CHAPTER 1

# Origin of Differential Equations

**A DIFFERENTIAL EQUATION** is an equation which involves derivatives. For example,

1) $\dfrac{dy}{dx} = x + 5$

2) $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = 0$

3) $xy' + y = 3$

4) $y''' + 2(y'')^2 + y' = \cos x$

5) $(y'')^2 + (y')^3 + 3y = x^2$

6) $\dfrac{\partial z}{\partial x} = z + x\dfrac{\partial z}{\partial y}$

7) $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = x^2 + y.$

If there is a single independent variable, as in 1) -5), the derivatives are ordinary derivatives and the equation is called an *ordinary differential equation.*

If there are two or more independent variables, as in 6) -7), the derivatives are partial derivatives and the equation is called a *partial differential equation.*

The *order* of a differential equation is the order of the highest derivative which occurs. Equations 1), 3), and 6) are of the first order; 2), 5), and 7) are of the second order; and 4) is of the third order.

The *degree* of a differential equation which can be written as a polynomial in the derivatives is the degree of the highest ordered derivative which then occurs. All of the above examples are of the first degree except 5) which is of the second degree.

A discussion of partial differential equations will be given in Chapter 28. For the present, only ordinary differential equations with a single dependent variable will be considered.

## ORIGIN OF DIFFERENTIAL EQUATIONS.

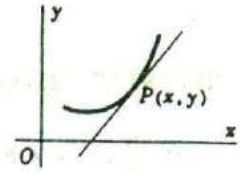
*a)* Geometric Problems. See Problems 1 and 2 below.

*b)* Physical Problems. See Problems 3 and 4 below.

*c)* Primitives. A relation between the variables which involves $n$ essential arbitrary constants, as $y = x^4 + Cx$ or $y = Ax^2 + Bx$ is called a *primitive*. The $n$ constants, always indicated by capital letters here, are called *essential* if they cannot be replaced by a smaller number of constants. See Problem 5.

In general, a primitive involving $n$ essential arbitrary constants will give rise to a differential equation, of order $n$, free of arbitrary constants. This equation is obtained by eliminating the $n$ constants between the $(n + 1)$ equations consisting of the primitive and the $n$ equations obtained by differentiating the primitive $n$ times with respect to the independent variable. See Problems 6-14 below.

## SOLVED PROBLEMS




1. A curve is defined by the condition that at each of its points $(x,y)$, its slope is equal to twice the sum of the coordinates of the point. Express the condition by means of a differential equation.

The differential equation representing the condition is $\frac{dy}{dx} = 2(x+y)$.

2. A curve is defined by the condition that the sum of the $x$- and $y$-intercepts of its tangents is always equal to 2. Express the condition by means of a differential equation.

The equation of the tangent at $(x,y)$ on the curve is $Y - y = \frac{dy}{dx}(X - x)$ and the $x$- and $y$-intercepts are respectively $X = x - y\frac{dx}{dy}$ and $Y = y - x\frac{dy}{dx}$. The differential equation representing the condition is $X + Y = x - y\frac{dx}{dy} + y - x\frac{dy}{dx} = 2$ or $x(\frac{dy}{dx})^2 - (x+y-2)\frac{dy}{dx} + y = 0$.

3. One hundred grams of cane sugar in water are being converted into dextrose at a rate which is proportional to the amount unconverted. Find the differential equation expressing the rate of conversion after $t$ minutes.

Let $q$ denote the number of grams converted in $t$ minutes. Then $(100-q)$ is the number of grams unconverted and the rate of conversion is given by $\frac{dq}{dt} = k(100-q)$, $k$ being the constant of proportionality.

4. A particle of mass $m$ moves along a straight line (the $x$-axis) while subject to 1) a force proportional to its displacement $x$ from a fixed point $O$ in its path and directed toward $O$ and 2) a resisting force proportional to its velocity. Express the total force as a differential equation.

The first force may be represented by $-k_1x$ and the second by $-k_2\frac{dx}{dt}$, where $k_1$ and $k_2$ are factors of proportionality.

The total force (mass $\times$ acceleration) is given by $m\frac{d^2x}{dt^2} = -k_1x - k_2\frac{dx}{dt}$.

5. In each of the equations *a)* $y = x^2 + A + B$, *b)* $y = Ae^{x+B}$, *c)* $y = A + \ln Bx$ show that only one of the two arbitrary constants is essential.

*a)* Since $A + B$ is no more than a single arbitrary constant, only one essential arbitrary constant is involved.

*b)* $y = Ae^{x+B} = Ae^xe^B$, and $Ae^B$ is no more than a single arbitrary constant.

*c)* $y = A + \ln Bx = A + \ln B + \ln x$, and $(A + \ln B)$ is no more than a single constant.

6. Obtain the differential equation associated with the primitive $y = Ax^2 + Bx + C$.

Since there are three arbitrary constants, we consider the four equations

$$y = Ax^2 + Bx + C, \qquad \frac{dy}{dx} = 2Ax + B, \qquad \frac{d^2y}{dx^2} = 2A, \qquad \frac{d^3y}{dx^3} = 0.$$

The last of these $\frac{d^3y}{dx^3}$, being free of arbitrary constants and of the proper order, is the required equation.

Note that the constants could not have been eliminated between the first three of the above equations. Note also that the primitive can be obtained readily from the differential equation by integration.

**7.** Obtain the differential equation associated with the primitive $x^2y^3 + x^3y^5 = C$.

Differentiating once with respect to $x$, we obtain $(2xy^3 + 3x^2y^2\frac{dy}{dx}) + (3x^2y^5 + 5x^3y^4\frac{dy}{dx}) = 0$ or, when $xy \neq 0$, $(2y + 3x\frac{dy}{dx}) + xy^2(3y + 5x\frac{dy}{dx}) = 0$ as the required equation.

When written in differential notation, these equations are

1) $(2xy^3\,dx + 3x^2y^2\,dy) + (3x^2y^5\,dx + 5x^3y^4\,dy) = 0$

and 2) $(2y\,dx + 3x\,dy) + xy^2(3y\,dx + 5x\,dy) = 0.$

Note that the primitive can be obtained readily from 1)by integration but not so readily from 2). Thus, to obtain the primitive when 2) is given, it is necessary to determine the factor $xy^2$ which was removed from 1).

**8.** Obtain the differential equation associated with the primitive $y = A\cos ax + B\sin ax$, $A$ and $B$ being arbitrary constants, and $a$ being a fixed constant.

Here $$\frac{dy}{dx} = -Aa\sin ax + Ba\cos ax$$

and $$\frac{d^2y}{dx^2} = -Aa^2\cos ax - Ba^2\sin ax = -a^2(A\cos ax + B\sin ax) = -a^2y.$$

The required differential equation is $\frac{d^2y}{dx^2} + a^2y = 0$.

**9.** Obtain the differential equation associated with the primitive $y = Ae^{2x} + Be^x + C$.

Here $\frac{dy}{dx} = 2Ae^{2x} + Be^x$, $\frac{d^2y}{dx^2} = 4Ae^{2x} + Be^x$, $\frac{d^3y}{dx^3} = 8Ae^{2x} + Be^x$.

Then $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} = 4Ae^{2x}$, $\frac{d^2y}{dx^2} - \frac{dy}{dx} = 2Ae^{2x}$, and $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} = 2(\frac{d^2y}{dx^2} - \frac{dy}{dx})$.

The required equation is $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$.

**10.** Obtain the differential equation associated with the primitive $y = C_1e^{3x} + C_2e^{2x} + C_3e^x$.

Here $\frac{dy}{dx} = 3C_1e^{3x} + 2C_2e^{2x} + C_3e^x$, $\frac{d^2y}{dx^2} = 9C_1e^{3x} + 4C_2e^{2x} + C_3e^x$,

and $\frac{d^3y}{dx^3} = 27C_1e^{3x} + 8C_2e^{2x} + C_3e^x$.

The elimination of the constants by elementary methods is somewhat tedious. If three of the equations are solved for $C_1$, $C_2$, $C_3$, using determinants, and these substituted in the fourth equation, the result may be put in the form (called the eliminant):

—2

$$\begin{vmatrix} e^{3x} & e^{2x} & e^{x} & y \\ 3e^{3x} & 2e^{2x} & e^{x} & y' \\ 9e^{3x} & 4e^{2x} & e^{x} & y'' \\ 27e^{3x} & 8e^{2x} & e^{x} & y''' \end{vmatrix} = e^{6x}\begin{vmatrix} 1 & 1 & 1 & y \\ 3 & 2 & 1 & y' \\ 9 & 4 & 1 & y'' \\ 27 & 8 & 1 & y''' \end{vmatrix} = e^{6x}(-2y''' + 12y'' - 22y' + 12y) = 0.$$

The required differential equation is $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0.$

**11.** Obtain the differential equation associated with the primitive $y = Cx^2 + C^2$.

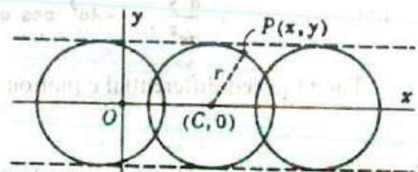
Since $\frac{dy}{dx} = 2Cx$, $C = \frac{1}{2x}\frac{dy}{dx}$ and $y = Cx^2 + C^2 = \frac{1}{2x}\frac{dy}{dx}x^2 + \frac{1}{4x^2}(\frac{dy}{dx})^2$.

The required differential equation is $(\frac{dy}{dx})^2 + 2x^3\frac{dy}{dx} - 4x^2y = 0.$

Note. The primitive involves one arbitrary constant of degree two and the resulting differential equation is of order one and degree two.

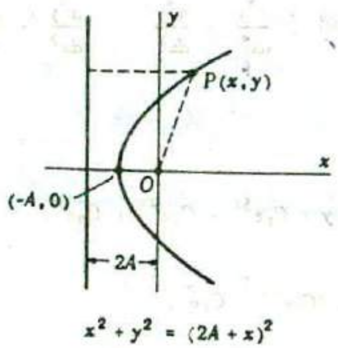
**12.** Find the differential equation of the family of circles of fixed radius $r$ with centres on the $x$ axis.

The equation of the family is $(x-C)^2 + y^2 = r^2$, $C$ being an arbitrary constant.



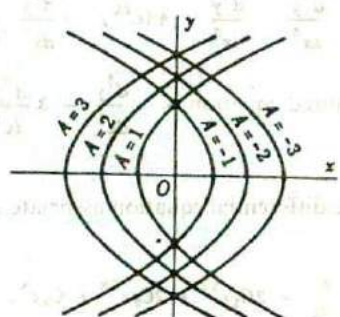

Then $(x-C) + y\frac{dy}{dx} = 0$, $x - C = -y\frac{dy}{dx}$ and the differential equation is $y^2(\frac{dy}{dx})^2 + y^2 = r^2$.

**13.** Find the differential equation of the family of parabolas with foci at the origin and axes along the $x$- axis.




$x^2 + y^2 = (2A + x)^2$

$y^2 = 4A(A + x)$




$y^2 = 4A(A + x)$

The equation of the family of parabolas is $y^2 = 4A(A + x)$.

Then $yy' = 2A$, $A = \frac{1}{2}yy'$ and $y^2 = 2yy'(\frac{1}{2}yy' + x)$.

The required equation is $y(\frac{dy}{dx})^2 + 2x\frac{dy}{dx} - y = 0.$

14. Form the differential equation representing all tangents to the parabola $y^2 = 2x$.

At any point $(A,B)$ on the parabola, the equation of the tangent is $y - B = (x-A)/B$ or, since $A = \frac{1}{2}B^2$, $By = x + \frac{1}{2}B^2$. Eliminating $B$ between this and $By' = 1$, obtained by differentiation with respect to $x$, we have as the required differential equation $2x(y')^2 - 2yy' + 1 = 0$.

## SUPPLEMENTARY PROBLEMS

15. Classify each of the following equations as to order and degree.

a) $dy + (xy - \cos x)dx = 0$ *Ans.* Order one; degree one

b) $L\dfrac{d^2Q}{dt^2} + R\dfrac{dQ}{dt} + \dfrac{Q}{C} = 0$ *Ans.* Order two; degree one

c) $y''' + xy'' + 2y(y')^2 + xy = 0$ *Ans.* Order three; degree one

d) $\dfrac{d^2v}{dx^2}\dfrac{dv}{dx} + x(\dfrac{dv}{dx})^2 + v = 0$ *Ans.* Order two; degree one

e) $(\dfrac{d^3w}{dv^3})^2 - (\dfrac{d^2w}{dv^2})^4 + vw = 0$ *Ans.* Order three; degree two

f) $e^{y'''} - xy'' + y = 0$ *Ans.* Order three; degree does not apply

g) $\sqrt{\rho' + \rho} = \sin\theta$ *Ans.* Order one; degree one

h) $y' + x = (y - xy')^{-3}$ *Ans.* Order one; degree four

i) $\dfrac{d^2\rho}{d\theta^2} = \sqrt[4]{\rho + (\dfrac{d\rho}{d\theta})^2}$ *Ans.* Order two; degree four

16. Write the differential equation for each of the curves determined by the given conditions.

a) At each point $(x,y)$ the slope of the tangent is equal to the square of the abscissa of the point.
*Ans.* $y' = x^2$

b) At each point $(x,y)$ the lenght of the subtangent is equal to the sum of the coordinates of the point.
*Ans.* $y/y' = x + y$ or $(x+y)y' = y$

c) The segment joining $P(x,y)$ and the point of intersection of the normal at $P$ with the $x$-axis is bisected by the $y$-axis.
*Ans.* $y + x\dfrac{dx}{dy} = \frac{1}{2}y$ or $yy' + 2x = 0$

d) At each point $(\rho,\theta)$ the tangent of the angle between the radius vector and the tangent is equal to 1/3 the tangent of the vectorial angle.
*Ans.* $\rho\dfrac{d\theta}{d\rho} = \dfrac{1}{3}\tan\theta$

e) The area bounded by the arc of a curve, the $x$-axis, and two ordinates, one fixed and one variable, is equal to twice the length of the arc between the ordinates.

Hint: $\int_a^x y\,dx = 2\int_a^x \sqrt{1+(y')^2}\,dx$. *Ans.* $y = 2\sqrt{1+(y')^2}$

17. Express each of the following physical statements in differential equation form.

a) Radium decomposes at a rate proportional to the amount $Q$ present. *Ans.* $dQ/dt = -kQ$

b) The population $P$ of a city increases at a rate proportional to the population and to the difference between 200,000 and the population.
*Ans.* $dP/dt = kP(200{,}000 - P)$

c) For a certain substance the rate of change of vapour pressure ($P$) with respect to temperature ($T$) is proportional to the vapour pressure and inversely proportional to the square of the temperature.
*Ans.* $dP/dT = kP/T^2$

d) The potential difference $E$ across an element of inductance $L$ is equal to the product of $L$ and the time rate of change of the current $i$ in the inductance.
*Ans.* $E = L\dfrac{di}{dt}$

e) Mass × acceleration = net force. *Ans.* $m\dfrac{dv}{dt} = F$ or $m\dfrac{d^2s}{dt^2} = F$

18. Obtain the differential equation associated with the given primitive, $A$ and $B$ being arbitrary constants.

a) $y = Ax$ *Ans.* $y' = y/x$

b) $y = Ax + B$ *Ans.* $y'' = 0$

c) $y = e^{x+A} = Be^x$ *Ans.* $y' = y$

d) $y = A\sin x$ *Ans.* $y' = y\cot x$

e) $y = \sin(x + A)$ *Ans.* $(y')^2 = 1 - y^2$

f) $y = Ae^x + B$ *Ans.* $y'' = y'$

g) $x = A\sin(y + B)$ *Ans.* $y'' = x(y')^3$

h) $\ln y = Ax^2 + B$
*Ans.* $xyy'' - yy' - x(y')^2 = 0$

19. Find the differential equation of the family of circles of variable radii $r$ with centres on the $x$-axis. (Compare with Problem 12.)
Hint: $(x - A)^2 + y^2 = r^2$, $A$ and $r$ being arbitrary constants. *Ans.* $yy'' + (y')^2 + 1 = 0$

20. Find the differential equation of the family of cardiods $\rho = a(1 - \cos\theta)$.
*Ans.* $(1 - \cos\theta)d\rho = \rho\sin\theta\, d\theta$

21. Find the differential equation of all straight lines at a unit distance from the origin.
*Ans.* $(xy' - y)^2 = 1 + (y')^2$

22. Find the differential equation of all circles in the plane.
Hint: Use $x^2 + y^2 - 2Ax - 2By + C = 0$. *Ans.* $[1 + (y')^2]y''' - 3y'(y'')^2 = 0$

CHAPTER 2

# Solutions of Differential Equations

**THE PROBLEM** in elementary differential equations is essentially that of recovering the primitive which gave rise to the equation. In other words, the problem of solving a differential equation of order $n$ is essentially of finding a relation between the variables involving $n$ independent arbitrary constants which together with the derivatives obtained from it satisfy the differential equation. For example:

| | Differential Equation | Primitive | |
|---|---|---|---|
| 1) | $\dfrac{d^3y}{dx^3} = 0$ | $y = Ax^2 + Bx + C$ | (Prob. 6, Chap. 1) |
| 2) | $\dfrac{d^3y}{dx^3} - 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} - 6y = 0$ | $y = C_1e^{3x} + C_2e^{2x} + C_3e^x$ | (Prob. 10, Chap.1) |
| 3) | $y^2(\dfrac{dy}{dx})^2 + y^2 = r^2$ | $(x-C)^2 + y^2 = r^2$ | (Prob. 12, Chap.1) |

**THE CONDITIONS** under which we can be assured that a differential equation is solvable are given by *Existence Theorems.*

For example, a differential equation of the form $y' = g(x,y)$ for which

*a)* $g(x,y)$ is continuous and single valued over a region $R$ of points $(x,y)$

*b)* $\dfrac{\partial g}{\partial y}$ exists and is continuous at all points in $R$,

admits an infinity of solutions $f(x,y,C) = 0$ ( $C$ an arbitrary constant) such that through each point of $R$ there passes one and only one curve of the family $f(x,y,C) = 0$.See Problem 5.

**A PARTICULAR SOLUTION** of a differential equation is one obtained from the primitive by assigning definite values to the arbitrary constants. For example, in 1) above $y = 0$ $(A = B = C = 0)$, $y = 2x + 5$ $(A = 0,\ B = 2,\ C = 5)$ and $y = x^2 + 2x + 3$ $(A = 1,\ B = 2,\ C = 3)$ are particular solutions.

Geometrically, the primitive is the equation of a family of curves and a particular solution is the equation of some one of the curves. These curves are called *integral curves* of the differential equation.

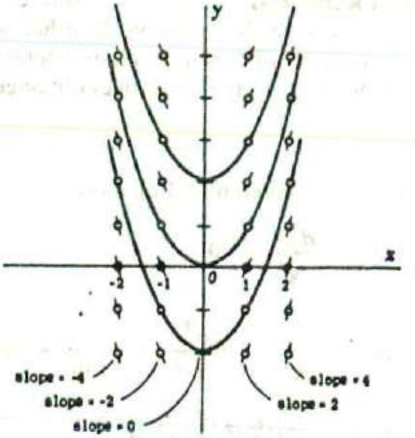
As will be seen from Problem 6, a given form of the primitive may not include all of the particular solutions. Moreover, as will be seen from Problem7, a differential equation may have solutions which cannot be obtained from the primitive by any manipulation of the arbitrary constant as in Problem 6. Such solutions, called *singular solutions,* will be considered in Chapter 10.

The primitive of a differential equation is usually called *the general solution* of the equation. Certain authors, because of the remarks in the paragraph above, call it *a general solution* of the equation.

**A DIFFERENTIAL EQUATION** $\frac{dy}{dx} = g(x,y)$ associates with each point $(x_0, y_0)$ in the region $R$ of the above existence theorem a direction $m = \left.\frac{dy}{dx}\right|_{(x_0, y_0)} = g(x_0, y_0)$.

The direction at each such point is that of the tangent to the curve of the family $f(x,y,C) = 0$ that is, the primitive, passing through the point.

The region $R$ with the direction at each of its points indicated is called a *direction field*. In the adjoining figure, a number of points with the direction at each is shown for the equation $dy/dx = 2x$. The integral curves of the differential equation are those curves having at each of their points the direction given by the equation. In this example, the integral curves are parabolas.




Such diagrams are helpful in that they aid in clarifying the relation between a differential equation and its primitive, but since the integral curves are generally quite complex, such a diagram does not aid materially in obtaining their equations.

## SOLVED PROBLEMS

**1.** Show by direct substitution in the differential equation and a check of the arbitrary constants that each primitive gives rise to the corresponding differential equation.

*a)* $y = C_1 \sin x + C_2 x$ $\qquad (1 - x\cot x)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$

*b)* $y = C_1 e^x + C_2 x e^x + C_3 e^{-x} + 2x^2 e^x$ $\qquad \frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 8e^x$

*a)* Substitute $y = C_1 \sin x + C_2 x$, $\frac{dy}{dx} = C_1 \cos x + C_2$, $\frac{d^2y}{dx^2} = -C_1 \sin x$ in the differential equation to obtain

$$\begin{aligned}(1 - x\cot x)(-C_1 \sin x) - x(C_1 \cos x + C_2) + (C_1 \sin x + C_2 x) &= \\ -C_1 \sin x + C_1 x \cos x - C_1 x \cos x - C_2 x + C_1 \sin x + C_2 x &= 0.\end{aligned}$$

The order of the differential equation (2) and the number of arbitrary constants (2) agree

*b)*

$$\begin{aligned}y &= C_1 e^x + C_2 x e^x + C_3 e^{-x} + 2x^2 e^x,\\ y' &= (C_1 + C_2)e^x + C_2 x e^x - C_3 e^{-x} + 2x^2 e^x + 4xe^x,\\ y'' &= (C_1 + 2C_2)e^x + C_2 x e^x + C_3 e^{-x} + 2x^2 e^x + 8xe^x + 4e^x,\\ y''' &= (C_1 + 3C_2)e^x + C_2 x e^x - C_3 e^{-x} + 2x^2 e^x + 12xe^x + 12e^x,\end{aligned}$$

and $y''' - y'' - y' + y = 8e^x$. The order of the differential equation and the number of arbitrary constants agree.

**2.** Show that $y = 2x + Ce^x$ is the primitive of the differential equation $\frac{dy}{dx} - y = 2(1 - x)$ and find the particular solution satisfied by $x = 0$, $y = 3$, i.e., the equation of the integral through (0,3).

Substitute $y = 2x + Ce^x$ and $\frac{dy}{dx} = 2 + Ce^x$ in the differential equation to obtain $2 + Ce^x - (2x + Ce^x) = 2 - 2x$. When $x = 0$, $y = 3$, $3 = 2\cdot 0 + Ce^0$ and $C = 3$. The particular solution is $y = 2x + 3e^x$.

3. Show that $y = C_1e^x + C_2e^{2x} + x$ is the primitive of the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2x - 3$ and find the equation of the integral curve through the points (0,0) and (1,0).

Substitute $y = C_1e^x + C_2e^{2x} + x$, $\frac{dy}{dx} = C_1e^x + 2C_2e^{2x} + 1$, $\frac{d^2y}{dx^2} = C_1e^x + 4C_2e^{2x}$ in the differential equation to obtain $C_1e^x + 4C_2e^{2x} - 3(C_1e^x + 2C_2e^{2x} + 1) + 2(C_1e^x + C_2e^{2x} + x) = 2x - 3$.

When $x = 0$, $y = 0$: $C_1 + C_2 = 0$. When $x = 1$, $y = 0$: $C_1e + C_2e^2 = -1$.

Then $C_1 = -C_2 = \frac{1}{e^2 - e}$ and the required equation is $y = x + \frac{e^x - e^{2x}}{e^2 - e}$

4. Show that $(y - C)^2 = Cx$ is the primitive of the differential equation $4x\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$ and find the equations of the integral curves through the point (1,2).

Here $2(y - C)\frac{dy}{dx} = C$ and $\frac{dy}{dx} = \frac{C}{2(y - C)}$.

Then $4x\frac{C^2}{4(y-C)^2} + 2x\frac{C}{2(y-C)} - y = \frac{C^2x + Cx(y-C) - y(y-C)^2}{(y-C)^2} = \frac{y[Cx - (y-C)^2]}{(y-C)^2} = 0.$

When $x = 1$, $y = 2$: $(2 - C)^2 = C$ and $C = 1, 4$.

The equations of the integral curves through (1,2) are $(y - 1)^2 = x$ and $(y - 4)^2 = 4x$.

5. The primitive of the differential equation $\frac{dy}{dx} = \frac{y}{x}$ is $y = Cx$. Find the equation of the integral curve through *a)* (1,2) and *b)* (0,0).

*a)* When $x = 1$, $y = 2$: $C = 2$ and the required equation is $y = 2x$.

*b)* When $x = 0$, $y = 0$: $C$ is not determined, that is all of the integral curves pass through the origin. Note that $g(x,y) = y/x$ is not continuous at the origin and hence the existence theorem assures one and only one curve of the family $y = Cx$ through each point of the plane except the origin.

6. Differentiating $xy = C(x - 1)(y - 1)$ and substituting for $C$, we obtain the differential equation

$$x\frac{dy}{dx} + y = C\{(x-1)\frac{dy}{dx} + y - 1\} = \frac{xy}{(x-1)(y-1)}\{(x-1)\frac{dy}{dx} + y - 1\}$$

or 1) $$x(x-1)\frac{dy}{dx} + y(y-1) = 0.$$

Now both $y = 0$ and $y = 1$ are solutions of 1), since, for each, $dy/dx = 0$ and 1) is satisfied. The first is obtained from the primitive by setting $C = 0$, but the second $y = 1$ cannot be obtained by assigning a finite value to $C$. Similarly, 1) may be obtained from the primitive in the form $Bxy = (x - 1)(y - 1)$. Now the solution $y = 1$ is obtained by setting $B = 0$ while the solution $y = 0$ cannot be obtained by assigning finite value to $B$. Thus, the given form of a primitive may not include all of the particular solutions of the differential equation. (Note that $x = 1$ is also a particular solution.)

7. Differentiating $y = Cx + 2C^2$, solving for $C = \frac{dy}{dx}$, and substituting in the primitive yields the differential equation

$$1) \quad 2(\frac{dy}{dx})^2 + x(\frac{dy}{dx}) - y = 0.$$

Since $y = -\frac{1}{8}x^2$, $\frac{dy}{dx} = -\frac{1}{4}x$ satisfies 1), $x^2 + 8y = 0$ is a solution of 1).

Now the primitive is represented by a family of straight lines and it is clear that the equation of a parabola cannot be obtained by manipulating the arbitrary constant. Such a solution is called a singular solution of the differential equation.

8. Verify and reconcile the fact that $y = C_1 \cos x + C_2 \sin x$ and $y = A\cos(x+B)$ are primitives of $\frac{d^2y}{dx^2} + y = 0$.

From $y = C_1\cos x + C_2 \sin x$, $y' = -C_1 \sin x + C_2 \cos x$ and

$$y'' = -C_1 \cos x - C_2 \sin x = -y \quad \text{or} \quad \frac{d^2y}{dx^2} + y = 0.$$

From $y = A\cos(x+B)$, $y' = -A\sin(x+B)$ and $y'' = -A\cos(x+B) = -y$.

Now $y = A\cos(x+B) = A(\cos x \cos B - \sin x \sin B)$

$$= (A\cos B)\cos x + (-A\sin B)\sin x = C_1\cos x + C_2 \sin x.$$

9. Show that $\ln x^2 + \ln\frac{y^2}{x^2} = A + x$ may be written as $y^2 = Be^x$.

$\ln x^2 + \ln\frac{y^2}{x^2} = \ln(x^2\frac{y^2}{x^2}) = \ln y^2 = A + x$. Then $y^2 = e^{A+x} = e^A \cdot e^x = Be^x$.

10. Show that $\operatorname{Arc}\sin x - \operatorname{Arc}\sin y = A$ may be written as $x\sqrt{1-y^2} - y\sqrt{1-x^2} = B$.

$\sin(\operatorname{Arc}\sin x - \operatorname{Arc}\sin y) = \sin A = B$.

Then $\sin(\operatorname{Arc}\sin x)\cos(\operatorname{Arc}\sin y) - \cos(\operatorname{Arc}\sin x)\sin(\operatorname{Arc}\sin y) = x\sqrt{1-y^2} - y\sqrt{1-x^2} = B$.

11. Show that $\ln(1+y) + \ln(1+x) = A$ may be written as $xy + x + y = C$.

$\ln(1+y) + \ln(1+x) = \ln(1+y)(1+x) = A$.

Then $(1+y)(1+x) = xy + x + y + 1 = e^A = B$ and $xy + x + y = B - 1 = C$.

12. Show that $\sinh y + \cosh y = Cx$ may be written as $y = \ln x + A$.

Here $\sinh y + \cosh y = \frac{1}{2}(e^y - e^{-y}) + \frac{1}{2}(e^y + e^{-y}) = e^y = Cx$.

Then $y = \ln C + \ln x = A + \ln x$.

## SUPPLEMENTARY PROBLEMS

Show that each of the following expressions is a solution of the corresponding differential equation. Classify each as a particular solution or general solution (primitive).

| | | | |
|---|---|---|---|
| 13. | $y = 2x^2$, | $xy' = 2y$. | Particular solution |
| 14. | $x^2 + y^2 = C$, | $yy' + x = 0$. | Primitive |
| 15. | $y = Cx + C^4$, | $y = xy' + (y')^4$. | Primitive |
| 16. | $(1-x)y^2 = x^3$, | $2x^3y' = y(y^2 + 3x^2)$. | Particular solution |
| 17. | $y = e^x(1+x)$, | $y'' - 2y' + y = 0$. | Particular solution |
| 18. | $y = C_1x + C_2e^x$, | $(x-1)y'' - xy' + y = 0$. | General solution |
| 19. | $y = C_1e^x + C_2e^{-x}$, | $y'' - y = 0$. | General solution |
| 20. | $y = C_1e^x + C_2e^{-x} + x - 4$, | $y'' - y = 4 - x$. | General solution |
| 21. | $y = C_1e^x + C_2e^{2x}$, | $y'' - 3y' + 2y = 0$. | General solution |
| 22. | $y = C_1e^x + C_2e^{2x} + x^2e^x$, | $y'' - 3y' + 2y = 2e^x(1-x)$. | General solution |

CHAPTER 3

# Equations of First Order and First Degree

**A DIFFERENTIAL EQUATION** of the first order and first degree may be written in the form

1) $$M(x,y)\,dx + N(x,y)\,dy = 0.$$

EXAMPLE 1. *a)* $\frac{dy}{dx} + \frac{y+x}{y-x} = 0$ may be written as $(y+x)\,dx + (y-x)\,dy = 0$ in which $M(x,y) = y+x$ and $N(x,y) = y-x$.

*b)* $\frac{dy}{dx} = 1 + x^2y$ may be written as $(1+x^2y)\,dx - dy = 0$ in which $M(x,y) = 1 + x^2y$ and $N(x,y) = -1$.

If $M(x,y)\,dx + N(x,y)\,dy$ is the complete differential of a function $\mu(x,y)$, that is, if

$$M(x,y)\,dx + N(x,y)\,dy = d\mu(x,y).$$

1) is called exact and $\mu(x,y) = C$ is its primitive or general solution.

EXAMPLE 2. $3x^2y^2\,dx + 2x^3y\,dy = 0$ is an exact differential equation since $3x^2y^2\,dx + 2x^3y\,dy = d(x^3y^2)$. Its primitive is $x^3y^2 = C$.

If 1) is not exact but

$$\xi(x,y)\{M(x,y)\,dx + N(x,y)\,dy\} = d\mu(x,y),$$

$\xi(x,y)$ is called an integrating factor of 1) and $\mu(x,y) = C$ is its primitive.

EXAMPLE 3. $3y\,dx + 2x\,dy = 0$ is not an exact differential equation but when multiplied by $\xi(x,y) = x^2y$, we have $3x^2y^2\,dx + 2x^3y\,dy = 0$ which is exact. Hence, the primitive of $3y\,dx + 2x\,dy = 0$ is $x^3y^2 = C$. See Example 2.

If 1) is not exact and no integrating factor can be found readily, it may be possible by a change of one or both of the variables to obtain an equation for which an integrating factor can be found.

EXAMPLE 4. The transformation $x = t - y,\ dx = dt - dy,\ (\text{i.e., } x + y = t),$

reduces the equation $$(x+y+1)\,dx + (2x+2y+3)\,dy = 0$$

to $$(t+1)(dt-dy) + (2t+3)\,dy = 0$$

or $$(t+1)\,dt + (t+2)\,dy = 0.$$

By means of the integrating factor $\frac{1}{t+2}$ the equation takes the form

$$dy + \frac{t+1}{t+2}dt = dy + dt - \frac{1}{t+2}dt = 0.$$

Then $$y + t - \ln(t+2) = C$$

and, since $t = x+y$, $2y + x - \ln(x+y+2) = C$.

Note. The transformation $x + y + 1 = t$ or $2x + 2y + 3 = 2s$ is also suggested by the form of the equation.

**A DIFFERENTIAL EQUATION** for which an integrating factor is found readily has the form

2) $$f_1(x)\cdot g_2(y)\,dx + f_2(x)\cdot g_1(y)\,dy = 0.$$

By means of the integrating factor $\dfrac{1}{f_2(x)\cdot g_2(y)}$ ,2) is reduced to

2′) $$\frac{f_1(x)}{f_2(x)}\,dx + \frac{g_1(y)}{g_2(y)}\,dy = 0$$

whose primitive is

$$\int \frac{f_1(x)}{f_2(x)}\,dx + \int \frac{g_1(y)}{g_2(y)}\,dy = C.$$

Equation 2) is typed as *Variables Seperable* and in 2′) the variables are separated.

EXAMPLE 5. When the differential equation

$$(3x^2y - xy)\,dx + (2x^3y^2 + x^3y^4)\,dy = 0$$

is put in the form $y(3x^2 - x)\,dx + x^3(2y^2 + y^4)\,dy = 0$

it is seen to be of the type Variables Separable. The integrating factor $\dfrac{1}{yx^3}$ reduces it to $(\dfrac{3}{x} - \dfrac{1}{x^2})\,dx +$ $(2y + y^3)\,dy = 0$ in which the variables are separated. Integrating, we obtain the primitive

$$3\ln x + \frac{1}{x} + y^2 + \frac{1}{4}y^4 = C.$$

**IF EQUATION** 1) admits a solution $f(x,y,C) = 0$ where $C$ is an arbitrary constant, there exist infinitely many integrating factors $\xi(x,y)$ such that

$$\xi(x,y)\{M(x,y)\,dx + N(x,y)\,dy\} = 0$$

is exact. Also, there exist transformations of the variables which carry 1) into the type Variables Separable. However, no general rule can be stated here for finding either an integrating factor or a transformation. Thus we are limited to solving certain types of differential equations of the first order and first degree, i.e., those for which rules may be laid down for determining either an integrating factor or an effective transformation.

Equations of the type Variables Separable, together with equations which can be reduced to this type by a transformation of the variables are considered in Chapter 4.

Exact differential equations and other types reducible to exact equations by means of integrating factors are treated in Chapter 5.

The linear equation of order one

3) $$\frac{dy}{dx} + P(x)\cdot y = Q(x)$$

and equations reducible to the form 3) by means of transformations are considered in Chapter 6.

These groupings are a matter of convenience. A given equation may fall into more than one group.

EXAMPLE 6. The equation $x\,dy - y\,dx = 0$ may be placed in any one of the groups since

*a)* by means of the integrating factor $1/xy$ the variables are separated; thus, $dy/y - dx/x = 0$ so that $\ln y - \ln x = \ln C$ or $y/x = C$.

*b)* by means of the integrating factor $1/x^2$ or $1/y^2$ the equation is made exact; thus, $\dfrac{x\,dy - y\,dx}{x^2} = 0$ and $\dfrac{y}{x} = C$ or $\dfrac{x\,dy - y\,dx}{y^2} = 0$ and $-\dfrac{x}{y} = C_1$, $\dfrac{y}{x} = -\dfrac{1}{C_1} = C$.

*c)* when written as $\dfrac{dy}{dx} - \dfrac{1}{x}y = 0$, it is a linear equation of order one.

Attention has been called to the fact that the form of the primitive is not unique. Thus, the primitive in Example 6 might be given as

*a)* $\ln y - \ln x = \ln C$, *b)* $y/x = C$, *c)* $y = Cx$, *d)* $x/y = K$, etc.

It is usual to accept any one of these forms with the understanding, already noted, that thereby certain particular solutions may be lost. There is an additional difficulty!

EXAMPLE 7. It is clear that $y = 0$ is a particular solution of $dy/dx = y$ or $dy - y\,dx = 0$. When $y \neq 0$ we may write $dy/y - dx = 0$ and obtain $\ln y - x = \ln C$ with $C \neq 0$ in turn, this may be written as $y = Ce^x$, $C \neq 0$. Thus, to include all solutions, we should write $y = 0$; $y = Ce^x$, $C \neq 0$. But note that $y = Ce^x$ free of the restrictions imposed on $y$ and $C$, includes *all* solutions.

This situation will arise repeatedly as we proceed but, as is customary, we shall refrain from pointing out the restrictions; that is, we shall write the primitive as $y = Ce^x$ with $C$ completely arbitrary. In defense, we offer the following observation. Let us multiply the given equation by $e^{-x}$ to obtain $e^{-x}dy - ye^{-x}dx = 0$ from which, by integration, we get $e^{-x}y = C$ or $y = Ce^x$. In this procedure, it has not been necessary to impose any restriction $y$ or $C$.

CHAPTER 4

# Equations of First Order and First Degree

## VARIABLES SEPARABLE AND REDUCTION TO VARIABLES SEPARABLE

**VARIABLES SEPARABLE.** The variables of the equation $M(x,y)\,dx + N(x,y)\,dy = 0$ are separable if the equation can be written in the form

1) $$f_1(x)\cdot g_2(y)\,dx + f_2(x)\cdot g_1(y)\,dy = 0.$$

The integrating factor $\dfrac{1}{f_2(x)\cdot g_2(y)}$, found by inspection, reduces 1) to the form

$$\frac{f_1(x)}{f_2(x)}\,dx + \frac{g_1(y)}{g_2(y)}\,dy = 0$$

from which the primitive may be obtained by integration.

For example, $(x-1)^2 y\,dx + x^2(y+1)\,dy = 0$ is of the form 1). The integrating factor $\dfrac{1}{x^2 y}$ reduces the equation to $\dfrac{(x-1)^2}{x^2}\,dx + \dfrac{(y+1)}{y}\,dy = 0$ in which the variables are separated.

See Problems 1-5.

**HOMOGENEOUS EQUATIONS.** A function $f(x,y)$ is called homogeneous of degree $n$ if

$$f(\lambda x, \lambda y) = \lambda^n f(x,y).$$

For example:

*a)* $f(x,y) = x^4 - x^3 y$ is homogeneous of degree 4 since

$$f(\lambda x, \lambda y) = (\lambda x)^4 - (\lambda x)^3(\lambda y) = \lambda^4(x^4 - x^3 y) = \lambda^4 f(x,y).$$

*b)* $f(x,y) = e^{y/x} + \tan\dfrac{y}{x}$ is homogeneous of degree 0 since

$$f(\lambda x, \lambda y) = e^{\lambda y/\lambda x} + \tan\frac{\lambda y}{\lambda x} = e^{y/x} + \tan\frac{y}{x} = \lambda^0 f(x,y).$$

*c)* $f(x,y) = x^2 + \sin x\,\cos y$ is not homogeneous since

$$f(\lambda x, \lambda y) = \lambda^2 x^2 + \sin(\lambda x)\cos(\lambda y) \neq \lambda^n f(x,y).$$

The differential equation $M(x,y)\,dx + N(x,y)\,dy = 0$ is called homogeneous if $M(x,y)$ and $N(x,y)$ are homogeneous and of the same degree. For example, $x\ln\dfrac{y}{x}\,dx + \dfrac{y^2}{x}\arcsin\dfrac{y}{x}\,dy = 0$ is homogeneous of degree 1, but

neither $(x^2+y^2)\,dx - (xy^2 - y^3)\,dy = 0$ nor $(x+y^2)\,dx + (x-y)\,dy = 0$ is a homogeneous equation.

The transformation $y = vx, \quad dy = v\,dx + x\,dv$
will reduce any homogeneous equation to the form

$$P(x,v)\,dx + Q(x,v)\,dv = 0$$

in which the variables are separable. After integrating, $v$ is replaced by $y/x$ to recover the original variables.

See Problems 6-11.

**EQUATIONS IN WHICH $M(x,y)$ AND $N(x,y)$ ARE LINEAR BUT NOT HOMOGENEOUS.**

*a)* The equation $(a_1x + b_1y + c_1)\,dx + (a_2x + b_2y + c_2)\,dy = 0$, $(a_1b_2 - a_2b_1 = 0)$ is reduced by the transformation

$$a_1x + b_1y = t, \qquad dy = \frac{dt - a_1\,dx}{b_1}$$

to the form

$$P(x,t)\,dx + Q(x,t)\,dt = 0$$

in which the variables are separable. See Problem 12.

*b)* The equation $(a_1x + b_1y + c_1)\,dx + (a_2x + b_2y + c_2)\,dy = 0$, $(a_1b_2 - a_2b_1 \neq 0)$, is reduced t the homogeneous form

$$(a_1x' + b_1y')\,dx' + (a_2x' + b_2y')\,dy' = 0$$

by the transformation

$$x = x' + h, \qquad y = y' + k$$

in which $x = h$, $y = k$ are the solutions of the equations

$$a_1x + b_1y + c_1 = 0 \quad \text{and} \quad a_2x + b_2y + c_2 = 0.$$

See Problems 13-14.

**EQUATIONS OF THE FORM $y \cdot f(xy)\,dx + x \cdot g(xy)\,dy = 0$.** The transformation

$$xy = z, \qquad y = \frac{z}{x}, \qquad dy = \frac{x\,dz - z\,dx}{x^2}$$

reduces an equation of this form to the form

$$P(x,z)\,dx + Q(x,z)\,dz = 0$$

in which the variables are separable.

See Problems 15-17.

**OTHER SUBSTITUTIONS.** Equations, not of the types discussed above, may be reduced to a form in which the variables are separable by means of a properly chosen transformation. No general rule of procedure can be given; in each case the form of the equation suggests the transformation.

See Problems 18-22.

## SOLVED PROBLEMS

**VARIABLES SEPARABLE.**

1. Solve $x^3 dx + (y+1)^2 dy = 0$.

The variables are separated. Hence, integrating term by term,

$$\frac{x^4}{4} + \frac{(y+1)^3}{3} = C_1 \quad \text{or} \quad 3x^4 + 4(y+1)^3 = C.$$

2. Solve $x^2(y+1)dx + y^2(x-1)dy = 0$.

The integrating factor $\dfrac{1}{(y+1)(x-1)}$ reduces the equation to $\dfrac{x^2}{x-1}dx + \dfrac{y^2}{y+1}dy = 0$.

Then, integrating $(x+1+\dfrac{1}{x-1})dx + (y-1+\dfrac{1}{y+1})dy = 0$,

$$\tfrac{1}{2}x^2 + x + \ln(x-1) + \tfrac{1}{2}y^2 - y + \ln(y+1) = C_2,$$

$$x^2 + y^2 + 2x - 2y + 2\ln(x-1)(y+1) = C_1,$$

and

$$(x+1)^2 + (y-1)^2 + 2\ln(x-1)(y+1) = C.$$

3. Solve $4x\,dy - y\,dx = x^2dy$ or $y\,dx + (x^2-4x)dy = 0$.

The integrating factor $\dfrac{1}{y(x^2-4x)}$ reduces the equation to $\dfrac{dx}{x(x-4)} + \dfrac{dy}{y} = 0$ in which the variables are separated.

The latter equation may be written as $\dfrac{\frac{1}{4}dx}{x-4} - \dfrac{\frac{1}{4}dx}{x} + \dfrac{dy}{y} = 0$ or $\dfrac{dx}{x-4} - \dfrac{dx}{x} + 4\dfrac{dy}{y} = 0$.

Integrating, $\ln(x-4) - \ln x + 4\ln y = \ln C$ or $(x-4)y^4 = Cx$.

4. Solve $\dfrac{dy}{dx} = \dfrac{4y}{x(y-3)}$ or $x(y-3)dy = 4y\,dx$.

The integrating factor $\dfrac{1}{xy}$ reduces the equation to $\dfrac{y-3}{y}dy = \dfrac{4}{x}dx$.

Integrating, $y - 3\ln y = 4\ln x + \ln C_1$ or $y = \ln(C_1x^4y^3)$.

This may be written as $C_1x^4y^3 = e^y$ or $x^4y^3 = Ce^y$.

5. Find the particular solution of $(1+x^3)dy - x^2y\,dx = 0$ satisfying the initial conditions $x = 1, y = 2$.

First find the primitive, using the integrating factor $\dfrac{1}{y(1+x^3)}$.

Then $\dfrac{dy}{y} - \dfrac{x^2}{1+x^3}dx = 0$, $\ln y - \frac{1}{3}\ln(1+x^3) = C_1$, $3\ln y = \ln(1+x^3) + \ln C$, $y^3 = C(1+x^3)$.

When $x = 1$, $y = 2$: $2^3 = C(1+1)$, $C = 4$ and the required particular solution is $y^3 = 4(1+x^3)$.

## HOMOGENEOUS EQUATIONS.

6. When $M\,dx + N\,dy = 0$ is homogeneous, show that the transformation $y = vx$ will separate the variables.

When $M\,dx + N\,dy = 0$ is homogeneous of degree $n$, we may write

$$M\,dx + N\,dy = x^n\{M_1(\tfrac{y}{x})dx + N_1(\tfrac{y}{x})dy\} = 0 \quad \text{whence} \quad M_1(\tfrac{y}{x})dx + N_1(\tfrac{y}{x})dy = 0.$$

The transformation $y = vx$, $dy = v\,dx + x\,dv$ reduces this to

$$M_1(v)\,dx + N_1(v)\{v\,dx + x\,dv\} = 0 \quad \text{or} \quad \{M_1(v) + vN_1(v)\}dx + xN_1(v)\,dv = 0$$

or, finally, $\dfrac{dx}{x} + \dfrac{N_1(v)\,dv}{M_1(v) + vN_1(v)} = 0$ in which the variables are separated.

7. Solve $(x^3+y^3)dx - 3xy^2dy = 0$.

The equation is homogeneous of degree 3. We use the transformation $y = vx,\ dy = v\,dx + x\,dv$ to obtain

$$1)\quad x^3\{(1+v^3)dx - 3v^2(v\,dx + x\,dv)\} = 0 \quad \text{or} \quad (1-2v^3)dx - 3v^2x\,dv = 0$$

in which the variables are separable.

Upon separating the variables, using the integrating factor $\dfrac{1}{x(1-2v^3)}$, $\dfrac{dx}{x} - \dfrac{3v^2\,dv}{1-2v^3} = 0$, and

$\ln x + \frac{1}{2}\ln(1-2v^3) = C_1$, $\quad 2\ln x + \ln(1-2v^3) = \ln C$, $\quad$ or $\quad x^2(1-2v^3) = C$.

Since $v = y/x$, the primitive is $x^2(1-2y^3/x^3) = C$ or $x^3 - 2y^3 = Cx$.

Note that the equation is of degree 3 and that after the transformation $x^3$ is a factor of the left member of 1). This factor may be removed when making the transformation.

8. Solve $x\,dy - y\,dx - \sqrt{x^2-y^2}\,dx = 0$.

The equation is homogeneous of degree 1. Using the transformation $y = vx,\ dy = v\,dx + x\,dv$ and dividing by $x$, we have

$$v\,dx + x\,dv - v\,dx - \sqrt{1-v^2}\,dx = 0 \quad \text{or} \quad x\,dv - \sqrt{1-v^2}\,dx = 0.$$

When the variables are separated, using the integrating factor $\dfrac{1}{x\sqrt{1-v^2}}$, $\dfrac{dv}{\sqrt{1-v^2}} - \dfrac{dx}{x} = 0$.

Then $\arcsin v - \ln x = \ln C$ or $\arcsin v = \ln(Cx)$ and returning to the original variables, using $v = y/x$, $\arcsin \dfrac{y}{x} = \ln(Cx)$ or $Cx = e^{\arcsin y/x}$.

9. Solve $(2x \sinh \frac{y}{x} + 3y \cosh \frac{y}{x})dx - 3x \cosh \frac{y}{x}\,dy = 0$.

The equation is homogeneous of degree 1. Using the standard transformation and dividing by $x$, we have

$$2 \sinh v\ dx - 3x \cosh v\ dv = 0.$$

Then, separating the variables, $2\,\dfrac{dx}{x} - 3\,\dfrac{\cosh v}{\sinh v}\,dv = 0$.

Integrating, $2\ln x - 3\ln\sinh v = \ln C$, $\quad x^2 = C\sinh^3 v$, $\quad$ and $\quad x^2 = C\sinh^3 \dfrac{y}{x}$.

10. Solve $(2x+3y)dx + (y-x)dy = 0$.

The equation is homogeneous of degree 1. The standard transformation reduces it to

$$(2+3v)dx + (v-1)(v\,dx + x\,dv) = 0 \quad \text{or} \quad (v^2+2v+2)dx + x(v-1)dv = 0.$$

Separating the variables, $\dfrac{dx}{x} + \dfrac{v-1}{v^2+2v+2}\,dv = \dfrac{dx}{x} + \dfrac{1}{2}\,\dfrac{2v+2}{v^2+2v+2}\,dv - \dfrac{2\,dv}{(v+1)^2+1} = 0$.

Integrating, $\ln x + \frac{1}{2}\ln(v^2+2v+2) - 2\arctan(v+1) = C_1$,

$\ln x^2(v^2+2v+2) - 4\arctan(v+1) = C$, $\quad$ and $\quad \ln(y^2+2xy+2x^2) - 4\arctan\dfrac{x+y}{x} = C$.

11. Solve $(1+2e^{x/y})dx + 2e^{x/y}(1-\frac{x}{y})dy = 0$.

The equation is homogeneous of degree 0 . The appearance of $x/y$ throughout the equation suggests the use of the transformation $x = vy$, $dx = v\,dy + y\,dv$.

Then $(1+2e^{v})(v\,dy + y\,dv) + 2e^{v}(1-v)dy = 0$, $\quad (v+2e^{v})dy + y(1+2e^{v})dv = 0$,

and
$$\frac{dy}{y} + \frac{1+2e^{v}}{v+2e^{v}}\,dv = 0.$$

Integrating and replacing $v$ by $x/y$, $\ln y + \ln(v+2e^{v}) = \ln C$ and $x + 2ye^{x/y} = C$.

**LINEAR BUT NOT HOMOGENEOUS.**

**12.** Solve $(x+y)dx + (3x+3y-4)dy = 0$.

The expressions $(x+y)$ and $(3x+3y)$ suggest the transformation $x+y = t$.

We use $y = t-x$, $dy = dt - dx$ to obtain $t\,dx + (3t-4)(dt-dx) = 0$ or $(4-2t)dx + (3t-4)dt = 0$

in which the variables are separable.

Then $2dx + \dfrac{3t-4}{2-t}\,dt = 2\,dx - 3\,dt + \dfrac{2}{2-t}\,dt = 0.$

Integrating and replacing by $x+y$, we have

$2x - 3t - 2\ln(2-t) = C_1$, $\quad 2x - 3(x+y) - 2\ln(2-x-y) = C_1$, and $\quad x + 3y + 2\ln(2-x-y) = C$.

**13.** Solve $(2x-5y+3)dx - (2x+4y-6)dy = 0$.

First solve $2x-5y+3 = 0$, $2x+4y-6 = 0$ simultaneously to obtain $x = h = 1$, $y = k = 1$.

The transformation $x = x' + h = x' + 1$, $\quad dx = dx'$
$y = y' + k = y' + 1$, $\quad dy = dy'$

reduces the given equation to $(2x' - 5y')dx' - (2x' + 4y')dy' = 0$

which is homogeneous of degree 1. (Note that this latter equation can be written down without carrying out the details of the transformation

Using the transformation $\quad y' = vx'$, $dy' = v\,dx' + x'\,dv$,

we obtain $\quad (2-5v)dx' - (2+4v)(v\,dx' + x'\,dv) = 0$, $\quad (2-7v-4v^2)dx' - x'(2+4v)dv = 0$,

and finally
$$\frac{dx'}{x'} + \frac{4}{3}\frac{dv}{4v-1} + \frac{2}{3}\frac{dv}{v+2} = 0.$$

Integrating, $\quad \ln x' + \frac{1}{3}\ln(4v-1) + \frac{2}{3}\ln(v+2) = \ln C_1$ or $\quad x'^3(4v-1)(v+2)^2 = C$.

Replacing $v$ by $y'/x'$, $\quad (4y'-x')(y'+2x')^2 = C$,

and replacing $x'$ by $x-1$ and $y'$ by $y-1$ we obtain the primitive $(4y-x-3)(y+2x-3)^2 = C$.

**14.** Solve $(x-y-1)dx + (4y+x-1)dy = 0$.

Solving $x-y-1 = 0$, $4y+x-1 = 0$ simultaneously, we obtain $x = h = 1$, $y = k = 0$.

The transformation $x = x' + h = x' + 1$, $\quad dx = dx'$
$y = y' + k = y'$ , $\quad dy = dy'$

reduces the given equation to $(x'-y')dx' + (4y'+x')dy' = 0$ which is homogeneous of degree 1. (Note that this transformation $x-1 = x'$, $y = y'$ could have been obtained by inspection, that is, by examining the terms $(x-y-1)$ and $(4y+x-1)$.)

Using the transformation $y' = vx', \quad dy' = v\,dx' + x'\,dv$

we obtain $$(1-v)dx' + (4v+1)(v\,dx' + x'\,dv) = 0.$$

Then $$\frac{dx'}{x'} + \frac{4v+1}{4v^2+1}dv = \frac{dx'}{x'} + \tfrac{1}{2}\,\frac{8v}{4v^2+1}dv + \frac{dv}{4v^2+1} = 0,$$

$$\ln x' + \tfrac{1}{2}\ln(4v^2+1) + \tfrac{1}{2}\arctan 2v = C_1, \qquad \ln x'^2(4v^2+1) + \arctan 2v = C,$$

$$\ln(4y'^2 + x'^2) + \arctan\frac{2y'}{x'} = C, \quad \text{and} \quad \ln[4y^2 + (x-1)^2] + \arctan\frac{2y}{x-1} = C.$$

**FORM** $y\,f(xy)\,dx + x\,g(xy)\,dy = 0.$

**15.** Solve $y(xy+1)dx + x(1+xy+x^2y^2)dy = 0.$

The transformation $xy = v, \quad y = v/x, \quad dy = \dfrac{x\,dv - v\,dx}{x^2}$

reduces the equation to $$\frac{v}{x}(v+1)dx + x(1+v+v^2)\frac{x\,dv - v\,dx}{x^2} = 0$$

which, after clearing of fractions and rearranging, becomes $v^3\,dx - x(1+v+v^2)dv = 0.$

Separating the variables, we have $\dfrac{dx}{x} - \dfrac{dv}{v^3} - \dfrac{dv}{v^2} - \dfrac{dv}{v} = 0.$

Then $$\ln x + \frac{1}{2v^2} + \frac{1}{v} - \ln v = C_1, \qquad 2v^2\ln\left(\frac{v}{x}\right) - 2v - 1 = Cv^2,$$

and $$2x^2y^2\ln y - 2xy - 1 = Cx^2y^2.$$

**16.** Solve $(y - xy^2)dx - (x + x^2y)dy = 0$ or $y(1-xy)dx - x(1+xy)dy = 0.$

The transformation $xy = v, \; y = v/x, \; dy = \dfrac{x\,dv - v\,dx}{x^2}$ reduces the equation to

$$\frac{v}{x}(1-v)dx - x(1+v)\frac{x\,dv - v\,dx}{x^2} = 0 \quad \text{or} \quad 2v\,dx - x(1+v)dv = 0.$$

Then $2\dfrac{dx}{x} - \dfrac{1+v}{v}dv = 0, \quad 2\ln x - \ln v - v = \ln C, \quad \dfrac{x^2}{v} = Ce^v,$ and $x = Cye^{xy}.$

**17.** Solve $(1 - xy + x^2y^2)dx + (x^3y - x^2)dy = 0$ or $y(1 - xy + x^2y^2)dx + x(x^2y^2 - xy)dy = 0.$

The transformation $xy = v, \; y = v/x, \; dy = \dfrac{x\,dv - v\,dx}{x^2}$ reduces the equation to

$$\frac{v}{x}(1 - v + v^2)dx + x(v^2 - v)\frac{x\,dv - v\,dx}{x^2} = 0 \quad \text{or} \quad v\,dx + x(v^2 - v)dv = 0.$$

Then $\dfrac{dx}{x} + (v-1)dv = 0, \quad \ln x + \tfrac{1}{2}v^2 - v = C,$ and $\ln x = xy - \tfrac{1}{2}x^2y^2 + C.$

## MISCELLANEOUS SUBSTITUTIONS.

18. Solve $\frac{dy}{dx} = (y-4x)^2$ or $dy = (y-4x)^2\,dx$.

The suggested transformation $y-4x = v$, $dy = 4dx + dv$ reduces the equation to

$$4\,dx + dv = v^2 dx \quad \text{or} \quad dx - \frac{dv}{v^2-4} = 0.$$

Then $x + \frac{1}{4}\ln\frac{v+2}{v-2} = C_1$, $\ln\frac{v+2}{v-2} = \ln C - 4x$, $\frac{v+2}{v-2} = Ce^{-4x}$, and $\frac{y-4x+2}{y-4x-2} = Ce^{-4x}$.

19. Solve $\tan^2(x+y)dx - dy = 0$.

The suggested transformation $x+y = v$, $dy = dv - dx$ reduces the equation to

$$\tan^2 v\,dx - (dv-dx) = 0, \quad dx - \frac{dv}{1+\tan^2 v} = 0, \quad \text{or} \quad dx - \cos^2 v\,dv = 0.$$

Integrating, $x - \frac{1}{2}v - \frac{1}{4}\sin 2v = C_1$ and $2(x-y) = C + \sin 2(x+y)$.

20. Solve $(2+2x^2y^{1/2})y\,dx + (x^2y^{1/2}+2)x\,dy = 0$.

The suggested transformation $x^2y^{1/2} = v$, $y = \frac{v^2}{x^4}$, $dy = \frac{2v}{x^4}dv - \frac{4v^2}{x^5}dx$ reduces the equation to

$$(2+2v)\frac{v^2}{x^4}dx + x(v+2)\left(\frac{2v}{x^4}dv - \frac{4v^2}{x^5}dx\right) = 0 \quad \text{or} \quad v(3+v)dx - x(v+2)dv = 0.$$

Then $\frac{dx}{x} - \frac{2}{3}\frac{dv}{v} - \frac{1}{3}\frac{dv}{v+3} = 0$, $3\ln x - 2\ln v - \ln(v+3) = \ln C_1$, $x^3 = C_1 v^2(v+3)$,

and $1 = C_1 xy(x^2y^{1/2}+3)$ or $xy(x^2y^{1/2}+3) = C$.

21. Solve $(2x^2+3y^2-7)x\,dx - (3x^2+2y^2-8)y\,dy = 0$.

The suggested transformation $x^2 = u$, $y^2 = v$ reduces the equation to

$$(2u+3v-7)du - (3u+2v-8)dv = 0$$

which is linear but not homogeneous.

The transformation $u = s+2$, $v = t+1$ yields the homogeneous equation $(2s+3t)ds - (3s+2t)dt = 0$ and the transformation $s = rt$, $ds = r\,dt + t\,dr$ yields $2(r^2-1)dt + (2r+3)t\,dr = 0$.

Separating the variables, we have $2\frac{dt}{t} + \frac{2r+3}{r^2-1}dr = 2\frac{dt}{t} - \frac{1}{2}\frac{dr}{r+1} + \frac{5}{2}\frac{dr}{r-1} = 0$.

Then $$4\ln t - \ln(r+1) + 5\ln(r-1) = \ln C,$$

$\frac{t^4(r-1)^5}{r+1} = \frac{(s-t)^5}{s+t} = \frac{(u-v-1)^5}{u+v-3} = \frac{(x^2-y^2-1)^5}{x^2+y^2-3} = C$, and $(x^2-y^2-1)^5 = C(x^2+y^2-3)$.

22. Solve $x^2(x\,dx + y\,dy) + y(x\,dy - y\,dx) = 0$.

Here $x\,dx + y\,dy = \frac{1}{2}d(x^2+y^2)$ and $x\,dy - y\,dx = x^2 d(y/x)$ suggests $x^2+y^2 = \rho^2$, $y/x = \tan\theta$ or $x = \rho\cos\theta$, $y = \rho\sin\theta$, $dx = -\rho\sin\theta\,d\theta + \cos\theta\,d\rho$, $dy = \rho\cos\theta\,d\theta + \sin\theta\,d\rho$.

The given equation takes the form $\rho^2\cos^2\theta\,(\rho\,d\rho) + \rho\sin\theta\,(\rho^2\,d\theta) = 0$

or $d\rho + \tan\theta\sec\theta\,d\theta = 0$.

Then $\rho + \sec\theta = C_1$, $\sqrt{x^2+y^2}\left(\frac{x+1}{x}\right) = C_1$ and $(x^2+y^2)(x+1)^2 = Cx^2$.

## SUPPLEMENTARY PROBLEMS

23. Determine whether or not each of the following functions is homogeneous and, when homogeneous, state the degree.

| | | | |
|---|---|---|---|
| *a)* $x^2 - xy$, | homo. of degree two. | *e)* $\arcsin xy$, | not homo. |
| *b)* $\dfrac{xy}{x+y^2}$, | not homo. | *f)* $xe^{y/x} + ye^{x/y}$, | homo. of degree one. |
| *c)* $\dfrac{xy}{x^2+y^2}$; | homo. of degree zero. | *g)* $\ln x - \ln y$ or $\ln\dfrac{x}{y}$, | homo. of degree zero. |
| | | *h)* $\sqrt{x^2 + 2xy + 3y^2}$ | homo. of degree one. |
| *d)* $x + y\cos\dfrac{y}{x}$, | homo. of degree one. | *i)* $x\sin y + y\sin x$, | not homo. |

Classify each of the equations below in one or more of the following categories:

(1) Variables separable
(2) Homogeneous equations
(3) Equations in which $M(x,y)$ and $N(x,y)$ are linear but not homogeneous
(4) Equations of the form $y\,f(xy)dx + x\,g(xy)dy = 0$
(5) None of the above apply.

24. $4y\,dx + x\,dy = 0$ *Ans.* (1); (2), of degree one
25. $(1+2y)dx + (4-x^2)dy = 0$ (1)
26. $y^2\,dx - x^2\,dy = 0$ (1); (2), of degree two
27. $(1+y)dx - (1+x)dy = 0$ (1); (3)
28. $(xy^2 + y)dx + (x^2y - x)dy = 0$ (4)
29. $(x\sin\frac{y}{x} - y\cos\frac{y}{x})dx + x\cos\frac{y}{x}\,dy = 0$ (2), of degree one
30. $y^2(x^2+2)dx + (x^3+y^3)(y\,dx - x\,dy) = 0$ (5)
31. $y\sqrt{x^2+y^2}\,dx - x(x + \sqrt{x^2+y^2})dy = 0$ (2), of degree two
32. $(x+y+1)dx + (2x+2y+1)dy = 0$ (3)

33. Solve each of the above equations (Problems 24-32) which fall in categories (1)-(4).

*Ans.* 24. $x^4y = C$ 28. $y = Cxe^{xy}$

25. $(1+2y)^2 = C\,\dfrac{2-x}{2+x}$ 29. $x\sin\dfrac{y}{x} = C$

26. $y = x + Cxy$ 31. $Cx - \sqrt{x^2+y^2} = x\ln(\sqrt{x^2+y^2} - x)$

27. $(1+y) = C(1+x)$ 32. $x + 2y + \ln(x+y) = C$

Solve each of the following equations.

34. $(1+2y)dx - (4-x)dy = 0$ *Ans.* $(x-4)^2(1+2y) = C$

35. $xy\,dx + (1+x^2)dy = 0$ Ans. $y^2(1+x^2) = C$

36. $\cot\theta\,d\rho + \rho\,d\theta = 0$ Ans. $\rho = C\cos\theta$

37. $(x+2y)dx + (2x+3y)dy = 0$ Ans. $x^2 + 4xy + 3y^2 = C$

38. $2x\,dy - 2y\,dx = \sqrt{x^2+4y^2}\,dx$ Ans. $1 + 4Cy - C^2x^2 = 0$

39. $(3y-7x+7)dx + (7y-3x+3)dy = 0$ Ans. $(y-x+1)^2(y+x-1)^5 = C$

40. $xy\,dy = (y+1)(1-x)dx$ Ans. $y+x = \ln Cx(y+1)$

41. $(y^2-x^2)dx + xy\,dy = 0$ Ans. $2x^2y^2 = x^4 + C$

42. $y(1+2xy)dx + x(1-xy)dy = 0$ Ans. $y = Cx^2e^{-1/xy}$

43. $dx + (1-x^2)\cot y\,dy = 0$ Ans. $\sin^2 y = C\dfrac{1-x}{1+x}$

44. $(x^3+y^3)dx + 3xy^2dy = 0$ Ans. $x^4 + 4xy^3 = C$

45. $(3x+2y+1)dx - (3x+2y-1)dy = 0$ Ans. $\ln(15x+10y-1) + \frac{5}{2}(x-y) = C$

In each of the following, find the particular solution indicated.

46. $x\,dy + 2y\,dx = 0$; when $x = 2,\ y = 1$. Ans. $x^2y = 4$

47. $(x^2+y^2)dx + xy\,dy = 0$ ; when $x = 1,\ y = -1$. Ans. $x^4 + 2x^2y^2 = 3$

48. $\cos y\,dx + (1+e^{-x})\sin y\,dy = 0$; when $x = 0,\ y = \pi/4$. Ans. $(1+e^x)\sec y = 2\sqrt{2}$

49. $(y^2+xy)dx - x^2dy = 0$ ; when $x = 1,\ y = 1$. Ans. $x = e^{1-x/y}$

50. Solve the equation of Problem 30 using the substitution $y = vx$.

Ans. $x^2y\ln x - y + x^3 - \frac{1}{2}y^3 = Cx^2y$

51. Solve $y' = -2(2x+3y)^2$ using the substitution $z = 2x+3y$.

Ans. $\dfrac{1+\sqrt{3}(2x+3y)}{1-\sqrt{3}(2x+3y)} = Ce^{4\sqrt{3}\,x}$

52. Solve $(x - 2\sin y + 3)dx + (2x - 4\sin y - 3)\cos y\,dy = 0$ using the substitution $\sin y = z$.

Ans. $8\sin y + 4x + 9\ln(4x - 8\sin y + 3) = C$

# Equations of First Order and First Degree

## EXACT EQUATIONS AND REDUCTION TO EXACT EQUATIONS

**THE NECESSARY AND SUFFICIENT CONDITION** that

1) $$M(x,y)\,dx + N(x,y)\,dy = 0$$

be exact is

2) $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

At times an equation may be seen to be exact after a regrouping of its terms. The equation in the reg-grouped form may then be integrated term by term.

For example, $(x^2 - y)dx + (y^2 - x)dy = 0$ is exact since

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(x^2 - y) = -1 = \frac{\partial}{\partial x}(y^2 - x) = \frac{\partial N}{\partial x}.$$

This may also be seen after regrouping thus: $x^2 dx + y^2 dy - (y\,dx + x\,dy) = 0.$

This equation may be integrated term by term to obtain the primitive $x^3/3 + y^3/3 - xy = C$. The equation $(y^2 - x)dx + (x^2 - y)dy = 0$, however, is not exact since $\frac{\partial M}{\partial y} = 2y \neq 2x = \frac{\partial N}{\partial x}$.

See also Problem 1.

**IF 1) IS THE EXACT DIFFERENTIAL** of the equation $\mu(x,y) = C$,

$$d\mu = \frac{\partial \mu}{\partial x}dx + \frac{\partial \mu}{\partial y}dy = M(x,y)\,dx + N(x,y)\,dy.$$

Then $\frac{\partial \mu}{\partial x}dx = M(x,y)\,dx$ and $\mu(x,y) = \int^x M(x,y)\,dx + \phi(y)$,

where $\int^x$ indicates that in the integrating $y$ is to be treated as a constant and $\phi(y)$ is the constant (with respect to $x$) of integration. Now

$$\frac{\partial \mu}{\partial y} = \frac{\partial}{\partial y}\{\int^x M(x,y)\,dx\} + \frac{d\phi}{dy} = N(x,y)$$

from which $\frac{d\phi}{dy} = \phi'(y)$ and, hence, $\phi(y)$ can be found. See Problems 2-3.

**INTEGRATING FACTORS.** If 1) is not exact, an integrating factor is sought.

*a)* If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ a function of $x$ alone, then $e^{\int f(x)dx}$ is an integrating factor of 1).

If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{M} = -g(y)$, a function of $y$ alone, then $e^{\int g(y)\,dy}$ is an integrating factor of 1).

See Problems 4-6.

*b)* If 1) is homogeneous and $Mx + Ny \neq 0$, then $\dfrac{1}{Mx + Ny}$ is an integrating factor.

See Problems 7-9.

*c)* If 1) can be written in the form $y\,f(xy)\,dx + x\,g(xy)\,dy = 0$, where $f(xy) \neq g(xy)$ then

$\dfrac{1}{xy\{f(xy) - g(xy)\}} = \dfrac{1}{Mx - Ny}$ is an integrating factor.

See Problems 10-12.

*d)* At times an integrating factor may be found by inspection, after regrouping the terms of the equation, by recognizing a certain group of terms as being a part of an exact differential. For example:

| Group of Terms | Integrating Factor | Exact Differential |
|---|---|---|
| $x\,dy - y\,dx$ | $\dfrac{1}{x^2}$ | $\dfrac{x\,dy - y\,dx}{x^2} = d(\dfrac{y}{x})$ |
| $x\,dy - y\,dx$ | $\dfrac{1}{y^2}$ | $-\dfrac{y\,dx - x\,dy}{y^2} = d(-\dfrac{x}{y})$ |
| $x\,dy - y\,dx$ | $\dfrac{1}{xy}$ | $\dfrac{dy}{y} - \dfrac{dx}{x} = d(\ln\dfrac{y}{x})$ |
| $x\,dy - y\,dx$ | $\dfrac{1}{x^2 + y^2}$ | $\dfrac{x\,dy - y\,dx}{x^2 + y^2} = \dfrac{\frac{x\,dy - y\,dx}{x^2}}{1 + (\frac{y}{x})^2} = d(\arctan\dfrac{y}{x})$ |
| $x\,dy + y\,dx$ | $\dfrac{1}{(xy)^n}$ | $\dfrac{x\,dy + y\,dx}{(xy)^n} = d\{\dfrac{-1}{(n-1)(xy)^{n-1}}\}$, if $n \neq 1$<br>$\dfrac{x\,dy + y\,dx}{xy} = d\{\ln(xy)\}$, if $n = 1$ |
| $x\,dx + y\,dy$ | $\dfrac{1}{(x^2 + y^2)^n}$ | $\dfrac{x\,dx + y\,dy}{(x^2 + y^2)^n} = d\{\dfrac{-1}{2(n-1)(x^2 + y^2)^{n-1}}\}$, if $n \neq 1$<br>$\dfrac{x\,dx + y\,dy}{x^2 + y^2} = d\{\frac{1}{2}\ln(x^2 + y^2)\}$, if $n = 1$ |

See Problems 13-19.

*e)* The equation $x^r y^s (my\,dx + nx\,dy) + x^\rho y^\sigma (\mu y\,dx + \nu x\,dy) = 0$ where $r, s, m, n, \rho, \sigma, \mu, \nu$ are constants and $m\nu - n\mu \neq 0$, has an integrating factor of the form $x^\alpha y^\beta$ The method of solution usually given consists of determining $\alpha$ and $\beta$ by means of certain derived formulas. In Problems 20-22, a procedure, essentially that used in deriving the formulas, is followed.

## SOLVED PROBLEMS

1. Show first by the use of 2) and then by regrouping of terms that each equation is exact, and solve.

*a)* $(4x^3y^3 - 2xy)dx + (3x^4y^2 - x^2)dy = 0$

*b)* $(3e^{3x}y - 2x)dx + e^{3x}dy = 0$

*c)* $(\cos y + y\cos x)dx + (\sin x - x\sin y)dy = 0$

*d)* $2x(ye^{x^2} - 1)dx + e^{x^2}dy = 0$

*e)* $(6x^5y^3 + 4x^3y^5)dx + (3x^6y^2 + 5x^4y^4)dy = 0$

*a)* By 2): $\dfrac{\partial M}{\partial y} = 12x^3y^2 - 2x = \dfrac{\partial N}{\partial x}$ and the equation is exact.

By inspection: $(4x^3y^3dx + 3x^4y^2dy) - (2xy\,dx + x^2dy) = d(x^4y^3) - d(x^2y) = 0.$

The primitive is $x^4y^3 - x^2y = C.$

*b)* By 2): $\dfrac{\partial M}{\partial y} = 3e^{3x} = \dfrac{\partial N}{\partial x}$ and the equation is exact.

By inspection: $(3e^{3x}y\,dx + e^{3x}dy) - 2x\,dx = d(e^{3x}y) - d(x^2) = 0.$

The primitive is $e^{3x}y - x^2 = C.$

*c)* By 2): $\dfrac{\partial M}{\partial y} = -\sin y + \cos x = \dfrac{\partial N}{\partial x}$ and the equation is exact.

By inspection: $(\cos y\,dx - x\sin y\,dy) + (y\cos x\,dx + \sin x\,dy)$

$= d(x\cos y) + d(y\sin x) = 0.$ The primitive is $x\cos y + y\sin x = C.$

*d)* By 2): $\dfrac{\partial M}{\partial y} = 2xe^{x^2} = \dfrac{\partial N}{\partial x}$ and the equation is exact.

By inspection: $(2xye^{x^2}dx + e^{x^2}dy) - 2x\,dx = d(ye^{x^2}) - d(x^2) = 0.$

The primitive is $ye^{x^2} - x^2 = C.$

*e)* By 2): $\dfrac{\partial M}{\partial y} = 18x^5y^2 + 20x^3y^4 = \dfrac{\partial N}{\partial x}$ and the equation is exact.

By inspection: $(6x^5y^3dx + 3x^6y^2dy) + (4x^3y^5dx + 5x^4y^4dy) = d(x^6y^3) + d(x^4y^5) = 0.$

The primitive is $x^6y^3 + x^4y^5 = C.$

2. Solve $(2x^3 + 3y)dx + (3x + y - 1)dy = 0.$

$\dfrac{\partial M}{\partial y} = 3 = \dfrac{\partial N}{\partial x}$ and the equation is exact.

*Solution* 1. Set $\mu(x,y) = \int^x (2x^3 + 3y)dx = \tfrac{1}{2}x^4 + 3xy + \phi(y).$

Then $\dfrac{\partial \mu}{\partial y} = 3x + \phi'(y) = N(x,y) = 3x + y - 1, \quad \phi'(y) = y - 1, \quad \phi(y) = \tfrac{1}{2}y^2 - y,$

and the primitive is $\tfrac{1}{2}x^4 + 3xy + \tfrac{1}{2}y^2 - y = C_1$ or $x^4 + 6xy + y^2 - 2y = C.$

*Solution* 2. Grouping the terms thus $2x^3dx + y\,dy - dy + 3(y\,dx + x\,dy) = 0$ and recalling that $y\,dx + x\,dy = d(xy)$, we obtain, by integration, $\tfrac{1}{2}x^4 + \tfrac{1}{2}y^2 - y + 3xy = C_1$ as before.

3. Solve $(y^2e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0.$

$$\frac{\partial M}{\partial y} = 2ye^{xy^2} + 2xy^3e^{xy^2} = \frac{\partial N}{\partial x}$$ and the equation is exact.

Set $\mu(x,y) = \int^x (y^2e^{xy^2} + 4x^3)dx = e^{xy^2} + x^4 + \phi(y).$

Then $\frac{\partial \mu}{\partial y} = 2xye^{xy^2} + \phi'(y) = 2xye^{xy^2} - 3y^2, \quad \phi'(y) = -3y^2, \quad \phi(y) = -y^3,$

and the primitive is $e^{xy^2} + x^4 - y^3 = C.$

The equation may be solved by regrouping thus $4x^3dx - 3y^2dy + (y^2e^{xy^2}dx + 2xye^{xy^2}dy) = 0$ and noting that $y^2e^{xy^2}dx + 2xye^{xy^2}dy = d(e^{xy^2}).$

4. Solve $(x^2 + y^2 + x)dx + xy\,dy = 0.$

$\frac{\partial M}{\partial y} = 2y, \ \frac{\partial N}{\partial x} = y;$ the equation is not exact.

However, $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y-y}{xy} = \frac{1}{x} = f(x) \quad \text{and} \quad e^{\int f(x)dx} = e^{\int dx/x} = e^{\ln x} = x$$
is an integrating factor. Introducing the integrating factor, we have

$$(x^3 + xy^2 + x^2)dx + x^2y\,dy = 0 \qquad \text{or} \qquad x^3dx + x^2dx + (xy^2dx + x^2y\,dy) = 0.$$

Then, noting that $xy^2dx + x^2y\,dy = d(\tfrac{1}{2}x^2y^2)$, we have for the primitive

$$\frac{x^4}{4} + \frac{x^3}{3} + \tfrac{1}{2}x^2y^2 = C_1 \qquad \text{or} \qquad 3x^4 + 4x^3 + 6x^2y^2 = C.$$

5. Solve $(2y^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0.$

$\frac{\partial M}{\partial y} = 8xy^3e^y + 2xy^4e^y + 6xy^2 + 1, \ \frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3;$ the equation is not exact.

However, $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 8xy^3e^y + 8xy^2 + 4$ and $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{4}{y} = -g(y).$$

Then $e^{\int g(y)dy} = e^{-4\int dy/y} = e^{-4\ln y} = 1/y^4$ is an integrating factor and, upon introducing it the equation takes the form

$$(2xe^y + 2\frac{x}{y} + \frac{1}{y^3})dx + (x^2e^y - \frac{x^2}{y^2} - 3\frac{x}{y^4})dy = 0 \quad \text{and is exact.}$$

Set $\mu(x,y) = \int^x (2xe^y + 2\frac{x}{y} + \frac{1}{y^3})dx = x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} + \phi(y).$

Then $\frac{\partial \mu}{\partial y} = x^2e^y - \frac{x^2}{y^2} - 3\frac{x}{y^4} + \phi'(y) = x^2e^y - \frac{x^2}{y^2} - 3\frac{x}{y^4}, \ \phi'(y) = 0, \ \phi(y) =$ constant, and

the primitive is $x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = C.$

—5

6. Solve $(2x^3y^2 + 4x^2y + 2xy^2 + xy^4 + 2y)dx + 2(y^3 + x^2y + x)dy = 0.$

$\frac{\partial M}{\partial y} = 4x^3y + 4x^2 + 4xy + 4xy^3 + 2,\ \frac{\partial N}{\partial x} = 2(2xy+1)$; the equation is not exact.

$\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = 2x$ and the integrating factor is $e^{\int 2x\,dx} = e^{x^2}$. When it is introduced, the given equation becomes

$$(2x^3y^2 + 4x^2y + 2xy^2 + xy^4 + 2y)e^{x^2}dx + 2(y^3 + x^2y + x)e^{x^2}dy = 0 \quad \text{and is exact.}$$

Set

$$\begin{aligned}\mu(x,y) &= \int^x (2x^3y^2 + 4x^2y + 2xy^2 + xy^4 + 2y)e^{x^2}dx \\ &= \int^x (2xy^2 + 2x^3y^2)e^{x^2}dx + \int^x (2y + 4x^2y)e^{x^2}dx + \int^x xy^4e^{x^2}dx \\ &= x^2y^2e^{x^2} + 2xye^{x^2} + \tfrac{1}{2}y^4e^{x^2} + \phi(y).\end{aligned}$$

Then $\frac{\partial \mu}{\partial y} = 2x^2ye^{x^2} + 2xe^{x^2} + 2y^3e^{x^2} + \phi'(y) = 2(y^3 + x^2y + x)e^{x^2}$, $\phi'(y) = 0$, and the primitive is $(2x^2y^2 + 4xy + y^4)e^{x^2} = C$.

7. Show that $\frac{1}{Mx + Ny}$, where $Mx + Ny$ is not identically zero, is an integrating factor of the homogeneous equation $M(x,y)dx + N(x,y)dy = 0$ of degree $n$. Investigate the case $Mx + Ny = 0$ identically.

We are to show that $\frac{M}{Mx+Ny}dx + \frac{N}{Mx+Ny}dy = 0$ is an exact equation, that is, that

$$\frac{\partial}{\partial y}\left(\frac{M}{Mx+Ny}\right) = \frac{\partial}{\partial x}\left(\frac{N}{Mx+Ny}\right).$$

$$\frac{\partial}{\partial y}\left(\frac{M}{Mx+Ny}\right) = \frac{(Mx+Ny)\frac{\partial M}{\partial y} - M(x\frac{\partial M}{\partial y} + N + y\frac{\partial N}{\partial y})}{(Mx+Ny)^2} = \frac{Ny\frac{\partial M}{\partial y} - MN - My\frac{\partial N}{\partial y}}{(Mx+Ny)^2}$$

and

$$\frac{\partial}{\partial x}\left(\frac{N}{Mx+Ny}\right) = \frac{(Mx+Ny)\frac{\partial N}{\partial x} - N(x\frac{\partial M}{\partial x} + M + y\frac{\partial N}{\partial x})}{(Mx+Ny)^2} = \frac{Mx\frac{\partial N}{\partial x} - MN - Nx\frac{\partial M}{\partial x}}{(Mx+Ny)^2}.$$

$$\frac{\partial}{\partial y}\left(\frac{M}{Mx+Ny}\right) - \frac{\partial}{\partial x}\left(\frac{N}{Mx+Ny}\right) = \frac{N(x\frac{\partial M}{\partial x} + y\frac{\partial M}{\partial y}) - M(x\frac{\partial N}{\partial x} + y\frac{\partial N}{\partial y})}{(Mx+Ny)^2} = \frac{N(nM) - M(nN)}{(Mx+Ny)^2} = 0$$

(by Euler's Theorem on homogeneous functions).

If $Mx + Ny = 0$ identically, then $\frac{M}{N} = -\frac{y}{x}$ and the differential equation reduces to $y\,dx - x\,dy = 0$ for which $1/xy$ is an integrating factor.

8. Solve $(x^4 + y^4)dx - xy^3dy = 0.$

The equation is homogeneous and $\frac{1}{Mx + Ny} = \frac{1}{x^5}$ is an integrating factor. Upon its introduction, the equation becomes $(\frac{1}{x} + \frac{y^4}{x^5})dx - \frac{y^3}{x^4}dy = 0$ and is exact.

Set $\mu(x,y) = \int^x (\frac{1}{x} + \frac{y^4}{x^5})dx = \ln x - \frac{1}{4}\frac{y^4}{x^4} + \phi(y)$.

Then $\frac{\partial\mu}{\partial y} = -\frac{y^3}{x^4} + \phi'(y) = -\frac{y^3}{x^4}$, $\phi'(y) = 0$, and the primitive is

$$\ln x - \frac{1}{4}\frac{y^4}{x^4} = C_1 \qquad \text{or} \qquad y^4 = 4x^4 \ln x + Cx^4.$$

Note. The same integrating factor is obtained by using the procedure of *a)* above. The equation may be solved by the method of Chapter 4.

9. Solve $y^2dx + (x^2 - xy - y^2)dy = 0$.

The equation is homogeneous and $\frac{1}{Mx + Ny} = \frac{1}{y(x^2 - y^2)}$ is an integrating factor.

Upon introducing it the given equation becomes $\frac{y}{x^2 - y^2}dx + \frac{x^2 - xy - y^2}{y(x^2 - y^2)}dy = 0$ which is exact.

Set $\mu(x,y) = \int^x \frac{y}{x^2 - y^2}dx = \frac{1}{2}\int^x (\frac{1}{x-y} - \frac{1}{x+y})dx = \frac{1}{2}\ln\frac{x-y}{x+y} + \phi(y)$.

Then $\frac{\partial\mu}{\partial y} = -\frac{x}{x^2 - y^2} + \phi'(y) = \frac{x^2 - xy - y^2}{y(x^2 - y^2)} = \frac{1}{y} - \frac{x}{x^2 - y^2}$, $\phi'(y) = \frac{1}{y}$, $\phi(y) = \ln y$,

and the primitive is $\frac{1}{2}\ln\frac{x-y}{x+y} + \ln y = \ln C_1$ or $(x-y)y^2 = C(x+y)$.

10. Show that $\frac{1}{Mx - Ny}$, when $Mx - Ny$ is not identically zero, is an integrating factor for the equation $M\,dx + N\,dy = yf_1(xy)dx + xf_2(xy)dy = 0$. Investigate the case $Mx - Ny = 0$ identically.

The equation $$\frac{y f_1(xy)}{xy\{f_1(xy) - f_2(xy)\}}dx + \frac{x f_2(xy)}{xy\{f_1(xy) - f_2(xy)\}}dy = 0$$ is exact

since

$$\frac{\partial}{\partial y}\{\frac{f_1}{x(f_1 - f_2)}\} = \frac{x(f_1 - f_2)\frac{\partial f_1}{\partial y} - f_1 x(\frac{\partial f_1}{\partial y} - \frac{\partial f_2}{\partial y})}{x^2(f_1 - f_2)^2} = \frac{-f_2\frac{\partial f_1}{\partial y} + f_1\frac{\partial f_2}{\partial y}}{x(f_1 - f_2)^2},$$

$$\frac{\partial}{\partial x}\{\frac{f_2}{y(f_1 - f_2)}\} = \frac{y(f_1 - f_2)\frac{\partial f_2}{\partial x} - f_2 y(\frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial x})}{y^2(f_1 - f_2)^2} = \frac{f_1\frac{\partial f_2}{\partial x} - f_2\frac{\partial f_1}{\partial x}}{y(f_1 - f_2)^2},$$

and

$$\frac{\partial}{\partial y}\{\frac{f_1}{x(f_1 - f_2)}\} - \frac{\partial}{\partial x}\{\frac{f_2}{y(f_1 - f_2)}\} = \frac{f_2(-y\frac{\partial f_1}{\partial y} + x\frac{\partial f_1}{\partial x}) + f_1(y\frac{\partial f_2}{\partial y} - x\frac{\partial f_2}{\partial x})}{xy(f_1 - f_2)^2}.$$

This is identically zero since $$y\frac{\partial f(xy)}{\partial y} = x\frac{\partial f(xy)}{\partial x}.$$

If $Mx - Ny = 0$, then $\frac{M}{N} = \frac{y}{x}$ and the equation reduces to $x\,dy + y\,dx = 0$ with solution $xy = C$.

11. Solve $y(x^2y^2 + 2)dx + x(2 - 2x^2y^2)dy = 0$.

The equation is of the form $y f_1(xy)dx + x f_2(xy)dy = 0$ and $\frac{1}{Mx - Ny} = \frac{1}{3x^3y^3}$ is an integrating factor.

Upon introducing it, the equation becomes $\frac{x^2y^2 + 2}{3x^3y^2}dx + \frac{2 - 2x^2y^2}{3x^2y^3}dy = 0$ and is exact.

Set $\mu(x,y) = \int^x (\frac{x^2y^2 + 2}{3x^3y^2})dx = \int^x (\frac{1}{3x} + \frac{2}{3x^3y^2})dx = \frac{1}{3}\ln x - \frac{1}{3x^2y^2} + \phi(y)$.

Then $\frac{\partial\mu}{\partial y} = \frac{2}{3x^2y^3} + \phi'(y) = \frac{2 - 2x^2y^2}{3x^2y^3}$, $\quad \phi'(y) = -\frac{2}{3y}$, $\quad \phi(y) = -\frac{2}{3}\ln y$,

and the primitive is $\frac{1}{3}\ln x - \frac{1}{3x^2y^2} - \frac{2}{3}\ln y = \ln C_1$ or $x = Cy^2e^{1/x^2y^2}$.

The equation may be solved by the method of Chapter 4.

12. Solve $y(2xy + 1)dx + x(1 + 2xy - x^3y^3)dy = 0$.

The equation is of the form $y f_1(xy)dx + x f_2(xy)dy = 0$ and $\frac{1}{Mx - Ny} = \frac{1}{x^4y^4}$ is an integrating factor.

Upon introducing it, the equation becomes $(\frac{2}{x^3y^2} + \frac{1}{x^4y^3})dx + (\frac{1}{x^3y^4} + \frac{2}{x^2y^3} - \frac{1}{y})dy = 0$ and is exact.

Set $\mu(x,y) = \int^x (\frac{2}{x^3y^2} + \frac{1}{x^4y^3})dx = -\frac{1}{x^2y^2} - \frac{1}{3x^3y^3} + \phi(y)$.

Then $\frac{\partial\mu}{\partial y} = \frac{2}{x^2y^3} + \frac{1}{x^3y^4} + \phi'(y) = \frac{1}{x^3y^4} + \frac{2}{x^2y^3} - \frac{1}{y}$, $\quad \phi'(y) = -\frac{1}{y}$, $\quad \phi(y) = -\ln y$,

and the primitive is $-\ln y - \frac{1}{x^2y^2} - \frac{1}{3x^3y^3} = C_1$ or $y = Ce^{-(3xy+1)/(3x^3y^3)}$.

13. Obtain an integrating factor by inspection for each of the following equations.

a) $(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$ (Problem 5)

b) $(x^2y^3 + 2y)dx + (2x - 2x^3y^2)dy = 0$ (Problem 11)

c) $(2xy^2 + y)dx + (x + 2x^2y - x^4y^3)dy = 0$ (Problem12)

a) When the equation is written in the form

$$y^4(2xe^y dx + x^2e^y dy) + 2xy^3dx - x^2y^2dy + y\,dx - 3x\,dy = 0$$

the term $y^4(2xe^y dx + x^2e^y dy) = y^4\cdot$ (an exact differential) suggests that $1/y^4$ is a possible integrating factor. To show that it is an integrating factor, we verify that its introduction produces an exact equation.

b) When the equation is written in the form $2(y\,dx + x\,dy) + x^2y^3dx - 2x^3y^2dy = 0$, the term $(y\,dx + x\,dy)$ suggests $1/(xy)^k$ as a possible integrating factor. An examination of the remaining terms shows that each will be an exact differential if $k = 3$, i.e., $1/(xy)^3$ is an integrating factor.

*c)* When the equation is written in the form $(x\,dy + y\,dx) + 2xy(x\,dy + y\,dx) - x^4y^3dy = 0$ the first two terms suggest $1/(xy)^k$. The third term will be an exact differential if $k = 4$; thus, $1/(xy)^4$ is an integrating factor.

**14.** Solve $y\,dx + x(1-3x^2y^2)dy = 0$ or $x\,dy + y\,dx - 3x^3y^2dy = 0$.

The terms $x\,dy + y\,dx$ suggest $1/(xy)^k$ and the last term requires $k = 3$. Upon introducing the integrating factor $\dfrac{1}{(xy)^3}$, the equation becomes $\dfrac{x\,dy + y\,dx}{x^3y^3} - \dfrac{3}{y}dy = 0$ whose primitive is $\dfrac{-1}{2x^2y^2} - 3\ln y = C_1$, $6\ln y = \ln C - \dfrac{1}{x^2y^2}$ or $y^6 = Ce^{-1/(x^2y^2)}$.

**15.** Solve $x\,dx + y\,dy + 4y^3(x^2+y^2)dy = 0$.

The last term suggests $1/(x^2+y^2)$ as an integrating factor.

Introducing it, the equation becomes $\dfrac{x\,dx + y\,dy}{x^2+y^2} + 4y^3dy = 0$ and is exact.

The primitive is $\frac{1}{2}\ln(x^2+y^2) + y^4 = \ln C_1$ or $(x^2+y^2)e^{2y^4} = C$.

**16.** Solve $x\,dy - y\,dx - (1-x^2)dx = 0$.

Here $1/x^2$ is the integrating factor, since all other possibilities suggests by $x\,dy - y\,dx$ render the last term inexact.

Upon introducing it, the equation becomes $\dfrac{x\,dy - y\,dx}{x^2} - (\dfrac{1}{x^2} - 1)dx = 0$ whose primitive is $\dfrac{y}{x} + \dfrac{1}{x} + x = C$ or $y + x^2 + 1 = Cx$.

**17.** Solve $(x + x^4 + 2x^2y^2 + y^4)dx + y\,dy = 0$ or $x\,dx + y\,dy + (x^2+y^2)^2dx = 0$.

An integrating factor suggested by the form of the equation is $\dfrac{1}{(x^2+y^2)^2}$. Using it, we have $\dfrac{x\,dx + y\,dy}{(x^2+y^2)^2} + dx = 0$ whose primitive is $-\dfrac{1}{2(x^2+y^2)} + x = C_1$ or $(C + 2x)(x^2+y^2) = 1$.

**18.** Solve $x^2\dfrac{dy}{dx} + xy + \sqrt{1-x^2y^2} = 0$ or $x(x\,dy + y\,dx) + \sqrt{1-x^2y^2}\,dx = 0$.

The integrating factor $\dfrac{1}{x\sqrt{1-x^2y^2}}$ reduces the equation to the form $\dfrac{x\,dy + y\,dx}{\sqrt{1-x^2y^2}} + \dfrac{dx}{x} = 0$ whose primitive is $\arcsin(xy) + \ln x = C$.

**19.** Solve $\dfrac{dy}{dx} = \dfrac{y - xy^2 - x^3}{x + x^2y + y^3}$ or $(x^3 + xy^2 - y)dx + (y^3 + x^2y + x)dy = 0$.

When the equation is written thus $(x^2+y^2)(x\,dx + y\,dy) + x\,dy - y\,dx = 0$, the terms $x\,dy - y\,dx$ suggest several possible integrating factors. By trial, we determine $1/(x^2+y^2)$ which reduces the given equation to the form $$x\,dx + y\,dy + \frac{x\,dy - y\,dx}{x^2+y^2} = x\,dx + y\,dy + \frac{\dfrac{x\,dy - y\,dx}{x^2}}{1+(\frac{y}{x})^2} = 0.$$

The primitive is $\frac{1}{2}x^2 + \frac{1}{2}y^2 + \arctan\frac{y}{x} = C_1$ or $x^2 + y^2 + 2\arctan\frac{y}{x} = C.$

**20.** Solve $x(4y\,dx + 2x\,dy) + y^3(3y\,dx + 5x\,dy) = 0.$

Suppose that the effect of multiplying the given equation by $x^\alpha y^\beta$ is to produce an equation

*A)* $$(4x^{\alpha+1}y^{\beta+1}dx + 2x^{\alpha+2}y^\beta dy) + (3x^\alpha y^{\beta+4}dx + 5x^{\alpha+1}y^{\beta+3}dy) = 0$$

each of whose two terms is an exact differential. Then the first term of *A)* is proportional to

*B)* $$d(x^{\alpha+2}y^{\beta+1}) = (\alpha+2)x^{\alpha+1}y^{\beta+1}dx + (\beta+1)x^{\alpha+2}y^\beta dy,$$

that is,

*C)* $$\frac{\alpha+2}{4} = \frac{\beta+1}{2} \quad \text{and} \quad \alpha - 2\beta = 0.$$

Also, the second term of *A)* is proportional to

*D)* $$d(x^{\alpha+1}y^{\beta+4}) = (\alpha+1)x^\alpha y^{\beta+4}dx + (\beta+4)x^{\alpha+1}y^{\beta+3}dy,$$

that is,

*E)* $$\frac{\alpha+1}{3} = \frac{\beta+4}{5} \quad \text{and} \quad 5\alpha - 3\beta = 7.$$

Solving $\alpha - 2\beta = 0$, $5\alpha - 3\beta = 7$ simultaneously, we find $\alpha = 2$, $\beta = 1$.

When these substitutions are made in *A)*, the equation becomes

$$(4x^3y^2dx + 2x^4y\,dy) + (3x^2y^5dx + 5x^3y^4dy) = 0.$$

The primitive is $x^4y^2 + x^3y^5 = C.$

**21.** Solve $(8y\,dx + 8x\,dy) + x^2y^3(4y\,dx + 5x\,dy) = 0.$

Suppose that the effect of multiplying the given equation by $x^\alpha y^\beta$ is to produce an equation

*A)* $$(8x^\alpha y^{\beta+1}dx + 8x^{\alpha+1}y^\beta dy) + (4x^{\alpha+2}y^{\beta+4}dx + 5x^{\alpha+3}y^{\beta+3}dy) = 0$$

each of whose two terms is an exact differential. Then the first is proportional to

*B)* $$d(x^{\alpha+1}y^{\beta+1}) = (\alpha+1)x^\alpha y^{\beta+1}dx + (\beta+1)x^{\alpha+1}y^\beta dy,$$

that is,

*C)* $$\frac{\alpha+1}{8} = \frac{\beta+1}{8} \quad \text{and} \quad \alpha - \beta = 0.$$

Also, the second term is proportional to

*D)* $$d(x^{\alpha+3}y^{\beta+4}) = (\alpha+3)x^{\alpha+2}y^{\beta+4}dx + (\beta+4)x^{\alpha+3}y^{\beta+3}dy,$$

that is,

*E)* $$\frac{\alpha+3}{4} = \frac{\beta+4}{5} \quad \text{and} \quad 5\alpha - 4\beta = 1.$$

Solving $\alpha - \beta = 0$, $5\alpha - 4\beta = 1$ simultaneously, we find $\alpha = 1$, $\beta = 1$.

When these substitutions are made in *A)*, the equation becomes

$$(8xy^2 dx + 8x^2 y\,dy) + (4x^3y^5 dx + 5x^4y^4 dy) = 0.$$

The primitive is $$4x^2y^2 + x^4y^5 = C.$$

*Note.* In this and the previous problem it was not necessary to write statements *B)* and *D)* since, after a little practice, the relations *C)* and *E)* may be obtained directly from *A)*.

**22.** Solve $x^3y^3(2y\,dx + x\,dy) - (5y\,dx + 7x\,dy) = 0.$

Multiplying the given equation by $x^\alpha y^\beta$, we have

*A)* $(2x^{\alpha+3}y^{\beta+4}dx + x^{\alpha+4}y^{\beta+3})dy - (5x^\alpha y^{\beta+1}dx + 7x^{\alpha+1}y^\beta dy) = 0.$

If the first term of *A)* is to be exact, then $\dfrac{\alpha+4}{2} = \dfrac{\beta+4}{1}$ and $\alpha - 2\beta = 4.$

If the second term of *A)* is to be exact, then $\dfrac{\alpha+1}{5} = \dfrac{\beta+1}{7}$ and $7\alpha - 5\beta = -2.$

Solving $\alpha - 2\beta = 4$, $7\alpha - 5\beta = -2$ simultaneously, we find $\alpha = -8/3$, $\beta = -10/3$.

Then, from *A)*, $(2x^{1/3}y^{2/3}dx + x^{4/3}y^{-1/3}dy) - (5x^{-8/3}y^{-7/3}dx + 7x^{-5/3}y^{-10/3}dy) = 0$, each of the two terms is exact, and the primitive is

$$\frac{3}{2}x^{4/3}y^{2/3} + 3x^{-5/3}y^{-7/3} = C_1, \quad x^{4/3}y^{2/3} + 2x^{-5/3}y^{-7/3} = C \text{ or } x^3y^3 + 2 = Cx^{5/3}y^{7/3}.$$

## SUPPLEMENTARY PROBLEMS

**23.** Select from the following equations those which are exact and solve.

*a)* $(x^2 - y)dx - x\,dy = 0$ *Ans.* $xy = x^3/3 + C$

*b)* $y(x-2y)dx - x^2dy = 0$

*c)* $(x^2+y^2)dx + xy\,dy = 0$

*d)* $(x^2+y^2)dx + 2xy\,dy = 0$ *Ans.* $xy^2 + x^3/3 = C$

*e)* $(x + y\cos x)dx + \sin x\,dy = 0$ *Ans.* $x^2 + 2y\sin x = C$

*f)* $(1 + e^{2\theta})d\rho + 2\rho e^{2\theta}d\theta = 0$ *Ans.* $\rho(1 + e^{2\theta}) = C$

*g)* $dx - \sqrt{a^2 - x^2}\,dy = 0$

*h)* $(2x + 3y + 4)dx + (3x + 4y + 5)dy = 0$ *Ans.* $x^2 + 3xy + 2y^2 + 4x + 5y = C$

*i)* $(4x^3y^3 + \frac{1}{x})dx + (3x^4y^2 - \frac{1}{y})dy = 0$ *Ans.* $x^4y^3 + \ln(x/y) = C$

*j)* $2(u^2 + uv)du + (u^2 + v^2)dv = 0$ *Ans.* $2u^3 + 3u^2v + v^3 = C$

*k)* $(x\sqrt{x^2+y^2} - y)dx + (y\sqrt{x^2+y^2} - x)dy = 0$ *Ans.* $(x^2+y^2)^{3/2} - 3xy = C$

*l)* $(x+y+1)dx - (x-y-3)dy = 0$

*m)* $(x+y+1)dx - (y-x+3)dy = 0$ *Ans.* $x^2 + 2xy - y^2 + 2x - 6y = C$

*n)* $\csc\theta \tan\theta\, dr - (r\csc\theta + \tan^2\theta)d\theta = 0$ *Ans.* $r\csc\theta = \ln\sec\theta + C$

*o)* $(y^2 - \frac{y}{x(x+y)} + 2)dx + [\frac{1}{x+y} + 2y(x+1)]dy = 0$ *Ans.* $\ln\frac{x+y}{x} + (x+1)(y^2+2) = C$

*p)* $(2xye^{x^2y} + y^2e^{xy^2} + 1)dx + (x^2e^{x^2y} + 2xye^{xy^2} - 2y)dy = 0$ *Ans.* $e^{x^2y} + e^{xy^2} + x - y^2 = C$

**24.** Solve the remaining problems above [*b*), *c*), *g*), *l*)] using the appropriate procedure of Chap.4.

*Ans.* *b)* $x/y = 2\ln x + C$ *g)* $y = \text{arc}\sin x/a + C$

*c)* $x^4 + 2x^2y^2 = C$ *l)* $\ln\sqrt{x^2+y^2-2x+4y+5} - \text{arc}\tan\frac{y+2}{x-1} = C$

**25.** For each of the following, obtain an integrating factor by inspection and solve.

*a)* $x\,dx + y\,dy = (x^2+y^2)dx$ *Ans.* $1/(x^2+y^2)$; $x^2+y^2 = Ce^{2x}$

*b)* $(2y-3x)dx + x\,dy = 0$ *Ans.* $x$; $x^2y = x^3 + C$

*c)* $(x-y^2)dx + 2xy\,dy = 0$ *Ans.* $1/x^2$; $y^2 + x\ln x = Cx$

*d)* $x\,dy - y\,dx = 3x^2(x^2+y^2)dx$ *Ans.* $1/(x^2+y^2)$; $\text{arc}\tan y/x = x^3 + C$

*e)* $y\,dx - x\,dy + \ln x\,dx = 0$ *Ans.* $1/x^2$; $y + \ln x + 1 = Cx$

*f)* $(3x^2+y^2)dx - 2xy\,dy = 0$ *Ans.* $1/x^2$; $3x^2 - y^2 = Cx$

*g)* $(xy-2y^2)dx - (x^2-3xy)dy = 0$ *Ans.* $1/xy^2$; $x/y + \ln(y^3/x^2) = C$

*h)* $(x+y)dx - (x-y)dy = 0$ *Ans.* $1/(x^2+y^2)$; $x^2+y^2 = Ce^{2\,\text{arc}\tan y/x}$

*i)* $2y\,dx - 3xy^2dx - x\,dy = 0$ *Ans.* $x/y^2$; $x^2/y - x^3 = C$

*j)* $y\,dx + x(x^2y-1)dy = 0$ *Ans.* $y/x^3$; $3y^2 - 2x^2y^3 = Cx^2$

*k)* $(y+x^3y+2x^2)dx + (x+4xy^4+8y^3)dy = 0$ *Ans.* $1/(xy+2)$; $\ln(xy+2)^3 + x^3 + 3y^4 = C$

**26.** For each of the following, obtain an integrating factor and solve.

*a)* $x\,dy - y\,dx = x^2e^x dx$ *Ans.* $y = Cx + xe^x$

*b)* $(1+y^2)dx = (x+x^2)dy$ *Ans.* $\text{arc}\tan y = \ln x/(x+1) + C$

*c)* $(2y-x^3)dx + x\,dy = 0$ *Ans.* $x^2y - x^5/5 = C$

*d)* $y^2dy + y\,dx - x\,dy = 0$ $y^2 + x = Cy$

*e)* $(3y^3-xy)dx - (x^2+6xy^2)dy = 0$ *Ans.* $3y^2 + x\ln(xy) = Cx$

*f)* $3x^2y^2dx + 4(x^3y-3)dy = 0$ *Ans.* $x^3y^4 - 4y^3 = C$

*g)* $y(x+y)dx - x^2dy = 0$ *Ans.* $x/y + \ln x = C$

*h)* $(2y+3xy^2)dx + (x+2x^2y)dy = 0$ *Ans.* $x^2y(1+xy) = C$

*i)* $y(y^2-2x^2)dx + x(2y^2-x^2)dy = 0$ *Ans.* $x^2y^2(y^2-x^2) = C$

**27.** Show that $\frac{1}{x^2}f(y/x)$ is an integrating factor of $x\,dy - y\,dx = 0$.

# Equations of First Order and First Degree

## LINEAR EQUATIONS AND THOSE REDUCIBLE TO THAT FORM

**THE EQUATION** 1) $\dfrac{dy}{dx} + yP(x) = Q(x)$,

whose left member is linear in both the dependent variables and its derivative, is called a *linear equation of the first order*.

For example,

$\dfrac{dy}{dx} + 3xy = \sin x$ is called linear while $\dfrac{dy}{dx} + 3xy^2 = \sin x$ is not.

Since $\dfrac{d}{dx}(ye^{\int P(x)\,dx}) = \dfrac{dy}{dx}e^{\int P(x)\,dx} + y\,P(x)\,e^{\int P(x)\,dx} = e^{\int P(x)\,dx}(\dfrac{dy}{dx} + yP(x))$,

$e^{\int P(x)\,dx}$ is an integrating factor of 1) and its primitive is

$$ye^{\int P(x)\,dx} = \int Q(x)\cdot e^{\int P(x)\,dx}\,dx + C.$$

See Problems 1-7.

**BERNOULLI'S EQUATION.** An equation of the form

$$\frac{dy}{dx} + yP(x) = y^n Q(x) \qquad \text{or} \qquad y^{-n}\frac{dy}{dx} + y^{-n+1}P(x) = Q(x)$$

is reduced to the form 1), namely, $\dfrac{dv}{dx} + v\{(1-n)P(x)\} = (1-n)Q(x)$, by the transformation

$$y^{-n+1} = v, \qquad y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dv}{dx}.$$

See Problems 8-12.

**OTHER EQUATIONS** may be reduced to the form 1) by means of appropriate transformations. As in previous chapters, no general rule can be stated; in each instance, the proper transformation is suggested by the form of the equation.

See Problems 13-18.

## SOLVED PROBLEMS

LINEAR EQUATIONS.

1. Solve $\dfrac{dy}{dx} + 2xy = 4x$.

$\int P(x)\,dx = \int 2x\,dx = x^2$ and $e^{\int P(x)\,dx} = e^{x^2}$ is an integrating factor.

Then $ye^{x^2} = \int 4xe^{x^2}dx = 2e^{x^2} + C$ or $y = 2 + Ce^{-x^2}$.

2. Solve $x\dfrac{dy}{dx} = y + x^3 + 3x^2 - 2x$ or $\dfrac{dy}{dx} - \dfrac{1}{x}y = x^2 + 3x - 2$.

$\int P(x)\,dx = -\int \dfrac{dx}{x} = -\ln x$ and $e^{-\ln x} = \dfrac{1}{x}$ is an integrating factor.

Then $y\frac{1}{x} = \int \frac{1}{x}(x^2+3x-2)\,dx = \int (x+3-\frac{2}{x})\,dx = \frac{1}{2}x^2+3x-2\ln x+C_1$ or

$$2y = x^3 + 6x^2 - 4x\ln x + Cx.$$

3. Solve $(x-2)\frac{dy}{dx} = y + 2(x-2)^3$ or $\frac{dy}{dx} - \frac{1}{x-2}y = 2(x-2)^2$.

$\int P(x)\,dx = -\int \frac{dx}{x-2} = -\ln(x-2)$ and an integrating factor is $e^{-\ln(x-2)} = \frac{1}{x-2}$.

Then $y(\frac{1}{x-2}) = 2\int (x-2)^2\cdot\frac{1}{x-2}\,dx = 2\int (x-2)\,dx = (x-2)^2 + C$ or $y = (x-2)^3 + C(x-2)$.

4. Solve $\frac{dy}{dx} + y\cot x = 5e^{\cos x}$. Find the particular solution, given the initial conditions: $x = \frac{1}{2}\pi,\ y = -4$.

An integrating factor is $e^{\int \cot x\,dx} = e^{\ln\sin x} = \sin x$ and

$$y\sin x = 5\int e^{\cos x}\sin x\,dx = -5e^{\cos x} + C.$$

When $x = \frac{1}{2}\pi,\ y = -4$: $(-4)(1) = -5(1) + C$ and $C = 1$. The particular solution is

$$y\sin x + 5e^{\cos x} = 1.$$

5. Solve $x^3\frac{dy}{dx} + (2-3x^2)y = x^3$ or $\frac{dy}{dx} + \frac{2-3x^2}{x^3}y = 1$.

$\int \frac{2-3x^2}{x^3}\,dx = -\frac{1}{x^2} - 3\ln x$ and an integrating factor is $\frac{1}{x^3e^{1/x^2}}$.

Then $\frac{y}{x^3e^{1/x^2}} = \int \frac{dx}{x^3e^{1/x^2}} = \frac{1}{2e^{1/x^2}} + C_1$ or $2y = x^3 + Cx^3e^{1/x^2}$

6. Solve $\frac{dy}{dx} - 2y\cot 2x = 1 - 2x\cot 2x - 2\csc 2x$.

An integrating factor is $e^{-\int 2\cot 2x\,dx} = e^{-\ln\sin 2x} = \csc 2x$.

Then $y\csc 2x = \int(\csc 2x - 2x\cot 2x\csc 2x - 2\csc^2 2x)\,dx = x\csc 2x + \cot 2x + C$

or

$$y = x + \cos 2x + C\sin 2x.$$

7. Solve $y\ln y\,dx + (x - \ln y)\,dy = 0$.

The equation, with $x$ taken as dependent variable, may be put in the form $\frac{dx}{dy} + \frac{1}{y\ln y}x = \frac{1}{y}$.

Then $e^{\int dy/(y\ln y)} = e^{\ln(\ln y)} = \ln y$ is an integrating factor.

Thus, $x\ln y = \int \ln y\,\frac{dy}{y} = \frac{1}{2}\ln^2 y + K$ and the solution is $2x\ln y = \ln^2 y + C$.

BERNOULLI'S EQUATION.

**8.** Solve $\frac{dy}{dx} - y = xy^5$ or $y^{-5}\frac{dy}{dx} - y^{-4} = x$.

The transformation $y^{-4} = v,\ y^{-5}\frac{dy}{dx} = -\frac{1}{4}\frac{dv}{dx}$ reduces the equation to

$-\frac{1}{4}\frac{dv}{dx} - v = x$ or $\frac{dv}{dx} + 4v = -4x$. An integrating factor is $e^{4\int dx} = e^{4x}$.

Then $ve^{4x} = -4\int xe^{4x}dx = -xe^{4x} + \frac{1}{4}e^{4x} + C$,

$y^{-4}e^{4x} = -xe^{4x} + \frac{1}{4}e^{4x} + C$, or $\frac{1}{y^4} = -x + \frac{1}{4} + Ce^{-4x}$.

**9.** Solve $\frac{dy}{dx} + 2xy + xy^4 = 0$ or $y^{-4}\frac{dy}{dx} + 2xy^{-3} = -x$.

The transformation $y^{-3} = v,\ -3y^{-4}\frac{dy}{dx} = \frac{dv}{dx}$ reduces the equation to $\frac{dv}{dx} - 6xv = 3x$.

Using the integrating factor $e^{-\int 6x\,dx} = e^{-3x^2}$, we have

$$ve^{-3x^2} = \int 3xe^{-3x^2}dx = -\frac{1}{2}e^{-3x^2} + C \quad \text{or} \quad \frac{1}{y^3} = -\frac{1}{2} + Ce^{3x^2}.$$

**10.** Solve $\frac{dy}{dx} + \frac{1}{3}y = \frac{1}{3}(1-2x)y^4$ or $y^{-4}\frac{dy}{dx} + \frac{1}{3}y^{-3} = \frac{1}{3}(1-2x)$.

The transformation $y^{-3} = v,\ -3y^{-4}\frac{dy}{dx} = \frac{dv}{dx}$ reduces the equation to $\frac{dv}{dx} - v = 2x - 1$

for which $e^{-x}$ is an integrating factor. Then, integrating by parts,

$$ve^{-x} = \int(2x-1)e^{-x}dx = -2xe^{-x} - e^{-x} + C \quad \text{or} \quad \frac{1}{y^3} = -1 - 2x + Ce^{x}.$$

**11.** Solve $\frac{dy}{dx} + y = y^2(\cos x - \sin x)$ or $y^{-2}\frac{dy}{dx} + y^{-1} = \cos x - \sin x$.

The transformation $y^{-1} = v,\ -y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$ reduces the equation to $\frac{dv}{dx} - v = \sin x - \cos x$

for which $e^{-x}$ is an integrating factor. Then

$$ve^{-x} = \int(\sin x - \cos x)e^{-x}dx = -e^{-x}\sin x + C \quad \text{or} \quad \frac{1}{y} = -\sin x + Ce^{x}.$$

**12.** Solve $x\,dy - \{y + xy^3(1 + \ln x)\}dx = 0$ or $y^{-3}\frac{dy}{dx} - \frac{1}{x}y^{-2} = 1 + \ln x$.

The transformation $y^{-2} = v,\ -2y^{-3}\frac{dy}{dx} = \frac{dv}{dx}$ reduces the equation to $\frac{dv}{dx} + \frac{2}{x}v = -2(1 + \ln x)$

for which $e^{\int 2\,dx/x} = x^2$ is an integrating factor. Then

$$vx^2 = -2\int(x^2 + x^2\ln x)dx = -\frac{4}{9}x^3 - \frac{2}{3}x^3\ln x + C \quad \text{or} \quad \frac{x^2}{y^2} = -\frac{2}{3}x^3(\frac{2}{3} + \ln x) + C.$$

MISCELLANEOUS SUBSTITUTIONS.

13. An equation of the form $f'(y)\frac{dy}{dx} + f(y)P(x) = Q(x)$ is a linear equation of the first order $\frac{dv}{dx} + vP(x) = Q(x)$ in the new variable $v = f(y)$. (Note that the Bernoulli equation

$y^{-n}\frac{dy}{dx} + y^{-n+1}P(x) = Q(x)$ or $(-n+1)y^{-n}\frac{dy}{dx} + y^{-n+1}(-n+1)P(x) = (-n+1)Q(x)$ is an example)

Solve $\frac{dy}{dx} + 1 = 4e^{-y}\sin x$ or $e^{y}\frac{dy}{dx} + e^{y} = 4\sin x$.

In the new variable $v = f(y) = e^{y}$, the equation becomes $\frac{dv}{dx} + v = 4\sin x$ for which $e^{x}$ is an integrating factor. Then

$$ve^{x} = 4\int e^{x}\sin x\,dx = 2e^{x}(\sin x - \cos x) + C \quad \text{or} \quad e^{y} = 2(\sin x - \cos x) + Ce^{-x}.$$

14. Solve $\sin y\frac{dy}{dx} = \cos x(2\cos y - \sin^2 x)$ or $-\sin y\frac{dy}{dx} + \cos y(2\cos x) = \sin^2 x\cos x$.

In the new variable $v = \cos y$, the equation becomes $\frac{dv}{dx} + 2v\cos x = \sin^2 x\cos x$ for which

$e^{2\int\cos x\,dx} = e^{2\sin x}$ is an integrating factor. Then

$$ve^{2\sin x} = \int e^{2\sin x}\sin^2 x\cos x\,dx = \tfrac{1}{2}e^{2\sin x}\sin^2 x - \tfrac{1}{2}e^{2\sin x}\sin x + \tfrac{1}{4}e^{2\sin x} + C$$

or $$\cos y = \tfrac{1}{2}\sin^2 x - \tfrac{1}{2}\sin x + \tfrac{1}{4} + Ce^{-2\sin x}.$$

15. Solve $\sin y\frac{dy}{dx} = \cos y(1 - x\cos y)$ or $\frac{\sin y}{\cos^2 y}\frac{dy}{dx} - \frac{1}{\cos y} = -x$.

Since $\frac{d}{dy}\left(\frac{1}{\cos y}\right) = \frac{\sin y}{\cos^2 y}$, we take $v = \frac{1}{\cos y}$ and obtain the equation $\frac{dv}{dx} - v = -x$.

Using the integrating factor $e^{-x}$, we obtain

$$ve^{-x} = \int -xe^{-x}dx = xe^{-x} + e^{-x} + C \quad \text{or} \quad v = \frac{1}{\cos y} = \sec y = x + 1 + Ce^{x}.$$

16. Solve $x\frac{dy}{dx} - y + 3x^3y - x^2 = 0$ or $x\,dy - y\,dx + 3x^3y\,dx - x^2dx = 0$.

Here $(x\,dy - y\,dx)$ suggest the transformation $\frac{y}{x} = v$.

Then $\frac{x\,dy - y\,dx}{x^2} + 3x^2\frac{y}{x}dx - dx = 0$ is reduced to $\frac{dv}{dx} + 3x^2v = 1$ for which $e^{x^3}$ is an integrating factor

Thus $ve^{x^3} = \int e^{x^3}dx + C$ or $y = xe^{-x^3}\int e^{x^3}dx + Cxe^{-x^3}$.

The indefinite integral here cannot be evaluated in terms of elementary functions.

**17.** Solve $(4r^2s - 6)dr + r^3ds = 0$ or $(r\,ds + s\,dr) + 3s\,dr = \dfrac{6}{r^2}\,dr$.

The first term suggests the substitution $rs = t$ which reduces the equation to

$$dt + 3\frac{t}{r}dr = \frac{6}{r^2}\,dr \quad \text{or} \quad \frac{dt}{dr} + \frac{3}{r}t = \frac{6}{r^2}.$$

The $r^3$ is an integrating factor and the solution is

$$tr^3 = r^4s = 3r^2 + C \quad \text{or} \quad s = \frac{3}{r^2} + \frac{C}{r^4}.$$

**18.** Solve $x\sin\theta\,d\theta + (x^3 - 2x^2\cos\theta + \cos\theta)dx = 0$ or $-\dfrac{x\sin\theta\,d\theta + \cos\theta\,dx}{x^2} + 2\cos\theta\,dx = x\,dx$.

The substitution $xy = \cos\theta$, $dy = -\dfrac{x\sin\theta\,d\theta + \cos\theta\,dx}{x^2}$ reduces the equation to

$$dy + 2xy\,dx = x\,dx \quad \text{or} \quad \frac{dy}{dx} + 2xy = x.$$

An integrating factor is $e^{x^2}$ and the solution is

$$ye^{x^2} = \frac{\cos\theta}{x}e^{x^2} = \int e^{x^2}x\,dx = \frac{1}{2}e^{x^2} + K \quad \text{or} \quad 2\cos\theta = x + Cxe^{-x^2}.$$

## SUPPLEMENTARY PROBLEMS

**19.** From the following equations, select those which are linear, state the dependent variable, and solve.

*a)* $dy/dx + y = 2 + 2x$

*b)* $d\rho/d\theta + 3\rho = 2$

*c)* $dy/dx - y = xy^2$

*d)* $x\,dy - 2y\,dx = (x-2)e^x dx$

*e)* $di/dt - 6i = 10\sin 2t$

*f)* $dy/dx + y = y^2e^x$

*g)* $y\,dx + (xy + x - 3y)dy = 0$

*h)* $(2s - e^{2t})ds = 2(se^{2t} - \cos 2t)dt$

*i)* $x\,dy + y\,dx = x^3y^6dx$

*j)* $dr + (2r\cot\theta + \sin 2\theta)d\theta = 0$

*k)* $y(1 + y^2)dx = 2(1 - 2xy^2)dy$

*l)* $yy' - xy^2 + x = 0$

*m)* $x\,dy - y\,dx = x\sqrt{x^2 - y^2}\,dy$

*n)* $\phi_1(t)\,dx/dt + x\phi_2(t) = 1$

*o)* $2\,dx/dy - x/y + x^3\cos y = 0$

*p)* $xy' = y(1 - x\tan x) + x^2\cos x$

*q)* $(2 + y^2)dx - (xy + 2y + y^3)dy = 0$

*r)* $(1 + y^2)dx = (\arctan y - x)dy$

*s)* $(2xy^5 - y)dx + 2x\,dy = 0$

*t)* $(1 + \sin y)dx = [2y\cos y - x(\sec y + \tan y)]dy$

*Ans.*

*a)* $y$; I.F., $e^x$; $y = 2x + Ce^{-x}$

*b)* $\rho$; I.F., $e^{3\theta}$; $3\rho = 2 + Ce^{-3\theta}$

*d)* $y$; I.F., $1/x^2$; $y = e^x + Cx^2$

*e)* $i$; I.F., $e^{-6t}$; $i = -\frac{1}{2}(3\sin 2t + \cos 2t) + Ce^{6t}$

*g)* $x$; I.F., $ye^y$; $xy = 3(y-1) + Ce^{-y}$

*j)* $r$; I.F., $\sin^2\theta$; $2r\sin^2\theta + \sin^4\theta = C$

*k)* $x$; I.F., $(1+y^2)^2$: $(1+y^2)^2 x = 2\ln y + y^2 + C$

*n)* $x$; I.F., $e^{\int \phi_2(t)\,dt/\phi_1(t)}$; $x\,e^{\int \phi_2(t)dt/\phi_1(t)} = \int \frac{1}{\phi_1(t)} e^{\int \phi_2(t)dt/\phi_1(t)}\,dt + C$

*p)* $y$; I.F., $\frac{1}{x\cos x}$; $y = x^2\cos x + Cx\cos x$

*q)* $x$; I.F., $1/\sqrt{2+y^2}$; $x = 2 + y^2 + C\sqrt{2+y^2}$

*r)* $x$; I.F., $e^{\arctan y}$; $x = \arctan y - 1 + Ce^{-\arctan y}$

*t)* $x$; I.F., $\sec y + \tan y$; $x(\sec y + \tan y) = y^2 + C$

**20.** From the remaining equations in Problem 19, solve those of the Bernoulli type.

*Ans. c)* $y^{-1} = v$; $1/y = 1 - x + Ce^{-x}$ *l)* $y^2 = v$; $y^2 = 1 + Ce^{x^2}$

*f)* $y^{-1} = v$; $(C+x)ye^x + 1 = 0$ *o)* $x^{-2} = v$; $x^{-2}y = \cos y + y\sin y + C$

*i)* $y^{-3} = v$; $2/y^3 = Cx^5 + 5x^3$ *s)* $y^{-4} = v$; $3x^2 = (4x^3 + C)y^4$

**21.** Solve the remaining equations, *h)* and *m)*, of Problem 19.

*Ans. h)* $s^2 - se^{2t} + \sin 2t = C$ *m)* $y = x\sin(y + C)$

**22.** Solve:

*a)* $xy' = 2y + x^3e^x$ subject to $y = 0$ when $x = 1$. *Ans.* $y = x^2(e^x - e)$

*b)* $L\frac{di}{dt} + Ri = E\sin 2t$, where $L, R, E$, are constants, subject to the condition $i = 0$ when $t = 0$.

*Ans.* $i = \frac{E}{R^2 + 4L^2}(R\sin 2t - 2L\cos 2t + 2Le^{-Rt/L})$

**23.** Solve:

*a)* $x^2\cos y\frac{dy}{dx} = 2x\sin y - 1$, using $\sin y = z$. *Ans.* $3x\sin y = Cx^3 + 1$

*b)* $4x^2yy' = 3x(3y^2+2) + 2(3y^2+2)^3$, using $3y^2 + 2 = z$. *Ans.* $4x^9 = (C - 3x^8)(3y^2+2)^2$

*c)* $(xy^3 - y^3 - x^2e^x)dx + 3xy^2dy = 0$, using $y^3 = vx$. *Ans.* $2y^3e^x = xe^{2x} + Cx$

*d)* $dy/dx + x(x+y) = x^3(x+y)^3 - 1$. *Ans.* $1/(x+y)^2 = x^2 + 1 + Ce^{x^2}$

*e)* $(y + e^y - e^{-x})dx + (1 + e^y)dy = 0$. *Ans.* $y + e^y = (x + C)e^{-x}$

# Geometric Applications

**IN CHAPTER 1** it was shown how the differential equation

1) $$f(x,y,y') = 0$$

of a family of curves

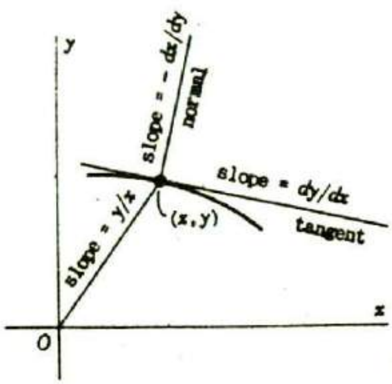
2) $$g(x,y,C) = 0$$

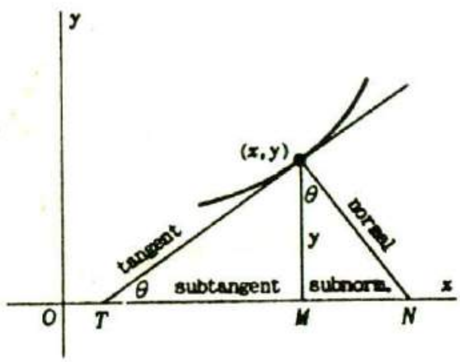
could be obtained. The differential equation expresses analytically a certain property common to every curve of the family.

Conversely, if a property whose analytic representation involves the derivative is given, the solution of the resulting differential equation represents a one parameter family of curves, all possessing the given property. Each curve of the family is called an *integral curve* of 1) and particular integral curves be singled out by giving additional properties, for example, a point through which the curve passes.

For convenience, the following properties of curves which involve the derivative, are listed.

**RECTANGULAR COORDINATES** Let $(x,y)$ be a general point of a curve $F(x,y) = 0$.




*a)* $\dfrac{dy}{dx}$ is the slope of the tangent to the curve at $(x,y)$.

*b)* $-\dfrac{dx}{dy}$ is the slope of the normal to the curve at $(x,y)$.

*c)* $Y-y = \dfrac{dy}{dx}(X-x)$ is the equation of the tangent at $(x,y)$, where $(X,Y)$ are the coordinates of any point on it.

*d)* $Y-y = -\dfrac{dx}{dy}(X-x)$ is the equation of the normal at $(x,y)$, where $(X,Y)$ are the coordinates of any point on it.

*e)* $x-y\dfrac{dx}{dy}$ and $y-x\dfrac{dy}{dx}$ are the $x$- and $y$- intercepts of the tangent.

*f)* $x + y\dfrac{dy}{dx}$ and $y + x\dfrac{dx}{dy}$ are the $x$- and $y$- intercepts of the normal.

*g)* $y\sqrt{1 + (\dfrac{dx}{dy})^2}$ and $x\sqrt{1 + (\dfrac{dy}{dx})^2}$ are the lengths of the tangent between $(x, y)$ and the $x$- and $y$- axes.

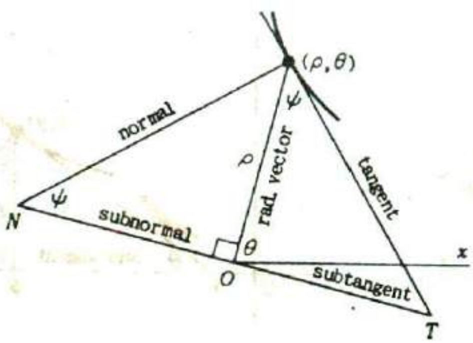
*h)* $y\sqrt{1 + (\dfrac{dy}{dx})^2}$ and $x\sqrt{1 + (\dfrac{dx}{dy})^2}$ are the lengths of the normal between $(x, y)$ and the $x$- and $y$- axes.

*i)* $y\dfrac{dx}{dy}$ and $y\dfrac{dy}{dx}$ are the lengths of the subtangent and subnormal.

*j)* $ds = \sqrt{(dx)^2 + (dy)^2} = dx\sqrt{1 + (\dfrac{dy}{dx})^2} = dy\sqrt{1 + (\dfrac{dx}{dy})^2}$ is an element of length of arc.

*k)* $y\,dx$ or $x\,dy$ is an element of area.

**POLAR COORDINATES**. Let $(\rho, \theta)$ be a general point on a curve $\rho = f(\theta)$.




*l)* $\tan\psi = \rho\dfrac{d\theta}{d\rho}$, where $\psi$ is the angle between the radius vector and the part of the tangent drawn toward the initial line.

*m)* $\rho\tan\psi = \rho^2\dfrac{d\theta}{d\rho}$ is the length of the polar subtangent.

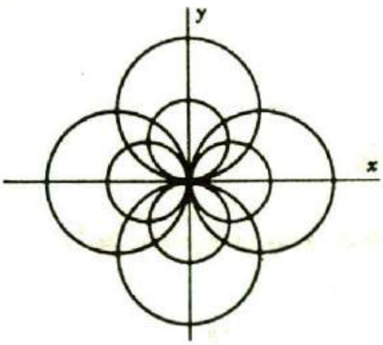
*n)* $\rho\cot\psi = \dfrac{d\rho}{d\theta}$ is the length of the polar subnormal.

*o)* $\rho\sin\psi = \rho^2\dfrac{d\theta}{ds}$ is the length of the perpendicular from the pole to the tangent.

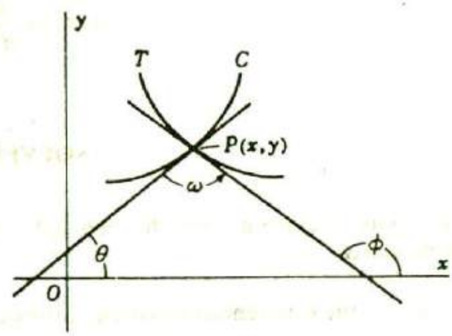
*p)* $ds = \sqrt{(d\rho)^2 + \rho^2(d\theta)^2} = d\rho\sqrt{1 + \rho^2(\dfrac{d\theta}{d\rho})^2} = d\theta\sqrt{(\dfrac{d\rho}{d\theta})^2 + \rho^2}$ is an element of length of arc.

*q)* $\frac{1}{2}\rho^2\,d\theta$ is an element of area.

**TRAJECTORIES**. Any curve which cuts every member of a given family of curves at the constant angle $\omega$ is called an $\omega$- *trajectory* of the family. A 90° trajectory of the family is commonly called an *orthogonal trajectory* of the family. For example, in Figure *(a)* below, the circles through the origin with centres on the $y$-axis are the orthogonal trajectories of the family of circles through the origin with centres on the $x$- axis.




*(a)*




*(b)*

In finding such trajectories, we shall use:

*A)* The integral curves of the differential equation

3) $$f(x,y,\ \frac{y'-\tan\omega}{1+y'\tan\omega}) = 0$$

are the $\omega$- trajectories of the family of integral curves of

1) $$f(x,y,y') = 0.$$

To prove this, consider the integral curve $C$ of 1) and an $\omega$- trajectory which intersect at $P(x,y)$, as shown in Figure *(b)* above. At each point of $C$ for which 1) defines a value of $y'$, we associate a triad of numbers $(x,y;y')$, the first two being the coordinates of the point and the third being the corresponding value of $y'$ given by 1). Similarly, with each point of $T$ for which there is a tangent line, we associate a triad $(x,y;y')$ the first two being the coordinates of the point and the third the slope of the tangent. To avoid confusion, since we are to consider the triads associated with $P$ as a point on $C$ and as a point on $T$, let us write the latter (associated with $P$ on $T$) as $(\bar{x},\bar{y};\bar{y}')$. Now, from the figure, $x=\bar{x}$, $y=\bar{y}$ at $P$ while $y'=\tan\theta$ and $\bar{y}'=\tan\phi$ are related by

$$y' = \tan\theta = \tan(\phi-\omega) = \frac{\tan\phi - \tan\omega}{1+\tan\phi\tan\omega} = \frac{\bar{y}'-\tan\omega}{1+\bar{y}'\tan\omega}.$$

Thus, at $P$ (a general point in the plane) on an $\omega$- trajectory, the relation

$$f(x,y,y') = f(\bar{x},\bar{y},\ \frac{\bar{y}'-\tan\omega}{1+\bar{y}'\tan\omega}) = 0$$

holds, or, dropping the dashes, $f(x,y,\ \frac{y'-\tan\omega}{1+y'\tan\omega}) = 0.$

*B)* The integral curves of the differential equation

4) $$f(x,y,-1/y') = 0$$

are the orthogonal trajectories of the family of integral curves of 1).

*C)* In polar coordinates, the integral curves of the differential equation

5) $$f(\rho, \theta, -\rho^2 \frac{d\theta}{d\rho}) = 0$$

are the orthogonal trajectories of the integral curves of

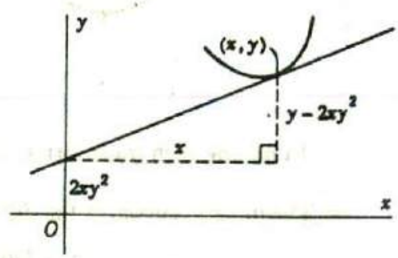
6) $$f(\rho, \theta, \frac{d\rho}{d\theta}) = 0.$$

## SOLVED PROBLEMS

1. At each point $(x,y)$ of a curve the intercept of the tangent on the $y$-axis is equal to $2xy^2$. Find the curve.

Using *e)*, the differential equation of the curve is

$$y - x\frac{dy}{dx} = 2xy^2 \quad \text{or} \quad \frac{y\,dx - x\,dy}{y^2} = 2x\,dx.$$




Integrating, $\frac{x}{y} = x^2 + C$ or $x - x^2y = Cy$.

The differential equation may also be obtained directly from the adjoining figure as $\frac{dy}{dx} = \frac{y - 2xy^2}{x}$.

2. At each point $(x,y)$ of a curve the subtangent is proportional to the square of the abscissa. Find the curve if it also passes through the point $(1,e)$.

Using *i)*, the differential equation is $y\frac{dx}{dy} = kx^2$ or $\frac{dx}{x^2} = k\frac{dy}{y}$, where $k$ is the proportionality factor.

Integrating, $k \ln y = -\frac{1}{x} + C$. When $x = 1$, $y = e$: $k = -1 + C$ and $C = k + 1$.

The required curve has equation $k \ln y = -\frac{1}{x} + k + 1$.

3. Find the family of curves for which the length of the part of the tangent between the point of contact $(x,y)$ and the $y$-axis is equal to the $y$-intercept of the tangent.

From *g)* and *e)*, we have $x\sqrt{1 + (\frac{dy}{dx})^2} = y - x\frac{dy}{dx}$ or *A)* $x^2 = y^2 - 2xy\frac{dy}{dx}$.

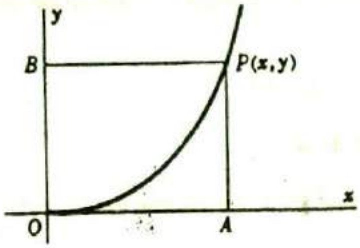
The transformation $y = vx$ reduces *A)* to

$$(1+v^2)dx + 2vx\,dv = 0 \quad \text{or} \quad \frac{dx}{x} + \frac{2v\,dv}{1+v^2} = 0.$$

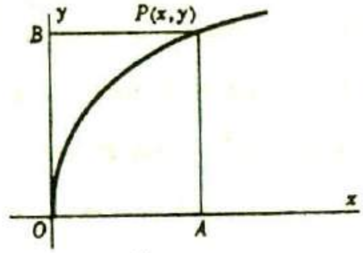
Integrating, $\ln x + \ln(1+v^2) = \ln C$.

Then $x(1 + \frac{y^2}{x^2}) = C$ or $x^2 + y^2 = Cx$ is the equation of the family.

4. Through any point $(x,y)$ of a curve which passes through the origin, lines are drawn parallel to the coordinate axes. Find the curve given that it divides the rectangle formed by the two lines and the axes into two areas, one of which is three times the other.




(a)




(b)

There are two cases illustrated in the figures.

*a)* Here 3(area $OAP$) = area $OPB$. Then $3\int_0^x y\,dx = xy - \int_0^x y\,dx$ or $4\int_0^x y\,dx = xy$.

To obtain the differential equation, we differentiate with respect to $x$.

Thus, $4y = y + x\dfrac{dy}{dx}$ or $\dfrac{dy}{dx} = \dfrac{3y}{x}$.

An integration yields the family of curves $y = Cx^3$.

*b)* Here area $OAP$ = 3(area $OPB$) and $4\int_0^x y\,dx = 3xy$.

The differential equation is $\dfrac{dy}{dx} = \dfrac{y}{3x}$, and the family of curves has equation $y^3 = Cx$.

Since the differential equation in each case was obtained by a differentiation extraneous solutions may have been introduced. It is necessary therefore to compute the areas as a check. In each of the above cases, the curves found satisfy the conditions. However, see Problem 5.

5. The areas bounded by the $x$-axis, a fixed ordinate $x = a$, a variable ordinate, and the part of a curve intercepted by the ordinates is revolved about the $x$-axis. Find the curve if the volume generated is proportional to *a)* the sum of the two ordinates, *b)* the difference of the two ordinates.

*a)* Let $A$ be the length of the fixed ordinate. The differential equation obtained by differentiating

1) $\pi\int_a^x y^2 dx = k(y + A)$ is $\pi y^2 = k\dfrac{dy}{dx}$. Integrating, we have 2) $y(C - \pi x) = k$.

When the value of $y$ given by 2) is used in computing the left member of 1), we find

$$3)\qquad \pi\int_a^x \frac{k^2\,dx}{(C - \pi x)^2} = \frac{k^2}{C - \pi x} - \frac{k^2}{C - \pi a} = k(y - A).$$

Thus, the solution is extraneous and no curve exists having the property *a)*.

*b)* Repeating the above procedure with 1′) $\pi\int_a^x y^2\,dx = k(y - A)$, we obtain the differential equation $\pi y^2 = k\dfrac{dy}{dx}$ whose solution is 2′) $y(C - \pi x) = k$.

It is seen from 3) that this equation satisfies 1′). Thus, the family of curves 2′) has the required property.

6. Find the curve such that at any point on it the angle between the radius vector and the tangent is equal to one-third the angle of inclination of the tangent.

Let $\theta$ denote the angle of inclination of the radius vector, $\tau$ the angle of inclination of the tangent, and $\psi$ the angle between the radius vector and the tangent.

Since $\psi = \tau/3 = (\psi + \theta)/3$, then $\psi = \frac{1}{2}\theta$ and $\tan\psi = \tan\frac{1}{2}\theta$.

Using $l)$, $\tan\psi = \rho\dfrac{d\theta}{d\rho} = \tan\frac{1}{2}\theta$ so that $\dfrac{d\rho}{\rho} = \cot\frac{1}{2}\theta\, d\theta$.

Integrating, $\ln\rho = 2\ln\sin\frac{1}{2}\theta + \ln C_1$ or $\rho = C_1\sin^2\frac{1}{2}\theta = C(1-\cos\theta)$.

7. The area of the sector formed by an arc of a curve and the radii vectors to the end points is one-half the length of the arc. Find the curve.

Let the radii vectors be given by $\theta = \theta_1$ and $\theta = \theta$.

Using $q)$ and $p)$, $\frac{1}{2}\int_{\theta_1}^{\theta}\rho^2\,d\theta = \frac{1}{2}\int_{\theta_1}^{\theta}\sqrt{(\frac{d\rho}{d\theta})^2 + \rho^2}\,d\theta$.

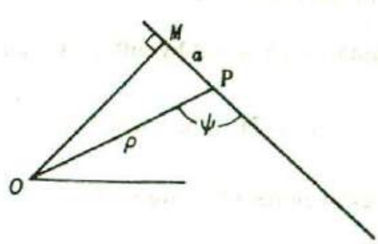
Differentiating with respect to $\theta$, we obtain the differential equation

$$\rho^2 = \sqrt{(\frac{d\rho}{d\theta})^2 + \rho^2} \quad \text{or} \quad 1) \quad d\rho = \pm\rho\sqrt{\rho^2 - 1}\,d\theta.$$

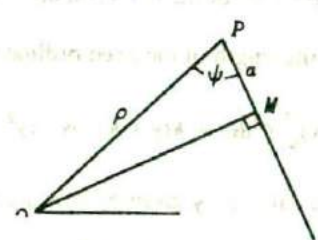
If $\rho^2 = 1$, 1) reduces to $d\rho = 0$. It is easily verified that $\rho = 1$ satisfies the condition of the problem.

If $\rho^2 \neq 1$, we write the equation in the form $\dfrac{d\rho}{\rho\sqrt{\rho^2 - 1}} = \pm\, d\theta$ and obtain the solution $\rho = \sec(C \pm \theta)$. Thus, the conditions are satisfied by the circle $\rho = 1$ and the family of curves $\rho = \sec(C + \theta)$. Note that the families $\rho = \sec(C + \theta)$ and $\rho = \sec(C - \theta)$ are the same.

8. Find the curve for which the portion of the tangent between the point of contact and the foot of the perpendicular through the pole to the tangent is one-third the radius vector to the point of contact.




(a)




(b)

In Figure $(a)$: $\rho = 3a = 3\rho\cos(\pi - \psi) = -3\rho\cos\psi$, $\cos\psi = -1/3$, und $\tan\psi = -2\sqrt{2}$.
In Figure $(b)$: $\rho = 3a = 3\rho\cos\psi$ and $\tan\psi = 2\sqrt{2}$.

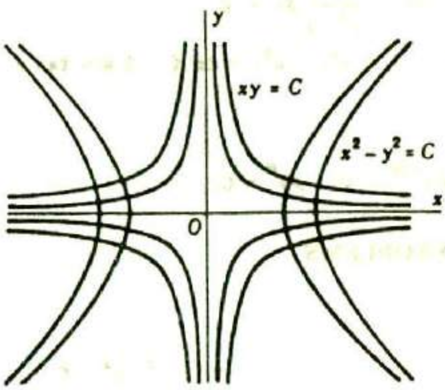
Using $l)$ and combining the two cases, $\tan\psi = \rho\dfrac{d\theta}{d\rho} = \pm 2\sqrt{2}$ or $\dfrac{d\rho}{\rho} = \pm\dfrac{d\theta}{2\sqrt{2}}$.

The required curves are the families $\rho = Ce^{\theta/2\sqrt{2}}$ and $\rho = Ce^{-\theta/2\sqrt{2}}$.

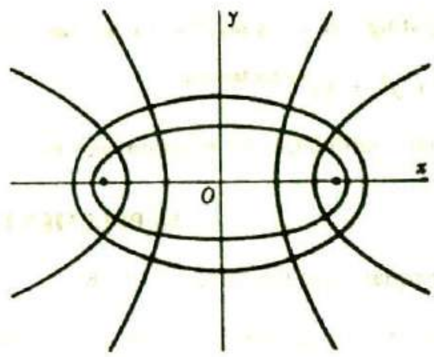
9. Find the orthogonal trajectories of the hyperbolas $xy = C$.

The differential equation of the given family is $x\frac{dy}{dx} + y = 0$, obtained by differentiating $xy = C$. The differential equation of the orthogonal trajectories, obtained replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, is $-x\frac{dx}{dy} + y = 0$ or $y\,dy - x\,dx = 0$.

Integrating, the orthogonal trajectories are the family of curves (hyperbolas) $y^2 - x^2 = C$.



Problem 9



Problem 10

10. Show that the family of confocal conics $\frac{x^2}{C} + \frac{y^2}{C-\lambda} = 1$, where $C$ is an arbitrary constant, is self-orthogonal.

Differentiating the equation of the family with respect to $x$ yields $\frac{x}{C} + \frac{yp}{C-\lambda} = 0$, where $p = \frac{dy}{dx}$. Solving this for $C$, we find $C = \frac{\lambda x}{x + yp}$ so that $C - \lambda = \frac{-\lambda py}{x + yp}$. When these replacements are made in the equation of the family, the differential equation of the family is found to be

$$(x + yp)(px - y) - \lambda p = 0.$$

Since this equation is unchanged when $p$ is replaced by $-1/p$, it is also the differential equation of the orthogonal trajectories of the given family.

11. Determine the orthogonal trajectories of the family of cardioids $\rho = C(1 + \sin\theta)$.

Differentiating with respect to $\theta$ to obtain $\frac{d\rho}{d\theta} = C\cos\theta$, solving for $C = \frac{1}{\cos\theta}\frac{d\rho}{d\theta}$, and substituting for $C$ in the given equation, the differential equation of the given family is

$$\frac{d\rho}{d\theta} = \frac{\rho\cos\theta}{1 + \sin\theta}.$$

The differential equation of the orthogonal trajectories, obtained by replacing $\frac{d\rho}{d\theta}$ by $-\rho^2\frac{d\theta}{d\rho}$ is

$$-\frac{d\theta}{d\rho} = \frac{\cos\theta}{\rho(1 + \sin\theta)} \quad \text{or} \quad \frac{d\rho}{\rho} + (\sec\theta + \tan\theta)d\theta = 0.$$

Then $\ln\rho + \ln(\sec\theta + \tan\theta) - \ln\cos\theta = \ln C$ or $\rho = \frac{C\cos\theta}{\sec\theta + \tan\theta} = C(1 - \sin\theta)$.

**12.** Determine the 45° trajectories of the family of concentric circles $x^2 + y^2 = C$.

The differential equation of the family of circles is $x + yy' = 0$.

The differential equation of the 45° trajectories, obtained by replacing $y'$ in the above equation by $\dfrac{y' - \tan 45^\circ}{1 + y'\tan 45^\circ} = \dfrac{y' - 1}{1 + y'}$, is $x + y\dfrac{y' - 1}{1 + y'} = 0$ or $(x + y)dy + (x - y)dx = 0$.

Using the transformation $y = vx$, this equation is reduced to

$$(v^2 + 1)dx + x(v + 1)dv = 0 \qquad \text{or} \qquad \frac{dx}{x} + \frac{v+1}{v^2+1}dv = 0.$$

Integrating, $\ln x + \frac{1}{2}\ln(v^2 + 1) + \arctan v = \ln K_1$, $\ln x^2(1 + v^2) = \ln K - 2\arctan v$,

and $x^2 + y^2 = Ke^{-2\arctan y/x}$.

In polar coordinates, the equation becomes $\rho^2 = Ke^{-2\theta}$ or $\rho e^{\theta} = C$.

## SUPPLEMENTARY PROBLEMS

**13.** Find the equation of the curve for which

*a)* the normal at any point $(x, y)$ passes through the origin. *Ans.* $x^2 + y^2 = C$

*b)* the slope of the tangent at any point $(x, y)$ is $\frac{1}{2}$ the slope of the line from the origin to the point. *Ans.* $y^2 = Cx$

*c)* the normal at any point $(x, y)$ and the line joining the origin to that point form an isosceles triangle having the $x$-axis as base. *Ans.* $y^2 - x^2 = C$

*d)* the part of the normal drawn at point $(x, y)$ between this point and the $x$-axis is bisected by the $y$-axis. *Ans.* $y^2 + 2x^2 = C$

*e)* the perpendicular from the origin to a tangent line of the curve is equal to the abscissa of the point of contact $(x, y)$. *Ans.* $x^2 + y^2 = Cx$

*f)* the arc length from the origin to the variable point $(x, y)$ is equal to twice the square root of the abscissa of the point. *Ans.* $y = \pm(\arcsin\sqrt{x} + \sqrt{x - x^2}) + C$

*g)* the polar subnormal is twice the sine of the vectorial angle. *Ans.* $\rho = C - 2\cos\theta$

*h)* the angle between the radius vector and the tangent is $\frac{1}{2}$ the vectorial angle. *Ans.* $\rho = C(1 - \cos\theta)$

*i)* the polar subtangent is equal to the polar subnormal. *Ans.* $\rho = Ce^{\theta}$

**14.** Find the orthogonal trajectories of each of the following families of curves.

| | | | |
|---|---|---|---|
| *a)* $x + 2y = C$ | *Ans* $y - 2x = K$ | *f)* $y = x - 1 + Ce^{-x}$ | *Ans* $x = y - 1 + Ke^{-y}$ |
| *b)* $xy = C$ | $x^2 - y^2 = K$ | *g)* $y^2 = 2x^2(1 - Cx)$ | $x^2 + 3y^2\ln(Ky) = 0$ |
| *c)* $x^2 + 2y^2 = C$ | $y = Kx^2$ | *h)* $\rho = a\cos\theta$ | $\rho = b\sin\theta$ |
| *d)* $y = Ce^{-2x}$ | $y^2 = x + K$ | *i)* $\rho = a(1 + \sin\theta)$ | $\rho = b(1 - \sin\theta)$ |
| *e)* $y^2 = x^3/(C - x)$ | $(x^2 + y^2)^2 = K(2x^2 + y^2)$ | *j)* $\rho = a(\sec\theta + \tan\theta)$ | $\rho = be^{-\sin\theta}$ |

## CHAPTER 8

# Physical Applications

**MANY OF THE APPLICATIONS** of this and later chapters will be concerned with the motion of a body along a straight line. If the body moves with varying velocity $v$ (that is with accelerated motion) its acceleration, given by $dv/dt$, is due to one or more forces acting in the direction of motion or in the opposite direction. The net force on the mass is the (algebraic) sum of the several forces.

EXAMPLE 1. A boat is moving subject to a force of 90 newtons on its sail and a resisting force (N) equal to 0.3 times its velocity ($ms^{-1}$) If the direction of motion is taken as positive, the net force (N) is $90 - 0 \cdot 3v$.

EXAMPLE 2. To the free end of a spring of negligible mass, hanging vertically, a mass is attached and brought to rest. There are two forces acting on the mass – gravity acting downward and a restoring force, called the spring force, opposing gravity. The two forces, being opposite in direction, are equal in magnitude since the mass is at rest. Thus, the net force is zero.

Newton's Second Law of Motion states in part that the product of the mass and acceleration is proportional to the net force on the mass. When the system of units described below is used, the factor of proportionality is $k = 1$ and we have

$$\text{mass} \times \text{acceleration} = \text{net force.}$$

**THE S.I. SYSTEM** is based on the fundamental units: the *kilogramme* (kg) *of mass*, the *metre* (m) *of length*, and the *second* (s) *of time*. The derived unit of *force* is the *newton* (N), defined by

$$1\text{N} = 1\text{kg ms}^{-2}$$

Hence,

$$\text{mass in kilogrammes} \times \text{acceleration in ms}^{-2} = \text{net force in newtons}$$

The acceleration $g$ of a freely falling body varies but slightly over the earth's surface. For convenience in computing, an approximate value $g = 9 \cdot 8\text{ms}^{-2}$ is used in the problems.

## SOLVED PROBLEMS

**1.** If the population of a country doubles in 50 years, in how many years will it treble under the assumption that the rate of increase is proportional to the number of inhabitants?

Let $y$ denote the population at time $t$ years and $y_0$ the population at time $t = 0$. Then

1) $\dfrac{dy}{dt} = ky$ or $\dfrac{dy}{y} = k\,dt$, where $k$ is the proportionality factor.

*First Solution*. Integrating 1), we have 2) $\ln y = kt + \ln C$ or $y = Ce^{kt}$.

At time $t = 0$, $y = y_0$ and, from 2), $C = y_0$. Thus, 3) $y = y_0e^{kt}$.

At $t = 50$, $y = 2y_0$. From 3), $2y_0 = y_0 e^{50k}$ or $e^{50k} = 2$.

When $y = 3y_0$, 3) gives $3 = e^{kt}$. Then $3^{50} = e^{50kt} = (e^{50k})^t = 2^t$ and $t = 79$ years.

*Second Solution.* Integrating 1) between the limits $t = 0$, $y = y_0$ and $t = 50$, $y = 2y_0$,

$$\int_{y_0}^{2y_0} \frac{dy}{y} = k\int_0^{50} dt, \qquad \ln 2y_0 - \ln y_0 = 50k \quad \text{and} \quad 50k = \ln 2.$$

Integrating 1) between the limits $t = 0$, $y = y_0$ and $t = t$, $y = 3y_0$,

$$\int_{y_0}^{3y_0} \frac{dy}{y} = k\int_0^{t} dt, \quad \text{and} \quad \ln 3 = kt.$$

Then $50 \ln 3 = 50kt = t \ln 2$ and $t = \dfrac{50 \ln 3}{\ln 2} = 79$ years.

2. In a certain culture of bacteria the rate of increase is proportional to the number present. *(a)* If it is found that the number doubles in 4 hours, how many may be expected at the end of 12 hours? *(b)* If there are $10^4$ at the end of 3 hours and $4 \cdot 10^4$ at the end of 5 hours, how many were there in the beginning?

Let $x$ denote the number of bacteria at time $t$ hours. Then

$$1) \qquad \frac{dx}{dt} = kx \quad \text{or} \quad \frac{dx}{x} = k\,dt.$$

*a) First Solution.* Integrating 1), we have 2) $\ln x = kt + \ln C$ or $x = Ce^{kt}$.

Assuming that $x = x_0$ at time $t = 0$, $C = x_0$ and $x = x_0 e^{kt}$.

At time $t = 4$, $x = 2x_0$. Then $2x_0 = x_0 e^{4k}$ and $e^{4k} = 2$.

When $t = 12$, $x = x_0 e^{12k} = x_0(e^{4k})^3 = x_0(2^3) = 8x_0$, that is, there are 8 times the original number.

*Second Solution.* Integrating 1) between the limits $t = 0$, $x = x_0$ and $t = 4$, $x = 2x_0$,

$$\int_{x_0}^{2x_0} \frac{dx}{x} = k\int_0^{4} dt, \qquad \ln 2x_0 - \ln x_0 = 4k \quad \text{and} \quad 4k = \ln 2.$$

Integrating 1) between the limits $t = 0$, $x = x_0$ and $t = 12$, $x = x$,

$$\int_{x_0}^{x} \frac{dx}{x} = k\int_0^{12} dt, \quad \text{and} \quad \ln\frac{x}{x_0} = 12k = 3(4k) = 3\ln 2 = \ln 8.$$

Then $x = 8x_0$, as before.

*b) First Solution.* When $t = 3$, $x = 10^4$. Hence, from 2), $10^4 = Ce^{3k}$ and $C = \dfrac{10^4}{e^{3k}}$.

When $t = 5$, $x = 4 \cdot 10^4$. Hence, $4 \cdot 10^4 = Ce^{5k}$ and $C = \dfrac{4 \cdot 10^4}{e^{5k}}$.

Equating the values of $C$, $\dfrac{10^4}{e^{3k}} = \dfrac{4 \cdot 10^4}{e^{5k}}$. Then $e^{2k} = 4$ and $e^k = 2$.

Thus, the original number is $C = \dfrac{10^4}{e^{3k}} = \dfrac{10^4}{8}$ bacteria.

*Second Solution*. Integrating 1) between the limits $t = 3,\ x = 10^4$ and $t = 5,\ x = 4\cdot 10^4$,

$$\int_{10^4}^{4\cdot 10^4} \frac{dx}{x} = k\int_3^5 dt, \qquad \ln 4 = 2k \quad \text{and} \quad k = \ln 2.$$

Integrating 1) between the limits $t = 0,\ x = x_0$ and $t = 3,\ x = 10^4$,

$$\int_{x_0}^{10^4} \frac{dx}{x} = k\int_0^3 dt, \qquad \ln\frac{10^4}{x_0} = 3k = 3\ln 2 = \ln 8 \quad \text{and} \quad x_0 = \frac{10^4}{8} \text{ as before.}$$

3. According to Newton's law of cooling, the rate at which a substance cools in moving air is proportional to the difference between the temperature of the substance and that of the air. If the temperature of the air is 300K and the substance cools from 370K to 340K in 15 minutes, find when the temperature will be 310K.

Let $T$ be the temperature of the substance at the time $t$ minutes.

Then $\quad \dfrac{dT}{dt} = -k(T - 300) \quad$ or $\quad \dfrac{dT}{T-300} = -k\,dt.$

(Note. The use of $-k$ here is optional. It will be found that $k$ is positive, but if $+k$ is used it will be found that $k$ is equally negative.)

Integrating between the limits $t = 0,\ T = 370$ and $t = 15,\ T = 340$,

$$\int_{370}^{340} \frac{dT}{T-300} = -k\int_0^{15} dt, \qquad \ln 40 - \ln 70 = -15k = \ln\frac{4}{7} \quad \text{and} \quad 15k = \ln\frac{7}{4} = 0.56.$$

Integrating between the limits $t = 0,\ T = 370$ and $t = t,\ T = 310$,

$$\int_{370}^{310} \frac{dT}{T-300} = -k\int_0^t dt, \qquad \ln 10 - \ln 70 = -kt, \quad 15kt = 15\ln 7, \quad t = \frac{15\ln 7}{0.56} = 52 \text{ min.}$$

4. A certain chemical dissolves in water at a rate proportional to the product of the amount undissolved and the difference between the concentration in a saturated solution and the concentration in the actual solution. In 100 grams of a saturated solution it is known that 50 grams of the substance are dissolved. If when 30 grams of the chemical are agitated with 100 grams of water, 10 grams are dissolved in 2 hours. how much will be dissolved in 5 hours?

Let $x$ denote the number of grams of the chemical undissolved after $t$ hours. At this time the concentration of the actual solution is $\dfrac{30-x}{100}$ an that of the saturated solution is $\dfrac{50}{100}$.

Then

$$\frac{dx}{dt} = kx\left(\frac{50}{100} - \frac{30-x}{100}\right) = kx\,\frac{x+20}{100} \quad \text{or} \quad \frac{dx}{x} - \frac{dx}{x+20} = \frac{k}{5}\,dt.$$

Integrating between $t = 0,\ x = 30$ and $t = 2,\ x = 30 - 10 = 20$,

$$\int_{30}^{20} \frac{dx}{x} - \int_{30}^{20} \frac{dx}{x+20} = \frac{k}{5}\int_0^2 dt, \quad \text{and} \quad k = \frac{5}{2}\ln\frac{5}{6} = -0.46.$$

Integrating between $t = 0,\ x = 30$ and $t = 5,\ x = x$,

$$\int_{30}^{x} \frac{dx}{x} - \int_{30}^{x} \frac{dx}{x+20} = \frac{k}{5}\int_0^5 dt, \qquad \ln\frac{5x}{3(x+20)} = k = -0.46, \quad \frac{x}{x+20} = \frac{3}{5}e^{-0.46}$$

$= 0.38$, and $x = 12$. Thus, the amount dissolved after 5 hours is $30 - 12 = 18$ grams.

5. A tank of volume $0{\cdot}5\ m^3$ is filled with brine containing 30 kg of dissolved salt. Water runs into the tank at the rate of $15 \times 10^{-5}\ m^3 s^{-1}$ and the mixture, kept uniform by stirring, runs out at the same rate. How much salt is in the tank after 1 hour?

Let $x$ be the number of kilogrammes of salt in the tank after $t$ seconds, the concentration then being $2x\ kg\,m^{-3}$. During the interval $dt$, $15 \times 10^{-5} dt$ cubic meters of water flow in and $15 \times 10^{-5}\, dt$ cubic meters of brine containing $2x \times 15 \times 10^{-5} dt = 3x \times 10^{-4}\, dt$ kilogrammes of salt flow out.

Thus, the change $dx$ of the amount of salt in the tank is $dx = -3x \times 10^{-4} dt$.

Integrating $x = Ce^{-3\times 10^{-4} t}$. At $t = 0$, $x = 30$, hence $C = 30$ and $x = 30\,e^{-3\times 10^{-4} t}$

When $t = 3{,}600$ seconds, $x = 30\,e^{-108 \times 10^{-2}} = 30\,e^{-1{\cdot}08} = 10$ kilogrammes.

6. The air in a certain room $50m \times 17.5m \times 4m$ tested $0{\cdot}2\%\,CO_2$. Fresh air containing $0{\cdot}05\%\,CO_2$ was then admitted by ventilators at the rate $4{\cdot}2\,m^3 s^{-1}$. Find the percentage $CO_2$ after 20 minutes.

Let $x$ denote the number of cubic meters of $CO_2$ in the room at time $t$, the concentration of $CO_2$ then being $x/3{,}500$ During the interval $dt$, the amount of $CO_2$ entering the room is $4{\cdot}2(\cdot 0005)\,dt\ m^3$ and the amount leaving is $4{\cdot}2\,\dfrac{x}{3{\cdot}500}\,dt\ m^3$

Hence the change $dx$ in the interval is $dx = 4{\cdot}2\left(0{\cdot}0005 - \dfrac{x}{3{,}500}\right)dt = (21\times 10^{-4} - 12 \times 10^{-4} x)\,dt$.

Integrating $\dfrac{1}{12} \cdot 10^4 \ln(21 \cdot 10^{-4} - 12 \cdot 10^{-4} x) = -t + \ln C_1$ and $x = 7/4 + Ce^{-12\times 10^{-4} t}$.

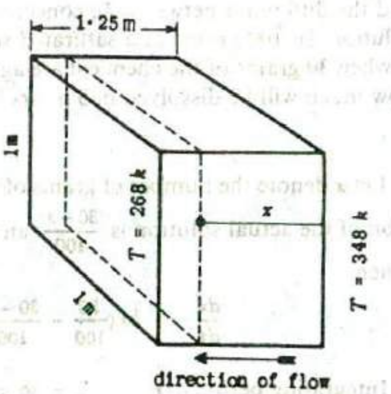
At $t = 0$, $x = 0{\cdot}002 \times 3{,}500 = 7$. Then $C = 21/4$ and $x = 7/4(1 + 3e^{-12\times 10^{-4} t})$.

When $t = 1{,}200$, $x = 7/4\,(1 + 3e^{-1{\cdot}44}) = 3{\cdot}06$. The percentage $CO_2$ is then $\dfrac{3{\cdot}06 \times 100}{3{\cdot}500} = 0{\cdot}09\%$

7. Under certain conditions the constant quantity $Q$ joules/second of heat flowing through a wall is given by

$$Q = -kA\,\frac{dT}{dx},$$

where $k$ is the conductivity of the material, $A(m^2)$ is the area of a face of the wall perpendicular to the direction of flow, and $T$ is the temperature $x$ (m) from that face such that $T$ decreases as $x$ increases. Find the number of joules of heat per hour flowing through 1 square metre of the wall of a refrigerator room $1{\cdot}25\ m$ thick for which $k = 1{\cdot}05$, if the temperature of the inner face is $268K$ and that of the outer face is $348K$



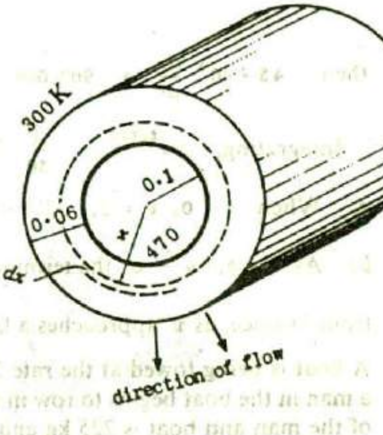

Let $x$ denote the distance of a point within the wall from the outer face.

Integrating $dT = -\dfrac{Q}{kA}\,dx$ from $x = 0$, $T = 348$ to $x = 1{\cdot}25$, $T = 268$,

$$\int_{348}^{268} dT = -\frac{Q}{kA}\int_0^{1.25} dx, \qquad 80 = \frac{Q}{kA}(1{\cdot}25) \text{ and } \qquad Q = \frac{80kA}{1{\cdot}25} = \frac{80 \times 1{\cdot}05}{1{\cdot}25} = 67{\cdot}2\ Js^{-1}$$

Thus the flow of heat per hour is $3{,}600Q = 2{\cdot}42 \times 10^5\,J$.

**8.** A steam pipe 0·2 m in diameter is protected with a covering 0·06 m thick for which $k = 0 \cdot 13$ *(a)* Find the heat loss per hour through a metre length of the pipe if the surface of the pipe is at 470 K and the outer surface of the covering is at 300 K *(b)* Find the temperature at a distance $x > 0 \cdot 1$m from the centre of the pipe.




At a distance $x > 0 \cdot 1$m from the centre of the pipe, heat is flowing across a cylindrical shell of surface area $2\pi x$ m$^2$ per m of length of pipe. From Problem 7,

$$Q = -kA\frac{dT}{dx} = -2\pi kx\frac{dT}{dx} \quad \text{or} \quad 2\pi k\, dT = -Q\frac{dx}{x}.$$

*a)* Integrating between the limits $T = 300,\ x = 0 \cdot 16$ and $T = 470,\ x = 0 \cdot 1$

$$2\pi k\int_{300}^{470} dT = -Q\int_{0\cdot16}^{0\cdot1}\frac{dx}{x},\quad 340\pi k = Q(\ln 0\cdot16 - \ln 0\cdot1) = Q\ln 1\cdot6 \quad \text{and} \quad Q = \frac{340\pi k}{\ln 1\cdot6}\ \text{J s}^{-1}$$

Thus, the heat loss per hour through a metre length of pipe is $3{,}600\ Q = 1 \cdot 03$ MJ

*b)* Integrating $2\pi k\, dT = -\frac{340\pi k}{\ln 1.6}\frac{dx}{x}$ between the limits $T = 300,\ x = 0 \cdot 16$ and $T = T,\ x = x$,

$$\int_{300}^{T} dT = -\frac{170}{\ln 1.6}\int_{0.16}^{x}\frac{dx}{x},\qquad T - 300 = -\frac{170}{\ln 1.6}\ln\frac{x}{0.16} \quad \text{and} \quad T = 300 + \frac{170}{\ln 1.6}\ln\frac{16}{x}$$

*Check*. When $x = 0 \cdot 1$, $T = 300 + \frac{170}{\ln 1.6}\ln 1.6 = 470$K. When $x = 0 \cdot 16$, $T = 300 + 0 = 300$ K

**9.** Find the time required for a cylindrical tank of radius 2·5 m and height 3 m to empty through a round hole of radius 25 mm in the bottom of the tank, given that water will issue from such a hole with velocity approximately $v = 2 \cdot 5\sqrt{h}$ ms$^{-1}$; $h$ being the depth of the water in the tank.

The volume of water which runs out per second may be thought of as that of a cylinder 25 mm in radius and of height $v$. Hence, the volume which runs out in time $dt$ sec is

$$\pi(0 \cdot 025)^2(2 \cdot 5\sqrt{h})dt.$$

Denoting by $dh$ the corresponding drop in the water level in the tank, the volume of water which runs out is also given by $(2 \cdot 5)^2\pi dh$. Hence,

$$\pi(0{,}025)^2(2{,}5\sqrt{h})dt = -\pi(2{,}5)^2 dh \quad \text{or} \quad dt = -\left(\frac{2{,}5}{0{,}025}\right)^2\frac{dh}{2{,}5\sqrt{h}} = -4\,000\frac{dh}{\sqrt{h}}$$

Integrating between $t = 0,\ h = 3$ and $t = t,\ h = 0$,

$$\int_0^t dt = -4000\int_3^0\frac{dh}{\sqrt{h}} \qquad t = -8\,000\sqrt{h}\,\Big|_3^0 = 8\,000\sqrt{3}\ \text{secs} = 3\text{hr}\ 34\text{min}$$

**10.** A ship of mass 45,000 Mg starts from rest under the force of a constant propeller thrust of 900,000 N. *a)* Find its velocity as a function of time $t$ given that the resistance in newtons is $150{,}000v$ with $v$ = velocity measured in m s$^{-1}$ *b)* Find the terminal velocity (i.e. $v$ when $t \to \infty$) in kilometres per hour.

Since mass (kg) × acceleration ($ms^{-2}$) = net force (N)
= impetus of propeller – resistance

then $45 \times 10^6 \frac{dv}{dt} = 900{,}000 - 15 \times 10^4 v$ or 1) $\frac{dv}{dt} + \frac{v}{300} = \frac{1}{50}$

Integrating, $ve^{t/300} = \frac{1}{50}\int e^{t/300}\,dt = 6e^{t/300} + C.$

*a)* When $t = 0,\ v = 0;\ C = -6$ and $v = 6(1 - e^{-t/300})$.

*b)* As $t \to \infty,\ v \to 6$ the terminal velocity is $6\ ms^{-1} = 21 \cdot 6\ km$ per hour. This may also be obtained from 1) since, as $v$ approaches a limiting value, $\frac{dv}{dt} \to 0$. Then $v = 6$ as before.

11. A boat is being towed at the rate 20 km per hour. At the instant ($t = 0$) that the towing line is cast off, a man in the boat begins to row in the direction of motion exerting a force of 90 N. If the combined mass of the man and boat is 225 kg and the resistance (N) is equal to $26{\cdot}25\ v$, where $v$ is measured in $ms^{-1}$ find the speed of the boat after 1/2 minute.

Since mass (kg) × acceleration ($ms^{-2}$) = net force (N)
= forward force – resistance

then $225\frac{dv}{dt} = 90 - 26{\cdot}25v$ or $\frac{dv}{dt} + \frac{7}{60}v = \frac{2}{5}$

Integrating, $ve^{7t/60} = \frac{2}{5}\int e^{7t/60}\,dt = \frac{120}{35}e^{7t/60} + C.$

When $t = 0,\ v = \frac{20{,}000}{3600} = \frac{50}{9},\quad C = \frac{134}{63}$ and $v = \frac{24}{7} + \frac{134}{63}e^{-7t/60}$.

When $t = 30,\quad v = \frac{24}{7} + \frac{134}{63}e^{-3.5} = 3{\cdot}5\ ms^{-1}$

12. A load is being pulled across the ice on a sled, the total mass including the sled being 35 kg. Under the assumption that the resistance offered by the ice to the runners is negligible and that the air offers a resistance in newtons equal to 70 times the velocity ($v\ ms^{-1}$) of the sled, find
*a)* the constant force (newtons) exerted on the sled which will give it a terminal velocity of 16 kilometres per hour, and
*b)* the velocity and distance ($s$ m) travelled at the end of 48 seconds.

Since mass (kg) × acceleration ($ms^{-2}$) = net force (N)
= forward force – resistance

then $35\frac{dv}{dt} = F - 70v$ or $\frac{dv}{dt} + 2v = \frac{1}{35}F$, where $F$ (N) is the forward force.

Integrating, $v = \frac{F}{70} + Ce^{-2t}$. When $t = 0,\ v = 0;$ then $C = -\frac{F}{70}$ and

*A)* $$v = \frac{F}{70}(1 - e^{-2t}).$$

*a)* As $t \to \infty,\ v = \frac{F}{70} = \frac{16\,000}{3\,600} = \frac{40}{9}$. The required force is $F = \frac{2\,800}{9} = 311$ N.

*b)* Substituting from *a)* in *A)*, $v = \frac{40}{9}(1 - e^{-2t})$.

When $t = 48$: $v = \frac{40}{9}(1 - e^{-96}) = \frac{40}{9}\ ms^{-1}$ and $s = \int_0^{48} v\,dt = \frac{40}{9}\int_0^{48}(1 - e^{-2t})dt = 211$ m.

13. A spring of negligible weight hangs vertically. A mass of $m$ kg is attached to the other end. If the mass is moving with velocity $v_0$ $ms^{-1}$ when the spring is unstretched, find the velocity $v$ as a function of the stretch $x$ m.

According to Hooke's law, the spring force (force opposing the stretch) is proportional to the stretch.

Net force on body = weight of body − spring force.

Then $m\dfrac{dv}{dt} = mg - kx$ or $m\dfrac{dv}{dx}\dfrac{dx}{dt} = mv\dfrac{dv}{dx} = mg - kx$, since $\dfrac{dx}{dt} = v$.

Integrating, $mv^2 = 2mgx - kx^2 + C$.

When $x = 0$, $v = v_0$. Then $C = mv_0^2$ and $mv^2 = 2mgx - kx^2 + mv_0^2$.

14. A parachutist is falling with speed 55 $ms^{-1}$ when his parachute opens. If the air resistance is $Wv^2/25$ N, where $W$ is the total weight of the man and parachute, find his speed as a function of the time $t$ after the parachute opened.

Net force on system = weight of system − air resistance.

Then $\dfrac{W}{g}\dfrac{dv}{dt} = W - \dfrac{Wv^2}{25}$ or $\dfrac{dv}{v^2 - 25} = -\dfrac{9.8}{25}dt$.

Integrating between the limits $t = 0$, $v = 55$ and $t = t$, $v = v$,

$$\int_{55}^{v} \frac{dv}{v^2 - 25} = -\frac{9\cdot 8}{25}\int_0^t dt, \qquad \frac{1}{10}\ln\frac{v-5}{v+5}\bigg|_{55}^{v} = -\frac{9\cdot 8t}{25}\bigg|_0^t .$$

$$\ln\frac{v-5}{v+5} - \ln\frac{5}{6} = -\frac{98}{25}t, \qquad \frac{v-5}{v+5} = \frac{5}{6}e^{-4t}, \text{ and } v = 5\,\frac{6+5e^{-4t}}{6-5e^{-4t}} .$$

Note that the parachutist quickly attains an approximately constant speed, that is, the terminal speed of 5 $ms^{-1}$

15. A body of mass $m$ kg falls from rest in a medium for which the resistance (N) is proportional to the square of the velocity ($ms^{-1}$) If the terminal velocity is 50 $ms^{-1}$ find

*a)* the velocity at the end of 2 seconds, and

*b)* the time required for the velocity to become 30 $ms^{-1}$

Let $v$ denote the velocity of the body at time seconds.
Net force on body = weight of body − resistance, and the equation of motion is 1) $m\dfrac{dv}{dt} = mg - Kv^2$.

Taking $g = 9\cdot 8ms^{-2}$ it is seen that some simplification is possible by choosing $K = -\dfrac{9\cdot 8}{16}mk^2$.

Then 1) reduces to $\dfrac{dv}{dt} = \dfrac{9\cdot 8}{16}(16 - k^2v^2)$ or $\dfrac{dv}{k^2v^2 - 16} = \dfrac{9\cdot 8}{16}dt$.

Integrating, $\ln\dfrac{kv-4}{kv+4} = -4.9\,kt + \ln C$ or $\dfrac{kv-4}{kv+4} = Ce^{-4.9kt}$.

When $t = 0$, $v = 0$. Then $C = -1$ and 2) $\dfrac{kv-4}{kv+4} = -e^{-4.9kt}$.

When $t \to \infty$, $v = 50$. Then $e^{-4.9kt} = 0$, $k = \dfrac{2}{25}$, and 2) becomes $\dfrac{v-50}{v+50} = -e^{-0.39t}$.

*a)* When $t = 2$, $\dfrac{v-50}{v+50} = -e^{-0.78} = -0.46$ and $v = 18{\cdot}5\,ms^{-1}$.

*b)* When $v = 30$, $e^{-0.39t} = 0.25 = e^{-1.39}$ and $t = 3.5$ secs.

**16.** A body of mass $m$ falls from rest in a medium for which the resistance (N) is proportional to the velocity $ms^{-1}$. If the specific gravity of the medium is one-fourth that of the body and if the terminal velocity is $7{\cdot}35\,ms^{-1}$ find *(a)* the velocity at the end of 3 sec and *(b)* the distance travelled in 3 sec.

Let $v$ denote the velocity of the body at time $t$ sec. In addition to the two forces acting as in Problem 15, there is a third force which results from the difference in specific gravities. This force is equal in magnitude to the weight of the medium which the body displaces and opposes gravity.

Net force on body = weight of body − bouyant force − resistance, and the equation of motion is

$$m\frac{dv}{dt} = mg - \frac{1}{4}mg - Kv = \frac{3}{4}mg - Kv.$$

Taking $g = 9{\cdot}8\,ms^{-2}$ and $K = 3mk$ the equation becomes $\dfrac{dv}{dt} = 3(2.45 - kv)$ or $\dfrac{dv}{2.45-kv} = 3dt$.
Integrating from $t = 0,\ v = 0$ to $t = t,\ v = v$,

$$-\frac{1}{k}\ln(2.45-kv)\Big|_0^v = 3t\Big|_0^t \quad -\ln(2.45-kv) + \ln 2.45 = 3kt \text{ and } kv = 2.45(1-e^{-3kt}).$$

When $t \to \infty$, $v = 7\cdot35$. Then $k = 1/3$ and 1) $v = 7.35(1-e^{-t})$.

*a)* When $t = 3$, $v = 7.35(1-e^{-3}) = 7ms^{-1}$.

*b)* Integrating $v = \dfrac{dx}{dt} = 7{\cdot}35(1-e^{-t})$ between $t = 0,\ x = 0$ and $t = 3,\ x = x$,

$$x\Big|_0^x = 7.35(t+e^{-t})\Big|_0^3 \quad \text{and} \quad x = 7.35(2+e^{-3}) = 15\,m$$

**17.** The gravitational pull on a mass $m$ at a distance $s$ metres from the centre of the earth is proportional to $m$ and inversely proportional to $s^2$. *a)* Find the velocity attained by the mass in falling from rest at a distance $5R$ from the centre to the earth's surface, where $R = 6500$ km is taken as the radius of the earth. *b)* What velocity would correspond to a fall from an infinite distance, that is, with what velocity must the mass be propelled vertically upward to escape the gravitational pull? (All other forces, including friction, are to be neglected.)

The gravitational force at a distance $s$ from the earth's centre is $km/s^2$. To determine $k$, note that the force is $mg$ when $s = R$; thus $mg = km/R^2$ and $k = gR^2$. The equation of motion is

$$1)\quad m\frac{dv}{dt} = m\frac{ds}{dt}\frac{dv}{ds} = mv\frac{dv}{ds} = -\frac{mgR^2}{s^2} \quad \text{or} \quad v\,dv = -gR^2\frac{ds}{s^2},$$

the sign being negative since $v$ increases as $s$ decreases.

*a)* Integrating 1) from $v = 0,\ s = 5R$ to $v = v,\ s = R$,

$$\int_0^v v\,dv = -gR^2\int_{5R}^{R}\frac{ds}{s^2}, \quad \tfrac{1}{2}v^2 = gR^2\Big(\frac{1}{R}-\frac{1}{5R}\Big) = \frac{4}{5}gR, \quad v^2 = \frac{8}{5}(9.8)(6\,500)(1\,000),$$

and $v = \sqrt{102}\times 10^3\,ms^{-1}$ or approximately $10\,kms^{-1}$.

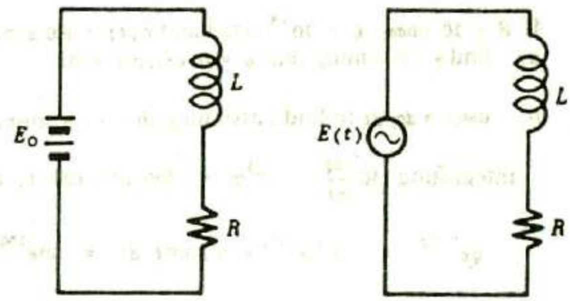
*b)* Integrating 1) from $v = 0,\ s \to \infty$ to $v = v,\ s = R$,

$$\int_0^v v\,dv = -gR^2 \int_\infty^R \frac{ds}{s^2}, \quad v^2 = 2gR, \quad v = 1000\sqrt{127}\ \text{ms}^{-1} \text{ or approximately } 11.3\ \text{kms}^{-1}.$$

**18.** One of the basic equations in electric circuits is

1) $L\dfrac{di}{dt} + Ri = E(t)$,

where $L$ (henries) is called the inductance, $R$ (ohms) the resistance, $i$ (amperes) the current, and $E$ (volts) the electromotive force or emf. (In this book $R$ and $L$ will be constants.)




(a) (b)

*a)* Solve 1) when $E(t) = E_0$ and the initial current is $i_0$.

*b)* Solve 1) when $L = 3$ henries, $R = 15$ ohms, is the 60 cycle sine wave of amplitude 110 volts, and $i = 0$ when $t = 0$.

*a)* Integrating $L\dfrac{di}{dt} + Ri = E_0$, $\quad ie^{Rt/L} = \dfrac{E_0}{L}\displaystyle\int e^{Rt/L}\,dt = \dfrac{E_0}{R}e^{Rt/L} + C$ or $i = \dfrac{E_0}{R} + Ce^{-Rt/L}$.

When $t = 0$, $i = i_0$. Then $C = i_0 - \dfrac{E_0}{R}$ and $i = \dfrac{E_0}{R}(1 - e^{-Rt/L}) + i_0e^{-Rt/L}$.

Note that as $t \to \infty$, $i = E_0/R$, a constant.

*b)* Integrating $3\dfrac{di}{dt} + 15i = E_0 \sin \omega t = 110 \sin 2\pi(60)t = 110 \sin 120\pi t$,

$$ie^{5t} = \frac{110}{3}\int e^{5t} \sin 120\pi t\,dt = \frac{110}{3}e^{5t}\,\frac{5 \sin 120\pi t - 120\pi \cos 120\pi t}{25 + 14400\pi^2} + C$$

or
$$i = \frac{22}{3}\,\frac{\sin 120\pi t - 24\pi \cos 120\pi t}{1 + 576\pi^2} + Ce^{-5t}.$$

When $t = 0$, $i = 0$. Then $C = \dfrac{22 \cdot 24\pi}{3(1 + 576\pi^2)}$

and
$$i = \frac{22}{3}\,\frac{\sin 120\pi t - 24\pi \cos 120\pi t + 24\pi e^{-5t}}{1 + 576\pi^2}.$$

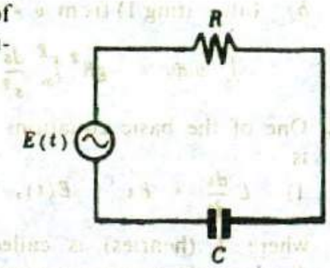
A more useful form is obtained by noting that the sum of the squares of the coefficients of the sine and cosine terms is the denominator of the fraction above. Hence, we may define

$$\sin\phi = \frac{24\pi}{(1 + 576\pi^2)^{1/2}} \quad \text{and} \quad \cos\phi = \frac{1}{(1 + 576\pi^2)^{1/2}}.$$

so that
$$i = \frac{22}{3(1 + 576\pi^2)^{1/2}}(\cos\phi \sin 120\pi t - \sin\phi \cos 120\pi t) + \frac{176\pi e^{-5t}}{1 + 576\pi^2}.$$

$$= \frac{22}{3(1 + 576\pi^2)^{1/2}}\sin(120\pi t - \phi) + \frac{176\pi e^{-5t}}{1 + 576\pi^2}.$$

19. If an electric circuit contains a resistance $R$ (ohms) and a condenser of capacitance $C$ (farads) in series, and an emf $E$(volts), the charge $q$ (coulombs) on the condenser is given by




$$R\frac{dq}{dt} + \frac{q}{C} = E.$$

If $R = 10$ ohms, $C = 10^{-3}$ farad and $E(t) = 100 \sin 120\pi t$ volts,

*a)* find $q$, assuming that $q = 0$ when $t = 0$.

*b)* use $i = dq/dt$ to find $i$ assuming that $i = 5$ amperes when $t = 0$.

Integrating $10\frac{dq}{dt} + 10^3 q = 100 \sin 120\pi t$, we have

$$qe^{100t} = 10\int e^{100t}\sin 120\pi t\, dt = 10e^{100t}\frac{100\sin 120\pi t - 120\pi\cos 120\pi t}{10{,}000 + 14{,}400\pi^2} + A$$

$$= e^{100t}\frac{10\sin 120\pi t - 12\pi\cos 120\pi t}{100 + 144\pi^2} + A,$$

and 1) $$q = \frac{1}{(100 + 144\pi^2)^{1/2}}\sin(120\pi t - \phi) + Ae^{-100t}$$

where $\sin\phi = \dfrac{12\pi}{(100 + 144\pi^2)^{1/2}}$ and $\cos\phi = \dfrac{10}{(100 + 144\pi^2)^{1/2}}$.

*a)* When $t = 0$, $q = 0$. Then $A = \dfrac{3\pi}{25 + 36\pi^2}$ and $q = \dfrac{1}{2(25 + 36\pi^2)^{1/2}}\sin(120\pi t - \phi) + \dfrac{3\pi e^{-100t}}{25 + 36\pi^2}$.

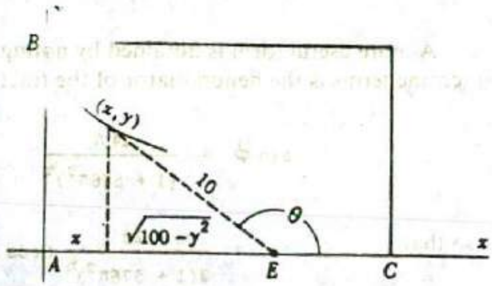
*b)* Differentiating 1) with respect to $t$, we obtain

$$i = \frac{dq}{dt} = \frac{60\pi}{(25 + 36\pi^2)^{1/2}}\cos(120\pi t - \phi) - 100Ae^{-100t}.$$

When $t = 0$, $i = 5$. Then $100A = \dfrac{60\pi}{(25 + 36\pi^2)^{1/2}}\cos\phi - 5 = \dfrac{300\pi}{25 + 36\pi^2} - 5$

and $$i = \frac{60\pi}{(25 + 36\pi^2)^{1/2}}\cos(120\pi t - \phi) - \left(\frac{300\pi}{25 + 36\pi^2} - 5\right)e^{-100t}.$$

20. A boy, standing in corner $A$ of a rectangular pool, has a boat in the adjacent corner $B$ on the end of a string 10 metres long. He walks along the side of the pool toward $C$ keeping the string taut. Locate the boy and boat when the latter is 6 metres from $AC$.




Choose the coordinate system so that $AC$ is along the $x$-axis and $AB$ is along the $y$-axis. Let $(x, y)$ be the position of the boat when the boy has reached $E$, and let $\theta$ denote the angle of inclination of the string.

Then $\tan\theta = \dfrac{dy}{dx} = \dfrac{-y}{\sqrt{100 - y^2}}$ or $dx = -\dfrac{\sqrt{100 - y^2}}{y}\,dy$.

Integrating, $x = -\sqrt{100-y^2} + 10\ln\frac{10+\sqrt{100-y^2}}{y} + C$

When the boat is at $B$, $x = 0$ and $y = 10$.

Then $C = 0$ and $x = -\sqrt{100-y^2} + 10\ln\frac{10+\sqrt{100-y^2}}{y} + C$ is the equation of the boat's path.

Now $AE = x + \sqrt{100-y^2} + 10\ln\frac{10+\sqrt{100-y^2}}{y} + C$ Hence, when the boat is 6 metres from $AC$ (i. e., $y = 6$), $x + 8 = 10\ln 3 = 11$.

The boy is 11 m from $A$ and the boat is 3 m from $AB$.

**21.** A substance $\gamma$ is being formed by the reaction of two substances $\alpha$ and $\beta$ in which $a$ grams of $\alpha$ and $b$ grams of $\beta$ form $(a+b)$ grams of $\gamma$. If initially there are $x_0$ grams of $\alpha$, $y_0$ grams of $\beta$, and none of $\gamma$ present and if the rate of formation of $\gamma$ is proportional to the product of the quantities of $\alpha$ and $\beta$ uncombined, express the amount ($z$ grams) of $\gamma$ formed as a function of time $t$.

The $z$ grams of $\gamma$ formed at time $t$ consists of $\frac{az}{a+b}$ grams of $\alpha$ and $\frac{bz}{a+b}$ grams of $\beta$.

Hence, at time $t$ there remain uncombined $(x_0 - \frac{az}{a+b})$ grams of $\alpha$ and $(y_0 - \frac{bz}{a+b})$ grams of $\beta$.

Then $$\frac{dz}{dt} = K(x_0 - \frac{az}{a+b})(y_0 - \frac{bz}{a+b}) = \frac{Kab}{(a+b)^2}(\frac{a+b}{a}x_0 - z)(\frac{a+b}{b}y_0 - z)$$

$$= k(A-z)(B-z), \quad \text{where} \quad k = \frac{Kab}{(a+b)^2}, \quad A = \frac{(a+b)x_0}{a} \quad \text{and} \quad B = \frac{(a+b)y_0}{b}.$$

There are two cases to be considered: 1) $A \neq B$, say $A > B$, and 2) $A = B$.

1) Here $$\frac{dz}{(A-z)(B-z)} = -\frac{1}{A-B}\cdot\frac{dz}{A-z} + \frac{1}{A-B}\cdot\frac{dz}{B-z} = k\,dt.$$

Integrating from $t = 0$, $z = 0$ to $t = t$, $z = z$, we obtain

$$\frac{1}{A-B}\ln\frac{A-z}{B-z}\Big|_0^z = kt\Big|_0^t, \quad \frac{1}{A-B}(\ln\frac{A-z}{B-z} - \ln\frac{A}{B}) = kt, \quad \frac{A-z}{B-z} = \frac{A}{B}e^{(A-B)kt},$$

and $$z = \frac{AB(1-e^{-(A-B)kt})}{A - Be^{-(A-B)kt}}.$$

2) Here $\frac{dz}{(A-z)^2} = k\,dt$. Integrating from $t = 0$, $z = 0$ to $t = t$, $z = z$, we obtain

$$\frac{1}{A-z}\Big|_0^z = kt\Big|_0^t, \quad \frac{1}{A-z} - \frac{1}{A} = kt, \quad \text{and} \quad z = \frac{A^2kt}{1+Akt}.$$

## SUPPLEMENTARY PROBLEMS

**22.** A body moves in a straight line so that its velocity exceeds by 2 its distance from a fixed point of the line. If $v = 5$ when $t = 0$ find the equation of motion. *Ans.* $x = 5e^t - 2$

**23.** Find the time required for a sum of money to double itself at 5% per annum compounded continuously. Hint: $dx/dt = 0.05x$, where $x$ is the amount after $t$ years. *Ans.* 13.9 years

**24.** Radium decomposes at a rate proportional to the amount present. If half the original amount disappears in 1600 years, find the percentage lost in 100 years. *Ans.* 4.2 %

**25.** In a culture of yeast the amount of active ferment grows at a rate proportional to the amount present. If the amount doubles in 1 hour, how many times the original amount may be anticipated at the end of 2·75 hours? *Ans.* 6.73 times the original amount

**26.** If, when the temperature of the air is 290 K, a certain substance cools from 370 K to 330K in 10 minutes, find the temperature after 40 minutes. *Ans.* 295 K

**27.** A tank contains 450 litres of brine made by dissolving 30 kg of salt in water. Salt water containing 1/9 kg of salt per litre runs in at the rate 9 l/min and the mixture, kept uniform by stirring, runs out at the rate 13·5 l/min. Find the amount of salt in the tank at the end of 1 hr. Hint: $dx/dt = 2 - 3x/(100-t)$. *Ans.* 18·7 kg

**28.** Find the time required for a square tank of side 2 m and depth 4 m to empty through a 22 mm circular hole in the bottom. (Assume, as in Prob. 9, $v = 2{\cdot}5\sqrt{h}\ \text{ms}^{-1}$) *Ans.* 171 min

**29.** A brick wall $(k = 0{\cdot}48)$ is 0·3 m thick. If the inner surface is 290 K and the outer is 270 K, find the temperature in the wall as a function of the distance from the outer surface and the heat loss per day through a square metre. *Ans.* $T = \frac{200}{3}x + 270$; $276 \times 10^4$ J

**30.** A man and his boat have a mass of 150 kg. If the force exerted by the oars in the direction of motion is 70 N and if the resistance (in N) to the motion is equal to thirty times the speed $(\text{ms}^{-1})$ find the speed 15 sec after the boat starts from rest. *Ans.* 2·3 $\text{ms}^{-1}$

**31.** A tank contains 0·5 $\text{m}^3$ of brine made by dissolving 40 kg of salt in water. Pure water runs into the tank at the rate $3 \times 10^{-4}\ \text{m}^3\text{s}^{-1}$ and the mixture, kept uniform by stirring, runs out at the same rate. The outflow runs into a second tank which contains 0·5 $\text{m}^3$ of pure water initially and the mixture, kept uniform by stirring, runs out at the same rate. Find the amount of salt in the second tank after 1 hr.

Hint: $\frac{dx}{dt} = 6 \times 10^{-4}(40e^{-0{\cdot}0006t} - x)$ for the second tank *Ans.* 10·4 kg

**32.** A funnel 0·24 m in diametre at the top and 24 mm in diameter at the bottom is 0·54 m deep. If initially full of water, find the time required to empty. *Ans.* 13·7 sec

**33.** Water is flowing into a vertical cylindrical tank of radius 2 m and height 4 m at the rate $0{\cdot}003\pi\ \text{m}^3\text{s}^{-1}$ and is escaping through a hole 24 mm in diameter in the bottom. Find the time required to fill the tank.

Hint: $\left(0{\cdot}003\pi - \frac{\pi(12)^2}{(1000)^2}\,2{\cdot}5\sqrt{h}\right)dt = 4\pi\,dh$. *Ans.* 106 min

**34.** A mass of 60 kg slides on a table. The friction is equal to sixty times the velocity, and the mass is subjected to a force 54 $\sin 2t$ N. Find the velocity as a function of $t$ if $v = 0$ when $t = 0$.

*Ans.* $v = 9/50\,(\sin 2t - 2\cos 2t + 2e^{-t})$

**35.** A steam pipe of diameter 24 cm has a jacket of insulating material $(K = 0{\cdot}1)$ 12 cm thick. The pipe is kept at 550 K and the outside of the jacket at 300 K. Find the temperature in the jacket at a distance $x$ m from the centre of the pipe and the heat loss per day per metre of pipe.

*Ans.* $T = 300 - 250\,(\ln x - \ln 0{\cdot}24)/(\ln 2)$; 19·6 MJ

**36.** The differential equation of a circuit containing a resistance $R$, capacitance $C$, and emf $e = E\sin\omega t$ is $R\,di/dt$. Assuming $R, C, E, \omega$ to be constants, find the current $i$ at time $t$.

*Ans.* $i = \frac{EC\omega}{1 + R^2C^2\omega^2}(\cos\omega t + RC\omega\sin\omega t) + C_1e^{-t/RC}$

CHAPTER 9

# Equations of First Order and Higher Degree

**A DIFFERENTIAL EQUATION** of the first order has the form $f(x, y, y') = 0$ or $f(x, y, p) = 0$, where for convenience $y' = \frac{dy}{dx}$ is replaced by $p$. If the degree of $p$ is greater than one, as in $p^2 - 3px + 2y = 0$, the equation is of first order and higher (here, second) degree.

The general first order equation of degree $n$ may be written in the form

1) $$p^n + P_1(x, y)p^{n-1} + \cdots\cdots + P_{n-1}(x, y)p + P_n(x, y) = 0.$$

It may be possible, at times to solve such equations by one or more of the procedures outlined below. In each case the problem is reduced to that of solving one or more equations of the first order and first degree.

**EQUATIONS SOLVABLE FOR** $p$. Here the left member of 1), considered as a polynomial in $p$, can be resolved into $n$ linear real factors, that is, 1) can be put in the form

$$(p - F_1)(p - F_2)\cdots\cdots(p - F_n) = 0,$$

where the $F$'s are funtions of $x$ and $y$.

Set each factor equal to zero and solve the resulting $n$ differential equations of first order and first degree

$$\frac{dy}{dx} = F_1(x, y), \quad \frac{dy}{dx} = F_2(x, y), \quad \cdots\cdots, \quad \frac{dy}{dx} = F_n(x, y)$$

to obtain

2) $$f_1(x, y, C) = 0, \quad f_2(x, y, C) = 0, \quad \cdots\cdots, \quad f_n(x, y, C) = 0.$$

The primitive of 1) is the product

3) $$f_1(x, y, C)\cdot f_2(x, y, C)\cdots\cdots f_n(x, y, C) = 0$$

of the $n$ solutions 2).

Note. Each individual solution of 2) may be written in any one of its several possible forms before being combined into the product 3). See Prob. 1–3.

**EQUATIONS SOLVABLE FOR** $y$, i.e., $y = f(x, p)$.

Differentiate with respect to $x$ to obtain

$$\frac{dy}{dx} = p = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}\frac{dp}{dx} = F(x, p, \frac{dp}{dx}),$$

an equation of the first order and first degree.

Solve $p = F(x, p, \frac{dp}{dx})$ to obtain $\phi(x, p, C) = 0$.

Obtain the primitive by eliminating $p$ between $y = f(x, p)$ and $\phi(x, p, C) = 0$, when possible, or express $x$ and $y$ separately as functions of the parameter $p$.

See Problems 4–7.

**EQUATIONS SOLVABLE FOR $x$**, i.e., $x = f(y,p)$.

Differentiate with respect to $y$ to obtain

$$\frac{dx}{dy} = \frac{1}{p} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial p}\frac{dp}{dy} = F(y,p,\frac{dp}{dy}),$$

an equation of the first order and first degree.

Solve $\frac{1}{p} = F(y,p,\frac{dp}{dy})$ to obtain $\phi(y,p,C) = 0$.

Obtain the primitive by eliminating $p$ between $x = f(y,p)$ and $\phi(y,p,C) = 0$, when possible, or express $x$ and $y$ separately as functions of the parameter $p$.

See Problems 8-10.

**CLAIRAUT'S EQUATION.** The differential equation of the form

$$y = px + f(p)$$

is called Clairaut's equation. Its primitive is

$$y = Cx + f(C)$$

and is obtained simply by replacing $p$ by $C$ in the given equation.

See Problems 11-16.

## SOLVED PROBLEMS

1. Solve $p^4 - (x+2y+1)p^3 + (x+2y+2xy)p^2 - 2xyp = 0$ or $p(p-1)(p-x)(p-2y) = 0$.

The solutions of the component equations of first order and first degree

$$\frac{dy}{dx} = 0, \quad \frac{dy}{dx} = 1, \quad \frac{dy}{dx} - x = 0, \quad \frac{dy}{dx} - 2y = 0$$

are respectively $y - C = 0, \quad y - x - C = 0, \quad 2y - x^2 - C = 0, \quad y - Ce^{2x} = 0.$

The primitive of the given equation is $(y-C)(y-x-C)(2y-x^2-C)(y-Ce^{2x}) = 0$.

2. Solve $xyp^2 + (x^2+xy+y^2)p + x^2 + xy = 0$ or $(xp+x+y)(yp+x) = 0$.

The solutions of the component equations $x\frac{dy}{dx} + x + y = 0$ and $y\frac{dy}{dx} + x = 0$

are respectively $2xy + x^2 - C = 0$ and $x^2 + y^2 - C = 0$.

The primitive of the given equation is $(2xy + x^2 - C)(x^2 + y^2 - C) = 0$.

3. Solve $(x^2+x)p^2 + (x^2+x-2xy-y)p + y^2 - xy = 0$ or $[(x+1)p-y][xp+x-y] = 0$.

The solutions of the component equations $(x+1)\frac{dy}{dx} - y = 0$ and $x\frac{dy}{dx} + x - y = 0$

are respectively $y - C(x+1) = 0$ and $y + x \ln Cx = 0$.

The primitive of the given equation is $[y - C(x+1)][y + x \ln Cx] = 0$.

4. Solve $16x^2 + 2p^2y - p^3x = 0$ or $2y = px - 16\dfrac{x^2}{p^2}$.

Differentiating the latter form with respect to $x$, $2p = p + x\dfrac{dp}{dx} - \dfrac{32x}{p^2} + \dfrac{32x^2}{p^3}\dfrac{dp}{dx}$.

Clearing of fractions and combining, $p(p^3 + 32x) - x(p^3 + 32x)\dfrac{dp}{dx} = 0$

or 1) $(p^3 + 32x)(p - x\dfrac{dp}{dx}) = 0.$

This equation is satisfied when $p^3 + 32x = 0$ or $p - x\dfrac{dp}{dx} = 0$. From the latter, $\dfrac{dp}{p} = \dfrac{dx}{x}$ and $p = Kx$. When this replacement for $p$ is made in the given equation, we have

$$16x^2 + 2K^2x^2y - K^3x^4 = 0 \qquad \text{or} \qquad 2 + C^2y - C^3x^2 = 0,$$

after replacing $K$ by $2C$.

The factor $p^3 + 32x$ of 1) will not be considered here since it does not contain the derivative $\dfrac{dp}{dx}$. Its significance will be noted in Chapter 10.

5. Solve $y = 2px + p^4x^2$.

Differentiating with respect to $x$, $p = 2x\dfrac{dp}{dx} + 2p + 2p^4x + 4p^3x^2\dfrac{dp}{dx}$

or $(p + 2x\dfrac{dp}{dx})(1 + 2p^3x) = 0.$

The factor $1 + 2p^3x$ is discarded as in Problem 4. From $p + 2x\dfrac{dp}{dx} = 0$, $xp^2 = C$.

In parametric form, we have $x = C/p^2$, $y = 2C/p + C^2$, the second relation being obtained by substituting $x = C/p^2$ in the differential equation.

Here $p$ may be eliminated without difficulty between the relation $xp^2 = C$ or $p^2 = C/x$ and the given equation. The latter may be put out in the form $y - p^4x^2 = 2px$ and squared to give $(y - p^4x^2)^2 = 4p^2x^2$. Then, substituting for $p^2$, we have $(y - C^2)^2 = 4Cx$.

6. Solve $x = yp + p^2$ or $y = \dfrac{x}{p} - p$.

Differentiating with respect to $x$, $p = \dfrac{1}{p} - \dfrac{x}{p^2}\dfrac{dp}{dx} - \dfrac{dp}{dx}$ or $p^3 - p + (x + p^2)\dfrac{dp}{dx} = 0$.

Then $(p^3 - p)\dfrac{dx}{dp} + x + p^2 = 0$ or $\dfrac{dx}{dp} + \dfrac{x}{p^3 - p} = -\dfrac{p}{p^2 - 1}$.

The latter is a linear equation for which $e^{\int dp/(p^3-p)} = \dfrac{\sqrt{p^2-1}}{p}$ is an integrating factor. Using it,

$$\frac{x\sqrt{p^2-1}}{p} = -\int\frac{dp}{\sqrt{p^2-1}} = -\ln(p + \sqrt{p^2-1}) + C$$

and $x = -\dfrac{p}{\sqrt{p^2-1}}\ln(p + \sqrt{p^2-1}) + \dfrac{Cp}{\sqrt{p^2-1}}$, $y = -p - \dfrac{1}{\sqrt{p^2-1}}\ln(p + \sqrt{p^2-1}) + \dfrac{C}{\sqrt{p^2-1}}$.

7. Solve $y = (2+p)x + p^2$.

Differentiating with respect to $x$, $p = 2 + p + (x+2p)\frac{dp}{dx}$ or $\frac{dx}{dp} + \frac{1}{2}x = -p$.

This is a linear equation having $e^{\frac{1}{2}\int dp} = e^{\frac{1}{2}p}$ as an integrating factor.

Then $$xe^{\frac{1}{2}p} = -\int pe^{\frac{1}{2}p}\,dp = -2pe^{\frac{1}{2}p} + 4e^{\frac{1}{2}p} + C$$

and $$x = 2(2-p) + Ce^{-\frac{1}{2}p}, \qquad y = 8 - p^2 + (2+p)Ce^{-\frac{1}{2}p}.$$

8. Solve $y = 3px + 6p^2y^2$.

Solving for $x$, $3x = \frac{y}{p} - 6py^2$. Then, differentiating with respect to $y$,

$$\frac{3}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 6y^2\frac{dp}{dy} - 12py \qquad \text{and} \qquad (1 + 6p^2y)(2p + y\frac{dp}{dy}) = 0.$$

The second factor equated to zero yields $py^2 = C$. Solving for $p$ and substituting in the original differential equation yields the primitive $y^3 = 3Cx + 6C^2$.

9. Solve $p^3 - 2xyp + 4y^2 = 0$ or $2x = \frac{p^2}{y} + \frac{4y}{p}$.

Differentiating with respect to $y$,

$$\frac{2}{p} = \frac{2p}{y}\frac{dp}{dy} - \frac{p^2}{y^2} + 4(\frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy}) \qquad \text{or} \qquad (p - 2y\frac{dp}{dy})(2y^2 - p^3) = 0.$$

Integrating $p - 2y\frac{dp}{dy} = 0$ and eliminating $p$ between the solution $p^2 = Ky$ and the original differential equation, we have $16y = K(K-2x)^2$. This may be put in the form $2y = C(C-x)^2$ by letting $K = 2C$.

10. Solve $4x = py(p^2-3)$.

Differentiating with respect to $y$,

$$\frac{4}{p} = p(p^2-3) + 3y(p^2-1)\frac{dp}{dy} \qquad \text{or} \qquad \frac{dy}{y} + \frac{3p(p^2-1)dp}{(p^2-4)(p^2+1)} = 0.$$

Integrating, by partial fractions, $\ln y + \frac{9}{10}\ln(p+2) + \frac{9}{10}\ln(p-2) + \frac{3}{5}\ln(p^2+1) = \ln C$.

Then $$y = \frac{C}{(p^2-4)^{9/10}(p^2+1)^{3/5}}, \qquad x = \frac{1}{4}\frac{Cp(p^2-3)}{(p^2-4)^{9/10}(p^2+1)^{3/5}}.$$

CLAIRAUT'S EQUATION.

11. Solve $y = px + \sqrt{4+p^2}$. The primitive is $y = Cx + \sqrt{4+C^2}$.

12. Solve $(y-px)^2 = 1 + p^2$.

Here $y = px \pm \sqrt{1+p^2}$.

The primitive is $(y - Cx - \sqrt{1+C^2})(y - Cx + \sqrt{1+C^2}) = 0$ or $(y - Cx)^2 = 1 + C^2$.

13. Solve $y = 3px + 6y^2p^2$. (See Problem 8.)

This may be reduced to the form of a Clairaut equation.

Multiply the equation by $y^2$ to obtain $y^3 = 3y^2px + 6y^4p^2$.

Using the transformation $y^3 = v,\ 3y^2p = \frac{dv}{dx}$ this becomes $v = x\frac{dv}{dx} + \frac{2}{3}(\frac{dv}{dx})^2$.

The primitive is $v = Kx + \frac{2}{3}K^2$ or $y^3 = Kx + \frac{2}{3}K^2$ or $y^3 = 3Cx + 6C^2$.

14. Solve $\cos^2 y\, p^2 + \sin x \cos x \cos y\, p - \sin y \cos^2 x = 0$.

The transformation $\sin y = u,\ \sin x = v,\ p\frac{\cos y}{\cos x} = \frac{du}{dv}$ reduces the equation to $u = v\frac{du}{dv} + (\frac{du}{dv})^2$.

Then $u = Cv + C^2$ or $\sin y = C \sin x + C^2$.

15. Solve $(px - y)(py + x) = 2p$.

The transformation $y^2 = u,\ x^2 = v,\ p = \frac{v^{1/2}}{u^{1/2}}\frac{du}{dv}$ reduces the equation to

$$(\frac{v}{u^{1/2}}\frac{du}{dv} - u^{1/2})(v^{1/2}\frac{du}{dv} + v^{1/2}) = 2\frac{v^{1/2}}{u^{1/2}}\frac{du}{dv} \qquad \text{or} \qquad (v\frac{du}{dv} - u)(\frac{du}{dv} + 1) = 2\frac{du}{dv}.$$

Then $u = v\frac{du}{dv} - \frac{2\frac{du}{dv}}{1 + \frac{du}{dv}}$, and $u = Cv - \frac{2C}{1+C}$ or $y^2 = Cx^2 - \frac{2C}{1+C}$.

16. Solve $p^2x(x-2) + p(2y - 2xy - x + 2) + y^2 + y = 0$.

The equation may be written as $(y - px + 2p)(y - px + 1) = 0$.

Each of $y = px - 2p$ and $y = px - 1$ is a Clairaut equation.

Thus the primitive is $(y - Cx + 2C)(y - Cx + 1) = 0$.

## SUPPLEMENTARY PROBLEMS

Find the primitive of each of the following.

17. $x^2p^2 + xyp - 6y^2 = 0$ *Ans.* $(y - Cx^2)(y - Cx^{-3}) = 0$

18. $xp^2 + (y - 1 - x^2)p - x(y-1) = 0$ *Ans.* $(2y - x^2 + C)(xy - x + C) = 0$

19. $xp^2 - 2yp + 4x = 0$ *Ans.* $Cy = x^2 + C^2$

20. $3x^4p^2 - xp - y = 0$ *Ans.* $xy = C(3Cx - 1)$

21. $8yp^2 - 2xp + y = 0$ *Ans.* $y^2 - Cx + 2C^2 = 0$

22. $y^2p^2 + 3px - y = 0$ *Ans.* $y^3 - 3Cx - C^2 = 0$

23. $p^2 - xp + y = 0$ *Ans.* $y = Cx - C^2$

24. $16y^3p^2 - 4xp + y = 0$ *Ans.* $y^4 = C(x - C)$

25. $xp^5 - yp^4 + (x^2+1)p^3 - 2xyp^2 + (x+y^2)p - y = 0$ *Ans.* $(y - Cx - C^3)(C^2x - Cy + 1) = 0$

26. $xp^2 - yp - y = 0$ *Ans.* $x = C(p+1)e^p, \quad y = Cp^2e^p$

27. $y = 2px + y^2p^3$ (Use $y^2 = z$.) *Ans.* $y^2 = 2Cx + C^3$

28. $p^2 - xp - y = 0$ *Ans.* $3x = 2p + C/\sqrt{p}, \quad 3y = p^2 - C/\sqrt{p}$

29. $y = (1+p)x + p^2$ *Ans.* $x = 2(1-p) + Ce^{-p}, \quad y = 2 - p^2 + C(1+p)e^{-p}$

30. $y = 2p + \sqrt{1+p^2}$ *Ans.* $x = 2\ln p + \ln(p + \sqrt{1+p^2}) + C, \quad y = 2p + \sqrt{1+p^2}$

31. $yp^2 - xp + 3y = 0$ *Ans.* $x = Cp^{1/2}(p^2+3)(p^2+2)^{-5/4}, \quad y = Cp^{3/2}(p^2+2)^{-5/4}$

CHAPTER 10

# Singular Solutions—Extraneous Loci

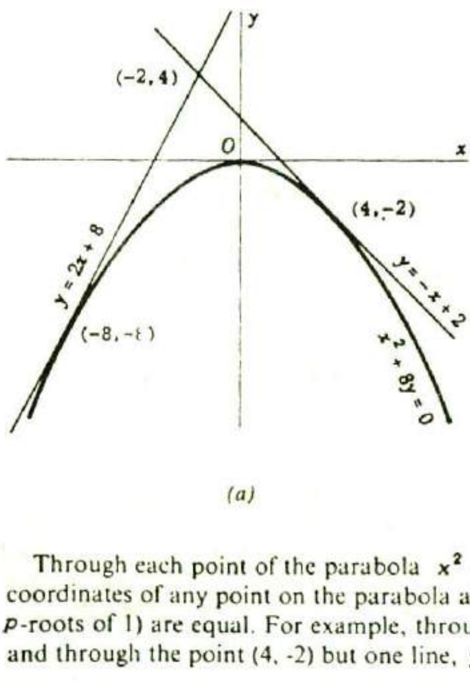
## THE DIFFERENTIAL EQUATION

1) $$y = px + 2p^2$$

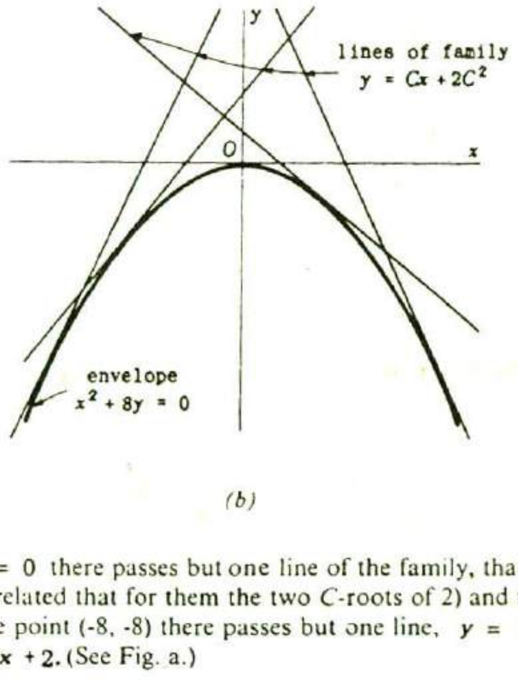
has as primitive the family of straight lines of equation

2) $$y = Cx + 2C^2.$$

With each point $(x, y)$ in the region of points for which $x^2 + 8y > 0$, equation 1) associates a pair of distinct real directions and equation 2) associates a pair of distinct real lines having the directions determined by 1). For example, when the coordinates (-2, 4) are substituted in 1), we have $4 = -2p + 2p^2$ or $p^2 - p - 2 = 0$ and then $p = 2, -1$. Similarly, when 2) is used, we obtain $C = 2, -1$. Thus, through the point (-2, 4) pass the lines $y = 2x + 8$ and $y = -x + 2$ of the family 2) whose slopes are given by 1). Points for which $x^2 + 8y < 0$ yield distinct imaginary $p$- and $C$-roots.




(a)




(b)

Through each point of the parabola $x^2 + 8y = 0$ there passes but one line of the family, that is, the coordinates of any point on the parabola are so related that for them the two $C$-roots of 2) and the two $p$-roots of 1) are equal. For example, through the point (-8, -8) there passes but one line, $y = 2x + 8$, and through the point (4, -2) but one line, $y = -x + 2$. (See Fig. a.)

It is easily verified that the line of 2) through a point of $x^2 + 8y = 0$ is tangent to the parabola there, that is, the direction of the parabola at any one of its points is given by 1). Thus, $x^2 + 8y = 0$ is a solution of 1). It is called a *singular solution* since it cannot be obtained from 2) by a choice of the arbitrary constant, that is, since it is not a particular solution. The corresponding curve, the parabola, is called an *envelope* of the family of lines 2). (See Fig. *b* above).

Summary and Extension:

A singular solution of a differential equation satisfies the differential equation but is not a particular solution of the equation.

At each point of its locus (envelope) the number of distinct directions given by the differential equation and the number of distinct curves given by the corresponding primitive are fewer than at points off the locus.

**THE SINGULAR SOLUTIONS** of a differential equation are to be found by expressing the conditions.

*a)* that the differential equation($p$-equation) have multiple roots, and

*b)* that the primitive ($C$-equation) have multiple roots.

In general, an equation of the first order does not have singular solutions; if it is of the first degree it cannot have singular solutions. Moreover, an equation $f(x,y,p) = 0$ cannot have singular solutions if $f(x,y,p)$ can be resolved into factors which are linear in $p$ and rational in $x$ and $y$.

The simplest expression, called the discriminant, involving the coefficients of an equation $F(X) = 0$ whose vanishing is the condition that the equation have multiple roots is obtained by eliminating $X$ between $F(X) = 0$ and $F'(X) = 0$. The discriminant of

$aX^2 + bX + c = 0$ is $b^2 - 4ac$,

of $aX^3 + bX^2 + cX + d = 0$ is $b^2c^2 + 18abcd - 4ac^3 - 4b^3d - 27a^2d^2$.

See Problem 1.

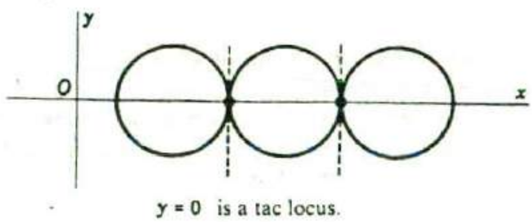
For the example above, the discriminants of the $p$- and $C$-equations are identical, being $x^2 + 8y$.

If $E(x,y) = 0$ is a singular solution of the differential equation $f(x,y,p) = 0$, whose primitive is $g(x,y,C) = 0$, then $E(x,y)$ is a factor of both discriminants. Each discriminant, however, may have other factors which give rise to other loci associated with the primitive. Since the equations of these loci generally do not satisfy the differential equation, they are called *extraneous*.

**EXTRANEOUS LOCI**. (Differential equation, $f(x,y,p) = 0$; primitive, $g(x,y,C) = 0$.)

*a) Tac Locus.*

Let $P$ be a point for which two or more of the $n$ distinct curves of the family $g(x,y,C) = 0$ through it have a common tangent at $P$. Now the number of distinct directions at $P$ is less than $n$ so that the $p$-discriminant must vanish there. The locus, if there is one, of all such points is called a *tac locus*. If $T(x,y) = 0$ is the equation of the tac locus, then $T(x,y)$ is a factor of the $p$-discriminant. In general, $T(x,y)$ is not a factor of the C-discriminant and $T(x,y) = 0$ does not satisfy the differential equation.



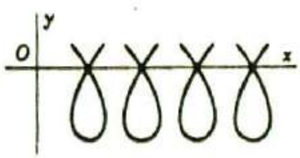

$y = 0$ is a tac locus.

*b) Nodal Locus.*

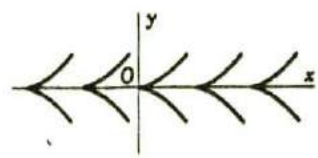
Let one of the curves of the family through $P$ have a node (a double point with distinct tangents) there. Since two of the $n$ values of $p$ are thus accounted for, there can be no more than $n-1$ distinct curves through $P$; hence, the $C$-discriminant must vanish at $P$. The locus, if there is one, of all such points is called a *nodal locus*. If $N(x,y) = 0$ is the equation of the nodal locus, then $N(x,y)$ is a factor of the $C$-discriminant. In general, $N(x,y)$ is not a factor of the $C$-discriminant and $N(x,y) = 0$ does not satisfy the differential equation.

*c) Cusp Locus.*

Let one of the curves of the family through $P$ have a cusp (a double point with coincident tangents) there. Since one of the $p$-roots is of multiplicity two, the $p$-discriminant must vanish at $P$. Moreover, as in the case of a node, there can be no more than $n-1$ curves through $P$ and the $C$-discriminant must vanish at $P$. The locus, if there is one, of all such points is a cusp locus. If $C(x,y) = 0$ is the equation of the cusp locus, then $C(x,y)$ is a factor of both the $p$- and $C$-discriminants. In general, $C(x,y) = 0$ does not satisfy the differential equation.



$y = 0$ is a nodal locus.



$y = 0$ is a cusp locus.

If the curves of the family $g(x,y,C) = 0$ are straight lines, there are no extraneous loci.
If the curves of the family are conics, there can be neither a nodal nor cusp locus.

**THE $p$-DISCRIMINANT RELATION.** The discriminant of the differential equation $f(x,y,p) = 0$, the p-discriminant, equated to zero includes as a factor

1) the equation of the envelope (singular solution) once. See Problems 2–4.
   (The singular solution satisfies the differential equation.)

2) the equation of the cuspidal locus once. See Problem 7.
   (The equation of the cuspidal locus does not satisfy the differential equation unless it is also a singular solution or particular solution.)

3) the equation of the tac locus twice. See Problem 5.
   (The equation of the tac locus does not satisfy the differential equation unless it is also a singular solution or particular solution.)

**THE $C$-DISCRIMINANT RELATION.** The discriminant of the primitive $g(x,y,C) = 0$, the $C$-discriminant, equated to zero includes as a factor.

1) the equation of the envelope or singular solution once.
2) the equation of the cuspidal locus three times.

3) the equation of the nodal locus twice. See Problem 6.
(The equation of the nodal locus does not satisfy the differential equation unless it is also a singular solution or particular solution).

When any locus falls in two of the categories, the multiplicity of its equation in a discriminant relation is the sum of the multiplicities for each category; thus, a cuspidal locus which is also an envelope is included twice in the $p$-discriminant and four times in the $C$-discriminant relation.

The identification of extraneous loci is, however, more than a mere counting of multiplicities of factors.

## SOLVED PROBLEMS

1. Find the discriminant relation for each of the following:

*a)* $p^3 + px - y = 0$, *b)* $p^3x - 2p^2y - 16x^2 = 0$, *c)* $y = C(x-C)^2$.

Note. These discriminant relations may be written readily using the formula given above. We give here a procedure which may be preferred.

*a)* We are to eliminate $p$ between $f(x,y,p) = p^3 + px - y = 0$ and $\frac{\partial f}{\partial p} = 3p^2 + x = 0$. This is best done by eliminating $p$ between

$3f - p\frac{\partial f}{\partial p} = 3p^3 + 3px - 3y - 3p^3 - px = 2px - 3y = 0$ and $\frac{\partial f}{\partial p} = 3p^2 + x = 0$. Solving the first for $p = \frac{3y}{2x}$ and substituting in the second, we find $3p^2 + x = \frac{27y^2}{4x^2} + x = 0$ or $4x^3 + 27y^2 = 0$.

Note. If $f(x,y,p) = 0$ is of degree $n$ in $p$, we eliminate $p$ between $nf - p\frac{\partial f}{\partial p} = 0$ and $\frac{\partial f}{\partial p} = 0$.

*b)* We are to eliminate $p$ between $3f - p\frac{\partial f}{\partial p} = 3p^3x - 6p^2y - 48x^2 - 3p^3x + 4p^2y = -2p^2y - 48x^2 = 0$ and $\frac{\partial f}{\partial p} = 3p^2x - 4py = 0$. From the latter we obtain $9p^4x^2 = 16p^2y^2$ or $9p^4x^2 - 16p^2y^2 = 0$ and from the former $p^2 = -24\frac{x^2}{y}$. Substituting for $p^2$, we obtain $x^2(2y^3 + 27x^4) = 0$.

*c)* Here $g(x,y,C) = C^3 - 2C^2x + Cx^2 - y = 0$ and we are to eliminate $C$ between

1) $3g - C\frac{\partial g}{\partial C} = 3C^3 - 6C^2x + 3Cx^2 - 3y - 3C^3 + 4C^2x - Cx^2 = -2C^2x + 2Cx^2 - 3y = 0$ and

2) $\frac{\partial g}{\partial C} = 3C^2 - 4Cx + x^2 = 0$.

Multiplying 1) by 3 and 2) by $2x$, and adding, we have $-2Cx^2 + 2x^3 - 9y = 0$.

Substituting $C = \frac{2x^3 - 9y}{2x^2}$ in 2) and simplifying we obtain $y(4x^3 - 27y) = 0$.

2. Solve $y = 2xp - yp^2$ and examine for singular solutions.

Solving for $2x = \frac{y}{p} + yp$ and differentiating with respect to $y$, we have

$$\frac{2}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} + p + y\frac{dp}{dy} \quad \text{or} \quad (p^2-1)(p + y\frac{dp}{dy}) = 0.$$

Integrating $p + y\dfrac{dp}{dy} = 0$ to obtain $py = C$ and substituting for $p = \dfrac{C}{y}$ in the given differential equation, we obtain the primitive $y^2 = 2Cx - C^2$.

The $p$- and $C$-discriminant relations are $x^2 - y^2 = 0$. Since both $y = x$ and $y = -x$ satisfy the given differential equation, they are singular solutions.

If $p$ is eliminated between the differential equation and the relation $p^2 - 1 = 0$, discarded in this solution, the equation of the envelope $x^2 - y^2 = 0$ is again obtained. The presence of such a factor implies the existence of a singular solution but not conversely. Hence, this procedure is not to be used in finding singular solutions.

The primitive represents a family of parabolas with principal axis along the $x$-axis. Each parabola is tangent to the line $y = x$ at the point $(C, C)$ and to the line $y = -x$ at the point $(C, -C)$. See Figure *(a)* below.

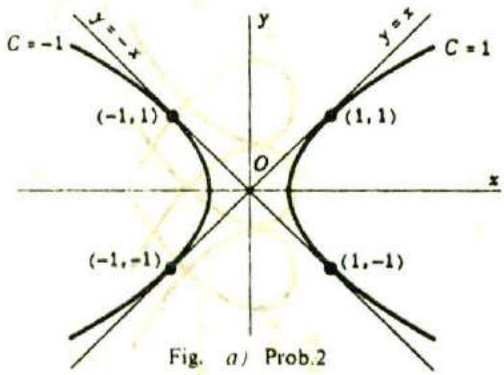



Fig. *a)* Prob.2

Family of parabolas $y^2 = 2Cx - C^2$,
envelope $y = \pm x$.

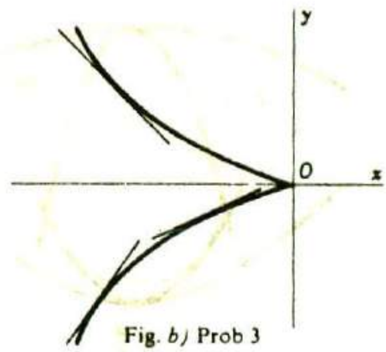



Fig. *b)* Prob 3

Family of straight lines $y = Cx + C^3$,
envelope $4x^3 + 27y^2 = 0$.

3. Examine $p^3 + px - y = 0$ for singular solutions.

This is a Clairaut equation, the primitive being $y = Cx + C^3$.

The $p$- and $C$-discriminant relation $4x^3 + 27y^2 = 0$ is a singular solution since it satisfies the differential equation.

The primitive represents a family of straight lines tangent to the semi-cubical parabola $4x^3 + 27y^2 = 0$, the envelope. See Figure *(b)* above.

4. Examine $6p^2y^2 + 3px - y = 0$ for singular solutions.

From Problem 13, Chapter 9, the primitive is $y^3 = 3Cx + 6C^2$.

Both the $p$- and $C$-discriminant relations are $3x^2 + 8y^3 = 0$. Since this satisfies the differential equation, it is a singular solution.

5. Solve $(x^2 - 4)p^2 - 2xyp - x^2 = 0$ and examine for singular solutions and extraneous loci.

Solving for $2y = xp - \frac{4}{x}p - \frac{x}{p}$ and differentiating with respect to $x$, we have

$$2p = p + x\frac{dp}{dx} + \frac{4p}{x^2} - \frac{4}{x}\frac{dp}{dx} - \frac{1}{p} + \frac{x}{p^2}\frac{dp}{dx} \quad \text{or} \quad (p^2x^2 - 4p^2 + x^2)(p - x\frac{dp}{dx}) = 0.$$

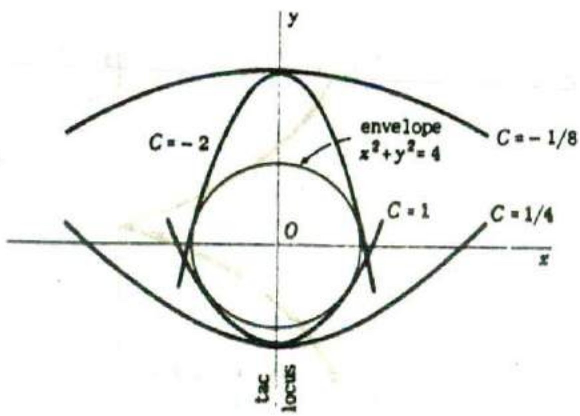
From $p - x\frac{dp}{dx} = 0$, $p = Cx$ and the primitive is $C^2(x^2 - 4) - 2Cy - 1 = 0$. The $p$-discriminant relation $x^2(x^2 + y^2 - 4) = 0$, and the $C$-discriminant relation is $x^2 + y^2 - 4 = 0$.

Now $x^2 + y^2 = 4$ occurs once in the $p$- and $C$-discriminant relations and satisfies the differential equation: it is a singular solution. Also $x = 0$ occurs twice in the $p$-discriminant relation, does not occur in the $C$-discriminant relation, and does not satisfy the differential equation; it is a tac locus.

The primitive represents a family of parabolas having the circle $x^2 + y^2 = 4$ as envelope, See Figure *(c)* below.

Note 1. The two parabolas through a point $P$ of the tac locus $x = 0$ have at $P$ a common tangent.

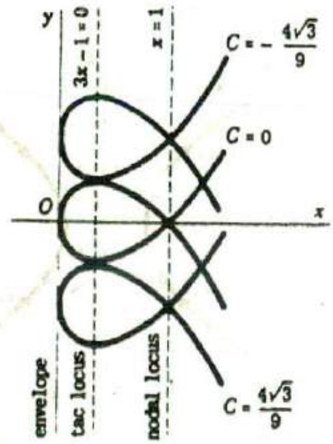
Note 2. A curve of the family meets the envelope in the points $(\pm\frac{\sqrt{4C^2-1}}{C}, -\frac{1}{C})$ ; hence, only those parabolas given by $C^2 \geq \frac{1}{4}$ touch the circle.




Family of parabolas

$C^2(x^2 - 4) - 2Cy - 1 = 0.$

Fig. *(c)* Prob. 5




Family of cubic curves

$(y+C)^2 = x(x-1)^2.$

Fig. *(d)* Prob. 6

**6.** Solve $4xp^2 - (3x-1)^2 = 0$ and examine for singular solutions and extraneous loci.

Solving for $p = \pm(\frac{3}{2}x^{1/2} - \frac{1}{2}x^{-1/2})$, we obtain by integration $y = \pm(x^{3/2} - x^{1/2}) + C_1$ or $(y+C)^2 = x(x-1)^2$. The $p$-discriminant relation is $x(3x-1)^2 = 0$, and the $C$-discriminant relation is $x(x-1)^2 = 0$.

Here $x = 0$ is common to the two relations and satisfies the differential equation, that is, $x = 0$, $\frac{dx}{dy} = 0$ satisfies the equation when written in the form $4x - (3x-1)^2(\frac{dx}{dy})^2 = 0$. It is a singular solution.

$3x - 1 = 0$ is a tac locus since it occurs twice in the $p$-discriminant relation, does not occur in the $C$-discriminant relation, and does not satisfy the differential equation.

$x - 1 = 0$ is a nodal locus since it occurs twice in the $C$-discriminant relation, does not occur in the p-discriminant relation, and does not satisfy the differential equation.

The primitive represents a family of cubics obtained by moving $y^2 = x(x-1)^2$ along the $y$-axis. These curves are tangent to the $y$-axis and have a double point at $x = 1$. Moreover, through each point on $x = 1/3$ pass two curves of the family having a common tangent there. See Figure *(d)* above.

7. Solve $9yp^2 + 4 = 0$ and examine for singular solutions and extraneous loci.

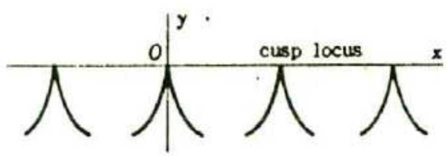
Solving for $9y = -4/p^2$ and differentiating with respect to $x$, we have

$$dx = \frac{8}{9}\frac{dp}{p^4} \quad \text{and} \quad x + C = -\frac{8}{27p^3}.$$

Eliminating $p$ between this latter relation and the differential equation, the primitive is $y^3 + (x+C)^2 = 0$.

The $p$-discriminant relation is $y = 0$, and the $C$-discriminant relation is $y^3 = 0$. Since $y = 0$ occurs once in the $p$-discriminant relation, three times in the $C$-discriminat relation, and does not satisfy the differential equation, it is a cusp locus.

The primitive represents the family of semi-cubical parabolas obtained by moving $y^3 + x^2 = 0$ along the $x$-axis. Each curve has a cusp at its intersection with the $x$-axis, and $y = 0$ is the locus of these cusps. See the figure below.




Family of semicubical parabolas

$$y^3 + (x+C)^2 = 0$$

8. Solve $x^3p^2 + x^2yp + 1 = 0$ and examine for singular solutions and extraneous loci.

Solving for $y = -\frac{1}{x^2p} - xp$ and differentiating with respect to $x$, we have

$$(1 - x^3p^2)(2p + x\frac{dp}{dx}) = 0.$$

From $2p + x\frac{dp}{dx} = 0$, $px^2 = C$ and, eliminating $p$ between this and the differential equation the primitive is $C^2 + Cxy + x = 0$.

The $p$-discriminant relation is $x^3(xy^2 - 4) = 0$, and the $C$-discriminant relation is $x(xy^2 - 4) = 0$.

$xy^2 - 4 = 0$ satisfies the differential equation and is a singular solution.

$x = 0$ is a particular solution $(C = 0)$. Note that it occurs three times in the $p$-discriminant relation and

once in the $C$-discriminant relation.

**9.** Examine $p^3x - 2p^2y - 16x^2 = 0$ for singular solutions and extraneous loci.

From Problem 4, Chapter 9, the primitive is $C^3x^2 - C^2y - 2 = 0$.

The $p$-discriminant relation is $x^2(2y^3 + 27x^4) = 0$, and the $C$-discriminant relation is $2y^3 + 27x^4 = 0$.

Since $2y^3 + 27x^4 = 0$ is common to the discriminant relations and satisfies the differential equation, it is a singular solution. At each point of the line $x = 0$, two parabolas of the family are tangent there (for $y < 0$, the parabolas are real). Thus $x = 0$ is a tac locus. Also $x = 0$ is a particular solution. Since it is obtained by letting $C \to \infty$, it is sometimes called an infinite solution. Note however that when the primitive is written as $x^2 - Ky - 2K^3 = 0$, this solution is obtained when $K = 0$.

## SUPPLEMENTARY PROBLEMS

Investigate for singular solutions and extraneous loci.

**10.** $y = px - 2p^2$. *Ans.* primitive, $y = Cx - 2C^2$; singular solution, $x^2 = 8y$.

**11.** $y^2p^2 + 3xp - y = 0$. *Ans.* prim., $y^3 + 3Cx - C^2 = 0$; s.s., $9x^2 + 4y^3 = 0$.

**12.** $xp^2 - 2yp + 4x = 0$. *Ans.* prim., $C^2x^2 - Cy + 1 = 0$; s.s., $y^2 - 4x^2 = 0$.

**13.** $xp^2 - 2yp + x + 2y = 0$. *Ans.* prim., $2x^2 + 2C(x - y) + C^2 = 0$; s.s., $x^2 + 2xy - y^2 = 0$.

**14.** $(3y-1)^2p^2 = 4y$. *Ans.* prim., $(x + C)^2 = y(y-1)^2$; s.s., $y = 0$; t.l., $y = 1/3$; n.l., $y = 1$.

**15.** $y = -xp + x^4p^2$. *Ans.* prim., $xy = C + C^2x$; s.s., $1 + 4x^2y = 0$; t.l., $x = 0$.

**16.** $2y = p^2 + 4xp$. *Ans.* prim., $(4x^3 + 3xy + C)^2 = 2(2x^2 + y)^3$; no s.s.; c.l., $2x^2 + y = 0$.

**17.** $y(3-4y)^2p^2 = 4(1-y)$. *Ans.* prim., $(x - C)^2 = y^3(1-y)$; s.s., $y = 1$; c.l., $y = 0$; t.l., $y = 3/4$.

**18.** $p^3 - 4x^4p + 8x^3y = 0$. *Ans.* prim., $y = Cx^2 - C^3$; s.s., $4x^6 - 27y^2 = 0$; t.l. $x = 0$.

**19.** $(p^2 + 1)(x-y)^2 = (x + yp)^2$. *Ans.* prim., $(x - C)^2 + (y - C)^2 = C^2$; s.s., $xy = 0$; t.l. $y = x$.
Hint: Use $x = \rho \cos \theta$, $y = \rho \sin \theta$.

# Applications of First Order and Higher Degree Equations

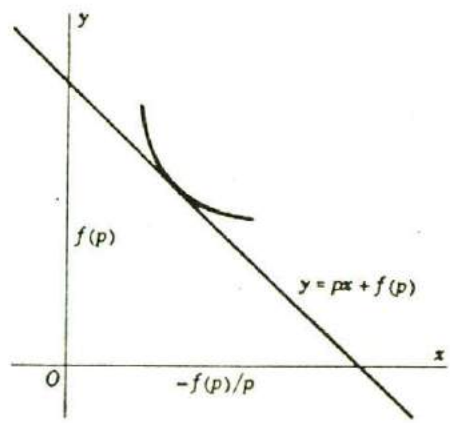
**IN FINDING THE EQUATION** of a curve having a given property, (for example, that its slope at any point is twice the abscissa of the point), we obtained in Chapter 7 a *family* of curves ($y = x^2 + C$) having the property. In this chapter the family of curves will frequently be a family of straight lines, In such cases, the curve in which we are most interested is the envelope of the family.

## SOLVED PROBLEMS

1. Find the curve for which:
   *a)* the sum of the intercepts of the tangent line on the coordinate axes is equal to $k$.

   *b)* the product of the intercepts of the tangent line on the coordinate axes is equal to $k$.

   *c)* the portion of the tangent line intercepted by the coordinate axes is of constant length $k$.




Let the equation of the tangent line be

$$y = px + f(p),$$

the $x$- and $y$- intercepts being $-f(p)/p$ and $f(p)$ respectively.

*a)* Since $f(p) - f(p)/p = k$, $f(p) = -kp/(1-p)$, and the equation of the tangent line is $y = px - \dfrac{kp}{1-p}$.

This is a Clairaut equation, the primitive being the family of lines $y = Cx - \dfrac{kC}{1-C}$ or $xC^2 - (x+y-k)C + y = 0$. The required curve, the envelope of the family, has equation $(x+y-k)^2 = 4xy$ or $x^{1/2} \pm y^{1/2} = k^{1/2}$. Note that this curve is an envelope (singular solution) since it satisfies the differential equation and cannot be obtained from the primitive by assigning a value to $C$.

*b)* Since $f(p)[-f(p)/p] = k$, $f(p) = \pm\sqrt{-kp}$, and the equation of the tangent line is $y = px \pm \sqrt{-kp}$. This is a Clairaut equation, the primitive being

$$y - Cx = \pm\sqrt{-Ck} \qquad \text{or} \qquad x^2C^2 + (k - 2xy)C + y^2 = 0.$$

The required curve, the envelope of the family, has equation $4xy = k$.

*c)* Since $[\{f(p)\}^2 + \{-f(p)/p\}^2]^{1/2} = k$, $f(p) = \pm kp/\sqrt{1+p^2}$, and the equation of the tangent line is $y = px \pm kp/\sqrt{1+p^2}$. The primitive of this equation is $y = Cx \pm kC/\sqrt{1+C^2}$.

Differentiating with respect to C we have $0 = x \pm k/(1+C^2)^{3/2}$.

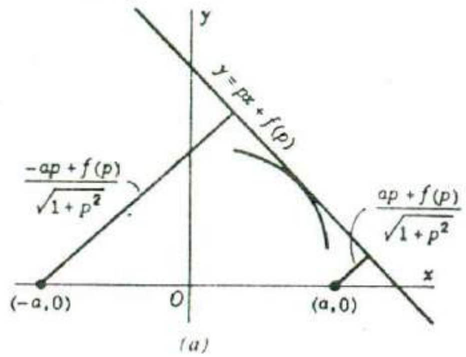
Then $x = \mp k/(1+C^2)^{3/2}$, $y = Cx \pm kC/(1+C^2)^{1/2} = \pm kC^3/(1+C^2)^{3/2}$, and the equation of the envelope is $x^{2/3} + y^{2/3} = k^{2/3}/(1+C^2) + k^{2/3}C^2/(1+C^2) = k^{2/3}$.

2. Find the curve for which:

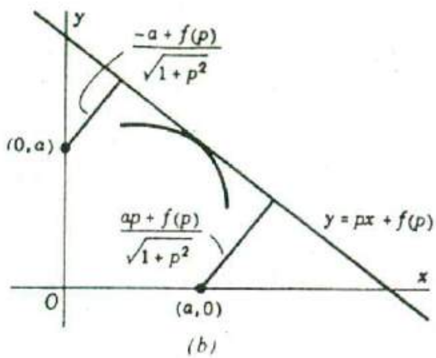
*a)* the sum of the distances of the points $(a,0)$ and $(-a,0)$ from the tangent line is equal to $k$.

*b)* the sum of the distances of the points $(a,0)$ and $(0,a)$ from the tangent line is equal to $k$.

Take $\dfrac{px - y + f(p)}{\sqrt{1+p^2}} = 0$ as the normal form of the equation of a tangent line.



*(a)*



*(b)*

*a)* The distances of the points $(a,0)$ and $(-a,0)$ from the line are $\dfrac{ap + f(p)}{\sqrt{1+p^2}}$ and $\dfrac{-ap + f(p)}{\sqrt{1+p^2}}$ respectively.

Thus, $\dfrac{2f(p)}{\sqrt{1+p^2}} = k$, $f(p) = \frac{1}{2}k\sqrt{1+p^2}$, and the equation of the tangent line is $y = px + \frac{1}{2}k\sqrt{1+p^2}$. The primitive of this Clairaut equation is

$$y = Cx + \tfrac{1}{2}k\sqrt{1+C^2} \quad \text{or} \quad (4x^2 - k^2)C^2 - 8xyC + 4y^2 - k^2 = 0.$$

The required curve, the envelope of this family of lines, has as equation $x^2 + y^2 = \frac{1}{4}k^2$.

*b)* The distances of the points $(a,0)$ and $(0,a)$ from the line are $\dfrac{ap + f(p)}{\sqrt{1+p^2}}$ and $\dfrac{-a + f(p)}{\sqrt{1+p^2}}$ respectively.

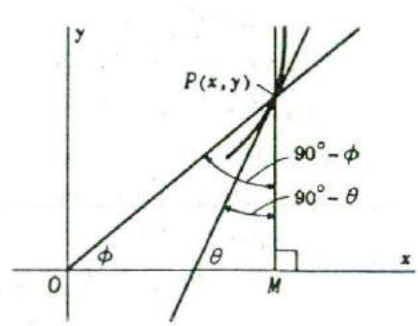
Thus, $\dfrac{-a + ap + 2f(p)}{\sqrt{1+p^2}} = k$, $f(p) = \frac{1}{2}[k\sqrt{1+p^2} - ap + a]$, and the equation of the tangent line is

$y = px + \frac{1}{2}[k\sqrt{1+p^2} - ap + a]$. The primitive is $y = Cx + \frac{1}{2}[k\sqrt{1+C^2} - aC + a]$.

Differentiating with respect to $C$, we have $0 = x + \frac{1}{2}[kC/\sqrt{1+C^2} - a]$.

Then $x = -\frac{1}{2}[kC/\sqrt{1+C^2} - a]$, $y = \frac{1}{2}[k/\sqrt{1+C^2} + a]$, and the envelope of the family of lines has equation $x^2 + y^2 - ax - ay = \frac{1}{4}(k^2 - 2a^2)$.

3. Find the curve such that the tangent line at any of its points $P$ bisects the angle between the ordinate at $P$ and the line joining $P$ and the origin.




Let $\theta$ be the angle of inclination of a tangent line and $\phi$ be the angle of inclination of $OP$. Then, if $M$ is the foot of the ordinate through $P$,
angle $OPM = 90° - \phi = 2(90° - \theta) = 180° - 2\theta$.

Now $\tan(90° - \phi) = \cot\phi = \tan(180° - 2\theta) = -\tan 2\theta$ and $\tan\phi \tan 2\theta = -1$.

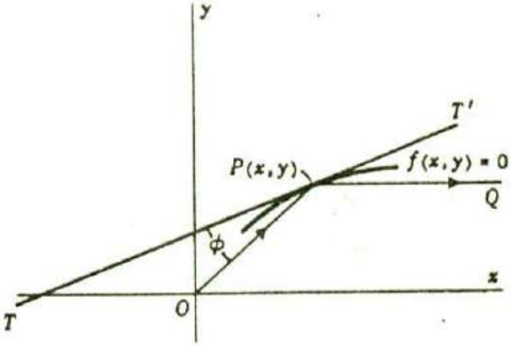
Since $\tan\phi = y/x$ and $\tan\theta = y' = p$, we obtain the differential equation of the curve $\frac{y}{x} \cdot \frac{2p}{1-p^2} = -1$ or $2y = xp - x/p$. Differentiating with respect to $x$, $2p = p - \frac{1}{p} + (x + \frac{x}{p^2})\frac{dp}{dx}$, $p(p^2+1) = x(p^2+1)\frac{dp}{dx}$, and $x\,dp - p\,dx = 0$.

Integrating, $\ln p = \ln x + \ln C$ or $p = Cx$. Substituting for $p$ in the differential equation, we obtain the family of parabolas $C^2x^2 - 2Cy - 1 = 0$.

4. Find the shape of a reflector such that light coming from a fixed source is reflected in parallel rays.

Let the fixed point be at the origin of coordinates and the reflected rays be parallel to the $x$-axis. The reflector is then a surface of revolution generated by revolving a curve $f(x,y) = 0$ about the $x$-axis.

Confining ourselves to the $xOy$ plane, let $P(x,y)$ be a point on the curve $f(x,y) = 0$, $TPT'$ be the tangent line at $P$, and $PQ$ be the reflected ray. Since the angle of incidence is equal to the angle of reflection, it follows that $\angle OPT = \phi = \angle QPT'$.




Now $p = \frac{dy}{dx} = \tan\angle OTP = \tan\phi$ and $\tan\angle TOP = \tan(\pi - 2\phi) = -\tan 2\phi = \frac{-2\tan\phi}{1-\tan^2\phi} = -\frac{y}{x}$;

hence, $\frac{y}{x} = \frac{2p}{1-p^2}$ or $2x = \frac{y}{p} - yp$.

Differentiating with respect to $y$, $\frac{2}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - p - y\frac{dp}{dy}$ and $\frac{dp}{p} = -\frac{dy}{y}$. Then, $p = \frac{C}{y}$.

Eliminating $p$ between this relation and the original differential equation, we have the family of curves $y^2 = 2Cx + C^2$. Thus, the reflector is a member of the family of paraboloids of revolution $y^2 + z^2 = 2Cx + C^2$.

## SUPPLEMENTARY PROBLEMS

5. Find the curve for which each of its tangent lines forms with the coordinate axes a triangle of constant area $a^2$. *Ans.* $2xy = a^2$

6. Find the curve for which the product of the distances of the points $(a,0)$ and $(-a,0)$ from the tangent lines is equal to *Ans.* $kx^2 = (k+a^2)(k-y^2)$

7. Find the curve for which the projection upon the $y$- axis of the perpendicular from the origin upon any tangent is equal to $k$. *Ans.* $x^2 = 4k(k-y)$

8. Find the curve such that the origin bisects the portion of the $y$-axis intercepted by the tangent and normal at each of its points. *Ans.* $x^2 + 2Cy = C^2$

9. Find the curves for which the distance of the tangent from the origin varies as the distance of the origin from the point of contact.

Hint: $\frac{\rho^2}{\sqrt{\rho^2 + (d\rho/d\theta)^2}} = k\rho$. *Ans.* $\rho = Ce^{\theta\frac{\sqrt{1-k^2}}{k}}$