APPENDIX 1

Mathematics for Two-Dimensional Computer Graphics

The key to understanding how geometric objects can be described and manipulated within a computer graphics system lies in understanding the interplay between geometry and numbers. While we have an innate geometric intuition which enables us to understand verbal descriptions such as *line*, *angle*, and *shape* and descriptions of the manipulation of objects (*rotating*, *shifting*, *distorting*, etc.), we also have the computer's ability to manipulate numbers. The problem then is to express our geometric ideas in numeric form so that the computer can do our bidding.

A *coordinate system* provides a framework for translating geometric ideas into numerical expressions. We start with our intuitive understanding of the concept of a two-dimensional plane.

A1.1 THE TWO-DIMENSIONAL CARTESIAN COORDINATE SYSTEM

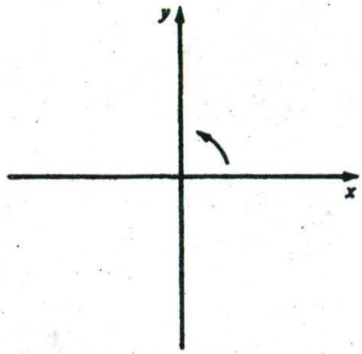
In a two-dimensional plane, we can pick any point and single it out as a reference point called the *origin*. Through the origin we construct two perpendicular number lines called *axes*. These are traditionally labeled the x axis and the y axis. An orientation or sense of the plane is determined by the positions of the positive sides of the x and y axes. If a counterclockwise rotation of 90° about the origin aligns the positive x axis with the positive y axis, the coordinate system is said to have a *right-handed* orientation [see Fig. A1-1(a)]; otherwise, the coordinate system is called *left-handed* [see Fig. A1-1(b)].

The system of lines perpendicular to the x axis and perpendicular to the y axis forms a rectangular grid over the two-dimensional plane. Every point P in the plane lies at the intersection of exactly one line perpendicular to the x axis and one line perpendicular to the y axis. The number pair (x, y) associated with the point P is called the *Cartesian coordinates* of P. In this way every point in the plane is assigned a pair of coordinates (see Fig. A1-2).

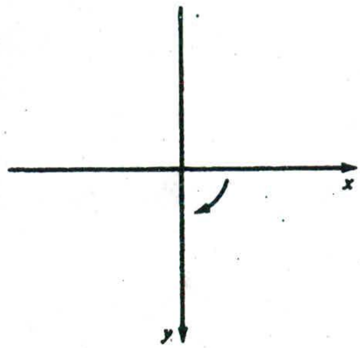
Measuring Distances in Cartesian System

The distance between any two points P_1 and P_2 with coordinates (x_1, y_1) and (x_2, y_2) can be found with the formula

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



(a) Right-handed.



(b) Left-handed.

Fig. A1-1

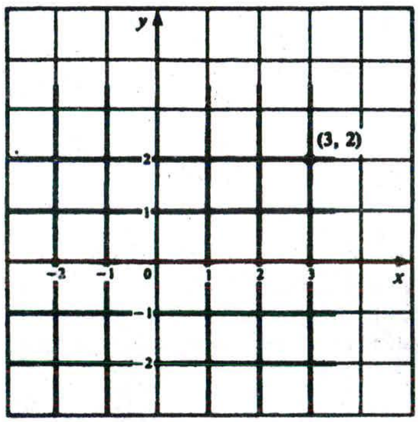


Fig. A1-2

The length of a line segment can be measured by finding the distance between the endpoints of the segment using the formula.

EXAMPLE 1. The length of the line segment joining points $P_0(-1, 2)$ and $P_1(3, 5)$ can be found by

$$D = \sqrt{(5-2)^2 + [3-(-1)]^2} = \sqrt{3^2 + 4^2} = 5$$

Measuring Angles in Cartesian System

The angles of a triangle can be measured in terms of the length of the sides of the triangle (see Fig. A1-3), by using the Law of Cosines, which is stated as

$$c^2 = a^2 + b^2 - 2ab(\cos\theta)$$

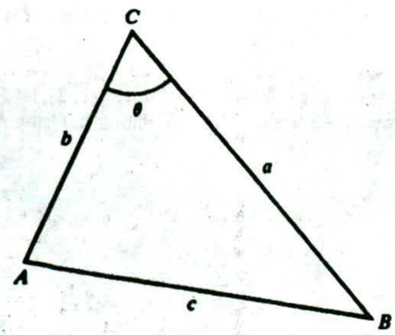


Fig. A1-3

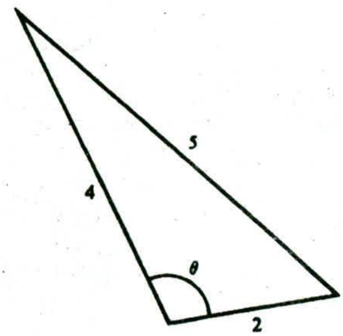


Fig. A1-4

EXAMPLE 2. Refer to Fig. A1-4. To find the angle θ, we use the Law of Cosines:

$$5^2 = 4^2 + 2^2 - 2(4)(2)\cos\theta \quad \text{or} \quad \cos\theta = \frac{-5}{16} \quad \text{so} \quad \theta = 108.21^\circ$$

The angle formed by two intersecting lines can be measured by forming a triangle and applying the Law of Cosines.

Describing a Line in Cartesian System

The line is a basic concept of geometry. In a coordinate system, the description of a line involves an equation which enables us to find the coordinates of all those points which make up the line. The fact that a line is straight is incorporated in the quantity called the *slope m* of the line. Here $m = \tan\theta$, where θ is the angle formed by the line and the positive x axis.

From Fig. A1-5 we see that $\tan\theta = \Delta y/\Delta x$. This gives an alternate formula for the slope: $m = \Delta y/\Delta x$.

EXAMPLE 3. The slope of the line passing through the points $P_0(-1, 2)$ and $P_1(3, 5)$ is found by

$$\Delta y = 5 - 2 = 3 \qquad \Delta x = 3 - (-1) = 4$$

so $m = \Delta y/\Delta x = \frac{3}{4}$. The angle θ is found by $\tan\theta = m = \frac{3}{4}$ or $\theta = 36.87^\circ$.

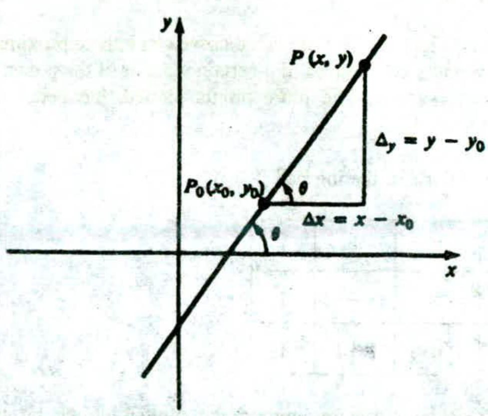


Fig. A1-5

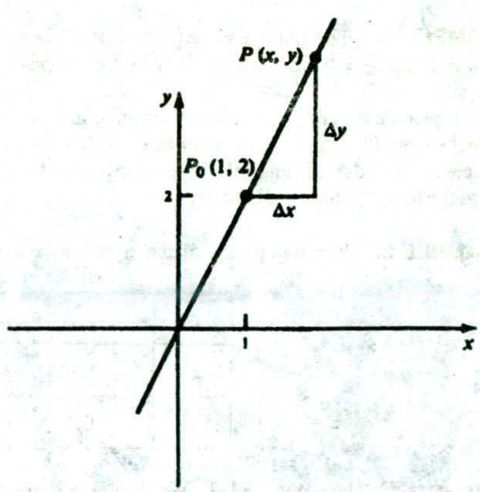


Fig. A1-6

The straightness of a line is expressed by the fact that the slope of the line is the same regardless of which two points are used to calculate it. This enables us to find the equation of a line.

EXAMPLE 4. To find the equation of the line whose slope is 2 and passes through the point $P_0(1, 2)$, let $P(x, y)$ be any point on the line. The slope is the same regardless of which two points are used in calculating it. Using P and P_0, we obtain

$$\Delta y = y - 2 \qquad \Delta x = x - 1$$

so

$$m = \frac{\Delta y}{\Delta x} \qquad \text{or} \qquad 2 = \frac{y-2}{x-1}$$

Solving, we have $y = 2x$ (see Fig. A1-6).

Every line has an equation which can be put in the form $y = mx + b$, where m is the slope of the line and the point $(0, b)$ is the y intercept of the line (the point where the line intercepts the y axis).

Curves and Parametric Equations

The equation of a curve is a mathematical expression which enables us to determine the coordinates of the points that make up the curve.

The equation of a circle of radius r whose center lies at the point (h, k) is

$$(x-h)^2 + (y-k)^2 = r^2$$

It is often more convenient to write the equation of a curve in parametric form; that is

$$x = f(t) \qquad y = g(t)$$

where parameter t might be regarded as representing the "moment" at which the curve arrives at the point (x, y).

The parametric equations of a line can be written in the form (Probs. A1.21 and A1.23)

$$x = at + x_0 \qquad y = bt + y_0$$

EXAMPLE 5. The parametric equation of a circle of radius r and center at the origin $(0, 0)$ can be written as $x = r\cos t$ and $y = r\sin t$, where t lies in the interval $0 \le t \le 2\pi$.

A geometric curve consists of an infinite number of points. Thus any plot of such a curve can only approximate its real shape. Plotting a curve requires the calculation of the x and y coordinates of a certain number of the points of the curve and the placing of these points on the coordinate system. The more points plotted, the better the approximation to the actual shape.

EXAMPLE 6. Plot five points of the equations $x = t$, $y = t^2$ for t in the interval $[-1, 1]$.

t	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
x	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
y	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1

Plotting (x, y) gives Fig. A1-7. We can approximate the actual curve by joining the plotted points by line segments.

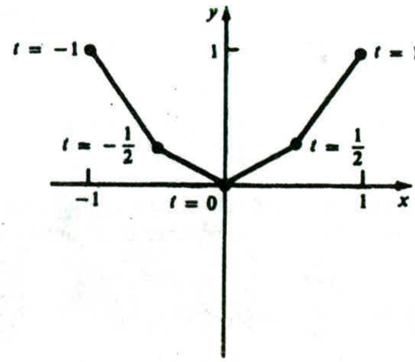


Fig. A1-7

A1.2 THE POLAR COORDINATE SYSTEM

The Cartesian coordinate system is only one of many schemes for attaching coordinates to the points of a plane. Another useful system is the *polar coordinate system*. To develop it, we pick any point in the plane and call it the origin. Through the origin we choose any *ray* (half-line) as the *polar axis*. Any point in the plane can be located at the intersection of a circle of radius r and a ray from the origin making an angle θ with the polar axis (see Fig. A1-8).

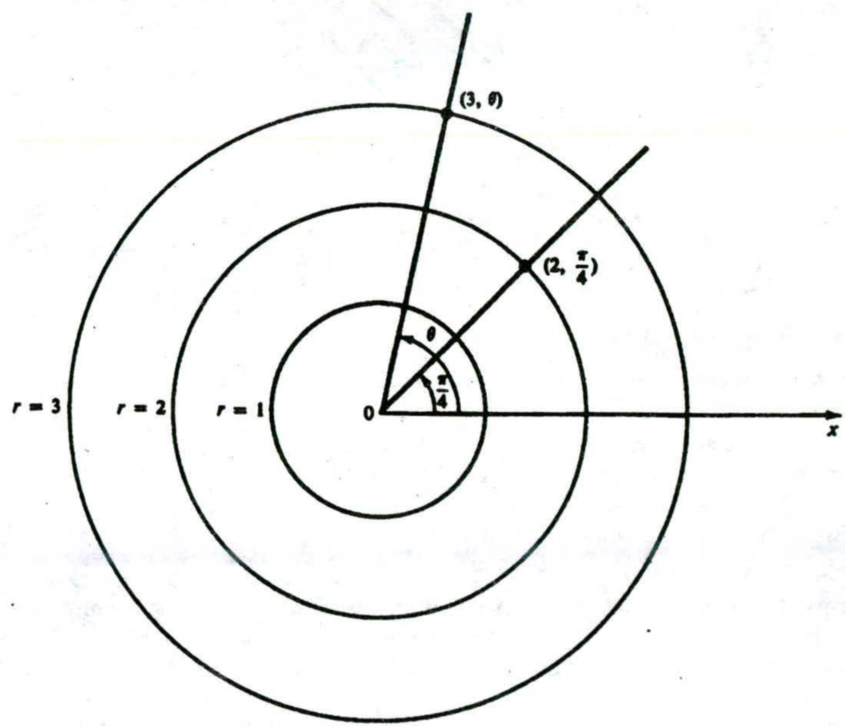


Fig. A1-8

The polar coordinates of a point are given by the pair (r, θ). The polar coordinates of a point are *not* unique. This is because the addition or subtraction of any multiple of 2π (360°) to θ describes the same ray as that described by θ.

Changing Coordinate Systems

How are the Cartesian coordinates of a point related to the polar coordinates of that point? If (r, θ) are the polar coordinates of point P, the Cartesian coordinates (x, y) are given by

$$x = r\cos\theta \qquad y = r\sin\theta$$

Conversely, the polar coordinates of a point whose Cartesian coordinates are known can be found by

$$r^2 = x^2 + y^2 \qquad \theta = \arctan\frac{y}{x}$$

A1.3 VECTORS

Vectors provide a link between geometric reasoning and arithmetic calculation. A vector is represented by a family of directed line segments that all have the same length or magnitude. That is, any two line segments pointing in the same direction and having the same lengths are considered to be the same vector, regardless of their location (see Fig. A1-9).

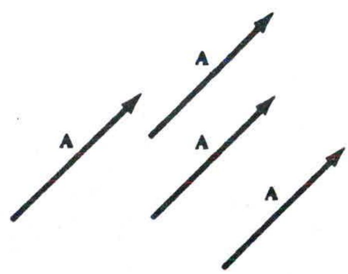


Fig. A1-9

Properties of Vectors

Vectors have special arithmetic properties:

1. If **A** is a vector, then −**A** is a vector with the same length as **A** but pointing in the opposite direction.
2. If **A** is a vector, then k**A** is a vector whose direction is the same as or opposite that of **A**, depending on the sign of the number k, and whose length is k times the length of **A**. This is an example of scalar multiplication.
3. Two vectors can be added together to produce a third vector by using the *parallelogram method* or the *head-to-tail method*. This is an example of vector addition.

In the parallelogram method, vectors **A** and **B** are placed tail to tail. Their sum **A** + **B** is the vector determined by the diagonal of the parallelogram formed by the vectors **A** and **B** (see Fig. A1-10).

In the head-to-tail method, the tail of **B** is placed at the head of **A**. The vector **A** + **B** is determined by the line segment pointing from the tail of **A** to the head of **B** (see Fig. A1-11).

Both methods of addition are equivalent, but the head-to-tail is easier to use when adding several vectors.

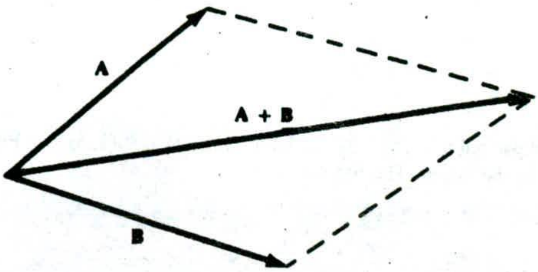


Fig. A1-10

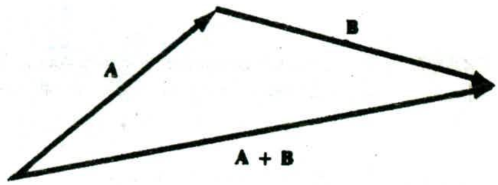


Fig. A1-11

Coordinate Vectors and Components

In a Cartesian coordinate system, vectors having lengths equal to 1 and pointing in the positive direction along the x and y coordinate axes are called the *natural coordinate vectors* and are designated as **I** and **J** (see Fig. A1-12).

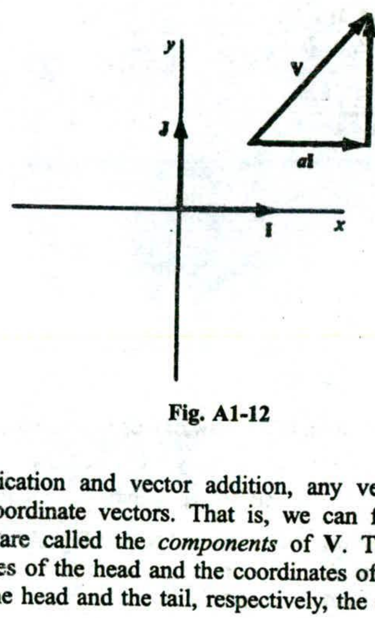


Fig. A1-12

By use of scalar multiplication and vector addition, any vector **V** can be written as a linear combination of the natural coordinate vectors. That is, we can find numbers a and b so that $\mathbf{V} = a\mathbf{I} + b\mathbf{J}$. The numbers $[a, b]$ are called the *components* of **V**. The components of a vector can be determined from the coordinates of the head and the coordinates of the tail of the vector. If (h_x, h_y) and (t_x, t_y) are the coordinates of the head and the tail, respectively, the components of **V** are given by

$$a = h_x - t_x \qquad b = h_y - t_y$$

Notice that if the tail of **V** is placed at the origin, the components of the vector are the coordinates of the head of **V**.

The introduction of components allows us to translate the geometric properties of vectors into computational properties. If the vector **A** has components $[x_1, y_1]$ and the vector **B** has components $[x_2, y_2]$, the length of **A**, denoted as $|\mathbf{A}|$, can be computed by

$$|\mathbf{A}| = \sqrt{x_1^2 + y_1^2}$$

To perform scalar multiplication by a number c, we have

$$c\mathbf{A} = cx_1\mathbf{I} + cy_1\mathbf{J}$$

and to perform vector addition

$$\mathbf{A} + \mathbf{B} = (x_1 + x_2)\mathbf{I} + (y_1 + y_2)\mathbf{J}$$

EXAMPLE 7. Find the components of the vector **A** whose tail is at $P_1(1, 2)$ and whose head is at $P_2(3, 5)$ (see Fig. A1-13). To find the components, we shift the tail of **A** to the origin. The head is at

$$x = 3 - 1 = 2 \qquad y = 5 - 2 = 3$$

Thus $\mathbf{A} = 2\mathbf{I} + 3\mathbf{J}$. The length of **A** is

$$|\mathbf{A}| = \sqrt{2^2 + 3^2} = \sqrt{13}$$

If $\mathbf{B} = -3\mathbf{I} + 2\mathbf{J}$, then $\mathbf{A} + \mathbf{B} = (2 - 3)\mathbf{I} + (3 + 2)\mathbf{J} = -\mathbf{I} + 5\mathbf{J}$.

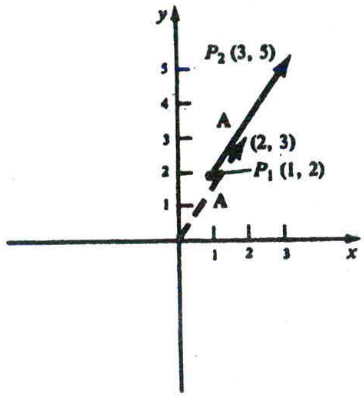


Fig. A1-13

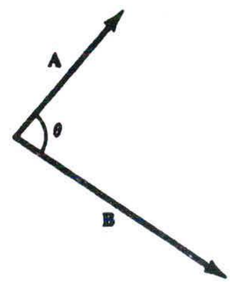


Fig. A1-14

The Dot Product

The dot product $\mathbf{A} \cdot \mathbf{B}$ is the translation of the Law of Cosines into the language of vectors. It is defined as

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta$$

where θ is the smaller angle between the vectors **A** and **B** (see Fig. A1-14). If **A** has components $[x_1, y_1]$ and **B** has components $[x_2, y_2]$, then $\mathbf{A} \cdot \mathbf{B} = x_1x_2 + y_1y_2$ (componentwise multiplication). (*Note*: since $\cos 90° = 0$, two nonzero vectors **A** and **B** are perpendicular if and only if $\mathbf{A} \cdot \mathbf{B} = 0$.)

EXAMPLE 8. To find the angle θ between the vectors $\mathbf{A} = 2\mathbf{I} + 3\mathbf{J}$ and $\mathbf{B} = \mathbf{J}$, we use the definition of the dot product to find

$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|}$$

$$\mathbf{A} \cdot \mathbf{B} = (2\mathbf{I} + 3\mathbf{J}) \cdot (0\mathbf{I} + \mathbf{J}) = 2 \cdot 0 + 3 \cdot 1 = 3$$

$$|\mathbf{A}| = \sqrt{2^2 + 3^2} = \sqrt{13} \qquad |\mathbf{B}| = \sqrt{0^2 + 1^2} = 1$$

So

$$\cos\theta = \frac{3}{\sqrt{13}} \qquad \text{and} \qquad \theta = 33.69°$$

A1.4 MATRICES

A *matrix* is a rectangular array or table of numbers, arranged in rows and columns. The notation a_{ij} is used to designate the matrix entry at the intersection of row i with column j (see Fig. A1-15).

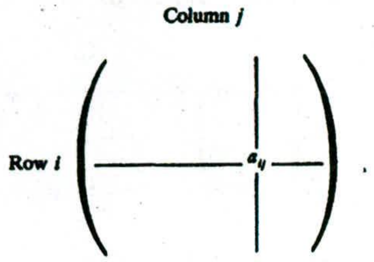


Fig. A1-15

The *size* or *dimension* of a matrix is indicated by the notation $m \times n$, where m is the number of rows in the matrix and n is the number of columns.

A matrix can be used as an organizational tool to represent the information content of data in tabular form. For example, a polygonal figure can be represented as an ordered array of the coordinates of its vertices. The geometric transformations used in computer graphics can also be represented by matrices.

Arithmetic Properties of Matrices

Examples of these properties are as follows.

1. *Scalar multiplication*. The matrix $k\mathbf{A}$ is the matrix obtained by multiplying every entry of **A** by the number k.
2. *Matrix addition*. Two $m \times n$ matrices **A** and **B** can be added together to form a new $m \times n$ matrix **C** whose entries are the sum of the corresponding entries of **A** and **B**. That is,

$$c_{ij} = a_{ij} + b_{ij}$$

3. *Matrix multiplication*. An $m \times p$ matrix **A** can be multiplied by a $p \times n$ matrix **B** to form an $m \times n$ matrix **C**. The entry c_{ij} is found by taking the dot product of the i row of **A** with the j column of **B** (see Fig. A1-16). So $c_{ij} = (\text{row } i) \cdot (\text{column } j) = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}$. Matrix multiplication is not commutative in general. So $\mathbf{AB} \neq \mathbf{BA}$. Matrix multiplication is also called *matrix concatenation*.

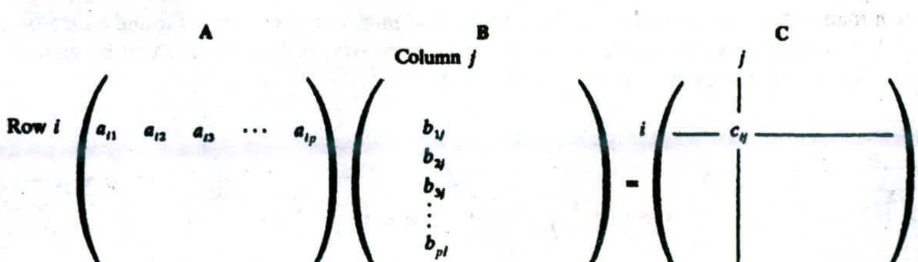


Fig. A1-16

4. *Matrix transpose*. The transpose of a matrix **A** is a matrix, denoted as $\mathbf{A}^T$, formed by exchanging the rows and columns of **A**. If **A** is an $m \times n$ matrix, then $\mathbf{A}^T$ is an $n \times m$ matrix. A matrix is said to be symmetrical if $\mathbf{A} = \mathbf{A}^T$.

Two basic properties of the transpose operation are (1) $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$ and (2) $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

EXAMPLE 9

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 5 \\ 1 & -1 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} -1 & 0 \\ 2 & 3 \\ 1 & 2 \end{pmatrix}$$

$$-2\mathbf{A} = -2\begin{pmatrix} 3 & 2 & 5 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} -6 & -4 & -10 \\ -2 & 2 & -4 \end{pmatrix}$$

$$\mathbf{AB} = \begin{pmatrix} 3 & 2 & 5 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 2 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} [3\cdot(-1)] + (2\cdot 2) + (5\cdot 1) & (3\cdot 0) + (2\cdot 3) + (5\cdot 2) \\ [1\cdot(-1)] + [(-1)\cdot 2] + (2\cdot 1) & (1\cdot 0) + [(-1)\cdot 3] + (2\cdot 2) \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 16 \\ -1 & 1 \end{pmatrix}$$

$$\mathbf{A}^T = \begin{pmatrix} 3 & 1 \\ 2 & -1 \\ 5 & 2 \end{pmatrix}$$

Matrix Inversion and the Identity Matrix

The $n \times n$ matrix whose entries along the main diagonal are all equal to 1 and all other entries are 0 is called the *identity matrix* and is denoted by **I** (Fig. A1-17).

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Fig. A1-17

If **A** is also an $n \times n$ matrix, then $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$. That is, multiplication by the identity matrix **I** leaves the matrix **A** unchanged. Therefore, multiplication by the identity matrix is analogous to multiplication of a real number by 1.

An $n \times n$ matrix **A** is said to be *invertible* or to have an *inverse* if there can be found an $n \times n$ matrix, denoted by $\mathbf{A}^{-1}$, such that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{AA}^{-1} = \mathbf{I}$. The inverse matrix, if there is one, will be unique.

A basic property of matrix inversion is $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

EXAMPLE 10

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{M} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

Then

$$\mathbf{AM} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 0\cdot(-2) & 1\cdot 0 + 0\cdot 1 \\ 2\cdot 1 + 1\cdot(-2) & 2\cdot 0 + 1\cdot 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\mathbf{MA} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 0\cdot 2 & 1\cdot 0 + 0\cdot 1 \\ -2\cdot 1 + 1\cdot 2 & -2\cdot 0 + 1\cdot 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So $\mathbf{MA} = \mathbf{AM} = \mathbf{I}$. Thus $\mathbf{M}$ must be $\mathbf{A}^{-1}$

A1.5 FUNCTIONS AND TRANSFORMATIONS

The concept of a *function* is at the very heart of mathematics and the application of mathematics as a tool for modeling the real world. Stated simply, a function is any process or program which accepts an input and produces a unique output according to a definite rule. Although a function is most often regarded in mathematical terms, this need not be the case. The concept can be usefully extended to include processes described in nonmathematical ways, such as a chemical formula, a recipe or a prescription, and such related concepts as a computer subroutine or a program module. All convey the idea of changing an input to an output. Some synonyms for the word function are *operator*, *mapping*, and *transformation*.

The quantities used as input to the function are collectively called the *domain* of the function. The outputs are called the *range* of the function. Various notations are used to denote functions.

EXAMPLE 11. Some examples of functions are:

1. The equation $f(x) = x^2 + 2x + 1$ is a numerical function whose domain consists of all real numbers and whose range consists of all real numbers greater than or equal to 0.
2. The relationship $T(\mathbf{V}) = 2\mathbf{V}$ is a transformation between vectors. The domain of T is all real vectors, as is the range. This function transforms each vector into a new vector which is twice the original one.
3. The expression $H(x, y) = (x, -y)$ represents a mapping between points of the plane. The domain consists of all points of the plane, as does the range. Each individual point is mapped to that point which is the reflection of the original point about the x axis.
4. If $\mathbf{A}$ is a matrix and $\mathbf{X}$ is a column matrix, the column matrix $\mathbf{Y}$ found by multiplying $\mathbf{A}$ and $\mathbf{X}$ can be regarded as a function $\mathbf{Y} = \mathbf{AX}$.

Graphs of Functions

If x and y are real numbers (scalars), the graph of a function $y = f(x)$ consists of all points in the plane whose coordinates have the form $[x, f(x)]$, where x lies in the domain of f. The graph of a function is the curve associated with the function, and it consists of an infinite number of points. In practice, plotting the graph of a function is done by computing a table of values and plotting the results. This gives an approximation to the actual graph of f.

EXAMPLE 12. Plot five points for the function $y = x^2$ over the interval $[-1, 1]$.

x	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
x^2	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1

Plotting the points (x, x^2) calculated in the table and joining these points with line segments gives an approximation to the actual graph of $y = x^2$. See Fig. A1-7 for the plot of the graph.

The *plotting resolution* is determined by the number of x values used in plotting the graph. The higher the plotting resolution, the better the approximation.

Composing Functions

If the process performed by a function H can be described by the successive steps of first applying a function G and then applying a function F to the results of G, we say that H is the *composition* of F and G. We write $H = F \circ G$. If the input to the function is denoted by x, the output $H(x)$ is evaluated by

$$H(x) = F[G(x)]$$

That is, first G operates on x; then the result $G(x)$ is passed to F as input.

Composition of functions is not commutative in general; that is, $F \circ G \neq G \circ F$.

The concept of composition is not restricted to only two functions but extends to any number of functions. For functions that are represented by matrices, composition of functions is equivalent to matrix multiplication; that is, $\mathbf{A} \circ \mathbf{B} = \mathbf{AB}$.

EXAMPLE 13

1. If $f(x) = x^2 + 2$ and $g(x) = 2x + 1$, then $f[g(x)] = [g(x)]^2 + 2 = (2x + 1)^2 + 2 = 4x^2 + 4x + 3$.
2. If

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} -5 & 4 \\ 2 & 2 \end{pmatrix}$$

then

$$\mathbf{A} \circ \mathbf{B} = \mathbf{AB} = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -5 & 4 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 10 \\ 4 & 4 \end{pmatrix}$$

The Inverse Function

The inverse of a function f (with respect to composition) is a function, denoted by f^{-1}, that satisfies the relationships $f^{-1} \circ f = i$ and $f \circ f^{-1} = i$, where i is the identity function $i(x) = x$. Applying the above compositions to an element x, we obtain the equivalent statements:

$$f^{-1}[f(x)] = x \qquad f[f^{-1}(x)] = x$$

The inverse operator thus "undoes" the work that f has performed.

Not every function has an inverse, and it is often very difficult to tell whether a given function has an inverse. One must often rely on geometric intuition to establish the inverse of an operator.

EXAMPLE 14. Let R be the transformation which rotates every point in the plane by an angle of 30° (in the positive or counterclockwise direction). Then it is clear that R^{-1} is the transformation that rotates every point by an angle of −30° (a rotation of 30° in the clockwise direction).

Solved Problems

A1.1 Find the distance between the points whose coordinates are (*a*) (5, 2) and (7, 3), (*b*) (−3, 1) and (5, 2), (*c*) (−3, −1) and (−5, −2), and (*d*) (0, 1) and (2, 0).

SOLUTION

(*a*) $D = \sqrt{(7-5)^2 + (3-2)^2} = \sqrt{2^2 + 1^2} = \sqrt{5}$

(*b*) $D = \sqrt{[5-(-3)]^2 + (2-1)^2} = \sqrt{(8)^2 + (1)^2} = \sqrt{65}$

(c) $D=\sqrt{[-5-(-3)]^2+[-2-(-1)]^2}=\sqrt{(-2)^2+(-1)^2}=\sqrt{5}$

(d) $D=\sqrt{(2-0)^2+(0-1)^2}=\sqrt{2^2+(-1)^2}=\sqrt{5}$

A1.2 Derive the equation for a straight line (see Fig. A1-5).

SOLUTION

A straight line never changes direction. We determine the direction of a line by the angle θ the line makes with the positive x axis. Then at any point P_0 on the line, the angle formed by the line and a segment through P parallel to the x axis is also equal to θ. Let $P_0(x_0, y_0)$ be a point on the line. Then if $P(x, y)$ represents any point on the line, drawing the right triangle with hypotenuse $\overline{P_0P}$, we find

$$\tan\theta=\frac{y-y_0}{x-x_0}$$

The quantity $\tan\theta$ is called the slope of the line and its traditionally denoted by m.

We rewrite the equation as

$$m=\frac{y-y_0}{x-x_0} \qquad \text{or} \qquad m=\frac{\Delta y}{\Delta x}$$

(The term Δy is often called the "rise" and Δx, the "run.") This can be solved for y in terms of x.

A1.3 Write the equation of the line whose slope is 2 and which passes through the point $(-1, 2)$.

SOLUTION

Let $P(x, y)$ represent any point on the line. Then

$$\Delta y=y-2 \qquad \Delta x=x-(-1)=x+1$$

and $m=2$. Using $\Delta y/\Delta x=m$, we find

$$\frac{y-2}{x+1}=2 \qquad \text{or} \qquad y-2=2(x+1)=2x+2$$

thus $y=2x+4$.

A1.4 Write the equation of the line passing through $P_1(1, 2)$ and $P_2(3, -2)$.

SOLUTION

Let $P(x, y)$ represent any point on the line. Then using P_1, we compute

$$\Delta y=y-2 \qquad \Delta x=x-1$$

To find the slope m, we use P_1 and P_2 to find

$$\Delta y=-2-2=-4 \qquad \Delta x=3-1=2$$

Then

$$m=\frac{\Delta y}{\Delta x}=-2 \qquad \text{so} \qquad \frac{y-2}{x-1}=-2$$

Then

$$y-2=-2x+2 \qquad \text{and} \qquad y=-2x+4$$

A1.5 Show that lines are parallel if and only if their slopes are equal.

SOLUTION

Refer to Fig. A1-18. Suppose that lines l_1 and l_2 are parallel. Then the alternate interior angles θ_1 and θ_2 are equal, and so are the slopes $\tan\theta_1$ and $\tan\theta_2$.

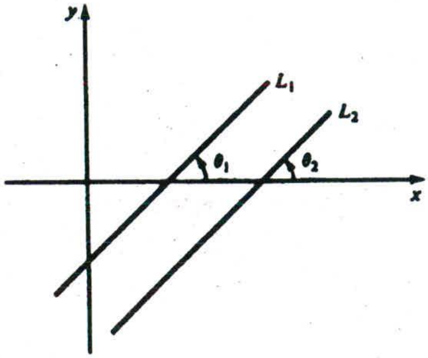


Fig. A1-18

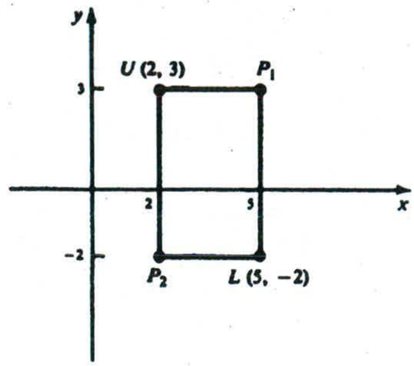


Fig. A1-19

Conversely, if the slopes $\tan\theta_1$ and $\tan\theta_2$ are equal, so are the alternate interior angles θ_1 and θ_2. Consequently lines l_1 and l_2 are parallel.

A1.6 Let $U(2, 3)$ and $L(5, -2)$ be the upper left and lower right corners, respectively, of a rectangle whose sides are parallel to the x and y axes. Find the coordinates of the remaining two vertices.

SOLUTION

Referring to Fig. A1-19, we see that the x coordinate of P_1 is the same as that of L, namely 5, and the y coordinate that of U, namely 3. So $P_1 = (5, 3)$. Similarly, $P_2 = (2, -2)$.

A1.7 Plot the points $A(1, 1)$, $B(-1, 1)$, and $C(-4, 2)$. Then (*a*) show that ABC is a right triangle and (*b*) find a fourth point D such that $ABCD$ is a rectangle (see Fig. A1-20).

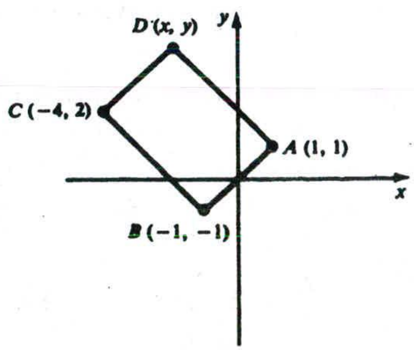


Fig. A1-20

SOLUTION

(*a*) Show that the Pythagorean theorem is satisfied:

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$$

Use the distance formula to compute the lengths of the sides of ABC:

$$\overline{AB} = \sqrt{[1-(-1)]^2 + [1-(-1)]^2} = \sqrt{2^2 + 2^2} = \sqrt{8}$$
$$\overline{BC} = \sqrt{[-1-(-4)]^2 + (-1-2)^2} = \sqrt{(3)^2 + (-3)^2} = \sqrt{18}$$
$$\overline{AC} = \sqrt{[1-(-4)]^2 + (1-2)^2} = \sqrt{5^2 + (-1)^2} = \sqrt{26}$$

So

$$\overline{AC}^2 = 26 = \overline{AB}^2 + \overline{BC}^2 = 8 + 18$$

(*b*) Let the unknown coordinates of D be denoted by (x, y). Use the fact that opposite sides of a rectangle are parallel to find x and y. Since parallel lines have equal slopes, compute the slopes of all four sides:

$$\text{Slope } \overline{AB} = \frac{-1-1}{-1-1} = \frac{-2}{-2} = 1 \qquad \text{Slope } \overline{CD} = \frac{y-2}{x-(-4)} = \frac{y-2}{x+4}$$
$$\text{Slope } \overline{BC} = \frac{-1-(2)}{-1-(-4)} = \frac{-3}{3} = -1 \qquad \text{Slope } \overline{DA} = \frac{y-1}{x-1}$$

Then, for $ABCD$ to be a rectangle

$$\text{Slope } \overline{CD} = \text{slope } \overline{AB} \qquad \text{Slope } \overline{DA} = \text{slope } \overline{BC}$$

or

$$\frac{y-2}{x+4} = 1 \quad \text{and} \quad \frac{y-1}{x-1} = -1$$

This leads to the equations

$$y - 2 = x + 4 \quad \text{and} \quad y - 1 = -x + 1$$

or

$$-x + y = 6 \quad \text{and} \quad x + y = 2$$

Solving, $x = -2$ and $y = 4$.

A1.8 Find the equation of a circle that has radius r and its center at the point (h, k).

SOLUTION

Refer to Fig. A1-21. If $P(x, y)$ is any point lying on the circle, its distance from the center of the circle must be equal to r. Using the distance formula to express this mathematically, we have

$$D = \sqrt{(x-h)^2 + (y-k)^2} = r$$

So $(x-h)^2 + (y-k)^2 = r^2$, which is the equation of the circle.

A1.9 Given any three points, not all lying on a line, find the equation of the circle determined by them.

SOLUTION

Refer to Fig. A1-22. Let $P_1(a_1, b_1)$, $P_2(a_2, b_2)$, and $P_3(a_3, b_3)$ be the coordinates of the points. Let r be the radius of the circle and (h, k) the center. Since each point is distance r from the center, then

$$(a_1 - h)^2 + (b_1 - k)^2 = r^2$$
$$(a_2 - h)^2 + (b_2 - k)^2 = r^2$$
$$(a_3 - h)^2 + (b_3 - k)^2 = r^2$$

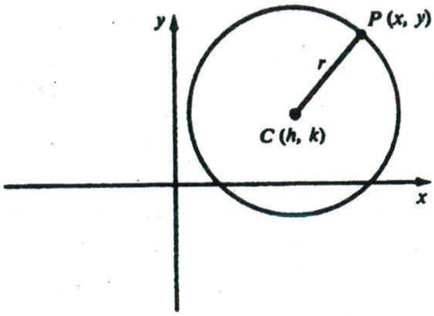


Fig. A1-21

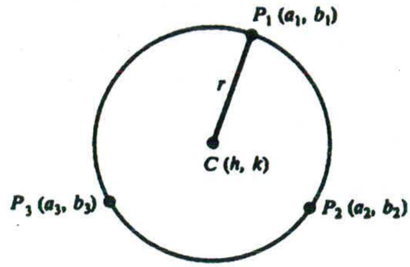


Fig. A1-22

This yields, after multiplying and collecting like terms:

$$(a_2 - a_1)h + (b_2 - b_1)k = \frac{a_2^2 - a_1^2}{2} + \frac{b_2^2 - b_1^2}{2}$$
$$(a_3 - a_2)h + (b_3 - b_2)k = \frac{a_3^2 - a_2^2}{2} + \frac{b_3^2 - b_2^2}{2}$$

These equations can be solved for h and k to yield

$$h = \frac{1}{2}\frac{[d_1^2(b_2 - b_3) + d_2^2(b_3 - b_1) + d_3^2(b_1 - b_2)]}{d}$$
$$k = \frac{-1}{2}\frac{[d_1^2(a_2 - a_3) + d_2^2(a_3 - a_1) + d_3^2(a_1 - a_2)]}{d}$$

Here, $d_1^2 = a_1^2 + b_1^2$, $d_2^2 = a_2^2 + b_2^2$, $d_3^2 = a_3^2 + b_3^2$, and $d = a_1(b_2 - b_3) + a_2(b_3 - b_1) + a_3(b_1 - b_2)$. Finally, r can be found:

$$r = \sqrt{(a_1 - h)^2 + (b_1 - k)^2}$$

A1.10 Find the equation of the circle passing through the three points $P_1(1, 2)$, $P_2(3, 0)$, and $P_3(0, -4)$.

SOLUTION

As in Prob. A1.9, we find

$$\begin{array}{lll} d_1^2 = a_1^2 + b_1^2 = 5 & a_2 - a_3 = 3 & b_2 - b_3 = 4 \\ d_2^2 = a_2^2 + b_2^2 = 9 & a_3 - a_1 = -1 & b_3 - b_1 = -6 \\ d_3^2 = a_3^2 + b_3^2 = 16 & a_1 - a_2 = -2 & b_1 - b_2 = 2 \end{array}$$

So

$$d = 1(4) + 3(-6) + 0(2) = -14$$

and

$$h = \frac{-1[5(4) + 9(-6) + 16(2)]}{28} = \frac{2}{28} = \frac{1}{14}$$
$$k = \frac{1[5(3) + 9(-1) + 16(-2)]}{28} = \frac{-26}{28} = \frac{-13}{14}$$

Therefore, the center of the circle is located at

$$\left(\frac{1}{14}, \frac{-13}{14}\right)$$

and the radius is calculated by

$$r = \sqrt{\left(1 - \frac{1}{14}\right)^2 + \left(2 + \frac{13}{14}\right)^2} = \frac{5}{14}\sqrt{74}$$

A1.11 Show that $x = r\cos t$, $y = r\sin t$ are the parametric equations of a circle of radius r whose center is at the origin.

SOLUTION

By Prob. A1.8 we must show that $x^2 + y^2 = r^2$. Using the trigonometric identity $\cos^2 t + \sin^2 t = 1$, we obtain

$$x^2 + y^2 = (r\cos t)^2 + (r\sin t)^2 = r^2\cos^2 t + r^2\sin^2 t = r^2(\cos^2 t + \sin^2 t) = r^2$$

A1.12 Show that the parametric equations

$$x = \frac{a + bt}{e + ft} \qquad y = \frac{c + dt}{e + ft}$$

are the equations of a line in the plane.

SOLUTION

We show that the slope

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

is a constant, independent of the parameter t. So

$$\begin{aligned}\frac{y_2 - y_1}{x_2 - x_1} &= \frac{(c + dt_2)/(e + ft_2) - (c + dt_1)(e + ft_1)}{(a + bt_2)/(e + ft_2) - (a + bt_1)/(e + ft_1)} \\ &= \frac{ce + det_2 + cft_1 + dft_2t_1 - ce - cft_2 - det_1 - dft_2t_1}{ae + aft_1 + bet_2 + bft_2t_1 - ae - aft_2 - ebt_1 - bft_2t_1} \\ &= \frac{de(t_2 - t_1) - cf(t_2 - t_1)}{be(t_2 - t_1) - af(t_2 - t_1)} = \frac{de - cf}{be - af}\end{aligned}$$

So if $be - af \neq 0$, the slope $\Delta y/\Delta x$ is constant, and so this is the equation of a line.

A1.13 Let the equations of a line be given by (Prob. A1.12)

$$x = \frac{1 + t}{1 - t} \qquad y = \frac{2 + t}{1 - t}$$

Then (*a*) plot the line for all values of t, (*b*) plot the line segment over the interval $[0, 2]$, and (*c*) find the slope of the line.

SOLUTION

Making a table of values, we have

t	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	2	3
x	0	$\frac{1}{3}$	1	3	-3	-2
y	$\frac{1}{2}$	1	2	5	-4	$-\frac{5}{2}$

The resulting line is shown in Fig. A1-23.

(*a*) We observe the following: (1) the line is undefined at $t = 1$, (2) $(x, y) \to (\infty, \infty)$ as $t \to 1^-$, (3) $(x, y) \to (-\infty, -\infty)$ as $t \to 1^+$, and (4) $(x, y) \to (-1, -1)$ as $t \to \pm\infty$ (see Fig. A1-23).

(*b*) The interval $[0, 2]$ includes the infinite point at $t = 1$. The corresponding region is the exterior line segment between points $P_1(1, 2)$ at $t = 0$ and $P_2(-3, -4)$ at $t = 2$ (see Fig. A1-24).

(*c*) From Prob. A1.12, the slope of the line is, with $a = 1$, $b = 1$, $c = 2$, $d = 1$, $e = 1$, and $f = -1$.

$$\frac{\Delta y}{\Delta x} = \frac{(1)(1) - (2)(-1)}{(1)(1) - (1)(-1)} = \frac{3}{2}$$

A1.14 Let $\mathbf{A} = 2\mathbf{I} + 7\mathbf{J}$, $\mathbf{B} = -3\mathbf{I} + \mathbf{J}$, and $\mathbf{C} = \mathbf{I} - 2\mathbf{J}$. Find (*a*) $2\mathbf{A} - \mathbf{B}$ and (*b*) $-3\mathbf{A} + 5\mathbf{B} - 2\mathbf{C}$.

SOLUTION

Perform the scalar multiplication and then the addition.

(*a*) $$\begin{aligned} 2\mathbf{A} - \mathbf{B} &= 2(2\mathbf{I} + 7\mathbf{J}) - (-3\mathbf{I} + \mathbf{J}) = (4\mathbf{I} + 14\mathbf{J}) + (3\mathbf{I} - \mathbf{J}) \\ &= (4 + 3)\mathbf{I} + (14 - 1)\mathbf{J} = 7\mathbf{I} + 13\mathbf{J} \end{aligned}$$

(*b*) $$\begin{aligned} -3\mathbf{A} + 5\mathbf{B} - 2\mathbf{C} &= -3(2\mathbf{I} + 7\mathbf{J}) + 5(-3\mathbf{I} + \mathbf{J}) - 2(\mathbf{I} - 2\mathbf{J}) \\ &= (-6\mathbf{I} - 21\mathbf{J}) + (-15\mathbf{I} + 5\mathbf{J}) + (-2\mathbf{I} + 4\mathbf{J}) \\ &= (-6 - 15 - 2)\mathbf{I} + (-21 + 5 + 4)\mathbf{J} = -23\mathbf{I} - 12\mathbf{J} \end{aligned}$$

A1.15 Find x and y such that $2x\mathbf{I} + (y - 1)\mathbf{J} = y\mathbf{I} + (3x + 1)\mathbf{J}$.

SOLUTION

Since vectors are equal only if their corresponding components are equal, we solve the equations (1) $2x = y$ and (2) $y - 1 = 3x + 1$. Substituting into equation (2), we have $(2x) - 1 = 3x + 1$ and $-2 = x$ and finally $y = 2x = 2(-2) = -4$, so $x = -2$ and $y = -4$.

A1.16 The tail of vector **A** is located at $P(-1, 2)$, and the head is at $Q(5, -3)$. Find the components of **A**.

SOLUTION

Translate vector **A** so that its tail is at the origin. In this position, the coordinates of the head will be the components of **A**.

Translating P to the origin is the same as subtracting -1 from the x component and 2 from the y component. Thus the new head of **A** will be located at point Q_1, whose coordinates (x_1, y_1) can be found by

$$x_1 = 5 - (-1) = 6 \qquad y_1 = -3 - 2 = -5$$

Thus $\mathbf{A} = 6\mathbf{I} - 5\mathbf{J}$.

A1.17 Given the vectors $\mathbf{A} = \mathbf{I} + 2\mathbf{J}$ and $\mathbf{B} = 2\mathbf{I} - 3\mathbf{J}$, find (*a*) the length, (*b*) the dot product, and (*c*) the angle θ between the vectors.

SOLUTION

(*a*) $|\mathbf{A}| = \sqrt{1^2 + 2^2} = \sqrt{5} \qquad |\mathbf{B}| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$

(*b*) $\mathbf{A} \cdot \mathbf{B} = (\mathbf{I} + 2\mathbf{J}) \cdot (2\mathbf{I} - 3\mathbf{J}) = (1 \cdot 2) + [2 \cdot (-3)] = 2 - 6 = -4$

(*c*) From the definition of the dot product, we can solve for $\cos\theta$:

$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{-4}{\sqrt{5}\sqrt{13}}$$

So $\theta = 119.74°$.

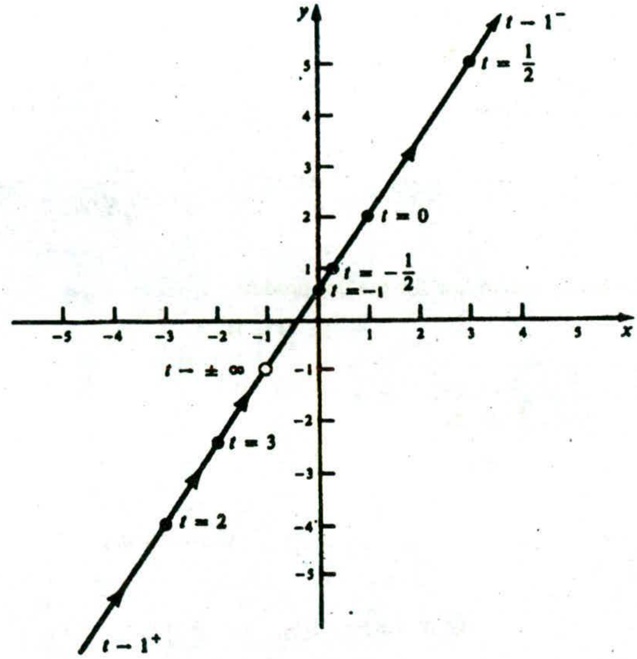


Fig. A1-23

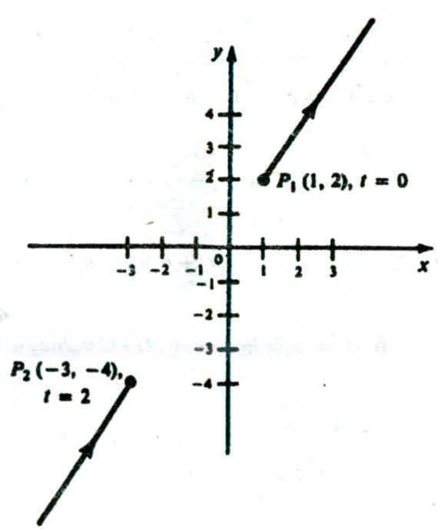


Fig. A1-24

A1.18 Find the unit vector $\mathbf{U_A}$ having the direction of $\mathbf{A} = 2\mathbf{I} - 3\mathbf{J}$.

SOLUTION

Since $\mathbf{U_A} = \dfrac{\mathbf{A}}{|\mathbf{A}|}$, it follows that

$$|\mathbf{A}| = \sqrt{2^2 + (-3)^2} = \sqrt{13} \qquad \text{and} \qquad \mathbf{U_A} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{2\mathbf{I} - 3\mathbf{J}}{\sqrt{13}} = \frac{2}{\sqrt{13}}\mathbf{I} - \frac{3}{\sqrt{13}}\mathbf{J}$$

A1.19 Show that the commutative law for the dot product

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$$

holds for any vectors **A** and **B**.

SOLUTION

Let

$$\mathbf{A} = a_1\mathbf{I} + a_2\mathbf{J} \qquad \mathbf{B} = b_1\mathbf{I} + b_2\mathbf{J}$$

So

$$\mathbf{A} \cdot \mathbf{B} = a_1b_1 + a_2b_2 \qquad \mathbf{B} \cdot \mathbf{A} = b_1a_1 + b_2a_2$$

Comparing both expressions, we see that they are equal.

A1.20 Show that the distributive law for the dot product

$$(\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{C}$$

holds for any vectors **A**, **B**, and **C**.

SOLUTION

Let

$$\mathbf{A} = a_1\mathbf{I} + a_2\mathbf{J} \qquad \mathbf{B} = b_1\mathbf{I} + b_2\mathbf{J} \qquad \mathbf{C} = c_1\mathbf{I} + c_2\mathbf{J}$$

So

$$\mathbf{A} + \mathbf{B} = (a_1 + b_1)\mathbf{I} + (a_2 + b_2)\mathbf{J}$$

and

$$(\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} = (a_1 + b_1)c_1 + (a_2 + b_2)c_2 = a_1c_1 + b_1c_1 + a_2c_2 + b_2c_2$$

On the other hand,

$$\mathbf{A} \cdot \mathbf{C} = a_1c_1 + a_2c_2 \qquad \mathbf{B} \cdot \mathbf{C} = b_1c_1 + b_2c_2$$

so

$$\mathbf{A} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{C} = a_1c_1 + a_2c_2 + b_1c_1 + b_2c_2$$

Comparing both expressions, we see that they are equal.

A1.21 Show that the equation of a line can be determined by specifying a vector **V** having the direction of the line and by a point on the line.

SOLUTION

Suppose that **V** has components $[a, b]$ and the point $P_0(x_0, y_0)$ is on the line (see Fig. A1-25). If $P(x, y)$ is any point on the line, the vector $\overline{\mathbf{P_0P}}$ has the same direction as **V**, and so, by the definition of a vector, it must be a (scalar) multiple of **V**, that is $\overline{\mathbf{P_0P}} = t\mathbf{V}$. The components of $\overline{\mathbf{P_0P}}$ are $[x - x_0, y - y_0]$ and those of $t\mathbf{V}$ are $t[a, b]$. Equating components, we obtain the parametric equations of the line:

$$x - x_0 = ta \qquad y - y_0 = tb \qquad \text{or} \qquad x = at + x_0 \qquad y = bt + y_0$$

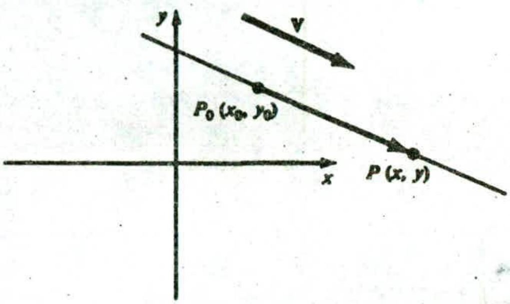


Fig. A1-25

The nonparametric form of the equation can be determined by eliminating the parameter t from both equations. So

$$\frac{x - x_0}{a} = t = \frac{y - y_0}{b}$$

Solving for y, we have

$$y = \frac{b}{a}x + \left(y_0 - \frac{b}{a}x_0\right)$$

A1.22 Find the (a) parametric and (b) nonparametric equation of the line passing through the point $P_0(1, 2)$ and parallel to the vector $\mathbf{V} = 2\mathbf{I} + \mathbf{J}$.

SOLUTION

As in Prob. A1.21, we find, with $a = 2$, $b = 1$, $x_0 = 1$, and $y_0 = 2$, that (a) $x = 2t + 1$, $y = t + 2$ and (b) with $b/a = \frac{1}{2}$, $y = \frac{1}{2}x + (2 - \frac{1}{2}) = \frac{1}{2}x + \frac{3}{2}$.

A1.23 Find the parametric equation of the line passing through points $P_1(1, 2)$ and $P_2(4, 1)$. What is the general form of the parametric equation of a line joining points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$?

SOLUTION

Refer to Fig. A1-26. Choosing $\mathbf{V} = \overline{\mathbf{P_1P_2}} = (4 - 1)\mathbf{I} + (1 - 2)\mathbf{J} = 3\mathbf{I} - 1\mathbf{J}$. Then as in Prob. A1.21, $x = 3t - 1$ and $y = -t - 2$. In the general case, the direction vector **V** is chosen, as above, to be $\overline{\mathbf{P_1P_2}} = (x_2 - x_1)\mathbf{I} + (y_2 - y_1)\mathbf{J}$. The equation of the line is then

$$x = x_1 + (x_2 - x_1)t \qquad y = y_1 + (y_2 - y_1)t$$

A1.24 Find the number c such that the vector $\mathbf{A} = \mathbf{I} + c\mathbf{J}$ is orthogonal to $\mathbf{B} = 2\mathbf{I} - \mathbf{J}$.

SOLUTION

Two nonzero vectors are orthogonal (perpendicular) if and only if their dot product is zero. So

$$\mathbf{A} \cdot \mathbf{B} = (\mathbf{I} + c\mathbf{J}) \cdot (2\mathbf{I} - \mathbf{J}) = (1 \cdot 2) + [c(-1)] = 2 - c$$

So **A** and **B** are orthogonal if $2 - c = 0$ or $c = 2$.

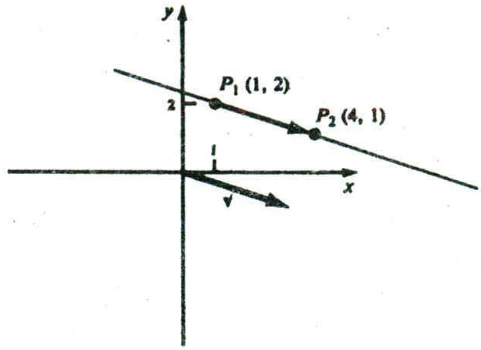


Fig. A1-26

A1.25 Compute:

(*a*) $\begin{pmatrix} 5 & 4 & 1 \\ 0 & -1 & 7 \end{pmatrix} + \begin{pmatrix} 2 & -1 & 3 \\ 2 & 0 & 1 \end{pmatrix}$

(*b*) $\begin{pmatrix} 5 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix} + \begin{pmatrix} 4 & 1 \\ 1 & 5 \end{pmatrix}$

(*c*) $3\begin{pmatrix} 5 & 4 & 1 \\ 0 & -1 & 7 \end{pmatrix}$

SOLUTION

(*a*) Adding corresponding entries, we obtain

$$\begin{pmatrix} 5 & 4 & 1 \\ 0 & -1 & 7 \end{pmatrix} + \begin{pmatrix} 2 & -1 & 3 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5+2 & 4-1 & 1+3 \\ 0+2 & -1+0 & 7+1 \end{pmatrix} = \begin{pmatrix} 7 & 3 & 4 \\ 2 & -1 & 8 \end{pmatrix}$$

(*b*) Since the matrices are of different sizes, we cannot add them.

(*c*) Multiplying each entry by 3, we have

$$3\begin{pmatrix} 5 & 4 & 1 \\ 0 & -1 & 7 \end{pmatrix} = \begin{pmatrix} 15 & 12 & 3 \\ 0 & -3 & 21 \end{pmatrix}$$

A1.26 Let

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 5 & -7 \\ 3 & -2 \end{pmatrix}$$

Find $2\mathbf{A} - 3\mathbf{B}$.

SOLUTION

First multiply, and then add:

$$2\mathbf{A} - 3\mathbf{B} = 2\begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} - 3\begin{pmatrix} 5 & -7 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} -15 & 21 \\ -9 & 6 \end{pmatrix} = \begin{pmatrix} 6-15 & 4+21 \\ 0-9 & 2+6 \end{pmatrix} = \begin{pmatrix} -9 & 25 \\ -9 & 8 \end{pmatrix}$$

A1.27 Determine the size of the following matrix multiplications $\mathbf{A} \cdot \mathbf{B}$, where the sizes of **A** and **B** are given as (*a*) (3×5), (5×2); (*b*) (1×2), (3×1); (*c*) (2×2), (2×1); and (*d*) (2×2), (2×2).

SOLUTION

(*a*) (3×2); (*b*) undefined, since the column size of **A** (2) and the row size of **B** (3) are not equal; (*c*) (2×1); (*d*) (2×2).

A1.28 Find the sizes of **A** and **B** so that **AB** and **BA** can both be computed. Show that, if both **A** and **B** are square matrices of the same size, both **AB** and **BA** are defined.

SOLUTION

Let the size of **A** be $(m \times n)$ and the size of **B** be $(r \times s)$. Then **AB** is defined only if $r = n$. Also, **BA** is defined only if $s = m$. Thus, if **A** is $(m \times n)$, then **B** must be $(n \times m)$. If **A** is square, say, $(n \times n)$, and **B** is also $(n \times n)$, then both **AB** and **BA** are defined.

A1.29 Given

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 5 & 1 \\ 6 & 3 \end{pmatrix}$$

find $\mathbf{A}^T$.

SOLUTION

Exchanging the rows and columns of **A**, we obtain

$$\mathbf{A}^T = \begin{pmatrix} 1 & 5 & 6 \\ 2 & 1 & 3 \end{pmatrix}$$

A1.30 Compute **AB** for

(*a*) $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -4 \\ 7 \end{pmatrix}$

(*b*) $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -4 & 5 \\ 7 & 6 \end{pmatrix}$

(*c*) $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -4 & 5 & 9 \\ 7 & 6 & 10 \end{pmatrix}$

SOLUTION

(*a*) Since **A** is (2×2) and **B** is (2×1), then **AB** is (2×1):

$$\mathbf{AB} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -4 \\ 7 \end{pmatrix} = \begin{pmatrix} 2 \cdot (-4) + 3 \cdot 7 \\ 1 \cdot (-4) + 2 \cdot 7 \end{pmatrix} = \begin{pmatrix} 13 \\ 10 \end{pmatrix}$$

(*b*) $$\mathbf{AB} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -4 & 5 \\ 7 & 6 \end{pmatrix} = \begin{pmatrix} 2 \cdot (-4) + 3 \cdot 7 & 2 \cdot 5 + 3 \cdot 6 \\ 1 \cdot (-4) + 2 \cdot 7 & 1 \cdot 5 + 2 \cdot 6 \end{pmatrix} = \begin{pmatrix} 13 & 28 \\ 10 & 17 \end{pmatrix}$$

(*c*) $$\mathbf{A} \cdot \mathbf{B} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -4 & 5 & 9 \\ 7 & 6 & 10 \end{pmatrix} = \begin{pmatrix} 2 \cdot (-4) + 3 \cdot 7 & 2 \cdot 5 + 3 \cdot 6 & 2 \cdot 9 + 3 \cdot 10 \\ 1 \cdot (-4) + 2 \cdot 7 & 1 \cdot 5 + 2 \cdot 6 & 1 \cdot 9 + 2 \cdot 10 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & 28 & 48 \\ 10 & 17 & 29 \end{pmatrix}$$

A1.31 Let

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 5 & 6 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 6 & 2 & 1 \\ 3 & 5 & 8 \end{pmatrix}$$

Find (*a*) **AB** and (*b*) **BA**.

SOLUTION

(*a*) $$\mathbf{AB} = \begin{pmatrix} 3 & 2 \\ 5 & 6 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 6 & 2 & 1 \\ 3 & 5 & 8 \end{pmatrix} = \begin{pmatrix} 3\cdot 6+2\cdot 3 & 3\cdot 2+2\cdot 5 & 3\cdot 1+2\cdot 8 \\ 5\cdot 6+6\cdot 3 & 5\cdot 2+6\cdot 5 & 5\cdot 1+6\cdot 8 \\ 2\cdot 6+1\cdot 3 & 2\cdot 2+1\cdot 5 & 2\cdot 1+1\cdot 8 \end{pmatrix} = \begin{pmatrix} 24 & 16 & 19 \\ 48 & 40 & 53 \\ 15 & 9 & 10 \end{pmatrix}$$

(*b*) $$\mathbf{BA} = \begin{pmatrix} 6 & 2 & 1 \\ 3 & 5 & 8 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 5 & 6 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 6\cdot 3+2\cdot 5+1\cdot 2 & 6\cdot 2+2\cdot 6+1\cdot 1 \\ 3\cdot 3+5\cdot 5+8\cdot 2 & 3\cdot 2+5\cdot 6+8\cdot 1 \end{pmatrix} = \begin{pmatrix} 30 & 25 \\ 50 & 44 \end{pmatrix}$$

A1.32 Find the inverse of $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

SOLUTION

We wish to find a matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ so that

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Multiplying, we have

$$\begin{pmatrix} p+2r & q+2s \\ 3p+4r & 3q+4s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So $p+2r=1$, $q+2s=0$, $3p+4r=0$, and $3q+4s=1$. Solving the first and third equations we find $p=-2$, $r=\frac{3}{2}$. Solving the second and fourth equations gives $q=1$ and $s=-\frac{1}{2}$. So

$$\mathbf{A}^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

A1.33 Let G be the function which multiplies a given vector by 2 and F be the function that adds the vector **b** to a given vector. Find (*a*) $F+G$, (*b*) $F \circ G$, (*c*) $G \circ F$, (*d*) F^{-1}, and (*e*) G^{-1}.

SOLUTION

If **v** is any vector, the functions F and G operate on **v** as $F(\mathbf{v}) = 2\mathbf{v}$ and $G(\mathbf{v}) = \mathbf{v}+\mathbf{b}$.

1. $(F+G)(\mathbf{v}) = F(\mathbf{v}) + G(\mathbf{v}) = (2\mathbf{v}) + (\mathbf{v}+\mathbf{b}) = 3\mathbf{v}+\mathbf{b}$.
2. $(F \circ G)(\mathbf{v}) = F[G(\mathbf{v})] = 2[G(\mathbf{v})] = 2[\mathbf{v}+\mathbf{b}] = 2\mathbf{v}+2\mathbf{b}$.
3. $(G \circ F)(\mathbf{v}) = G[F(\mathbf{v})] = [F(\mathbf{v})]+\mathbf{b} = 2\mathbf{v}+\mathbf{b}$.
4. We can guess that $F^{-1}(\mathbf{v}) = \frac{1}{2}\mathbf{v}$. To check this, we set $F^{-1}[F(\mathbf{v})] = \frac{1}{2}[F(\mathbf{v})] = \frac{1}{2}[2\mathbf{v}] = \mathbf{v}$ and $F[F^{-1}(\mathbf{v})] = 2[F^{-1}(\mathbf{v})] = 2[(\frac{1}{2})\mathbf{v}] = \mathbf{v}$.
5. We can verify that $G^{-1}(\mathbf{v}) = \mathbf{v}-\mathbf{b}$: $G^{-1}[G(\mathbf{v})] = G^{-1}(\mathbf{v}+\mathbf{b}) = (\mathbf{v}+\mathbf{b})-\mathbf{b} = \mathbf{v}$ and $G[G^{-1}(\mathbf{v})] = G^{-1}(\mathbf{v})+\mathbf{b} = (\mathbf{v}-\mathbf{b})+\mathbf{b} = \mathbf{v}$.

A1.34 Show that $\mathbf{A} \circ \mathbf{B} = \mathbf{AB}$ for any two matrices (that can be multiplied together).

SOLUTION

The terms $\mathbf{A} \circ \mathbf{B}$ and **AB** produce the same effect on any column matrix **X**, i.e., $(\mathbf{A} \circ \mathbf{B})(\mathbf{X}) = \mathbf{ABX}$.

Recall that any matrix function $\mathbf{A(X)}$ is defined by $\mathbf{A(X) = AX}$. So

$$(\mathbf{A} \circ \mathbf{B})(\mathbf{X}) = \mathbf{A}[\mathbf{B}(\mathbf{X})] = \mathbf{A}(\mathbf{BX}) = \mathbf{ABX}$$

A1.35 Given that $\mathbf{A}$ is a 2×2 matrix and $\mathbf{b}$ is a vector, show that the function $F(\mathbf{X}) = \mathbf{AX} + \mathbf{b}$, called an affine transformation, can be considered as either a transformation between vectors or as a mapping between points of the plane.

SOLUTION

Suppose that

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

and $\mathbf{b}$ has components $[b_1, b_2]$. If $\mathbf{X}$ is a vector with components $[x_1, x_2]$, then

$$\mathbf{AX} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

can be identified with the vector having components $[a_{11}x_1 + a_{12}x_2, a_{21}x_1 + a_{22}x_2]$. And so $\mathbf{AX} + \mathbf{b}$ is a vector.

If $X = (x_1, x_2)$ is a point of the plane, then as a point mapping, $F(X) = [f_1(X), f_2(X)]$, where the coordinate functions f_1 and f_2 are

$$f_1(X) = a_{11}x_1 + a_{12}x_2 + b_1 \qquad \text{and} \qquad f_2(X) = a_{21}x_1 + a_{22}x_2 + b_2$$

A1.36 Show that for any 2×2 matrix $\mathbf{A}$ and any vector $\mathbf{b}$ the transformation $F(\mathbf{X}) = \mathbf{AX} + \mathbf{b}$ transforms lines into lines.

SOLUTION

Let $x = at + x_0$ and $y = bt + y_0$ be the parametric equations of a line. With $\mathbf{X} = (x, y)$ then

$$\mathbf{AX} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} at + x_0 \\ bt + y_0 \end{pmatrix} = \begin{pmatrix} a_{11}at + a_{11}x_0 + a_{12}bt + a_{12}y_0 \\ a_{21}at + a_{21}x_0 + a_{22}bt + a_{22}y_0 \end{pmatrix}$$

So

$$F(\mathbf{X}) = \mathbf{AX} + \mathbf{b} = \begin{pmatrix} t(a_{11}a + a_{12}b) + (a_{11}x_0 + a_{12}y_0 + b_1) \\ t(a_{21}a + a_{22}b) + (a_{21}x_0 + a_{22}y_0 + b_2) \end{pmatrix}$$

This can be recognized as the parametric equation of a line (Prob. A1.21) passing through the point with coordinates $(a_{11}x_0 + a_{12}y_0 + b_1, a_{21}x_0 + a_{22}y_0 + b_2)$ and having the direction of the vector $\mathbf{v}$ with components $[a_{11}a + a_{12}b, a_{21}a + a_{22}b]$.

A1.37 Show that the transformation $F(\mathbf{X}) = \mathbf{AX} + \mathbf{b}$ transforms a line passing through points P_1 and P_2 into a line passing through $F(P_1)$ and $F(P_2)$.

SOLUTION

As in Prob. A1.23, the parametric equation of the line passing through P_1 and P_2 can be written as

$$x = x_1 + (x_2 - x_1)t \qquad y = y_1 + (y_2 - y_1)t$$

As in Prob. A1.36 with $a = x_2 - x_1$ and $b = y_2 - y_1$, we find that F transforms this line into another line.

Now when $t = 0$, this line passes through the point

$$(a_{11}x_1 + a_{12}y_1 + b_1, a_{21}x_1 + a_{22}y_1 + b_2) = F(P_1)$$

and when $t = 1$, it passes through the point

$$(a_{11}a + a_{12}b + a_{11}x_1 + a_{12}y_1 + b_1, a_{21}a + a_{22}b + a_{21}x_1 + a_{22}y_1 + b_2) = F(P_2)$$

APPENDIX 2

Mathematics for Three-Dimensional Computer Graphics

A2.1 THREE-DIMENSIONAL CARTESIAN COORDINATES

The three-dimensional Cartesian (rectangular) coordinate system consists of a reference point, called the *origin*, and three mutually perpendicular lines passing through the origin. These mutually perpendicular lines are taken to be number lines and are labeled the x, y, and z coordinate axes. The labels are placed on the positive ends of the axes (see Fig. A2-1).

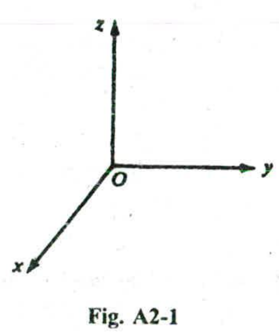


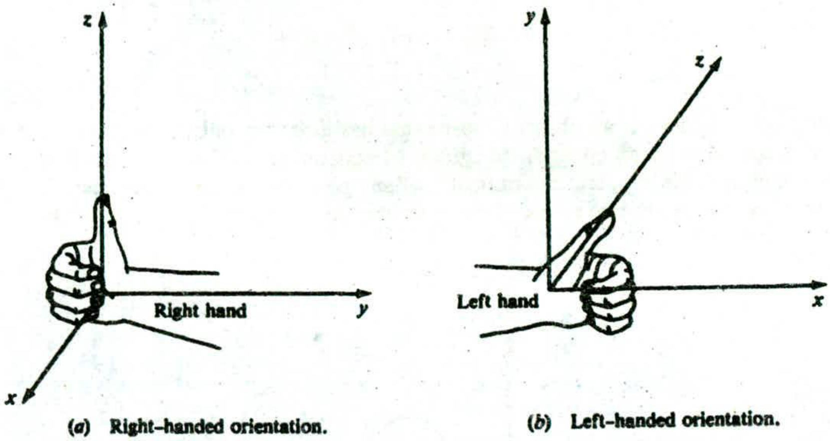
Fig. A2-1

Orientation

The labeling of the x, y, and z axes is arbitrary. However, any labeling falls into one of two classifications, called *right-* and *left-handed orientation*. The orientation is determined by the *right-hand rule*.

The Right-Hand Rule

A labeling of the axes is a *right-handed orientation* if whenever the fingers of the right hand are aligned with the positive x axis and are then rotated (through the smaller angle) toward the positive y axis, then the thumb of the right-hand points in the direction of the positive z axis. Otherwise, the orientation is a *left-handed orientation* (see Fig. A2-2).



(*a*) Right-handed orientation. (*b*) Left-handed orientation.

Fig. A2-2

Cartesian Coordinates of Points in Three-dimensional Space

Any point P in three-dimensional space can have coordinates (x, y, z) associated with it as follows:

1. Let the x coordinate be the directed distance that P is above or below the yz plane.
2. Let the y coordinate be the directed distance that P is above or below the xz plane.
3. Let the z coordinate be the directed distance that P is above or below the xy plane.

See Fig. A2-3.

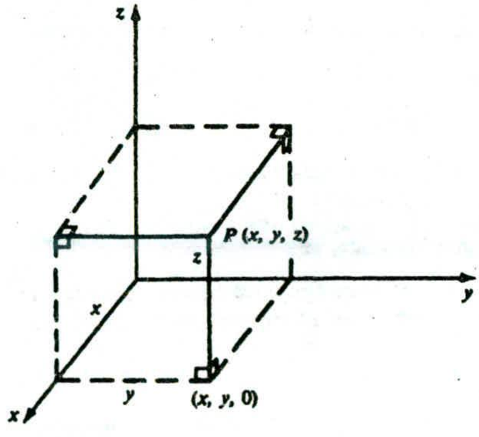


Fig. A2-3

Distance Formula

If $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$ are any two points in space, the distance D between these points is given by the distance formula:

$$D = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}$$

A2.2 CURVES AND SURFACES IN THREE-DIMENSIONS

Curves

A three-dimensional curve is an object in space that has direction only, much like a thread (see Fig. A2-4). A curve is specified by an equation (or group of equations) that has only one free (independent) variable or parameter, and the x, y, and z coordinates of any point on the curve are determined by this free variable or parameter. There are two types of curve description, nonparametric and parametric.

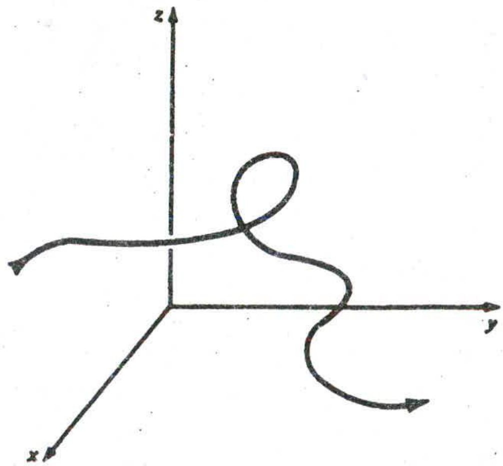


Fig. A2-4

1. *Nonparametric curve description.*

   (*a*) *Explicit form.* The equation for curve C are given in terms of a variable, say, x, as

   $$C: \quad y = f(x) \qquad z = g(x)$$

   That is, y and z can be calculated explicitly in terms of x. Any point P on the curve has coordinates $P[x, f(x), g(x)]$.

   (*b*) *Implicit form.* The equations of the curve are $F(x, y, z) = 0$ and $G(x, y, z) = 0$. Here, y and z must be solved in terms of x.

2. *Parametric curve description.* The three equations for determining the coordinates of any point on the curve are given in terms of an independent parameter, say, t, in a parameter range $[a, b]$, which may be infinite:

$$C: \quad \begin{aligned} x &= f(t) \\ y &= g(t), \\ z &= h(t) \end{aligned} \qquad a \le t \le b$$

Any point P on the curve has coordinates $[f(t), g(t), h(t)]$.

- *Equations of a straight line*. The equations of a line L determined by two points $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$ are given by:

  nonparametric form

$$L: \begin{cases} y = m_1 x + b_1 = \left(\dfrac{y_1 - y_0}{x_1 - x_0}\right)x + \left(\dfrac{y_0 x_1 - y_1 x_0}{x_1 - x_0}\right) \\ z = m_2 x + b_2 = \left(\dfrac{z_1 - z_0}{x_1 - x_0}\right)x + \left(\dfrac{z_0 x_1 - z_1 x_0}{x_1 - x_0}\right) \end{cases}$$

  parametric form

$$x = x_0 + (x_1 - x_0)t \qquad y = y_0 + (y_1 - y_0)t \qquad z = z_0 + (z_1 - z_0)t$$

  Note that when $t = 0$, then $x = x_0$, $y = y_0$, and $z = z_0$. When $t = 1$, then $x = x_1$, $y = y_1$, and $z = z_1$. Thus, when the parameter t is restricted to the range $0 \leq t \leq 1$, the parametric equations describe the line segment $\overline{P_0P_1}$.

Surfaces

A surface in three-dimensional space is an object that has breadth and width, much like a piece of cloth (see Fig. A2-5).

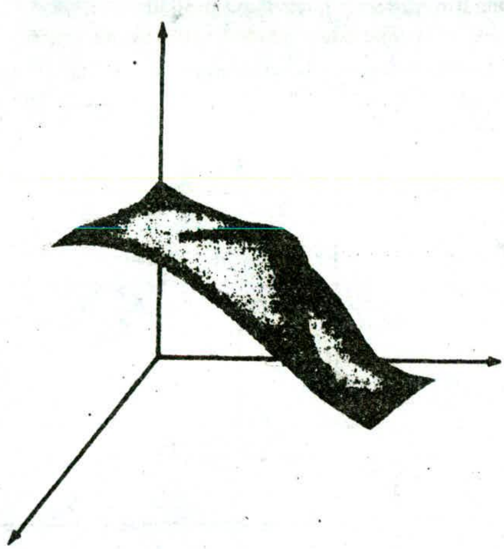


Fig. A2-5

A surface is specified by an equation (or group of equations) that has two free (or independent) variables or parameters. There are two types of surface description, nonparametric and parametrics.

1. *Nonparametric surface description*

   (*a*) *Explicit form*. The z coordinate of any point on the surface S is given in terms of two free variables x and y, that is, $z = f(x, y)$. Any point P on the surface has coordinates $[x, y, f(x, y)]$.

(*b*) *Implicit form*. The equation of the surface is given in the form $F(x, y, z) = 0$. Here, z is to be solved in terms of x and y. There is no restriction as to which variables are free. The convention is to represent z in terms of x and y, but nothing disallows a representation of x in terms of y and z or y in terms of x and z.

2. *Parametric description*. The three equations for determining the coordinates of any point on the surface S are described in terms of parameters, say, s and t, and in parameter ranges $[a, b]$ and $[c, d]$, which may be infinite:

$$S: \begin{array}{ll} x = f(s, t), & a \le s \le b \\ y = g(s, t), & c \le t \le d \\ z = h(s, t) & \end{array}$$

The coordinates of any point P on the surface have the form $[f(s, t), g(s, t), h(s, t)]$.

- *Equations of a plane*. The equation of a plane can be written in explicit form as $z = ax + by + c$ or in implicit form as $Ax + By + Cz + D = 0$ (see Prob. A2.8). The equation of a plane is linear in the variables x, y, and z. A plane divides three-dimensional space into two separate regions. The implicit form of the equation of a plane can be used to determine whether two points are on the same or opposite sides of the plane. Given the implicit equation of the plane $Ax + By + Cz + D = 0$, let $f(x, y, z) = Ax + By + Cz + D$. The two sides of the plane R^+, R^- are determined by the sign of $f(x, y, z)$; that is, point $P(x_0, y_0, z_0)$ lies in region R^+ if $f(x_0, y_0, z_0) > 0$ and in region R^- if $f(x_0, y_0, z_0) < 0$. If $f(x_0, y_0, z_0) = 0$, the point lies on the plane. The equations $x = 0$, $y = 0$, and $z = 0$ represent the yz, xz, and xy planes, respectively.
- *Quadric surfaces*. Quadric surfaces have the (implicit) form $Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$. The basic quadric surfaces are described in Chap. 9.
- *Cylinder surfaces*. In two dimensions, the equation $y = f(x)$ represents a (planar) curve in the xy plane. In three dimensions, the equation $y = f(x)$ is a surface. That is, the variables x and z are free. This type of surface is called a *cylinder surface* (see Fig. A2- 6).

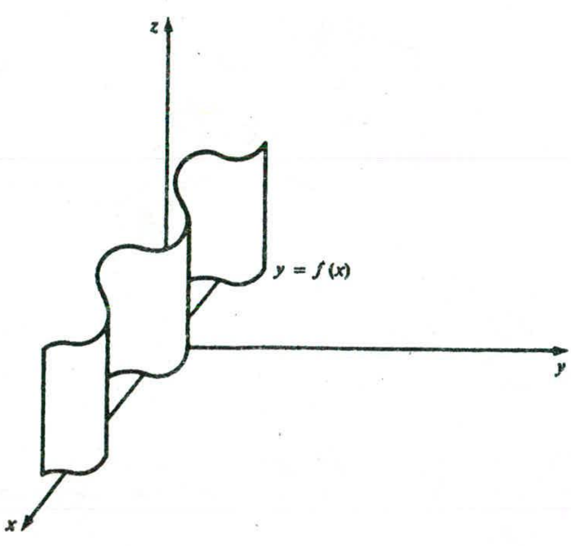


Fig. A2-6

EXAMPLE 1. The equation $x^2 + y^2 = 1$ is a circle in the xy plane. However, in three dimensions, it represents a cylinder (see Fig. A2-7).

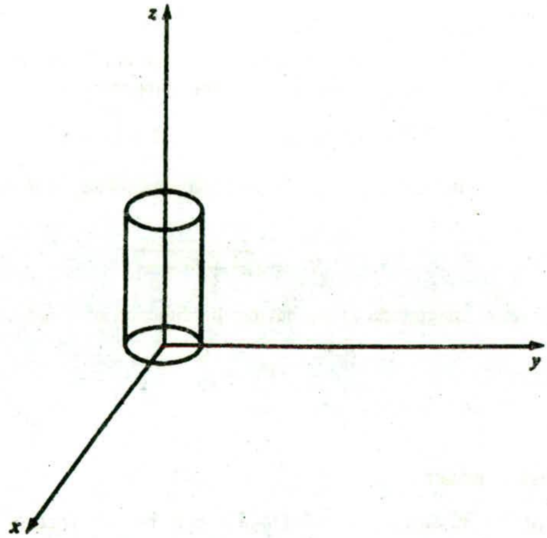


Fig. A2-7

A2.3 VECTORS IN THREE-DIMENSIONS

The definition of a vector and the concepts of magnitude, scalar multiplication, and vector addition are completely analogous to the two-dimensional case in App. 1.

In three-dimensions, there are three natural coordinate vectors **I**, **J**, and **K**. These vectors are unit vectors (magnitude 1) having the direction of the positive x, y, and z axes, respectively. Any vector **V** can be resolved into components in terms of **I**, **J**, and **K**: $\mathbf{V} = a\mathbf{I} + b\mathbf{J} + c\mathbf{K}$.

The components $[a, b, c]$ of vectors **V** are also the Cartesian coordinates of the head of the vector **V** when the tail of **V** is placed at the origin of the Cartesian coordinate system (see Fig. A2-8).

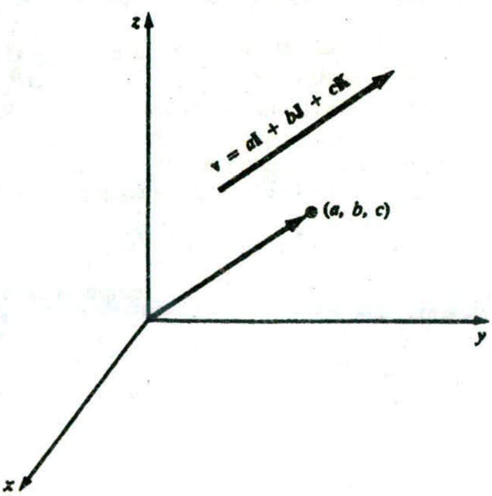


Fig. A2-8

EXAMPLE 2. Let $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$ be two points in space. The directed line segment $\overline{P_0P_1}$ defines a vector whose tail is at P_0 and head is at P_1.

To find the components of $\overline{P_0P_1}$, we must translate so that the tail P_0 is placed at the origin. The head of the vector will then be at the point $(x_1 - x_0, y_1 - y_0, z_1 - z_0)$. The components of $\overline{P_0P_1}$ are then

$$\overline{P_0P_1} = (x_1 - x_0)\mathbf{I} + (y_1 - y_0)\mathbf{J} + (z_1 - z_0)\mathbf{K}$$

Vector addition and scalar multiplication can be performed componentwise, as in App. 1. The magnitude of a vector $\mathbf{V}$, $|\mathbf{V}|$, is given by the formula

$$|\mathbf{V}| = \sqrt{a^2 + b^2 + c^2}$$

For any vector $\mathbf{V}$, a *unit vector* (magnitude 1) $\mathbf{U_V}$ having the direction of $\mathbf{V}$ can be written as

$$\mathbf{U_V} = \frac{\mathbf{V}}{|\mathbf{V}|}$$

The Dot and the Cross Product

Let $\mathbf{V}_1 = a_1\mathbf{I} + b_1\mathbf{J} + c_1\mathbf{K}$ and $\mathbf{V}_2 = a_2\mathbf{I} + b_2\mathbf{J} + c_2\mathbf{K}$ be two vectors.

The *dot* or *scalar product* of two vectors is defined geometrically as $\mathbf{V}_1 \cdot \mathbf{V}_2 = |\mathbf{V}_1||\mathbf{V}_2| \cos\theta$, where θ is the smaller angle between $\mathbf{V}_1$ and $\mathbf{V}_2$ (when the vectors are placed tail to tail). The component form of the dot product can be shown to be

$$\mathbf{V}_1 \cdot \mathbf{V}_2 = a_1a_2 + b_1b_2 + c_1c_2$$

Note that the dot product of two vectors is a number and the order of the dot product is immaterial: $\mathbf{V}_1 \cdot \mathbf{V}_2 = \mathbf{V}_2 \cdot \mathbf{V}_1$. This formula enables us to calculate the angle θ between two vectors from the formula

$$\cos\theta = \frac{\mathbf{V}_1 \cdot \mathbf{V}_2}{|\mathbf{V}_1||\mathbf{V}_2|} = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Note that two vectors are *perpendicular* (*orthogonal*) (i.e., $\theta = 90°$) if and only if their dot product $\mathbf{V}_1 \cdot \mathbf{V}_2 = 0$. This provides a rapid test for determining whether two vectors are perpendicular. (Equivalently, we say that two vectors are *parallel* if they are scalar multiples of each other, i.e, $\mathbf{V}_1 = k\mathbf{V}_2$ for some number k.)

The *cross product* of two vectors, denoted $\mathbf{V}_1 \times \mathbf{V}_2$, produces a new vector defined geometrically as follows: $\mathbf{V}_1 \times \mathbf{V}_2$ is a vector whose magnitude is $|\mathbf{V}_1 \times \mathbf{V}_2| = |\mathbf{V}_1||\mathbf{V}_2| \sin\theta$, where θ is the angle between $\mathbf{V}_1$ and $\mathbf{V}_2$ and whose direction is determined by the right-hand rule: $\mathbf{V}_1 \times \mathbf{V}_2$ is a vector perpendicular to both $\mathbf{V}_1$ and $\mathbf{V}_2$ and whose direction is that of the thumb of the right hand when the fingers are aligned with $\mathbf{V}_1$ and rotated toward $\mathbf{V}_2$ through the smaller angle (see Fig. A2-9).

From this definition, we see that the order in which the cross product is performed is relevant. In fact:

$$\mathbf{V}_1 \times \mathbf{V}_2 = -(\mathbf{V}_2 \times \mathbf{V}_1)$$

Note also that $\mathbf{V} \times \mathbf{V} = \mathbf{0}$ for any vector $\mathbf{V}$, since $\theta = 0°$. The component form for the cross product can be calculated as a determinant as follows:

$$\mathbf{V}_1 \times \mathbf{V}_2 = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}\mathbf{I} - \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}\mathbf{J} + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\mathbf{K}$$

$$= (b_1c_2 - b_2c_1)\mathbf{I} + (c_1a_2 - c_2a_1)\mathbf{J} + (a_1b_2 - a_2b_1)\mathbf{K}$$

EXAMPLE 3. For a right-handed Cartesian coordinate system, we have $\mathbf{I} \times \mathbf{J} = \mathbf{K}$, $\mathbf{J} \times \mathbf{K} = \mathbf{I}$, $\mathbf{I} \times \mathbf{K} = -\mathbf{J}$.

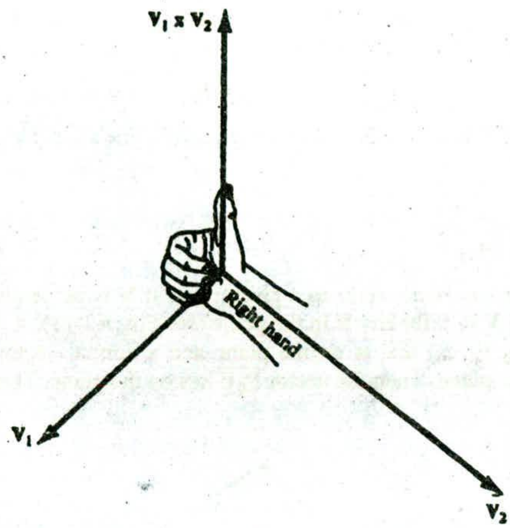


Fig. A2-9

The Vector Equation of a Line

A line L in space is determined by its direction and a point $P_0(x_0, y_0, z_0)$ that the line passes through. If the direction is specified by a vector $\mathbf{V} = a\mathbf{I} + b\mathbf{J} + c\mathbf{K}$ and if $P(x, y, z)$ is any point on the line, the direction of the vector $\overline{P_0P}$ determined by the points P_0, and P is parallel to the vector $\mathbf{V}$ (see Fig. A2-10). Thus, $\overline{P_0P} = t\mathbf{V}$ for some number t.

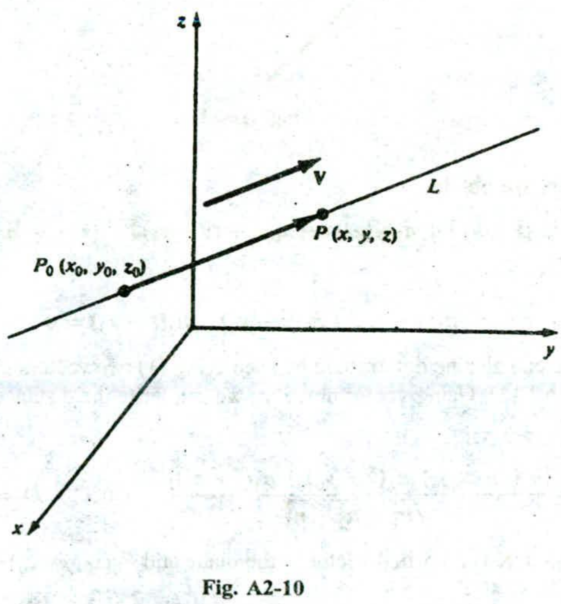


Fig. A2-10

In component form, we find that $(x - x_0)\mathbf{I} + (y - y_0)\mathbf{J} + (z - z_0)\mathbf{K} = ta\mathbf{I} + tb\mathbf{J} + tc\mathbf{K}$. Comparison of components leads to the parametric equations:

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

In Probs. A2.5 and A2.6 it is shown how the equations of a line are determined when given two points on the line.

The Vector Equation of a Plane

A vector **N** is said to be a *normal vector* to a given plane if **N** is perpendicular to any vector **V** which lies on the plane; that is, $\mathbf{N} \cdot \mathbf{V} = 0$ for any **V** in the plane (see Fig. A2-11). A plane is uniquely determined by specifying a point $P_0(x_0, y_0, z_0)$ that is on the plane and a normal vector $\mathbf{N} = n_1\mathbf{I} + n_2\mathbf{J} + n_3\mathbf{K}$. Let $P(x, y, z)$ be any point on the plane. Then the vector $\overline{\mathbf{P_0P}}$ lies on the plane. Therefore, **N** is perpendicular to it. So $\mathbf{N} \cdot \overline{\mathbf{P_0P}} = 0$.

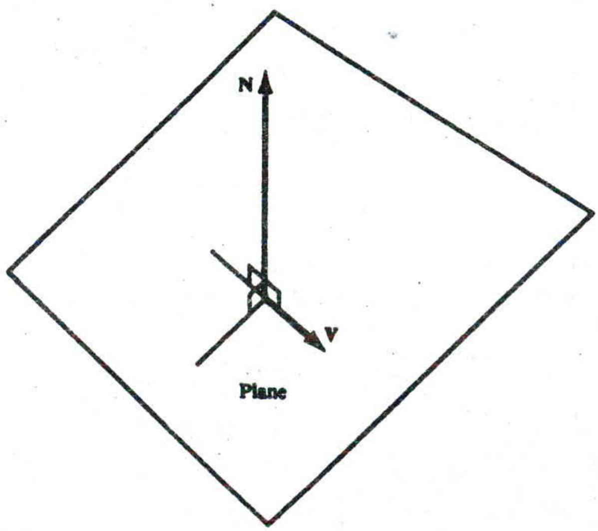


Fig. A2-11

In component form, we obtain

$$[n_1\mathbf{I} + n_2\mathbf{J} + n_3\mathbf{K}] \cdot [(x - x_0)\mathbf{I} + (y - y_0)\mathbf{J} + (z - z_0)\mathbf{K}] = 0$$

or

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$

The equation of a plane can also be determined by specifying (1) two vectors and a point (Prob. A2.10) and (2) three points (Prob. A2.11). Using vector notation, we can write the distance D from a point $\bar{P}(\bar{x}, \bar{y}, \bar{z})$ to a plane as

$$D = \frac{[n_1(\bar{x} - x_0) + n_2(\bar{y} - y_0) + n_3(\bar{z} - z_0)]}{\sqrt{n_1^2 + n_2^2 + n_3^2}} \qquad \text{or} \qquad D = \frac{|\mathbf{N} \cdot \overline{\mathbf{P_0\bar{P}}}|}{|\mathbf{N}|}$$

where $\mathbf{N} = n_1\mathbf{I} + n_2\mathbf{J} + n_3\mathbf{K}$ is a normal vector to the plane and $P_0(x_0, y_0, z_0)$ is a point on the plane (Prob. A2.13).

A2.4 HOMOGENEOUS COORDINATES

The Two-dimensional Projective Plane

The *projective plane* was introduced by geometers in order to study the geometric relationships of figures under perspective transformations.

The two-dimensional projective plane $\mathbf{P}_3$ is defined as follows.

In three-dimensional Cartesian space, consider the set of all lines through the origin and the set of all planes through the origin. In the projective plane, a line through the origin is called a *point* of the projective plane, while a plane through the origin is called a *line* of the projective plane.

To see why this is "natural" from the point of view of a perspective projection, consider the perspective projection onto the plane $z = 1$ using the origin as the center of projection. Then a line through the origin projects onto a point of the plane $z = 1$, while a plane through the origin projects onto a line in the plane $z = 1$ (Fig. A2-12).

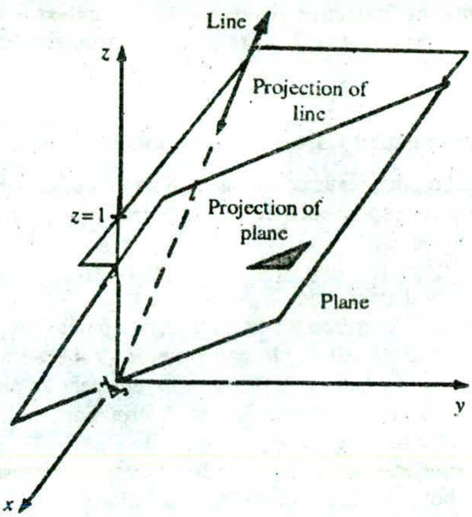


Fig. A2-12

In this projection, lines through points $(x, y, 0)$ in the plane project to infinity. This leads to the notion of ideal points, discussed later.

Homogeneous Coordinates of Points and Lines of the Projective Plane

If (a, b, c) is any point in Cartesian three-dimensional space, this point determines a line through the origin whose equations are

$$x = at$$
$$y = bt \qquad \text{(where } t \text{ is a number)}$$
$$z = ct$$

That is, any other point (at, bt, ct) determines the same line. So two points (a_1, b_1, c_1) and (a_2, b_2, c_2), are on the same line through the origin if there is a number t so that

$$a_2 = a_1 t \qquad b_2 = b_1 t \qquad c_2 = c_1 t \tag{A2.1}$$

We say that two triples, (a_1, b_1, c_1) and (a_2, b_2, c_2), are equivalent (i.e., define the same line through the origin) if there is some number t so that the equations (*A2.1*) hold. We write $(a_1, b_1, c_1) \sim (a_2, b_2, c_2)$. The equivalence classes of all triples equivalent to (a, b, c), written as $[a, b, c]$, are the points of the projective plane. Any representative (a_1, b_1, c_1) equivalent to (a, b, c) is called the *homogeneous coordinate* of the point $[a, b, c]$ in the projective plane.

The points of the form $(a, b, 0)$ are called *ideal points* of the projective plane. This arises from the fact that lines in the plane $z = 0$ project to infinity. In a similar manner, any plane through the origin has an equation $n_1x + n_2y + n_3z = 0$. Note that any multiple $kn_1x + kn_2y + kn_3z = 0$ defines the same plane.

Any triple of numbers (n_1, n_2, n_3) defines a plane through the origin. Two triples are equivalent, $(n_1, n_2, n_3) \sim (d_1, d_2, d_3)$ (i.e., define the same plane), if there is a number k so that $d_1 = kn_1$, $d_2 = kn_2$, and $d_3 = kn_3$. The equivalence classes of all triples, $[n_1, n_2, n_3]$, are the lines of the projective plane. Any representative (d_1, d_2, d_3) of the equivalence class $[n_1, n_2, n_3]$ is called the *homogeneous line coordinate* of this line in the projective plane.

The ambiguity of whether a triple (a, b, c) represents a point or a line of the projection plane is exploited as the Duality Principle of Projective Geometry. If the context is not clear, one usually writes (a, b, c) to indicate a (projective) point and $[a, b, c]$ to indicate a (projective) line.

Correlation between Homogeneous and Cartesian Coordinates

If (x_1, y_1, z_1), $z_1 \neq 0$ are the homogeneous coordinates of a point of the projective plane, the equations $x = x_1/z_1$ and $y = y_1/z_1$ define a correspondence between points $P_1(x_1, y_1, z_1)$ of the projective plane and points $P(x, y)$ of the Cartesian plane.

There is no Cartesian point corresponding to the ideal point $(x_1, y_1, 0)$. However, it is convenient to consider it as defining an infinitely distant point.

Also, any Cartesian point $P(x, y)$ corresponds to a projective point $P(x_1, y_1, z_1)$ whose homogeneous coordinates are $x_1 = x$, $y_1 = y$, and $z_1 = 1$. This correspondence between Cartesian coordinates and homogeneous coordinates is exploited when using matrices to represent graphics transformations. The use of homogeneous coordinates allows the translation transformation and the perspective projection transformation to be represented by matrices (Chaps. 6 and 7).

To conform to the use of homogeneous coordinates, 2×2 matrices representing transformations of the plane can be augmented to use homogeneous coordinates as follows:

$$\mathbf{AX} = \begin{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Finally, note that even though we have a correspondence between the points of the projective plane and those of the Cartesian plane, the projective plane and the Cartesian plane have different topological properties which must be taken into account in work with homogeneous coordinates in advanced applications.

Three-dimensional Projective Plane and Homogeneous Coordinates

Everything stated about the two-dimensional projective plane and homogeneous coordinates may be generalized to the three-dimensional case. For example, if $P_1(x_1, y_1, z_1, w_1)$ are the homogeneous coordinates of a point in the three-dimensional projective plane, the corresponding three-dimensional Cartesian point $P(x, y, z)$ is, for $w_1 \neq 0$,

$$x = \frac{x_1}{w_1} \qquad y = \frac{y_1}{w_1} \qquad z = \frac{z_1}{w_1}$$

In addition, if $P(x, y, z)$ is a Cartesian point, it corresponds to the projective point $P(x, y, z, 1)$. Finally, 3×3 matrices can be augmented to use homogeneous coordinates:

$$\begin{pmatrix} \begin{pmatrix} & & \\ & 3 \times 3 & \\ & & \end{pmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ 0 \quad 0 \quad 0 & 1 \end{pmatrix}$$

Solved Problems

A2.1 Describe the space curve whose parametric equations are $x = \cos t$, $y = \sin t$, and $z = t$.

SOLUTION

Noting that $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$ (see Fig. A2-13), we find that the x, y variables lie on a unit circle, while the z coordinate varies. The curve is a (cylindrical) spiral.

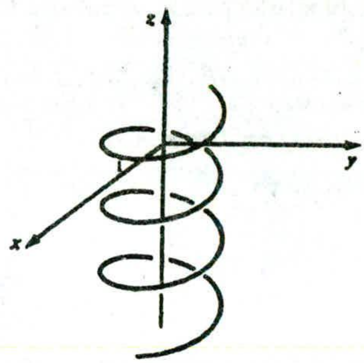


Fig. A2-13

A2.2 Find the equation of a sphere of radius r centered at the origin $(0, 0, 0)$.

SOLUTION

Let $P(x, y, z)$ be any point on the sphere. Then the distance D between this point and the center of the sphere is equal to the length of the radius r. The distance formula yields

$$\sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = r \qquad \text{or} \qquad x^2 + y^2 + z^2 = r^2$$

This is the (implicit) equation of the sphere.

A2.3 Show that $\mathbf{V} \cdot \mathbf{V} = |\mathbf{V}|^2$ for any vector $\mathbf{V}$.

SOLUTION

If $\mathbf{V} = a\mathbf{I} + b\mathbf{J} + c\mathbf{K}$, then

$$\mathbf{V} \cdot \mathbf{V} = (a\mathbf{I} + b\mathbf{J} + c\mathbf{K}) \cdot (a\mathbf{I} + b\mathbf{J} + c\mathbf{K}) = a^2 + b^2 + c^2 = |\mathbf{V}|^2$$

A2.4 Let $\mathbf{V}_1 = 2\mathbf{I} - \mathbf{J} + \mathbf{K}$ and $\mathbf{V}_2 = \mathbf{I} + \mathbf{J} - \mathbf{K}$. Find (*a*) the angle between $\mathbf{V}_1$ and $\mathbf{V}_2$, (*b*) a vector perpendicular to both $\mathbf{V}_1$ and $\mathbf{V}_2$, and (*c*) a unit vector perpendicular to both $\mathbf{V}_1$ and $\mathbf{V}_2$.

SOLUTION

(*a*) We use the formula

$$\cos\theta = \frac{\mathbf{V}_1 \cdot \mathbf{V}_2}{|\mathbf{V}_1||\mathbf{V}_2|}$$

Now

$$|\mathbf{V}_1| = \sqrt{2^2 + (-1)^2 + (1)^2} = \sqrt{6} \qquad |\mathbf{V}_2| = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}$$

and

$$\mathbf{V}_1 \cdot \mathbf{V}_2 = (2)(1) + (-1)(1) + (1)(-1) = 0$$

Thus $\cos\theta = 0$, and so $\theta = 90^\circ$. So the vectors are perpendicular.

(*b*) The vector $\mathbf{V}_1 \times \mathbf{V}_2$ is perpendicular to both $\mathbf{V}_1$ and $\mathbf{V}_2$. So

$$\mathbf{V}_1 \times \mathbf{V}_2 = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ 2 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}\mathbf{I} - \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}\mathbf{J} + \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}\mathbf{K} = -2\mathbf{I} - \mathbf{J} + 3\mathbf{K}$$

(*c*) Since $\mathbf{V}_1 \times \mathbf{V}_2$ is perpendicular to both $\mathbf{V}_1$ and $\mathbf{V}_2$, we find a unit vector having the direction of $\mathbf{V}_1 \times \mathbf{V}_2$. This is

$$\mathbf{U}_{\mathbf{V}_1 \times \mathbf{V}_2} = \frac{\mathbf{V}_1 \times \mathbf{V}_2}{|\mathbf{V}_1 \times \mathbf{V}_2|}$$

From part (*b*), we have

$$|\mathbf{V}_1 \times \mathbf{V}_2| = \sqrt{(-2)^2 + (-1)^2 + (3)^2} = \sqrt{14}$$

So

$$\mathbf{U}_{\mathbf{V}_1 \times \mathbf{V}_2} = \frac{-2}{\sqrt{14}}\mathbf{I} - \frac{1}{\sqrt{14}}\mathbf{J} + \frac{3}{\sqrt{14}}\mathbf{K}$$

A2.5 Find the equation of the line passing through two points $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$.

SOLUTION

To find the equation of a line, we need to know a point on the line and a vector having the direction of the line. The vector determined by P_0 and P_1, $\overline{\mathbf{P_0P_1}}$ clearly has the direction of the line (see Fig. A2-14), and point P_0 lies on the line, so with direction vector

$$\overline{\mathbf{P_0P_1}} = (x_1 - x_0)\mathbf{I} + (y_1 - y_0)\mathbf{J} + (z_1 - z_0)\mathbf{K}$$

and point $P_0(x_0, y_0, z_0)$, the equation is

$$x = x_0 + (x_1 - x_0)t \qquad y = y_0 + (y_1 - y_0)t \qquad z = z_0 + (z_1 - z_0)t$$

A2.6 Find the equation of the line passing through $P_0(1, -5, 2)$ and $P_1(6, 7, -3)$.

SOLUTION

From Prob. A2.5, the direction vector is

$$\overline{\mathbf{P_0P_1}} = (6 - 1)\mathbf{I} + [7 - (-5)]\mathbf{J} + (-3 - 2)\mathbf{K} = 5\mathbf{I} + 12\mathbf{J} - 5\mathbf{K}.$$

Using point $P_0(1, -5, 2)$, we have $x = 1 + 5t$, $y = -5 + 12t$, and $z = 2 - 5t$.

A2.7 Let line segment L_1 be determined by points $P_1(a_1, b_1, c_1)$ and $P_2(a_2, b_2, c_2)$. Let line segment L_2

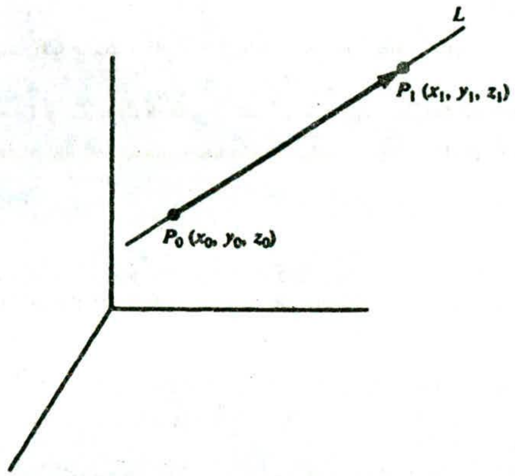


Fig. A2-14

be determined by points $Q_1(u_1, v_1, w_1)$ and $Q_2(u_2, v_2, w_2)$. How can we determine whether the line segments intersect?

SOLUTION

The parametric equations of L_1 are (Prob. A2.5)

$$x = a_1 + (a_2 - a_1)s$$
$$y = b_1 + (b_2 - b_1)s$$
$$z = c_1 + (c_2 - c_1)s$$

The equations of L_2 are

$$x = u_1 + (u_2 - u_1)t$$
$$y = v_1 + (v_2 - v_1)t$$
$$z = w_1 + (w_2 - w_1)t$$

Equating, we find

$$(u_2 - u_1)t - (a_2 - a_1)s = a_1 - u_1$$
$$(v_2 - v_1)t - (b_2 - b_1)s = b_1 - v_1$$
$$(w_2 - w_1)t - (c_2 - c_1)s = c_1 - w_1$$

Using the first two equations, we solve for s and t:

$$t = \frac{(b_1 - v_1)(a_2 - a_1) - (a_1 - u_1)(b_2 - b_1)}{(a_2 - a_1)(v_2 - v_1) - (b_2 - b_1)(u_2 - u_1)}$$

$$s = \frac{(b_1 - v_1)(u_2 - u_1) - (a_1 - u_1)(v_2 - v_1)}{(a_2 - a_1)(v_2 - v_1) - (b_2 - b_1)(u_2 - u_1)}$$

We now substitute the s value into equation L_1 and the t value into equation L_2. If all three corresponding numbers x, y, and z are the same, the lines intersect; if not, the lines do not intersect. Next, if both $0 \le s \le 1$ and $0 \le t \le 1$, the intersection point is on the line segments L_1 and L_2, between P_1 and P_2 and Q_1 and Q_2.

A2.8 Show that the equation of a plane has the implicit form $Ax + By + Cz + D = 0$, where A, B, and C are the components of the normal vector.

SOLUTION

The equation of a plane with normal vector $\mathbf{N} = A\mathbf{I} + B\mathbf{J} + C\mathbf{K}$ and passing through a point $P_0(x_0, y_0, z_0)$ is

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0 \quad \text{or} \quad Ax + By + Cz + (-Ax_0 - By_0 - Cz_0) = 0$$

Calling the quantity $D = (-Ax_0 - By_0 - Cz_0)$ yields the equation of the plane:

$$Ax + By + Cz + D = 0$$

A2.9 Given the plane $5x - 3y + 6z = 7$: (*a*) find the normal vector to the plane, and (*b*) determine whether $P_1(1, 5, 2)$ and $P_2(-3, -1, 2)$ are on the same side of the plane.

SOLUTION

Write the equation in implicit form as $5x - 3y + 6z - 7 = 0$.

(*a*) From Prob. A2.8, the coefficients 5, −3, and 6 are the components of a normal vector, that is, $\mathbf{N} = 5\mathbf{I} - 3\mathbf{J} + 6\mathbf{K}$.

(*b*) Let $f(x, y, z) = 5x - 3y + 6z - 7$. The plane has two sides, R^+ where $f(x, y, z)$ is positive and R^- where $f(x, y, z)$ is negative. Now for point $P_1(1, 5, 2)$, we have

$$f(1, 5, 2) = 5(1) - 3(5) + 6(2) - 7 = -5$$

and for point $P_2(-3, -1, 2)$,

$$f(-3, -1, 2) = 5(-3) - 3(-1) + 6(2) - 7 = -7$$

Since both $f(1, 5, 2)$ and $f(-3, -1, 2)$ are negative, P_1 and P_2 are on the same side of the plane.

A2.10 Find the equation of a plane passing through the point $P_0(1, -1, 1)$ and containing the vectors $\mathbf{V}_1 = \mathbf{I} - \mathbf{J} + \mathbf{K}$ and $\mathbf{V}_2 = -\mathbf{I} + \mathbf{J} + 2\mathbf{K}$ (see Fig. A2-15).

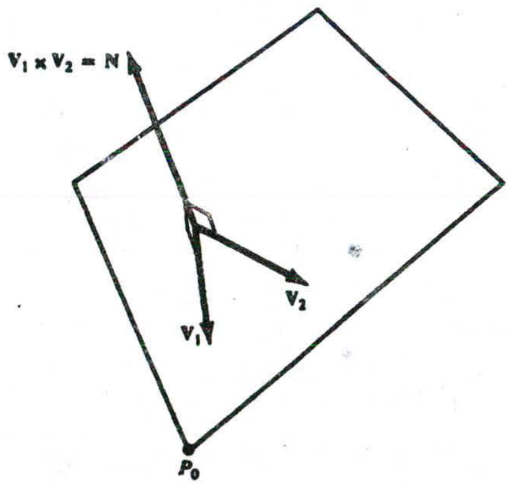


Fig. A2-15

SOLUTION

To find the equation of a plane, we need to find a normal vector perpendicular to the plane. Since $\mathbf{V}_1$ and $\mathbf{V}_2$ are to lie on the plane, the cross product $\mathbf{V}_1 \times \mathbf{V}_2$ perpendicular to both $\mathbf{V}_1$ and $\mathbf{V}_2$ can be chosen to be the normal vector **N** (see Fig. A2-15). So

$$\mathbf{N} = \mathbf{V}_1 \times \mathbf{V}_2 = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ 1 & -1 & 1 \\ -1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix}\mathbf{I} - \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}\mathbf{J} + \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}\mathbf{K} = -3\mathbf{I} - 3\mathbf{J} + 0\mathbf{K}$$

So with $\mathbf{N} = -3\mathbf{I} - 3\mathbf{J}$ and the point $P_0(1, -1, 1)$, the equation of the plane is

$$-3(x-1) - 3[y - (-1)] + 0(z-1) = 0 \qquad \text{or} \qquad -3x - 3y = 0$$

Finally, $x + y = 0$ is the equation of the plane. This is an example of a cylinder surface, since z is a free variable and $y = -x$.

A2.11 Find the equation of the plane determined by the three points $P_0(1, 5, -7)$, $P_1(2, 6, 1)$, and $P_2(0, 1, 2)$ (see Fig. A2-16).

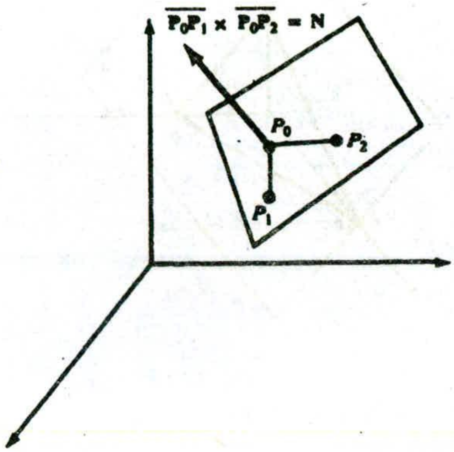


Fig. A2-16

SOLUTION

To find the equation of a plane, we must know a point on the plane and a normal vector perpendicular to the plane.

To find the normal vector, we observe that the vectors $\overline{\mathbf{P_0P_1}}$ and $\overline{\mathbf{P_0P_2}}$ lie on the plane, and so the cross product will be a vector perpendicular to both these vectors and so would be our choice for the normal vector; that is,

$$\mathbf{N} = \overline{\mathbf{P_0P_1}} \times \overline{\mathbf{P_0P_2}}$$

Now

$$\overline{\mathbf{P_0P_1}} = (2-1)\mathbf{I} + (6-5)\mathbf{J} + (1-(-7))\mathbf{K} = \mathbf{I} + \mathbf{J} + 8\mathbf{K}$$

and

$$\overline{\mathbf{P_0P_2}} = (0-1)\mathbf{I} + (1-5)\mathbf{J} + (2-(-7))\mathbf{K} = -\mathbf{I} - 4\mathbf{J} + 9\mathbf{K}$$

So

$$\overline{\mathbf{P_0P_1}} \times \overline{\mathbf{P_0P_2}} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ 1 & 1 & 8 \\ -1 & -4 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 8 \\ -4 & 9 \end{vmatrix}\mathbf{I} - \begin{vmatrix} 1 & 8 \\ -1 & 9 \end{vmatrix}\mathbf{J} + \begin{vmatrix} 1 & 1 \\ -1 & -4 \end{vmatrix}\mathbf{K} = 41\mathbf{I} - 17\mathbf{J} - 3\mathbf{K}$$

So $\mathbf{N} = 41\mathbf{I} - 17\mathbf{J} - 3\mathbf{K}$, and with point $P_0(1, 5, -7)$, the equation of the plane is

$$41(x-1) - 17(y-5) - 3[z-(-7)] = 0 \qquad \text{or} \qquad 41x - 17y - 3z + 23 = 0$$

A2.12 Show that the equation of the plane that has x, y, and z intercepts $A(a, 0, 0)$, $B(0, b, 0)$, and $C(0, 0, c)$, respectively, is (see Fig. A2-17) $x/a + y/b + z/c = 1$.

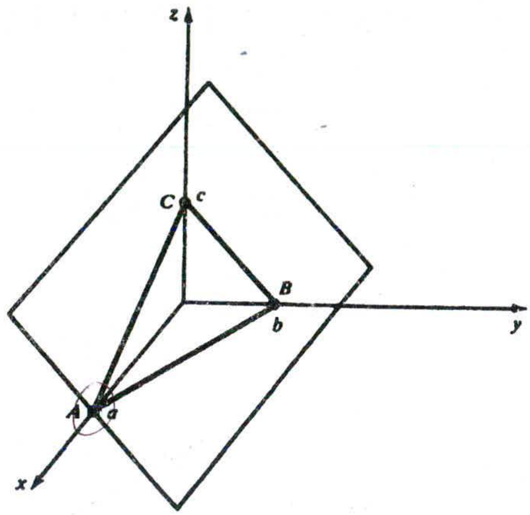


Fig. A2-17

SOLUTION

As in Prob. A2.11, we form the vectors $\overline{\mathbf{AB}} = -a\mathbf{I} + b\mathbf{J}$ and $\overline{\mathbf{AC}} = -a\mathbf{I} + c\mathbf{K}$. The normal vector to the plane is then

$$\mathbf{N} = \overline{\mathbf{AB}} \times \overline{\mathbf{AC}} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = \begin{vmatrix} b & 0 \\ 0 & c \end{vmatrix}\mathbf{I} - \begin{vmatrix} -a & 0 \\ -a & c \end{vmatrix}\mathbf{J} + \begin{vmatrix} -a & b \\ -a & 0 \end{vmatrix}\mathbf{K} = bc\mathbf{I} + ac\mathbf{J} + ab\mathbf{K}$$

The equation of the plane with this normal vector and passing through $A(a, 0, 0)$ is

$$bc(x-a) + ac(y-0) + ab(z-0) = 0 \qquad \text{or} \qquad bcx + acy + abz = abc$$

Dividing both sides by abc, we have $x/a + y/b + z/c = 1$.

A2.13 Find the distance from a point $P_1(x_1, y_1, z_1)$ to a given plane (see Fig. A2-18).

SOLUTION

Let $\mathbf{N} = n_1\mathbf{I} + n_2\mathbf{J} + n_3\mathbf{K}$ be the normal vector to the plane, and let $P_0(x_0, y_0, z_0)$ be any point on the plane. The equation of the plane is

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$

The distance D from $P_1(x_1, y_1, z_1)$ to the plane is measured along the perpendicular or normal to the plane.

Let L_N be the line through $P_1(x_1, y_1, z_1)$ and having the direction of the normal vector $\mathbf{N}$. The equation of

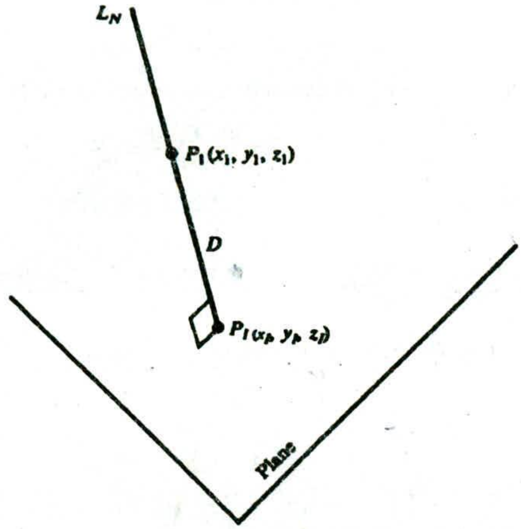


Fig. A2-18

L_N is

$$L_N: \quad \begin{aligned} x &= x_1 + n_1 t \\ y &= y_1 + n_2 t \\ z &= z_1 + n_3 t \end{aligned}$$

We first find the intersection point $P_I(x_I, y_I, z_I)$ of the line L_N with the plane. The distance from the point $P_1(x_1, y_1, z_1)$ to the plane will be the same as the distance from the point $P_1(x_1, y_1, z_1)$ to the intersection point $P_I(x_I, y_I, z_I)$.

Substituting the equations of the line L_N into the equation of the plane, we find

$$n_1(x_1 + n_1 t - x_0) + n_2(y_1 + n_2 t - y_0) + n_3(z_1 + n_3 t - z_0) = 0$$

Solving for t, we have

$$t = -\frac{n_1(x_1 - x_0) + n_2(y_1 - y_0) + n_3(z_1 - z_0)}{n_1^2 + n_2^2 + n_3^2}$$

Calling this number t_I, we find that the coordinates of P_I are

$$x_I = x_1 + n_1 t_I \qquad y_I = y_1 + n_2 t_I \qquad z_I = z_1 + n_3 t_I \tag{A2.2}$$

The distance D from $P(x_1, y_1, z_1)$ to $P_I(x_I, y_I, z_I)$ is

$$D = \sqrt{(x_I - x_1)^2 + (y_I - y_1)^2 + (z_I - z_1)^2}$$

From equation (*A2.2*), we obtain

$$x_I - x_1 = n_1 t_I \qquad y_I - y_1 = n_2 t_I \qquad z_I - z_1 = n_3 t_I$$

Substitution into the formula for D yields

$$D = \sqrt{(n_1 t_I)^2 + (n_2 t_I)^2 + (n_3 t_I)^2} = |t_I|\sqrt{n_1^2 + n_2^2 + n_3^2}$$

or, substituting for t_I

$$D = \frac{|n_1(x_1 - x_0) + n_2(y_1 - y_0) + n_3(z_1 - z_0)|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$$

We can rewrite this in vector form by observing that

$$|\mathbf{N}| = \sqrt{n_1^2 + n_2^2 + n_3^2}$$

and that $(x_1 - x_0, y_1 - y_0, z_1 - z_0)$ are the components of the vector $\overline{\mathbf{P_0P_1}}$. So

$$D = \frac{|\mathbf{N} \cdot \overline{\mathbf{P_0P_1}}|}{|\mathbf{N}|} = \frac{d}{|\mathbf{N}|}$$

where $d = |\mathbf{N} \cdot \overline{\mathbf{P_0P_1}}|$.

A2.14 Find the projection $\mathbf{V_p}$ of a vector $\mathbf{V}$ onto a given plane in the direction of the normal vector $\mathbf{N}$.

SOLUTION

From Fig. A2-19, by the definition of (head-to-tail) vector addition (see App. 1), we have

$$\mathbf{V_p} + k\mathbf{N} = \mathbf{V} \qquad \text{or} \qquad \mathbf{V_p} = \mathbf{V} - k\mathbf{N}$$

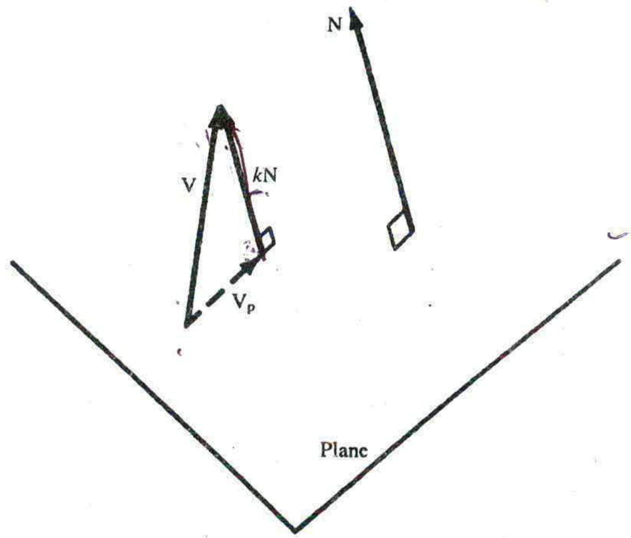


Fig. A2-19

To find the number k, we use the fact that $\mathbf{V_p}$ lies on the plane, so $\mathbf{N}$ is perpendicular to $\mathbf{V_p}$, i.e., $\mathbf{V_p} \cdot \mathbf{N} = 0$. So

$$0 = \mathbf{V_p} \cdot \mathbf{N} = \mathbf{V} \cdot \mathbf{N} - k(\mathbf{N} \cdot \mathbf{N}) \qquad \text{or} \qquad k = \frac{\mathbf{V} \cdot \mathbf{N}}{\mathbf{N} \cdot \mathbf{N}} = \frac{\mathbf{V} \cdot \mathbf{N}}{|\mathbf{N}|^2} \qquad (\text{since } \mathbf{N} \cdot \mathbf{N} = |\mathbf{N}|^2)$$

Then

$$\mathbf{V_p} = \mathbf{V} - \left(\frac{\mathbf{V} \cdot \mathbf{N}}{|\mathbf{N}|^2}\right)\mathbf{N} \tag{A2.3}$$

A2.15 Let a plane be determined by the normal vector $\mathbf{N} = \mathbf{I} - \mathbf{J} + \mathbf{K}$ and a point $P_0(2, 3 - 1)$.

(*a*) Find the distance from point $P_1(5, 2, 7)$ to the plane.

(*b*) Let $\mathbf{V} = 2\mathbf{I} + 3\mathbf{J} - \mathbf{K}$ be a vector. Find the projection of $\mathbf{V}_\mathbf{p}$ (in the direction of the normal) onto the plane.

SOLUTION

(*a*) The vector $\overline{\mathbf{P}_0\mathbf{P}_1} = 3\mathbf{I} - \mathbf{J} + 8\mathbf{K}$. From Prob. A2.13 we have

$$D = \frac{|\mathbf{N} \cdot \overline{\mathbf{P}_0\mathbf{P}_1}|}{|\mathbf{N}|} = \frac{|(1)(3) + (-1)(-1) + (1)(8)|}{\sqrt{(1)^2 + (-1)^2 + (1)^2}} = \frac{12}{\sqrt{3}} = 4\sqrt{3}$$

(*b*) From Prob. A2.14, the projection vector $\mathbf{V}_\mathbf{p}$ is given by

$$\mathbf{V}_\mathbf{p} = \mathbf{V} - \left(\frac{\mathbf{V} \cdot \mathbf{N}}{|\mathbf{N}|^2}\right)\mathbf{N}$$

Now

$$\frac{\mathbf{V} \cdot \mathbf{N}}{|\mathbf{N}|^2} = \frac{(2)(1) + (3)(-1) + (-1)(1)}{(1)^2 + (-1)^2 + (1)^2} = \frac{-2}{3}$$

So

$$\mathbf{V}_\mathbf{p} = (2\mathbf{I} + 3\mathbf{J} - \mathbf{K}) - (-\tfrac{2}{3})(\mathbf{I} - \mathbf{J} + \mathbf{K})$$
$$= (2\mathbf{I} + 3\mathbf{J} - \mathbf{K}) - (-\tfrac{2}{3}\mathbf{I} + \tfrac{2}{3}\mathbf{J} - \tfrac{2}{3}\mathbf{K}) = \tfrac{8}{3}\mathbf{I} + \tfrac{7}{3}\mathbf{J} - \tfrac{1}{3}\mathbf{K}$$

A2.16 Given vectors **A** and **B** that are placed tail to tail we define the perpendicular projection of **A** onto **B** to be the vector **V** shown in Fig. A2-20. Find a formula for computing **V** from **A** and **B**.

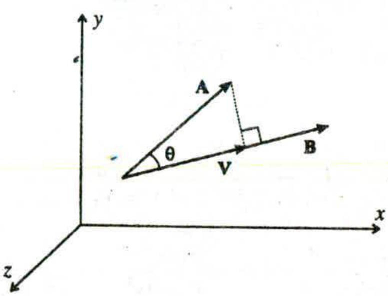


Fig. A2-20

SOLUTION

We first find (see Fig. A2-20)

$$|\mathbf{V}| = |\mathbf{A}| \cos(\theta) = |\mathbf{A}| \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{B}|}$$

Using the unit vector

$$\mathbf{V} = |\mathbf{V}|\mathbf{U}_\mathbf{V}$$

Since **V** and **B** have the same direction, we have $\mathbf{U}_\mathbf{V} = \mathbf{U}_\mathbf{B}$. Hence

$$\mathbf{V} = |\mathbf{V}|\mathbf{U}_\mathbf{B} = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{B}|} \frac{\mathbf{B}}{|\mathbf{B}|} = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{B}|^2}\mathbf{B}$$

A2.17 Let $(3, 1, -3)$ be the coordinate of point A. Find a point B on the line $y = 2x$ in the xy plane such that the line connecting A and B is perpendicular to $y = 2x$.

SOLUTION 1

Since point B is on $y = 2x$, it has coordinates $(x, 2x, 0)$. We introduce two vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ (see Fig. A2-21):

$$\mathbf{V}_1 = x\mathbf{I} + 2x\mathbf{J}$$
$$\mathbf{V}_2 = (3 - x)\mathbf{I} + (1 - 2x)\mathbf{J} - 3\mathbf{K}$$

The line connecting A and B is perpendicular to $y = 2x$ if

$$\mathbf{V}_1 \cdot \mathbf{V}_2 = 0$$

or

$$x(3 - x) + 2x(1 - 2x) = 0$$

This yields $x(1 - x) = 0$. Since the angle between the line $y = 2x$ and the line from the origin to point A is verifiably not 90°, we have $x \neq 0$. Hence $x = 1$, and the coordinates of point B are $(1, 2, 0)$.

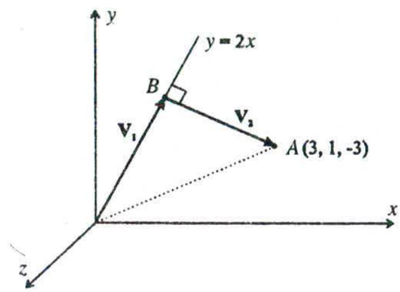


Fig. A2-21

SOLUTION 2

Referring to Fig. A2-21, let $\mathbf{A}$ be a vector whose tail is at the origin and whose head is at point A. From Prob. A2.16, we can see that $\mathbf{V}_1$ is simply the perpendicular projection of $\mathbf{A}$ onto $\mathbf{V}_1$ itself. Since $\mathbf{A} = 3\mathbf{I} + \mathbf{J} - 3\mathbf{K}$ and $\mathbf{V}_1 = x\mathbf{I} + 2x\mathbf{J}$, the projection $\mathbf{V} = \mathbf{V}_1$ is given by

$$\mathbf{V}_1 = \frac{\mathbf{A} \cdot \mathbf{V}_1}{|\mathbf{V}_1|^2}\mathbf{V}_1 = \frac{3x + 2x}{x^2 + 4x^2}(x\mathbf{I} + 2x\mathbf{J}) = \mathbf{I} + 2\mathbf{J}$$

This means that the coordinates of point B are $(1, 2, 0)$.

A2.18 Let $a = |\mathbf{A}|$ and $b = |\mathbf{B}|$. Show that the vector

$$\mathbf{C} = \frac{a\mathbf{B} + b\mathbf{A}}{a + b}$$

bisects the angle between $\mathbf{A}$ and $\mathbf{B}$.

SOLUTION 1

$$\mathbf{C} = \frac{a\mathbf{B} + b\mathbf{A}}{a + b} = \frac{ab\mathbf{U}_\mathbf{B} + ba\mathbf{U}_\mathbf{A}}{a + b} = \frac{ab}{a + b}(\mathbf{U}_\mathbf{B} + \mathbf{U}_\mathbf{A})$$

Since vector $\mathbf{C}$ is in the direction of the diagonal line of the diamond figure formed by the two unit vectors $\mathbf{U}_\mathbf{A}$ and $\mathbf{U}_\mathbf{B}$ (see Fig. A2-22), it bisects the angle between $\mathbf{U}_\mathbf{A}$ and $\mathbf{U}_\mathbf{B}$, which is also the angle between $\mathbf{A}$ and $\mathbf{B}$.

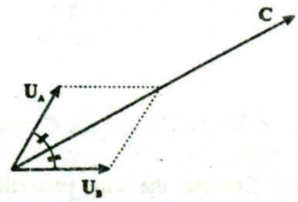


Fig. A2-22

SOLUTION 2

Let α be the angle between **A** and **C**, β be the angle between **B** and **C**, and $c = |\mathbf{C}|$. We have

$$\cos(\alpha) = \frac{\mathbf{A}\cdot\mathbf{C}}{ac} = \frac{\mathbf{A}\cdot\dfrac{a\mathbf{B}+b\mathbf{A}}{a+b}}{ac} = \frac{a\mathbf{A}\cdot\mathbf{B}+b\mathbf{A}\cdot\mathbf{A}}{ac(a+b)} = \frac{\mathbf{A}\cdot\mathbf{B}+ba}{c(a+b)}$$

and

$$\cos(\beta) = \frac{\mathbf{B}\cdot\mathbf{C}}{bc} = \frac{\mathbf{B}\cdot\dfrac{a\mathbf{B}+b\mathbf{A}}{a+b}}{bc} = \frac{a\mathbf{B}\cdot\mathbf{B}+b\mathbf{A}\cdot\mathbf{B}}{bc(a+b)} = \frac{ab+\mathbf{A}\cdot\mathbf{B}}{c(a+b)}$$

Comparing the two expressions we get $\cos(\alpha) = \cos(\beta)$, or $\alpha = \beta$.

A2.19 Prove the formula $\mathbf{V}_1 \cdot \mathbf{V}_2 = |\mathbf{V}_1||\mathbf{V}_2|\cos(\theta)$, where $\mathbf{V}_1$ and $\mathbf{V}_2$ are two vectors and θ is the smaller angle between $\mathbf{V}_1$ and $\mathbf{V}_2$ (when the vectors are placed tail to tail).

SOLUTION

Since $\mathbf{V}\cdot\mathbf{V} = |\mathbf{V}|^2$ for any vector **V** (see Prob. A2.3), we have (see Fig. A2- 23):

$$\begin{aligned} |\mathbf{V}_1 - \mathbf{V}_2|^2 &= (\mathbf{V}_1 - \mathbf{V}_2)\cdot(\mathbf{V}_1 - \mathbf{V}_2) \\ &= \mathbf{V}_1\cdot(\mathbf{V}_1 - \mathbf{V}_2) - \mathbf{V}_2\cdot(\mathbf{V}_1 - \mathbf{V}_2) && \text{(Prob. A1.20)} \\ &= \mathbf{V}_1\cdot\mathbf{V}_1 - 2\mathbf{V}_1\cdot\mathbf{V}_2 + \mathbf{V}_2\cdot\mathbf{V}_2 && \text{(Prob. A1.19)} \\ &= |\mathbf{V}_1|^2 - 2\mathbf{V}_1\cdot\mathbf{V}_2 + |\mathbf{V}_2|^2 \end{aligned}$$

On the other hand, using the Law of Cosines (see Sect. A1.1), we have

$$|\mathbf{V}_1 - \mathbf{V}_2|^2 = |\mathbf{V}_1|^2 + |\mathbf{V}_2|^2 - 2|\mathbf{V}_1||\mathbf{V}_2|\cos(\theta)$$

Comparing the two expressions we get $\mathbf{V}_1 \cdot \mathbf{V}_2 = |\mathbf{V}_1||\mathbf{V}_2|\cos(\theta)$.

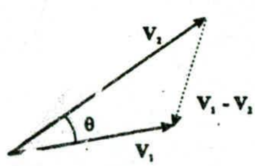


Fig. A2-23

A2.20 Use vectors to show that, if the two diagonals of a rectangle area perpendicular to each other, the rectangle is a square.

SOLUTION

Let the lower left corner of the rectangle be at the origin and the upper right corner be at (x, y) or $(x, y, 0)$. The two diagonals of the rectangle can be expressed as $\mathbf{V}_1 = x\mathbf{I} + y\mathbf{J}$ and $\mathbf{V}_2 = x\mathbf{I} - y\mathbf{J}$. When the two diagonals are perpendicular to each other, we have $xx - yy = 0$, or $x = y$. Hence the rectangle is a square.

A2.21 (*a*) What three-dimensional line determines the homogeneous coordinate point $(1, 5, -1)$? (*b*) Do the homogeneous coordinates $(1, 5, -1)$ and $(-2, -10, -3)$ represent the same projective point?

SOLUTION

(*a*) The line passes through the origin $(0, 0, 0)$ and the Cartesian point $(1, 5, -1)$. So $x = t$, $y = 5t$, and $z = -t$ is the equation of the line.

(*b*) The homogeneous coordinates represent the same projective point if and only if the coordinates are proportional, i.e., there is some number t so that $-2 = (1)t$, $-10 = (5)t$, and $-3 = (-1)t$. Since there is no such number, these coordinates represent different projective points.

Answers to Supplementary Problems

Chapter 2

2.42 No, since there is a change in aspect ratio ($5/3.5 \neq 6/4$).

2.43 Yes, since $5.25/3.5 = 6/4 = 1.5$.

2.44 Present the image at an aspect ratio that is lower than the original.

2.45

```
int i, j, c, rgb[3];
for (j = 0; j < height; j++)
  for (i = 0; i < width; i++) {
    getPixel(i, j, rgb);
    c = 0.299*rgb[0] + 0.587*rgb[1] + 0.144*rgb[2];
    setPixel(i, j, c);
  }
```

Chapter 3

3.35

(a)

$y = 4x + 3$	x
11	2
31	7
7	1

(b)

$y = 1x + 0$	x
2	2
7	7
1	1

(c)

$y = -3x - 4$	x
−10	2
−25	7
−7	1

(d)

$y = -2x + 1$	x
−3	2
−13	7
−1	1

3.36

1. Compute the initial values. Prior to passing the variables to the line plotting routine, we exhange x and y coordinates, (x, y) giving (y, x).

$$dx = y_1 - y_2 \quad Inc_1 = 2dy \qquad d = Inc_1 - dx$$
$$dy = x_1 - x_2 \quad Inc_2 = 2(dy - dx)$$

2. Set (x, y) equal to the lower left-hand endpoint and x_{end} equal to the largest value of x. If $dx < 0$, then $y = x_2, x = y_2, x_{end} = y_1$. If $dx > 0$, then $y = x_1, x = y_1, x_{end} = y_2$.
3. Plot a point at the current (y, x) coordinates. Note the coordinate values are exchanged before they are passed to the plot routine.
4. Test to determine whether the entire line has been drawn. If $x = x_{end}$, stop.
5. Compute the location of the next pixel. If $d < 0$, then $d = d + Inc_1$. If $d \geq 0$, then $d = d + Inc_2$, $y = y + 1$.
6. Increment x: $x = x + 1$.
7. Prior to plotting, the (x, y) coordinates are again exchanged. Plot a point at the current (x, y) coordinates.
8. Go to step 4.

3.37

1. Set the initial values: (x_1, y_1) = start of line; (x_3, y_3) = end of line; $\alpha = \tan^{-1}((y_3 - y_1)/(x_3 - x_1))$; d = length of dash; c = length of blank.
2. Test to see whether the entire line has been drawn. If $x_1 \geq x_3$, stop.

3. Compute end of dash:

$$x_2 = x_1 + d\cos(\alpha)$$
$$y_2 = y_1 + d\sin(\alpha)$$

4. Send (x_1, y_1) and (x_2, y_2) to the line routine and plot dash.
5. Compute the starting point of the next dash:

$$x_1 = x_2 + c\cos(\alpha)$$
$$y_1 = y_2 + c\sin(\alpha)$$

6. Go to step 2.

3.38 See Fig. S-1. Solving for $\theta = \pi/4$:

$$x = 2\cos(\pi/4) + 0 = 1.414 \qquad y = 1\sin(\pi/4) + 0 = 0.7071$$

Solving for $\theta = 3\pi/4$:

$$x = 2\cos(3\pi/4) + 0 = -1.414 \qquad y = 1\sin(3\pi/4) + 0 = 0.7071$$

Solving for $\theta = 5\pi/4$:

$$x = 2\cos(5\pi/4) + 0 = -1.414 \qquad y = 1\sin(5\pi/4) + 0 = -0.7071$$

Solving for $\theta = 7\pi/4$:

$$x = 2\cos(7\pi/4) + 0 = 1.414 \qquad y = 1\sin(7\pi/4) + 0 = -0.7071$$

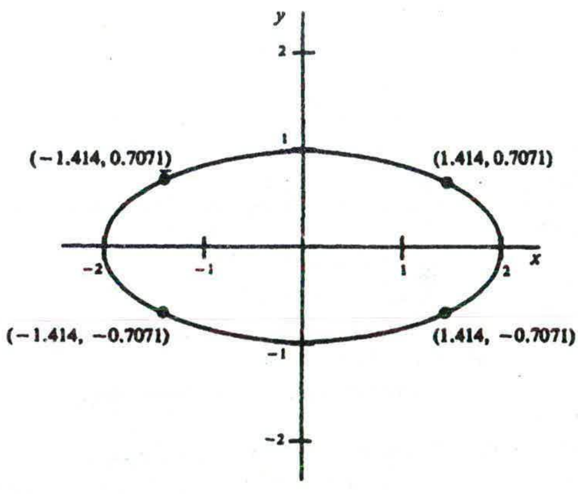


Fig. S-1

3.39 (*a*) Step 3 should be changed to read

$$x = a\cos(\theta) - b\sin\left(\theta + \frac{\pi}{4}\right) + h$$

$$y = b\sin(\theta) + a\cos\left(\theta + \frac{\pi}{4}\right) + k$$

(*b*) Step 3 should be changed to read

$$x = a\cos(\theta) - b\sin\left(\theta + \frac{\pi}{9}\right) + h$$

$$y = b\sin(\theta) + a\cos\left(\theta + \frac{\pi}{9}\right) + k$$

(*c*) Step 3 should be changed to read

$$x = a\cos(\theta) - b\sin\left(\theta + \frac{\pi}{2}\right) + h$$

$$y = b\sin(\theta) + a\cos\left(\theta + \frac{\pi}{2}\right) + k$$

Note that rotating an ellipse $\pi/2$ requires only that the major and minor axes be interchanged. Therefore, the rotation could also be accomplished by changing step 3 to read

$$x = b\cos\theta \qquad y = a\sin(\theta)$$

3.40
1. Set initial variables: a = radius, (h, k) = coordinates of sector center, θ_1 = starting angle, θ_2 = ending angle, and i = step size.
2. Plot line from sector center to coordinates of start of arc: plot (h, k) to $(a\cos(\theta_1) + h, a\sin(\theta_1) + k)$.
3. Plot line from sector center to coordinates of end of arc: plot (h, k) to $(a\cos(\theta_2) + h, a\sin(\theta_2) + h)$.
4. Plot arc.

3.41 When a region is to be filled with a pattern, the fill algorithm must look at a table containing the pattern before filling each pixel. The correct value for the pixel is taken from the table and placed in the pixel examined by the fill algorithm.

3.42 The human brain tends to compensate for deficiencies in models. For example, although the cube shown in Fig. S-2 is lacking the visual cue, convergence, it is perceived as a cube. When the choice of aliasing is inconsistent, the brain either cannot decode the model or can decode it only with difficulty because there is no one rule that can be learned to compensate for the inconsistencies of the models.

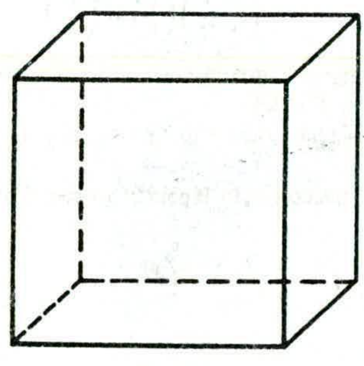


Fig. S-2

3.43
1. Initialize the edge list. For each nonhorizontal edge, find $1/m(= \Delta x/\Delta y)$, y_{max}, y_{min}, and the x coordinate of the edge's lower endpoint.
2. Begin with the first scan line y.
3. If y is beyond the last scan line, stop.
4. Activate all edges with $y_{min} = y$ and delete all edges for which $y > y_{max}$.
5. Sort the intersection points by x value.
6. Fill the pixels between and including each pair of intersection points.
7. Increment x by $1/m$ for each active edge.
8. Increment y by 1 and go to step 3.

3.44 Overstrike can be eliminated by checking each pixel before writing to it. If the pixel has already been written to, no point will be written. Or better yet, design scan-conversion algorithms that do not result in overstrike.

Chapter 4

4.19

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{and} \quad R_{-\theta} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Also

$$R_\theta \cdot R_{-\theta} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$
$$= \begin{pmatrix} (\cos^2\theta + \sin^2\theta) & (\cos\theta\sin\theta - \sin\theta\cos\theta) \\ (\sin\theta\cos\theta - \cos\theta\sin\theta) & (\sin^2\theta + \cos^2\theta) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore, R_θ and $R_{-\theta}$ are inverse, so $R_{-\theta} = R_\theta^{-1}$. In other words, the inverse of a rotation by θ degrees is a rotation in the opposite direction.

4.20 Magnification and reduction can be achieved by a uniform scaling of s units in both the X and Y directions. If $s > 1$, the scaling produces magnification. If $s < 1$, the result is a reduction. The transformation can be written as

$$(x, y) \longmapsto (sx, sy)$$

In matrix form, this becomes

$$\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} sx \\ sy \end{pmatrix}$$

(*a*) Choosing $s = 2$ and applying the transformation to the coordinates of the points A, B, C yields the new coordinates $A'(0, 0)$, $B'(2, 2)$, $C'(10, 4)$.

(*b*) Here, $s = \frac{1}{2}$ and the new coordinates are $A''(0, 0)$, $B''(\frac{1}{2}, \frac{1}{2})$, $C''(\frac{5}{2}, 1)$.

4.21 The line $y = x$ has slope 1 and y intercept $(0, 0)$. If point P has coordinates (x, y), then following Prob. 4.10 we have

$$M_L \cdot P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix} \quad \text{or} \quad M_L(x, y) = (y, x)$$

4.22 The rotation matrix is

$$R_{45^\circ} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The translation matrix is

$$T_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix of vertices $[A \quad B \quad C]$ is

$$V = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

(*a*)

$$T_{\mathrm{I}} \cdot R_{45^\circ} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T_{\mathrm{I}} \cdot R_{45^\circ} \cdot V = \begin{pmatrix} \left(\frac{\sqrt{2}}{2}+1\right) & \left(-\frac{\sqrt{2}}{2}+1\right) & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \sqrt{2} \\ 1 & 1 & 1 \end{pmatrix}$$

So the transformed vertices are $A'\left(\frac{\sqrt{2}}{2}+1, \frac{\sqrt{2}}{2}\right)$, $B'\left(-\frac{\sqrt{2}}{2}+1, \frac{\sqrt{2}}{2}\right)$, and $C'(1, \sqrt{2})$.

(*b*)

$$R_{45^\circ} \cdot T_{\mathrm{I}} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad R_{45^\circ} \cdot T_{\mathrm{I}} \cdot V = \begin{pmatrix} \sqrt{2} & 0 & \frac{\sqrt{2}}{2} \\ \sqrt{2} & \sqrt{2} & \frac{3\sqrt{2}}{2} \\ 1 & 1 & 1 \end{pmatrix}$$

The transformed coordinates are $A''(\sqrt{2}, \sqrt{2})$, $B''(0, \sqrt{2})$, and $C''(\sqrt{2}/2, 3\sqrt{2}/2)$. From this we see that the order in which the transformations are applied is important in the formation of composed or concatenated transformations (see Fig. S-3). Figure S-3(*b*) represents the triangle of Fig. S-3(*a*) after application of the transformation $T_{\mathrm{I}} \cdot R_{45^\circ}$; Fig. S-3(*c*) represents the same triangle after the transformation $R_{45^\circ} \cdot T_{\mathrm{I}}$.

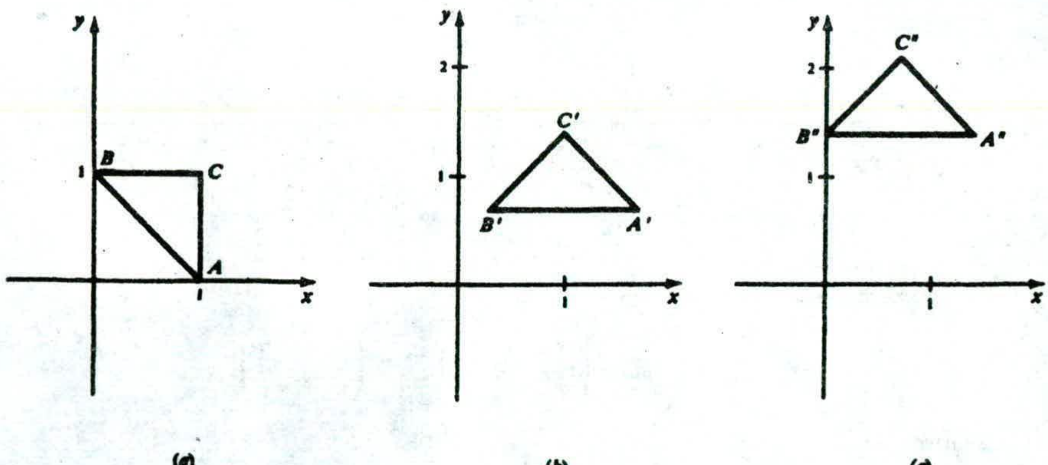


Fig. S-3

4.23 To determine the coordinates of the displaced object from the observer's point of view, we must find the coordinates of the object with respect to the observer's coordinate system. In our case we have performed an object translation $T_{\mathbf{v}}$ and a coordinate system translation $\bar{T}_{\mathbf{v}}$. The result is found by the composition $\bar{T}_{\mathbf{v}} \cdot T_{\mathbf{v}}$ (or $T_{\mathbf{v}} \cdot \bar{T}_{\mathbf{v}}$):

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+a \\ y+b \end{pmatrix} \mapsto \begin{pmatrix} x+a-a \\ y+b-b \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

So the coordinates have remained the same.

4.24 We express the general form of an equation in the $x'y'$ coordinate system as $F(x', y') = 0$. Writing the coordinate transformation in equation form as

$$x' = q(x, y) \qquad y' = r(x, y)$$

and substituting this into the expression for F, we get

$$F(q(x, y), r(x, y)) = 0$$

which is an equation in xy coordinates.

4.25

SOLUTION 1

Let

$$V_1 = t_{x_1}\mathbf{I} + t_{y_1}\mathbf{J} \qquad \text{and} \qquad V_2 = t_{x_2}\mathbf{I} + t_{y_2}\mathbf{J}$$

We have

$$T_{V_1} \cdot T_{V_2}(x, y) = T_{V_1}(x + t_{x_2}, y + t_{y_2}) = (x + t_{x_2} + t_{x_1}, y + t_{y_2} + t_{y_1})$$

and

$$T_{V_2} \cdot T_{V_1}(x, y) = T_{v_2}(x + t_{x_1}, y + t_{y_1}) = (x + t_{x_1} + t_{x_2}, y + t_{y_1} + t_{y_2})$$

Since

$$V_1 + V_2 = (t_{x_1} + t_{x_2})\mathbf{I} + (t_{y_1} + t_{y_2})\mathbf{J}$$

we also have

$$T_{V_1+V_2}(x, y) = (x + t_{x_1} + t_{x_2}, y + t_{y_1} + t_{y_2})$$

Therefore

$$T_{V_1} \cdot T_{V_2} = T_{V_2} \cdot T_{V_1} = T_{V_1+V_2}$$

SOLUTION 2

Let

$$V_1 = t_{x_1}\mathbf{I} + t_{y_1}\mathbf{J} \qquad \text{and} \qquad V_2 = t_{x_2}\mathbf{I} + t_{y_2}\mathbf{J}$$

and express the translation transformations in matrix form

$$T_{V_1} = \begin{pmatrix} 1 & 0 & t_{x_1} \\ 0 & 1 & t_{y_1} \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad T_{V_2} = \begin{pmatrix} 1 & 0 & t_{x_2} \\ 0 & 1 & t_{y_2} \\ 0 & 0 & 1 \end{pmatrix}$$

we have

$$T_{V_1} \cdot T_{V_2} = \begin{pmatrix} 1 & 0 & t_{x_1} \\ 0 & 1 & t_{y_1} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & t_{x_2} \\ 0 & 1 & t_{y_2} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_{x_2} + t_{x_1} \\ 0 & 1 & t_{y_2} + t_{y_1} \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$T_{V_2} \cdot T_{V_1} = \begin{pmatrix} 1 & 0 & t_{x_2} \\ 0 & 1 & t_{y_2} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & t_{x_1} \\ 0 & 1 & t_{y_1} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_{x_1} + t_{x_2} \\ 0 & 1 & t_{y_1} + t_{y_2} \\ 0 & 0 & 1 \end{pmatrix}$$

Also since

$$V_1 + V_2 = (t_{x_1} + t_{x_2})\mathbf{I} + (t_{y_1} + t_{y_2})\mathbf{J}$$

we have

$$T_{V_1+V_2} = \begin{pmatrix} 1 & 0 & t_{x_1} + t_{x_2} \\ 0 & 1 & t_{y_1} + t_{y_2} \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore

$$T_{V_1} \cdot T_{V_2} = T_{V_2} \cdot T_{V_1} = T_{V_1+V_2}$$

4.26

SOLUTION 1

Since

$$\begin{aligned} S_{a,b} \cdot S_{c,d}(x, y) &= S_{a,b}(cx, dy) = (acx, bdy) \\ S_{c,d} \cdot S_{a,b}(x, y) &= S_{c,d}(ax, by) = (cax, dby) \\ S_{ac,bd}(x, y) &= (acx, bdy) \end{aligned}$$

we have $S_{a,b} \cdot S_{c,d} = S_{c,d} \cdot S_{a,b} = S_{ac,bd}$.

SOLUTION 2

Express the scaling transformations in matrix form

$$S_{a,b} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \qquad S_{c,d} = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}, \qquad \text{and} \qquad S_{ac,bd} = \begin{pmatrix} ac & 0 \\ 0 & bd \end{pmatrix}$$

Since

$$S_{a,b} \cdot S_{c,d} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} ac & 0 \\ 0 & bd \end{pmatrix}$$

and

$$S_{c,d} \cdot S_{a,b} = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} ca & 0 \\ 0 & db \end{pmatrix}$$

we have $S_{a,b} \cdot S_{c,d} = S_{c,d} \cdot S_{a,b} = S_{ac,bd}$.

4.27 Express the rotation transformations in matrix form

$$R_\alpha = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \qquad \text{and} \qquad R_\beta = \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix}$$

we have

$$\begin{aligned} R_\alpha \cdot R_\beta &= \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \cdot \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) & -\cos(\alpha)\sin(\beta) - \sin(\alpha)\cos(\beta) \\ \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) & -\sin(\alpha)\sin(\beta) + \cos(\alpha)\cos(\beta) \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} R_\beta \cdot R_\alpha &= \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix} \cdot \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\beta)\cos(\alpha) - \sin(\beta)\sin(\alpha) & -\cos(\beta)\sin(\alpha) - \sin(\beta)\cos(\alpha) \\ \sin(\beta)\cos(\alpha) + \cos(\beta)\sin(\alpha) & -\sin(\beta)\sin(\alpha) + \cos(\beta)\cos(\alpha) \end{pmatrix} \end{aligned}$$

Using trigonometric identities

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

and

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

we have

$$R_\alpha \cdot R_\beta = R_\beta \cdot R_\alpha = \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix} = R_{\alpha+\beta}$$

4.28 First express scaling and rotation in matrix form

$$S_{s_x,s_y} = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \quad \text{and} \quad R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

we have

$$S_{s_x,s_y} \cdot R_\theta = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \cdot \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} = \begin{pmatrix} s_x\cos(\theta) & -s_x\sin(\theta) \\ s_y\sin(\theta) & s_y\cos(\theta) \end{pmatrix}$$

and

$$R_\theta \cdot S_{s_x,s_y} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \cdot \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} = \begin{pmatrix} \cos(\theta)s_x & -\sin(\theta)s_y \\ \sin(\theta)s_x & \cos(\theta)s_y \end{pmatrix}$$

In order to satisfy

$$S_{s_x,s_y} \cdot R_\theta = R_\theta \cdot S_{s_x,s_y}$$

we need

$$s_y\sin(\theta) = \sin(\theta)s_x$$

This yields $\theta = n\pi$, where n is an integer, or $s_y = s_x$, which means that the scaling transformation is uniform.

4.29 No, since

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} = \begin{pmatrix} 1+ab & a \\ b & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a \\ b & ba+1 \end{pmatrix} \neq \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix}$$

4.30 A rotation followed by a simultaneous shearing can be expressed as

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \cdot \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix} = \begin{pmatrix} \cos(\theta) - b\cdot\sin(\theta) & a\cdot\cos(\theta) - \sin(\theta) \\ \sin(\theta) + b\cdot\cos(\theta) & a\cdot\sin(\theta) + \cos(\theta) \end{pmatrix}$$

On the other hand, a simultaneous shearing followed by a rotation can be expressed as

$$\begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\theta) + a\cdot\sin(\theta) & -\sin(\theta) + a\cdot\cos(\theta) \\ b\cdot\cos(\theta) + \sin(\theta) & -b\cdot\sin(\theta) + \cos(\theta) \end{pmatrix}$$

In order for the two composite transformation matrices to be the same, we need

$$\cos(\theta) - b\cdot\sin(\theta) = \cos(\theta) + a\cdot\sin(\theta)$$

or

$$-b\cdot\sin(\theta) = a\cdot\sin(\theta)$$

which means $\theta = n\pi$, where n is an integer, or $a = -b$.

4.31 Consider the following sequence of rotate–scale–rotate transformations

$$R_\alpha \cdot S_{s_x,s_y} \cdot R_\beta = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \cdot \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \cdot \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix}.$$

$$= \begin{pmatrix} \cos(\alpha)\cos(\beta)s_x - \sin(\alpha)\sin(\beta)s_y & -\cos(\alpha)\sin(\beta)s_x - \sin(\alpha)\cos(\beta)s_y \\ \sin(\alpha)\cos(\beta)s_x + \cos(\alpha)\sin(\beta)s_y & -\sin(\alpha)\sin(\beta)s_x + \cos(\alpha)\cos(\beta)s_y \end{pmatrix}$$

By equating the composite transformation matrix on the right-hand side to the matrix for a simultaneous shearing transformation

$$\begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix}$$

we have four equations that can be solved for parameters α, β, s_x and s_y.

4.32 Consider the following sequence of shearing and scaling transformations:

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} = \begin{pmatrix} s_x + a \cdot b \cdot s_y & a \cdot s_y \\ b \cdot s_y & s_y \end{pmatrix}$$

By equating the composite transformation matrix on the right to

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

we have

$$a = -\frac{\sin(\theta)}{\cos(\theta)}, \qquad b = \frac{\sin(\theta)}{\cos(\theta)}, \qquad s_x = \frac{1}{\cos(\theta)}, \qquad s_y = \cos(\theta).$$

4.33 Consider the following sequence of shearing transformations:

$$\begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 + a_1 b & (1 + a_1 b)a_2 + a_1 \\ b & ba_2 + 1 \end{pmatrix}$$

By equating the composite transformation matrix on the right to

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta \end{pmatrix}$$

we have

$$a_1 = a_2 = \frac{\cos(\theta) - 1}{\sin(\theta)} \qquad \text{and} \qquad b = \sin(\theta)$$

4.34 Let CTM_n be the composite transformation matrix representing the concatenation of n basic transformation matrices. We prove, by mathematical induction on n, that CTM_n is always in the following form

$$\begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix}$$

$n = 1$: The basis case is true since CTM_1 is simply a basic transformation matrix, which fits into the given template.

$n = k$: Suppose that CTM_k is indeed in the specified form.

$$\text{CTM}_k = \begin{pmatrix} a_k & b_k & c_k \\ d_k & e_k & f_k \\ 0 & 0 & 1 \end{pmatrix}$$

$n = k + 1$: We now show that CTM_{k+1} is in the same form:

$$\text{CTM}_{k+1} = \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix} \cdot \text{CTM}_k = \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_k & b_k & c_k \\ d_k & e_k & f_k \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} a \cdot a_k + b \cdot d_k & a \cdot b_k + b \cdot e_k & a \cdot c_k + b \cdot f_k + c \\ d \cdot a_k + e \cdot d_k & d \cdot b_k + e \cdot e_k & d \cdot c_k + e \cdot f_k + f \\ 0 & 0 & 1 \end{pmatrix}$$

4.35 Let $P'_1(x'_1, y'_1)$ be the transformation of $P_1(x_1, y_1)$ and $P'_2(x'_2, y'_2)$ be the transformation of $P_2(x_2, y_2)$. Also let the composite transformation be expressed as

$$\begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix}$$

we have

$$x'_1 = ax_1 + by_1 + c \qquad y'_1 = dx_1 + ey_1 + f$$

and

$$x'_2 = ax_2 + by_2 + c \qquad y'_2 = dx_2 + ey_2 + f$$

Now consider an arbitrary point $P(x, y)$ on the line from P_1 to P_2. We want to show that the transformed P, denoted by $P'(x', y')$, where $x' = ax + by + c$ and $y' = dx + ey + f$, is on the line between P'_1 and P'_2. In other words, we want to show

$$\frac{y'_2 - y'_1}{x'_2 - x'_1} = \frac{y'_2 - y'}{x'_2 - x'}$$

which is

$$\frac{dx_2 + ey_2 + f - dx_1 - ey_1 - f}{ax_2 + by_2 + c - ax_1 - by_1 - c} = \frac{dx_2 + ey_2 + f - dx - ey - f}{ax_2 + by_2 + c - ax - by - c}$$

and is

$$\frac{d + e\dfrac{y_2 - y_1}{x_2 - x_1}}{a + b\dfrac{y_2 - y_1}{x_2 - x_1}} = \frac{d + e\dfrac{y_2 - y}{x_2 - x}}{a + b\dfrac{y_2 - y}{x_2 - x}}$$

Since (x, y) satisfies

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2 - y}{x_2 - x}$$

we have established the equality that shows P' being on the line between P'_1 and P'_2.

Chapter 5

5.20 From Prob. 5.1 we need only identify the appropriate parameters.

(*a*) The window parameters are $wx_{\min} = 0$, $wx_{\max} = 1$, $wy_{\min} = 0$, and $wy_{\max} = 1$. The viewport parameters are $vx_{\min} = 0$, $vx_{\max} = 199$, $vy_{\min} = 0$, and $vy_{\max} = 639$. Then $s_x = 199$, $s_y = 639$, and

$$W = \begin{pmatrix} 199 & 0 & 0 \\ 0 & 639 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(*b*) The parameters are the same, but the device y coordinate is now $639 - y$ (see Prob. 2.8) instead of the y

value computed by W in (a)

$$W = \begin{pmatrix} 199 & 0 & 0 \\ 0 & -639 & 639 \\ 0 & 0 & 1 \end{pmatrix}$$

5.21 If $s_x = s_y$, then

$$\frac{vx_{\max} - vx_{\min}}{wx_{\max} - wx_{\min}} = \frac{vy_{\max} - vy_{\min}}{wy_{\max} - wy_{\min}} \qquad \text{or} \qquad \frac{wy_{\max} - wy_{\min}}{wx_{\max} - wx_{\min}} = \frac{vy_{\max} - vy_{\min}}{vx_{\max} - vx_{\min}}$$

Inverting, we have $a_w = a_v$.

A similar argument shows that if the aspect ratios are equal, $a_w = a_v$, the scale factors are equal, $s_x = s_y$.

5.22 We form N by composing (1) a translation mapping the center $(1, 1)$ to the center $(\frac{1}{2}, \frac{1}{2})$ and (2) a scaling about $C(\frac{1}{2}, \frac{1}{2})$ with uniform scaling factor $s = \frac{1}{10}$, so

$$N = S_{1/10,1/10,C} \cdot T_{\mathbf{v}}, \qquad \text{where} \qquad \mathbf{v} = -\tfrac{1}{2}\mathbf{I} - \tfrac{1}{2}\mathbf{J}$$

$$= \begin{pmatrix} \frac{1}{10} & 0 & \frac{9}{20} \\ 0 & \frac{1}{10} & \frac{9}{20} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{10} & 0 & \frac{2}{5} \\ 0 & \frac{1}{10} & \frac{2}{5} \\ 0 & 0 & 1 \end{pmatrix}$$

5.23 Let the clipping region be a circle with center at $O(h, k)$ and radius r. We reduce the number of candidates for clipping by assigning region codes as in the Cohen–Sutherland algorithm. To do this, we use the circumscribed square with lower left corner at $(h - r, k - r)$ and upper right corner at $(h + r, k + r)$ to preprocess the line segments. However, we now have only two clipping categories—not displayed and candidates for clipping. Next, we decide which line segments are to be displayed. Since the (nonparametric) equation of the circle is $(x - h)^2 + (y - k)^2 = r^2$, the quantity $K(x, y) = (x - h)^2 + (y - k)^2 - r^2$ determines whether a point $P(x, y)$ is inside, on, or outside the circle. So, if $K \leq 0$ for both endpoints P_1 and P_2 of a line segment, both points are inside or on the circle and so the line segment is displayed. If $K > 0$ for either P_1 or P_2 or both, we calculate the intersection(s) of the line segment and the circle. Using parametric representations, we find (App. 1, Prob. A1.24) that the intersection parameter is

$$t_I = \frac{-S \pm \sqrt{S^2 - L^2C}}{L^2}$$

where

$$L^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$
$$S = (x_1 - h)(x_2 - x_1) + (y_1 - k)(y_2 - y_1)$$
$$C = (x_1 - h)^2 + (y_1 - k)^2 - r^2$$

If $0 \leq t_I \leq 1$, the actual intersection point(s) $I(\bar{x}, \bar{y})$ is (are)

$$\bar{x} = x_1 + t_I(x_2 - x_1) \qquad \bar{y} = y_1 + t_I(y_2 - y_1)$$

So, if $K > 0$ for either P_1 or P_2 (or both), we first relabel the endpoints so that P_1 satisfies $K > 0$. Next we calculate t_I. The following situations arise:

1. $S^2 - L^2C < 0$. Then t_I is undefined and no intersection takes place. The line segment is not displayed.
2. $S^2 - L^2C = 0$. There is exactly one intersection. If $t_I > 1$ or $t_I < 0$, the intersection point is on the extended line, and so there is no actual intersection. The line is not displayed. If $0 < t_I \leq 1$, $\overline{P_1P_2}$ is tangent to the circle at point I, so only I is displayed.
3. $S^2 - L^2C > 0$. Then there are two values for t_I, t_I^+, and t_I^-. If $0 < t_I^-, t_I^+ \leq 1$, the line segment $\overline{I^-I^+}$ is displayed and the segments (assuming $t_I^+ > t_I^-$) $\overline{P_1I^-}$ and $\overline{I^+P_2}$ are clipped. If only one value, say, t_I^+, satisfies $0 < t_I^+ \leq 1$, there is one actual intersection and one apparent intersection. Since in this case P_2 is either point I^+ or inside the circle, $\overline{P_1I^+}$ is clipped and $\overline{I^+P_2}$ is displayed. If $t_I^+, t_I^- > 1$ or $t_I^+, t_I^- < 0$, then $\overline{P_1P_2}$ is not displayed.

5.24 Following the logic of the Sutherland–Hodgman algorithm as described in Prob. 5.14, we first clip the "polygon" P_1P_2 against edge $\overline{AB}$ of the window:

1. $\overline{AB}$. We first determine which side of $\overline{AB}$ the points P_1 and P_2 lie. Calculating the quantity (see Prob. 5.13), we have

$$\bar{C} = (x_2 - x_1)(y - y_1) - (y_2 - y_1)(x - x_1)$$

With point $A = (x_1, y_1)$ and point $B = (x_2, y_2)$, we find $\bar{C} = 8$ for point P_1 and $\bar{C} = 2$ for point P_2. So both points lie on the left of $\overline{AB}$. Consequently, the algorithm will output both P_1 and P_2.

2. $\overline{BC}$. Setting point $B = (x_1, y_1)$ and $\bar{C} = (x_2, y_2)$, we calculate $\bar{C} = 13$ for point P_1 and $\bar{C} = -3$ for point P_2. Thus P_1 is to the left of $\overline{BC}$ and P_2 is to right of $\overline{BC}$. We now find the intersection point I_1 of $\overline{P_1P_2}$ with the extended line $\overline{BC}$. From Prob. A2.7 in App. 2, we have $I_1 = (4\frac{11}{16}, 3\frac{5}{8})$. Following the algorithm, points P_1 and I_1 are passed on to be clipped.

3. $\overline{CD}$. Proceedings as before, we find that $\bar{C} = 2$ for point P_1 and $\bar{C} = 6\frac{7}{8}$ for point I_1. So both points lie to the left of $\overline{CD}$ and consequently are passed on.

4. $\overline{DA}$. Setting point $D = (x_1, y_1)$ and $A = (x_2, y_2)$, we find $\bar{C} = -3$ for P_1 and $\bar{C} = 10$ for I_1. Then P_1 lies to the right of $\overline{DA}$ and I_1 to the left. The intersection point of $\overline{P_1I_1}$ with the extended edge $\overline{DA}$ is $I_2 = (\frac{5}{16}, 2\frac{3}{8})$. The clipped line is the segment $\overline{I_1I_2}$.

Chapter 6

6.9 From Prob. 6.2, we identify the parameters

$$\begin{aligned} \mathbf{V} &= a\mathbf{I} + b\mathbf{J} + c\mathbf{K} = \mathbf{I} + \mathbf{J} + \mathbf{K} \\ |\mathbf{V}| &= \sqrt{a^2 + b^2 + c^2} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3} \\ \lambda &= \sqrt{b^2 + c^2} = \sqrt{1^2 + 1^2} = \sqrt{2} \end{aligned}$$

Then

$$A_{\mathbf{V}} = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & \frac{-1}{\sqrt{2}\sqrt{3}} & \frac{-1}{\sqrt{2}\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

6.10 From Prob. 6.5, $A_{\mathbf{V},\mathbf{N}} = A_{\mathbf{N}}^{-1} \cdot A_{\mathbf{V}}$. We find $A_{\mathbf{V}}$ first. From Prob. 6.2 we identify the parameters $|\mathbf{V}| = \sqrt{3}$, $\lambda = \sqrt{2}$, $a = 1$, $b = 1$, $c = 1$. So

$$A_{\mathbf{V}} = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & \frac{-1}{\sqrt{2}\sqrt{3}} & \frac{-1}{\sqrt{2}\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For $A_{\mathbf{N}}^{-1}$, we have $|\mathbf{N}| = \sqrt{6}$, $\lambda = \sqrt{2}$, $a = 2$, $b = -1$, and $c = -1$. So

$$A_{\mathbf{N}}^{-1} = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{6}} & 0 & \frac{2}{\sqrt{6}} & 0 \\ \frac{2}{\sqrt{2}\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & 0 \\ \frac{2}{\sqrt{2}\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note that $\mathbf{V}' = A_{\mathbf{V},\mathbf{N}} \cdot \mathbf{V} = A_{\mathbf{N}}^{-1} \cdot A_{\mathbf{V}} \cdot \mathbf{V} = \sqrt{2}\mathbf{I} - \frac{\sqrt{2}}{2}\mathbf{J} - \frac{\sqrt{2}}{2}\mathbf{K}$ so that $\mathbf{V}' = \frac{\sqrt{2}}{2}\mathbf{N}$. In other words, the image of $\mathbf{V}$ under $A_{\mathbf{V},\mathbf{N}}$ is not the vector $\mathbf{N}$, but a vector that has the direction of $\mathbf{N}$.

6.11 This follows from comparing the matrices $A_{\mathbf{V}}^{-1}$ with $A_{\mathbf{V}}^{T}$ from Prob. 6.2.

6.12 If we place vectors $\mathbf{V}$ and $\mathbf{N}$ at the origin, then from App. 2, $\mathbf{V} \times \mathbf{N}$ is perpendicular to both $\mathbf{V}$ and $\mathbf{N}$. If θ is the angle between $\mathbf{V}$ and $\mathbf{N}$, then a rotation of $\theta°$ about the axis L whose direction is that of $\mathbf{V} \times \mathbf{N}$ and which passes through the origin will align $\mathbf{V}$ with $\mathbf{N}$. So $A_{\mathbf{V},\mathbf{N}} = R_{\theta,L}$.

6.13 As in the two-dimensional case in Chap. 4, we reduce the problem of scaling with respect to an arbitrary point P_0 to scaling with respect to the origin by translating P_0 to the origin, performing the scaling about the origin and then translating back to P_0. So

$$S_{s_x,s_y,s_z,P_0} = T_{-P_0}^{-1} \cdot S_{s_x,s_y,s_z} \cdot T_{-P_0}$$

Chapter 7

7.16 From Prob. 7.5, we need to evaluate the parameters (a, b, c), (n_1, n_2, n_3), (d, d_0, d_1) to construct the transformation. From the equations in Prob. 7.6, part (*b*) [denoting the principal vanishing points as $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, and $P_3(x_3, y_3, z_3)$], we find $a = x_2$ (or x_3), $b = y_1$ (or y_3), and $c = z_1$ (or z_2). Also

$$n_1 = \frac{d}{x_1 - a} \qquad n_2 = \frac{d}{y_2 - b} \qquad n_3 = \frac{d}{z_3 - c}$$

To find d, d_0, and d_1, we note (App. 2, Prob. A2.13) that the distance D from the point $C(a, b, c)$ to the plane can be expressed as $D = |d|/|\mathbf{N}|$, where $|\mathbf{N}|$ is the magnitude of $\mathbf{N}$. Since we need only find the direction of the normal $\mathbf{N}$, we can assume $|\mathbf{N}| = 1$. Then $d = \pm D$. The choice $\pm$, based on the definition of d in Prob. 7.5, is dependent on the direction of the normal vector $\mathbf{N}$, the reference point R_0, and the center of projection C. Since these are not all specified, we are free to choose, and we shall choose the + sign, that is, $d = D$. Finally

$$d_1 = n_1 a + n_2 b + n_3 c \qquad \text{and} \qquad d_0 = d + d_1$$

7.17 We use the coordinate matrix $\mathbf{V}$ constructed in Prob. 7.1 to represent the unit cube.

(*a*) From Problem 7.14, the isometric projection matrix *Par* is applied to the coordinate matrix $\mathbf{V}$:

$$Par \cdot \mathbf{V} = \begin{pmatrix} 0 & \sqrt{\frac{2}{3}} & \frac{3}{2}\sqrt{\frac{2}{3}} & \frac{1}{2}\sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & \frac{1}{2}\sqrt{\frac{2}{3}} & \frac{3}{2}\sqrt{\frac{2}{3}} & 2\sqrt{\frac{2}{3}} \\ 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & \frac{-\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

This is the matrix of the projected vertices, which can now be read off (see also Fig. S-4).

$$A' = (0, 0, 0) \qquad E' = \left(\sqrt{\frac{2}{3}}, 0, 0\right)$$

$$B' = \left(\sqrt{\frac{2}{3}}, 0, 0\right) \qquad F' = \left(\frac{1}{2}\sqrt{\frac{2}{3}}, -\frac{\sqrt{2}}{2}, 0\right)$$

$$C' = \left(\frac{3}{2}\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{2}, 0\right) \qquad G' = \left(\frac{3}{2}\sqrt{\frac{2}{3}}, -\frac{\sqrt{2}}{2}, 0\right)$$

$$D' = \left(\frac{1}{2}\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{2}, 0\right) \qquad H' = \left(2\sqrt{\frac{2}{3}}, 0, 0\right)$$

(*b*) To produce a dimetric drawing, we proceed, as in part (*a*), by using the dimetric transformation *Par* from Prob. 7.15. Choosing the projection ratio of $\frac{1}{2}$:1:1 (i.e., $l = \frac{1}{2}$), we have

$$Par = \begin{pmatrix} \frac{\sqrt{2}}{3} & \frac{\sqrt{14}}{6} & \frac{\sqrt{14}}{6} & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The projected image coordinates are found by multiplying the matrices *Par* and **V**:

$$Par \cdot \mathbf{V} = \begin{pmatrix} 0 & \frac{\sqrt{2}}{3} & \frac{2\sqrt{2}+\sqrt{14}}{6} & \frac{\sqrt{14}}{6} & \frac{\sqrt{14}}{3} & \frac{\sqrt{14}}{6} & \frac{2\sqrt{2}+\sqrt{14}}{6} & \frac{\sqrt{2}+\sqrt{14}}{3} \\ 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & \frac{-\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

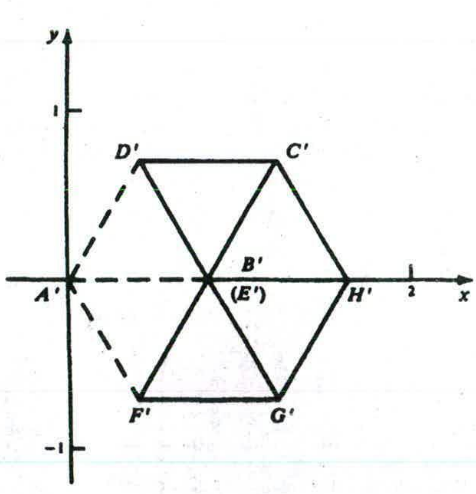


Fig. S-4

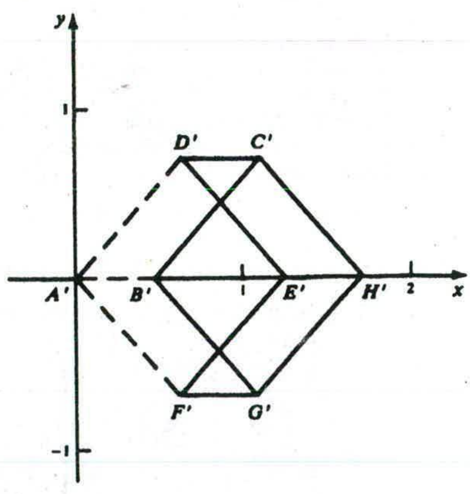


Fig. S-5

The image coordinates are (see Fig. S-5)

$$A' = (0, 0, 0) \qquad E' = \left(\frac{\sqrt{14}}{3}, 0, 0\right)$$

$$B' = \left(\frac{\sqrt{2}}{3}, 0, 0\right) \qquad F' = \left(\frac{\sqrt{14}}{6}, \frac{-\sqrt{2}}{2}, 0\right)$$

$$C' = \left(\frac{2\sqrt{2}+\sqrt{14}}{6}, \frac{\sqrt{2}}{2}, 0\right) \qquad G' = \left(\frac{2\sqrt{2}+\sqrt{14}}{6}, \frac{-\sqrt{2}}{2}, 0\right)$$

$$D' = \left(\frac{\sqrt{14}}{6}, \frac{\sqrt{2}}{2}, 0\right) \qquad H' = \left(\frac{\sqrt{2}+\sqrt{14}}{3}, 0, 0\right)$$

7.18 Since the planes we seek are to be located at the origin, we need only find the normal vectors of these planes so that orthographic projections onto these planes produce isometric projections. In Prob. 7.14, we rotated the *xyz* triad first about the *x* axis and then about the *y* axis to produce an isometric projection onto the *xy* plane. Equivalently, we could have tilted the *xy* plane (and its normal vector **K**) to a new position, thus yielding a new view plane which produces an isometric projection with respect to the (unrotated) *xyz* triad. Using this approach to find all possible view planes, we shall use the equations from Prob. 7.14 to find the appropriate rotation angles. From the equations

$$\sin^2\theta_x - \cos^2\theta_x = 0 \qquad \cos^2\theta_y = \tfrac{1}{2}[\sin^2\theta_y + 1]$$

we find the solutions

$$\sin\theta_x = \pm\frac{\sqrt{2}}{2}, \qquad \cos\theta_x = \pm\frac{\sqrt{2}}{2} \qquad \text{and} \qquad \sin\theta_y = \pm\sqrt{\frac{1}{3}}, \qquad \cos\theta_y = \pm\sqrt{\frac{2}{3}}$$

From Chap. 6, Prob. 6.1, part (*b*), the matrix that produces the tilting is

$$R_{\theta_x,\mathbf{I}} \cdot R_{\theta_y,\mathbf{J}} = \begin{pmatrix} \cos\theta_y & 0 & \sin\theta_y \\ \sin\theta_x \sin\theta_y & \cos\theta_x & -\sin\theta_x \cos\theta_y \\ -\cos\theta_x \sin\theta_y & \sin\theta_x & \cos\theta_x \cos\theta_y \end{pmatrix}$$

Applying this to the vector $\mathbf{K} = (0, 0, 1)$, we find the components of the tilted vector to be

$$x = \sin\theta_y \qquad y = -\sin\theta_x \cos\theta_y \qquad z = \cos\theta_x \cos\theta_y$$

Substituting the values found above, we have eight candidates for the normal vector $\mathbf{N} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$, where

$$x = \pm\sqrt{\frac{1}{3}} \qquad y = \pm\sqrt{\frac{1}{3}} \qquad z = \pm\sqrt{\frac{1}{3}}$$

However, both **N** and −**N** define normals to the same plane. So we finally have four solutions. These are the view planes (through the origin) with normals

$$\mathbf{N}_1 = \sqrt{\frac{1}{3}}(\mathbf{I}+\mathbf{J}+\mathbf{K}) \qquad \mathbf{N}_3 = \sqrt{\frac{1}{3}}(\mathbf{I}-\mathbf{J}+\mathbf{K})$$

$$\mathbf{N}_2 = \sqrt{\frac{1}{3}}(-\mathbf{I}+\mathbf{J}+\mathbf{K}) \qquad \mathbf{N}_4 = \sqrt{\frac{1}{3}}(\mathbf{I}+\mathbf{J}-\mathbf{K})$$

Chapter 8

8.11 Referring to Fig. 8-7 (and Prob. 8.4) we call $\overline{\mathbf{CR}}$ the vector having the direction of the line from the center of projection *C* to the window corner *R*. Similarly, we call $\overline{\mathbf{CL}}$ the vector to the window corner *L*. Then:

1. *Top plane*—determined by the vectors $\mathbf{I_p}$ and $\overline{\mathbf{CR}}$ and the reference point R_f
2. *Bottom plane*—determined by the vectors $\mathbf{I_p}$ and $\overline{\mathbf{CL}}$ and the reference point L_f
3. *Right side plane*—determined by the vectors $\mathbf{J_q}$ and $\overline{\mathbf{CR}}$ and the reference point R_f

4. *Left side plane*—determined by the vectors $\mathbf{J_q}$ and $\overline{\mathbf{CL}}$ and the reference point L_f
5. *Front (near) plane*—determined by the (view plane) normal vector **N** and the reference point P_f
6. *Back (far) plane*—determined by the normal vector **N** and the reference point P_b

8.12 Suppose that the plane passes through point $R_0(x_0, y_0, z_0)$ and has a normal vector $\mathbf{N} = n_1\mathbf{I} + n_2\mathbf{J} + n_3\mathbf{K}$. Let the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ determine a line segment. From App. 2, the equation of the plane is

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$

and the parametric equation of the line is

$$x = x_1 + (x_2 - x_1)t \qquad y = y_1 + (y_2 - y_1)t \qquad z = z_1 + (z_2 - z_1)t$$

Substituting these equations into the equations of the plane, we obtain

$$n_1[x_1 + (x_2 - x_1)t - x_0] + n_2[y_1 + (y_2 - y_1)t - y_0] + n_3[z_1 + (z_2 - z_1)t - z_0] = 0$$

Solving this for t yields the parameter value t_I at the time of intersection:

$$t_I = -\frac{n_1(x_1 - x_0) + n_2(y_1 - y_0) + n_3(z_1 - z_0)}{n_1(x_2 - x_1) + n_2(y_2 - y_1) + n_3(z_2 - z_1)}$$

We can rewrite this using vector notation as

$$t_I = -\frac{\mathbf{N} \cdot \overline{R_0P_1}}{\mathbf{N} \cdot \overline{P_1P_2}}$$

The intersection points $I(x_I, y_I, z_I)$ can be found from the parametric equations of the line:

$$x_I = x_1 + (x_2 - x_1)t_I \qquad y_I = y_1 + (y_2 - y_1)t_I \qquad z_I = z_1 + (z_2 - z_1)t_I$$

If $0 \le t_I \le 1$, the intersection point I is on the line segment from P_1 to P_2; if not, the intersection point is on the extended line.

Chapter 9

9.9 Referring to Fig. 7-12 in Chap. 7, we define a vertex list as

$$V = \{ABCDEFGH\}$$

and an explicit edge list is:

$$E = \{\overline{AB}, \overline{AD}, \overline{AF}, \overline{BC}, \overline{BG}, \overline{CD}, \overline{CH}, \overline{DE}, \overline{EF}, \overline{EH}, \overline{FG}, \overline{GH}\}$$

The cube can be drawn by drawing the edges in list E. Referring to Prob. 9.2, we note that a typical polygon, say, P_1, can be represented in terms of its edges as

$$P_1 = \{\overline{AB}, \overline{AD}, \overline{BC}, \overline{CD}\}$$

The polygons sharing a specific edge can be identified by extending the edge's representation to include pointers to those polygons. For example:

$$\overline{AB} \to P_1, P_4 \qquad \overline{AD} \to P_1, P_3$$

9.10 The knot set can be represented as $t_0, t_0 + L, t_0 + 2L, \ldots$. On the interval $t_i = t_0 + (i - 1)L$ to $t_{i+2} = t_0 + (i + 1)L$, we have

$$B_{i,1}(x) = \frac{x - [t_0 + (i - 1)L]}{(t_0 + iL) - [t_0 + (i - 1)L]} B_{i,0}(x) + \frac{[t_0 + (i + 1)L] - x}{[t_0 + (i + 1)L] - (t_0 + iL)} B_{i+1,0}(x)$$

On the interval $[t_i, t_{i+1}]$, that is, $t_0 + (i - 1)L \le x \le t_0 + iL$, we have $B_{i,0}(x) = 1$ and $B_{i+1,0}(x) = 0$. On the interval $[t_{i+1}, t_{i+2}]$, that is, $t_0 + iL \le x \le t_0 + (i + 1)L$, we have $B_{i,0}(x) = 0$ and $B_{i+1,0}(x) = 1$. Elsewhere both

$B_{i,0}(x) = 0$ and $B_{i+1,0}(x) = 0$. So

$$B_{i,1}(x) = \begin{cases} \dfrac{x - t_0 + (i-1)L}{L} & \text{on} \quad t_0 + (i-1)L \le x \le t_0 + iL \\ \dfrac{t_0 + (i+1)L - x}{L} & \text{on} \quad t_0 + iL \le x \le t_0 + (i+1)L \\ 0 & \text{elsewhere} \end{cases}$$

9.11 From the definition of a B-spline, $B_{i,3}(x)$ is nonzero only if $t_i \le x \le t_{i+4}$. In terms of the given knot set, this equates to $i \le x \le i + 4$. With $x = 5.5$, $B_{i,3}(5.5)$ is nonzero for $i = 2$, 3, 4, and 5. Now

$$B_{i,3}(5.5) = \frac{(5.5) - i}{(i+3) - i} B_{i,2}(5.5) + \frac{(i+4) - (5.5)}{(i+4) - (i+1)} B_{i+1,2}(5.5)$$

or

$$B_{i,2}(5.5) = \frac{(5.5) - i}{3} B_{i,2}(5.5) + \frac{i - (1.5)}{3} B_{i+1,2}(5.5)$$

Starting with $i = 2$,

$$B_{2,3}(5.5) = \frac{3.5}{3} B_{2,2}(5.5) + \frac{0.5}{3} B_{3,2}(5.5)$$

Now $B_{2,2}(x)$ is nonzero if $2 \le x \le 5$. Thus $B_{2,2}(5.5) = 0$, and so $B_{2,3}(5.5) = (0.5/3)\, B_{3,2}(5.5)$. Because $B_{3,2}(x)$ is nonzero for $3 \le x \le 6$, we find that

$$B_{3,2}(5.5) = \frac{(5.5) - 3}{5 - 3} B_{3,1}(5.5) + \frac{6 - (5.5)}{6 - 4} B_{4,1}(5.5)$$

Now $B_{3,1}(x)$ is nonzero if $3 \le x \le 5$. So $B_{3,1}(5.5) = 0$. Now $B_{4,1}(x)$ is nonzero if $4 \le x \le 6$. Thus

$$B_{3,2}(5.5) = \frac{0.5}{2} B_{4,1}(5.5)$$

Now

$$B_{4,1}(5.5) = \frac{(5.5) - 4}{5 - 4} B_{4,0}(5.5) + \frac{6 - (5.5)}{6 - 5} B_{5,0}(5.5)$$

Since $B_{4,0}(x)$ is nonzero if $4 \le x \le 5$, we find that $B_{4,0}(5.5) = 0$. So

$$B_{4,1}(5.5) = \frac{0.5}{1} B_{5,0}(5.5)$$

However, $B_{5,0}(x) = 1$ if $5 \le x \le 6$. So $B_{5,0}(5.5) = 1$, $B_{4,1}(5.5) = 0.5(1) = 0.5$, and $B_{3,2}(5.5) = (0.5/2)(0.5) = 0.25/2$, and finally

$$B_{2,3}(5.5) = \frac{0.5}{3}\left(\frac{0.25}{2}\right) = \frac{0.125}{6} = 0.0208333$$

The computations for $B_{3,3}(5.5)$, $B_{4,3}(5.5)$, and $B_{5,3}(5.5)$ are carried out in the same way.

Chapter 10

10.25 The properties of parallel projection can be used to simplify calculations if the objects to be projected are transformed into "new objects" whose parallel projection results in the same image as the perspective projection of the original object.

10.26 Since the Z-buffer algorithm changes colors at a pixel only if $Z(x, y) < Z_{\text{buf}}(x, y)$, the first polygon written will determine the color of the pixel (see Prob. 10.7).

10.27 A priority flag could be assigned to break the tie resulting in applying the Z-buffer algorithm.

10.28 A system that distinguishes 2^{24} depth values would require three bytes of memory to represent each z value. Thus $3 \times 1024 \times 768 = 2304\,\text{K}$ of memory would be needed.

10.29 The scan-line method can take advantage of (*a*) scan-line coherence, (*b*) edge coherence, (*c*) area coherence, and (*d*) spatial coherence.

10.30 Scan-line coherence is based on the assumption that if a pixel belongs to the scan-converted image of an object, the pixels next to it will (most likely) also belong to this object.

10.31 Since this figure is a polygon, we need only find the maximum and minimum coordinate values of the vertices A, B, and C. Then

$$x_{\min} = 0 \qquad x_{\max} = 2$$
$$y_{\min} = 0 \qquad y_{\max} = 2$$
$$z_{\min} = 1 \qquad z_{\max} = 2$$

The bounding box is shown in Fig. 10-26.

10.32 Horizontal line segments ($y_{\min} = y_{\max}$) lie on only one scan line; they are automatically displayed when nonhorizontal edges are used in the scan-conversion process.

10.33 We search the z coordinates of the vertices of the polygon for the largest value, $z_{\max}$. The depth of the polygon is then $z_{\max}$.

10.34 Area coherence is exploited by classifying polygons with respect to a given screen area as either a surrounding polygon, an intersecting polygon, a contained polygon, or a disjoint polygon. The key fact is that a polygon is not visible if it is in back of a surrounding polygon.

10.35 When using a hidden-surface algorithm to eliminate hidden lines, we set the fill color of the polygons, determined by the lines, to the background color.

Chapter 11

11.25 Let W, H, D, and P be defined in the same way as in Prob. 11.3. We can calculate saturation using $D \times P/(W \times H + D \times P)$.

11.26 The color-sensitive cones in our eyes do not respond well to low intensity light. On the other hand, the rods that are sensitive to low intensity light are color blind.

11.27 Yes. We can use

$$X = \frac{x}{1-x-y}Z, \qquad Y = \frac{y}{1-x-y}Z, \qquad Z = Z$$

11.28 The Y in CMY means yellow, whereas the Y in YIQ represents luminance.

11.29

$$\mathbf{N} = \frac{x - x_c}{R}\mathbf{I} + \frac{y - y_c}{R}\mathbf{J} + \frac{z - z_c}{R}\mathbf{K}$$

11.30 The parametric representation for the target area is

$$\begin{cases} x = \theta & 0 \le \theta \le 3.0 \\ y = 1.2\cos(\varphi) & 0 \le \varphi \le \pi/2 \\ z = 1.2\sin(\varphi) & \end{cases}$$

Note the relationship between the corner points:

$$\begin{array}{lll} u = 0, & w = 0 \rightarrow \theta = 0, & \varphi = \pi/2 \\ u = 1, & w = 0 \rightarrow \theta = 3, & \varphi = \pi/2 \\ u = 0, & w = 2 \rightarrow \theta = 0, & \varphi = 0 \\ u = 1, & w = 2 \rightarrow \theta = 3, & \varphi = 0 \end{array}$$

Substitute these into $\theta = Au + B$ and $\varphi = Cw + D$, we get

$$A = 3, \qquad B = 0, \qquad C = -\pi/4, \qquad D = \pi/2$$

Hence the mapping functions are

$$\theta = 3u \qquad \text{and} \qquad \varphi = \frac{\pi}{2} - \frac{\pi}{4}w$$

The inverse mapping functions are

$$u = \frac{\theta}{3} \qquad \text{and} \qquad w = \frac{\pi/2 - \varphi}{\pi/4}$$

Chapter 12

12.27 Let t_1 be the time required for the light ray to travel from A to P, and t_2 be the time required for the light ray to travel from P to B (see Fig. 12-21). We have

$$t_1 = \frac{\sqrt{x^2 + y_A^2}}{c_1} \qquad t_2 = \frac{\sqrt{(x_B - x)^2 + y_B^2}}{c_2}$$

To locate P (i.e., to determine x) such that the total travel time $t = t_1 + t_2$ is minimal, we find

$$\frac{dt}{dx} = \frac{x}{c_1\sqrt{x^2 + y_A^2}} - \frac{x_B - x}{c_2\sqrt{(x_B - x)^2 + y_B^2}}$$

or

$$\frac{dt}{dx} = \frac{\sin(\alpha)}{c_1} - \frac{\sin(\beta)}{c_2}$$

Notice that $0 \le x \le x_B$ and

$$\left(\frac{dt}{dx}\right)_0 < 0 \qquad \left(\frac{dt}{dx}\right)_{x_B} > 0$$

These suggest that t reaches a minimum when

$$\frac{\sin(\alpha)}{c_1} - \frac{\sin(\beta)}{c_2} = 0$$

Hence

$$\frac{\sin(\alpha)}{\sin(\beta)} = \frac{c_1}{c_2}$$

12.28 Yes, these two procedures produce identical results when invoked with the same call. However, for a given depth value the procedure in this question involves one more level of recursion than the one in the text. Hence the procedure in Sect. 12.2 has better execution efficiency.

12.29 Find two points on the line

$$x = 0 \rightarrow y = 0 - 6 = -6$$
$$x = 1 \rightarrow y = 2 - 6 = -4$$

Use (0, −6) to be the starting point, we have

$$\mathbf{s} = -6\mathbf{J}$$
$$\mathbf{d} = (1 - 0)\mathbf{I} + (-4 - (-6))\mathbf{J} = \mathbf{I} + 2\mathbf{J}$$

and the parametric vector equation for the line is

$$\mathbf{L}(t) = \mathbf{s} + t\mathbf{d} \qquad \text{where } -\infty < t < +\infty$$

12.30 Intersection of a ray with the yz plane can be determined by solving the following for t:

$$\mathbf{s} + t\mathbf{d} = y_i\mathbf{J} + z_i\mathbf{K}$$

With $\mathbf{s} = x_s\mathbf{I} + y_s\mathbf{J} + z_s\mathbf{K}$ and $\mathbf{d} = x_d\mathbf{I} + y_d\mathbf{J} + z_d\mathbf{K}$, we have

$$\begin{cases} x_s + tx_d = 0 \\ y_s + ty_d = y_i \\ z_s + tz_d = z_i \end{cases}$$

When $x_d = 0$, the ray is parallel to the plane (no intersection). When $x_s = 0$, the ray originates from the plane (no intersection). Otherwise, we calculate t using the first equation

$$t = -\frac{x_s}{x_d}$$

If $t < 0$, the negative extension of the ray intersects the plane. On the other hand, if $t > 0$, the ray itself intersects the plane and the coordinates y_i and z_i of the intersection point can be calculated from the second and third equations.

12.31 Let $\mathbf{s} = x_s\mathbf{I} + y_s\mathbf{J} + z_s\mathbf{K}$ and $\mathbf{d} = x_d\mathbf{I} + y_d\mathbf{J} + z_d\mathbf{K}$. Substitute $x_s + tx_d$ and $y_s + ty_d$ for x and y, respectively

$$(x_s + tx_d)^2 + (y_s + ty_d)^2 - R^2 = 0$$

Expand and regroup terms

$$(x_d^2 + y_d^2)t^2 + 2(x_s x_d + y_s y_d)t + (x_s^2 + y_s^2) - R^2 = 0$$

or

$$At^2 + 2Bt + C = 0$$

where

$$A = x_d^2 + y_d^2 \qquad B = x_s x_d + y_s y_d \qquad C = x_s^2 + y_s^2 - R^2$$

When $A = 0$, the ray is parallel to the z axis and does not intersect the cylinder (the entire ray is on the cylinder if the starting point is on the cylinder). Otherwise, the solution for the quadratic equation is

$$t = \frac{-B \pm \sqrt{B^2 - AC}}{A}$$

with

$$B^2 - AC \begin{cases} < 0 & \text{no intersection} \\ = 0 & \text{ray (or its negative extension) touching cylinder} \\ > 0 & \text{two (possible) intersection points} \end{cases}$$

The last case ($B^2 - AC > 0$) produces two t values: t_1 and t_2. If $t_1 < 0$ and $t_2 < 0$, the negative extension of the ray intersects the cylinder (no intersection by the ray). If one of the two values is 0, the ray starts from a point on the cylinder and intersects the cylinder only if the other value is positive. If t_1 and t_2 differ in signs, the ray originates from inside the cylinder and intersects the cylinder once. If both values are positive, the ray intersects the cylinder twice (first enters and then exits), and the smaller value corresponds to the intersection point that is closer to the starting point of the ray.

12.32 Substitute B in $At^2 + Bt + C = 0$ with D we have

$$At^2 + Dt + C = 0$$

and the solution is

$$t = \frac{-D \pm \sqrt{D^2 - 4AC}}{2A}$$

Now let $D = 2B$, the above equation becomes

$$At^2 + 2Bt + C = 0$$

and the solution is

$$t = \frac{-2B \pm \sqrt{(2B)^2 - 4AC}}{2A} = \frac{-2B \pm 2\sqrt{B^2 - AC}}{2A} = \frac{-B \pm \sqrt{B^2 - AC}}{A}$$

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