15 CURVATURE

15.1. Definitions.

Let P be a given point on a curve, and Q be a point on the curve near P. Let the arc AP measured from some fixed point A on the curve be s, and the arc AQ be $s + \Delta s$; then the arc $PQ = \Delta s$. Let $\overline{TPL}$, $\overline{MRQ}$ be the tangents to the curve at P and Q, and let $m\angle PTM = \psi$ and $m\angle RMX = \psi + \Delta\psi$; then $m\angle QRL = \Delta\psi$. Thus, $\Delta\psi$ is the change in the inclination of the tangent line as the point of contact of the tangent line describes the arc PQ $(= \Delta s)$.

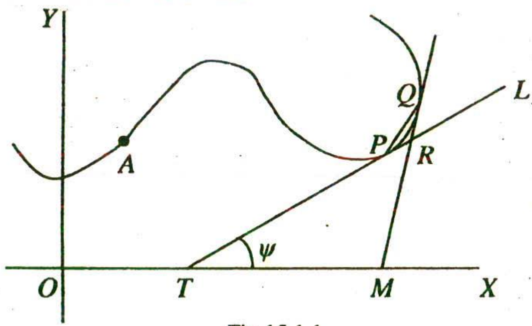


Fig 15.1.1

$\dfrac{\Delta\psi}{\Delta s}$ is called the average curvature of the arc PQ.

The curvature at P (denoted by χ) is the limiting value, when it exists, of the average curvature when $Q \to P$ (from either side) along the curve, *i.e.*, curvature at P

$$\chi = \underset{\Delta s \to 0}{Lt} \frac{\Delta\psi}{\Delta s} = \frac{d\psi}{ds}.$$

Thus, *the curvature is the rate of change of direction of the curve with respect to the arc, or roughly speaking, the curvature is the "rate at which the curve curves".*

The reciprocal of the curvature at any point P is called the radius of curvature at P, and is denoted by ρ. Thus,

$$\rho = \frac{ds}{d\psi}.$$

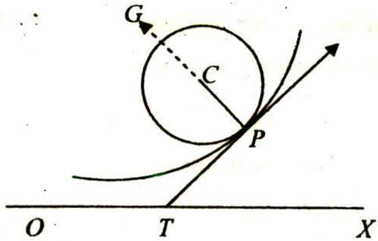


Fig 15.1.2

If a length PC equal to ρ is measured from P along the positive direction of the normal, the point C is called the *centre of curvature* at P, and the circle with centre C and radius CP (*i.e.*, ρ) is called the *circle of curvature at P*.

Any chord of this circle through the point of contact is called a *chord of curvature*.

Note 1. The line PG which makes an angle $+\frac{1}{2}\pi$ with the positive direction of the tangent (*i.e.*, the direction in which s increases) is called the *positive direction of the normal at P*. To avoid ambiguities we make the convention that ρ is positive or negative according as C is on the positive or negative side of the normal.

Note 2. The above formula is convenient only when the equation of the curve is given in terms of s and ψ (*i.e.*, when the *intrinsic equation* of the curve is given). So in the next article we shall obtain different transformations of the above formula for the radius of curvature for different forms of the above formula for the radius of curvature for different forms of the equations of the curve, and henceforth whenever we require the curvature of a curve we shall take the reciprocals of those radii of curvature.

Note 3. Since the radius of curvature of a circle is equal to its radius (See *Ex. 1, § 15.6*), it follows that the radius of curvature at any point P is the radius of a circle which has the same curvature at P as the curve has, and this explains the nomenclature of the above circle. Since, the curve and the circle of curvature at any point $P\ (x, y)$ have the same tangent and the same curvature, hence x, y, y', y'' *have the same values at P for the circle of curvature and the curve.* [See *Art. 15.2*]

15.2. Formulae for radius of Curvature.

(A) For the **Cartesian equation** $y = f(x)$.

We know $\dfrac{dy}{dx} = \tan\psi$.

$\therefore$ differentiating with respect to x,

$$\frac{d^2y}{dx^2} = \sec^2\psi\frac{d\psi}{dx} = \sec^2\psi\frac{d\psi}{ds}\cdot\frac{ds}{dx}$$

$$= \sec^3\psi\frac{d\psi}{ds}. \quad \left[\because \frac{dx}{ds} = \cos\psi\right]$$

$$\therefore \quad \rho = \frac{ds}{d\psi} = \sec^3\psi\bigg/\frac{d^2y}{dx^2}.$$

Since $\quad \sec\psi = \left(1+\tan^2\psi\right)^{\frac{1}{2}} = \left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}.$

$$\therefore \quad \rho = \frac{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}} = \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2} \qquad \ldots(1)$$

where $y_2 \neq 0$.

Note 1. Making the convention of attaching positive sign to $\left(1+y_1^2\right)^{\frac{3}{2}}$, ρ is positive or negative according as y_2 is positive or negative.

Note 2. *The above formula fails* when at any point y_1 becomes infinite, *i.e.*, when the tangent at the point is parallel to the y-axis (For *illustration*, see *Ex. 4, § 15.6*). In such cases the following formula, *for the equation of the curve as* $x = \phi(y)$, would be found useful.

$\dfrac{dx}{dy} = \cot\psi, \quad \therefore \quad$ differentiating with respect to y,

$$\frac{d^2x}{dy^2} = -\operatorname{cosec}^2\psi\frac{d\psi}{dy} = -\operatorname{cosec}^2\psi\cdot\frac{d\psi}{ds}\cdot\frac{ds}{dy}$$

$$= -\operatorname{cosec}^3\psi\cdot\frac{1}{\rho}. \qquad \left[\because \frac{dy}{ds} = \sin\psi\right]$$

$$\therefore \quad \rho = -\operatorname{cosec}^3\psi\bigg/\frac{d^2x}{dy^2}$$

Since $\operatorname{cosec}^2 \psi = 1 + \cot^2 \psi = 1 + \left(\frac{dx}{dy}\right)^2$.

$\therefore$ considering the magnitude only of the radius of curvature

$$\rho = \frac{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2x}{dy^2}} = \frac{\left(1 + x_1^2\right)^{\frac{3}{2}}}{x_2} \qquad \ldots \quad (1)$$

where $x_2 \neq 0$.

(B) For the Parametric equations $\mathbf{x} = \phi(\mathbf{t}),\ \mathbf{y} = \psi(\mathbf{t})$.

Here, $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{y'}{x'} \qquad (x' \neq 0)$,

where dashes denote differentiations with respect to t.

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{y'}{x'}\right) = \frac{d}{dt}\left(\frac{y'}{x'}\right) \cdot \frac{dt}{dx} = \frac{x'y'' - y'x''}{x'^3}.$$

Then substituting the values of $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ in the formula (1) above, we get

$$\rho = \frac{\left(x'^2 + y'^2\right)^{\frac{3}{2}}}{x'y'' - y'x''}. \qquad \ldots \quad (2)$$

where dashes denote differentiations with respect to t, and where

$$x'y'' - y'x'' \neq 0.$$

(C) For the Implicit equation $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$.

Here, $\frac{dy}{dx} = -\frac{f_x}{f_y}$, $\left(f_y \neq 0\right)$, *i.e.*, $f_x + f_y \frac{dy}{dx} = 0$

Differentiating this with respect to x.

$$f_{xx} + f_{xy} \frac{dy}{dx} + \left(f_{yx} + f_{yy} \frac{dy}{dx}\right) \frac{dy}{dx} + f_y \frac{d^2y}{dx^2} = 0$$

$$\text{or,} \quad f_{xx} + 2f_{xy} \frac{dy}{dx} + f_{yy}\left(\frac{dy}{dx}\right)^2 + f_y \frac{d^2y}{dx^2} = 0$$

$\left[\because \text{ we assume here } f_{xy} = f_{yx}\right]$

whence, replacing $\dfrac{f_x}{f_y}$ for $\dfrac{dy}{dx}$ and simplifying,

$$\frac{d^2y}{dx^2} = -\frac{f_{xx}f_y^{\,2} - 2f_{xy}f_xf_y + f_{yy}f_x^{\,2}}{f_y^{\,3}}.$$

Substituting the values of $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$ in the formula (1) above, and considering the magnitude of ρ only, we get

$$\rho = \frac{\left(f_x^{\,2} + f_y^{\,2}\right)^{\frac{3}{2}}}{f_{xx}f_y^{\,2} - 2f_{xy}f_xf_y + f_{yy}f_x^{\,2}}. \qquad \ldots (3)$$

where $f_{xx}f_y^{\,2} - 2f_{xy}f_xf_y + f_{yy}f_x^{\,2} \neq 0$.

(D) For the **polar equation** $\mathbf{r = f(\theta)}$.

$$\rho = \frac{ds}{d\psi} = \frac{ds}{d\theta}\cdot\frac{d\theta}{d\psi} = \frac{ds}{d\theta}\Big/\frac{d\psi}{d\theta} \qquad \ldots (4)$$

Now, $\psi = \theta + \phi = \theta + \tan^{-1}\dfrac{r}{r_1}$, where $r_1 = \dfrac{dr}{d\theta}$.

$$\therefore \quad \frac{d\psi}{d\theta} = 1 + \frac{1}{1 + \dfrac{r^2}{r_1^{\,2}}}\cdot\frac{r_1^{\,2} - rr_2}{r_1^{\,2}}.$$

$$= 1 + \frac{r_1^{\,2} - rr_2}{r^2 + r_1^{\,2}} = \frac{r^2 + 2r_1^{\,2} - rr_2}{r^2 + r_1^{\,2}} \qquad \ldots (5)$$

Again, $\dfrac{ds}{d\theta} = \sqrt{r^2 + r_1^{\,2}}$ $\qquad \ldots (6)$

[See *Art. 14.13*]

$\therefore$ from (4), (5) and (6), we get

$$\rho = \frac{\left(r^2 + r_1^{\,2}\right)^{\frac{3}{2}}}{r^2 + 2r_1^{\,2} - rr_2} \qquad \ldots (7)$$

Cor. For the Polar equation $\mathbf{u = f(\theta)}$, where $u = 1/r$.

Since $u = 1/r$, $\therefore$ $r_1 = -\dfrac{u_1}{u^2}$; $r_2 = -\dfrac{uu_2 - 2u_1^2}{u^3}$.

Substituting these values of r, r_1, r_2 in the formula (7) above, we get

$$\rho = \frac{\left(u^2 + u_1^2\right)^{\frac{3}{2}}}{u^3\left(u + u_2\right)}, \qquad \ldots (7a)$$

where $u^3\left(u + u_2\right) \neq 0$.

(E) For the **Pedal equation $\mathbf{p = f(r)}$.**

We have $p = r \sin\phi$.

Differentiating with respect to r.

$$\frac{dp}{dr} = \sin\phi + r\cos\phi\frac{d\phi}{dr}$$

$$= r\frac{d\theta}{ds} + r\frac{d\phi}{dr}\cdot\frac{dr}{ds} = r\frac{d\theta}{ds} + r\frac{d\phi}{ds}$$

$$= r\frac{d}{ds}(\theta + \phi) = r\frac{d\psi}{ds} = r\cdot\frac{1}{\rho}. \quad [\because \theta + \phi = \psi.]$$

$$\therefore \ \boldsymbol{\rho = r\frac{dr}{dp}}. \qquad \ldots (8)$$

(F) For the **Tangential polar equation $\mathbf{p = f(\psi)}$.**

When the tangential polar equation, *i.e.*, the relation between p and ψ of a curve is given,

$$\frac{dp}{d\psi} = \frac{dp}{dr}\cdot\frac{dr}{ds}\cdot\frac{ds}{d\psi} = \frac{dp}{dr}\cdot\cos\phi\cdot\rho = \frac{dp}{dr}\cdot\cos\phi\cdot r\frac{dr}{dp}$$

$$= r\cos\phi.$$

$$\therefore \quad p^2 + \left(\frac{dp}{d\psi}\right)^2 = r^2\sin^2\phi + r^2\cos^2\phi = r^2.$$

Differentiating with respect to p,

$$2p + 2\frac{dp}{d\psi}\cdot\frac{d^2p}{d\psi^2}\cdot\frac{d\psi}{dp} = 2r\frac{dr}{dp}.$$

$$\therefore \ \boldsymbol{\rho = p + \frac{d^2p}{d\psi^2}} \qquad \ldots (9)$$

Alternative Method :

If p be the length of the perpendicular from the origin on the tangent at (x, y), *viz.*, $Y - y_1 X + xy_1 - y = 0$,

then $$p = \frac{xy_1 - y}{\sqrt{1 + y_1{}^2}} = \frac{x \tan\psi - y}{\sqrt{1 + \tan^2\psi}}.$$

$$\therefore \quad p = x\sin\psi - y\cos\psi.$$

$$\therefore \quad \frac{dp}{d\psi} = \frac{dx}{d\psi}\sin\psi - x\cos\psi + \frac{dy}{d\psi}\cos\psi + y\sin\psi.$$

$$= x\cos\psi + y\sin\psi.$$

$$\left[\text{Since } \frac{dx}{d\psi} = \frac{dx}{ds}\cdot\frac{ds}{d\psi} = \rho\cos\psi.\ \frac{dy}{d\psi} = \frac{dy}{ds}\cdot\frac{ds}{d\psi} = \rho\sin\psi\right]$$

Similarly, $$\frac{d^2p}{d\psi^2} = \frac{dx}{d\psi}\cos\psi - x\sin\psi + \frac{dy}{d\psi}\sin\psi + y\cos\psi$$
$$= \rho\cos^2\psi - x\sin\psi + \rho\sin^2\psi + y\cos\psi$$
$$= \rho - (x\sin\psi - y\cos\psi) = \rho - p.$$

Hence, the result follows.

15.3. A Theorem on curvature.

If a circle be drawn touching a curve at P and cutting it at another point P_1, then, as $P_1 \to P$, the circle tends to the circle of curvature.

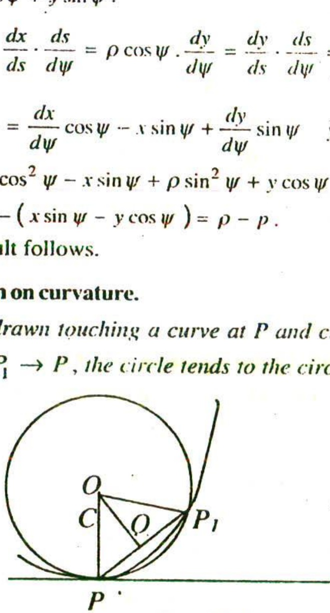


Fig 15.3.1

Let C be the centre of curvature at P, and let $y = f(x)$ be the equation of the curve, where $f'(x)$ and $f''(x)$ exist.

Let $O(\xi, \eta)$ be the centre of a circle touching the curve at P and cutting it again at P_1, and let r be its radius; also let $Q(x, y)$ be any point on the arc PP_1 of the curve.

$$\therefore \quad OQ^2 = (\xi - x)^2 + (\eta - y)^2 \equiv F(x).$$

Since $OP^2 = OP_1^2 = r^2$, it follows that $F(x)$ has the same value both at P and P_1. Hence by Rolle's Theorem, there exists a point $Q_1(x_1, y_1)$ between P and P_1 such that $F'(x_1) = 0$,

$$i.e., \ (\xi - x_1) + (\eta - y_1)\left(\frac{dy}{dx}\right)_{x_1} = 0,$$

which is evidently the condition that $\overline{OQ_1}$ is the normal to the curve at Q_1. Now let $P_1 \to P$ then Q_1 also $\to P$ and hence by Art. 15.8, O, the point of intersection of the normals at Q and P, tends to C, the centre of curvature and thus r also tends to CP, *i.e.*, ρ.

Thus, the circle tends to the circle of curvature.

15.4. Curvature at the origin.

(i) Method of substitution.

Radius of curvature at the origin can be found by substituting $x = 0$, $y = 0$ in the value of ρ obtained from Art. 15.2, or by directly substituting the values of $(y_1)_0$ and $(y_2)_0$ in the formula.

(ii) *Method of Expansion.*

In some cases the above method fails, or becomes laborious. In such cases, the values of $(y_1)_0$ and $(y_2)_0$ can be easily obtained in the following way by assuming the equation of the curve to be $y = f(x)$, and writing for y in the given equation its expansion by Maclaurin's theorem, viz., $xf'(0) + \frac{x^2}{2!} f''(0) + \ldots$ [$f(0)$ being zero here, since the curve passes through the origin], i.e., $px + qx^2/2! + \ldots$, where p, q stand for $f'(0)$, $f''(0)$, i.e., $(y_1)_0$, $(y_2)_0$, and then equating coefficients of like powers of x in the identity obtained.

This is illustrated in Example 9 of Art. 15.6.

(iii) Newton's Formula.

If the curve passes through the origin, and the axis of x *is the tangent at the origin*, we have

$$x = 0, \ y = 0, \ (y_1)_0, \ i.e., \ p = 0.$$

$\therefore$ by Maclaurin's Theorem,

$$y = qx^2/2! + \ldots$$

Dividing by $x^2/2!$ and taking limits as $x \to 0$, we get

$$Lt \left(2y/x^2\right) = q.$$

It should be noted here that as $x \to 0$, y also $\to 0$,

But from formula of Art. 15.2. at the origin $\rho = \dfrac{\left(1+p^2\right)^{\frac{3}{2}}}{q} = \dfrac{1}{q}$,

$$\therefore\ \rho = \underset{\substack{x \to 0 \\ y \to 0}}{Lt} \frac{x^2}{2y} \qquad \ldots (1)$$

Similarly, *if a curve passes through the origin, and the axis of y is the tangent there,* we have at the origin

$$\rho = \underset{\substack{x \to 0 \\ y \to 0}}{Lt} \frac{y^2}{2x} \qquad \ldots (2)$$

Geometrically :

Let the x-axis be the tangent at the origin.

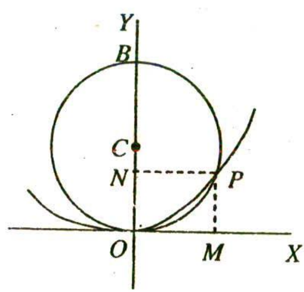


Fig.15.4.1

Draw a circle touching the curve at O, and passing through a point P (x, y) near O on the curve. Now, when $P \to O$, along the curve, the limiting position of the circle is the circle of the curvature. (Art. 15.3)

Let $\overline{OB}$ be the diameter of the circle, and draw $\overline{PN}$ perpendicular to it, and $\overline{PM}$ perpendicular to $\overline{OX}$. Let r be the radius of the circle.

Then, $ON.NB = PN^2$, *i.e.*, $ON(OB - ON) = PN^2$.

$$\therefore \quad OB = \frac{PN^2}{ON} + ON = \frac{PN^2 + ON^2}{ON} = \frac{OP^2}{ON}.$$

i.e., $$2r = \frac{OP^2}{PM} = \frac{x^2 + y^2}{y} = \frac{x^2}{y} + y.$$

In the limit when $P \to O$, $x \to 0$, $y \to 0$, $r \to \rho$, and hence we get as before

$$\rho = \frac{1}{2} Lt \frac{x^2}{y}.$$

Similarly, *when the y-axis is the tangent at the origin,*

we obtain $$\rho = \frac{1}{2} Lt \frac{y^2}{x}.$$

Analytically :

The equation of the circle passing through the origin and having the x-axis as the tangent at the origin is

$$x^2 + y^2 - 2fy = 0. \qquad \ldots (3)$$

If r be the radius of the circle, then $r = f$.

Since (3) passes through the point (x, y) on the curve,

$$\therefore \quad x^2 + y^2 - 2fy = 0, \text{ whence } f = (x^2 + y^2)/(2y).$$

$$\therefore \quad \rho = Lt\, r = Lt\, f = Lt \frac{x^2 + y^2}{2y} = Lt \frac{x^2}{2y}.$$

General Case :

If $ax + by = 0$ *be the tangent at the origin*, then proceeding as above, we get

$$OB = \frac{OP^2}{PM} = \frac{x^2 + y^2}{(ax + by)/\sqrt{(a^2 + b^2)}}.$$

$$\therefore \quad \rho = \tfrac{1}{2}\sqrt{a^2 + b^2}\,.\ \underset{\substack{x \to 0 \\ y \to 0}}{Lt} \frac{x^2 + y^2}{ax + by}. \qquad \ldots (4)$$

Note. It should be noted that as $x \to 0$, $y \to 0$, $y/x \to \left(-\frac{a}{b}\right)$, the '$m$' of the tangent line $ax + by = 0$. Here, it is supposed that $a \neq 0$, $b \neq 0$.

15.5. Chord of curvature through the origin (pole).

Let PQ be a chord passing through the origin O of the circle of curvature at P on the given curve, and let C be the centre of curvature and PT be the tangent at P.

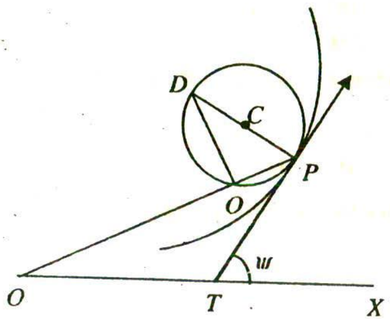


Fig 15.5.1

Join PC; produce it to D ; join DQ.

Then $\angle PQD =$ a rt. $\angle$, being in a semi-circle.

$$\angle OPT = \phi \text{ and } \angle PTX = \psi .$$

From ΔPQD, **chord PQ** $= PD \cos DPQ$

$$= 2\rho \cos\left(\tfrac{1}{2}\pi - \phi\right)$$

$$= \mathbf{2\rho \sin \phi}$$

$$= 2 . r \frac{dr}{dp} \cdot \frac{p}{r}$$

$$= \mathbf{2p \frac{dr}{dp}}.$$

Note 1. From above it is clear that the *chord of curvature through the origin* can be easily obtained when the pedal equation of the curve is given.

Note 2. If the chord PQ, instead of passing through the origin, makes an angle α with the tangent PT, *i.e.*, $\angle QPT = \alpha$, then obviously $\angle PDQ = \alpha$, and hence $PQ = 2\rho \sin \alpha$,

Hence, *the chord of curvature parallel to the x-axis is* $2\rho \sin\psi$

$$(\because \text{ here } \angle PDQ = \psi)$$

and *the chord of curvature parallel to y-axis is* $2\rho\cos\psi$

$$\left(\because \text{ here } \angle PDQ = \tfrac{1}{2}\pi - \psi\right).$$

15.6. Illustrative Examples.

Ex. 1. *Show that a circle is a curve of uniform curvature and its radius of curvature at every point is constant, being equal to the radius of the circle.*

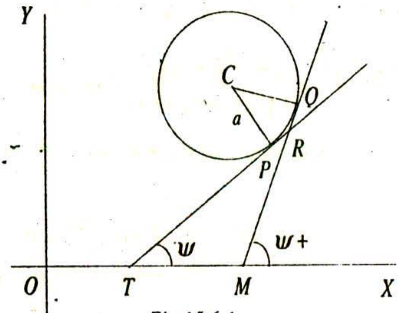


Fig 15.6.1

Let C be the centre of a circle of radius a. Let P be the gi[illegible]nt, Q a point near it, and let PT, QM be tangents at P, Q and let $\angle$ [illegible] $= \psi$, $\angle QMX = \psi + \Delta\psi$; join CP, CQ.

$$\therefore \quad \angle PCQ = \angle PRM = \Delta\psi .$$

$\therefore \quad \dfrac{\Delta\psi}{\Delta s} = \dfrac{\text{angle}PCQ}{\Delta s} = \dfrac{\Delta s/a}{\Delta s} = \dfrac{1}{a}$, since $\angle PCQ$ is measured in radian.

$\therefore$ as in Art. 15.1,

$$\text{curvature} = \underset{\Delta s \to 0}{Lt} \frac{\Delta\psi}{\Delta s} = \underset{\Delta s \to 0}{Lt} \frac{1}{a} = \frac{1}{a} \text{ (constant) and hence } \rho = a .$$

Ex. 2. *Find the radius of curvature at the point* (s, ψ) *of the curve*

$$s = a\sec\psi\tan\psi + a\log(\sec\psi + \tan\psi).$$

Here, $$\rho = \frac{ds}{d\psi} = a\left(\sec\psi.\sec^2\psi + \tan^2\psi\sec\psi\right)$$

$$+ a\cdot\frac{1}{\sec\psi + \tan\psi}\cdot\sec\psi\left(\sec\psi + \tan\psi\right)$$

$$= a\sec\psi\left(\sec^2\psi + \sec^2\psi - 1\right) + a\sec\psi = 2a\sec^3\psi$$

Ex. 3. *Find the radius of curvature at the point* (x, y) *on the curve* $y = a \log \sec(x/a)$.

Here, $\quad y_1 = a \cdot \dfrac{1}{\sec(x/a)} \cdot \sec(x/a)\tan(x/a) \cdot \dfrac{1}{a} = \tan(x/a)$,

$\therefore \quad y_2 = (1/a)\sec^2(x/a)$.

Also, $\quad 1 + y_1^2 = 1 + \tan^2(x/a) = \sec^2(x/a)$,

$$\therefore \quad \rho = \frac{\left(1 + y_1^2\right)^{\frac{3}{2}}}{y_2} = \frac{\left\{\sec^2(x/a)\right\}^{\frac{3}{2}}}{(1/a)\sec^2(x/a)} = a\sec(x/a).$$

Ex. 4. *Find the radius of curvature of the parabola* $y^2 = 4x$ *at the vertex* $(0, 0)$.

The tangent at the vertex being the y-axis, $\dfrac{dy}{dx}$ at the vertex $(0, 0)$ is infinite. Hence, formula (1) of Art. 15.2 being not applicable, let us apply formula (1a). [See *Note 2, Art. 15.2.*]

Hence, $\quad \dfrac{dx}{dy} = \frac{1}{2}y; \qquad \dfrac{d^2x}{dy^2} = \frac{1}{2}$.

$\therefore$ at the vertex, $x_1 = 0$, $x_2 = \frac{1}{2}$.

$$\therefore \quad \textit{at the vertex, } \rho = \frac{\left(1 + x_1^2\right)^{\frac{3}{2}}}{x_2} = \frac{1}{\frac{1}{2}} = 2.$$

Ex. 5. *Find the radius of curvature at the point* 'θ' *on the cycloid* $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$. [*C.P. 1944, V.P. 2000, '96*]

Here, $\quad x' = a(1 + \cos\theta)$, $y' = a\sin\theta$,

$\quad x'' = -a\sin\theta$, $y'' = a\cos\theta$,

$\therefore$ by formula (2) of Art. 15.2,

$$\rho = \frac{\left\{a^2(1+\cos\theta)^2 + a^2\sin^2\theta\right\}^{\frac{3}{2}}}{a^2\cos\theta(1+\cos\theta) + a^2\sin^2\theta} = a \cdot \frac{8\cos^3\frac{1}{2}\theta}{2\cos^2\frac{1}{2}\theta}$$

$$= 4a\cos\tfrac{1}{2}\theta.$$

Note. ρ can also be obtained by using formula (1) of Art. 15.2 by first obtaining the values of y_1 and y_2 in terms of θ.

Ex. 6. *Find the radius of curvature at the point (r, θ) on the cardioide $r = a(1-\cos\theta)$, and show that it varies as $\sqrt{r}$.* [V. P. 2002; C. P.007]

Here, $r_1 = a\sin\theta,\quad r_2 = a\cos\theta.$

$\therefore$ by applying formula (7) of Art. 15.2,

$$\rho = \frac{\{a^2(1-\cos\theta)^2 + a^2\sin^2\theta\}^{\frac{3}{2}}}{a^2(1-\cos\theta)^2 + 2a^2\sin^2\theta - a^2\cos\theta(1-\cos\theta)}$$

$$= \frac{a(2-2\cos\theta)^{\frac{3}{2}}}{3(1-\cos\theta)} = \frac{2^{\frac{3}{2}}a}{3}(1-\cos\theta) = \frac{2^{\frac{3}{2}}a}{3}\left(2\sin^2\frac{\theta}{2}\right)^{\frac{1}{2}}$$

$$= \frac{4}{3}\sin\frac{1}{2.}\theta. \qquad \ldots(1)$$

Since $\quad r = a(1-\cos\theta) = a2\sin^2\frac{1}{2}\theta,$

$\therefore\quad \sin\frac{1}{2}\theta = \sqrt{(r/2a)}.$

Hence, from (1), $\rho = \frac{2}{3}\sqrt{2a}.\sqrt{r}. \qquad \therefore \qquad \rho \propto \sqrt{r}.$

Note. In the cases where it is easier to transform a polar equation into a pedal one, to find the radius of curvature, it is convenient to transform the polar equation into the pedal form first, and then use formula (8) of Art. 15.2.

Ex. 7. *Find the radius of curvature at the point (p, r) of the curve $r^{m+1} = a^m p$.*

We have $p = \dfrac{r^{m+1}}{a^m}. \qquad \therefore \qquad \dfrac{dp}{dr} = \dfrac{(m+1)r^m}{a^m}.$

$$\therefore \quad \frac{dr}{dp} = \frac{a^m}{(m+1)r^m}.$$

$$\therefore \quad \rho = r\frac{dr}{dp} = r\cdot\frac{a^m}{(m+1)r^m} = \frac{a^m}{(m+1)r^{m-1}}.$$

Ex. 8. *Find the radius of curvature at the origin for the curve*

$$x^3 + y^3 - 2x^2 + 6y = 0.$$

Here, $y = 0$, *i.e.*, the x-axis is the tangent at the origin,

$$\therefore \quad \text{at the origin } Lt\frac{x^2}{y} = 2\rho.$$

· Dividing the equation of the curve by y, we have

$$x \cdot \frac{x^2}{y} + y^2 - 2\frac{x^2}{y} + 6 = 0.$$

Now, taking limits as $x \to 0$, and $y \to 0$, we have

$$-2.2\rho + 6 = 0, \qquad \text{or,} \qquad \rho = \tfrac{3}{2}.$$

Ex. 9. *Find the radius of curvature at the origin of the conic*

$$y - x = x^2 + 2xy + y^2. \qquad [\textit{ C. P. 1948 }]$$

First Method : Differentiating the equation successively with respect to x,

$$y_1 - 1 = 2(x + xy_1 + y + yy_1),$$

and $$y_2 = 2\left(1 + xy_2 + 2y_1 + yy_2 + y_1^{\,2}\right).$$

$\therefore$ at the origin, *i.e.*, when $x = 0$, $y = 0$, $y_1 = 1$ and $y_2 = 8$.

$\therefore$ at the origin, $\rho = \dfrac{\left(1 + y_1^{\,2}\right)^{\frac{3}{2}}}{y_2} = \dfrac{(1+1)^{\frac{3}{2}}}{8} = \dfrac{\sqrt{8}}{8} = \dfrac{\sqrt{2}}{4} = 0.35$ nearly.

Second Method :

Putting $y = px + \dfrac{qx^2}{2!} + \ldots\ldots.$ on both sides of the equation, we have

$$(p-1)x + \frac{qx^2}{2!} + \text{higher powers of } x = \left(1 + 2p + p^2\right)x^2$$
$$+ \text{ higher powers of } x.$$

Equating coefficients of x and x^2 on both sides,

$$p - 1 = 0, \ \textit{i.e.}, \ p = 1,$$

and $$\tfrac{1}{2} q = 1 + 2p + p^2. \qquad \therefore \quad q = 8.$$

Since here p and q are the values of y_1, y_2 at the origin, using the formula $\rho = \dfrac{\left(1 + y_1^{\,2}\right)^{\frac{3}{2}}}{y_2}$, we get ρ at the origin.

Third Method (Newtonian Method) :

Since $y - x = 0$ is the tangent at the origin here, by the formula for the Newtonian method at the origin,

$$\rho = \tfrac{1}{2}\sqrt{2} \,.\, Lt \frac{x^2 + y^2}{y - x} = \tfrac{1}{2}\sqrt{2} \,.\, Lt \frac{x^2 + y^2}{x^2 + 2xy + y^2}$$

(from the equation of the curve)

$$=\tfrac{1}{2}\sqrt{2}\,.\,Lt\,\frac{1+(y/x)^2}{1+2(y/x)+(y/x)^2}$$

(on dividing the numerator and denominator by x^2)

$$=\tfrac{1}{2}\sqrt{2}\,.\,\frac{1+1}{1+2+1}=\tfrac{1}{4}\sqrt{2}\,.$$

since $Lt\,(y/x)$, being the value of 'm' of the tangent at the origin, *viz*. $y-x=0$, is equal to 1.

Ex. 10. *Show that the chord of curvature through the pole of the curve* $r^m=a^m\cos m\theta$ *is* $\dfrac{2r}{m+1}$.

Taking logarithm of the given equation,

$$m\log r=m\log a+\log\cos m\theta.$$

Differentiating with respect to θ, we have

$$\frac{1}{r}\frac{dr}{d\theta}=\frac{-\sin m\theta}{\cos m\theta}=-\tan m\theta\,.$$

$$\therefore\quad \cot\phi=\cot\left(\tfrac{1}{2}\pi+m\theta\right),\ i.e.,\ \phi=\tfrac{1}{2}\pi+m\theta\,.$$

$$\therefore\quad p=r\sin\phi=r\cos m\theta=r.r^m/a^m=r^{m+1}/a^m\,.$$

$$\therefore\quad \frac{dp}{dr}=\frac{(m+1)r^m}{a^m}.$$

$$\therefore\quad \text{chord of curvature}=2\rho\sin\phi=2r\frac{dr}{dp}\cdot\frac{p}{r}=2\frac{dr}{dp}\cdot p$$

$$=2.\frac{a^m}{(m+1)r^m}\cdot\frac{r^{m+1}}{a^m}=\frac{2r}{m+1}.$$

Ex. 11. *For any curve prove that*

$$\frac{1}{\rho^2}=\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2.$$

We have $\dfrac{dx}{ds}=\cos\psi$. $\therefore\ \dfrac{d^2x}{ds^2}=-\sin\psi\dfrac{d\psi}{ds}=-\sin\psi.\dfrac{1}{\rho}\ \ldots$ (1)

and $\dfrac{dy}{ds} = \sin\psi$. $\quad \therefore \dfrac{d^2y}{ds^2} = \cos\psi \dfrac{d\psi}{ds} = \cos\psi \cdot \dfrac{1}{\rho} \quad \ldots \quad (2)$

Now, squaring (1) and (2) and adding, the required relation follows.

Ex. 12. *For any curve prove that*

$$\rho = \frac{r}{\sin\phi\left(1+\dfrac{d\phi}{d\theta}\right)}, \textit{ where } \tan\phi = r\frac{d\theta}{dr}.$$

$$\sin\phi\left(1+\frac{d\phi}{d\theta}\right) = \sin\phi + \frac{d\phi}{d\theta}\sin\phi = r\frac{d\theta}{ds} + \frac{d\phi}{d\theta}\cdot r\frac{d\theta}{ds}$$

$$= r\left(\frac{d\theta}{ds} + \frac{d\phi}{d\theta}\cdot\frac{d\theta}{ds}\right) = r\left(\frac{d\theta}{ds} + \frac{d\phi}{ds}\right)$$

$$= r\frac{d}{ds}(\theta+\phi) = r\frac{d\psi}{ds} \qquad (\because \theta+\phi=\psi).$$

$$\therefore \quad \text{right side} = \frac{r}{r\dfrac{d\psi}{ds}} = \frac{ds}{d\psi} = \rho.$$

EXAMPLES - XV (A)

1. Find the radius of curvature at any point (s, ψ) on the following curves:

(i) $s = a\psi$. (ii) $s = 4a\sin\psi$.

(iii) $s = c\tan\psi$. (iv) $s = 8a\sin^2\frac{1}{6}\psi$.

(v) $s = a\left(e^{m\psi} - 1\right)$ (vi) $s = c \log \sec\psi$

(vii) $s = m\left(\sec^2\psi - 1\right)$. (viii) $s = a \log\tan\left(\frac{1}{4}\pi + \frac{1}{2}\psi\right)$

2. Find the radius of curvature at any point (x, y) for the curves (i) to (viii), and at the points indicated for the curves (ix) to (xiv) :

(i) $y^2 = 4ax$. (ii) $e^{y/a} = \sec(x/a)$.

(iii) $y = \log\sin x$. (iv) $ay^2 = x^3$.

(v) $xy = c^2$. (vi) $y = \frac{1}{2}a\cdot\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$.

(vii) $x^2/a^2 + y^2/b^2 = 1$.   (viii) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

[*C.P 1943, B.P. '93*]

(ix) $y = x^3 - 2x^2 + 7x$ at the origin.

(x) $y = 4\sin x - \sin 2x$ at $x = \frac{1}{2}\pi$.

(xi) $9x^2 + 4y^2 = 36x$ at $(2, 3)$.   (xii) $y = e^{-x^2}$ at $(0, 1)$

(xiii) $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at the point where $y = x$ cuts it.

(xiv) $y = xe^{-x}$ at its maximum point.   [*C.P 1988, '96*]

3. Find the radius of curvature at any point of the curves (i) to (vi), and the points indicated for the curves (vii) and (viii) :

(i) $x = a\cos\theta$, $y = a\sin\theta$.   (ii) $x = at^2$, $y = 2at$.

(iii) $x = a\cos\phi$, $y = b\sin\phi$.

(iv) $x = a\sec\phi$, $y = b\tan\phi$

(v) $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$.

(vi) $x = a\sin 2\theta\,(1 + \cos 2\theta)$, $y = a\cos 2\theta\,(1 - \cos 2\theta)$.

(vii) $x = a\cos^3\theta$, $y = a\sin^3\theta$ at $\theta = \frac{1}{4}\pi$. [*B.P. 1999*]

(viii) $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ at $\theta = 0$.

4. Find the radius of curvature at any point (r, θ) for the curves (i) to (xi), and at the points indicated for the curves (xii) to (xvi) :

(i) $r = a\theta$ (ii) $r = a\cos\theta$.   (iii) $r = a\sec^2\frac{1}{2}\theta$.

(iv) $r = a(1 - \cos\theta)$.   [*C.P. 1999, B.P. '89, '91, 98*]

(v) $r^2 = a^2\cos 2\theta$.   [*C.P. 1992*]

(vi) $r = a\,e^{\theta\cot\alpha}$.   (vii) $r^3 = a^3\cos 3\theta$.

(viii) $r = a + b\cos\theta$.   (ix) $r^m = a^m\cos m\theta$.

(x) $r^2\cos 2\theta = a^2$. [*V. P. 1998*]   (xi) $r = a\sec 2\theta$.

(xii) $r = 2a\cos\theta - a$ at $\theta = 0$.

(xiii) $r = a\sin n\theta$ at the origin.

(xiv) $r = l/(1 + e\cos\theta)$ at $\theta = \pi$, $[e < 1]$.   [*C. P. 2005, 06*]

(xv) $r^2 = a^2 \cos 2\theta$ at $\theta = 0$.

(xvi) $r = a(\theta + \sin\theta)$ at $\theta = 0$. [*C..P. 1989*]

5. Find the radius of curvature at any point (p, r) on the following curves whose pedal equations are

(i) $p = r\sin\alpha$. (ii) $r^2 = 2ap$. (iii) $p^2 = ar$. [*C..P. 1982*]

(iv) $pr = a^2$. (v) $r^3 = 2ap^2$. (vi) $r^3 = a^2 p$.

(vii) $\dfrac{a^2b^2}{p^2} + r^2 = a^2 + b^2$. [*C..P. 1998*]

6. Find the radius of curvature at any point on the curves :

(i) $p = a(1 + \sin\psi)$. (ii) $p = a\operatorname{cosec}\psi$.

(iii) $p^2 + a^2\cos 2\psi = 0$.

7. Find the radius of curvature at the origin of the following curves :

(i) $y = x^4 - 4x^3 - 18x^2$. (ii) $2x^2 - xy + y^2 - y = 0$.

(iii) $3x^2 + 4y^2 = 2x$. (iv) $3x^2 + xy + y^2 - 4x = 0$.

(v) $3x^4 - 2y^4 + 5x^2y + 2xy - 2y^2 + 4x = 0$.

(vi) $4x^4 + 3y^3 - 8x^2y + 2x^2 - 3xy - 6y^2 - 8y = 0$.

(vii) $x^3 + y^3 = 3axy$. (viii) $x^2 + 6y^2 + 2x - y = 0$.

(ix) $x^4 + y^2 = 6a(x + y)$. [*C. P. 2006*]

(x) $s\,x^2 + y^2 + 6x + 8y = 0$.

(xi) $y^2 = x^2(a + x)/(a - x)$.

(xii) $ax + by + a'x^2 + 2h'xy + b'y^2 = 0$.

(xiii) $y^2 - 2xy - 3x^2 - 4x^3 - x^2y^2 = 0$.

(xiv) $y^2 - 3xy - 4x^2 + 5x^3 + x^4y - y^5 = 0$.

8. Show that the chord of curvature through the pole for the curve $p = f(r)$ is given by $2f(r)/f'(r)$.

9. Find the chord of curvature through the pole of the curves :

(i) $r = a(1 + \cos\theta)$. (ii) $r^2 = a^2\cos 2\theta$.

(iii) $r^2\cos 2\theta = a^2$. (iv) $r = a\,e^{\theta\cot\alpha}$.

(v) $r^n = a^n \sin n\theta$.

10. Show that the chord of curvature parallel to the axis of y for the curve
   (i) $y = a \log\sec(x/a)$ is constant.
   (ii) $y = c\cosh(x/c)$ is double of the ordinate.

11. Show that in a parabola the chords of curvature
   (i) through the focus, and
   (ii) parallel to the axis are each equal to four times the focal distance of the point.

12. Show that for the ellipse $x^2/a^2 + y^2/b^2 = 1$, the radius of curvature at an extremity of the major axis is equal to half the latus rectum. [C.P. 1990, '94]

13. If C be the centre of the ellipse $x^2/a^2 + y^2/b^2 = 1$, show that at any point P,

$$\rho = \frac{CD^3}{ab} = \frac{a^2b^2}{p^3},$$

where CD is the semi-diameter conjugate to CP, and p is the perpendicular from the centre on the tangent at P.

14. If ρ_1 and ρ_2 be the radii of curvature at the ends P and D of conjugate diameters of the ellipse $x^2/a^2 + y^2/b^2 = 1$, then

$$\rho_1^{\frac{2}{3}} + \rho_2^{\frac{2}{3}} = \left(a^2 + b^2\right)/(ab)^{\frac{2}{3}}.$$ [C.P. 1988]

15. Prove that the radius of curvature of the catenary $y = a\cosh(x/a)$ at any point is equal in length to the portion of the normal intercepted between the curve and the axis of x.

16. Show that for the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ the radius of curvature at any point is twice the portion of the normal intercepted between the curve and the axis of x. [C. P. 2004]

17. Show that in a parabola the radius of curvature is twice the part of the normal intercepted between the curve and the directrix.

18. If ρ_1 and ρ_2 be the radii of curvature at the ends of a focal chord of the parabola $y^2 = 4ax$, then show that

$$\rho_1^{-\frac{2}{3}} + \rho_2^{-\frac{2}{3}} = (2a)^{-\frac{2}{3}}.$$ [V. P. '99, C..P. 1986, 2006]

19. Show that in any curve

(i) $\rho = \left\{\left(\dfrac{dx}{d\psi}\right)^2 + \left(\dfrac{dy}{d\psi}\right)^2\right\}^{\frac{1}{2}}$.

(ii) $\dfrac{1}{\rho} = \dfrac{d^2x}{ds^2}\Big/\dfrac{dy}{ds} = \dfrac{d^2y}{ds^2}\Big/\dfrac{dx}{ds}$

(iii) $\dfrac{1}{\rho^3} = \dfrac{d^2x}{ds^2}\cdot\dfrac{d^3y}{ds^3} - \dfrac{d^2y}{ds^2}\cdot\dfrac{d^3x}{ds^3}$.

20. Show that in the curve for which

(i) $y = a\cos^m\psi$, ρ is m times the normal;

(ii) $y = ae^{m\psi}$, ρ is m times the tangent.

21. Show that

(i) for the cycloid for which $s^2 = 8ay$,

$$\rho = 4a\sqrt{\{1 - y/(2a)\}};$$

(ii) for the catenary for which $y^2 = c^2 + s^2$, $\rho = y^2/c$.

22. Prove that in any curve

(i) $\dfrac{1}{\rho} = \left\{\dfrac{1}{r} - \dfrac{1}{r}\left(\dfrac{dr}{ds}\right)^2 - \dfrac{d^2r}{ds^2}\right\} \div \left\{1 - \left(\dfrac{dr}{ds}\right)^2\right\}^{\frac{1}{2}}$;

(ii) $\rho = r\dfrac{d\theta}{ds} \div \left\{r\left(\dfrac{d\theta}{ds}\right)^2 - \dfrac{d^2r}{ds^2}\right\}$.

23. Show that the radius of curvature at any point of the equiangular spiral subtends a right angle at the pole.

24. Show that at the points in which the curves $r = a\theta$ and $r\theta = a$ intersect, their curvatures are in the ratio 3:1.

25. Show that when the angle between the tangent to a curve and the radius vector of the point of contact has a maximum or minimum value, $\rho = r^2/p$.

26. Prove that in any curve

$$\frac{d\rho}{ds} = \frac{3y_1 y_2^2 - y_3\left(1 + y_1^2\right)}{y_2^2}$$

and show that at every point of a circle

$$3y_1 y_2^2 = y_3\left(1 + y_1^2\right).$$

ANSWERS

[In the following examples, generally the magnitudes of the radii of curvature are given.]

1. (i) a. (ii) $4a\cos\psi$. (iii) $c\sec^2\psi$. (iv) $\frac{4}{3}a\sin\frac{1}{3}\psi$.
(v) $am\,e^{m\psi}$. (vi) $c\tan\psi$. (vii) $3m\sec^3\psi\tan\psi$.
(viii) $a\sec\psi$.

2. (i) $2(x+a)^{\frac{3}{2}}/\sqrt{a}$. (ii) $a\sec(x/a)$. (iii) $\operatorname{cosec} x$.
(iv) $(4a+9x)^{\frac{3}{2}}x^{\frac{1}{2}}/(6a)$. (v) $\left(x^2+y^2\right)^{\frac{3}{2}}/\left(2c^2\right)$
(vi) y^2/a. (vii) $\left(b^4x^2+a^4y^2\right)^{\frac{3}{2}}/a^4b^4$.
(viii) $3(axy)^{\frac{1}{3}}$. (ix) $\frac{1}{2}\left(125\sqrt{2}\right)$. (x) $\frac{5}{4}\sqrt{5}$.
(xi) $\frac{4}{3}$. (xi) $\frac{1}{2}$. (xiii) $a/\sqrt{2}$. (xiv) c.

3. (i) a. (ii) $2a\left(t^2+1\right)^{\frac{3}{2}}$. (iii) $\left(a^2\sin^2\phi+b^2\cos^2\phi\right)^{\frac{3}{2}}/ab$.
(iv) $\left(a^2\tan^2\phi+b^2\sec^2\phi\right)^{\frac{3}{2}}/ab$. (v) at. (vi) $4a\cos 3\theta$.
(vii) $\frac{3}{2}a$. (viii) $4a$.

4. (i) $\left(r^2+a^2\right)^{\frac{3}{2}}/\left(r^2+2a^2\right)$. (ii) $\frac{1}{2}a$.
(iii) $2a\sec^3\frac{1}{2}\theta$. (iv) $\frac{2}{3}\sqrt{2ar}$. (v) $a^2/3r$.
(vi) $r\operatorname{cosec}\alpha$. (vii) $a^3/4r^2$.
(viii) $\dfrac{\left(a^2+2ab\cos\theta+b^2\right)^{\frac{3}{2}}}{a^2+3ab\cos\theta+2b^2}$ (ix) $\dfrac{a^m}{m+1}\cdot\dfrac{1}{r^{m-1}}$.

(x) r^3/a^2. (xi) $r\left(4r^2-3a^2\right)^{\frac{3}{2}}/3a^3$. (xii) $\frac{1}{3}a$.

(xiii) $\frac{1}{2}na$. (xiv) l. (xv) $\frac{1}{3}a$ (xvi) a.

5. (i) $\operatorname{cosec}\alpha$. (ii) a. (iii) $2\sqrt{r^3/a}$.

(iv) r^3/a^2. (v) $\frac{2}{3}\sqrt{(2ar)}$. (vi) $a^2/3r$.

(vii) a^2b^2/p^3.

6. (i) a. (ii) $2a\operatorname{cosec}^3\psi$. (iii) a^4/p^3.

7. (i) $\frac{1}{36}$: (ii) $\frac{1}{4}$. (iii) $\frac{1}{4}$. (iv) 2. (v) 1.

(vi) 2. (vii) $\frac{3}{2}a$, $\frac{3}{2}a$. (viii) $\frac{1}{10}\sqrt{5}$.

(ix) $6a\sqrt{2}$. (x) 5. (xi) $\pm a\sqrt{2}$.

(xii) $\frac{1}{2}\cdot\dfrac{\left(a^2+b^2\right)^{\frac{3}{2}}}{a'b^2-2h'ab+b'a^2}$. (xiii) $5\sqrt{10}$; $\sqrt{2}$. (xiv) $\sqrt{2}$, $\frac{17}{2}\sqrt{17}$.

9. (i) $\frac{4}{3}r$. (ii) $\frac{2}{3}r$. (iii) $2r$. (iv) $2r$. (v) $2r/(n+1)$

15.7. Centre of Curvature.

Let $(\bar{x}, \bar{y})$ be the co-ordinates of the centre of curvature C corresponding to any point $P(x, y)$ on the curve.

Since $C(\bar{x}, \bar{y})$ lies on the normal at P, *viz.*,

$$(X-x)+(Y-y)y_1=0,$$

$$\therefore\quad (\bar{x}-x)+(\bar{y}-y)y_1=0. \qquad \ldots(1)$$

Again, since $PC=\rho$, *i.e.*, $PC^2=\rho^2$,

$$\therefore\quad (\bar{x}-x)^2+(\bar{y}-y)^2=\rho^2=\frac{\left(1+y_1^{\,2}\right)^3}{y_2^{\,2}} \qquad \ldots(2)$$

Substituting $-(\bar{y}-y)y_1$ for $(\bar{x}-x)$ from (1) in (2) we get

$$(\bar{y}-y)^2\left(1+y_1^{\,2}\right)=\rho^2=\frac{\left(1+y_1^{\,2}\right)^3}{y_2^{\,2}}, \qquad \ldots(3)$$

i.e., $(\bar{y}-y)^2=\dfrac{\left(1+y_1^{\,2}\right)^2}{y_2^{\,2}}$,

$$\therefore\quad (\bar{y}-y)=\frac{1+y_1^2}{y_2}. \qquad \ldots (4)$$

Again, from (1),

$$\bar{x}-x=-(\bar{y}-y)y_1=-\frac{y_1(1+y_1^2)}{y_2} \qquad \ldots (5)$$

$\therefore$ from (4) and (5), we get

$$\bar{x}=x-\frac{y_1(1+y_1^2)}{y_2},\quad \bar{y}=y+\frac{1+y_1^2}{y_2} \qquad \ldots (6)$$

Cor. Hence *the equation of the circle of curvature* is

$$(x-\bar{x})^2+(y-\bar{y})^2=\rho^2.$$

Note 1. According to our convention, we take the positive sign only in (4); for, if y_2 is positive, ρ is positive and hence $\bar{y}-y$ is positive. Similarly if y_2 is negative, ρ is negative and hence $\bar{y}-y$ is negative.

Note 2. Since the normal at P makes an angle $(\frac{1}{2}\pi+\psi)$ with the x-axis, it follows from the definition of the centre of curvature that

$$\frac{\bar{x}-x}{\cos\left(\frac{1}{2}\pi+\psi\right)}=\frac{\bar{y}-y}{\sin\left(\frac{1}{2}\pi+\psi\right)}=\rho,$$

i.e., $\bar{x}=x-\rho\sin\psi,\quad \bar{y}=y+\rho\cos\psi,$ $\qquad \ldots (7)$

Now, since $\tan\psi=y_1$, $\sin\psi=\dfrac{y_1}{\sqrt{1+y_1^2}}$ and $\cos\psi=\dfrac{1}{\sqrt{1+y_1^2}}$.

Thus substituting the values of ρ, $\sin\psi$, $\cos\psi$ in terms of y_1 and y_2 in (7), values (6) of $\bar{x},\bar{y}$ can be obtained.

Note 3. By writing the relation (1) as $(\bar{x}-x)x_1+(\bar{y}-y)=0$, and using the values $\rho^2=(1+x_1^2)^3/x_2^2$ of (from *Art. 15.2, Sec. A*) we can similarly obtain

$$\bar{x}=x+\frac{1+x_1^2}{x_2},\quad \bar{y}=y-\frac{x_1(1+x_1^2)}{x_2} \qquad \ldots (8)$$

where $x_1=\dfrac{dx}{dy}$, $x_2=\dfrac{d^2x}{dy^2}$.

This form is useful when y_1 becomes infinite.

Note 4. The centre of curvature can also be obtained *geometrically* as follows:

Let $C(\bar{x}, \bar{y})$ be the centre of curvature corresponding to the point $P(x, y)$ on the curve.

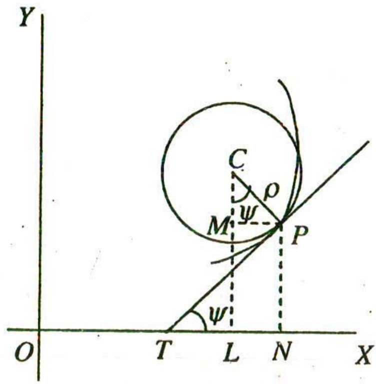


Fig 15.7.1

Then $PC = \rho$

Let PT be the tangent at P, so that $\angle PTX = \psi$.

Draw PN, CL perpendiculars on OX, PM perpendicular on CL.

Then $\quad \angle PCM = \psi$

$$\therefore \quad \bar{x} = OL = ON - MP$$

$$= x - PC \sin PCM$$

$$= x - \rho \sin \psi, \qquad \ldots(1)$$

$$\bar{y} = LC = LM + MC = PN + MC = y + PC \cos PCM$$

$$= y + \rho \cos \psi \qquad \ldots(2)$$

Since $\tan \psi = y_1$, $\sin \psi = \dfrac{y_1}{\sqrt{1+y_1^2}}$ and $\cos \psi = \dfrac{1}{\sqrt{1+y_1^2}}$.

Now substituting the values of ρ, $\sin \psi$, $\cos \psi$ in (1) and (2), the required values of $\bar{x}$ and $\bar{y}$ are obtained.

15.8. Property of the Centre of curvature.

The centre of curvature C for a point P on a curve is the limiting position of the intersection of the normal to the curve at P with a neighbouring normal at Q, as $Q \to P$ along the curve.

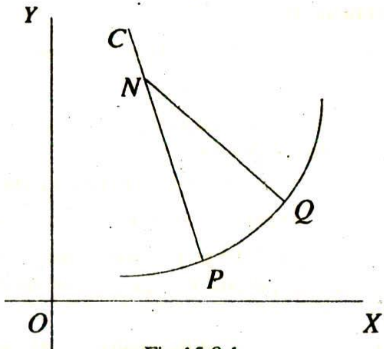


Fig 15.8.1

Let $P(x, y)$ be the given point and $Q(x + \Delta x, y + \Delta y)$ be a point near P on the curve $y = f(x)$; let us suppose y_1, y_2 exist at P and $y_2 \neq 0$. The normal at P is

$$(Y - y)y_1 + (X - x) = 0, \qquad \ldots (1)$$

or, $$(Y - y)\phi(x) + (X - x) = 0, \qquad \ldots (2)$$

putting $y_1 = \phi(x)$

$\therefore$ the normal at Q is

$$(Y - y - \Delta y)\phi(x + \Delta x) + (X - x - \Delta x) = 0 \qquad \ldots (3)$$

Suppose the normals at P, Q, *i.e.*, (2) and (3) intersect at $N(\xi, \eta)$; and let $(\bar{x}, \bar{y})$ be the point C to which N tends as $Q \to P$.

Subtracting (2) from (3) and putting η for Y, we have

$$(\eta - y)\{\phi(x + \Delta x) - \phi(x)\} - \Delta y\phi(x + \Delta x) - \Delta x = 0 \qquad \ldots (4)$$

Dividing by Δx, and making $\Delta x \to 0$ and noting that in that case $\eta \to \bar{y}$, we have

$$(\bar{y} - y)\phi'(x) - y_1\phi(x) - 1 = 0,$$

i.e., $$(\bar{y} - y)y_2 - y_1^2 - 1 = 0 \qquad \ldots (5)$$

Again, since $(\bar{x}, \bar{y})$ is a point on (1),

$$\therefore \quad (\bar{y} - y)y_1 + (\bar{x} - x) = 0 \qquad \ldots (6)$$

The value of $(\bar{x}, \bar{y})$ obtained from (5) and (6) are identical with those of the co-ordinates of the centre of curvature obtained in Art. 15.7.

Hence, $(\bar{x}, \bar{y})$, *i.e.*, C is the centre of curvature.

15.9. Evolute and Involute.

The locus of the centre of curvature of a given curve is called its *Evolute.*

If the evolute itself be regarded as the original curve, a curve of which it is the evolute, is called an *Involute.*

Formulæ (6) and (8) of Art. 15.7 give the co-ordinates of any point $(\bar{x}, \bar{y})$ on the evolute, experssed in terms of the co-ordinates of the corresponding point (x, y) of the given curve; since y is a function of x, these formulæ give us *the parametric equations of the evolute in terms of the parameter x.*

Ordinary cartesian equation of the evolute is obtained by eliminating x and y between the two expressions for $\bar{x}, \bar{y}$ and the equation of the curve. [See *Art. 15.11, Ex. 2.*]

15.10 Properties of the Evolute

(I) *The normal at any point to the given curve is the tangent of the evolute at the corresponding point of the evolute.*

Let $(\bar{x}, \bar{y})$ be the centre of curvature corresponding to the point (x, y) on the curve. Then from Note 2. Art. 15.7.

$$\bar{x} = x - \rho \sin \psi, \quad \bar{y} = y + \rho \cos \psi,$$

$$\therefore \quad \frac{d\bar{x}}{dx} = 1 - \rho \cos \psi \frac{d\psi}{dx} - \sin \psi \frac{d\rho}{dx}$$

$$= 1 - \frac{ds}{d\psi} \cdot \frac{dx}{ds} \cdot \frac{d\psi}{dx} - \sin \psi \frac{d\rho}{dx} = -\sin \psi \frac{d\rho}{dx}.$$

Thus, $$\frac{d\bar{x}}{dx} = -\sin \psi \frac{d\rho}{dx} \qquad \ldots (1)$$

Similarly, $$\frac{d\bar{y}}{dx} = \cos \psi \frac{d\rho}{dx} \qquad \ldots (2)$$

$\therefore$ dividing (2) by (1)

$\dfrac{d\bar{y}}{d\bar{x}} = -\cot \psi = -\dfrac{dx}{dy}$, which is the '$m$' of the normal at (x, y).

$\therefore$ 'm' of the tangent to the evolute at $(\bar{x}, \bar{y})$ = 'm' of the normal to the given curve at the corresponding point (x, y), and since both tangent to the evolute and the normal to the curve pass through the same point $(\bar{x}, \bar{y})$, they are identical. Hence the result.

(II) Length of an arc of the Evolute

The length of an arc of the evolute of a curve is the difference between the radii of curvature of the given curve, which are tangents to this arc of the evolute at its extremities.

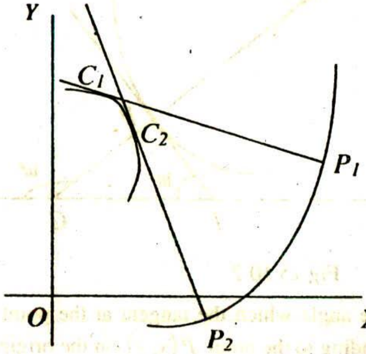


Fig 15.10.1

Let $\bar{s}$ be the length of the arc of the evolute measured from some fixed point on it up to the centre of curvature $(\bar{x}, \bar{y})$. Then from (1) and (2) above, we have

$$\left(\frac{d\bar{x}}{dx}\right)^2 + \left(\frac{d\bar{y}}{dx}\right)^2 = \left(\frac{d\rho}{dx}\right)^2.$$

Also we have $\left(\frac{d\bar{x}}{dx}\right)^2 + \left(\frac{d\bar{y}}{dx}\right)^2 = \left(\frac{d\bar{s}}{dx}\right)^2.$

$\therefore \quad \frac{d\bar{s}}{dx} = \frac{d\rho}{dx}$, *i.e.*, $\frac{d}{dx}(\bar{s} - \rho) = 0$. Hence [*by Art. 9.7, Ex.1*]

$\bar{s} - \rho = C$ (a constant), *i.e.*, $\bar{s} = \rho + C$.

Hence, $\bar{s}_1 - \bar{s}_2 = \rho_1 - \rho_2$, where ρ_1, ρ_2 are the values of ρ at the two points P_1, P_2 on the curve and $\bar{s}_1, \bar{s}_2$ are the values of $\bar{s}$ of the corresponding points C_1, C_2 on the evolute.

Thus, the arc $C_1 C_2$ of the evolute $= P_1 C_1 - P_2 C_2$.

Hence, if a string is wrapped round the curve $C_1 C_2$, it is clear that when the string is unwrapped, being kept tight all the time, the point on the thread which was at P_2 will describe the curve $P_2 P_1$

(III) Radius of curvature of the Evolute

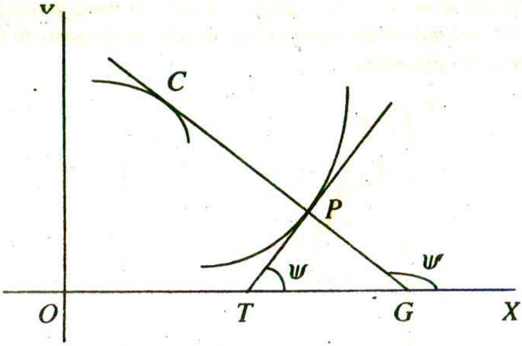


Fig 15.10.2

Let ψ' be the angle which the tangent at the point $C(\bar{x}, \bar{y})$ on the evolute [corresponding to the point $P(x, y)$ on the original curve] makes with the x-axis, then is the angle which the normal at on the given curve makes with the x-axis.

$$\therefore \quad \psi' = \tfrac{1}{2}\pi + \psi. \quad \therefore \frac{d\psi}{d\psi'} = 1 \text{; also from (II) above } \frac{ds}{d\rho} = 1.$$

Let $\bar{\rho}$ be the radius of curvature of the evolute at $(\bar{x}, \bar{y})$.

$$\therefore \quad \bar{\rho} = \frac{d\bar{s}}{d\psi'} = \frac{ds}{d\rho} \cdot \frac{d\rho}{d\psi} \cdot \frac{d\psi}{d\psi'} = \frac{d\rho}{d\psi} = \frac{d}{d\psi}\left(\frac{ds}{d\psi}\right) = \frac{\mathbf{d^2s}}{\mathbf{d\psi^2}}$$

15.11. Illustrative Examples.

Ex. 1. *Find the centre of curvature at any point* (x, y) *on the parabola* $y^2 = 4ax$.

Here, $\quad y_1 = \sqrt{\dfrac{a}{x}}, \quad y_2 = -\dfrac{1}{2} \cdot \dfrac{\sqrt{a}}{x^{\frac{3}{2}}}.$

$$\therefore \quad y_1\left(1 + y_1^2\right) = \sqrt{\frac{a}{x}}\left(1 + \frac{a}{x}\right) = \frac{\sqrt{a}(x+a)}{x^{\frac{3}{2}}}.$$

If $(\bar{x}, \bar{y})$ be the centre of curvature, we have

$$\bar{x} = x - \frac{y_1\left(1 + y_1^2\right)}{y_2} = x + 2(x + a) = 3x + 2a.$$

$$\bar{y} = y + \frac{1+y_1{}^2}{y_2} = y - \frac{2\sqrt{x}\,(x+a)}{\sqrt{a}}$$

$$= 2\sqrt{ax} - \frac{2\sqrt{x}\,(x+a)}{\sqrt{a}} \quad (\because y^2 = 4ax) = -\frac{2}{\sqrt{a}} x^{\frac{3}{2}}$$

Ex. 2. *Find the evolute of the parabola* $y^2 = 4ax$.

As proved above, its centre of curvature $(\bar{x}, \bar{y})$ at any point (x, y) is given by $\bar{x} = 3x + 2a$ ...(1)

$$\bar{y} = -\frac{2}{\sqrt{a}} x^{\frac{3}{2}} \qquad \text{...(2)}$$

From (1), $x = -\dfrac{\bar{x} - 2a}{3}$

$\therefore$ From (2), $\bar{y} = -\dfrac{2}{\sqrt{a}}\left(\dfrac{\bar{x} - 2a}{3}\right)^{\frac{3}{2}}$

$\therefore$ squaring and writing x, y for $\bar{x}, \bar{y}$, the required evolute is give by $27ay^2 = 4(x - 2a)^3$

Ex. 3. *Find the equation of the circle of curvature at the point* (3, 1) *the curve* $y = x^2 - 6x + 10$.

Here, $y_1 = 2x - 6$; $y_2 = 2$.

$\therefore$ at the point (3, 1), $y_1 = 0$, $y_2 = 2$.

If $(\bar{x}, \bar{y})$ be the centre and ρ the radius of curvature at (3, 1),

$$\bar{x} = x - \frac{y_1\left(1 + y_1{}^2\right)}{y_2} = 3, \quad \bar{y} = y + \frac{1 + y_1{}^2}{y_2} = 1 + \tfrac{1}{2} = \tfrac{3}{2}.$$

Also, $\rho = \dfrac{\left(1 + y_1{}^2\right)^{\frac{3}{2}}}{y_2} = \tfrac{1}{2}$.

$\therefore$ the equation of the required circle of curvature is

$$(x-3)^2 + \left(y - \tfrac{3}{2}\right)^2 = \tfrac{1}{4},$$

or: $x^2 + y^2 - 6x - 3y + 11 = 0$.

15.12 Miscellaneous Worked Out Examples

Ex. 1. Show that the radius of curvature of the curve $y = c\cdot\cosh\left(\frac{x}{c}\right)$ is $\frac{y^2}{c}$. [*C. P. 1980, 95*]

Solution : $\because \quad y = c\cdot\cos\text{h}\left(\frac{x}{c}\right)$ . . . (1)

$$y_1 = \frac{dy}{dx} = c\cdot\frac{1}{c}\cdot\sin\text{h}\left(\frac{x}{c}\right) = \sin\text{h}\left(\frac{x}{c}\right)$$

and $y_2 = \frac{d^2y}{dx^2} = \frac{1}{c}\cdot\cos\text{h}\left(\frac{x}{c}\right) = \frac{y}{c^2}$ [From (1)]

Radius of curvature $\rho = \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2} = \frac{\left\{1+\sin\text{h}^2\left(\frac{x}{c}\right)\right\}^{\frac{3}{2}}}{\frac{y}{c^2}}$

$$= \frac{c^2\left\{\cos\text{h}^2\left(\frac{x}{c}\right)\right\}^{\frac{3}{2}}}{y} \qquad [\cos h^2 x - \sin h^2 x = 1]$$

$$= \frac{c^2\left\{\cos\text{h}\left(\frac{x}{c}\right)\right\}^{3}}{y} = \frac{c^2y^3}{yc^3} = \frac{y^2}{c}.$$

Ex. 2. (*i*) For the tractrix $s = c\log\sec\psi$, prove that $\rho = c\tan\psi$ and in case of the equiangular spiral $s = a\left(e^{m\psi}-1\right)$, $\rho = m\,a\,e^{m\psi}$.

[*B. P. 1995*]

(*ii*) For the curve $s = a\sec\psi + \log(\sec\psi+\tan\psi)$, show that $\rho = \sec\psi(1+a\tan\psi)$. [*C. P. 1995*]

(*iii*) Find the radius of curvature of $s = a\sec\psi + \log(\sec\psi\tan\psi)$ at any position ψ. [*C. P. 2001*]

Solution : (*i*) $s = c\log\sec\psi$

$$\rho = \frac{ds}{d\psi} = c\cdot\frac{\sec\psi\tan\psi}{\sec\psi} = c\cdot\tan\psi$$

Again , $s = a\left(e^{m\psi} - 1\right)$

$$\therefore \ \rho = \frac{ds}{d\psi} = m\,a\,e^{m\psi}$$

(*ii*) $s = a\sec\psi + \log(\sec\psi + \tan\psi)$

$$\rho = \frac{ds}{d\psi} = a\sec\psi \ \tan\psi + \frac{\sec\psi \ \tan\psi + \sec^2\psi}{\sec\psi + \tan\psi}$$

$$= a\sec\psi \ \tan\psi + \frac{\sec\psi\left(\sec\psi + \tan\psi\right)}{\sec\psi + \tan\psi} = \sec\psi\,(1 + a\tan\psi).$$

(*iii*) $s = a\sec\psi + \log\left(\sec\psi \cdot \tan\psi\right)$

$$= a\sec\psi + \log\sec\psi + \log\tan\psi$$

$$\rho = \frac{ds}{d\psi} = a\sec\psi\tan\psi + \frac{\sec\psi\tan\psi}{\sec\psi} + \frac{\sec^2\psi}{\tan\psi}$$

$$= a\sec\psi\tan\psi + \tan\psi + \cot\psi.$$

Ex. 3. Find the radius of curvature for the curve

$$x = a(\theta + \sin\theta), \quad y = a\left(1 - \cos\theta\right) \text{ at } \theta = 0.$$

[*C. P. 1987, 91, 97, 2000, 2008*]

Solution : $\because \ x = a(\theta + \sin\theta), \ x' = a\left(1 + \cos\theta\right), \quad x'' = -a\sin\theta$

$\because \ y = a(1 - \cos\theta), \quad y' = a\sin\theta, \quad y'' = a\cos\theta$

At $\theta = 0, \ x' = 2a; \ x'' = 0, \ y' = 0, \ y'' = a$

$$\therefore \text{ At } \theta = 0, \ \rho = \frac{\left(x'^2 + y'^2\right)^{\frac{3}{2}}}{x'y'' - y'x''} = \frac{\left(4a^2 + 0\right)^{\frac{3}{2}}}{2a \cdot a - 0} = \frac{8a^3}{2a^2} = 4a.$$

Ex. 4. Find the radius of curvature at any point 't' on the curve

$$x = a(\cos t + t\sin t), \ y = a(\sin t - t\cos t).$$

[*C. P. 2002*]

Solution : Here, $x = a(\cos t + t\sin t)$

$\therefore \ x' = a\,t\cos t, \quad x'' = a(\cos t - t\sin t)$

and $y = a(\sin t - t\cos t)$

$\therefore \ y' = a\,t\sin t, \quad y'' = a(\sin t + t\cos t)$

Now,

$$\rho = \frac{\left(x'^2 + y'^2\right)^{\frac{3}{2}}}{x'y'' - y'x''} = \frac{\left\{a^2t^2\left(\sin^2 t + \cos^2 t\right)\right\}^{\frac{3}{2}}}{a^2t\left\{\sin t \cos t + t\cos^2 t - \sin t \cos t + t\sin^2 t\right\}}$$

$$= \frac{a^3t^3}{a^2t^2} = a\,t\,.$$

Ex. 5. (*i*) Find the least value of the radius of curvature of the curve $x = 5t, \quad y = 5\log\sec t$. [*C. P. 1983*]

(*ii*) Find the radius of curvature of the parabola $y^2 = 16x$ at the end of its latus rectum. [*B. P. 1988, 1997*]

Solution : (*i*) Here $x = 5t$ and $y = 5\log\sec t$.

$$\therefore \quad x' = 5, \quad x'' = 0, \quad y' = 5\tan t, y'' = 5\sec^2 t$$

$$\rho = \frac{(x'^2 + y'^2)^{\frac{3}{2}}}{x'y'' - y'x''} = \frac{\left\{25\left(1 + \tan^2 t\right)\right\}^{\frac{3}{2}}}{25\sec^2 t - 0} = \frac{125\sec^3 t}{25\sec^2 t}$$

or, $\rho = 5\sec t$

Since numerically the least value of sec t is 1, least value of ρ is 5.

(*ii*) Equation of the parabola is $y^2 = 16x$.

One end of latus rectum is at $(4, \ 8)$.

Now, $2y\dfrac{dy}{dx} = 16$

i.e., $\dfrac{dy}{dx} = y_1 = \dfrac{8}{y}$

$$y_2 = \frac{d^2y}{dx^2} = -\frac{8}{y^2}\cdot\frac{dy}{dx} = -\frac{8}{y^2}\left(\frac{8}{y}\right) = -\frac{64}{y^3}$$

At the point $(4, \ 8)$, $y_1 = 1, \quad y_2 = -\dfrac{1}{8}$.

$$\rho = \left|\frac{\left(1 + y_1^2\right)^{\frac{3}{2}}}{y_2}\right| = \left|\frac{(1+1)^{\frac{3}{2}}}{-\frac{1}{8}}\right| = 8\cdot 2\sqrt{2} = 16\sqrt{2}\,.$$

EXAMPLES - XV (B)

1. Find the centres of curvature of the following curves at the points indicated:

 (i) $xy = 12$ at $(3, 4)$. [*C. P. 1934, 2008*]

 (ii) $y = x^3 + 2x^2 + x + 1$ at $(0, 1)$.

 (iii) $xy = x^2 + 4$ at $(2, 4)$.

 (iv) $y = \sin^2 x$ at $(0, 0)$.

 (v) $x = e^{-2t} \cos 2t$, $y = e^{-2t} \sin 2t$ at $t = 0$.

2. Determine the centres of curvature of the following curves at any point (x, y):

 (i) $x^2 = 4ay$. (ii) $a^2 y = x^3$. (iii) $x^2/a^2 + y^2/b^2 = 1$.

 (iv) $xy = a^2$. (v) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

 (vi) $y = \frac{1}{2} a \left(e^{x/a} + e^{-x/a} \right)$.

 (vii) $x = a \cos \phi$, $y = b \sin \phi$. (viii) $x = at^2$, $y = 2at$.

 (ix) $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.

 (x) $x = a(\cos t + t \sin t)$, $y = a(\sin t - t \cos t)$.

3. Find the evolutes of the curves (iii), (iv), (v), (ix), (x) of Ex. 2, above.

4. If (α, β) be the co-ordinates of the centre of curvature of the parabola

 $\sqrt{x} + \sqrt{y} = \sqrt{a}$, at (x, y), then show that

 $$\alpha + \beta = 3(x + y).$$

5. Show that the co-ordinates $(\bar{x}, \bar{y})$ of the centre of curvature at any point (x, y) on a curve are given by

 (i) $\bar{x} = x - \dfrac{dy}{d\psi}$, $\bar{y} = y + \dfrac{dx}{d\psi}$,

 (ii) $\bar{x} = x + \rho^2 x''$, $\bar{y} = y + \rho^2 y''$,

 where dashes denote differentiations with respect to the arc s.

6. Prove that the distance r_1 between the pole and the centre of curvature corresponding to any point on the curve $r = f(\theta)$ is given by

 $$r_1^2 = r^2 + \rho^2 - 2p\rho,$$

 where ρ and p have the usual significance

7. For the equlangular spiral $r = ae^{\theta \cot \alpha}$, prove that the centre of curvature is at the point where the perpendicular to the radius vector through the pole intersects the normal.

8. Find the circle of curvature of the curves :

(i) $y = x + 4/x$ at $(2, 4)$. [C. P. 2006]

(ii) $y = x^3 + 2x^2 + x + 1$ at $(0, 1)$.

(iii) $x = e^y$ at the point where it crosses the x-axis.

(iv) $y^2 = 4x$ at at the ends of the latus rectum.

(v) $x + y = ax^2 + by^2 + cx^3$ at the origin.

ANSWERS

1. (i) $\left(7\frac{1}{6}, 7\frac{1}{8}\right)$. (ii) $\left(-\frac{1}{2}, \frac{3}{2}\right)$. (iii) $(2, 5)$.
(iv) $\left(0, \frac{1}{2}\right)$. (v) $(0, -1)$.

2. (i) $\left(-\dfrac{x^3}{4a^2}, 2a + \dfrac{3x^2}{4a}\right)$. (ii) $\left\{\dfrac{x}{2}\left(1 - \dfrac{9x^4}{a^4}\right), \left(\dfrac{5}{2}\dfrac{x^3}{a^2} + \dfrac{a^2}{6x}\right)\right\}$.

(iii) $\left(\dfrac{a^2 - b^2}{a^4}x^3, -\dfrac{a^2 - b^2}{b^4}y^3\right)$. (iv) $\left(\dfrac{3}{2}x + \dfrac{y^3}{2a^2}, \dfrac{3}{2}y + \dfrac{x^3}{2a^2}\right)$.

(v) $\left(x + 3x^{\frac{1}{3}}y^{\frac{2}{3}}, y + 3x^{\frac{2}{3}}y^{\frac{1}{3}}\right)$ (vi) $\left(x - \dfrac{y\sqrt{y^2 - a^2}}{a}, 2y\right)$.

(vii) $\left(\dfrac{a^2 - b^2}{a}\cos^3\phi, \dfrac{b^2 - a^2}{b}\sin^3\phi\right)$. (viii) $\left\{a\left(2 + 3t^2\right), -2at^3\right\}$.

(ix) $\{a(\theta + \sin\theta), -a(1 - \cos\theta)\}$. (x) $\{a\cos t, a\sin t\}$.

3. (i) $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = \left(a^2 - b^2\right)^{\frac{2}{3}}$.

(ii) $(x + y)^{\frac{2}{3}} - (x - y)^{\frac{2}{3}} = (4a)^{\frac{2}{3}}$.

(iii) $(x + y)^{\frac{2}{3}} + (x - y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$.

(iv) $x = a(\theta + \sin\theta)$, $y = -a(1 - \cos\theta)$. (v) $x^2 + y^2 = a^2$.

4. (i) $x^2 + y^2 - 4a - 10y + 28 = 0$. (ii) $x^2 + y^2 + x - 3y + 2 = 0$.

(iii) $x^2 + y^2 - 6x + 4y + 5 = 0$. (iv) $x^2 + y^2 - 10x \pm 4y - 3 = 0$.

(v) $(a + b)\left(x^2 + y^2\right) = 2(x + y)$.

16 ASYMPTOTES

16.1. In some cases a curve may have a branch or branches extending beyond the finite region. In this case if P be a point on such a branch of the curve, having co-ordinates (x, y), and if P moves along the curve so that at least one of x and y tends to $+\infty$ or to $-\infty$, then P is said to tend to infinity, and this we denote by $P \to \infty$.

Definition. *If P be a point on a branch of a curve extending beyond the finite region, and a straight line exists at a finite distance from the origin from which the distance of P gradually diminishes and ultimately tends to zero as $P \to \infty$ (moving along the curve), then such a straight line is called on* **asymptote** *of the curve.*

16.2. Asymptotes not parallel to y-axis.

If $y = mx + c$ be an asymptote corresponding to an infinite branch of a curve, where m and c are both finite (including zero), then

$$m = \underset{x\to\infty}{Lt} \frac{y}{x} \text{ and } c = \underset{x\to\infty}{Lt} (y - mx)$$

where (x, y) are the co-ordinates of a point P on the branch of the curve.

The distance of the point P from the straight line $y - mx - c = 0$ is given by

$$d = \frac{y - mx - c}{\sqrt{1 + m^2}}, \text{ and if } y = mx + c \text{ be an asymptote,}$$

$d \to 0$ as $x \to \infty$ and since m is finite here,

$$\underset{x\to\infty}{Lt} (y - mx - c) = 0, \text{ or, } \underset{x\to\infty}{Lt} (y - mx) = c.$$

Again, denoting $y - mx - c$, *i.e.*, $d\sqrt{1 + m^2}$ by u,

$$\frac{y}{x} - m = \frac{c + u}{x}.$$

Now making $x \to \infty$, since $u \to 0$ in this case, and c is finite.

$$\underset{x\to\infty}{Lt} \left(\frac{y}{x} - m \right) = 0, \text{ or, } \underset{x\to\infty}{Lt} \frac{y}{x} = m.$$

Accordingly, to find asymptotes (which are not parallel to the y-axis) of a curve $y = f(x)\{\text{or } F(x, y) = 0\}$, we first of all find out $\underset{x\to\infty}{Lt} \frac{y}{x}$ from the equation to the curve, which may have several finite values (inclusive of zero). Corresponding to any such value (m say), we next proceed to find $\underset{x \to \infty}{Lt}(y - mx)$, using the equation to the curve.

If this limit is found to be finite, say c, then $y = mx + c$ is an asymptote.

[See *Ex. 7*, § *16.8*]

Note. An alternative definition of a rectilinear asymptote is sometimes given as follows : *If P be a point on a branch of a curve extending to infinity and if a straight line at a finite distance from the origin exists towards which the tangent line to the curve at P approaches as a limit when $P \to \infty$, then the straight line is an asymptote of the curve.*

With this definition also, we can prove the results of the above article; for, the equation of the tangent at $P(x, y)$ to the curve is

$$Y - y = \frac{dy}{dx}(X - x), \text{ or, } Y = \frac{dy}{dx}X + \left(y - x\frac{dy}{dx}\right),$$

and as $x \to \infty$, if this tends to $Y = mX + c$, where m and c are finite, clearly

$$m = \underset{x \to \infty}{Lt}\frac{dy}{dx}, \text{ and } c = \underset{x \to \infty}{Lt}\left(y - x\frac{dy}{dx}\right) = \underset{x \to \infty}{Lt}(y - mx).$$[1]

It should be noted that when $P \to \infty$, if the tangent line tends to a straight line as its limiting position, that line is an asymptote. The converse, however, is not true, *i.e.*, even if the tangent line has no definite limiting position when $P \to \infty$, there may be an asymptote. [See *Ex. 8*, § *16.8*]

16.3. Asymptotes parallel to y-axis.

The necessary and sufficient condition that the straight line $x = a$ is an asymptote to the curve $y = f(x)$ is that $|f(x)| \to \infty$ when either $x \to a + 0$ or $x \to a - 0$.

For, suppose $x \to a - 0$. Since $|y| \to \infty$ in this case, P being the point (x, y) on the curve, $P \to \infty$ in this case ; [conversely, if $P \to \infty$ in this case, $|y| \to \infty$, and hence the necessity of the condition]. Now the perpendicular distance of P from the line $x = a$ is $x - a$ (the axes being rectangular), and $|x - a| \to 0$ as $x \to a - 0$. hence, $x = a$ is an asymptote. Similarly, for the case when $x \to a + 0$.

Thus to find asymptotes parallel to y-axis, we may write z for $1/y$ in the equation to the curve, and then make $z \to 0$. If then the result leads to

[1] A rigorous proof of this last equality requires the use of integration.

a finite value or values of x of the type $x = a$, these will give us the corresponding asymptotes parallel to y-axis. [See *Ex. 7*, § *16.8*]

Note. In a similar way we may get *asymptotes parallel to x-axis* thus if as $y \to b \pm 0$, $|x| \to \infty$, (where x, y is a point on the curve) then $y = b$ is an asymptote.

16.4. Asymptotes of algebraic curves.

The most useful case of determination of asymptotes is for algebraic curves. The general form of the equation of an algebraic curve of the nth degree is, arranging in groups of homogeneous terms,

$$\left(a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \dots\dots\dots\dots + a_ny^n\right)$$
$$+\left(b_0x^{n-1} + b_1x^{n-2}y + \dots\dots\dots\dots + b_{n-1}y^{n-1}\right)$$
$$+\left(c_0x^{n-2} + c_1x^{n-3}y + \dots\dots\dots\dots + c_{n-2}y^{n-2}\right)$$
$$+ \dots\dots\dots = 0, \qquad \dots (1)$$

which can also be written as

$$x^n\phi_n\left(\frac{y}{x}\right) + x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right) + x^{n-2}\phi_{n-2}\left(\frac{y}{x}\right) + \dots\dots = 0, \qquad \dots (2)$$

where ϕ_r is an algebraic polynomial of degree r.

For asymptotes of this curve, we proceed to prove the following rules:

Rule 1. Asymptotes not parallel to y-axis *will all be given by* $y = mx + c$, *where m is any of the real finite roots of* $\phi_n(m) = 0$ *and for each such values of m,* $c = -\phi_{n-1}(m)/\phi_n'(m)$ *provided it gives a definite value of* c.

***Proof*:**

The equation (2) of the curve can be put in the form

$$\phi_n\left(\frac{y}{x}\right) + \frac{1}{x}\phi_{n-1}\left(\frac{y}{x}\right) + \frac{1}{x^2}\phi_{n-2}\left(\frac{y}{x}\right) + \dots\dots = 0 \qquad \dots (3)$$

Now if $y = mx + c$ be an asymptote, where m and c are finite, $\underset{x \to \infty}{Lt}(y/x) = m$ (See § *16.2*). Hence from (3), making $x \to \infty$, since m is finite, and the functions $\phi_n(m)$, $\phi_{n-1}(m)$, etc. which are algebraic polynomials in m are accordingly finite, we get $\phi_n(m) = 0$.

Again, since in this case $\underset{x \to \infty}{Lt}(y - mx) = c$ (See § *16.2*) we can write $y - mx = c + u$, where u is a function of x such that $u \to 0$ when

$x \to \infty$. Thus, $\frac{y}{x} = m + \frac{c+u}{x}$.

From (3), now we get

$$\phi_n \doteq \left(m + \frac{c+u}{x}\right) + \frac{1}{x}\phi_{n-1}\left(m + \frac{c+u}{x}\right)$$

$$+ \frac{1}{x^2}\phi_{n-2}\left(m + \frac{c+u}{x}\right) + \ldots = 0$$

Expanding each term by Taylor's theorem, since the functions ϕ_r are all algebraic polynomials and will each lead to a finite series, and remembering that $\phi_n(m) = 0$, we get

$$\left\{\frac{c+u}{x}\phi_n'(m) + \frac{(c+u)^2}{x^2 2!}\phi_n''(m) + \frac{(c+u)^3}{x^3 3!} + \phi_n'''(m) + \ldots\right\}$$

$$+ \frac{1}{x}\left\{\phi_{n-1}(m) + \frac{c+u}{x}\phi_{n-1}'(m) + \frac{(c+u)^2}{x^2 2!}\phi_{n-1}''(m) + \ldots\right\}$$

$$+ \frac{1}{x^2}\left\{\phi_{n-2}(m) + \frac{c+u}{x}\phi_{n-2}'(m) + \ldots\right\}$$

$$+ \ldots\ldots = 0. \qquad \ldots \quad (4)$$

Now multiplying throughout by x and making $x \to \infty$, we get ($\because u \to 0$ now).

$$c\phi_n'(m) + \phi_{n-1}(m) = 0, \text{ or, } c = -\phi_{n-1}(m)/\phi_n'(m).$$

Each finite root of $\phi_n(m) = 0$ will give one value of c (provided $\phi_n'(m) \neq 0$ for this value), and accordingly we get the corresponding asymptote $y = mx + c$.

Special cases

If any value of m satisfying $\phi_n(m) = 0$ (say $m = m_1$) makes $\phi_n'(m) = 0$ also (which requires m_1 to be a multiple root $\phi_n(m) = 0$ of as we know from the theory of equations), and if $\phi_{n-1}(m) \neq 0$ for this value, then $c \to \infty$ as $m \to m_1$. Accordingly there is no asymptote corresponding to this value of m.

If for $m = m_1$, we get $\phi_n(m)$, $\phi'_n(m)$ $\phi_{n-1}(m)$, each = 0, then from (4), multiplying throughout by x^2, and making $x \to \infty$, we derive

$$\tfrac{1}{2}c^2\phi''_n(m) + c\phi'_{n-1}(m) + \phi_{n-2}(m) = 0$$

giving two values (say c_1, c_2) of c in general [provided $\phi''_n(m_1) \neq 0$], and thereby giving two parallel asymptotes of the type $y = m_1x + c_1$, $y = m_2x + c_2$.

If $\phi''_n(m_1)$ is also zero (*i.e.*, if m_1 is a triple root of $\phi_n(m) = 0$, and if $\phi'_{n-2}(m_1)$, $\phi_{n-2}(m_1)$ are also identically zero, we shall, proceeding in a similar manner, get three parallel asymptotes in general corresponding to $m = m_1$; and so on.

Rule II. Asymptotes parallel to y-axis *exist only when a_n (the coefficient of the highest power of y, i.e., of y^n) is zero, and in this case the coefficient of the highest available power of y in the equation (provided it involves x, and is not merely a constant) equated to zero will give us those asymptotes.*

A similar rule will apply to asymptotes parallel to x-axis.

Proof : *After dividing by y^n, and replacing $1/y$ by z, the equation (1) of the curve can be written in ascending powers of z in the form*

$$a_n + z\left(a_{n-1}x + b_{n-1}\right) + z^2\left(a_{n-2}x^2 + b_{n-2}x + c_{n-2}\right) + \ldots = 0 \qquad \ldots (5)$$

This will have an asymptote parallel to y-axis of the type $x = a$ where a is finite, provided $z \to 0$ when $x \to a + 0$ or $a - 0$. [See *Ex. § 16.3*]

Hence making $z \to 0$, since x now tends to a finite value, we must have the necessary condition $a_n = 0$.

Assuming this to be satisfied, we get from (5), dividing by z and making $z \to 0$,

$$a_{n-1}x + b_{n-1} = 0 \qquad \ldots (6)$$

giving a finite value of x (provided a_{n-1} is not zero) which makes $|y| \to \infty$ and thus represents an asymptote.

In case a_{n-1} is also zero along with a_n in order that we may have an asymptote parallel to y-axis, since x is to be finite, we must have, from (6), $b_{n-1} = 0$. Hence, from (5), dividing by z^2 and making $z \to 0$ now we get the asymptotes parallel to y-axis (two in this case) given by

$$a_{n-2}x^2 + b_{n-2}x + c_{n-2} = 0$$

provided this gives finite values of x. In case a_{n-2}, b_{n-2}, c_{n-2} are all identically zero, we proceed in similar manner with the coefficient of z^3 in (5), *i.e.*, the coefficient of y^{n-2} in the original equation (1), and so on, proving the rule.

Note. By interchanging y and x in arranging the given equation (1), and proceeding in a similar manner, (making $1/x \to 0$) we can prove the corresponding rule for finding the asymptotes parallel to the x-axis.

16.5. Working rule for asymptotes of algebraic curves.

For an algebraic curve of the nth degree with equation given by (1) of the previous article, first of all see if the term involving y^n is absent, in which case, the coefficient of the highest power of y involved in the equation (unless it is merely a constant independent of x) equated to zero will give asymptotes parallel to the y-axis.

Similarly, if the term involving x^n is absent, the coefficient of the highest available power of x equated to zero will in general give asymptotes parallel to the x-axis.

Next, replacing x by 1 and y by m in the homogeneous n^{th} degree terms, get $\phi_n(m)$ [as is apparent from the alternative form $x^n\phi^n(y/x)$]. Similarly, get $\phi_{n-1}(m)$ from the $(n-1)^{th}$ degree terms, and if necessary (see later); $\phi_{n-2}(m)$ from the $(n-2)^{th}$ degree terms, and so on. Now equating $\phi_n(m)$ to zero, obtain the real finite roots m_1, m_2, etc. which will indicate the directions of the corresponding asymptotes (repeated roots giving in general a set of parallel asymptotes).

For each non-repeated root (m_1, say), a definite value c_1 of

$$c = -\phi_{n-1}(m)/\phi_n'(m)$$

is obtained, and the corresponding asymptote $y = m_1 x + c_1$ is determined. For repeated roots the several values of c may be obtained as explained under the head 'Special cases' of Rule I.

16.6. Alternative method of finding asymptotes of algebraic curves.

Let the equation to an algebraic curve be

$$P_n + P_{n-1} + P_{n-2} + \ldots = 0, \qquad \ldots \quad (1)$$

where $P_n [\equiv a_0x^n + a_1x^{n-1}y + \ldots\ldots + a_ny^n = x^n\varphi_n(y/x)]$ consists of homogeneous terms of degree n, P_{n-1} is homogeneous of degree $n-1$, and so on. Now $m_1, m_2, m_3, \ldots\ldots$ being the roots of $\phi_n(m) = 0$, we know from the theory of equations that $m - m_1,\ m - m_2, \ldots\ldots$ are factors of $\phi_n(m)$ and accordingly $P_n \equiv a_n(y - m_1x)(y - m_2x)\ldots$ The possible asymptotes are parallel to $y - m_1x = 0,\ y - m_2x = 0$, etc., as proved in § 16.4, and their directions are thus all easily found from the factors of P_n.

CASE I. Let $y - m_1x$ be a non-repeated factor of P_n's, Equation (1) can then be written as

$$(y - m_1x)Q_{n-1} + F_{n-1} = 0$$

or, $$y - m_1x + (F_{n-1}/Q_{n-1}) = 0 \quad \ldots(2)$$

where $Q_{n-1}\,[=(y-m_2x)(y-m_3x)\ldots..]$ is a homogeneous expression of degree $n-1$ which does not contain $y - m_1x$ as a factor, and $F_{n-1}\,[= P_{n-1} + P_{n-2} + \ldots..]$ consists of $(n-1)^{th}$ and lower degree terms.

Now the asymptote parallel to $y - m_1x = 0$ is $y - m_1x = c_1$, where $c_1 = \underset{x\to\infty}{Lt}(y - m_1x) = \underset{x\to\infty}{Lt}(-F_{n-1}/Q_{n-1})$ [*from* (2) *above*], it being remembered that $\underset{x\to\infty}{Lt}(y/x) = m_1$ in this case [See § *16.2*]. In other words, the particular asymptote in question is

$$\mathbf{y - m_1x + \underset{x\to\infty}{Lt}(-F_{n-1}/Q_{n-1}) = 0},$$

where in determining the limit involved, we are to put $y = m_1x$ and then make $x \to \infty$.

CASE II. Let P_n have a repeated factor $(y - m_rx)$. The equation (1) can then be written as

$$(y - m_rx)^2 Q_{n-2} + P_{n-1} + F_{n-2} = 0, \quad \ldots(3)$$

where Q_{n-2} is a homogeneous expression of degree $n-2$ and $F_{n-2}\,[= P_{n-2} + P_{n-3} + \ldots..]$ consists of $(n-2)^{th}$ and lower degree terms.

Now the asymptotes parallel to $y - m_rx = 0$ will be $y - m_rx = c_r$, where $c_r = \underset{x\to\infty}{Lt}(y - m_rx)$, and this from (3) is given by

$$c_r^{\,2} + \underset{x\to\infty}{Lt}\frac{P_{n-1} + F_{n-2}}{Q_{n-2}} = 0,$$

it being remembered that $\underset{x\to\infty}{Lt}(y/x) = m_r$ here.

If P_{n-1} does not contain $y - m_r x$ as a factor, then it is easily seen that c_r^2, as given above, does not tend to a finite limit, and accordingly there are no asymptotes parallel to $y = m_r x$.

If, on the other hand, P_{n-1} has a factor $y - m_r x$, assuming $P_{n-1} = (y - m_r x) R_{n-2}$, we can write (3) in the form

$$(y - m_r x)^2 + (y - m_r x)\frac{R_{n-2}}{Q_{n-2}} + \frac{F_{n-2}}{Q_{n-2}} = 0,$$

and arguing as before, the required asymptotes will be given by

$$(y - m_r x)^2 + (y - m_r x) \underset{x\to\infty}{Lt} \frac{R_{n-2}}{Q_{n-2}} + \underset{x\to\infty}{Lt} \frac{F_{n-2}}{Q_{n-2}} = 0,$$

it being remembered that in proceeding to determine the limits we are to use $\underset{x\to\infty}{Lt} (y/x) = m_r$ here.

The two parallel asymptotes corresponding to the two repeated factors of P_n are thus obtained.

Similarly, we may proceed in cases of factors of P_n repeated more than twice.

Note. If, in P_n the term involving y^n be absent, that is, $a_n = 0$ clearly P_n will have a factor x, and corresponding to this there will be in general an asymptote parallel to y-axis, *i.e.*, parallel to $x = 0$, $\phi_n(m)$ (which is in general of degree n) will have its degree lower than n in this case. If, for instance, x^2 be a factor of P_n, $\phi_n(m)$ will be of degree $n-2$, as $x^2 y^{n-2}$ will be the term involving the highest power of y in P_n. In this case, there will be in general two asymptotes parallel to y-axis (*i.e.*, $x = 0$) and $n-2$ asymptotes corresponding to the roots of $\phi_n(m) = 0$, *i.e.*, corresponding to the other factor of P_n.

Thus, *all the possible directions of the asymptotes of the algebraic curve* (including those parallel to y-axis) *will be indicated by the factors of* , and the asymptotes may be very effectively determined by the method of the present article. [For *illustrations*, see *Ex. 1-5, § 16.8.*]

A special case (*Asymptotes by inspection*).

If the equation to an algebraic curve can be put in the form $F_n + F_{n-2} = 0$, *where* F_n *consists of* n^{th} *and lower degree terms which* *can be expressed as a product of n linear fr ctors none of which is repeated, and F_{n-2} consists of terms at most of degree $n-2$, then all the asymptotes are give by $F_n = 0$.*

For, let $F_n = (a_1 x + b_1 y + c_1)(a_2 x + b_2 y + c_2) \ldots$

$$(a_n x + b_n y + c_n)$$

$$= (a_1 x + b_1 y + c_1) Q_{n-1} \text{ (say)},$$

where Q_{n-1} is of degree $n-1$.

The equation of the curve can then be written as

$$a_1 x + b_1 y + c_1 + F_{n-2} / Q_{n-1} = 0,$$

and the asymptote parallel to $a_1 x + b_1 y = 0$ is, as shown above,

$$a_1 x + b_1 y + c_1 + \underset{x \to \infty}{Lt} (F_{n-2} / Q_{n-1}) = 0$$

where, in calculating the limit of the last term, we are to put

$$y = -(a_1 / b_1) x,$$

and then make $x \to \infty$, and this limit is easily seen to be zero, since F_{n-2} is at most of degree $n-2$, and Q_{n-1} is of degree $n-1$.

Thus, $a_1 x + b_1 y + c_1 = 0$ is an asymptote. Similarly, each of $a_2 x + b_2 y + c_2 = 0$, etc. will be an asymptote. As there are n asymptotes here, $F_n = 0$ represent all the asymptotes.

Note. If in the above case, F_n consists of real linear factors, some repeated, and some non-repeated, the non-repeated linear factors equated to zero will be asymptotes to the curve. The asymptotes corresponding to the repeated factors, however, will have to be obtained as in the general case.

16.7. Asymptote of Polar curves.

Let $r = f(\theta)$ be the polar equation to a curve. This may be written as

$$u = \frac{1}{r} = \frac{1}{f(\theta)} = F(\theta) \text{ (say)}. \qquad \ldots (1)$$

P being any point (r, θ) on the curve, $P \to \infty$ as $r \to \infty$ which requires $F(\theta) \to 0$. Let the solutions of $F(\theta) = 0$ be $\theta = \alpha, \beta, \gamma, \ldots$ etc. Then these are the only directions along which the branches of the curve tend to infinity. Consider the branch corresponding to $\theta = \alpha$. Let the straight line $r\cos\left(\theta-\alpha_1\right)=p\ldots(2)$ be the asymptote to this branch.

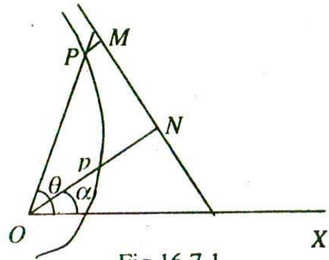


Fig 16.7.1

Then $p = ON$, Where ON is the perpendicular from the pole O on the line, and $\angle NOX = \alpha_1$. Let OP produced meet this line at Q. If PM be the perpendicular from P on the line, then

$$PM = PQ\cos QPM = \left(OQ - OP\right)\cos QON$$

$$= \left\{p\sec\left(\theta-\alpha_1\right) - f(\theta)\right\}\cos\left(\theta-\alpha_1\right)$$

[*From (1) & (2)*]

$$= p - f(\theta)\cos\left(\theta-\alpha_1\right).$$

Now since (2) is an asymptote, $PM \to 0$ as $P \to \infty$, *i.e.*, as $\theta \to \alpha$ for the branch in question.

$$\therefore\quad \underset{\theta\to\alpha}{Lt}\left\{p - f(\theta)\cos\left(\theta-\alpha_1\right)\right\} = 0$$

or, $\underset{\theta\to\alpha}{Lt} f(\theta)\cos\left(\theta-\alpha_1\right) = p$ and as p is finite, and $f(\theta)\to\infty$ as $\theta \to a$, $\underset{\theta\to\alpha}{Lt}\cos\left(\theta-\alpha_1\right)=0$;

$$\therefore\quad \alpha-\alpha_1 = \tfrac{1}{2}\pi \text{ or, } \alpha_1 = \alpha - \tfrac{1}{2}\pi$$

Again, $p = \underset{\theta\to\alpha}{Lt} f(\theta)\cos\left(\theta-\alpha_1\right) = \underset{\theta\to\alpha}{Lt}\dfrac{\cos\left(\theta-\alpha_1\right)}{F(\theta)}$

Which $\left(\text{being of the form } \dfrac{0}{0}\right) = \underset{\theta\to\alpha}{Lt}\dfrac{-\sin\left(\theta-\alpha_1\right)}{F'(\theta)}$

$$= -\frac{\sin\left(\alpha-\alpha_1\right)}{F'(\alpha)} = -\frac{1}{F'(\alpha)}.$$

Hence, (2) reduces to $r\cos\left(\theta-\alpha+\frac{1}{2}\pi\right)=-1/F'(\alpha)$,

$$\text{or, } \mathbf{r\sin(\theta-\alpha)=1/F'(\alpha)}$$

which is the required asymptote.

Similarly, the other possible asymptotes corresponding to the other branches are $r\sin(\theta-\beta)=1/F'(\beta)$, $r\sin(\theta-\gamma)=1/F'(\gamma)$, etc.

16.8. Illustrative Examples.

Ex. 1. *Find the asymptotes of the cubic*

$$x^3-2y^3+xy(2x-y)+y(x-y)+1=0 \qquad [\textit{C. P. 1949, '97}]$$

The curve being an algebraic curve of the third degree, since the terms involving x^3 and y^3 are both present, there are no asymptotes parallel to either the x-axis or the y-axis in this case.

To find the asymptotes of the type $y=mx+c$, which are oblique, considering respectively the third and second degree terms (putting 1 for x and m for y), we get here

$$\phi_n(m)=1-2m^3+m(2-m)=(1-m)(1+m)(1+2m),$$

and $\quad \phi_{n-1}(m)=m(1-m)$

Now, $\quad \phi_n(m)=0$ gives $m=1,-1,-\frac{1}{2}$,

Also, $c=\dfrac{\phi_{n-1}(m)}{\phi'_n(m)}=\dfrac{m(1-m)}{-6m^2+2-2m}$, and thus for $m=1$, $c_1=0$; for $m=-1$, $c_2=-1$; and for $m=-\frac{1}{2}$, $c_3=\frac{1}{2}$.

Hence the required asymptotes are

$$y=x,\ y=-x-1 \text{ and } y=-\tfrac{1}{2}x+\tfrac{1}{2},$$

$$\textit{i.e.,}\quad x-y=0,\ \ x+y+1=0 \text{ and } x+2y=1.$$

Note. It may be noted that *the equation to determine* m *and* c *might be obtained in practice by putting* $y=mx+c$ *in the given equation, and then equating to zero the coefficients of the two highest powers of* x.

Alternative method :

Writing the highest degree terms in factorised form, the equation can be written as

$$(x-y)(x+y)(x+2y)+y(x-y)+1=0.$$

Hence the possible asymptotes are parallel to $x - y = 0$, $x + y = 0$ and $x + 2y = 0$, and these asymptotes are respectively

$$x - y + \underset{\substack{x \to \infty \\ y = x}}{Lt} \frac{y(x-y)+1}{(x+y)(x+2y)} = 0 \qquad \ldots \qquad (1)$$

$$x + y + \underset{\substack{x \to \infty \\ y = -x}}{Lt} \frac{y(x-y)+1}{(x-y)(x+2y)} = 0 \qquad \ldots \qquad (2)$$

and

$$x + 2y + \underset{\substack{x \to \infty \\ y = -\frac{1}{2}x}}{Lt} \frac{y(x-y)+1}{(x-y)(x+y)} = 0 \qquad \ldots \qquad (3)$$

The limit, involved in (1), $= \underset{x \to \infty}{Lt} \dfrac{x(x-x)+1}{(x+x)(x+2x)} = 0$;

that in (2), $= \underset{x \to \infty}{Lt} \dfrac{-x(x+x)+1}{(x+x)(x-2x)} = \underset{x \to \infty}{Lt} \dfrac{-2x^2+1}{-2x^2} = 1$

and that in (3), $= \underset{x \to \infty}{Lt} \dfrac{-\frac{1}{2}x\left(x+\frac{1}{2}x\right)+1}{\left(x+\frac{1}{2}x\right)\left(x-\frac{1}{2}x\right)} = \underset{x \to \infty}{Lt} \dfrac{-\frac{3}{4}x^2+1}{\frac{3}{4}x^2} = -1$

Hence the asymptotes are

$$x - y = 0, \quad x + y + 1 = 0, \quad x + 2y - 1 = 0.$$

Ex. 2. *Find the asymptotes of* $2x(y-5)^2 = 3(y-2)(x-1)^2$

As the curve is algebraic, arranging the terms in descending degrees the equation can be written as

$$xy(2y-3x)+2x(3x-7y)+38x-3y+6=0 \qquad \ldots \qquad (1)$$

The possible asymptotes are parallel to $x = 0$, $y = 0$ and $2y - 3x = 0$. The asymptote parallel to $x = 0$; *i.e.*, to the y-axis, is (equating to zero the coefficient of y^2, the highest available power of y in (1), since the term involving y^3 is absent here) $2x = 0$, *i.e.*, $x = 0$, the y-axis itself.

The asymptote parallel to $y = 0$, (in a similar manner, since x^3 term is absent here), is $-3y + 6 = 0$, or, $y = 2$.

The third asymptote is

$$2y - 3x + \underset{\substack{x \to \infty \\ y = \frac{3}{2}x}}{Lt} \frac{2x(3x-7y)+38x-3y+6}{xy} = 0$$

and since the limit involved $= \underset{x\to\infty}{Lt} \dfrac{2x\left(3x-\frac{21}{2}x\right)+\left(38-\frac{9}{2}\right)x+6}{\frac{3}{2}x^2} = -10$,

the asymptote is $2y-3x-10=0$.

Hence the required asymptotes are $x=0$, $y=2$, $2y=3x+10$.

Ex. 3. *Determine the asymptotes of*

$$x^3+x^2y-xy^2-y^3+2xy+2y^2-3x+y=0 \qquad [\textit{C. P. 2008}]$$

Writing the equation as

$$(x+y)^2(x-y)+2y(x+y)-3x+y=0, \qquad \ldots(1)$$

we note that there are presumably two parallel asymptotes parallel to $x+y=0$, and one parallel to $x-y=0$.

The asymptotes parallel to $x+y=0$ are given by

$$(x+y)^2+2(x+y).\underset{\substack{x\to\infty\\ y=-x}}{Lt}\frac{y}{x-y}-\underset{\substack{x\to\infty\\ y=-x}}{Lt}\frac{3x-y}{x-y}=0 \qquad \ldots(2)$$

provided the limits involved exist.

Now, $\quad \underset{\substack{x\to\infty\\ y=-x}}{Lt}\dfrac{y}{x-y}=\underset{x\to\infty}{Lt}\dfrac{-x}{x+x}=-\frac{1}{2}$

and $\quad \underset{\substack{x\to\infty\\ y=-x}}{Lt}\dfrac{3x-y}{x-y}=\underset{x\to\infty}{Lt}\dfrac{3x+x}{x+x}=2$.

Hence, the asymptotes from (2) are

$(x+y)^2-(x+y)-2=0$, or, $(x+y+1)(x+y-2)=0$.

Again, the asymptote parallel to $x-y=0$ is given from (1) by

$$x-y+\underset{\substack{x\to\infty\\ y=x}}{Lt}\frac{2y(x+y)-3x+y}{(x+y)^2}=0,$$

i.e., $x-y+\underset{x\to\infty}{Lt}\dfrac{2x.(x+x)-3x+x}{(x+x)^2}=0$, *i.e.*, $x-y+1=0$.

Thus, the required asymptotes are

$$x+y+1=0,\ x+y-2=0 \text{ and } x-y+1=0.$$

Ex. 4. *Find the asymptotes of the Folium of Descartes*

$$x^3 + y^3 = 3axy. \qquad [\text{B.P. } 1993]$$

The equation can be written as $(x+y)(x^2 - xy + y^2) = 3axy$, and since the highest degree terms have got only one real linear factor $x+y$, (the linear factors of $x^2 - xy + y^2$ being clearly imaginary), there is only one possible asymptote here, which is parallel to $x + y = 0$. The asymptotes in question is

$$x + y = . \underset{\substack{x \to \infty \\ y = -x}}{Lt} \frac{3axy}{x^2 - xy + y^2} = \underset{x \to \infty}{Lt} \frac{-3ax^2}{x^2 + x^2 + x^2} = -a,$$

i.e., $x + y + a = 0$.

Ex. 5. *Find the asymptotes of* $x(x-y)^2 - 3(x^2 - y^2) + 8y = 0$.

The possible asymptotes here are one parallel to $x = 0$ and a pair parallel to $x - y = 0$.

The first one, which is parallel to y-axis, is found by equating the coefficient of y^2 to zero, (the term involving y^3 being absent, as it should be under the circumstances), namely, $x + 3 = 0$.

The other two are given by

$$(x-y)^2 - 3(x-y). \underset{\substack{x \to \infty \\ y = x}}{Lt} \frac{x+y}{x} + 8 \underset{\substack{x \to \infty \\ y = x}}{Lt} \frac{y}{x} = 0,$$

i.e., $(x-y)^2 - 3(x-y).2 + 8 = 0$, or, $(x-y-4)(x-y-2) = 0$.

Thus, the required asymptotes are

$$x + 3 = 0, \qquad x - y = 4 \text{ and } x - y = 2.$$

Ex. 6. *Prove that the asymptotes of the cubic*

$$(x^2 - y^2)y - 2ay^2 + 5x - 7 = 0$$

form a triangle of area a^2.

The equation to the curve may be written as

$$y(x^2 - y^2 - 2ay) + 5x - 7 = 0$$

or $\quad y\{x^2 - (y+a)^2\} + a^2 y + 5x - 7 = 0,$

i.e., $y(x+y+a)(x-y-a) + ay^2 + 5x - 7 = 0$

which is of the form $F_3 + F_1 = 0$, F_3 having three non-repeated linear factors, and so the required asymptotes are given by equating these factors to zero, namely,

$$y=0,\ x+y+a=0 \text{ and } x-y-a=0.$$

By solving in pairs, their points of intersection are easily seen to be

$$(-a,0),\quad (a,0) \text{ and } (0,-a).$$

The area of the triangle with these as vertices is

$$\frac{1}{2}\begin{vmatrix} a & 0 & 1 \\ -a & 0 & 1 \\ 0 & -a & 1 \end{vmatrix}=\frac{1}{2}.2a^2=a^2.$$

Ex. 7. *Find the asymptotes, if any, of the curve*

$$y=a\log\sec(x/a).$$

This is not an algebraic curve. To find its asymptotes, if any, which are not parallel to y-axis, we know that $y=mx+c$ will be an asymptote, where

$m=\underset{x\to\infty}{Lt}\frac{y}{x}$ and $c=\underset{x\to\infty}{Lt}(y-mx)$.

Now in the curve, $m=\underset{x\to\infty}{Lt}\frac{y}{x}=\underset{x\to\infty}{Lt}\frac{a\log\sec(x/a)}{x}$

which limit does not exist.

Hence there is no asymptote non-parallel to y-axis in this case.

To find if there be any asymptote parallel to y-axis, we notice that $y\to\infty$ when $x/a\to 2n\pi\pm\frac{1}{2}\pi$, and accordingly the asymptotes parallel to y-axis are (See § *16.3*)

$$x=\left(2n\pi\pm\tfrac{1}{2}\pi\right)a$$

which are the only asymptotes of the given curve.

Ex. 8. *An asymptote is defined in the following two ways :*

(A) *An asymptote is a straight line, the distance of which from a point on a curve diminishes without limit as the point on the curve moves to an infinite distance from the origin.*

(B) *An asymptote to a curve is the limiting position of the tangent when the point of contact moves to an infinite distance from the origin,*

Consider the two curves

(i) $y=ax+b+\dfrac{c}{x}$, (ii) $y=ax+b+\dfrac{c+\sin x}{x}$.

Show that for the first curve, an asymptote exists according to both the definitions, but for the second curve, an asymptote exists according to the first definition, but not according to the second.

Let us consider the first definition. According to this it has been proved (§ *16.2*) that the straight line $y = mx + c$ will be an asymptote to a curve, where $m = \underset{x\to\infty}{Lt} \dfrac{y}{x}$ and $c = \underset{x\to\infty}{Lt}(y - mx)$, (x, y) being a point on the curve. provided the limits exist.

Now for the curve (i),

$$m = \underset{x\to\infty}{Lt} \frac{y}{x} = \underset{x\to\infty}{Lt}\left(a + \frac{b}{x} + \frac{c}{x^2}\right) = a,$$

and
$$c = \underset{x\to\infty}{Lt}(y - mx) = \underset{x\to\infty}{Lt}(y - ax) = \underset{x\to\infty}{Lt}\left(b + \frac{c}{x}\right) = b.$$

Accordingly the asymptote exists, given by $y = ax + b$.

Again, for the curve (ii),

$$m = \underset{x\to\infty}{Lt} \frac{y}{x} = \underset{x\to\infty}{Lt}\left(a + \frac{b}{x} + \frac{c + \sin x}{x^2}\right) = a \qquad [\because |\sin x| \le 1]$$

and
$$c = \underset{x\to\infty}{Lt}(y - mx) = \underset{x\to\infty}{Lt}(y - ax) = \underset{x\to\infty}{Lt}\left(b + \frac{c + \sin x}{x}\right) = b.$$

Thus, the asymptote exists here also, given by $y = ax + b$.

Next, consider the send definition.

For curve (i), $\dfrac{dy}{dx} = a - \dfrac{c}{x^2}$, and so the equation to the tangent line at (x, y) is $Y - y = \left(a - \dfrac{c}{x^2}\right)(X - x)$

$$\text{or, } Y = \left(a - \frac{c}{x^2}\right)X + y - x\left(a - \frac{c}{x^2}\right) = \left(a - \frac{c}{x^2}\right)X + \left(b + \frac{2c}{x}\right)$$

As $x \to \infty$, the equation becomes $Y = aX + b$, which is then the definite straight line towards which the tangent line approaches, as the point of contact (x, y) moves to an infinite distance. Hence this is the asymptote.

For curve (ii), $\dfrac{dy}{dx} = a + \dfrac{x\cos x - (c + \sin x)}{x^2}$

and the equation to the tangent line at (x, y) is

$$Y-y=\left\{a+\frac{x\cos x-(c+\sin x)}{x^2}\right\}(X-x)$$

or substituting the value of y from (ii),

$$Y=\left(a+\frac{\cos x}{x}-\frac{c+\sin x}{x^2}\right)X+\left\{\frac{2(c+\sin x)}{x^2}-\cos x+b\right\}.$$

Now, as $x\to\infty$, $\cos x$ does not tend to any definite limit. Hence the tangent line at (x, y) does not tend to any definite limiting position and so the asymptote does not exist in this case, according to the second definition.

Ex. 9. *Find the asymptotes, if any, of the curve* $(r-a)\sin\theta=b$.

The equation can be written as $u=\dfrac{\sin\theta}{b+a\sin\theta}=F(\theta)(say)$.

The directions in which $r\to\infty$ are given by $u=0$, or $\sin\theta=0$, giving $\theta=n\pi$.

Now $F'(\theta)$, *i.e.*, $\dfrac{du}{d\theta}=\dfrac{\cos\theta(b+a\sin\theta)-\sin\theta.a.\cos\theta}{(b+a\sin\theta)^2}$

$$=\frac{b\cos\theta}{(b+a\sin\theta)^2},$$

and for $\theta=n\pi$, this $=\dfrac{b\cos n\pi}{b^2}=\dfrac{\cos n\pi}{b}$

Hence, as in § *16.7*, the required asymptote is given by

$$r\sin(\theta-n\pi)=1/F'(n\pi)=b\sec n\pi$$

which, whether n is even or odd, reduces to

$$r\sin\theta=b.$$

16.9 Miscellaneous Worked Out Examples

Ex. 1. Find the asymptotes of :

(*i*) $x^2-4y^2=1$. [*C. P. 1982, 94*]

(*ii*) $x^2-y^2=9$. [*C. P. 1998*]

(*iii*) $x^2y^2=a^2(x^2+y^2)$. [*B. P. 1990*]

(*iv*) $xy-3x-4y=0$. [*C. P. 2000*]

Solution :

(*i*) $x^2-4y^2-1=0$

or, $(x+2y)(x-2y)-1=0$ or, $F_2+F_0=0$

where $F_2=(x+2y)(x-2y)$ is of degree 2 and $F_0=-1$, which is of degree 0.

So, asymptotes of the given curve are given by $F_2=0$

i.e., $(x+2y)(x-2y)=0$

Hence, the asymptotes are $x+2y=0$, $x-2y=0$

(*ii*) $x^2-y^2-9=0$

or, $F_2+F_0=0$

where $F_2=(x+y)(x-y)$ *is* of degree 2 in x and y and $F_0=-9$, is of degree 0.

Hence the asymptotes are given by $F_2=0$.

i.e., $x+y=0$ and $x-y=0$.

(*iii*) $x^2y^2-a^2x^2-a^2y^2=0$ . . . (1)

This is a fourth degree equation in x and y ; So the curve represented by (1) may have *at most four* asymptotes.

Here, terms containing x^4 and y^4 are absent, so the curve has asymptotes parallel to x-axis and y-axis.

The coefficient of highest available power of x, *i.e.* of x^2 is y^2-a^2 and the coefficient of y^2 is x^2-a^2.

So, asymptotes parallel to the x-axis are : $y^2-a^2=0$

i.e., $y\pm a=0$.

and asymptotes parallel to the y-axis are : $x^2-a^2=0$

i.e., $x\pm a=0$.

(*iv*) $xy-3x-4y=0$ . . . (1)

As in the earlier problem, the asymptote parallel to the x-axis is $y-3=0$ and the asymptote parallel to y-axis is $x-4=0$.

Ex. 2. Find the asymptotes of :

(*i*) $x^3+2x^2y+xy^2-x+1=0$ [*C. P. 1992*]

(*ii*) $(x+y)^2(x+2y+2)=x+9y+2$. [*C. P. 2000*]

(*iii*) $4x^3-3xy^2-y^3+2x^2-xy-y^2-1=0$. [*C. P. 1998, 2001*]

Solution : (*i*) $x^3+2x^2y+xy^2-x+1=0$ . . . (1)

or, $x(x+y+1)(x+y-1)+1=0$

or, $F_3+F_0=0$

where, $F_3=x(x+y+1)(x+y-1)$ is of degree 3 and it has three non-repeated linear factors and $F_0=1$, which is of degree 0.

The asymptotes of the curve (1) are given by

$F_3=0$, or, $x(x+y+1)(x+y-1)=0$

i.e., $x=0$, $x+y+1=0$, $x+y-1=0$

(*ii*) $(x+y)^2(x+2y+2)=x+9y+2$

or, $(x+y)^2(x+2y+2)=(x+2y+2)+7y$

or, $(x+y)^2(x+2y+2)-(x+2y+2)-7y=0$

or, $(x+2y+2)\{(x+y)^2-1\}-7y=0$

or, $(x+2y+2)(x+y+1)(x+y-1)-7y=0$

or, $F_3+F_1=0$

where $F_3=(x+2y+2)(x+y+1)(x+y-1)$ which is of degree 3 and it has three *non-repeated linear factors*, while $F_1=-7y$, which is of degree 1. Hence, the asymptotes of the curve are given by

$$x+2y+2=0,\ x+y+1=0 \text{ and } x+y-1=0.$$

(*iii*) $4x^3-3xy^2-y^3+2x^2-xy-y^2-1=0$ . . . (1)

This is a third degree curve, so it may have three asymptotes at most. Since the terms involving x^3 and y^3 are both present, it has no asymptotes parallel to *x*-axis and *y*-axis.

Equation (1) can be be written as

$$4x^3-4xy^2+xy^2-y^3+2x^2-xy-y^2-1=0$$

or, $4x(x+y)(x-y)+y^2(x-y)+2x^2-xy-y^2-1=0$

or, $(x-y)(2x+y)^2+2x^2-2xy+xy-y^2-1=0$

or, $(x-y)(2x+y)^2+2x(x-y)+y(x-y)-1=0$

or, $(x-y)(2x+y)(2x+y+1)-1=0$ or, $F_3+F_0=0$

where F_3 is of degree three and it has three different linear factors, while $F_0 = -1$, which is of degree 0.

Hence, the asymptotes of the curve (1) are given by

$$(x-y)(2x+y)(2x+y+1)=0$$

i.e., $x-y=0$, $2x+y=0$ and, $2x+y+1=0$.

EXAMPLES - XVI

Find the asymptotes of the following curves [*Ex. 1-31*]

1. $y^2 - x^2 - 2x - 2y - 3 = 0$.

2. $y^3 - 6xy^2 + 11x^2y - 6x^3 + y^2 - x^2 + 2x - 3y - 1 = 0$.

3. $x^3 + 3x^2y - xy^2 - 3y^3 + x^2 - 2xy + 3y^2 + 4x + 5 = 0$.

4. $3x^3 + 2x^2y - 7xy^2 + 2y^3 - 14xy + 7y^2 + 4x + 5y = 0$. [*C.P. 1943*]

5. $x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy - 5y + 6 = 0$.

6. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

7. $x^3 - y^3 = 3y(x+y)$.

8. $x^4 - y^4 + 3x^2y + 3xy^2 + xy = 0$.

9. $4x^4 - 5x^2y^2 + y^4 + y^3 - 3x^2y + 5x - 8 = 0$

10. (i) $xy - 2y - 3x = 0$. (ii) $y^2\left(x^2 - a^2\right) = x$.

(iii) $x\left(x^2 + y^2\right) + a\left(x^2 - y^2\right) = 0$.

11. $x^2y^2 - 4(x-y)^2 + 2y - 3 = 0$.

12. $x^2y^2 - x^2y - xy^2 + x + y + 1 = 0$. [*C.P. 1937*]

13. $y^2x^2 - 3yx^2 - 5xy^2 + 2x^2 + 6y^2 - x - 3y + 2 = 0$.

14. $x^4 - x^2y^2 + x^2 + y^2 - a^2 = 0$.

15. $y^3 - yx^2 + y^2 + x^2 - 4 = 0$.

16. $x^2(x-y)^2 - a^2\left(x^2 + y^2\right) = 0$. [*C.P. 1945*]

17. $x^3 - 4xy^2 - 3x^2 + 12yx - 12y^2 + 8x + 2y + 4 = 0$.

18. $x^3 + 3x^2y - 4y^3 - x + y + 3 = 0$. [*C.P. 2003*]

19. $y^3 - xy^2 - x^2y + x^3 + x^2 - y^2 - 1 = 0$. [*C.P. 1939*]

20. $y^3 + x^2y + 2xy^2 - y + 1 = 0$. [*C.P. 1941, '44, '87, '90, '96*]

21. $(x^2 - y^2)^2 - 8(x^2 + y^2) + 8x - 16 = 0$.

22. $y(y-x)^2(y-2x) + 3x^2(y-x) - 2x^2 = 0$.

23. $x^2(x+y)(x-y)^2 + 2x^3(x-y) - 4y^3 = 0$.

24. $(x+y)(x-2y)(x-y)^2 + 3xy(x-y) + x^2 + y^2 = 0$.

25. $(x+y)^3(x-y)^2 - 2(x+y)^2(x-y)^2 - 2(x^2+y^2)(x+y)$
$$+ 2(x-y)^2 + 4(x-y) = 0.$$

26. $y^3 - 5xy^2 + 8x^2y - 4x^3 - 4y^2 + 12xy - 8x^2 + 3y - 3x + 2 = 0$.

27. (i) $xy(x^2 - y^2) = x^2 + y^2$.

    (ii) $xy(x^2 + y^2) = x^2 - y^2$.

28. $(x^2 - y^2)(x^2 - 9y^2) + 3xy - 6x - 5y + 2 = 0$.

29. (i) $x^3 - 6xy^2 + 11xy^2 - 6y^3 + 2x - y + 1 = 0$.

    (ii) $x^4 - 5x^2y^2 + 4y^4 + x^2 - 2y^2 + 2x + y + 7 = 0$.

30. $(x - y + 1)(x + y + 1)(x - 2y + 3) = 2x - 5y + 1$.

31. (i) $y = \tan x$. (ii) $y = e^{ax}$.

    (iii) $y = e^{-x^2}$. (iv) $y = \log x$.

32. Show that the asymptotes of the curve
$$x^2y^2 = a^2(x^2 + y^2) \text{ form a square of side } 2a.$$

33. Show that the asymptotes of the curve
$$x^2y^2 - a^2(x^2 + y^2) - a^3(x + y) + a^4 = 0$$
form a square, two of whose angular points lie on the curve.
[*C.P. 1947*]

34. Show that the finite points of intersection of the asymptotes of $xy(x^2 - y^2) + a(x^2 + y^2) - a^3 = 0$ with the curve lie on a circle whose centre is at the origin.

35. Find the equation of the cubic which has the same asymptotes as the curve $2x(y-3)^2 = 3y(x-1)^2$ and which touches the axis of x at

the origin and goes through the point (1, 1).

36. If any of the asymptotes of the curve

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \left(h^2 > ab\right)$$

passes through the origin, prove that

$$af^2 + bg^2 = 2fgh.$$

37. If the equation of a curve can be put in the form

$$y = ax + b + \phi(x), \text{ where } \phi(x) \to 0 \text{ as } x \to \infty,$$

then show that $y = ax + b$ is an asymptote of the curve. Apply this method in determining the asymptotes of the curve $x^2 y - x^3 - x^2 - 3x + 2 = 0$.

38. 'An asymptote is sometimes defined as a straight line which cuts the curve in two points at infinity without being itself at infinity.' Comment on this definition. Attempt a correct definition and use it to obtain the asymptotes of $(x + y)^2 (x + 2y + 2) = x + 9y - 2$.

39. Find the asymptotes of :

(i) $r = a(\cos\theta + \sec\theta)$. (ii) $r\cos\theta = 2a\sin\theta$.

(iii) $r = a\sec\theta + b\tan\theta$. (iv) $r\cos\theta = a\sin^2\theta$.

(v) $r = a\,\mathrm{cosec}\,\theta + b$. (vi) $r\sin n\theta = a$.

(vii) $r\theta = a$. (viii) $r^n \sin n\theta = a^n \ (n > 1)$.

40. Show that there is an infinite series of parallel asymptotes to the curve

$$r = \frac{a}{\theta\sin\theta} + b.$$

41. Show that all the asymptotes of the curve $r\tan n\theta = a$ touch the circle $r = a/n$.

42. Show that the curve $r = a\sec n\theta + b\tan n\theta$ has two sets of asymptotes, members of each set touching a fixed circle.

ANSWERS

1. $y - x = 2,\ y + x = 0$. **2.** $y = x,\ y = 2x + 3,\ y = 3x - 4$.

3. $4x + 12y + 9 = 0,\ 2x + 2y - 3 = 0,\ 4x - 4y + 1 = 0$.

4. $6y - 6x + 7 = 0,\ 2y - 6x + 3 = 0,\ 6y + 3x + 5 = 0$.

5. $x + 2y = 0,\ x + y - 1 = 0,\ x - y + 1 = 0$.

6. $y = \pm \frac{b}{a} x$. 7. $x - y = 2$.

8. $x + y = 0$, $2x - 2y + 3 = 0$.

9. $3x - 3y - 1 = 0$, $3x + 3y + 1 = 0$, $12x - 6y - 1 = 0$, $12x + 6y + 1 = 0$.

10. (i) $x = 2$, $y = 3$. (ii) $y = 0$, $x = \pm a$.

(iii) $x = a$.

11. $x = \pm 2$; $y = \pm 2$. 12. $x = 0$, $x = 1$, $y = 0$, $y = 1$.

13. $x = 2$, $x = 3$, $y = 1$, $y = 2$. 14. $x = \pm 1$; $y = \pm x$.

15. $y = 1$, $y = x - 1$, $y = -x - 1$. 16. $x = \pm a$; $x - y = \pm a\sqrt{2}$.

17. $x + 3 = 0$; $x - 2y = 0$; $x + 2y = 6$.

18. $y = x$; $x + 2y - 1 = 0$; $x + 2y + 1 = 0$.

19. $y = \pm x$; $y = x + 1$. 20. $y = 0$, $x + y = \pm 1$.

21. $y = x \pm 2$; $y = -x \pm 2$.

22. $2y + 3 = 0$; $x - y + 1 = 0$; $x - y + 2 = 0$; $4x - 2y - 3 = 0$.

23. $x = \pm 2$; $x - y + 2 = 0$; $x - y - 1 = 0$; $x + y + 1 = 0$.

24. $x - y - 2 = 0$; $2x - 2y + 1 = 0$; $2x + 2y - 1 = 0$; $x - 2y + 2 = 0$.

25. $x + y - 2 = 0$; $x + y = \pm 1$; $x - y = \pm 1$.

26. $y = x$; $y = 2x + 1$; $y = 2x + 3$.

27. (i) $x = 0$, $y = 0$, $x + y = 0$, $x - y = 0$. (ii) $x = 0$, $y = 0$.

28. $x + y = 0$, $x - y = 0$, $x + 3y = 0$, $x - 3y = 0$.

29. (i) $x - y = 0$, $x - 2y = 0$, $x - 3y = 0$.

(ii) $x + 2y = 0$, $x - 2y = 0$, $x + y = 0$, $x - y = 0$.

30. $x - y + 1 = 0$, $x + y + 1 = 0$, $x - 2y + 3 = 0$.

31. (i) $x = (2n + 1)\frac{1}{2}\pi$, where n is zero or any integer positive or negative.

(ii) $y = 0$. (iii) $y = 0$. (iv) $x = 0$.

35. $2xy^2 - 3x^2y - 6xy + 7y = 0$. 37. $y = x + 1$.

38. $x + 2y + 2 = 0$; $x + y = \pm 2\sqrt{2}$.

39. (i) $r\cos\theta = a$. (ii) $r\cos\theta = \pm 2a$. (iii) $r\cos\theta = a \pm b$.

(iv) $r\cos\theta = a$. (v) $r\sin\theta = a$.

(vi) $r\sin\left(\theta - \frac{m\pi}{n}\right) = \frac{a}{n}\sec m\pi$, where m is an integer.

(vii) $r\sin\theta = a$ (viii) $\theta = \frac{m\pi}{n}$, where m is an integer.

17 SEC. A. ENVELOPE OF STRAIGHT LINES

17.1. Introduction.

Let us consider the equation

$$x \cos \alpha + y \sin \alpha = a.$$

This represents a straight line; by giving different values to α, we shall obtain the equations of different straight lines but all these different straight lines have one characteristic feature common to each of them, *viz.*, each straight line is at the same distance a from the origin. On account of this property these straight lines are said to constitute a *family* and α which is a constant for one but different for different lines, and whose different values give different members of the family, is called the *parameter of the family.* It should be noted that the position of the line varies with a.

As we have a family of straight lines, we have a family of curve. Thus the equation

$$(x - \alpha)^2 + y^2 = r^2$$

represents a family of circles for different values of α, all the individual members of the family having the common characteristic, *viz.*, they are of equal radii 'r' and their centres lie on x-axis. Here is the parameter of the family.

In general, *the equation of a family of curves is represented by* $F(x, y, \alpha) = 0$, when α is the parameter.

17.2. Definition of Envelope.

If each of the members of the family of curves $C \equiv F(x, y, \alpha) = 0$ touches a fixed curve E, then E is called the *envelope* of the family of curves C. The curve E also, at each point, is touched by some member of the family C.

Illustration :

We know that $x \cos \alpha + y \sin \alpha = a$ touches the circle $x^2 + y^2 = a^2$ at $(a \cos \alpha, a \sin \alpha)$. Thus, each of the members of the family of straight lines $C \equiv x \cos \alpha + y \sin \alpha = a$ (for different values of α) touches the fixed circle $E \equiv x^2 + y^2 = a^2$, and hence *the circle* $x^2 + y^2 = a^2$ *is the envelope of the family of straight lines* $x \cos \alpha + y \sin \alpha = a$; also the circle $x^2 + y^2 = a^2$ at each point $(a \cos \alpha, a \sin \alpha)$ obtained by varying values a, is touched by some *member of the family of straight lines.*

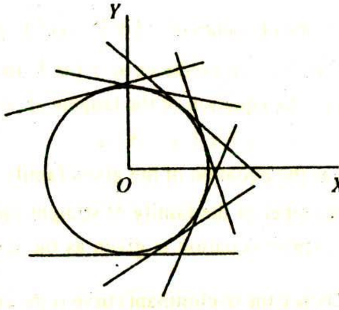


Fig 17.2.1

In the present section we shall confine ourselves to the determination of the simplest type of envelope, *i.e.*, the envelopes of straight lines.

17.3. Envelope of straight lines.

The equation of the envelope of the family of straight lines $F(x, y, \alpha) \equiv y - f(\alpha).x - \phi(\alpha) = 0$ *(α being the parameter) is the α-eliminant of* $\mathbf{F} = \mathbf{0}$ *and* $\frac{\partial \mathbf{F}}{\partial \alpha} = \mathbf{0}$.

From $F = 0$ and $\frac{\partial F}{\partial \alpha} = 0$, we have respectively

$$y = f(\alpha).x + \phi(\alpha) \qquad \ldots (1)$$

and $$0 = f'(\alpha).x + \phi'(\alpha) \qquad \ldots (2)$$

$\therefore$ from (2), $$x = -\frac{\phi'(\alpha)}{f'(\alpha)} = g(\alpha), \text{ say} \qquad \ldots (3)$$

and from (1), $$y = \frac{f'(\alpha)\phi(\alpha) - f(\alpha)\phi'(\alpha)}{f'(\alpha)} = h(\alpha), \text{say} \qquad \ldots (4)$$

Hence the curve (*i.e.*, the envelope) whose equation is obtained by eliminating α between (1) and (2) is the same as the curve whose equation is given parametrically as

$$\left.\begin{array}{l} x = g(\alpha) \\ y = h(\alpha) \end{array}\right\}, \qquad \ldots (5)$$

Now, the equation of the tangent at the point 'α' on the curve (5) is

$$y - h(\alpha) = \frac{h'(\alpha)}{g'(\alpha)}\{x - g(\alpha)\}, \qquad \ldots (6)$$

Substituting from (3) and (4) values of $g(\alpha)$, $h(\alpha)$, $g'(\alpha)$, $h'(\alpha)$ in (6) and noting that $h'(\alpha)/g'(\alpha)$ reduces to $f(\alpha)$, and simplifying, we get the equation (6), *i.e.*, the equation of the tangent at 'α' on the curve (5) as

$$y = f(\alpha).x + \phi(\alpha)$$

which is the same as the equation of hte given family of straight lines.

Thus, every member of the family of straight lines $F(x, y, \alpha) = 0$ touches the curve whose equation is given as the α-eliminant of $F = 0$ and $\dfrac{\partial F}{\partial \alpha} = 0$, and hence the α-eliminant curve is the envelope of the family of straight lines.

Cor. 1. From the definition, it at once follows that *every curve, is the envelope of its tangents.*

Cor. 2. Since we have seen that normals at different points on a curve touch the evolute at the corresponding points, it follows that **the evolute of a curve is the envelope of its normals.**

Thus, if $N(x, y, \alpha) = 0$ be the equation of the normal of a curve at a point with parameter a, the evolute is obtained by *eliminating* a between $N(x, y, \alpha) = 0$ (1) and $\dfrac{\partial N}{\partial \alpha} = 0$ (2). Since the evolute is the locus of centres of curvature, the co-ordinates of the centre of curvature are obtained in parametric form by *solving* the above two equations for x and y in terms of a.

The above methods of determining the evolute and centre of curvature are much simpler than the methods already given in chapter XV (curvature). [See *Ex. 4* and *Ex. 5, Art. 17.4.*]

17.4. Illustrative Examples.

Ex. 1. *Find the envelope of the straight line* $y = mx + \dfrac{a}{m}$, *m being the variable parameter* $(m \neq 0)$. [*C.P. 1994, 2008 V. P. '95*]

Here, $$mx + \frac{a}{m} - y = 0 \qquad \ldots (1)$$

Differentiating with respect to m,

$$x - \frac{a}{m^2} = 0. \qquad \therefore \qquad m^2 = \frac{a}{x}, \qquad \therefore \qquad m = \pm\sqrt{\frac{a}{x}}.$$

Substituting these values of m in (1),

$$\pm\left(\sqrt{\frac{a}{x}}\cdot x + a\Big/\sqrt{\frac{a}{x}}\right) - y = 0,$$

i.e., $\pm 2\sqrt{ax} = y$, or, $y^2 = 4ax$ (parabola)

which is the required envelope.

Ex. 2. *Find the envelope of the family of straight lines*

$$A\alpha^2 + B\alpha + C = 0,$$

where α *is the variable parameter, and A, B, C are linear function of x, y.*

We have $A\alpha^2 + B\alpha + C = 0$ ...(1)

Differentiating this with respect to α, we have

$$2A\alpha + B = 0, \text{ i.e., } \quad \alpha = -B/(2A).$$

Substituting this value of α in (1), we get

$$A\cdot\frac{B^2}{4A^2} - \frac{B^2}{2A} + C = 0, \qquad \text{or,} \qquad B^2 = 4AC.$$

Thus, the envelope of the family of straight lines $\mathbf{A\alpha^2 + B\alpha + C = 0}$ is the curve $\mathbf{B^2 = 4AC}$.

Note. When the parameter occurs as a quadratic in any equation the above result is sometimes used in determining the envelope.

Ex. 3. *Find the envelope of the straight lines*

$$\frac{x}{a} + \frac{y}{b} = 1,$$

where a and b are variable parameters, connected by the relation $a + b = c$, *c being a non-zero constant.* [*C. P. 1998, 2006*]

Since $a + b = c$, $\therefore$ $b = c - a$.

$\therefore$ the equation of the straight lines becomes

$$\frac{x}{a} + \frac{y}{c-a} = 1, \qquad \text{or,} \qquad (c-a)x + ay = a(c-a),$$

or, $a^2 + a(y - x - c) + cx = 0$.

Since it is in the form $A\alpha^2 + B\alpha + C = 0$, its envelope is

$$B^2 = 4AC, \quad \text{i.e.,} \quad (y - x - c)^2 = 4cx,$$

which represents a parabola.

The above equation can be written as

$$x^2+y^2+c^2=2xy+2cx+2cy.$$

or, $\sqrt{x}+\sqrt{y}=\sqrt{c}$ (which represents a parabola).

Otherwise :

The elimination of a and b can also be performed thus :

Differentiating the equation of the line and the given relation with respect to a, we get

$$-\frac{x}{a^2}-\frac{y}{b^2}\frac{db}{da}=0 \text{ and } 1+\frac{db}{da}=0.$$

On equating the values of $\frac{db}{da}$ from these two, we get

$$\frac{a}{\sqrt{x}}=\frac{b}{\sqrt{y}}=\frac{a+b}{\sqrt{x}+\sqrt{y}}=\frac{c}{\sqrt{x}+\sqrt{y}},$$

$$\therefore \quad a=\frac{c\sqrt{x}}{\sqrt{x}+\sqrt{y}}, \quad b=\frac{c\sqrt{y}}{\sqrt{x}+\sqrt{y}}.$$

Substituting these values of a and b in the equation of the line, we get

$$\left(\sqrt{x}+\sqrt{y}\right)^2=c, \text{ i.e., } \sqrt{x}+\sqrt{y}=\sqrt{c}.$$

Ex. 4. *Find the evolute of the parabola* $y^2=4ax$ *(evolute being regarded as the envelope of its normals).*

The equation of the normal to the parabola at any point 'm' is

$$y=mx-2am-am^3. \qquad \ldots(1)$$

Let us find the envelope of this, m being the parameter. Differentiating (1) with respect to m,

$$0=x-2a-3am^2, \text{ or, } m^2=(x-2a)/(3a). \qquad \ldots(2)$$

From (1), $y=m\left(x-2a-am^2\right)=m\left(3am^2-am^2\right)=2am^3$,

$$\therefore \quad y^2=4a^2m^6=4a^2\cdot\frac{(x-2a)^3}{27a^3} \text{ from (2).}$$

i.e., $27ay^2=4(x-2a)^3$,

which is the envelope of the normals, *i.e.*, the required evolute of the parabola.

Ex. 5. *Find the centre of curvature of the ellipse* $\frac{x}{a^2}+\frac{y}{b^2}=1$.

The normal at the point 'ϕ' is

$$ax\sec\phi - by\,\mathrm{cosec}\,\phi = a^2-b^2. \qquad \ldots \quad (1)$$

Differentiating this partially with respect to ϕ,

$$ax\sec\phi\tan\phi + by\,\mathrm{cosec}\,\phi\cot\phi = 0. \qquad \ldots \quad (2)$$

Solving for x and y from (1) and (2), we easily get

$$x=\frac{a^2-b^2}{a}\cos^3\phi,\quad y=\frac{a^2-b^2}{b}\sin^3\phi.$$

EXAMPLES - XVII (A)

Find the envelopes of the following families of straight lines (*Ex. 1--9*):

1. $x\cos\alpha + y\sin\alpha = a$, parameter α.

2. $ax\sec\alpha - by\,\mathrm{cosec}\,\alpha = a^2-b^2$, parameter α.

3. $x\cos 3\theta + y\sin 3\theta = a(\cos 2\theta)^{\frac{3}{2}}$, parameter θ.

4. $x\cos\alpha + y\sin\alpha = a\cos\alpha\sin\alpha$, parameter α. [*V.P. 1998*]

5. $y = mx + a\sqrt{1+m^2}$, parameter m. [*C.P. 1993, 2004, '06 V. P. '96*]

6. $y = mx + \sqrt{a^2m^2+b^2}$, parameter m.
[*C.P. 1990, '97, B.P. '86, '94, '96, V.P. 2001*]

7. $x\sec^2\theta + y\,\mathrm{cosec}^2\theta = a$, parameter θ.

8. $x\sqrt{\cos\theta} + y\sqrt{\sin\theta} = a$, parameter θ.

9. $x\cos^n\theta + y\sin^n\theta = a$, parameter θ.

10. Find the envelopes of the straight line

$$\frac{x}{a}+\frac{y}{b}=1,$$

where the parameters a and b are connected by the relations

(i) $a^2+b^2=c^2$ [*C.P. 1989, '96, 2001, '03 B.P. 1989, '95, V.P. '99, '97*]

(ii) $ab=c^2$, [*B.P. 1988, '93*]

c being a constant.

11. Find the envelopes of the straight line

$$\frac{x}{l} + \frac{y}{m} = 1,$$

where l and m are parameters connected by the relation

$l/a + m/b = 1$, a and b being constants.

12. Find the envelopes of straight lines at right angles to the radii of the following curves drawn through their extremities :

(i) $r = a\left(1 + \cos\theta\right)$. (ii) $r^2 = a^2 \cos 2\theta$. (iii) $r = ae^{m\theta}$.

13. From any point P on a parabola, PM and PN are drawn perpendiculars to the axis and tangent at the vertex ; show that the envelope of MN is another parabola. [*C.P. 1996, '99, 2005*]

14. Show that the envelope of straight lines which join the extremities of a pair of conjugate diameters of an ellipse is a similar ellipse.

15. If PM, PN be the perpendiculars drawn from any point P on the curve $y = ax^3$ upon the co-ordinate axes, show that the envelope of MN is

$$27y + 4ax^3 = 0.$$

16. From any point P on the ellipse $x^2/a^2 + y^2/b^2 = 1$, perpendiculars PM and PN are drawn upon the co-ordinate axes. Show that MN always touches the curve $\left(x/a\right)^{\frac{2}{3}} + \left(y/b\right)^{\frac{2}{3}} = 1$.

17. Find the envelope, when t varies, of

$$\left(a_1t^2 + 2a_2t + a_3\right)x + \left(b_1t^2 + 2b_2t + b_3\right)y + \left(c_1t^2 + 2c_2t + c_3\right) = 0.$$

18. Find the evolutes of the following curves (evolute being regarded as envelope of normals) :

(i) $x = a\cos\phi$, $y = b\sin\phi$. (ii) $x = at^2$, $y = 2at$.

(iii) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. (iv) $x^2/a^2 + y^2/b^2 = 1$.

(v) $x = a\left(\theta - \sin\theta\right)$, $y = a\left(1 - \cos\theta\right)$.

(vi) $x = a\left(\cos t + t\sin t\right)$, $y = a\left(\sin t - t\cos t\right)$.

19. Two particles P, Q move along parallel straight lines one with uniform velocity u and the other with the same initial velocity u but with uniform acceleration f. Show that the line joining them always touches a fixed hyperbola.

20. Show that the radius of curvature of the envelope of the line $x\cos\alpha + y\sin\alpha = f(\alpha)$ is $f(\alpha) + f''(\alpha)$.

ANSWERS

1. $x^2 + y^2 = a^2$.

2. $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$.

3. $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

4. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

5. $x^2 + y^2 = a^2$.

6. $x^2/a^2 + y^2/b^2 = 1$.

7. $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

8. $x^{\frac{4}{3}} + y^{\frac{4}{3}} = a^{\frac{4}{3}}$.

9. $x^{\frac{2}{2-n}} + y^{\frac{2}{2-n}} = a^{\frac{2}{2-n}}$.

10. (i) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$. (ii) $4xy = c^2$.

11. $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$.

12. (i) a circle through the pole ; (ii) a rectangular hyperbola ;
(iii) an equiangular spiral.

17. $(a_1x + b_1y + c_1)(a_3x + b_3y + c_3) = (a_2x + b_2y + c_2)^2$.

18. (i) $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$. (ii) $27ay^2 = 4(x - 2a)^3$.
(iii) $(x + y)^{\frac{2}{3}} + (x - y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$. (iv) same as (i).
(v) θ-eliminent of $x = a(\theta + \sin\theta)$, $y = -a(1 - \cos\theta)$.
(vi) $x^2 + y^2 = a^2$.

SEC. B. ENVELOPE OF CURVES

17.5. If a curve E exists, which touches each member of a family of curves $C\ [\equiv f(x,\ y,\ \alpha)=0]$, the curve E is called the Envelope of the curves C. Since E is the locus of the points of contact of the family of curves $f(x,\ y,\ \alpha)=0$ the point where the curve $f(x,\ y,\ \alpha)=0$ for a particular value of α. touches E, depends upon that value of α. Accordingly the co-ordinates of any point on E are functions of the parameter α [being of the forms $x=\phi(\alpha),\ y=\psi(\alpha)$] and they satisfy the equation $f(x,\ y,\ \alpha)=0$ of the enveloping curve which touches E at that point.

The equation of the tangent to a C-curve at (x, y) is

$$(X-x)\frac{\partial f}{\partial x}+(Y-y)\frac{\partial f}{\partial y}=0 \qquad \ldots (1)$$

and that of the tangent to the E-curve at $(x,\ y)$ is

$$\frac{X-x}{\frac{dx}{d\alpha}}=\frac{Y-y}{\frac{dy}{d\alpha}}, \qquad \ldots (2)$$

[since equation of E is of the form $x=\phi(\alpha),\ y=\psi(\alpha)$]

or, $$(X-x)\frac{dy}{d\alpha}-(Y-y)\frac{dx}{d\alpha}=0 \qquad \ldots (3)$$

Since the lines (1) and (3) are coincident, coefficients of X and Y in the above two equations are proportional.

$$\therefore \qquad \frac{\frac{\partial f}{\partial x}}{\frac{dy}{d\alpha}}=\frac{\frac{\partial f}{\partial y}}{-\frac{dx}{d\alpha}}, \quad i.e., \quad \frac{\partial f}{\partial x}\frac{dx}{d\alpha}+\frac{\partial f}{\partial y}\frac{dy}{d\alpha}=0 \qquad \ldots (4)$$

Now, differentiating $f(x,\ y,\ \alpha)=0$ with respect to α, remembering that x and y are now functions of α, we get

$$\frac{\partial f}{\partial \alpha}+\frac{\partial f}{\partial x}\frac{dx}{d\alpha}+\frac{\partial f}{\partial y}\frac{dy}{d\alpha}=0 \qquad \ldots (5)$$

$$\therefore \qquad \text{from (4)}, \ \frac{\partial f}{\partial \alpha}=0 \qquad \ldots (6)$$

Hence the equation of the envelope, in case an envelope exists, is to be found by eliminating the parameter α between the equations

$$\left.\begin{aligned} f(x, y, \alpha) &= 0 \\ \text{and} \qquad \frac{\partial f}{\partial \alpha} &= 0. \end{aligned}\right\} \qquad \ldots (7)$$

Cor. It is shown in Art. 18.1 (iii) that the circle on the radius vector of a curve as diameter touches the *pedal of the curve, so the pedal of a curve can be obtained as the envelope of the circles described on the radius vectors of the curve as diameters.* [See *Art. 17.9, Ex. 6.*]

Note. It should be noted that the α-eliminant between $f(x, y, \alpha) = 0$ and $\dfrac{\partial f}{\partial \alpha} = 0$ may contain other loci, besides the envelope, for instance nodal locus, cuspidal locus, tac-locus, etc., in case the family of curves C has singular points.

17.6. *The envelope is, in general, the locus of the ultimate points of intersection of neighbouring curves of a family.*

The co-ordinates of the point of intersection P of two neighbouring curves of the family must satisfy

$$f(x, y, \alpha) = 0 \text{ and } f(x, y, \alpha + \Delta\alpha) = 0,$$

$$\text{i.e., } f(x, y, \alpha) = 0 \textit{ and } \frac{f(x, y, \alpha + \Delta\alpha) - f(x, y, \alpha)}{\Delta\alpha} = 0,$$

the second relation by Mean Value Theorem becomes

$$\frac{\partial}{\partial \alpha} f(x, y, \alpha + \theta\,\Delta\alpha) = 0, \qquad \text{where } 0 < \theta < 1.$$

Now as $\Delta\alpha \to 0$, P satisfies $f = 0, \dfrac{\partial f}{\partial \alpha} = 0.$

$\therefore$ the required locus is the α-eliminant of $f = 0, \dfrac{\partial f}{\partial \alpha} = 0.$

Note 1. Although the above theorem is generally true, but it is not always true. For example, consider the *family of semi-cubical parabolas* $y = (x - \alpha)^3$. Here for different values of α, we have different semi-cubical parabolas, no two of which intersect but every one of which touches the x-axis. So here the x-axis is the envelope, although no two members of the family intersect. From the graphs of the curves the whole thing becomes at once clear.

Note 2. *Alternative definition of Envelope.*

The points of intersection of the curves $f(x, y, \alpha)=0$ and $\frac{\partial}{\partial \alpha} f(x, y, \alpha)=0$ (α being given) are called *characteristic points* of the family $f(x, y, \alpha)=0$ (for the given α) if these points exist (*i.e.*, if $f=0$ and $\frac{\partial f}{\partial \alpha}=0$ intersect) and if those points are not singular points of $f(x, y, \alpha)=0$. *The locus of the characteristic points of a family of curves is sometimes called the envelope of the family.*

17.7. Envelope of a special family.

If the curve $f(x, y, \alpha)=0$ be algebraic, the α-eliminant of $f=0$, $\frac{\partial f}{\partial \alpha}=0$ is the condition that $f=0$ (considered as an equation in α) has equal roots. [*Theory of Equation*]

Thus, if $f(x, y, \alpha) \equiv A(x, y)\alpha^2 + B(x, y)\alpha + C = 0$ [*i.e.*, if $f(x, y, \alpha)=0$ be a quadratic in α, the parameter], the envelope of the family is given by

$$B^2 = 4AC \cdot$$

[For *illustration* see *Ex.* 2 *of Art. 17.9*]

17.8. Envelope of two-parameter family.

If the equation of a family of curves involves two parameters α and β connected by a given equation, then we can proceed by two methods in finding out the envelope. Suppose the equation of the family is

$$f(x, y, \alpha, \beta)=0 \qquad \ldots(1)$$

where α and β are connected by the equation

$$\phi(\alpha, \beta)=0 \qquad \ldots(2)$$

First Method : Suppose we can solve $\phi(\alpha, \beta)=0$ for β in terms of α; then we substitute this value of β in (1) and now (1) reduces to one parameter family and we eliminate α between $f=0$ and $\frac{\partial f}{\partial \alpha}=0$.

Second Method : For a particular point (x, y) on the envelope

$$\frac{\partial f}{\partial \alpha} d\alpha + \frac{\partial f}{\partial \beta} d\beta = 0 \qquad \ldots(3)$$

and from (2), $$\frac{\partial \phi}{\partial \alpha} d\alpha + \frac{\partial \phi}{\partial \beta} d\beta = 0 \qquad \ldots (4)$$

Eliminating $d\alpha$, $d\beta$ between these two equations (we may regard α as the independent variable and β the dependent variable), we get

$$\frac{\dfrac{\partial f}{\partial \alpha}}{\dfrac{\partial \phi}{\partial \alpha}} = \frac{\dfrac{\partial f}{\partial \beta}}{\dfrac{\partial \phi}{\partial \beta}} \qquad \ldots (5)$$

If we eliminate α and β from (1), (2) and (5), we obtain the equation of the envelope.

Note. This method can be extended to obtain the envelope of a family depending upon n parameters which are connected by $(n-1)$ equations.

17.9. Illustrative Examples.

Ex. 1. *Find the envelope of the family of ellipses*

$$\frac{x^2}{\alpha^2} + \frac{y^2}{(a-\alpha)^2} = 1,$$

α *being the parameter.*

We have $\quad x^2\alpha^{-2} + y^2(a-\alpha)^{-2} = 1. \qquad \ldots (1)$

Differentiating with respect to α, we have

$$\frac{x^2}{\alpha^3} = \frac{y^2}{(a-\alpha)^3}, \qquad \ldots (2)$$

$$\therefore \quad \frac{\dfrac{x^2}{\alpha^2}}{\alpha} = \frac{\dfrac{y^2}{(a-\alpha)^2}}{a-\alpha} = \frac{\dfrac{x^2}{\alpha^2} + \dfrac{y^2}{(a-\alpha)^2}}{a} = \frac{1}{a} \quad [\text{by } (1)]$$

i.e., $$\frac{x^2}{\alpha^3} = \frac{y^2}{(a-\alpha)^3} = \frac{1}{a}. \quad \therefore \quad \frac{x^{\frac{2}{3}}}{\alpha} = \frac{y^{\frac{2}{3}}}{a-\alpha} = \frac{1}{a^{\frac{1}{3}}} = \frac{x^{\frac{2}{3}} + y^{\frac{2}{3}}}{a}$$

$\therefore \quad x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, which is the required α-eliminant between (1) and (2), is the equation of the required envelope.

Ex. 2. *Find the envelope of the system of parabolas*

$$\lambda x^2 + \lambda^2 y = 1, \ \lambda \text{ being the parameter.}$$

Since the equation of the family is

$$\lambda^2 y+\lambda x^2-1=0,$$

and since it is a quadratic in λ, the parameter, by Art. 17.7, its envelope is

$$x^4+4y=0.$$

Ex. 3. *Prove that the envelope of the paths of projectiles in vacuum from the same point with the same velocity in the same vertical plane is a parabola with the point of projection as focus.*

The equation of the path of the projectile with the point of projection O as origin and the horizontal and the vertical lines through O as axes of x and y is

$$y=x\tan\alpha-\tfrac{1}{2}g\frac{x^2}{u^2\cos^2\alpha}=x\tan\alpha-\tfrac{1}{2}g\frac{x^2}{u^2}\left(1+\tan^2\alpha\right)$$

[See Authors' *Dynamics* : Art. 11.5]

$$=mx-\tfrac{1}{2}g\frac{x^2}{u^2}\left(1+m^2\right), \text{ where } \tan\alpha=m,$$

i.e., $m^2\cdot\tfrac{1}{2}g\dfrac{x^2}{u^2}-mx+y+\tfrac{1}{2}g\dfrac{x^2}{u^2}=0.$

Here α, and hence $\tan\alpha$, *i.e.*, m being the variable parameter, the equation of the envelope is, by Art. 17.7,

$$x^2=4.\tfrac{1}{2}g\frac{x^2}{u^2}\left(y+\tfrac{1}{2}g\frac{x^2}{u^2}\right),$$

i.e., $\dfrac{u^2}{2g}=y+\tfrac{1}{2}g\dfrac{x^2}{u^2}$, $\quad\therefore\quad x^2=-\dfrac{2u^2}{g}\left(y-\dfrac{u^2}{2g}\right).$

Transferring the origin to the point $\left(0,\dfrac{u^2}{2g}\right)$, the equation of the envelope is $x^2=-\dfrac{2u^2}{g}y$, which is a parabola with its vertex on the y-axis at the point $\left(0,\dfrac{u^2}{2g}\right)$ and its concavity turned downwards and latus rectum '$4a$'$=\dfrac{2u^2}{g}$ and hence, 'a' being equal to $\dfrac{u^2}{g}$, the focus is at the origin.

Ex. 4. *Find the envelope of circles whose centres lie on the rectangular hyperbola $xy = c^2$ and which pass through its centre.*

Let the equation of a circle having centre at (α, β) and passing through the centre of $xy = c^2$, which is the origin here, be

$$x^2 + y^2 - 2\alpha x - 2\beta y = 0 \quad \ldots(1) \quad \text{where } \alpha\beta = c^2 \quad \ldots(2)$$

This is the case of two-parameter family, where the parameters are connected by a given relation.

Following the first method of Art. 17.8, *i.e.*, eliminating β between (1) and (2), the equation of the circle becomes

$$x^2 + y^2 - 2\alpha x - 2\frac{c^2}{\alpha} y = 0, \text{ since from (2), } \beta = \frac{c^2}{\alpha},$$

or, $2\alpha^2 x - \alpha(x^2 + y^2) + 2c^2 y = 0$.

$\therefore$ by Art. 17.7, the required envelope is

$$(x^2 + y^2)^2 = 4.2x.2c^2 y = 16c^2 xy.$$

Note. By transformation to polars, this equation can be shown to be transformed to $r^2 = 8c^2 \cos 2\phi$, where $\phi = \frac{1}{4}\pi - \theta$; *i.e.*, the required envelope is a lemniscate.

Ex. 5. *Find the envelope of the parabola*

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1,$$

where $ab = k^2$, a and b being variable parameters.

We have $$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1, \quad \ldots(1)$$

$$ab = k^2. \quad \ldots(2)$$

We apply here the second method of Art. 17.8. Taking differentials of both (1) and (2) with respect to a and b, we have

$$\frac{\sqrt{x}}{a^{\frac{3}{2}}} da + \frac{\sqrt{y}}{b^{\frac{3}{2}}} db = 0, \quad \ldots(3)$$

$$\frac{da}{a} + \frac{db}{b} = 0. \quad \ldots(4)$$

From (3) and (4)

$$\frac{\frac{\sqrt{x}}{a^{\frac{3}{2}}}}{\frac{1}{a}} = \frac{\frac{\sqrt{y}}{b^{\frac{3}{2}}}}{\frac{1}{b}}, \qquad or, \quad \frac{\sqrt{\frac{x}{a}}}{1} = \frac{\sqrt{\frac{y}{b}}}{1} = \frac{\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}}}{2} = \frac{1}{2} \text{ from (1)}$$

$$\therefore \quad \frac{\sqrt{\frac{x}{a}} \cdot \sqrt{\frac{y}{b}}}{1.1} = \frac{1}{4}, \qquad i.e., \quad \frac{\sqrt{xy}}{\sqrt{ab}} = \frac{1}{4}.$$

Squaring and using (2), we get (a, b)-eliminant, $16xy = k^2$ and hence, this is the required envelope. This obviously represents a hyperbola.

Ex. 6. *Find the pedal of the cardioide* $r = a\left(1 + \cos\theta\right)$ *with respect to the pole* ***(origin)***.

We shall here find out the first positive pedal by considering it as the envelope of the circles described on the radii vectors as diameters.

[See *Cor., Art. 17.5*]

Let (ρ, α) be the polar co-ordinates of any point on the cardioide.

Then $$\rho = a(1 + \cos\theta), \qquad \ldots (1)$$

Again, the equation of the circle on the radius vector ρ *as diameter is*

$$r = \rho\cos\left(\theta - \alpha\right), \qquad \ldots (2)$$

or, $$r = a\left(1 + \cos\alpha\right)\cos\left(\theta - \alpha\right) \textit{ from (1)} \qquad \ldots (3)$$

Here α is the parameter.

Differentiating (3) with respect to α,

$$0 = -\sin\alpha\cos\left(\theta - \alpha\right) + \left(1 + \cos\alpha\right)\sin\left(\theta - \alpha\right);$$

$$\therefore \quad \sin\alpha\cos\left(\theta - \alpha\right) - \cos\alpha\,\sin\left(\theta - \alpha\right) = \sin\left(\theta - \alpha\right),$$

i.e., $\sin\left(2\alpha - \theta\right) = \sin\left(\theta - \alpha\right)$,

i.e., $2\alpha - \theta = \theta - \alpha$, *i.e.,* $\alpha = \frac{2}{3}\theta$, $\qquad \ldots (4)$

Substituting this value of α in (3), we have the required envelope as

$$r = a\left(1 + \cos\tfrac{2}{3}\alpha\right)\cos\tfrac{1}{3}\theta = 2a\cos^3\tfrac{1}{3}\theta. \quad \text{or, } r^{\frac{1}{3}} = \left(2a\right)^{\frac{1}{3}}\cos\tfrac{1}{3}\theta.$$

Ex. 7. *Show that the pedal equation of the envelope of the line*

$$x\cos 2\alpha + y\sin 2\alpha = 2a\cos\alpha,$$

where α *is the parameter, is* $p^2 = \frac{4}{3}\left(r^2 - a^2\right)$.

Let (x, y) be the co-ordinates of any point P on the envelope.

Then x, y satisfy the equations $f(x, y, \alpha)=0$, $\dfrac{\partial f}{\partial \alpha}=0$,

i.e., $$x\cos 2\alpha + y\sin 2\alpha = 2a\cos\alpha, \qquad \ldots(1)$$

$$x\sin 2\alpha - y\cos 2\alpha = a\sin\alpha, \qquad \ldots(2)$$

From the definition of the envelope, it follows that (1) is the tangent to the envelope at $P(x, y)$.

Let p be the length of the perpendicular from the origin O upon the tangent (1) to the envelope at P and r be the distance of P from O,

$$\therefore \quad p^2 = 4a^2\cos^2\alpha, \qquad \ldots(3)$$

$$r^2 = x^2 + y^2 = 4a^2\cos^2\alpha + a^2\sin^2\alpha,$$

[*Squaring and adding (1) and (2)*]

$$= 3a^2\cos^2\alpha + a^2. \qquad \ldots(4)$$

Eliminating α between (3) and (4), the required pedal equation of the envelope is obtained.

EXAMPLES - XVII (B)

1. Find the envelopes of the following curves, α being the parameter :

   (i) circles $(x-\alpha)^2 + y^2 - 4\alpha = 0$,

   (ii) parabolas $\alpha y^2 = 2x + 12\alpha^3$,

   (iii) ellipses $x^2 + \alpha^2 y^2 = 4\alpha$.

2. Find the envelopes of the family of curves, θ being the parameter.

   (i) $x^2\cos\theta + y^2\sin\theta = a^2$.

   (ii) $P(x, y)\cos\theta + Q(x, y)\sin\theta = R(x, y)$.

   (iii) $A(x, y)\cos^m\theta + B(x, y)\sin^m\theta = C(x, y)$.

3. Find the envelope of the family of curves

   $$L\lambda^3 + 3M\lambda^2 + 3N\lambda + P = 0$$

   where λ is a parameter and L, M, N, P are functions of x and y.

4. Show that the envelope of the family of ellipses, (α being the parameter)

   $$a^2x^2\sec^4\alpha + b^2y^2\operatorname{cosec}\alpha = (a^2 - b^2).$$

   is the evolute of the ellipse $x^2/a^2 + y^2/a^2 = 1$.

5. Find the envelopes of the family of circles which are described on the double ordinates of

   (i) the parabola $y^2 = 4ax$ as diameters,

   (ii) the ellipse $x^2/a^2 + y^2/a^2 = 1$ as diameters.

6. Find in each case the envelope of circles described upon OP as diameters, where O is the origin and P is a point on

   (i) the circles $x^2 + y^2 = 2ax$,

   (ii) the parabolas $y^2 = 4ax$,

   (iii) the ellipses $b^2x^2 + a^2y^2 = a^2b^2$.

   (iv) the rectangular hyperbolas $xy = c^2$.

7. If the centre of a circle lies upon the parabola $y^2 = 4ax$ and the circle passes through the vertex of the parabola, show that the envelope of the circle is $y^2(2a + x) + x^3 = 0$.

8. Find the envelopes of the family of ellipses $\dfrac{x^2}{\alpha^2} + \dfrac{y^2}{\beta^2} = 1$.

   (i) whose sum of semi-axes is constant $(= c)$,

   (ii) whose area is constant $(= \pi c^2)$.

9. Show that the envelope of the circles $x^2 + y^2 - 2\alpha x - 2\beta y + \beta^2 = 0$, where α, β are parameters and whose centres lie on the parabola $y^2 = 4ax$, is $x\left(x^2 + y^2 - 2ax\right) = 0$.

10. Find the envelopes of

    (i) of the family of ellipses $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and

    (ii) the family of parabolas $\sqrt{\dfrac{x}{a}} + \sqrt{\dfrac{y}{b}} = 1$,

    where $a^n + b^n = c^n$ (a, b being the parameters).

11. Show that the envelope of the ellipses

$$\frac{(x-\alpha)^2}{a^2} + \frac{(y-\beta)^2}{b^2} = 1,$$

where the parameters α, β are connected by the relation

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} = 1, \text{ is the ellipse } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 4.$$

12. Find the envelope of the family of curves $\frac{x^n}{a^n}+\frac{y^n}{b^n}=1$, where the parameters a and b are connected by the equation $a^p+b^p=c^p$.

13. Find the pedal with respect to the pole of the curve $r^2=a^2\cos 2\theta$.

14. Find the envelope of the circles described on the radii vectors of the curve $r^m=a^m\cos m\theta$ as diameters.

15. Show that the pedal equation of the envelope of the line $x\cos m\alpha+y\sin m\alpha=a\cos n\alpha$, $(m\neq n)$, where α is the parameter, is $p^2=\frac{m^2r^2-n^2a^2}{m^2-n^2}$.

16. Given that the astroid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}}$ is the envelope of the family of the ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a and b are parameters, show that a and b are connected by the relation $a+b=c$.

ANSWERS

1. (i) $y^2-4x-4=0$. (ii) $y^2=\pm 18x$. (iii) $xy=\pm 2$.
2. (i) $x^4+y^4=a^4$. (ii) $P^2+Q^2=R^2$.
   (iii) $A^{\frac{2}{2-m}}+B^{\frac{2}{2-m}}=C^{\frac{2}{2-m}}$.
3. $(MN-LP)^2=4(MP-N^2)(LN-M^2)$.
5. (i) $y^2=4a(x+a)$. (ii) $\frac{x^2}{a^2+b^2}+\frac{y^2}{b^2}=1$.
6. (i) $(x^2+y^2-ax)^2=a^2(x^2+y^2)$.
   (ii) $x(x^2+y^2)+ay^2=0$. (iii) $a^2x^2+b^2y^2=(x^2+y^2)^2$.
   (iv) $(x^2+y^2)^2=4c^2xy$.
8. (i) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}}$. (ii) $2xy=c^2$.
10. (i) $x^{\frac{2n}{n+2}}+y^{\frac{2n}{n+2}}=c^{\frac{2n}{n+2}}$. (ii) $x^{\frac{n}{2n+1}}+y^{\frac{n}{2n+1}}=c^{\frac{n}{2n+1}}$
12. $x^{\frac{np}{n+p}}+y^{\frac{np}{n+p}}=c^{\frac{np}{n+p}}$ 13. $r^{\frac{2}{3}}=a^{\frac{2}{3}}\cos\frac{2}{3}\theta$
14. $r^n=a^n\cos n\theta$ where $n=m/(m+1)$.

18 ASSOCIATED LOCI

18.1. Pedal curve.

The locus of the foot of the perpendicular drawn from a fixed point on the tangent to a curve, is called *the pedal of the curve* with regard to the fixed point.

(*i*) *To find the pedal with regard to the origin of any curve whose cartesian equation is given.*

Let the equation of the curve be $f(x, y) = 0$. ...(1)

Let $x \cos \alpha + y \sin \alpha = p$ be the equation of the tangent PT to the curve at any point P.

Now, the condition that the line $x \cos \alpha + y \sin \alpha = p$ should touch the curve is of the form $\phi(p, \alpha) = 0$. ...(2)

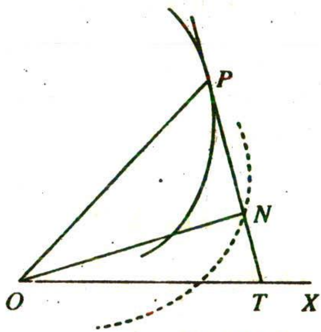


Fig 18.1.1

Since (p, α) are the polar co-ordinates of the foot of the perpendicular N on the tangent PT, hence in (2), if r, θ are written for p, α, *the polar equation* of the locus of N, *i.e., of the pedal curve* will be obtained as

$$\phi(r, \theta) = 0 \qquad \text{...(3)}$$

which can now be transformed into cartesian.

Alternative Method :

Let (x, y) be the co-ordinates of P; then the equation of the tangent PT is

$$Y - y = \frac{dy}{dx}(X - x) \qquad \text{...(1)}$$

and the equation of ON, which passes through the origin and is perpendicular to PT, is

$$X + \frac{dy}{dx} Y = 0. \qquad \ldots (2)$$

Hence the locus of N, (*i.e.*, the pedal) which is the intersection of (1) and (2) is obtained by eliminating x and y from (1) and (2) and from the equation of the curve $f(x, y) = 0$.

(*ii*) *To find the pedal with regard to the pole of any curve whose polar equation is given.*

Let the polar equation of the curve be $f(r, \theta) = 0$, $\ldots (1)$

and let (r_1, θ_1) be the polar co-ordinates of the foot of the perpendicular N drawn from O on the tangent at $P(r, \theta)$.

Now f denoting $\angle OPN$, $\tan \phi = r \dfrac{d\theta}{dr}$. $\ldots (2)$

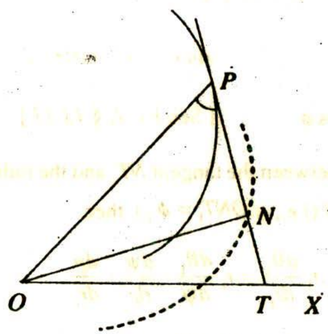


Fig 18.1.2

Also $\theta = \angle XOP = \angle XON + \angle NOP = \theta_1 + \frac{1}{2}\pi - \phi$ $\ldots (3)$

and since $ON = OP \sin \phi$, $r_1 = r \sin \phi$, $\ldots (4)$

If r, θ, ϕ be eliminated from (1), (2) (3) and (4), a relation between r and θ_r will be obtained, and from this relation, by dropping the suffixes, we get the required polar equation of the pedal.

(*iii*) *The circle on radius vector as diameter touches the pedal.*

$\angle XON = \angle PTX - \angle ONT$. [See *Fig. of* (*ii*)]

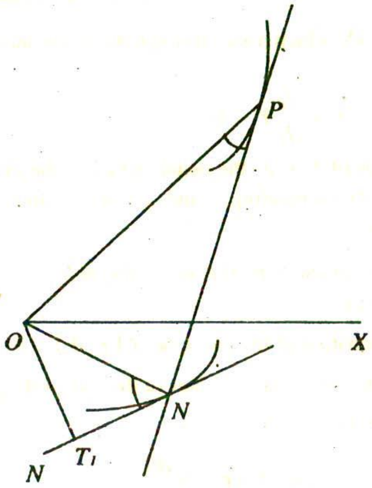


Fig 18.1.3

$\therefore \quad \theta_1 = \psi - \frac{1}{2}\pi$; also $p = ON = r_1$.

Again, $\dfrac{dp}{d\psi} = r\cos\phi$. [See *Ex. 7, § 13.17*]

If ϕ_1 be the angle between the tangent NT_1 and the radius vector ON of the pedal at any point N (*i.e.*, $\angle ONT_1 = \phi_1$) then

$$\tan\phi_1 = r_1 \frac{d\theta_1}{dr_1} = r_1 \frac{d\theta_1}{d\psi}\cdot\frac{d\psi}{dp}\cdot\frac{dp}{dr_1}$$

$$= p\frac{d\psi}{dp} = \frac{r\sin\phi}{r\cos\phi} = \tan\phi.$$

$\therefore \quad \phi_1 = \phi$, *i.e.*, $\angle ONT_1 = \angle OPN$.

$\therefore \quad T_1N$ touches the circle passing through OPN.

Hence the result.

(*iv*) *If* p_1 *be the perpendicular from the pole on the tangent to the pedal, then* $p_1 r = p^2$.

Draw OT_1 perpendicular from O on the tangent NT_1 to the pedal.

Since $\angle OPN = \angle ONT_1$, $\therefore$ Δ^s OPN, ONT_1 are similar.

$$\therefore \quad \frac{OP}{ON} = \frac{ON}{OT_1}, \text{ i.e., } \frac{r}{p} = \frac{p}{p_1}, \text{ i.e., } p_1 r = p^2.$$

(v) *To find the pedal of a curve when its pedal equation is given.*

Let the pedal equation of the curve be $p = f(r)$ ...(1)

and let p, r denote the usual entities of the original curve and p_1, r_1 the corresponding things of the pedal curve.

Then $p = r_1$; also from above, $r = \dfrac{p^2}{p_1} = \dfrac{r_1^2}{p_1}$.

Hence from equation (1), we get

$$r_1 = f\left(\frac{r_1^2}{p_1}\right).$$

$\therefore$ the pedal equation of the pedal curve is

$$r = f\left(\frac{r^2}{p}\right).$$

Note. If there be a series of curves designated as

$$P, P_1, P_2, \ldots\ldots, P_n$$

such that each is the pedal of the one which immediately precedes it, then $P_1, P_2, \ldots\ldots, P_n$ are called the *first*, the *second*, ..., the n^{th} *positive pedal* of P. Also regarding any one curve of the series, say P_3, as the original curve, the preceding curves P_2, P_1, P are called respectively the *first*, *second* and *third negative pedals of* P_3.

18.2. Inverse curve.

If on the radius vector OP (or OP produced) from the origin O to any point P moving on a curve, a second point Q be taken such that $OP \cdot OQ =$ a constant. say k^2, then the locus of Q is called the *inverse of the curve* along which P moves, with respect to a circle of radius k and centre O, or briefly with respect to O.

(i) ***To find the inverse of a given curve whose cartesian equation is given.***

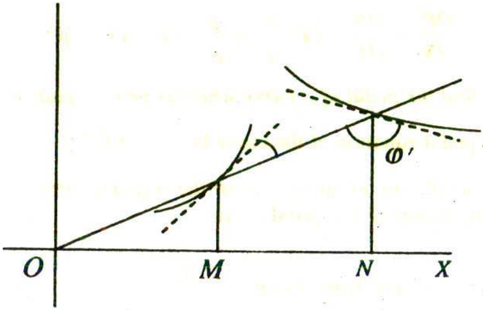


Fig 18.2.1

Let (x, y) be the co-ordinates of any point P on the curve $f(x, y) = 0$ and let $Q(x', y')$ be a point on OP such that $OP \,.\, OQ = k^2$. Draw PM, QN perpendiculars on OX.

Now, $\dfrac{x}{x'} = \dfrac{OM}{ON} = \dfrac{OP}{OQ}$ $\left(\because \Delta^s\ OPM,\ OQN \text{ are similar}\right)$

$$= \frac{OP \,.\, OQ}{OQ^2} = \frac{k^2}{x'^2 + y'^2}.$$

$$\therefore \quad x = \frac{k^2 x'}{x'^2 + y'^2}. \qquad \text{Similarly, } y = \frac{k^2 y'}{x'^2 + y'^2}.$$

Since $f(x, y) = 0$. $\therefore$ $f\left(\dfrac{k^2 x'}{x'^2 + y'^2}, \dfrac{k^2 y'}{x'^2 + y'^2}\right) = 0.$

Hence, by dropping the dashes, the equation of the inverse curve is

$$f\left(\frac{k^2 x}{x^2 + y^2}, \frac{k^2 y}{x^2 + y^2}\right) = 0,$$

i.e., the equation of the inverse of a curve is obtained by writing $k^2 x/\left(x^2 + y^2\right)$, $k^2 y/\left(x^2 + y^2\right)$ for x, y in the cartesian equation of hte curve.

(*ii*) *To find the inverse of a given curve whose polar equation is given.*

Let $f(r,\theta)=0$ be the equation of the given curve and let (r,θ) be the co-ordinates of P and (r',θ) be the co-ordinates of Q.

Since $OP.OQ=k^2$, $\therefore\ rr'=k^2$. $\therefore\ r=k^2/r'$.

Again, since $f(r,\theta)=0$, $\therefore\ f\left(\dfrac{k^2}{r'},\theta\right)=0$.

Hence the polar equation of the inverse curve is

$$f\left(\frac{k^2}{r},\ \theta\right)=0.$$

Thus, the equation of the inverse of a curve is obtained by writing k^2/r for r in the polar equation of the curve.

(*iii*) *Tangents to a curve and its inverse are inclined to the radius vector at supplementary angles.*

Let ϕ,ϕ' denote the angles between the tangents and radius vetor at the corresponding points of a curve and its inverse.

Then, $\tan\phi=\dfrac{rd\theta}{dr}$, $\tan\phi'=r'\dfrac{d\theta}{dr'}$.

Now, $(\because rr'=k^2)$, $\dfrac{dr'}{d\theta}=\dfrac{d}{d\theta}\left(\dfrac{k^2}{r}\right)=-\dfrac{k^2}{r^2}\cdot\dfrac{dr}{d\theta}$.

$$\therefore\quad \tan\phi'=r'\frac{d\theta}{dr'}=\frac{k^2}{r}\left(-\frac{r^2}{k^2}\right)\frac{d\theta}{dr}=-r\frac{d\theta}{dr}=-\tan\phi$$

$$=\tan(\pi-\phi).$$

$\therefore\quad \phi'=\pi-\phi$, *i.e.*, $\phi+\phi'=\pi$.

(iv) *To find the inverse of a curve when its pedal equation is given.*

Let the pedal equation of the given curve be

$$p=f(r).$$

Let p,r,ϕ denote the usual entities of the original curve, and let p',r',ϕ' denote the corresponding things of the inverse.

Then $\quad rr' = k^2, \quad i.e., \quad r = k^2/r'.$

Also, $\quad \dfrac{p'}{r'} = \sin\phi' = \sin(\pi - \phi) = \sin\phi = \dfrac{p}{r}.$

$\therefore \quad p = \dfrac{r}{r'}p' = \dfrac{rr'}{r'^2}p' = \dfrac{k^2}{r'^2}p'.$

$\therefore$ from equation (1), we get

$$\frac{k^2}{r'^2}p' = f\left(\frac{k^2}{r'}\right).$$

Hence the pedal equation of the inverse curve is

$$p = \frac{r^2}{k^2} f\left(\frac{k^2}{r}\right).$$

18.3. Polar reciprocal.

If on the perpendicular ON (or ON produced) from the origin on the tangent at any point P on a curve, a second point Q be taken such that $ON.OQ$ = a constant (say k^2), then the locus of Q is called *the polar reciprocal of the given curve* with respect to a circle of radius k and centre O.

From the definition, it follows that *the polar reciprocal of a curve is the inverse of its pedal.* Hence the equation of the polar reciprocal of a curve can be obtained by the first finding the pedal of the curve and then its inverse.

Let NN_1 be the tangent to the pedal at N and let QM be the tangent to the polar reciprocal at Q meeting OP produced at M.

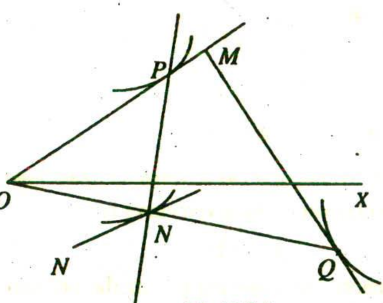


Fig 18.3.1

Now, $\phi = \angle OPN = \angle ONN_1$ [by § 18.1 (iii)]. Since QM is the tangent to the inverse of the pedal, hence by §18.2 (iii),

$$\angle OQM = \angle ONN_1 = \angle OPN.$$

Hence the quadrilateral $PMQN$ is cyclic.

$$\therefore OM \,.\, OP = OQ \,.\, ON = k^2$$

Also, $\because \angle PNQ = 90^0$, $\angle PMQ = 90^0$, *i.e.*, OM is perpendicular to QM. Hence the locus of P, *i.e.*, the original curve is the polar reciprocal of the locus of Q, *i.e.*, of the polar reciprocal. Thus, *the polar reciprocal of the polar reciprocal of a curve is the curve itself.*

18.4 Illustrative Examples.

Ex. 1. *Find the Pedal of the parabola* $y^2 = 4ax$ *with respect to the vertex.*

The condition that $X\cos\alpha + Y\sin\alpha = p$ will touch the parabola $y^2 = 4ax$ is obtained by comparing the equation with the equation of the tangent at (x, y) to the parabola, *i.e.*, with

$$Yy = 2a(X + x), \text{ or } -2aX + Yy = 2ax.$$

Hence

$$-\frac{2a}{\cos\alpha} = \frac{y}{\sin\alpha} = \frac{2ax}{p}, \quad \therefore \quad y = -2a\tan\alpha, \quad x = -p\sec\alpha.$$

Since $y^2 = 4ax$, $\qquad \therefore\ 4a^2\tan^2\alpha = -4a \cdot p\sec\alpha$.

Hence the required condition of tangency is $p + a\sin\alpha\tan\alpha = 0$.

$\therefore$ the polar equation of the pedal is

$$r + a\sin\theta\tan\theta = 0$$

or, $\quad r^2 + ar\sin\theta\tan\theta = 0$.

Writing $r^2 = x^2 + y^2$, $r\sin\theta = y$ and $\tan\theta = y/x$, we get cartesian equation of the pedal as

$$x\left(x^2 + y^2\right) + ay^2 = 0$$

Alternatively :

$y = mx + a/m \quad \ldots \quad$ (1) is a tangent to the parabola $y^2 = 4ax$

$y = -(1/m)x \quad \ldots \quad$ (2) is a equation of the perpendicular from the origin on the above tangent.

$\therefore$ the locus of the point intersection of (1) and (2), *i.e.*, the locus of the foot of the perpendicular, *i.e.*, the equation of the pedal is obtained by eliminating m between (1) and (2).

From (2), $m=-\dfrac{x}{y}$; substituting in (1), $y=-\dfrac{x^2}{y}-\dfrac{ay}{x}$,

i.e., $xy^2+x^3+ay^2=0$. Hence the result.

Ex. 2. *Show that the pedal of the circle* $r=2a\cos\theta$ *with respect to the origin is the cardioide* $r=a(1+\cos\theta)$.

Since the given equation is $r=2a\cos\theta$, ... (1)

$$\therefore \quad \tan\phi=r\Big/\frac{dr}{d\theta}=\frac{2a\cos\theta}{-2a\sin\theta}=-\cot\theta=\tan\left(\tfrac{1}{2}\pi+\theta\right),$$

$$\therefore \quad \phi=\tfrac{1}{2}\pi+\theta \qquad \text{... (2)}$$

Let (r_1, θ_1) be the co-ordinates of the foot of the perpendicular; then as in (2) of § 18.1,

$$\theta=\theta_1+\tfrac{1}{2}\pi-\phi=\theta_1+\tfrac{1}{2}\pi-\left(\tfrac{1}{2}\pi+\theta\right);$$

$$\therefore \quad \theta=\tfrac{1}{2}\theta_1 \qquad \text{... (3)}$$

Again, $r_1=r\sin\phi=2a\cos\theta\cdot\sin\left(\tfrac{1}{2}\pi+\theta\right)$

[*from (1) and* (2)]

$$=2a\cos^2\theta=2a\cos^2\tfrac{1}{2}\theta_1=a\left(1+\cos\theta_1\right)$$

Hence, the required locus is $r=a(1+\cos\theta)$.

Ex. 3. *Show that the inverse of the straight line* $ax+by+c=0$ *is a circle.*

Writing $\dfrac{k^2x}{x^2+y^2}$ and $\dfrac{k^2y}{x^2+y^2}$ for x, y in the given equation of the straight line, the equation of the inverse is

$$a\frac{k^2x}{x^2+y^2}+b\frac{k^2y}{x^2+y^2}+c=0$$

or, $c\left(x^2+y^2\right)+ak^2x+bk^2y=0$

which obviously represents a circle.

Ex. 4. *Find the inverse of the parabola* $r = l / (1+\cos\theta)$.

Writing k^2/r for r in the equation of the parabola, the equation of the inverse is

$$\frac{k^2}{r} = \frac{l}{1+\cos\theta}, \text{ or, } r = \frac{k^2}{l}(1+\cos\theta) = a(1+\cos\theta)$$

$$\left[\text{where } a = \frac{k^2}{l}\right]$$

which represents a cardioide.

Ex. 5. *Find the polar reciprocal of the parabola* $y^2 = 4ax$ *with respect to the vertex.*

The pedal of the parabola with respect to the vertex is

$$x(x^2+y^2)+ay^2 = 0 \qquad \text{[See } Ex. \ 1\text{]}$$

Its inverse is

$$\frac{k^2 x}{x^2+y^2}\left\{\frac{k^4x^2}{(x^2+y^2)^2}+\frac{k^4y^2}{(x^2+y^2)^2}\right\}+a\frac{k^4y^2}{(x^2+y^2)^2} = 0.$$

or, $k^2x + ay^2 = 0$, *i.e.*, $y^2 = -(k^2/a)x$

which represents a parabola.

Ex. 6. *Find the polar reciprocal of the curve* $p = f(r)$.

By Art. 18.1(v), the pedal equation of its pedal is $r = f\left(\dfrac{r^2}{p}\right)$.

Now, to obtain its inverse, writing $\dfrac{k^2}{r'}$ for r and $\dfrac{k^2}{r'^2}p'$ for p [see § *18.2 (iv)*], we get

$$\frac{k^2}{r'} = f\left(\frac{k^4}{r'^2}\frac{r'^2}{k^2p'}\right).$$

Hence on simplifying and dropping the dashes, the pedal equation of the polar reciprocal is

$$\frac{k^2}{r} = f\left(\frac{k^2}{p}\right).$$

EXAMPLES - XVIII

1. Find the pedals of

   (i) the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with respect to the centre and focus.

   (ii) the parabola $y^2 = 4ax$ with respect to the focus.

2. Find the equations of the pedals of the following curves with respect to the origin :

   (i) $x^n + y^n = a^n$. (ii) $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$.

   (iii) $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$.

3. Show that the first positive pedal of the rectangular hyperbola $x^2 - y^2 = a^2$ with respect to the centre is the lemniscate $r^2 = a^2 \cos 2\theta$.

4. Find the pedals with respect to the pole of the curves :

   (i) $r^2 \cos 2\theta = a^2$. (ii) $r^2 = a^2 \cos 2\theta$.

   (iii) $r = a(1 + \cos\theta)$. [*C. P. 2006*]

   (iv) $r = ae^{\theta \cot \alpha}$.

   (v) $r^m = a^m \cos m\theta$.

5. Show that the pedal of a circle with respect to any point is the curve $r = a + b\cos\theta$, where a is the radius of the circle and b the distance of the centre from the origin.

6. Find the inverses of the following curves with respect to the origin.

   (i) $x^2 + y^2 = a^2$. (ii) $x^2/a^2 + y^2/b^2 = 1$.

   (iii) $r = a(1 + \cos\theta)$. (iv) $r = ae^{\theta \cot \alpha}$.

7. Show that the inverses of the lines $2x + 3y = 4$ and $3x - 2y = 6$ are a pair of orthogonal circles.

8. Show that the inverse of the conic $r = \frac{l}{1 + e\cos\theta}$ with regard to the focus is a curve of the form $r = a + b\cos\theta$.

9. Show that the inverse of a rectangular hyperbola is a lemniscate, and conversely.

10. Find the polar reciprocals, with regard to a circle of radius k and centre at the origin, of the curves :

(i) $x^2/a^2 + y^2/b^2 = 1$. (ii) $y^2 = 4ax$.

(iii) $r = a\cos\theta$. (iv) $r = a(1+\cos\theta)$.

(v) $r^m = a^m \cos m\theta$.

ANSWERS

1. (i) $(x^2+y^2)^2 = a^2x^2 + b^2y^2$; $x^2+y^2 = a^2$. (ii) $x = 0$.

2. (i) $a^m(x^m + y^m) = (x^2+y^2)^m$, where $m = n/(n-1)$.

(ii) $(x^2+y^2)(ax+by) = abxy$.

(iii) $(ax)^n + (by)^n = (x^2+y^2)^n$, where $n = m/(m-1)$.

4. (i) $r^2 = a^2\cos 2\theta$ (ii) $r^{\frac{2}{3}} = a^{\frac{2}{3}}\cos\frac{2}{3}\theta$.

(iii) $r^{\frac{1}{3}} = (2a)^{\frac{1}{3}}\cos\frac{1}{3}\theta$.

(iv) $r = a_1 e^{\theta\cot\alpha}$, where $a_1 = a\sin\alpha \,.\, e^{\left(\frac{1}{2}\pi-\alpha\right)\cot\alpha}$.

(v) $r^n = a^n\cos n\theta$, where $n = m/(m+1)$.

6. (i) $(x^2+y^2) = k^4/a^2$.

(ii) $(x^2+y^2)^2 = k^4(x^2/a^2 + y^2/b^2)$.

(iii) $r = \dfrac{b}{1+\cos\theta}$, where $b = k^2/a$.

(iv) $r = a_1 e^{-\theta\cot\alpha}$, where $a_1 = k^2/a$.

10. (i) $a^2x^2 + b^2y^2 = k^4$.

(ii) $ay^2 + k^2x = 0$. (iii) $r = \dfrac{b}{1+\cos\theta}$, where $b = 2k^2/a$.

(iv) $r^{\frac{1}{3}}\cos\frac{1}{3}\theta = (k^2/2a)^{\frac{1}{3}}$.

(v) $r^n\cos n\theta = (k^2/a)^n$, where $n = m/(m+1)$

19 CONCAVITY AND CONVEXITY POINT OF INFLEXION

19.1. Concavity and Convexity (with respect to a given point).

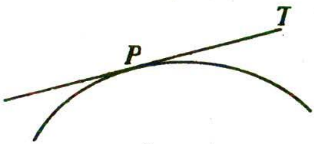


Fig (i)

Fig(ii)

Fig 19.1.1

Let PT be the tangent to a curve at P. Then the curve at P is said to be *concave* or *convex* with respect to a point A not lying on PT), according as a small portion of the curve in the immediate neighbourhood of P (on both side of it) lies entirely on the same side of PT as A [as in Fig. (i)], or on opposite sides of PT with respect to A [as in Fig. (ii)].

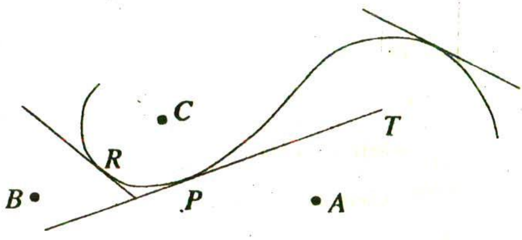


Fig 19.1.2

Thus, in Fig. 19.1.2, the curve at P is convex with respect to A, and concave with respect to B or C. The curve at Q is concave with respect to A. Again, the curve at R is convex to B and concave to C.

Note. A curve at a point P on it is **Convex** or **Concave** *with respect to a given line* according as it is convex or concave with respect to the foot of the perpendicular from P on the line.

19.2. Point of Inflexion.

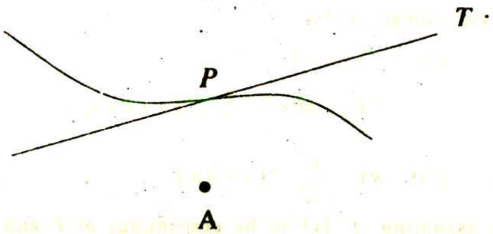


Fig 19.2.1

In some curves, at a particular point P on it, the tangent line crosses the curve, as in Fig. 19.2.1. At this point, clearly the curve, on one side of P, is convex, and on the other side it is concave to *any* point A (not lying on the tangent line). Such a point on a curve is defined to be a *point of inflexion* (or a point of contrary flexure).

19.3. Analytical Test of Concavity or Convexity

(with respect to the x-axis).

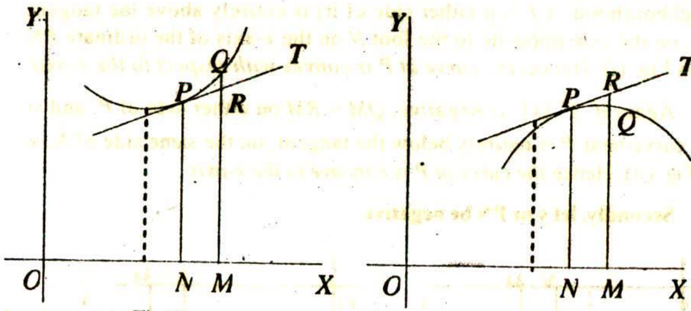


Fig (*i*)

Fig (*ii*)

Fig 19.3.1

Let $P(x, y)$ be a point on the curve $y = f(x)$, Q a neighbouring point whose abscissa is $x + h$ (h being small, positive or negative). Let PT be the tangent at P, and let the ordinate QM of Q intersect PT at R.

The equation to PT is

$$Y - y = f'(x)(X - x)$$

and abscissa X of R being $x + h$, its ordinate

$$RM = Y = y + hf'(x).$$

Also the ordinate of Q is

$$QM = f(x+h)$$

$$= f(x) + hf'(x) + \frac{h^2}{2!} f''(x + \theta h), \; 0 < \theta < 1.$$

$$\therefore \quad QM - RM = \frac{h^2}{2!} f''(x + \theta h). \qquad \ldots (1)$$

Now assuming $f''(x)$ to be continuous at P and $\neq 0$ there, $f''(x + \theta h)$ has the same sign as that of $f''(x)$ when $|h|$ is sufficiently small.

Hence from (1), $QM - RM$ has the same sign as that of $f''(x)$, for positive as well as negative values of h, provided it is sufficiently small in magnitude.

Firstly, let the ordinate PN or y be positive.

Then *if* $f''(x)$ $\left(\text{or } \frac{d^2y}{dx^2} \text{ at } P\right)$ *is positive,* from (1) $QM > RM$ for Q on either side of P in its neighbourhood, and so the curve in the neighbourhood of P (on either side of it) is entirely above the tangent, *i.e.*, on the side opposite to the foot N on the x-axis of the ordinate PN, as in Fig. (i). Hence, *the curve at P is convex with respect to the x-axis.*

Again *if* $f''(x)$ *is negative*, $QM < RM$ on either side of P, and so the curve near P is entirely below the tangent, on the same side of N, as in Fig. (ii). Hence *the curve at P is concave to the x-axis.*

Secondly, let y or PN be negative.

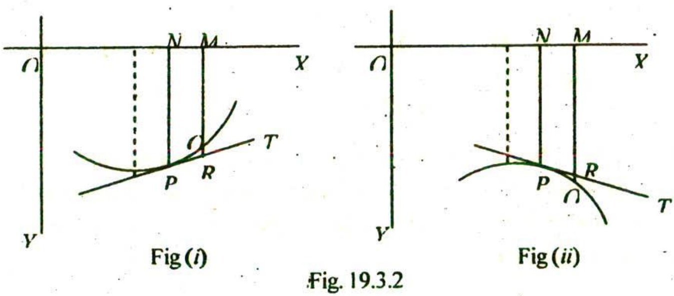


Fig. 19.3.2

If $f''(x)$ is positive, from (1), as before, $QM > RM$ on either side of P, and as both are negative, QM is numerically less than RM, as in Fig. (i) of Fig 19.3.2. The curve, therefore, at P lies on the same side as N with respect to the tangent PT. Hence, *the curve at P is concave with respect to the* x-axis.

If $f''(x)$ is negative, we similarly *get the curve at P convex with respect to the x-axis*, as in *Fig. (ii)* of *Fig 16.3.2*. Combining the two cases, we get the following criterion for convexity or concavity of a curve at a point with respect to the x-axis :

If $y\dfrac{d^2y}{dx^2}$ is positive at P, the curve at P is convex *to the x-axis.*

If $y\dfrac{d^2y}{dx^2}$ is negative at P, the curve at P is concave *to the x-axis.*

Note. At a point where the tangent is parallel to the y-axis, $\dfrac{dy}{dx}$ is infinite. At such a point, instead of considering with respect to the x-axis, we investigate convexity or concavity of the curve with respect to the y-axis. The criterion, as obtained by a method similar to above, is as follows :

The curve at P is *convex or concave with respect to the y-axis* according as $x\dfrac{d^2x}{dy^2}$ is positive or negative at P.

19.4. Analytical condition for Point of Inflexion.

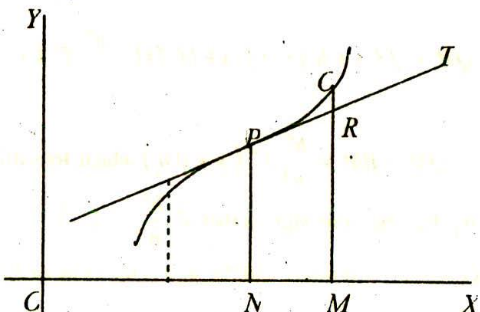


Fig 19.4.1

In the above investigation, let $f''(x) = 0$ at P, and $f'''(x) \neq 0$.

Then, $\quad QM = f(x+h) = f(x) + hf'(x) + \dfrac{h^3}{3!}f'''(x+\theta h)$.

$$\therefore \quad QM - RM = \frac{h^3}{3!} f'''(x + \theta h),$$

and the sign of this for sufficiently small $|h|$ is the same as that of $\frac{h^3}{3!} f'''(x)$, which has got opposite signs for positive and negetive values of h, whatever be the sign of $f'''(x)$ at P. Thus, near P the curve is above the tangent on one side of P, and below the tangent on the other side, as in the above figure. Hence, P is a point of inflexion.

Thus, the condition that P is a point of inflexion on the curve $y = f(x)$ is that, at P,

$$\frac{d^2 y}{dx^2} = 0 \text{ and } \frac{d^3 y}{dx^3} \neq 0.$$

Note. If $\frac{dy}{dx}$ is infinite at P, the condition that P is a point on inflexion is that, at P,

$$\frac{d^2 x}{dy^2} = 0 \text{ and } \frac{d^3 x}{dy^3} \neq 0.$$

19.5. A more general criterion.

Suppose that at P, $f''(x) = f'''(x) = \ldots\ldots = f^{n-1}(x) = 0$ and $f^n(x) \neq 0$.

Then, $QM = f(x + h) = f(x) + hf'(x) + \frac{h^n}{n!} f^n(x + \theta h).$

$$[0 < \theta < 1]$$

$\therefore \quad QM - RM = \frac{h^n}{n!} f^n(x + \theta h)$ which, for sufficiently small values of $|h|$, has the same sign as that of $\frac{h^n}{n!} f^n(x)$.

If n is even, h^n is positive and the sign is the same as that of $f^n(x)$ or $\frac{d^n y}{dx^n}$ at P for both positive and negative values of h. Considering both the cases when y of P is positive and negative, we find that the curve at P is *convex or concave with respect to the x-axis according as* $y \frac{d^n y}{dx^n}$ is positive or negative.

If n is odd, $\dfrac{h^n}{n!}f^n(x)$ will have opposite signs for positive or negative values of h, whatever be the sign of $f^n(x)$. Hence, Q lies on opposite sides of the tangent for positive and negative values of h. Thus, P *is a point of inflexion.*

Note. Since from (1) Art. 19.3, $QM - RM = \dfrac{h^2}{2!}f''(x+\theta h)$, if $f''(x+\theta h)$ has opposite signs for opposite signs of h when $|h|$ is sufficiently small, $QM > RM$ on one side, and $QM < RM$ on the other side of P on the curve in the immediate neighbourhood, and thus the tangent at P crosses the curve at P, and so P is a point of inflexion. Thus, since q is positive and numerically less than 1, *an alternative criterion for a point of inflexion is that $f''(x+\theta h)$ should have opposite signs for opposite signs of h when h is numerically sufficiently small*; in other words, $f''(x)$ changes sign in passing through P from one side to the other.

19.6. Illustrative Examples.

Ex. 1. *Examine the curve $y = \sin x$ regarding its convexity or concavity to the x-axis, and determine its point of inflexion, if any.*

As $y = \sin x$, $\dfrac{dy}{dx} = \cos x$ and $\dfrac{d^2y}{dx^2} = -\sin x$. Hence, $y\dfrac{d^2y}{dx^2} = -\sin^2 x$ which is negative for all values of x excepting those which make $\sin x = 0$, *i.e.*, for $x = k\pi$, k being any integer, positive or negative.

Thus, the curve is concave to the x-axis at every point, excepting at points where it crosses the x-axis.

At these points, given by $x = k\pi$, $\dfrac{d^2y}{dx^2} = 0$, and $\dfrac{d^2y}{dx^2} = -\cos x \neq 0$.

Hence, those points where the curve crosses the x-axis are points of inflexion.

Ex. 2. *Show that the curve $y^3 = 8x^2$ is concave to the foot of the ordinate everywhere except at the origin.*

From the given equation, $y = 2x^{\frac{2}{3}}$,

$$\therefore \frac{dy}{dx} = \frac{4}{3}x^{-\frac{1}{3}},\ \frac{d^2y}{dx^2} = -\frac{4}{9}x^{-\frac{4}{3}}.$$

$$\therefore y\frac{d^2y}{dx^2} = -\frac{8}{9}x^{-\frac{2}{3}} = -\frac{8}{9x^{\frac{2}{3}}}.$$

Thus, excepting at the origin, $x^{\frac{2}{3}}$ being positive for all values of x, $y\dfrac{d^2y}{dx^2}$ is negative.

Hence, the curve is concave everywhere to the foot of the ordinate excepting at the origin.

Ex. 3. *Prove that* $\left(a-2, -2/e^2\right)$ *is a point of inflexion of the curve* $y=(x-a)e^{x-a}$.

Here, at points on the curve,

$$\frac{dy}{dx}=e^{x-a}+(x-a)e^{x-a}=(1+x-a)e^{x-a},$$

$$\frac{d^2y}{dx^2}=e^{x-a}+(1+x-a)e^{x-a}=(2+x-a)e^{x-a}.$$

and, similarly, $\dfrac{d^3y}{dx^3}=(3+x-a)e^{x-a}$

Hence, at $x=a-2$, (where $-2e^{-2}$), $\dfrac{d^2y}{dx^2}=0$ and $\dfrac{d^3y}{dx^3}=e^{-2}\neq 0$.

Hence, the point $\left(a-2, -2/e^2\right)$ is a point of inflexion.

Ex. 4. *Find if there is any point of inflexion on the curve*

$$y-3=6(x-2)^5.$$

Here, $\dfrac{dy}{dx}=30(x-2)^4$, $\dfrac{d^2y}{dx^2}=120(x-2)^3$.

Thus, $\dfrac{d^2y}{dx^2}=0$ when $x=2$ (and so $y=3$).

In the neighbourhood of this point, where $x=2+h$ (h numerically small, positive or negative),

$\dfrac{d^2y}{dx^2}=120h^3$, which has opposite signs for positive and negative values of h. Hence, $\dfrac{d^2y}{dx^2}$ changes sign in passing through $x=2$.

Thus, (2, 3) is the only point of inflexion.

Alternatively

Here, $\dfrac{d^3y}{dx^3}=360(x-2)^2$, $\dfrac{d^4y}{dx^4}=720(x-2)$, $\dfrac{d^5y}{dx^5}=720$.

Thus at $x=2$, $\dfrac{d^2y}{dx^2}=\dfrac{d^3y}{dx^3}=\dfrac{d^4y}{dx^4}=0$, and $\dfrac{d^5y}{dx^5}\neq 0$.
(which is of *odd* order)

Hence $x=2$ gives the point of inflexion.

EXAMPLES-XIX

1. Prove that the curve $y=e^x$ is convex to the x-axis at every point.
2. Prove that the curve $y=\cos^{-1}x$ is everywhere concave to the y-axis excepting where it crosses the y-axis.
3. Prove that the curve $y=\log x$ is convex to the foot of the ordinate in the range $0<x<1$, and concave where $x>1$. Prove also that the curve is convex everywhere to the y-axis.
4. Show that the curve $(y-a)^3=a^3-2a^2x+ax^2$, where $a>0$, is always concave to the x-axis. How is it situated with respect to the y-axis ?
5. Show that the origin is a point of inflexion on the curves :

   (i) $y=x^2\log(1-x)$, (ii) $y=x\cos 2x$.
6. Find the points of inflexion, if any, on the curves :

   (i) $y=\dfrac{x}{(x+1)^2+1}$. (ii) $y^2=x(x+1)^2$.

   (iii) $c^2y=(x-a)^3$. (iv) $y=ae^{-8x^2}$.
7. Show that the points of inflexion on the curve $y^2=(x-a)^2(x-b)$ lie on the line $3x+a=4b$.
8. Show that the curve $y(x^2+a^2)=a^2x$ has three points of inflexion which lie on a straight line.

ANSWERS

4. Concave where $0<x<a$, convex everywhere else.

6. (i) $(-2,-1)$, $\left(1+\sqrt{3}, \dfrac{\sqrt{3}-1}{4}\right)$, $\left(1-\sqrt{3}, \dfrac{-1-\sqrt{3}}{4}\right)$.

   (ii) $\left(\frac{1}{3}, \pm\frac{4}{9}\sqrt{3}\right)$, (iii) $(a, 0)$,. (iv) $\left(\pm\frac{1}{2}, ae^{-2}\right)$.

20 On Some Well-Known Curves

20.1. We give below diagrams, equations, and a few characteristics of some well-known curves which have been used in the preceding pages in obtaining their properties. The student is supposed to be familiar with conic sections and graphs of circular functions, so they are not given here.

20.2. Cycloid.

The *cycloid* is the curve traced out by a point on the circumference of a circle which rolls (without sliding) on a straight line.

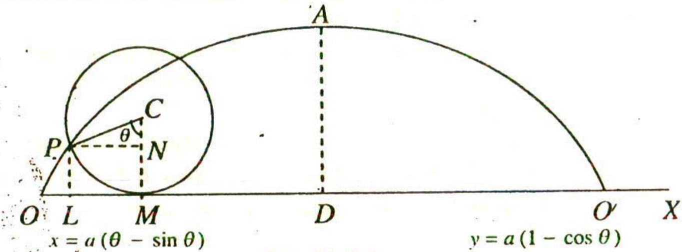


Fig 20.2.1

Let P be the point on the circle MP, called the *generating circle*, which traces out the cycloid. Let the line OMX on which the circle rolls be taken as x-axis and the point O on OX, with which P was in contact when the circle began rolling, be taken as origin.

Let a be the radius of the generating circle and C its centre, P the point (x, y) on it, and let $\angle PCM = \theta$. Then θ is the angle through which the circle turns as the point P traces out the locus.

$$\therefore \quad OM = \text{arc } PM = a\theta .$$

Let PL be drawn perpendicular to OX.

$$\therefore \quad x = OL = OM - LM = a\theta - PN = a\theta - a\sin\theta$$

$$= a\left(\theta - \sin\theta\right).$$

$$y = PL = NM = CM - CN = a - a\cos\theta$$

$$= a\left(1 - \cos\theta\right).$$

Thus, the *parametric equations of the cycloid with the starting point as the origin* and *the line on which the circle rolls,* called the *base, as the x-axis* are

$$\mathbf{x = a(\theta - \sin\theta),\ \ y = a(1 - \cos\theta).} \qquad \ldots(1)$$

The point A at the greatest distance from the base OX is called the *vertex*. Thus, for the vertex, y, *i.e.*, $a(1-\cos\theta)$ is maximum. Hence, $\cos\theta = -1$, *i.e.*, $\theta = \pi$.

$\therefore\ \ AD = a(1-\cos\pi) = 2a.$

$\therefore$ **vertex is $(\mathbf{a\pi}, \mathbf{2a})$.**

For O and O', $y = 0$, $\quad \therefore \cos\theta = 1. \qquad \therefore \theta = 0$ and 2π.

As the circle rolls on, arches like OAO' are generated over and over again, and any single arch is called a cycloid.

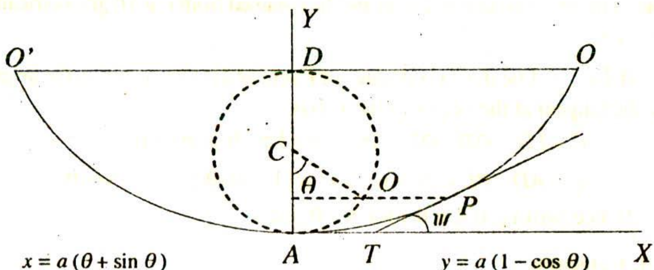


Fig 20.2.2

Since the vertex is the point $(a\pi, 2a)$ the equation of the cycloid with the vertex as the origin and the tangent at the vertex as the x-axis can be obtained from the previous equation by transferring the origin to $(a\pi, 2a)$ and turning the axes through π, , *i.e.*, by writing

$$a\pi + x'\cos\pi - y'\sin\pi \quad \text{and} \quad 2a + x'\sin\pi + y'\cos\pi$$

for x and y respectively.

Hence, $\quad a(\theta - \sin\theta) = a\pi - x'$

or, $\quad x' = a(\pi - \theta) + a\sin\theta$

$\qquad\quad = a(\theta' + \sin\theta')$, where $\theta' = \pi - \theta$

and $\quad a(1-\cos\theta) = 2a - y'$

or, $\quad y' = 2a - a + a\cos\theta = a + a\cos\theta$

$\qquad\quad = a - a\cos(\pi - \theta) = a(1-\cos\theta').$

Hence, (replacing θ' by θ) the *equation of the cycloid with the vertex as the origin and the tangent at the vertex as the x-axis* are

$$\mathbf{x = a(\theta + \sin\theta),\ \ y = a(1 - \cos\theta)} \qquad \ldots(2)$$

In this equation, $\theta = 0$ *for the vertex*, $\theta = \pi$ *for* O, and $\theta = -\pi$ *for* O'. The *characteristic properties* are

(i) For the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, radius of curvature = twice the length of the normal, (the centre of curvature and the x-axis being on the same side of the curvature).

(ii) The evolute of the cycloid is an equal cycloid.

(iii) For the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, $\psi = \frac{1}{2}\theta$ and $s^2 = 8ay$, s being measured from the vertex.

Note. The above equation (2) can also be obtained from Fig. (i) geometrically as follows :

If (x', y') be the co-ordinates of P referred to the vertex as the origin and the tangent at the vertex as the x-axis.

$$x' = LD = OD - OL = a\pi - x = a(\pi - \theta) + a\sin\theta,$$

$$y' = AD - PL = 2a - y = 2a - a(1 - \cos\theta) = a(1 + \cos\theta),$$

Hence, writing θ' (or θ) for $\pi - \theta$, etc.

20.3. Catenary.

The *catenary* is the curve in which a uniform heavy flexible string will hang under the action of gravity when suspended from two points. It is also called the *chainette*.

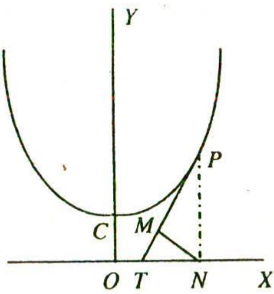


Fig 20.3.1

Its equation, as shown in books on Statics, is

$$\mathbf{y = c\cosh\frac{x}{c} = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right).}$$

C is called the *vertex*, $OC = c$, OX is called the *directrix*.

The *characteristic properties* are

(i) The perpendicular from the foot of the ordinate upon the tangent at any point is of constant length.

(ii) Radius of curvature at any point = length of the normal at the point (the centre of curvature and the x-axis being on the opposite sides of the curve).

(iii) $y^2 = c^2 + s^2$, s being measured from the vertex C.

(iv) $s = c \tan \psi$, $y = c \sec \psi$.

(v) $x = c \log(\sec \psi + \tan \psi)$.

20.4. Tractrix.

Its equation is

$$x = \sqrt{a^2 - y^2} + \frac{a}{2} \log \frac{a - \sqrt{a^2 - y^2}}{a + \sqrt{a^2 - y^2}},$$

or, $$x = a\left(\cos t + \log \tan \tfrac{1}{2} t\right), \quad y = a \sin t.$$

Here, $OA = a$.

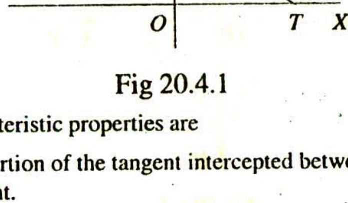


Fig 20.4.1

The characteristic properties are

(i) The portion of the tangent intercepted between the curve and the x-axis is constant.

(ii) The radius of curvature varies inversely as the normal (the centre of curvature and the x-axis being on the opposite sides of the curve).

(iii) The evolute of the tractrix is the catenary

$$y = a \cosh(x/a).$$

20.5. Four cusped Hypo-cycloid.

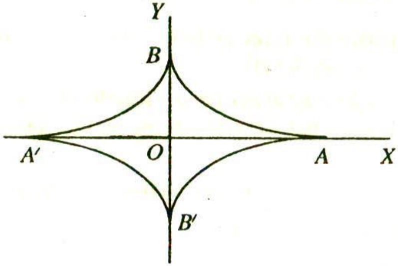


Fig 20.5.1

Its equation is $\left(\frac{\mathbf{x}}{\mathbf{a}}\right)^{\frac{2}{3}} + \left(\frac{\mathbf{y}}{\mathbf{b}}\right)^{\frac{2}{3}} = \mathbf{1}$

or, $\quad \mathbf{x} = \mathbf{a}\cos^3\phi,\ \mathbf{y} = \mathbf{b}\sin^3\phi.$

Here, $OA = OA' = a$; $OB = OB' = b$.

The astroid is a special case of this when $a = b$.

20.6. Astroid.

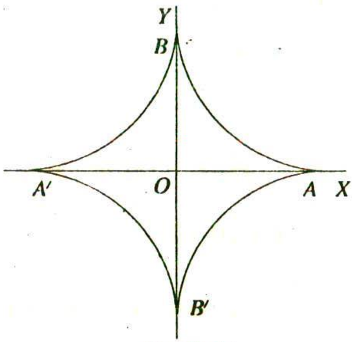


Fig 20.6.1

Its equation is $\mathbf{x}^{\frac{2}{3}} + \mathbf{y}^{\frac{2}{3}} = \mathbf{a}^{\frac{2}{3}}$

or, $\quad \mathbf{x} = \mathbf{a}\cos^3\theta,\ \mathbf{y} = \mathbf{a}\sin^3\theta$

Here, $OA = OB = OA' = OB' = a$.

The whole figure lies completely within a circle of radius a and centre O. The points A, A', B, B' are called cusps. It is a special type of a four-cusped hypo-cycloid. [See § *20.5*]

The *characteristic property* of this curve is that the tangent at any point to the curve intercepted between the axes is of constant length.

20.7. Evolutes of Parabola and Ellipse.

(i) The equation of the evolute of the parabola

$$y^2 = 4ax \text{ is}$$

$$27ay^2 = 4(x - 2a)^3.$$

This curve is called a *semi-cubical parabola.*

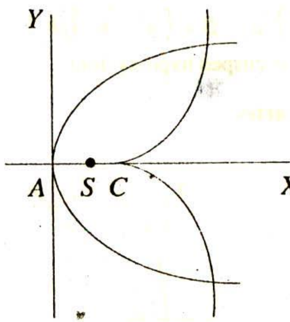


Fig 20.7.1

Transferring the origin to $(2a, 0)$, its equation assumes the form $y^2 = kx^3$ where $k = 4/(27a)$, which is the standard equation of the semi-cubical parabola with its vertex at the origin.

Hence, the vertex C of the evolute is $(2a, 0)$.

(ii) The equation of the evolute of the ellipse

$$x^2/a^2 + y^2/b^2 = 1 \text{ is}$$

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}},$$

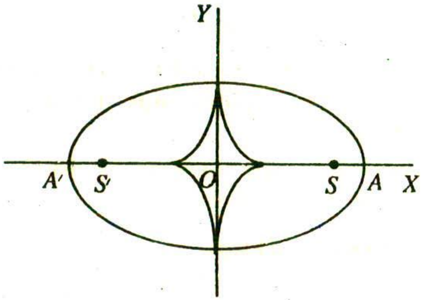


Fig 20.7.2

which can be written in the form

$$\left(\frac{x}{\alpha}\right)^{\frac{2}{3}} + \left(\frac{y}{\beta}\right)^{\frac{2}{3}} = 1$$

where $\alpha = \left(a^2 - b^2\right)/a, \ \ \beta = \left(a^2 - b^2\right)/b$.

Hence, it is a four-cusped hypo-cycloid.

20.8. Folium of Descartes.

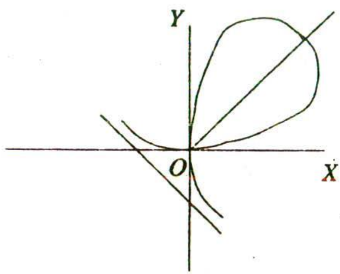


Fig 20.8.1

Its equation is $\mathbf{x^3 + y^3 = 3axy}$

It is symmetrical about the line $y = x$.

The axes of co-ordinates are tangents at the origin, and there is a loop in the first quadrant.

It has an asymptote $x + y + a = 0$ and its radii of curvature at the origin are each $= \frac{3}{2}a$.

20.9. Logarithmic and exponential curves.

(i) The equation of the Logarithmic curve is $y = \log x$. x is always positive; $y = 0$ when $x = 1$ and as x becomes smaller and smaller, y, being negative, becomes numerically larger and larger. For $x > 0$, *the curve is continuous.*

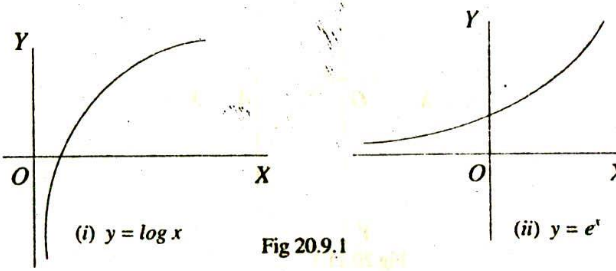


Fig 20.9.1

(ii) The equation of the Exponential curve is $y = e^x$. x may be positive or negative but y is always positive, and y becomes smaller and smaller,

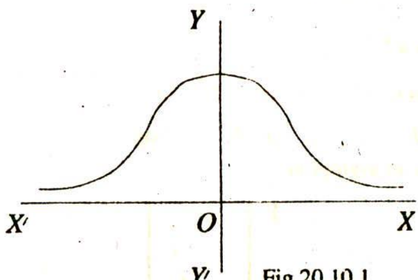


Fig 20.10.1

as x, being negative, becomes numerically larger and larger. *The curve is continuous for all values of* x.

20.10. Probability Curve.

The equation of the probability curve is $y = e^{-x^2}$.

The x-axis is an asymptote.

The area between the curve and the asymptote is

$$= 2\int_0^\infty e^{-x^2}\,dx = 2.\tfrac{1}{2}\sqrt{\pi} = \sqrt{\pi}.$$

20.11. Cissoid of Diocles.

Its cartesian equation is $y^2(2a - x) = x^3$.

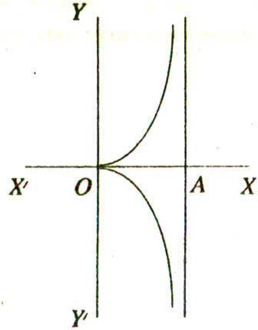


Fig 20.11.1

$OA = 2a$; $x = 2a$ **is an asymptote.**

Its polar equation is $r = \dfrac{2a \sin^2 \theta}{\cos \theta}$.

20.12. Strophoid.

The equation of the curve is $y^2 = x^2 \cdot \dfrac{a + x}{a - x}$.

$OA = OB = a$. $OCBP$ is a loop.

$x = a$ is an asymptote.

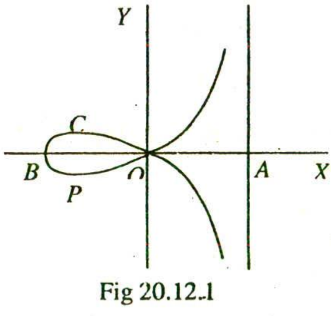


Fig 20.12.1

The curve $y^2 = x^2 \cdot \dfrac{a - x}{a + x}$ is similar, just the reverse of strophoid, the loop being on the right side of the origin and the asymptote on the left side.

20.13. Witch of Agnesi.

The equation of the curve is $xy^2 = 4a^2(2a - x)$.

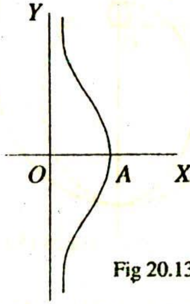


Fig 20.13.1

Here, $OA = 2a$.

This curve was first discussed by the Italian lady mathematician Maria Gactaua Agnesi, Professor of Mathematics at Bologna.

20.14. Logarithmic (or Equiangular) spiral.

Its equation is $\mathbf{r} = \mathbf{a}e^{\theta \cot \alpha}$ (or, $r = ae^{m\theta}$),

where $\cot \alpha$ or m is constant.

Characteristic Properties :

(i) *The tangent at any point makes constant angle with the radius vector ($\phi = \alpha$).*

(ii) Its pedal, inverse, polar reciprocal and evolute are all equiangular sprials.

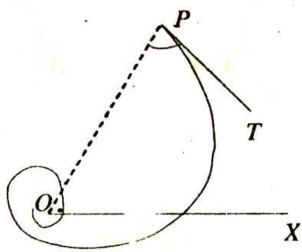


Fig 20.14.1

(iii) The radius of curvature subtends a right angle at the pole.

Note. Because of the property (i), the spiral is called *equiangular.*

20.15. Spiral of Archimedes.

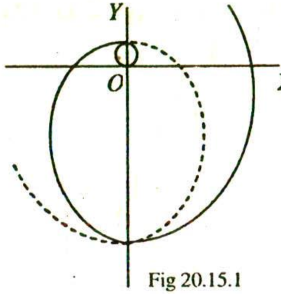


Fig 20.15.1

Its equation $\mathbf{r} = \mathbf{a}\theta$.

Its *characteristic property* is that its polar subnormal is constant.

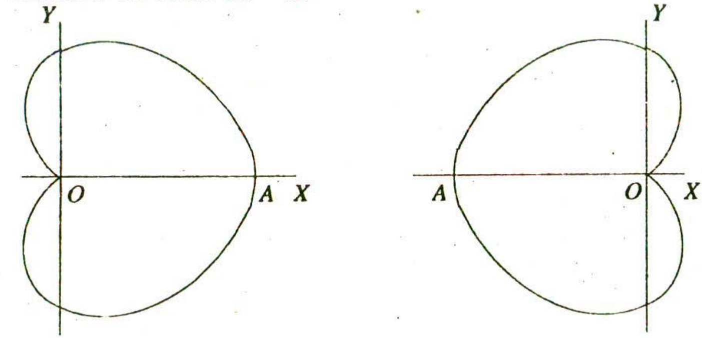
20.16. Cardioide.

Its equation is (i) $\mathbf{r} = \mathbf{a}(1 + \cos\theta)$, or (ii) $\mathbf{r} = \mathbf{a}(1 - \cos\theta)$.

In (i), $\theta = 0$ for A, and $\theta = \pi$ for O.

In (ii), $\theta = \pi$ for A, and $\theta = 0$ for O.

In both cases, the curve is symmetrical about the initial line, which divides the whole curve into two equal halves and for the upper half, θ varies from 0 to π, and $OA = 2a$.



(*i*) $r = a(1 + \cos\theta)$ (*ii*) $r = a(1 - \cos\theta)$

Fig 20.16.1

The curve (ii) is really the same as (i) turned through 180°.

The curve passes through the origin, its tangent there being the initial line, and tangent at A is perpendicular to the initial line.

The evolute of the cardioide is a cardioide.

Note. Because of its shape like human hea.t, its is called a cardioide. The cardioide $r = a(1+\cos\theta)$ is the pedal of the circle $r = 2a\cos\theta$ with respect to a point on the circumference of the circle, and inverse of the parabola $r = a/(1+\cos\theta)$

20.17. Limacon.

The equation of the curve is $\mathbf{r = a + b}\cos\theta$.

When $a > b$, we have the outer curve, and when $a < b$, we have the inner curve with the loop.

When $a = b$, the curve reduces to a cardioide. [See *fig. in* § *20.16*]

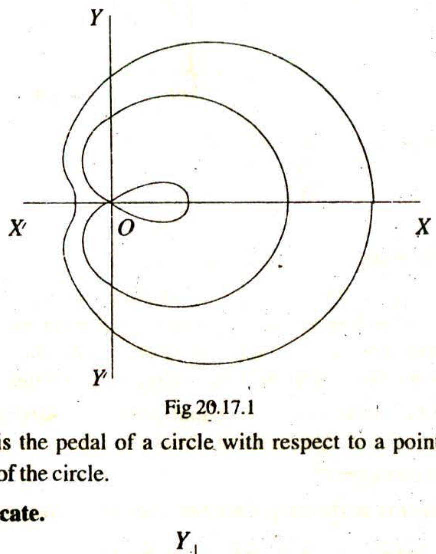


Fig 20.17.1

Limacon is the pedal of a circle with respect to a point outside the circumference of the circle.

20.18. Lemniscate.

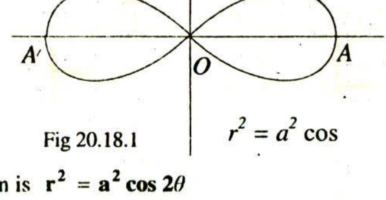


Fig 20.18.1

Its equation is $\mathbf{r^2 = a^2 \cos 2\theta}$

or, $(x^2+y^2)^2 = a^2(x^2-y^2)$.

It consists of two equal loops each symmetrical about the initial line, which divides each loop into two equal halves.

Here, $OA = OA' = a$.

The tangents at the origin are $y = \pm x$.

For the upper half of the right-hand loop, θ varies from 0 to $\frac{1}{4}\pi$.

A *characteristic property* of it is that the product of the distances of any point on it from each of the points $(\pm a/\sqrt{2}, 0)$ is constant.

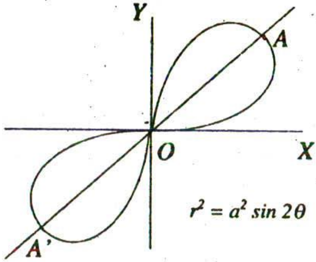


Fig 20.18.2

The lemniscate is the pedal of the rectangular hyperbola $r^2 \cos 2\theta = a^2$. The curve represented by $r^2 = a^2 \sin 2\theta$ is also sometimes called lemniscate or *rose lemniscate*, to distinguish it from the first lemniscate which is sometimes called *lemniscate of Bernoulli* after the name of the mathematician J. Bernoulli who first studied its properties.

The curve consists of two equal loops, situated in the first and third quardants, and symmetrical about the line $y = x$. It is the first curve turned through 45°.

The tangents at the origin are the axes of x and y.

20.19. Rose-Petals ($r = a \sin n\theta$, $r = a \cos n\theta$).

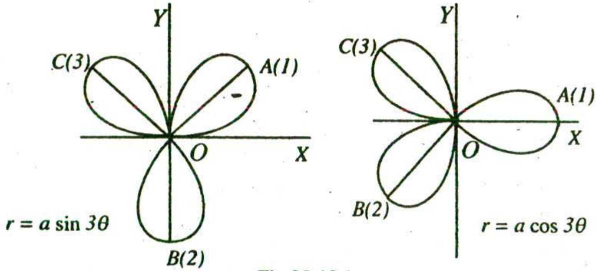


Fig 20.19.1

The curve represented by $r = a \sin 3\theta$, or, $r = a \cos 3\theta$ is called a *three-leaved rose*, each consisting of three equal loops. The order in which the loops are described is indicated in the figures by numbers. In each case, $OA = OB = OC = a$, and $\angle AOB = \angle BOC = \angle COA = 120^0$.

The curve represented by $r = a \sin 2\theta$, or, $r = a \cos 2\theta$ is called a *four-leaved rose*, each consisting of four equal loops. In each case, $OA = OB = OC = a$ and $\angle AOB = \angle BOC = \angle COD = \angle DOA = 90^0$.

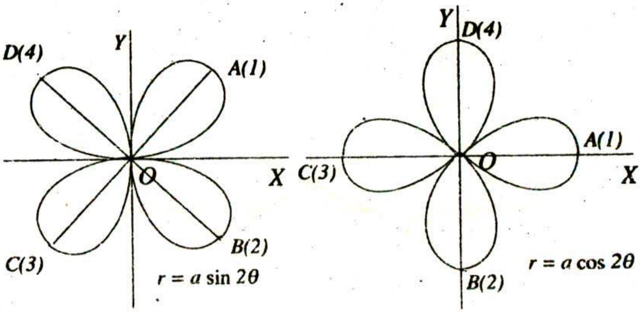


Fig 20.19.2

The class of curves represented by $r = a \sin n\theta$, or, $r = a \cos n\theta$ where n is a positive integer is called *rose-petal*, there being n or *2n equal loops* according as *n is odd* or *even*, all being arranged symmetrically about the origin and lying entirely within a circle whose centre is the pole and radius a.

20.20. Sine Spiral $\left(r^n = a^n \sin n\theta,\ \text{or},\ r^n = a^n \cos n\theta\right)$.

The class of curves represented by (i) $r^n = a^n \sin n\theta$, or (ii) $r^n = a^n \cos n\theta$ is called *sine spiral* and embraces several important and well-known curves as particular cases.

Thus, for the values $n = -1, 1, -2, 2, -\frac{1}{2}$, and $\frac{1}{2}$, the sine spiral is respectively a straight line, a circle, a rectangular hyperbola, a lemniscate, a parabola and a cardioide.

For (i) $\phi = n\theta$; for (ii) $\phi = \frac{1}{2}\pi + n\theta$

The pedal equation in both the cases is

$$p = r^{n+1} / a^n.$$

21.1. Double Points.

If two branches of a plane curve pass through a point P, that is, two tangents at P can be drawn to the curve, then the point is called a *Double point on the curve.*

21.2. Classification of Double Points.

Node : If the tangents at a double point P on a plane curve be real and distinct, the double point is called a *Node.*

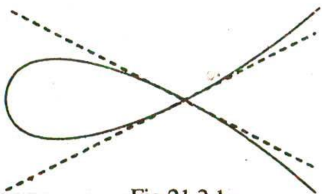


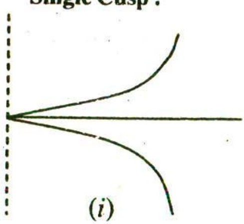
Fig 21.2.1

Cusp : If the tangents at a double point P on the plance curve be real but coincident and the curve has real branches in the neighbourhood of P, then the double point is called a *Cusp.*

Isolated Point : If the tangents at a double point P on a plane curve be either non-real or real , coincident but the curve has no real branches in the neighbourhood of P, then the double point is called an *Isolated point.*

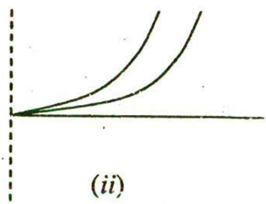
21.3. Different types of cusps.

Single Cusp :



(i)

Single cusp,first species

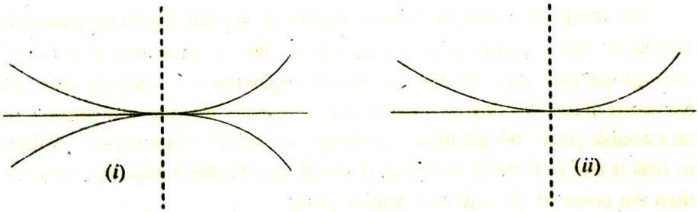


(ii)

Single cusp,second species

If P be a cusp on a plane curve and both the branches of the curve lie on the same side of the normal at P, then the cusp is called a *Single Cusp.*

Double Cusp :



Double cusp, first species Fig 21.3.2 Double cusp, second species

If a plane curve has branches on either side of the normal at a cusp P, then the cusp is called a *Double Cusp*.

21.4. Species of a Cusp.

Cusp of the Fist Species (or Keratoid Cusp) :

If the branches of a curve at a cusp P (single or double) lie on either side of the tangent at P, then the cusp is of the first species or it is called a *Keratoid Cusp*.

Cusp of the Second Species (or Ramphoid Cusp) :

If the branches of a curve at a cusp P (single or double) lie on one side of the tangent at P, then the cusp is of the second species or it is called a *Ramphoid Cusp*.

Osculinflexion :

If a curve has double cusp at P and it is Keratoid on one side of the normal at P and Ramphoid on the other, then P is called a *Point of osculinflexion*.

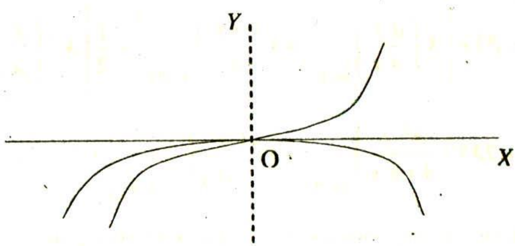


Fig 21.4.1
Double cusp, change of species
Osculinflexion

21.5. Search for double points.

Let the given curve be represented by a rational algebraic equation. Let (α, β) be a double point on the curve. We shift the origin to (α, β) through parallel axes. In the transformed equation the constant term and the terms of the first degree must be absent in order that the new origin may be a double point. We get three equations in α and β. Take any two of them to find α and β. If these values of α and β satisfy the remaining equation, then the point (α, β) will be a double point.

If no such values of α, β are available to satisfy all these three equations, then we say there is no double point on the curve.

21.6. Conditions for existence of double points on an algebraic curve.

Let $f(x, y) = 0$ ... (1)

be a rational algebraic curve.

If we shift our origin to the point (α, β) through parallel axes, the above equation transforms into

$$f(X + \alpha, Y + \beta) = 0 \quad \ldots (2)$$

By Taylor's therom, we get

$$f(X+\alpha, Y+\beta) = f(\alpha, \beta) + X\left(\frac{\partial f}{\partial x}\right)_{(\alpha,\beta)} + Y\left(\frac{\partial f}{\partial y}\right)_{(\alpha,\beta)}$$

$$+\frac{1}{2}\left[X^2\left(\frac{\partial^2 f}{\partial x^2}\right)_{(\alpha,\beta)} + 2XY\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(\alpha,\beta)} + Y^2\left(\frac{\partial^2 f}{\partial y^2}\right)_{(\alpha,\beta)}\right] + \cdots$$

Therefore, the equation (2) becomes

$$f(\alpha, \beta) + \left[X\left(\frac{\partial f}{\partial x}\right)_{(\alpha,\beta)} + Y\left(\frac{\partial f}{\partial y}\right)_{(\alpha,\beta)}\right] + \frac{1}{2}\left[X^2\left(\frac{\partial^2 f}{\partial x^2}\right)_{(\alpha,\beta)}\right.$$

$$\left. + 2XY\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(\alpha,\beta)} + Y^2\left(\frac{\partial^2 f}{\partial y^2}\right)_{(\alpha,\beta)}\right] + \cdots = 0 \quad \ldots (3)$$

(α, β) will be a double point on (1) if origin is a double point on (3) which requires

$$f(x, y) = 0, \left(\frac{\partial f}{\partial x}\right)_{(\alpha,\beta)} = 0 \text{ and } \left(\frac{\partial f}{\partial y}\right)_{(\alpha,\beta)} = 0.$$

If these equations be consistent, the tangents at the new origin are given by

$$X^2\left(\frac{\partial^2 f}{\partial x^2}\right)_{(\alpha,\beta)}+2XY\left(\frac{\partial^2 f}{\partial x\partial y}\right)_{(\alpha,\beta)}+Y^2\left(\frac{\partial^2 f}{\partial y^2}\right)_{(\alpha,\beta)}=0 \qquad \dots (4)$$

In general, the double point will be a node, cusp or an isolated point according, as

$$\left(\frac{\partial^2 f}{\partial x\partial y}\right)_{(\alpha,\beta)} >, =, \text{ or } < \left(\frac{\partial^2 f}{\partial x^2}\right)_{(\alpha,\beta)}\left(\frac{\partial^2 f}{\partial y^2}\right)_{(\alpha,\beta)}.$$

21.7. Illustrative Examples.

Ex. 1. *Determine the existence and nature of the double points on the curve* $(x-2)^2=y(y-1)^2$.

The given curve is

$$f(x,y)\equiv(x-2)^2-y(y-1)^2=0 \qquad \dots (1)$$

$$\left(\frac{\partial f}{\partial x}\right)=0 \text{ gives } 2(x-2)=0,\ i.e.,\ x=2, \qquad \dots (2)$$

$$\left(\frac{\partial f}{\partial y}\right)=0 \text{ gives } 2y(y-1)+(y-1)^2=0,$$

i.e., $(y-1)(3y-1)=0$,

i.e., $y=1, \frac{1}{3}$. ... (3)

Equations (2) and (3) say that the possible points are $(2,1)$, $(2,\frac{1}{3})$.

The point $(2,1)$ satisfies equation (1) but the point $(2,\frac{1}{3})$ does not. So, there is a double point at the point $(2,1)$.

Let us shift the origin to $(2,1)$ through parallel axes. Equation (1) becomes $X^2=(Y+1)Y^2$, *i.e.*, $Y^3+Y^2-X^2=0$.

The tangents at the new origin are given by

$Y^2-X^2=0$, *i.e.*, $Y=\pm X$,

which are real and distinct.

Hence, there is a node on the given curve.

Ex. 2. *Examine the character of the origin on the curve*

$$y^2=2x^2y+x^3y+x^3.$$

The given equation is

$$f(x, y) \equiv x^3y + x^3 + 2x^2y - y^2 = 0 \qquad \ldots (1)$$

Now, $\dfrac{\partial f}{\partial x} = 3x^2y + 3x^2 + 4xy$;

$$\frac{\partial f}{\partial y} = x^3 + 2x^2 - 2y.$$

At $(0,0)$, $f(x, y) = 0$, $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$.

So, there is a double point at $(0,0)$ on the given curve.

Tangents at $(0,0)$ are given by $y^2 = 0$, that is, $y = 0$, $y = 0$ which are real but repeated. The double point is either a cusp or an isolated point.

Take a point $P(x, y)$ on the curve in the neighbourhood of $(0,0)$.

Distance of P from the tangent $y = 0$ is given by

$$p = y \qquad \ldots (2)$$

Eliminating y between (1) and (2) we get

$$x^3p + x^3 + 2x^2p - p^2 = 0$$

i.e., $$p^2 - \left(x^3 + 2x^2\right)p - x^3 = 0 \qquad \ldots (3)$$

which is a quadratic in p whose discriminant is

$$\left(x^3 + 2x^2\right)^2 + 4x^3 = 4x^3 + 4x^4 + 4x^5 + x^6$$

$$\approx 4x^3, \text{ since } x \text{ is small.}$$

It is positive if $x > 0$, that is, (3) can have two real roots if $x > 0$, that is, (1) has two real branches on the right of the y-axis in the neighbourhood of (0, 0). So, there is a single cusp at the origin.

The product of the roots of (3) is $-x^3 < 0$ for $x > 0$, that is, the roots of (3) are opposite signs, that is, the two branches of the curve lie on opposite sides of the tangent $y = 0$. The cusp is of the first species.

Hence there is a single cusp of the first species at the origin.

Ex. 3. *Is origin a double point on the curve* $y^2 = 2x^2y + x^4y - 2x^4$? *If so, state its nature.*

The given curve is

$$f(x, y) \equiv x^4y - 2x^4 + 2x^2y - y^2 = 0 \qquad \ldots (1)$$

Now, $\dfrac{\partial f}{\partial x} = 4x^3y - 8x^3 + 4xy$;

$$\frac{\partial f}{\partial y} = x^4 + 2x^2 - 2y.$$

At $(0,0)$, we see that $f(x, y) = 0$, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

So, there is a double point at (0, 0).

The tangents at origin are given by $y^2 = 0$, that is, $y = 0$, $y = 0$ which are real and coincident. The double point may be a cusp or an isolated point.

We take a point $P(x, y)$ on the curve in the neighbourhood of the origin. Distance of P from the tangent $y = 0$ is given by

$$p = y \qquad \dots (2)$$

Eliminating y between (1) and (2) we get

$$x^4 p - 2x^4 + 2x^2 p - p^2 = 0,$$

i.e., $$p^2 - \left(2x^2 + x^4\right)p + 2x^4 = 0 \qquad \dots (3)$$

which is a quadratic in p whose discriminant is

$$\left(2x^2 + x^4\right)^2 - 8x^4 = -4x^4 + 4x^6 + x^8 \approx -4x^4 < 0 \text{ for all } x,$$

that is, the roots of (3) are complex, that is, there is no real branch of the curve in the neighbourhood of (0, 0).

Hence, the origin is an isolated point on the curve.

Ex. 4. *Search for double points on the curve*

$$x^2 y + x^3 y + 5x^4 = y^2.$$

The curve is $f(x, y) \equiv 5x^4 + x^3 y + x^2 y - y^2 = 0 \qquad \dots (1)$

$\frac{\partial f}{\partial x} = 0$ gives $20x^3 + 3x^2 y + 2xy = 0$,

i.e., $x(20x^2 + 3xy + 2y) = 0. \qquad \dots (2)$

$\frac{\partial f}{\partial y} = 0$ gives $x^3 + x^2 - 2y = 0$,

i.e., $$y = \frac{x^2 + x^3}{2} \qquad \dots (3)$$

Eliminating y between (2) and (3) we get

$$x\left\{20x^2 + \frac{3x\left(x^2 + x^3\right)}{2} + x^2 + x^3\right\} = 0,$$

i.e., $x^3\{40 + 3x(1 + x) + 2(1 + x)\} = 0$

i.e., $x^3(3x^2 + 5x + 42) = 0$.

$$i.e, \quad x = 0 \text{ or } \frac{-5 \pm \sqrt{25-504}}{6},$$

$$= 0 \text{ or } \frac{-5 \pm i\sqrt{479}}{6}.$$

Only $x = 0$ is acceptable as the other two values of x are complex.

When $x = 0$, we get $y = 0$.

We see that (1) is satisfied with $x = 0$, $y = 0$.

So (0, 0) $(0, 0)$ is a double point on the given curve.

Tangents at (0, 0) are given by $y^2 = 0$, that is, $y = 0$, $y = 0$ which are real but repeated, that is, the double point is either a cusp or an isolated point.

Let us take a point $P(x, y)$ on the curve in the neighbourhood of (0, 0). Its distance from the tangent $y = 0$ is given by

$$p = y \qquad \ldots (4)$$

Eliminating y between (1) and (4) we get

$$x^2 p + x^3 p + 5x^4 = p^2,$$

that is, $$p^2 - (x^2 + x^3)p - 5x^4 = 0 \qquad \ldots (5)$$

which is a quadratic in p and so two branches of the curve exist in the neighbourhood of (0, 0) depending on the roots of (5).

Its discriminant is $(x^2 + x^3) + 20x^4 \approx 21x^4 > 0$ for positive as well as negative values of x, that is, the curve has real branches on either side of the y-axis. The origin is, therefore, a double cusp.

Product of the roots of (5) is $-5x^4 < 0$ for all x. The roots are of opposite signs implying that the branches lie on either side of the x-axis, that is, the cusp is of the first species.

Hence, the origin is a double cusp of the first species.

EXAMPLES - XXI

1. Examine the character of the origin on each of the following curves $(a > 0)$:

(i) $ay^2 = x^3$. (ii) $a(y - x)^2 = x^3$. (iii) $(2a - x)y^2 = x^3$.

(iv) $x^4 - 4x^2 y - 2xy^2 + 4y^2 = 0$.

(v) $x^4 - ax^2 y + axy^2 + a^2 y^2 = 0$.

(vi) $x^4 - 3x^2y - 3xy^2 + 9y^2 = 0$.

(vii) $x^4 - 2ax^2y - axy^2 + a^2y^2 = 0$.

(viii) $y^2 = 2x^2y + x^4y + x^4$.

(ix) $x^4 + x^3y + 5x^3 - 2x^2y + x^2 - 3xy + 2y^2 = 0$

(x) $5x^2y + x^3y - 5x^4 = y^2$.

(xi) $(2x + y)^2 - 6xy(2x + y) - 7x^3 = 0$.

2. Search for double points on each of the following curves :

(i) $x^4 + y^3 + 2x^2 + 3y^2 = 0$.

(ii) $y(y - 6) = x^2(x - 2)^3 - 9$.

(iii) $x^3 - y^2 - 7x^2 + 4y + 15x - 13 = 0$.

(iv) $x^4 - 2y^3 - 3y^2 - 2x^2 + 1 = 0$.

(v) $x(x^2 + y^2) = ay^2$. (vi) $x^3 + y^3 - 3axy = 0$.

(vii) $y^2 = (x - 1)(x - 2)^2$. (viii) $y^2 = (x - 2)^2(x - 5)$.

(ix) $x^3 + 2x^2 + 2xy - y^2 + 5x - 2y = 0$.

(x) $(x + y)^3 = y(y - x + 2)^2$.

(xi) $(2y + x + 1)^2 = 4(1 - x)^5$.

(xii) $xy^2 + 2a^2y - ax^2 - 3a^2x - 3a^3 = 0$.

(xiii) $x^4 + 4x^3 + 2y^3 + 4x^2 + 3y^2 - 1 = 0$.

(xiv) $y^4 - 8y^3 - 12xy^2 + 16y^2 + 48xy + 4x^2 - 64x = 0$.

3. Show that the curve $(xy+1)^2 + (x-1)^3(x-2) = 0$ has a single cusp of the first species at $(1, -1)$.

4. Prove that the curve $ay^2 = (x-a)^2(x-b)$ has, at $x = a$, an isolated point if $a < b$, a node if $a > b$, and a cusp if $a = b$.

5. Examine the nature of the point $(0, -1)$ on the curve $x^4 - 2x^2y - xy^2 - 2x^2 - 2xy + y^2 - x + 2y + 1 = 0$.

6. Examine the nature of the point on the curve $y - 2 = x\left(1 + x + x\sqrt{x}\right)$ where it cuts the y-axis.

7. Examine the nature of the point $(-1, -2)$ on the curve $x^3 + 2x^2 + 2xy - y^2 + 5x - 2y = 0$.

8. Show that each of the curves

$$(x\cos\alpha - y\sin\alpha - b)^3 = c(x\sin\alpha + y\cos\alpha)^2$$

where α is a variable, has a cusp and that all cusps lie on a circle.

9. Show that each of the curves

$$x^3 - 2t^2x^2 - y^2 + t^4x + 4ty - 4t^2 = 0$$

where t is a variable, has a node and that all these nodes lie on a parabola.

ANSWERS

1. (i) Single cusp of the first species.
   (ii) Single cusp of the first species.
   (iii) Single cusp of the first species.
   (iv) Single cusp of the second species.
   (v) Isolated point. (vi) Isolated point.
   (vii) Single cusp of the second species.
   (viii) Double cusp of the first species. (ix) Node.
   (x) Double cusp of the second species.
   (xi) Single cusp of the first species.
2. (i) Isolated point at $(0, 0)$.
   (ii) Isolated point at $(0, 3)$ and a single cusp of the first species at $(2, 3)$
   (iii) Node at $(3, 2)$.
   (iv) Nodes at $(0, -1)$; $(1, 0)$ and $(-1, 0)$
   (v) Single cusp of the first species at $(0, 0)$.
   (vi) Node at $(0, 0)$. (vii) Node at $(2, 0)$.
   (viii) Isolated point at $(2, 0)$.
   (ix) Single cusp of the first species at $(-1, -2)$.
   (x) Single cusp of the first species at $(1, -1)$.
   (xi) Single cusp of the first species at $(1, -1)$.
   (xii) Single cusp of the first species at $(-a, 0)$.
   (xiii) Nodes at $(0, -1)$, $(-1, 0)$ and $(-2, -1)$.
   (xiv) Node at $(2, 2)$.
5. Single cusp of the second species.
6. Single cusp of the second species.
7. Single cusp of the first species.