \theta_2 < 1].$$

Adding these two and using relation (2) and dividing by Δx, we get

$$\frac{\partial}{\partial x} f(x + \theta_1 \Delta x, y + \Delta y) + \frac{\Delta y}{\Delta x}\frac{\partial}{\partial y} f(x, y + \theta_2 \Delta y) = 0 \ldots \quad (3)$$

Since y is a differentiable function of x, when $\Delta x \to 0$, $\Delta y \to 0$, and since f_x and f_y are continuous, we get, by making $\Delta x \to 0$ in (3),

$$\frac{\partial f}{\partial x}+\frac{dy}{dx}\cdot\frac{\partial f}{\partial y}=0.$$

$$\therefore \quad \frac{dy}{dx}=-\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}=-\frac{f_x}{f_y} \qquad (f_y \neq 0) \qquad \ldots \quad (4)$$

12.8. Total Differential Coefficient.

Let $u=f(x,y)$, where $x=\phi(t)$, $y=\psi(t)$.

Then usually u is a function of t in this case.

To obtain the value of $\dfrac{du}{dt}$.

Let us suppose that f_x, f_y as also $\phi'(t)$, $\psi'(t)$ are continuous.

When t changes to $t+\Delta t$, let x and y change to $x+\Delta x$, $y+\Delta y$.

Now, $u=f\{\phi(t), \psi(t)\}\equiv F(t)$, say.

$$\therefore \quad \frac{du}{dt}=\underset{\Delta t\to 0}{Lt}\frac{F(t+\Delta t)-F(t)}{\Delta t}$$

$$=\underset{\Delta t\to 0}{Lt}\frac{f\{\phi(t+\Delta t), \psi(t+\Delta t)\}-f\{\phi(t), \psi(t)\}}{\Delta t}$$

$$=\underset{\Delta t\to 0}{Lt}\frac{f(x+\Delta x, y+\Delta y)-f(x,y)}{\Delta t}$$

$$=\underset{\Delta t\to 0}{Lt}\left\{\frac{f(x+\Delta x, y+\Delta y)-f(x, y+\Delta y)}{\Delta x}\cdot\frac{\Delta x}{\Delta t}\right.$$

$$\left.+\frac{f(x, y+\Delta y)-f(x,y)}{\Delta y}\cdot\frac{\Delta y}{\Delta t}\right\}$$

But, by the Mean Value Theorem, [see § 9.2]

$$f(x+\Delta x, y+\Delta y)-f(x, y+\Delta y)=\Delta x.f_x(x+\theta\Delta x, y+\Delta y)$$

and $f(x, y+\Delta y)-f(x,y)=\Delta y.f_y(x, y+\theta'\Delta y)$.

where θ and θ' each lies between 0 and 1.

When $\Delta t\to 0, \Delta x\to 0$ and $\Delta y\to 0$; also $\dfrac{\Delta x}{\Delta t}\to\phi'(t)$ and

$\dfrac{\Delta y}{\Delta t}\to\psi'(t)$.

Also $f_x(x+\theta\,\Delta x, y+\Delta y) \to f_x(x, y)$;

$f_y(x, y+\theta'\,\Delta y) \to f_y(x, y)$.

$$\therefore \qquad \frac{du}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}. \qquad \ldots (1)$$

$$i.e., \qquad \frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} \qquad \ldots (2)$$

Note 1. As a particular case, if $u = f(x, y)$, where y is a function of x.

$$\frac{du}{dt} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$\frac{du}{dx}$ is called the *total differential coefficient of* u, to distinguish it from its partial differntial coefficient.

Note 2. The above result can easily be extended to the case when u is a function of three or more variables.

Thus, If $u = f(x, y, z)$, where x, y, z are all function of t.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

12.9. Differentials.

We have already defined the differential of a function of a single independent variable; we now give the corresponding definition of the differential of a function of two independent variables x, y. Thus, if $u = f(x, y)$, we define du by the relation

$$du = f_x\,\Delta x + f_y\,\Delta y$$

Putting $u = x$ and $u = y$ in turn, we obtain

$$dx = \Delta x, \qquad dy = \Delta y \qquad \ldots (1)$$

so that

$$\mathbf{du = f_x\,dx + f_y\,dy} \qquad \ldots (2)$$

Multiplying both sides of the relation (1) or Art. 12.8 by dt, and noting that, since x, y are each function of t,

$$du = \frac{du}{dt}dt, \qquad dx = \frac{dx}{dt}dt, \qquad dy = \frac{dy}{dt}dt,$$

we get

$$du = f_x\,dx + f_y\,dy$$

which is same in form as (2) above. But x, y here are not independent, but each is a function of t.

Hence, *the formula* $du = f_x\, dx + f_y\, dy$ *is true whether the variables* x *and* y *are independent or not.*

This remark is of great importance in applications

Similarly, if $u = f(x, y, z)$,

$$\mathbf{du = f_x\, dx + f_y\, dy + f_z\, dz},$$

whether x, y, z are independent or not.

Note. It should be noted that the relations (1) above are true only when x and y are independent variables. If x and y are not independent but functions of a third independent variable t, say, $x=\phi(t)$ $y=\psi(t)$, then $dx=\phi'(t)dt$ and $dy=\psi'(t)dt$, where $dt=\Delta t$.

12.10. Exact (or Perfect) Differential.

The expression

$$\phi(x, y)dx+\psi(x, y)dy \qquad \ldots \quad (1)$$

is called an exact (or perfect) differential if a function u of x, y exists such that its differential

$$du, \text{ i.e., } \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \qquad \ldots \quad (2)$$

is equal to (1) for all values of dx and dy.

Hence, comparing (1) and (2), we see that if (1) be an exact differential it is necessary that

$$\frac{\partial u}{\partial x} = \phi(x, y) \text{ and } \frac{\partial u}{\partial y} = \psi(x, y).$$

Differentiating these relations with respect to y and x respectively, we have

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial \phi}{\partial y} \text{ and } \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial \psi}{\partial x}$$

Since, in all ordinary cases, $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$, hence, in all ordinary cases, in order that (1) *may be an exact differential* it is necessary that

$$\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x},$$

It can be easily shown that this condition is, in general, also sufficient.

12.11. Partial Derivatives of a Function of two Functions.

If $u = f(x_1, x_2)$

where $x_1 = \phi_1(x, y)$, $x_2 = \phi_2(x, y)$, and x, y are independent variables, then

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial x} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial x},$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial y} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial y}.$$

We have

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 \qquad \dots \quad (1)$$

$$dx_1 = \frac{\partial x_1}{\partial x} dx + \frac{\partial x_1}{\partial y} dy, \quad dx_2 = \frac{\partial x_2}{\partial x} dx + \frac{\partial x_2}{\partial y} dy.$$

When values of x_1, x_2 in terms of x, y of are substituted u becomes a function of x, y; hence

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy. \qquad \dots \quad (2)$$

Now, substituting the values of dx_1, dx_2 in (1), we get

$$du = \left(\frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial x} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial x} \right) dx + \left(\frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial y} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial y} \right) dy$$

Comparing this with (2), since dx, dx are independent, the required relations follow.

Note. The above result admits of easy generalization to the cases of more than two variables. Thus, If $u = f(x_1, x_2, x_3)$, where $x_1 = \phi_1(x, y, z)$, $x_2 = \phi_2(x, y, z)$, $x_3 = \phi_3(x, y, z)$ and x, y, z are independent variables, then

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial x} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial x} + \frac{\partial u}{\partial x_3} \frac{\partial x_3}{\partial x},$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial y} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial y} + \frac{\partial u}{\partial x_3} \frac{\partial x_3}{\partial y},$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial z} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial z} + \frac{\partial u}{\partial x_3} \frac{\partial x_3}{\partial z}.$$

12.12. Euler's Theorem on Homogeneous Functions (*generalisation*).

If $f(x, y, z)$ be a homogeneous function in x, y, z of degree n, having continuous partial derivatives,

then $$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=nf.$$

***Proof*:** Since $f(x, y, z)$ is a homogeneous function of degree n.

$$\therefore \quad f(tx, ty, tz)=t^n f(x, y, z) \quad \ldots \quad (1)$$

for all values of t.

Putting $tu=u,\ ty=u,\ tz=w,$

differentiating both sides of (1) with respect to t, we have

$$\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial t}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial t}+\frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial t}=nt^{n-1}f(x, y, z)$$

$$\therefore \quad x\frac{\partial f}{\partial u}+y\frac{\partial f}{\partial v}+z\frac{\partial f}{\partial w}=nt^{n-1}f(x, y, z)$$

Putting $t=1$ in (2),

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=nf$$

Note 1. The above method of proof is applicable to a function of any number of independent variables.

Note 2. The above result can also be established as in the case of two independent variables, *i.e.*, by writing

$$f(x, y, z)=x^n f\left(\frac{y}{x}, \frac{z}{x}\right)=x^n f(u, v), \text{ where } u=\frac{y}{x},\ y=\frac{z}{x}$$

and then obtaining $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$.

12.13. Converse of Euler's Theorem.

If $f(x, y, z)$ admits of continuous partial derivatives and satisfies the relation

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=nf(x, y, z)$$

where n is a positive integer, prove that $f(x, y, z)$ is a homogenous function of degree n. [*C. H. 1960*]

***Proof*:** Put $\xi=\frac{x}{z},\ \eta=\frac{y}{z},\ \zeta=z;$

then $x = \xi\zeta$, $y = \eta\zeta$, $z = \zeta$. Suppose, when expressed in terms of ξ, η, ζ

$$f(x, y, z) = v(\xi, \eta, \zeta)$$

Then $$x\frac{\partial f}{\partial x} = x\left(\frac{\partial v}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \zeta}\cdot\frac{\partial \zeta}{\partial x}\right)$$

$$= x\left(\frac{\partial v}{\partial \xi}\cdot\frac{1}{z} + \frac{\partial v}{\partial \eta}\cdot 0 + \frac{\partial v}{\partial \zeta}\cdot 0\right)$$

$$= \xi\frac{\partial v}{\partial \xi}.$$

Similarly, $$y\frac{\partial f}{\partial y} = \eta\frac{\partial v}{\partial \eta}$$

$$z\frac{\partial f}{\partial z} = z\left(\frac{\partial v}{\partial \xi}\cdot\frac{\partial \xi}{\partial z} + \frac{\partial v}{\partial \eta}\cdot\frac{\partial \eta}{\partial z} + \frac{\partial v}{\partial \zeta}\cdot\frac{\partial \zeta}{\partial z}\right)$$

$$= z\left(-\frac{\partial v}{\partial \xi}\cdot\frac{x}{z^2} - \frac{\partial v}{\partial \eta}\cdot\frac{y}{z^2} + \frac{\partial v}{\partial \zeta}\cdot 1\right)$$

$$= -\xi\frac{\partial v}{\partial \xi} - \eta\frac{\partial v}{\partial \eta} + \zeta\frac{\partial v}{\partial \zeta}.$$

Hence the given relation reduces to

$$\zeta\frac{\partial v}{\partial \zeta} = nv, \qquad \text{or,} \qquad \frac{1}{v}\frac{\partial v}{\partial \zeta} = \frac{n}{\zeta},$$

whence $\log v = n\log\zeta + \text{a constant}$,

where the constant is independent of ζ, but may depend on ξ and η; let this constant by denoted by $\log\phi(\xi, \eta)$.

Then $$v = \zeta^n\phi(\xi, \eta),$$

i.e., $$f(x, y, z) = z^n\phi\left(\frac{x}{z}, \frac{y}{z}\right)$$

which, according to the definition of a homogeneous function, shows that $f(x, y, z)$ is a homogeneous function of degree n.

Note. *If n be any rational number, the proof and the result remain unchanged.*

12.14. Illustrative Examples.

Ex. 1. *If* $u = \tan^{-1}\dfrac{x^3+y^3}{x-y}$, *show that*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u. \quad \text{[C.P. 1996, '98, B.P. '95, V.P. '99, 2002]}$$

From the given relation, we get

$$\tan u = \frac{x^3+y^3}{x-y} = \frac{x^3\{1+(y/x)^3\}}{x\{1-(y/x)\}} = x^2\phi\left(\frac{y}{x}\right)$$

$\therefore$ $\tan u$ is a homogeneous function of degree 2.

Let $v = \tan u$; $\therefore$ by Euler's Theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 2v$$

$$\therefore \quad x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u} = 2\sin u\cos u = \sin 2u.$$

Ex. 2. *If* Δ *be the area of a triangle ABC, show that*

$$d\Delta = R(\cos A\ da + \cos B\ db + \cos C\ dc)$$

where R is the circum-radius of the triangle.

From trigonometry, we have

$$\Delta^2 = \tfrac{1}{16}\left(2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4\right).$$

Thus, Δ is a function of the three independent variables a, b, c,. Hence, taking differentials of both sides,

$$2d\Delta = \tfrac{1}{16}\left\{4a\left(b^2+c^2-a^2\right)da + 4b\left(c^2+a^2-b^2\right)db + 4c\left(a^2+b^2-c^2\right)dc\right\}$$

$$= \tfrac{1}{16}(4a\cdot 2bc\cos A\ da + 4b\cdot 2ca\cos B\ db + 4c\cdot 2ab\cos C\ dc)$$

$$= \tfrac{1}{2}abc(\cos A\ da + \cos B\ db + \cos C\ dc)$$

$$\therefore d\Delta = \frac{abc}{4\Delta}(\cos A\ da + \cos B\ db + \cos C\ dc)$$

$$= R(\cos A\ da + \cos B\ db + \cos C\ dc)$$

Ex. 3. *If* $P\ dx + Q\ dy + R\ dz$ *can be made a perfect differential of some function of* x, y, z *on multiplication by a factor, prove that*

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

[*C. H. 1949, 1954*]

Suppose u is a function of x, z and

$$\mu (P \ dx + Q \ dy + R \ dz) = du , \qquad \ldots \quad (1)$$

where μ is some function of x, y, z.

Also, $$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz , \qquad \ldots \quad (2)$$

since u is a function of x, y, z.

Comparing (1) and (2),

$$\frac{\partial u}{\partial x} = \mu P \qquad \ldots \quad (3)$$

$$\frac{\partial u}{\partial y} = \mu Q \qquad \ldots \quad (4)$$

$$\frac{\partial u}{\partial z} = \mu R \qquad \ldots \quad (5)$$

$$\therefore \ \frac{\partial^2 u}{\partial y \partial x} = \mu \frac{\partial P}{\partial y} + P \frac{\partial \mu}{\partial y} = \mu \frac{\partial}{\partial x} + \frac{\partial \mu}{\partial x} \qquad \ldots \quad (6)$$

(on differentiating (3) with respect to y)

(on differentiating (4) with respect to x, assuming $\left. \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} \right)$.

Similarly,

$$\frac{\partial^2 u}{\partial z \partial y} = \mu \frac{\partial Q}{\partial z} + Q \frac{\partial \mu}{\partial z} = \mu \frac{\partial R}{\partial y} + R \frac{\partial \mu}{\partial y} , \qquad \ldots \quad (7)$$

$$\frac{\partial^2 u}{\partial x \partial z} = \mu \frac{\partial R}{\partial x} + R \frac{\partial \mu}{\partial x} = \mu \frac{\partial P}{\partial z} + P \frac{\partial \mu}{\partial z} , \qquad \ldots \quad (8)$$

From (6), (7), (8), we get on re-arranging

$$\mu \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) = Q \frac{\partial \mu}{\partial x} - P \frac{\partial \mu}{\partial y} \qquad \ldots \quad (9)$$

$$\mu \left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} \right) = R \frac{\partial \mu}{\partial y} - Q \frac{\partial \mu}{\partial z} \qquad \ldots \quad (10)$$

$$\mu \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) = P \frac{\partial \mu}{\partial z} - R \frac{\partial \mu}{\partial x} \qquad \ldots \quad (11)$$

Multiplying (9) by R. (10) by P, (11) by Q and adding together, we get the required result.

Note. If $P\ dx + Q\ dy + R\ dz$ be *itself a perfect differential*, then we easily deduce the conditions that

$$\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} = \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} = \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 0.$$

Ex. 4. *If V be a function of x and y, prove that*

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2},$$

where $x = r\cos\theta$, $y = r\sin\theta$. [*C. H. 1953*]

$$\therefore \quad r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\frac{y}{x}.$$

Hence, $\dfrac{\partial x}{\partial r} = \cos\theta$; $\dfrac{\partial r}{\partial x} = \dfrac{x}{\sqrt{x^2 + y^2}} = \dfrac{r\cos\theta}{r} = \cos\theta$.

$\dfrac{\partial y}{\partial r} = \sin\theta$; $\dfrac{\partial r}{\partial y} = \dfrac{y}{\sqrt{x^2 + y^2}} = \dfrac{r\sin\theta}{r} = \sin\theta$.

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + \dfrac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{x^2}{x^2 + y^2}\cdot\frac{y}{x^2} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}.$$

Similarly, $\dfrac{\partial \theta}{\partial y} = \dfrac{\cos\theta}{r}$.

Since V is a function of (x, y), and x and y are functions of r, θ, so V is a function of (r, θ).

Hence, $$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial V}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}.$$

$$= \cos\theta\,.\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\cdot\frac{\partial V}{\partial \theta}. \qquad \ldots (1)$$

$$\frac{\partial V}{\partial y} = \frac{\partial V}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial V}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}$$

$$= \sin\theta\,.\frac{\partial V}{\partial r} + \frac{\cos\theta}{r}\cdot\frac{\partial V}{\partial \theta} \qquad \ldots (2)$$

Thus we have the following *equivalence of Cartesian and polar operators :*

$$\frac{\partial}{\partial x} \equiv \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right).$$

$$\frac{\partial}{\partial y} \equiv \left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \frac{\partial}{\partial \theta} \right);$$

$$\therefore \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) = \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta \frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}\right)$$

$$= \cos\theta \left(\cos\theta \frac{\partial^2 V}{\partial r^2} - \frac{\sin\theta}{r}\frac{\partial^2 V}{\partial r \partial \theta} + \frac{\partial V}{\partial \theta}\frac{\sin\theta}{r^2} \right)$$

$$- \frac{\sin\theta}{r}\left(\cos\theta \frac{\partial^2 V}{\partial \theta \partial r} - \sin\theta \frac{\partial V}{\partial r} - \frac{1}{r}\frac{\partial V}{\partial \theta}\cos\theta - \frac{1}{r}\sin\theta \frac{\partial^2 V}{\partial \theta^2} \right)$$

$$\therefore \frac{\partial^2 V}{\partial x^2} = \cos^2\theta \frac{\partial^2 V}{\partial r^2} - \frac{2\sin\theta \cos\theta}{r}\frac{\partial^2 V}{\partial r \partial \theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\sin^2\theta}{r}\frac{\partial V}{\partial r}$$

$$+ \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial \theta}$$

Similarly,

$$\frac{\partial^2 V}{\partial y^2} = \sin^2\theta \frac{\partial^2 V}{\partial r^2} + \frac{2\sin\theta \cos\theta}{r}\frac{\partial^2 V}{\partial r \partial \theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\cos^2\theta}{r}\frac{\partial V}{\partial r}$$

$$- \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial \theta}.$$

$$\therefore \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial r^2} - \frac{1}{r}\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r}\frac{\partial V}{\partial r}.$$

12.15. Miscellaneous Worked Out Examples

Ex. 1. (*i*) If $u = \sqrt{xy}$, find the value of $\dfrac{\delta^2 u}{\delta x^2} + \dfrac{\delta^2 u}{\delta y^2}$. [*C. P. 1986*]

(*ii*) If $u = x^y$, prove that $\dfrac{\delta^2 u}{\delta x \delta y} = \dfrac{\delta^2 u}{\delta y \delta x}$. [*C. P. 1986, 2001, 2008*]

(*iii*) If $f(x, y) = x^3 y + e^{xy^2}$, show that $\dfrac{\delta^2 f}{\delta x \delta y} = \dfrac{\delta^2 f}{\delta y \delta x}$. [*C. P. 1993*]

Solution : (*i*) Here, $u = x^{\frac{1}{2}} \cdot y^{\frac{1}{2}}$

$$\frac{\delta u}{\delta x} = \frac{1}{2} x^{-\frac{1}{2}} y^{\frac{1}{2}} \text{ and } \frac{\delta^2 u}{\delta x^2} = \frac{1}{2}\left(-\frac{1}{2}\right) x^{-\frac{3}{2}} \cdot y^{\frac{1}{2}} = -\frac{1}{4} x^{-\frac{3}{2}} \cdot y^{\frac{1}{2}}$$

$$\frac{\delta u}{\delta y}=\frac{1}{2}x^{\frac{1}{2}}\cdot y^{-\frac{1}{2}} \text{ and } \frac{\delta^2 u}{\delta y^2}=-\frac{1}{4}x^{\frac{1}{2}}\cdot y^{-\frac{3}{2}}$$

$$\therefore \frac{\delta^2 u}{\delta x^2}+\frac{\delta^2 u}{\delta y^2}=-\frac{1}{4}\left(\frac{\sqrt{y}}{x\sqrt{x}}+\frac{\sqrt{x}}{y\sqrt{y}}\right)=-\frac{1}{4}\left(\frac{x^2+y^2}{xy\sqrt{xy}}\right)$$

(*ii*) $u=x^y$

or, $\dfrac{\delta u}{\delta y}=x^y\cdot\log_e^x$

and, $\dfrac{\delta^2 u}{\delta x\delta y}=y\cdot x^{y-1}\cdot\log_e^x+x^y\cdot\dfrac{1}{x}$

$$=y\cdot x^{y-1}\cdot\log_e^x+x^{y-1} \qquad \dots \quad (1)$$

Again, $\dfrac{\delta u}{\delta x}=y\cdot x^{y-1}$

and $\dfrac{\delta^2 u}{\delta y\delta x}=1\cdot x^{y-1}+y\cdot x^{y-1}\cdot\log_e^x \qquad \dots \quad (2)$

From (1) and (2), $\dfrac{\delta^2 u}{\delta x\delta y}=\dfrac{\delta^2 u}{\delta y\delta x}$

(*iii*) Here, $f(x,\ y)=x^3y+e^{xy^2}$

$\therefore \dfrac{\delta f}{\delta y}=x^3+e^{xy^2}(2xy)$

and $\dfrac{\delta^2 f}{\delta x\delta y}=3x^2+e^{xy^2}\cdot y^2(2xy)+e^{xy^2}(2y)$

$$=3x^2+2ye^{xy^2}\left(xy^2+1\right) \qquad \dots \quad (1)$$

Again, $\dfrac{\delta f}{\delta x}=3x^2y+e^{xy^2}(y^2)$

and $\dfrac{\delta^2 f}{\delta y\delta x}=3x^2+e^{xy^2}(2xy)y^2+e^{xy^2}\cdot 2y=3x^2+2ye^{xy^2}\left(xy^2+1\right)$

$$\dots \quad (2)$$

From (1) and (2) $\dfrac{\delta^2 f}{\delta x\delta y}=\dfrac{\delta^2 f}{\delta y\delta x}$.

Ex. 2. (*i*) If $u=r^3,\ x^2+y^2+z^2=r^2$, then prove that

$$\frac{\delta^2 u}{\delta x^2}+\frac{\delta^2 u}{\delta y^2}+\frac{\delta^2 u}{\delta z^2}=12r \qquad [\text{ C. P. 1991 }]$$

(*ii*) If $u = x^2 \tan^{-1}\dfrac{y}{x} - y^2 \tan^{-1}\dfrac{x}{y}$, prove that

$$\frac{\delta^2 u}{\delta x \delta y} = \frac{x^2 - y^2}{x^2 + y^2}.$$ [*C. P. 1998*]

(*iii*) If $u = x \log y$, show that $u_{xy} = y_{yx}$. [*C. P. 1983, 96, 2003*]

Solution : (*i*) $\because \quad r^2 = x^2 + y^2 + z^2$

$$2r\frac{\delta r}{\delta x} = 2x, \text{ i.e., } \frac{\delta r}{\delta x} = \frac{x}{r}$$

similarly, $\dfrac{\delta r}{\delta y} = \dfrac{y}{r}$ and $\dfrac{\delta r}{\delta z} = \dfrac{z}{r}$

$$\because \quad u = r^3, \quad \frac{\delta u}{\delta x} = 3r^2 \cdot \frac{\delta r}{\delta x} = 3r \cdot x$$

$$\therefore \quad \frac{\delta^2 u}{\delta x^2} = 3r \cdot 1 + x \cdot 3\frac{\delta r}{\delta x} = 3r + \frac{3x^2}{r}$$

similarly, $\dfrac{\delta^2 u}{\delta y^2} = 3r + \dfrac{3y^2}{r}$ and $\dfrac{\delta^2 u}{\delta z^2} = 3r + \dfrac{3z^2}{r}$

$$\therefore \quad \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} + \frac{\delta^2 u}{\delta z^2} = 9r + \frac{3}{r}\left(x^2 + y^2 + z^2\right) = 9r + \frac{3}{r} \cdot r^2 = 12r$$

(*ii*) $u = x^2 \tan^{-1}\dfrac{y}{x} - y^2 \tan^{-1}\dfrac{x}{y}$,

$$\frac{\delta u}{\delta y} = x^2 \cdot \frac{1}{1+\left(\dfrac{y}{x}\right)^2} \cdot \frac{1}{x} - 2y\tan^{-1}\frac{x}{y} - y^2 \times \frac{1}{1+\left(\dfrac{x}{y}\right)^2} \times \left(-\frac{x}{y^2}\right)$$

$$= \frac{x^3}{x^2 + y^2} - 2y\tan^{-1}\frac{x}{y} + \frac{xy^2}{x^2 + y^2} = x - 2y\tan^{-1}\frac{x}{y}$$

$$\therefore \quad \frac{\delta^2 u}{\delta x \delta y} = 1 - 2y \cdot \frac{1}{1+\left(\dfrac{x}{y}\right)^2} \cdot \frac{1}{y} = 1 - \frac{2y^2}{x^2 + y^2} = \frac{x^2 - y^2}{x^2 + y^2}$$

(*iii*) $u = x \log y \quad \therefore \quad u_y = \dfrac{x}{y}$ and $u_{xy} = \dfrac{1}{y}$

Again $u_x = 1 \cdot \log y$ and $u_{yx} = \dfrac{1}{y} \qquad \therefore \quad u_{xy} = u_{yx}$

Ex. 3. (*i*) Show that $f(x, y) = \tan^{-1}\frac{y}{x} + \sin^{-1}\frac{x}{y}$ is a homogeneous function of x, y. Determine the degree of homogeneity. Hence, or otherwise, find the value of $x\frac{\delta f}{\delta x} + y\frac{\delta f}{\delta y}$. [*C. P. 1991, B.P. 1996*]

(*ii*) Examine whether $f(x, y) = x^{-\frac{1}{3}} \cdot y^{\frac{4}{3}} \cdot \tan\frac{y}{x}$ is a homogeneous function of (x, y). If so, find its degree. [*C.P. 1993*]

(*iii*) Examine whether the function $u(x, y) = \frac{x^3 + y^3}{x^2 + y^2}$ is a homogeneous function of x and y. [*B. P. 1995*]

Solution : (*i*) $f(x, y) = \tan^{-1}\frac{y}{x} + \sin^{-1}\frac{x}{y} = \tan^{-1}\frac{y}{x} + \operatorname{cosec}^{-1}\frac{y}{x}$

$$= x^0\left\{\tan^{-1}\left(\frac{y}{x}\right) + \operatorname{cosec}^{-1}\left(\frac{y}{x}\right)\right\} = x^0\phi\left(\frac{y}{x}\right),$$

Hence, $f(x, y)$ is a homogeneous function of x, y of degree 0.
By Euler's theorem on homogeneous functions,

$$x\frac{\delta f}{\delta x} + y\frac{\delta f}{\delta y} = 0 \times f = 0. \qquad [\because \text{ here } n = 0].$$

(*ii*) $f(x, y) = x^{-\frac{1}{3}} \cdot y^{\frac{4}{3}} \cdot \tan\frac{y}{x} = x^{\frac{4}{3}} \cdot x^{-\frac{1}{3}} \cdot \frac{y^{\frac{4}{3}}}{x^{\frac{4}{3}}}\tan\left(\frac{y}{x}\right)$

$$= x^1 \cdot \left(\frac{y}{x}\right)^{\frac{4}{3}} \cdot \tan\left(\frac{y}{x}\right) = x^1 \cdot \phi\left(\frac{y}{x}\right)$$

Hence, $f(x, y)$ is a homogeneous function of x, y of degree 1.

$$(iii)\ u(x, y) = \frac{x^3\left(1 + \frac{y^3}{x^3}\right)}{x^2\left(1 + \frac{y^2}{x^2}\right)} = x \times \left\{\frac{1 + \left(\frac{y}{x}\right)^3}{1 + \left(\frac{y}{x}\right)^2}\right\} = x^1 \times \phi\left(\frac{y}{x}\right).$$

So, $u(x, y)$ is a homogeneous function of x, y of degree 1.

Ex. 4. (*i*) If $u = x\sin^{-1}\left(\frac{y}{x}\right) + y\tan^{-1}\left(\frac{x}{y}\right)$ find the value of $x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y}$ at $(1, 1)$. [*C. P. 1988*] (*ii*) If $u = x \cdot \phi\left(\frac{y}{x}\right) + \psi(x)$, prove that $x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = x\phi\left(\frac{y}{x}\right)$.

[*C. P. 1988*]

(*iii*) If $u = \frac{x^2 + y^2}{\sqrt{x+y}}$, $(x, y) \neq (0, 0)$ and $x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = k\,u$ find the value of k .

[*C. P. 1997, 2002, 2004*]

Solution : (*i*) $u = x\sin^{-1}\left(\frac{y}{x}\right) + y\tan^{-1}\left(\frac{x}{y}\right)$

$$= x\left\{\sin^{-1}\left(\frac{y}{x}\right) + \frac{y}{x}\cot^{-1}\left(\frac{y}{x}\right)\right\} = x^1 \cdot \phi\left(\frac{y}{x}\right)$$

So, u is a homogeneous function of x and y of degree 1.

By Euler's theorem, $x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = 1 \cdot u$

$$= x\sin^{-1}\left(\frac{y}{x}\right) + y\tan^{-1}\left(\frac{x}{y}\right)$$

At (1, 1), $x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = 1.\sin^{-1}(1) + 1.\tan^{-1}(1) = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$.

(*ii*) Let $v = x \cdot \phi\left(\frac{y}{x}\right)$ and $w = \psi\left(\frac{y}{x}\right)$

Then v is a homogeneous function of x, y of degree 1, and w is a homogeneous function of x, y of degree 0.

So, $x\frac{\delta v}{\delta x} + y\frac{\delta v}{\delta y} = 1 \cdot v = v$ ... ... (1)

and $x\frac{\delta w}{\delta x} + y\frac{\delta w}{\delta y} = 0 \cdot w = 0$ ... ... (2)

$\because$ $u = v + w$, $\frac{\delta u}{\delta x} = \frac{\delta v}{\delta x} + \frac{\delta w}{\delta x}$

or, $x\frac{\delta u}{\delta x} = x\frac{\delta v}{\delta x} + x\frac{\delta w}{\delta x}$

similarly, $y\frac{\delta u}{\delta y} = y\frac{\delta v}{\delta y} + y\frac{\delta w}{\delta y}$

$$\therefore \quad x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = \left(x\frac{\delta v}{\delta x} + y\frac{\delta v}{\delta y}\right) + \left(x\frac{\delta w}{\delta x} + y\frac{\delta w}{\delta y}\right) = v + 0 = v = x \cdot \phi\left(\frac{y}{x}\right)$$

$$(iii)\ u = \frac{x^2 + y^2}{\sqrt{x+y}} = \frac{x^2\left\{1+\left(\frac{y}{x}\right)^2\right\}}{x^{\frac{1}{2}}\left\{\sqrt{1+\left(\frac{y}{x}\right)}\right\}} = x^{\frac{3}{2}} \cdot \phi\left(\frac{y}{x}\right)$$

So, u is a homogeneous function of x and y of degree $\frac{3}{2}$. Hence, by Euler's Theorem,

$$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = \frac{3}{2}u \qquad \ldots (1)$$

But given that, $x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = k\,u \qquad \ldots (2)$

From (1) and (2), $k = \frac{3}{2}$.

Ex. 5. (*i*) If $V = 2\cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that

$$x\frac{\delta V}{\delta x} + y\frac{\delta V}{\delta y} + \cot\frac{V}{2} = 0.$$ [*C. P. 1994, 2008*]

(*ii*) If $u = xy\,f\left(\frac{y}{x}\right)$, prove that $x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = 2\,u$

by Euler's theorem,. [*C. P. 1984*]

(*iii*) Verify that $x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} + z\frac{\delta u}{\delta z} = 0$, if $u = \frac{y}{z} + \frac{z}{x} + \frac{x}{y}$.

[*C. P. 1987, 2005*]

(*iv*) If $V = \log\frac{x^3+y^3}{x^2+y^2}$, show that $x\frac{\delta V}{\delta x} + y\frac{\delta V}{\delta y} = 1$. [*C. P. 1978*]

(*v*) Find that value of, $x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y}$, if $u = \frac{x^2+y^2}{x^2-xy}$. [*C. P. 1990*]

Solution : (*i*) $V = 2\cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$

$$\text{or, } \cos\frac{V}{2} = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x\left\{1+\left(\frac{y}{x}\right)\right\}}{x^{\frac{1}{2}}\left\{1+\sqrt{\frac{y}{x}}\right\}} = x^{\frac{1}{2}}\phi\left(\frac{y}{x}\right)$$

$\therefore \quad \cos\left(\dfrac{V}{2}\right)$ is a homogeneous function of x, y of degree $\dfrac{1}{2}$.

Hence by Euler's theorem,

$$x\frac{\delta}{\delta x}\left(\cos\frac{V}{2}\right)+y\frac{\delta}{\delta y}\left(\cos\frac{V}{2}\right)=\frac{1}{2}\cos\frac{V}{2}$$

or, $x\left(-\dfrac{1}{2}\sin\dfrac{V}{2}\right)\dfrac{\delta V}{\delta x}+y\left(-\dfrac{1}{2}\sin\dfrac{V}{2}\right)\dfrac{\delta V}{\delta y}=\dfrac{1}{2}\cos\dfrac{V}{2}$

or, $x\dfrac{\delta V}{\delta x}+y\dfrac{\delta V}{\delta y}+\cot\dfrac{V}{2}=0$

(*ii*) $u=xy\,f\left(\dfrac{y}{x}\right)=x^2\cdot\left(\dfrac{y}{x}\right)\cdot f\left(\dfrac{y}{x}\right)=x^2\cdot\phi\left(\dfrac{y}{x}\right)$

Hence, u is a homogeneous function of x, y of degree 2.
By Euler's theorem

$$x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}=2\,u$$

(*iii*) $u=\dfrac{y}{z}+\dfrac{z}{x}+\dfrac{x}{y}=\dfrac{xy^2+yz^2+zx^2}{xyz}$

$$=\frac{x^3\left\{\left(\dfrac{y}{x}\right)^2+\dfrac{y}{x}\left(\dfrac{z}{x}\right)^2+\dfrac{z}{x}\right\}}{x^3\left\{\dfrac{z}{x}\cdot\dfrac{y}{x}\right\}}=x^0\;\phi\left(\frac{y}{x},\;\frac{z}{x}\right).$$

Hence, u is a homogeneous function of x, y, z of degree 0; by Euler's theorem,

$$x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}+z\frac{\delta u}{\delta z}=0\cdot u=0$$

(*iv*) $V=\log\dfrac{x^3+y^3}{x^2+y^2}$

or, $e^V=\dfrac{x^3+y^3}{x^2+y^2}=\dfrac{x^3\left\{1+\left(\dfrac{y}{x}\right)^3\right\}}{x^2\left\{1+\left(\dfrac{y}{x}\right)^2\right\}}=x\cdot\phi\left(\dfrac{y}{x}\right)$

Thus e^V is a homogeneous function of x, y of degree 1.

$$\therefore \quad x\frac{\delta}{\delta x}\left(e^V\right)+y\frac{\delta}{\delta y}\left(e^V\right)=1\cdot e^V$$

$$\text{or, } x\cdot e^V\cdot\frac{\delta V}{\delta x}+y\cdot e^V\cdot\frac{\delta V}{\delta y}=e^V$$

$$\therefore \quad x\frac{\delta V}{\delta x}+y\frac{\delta V}{\delta y}=1$$

(*v*) Here, $u=\dfrac{x^2\left\{1+\left(\dfrac{y}{x}\right)^2\right\}}{x^2\left\{1-\left(\dfrac{y}{x}\right)\right\}}=x^0\,\phi\left(\dfrac{y}{x}\right)$

So, u is a homogeneous function of $(x,\ y)$ of deg :e 0

By Euler's theorem, $x\dfrac{\delta u}{\delta x}+y\dfrac{\delta u}{\delta y}=0\cdot u=0$.

Ex. 6. (*i*) If u be a homogeneous function of x and y of degree n having continuous partial derivatives, then prove that

$$\left(x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}\right)^2 u=n(\ \ -1)u.$$

where, $\left(x\dfrac{\delta u}{\delta x}+y\dfrac{\delta u}{\delta y}\right)^2=x^2\dfrac{\delta^2 u}{\delta x^2}+2\ y\dfrac{\delta^{\ }}{\delta x\delta y}-+y^2\dfrac{\delta^2 u}{\delta y^2}$.

[*C. P. 1985, '98, '2001; B. P. 1993*]

(*ii*) If $u=\dfrac{x^2y^2}{x+y}$, apply Euler's theorem to find the value of $x\dfrac{\delta u}{\delta x}+y\dfrac{\delta u}{\delta y}$ and hence deduce that $x^2\dfrac{\delta^2 u}{\delta x^2}+2xy\dfrac{\delta^2 u}{\delta x\delta y}+y^2\dfrac{\delta^2 u}{\delta y^2}=6u$.

[*C. P. 1993, 2004*]

(*iii*) If $u=x\,\phi\left(\dfrac{y}{x}\right)+\psi\left(\dfrac{y}{x}\right)$, prove that $\left(x\dfrac{\delta}{\delta x}+y\dfrac{\delta}{\delta y}\right)^2 u=0$.

[*C. P. 1990*]

(*iv*) If $f(x,\ y)=\left(x^2+y^2\right)^{\frac{1}{3}}$, use Euler's theorem to find the value of $x\dfrac{\delta f}{\delta x}+y\dfrac{\delta f}{\delta y}$ and hence prove that $x^2\dfrac{\delta^2 f}{\delta x^2}+2xy\dfrac{\delta^2 f}{\delta x\delta y}+y^2\dfrac{\delta^2 f}{\delta y^2}+\dfrac{2}{9}f=0$.
[*C. P. 1995*]

(*v*) If $u=\tan^{-1}\dfrac{x^3+y^3}{x-y}$, prove that

(*a*) $x\dfrac{\delta u}{\delta x}+y\dfrac{\delta u}{\delta y}=\sin 2u$. [*B. P. 1995, C. P. 1984, 89, 96, 98*]

(*b*) $x^2\dfrac{\delta^2 u}{\delta x^2}+2xy\dfrac{\delta^2 u}{\delta x\delta y}+y^2\dfrac{\delta^2 u}{\delta y^2}=\left(1-4\sin^2 u\right)\sin 2u$.

[*C. P. 1996, '98, 2003, B. P. '95*]

Solution : (*i*) Since u is a homogeneous function of degree n, we have by Euler's theorem,

$$x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}=nu \qquad \ldots \quad \ldots \quad (1)$$

Differentiating (1) *w.r.t.* x partially,

$$\frac{\delta u}{\delta x}+x\frac{\delta^2 u}{\delta x^2}+y\frac{\delta^2 u}{\delta x\delta y}=n\cdot\frac{\delta u}{\delta x} \qquad \ldots \quad \ldots \quad (2)$$

Differentiating (1) *w.r.t.* y partially,

$$x\frac{\delta^2 u}{\delta y\delta x}+\frac{\delta u}{\delta y}+y\frac{\delta^2 u}{\delta y^2}=n\cdot\frac{\delta u}{\delta y} \qquad \ldots \quad \ldots \quad (3)$$

Multiplying (2) by x and (3) by y and adding

$$\left(x^2\frac{\delta^2 u}{\delta x^2}+2xy\frac{\delta^2 u}{\delta x\delta y}+y^2\frac{\delta^2 u}{\delta y^2}\right)+\left(x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}\right)=n\left(x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}\right),$$

where we have taken $\dfrac{\delta^2 u}{\delta x\delta y}=\dfrac{\delta^2 u}{\delta y\delta x}$, since the partial derivatives ar continuous.

$$\text{or, } x^2\frac{\delta^2 u}{\delta x^2}+2xy\frac{\delta u^2}{\delta x\delta y}+y^2\frac{\delta^2 u}{\delta y^2}+nu=n\cdot nu$$

$$\text{or, } \left(x\frac{\delta}{\delta x}+y\frac{\delta}{\delta y}\right)^2 u=n(n-1)u$$

(*ii*) $u = \dfrac{x^2y^2}{x+y} = \dfrac{x^4\left(\dfrac{y}{x}\right)^2}{x\left(1+\dfrac{y}{x}\right)} = x^3\phi\left(\dfrac{y}{x}\right)$

So, u is a homogeneous function of x and y of degree 3.

By Euler's theorem,

$$x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}=3u \quad \text{... ... (1)}$$

Differentiating (1) *w.r.t.* x partially,

$$x\frac{\delta^2 u}{\delta x^2}+1\cdot\frac{\delta u}{\delta x}+y\frac{\delta^2 u}{\delta x\delta y}=3\frac{\delta u}{\delta x}$$

$$\text{or, } x^2\frac{\delta^2 u}{\delta x^2}+xy\frac{\delta^2 u}{\delta x\delta y}+x\frac{\delta u}{\delta x}=3x\frac{\delta u}{\delta x} \quad \text{... ... (2)}$$

Differentiating (1) *w.r.t.* y partially,

$$x\frac{\delta^2 u}{\delta x\delta y}+y\frac{\delta^2 u}{\delta y^2}+\frac{\delta u}{\delta y}=3\frac{\delta u}{\delta y}$$

$$\text{or, } xy\frac{\delta^2 u}{\delta x\delta y}+y^2\frac{\delta^2 u}{\delta y^2}+y\frac{\delta u}{\delta y}=3y\frac{\delta u}{\delta y} \quad \text{... ... (3)}$$

$$\left(\because \ \frac{\delta^2 u}{\delta y\delta x}=\frac{\delta^2 u}{\delta x\delta y}\right),$$

Adding (2) and (3)

$$x^2\frac{\delta^2 u}{\delta x^2}+2xy\frac{\delta^2 u}{\delta x\delta y}+y^2\frac{\delta^2 u}{\delta y^2}=2\left(x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}\right)$$

$$=2\times 3u=6u.$$

(*iii*) $u = x\phi\left(\dfrac{y}{x}\right)+\psi\left(\dfrac{y}{x}\right)$

$= V + W$

where, $V = x\phi\left(\dfrac{y}{x}\right)$, a homogeneous function of x and y of degree 1,

and $W = \psi\left(\dfrac{y}{x}\right)$, a homogeneous function of x and y of degree 0.

By Euler's theorem,

$$x\frac{\delta V}{\delta x}+y\frac{\delta V}{\delta y}=1\cdot V=V$$

and

$$x\frac{\delta W}{\delta x}+y\frac{\delta W}{\delta y}=0\cdot W=0$$

$$x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}=\left(x\frac{\delta V}{\delta x}+y\frac{\delta V}{\delta y}\right)+\left(x\frac{\delta W}{\delta x}+y\frac{\delta W}{\delta y}\right)$$

$$=V+0=V. \qquad \dots \qquad \dots \qquad (1)$$

Differentiating (1) *w.r.t.* x. partially and then multiplying both the sides by x, we get

$$x^2\frac{\delta^2 u}{\delta x^2}+xy\frac{\delta^2 u}{\delta x\delta y}+x\frac{\delta u}{\delta x}=x\frac{\delta V}{\delta x} \qquad \dots \qquad \dots \qquad (2)$$

Similarly, $xy\dfrac{\delta^2 u}{\delta x\delta y}+y^2\dfrac{\delta^2 u}{\delta y^2}+y\dfrac{\delta u}{\delta y}=y\dfrac{\delta V}{\delta y}$ ... ... (3)

$$\because \quad \frac{\delta^2 u}{\delta y\delta x}=\frac{\delta^2 u}{\delta x\delta y}$$

Adding (2) and (3)

$$x^2\frac{\delta^2 u}{\delta x^2}+2xy\frac{\delta^2 u}{\delta x\delta y}+y^2\frac{\delta^2 u}{\delta y^2}+\left(x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}\right)$$

$$=x\frac{\delta V}{\delta x}+y\frac{\delta V}{\delta y}$$

or, $x^2\dfrac{\delta^2 u}{\delta x^2}+2xy\dfrac{\delta^2 u}{\delta x\delta y}+y^2\dfrac{\delta^2 u}{\delta y^2}=0$ [by (1)]

(*iv*) $f(x, y)=\left(x^2+y^2\right)^{\frac{1}{3}}=x^{\frac{2}{3}}\left\{1+\left(\dfrac{y}{x}\right)^2\right\}^{\frac{1}{3}}=x^{\frac{2}{3}}\phi\left(\dfrac{y}{x}\right)$

Thus $f(x, y)$ is a homogeneous function of x and y of degree $\frac{2}{3}$.
By Euler's theorem,

$$x\frac{\delta f}{\delta x}+y\frac{\delta f}{\delta y}=\frac{2}{3}f \qquad \dots \qquad \dots \qquad (1)$$

Differentiating *w.r.t.* x partially

$$x\frac{\delta^2 f}{\delta x^2}+\frac{\delta f}{\delta x}+y\frac{\delta^2 f}{\delta x\delta y}=\frac{2}{3}\frac{\delta f}{\delta x}.$$

$$\text{or, } x^2\frac{\delta^2 f}{\delta x^2}+xy\frac{\delta^2 f}{\delta x\delta y}+x\frac{\delta f}{\delta x}=\frac{2}{3}x\frac{\delta f}{\delta x} \quad \ldots \quad \ldots \quad (2)$$

$$\text{Similarly, } xy\frac{\delta^2 f}{\delta x\delta y}+y^2\frac{\delta^2 f}{\delta y^2}+y\frac{\delta f}{\delta y}=\frac{2}{3}y\frac{\delta f}{\delta y} \quad \ldots \quad \ldots \quad (3)$$

Adding (2) and (3),

$$\text{or, } x^2\frac{\delta^2 f}{\delta x^2}+2xy\frac{\delta^2 f}{\delta x\delta y}+y^2\frac{\delta^2 f}{\delta y^2}+\left(x\frac{\delta f}{\delta x}+y\frac{\delta f}{\delta y}\right)$$

$$=\frac{2}{3}\left(x\frac{\delta f}{\delta x}+y\frac{\delta f}{\delta y}\right)$$

$$x^2\frac{\delta^2 f}{\delta x^2}+2xy\frac{\delta^2 f}{\delta x\delta y}+y^2\frac{\delta^2 f}{\delta y^2}+\frac{2}{3}f=\frac{2}{3}\cdot\frac{2}{3}f$$

$$\text{i.e., } x^2\frac{\delta^2 f}{\delta x^2}+2xy\frac{\delta^2 f}{\delta x\delta y}+y^2\frac{\delta^2 f}{\delta y^2}=\frac{4}{9}f-\frac{2}{3}f$$

$$\therefore \quad x^2\frac{\delta^2 f}{\delta x^2}+2xy\frac{\delta^2 f}{\delta x\delta y}+y^2\frac{\delta^2 f}{\delta y^2}+\frac{2}{9}f=0$$

(*v*) $u=\tan^{-1}\dfrac{x^3+y^3}{x-y}$

$$\text{or, } \tan u=\frac{x^3+y^3}{x-y}=\frac{x^3\left\{1+\left(\frac{y}{x}\right)\right\}^3}{x\left\{1-\left(\frac{y}{x}\right)\right\}}=x^2\phi\left(\frac{y}{x}\right)$$

So, $\tan u$ is a homogeneous function of x and y of degree 2.

By Euler's theorem,

$$x\cdot\frac{\delta}{\delta x}(\tan u)+y\frac{\delta}{\delta y}(\tan u)=2\tan u$$

$$\text{or, } x\sec^2 u\frac{\delta u}{\delta x}+y\sec^2 u\frac{\delta u}{\delta y}=2\tan u$$

$$\therefore \quad x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}=2\frac{\sin u}{\cos u}\cdot\cos^2 u=2\sin u\cos u=\sin 2u \qquad \cdots \qquad (1)$$

Differentiating (1) *w.r.t.* x., partially,

$$x\frac{\delta^2 u}{\delta x^2}+1\cdot\frac{\delta u}{\delta y}+y\frac{\delta^2 u}{\delta x\delta y}=2\cos 2u\cdot\frac{\delta u}{\delta x}$$

$$\text{or, } x^2\frac{\delta^2 u}{\delta x^2}+xy\frac{\delta^2 u}{\delta x\delta y}+x\frac{\delta u}{\delta x}=2x\cdot\cos 2u\frac{\delta u}{\delta x} \qquad \cdots \qquad (2)$$

Differentiating (1) *w.r.t.* y., partially and then multiplying by y

$$xy\frac{\delta^2 u}{\delta x\delta y}+y^2\frac{\delta^2 u}{\delta y^2}+y\frac{\delta u}{\delta y}=2y\cdot\cos u\frac{\delta u}{\delta y} \qquad \cdots \qquad (3)$$

$$\because \quad \frac{\delta^2 u}{\delta y\delta x}=\frac{\delta^2 u}{\delta x\delta y}$$

Adding (2) and (3),

$$x^2\frac{\delta^2 u}{\delta x^2}+2xy\frac{\delta^2 u}{\delta x\delta y}+y^2\frac{\delta^2 u}{\delta y^2}=(2\cos 2u-1)\left(x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}\right)$$

$$=(2\cos 2u-1)\sin 2u \qquad [\text{ From (1) }]$$

$$=\sin 2u\left\{2\left(1-2\sin^2 u\right)-1\right\}$$

$$=\sin 2u\left(1-4\sin^2 u\right).$$

Ex. 7. If $f(x,\ y)=\dfrac{\left(x^2+y^2\right)^n}{2n(2n-1)}+x\phi\left(\dfrac{y}{x}\right)+\psi\left(\dfrac{y}{x}\right)$, then using Euler's theorem on homogeneous functions, show that

$$x^2\frac{\delta^2 f}{\delta x^2}+2xy\frac{\delta^2 f}{\delta x\delta y}+y^2\frac{\delta^2 f}{\delta y^2}=\left(x^2+y^2\right)^n. \qquad [\textit{ B. P. 1999 }]$$

Solution : Let $f(x,\ y)=u(x,\ y)+v(x,\ y)+w(x,\ y)$

where, $u(x.\ y)=\dfrac{\left(x^2+y^2\right)^n}{2n(2n-1)}=\dfrac{x^{2n}\left\{1+\left(\dfrac{y}{x}\right)^2\right\}^n}{2n(2n-1)}=\dfrac{x^{2n}F\left(\dfrac{y}{x}\right)}{2n(2n-1)}$,

a homogeneous function of x and y of degree $2n$,

$v(x, y) = x\,\phi\left(\frac{y}{x}\right)$, a homogeneous function of x and y of degree 1,

and $w(x, y) = \psi\left(\frac{y}{x}\right)$, a homogeneous function of x and y of degree 0.

We know, if $f(x, y)$ is a homogeneous function of x and y of degree n then

$$x^2\frac{\delta^2 f}{\delta x^2} + 2xy\frac{\delta^2 f}{\delta x\delta y} + y^2\frac{\delta^2 f}{\delta y^2} = n(n-1)f$$

Using this general result to $u(x, y)$, $v(x, y)$, $w(x, y)$,

$$x^2\frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta^2 u}{\delta x\delta y} + y^2\frac{\delta^2 u}{\delta y^2} = 2n(2n-1)u \quad \cdots \quad \cdots \quad (1)$$

$$x^2\frac{\delta^2 v}{\delta x^2} + 2xy\frac{\delta^2 v}{\delta x\delta y} + y^2\frac{\delta^2 v}{\delta y^2} = 1\cdot\,(1-1)v = 0 \quad \cdots \quad \cdots \quad (2)$$

$$x^2\frac{\delta^2 w}{\delta x^2} + 2xy\frac{\delta^2 w}{\delta x\delta y} + y^2\frac{\delta^2 w}{\delta y^2} = 0(0-1)w = 0 \quad \cdots \quad \cdots \quad (3)$$

Adding (1), (2), (3),

$$\left(x^2\frac{\delta^2}{\delta x^2} + 2xy\frac{\delta^2}{\delta x\delta y} + y^2\frac{\delta^2}{\delta y^2}\right)(u+v+w) = 2n(2n-1)u$$

$$\text{or, } x^2\frac{\delta^2 f}{\delta x^2} + 2xy\frac{\delta^2 f}{\delta x\delta y} + \cdot\cdot 2\frac{\delta^2 f}{\delta y^2} = 2n(2n-1)\times\frac{\left(x^2+y^2\right)^n}{2n(2n-1)}$$

$$= \left(x^2+y^2\right)^n.$$

Ex. 8. (*i*) If $z = f(x, y)$ anα $x = (u+v)^2$, $y = (u-v)^2$ then prove that $u\frac{\delta z}{\delta u} + v\frac{\delta z}{\delta v} = 2\left(x\frac{\delta z}{\delta x} + y\frac{\delta z}{\delta y}\right)$. [*C. P. 1992*]

(*ii*) If V is a function of$(x, y)$and $x = e^u\cdot\cos t$, $y = e^u\cdot\sin t$, show that

$$\left(\frac{\delta V}{\delta u}\right)^2 + \left(\frac{\delta V}{\delta t}\right)^2 = \left(x^2+y^2\right)\left\{\left(\frac{\delta V}{\delta x}\right)^2 + \left(\frac{\delta V}{\delta y}\right)^2\right\}.$$

[*C. P. 1993, 2003, 2008*]

(*iii*) If z is a function of x, y and $x = e^{u} + e^{-v}$ and $y = e^{-u} - e^{v}$, prove that $\frac{\delta z}{\delta u} - \frac{\delta z}{\delta v} = x\frac{\delta z}{\delta x} - y\frac{\delta z}{\delta y}$. [*C. P. 1997, 2000*]

Solution : (*i*) $$\frac{\delta z}{\delta u} = \frac{\delta z}{\delta x}\cdot\frac{\delta x}{\delta u} + \frac{\delta z}{\delta y}\cdot\frac{\delta y}{\delta u}$$

$$= 2(u+v)\frac{\delta z}{\delta x} + 2(u-v)\frac{\delta z}{\delta y} \qquad \cdots \quad \cdots \quad (1)$$

$$\frac{\delta z}{\delta v} = \frac{\delta z}{\delta x}\cdot\frac{\delta x}{\delta v} + \frac{\delta z}{\delta y}\cdot\frac{\delta y}{\delta v}$$

$$= 2(u+v)\frac{\delta z}{\delta x} - 2(u-v)\frac{\delta z}{\delta y} \qquad \cdots \quad \cdots \quad (2)$$

Adding (1) and (2),

$$u\cdot\frac{\delta z}{\delta u} + v\frac{\delta z}{\delta v} = 2\left\{u^2 + uv + uv + v^2\right\}\frac{\delta z}{\delta x} + 2\left\{u^2 - uv - uv + v^2\right\}\frac{\delta z}{\delta y}$$

$$= 2(u+v)^2\frac{dz}{dx} + 2(u-v)^2\frac{dz}{dy}$$

$$= 2\left\{x\frac{dz}{dx} + y\frac{dz}{dy}\right\}$$

Let $V = f(x, y)$. $x = e^{u}$ cost, $y = e^{u}\sin t$

$$\frac{\delta V}{\delta u} = \frac{\delta V}{\delta x}\cdot\frac{\delta x}{\delta u} + \frac{\delta V}{\delta y}\cdot\frac{\delta y}{\delta u}$$

$$= \frac{\delta V}{\delta x}\cdot e^{u}\cos t + \frac{\delta V}{\delta y}\cdot e^{u}\sin t$$

$$= x\frac{\delta V}{\delta x} + y\frac{\delta V}{\delta y} \qquad \cdots \quad \cdots \quad (1)$$

$$\frac{\delta V}{\delta t} = \frac{\delta V}{\delta x}\cdot\frac{\delta x}{\delta y} + \frac{\delta V}{\delta y}\cdot\frac{\delta y}{\delta t}$$

$$= \frac{\delta V}{\delta x}\left(-e^{u}\sin t\right) + \frac{\delta V}{\delta y}\cdot e^{u}\cos t$$

$$= -y\frac{\delta V}{\delta x} + x\frac{\delta V}{\delta v} \qquad \cdots \quad \cdots \quad (2)$$

$$\therefore\quad \left(\frac{\delta V}{\delta u}\right)^2+\left(\frac{\delta V}{\delta t}\right)^2=\left(x\frac{\delta V}{\delta x}+y\frac{\delta V}{\delta y}\right)^2+\left(x\frac{\delta V}{\delta y}-y\frac{\delta V}{\delta x}\right)^2$$

$$=\left(x^2+y^2\right)\left\{\left(\frac{\delta V}{\delta x}\right)^2+\left(\frac{\delta V}{\delta y}\right)^2\right\}.$$

(*iii*) Let, $z=f(x,\ y)$, where, $x=e^u+e^{-v},\ y=e^{-u}-e^v$

$$\frac{\delta z}{\delta u}=\frac{\delta z}{\delta x}\cdot\frac{\delta x}{\delta u}+\frac{\delta z}{\delta y}\cdot\frac{\delta y}{\delta u}=e^u\frac{\delta z}{\delta x}-e^{-u}\frac{\delta z}{\delta y}\qquad\ldots\quad\ldots\quad(1)$$

$$\frac{\delta z}{\delta v}=\frac{\delta z}{\delta x}\cdot\frac{\delta x}{\delta v}+\frac{\delta z}{\delta y}\cdot\frac{\delta y}{\delta v}=-e^{-v}\frac{\delta z}{\delta y}-e^v\frac{\delta z}{\delta y}\qquad\ldots\quad\ldots\quad(2)$$

From (1) and (2),

$$\frac{\delta z}{\delta u}-\frac{\delta z}{\delta v}=\left(e^u+e^{-v}\right)\frac{\delta z}{\delta x}+\left(e^v-e^{-u}\right)\frac{\delta z}{\delta y}$$

$$=x\frac{\delta z}{\delta x}-y\frac{\delta z}{\delta y}.$$

Ex. 9. (*i*) If $z=e^{xy^2},\ \ x=t\cos t,\ y=t\sin t$, obtain $\dfrac{dz}{dt}$ at $t=\dfrac{\pi}{2}$.

[*C. P. 2001*]

(*ii*) If $u=f(x,\ y)$ and $x=r\cos\theta,\ y=r\sin\theta$, prove that

$$\left(\frac{\delta u}{\delta x}\right)^2+\left(\frac{\delta u}{\delta y}\right)^2=\left(\frac{\delta u}{\delta r}\right)^2+\frac{1}{r^2}\left(\frac{\delta u}{\delta\theta}\right)^2.$$

[*C. P. 1990, 2001, 2008*

Solution : (*i*) $\because\quad z=e^{xy^2}$ and $x=t\cos t,\ \ y=t\sin t$,

$$\frac{\delta z}{\delta t}=\frac{\delta z}{\delta x}\cdot\frac{dx}{dt}+\frac{\delta z}{\delta y}\cdot\frac{dy}{dt}$$

$$=e^{xy^2}\cdot y^2(\cos t-t\sin t)+e^{xy^2}\cdot 2xy(\sin t+t\cos t)$$

$$=ye^{xy^2}\left\{y(\cos t-t\sin t)+2x(\sin t+t\cos t)\right\}\cdot$$

when $t=\dfrac{\pi}{2}$, $\quad x=\dfrac{\pi}{2}\cdot\cos\dfrac{\pi}{2}=0$ and $y=\dfrac{\pi}{2}\sin\dfrac{\pi}{2}=\dfrac{\pi}{2}$.

$\therefore\ \ e^{xy^2}=e^0=1$

Hence, at $t = \frac{\pi}{2}$

$$\frac{\delta z}{\delta t} = 1 \cdot \frac{\pi}{2}\left\{\frac{\pi}{2}\left(0 - \frac{\pi}{2}\right) + 0\right\} = -\frac{\pi^3}{8}$$

(*ii*) Here, $u = f(x, \; y)$ where, $x = r\cos\theta, \;\; y = r\sin\theta$

$$\frac{\delta u}{\delta r} = \frac{\delta u}{\delta x}\cdot\frac{\delta x}{\delta r} + \frac{\delta u}{\delta y}\cdot\frac{\delta y}{\delta r} = \frac{\delta u}{\delta x}\cdot\cos\theta + \frac{\delta u}{\delta y}\cdot\sin\theta \quad \cdots \quad \cdots \quad (1)$$

$$\frac{\delta u}{\delta \theta} = \frac{\delta u}{\delta x}\cdot\frac{\delta x}{\delta \theta} + \frac{\delta u}{\delta y}\cdot\frac{\delta y}{d\theta} = \frac{\delta u}{\delta x}(-r\sin\theta) + \frac{\delta u}{\delta y}\cdot r\cos\theta$$

$$\text{or, } \frac{1}{r}\cdot\frac{\delta u}{\delta \theta} = -\frac{\delta u}{\delta x}\cdot\sin\theta + \frac{\delta u}{\delta y}\cdot\cos\theta \quad \cdots \quad \cdots \quad (2)$$

From (1) and (2),

$$\left(\frac{\delta u}{\delta r}\right)^2 + \frac{1}{r^2}\left(\frac{\delta u}{\delta \theta}\right)^2 = \left(\frac{\delta u}{\delta x}\cos\theta + \frac{\delta u}{\delta y}\sin\theta\right)^2 + \left(\frac{\delta u}{\delta y}\cos\theta - \frac{\delta u}{\delta x}\sin\theta\right)^2$$

$$= \left\{\left(\frac{\delta u}{\delta x}\right)^2 + \left(\frac{\delta u}{\delta y}\right)^2\right\}^2 \left(\cos^2\theta + \sin^2\theta\right)$$

$$\therefore \quad \left(\frac{\delta u}{\delta x}\right)^2 + \left(\frac{\delta u}{\delta y}\right)^2 = \left(\frac{\delta u}{\delta r}\right)^2 + \frac{1}{r^2}\left(\frac{\delta u}{\delta \theta}\right)^2.$$

Ex. 10. (*i*) Show that $\lim\limits_{(x, \; y) \to (0, \; 0)} \frac{2xy}{x^2 + y^2}$ does not exist.

[*B. P. 2001*]

(*ii*) If $f(x, \; y) = \frac{x^2 y^2}{x^2 + y^2}, \qquad (x, \; y) \neq 0$

$\qquad = 0, \qquad x^2 + y^2 = 0$

Show that $f_{xy}(0, \; 0) = f_{yx} \; (0, \; 0)$. [*C. P. 1995,2006, B. P. 1999*]

(*iii*) If $f_{xy}(x, \; y) = xy \; \frac{x^2 - y^2}{x^2 + y^2}$, when both $x, \;\; y \neq 0$

$f(0, \;\; 0) = 0$.

show that $f_{xy} \; (0, \;\; 0) \neq f_{yx}(0, \; 0)$. [*C. P. 1988, 1999, 2004, 2007*]

Solution : (*i*) $f(x, y)=\dfrac{2xy}{x^2+y^2}$

Let, $y = mx$

then $f(x, y)=\dfrac{2x\cdot mx}{x^2+m^2x^2}=\dfrac{2m}{1+m^2}$

Thus $\lim\limits_{(x, y)\to(0, 0)} f(x, y)$ will assume different values for different modes of aproach of (x, y) towards the origin.

Hence, $\lim\limits_{(x, y)\to(0, 0)} f(x, y)$ does not exist.

(*ii*) Here, $f(x, y)=\dfrac{x^2y^2}{x^2+y^2}$, when $x^2+y^2\neq 0$

$=0$, when $x^2+y^2=0$

$$f_y(h, 0)=\lim_{k\to 0}\frac{f(h, k)-f(h, 0)}{k}$$

$$=\lim_{k\to 0}\frac{\dfrac{h^2k^2}{h^2+k^2}-0}{k}$$

$$=\lim_{k\to 0}\frac{h^2k}{h^2+k^2}=0$$

$$f_y(0, 0)=\lim_{k\to 0}\frac{f(0+k)-f(0, 0)}{k}$$

$$=\lim_{k\to 0}\frac{0-0}{k}=0$$

$$\therefore\ f_{xy}(0, 0)=\lim_{h\to 0}\frac{f_y(h, 0)-f_y(0,0)}{h}$$

$$=\lim_{h\to 0}\frac{0-0}{h}=0. \quad \dots \quad \dots \quad (1)$$

Again, $f_x(0, k)=\lim\limits_{h\to 0}\dfrac{f(h, k)-f(0, k)}{h}$

$$=\lim_{h\to 0}\frac{\dfrac{h^2k^2}{h^2+k^2}-0}{h}=\lim_{h\to 0}\frac{hk^2}{h^2+k^2}=0$$

and $f_x(0,\ 0) = \lim_{h \to 0} \dfrac{f(h,\ 0) - f(0,\ 0)}{h}$

$$= \lim_{h \to 0} \frac{0-0}{h} = 0$$

$$f_{yx}(0,\ 0) = \lim_{k \to 0} \lim_{k \to 0} \frac{f_x(0,\ k) - f_x(0,\ 0)}{k}$$

$$= \frac{0-0}{k} = 0 \qquad \ldots \qquad \ldots \qquad (2)$$

From (1) and (2), it follows that $f_{xy}(0,\ 0) = f_{yx}\ (0,\ 0)$

(*iii*) Here, $f(x,\ y) = xy\dfrac{x^2 - y^2}{x^2 + y^2}$, when x and y both $\neq 0$,

$f(0,\ 0) = 0$

$$f_y(h,\ 0) = \lim_{k \to 0} \frac{f(h,\ k) - f(h,\ 0)}{k}$$

$$= \lim_{k \to 0} \frac{hk\left(\dfrac{h^2 - k^2}{h^2 + k^2}\right) - 0}{k}$$

$$= \lim_{k \to 0} \frac{h\left(h^2 - k^2\right) - 0}{k} \qquad [\ k \neq 0\ ]$$

$$= h$$

and $f_y(0,\ 0) = \lim_{k \to 0} \dfrac{f(0,\ k) - f(0,\ 0)}{k}$

$$= \lim_{k \to 0} \frac{0-0}{k} = 0$$

$$\therefore\ f_{xy}(0,\ 0) = \lim_{h \to 0} \frac{f_y(h,\ 0) - f_y(0,\ 0)}{h}$$

$$= \lim_{h \to 0} \frac{h-0}{h} = 1 \qquad \ldots \qquad \ldots \qquad (1)$$

Again, $f_x(0,\ k) = \lim\limits_{h \to 0} \dfrac{f(h,\ k) - f(0,\ k)}{h}$

$$= \lim_{h \to 0} \frac{hk\left(\dfrac{h^2 - k^2}{h^2 + k^2}\right) - 0}{h}$$

$$= \lim_{h \to 0} \frac{k\left(h^2 - k^2\right)}{h^2 + k^2} \qquad [\ h \neq 0\ ]$$

$$= \frac{-k^3}{k^2} = -k$$

$$f_x(0,\ 0) = \lim_{h \to 0} \frac{f(h,\ 0) - f(0,\ 0)}{h}$$

$$= \lim_{h \to 0} \frac{0 - 0}{h} = 0$$

$$\therefore\ f_{yx}(0,\ 0) = \lim_{k \to 0} \frac{f_x(0,\ k) - f_x(0,\ 0)}{k}$$

$$= \lim_{k \to 0} \frac{-k - 0}{k} = -1 \qquad \ldots \quad \ldots \quad (2)$$

From (1) and (2), we conclude that

$f_{xy}(0,\ 0) \neq f_{yx}(0,\ 0)$.

EXAMPLE - XII (B)

1. Verify Euler's theorem for the following functions

   (i) $u = ax^2 + 2hxy + by^2$. (ii) $u = x^3 + y^3 + 3x^2y + 3xy^2$

   (iii) $u = \dfrac{x - y}{x + y}$ (iv) $u = \sin \dfrac{x^2 + y^2}{xy}$

   (v) $u = \left(x^{\frac{1}{4}} + y^{\frac{1}{4}} \right) \Big/ \left(x^{\frac{1}{5}} + y^{\frac{1}{5}} \right)$.

2. Find $\dfrac{dy}{dx}$ in the following cases :

   (i) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. (ii) $x^y + y^x = a^b$. [*C. H. 1944*]

(iii) $(\cos x)^y = (\sin y)^x$. (iv) $e^x + e^y = 2xy$.

(v) $y^x + x^y = (x + y)^{x+y}$.

3. (i) If $u = \phi(H_n)$, where H_n is a homogeneous function of degree n in x, y, z, then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = n\frac{F(u)}{F'(u)}$$

where $F(u) = H_n$.

(ii) If $u = \cos^{-1}\left\{(x + y)/\left(\sqrt{x} + \sqrt{y}\right)\right\}$, then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \tfrac{1}{2}\cot u = 0.$$ [*B.P. 1994, V.P. '96, 2001, 2008*]

4. If $V = \sin^{-1}\left\{\left(x^2 + y^2\right)/(x + y)\right\}$, then show that

$$xV_x + yV_y = \tan V.$$

5. If $u = x\phi(y/x) + \psi(y/x)$, then show that

(i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = x\phi(y/x)$.

(ii) $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = 0$.

6. (i) If $v = f(u)$, u being a homogeneous function of degree n in x, y, show that

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nu\frac{\partial v}{\partial u}.$$ [*C. P. 1948*]

(ii) If V be a homogeneous function in x, y, z of degree n, prove that $\dfrac{\partial V}{\partial x}, \dfrac{\partial V}{\partial y}$ and $\dfrac{\partial V}{\partial z}$ are each a homogeneous function in x, y, z of degree (n—1). [*C. P. 2006*]

(iii) If V be a homogeneous function of degree n in x, y, z and if $V = f(X, Y, Z)$, where X, Y, Z are respectively $\dfrac{\partial V}{\partial x}, \dfrac{\partial V}{\partial y}, \dfrac{\partial V}{\partial z}$, show that $X\dfrac{\partial V}{\partial X} + Y\dfrac{\partial V}{\partial Y} + Z\dfrac{\partial V}{\partial Z} = \dfrac{n}{n-1}V$.

7. (i) If H be a homogeneous function of degree n in x and y and if $u = \left(x^2 + y^2\right)^{-\frac{1}{2}n}$, show that

$$\frac{\partial}{\partial x}\left(H\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(H\frac{\partial u}{\partial y}\right)=0.$$

(ii) If H be a homogeneous function in x, y, z of degree n and if $u=\left(x^2+y^2+z^2\right)^{-\frac{1}{2}(n+1)}$, show that

$$\frac{\partial}{\partial x}\left(H\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(H\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(H\frac{\partial u}{\partial z}\right)=0.$$

8. If $x=r\cos\theta$, $y=r\sin\theta$, prove that

(i) $dx^2+dy^2=dr^2+r^2\,d\theta^2$; [B.P. 2001]

(ii) $x\,dy-y\,dx=r^2\,d\theta$. [B.P. 2001]

9. If $\phi(x, y)=0$, $\psi(x, z)=0$, prove that

$$\frac{\partial\psi}{\partial x}\cdot\frac{\partial\phi}{\partial y}\cdot\frac{\partial y}{\partial z}=\frac{\partial\phi}{\partial x}\cdot\frac{\partial\psi}{\partial z}.$$

10. Express Δ, the area of Δ ABC, as a function of a, b, C and hence show that

$$\frac{d\Delta}{\Delta}=\frac{da}{a}+\frac{db}{b}+\cot C\,dC.$$

11. If $x^2+y^2+z^2-2xyz=1$, show that

$$\frac{dx}{\sqrt{1-x^2}}+\frac{dy}{\sqrt{1-y^2}}+\frac{dz}{\sqrt{1-z^2}}=0.$$

12. (i) If $ax^2+by^2+cz^2=1$ and $lx+my+nz=0$, show that

$$\frac{dx}{bny-cmz}=\frac{dy}{clz-anx}=\frac{dz}{amx-bly}.$$

(ii) If $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1$,

prove that $\dfrac{x\left(b^2-c^2\right)}{dx}+\dfrac{y\left(c^2-a^2\right)}{dy}+\dfrac{z\left(a^2-b^2\right)}{dz}=0.$

13. The radius of a right circular cone is measured as 5 cm with a possible error of 0·01 cm, and altitude as 8 cm with a possible error of 0·024 cm. Find the possible relative error and percentage error in the volume as calculated from these measurements.

14. The side a of a triangle ABC is calculated from b, c, A. If there be small errors db, dc, dA in the measured values of b, c, A., show that the error in the calculated value of a is given by

$$da = \cos B.dc + \cos C.db + b \sin C.dA \cdot$$

15. If $f(p, t, v) = 0$, show that

$$\left(\frac{dp}{dt}\right)_{v\text{ const}} \times \left(\frac{dt}{dv}\right)_{p\text{ const}} \times \left(\frac{dv}{dp}\right)_{t\text{ const}} = -1.$$

16. If $u = F(y - z, z - x, x - y)$, prove that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

[*C.P. 1983, '88, '94, '98, 2002, 2007; V.P. '95*]

17. If $u = F\left(x^2 + y^2 + z^2\right) f(xy + yz + zx)$, prove that

$$(y - z)\frac{\partial u}{\partial x} + (z - x)\frac{\partial u}{\partial y} + (x - y)\frac{\partial u}{\partial z} = 0.$$

18. If $u = f\left(x^2 + 2yz,\, y^2 + 2zx\right)$, prove that

$$\left(y^2 - zx\right)\frac{\partial u}{\partial x} + \left(x^2 - yz\right)\frac{\partial u}{\partial y} + \left(z^2 - xy\right)\frac{\partial u}{\partial z} = 0$$

[*C.P. 1981, '95, 2008; C.H. 1947*]

19. If $F\left(v^2 - x^2,\, v^2 - y^2,\, v^2 - z^2\right) = 0$,

where v is a function of x, y, z, show that

$$\frac{1}{x}\frac{\partial v}{\partial x} + \frac{1}{y}\frac{\partial v}{\partial y} + \frac{1}{z}\frac{\partial v}{\partial z} = \frac{1}{v}.$$

20. If u be a homogeneous function of x and y of n dimensions, prove that

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 u = n(n-1)u,$$

where $\left(x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}\right)^2 u = x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2}$;

[*V.P. 1998, C.P. '85, B.P. '93, C.H. '46*]

21. If $u = x\phi(x+y) + y\psi(x+y)$

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0.$$

22. If V be a function of r alone, where $r^2 = x^2 + y^2 + z^2$,

show that $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = \dfrac{\partial^2 V}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial V}{\partial r}$.

ANSWERS

2. (i) $-\dfrac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$. (ii) $-\dfrac{yx^{y-1} + y^x \log y}{xy^{x-1} + x^y \log x}$.

(iii) $\dfrac{y \tan x + \log \sin y}{\log \cos x - x \cot y}$. (iv) $\dfrac{e^x - 2y}{2x - e^x}$.

(v) $-\dfrac{y^x \log y + yx^{y-1} - (x+y)^{x+y}\{\log(x+y)+1\}}{x^y \log x + xy^{x-1} - (x+y)^{x+y}\{\log(x+y)+1\}}$

13. 0.007 (relative error); 0.7 (percentage error).

13 EXTREMA OF FUNCTIONS OF TWO OR MORE VARIABLES

13.1. Extrema with two variables.

A function $f(x, y)$ of two independent variables x, y is said to be maximum for $x = a$, $y = b$ provided $f(a, b)$ is greater than every other values assumed by $f(x, y)$ in the immediate neighbourhood of $x = a$, $y = b$. Similarly, a minimum value of $f(x, y)$ is defined to be the value which is less than every other values in the immediate neighbourhood. A formal definition is as follows :

A function $f(x, y)$ is said to have a maximum value at a point (a, b) of the domain of $f(x, y)$, $f(x, y)$, provided we can find a positive quantity δ such that for all values of x, y in $a-\delta < x < a+\delta$ and $b-\delta < y < b+\delta$, $(x \neq a,\ y \neq b)$ $f(a, b) > f(x, y)$;

i.e., if $f(a+h,\ b+k) - f(a, b) < 0$,

for $|h| < \delta$ and $|k| < \delta$.

Similarly, *the function $f(x, y)$ has a minimum value at a point (c, d), provided we can find a positive quantity δ' such that for all values of x, y in $c-\delta' < x < c+\delta'$, $d-\delta' < y < d+\delta'$, $(x \neq c,\ y \neq d)$ $f(c, d) < f(x, y)$;*

i.e., if $f(c+h,\ d+k) - f(c, d) > 0$,

for $|h| < \delta'$ and $|k| < \delta'$.

13.2. Necessary conditions for Maximum and Minimum of extrema with two variables.

If a function $f(x, y)$ be a maximum or a minimum at $x = a$, $y = b$ and if the first partial derivatives $f_x(a, b)$ and $f_y(a, b)$ exist, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Proof. If $f(a, b)$ be a maximum or a minimum value of $f(x, y)$, then clearly it is also a maximum or a minimum value of the function $f(x, b)$ of one variable x for $x = a$ and so its derivative $f_x(a, b)$ for $x = a$ must be zero, provided it exists.

Similarly, $f_y(a, b) = 0$.

13.3. Determination of Maxima and Minima of extrema with two variables.

If (a, b) be a point in the domain in which the function $f(x, y)$ is defined, and if $f_x(a,b)=0$, $f_y(a,b)=0.$ and $f_{xx}(a,b)\cdot f_{yy}(a,b)-\{f_{xy}(a,b)\}^2>0$, then $f(a,b)$ is a maximum or a minimum according as $f_{xx}(a,b)<$ or >0 (and consequently $f_{yy}(a,b)<$ or >0). But if $f_{xx}(a,b)\cdot f_{yy}(a,b)-\{f_{xy}(a,b)\}^2<0$, $f(a,b)$ is neither a maximum nor a minimum and if $f_{xx}(a,b)\cdot f_{yy}(a,b)-\{f_{xy}(a,b)\}^2=0$ further analysis is necessary.

Proof of these results is beyond the scope of this elementary treatise.

Note. Points where $f_x=0$ and $f_y=0$ are called stationary points. These points may be a maximum or a minimum but in certain cases it may so happen that the points is a maximum in respect of one variable while a minimum in respect of the other variable.

13.4. Illustrative Examples.

Ex. 1. *Examine for extreme values of the function*

$$x^2+y^2+(x+y+1)^2, \qquad \text{[C.P. 1995]}$$

Let $f(x,y)=x^2+y^2+(x+y+1)^2$,

$\therefore\quad f_x=2x+2(x+y+1)=4x+2y+2,$

$f_y=2y+2(x+y+1)=2x+4y+2,$

$f_{xx}=4,\ f_{yy}=4,\ f_{xy}=2.$

The equations $f_x=0$, $f_y=0$ are equivalent to

$$2x+y+1=0 \quad \text{and} \quad x+2y+1=0.$$

These give, $\dfrac{x}{1-2}=\dfrac{y}{1-2}=\dfrac{1}{4-1}$, or, $\dfrac{x}{-1}=\dfrac{y}{-1}=\dfrac{1}{3}$,

or, $x=-\dfrac{1}{3},\ y=-\dfrac{1}{3}$,

The function may have an extreme value at $\left(-\frac{1}{3}, -\frac{1}{3}\right)$.

Now, at $\left(-\frac{1}{3},-\frac{1}{3}\right)$, $f_{xx}\cdot f_{yy}-(f_{xy})^2=4.4-2^2=12>0.$

Also, $f_{xx}>0.$

Therefore, $f(x,y)$ is a minimum at $\left(-\frac{1}{3}, -\frac{1}{3}\right)$.

Ex. 2. *Find all the maxima and minima of the function*

$$4x^2 - xy + 4y^2 + x^3y + xy^3 - 4.$$

Let $f(x, y) = 4x^2 - xy + 4y^2 + x^3y + xy^3 - 4$

We have $f_x = 8x - y + 3x^2y + y^3$

$$f_y = -x + 8y + x^3 + 3xy^2$$

$$f_{xx} = 8 + 6xy, \quad f_{yy} = 8 + 6xy, \quad f_{xy} = 3x^2 + 3y^2 - 1.$$

The equations $f_x = 0$, $f_y = 0$ give

$$8x - y + 3x^2y + y^3 = 0 \qquad \ldots (1)$$

$$-x + 8y + x^3 + 3xy^2 = 0 \qquad \ldots (2)$$

Adding (1) and (2) we get

$$7(x+y) + (x+y)^3 = 0, \quad \text{or,} \quad (x+y)\{(x+y)^2 + 7\} = 0,$$

or, $(x+y) = 0$, $\because (x+y)^2 + 7 > 0$

$$\therefore \quad y = -x. \qquad \ldots (3)$$

From (1) and (3) we get

$$9x - 4x^2 = 0, \quad \text{or,} \quad x(4x^2 - 9) = 0.$$

$$\therefore \quad x = 0, \ +\tfrac{3}{2}, \ -\tfrac{3}{2}.$$

The corresponding values of $y = 0, \ -\frac{3}{2}, \ \frac{3}{2}$

The function has three stationary points $(0, 0)$, $\left(\frac{3}{2}, -\frac{3}{2}\right)$, $\left(-\frac{3}{2}, \frac{3}{2}\right)$,

At $(0, 0)$, $f_{xx} = 8$, $f_{yy} = 8$, $f_{xy} = -1$,

$$f_{xx} f_{yy} - (f_{xy})^2 = 64 - 1 = 63 > 0 \text{ and } f_{xx} = 8 > 0,$$

$\therefore$ the function is minimum at $(0, 0)$.

At $\left(\frac{3}{2}, -\frac{3}{2}\right)$, $f_{xx} = 8 - \frac{27}{2} = -\frac{11}{2}$,

$$f_{yy} = 8 - \frac{27}{2} = -\frac{11}{2}, \quad f_{xy} = 3\left(\frac{9}{4} + \frac{9}{4}\right) - 1 = \frac{25}{2}$$

$$\therefore \qquad f_{xx} f_{yy} - (f_{xy})^2 = \frac{121}{4} - \frac{625}{4} = -126 < 0,$$

so that the function is neither a maximum nor a minimum

$$\text{At } \left(-\frac{3}{2},-\frac{3}{2}\right), f_{xx} = 8-\frac{27}{2} = -\frac{11}{2},$$

$$\therefore\ f_{xx}f_{yy}-(f_{xy})^2 = -126 < 0,$$

so that the function is neither a maximum nor a minimum

The points $\left(\frac{3}{2},-\frac{3}{2}\right)$ and $\left(-\frac{3}{2},\frac{3}{2}\right)$ where the function is neither a maximum nor a minimum are called Saddle Points.

Ex. 3. *Show that* $f(x,y) = y^2+2x^2y+2x^4$ *has a minimum at* $(0,0)$

We have $f_x = 4xy+8x^3$

$$f_y = 2y+2x^2$$

$$f_{xx} = 4y+24x^2,\ f_{yy} = 2,\ f_{xy} = 4x$$

At $(0,0), f_x = 0,\ f_y = 0,\ f_{xy} = 0,$

Since $f_x = 0,\ f_y = 0$. $(0,0)$ is a stationary point of $f(x,y)$.

Again $f_{xx}f_{yy}-(f_{xy})^2 = 0$. So further analysis is necessary.

We have $f(0,0) = 0$ and $f(x,y) = (y+x^2)^2+x^4$, which is positive for positive as well as negative values of x.

Hence, $f(x,y)$ is minimum at $(0,0)$.

EXAMPLES- XIII (A)

1. Examine the following for extreme values :

(i) x^3+y^3-3axy. (ii) $x^4+y^4-2x^2+4xy-2y^2$

(iii) $xy+\dfrac{a^2}{x}+\dfrac{a^2}{y}$ (iv) $x^3y^2(1-x-y)$.

(v) $\sin x\sin y\sin(x+y)$. (vi) $x^2y^2-5x^2-8xy-5y^2$.

(vii) $x^4+2x^2y-x^2+3y^2$. (viii) $2(x-y)^2-x^4-y^4$.

(ix) $y^2+x^2y+2x^4$. (x) x^2+xy+y^2-4x+y.

(xi) $xy(6a-x-y)$. (xii) $(x^2+y^2)e^{6x+2x^2}$.

(xiii) $(x^2+y^2)^2-8(x^2-y^2)$. (xiv) $x^2+y^2+(ax+by+c)^2$.

(xv) $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.

2. Show that the function $f(x, y) = x^2 + 2xy + y^2 + x^3 + y^3 + x^7$ has neither a maximum nor a minimum at the origin.

3. Show that the function $(x + y)^4 + (x - 3)^6$ has a minimum at $(3, -3)$.

4. Show that the function $f(x, y) = y^2 + 3x^2y + 5x^4$ has a minimum at $(0, 0)$.

5. Show that the function $f(x, y) = 4x^2y - y^2 - 8x^4$ has a minimum at $(0, 0)$.

6. Show that the function $f(x, y) = 3x^3 + 4x^2y - 3xy^2 - 4y$ is neither a maximum nor a minimum at $(0, 0)$.

ANSWERS

1. (i) Max. at (a, a) if $a < 0$ and min. at (a, a) if $a > 0$.

(ii) Min. at $\left(\sqrt{2}, -\sqrt{2}\right)$ and $\left(-\sqrt{2}, \sqrt{2}\right)$.

(iii) Min. at $\left(a^{\frac{2}{3}}, a^{\frac{2}{3}}\right)$ (iv) Max. at $\left(\frac{1}{2}, \frac{1}{3}\right)$.

(v) Max. at $\left(n'\pi + \frac{\pi}{3}, n'\pi + \frac{\pi}{3}\right)$ and

Min. at $\left(n'\pi - \frac{\pi}{3}, n'\pi - \frac{\pi}{3}\right)$, n' being any integer or 0.

(vi) Max. at $(0, 0)$, no min. (vii) No max., min. at $\left(\pm\frac{\sqrt{3}}{2}, -\frac{1}{4}\right)$.

(viii) Max. at $\left(\sqrt{2}, -\sqrt{2}\right)$ and $\left(-\sqrt{2}, \sqrt{2}\right)$.

(ix) Neither a max, nor a min. (x) Max. at $\left(-\frac{7}{3}, -2\right)$.

(xi) At $(2a, 2a)$ maximum if $a > 0$ and min. if $a < 0$.

(xii) Min. at $(0, 0)$ and $(-1, 0)$. (xiii) Min. at $(2, 0)$ and $(-2, 0)$.

(xiv) Min. at $\left(\frac{-ac}{1 + a^2 + b^2}, \frac{-bc}{1 + a^2 + b^2}\right)$.

(xv) Max. at $(4, 0)$ and min. at $(6, 0)$.

13.5. Lagrange's method of undetermined multipliers.

Let $u = f(x_1, x_2, x_3, \ldots x_n)$ ... (1) be a function of n variables which are connected by m equations.

$$\left.\begin{array}{l} g_1(x_1, x_2, x_3, \ldots x_n) = 0 \\ g_2(x_1, x_2, x_3, \ldots x_n) = 0 \\ \ldots \quad \ldots \\ g_m(x_1, x_2, x_3, \ldots x_n) = 0 \end{array}\right\} \quad \ldots (2)$$

We have $m\,(< n)$ equations in n variables given in (2), so only $n - m$ variables are independent.

For u to be maximum or minimum, we should have

$$\left.\begin{array}{l} du = \dfrac{\partial u}{\partial x_1} dx_1 + \dfrac{\partial u}{\partial x_2} dx_2 + \ldots + \dfrac{\partial u}{\partial x_n} dx_n = 0. \\ \text{Also the equations in (2) give} \\ dg_1 = \dfrac{\partial g_1}{\partial x_1} dx_1 + \dfrac{\partial g_1}{\partial x_2} dx_2 + \ldots + \dfrac{\partial g_1}{\partial x_n} dx_n = 0. \\ \ldots \quad \ldots \quad \ldots \\ dg_m = \dfrac{\partial g_m}{\partial x_1} dx_1 + \dfrac{\partial g_m}{\partial x_2} dx_2 + \ldots + \dfrac{\partial g_m}{\partial x_n} dx_n = 0 \end{array}\right\} \quad \ldots (3)$$

Multiplying the equations in (3) by $1, \lambda_1, \lambda_2, \ldots, \lambda_m$ respectively and adding we get

$$F_1\, dx_1 + F_2\, dx_2 + \ldots + F_n\, dx_n = 0 \quad \ldots (4)$$

where $F_k = \dfrac{\partial u}{\partial x_k} + \lambda_1 \dfrac{\partial g_1}{\partial x_k} + \lambda_2 \dfrac{\partial g_2}{\partial x_k} + \ldots + \lambda_m \dfrac{\partial g_m}{\partial x_k}$,

$(k = 1, 2, \ldots, n)$.

The m quantities $\lambda_1, \lambda_2, \ldots, \lambda_m$ are at our choice. Let us choose these quantities so as to satisfy m linear equations

$$F_1 = 0, \quad F_2 = 0, \ldots F_m = 0 \quad \ldots (5)$$

Then the equation (4) becomes

$$F_{m+1}\, dx_{m+1} + F_{m+2}\, dx_{m+2} + \ldots + F_n\, dx_n = 0 \quad \ldots (6)$$

We have already noted that only $n-m$ variables are independent. To be specific let $x_{m+1}, x_{m+2}, \ldots, x_n$ be these independent variables.

Since the quantities $dx_{m+1}, dx_{m+2}, \ldots, dx_n$ are all independent, their coeficients must be separately zero.

Thus we get $n-m$ extra equations

$$F_{m+1}=0,\ F_{m+2}=0, \ldots, F_n=0 \qquad \ldots (7)$$

Thus from (2), (5) and (7) we have $n+m$ equations

$$g_1=0,\quad g_2=0, \ldots g_m=0$$
$$F_1=0,\quad F_2=0, \ldots\ F_n=0.$$

From these equations we can find the multipliers $\lambda_1, \lambda_2, \ldots, \lambda_m$. and variables $x_1, x_2, \ldots, x_n$ for which u is maximum or minimum.

10.06. Illustrative Examples

Ex. 1. *Find the minimum value of* $x^2+y^2+z^2$, *subject ot the condition*

$$2x+3y+5z=30$$

Let $u=x^2+y^2+z^2$ $\qquad \ldots (1)$

and $2x+3y+5z=30$ $\qquad \ldots (2)$

For a maximum or a minimum value of u, $du=0$,

i.e., $2xdx+2ydy+2zdz=0$,

i.e., $xdx+ydy+zdz=0$. $\qquad \ldots (3)$

Taking differentials we get, from equation (2),

$$2dx+3dy+5dz=0. \qquad \ldots (4)$$

Multiplying (3) by 1, (4) by λ and then adding we get

$$(x+2\lambda)dx+(y+3\lambda)dy+(z+5\lambda)dz=0$$

Equating the coefficients of dx, dy, dz to zero we get,

$$x+2\lambda=0, \qquad \ldots (5)$$
$$y+3\lambda=0, \qquad \ldots (6)$$
$$z+5\lambda=0, \qquad \ldots (7)$$

From the equation (5), (6) and (7) we get

$$\frac{x}{2}=\frac{y}{3}=\frac{z}{5}=\frac{2x+3y+5z}{2.2+3.3+5.5}=\frac{30}{38}=\frac{15}{19},$$

i.e., $x=\frac{30}{19}$, $y=\frac{45}{19}$, $z=\frac{75}{19}$.

From the equation (5), $\lambda=-\frac{15}{19}$.

$\therefore$ u has an extreme value when $x=\frac{30}{19}$, $y=\frac{45}{19}$, $z=\frac{75}{19}$.

From the equation (2), one of the three variables, say, z can be expressed as a function of two independent variables x, y, that is, $z=6-\frac{2}{5}x-\frac{3}{5}y$.

$$\therefore \quad \frac{\partial z}{\partial x}=-\frac{2}{5}, \frac{\partial z}{\partial y}=-\frac{3}{5}.$$

Now, $$\frac{\partial u}{\partial x}=2x+2z\frac{\partial z}{\partial x}=2x-\frac{4z}{5},$$

$$\frac{\partial u}{\partial y}=2y+2z\frac{\partial z}{\partial y}=2y-\frac{6z}{5},$$

$$\frac{\partial^2 u}{\partial x^2}=2-\frac{4}{5}\frac{\partial z}{\partial x}=2+\frac{8}{25}=\frac{58}{25},$$

$$\frac{\partial^2 u}{\partial x\partial y}=-\frac{6}{5}\frac{\partial z}{\partial x}=\frac{12}{25},$$

$$\frac{\partial^2 u}{\partial y^2}=2-\frac{6}{5}\frac{\partial z}{\partial y}=2+\frac{18}{25}=\frac{68}{25}.$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial^2 u}{\partial y^2}-\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2=\frac{58}{25}\cdot\frac{68}{25}-\left(\frac{12}{25}\right)^2=\frac{152}{25}>0.$$

Also $$\frac{\partial^2 u}{\partial x^2}>0.$$

$\therefore$ u is minimum when $x=\frac{30}{19}$, $y=\frac{45}{19}$, $z=\frac{75}{19}$ and this minimum value of u is

$$\frac{30^2+45^2+75^2}{19^2}=\frac{8550}{361}.$$

Ex. 2. *Find the maximum value of* $x^2y^3z^4$ *subject to the condition* $x+y+z=18$.

Let $u = x^2y^3z^4$.

$\therefore \quad \log u = 2\log x + 3\log y + 4\log z$.

Taking differentials we get

$$\frac{du}{u} = \frac{2}{x}dx + \frac{3}{y}dy + \frac{4}{z}dz. \qquad \ldots(1)$$

For a maximum or minimum value of u, $du = 0$.

$$\therefore \quad \frac{2}{x}dx + \frac{3}{y}dy + \frac{4}{z}dz = 0 \qquad \ldots(2)$$

From the relation $x + y + z = 18$ $\qquad \ldots(3)$

we get $\qquad dx + dy + dz = 0 \qquad \ldots(1)$

Multiplying (2) by 1 and (4) by λ and adding we get

$$\left(\frac{2}{x}+\lambda\right)dx + \left(\frac{3}{y}+\lambda\right)dy + \left(\frac{4}{z}+\lambda\right)dz = 0.$$

Equating the coefficients of dx, dy, dz to zero we get

$$\frac{2}{x} + \lambda = 0. \qquad \ldots(5)$$

$$\frac{3}{y} + \lambda = 0. \qquad \ldots(6)$$

$$\frac{4}{z} + \lambda = 0. \qquad \ldots(7)$$

From (5), (6) and (7) we get

$$\frac{2}{x} = \frac{3}{y} = \frac{4}{z} = \frac{2+3+4}{x+y+z} = \frac{9}{18} = \frac{1}{2}.$$

i.e., $\qquad x = 4,\ y = 6,\ z = 8$, and $\lambda = -\frac{1}{2}$.

From (3) one of the three variables, say, z can be expressed as a function of two independent variables x, y,

i.e., $\qquad z = 18 - x - y$.

$$\therefore \quad \frac{\partial z}{\partial x} = -1, \quad \frac{\partial z}{\partial y} = -1.$$

Now $\dfrac{\partial u}{\partial x} = 2xy^3z^4 + 4x^2y^3z^3\dfrac{\partial z}{\partial x}$

$$= 2xy^3z^4 - 4x^2y^3z^3.$$

$$\frac{\partial u}{\partial y} = 3x^2y^2z^4 + 4x^2y^3z^3\frac{\partial z}{\partial y}$$

$$= 3x^2y^2z^4 - 4x^2y^3z^3,$$

$$\frac{\partial^2 u}{\partial x^2} = 2y^3z^4 + 8xy^3z^3\frac{\partial z}{\partial x} - 8xy^3z^3 - 12x^2y^3z^2\frac{\partial z}{\partial x}$$

$$= 2y^3z^4 - 8xy^3z^3 - 8xy^3z^3 + 12x^2y^3z^2$$

$$= 2y^3z^4 - 16xy^3z^3 + 12x^2y^3z^2$$

$$= 2y^3z^2\left(z^2 - 8xz + 6x^2\right).$$

$$\frac{\partial^2 u}{\partial x\partial y} = 6xy^2z^4 + 8xy^3z^3\frac{\partial z}{\partial y} - 12x^2y^2z^3 - 12x^2y^3z^2\frac{\partial z}{\partial y}$$

$$= 6xy^2z^4 - 8xy^3z^3 - 12x^2y^2z^3 + 12x^2y^3z^2$$

$$= 2xy^2z^2\left(3z^2 - 4yz - 6xz + 6xy\right).$$

$$\frac{\partial^2 u}{\partial y^2} = 6x^2yz^4 - 12x^2y^2z^3 - 12x^2y^2z^3 + 12x^2y^3z^2$$

$$= 6x^2yz^2\left(z^2 - 4yz + 2y^2\right)$$

For $x = 4, \quad y = 6, \quad z = 8.$

$$\frac{\partial^2 u}{\partial x^2} = -192y^3z^2, \quad \frac{\partial^2 u}{\partial x\partial y} = -96xy^2z^2, \quad \frac{\partial^2 u}{\partial y^2} = -336x^2yz^2.$$

$$\frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x\partial y}\right)^2 = (-192)y^3z^2(-336)x^2yz^2 - \left(96xy^2z^2\right)^2.$$

$$= 192\cdot 336x^2y^4z^4 - 96\cdot 96x^2y^4z^4 > 0.$$

Also $\dfrac{\partial^2 u}{\partial x^2} < 0.$

$\therefore$ u is maximum when $x=4$, $y=6$, $z=8$ and the maximum value of u is $4^2.6^3.8^4$.

Ex. 2. ***Prove that of all rectangular parallelopiped of same volume, the cube has the least surface-area.***

Let x, y, z be the length of three coterminous edges of the rectangular parallelopiped. The volume is given by

$$xyz = a^3 \text{ (given)}. \qquad \ldots (1)$$

If S be the surface-area, then

$$S = 2(yz + zx + xy) \qquad \ldots (2)$$

and we are to minimize S subject to the condition (1).

For a maximum or a minimum value of S, $dS = 0$.

$$\therefore \quad (y+z)dx + (z+x)dy + (x+y)dz = 0 \qquad \ldots (3)$$

Also, from (1), $yzdx + zxdy + xydz = 0$. $\qquad \ldots (4)$

Multiplying (3) by 1, (4) by λ and adding we get

$$(y+z+\lambda yz)dx + (z+x+\lambda zx)dy + (x+y+\lambda xy)dz = 0$$

Equating the coefficients of dx, dy, dz to zero we get

$$y+z+\lambda yz = 0 \qquad \ldots (5)$$

$$z+x+\lambda zx = 0 \qquad \ldots (6)$$

$$x+y+\lambda xy = 0 \qquad \ldots (7)$$

$$\therefore \quad \frac{1}{y}+\frac{1}{z}=\frac{1}{z}+\frac{1}{x}=\frac{1}{x}+\frac{1}{y} \quad (=-\lambda)$$

$$\frac{1}{x}=\frac{1}{y}=\frac{1}{z} \quad i.e., \; x=y=z=(xyz)^{\frac{1}{3}}=a,$$

that is, all the edges are equal.

From (1), one of the variables, say, z can be expressed as a function of the independent variables x, y.

(1) gives $yz + xy\dfrac{\partial z}{\partial x} = 0$, $\quad$ *i.e.*, $\dfrac{\partial z}{\partial x} = -\dfrac{z}{x}$.

Also $\quad \dfrac{\partial z}{\partial y} = -\dfrac{z}{y}$.

Now $$\frac{\partial S}{\partial x}=2\left(y\frac{\partial z}{\partial x}+z+x\frac{\partial z}{\partial x}+y\right)$$

$$=2\left\{(y+z)+(x+y)\left(-\frac{z}{x}\right)\right\}$$

$$=\frac{2}{x}(xy+xz-zx-yz)=2\frac{(x-z)y}{x}=2\left(y-\frac{yz}{x}\right).$$

$$\frac{\partial^2 S}{\partial x^2}=\frac{2yz}{x^2}-\frac{2y}{x}\frac{\partial z}{\partial x}$$

$$=\frac{2yz}{x^2}+\frac{2y}{x}\cdot\frac{z}{x}=\frac{4yz}{x^2}.$$

Similarly $$\frac{\partial^2 S}{\partial x^2}=\frac{4zx}{y^2},$$

$$\frac{\partial^2 S}{\partial x\partial y}=2\left(1-\frac{z}{x}-\frac{y}{x}\cdot\frac{\partial z}{\partial y}\right)$$

$$=2\left(1-\frac{z}{x}+\frac{y}{x}\cdot\frac{z}{y}\right)=2.$$

For $x=a,\ y=a,\ z=a,$

$$\frac{\partial^2 S}{\partial x^2}=4,\quad \frac{\partial^2 S}{\partial y^2}=4,\quad \frac{\partial^2 S}{\partial x\partial y}=2.$$

$$\therefore\quad \frac{\partial^2 S}{\partial x^2}\frac{\partial^2 S}{\partial y^2}-\left(\frac{\partial^2 S}{\partial x\partial y}\right)^2=16-4=12>0.$$

Also $$\frac{\partial^2 S}{\partial x^2}>0.$$

Therefore, when $x=y=z$, that is, when the rectangular parallelopiped is a cube, its surface-area is minimum.

EXAMPLES - XIII (B)

1. Find the minimum values of $x^2 + y^2 + z^2$, when
   (i) $x + y + z = 15$, (ii) $yz + zx + xy = 12$., (iii) $xyz = 8$.
2. Find the extreme values of $yz + 3zx + 2xy$ where $x + y + z = 1$.
3. Find the extreme values of xy where $x^2 + xy + y^2 = 1$.
4. Determine the maximum and minimum values of $7x^2 + 8xy + y^2$ when $x^2 + y^2 = 1$.
5. Find the minimum distance of the point $(1, 2, 3)$ from the plane $x + y - 4z = 9$.
6. Which point of the sphere $x^2 + y^2 + z^2 = 1$ is at a maximum distance form the point $(2, 1, 3)$
7. If x, y, z are the angles of a triangle, then prove that the functions
   (i) $\sin x \sin y \sin z$;
   (ii) $\cos x \cos y \cos z$; are both maximum at $x = y = z = \dfrac{\pi}{3}$.
8. Find the maximum value of the function $x^2y^2z^2$, subject to the condition $x^2 + y^2 + z^2 = c^2$.
9. Find the rectangular parallelopiped of maximum volume that can be inscribed in the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.
10. Divide the number 27 into three parts x, y, z such that $2yz + 3zx + 4xy$ is maximum.

ANSWERS

1. (i) 75 for $(5, 5, 5)$; (ii) 12 for $(2, 2, 2)$, $(-2, -2, -2)$;
   (ii) 12 for $(2, 2, 2)$, $(-2, -2, 2)$, $(-2, 2, -2)$, $(2, -2, -2)$.
2. $\frac{3}{4}$ (max.) at $\left(\frac{1}{2}, 0, \frac{1}{2}\right)$; no min.
3. -1 (min.) at $(1, -1)$ and $(-1, 1)$;

   $\dfrac{1}{3}$ (max.). at $\left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$ and $\left(-\dfrac{1}{\sqrt{3}}, -\dfrac{1}{\sqrt{3}}\right)$. **4.** 9, -1. **5.** $\sqrt{18}$.

6. $\left(-\dfrac{2}{\sqrt{14}}, -\dfrac{1}{\sqrt{14}} -\dfrac{3}{\sqrt{14}}\right)$. **8.** $\dfrac{c^6}{27}$. **9.** $\dfrac{8abc}{3\sqrt{3}}$. **10.** $\frac{270}{23}, \frac{243}{23}, \frac{162}{23}$.

14 TANGENT AND NORMAL

14.1. We shall now consider certain properties of curves represented by continuous functions . If the equation of the curve is given in the explicit form $y = f(x)$, we shall assume that $f(x)$ has a derivative at every point, except, in some cases, at isolated points. If the equation of the curve is given in the implicit form $f(x, y) = 0$, we shall assume that the function $f(x, y)$ possesses continuous partial derivatives f_x and f_y which are not simultaneously zero. When the equation of the curve is given in the parametric form $x = \phi(t)$, $y = \psi(t)$, we shall assume that $\phi'(t)$ and $\psi'(t)$ are not simultaneously zero.

14.2. Equation of the tangent.

Def. *The tangent at P to a given curve is defined as the limiting position of the secant* $\overline{PQ}$ (when such a limit exists) *as the point Q approaches P along the curve* (whether Q is taken on one side or the other of the point P).

(i) *Let the equation of the curve be $y = f(x)$* and let the given point P on the curve be (x, y) and any other neighbouring point Q on the curve be $(x+\Delta x, + y+\Delta y)$

The equation of the secant $\overline{PQ}$ is (X, Y denoting the current co-ordinates)

$$Y - y = \frac{y+\Delta y - y}{x+\Delta x - x}(X - x) = \frac{\Delta y}{\Delta x}(X - x)$$

$\therefore$ the equation of the tangent P is

$$Y - y = \underset{\Delta x \to 0}{Lt}\frac{\Delta y}{\Delta x}(X - x) = \frac{dy}{dx}(X - x)$$

provided dy/dx is finite.

Thus, *the tangent to the curve* $y = f(x)$ at (x, y) (not parallel to the y-axis) is

$$\mathbf{Y - y = \frac{dy}{dx}(X - x)}. \qquad \ldots \qquad (1)$$

(ii) *When the equation of the curve is* $f(x, y) = 0$.

Since $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}$, $(f_y \neq 0)$, *the equation of the tangent to the curve at* (x, y) *is*

$$(\mathbf{X}-\mathbf{x})\mathbf{f}_{\mathbf{x}}+(\mathbf{Y}-\mathbf{y})\mathbf{f}_{\mathbf{y}}=\mathbf{0}. \qquad \ldots \quad (2)$$

(iii) *When the equation of the curve is* $x=\phi(t),\ y=\psi(t)$.

since $$\frac{dy}{dx}=\frac{dy}{dt}\bigg/\frac{dx}{dt}=\frac{\psi'(t)}{\phi'(t)}, \quad \phi'(t)\neq 0,$$

the equation of the tangent at the point 't' is

$$\left.\begin{aligned} Y-\psi(t)&=\frac{\psi'(t)}{\phi'(t)}\{X-\phi(t)\},\\ \textit{i.e., } \psi'(t)X-\phi'(t)Y&=\phi(t)\psi'(t)-\psi(t)\phi'(t), \end{aligned}\right\} \qquad \ldots \quad (3)$$

Note 1. When the left-hand and right-hand derivatives at (x, y) are infinite, with equal or opposite signs, the tangent at (x, y) can be conveniently obtained by using the *alternatives form of the equation of the tangent* $X-x=(Y-y)(dx/dy)$ which can be easily established as before.

[See *Ex. 32, Examples XIV(A)*]

Note 2. In the notation of Co-ordinate Geometry, the equation of the tangent to the curve $y=f(x)$ at (x_1, y_1) can be written as

$$y-y_1=f'(x_1)(x-x_1).$$

In the application of Differential Calculus to the theory of plane curves, for the sake of convenience, the current co-ordinates in the equation of the tangent and normal are usually denoted by (X, Y) while those of any particular point are denoted by (x, y). The current co-ordinates in the equation of the curve are however, as usual, denoted by (x_1, y_1).

14.3. Geometrical meaning of $\dfrac{dy}{dx}$

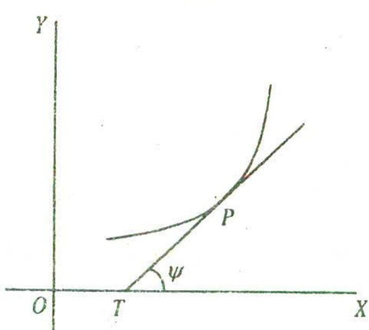


Fig 14.3.1

The equation (1) of the tangent can be written as

$$Y = \frac{dy}{dx} \cdot X + \left(y - x\frac{dy}{dx} \right)$$

which being of the form $y = mx + c$, the standard equation of a straight line, we conclude that

$\frac{dy}{dx}$ is the 'm' of the tangent at (x, y).

If ψ be the angle which the positive direction of the tangent at P makes with the positive direction of the x-axis, then $\tan\psi = m = \frac{dy}{dx}$.

Hence, *the direvative* $\frac{dy}{dx}$ *at* (x, y) *is equal to the trigonometrical tangent of the angle which the tangent to the curve at* (x, y) *makes with the positive direction of the* x-axis. [See Art. 7.14]

Note 1. It is customary to denote the angle which the tangent at any point on a curve makes with the x-axis by ψ.

Note 2. *The positive direction of the tangent* is the direction of the arc-length s increasing. Henceforth, this direciton will be spoken of as the direction of the tangent or simply as the tangent.

Note 3. $\tan\psi$, *i.e.*, $\frac{dy}{dx}$ is also called the **gradient** of the curve at the point $P(x, y)$.

Note 4. *The tangent at* (x, y) *is parallel to the x-axis* if $\psi = 0$, *i.e.*, if $\tan\psi = 0$, i.e., if $\frac{dy}{dx} = 0$.

The tangent at (x, y). *is perpendicular to the x-axis* (*i.e.*, parallel to the y-axis) if $\psi = \frac{1}{2}\pi$, *i.e.*, if $\cot\psi = 0$,

i.e., if $1 \Big/ \frac{dy}{dx} = 0$ or, $\frac{dx}{dy} = 0$.

14.4 Tangent at the origin.

If a curve passing through the origin be given by a rational integral algebraic equation, the equation of the tangent (or tangents) at the origin is obtained by equating to zero, the terms of the lowest degree in the equation.

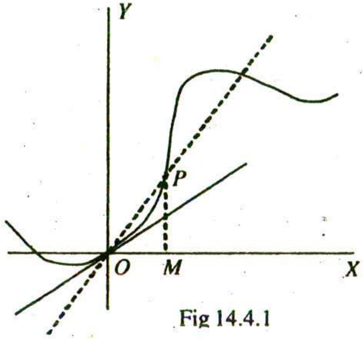


Fig 14.4.1

Let the equation of a curve of the n-th degree passing through the origin be

$$a_1x + b_1y + a_2x^2 + b_2xy + c_2y^2 + \cdots + a_nx^n + \cdots + k_ny^n = 0 \quad \ldots \quad (1)$$

Let $P(x, y)$ be a point on the curve near the origin O. The equation of the secant $\overline{OP}$ is $Y = \frac{y}{x}X$.

$\therefore$ the equation of the tangent at O is

$$Y = \underset{\substack{x \to 0 \\ y \to 0}}{Lt} \frac{y}{x} \cdot X = mX \text{ (say)}. \quad \ldots \quad (2)$$

Thus, the 'm' of the tangent at the origin is $\underset{\substack{x \to 0 \\ y \to 0}}{Lt} \frac{y}{x}$.

Case I. Let us suppose that m is finite, *i. e.*, the y-axis is not the tangent at the origin.

(i) Let us suppose $b_1 \neq 0$

Dividing (1) by x, we get

$$a_1 + b_1\frac{y}{x} + a_2x + b_2y + c_2y \cdot \frac{y}{x} + \cdots\cdots = 0$$

Now, let $x \to 0$, $y \to 0$, then $Lt(y/x) = m$.

$\therefore \quad a_1 + b_1m = 0$, the other terms vanishing.

$\therefore \quad m = -a_1/b_1$. $\quad \ldots \quad (3)$

From (2) and (3), the equation of the tangent at the origin is $a_1X + b_1Y = 0$,

or, taking x and y as current co-ordinates, $a_1x + b_1y = 0$.

(ii) If $b_1 = 0$, then from (3) it follows that $a_1 = 0$; now in this case, let us suppose that b_2 and c_2 are not both zero. Then, the equation of the curve (1) can be written as

$$a_2x^2 + b_2xy + c_2y^2 + a_3x^3 + \cdots = 0 \qquad \ldots \qquad (4)$$

Dividing by x^2, $\quad a_2 + b_2\dfrac{y}{x} + c_2\left(\dfrac{y}{x}\right)^2 + a_3x + \cdots\cdots = 0$.

When $x \to 0$, $y \to 0$, we have

$$a_2 + b_2m + c_2m^2 = 0, \text{ the other terms vanishing.} \qquad \ldots \qquad (5)$$

From (5) it is clear that there are two values of m and hence, there are two tangents at the origin and their equation, which is obtained by eliminating m between (2) and (5), is

$$a_2X^2 + b_2XY + c_2Y^2 = 0$$

or, taking x and y as current co-ordinates,

$$a_2x^2 + b_2xy + c_2y^2 = 0.$$

If $a_1 = b_1 = a_2 = b_2 = c_2 = 0$, it can be shown similarly that the rule holds good then also; and so on.

CASE II. When the tangent at the origin is the y-axis, then $Lt\,(x/y)$, as x and y both $\to 0$, being the tangent of the inclination of the tangent at the origin to the y-axis, is zero. Hence, dividing throughout the equation of the curve by y, and assuming $a_1 \neq 0$, and making x and y both approach zero, we find $b_1 = 0$. Hence, the equation of the curve now being

$$a_1x + a_2x^2 + b_2xy + c_2y^2 + \ldots.. = 0$$

we see that the theorem is still true in this case.

Illustration : If the equation of a curve be $x^2 - y^2 + x^3 + 3x^2y - y^3 = 0$ the tangents at the origin are given by $x^2 - y^2 = 0$, *i.e.*, $x + y = 0$ and $x - y = 0$.

14.5. Equation of the normal.

Definition. *The normal at any point of a curve is the straight line through that point drawn perpendicular to the tangent at that point.*

Let any line (not parallel to the co-ordinate axes) through the point be (x, y) be

$$Y - y = m(X - x).$$

This will be perpendicular to the tangent (not parallel to the co-ordinate axes) to the curve $y = f(x)$ at (x, y).

i.e., to $Y - y = \dfrac{dy}{dx}(X - x)$ if $m \cdot \dfrac{dy}{dx} = -1$, *i.e.*, if $m = -1\Big/\dfrac{dy}{dx}$.

Substituting this value of m in the above equation, we see that *the normal to the curve* $y = f(x)$ *at* (x, y) (when not parallel to the co-ordinate axes) is

$$\frac{\mathbf{dy}}{\mathbf{dx}}(\mathbf{Y} - \mathbf{y}) + (\mathbf{X} - \mathbf{x}) = \mathbf{0} \qquad \dots \qquad (1)$$

Similarly, if the equation of the curve is $f(x, y) = 0$, the equation of the normal at is

$$\frac{\mathbf{X} - \mathbf{x}}{\mathbf{f_x}} = \frac{\mathbf{Y} - \mathbf{y}}{\mathbf{f_y}} \qquad \dots \qquad (2)$$

and if the parametric equations of the curve are $x = \phi(t)$, $y = \psi(t)$, the equation of the normal at the point 't' is

$$\phi'(t)X + \psi'(t)Y = \phi(t)\,\phi'(t) + \psi(t)\psi'(t) \qquad \dots \qquad (3)$$

Note 1. When the tangents are parallel to $\overleftrightarrow{OX}$ and $\overleftrightarrow{OY}$ the nor mals are $X = x$ and $Y = y$ respectively.

Note 2. The *positive direction of the normal* makes an angle $+\frac{1}{2}\pi$ with the tangent, or $\frac{1}{2}\pi + \psi$ with the x-axis.

14.6. Angle of intersection of two curves.

The *angle of intersection of two curves* is the angle between the tangents to the two curves at their common point of intersection.

Suppose the two curves $f(x, y) = 0$, $\phi(x, y) = 0$ intersect at the point (x, y).

The tangents to the curves at (x, y) are

$$Xf_x + Yf_y - (xf_x + yf_y) = 0. \qquad [\text{ by § } 14.2(2)\,]$$

$$X\phi_x + Y\phi_y - (x\phi_x + y\phi_y) = 0.$$

The angle α at which these lines cut is given by

$$\tan\alpha = \frac{f_x\phi_y \sim \phi_x f_y}{f_x\phi_x + f_y\phi_y}.$$

Hence, *if the curves touch at* (x, y), $\alpha = 0$, *i.e.*, $\tan\alpha = 0$

i.e., $f_x\phi_y = \phi_x f_y$ *i.e.*, $f_x/\phi_x = f_y/\phi_y$, and *if they cut orthogonally at at* (x, y), $\alpha = \frac{1}{2}\pi$, *i.e.*, $\cot\alpha = 0$.

i.e., $f_x\phi_x + f_y\phi_y = 0$.

Note. If the equation of the curves are given in the forms $y = f(x)$, $y = \phi(x)$ the angle of their intersection is given by $\tan^{-1}\dfrac{f'(x) - \phi'(x)}{1 + f'(x)\phi'(x)}$.

Hence, the curves cut orthogonally if $f'(x)\,\phi'(x) = -1$.

14.7. Cartesian Subtangent and Subnormal.

Let the tangent and normal at any point $P(x, y)$ on a curve meet the x-axis in T and N respectively and let $\overline{PM}$ be drawn perpendicular to $\overleftrightarrow{OX}$.

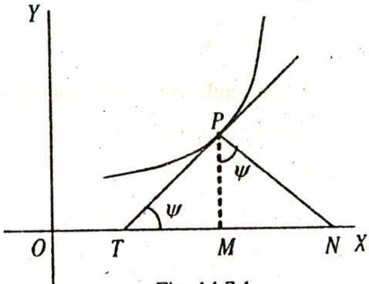


Fig. 14.7.1

Then, *TM* is called the *subtangent*, and *MN* the *subnormal* at *P*.

In the right-angled triangles *PTM*, *PNM*,

since $\angle NPM = \angle PTM = \psi$, and $PM = y$,

$$\textit{subtangent} = TM = y \cot\psi = \mathbf{y}\Big/\left(\frac{\mathbf{dy}}{\mathbf{dx}}\right),$$

$$\textit{subnormal} = MN = y \tan\psi = \mathbf{y}\,\frac{\mathbf{dy}}{\mathbf{dx}}.$$

Note. *PT* and *PN* are often called as the *length of the tangent* and the *length of the normal* (or sometimes simply tangent and normal) respectively.

Thus, from Δ^s *PTM* and *PNM*,

$$PT = y \operatorname{cosec}\psi = y\sqrt{1+\cot^2\psi} = y\sqrt{1+(1/y_1)^2} = \frac{1}{y_1}\left(y\sqrt{1+y_1^{\,2}}\right)$$

$$PN = y \sec\psi = y\sqrt{1+\tan^2\psi} = y\sqrt{1+y_1^{\,2}}.$$

14.8. Proof of $\underset{Q \to P}{\mathrm{Lt}} \dfrac{\text{chord}PQ}{\text{arc}PQ} = 1$.

Let $\overline{PP_1}, \overline{P_1P_2}, \ldots, \overline{P_{n-1}Q}$ be the sides of an open polygon inscribed in arc PQ of the curve $y = f(x)$. If the sum of the n sides $\Sigma \overleftrightarrow{PP_1}$ tends to a definite limit when $n \to \infty$ and the length of each side tends to zero, *that limit is defined as the length of the arc PQ.*

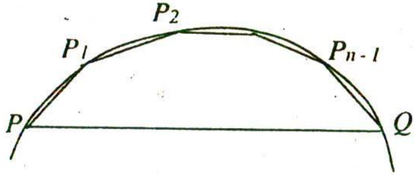


Fig. 14.8.1

Let $\theta_1, \theta_2, \ldots, \theta_n$ be the angles which the sides make with the chord $\overline{PQ}$, and let $f'(x)$ *be continuous* throughout PQ.

Projecting the sides on $\overline{PQ}$, we have

$$PQ = \text{proj.}\, PP_1 + \text{proj.}\, P_1P_2 + \ldots + \text{proj.}\, P_{n-1}Q$$

$$= PP_1 \cos\theta_1 + P_1P_2 \cos\theta_2 + \ldots + P_{n-1}Q \cos\theta_n$$

$\therefore$ it follows that $PQ < PP_1 + P_1P_2 + \ldots + P_{n-1}Q$

and $> \left(PP_1 + P_1P_2 + \ldots + P_{n-1}Q\right)\cos\theta$

where θ is numerically the greatest of the angles $\theta_1, \theta_2, \ldots, \theta_n$.

$$\text{Hence, } \cos\theta < \frac{PQ}{PP_1 + P_1P_2 + \ldots + P_{n-1}Q} < 1.$$

Since the chords $\overline{PP_1}, \overline{P_1P_2}, \ldots, \overline{P_{n-1}Q}$ as well as $\overline{PQ}$ are parallel to the tangents to the arcs at points between their respective extremities (by the Mean Value Theorem), it follows from the continuity of $f'(x)$ that the numerical value of θ can be made as small as we please by taking Q sufficiently near to P, and, in the limiting position, $\cos\theta \to 1$ and $\Sigma PP_1 \to \text{arc } PQ$.

$$\therefore \quad \underset{Q \to P}{Lt} \frac{\text{chord } PQ}{\text{arc } PQ} = 1$$

14.9. Derivative of arc-length (Cartesian).

Let $P(x, y)$ be the given point, and $Q(x + \Delta x,\ y + \Delta y)$ be any point near P on the curve.

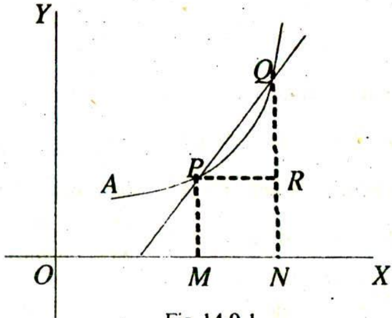


Fig 14.9.1

Let s denote the length of the arc AP measured from a fixed point A on the curve, and let $s + \Delta s$ denote the arc AQ. so that arc $PQ = \Delta s$. Here, s is obviously a function of x, and hence of y. We shall *assume* the fundamental limit

$$\underset{Q \to P}{Lt} \frac{\text{chord } PQ}{\text{arc } PQ} = 1$$

From the figure, $(\text{chord } PQ)^2 = PR^2 + QR^2 = (\Delta x)^2 + (\Delta y)^2$.

$$\therefore \quad \left(\frac{\text{chord } PQ}{\Delta s}\right)^2 \cdot \left(\frac{\Delta s}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2.$$

Now let $Q \to P$ as a limiting position; then $\Delta x \to 0$ and we have

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2, \qquad \ldots \quad (1)$$

$$\text{or} \quad \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \qquad \ldots \quad (2)$$

Since $\dfrac{ds}{dy} = \dfrac{ds}{dx} \cdot \dfrac{dx}{dy}$, we get, on multiplying both sides of (2) by $\dfrac{dx}{dy}$,

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \qquad \ldots \quad (3)$$

Cor. Multiplying both sides of (1), (2) and (3) by dx^2, dx, dy, we get the corresponding differential form

$$\mathbf{ds^2 = dx^2 + dy^2};$$

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx; \quad ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\cdot dy.$$

14.10 Values of $\sin\psi$, $\cos\psi$.

From ΔPQR [See *Fig.*, *§ 14.9*], $\sin QPR = \dfrac{RQ}{PQ} \doteq \dfrac{\Delta y}{\Delta s}\cdot\dfrac{\Delta s}{PQ}$.

In the limiting position when $Q \to P$, the secant PQ becomes the tangent at P, $\angle QPR \to \psi$ and $\Delta s \to 0$ and $\dfrac{\Delta s}{PQ} = \dfrac{arc\ PQ}{chord\ PQ} \to 1$.

$$\therefore \quad \sin\psi = \underset{\Delta s\to 0}{Lt}\frac{\Delta y}{\Delta s} = \frac{\mathbf{dy}}{\mathbf{ds}}. \qquad \ldots \quad (1)$$

$$\text{Similarly, } \cos\psi = \underset{\Delta s\to 0}{Lt}\frac{\Delta x}{\Delta s} = \frac{\mathbf{dx}}{\mathbf{ds}}. \qquad \ldots \quad (2)$$

Since $\tan\psi = \dfrac{dy}{dx}$ and $\cot\psi = \dfrac{dx}{dy}$, we get, from (2) and (3) of Art. 14.9, $\dfrac{ds}{dx} = \sec\psi.\ \dfrac{ds}{dy} = \operatorname{cosec}\psi$,

whence also $\cos\psi$, $\sin\psi$ are obtained.

$$\therefore \quad \left(\frac{\mathbf{dx}}{\mathbf{ds}}\right)^2 + \left(\frac{\mathbf{dy}}{\mathbf{ds}}\right)^2 = \mathbf{1}. \qquad \ldots \quad (3)$$

Cor. If $x = \phi(t)$, $y = \psi(t)$, $\dfrac{dx}{dt} = \dfrac{dx}{ds}\cdot\dfrac{ds}{dt}$; $\dfrac{dy}{dt} = \dfrac{dy}{ds}\cdot\dfrac{ds}{dt}$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left\{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right\}\left(\frac{ds}{dt}\right)^2$$

$$\therefore \quad \left(\frac{\mathrm{dx}}{\mathrm{dt}}\right)^2 + \left(\frac{\mathrm{dy}}{\mathrm{dt}}\right)^2 = \left(\frac{\mathrm{ds}}{\mathrm{dt}}\right)^2 \qquad \ldots \quad (4)$$

Note. Relations (2) and (3) of Art. 14.9 can also be deduced from the values of $\sin\psi$, $\cos\psi$, $\tan\psi$.

14.11. Illustrative Examples.

Ex. 1. *Find the equation of the tangent at* (x, y) *to the curve*

$$(x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}} = 1.$$

Here the equation of the curve is $f(x,y) \equiv (x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}} - 1 = 0$.

The equation of the tangent is

$$(X - x)f_x + (Y - y)f_y = 0,$$

$$i.e., \quad (X - x).\tfrac{2}{3}x^{-\frac{1}{3}}/a^{\frac{2}{3}} + (Y - y)\tfrac{2}{3}y^{-\frac{1}{3}}/b^{\frac{2}{3}} = 0,$$

$$i.e., \quad Xx^{-\frac{1}{3}}/a^{\frac{2}{3}} + Yy^{-\frac{1}{3}}/b^{\frac{2}{3}} = (x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}},$$

$$i.e., \quad Xx^{-\frac{1}{3}}/a^{\frac{2}{3}} + Yy^{-\frac{1}{3}}/b^{\frac{2}{3}} = 1.$$

Note. The equation of the tangent should be simplified as much as possible as in the above example.

Ex. 2. *Find the angle of intersection of the curves* $x^2 - y^2 = a^2$ *and* $x^2 + y^2 = a^2\sqrt{2}$. [*Patna, 1940*]

Adding and subtracting the equations of the two curves, we find their common points of intersection given by $2x^2 = a^2(\sqrt{2} + 1)$, *i.e.*, $x = \pm a\sqrt{(\sqrt{2}+1)}/\sqrt{2}$ and $2y^2 = a^2(\sqrt{2} - 1)$, *i.e.*, $y = \pm a\sqrt{(\sqrt{2}-1)}/\sqrt{2}$.

Since the equations of the curves can be written as

$$f(x,y) \equiv x^2 - y^2 - a^2 = 0 \text{ and } \phi(x,y) \equiv x^2 + y^2 - a^2\sqrt{2} = 0$$

hence if α be the angle of intersection of the curves at (x, y), we have, by Art. 14.6,

$$\tan\alpha = \frac{2x.2y - (2x)(-2y)}{2x.2x + (-2y)(2y)} = \frac{\pm 2xy}{x^2 - y^2} = 1,$$

on substituting the values of x and y found above. Hence, $\alpha = \frac{1}{4}\pi$.

$\therefore$ the curves intersect at an angle of 45°.

Ex. 3. *Find the condition that the conics*

$$ax^2 + by^2 = 1 \quad \textit{and} \quad a_1x^2 + b_1y^2 = 1$$

shall cut orthogonally.

The equations of the conics are

$$f(x,y) \equiv ax^2 + by^2 - 1 = 0, \qquad \ldots \quad (1)$$

$$\phi(x,y) \equiv a_1x^2 + b_1y^2 - 1 = 0 \qquad \ldots \quad (2)$$

Now, the condition that they should cut orthogonally at (x, y) is, by § 14.6.

$$f_x \phi_x + f_y \phi_y = 0,$$

i.e., $2ax.2a_1x + 2by.2b_1y = 0$,

i.e., $aa_1x^2 + bb_1y^2 = 0$ ... (3)

Since the point (x, y) is common to both (1) and (2), the required condition is obtained by eliminating x, y from (1), (2) and (3).

Subtracting (2) from (1), $(a - a_1)x^2 + (b - b_1)y^2 = 0$ ... (4)

Comparing (3) and (4), we get

$$\frac{a - a_1}{aa_1} = \frac{b - b_1}{bb_1}, \text{ or, } \frac{1}{a_1} - \frac{1}{a} = \frac{1}{b_1} - \frac{1}{b}$$

which is the required condition.

Ex. 4. *If* $x\cos\alpha + y\sin\alpha = p$ *touches the curve*

$$\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1,$$

show that $(a\cos\alpha)^{\frac{m}{m-1}} + (b\sin\alpha)^{\frac{m}{m-1}} = p^{\frac{m}{m-1}}$. [C.P. 1996]

The equation of the tangent to the given curve at (x, y) by formula (2) of Art. 14.2 is

$$(X - x)\frac{mx^{m-1}}{a^m} + (Y - y)\frac{my^{m-1}}{b^m} = 0,$$

i.e., $X x^{m-1}/a^m + Y y^{m-1}/b^m = x^m/a^m + y^m/b^m = 1$ ... (1)

If $X\cos\alpha + Y\sin\alpha = p$ ... (2)

touches the given curve, equations (1) and (2) must be identical.

Hence, $$\frac{x^{m-1}/a^m}{\cos\alpha} = \frac{y^{m-1}/b^m}{\sin\alpha} = \frac{1}{p},$$

i.e., $$\frac{x^{m-1}/a^{m-1}}{a\cos\alpha} = \frac{y^{m-1}/b^{m-1}}{b\sin\alpha} = \frac{1}{p},$$

$$\therefore \left(\frac{x}{a}\right)^{m-1} = \frac{a\cos\alpha}{p}, \left(\frac{y}{b}\right)^{m-1} = \frac{b\sin\alpha}{p}.$$

$$\therefore \left(\frac{a\cos\alpha}{p}\right)^{\frac{m}{m-1}} + \left(\frac{b\sin\alpha}{p}\right)^{\frac{m}{m-1}} = \left(\frac{x}{a}\right)^m + \left(\frac{y}{b}\right)^m = 1$$

i.e., $(a\cos\alpha)^{\frac{m}{m-1}} + (b\sin\alpha)^{\frac{m}{m-1}} = p^{\frac{m}{m-1}}$.

Ex. 5. *If x_1, y_1 be the parts of the axes of x and y intercepted by the tangent at any point (x, y) to the curve $(x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}} = 1$, show that $x_1^2/a^2 + y_1^2/b^2 = 1$.*

The equation of the tangent at (x, y) to the given curve is, as in Ex.1.

$$X\,x^{-\frac{1}{3}}\big/a^{\frac{2}{3}} + Y\,y^{-\frac{1}{3}}\big/b^{\frac{2}{3}} = 1.$$

Where it meets the x-axis, $Y = 0$, hence $X = a^{\frac{2}{3}}x^{\frac{1}{3}}$, *i.e.*, $x_1 = a^{\frac{2}{3}}x^{\frac{1}{3}}$, and where it meets the y-axis, $X = 0$, hence $Y = a^{\frac{2}{3}}y^{\frac{1}{3}}$, *i.e.*, $y_1 = b^{\frac{2}{3}}y^{\frac{1}{3}}$.

$$\therefore\ x_1^2/a^2 + y_1^2/b^2 = a^{\frac{4}{3}}x^{\frac{2}{3}}\big/a^2 + b^{\frac{4}{3}}y^{\frac{2}{3}}\big/b^2 = (x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}} = 1.$$

EXAMPLES - XIV(A)

1. Find the equation of the tangent at the point (x, y) on each of the following curves :

(i) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$ (ii) $\dfrac{x^m}{a^m} + \dfrac{y^m}{b^m} = 1.$

(iii) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ (iv) $x^3 - 3axy + y^3 = 0.$

(v) $(x^2 + y^2)^2 = a^2(x^2 - y^2).$

2. (i) Find the equation of the tangent at the point θ on each of the following curves :

(*a*) $x = a\cos\theta,\quad y = b\sin\theta.$

(*b*) $x = a\cos^3\theta,\quad y = b\sin^3\theta.$

(*c*) $x = a(\theta + \sin\theta),\ \ y = a(1 - \cos\theta).$

(ii) Find the equation of the normal at 't' on the curve

$x = a(2\cos t + \cos 2t),\quad y = a(2\sin t - \sin 2t).$

3. (i) Find the tangent at the point $(1, -1)$ to the curve

$x^3 + xy^2 - 3x^2 + 4x + 5y + 2 = 0.$

(ii) Show that the tangent at (a, b) to the curve

$(x/a)^3 + (y/b)^3 = 2$ is $x/a + y/b = 2.$ [*C.P. 1943*]

(iii) Show that the normal at the point $\theta = \frac{1}{4}\pi$ on the curve $x = 3\cos\theta - \cos^3\theta,\ y = 3\sin\theta - \sin^3\theta$ passes through the origin.

4. (i) Find the tangent and the normal to the curve

$$y(x-2)(x-3)-x+7=0$$

at the point where it cuts the x-axis.

(ii) Show that of the tangents at the points where the curve $y=(x-1)(x-2)(x-3)$ is met by the x-axis, two are parallel, and the third makes an angle of 135° with the x-axis.

(iii) Find the tangent to the curve $xy^2=4(4-x)$ at the point where it is cut by the line $y=x$.

5. (i) Find where the tangent is parallel to the x-axis for the curves :

(a) $y=x^3-3x^2-9x+15$ (b) $ax^2+2hxy+by^2=1$.

(ii) Find where the tangent is perpendicular to the x-axis for the curves

(a) $y^2=x^2(a-x)$. (b) $ax^2+2hxy+by^2=1$.

(c) $y=(x-3)^2(x-2)$. [C.P. 1935]

(iii) Show that the tangents to the curve

$$3x^2+4xy+5y^2-4=0$$

at the points in which it is intersected by the lines

$$3x+2y=0 \text{ and } 2x+5y=0$$

are parallel to the axes of co-ordinates.

(iv) Find at what points on the curve

$$y=2x^3-15x^2+34x-20$$

the tangents are parallel to $y+2x=0$.

(v) Find the points on the curve $y=x^2+3x+4$ the tangents at which pass through the origin.

6. Show that the tangent to the curve $x^3+y^3=3axy$ at the point other than the origin, where it meets the parabola $y^2=ax$, is parallel to the y-axis.

7. Prove that all points of the curve

$$y^2=4a\{x+a\sin(x/a)\}$$

at which the tangent is parallel to the x-axis lie on a parabola.

[C.P. 1998]

8. Tangents are drawn from the origin to the curve $y=\sin x$. Prove that their points of contact lie on $x^2y^2=x^2-y^2$.

9. (i) Show that the curve $(x/a)^n+(y/b)^n=2$ touches the straight line $x/a+y/b=2$ at the point (a,b), whatever be the value of n.

(ii) Prove that $\frac{x}{a}+\frac{y}{b}=1$ touches the curve $\frac{x}{a}+\log\left(\frac{y}{b}\right)=0$.

10. (i) If $lx+my=1$ touches the curve $(ax)^n+(by)^n=1$, show that $(l/a)^{\frac{n}{n-1}}+(m/b)^{\frac{n}{n-1}}=1$. [B.P. 1989]

(ii) If $lx+my=1$ is a normal to the parabola $y^2=4ax$, then show that $al^3+2alm^2=m^2$. [V.P. 1999]

11. Prove that the condition that $x\cos\alpha+y\sin\alpha=p$ should touch $x^m y^n=a^{m+n}$ is $p^{m+n}m^m n^n=(m+n)^{m+n}a^{m+n}\sin^n\alpha\cos^m\alpha$.

12. Find the angles of intersection of the following curves :

(i) $x^2-y^2=2a^2$ and $x^2+y^2=4a^2$.

(ii) $x^2=4y$ and $y(x^2+4)=8$.

(iii) $y=x^3$ and $6y=7-x^2$.

13. (i) Prove that the curves $\frac{x^2}{a}+\frac{y^2}{b}=1$ and $\frac{x^2}{a'}+\frac{y^2}{b'}=1$ will cut orthogonally if $a-b=a'-b'$. [C.P. 1980, '90,2007 V.P. 2000]

(ii) Find the condition that the curves $ax^3+by^3=1$ and $a'x^3+b'y^3=1$ should cut orthogonally.

(iii) Show that the curves $x^3-3xy^2=-2$ and $3x^2y-y^3=2$ cut orthogonally. [C. P. 2006]

14. (i) Prove that the sum of the intercepts of the tangent to the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ upon the co-ordinate axes is constant.

[B.P. 1993]

(ii) Find the abscissa of the point on each of the curves

(*a*) $ay^2=x^3$,

(*b*) $\sqrt{xy}=a+x$, the normal at which cuts off equal intercepts from the co-ordinate axes.

15. Show that the portion of the tangent at any point on the following curves intercepted between the axes is of constant length.

(i) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ [C.P. 1940]

(ii) $x=a\cos^3\theta$, $y=a\sin^3\theta$.

16. If the tangent at (x_1, y_1) to the curve $x^3+y^3=a^3$ meets the curve again in (x_2, y_2), show that $x_2/x_1+y_2/y_1=-1$.

17. (i) Show that at any point on the parabola $y^2 = 4ax$, the subnormal is constant and the subtangent varies as the abscissa of the point of contact.

(ii) Show that at any point on the hyperbola $xy = c^2$, the subtangent varies as the abscissa and the subnormal varies as the cube of the ordinate of the point of contact.

18. Prove that the subtangent is of constant length for the curve $\log y = x \log a$.

19. Show that for the curve $by^2 = (x + a)^3$ the square of the subtangent varies as the subnormal. [*C. P. 2006*]

20. Show that at any point on the curve $x^{m+n} = k^{m-n} y^{2n}$ the m^{th} power of the subtangent varies as the n^{th} power of the subnormal.
[*C.P. 1995, '97, 2002, 2004*]

21. For the curve $x^m y^n = a^{m+n}$, show that the subtangent at any point varies as the abscissa of the point.

22. Show that for any curve the rectangle contained by the subtangent and subnormal is equal to the square on the corresponding ordinate.
[*C. P. 2005*]

23. Find the lengths of the subtangent, subnormal, tangent and normal of the curves.

(i) $x = a(\theta + \sin\theta),\ \ y = a(1 - \cos\theta)$ at 'θ'

(ii) $x = a(\cos t + t\sin t),\ \ y = a(\sin t - t\cos t)$ at 't'.

[*C. P. 2006*]

24. Find the value of n so that the subnormal at any point on the curve $xy^n = a^{n+1}$ may be constant.

25. Show that in any curve

$$\frac{\text{subnormal}}{\text{subtangent}} = \left(\frac{\text{length of normal}}{\text{length of tangent}}\right)^2$$

26. Show that the length of the tangent at any point on the following curves is constant :

(i) $x = \sqrt{a^2 - y^2} + \dfrac{a}{2}\log\dfrac{a - \sqrt{a^2 - y^2}}{a + \sqrt{a^2 - y^2}}$.

(ii) $x = a\left(\cos t + \log\tan\frac{1}{2}t\right),\ \ y = a\sin t$.

(iii) $s = a\log(a/y)$.

27. (i) If p_1 and p_2 be the perpendiculars from the origin on the tangent and normal respectively at any point (x, y) on the curve, then show that

$$p_1 = x\sin\psi - y\cos\psi, \quad p_2 = x\cos\psi + y\sin\psi,$$

where, as usual, $\tan\psi = dy/dx$.

(ii) If, in the above case, the curve be $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ show that $4{p_1}^2 + {p_2}^2 = a^2$.

28. In the curve $x^m y^n = a^{m+n}$, show that the portion of the tangent intercepted between the axes is divided at its point of contact into segments which are in a constant ratio.

29. (i) In the catenary $y = c\cosh(x/c)$ show that the length of the perpendicular from the foot of the ordinate on the tangent is of constant length. [*C. P. 1943*]

(ii) Show that for the catenary $y = c\cosh(x/c)$ the length of the normal at any point is y^2/c.

30. Prove that the equation ofthe tangent to the curve $x = af(t)/\psi(t)$, $y = a\phi(t)/\psi(t)$ may be written in the form

$$\begin{vmatrix} x & y & a \\ f(t) & \phi(t) & \psi(t) \\ f'(t) & \phi'(t) & \psi'(t) \end{vmatrix} = 0$$

31. Find the equation of the tangent at the origin of the curve

$$\begin{aligned} y &= x^2\sin(1/x) \quad \text{for } x \neq 0 \\ &= 0 \qquad\qquad\quad \text{for } x = 0. \end{aligned}$$

32. Show that for the curve $y = x^{\frac{2}{3}}$ the tangent at the origin is $x = 0$, although dy/dx does not exist there.

33. If α and β be the intercepts on the axes of x and y cut off by the tangent to the curve $(x/a)^n + (y/b)^n = 1$ thén show that28

$$(a/\alpha)^{\frac{n}{n-1}} + (b/\beta)^{\frac{n}{n-1}} = 1$$

34. Find $\dfrac{ds}{dx}$ for the following curves :

(i) $y^2 = 4ax$. (ii) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

(iii) $y = \frac{1}{2}a(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$.

(iv) $x = a(1 - \cos\theta)$, $y = a(\theta + \sin\theta)$

35. If for the ellipse $x^2/a^2 + y^2/b^2 = 1$, $x = a\sin\phi$, show that

$$\frac{ds}{d\phi} = a\sqrt{1-e^2\sin^2\phi}\,.$$

36. Two curves are defined as follows :

(i) $x = t^3$,

$y = t^3 \sin(1/t)$, for $t \neq 0$

$= 0$ for $t = 0$.

(ii) $x = 2t + t^2 \sin(1/t)$, $y = t^2 \sin(1/t)$, for $t \neq 0$

$x = 0$, $y = 0$ for $t = 0$

show that, for the first curve, although dx/dt, dy/dt are *continuous* for $t = 0$, the *curve has no tangent* at the point; and for the second curve, although dx/dt, dy/dt *are not continuous* for $t = 0$, the *curve has a tangent* at the point.

ANSWERS

1. (i) $\dfrac{Xx}{a^2} + \dfrac{Yy}{b^2} = 1$. (ii) $\dfrac{Xx^{m-1}}{a^m} + \dfrac{Yy^{m-1}}{b^m} = 1$.

(iii) $Xx^{-\frac{1}{3}} + Yy^{-\frac{1}{3}} = a^{\frac{2}{3}}$. (iv) $X\left(x^2 - ay\right) + Y\left(y^2 - ax\right) = axy$.

(v) $X\left\{2x\left(x^2 + y^2\right) - a^2x\right\} + Y\left\{2y\left(x^2 + y^2\right) + a^2y\right\} = a^2\left(x^2 - y^2\right)$.

2. (i) (a) $\dfrac{X}{a}\cos\theta + \dfrac{Y}{b}\sin\theta = 1$.

(b) $bX\sin\theta + aY\cos\theta = ab\sin\theta\cos\theta$.

(c) $X\sin\frac{1}{2}\theta - Y\cos\frac{1}{2}\theta = a\theta\sin\frac{1}{2}\theta$.

(ii) $X\cos\frac{1}{2}t - Y\sin\frac{1}{2}t = 3a\cos\frac{3}{2}t$.

3. (i) $2x + 3y + 1 = 0$.

4. (i) Tangent $x - 20y - 7 = 0$; normal $20x + y - 140 = 0$.

(iii) $x + y - 4 = 0$

5. (i) (a) $(3, -12)$, $(-1, 20)$.

(b) Where $ax + hy = 0$ intersects the curve.

(ii) (a) $(0, 0)$, $(a, 0)$.

(b) Where $hx + by = 0$ intersects the curve.

(c) No such point exists

(iv) $(2, 4)$; $(3, 1)$. (v) $(2, 14)$; $(-2, 2)$.

12. (i) $\frac{1}{3}\pi$. (ii) $\tan^{-1} 3$. (iii) $\frac{1}{2}\pi$

13. (ii) $aa'(b-b')^{\frac{4}{3}} + bb'(a-a')^{\frac{4}{3}} = 0$.

14. (ii) (a) $\frac{4}{9}a$ (b) $\pm a/\sqrt{2}$.

23. (i) $a\sin\theta,\ 2a\sin^2\frac{1}{2}\theta\tan\frac{1}{2}\theta,\ 2a\sin\frac{1}{2}\theta,\ 2a\sin\frac{1}{2}\theta\tan\frac{1}{2}\theta$.

(ii) $y\cot t,\ y\tan t,\ y\operatorname{cosec} t,\ y\sec t$.

24. –2. **31.** $y = 0$.

34. (i) $\sqrt{\frac{a+x}{x}}$. (ii) $\left(\frac{a}{x}\right)^{\frac{1}{3}}$. (iii) $\frac{y}{a}$. (iv) $\sqrt{\left(\frac{2a}{x}\right)}$.

14.12. Angle between Radius Vector and Tangent.

If ϕ be the angle between the tangent and radius vector at any point on the curve $r = f(\theta)$, then

$$\tan\Phi = \frac{r\,d\theta}{dr},\ \sin\Phi = \frac{r\,d\theta}{ds},\ \cos\Phi = \frac{dr}{ds}$$

Let $P(r,\theta)$ be the given point on the curve $r = f(\theta)$, and $Q(r+\Delta r, \theta+\Delta\theta)$ be a point on the curve in the neighbourhood of P. Let QP be the secant through Q, P. Draw PN perpendicular on OQ.

Then, $\angle PON = \Delta\theta$, $PN = r\sin\Delta\theta$, $ON = r\cos\Delta\theta$.

Let ϕ be the angle which the tangent PT *at* P makes with the radius vector OP, *i.e.*, $\angle OPT = \phi$

From the right-angled $\Delta\ PQN$,

$$\tan PQN = \frac{PN}{NQ} = \frac{PN}{OQ - ON} = \frac{r\sin\Delta\theta}{r+\Delta r - r\cos\Delta\theta}$$

$$= \frac{r\sin\Delta\theta}{r(1-\cos\Delta\theta)+\Delta r} = \frac{r\sin\Delta\theta}{2r\sin^2\frac{1}{2}\Delta\theta + \Delta r}$$

$$= \frac{r\cdot(\sin\Delta\theta/\Delta\theta)}{\frac{1}{2}r\Delta\theta\cdot\left(\sin\frac{1}{2}\Delta\theta/\frac{1}{2}\Delta\theta\right)^2 + (\Delta r/\Delta\theta)}$$

[*i.e.*, dividing both numerator and denominator by $\Delta\theta$]

Now let $Q \to P$, then $\Delta\theta \to 0$, and secant QP becomes the tangent PT, and $\angle PQN \to \angle OPT$, *i.e.*, ϕ.

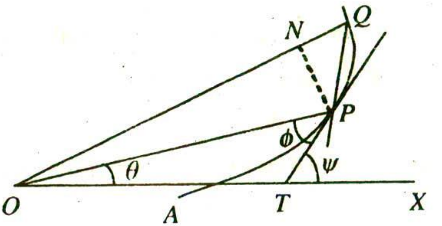


Fig 14.12.1

$$\therefore \quad \tan\phi = \underset{\Delta\theta\to 0}{Lt} \frac{r\cdot(\sin\Delta\theta/\Delta\theta)}{\frac{1}{2}r\Delta\theta\cdot\left(\sin\frac{1}{2}\Delta\theta/\frac{1}{2}\Delta\theta\right)^2+(\Delta r/\Delta\theta)}$$

$$= r\bigg/\frac{dr}{d\theta} \;(\text{ i.e., } = r/r') = r\frac{d\theta}{dr}$$

[since $Lt(\sin\Delta\theta/\Delta\theta)$ and $Lt\left(\sin\frac{1}{2}\Delta\theta/\frac{1}{2}\Delta\theta\right)$ are each equal to 1].

Now, let s denote the length of the arc AP measured from a fixed point A on the curve, and let $s+\Delta s$ denote the arc AQ., so that the arc $PQ = \Delta s$. Here s is obviously a function of θ, and hence of r.

$$\sin PQN = \frac{PN}{PQ} = \frac{r\sin\Delta\theta}{\Delta\theta}\cdot\frac{\Delta\theta}{\Delta s}\cdot\frac{\Delta s}{PQ}$$

Now let $Q\to P$, then $\Delta\theta\to 0$, $\Delta s\to 0$ and then $\Delta s/PQ = (\text{arc } PQ)/(\text{chord } PQ)\to 1$.

$$\therefore \quad \sin\phi = \underset{\Delta s\to 0}{Lt}\; r\frac{\Delta\theta}{\Delta s} = r\frac{d\theta}{ds}$$

$$\text{Again, } \cos PQN = \frac{QN}{PQ} = \frac{OQ-ON}{PQ} = \frac{(r+\Delta r)-r\cos\Delta\theta}{PQ}$$

$$= \frac{r(1-\cos\Delta\theta)+\Delta r}{PQ} = \frac{r\cdot 2\sin^2\frac{1}{2}\Delta\theta+\Delta r}{PQ}$$

$$= \frac{1}{2}r\Delta\theta\cdot\left(\frac{\sin\frac{1}{2}\Delta\theta}{\frac{1}{2}\Delta\theta}\right)^2\cdot\frac{\Delta\theta}{\Delta s}\cdot\frac{\Delta s}{PQ}+\frac{\Delta r}{\Delta s}\cdot\frac{\Delta s}{PQ}$$

Now let $Q\to P$. Then, as before,

$$\cos\phi = \underset{\Delta s\to 0}{Lt}\;\frac{\Delta r}{\Delta s} = r\frac{dr}{ds}$$

Otherwise :

$$\cos\phi = \cot\phi\cdot\sin\phi = \frac{dr}{r\,d\theta}\cdot\frac{r\,d\theta}{ds} = \frac{dr}{d\theta}\cdot\frac{d\theta}{ds} = \frac{dr}{ds}$$

Cor. 1. Form ΔOPT, $\angle PTX = \angle POT + \angle OPT$

$$\therefore \quad \psi = \theta + \Phi .$$

Cor. 2. $$\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2 = \cos^2\phi + \sin^2\phi = 1$$

14.13. Derivative of arc-length (*Polar*).

With the notations and the figure of the previous article, we have

$$PQ^2 = PN^2 + QN^2 = (r\sin\Delta\theta)^2 + (r + \Delta r - r\cos\Delta\theta)^2$$
$$= (r\sin\Delta\theta)^2 + \left(r\cdot 2\sin^2\tfrac{1}{2}\Delta\theta + \Delta r\right)^2.$$

Dividing both sides by $(\Delta\theta)^2$ we get

$$\left(\frac{PQ}{\Delta s}\cdot\frac{\Delta s}{\Delta\theta}\right)^2 = r^2\left(\frac{\sin\Delta\theta}{\Delta\theta}\right)^2 + \left\{\tfrac{1}{2}r\,\Delta\theta\cdot\left(\frac{\sin\frac{1}{2}\Delta\theta}{\frac{1}{2}\Delta\theta}\right)^2 + \left(\frac{\Delta r}{\Delta\theta}\right)\right\}^2$$

$\therefore$ in the limiting position, when $Q \to P$ and $\Delta\theta \to 0$.

$$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2, \qquad \ldots \quad (1)$$

i.e., $$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \qquad \ldots \quad (2)$$

Multiplying both sides of (2) by $\dfrac{d\theta}{dr}$, we get

$$\frac{ds}{dr} = \sqrt{1 + \left(r\frac{d\theta}{dr}\right)^2} \qquad \ldots \quad (3)$$

Cor. Multiplying (1), (2) and (3) by $d\theta^2$, $d\theta$, dr, we get the corresponding differential forms

$$ds^2 = dr^2 + r^2 d\theta^2.$$

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

$$ds = \sqrt{1 + \left(r\frac{d\theta}{dr}\right)^2}\, dr.$$

Note. Relations (2) and (3) can also be deduced from the values of $\sin\phi$, $\cos\phi$, $\tan\phi$.

14.14. Angle of intersection of two curves (*Polar*).

Suppose two curves $r = f(\theta)$, $r = \phi(\theta)$ intersect at the point P, and let $\overline{PT_1}$, $\overline{PT_2}$ be the tangents at P to the two curves, and let $\angle OPT_1 = \phi_1$, $\angle OPT_2 = \phi_2$

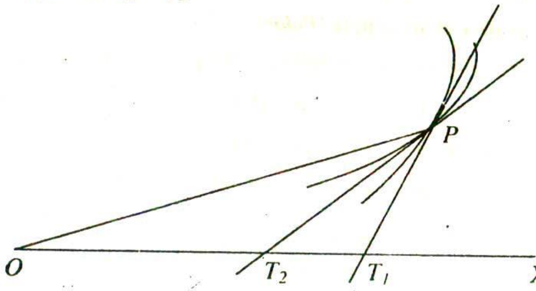


Fig 14.14.1

Then, if α be the angle between the two curves, $\alpha = \phi_1 - \phi_2$.

$$\therefore \quad \tan\alpha = \frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1 \tan\phi_2}$$

Since $\tan\phi_1 = r/r' = f(\theta)/f'(\theta)$ and $\tan\phi_2 = r/r' = \phi(\theta)/\phi'(\theta)$, we get

$$\tan\alpha = \frac{f(\theta)\phi'(\theta) - f'(\theta)\phi(\theta)}{f'(\theta)\phi'(\theta) + f(\theta)\phi(\theta)}.$$

14.15. Polar Subtangent and Subnormal.

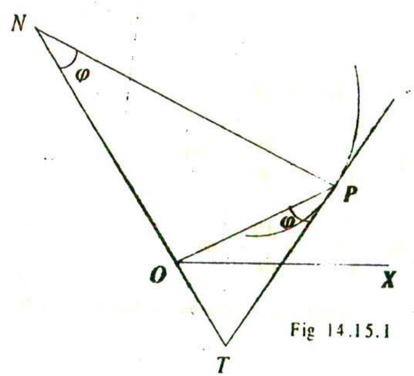


Fig 14.15.1

Let P be any point on the curve $r = f(\theta)$, and let the tangent PT and normal PN at P meet the line drawn through the pole O perpendicular to the radius vector $\overline{OP}$ in T and N respectively.

Then OT is called the *polar subtangent* and ON is called the *polar subnormal.*

Since, $\angle OPT = \phi$, $OT = OP \tan\phi = r.r\dfrac{d\theta}{dr}$.

$\therefore$ **polar subtangent** $= \mathbf{r^2 \dfrac{d\theta}{dr}}$.

Again, $ON = OP \tan OPN = r\cot\phi = r\cdot\dfrac{dr}{r\,d\theta}$.

$\therefore$ **polar subnormal** $= \mathbf{\dfrac{dr}{d\theta}}$.

Note. If $u = \dfrac{1}{r}$, $\dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}$,

$\therefore$ *polar subtangent* $= -\dfrac{1}{r^2}\dfrac{d\theta}{dr} = -\dfrac{d\theta}{du}$

14.16. Perpendicular from the pole on Tangent.

Let p be the length of the perpendicular ON from the pole O on the tangent PT at any point P.

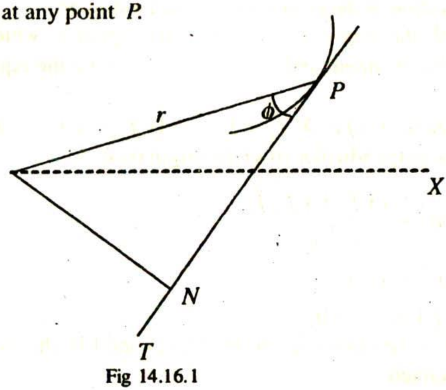


Fig 14.16.1

Then from ΔOPN, $ON = OP\sin\phi$.

$$\therefore \quad \mathbf{p = r\sin\phi}. \qquad \ldots \quad (1)$$

Again, $\dfrac{1}{p^2} = \dfrac{1}{r^2}\operatorname{cosec}^2\phi = \dfrac{1}{r^2}\left(1+\cot^2\phi\right)$

$$= \frac{1}{r^2}\left\{1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right\}$$

$$\therefore \quad \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \qquad (2)$$

The symbol u is generally used to denote $1/r$, the reciprocal of the radius vector.

$$\therefore \quad \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}.$$

Hence, the relation (2) becomes

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2. \qquad \ldots \quad (3)$$

14.17. The (p, r) or Pedal eqation of a curve.

The relation between the perpendicular (p) on the tangent at any point P on a cūrve and the radius vector (r) of the point of contaet P, from some given point O, is called the (p, r) or *pedal equation of the curve* with regard to O. Such equations are found very useful in the application of the principles of Statics and Dynamics.

(*i*) *Pedal equation deduced from Cartesian equation.*

Let us take the origin as the point with regard to which the pedal equation is to be obtained, and let $f(x, y) = 0$ be the equation of the curve.

The tangent at (x, y) is $X f_x + Y f_y - (x f_x + y f_y) = 0$.

If p be the perpendicular from the origin on it,

$$p^2 = \frac{(x f_x + y f_y)^2}{f_x^2 + f_y^2} \qquad \ldots \quad (1)$$

Also, $\quad r^2 = x^2 + y^2 \qquad \ldots \quad (2)$

and $\quad f(x, y) = 0 \qquad \ldots \quad (3)$

If x and y be eliminated from (1), (2) and (3), the required pedal equation is obtained.

(*ii*) *Pedal equation deduced from polar equation.*

Let us take the pole as the point with regard to which the pedal equation is to be obtained, and let $f(r, \theta) = 0$ be the eqation of the curve.

Let p be the perpendicular from the pole on the tangent at (r, θ); then

$$f(r,\theta) = 0, \qquad \ldots \quad (1)$$

$$\tan\phi = \frac{r\,d\theta}{dr}, \qquad \ldots \quad (2)$$

$$p = r\sin\phi. \qquad \ldots \quad (3)$$

If θ and ϕ be eliminated from (1), (2) and (3), the required pedal equation is obtained.

Note 1. When in any case nothing is mentioned about the given point with regard to which the pedal equation is to be obtained, the given point is to be taken as the origin in the Cartesian system and the pole in the Polar system.

Note 2. In some elementary cases, pedal equations can be easily obtained from geometrical properties. [See Ex. 6.of Art. 14.18]

14.18. Illustrative Examples.

Ex. 1. *Obtain the values of* $\sin\phi$, $\cos\phi$, $\tan\phi$ *and arc-differential in polar co-ordinates by transformation from Cartesian system.*

Since $\quad x = r\cos\theta,\ y = r\sin\theta,$

$\therefore \quad dx = \cos\theta\,dr - r\sin\theta\,d\theta$

and $\quad dy = \sin\theta\,dr + r\cos\theta\,d\theta,$

$\therefore \quad dx^2 + dy^2 = dr^2 + r^2\,d\theta^2,$

i.e. $\quad ds^2 = dr^2 + r^2\,d\theta^2. \qquad \ldots \quad (1)$

Also, $\quad x\,dy - y\,dx = r^2\,d\theta \qquad \ldots \quad (2)$

Again, since $\quad x^2 + y^2 = r^2, \quad \therefore\ x\,dx + y\,dy = r\,dr \qquad \ldots \quad (3)$

Now, $\quad \psi = \theta + \phi, \quad \therefore\ \phi = \psi - \theta.$

$\therefore \quad \cos\phi = \cos\psi\cos\theta + \sin\psi\sin\theta$

$$= \frac{dx}{ds}\cdot\frac{x}{r} + \frac{dy}{ds}\cdot\frac{y}{r} = \frac{x\,dx + y\,dy}{r\,ds} = \frac{dr}{ds}. \qquad [\,by\ (3)\,]$$

Again, $\sin\phi = \sin\psi\cos\theta - \cos\psi\sin\theta$

$$= \frac{dy}{ds}\cdot\frac{x}{r} - \frac{dx}{ds}\cdot\frac{y}{r} = \frac{x\,dy - y\,dx}{r\,ds} = \frac{r\,d\theta}{ds}. \qquad [\,by\ (2)\,]$$

$$\tan\phi = \sin\phi \div \cos\phi = \frac{r\,d\theta}{ds} \div \frac{dr}{ds} = \frac{r\,d\theta}{dr}.$$

Ex. 2. *Find the angle of intersection of the curves*

$$r = \sin\theta + \cos\theta \quad \textit{and} \quad r = 2\sin\theta.$$

Here, $\tan\phi_1 = \dfrac{r}{r'} = \dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta} = \dfrac{1+\tan\theta}{1-\tan\theta} = \tan\left(\frac{1}{4}\pi+\theta\right).$

$$\tan\phi_2 = \frac{r}{r'} = \frac{2\sin\theta}{2\cos\theta} = \tan\theta.$$

$\therefore \quad \phi_1 = \frac{1}{4}\pi+\theta, \quad \phi_2 = \theta.$

$\therefore$ angle of intersection $= \phi_1 - \phi_2 = \frac{1}{4}\pi.$

Ex. 3. *Prove that the curves*

$$r^n = a^n \cos n\theta \quad \textit{and} \quad r^n = b^n \sin n\theta$$

cut orthogonally.

Taking logarithm of the 1st equation,

$$n\log r = n\log a + \log\cos n\theta.$$

Differentiating with respect to θ,

$$n\frac{1}{r}\frac{dr}{d\theta} = -\frac{n\sin n\theta}{\cos n\theta}.$$

$\therefore \quad \cot\phi_1 = -\tan n\theta = \cot\left(\frac{1}{2}\pi+n\theta\right).$

Similarly, from the 2nd equation, we get

$$\cot\phi_2 = \cot n\theta.$$

$\therefore \quad \phi_1 = \frac{1}{2}\pi+n\theta; \quad \phi_2 = n\theta.$

$\therefore$ angle of intersection $= \phi_1 - \phi_2 = \frac{1}{2}\pi.$

Ex. 4. *Find the pedal equation of the parabola* $y^2 = 4ax$ *with regard to its vertex.*

Differentiating the given equation, $yy_1 = 2a$, $\therefore\ y_1 = 2a/y$.

$\therefore$ the tangent at (x, y) is $Y - y = (2a/y)(X-x)$,

i.e., $2aX - yY + 2ax = 0 \qquad (\because\ y^2 = 4ax)$

$$\therefore \quad p^2 = \frac{4a^2x^2}{4a^2+y^2} = \frac{4a^2x^2}{4a^2+4ax} = \frac{ax^2}{x+a} \qquad \ldots \quad (1)$$

and $r^2 = x^2+y^2 = x^2+4ax. \qquad \ldots \quad (2)$

From (1) and (2),

$$ax^2 - p^2x - ap^2 = 0 \qquad \ldots \quad (3)$$

and $x^2 + 4ax - r^2 = 0 \qquad \ldots \quad (4)$

By eliminating x between (3) and (4) the required relation between p and r will be obtained.

By cross-multiplication,

$$\frac{x^2}{p^2r^2+4a^2p^2}=\frac{x}{ar^2-ap^2}=\frac{1}{4a^2+p^2}$$

$$\therefore \quad \left(p^2r^2+4a^2p^2\right)\left(4a^2+p^2\right)=\left(ar^2-ap^2\right)^2$$

is the required pedal equation.

Ex. 5. *Find the pedal equation of* $r^m = a^m \cos m\theta$.

Taking logarithm of the given equation,

$$m \log r = m \log a + \log \cos m\theta .$$

Differentiating with respect to θ,

$$m\cdot\frac{1}{r}\frac{dr}{d\theta}=-\frac{m\sin m\theta}{\cos m\theta}.$$

$$\therefore \quad \cot\phi = -\tan m\theta = \cot\left(\tfrac{1}{2}\pi + m\theta\right),$$

$$\therefore \quad \phi = \tfrac{1}{2}\pi + m\theta .$$

Again, $p = r\sin\phi = r\sin\left(\tfrac{1}{2}\pi + m\theta\right) = r\cos m\theta$

$= r\cdot\dfrac{r^m}{a^m}$ from the equation of the curve.

$\therefore \quad r^{m+1} = a^m p$ is the required pedal equation.

Ex. 6. *Find geometrically the pedal equation of an ellipse with respect to a focus.*

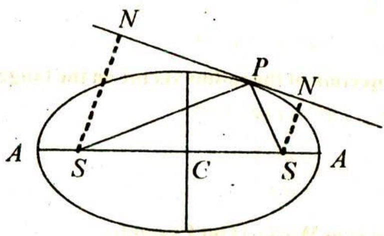


Fig 14.18.1

SN, S'N' are drawn perpendiculars on the tangent at any point *P* on the ellipse. $SP = r$, $S'P = r'$, $SN = p$, $S'N' = p'$. We know from Co-ordinate Geometry that

$$r + r' = 2a \quad \text{and} \quad pp' = b^2,$$

Since $\angle SPN = \angle S'PN'$, $\therefore$ Δ^s SPN, $S'PN'$ are similar.

$$\therefore \quad \frac{r}{p}=\frac{r'}{p'}=\sqrt{\frac{rr'}{pp'}}=\sqrt{\frac{r(2a-r)}{b^2}}$$

$$\therefore \quad \frac{r^2}{p^2} = \frac{r(2a-r)}{b^2}, \quad \text{or,} \quad \frac{b^2}{p^2} = \frac{2a}{r} - 1,$$

which is the required pedal equation.

Noting that the semi-latus rectum l of the ellipse $= b^2/a$, the above equation may be written as $\dfrac{l}{p^2} = \dfrac{2}{r} - \dfrac{1}{a}$.

Ex. 7. *Find the geometrical meaning of* $\dfrac{dp}{d\psi}$, *and hence deduce*

$$\mathbf{r}^2 = \mathbf{p}^2 + \left(\frac{\mathbf{dp}}{\mathbf{d}\psi}\right)^2.$$

We have $p = r \sin\phi$.

Differentiating with respect to ψ.

$$\frac{dp}{d\psi} = r\cos\phi \frac{d\phi}{d\psi} + \sin\phi \frac{dr}{d\psi}$$

$$= r\cos\phi \frac{d\phi}{d\psi} + \cos\phi \,.\, r\frac{d\theta}{dr} \cdot \frac{dr}{d\psi} \qquad \left[\because \frac{\sin\phi}{\cos\phi} = \frac{r\,d\theta}{dr}\right]$$

$$= r\cos\phi \frac{d}{d\psi}(\theta + \phi)$$

$$= \mathbf{r}\cos\phi \qquad (\because \ \theta + \phi = \psi).$$

$= PN$ (See *fig.*, § *13.16*)

= projection of the radius vector on the tangent.

From ΔOPN, $OP^2 = ON^2 + PN^2$.

$$\therefore \quad r^2 = p^2 + \left(\frac{dp}{d\psi}\right)^2.$$

14.19 Miscellaneous Worked Out Examples

Ex. 1. (*i*) At what point is the tangent to the parabola $y = x^2$ parallel to the straight line $y = 4x - 5$? [*C. P. 1982*]

(*ii*) Find the points on the curve $y = x^2$ [illegible] the tangents at which pass through the origin. [*C. P. 1991, 2002*]

Solution : $\because \quad y = x^2, \quad \ldots \quad (1)$

$\dfrac{dy}{dx} = 2x =$ gradient of the tangent to the curve (1) at (x, y).

If the tangent is parallel to the straight line $y = 4x - 5$, then $2x = 4$, *i.e.*, $x = 2$ and so $y = x^2 = 4$.

Hence the required point is $(2, 4)$.

(*ii*) Here, equation of the curve is $y = x^2 - 4x + 9$ ... (1)

Equation of the tangent at (x, y) is

$$Y - y = 2(x - 2)(X - x),$$

(X, Y) being current coordinates.

If this tangent passes through the origin $(0, 0)$, then

$$0 - y = 2(x - 2)(0 - x)$$

i.e., $y = 2x^2 - 4x$

or, $x^2 - 4x + 9 = 2x^2 - 4x$ [From (1)]

or, $x^2 = 9$, *i.e.*, $x = \pm 3$

when $x = 3$, $y = 6$ and when $x = -3$, $y = 30$

Hence the required points are $(3, 6)$ and $(-3, 30)$.

Ex. 2. Find the slope of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ at the point (x_1, y_1), and hence obtain the equations of the tangent and normal at the point. Also deduce that the portion of the tangent at (x_1, y_1) intercepted between the axes is of constant length.

[*C. P. 1983, '91.; B. P. 1995*]

Solution : Equation of the curve is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ ... (1)

$$\frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}} = \text{slope of the curve at the point } (x_1, y_1).$$

Equation of the tangent at (x_1, y_1) is

$$Y - y_1 = -\frac{y_1^{1/3}}{x_1^{1/3}}(X - x_1)$$

or, $\dfrac{X}{x_1^{1/3}} + \dfrac{Y}{y_1^{1/3}} = x_1^{\frac{2}{3}} + y_1^{\frac{2}{3}} = a^{\frac{2}{3}}$, [From (1)] ... (2)

is the equation of the tangent.

Similarly, equation of the normal at (x_1, y_1) is

$$Y - y_1 = \frac{x_1^{1/3}}{y_1^{1/3}}(X - x_1)$$

or, $x_1^{\frac{1}{3}}(X - x_1) = y_1^{\frac{1}{3}}(Y - y_1)$

Equation (2) can be written in the form

$$\frac{X}{x_1^{\frac{1}{3}}a^{\frac{2}{3}}} + \frac{Y}{y_1^{\frac{1}{3}}a^{\frac{2}{3}}} = 1,$$

so that the intercepts on the axes are $x_1^{\frac{1}{3}}a^{\frac{2}{3}}$ and $y_1^{\frac{1}{3}}a^{\frac{2}{3}}$ respectively.

Hence the length of the tangent intercepted between the axes

$$= \sqrt{\left\{\left(x_1^{\frac{1}{3}}a^{\frac{2}{3}}\right)^2 + \left(y_1^{\frac{1}{3}}a^{\frac{2}{3}}\right)^2\right\}}$$

$$= \sqrt{\left\{a^{\frac{4}{3}}\left(x_1^{\frac{2}{3}} + y_1^{\frac{2}{3}}\right)\right\}} = \left\{\left(a^{\frac{4}{3}} \cdot a^{\frac{2}{3}}\right)\right\}^{\frac{1}{2}} = a \text{, a constant.}$$

Ex. 3. If $p = x\cos\alpha + y\sin\alpha$ touches the curve $\left(\frac{x}{a}\right)^{\frac{n}{n-1}} + \left(\frac{y}{b}\right)^{\frac{n}{n-1}} = 1$, then prove that $p^n = (a\cos\alpha)^n + (b\sin\alpha)^n$. [C. P. 2001]

Solution : Equation of the curve is

$$\left(\frac{x}{a}\right)^{\frac{n}{n-1}} + \left(\frac{y}{b}\right)^{\frac{n}{n-1}} = 1 \qquad \ldots \quad (1)$$

or, $\frac{1}{a}\cdot\left(\frac{x}{a}\right)^{\frac{1}{n-1}} + \frac{1}{b}\cdot\left(\frac{y}{b}\right)^{\frac{1}{n-1}} \cdot \frac{dy}{dx} = 0$

i.e., $\frac{dy}{dx} = -\frac{b}{a}\cdot\left(\frac{x}{a}\right)^{\frac{1}{n-1}} \cdot \left(\frac{y}{b}\right)^{\frac{-1}{n-1}}$

Equation of the tangent of the curve (1) at any point (x, y) is

$$Y - y = -\frac{b}{a}\cdot\left(\frac{x}{a}\right)^{\frac{1}{n-1}} \cdot \left(\frac{y}{b}\right)^{\frac{-1}{n-1}}(X - x),$$

(X, Y) being current coordinates.

$$\text{or, } b\left(\frac{x}{a}\right)^{\frac{1}{n-1}}\cdot X+a\left(\frac{y}{b}\right)^{\frac{1}{n-1}}Y=bx\left(\frac{x}{a}\right)^{\frac{1}{n-1}}+ay\left(\frac{y}{b}\right)^{\frac{1}{n-1}}$$

$$=ab\left\{\left(\frac{x}{a}\right)^{\frac{n}{n-1}}+\left(\frac{y}{b}\right)^{\frac{n}{n-1}}\right\}$$

$$\text{or, } b\left(\frac{x}{a}\right)^{\frac{1}{n-1}}X+a\left(\frac{y}{b}\right)^{\frac{1}{n-1}}Y=ab \qquad \ldots \quad (2)$$

But $X\cos\alpha+Y\sin\alpha=p$ $\qquad \ldots \quad (3)$

touches the given curve. So, equations (2) and (3) should be identical.

$$\therefore \quad \frac{\cos\alpha}{b\left(\frac{x}{a}\right)^{\frac{1}{n-1}}}=\frac{\sin\alpha}{a\left(\frac{y}{b}\right)^{\frac{1}{n-1}}}=\frac{p}{ab}$$

$$\therefore \quad a\cos\alpha=p\cdot\left(\frac{x}{a}\right)^{\frac{1}{n-1}},\quad b\sin\alpha=p\cdot\left(\frac{y}{b}\right)^{\frac{1}{n-1}}.$$

Hence $(a\cos\alpha)^n+(b\sin\alpha)^n=p^n\left\{\left(\frac{x}{a}\right)^{\frac{n}{n-1}}+\left(\frac{y}{b}\right)^{\frac{n}{n-1}}\right\}$

$$=p^n \quad [\text{ From (1) }]$$

Ex. 4. (*i*) Find the length of the cartesian subtangent of the curve $y=e^{-\frac{x}{2}}$. [*C. P. 1983*]

(*ii*) For the parabola $y^2=4ax$, show that the subtangent is bisected at the vertex and that the subnormal is constant. [*B. P. 1993*]

Solution : (*i*) Equation of the curve is $y=e^{-\frac{x}{2}}$.

$$y_1=\frac{dy}{dx}=-\frac{1}{2}e^{-\frac{x}{2}}.$$

Length of cartesian subtangent $=\left|\frac{y}{y_1}\right|=\left|\frac{e^{-\frac{x}{2}}}{-\frac{1}{2}e^{-\frac{x}{2}}}\right|=|-2|=2$ units.

(*ii*) Equation of the parabola is $y^2 = 4ax$ ... (1)

$$2y\frac{dy}{dx} = 4a,\ \text{i.e.,}\ \frac{dy}{dx} = \frac{2a}{y}.$$

Equation of the tangent at any point $P(x,\ y)$ on the parabola is

$$Y - y = \frac{2a}{y}(X - x)$$

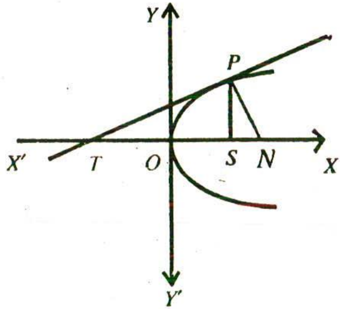
or, $\quad 2a\,X - yY = -y^2 + 2ax = -2a\,x$ ... (2)

If this tangent meets the $x-axis$ at $T(\alpha, 0)$, then

$2a\alpha - y0 = -2ax$

i.e., $\alpha = -x$.

So, coordinates of $T(-x, 0)$ and coordinates of $S\,(x, 0)$.



Evidently, the subtangent ST is bisected at the origin O,

since $|OS| = |OT| = x$.

Also, length of cartesian Subnormal

$$= \left| y\frac{dy}{dx} \right| = \left| y\frac{2a}{y} \right| = 2a,\ \text{(constant)}.$$

Ex. 5. (*i*) Find the length of the polar subtangent for the curve $r = a(1 + \cos\theta)$ at $\theta = \frac{\pi}{2}$. [*C. P. 1990*]

(*ii*) Find the length of perpendicular drawn from the pole upon the tangent to the cardioide $r = a(1 + \cos\theta)$ at the point whose vectorial angle is $\frac{\pi}{3}$. [*C. P. 1992*]

Solution : (*i*) Here, $r = a(1 + \cos\theta)$

$$\therefore\ \frac{dr}{d\theta} = -a\sin\theta$$

Length of polar subtangent $= \left| r^2 \frac{d\theta}{dr} \right|$

$$= \left| \frac{a^2(1 + \cos\theta)^2}{-a\sin\theta} \right| = a,\ \text{when}\ \theta = \frac{\pi}{2}.$$

(*ii*) Here, $r = a(1+\cos\theta)$, $\dfrac{dr}{d\theta} = -a\sin\theta$.

$$\tan\phi = r\frac{d\theta}{dr} = \frac{a(1+\cos\theta)}{-a\sin\theta} = -\cot\frac{\theta}{2} = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right). \quad \therefore\ \phi = \frac{\pi}{2}+\frac{\theta}{2}.$$

Now, length of perpendicular from pole upon the tangent at any point $(r,\ \theta)$ is,

$$p = r\sin\phi = r\sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right) = r\cos\frac{\theta}{2}$$

At the point where, $\theta = \dfrac{\pi}{3}$, $r = a(1+\cos\dfrac{\pi}{3}) = \dfrac{3}{2}a$

and $\cos\dfrac{1}{2}\theta = \cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$ and hence $p = \dfrac{3}{2}a \times \dfrac{\sqrt{3}}{2} = \dfrac{3\sqrt{3}a}{4}$.

Ex. 6. (*i*) Find the pedal equation of the following curves :

(*a*) $r = e^{\theta}$. [*C. P. 1983, 92, 2000, '02*]

(*b*) $r = a(1-\cos\theta)$. [*C. P. 1988*]

(*c*) $r = 5\cdot e^{\theta\cot\frac{\pi}{7}}$. [*C. P. 1989*]

(*ii*) Show that the pedal eqation of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ with respect to a focus is $\dfrac{b^2}{p^2} = \dfrac{2a}{r} - 1$. [*C. P. 1985, B. P. 1993*]

Solution : (*i*) (*a*) $r = e^{\theta}$ . . . (1)

or, $\log r = \theta \log e = \theta$

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = 1 \quad \text{or, } \cot\phi = 1 = \cot\frac{\pi}{4}, \quad \therefore\ \phi = \frac{\pi}{4}$$

$$p = r\sin\phi = r\sin\frac{\pi}{4} = \frac{r}{\sqrt{2}}$$

$\therefore$ $2p^2 = r^2$ is the required pedal equation.

(*b*) $r = a(1-\cos\theta) = 2a\sin^2\dfrac{\theta}{2}$ . . . (1)

$$\log r = \log 2a + 2\log\sin\frac{\theta}{2}$$

$$\text{or, } \frac{1}{r}\cdot\frac{dr}{d\theta} = 2\cdot\frac{\frac{1}{2}\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \cot\frac{\theta}{2} \qquad \therefore \; \cot\phi = \cot\frac{\theta}{2}, \text{ i.e., } \phi = \frac{\theta}{2}.$$

Now, $p = r\sin\phi = r\sin\dfrac{\theta}{2}$

or, $p^2 = r^2\sin^2\dfrac{\theta}{2} = r^2\cdot\dfrac{r}{2a}$ [From (1)]

or, $2ap^2 = r^3$ is the required pedal equation.

(c) $r = 5\cdot e^{\theta\cot\frac{\pi}{7}}$ . . . (1)

or, $\log r = \log 5 + \theta\cot\dfrac{\pi}{7}$ or, $\dfrac{1}{r}\cdot\dfrac{dr}{d\theta} = 0 + 1\cdot\cot\dfrac{\pi}{7} = \cot\dfrac{\pi}{7}$

or, $\cot\phi = \cot\dfrac{\pi}{7}$, i.e., $\phi = \dfrac{\pi}{7}$

Now, $p = r\sin\phi = r\sin\dfrac{\pi}{7}$

Hence, $p = r\cdot\sin\dfrac{\pi}{7}$ is the required pedal equation.

(ii) The polar equation of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

with focus as pole is given by $\dfrac{l}{r} = 1 + e\cos\theta$ . . . (1)

where, $l =$ semi-latus rectum $= \dfrac{b^2}{a}$ and $e =$ eccentricity $= \sqrt{\dfrac{a^2 - b^2}{a^2}}$.

From (1) on differentiation

$$-\frac{l}{r^2}\cdot\frac{dr}{d\theta} = -e\sin\theta \qquad \text{or, } \frac{1}{r^4}\cdot\left(\frac{dr}{d\theta}\right)^2 = \frac{e^2}{l^2}\cdot\sin^2\theta$$

or, $\dfrac{e^2\sin^2\theta}{l^2} = \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2 = \dfrac{1}{p^2} - \dfrac{1}{r^2}$ [Vide Art. 14·16]

or, $e^2\sin^2\theta = l^2\left(\dfrac{r^2 - p^2}{r^2p^2}\right)$ . . . (2)

Also, from (1)

$$e^2\cos^2\theta = \left(\frac{l-r}{r}\right)^2 \qquad \text{. . . (3)}$$

Adding (2) and (3)

$$e^2 = \frac{l^2\left(r^2 - p^2\right)}{r^2 p^2} + \frac{(l-r)^2}{r^2}$$

or, $\dfrac{l^2}{p^2} - \dfrac{l^2}{r^2} + \dfrac{l^2}{r^2} - \dfrac{2l}{r} + 1 - e^2 = 0$

or, $\dfrac{l^2}{p^2} - \dfrac{2l}{r} + \dfrac{l}{a} = 0$ $\left(\because\ 1 - e^2 = \dfrac{b^2}{a^2} = \dfrac{l}{a}\right)$

or, $\dfrac{2a}{r} - 1 = \dfrac{al}{p^2} = \dfrac{b^2}{p^2}$

Hence, the required pedal equation is $\dfrac{b^2}{p^2} = \dfrac{2a}{r} - 1$.

EXAMPLES - XIV (B)

1. Find $\dfrac{ds}{d\theta}$ for the following curves :
   (i) $r = a\left(1 + \cos\theta\right)$. (ii) $r = ae^{\theta \cot \alpha}$.
   (iii) $r^2 = a^2 \cos 2\theta$. (iv) $r^n = a^n \cos n\theta$.

2. Find $\dfrac{ds}{dr}$ for the curves :
   (i) $r = a\theta$; (ii) $r = a/\theta$.

3. Show that in the equiangular spiral $r = a e^{\theta \cot \alpha}$, the tangent is inclined at a constant angle to the radius vector. [*C. P. 2006*]

4. Show that for $\log r = a\theta + b$, $p \propto r$.

5. Find ϕ in terms of θ for the following curves :
   (i) Cardioide $r = a\left(1 - \cos\theta\right)$.
   (ii) Parabola $r = 2a/\left(1 - \cos\theta\right)$.
   (iii) Hyperbola $r^2 \cos 2\theta = a^2$.
   (iv) Lemniscate $r^2 = a^2 \cos 2\theta$.

6. Find the angle of intersection of the following curves :
   (i) $r = a\sin 2\theta$, $r = a\cos 2\theta$.
   (ii) $r = 6\cos\theta$, $r = 2\left(1 + \cos\theta\right)$
   (iii) $r^2 = 16 \sin 2\theta$, $r^2 \sin 2\theta = 4$.

7. Show that the following curves cut orthogonally :

(i) $r = a(1 + \cos\theta)$, $r = b(1 - \cos\theta)$.

(ii) $r = a/(1 + \cos\theta)$, $r = b/(1 - \cos\theta)$.

8. Show that the curves

$$r^n = a^n \sec(n\theta + \alpha),\ r^n = b^n \sec(n\theta + \beta).$$

intersect at an angle which is independent of a and b.

9. Prove that

$$\tan\phi = \left(x\frac{dy}{dx} - y\right)\Big/\left(x + y\frac{dy}{dx}\right),$$

where ϕ is the angle which the tangent to a curve makes with the radius vector drawn from the origin. [C. P. 1931, 2006]

[*Use* $\phi = \psi - \theta$, $\tan\theta = y/x$]

10. Show that for the curve $r\theta = a$, the polar subtangent is constant and for the curve $r = a\theta$, the polar subnormal is constant.

11. Show that for the curve $r = e^{\theta}$, the polar subtangent is equal to the polar subnormal. [C. P. 2007]

12. Find the polar subtangent of

(i) $r = a e^{\theta \cot\alpha}$ (ii) $r = a(1 - \cos\theta)$.

(iii) $r = 2a/(1 - \cos\theta)$. (iv) $r = l/(1 + e\cos\theta)$.

13. Show that the locus of the extremity of the polar subtangent of the curve $u + f(\theta) = 0$ is $u = f'\left(\frac{1}{2}\pi + \theta\right)$.

14. Prove that the locus of the extremity of the polar subnormal of the curve $r = f(\theta)$ is $r = f'\left(\theta - \frac{1}{2}\pi\right)$.

Hence deduce that the locus of the extremity of the polar subnormal of the equiangular spiral $r = a e^{\theta\cot\alpha}$ is another equiangular spiral.

15. Show that the pedal equation of the ellipse

$$x^2/a^2 + y^2/b^2 = 1.$$

with regard to the centre is $a^2b^2/p^2 = a^2 + b^2 - r^2$.

[C.P. 1988, '93, 2007, B.P. '95, V.P. '95]

16. (i) Show that the pedal equation of the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is $r^2 + 3p^2 = a^2$. [C. P. 2006]

(ii) Show that the pedal equation of the parabola $y^2 = 4a(x + a)$ is $p^2 = ar$. [C.P. 1931, '93, '97, V.P. 2000]

17. Show geometrically that the pedal equation of a circle with regard to a point on the circumference is $pd = r^2$, where d is the diameter of the circle.

18. Show that the pedal equation of

(i) the cardioide $r = a(1 + \cos\theta)$ is $r^3 = 2ap^2$.

[*B.P. 1997, V.P. '99*]

(ii) the parabola $r = 2a/(1 - \cos\theta)$ is $p^2 = ar$.

[*B.P. 1992, '94, V.P. 2002*]

(iii) the hyperbola $r^2 \cos 2\theta = a^2$ is $pr = a^2$.

(iv) the lemniscate $r^2 = a^2 \cos 2\theta$ is $r^3 = a^2 p$.

[*C.P. 1998, 2001, 2008*]

(v) the equiangular spiral $r = a\, e^{\theta \cot \alpha}$ is $p = r \sin \alpha$.

(vi) the class of curves $r^n = a^n \sin n\theta$ is $r^{n+1} = a^n p$.

(vii) the reciprocal spiral $r\theta = a$ is $p^2 \left(a^2 + r^2\right) = a^2 r^2$.

[*C. P. 1938*]

ANSWERS

1. (i) $2a \cos \frac{1}{2}\theta$. (ii) $r \operatorname{cosec} \alpha$. (iii) a^2/r. (iv) $a \sec^{\frac{n-1}{n}} n\theta$.

2. (i) $\sqrt{r^2 + a^2}/a$. (ii) $-\sqrt{r^2 + a^2}/r$.

5. (i) $\frac{1}{2}\theta$. (ii) $\pi - \frac{1}{2}\theta$. (iii) $\frac{1}{2}\pi - 2\theta$. (iv) $\frac{1}{2}\pi + 2\theta$.

6. (i) $\tan^{-1}\frac{4}{3}$. (ii) $\frac{1}{6}\pi$ (iii) $\frac{2}{3}\pi$.

12. (i) $r \tan \alpha$. (ii) $\left(2a \sin^3 \frac{1}{2}\theta\right) / \left(\cos \frac{1}{2}\theta\right)$.

(iii) $2a \operatorname{cosec} \theta$. (iv) $l / (e \sin \theta)$.