CHAPTER 23

Applications of Total and Simultaneous Equations

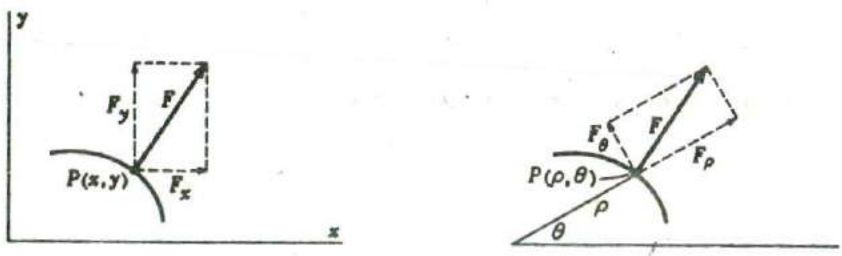
WHEN A MASS m moves in a plane subject to a force F, its acceleration continues to satisfy Newton's Second Law of Motion: mass × acceleration = force.

To obtain the equations of motion, when rectangular coordinates are used, consider the components of the vectors force and acceleration along the axes. The components of acceleration a_x and a_y are given by

$$a_x = \frac{d^2x}{dt^2}, \qquad a_y = \frac{d^2y}{dt^2}$$

and, denoting the components of the force by F_x and F_y, the equations of motion are

$$m\frac{d^2x}{dt^2} = F_x, \qquad m\frac{d^2y}{dt^2} = F_y.$$



COMPONENTS OF F IN RECTANGULAR AND POLAR COORDINATES.

In polar coordinates, the corresponding equations are

$$m\left\{\frac{d^2\rho}{dt^2} - \rho\left(\frac{d\theta}{dt}\right)^2\right\} = F_\rho, \qquad m\left\{2\,\frac{d\rho}{dt}\,\frac{d\theta}{dt} + \rho\,\frac{d^2\theta}{dt^2}\right\} = F_\theta,$$

where F_ρ and F_θ are the radial and transverse components of force, i.e., the components along the radius vector at P and a line perpendicular to it.

SOLVED PROBLEMS

1. Find the family of curves orthogonal to the surfaces $x^2 + 2y^2 + 4z^2 = C$.

Since $x^2 + 2y^2 + 4z^2 = C$ is the primitive of the total differential equation

$$x\,dx + 2y\,dy + 4z\,dz = 0,$$

the differential equation of the family of orthogonal curves is

$$\frac{dx}{x} = \frac{dy}{2y} = \frac{dz}{4z}.$$ (See Chapter 22, Problem 31.)

Solving $\frac{dx}{x} = \frac{dy}{2y}$, we have $y = Ax^2$. Solving $\frac{dy}{2y} = \frac{dz}{4z}$, we have $z = By^2$.

The required family of curves has equations $y = Ax^2,\ z = By^2$.

2. Show that there is no family of surfaces orthogonal to the system of curves

$$x^2 - y^2 = ay, \quad x + y = bz.$$

Differentiating the given equations and eliminating the constants, we have

$$2x\,dx - 2y\,dy = \frac{x^2 - y^2}{y}\,dy, \qquad dx + dy = \frac{x+y}{z}\,dz.$$

The first can be written as $\frac{dx}{x^2+y^2} = \frac{dy}{2xy}$. Solving it for dx, $dx = \frac{x^2+y^2}{2xy}\,dy$, and substituting in the second, we have $\left(\frac{x^2+y^2}{2xy} + 1\right)dy = \frac{x+y}{z}\,dz$ or $\frac{dy}{2xy} = \frac{dz}{(x+y)z}$.

Thus, the differential equations in symmetric form of the given family of curves are

$$\frac{dx}{x^2+y^2} = \frac{dy}{2xy} = \frac{dz}{(x+y)z}.$$

Since the equation $(x^2+y^2)dx + 2xy\,dy + (x+y)z\,dz = 0$ does not satisfy the condition of integrability, there is no family of surfaces cutting the curves orthogonally.

3. The x-component of the acceleration of a particle of unit mass, moving in a plane, is equal to its ordinate and the y-component is equal to twice its abscissa. Find the equation of its path, given the initial conditions $x = y = 0,\ dx/dt = 2,\ dy/dt = 4$ when $t = 0$.

The equations of motion are $\frac{d^2x}{dt^2} = y,\quad \frac{d^2y}{dt^2} = 2x.$

Differentiating the first twice and substituting from the second, $\frac{d^4x}{dt^4} = \frac{d^2y}{dt^2} = 2x$ and

$$x = C_1e^{at} + C_2e^{-at} + C_3\cos at + C_4\sin at, \quad \text{where } a^4 = 2.$$

Then,

$$y = \frac{d^2x}{dt^2} = a^2(C_1e^{at} + C_2e^{-at} - C_3\cos at - C_4\sin at),$$

$$\frac{dx}{dt} = a(C_1e^{at} - C_2e^{-at} - C_3\sin at + C_4\cos at),$$

and

$$\frac{dy}{dt} = a^3(C_1e^{at} - C_2e^{-at} + C_3\sin at - C_4\cos at).$$

Using the initial conditions: $C_1 + C_2 + C_3 = 0,\ C_1 + C_2 - C_3 = 0,\ C_1 - C_2 + C_4 = \frac{2}{a},\ C_1 - C_2 - C_4 = \frac{4}{a^3}$.

Then $C_1 = -C_2 = \frac{a^2+2}{2a^3}$, $C_3 = 0$, and $C_4 = \frac{a^2-2}{a^3}$.

The parametric equations of the path are:

$$x = \tfrac{1}{4}(2+\sqrt{2})\sqrt[4]{2}\,(e^{\sqrt[4]{2}t} - e^{-\sqrt[4]{2}t}) - \tfrac{1}{2}(2-\sqrt{2})\sqrt[4]{2}\sin\sqrt[4]{2}t,$$

$$y = \tfrac{1}{4}(2+\sqrt{2})\sqrt[4]{8}\,(e^{\sqrt[4]{2}t} - e^{-\sqrt[4]{2}t}) + \tfrac{1}{2}(2-\sqrt{2})\sqrt[4]{8}\sin\sqrt[4]{2}t.$$

4. A particle of mass m is repelled from the origin O by a force varying inversely as the cube of the distance ρ from O. If it starts at $\rho = a$, $\theta = 0$ with velocity v_0, perpendicular to the initial line, find the equation of the path.

The radial and transverse components of the repelling force are: $F_\rho = \dfrac{K}{\rho^3} = \dfrac{mk^2}{\rho^3}$, $F_\theta = 0$.

Hence, $m\left(\dfrac{d^2\rho}{dt^2} - \rho\left(\dfrac{d\theta}{dt}\right)^2\right) = \dfrac{mk^2}{\rho^3}$, $\qquad m\left(2\dfrac{d\rho}{dt}\dfrac{d\theta}{dt} + \rho\dfrac{d^2\theta}{dt^2}\right) = 0$

or 1) $\dfrac{d^2\rho}{dt^2} - \rho\left(\dfrac{d\theta}{dt}\right)^2 = \dfrac{k^2}{\rho^3}$, $\qquad$ 2) $\rho\dfrac{d^2\theta}{dt^2} + 2\dfrac{d\rho}{dt}\dfrac{d\theta}{dt} = 0$.

Integrating 2), $\rho^2\dfrac{d\theta}{dt} = C_1$. When $t = 0$, $\rho = a$ and $\rho\dfrac{d\theta}{dt} = v_0$; then $C_1 = av_0$ and $\dfrac{d\theta}{dt} = \dfrac{av_0}{\rho^2}$.

Substituting for $\dfrac{d\theta}{dt}$ in 1), $\dfrac{d^2\rho}{dt^2} = \dfrac{a^2v_0^2}{\rho^3} + \dfrac{k^2}{\rho^3}$. Multiplying by $2\dfrac{d\rho}{dt}$,

$$2\frac{d\rho}{dt}\frac{d^2\rho}{dt^2} = 2\frac{a^2v_0^2 + k^2}{\rho^3}\frac{d\rho}{dt} \quad \text{and} \quad \left(\frac{d\rho}{dt}\right)^2 = -\frac{a^2v_0^2 + k^2}{\rho^2} + C_2.$$

When $t = 0$, $\rho = a$ and $\dfrac{d\rho}{dt} = 0$; then $C_2 = \dfrac{a^2v^2 + k^2}{a^2}$ and

$$\left(\frac{d\rho}{dt}\right)^2 = (a^2v_0^2 + k^2)\left(\frac{1}{a^2} - \frac{1}{\rho^2}\right) = (a^2v_0^2 + k^2)\frac{\rho^2 - a^2}{a^2\rho^2}.$$

Dividing by $\left(\dfrac{d\theta}{dt}\right)^2 = \dfrac{a^2v_0^2}{\rho^4}$, $\left(\dfrac{d\rho}{d\theta}\right)^2 = \dfrac{(a^2v_0^2 + k^2)\rho^2(\rho^2 - a^2)}{a^4v_0^2}$ and $\dfrac{d\rho}{\rho\sqrt{\rho^2 - a^2}} = \dfrac{\sqrt{a^2v_0^2 + k^2}}{a^2v_0}\,d\theta$.

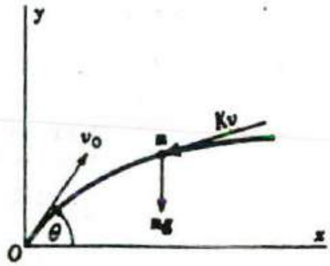
Integrating, $\dfrac{1}{a}\,\text{arc sec}\,\dfrac{\rho}{a} = \dfrac{\sqrt{a^2v_0^2 + k^2}}{a^2v_0}\,\theta + C_3$.

When $t = 0$, $\rho = a$ and $\theta = 0$; then $C_3 = 0$ and $\rho = a\sec\dfrac{\sqrt{a^2v_0^2 + k^2}}{av_0}\,\theta$.

5. A projectile of mass m is fired into the air with initial velocity v_0 at an angle θ with the ground. Neglecting all forces except gravity and the resistance of the air, assumed proportional to the velocity, find the position of the projectile at time t.

In its horizontal motion, the projectile is affected only by the x-component of the resistance. Hence,

1) $m\dfrac{d^2x}{dt^2} = -K\dfrac{dx}{dt} = -mk\dfrac{dx}{dt}$ $\quad$ or $\quad$ $\dfrac{d^2x}{dt^2} = -k\dfrac{dx}{dt}$.



In its vertical motion, the projectile is affected by gravity and by the y-component of the resistance. Hence,

2) $m\dfrac{d^2y}{dt^2} = -mg - mk\dfrac{dy}{dt}$ $\quad$ or $\quad$ $\dfrac{d^2y}{dt^2} = -g - k\dfrac{dy}{dt}$.

Integrating 1). $\dfrac{dx}{dt} = -kx + C_1$ and $x = \dfrac{1}{k}C_1 + C_2e^{-kt}$.

Integrating 2). $\dfrac{dy}{dt} = -gt - ky + K_1$ and $y = \dfrac{1}{k}K_1 + K_2e^{-kt} - g(\dfrac{1}{k}t - \dfrac{1}{k^2})$.

Using the initial conditions $x = y = 0$, $\dfrac{dx}{dt} = v_0 \cos\theta$, $\dfrac{dy}{dt} = v_0 \sin\theta$ when $t = 0$:

$$C_1 = v_0 \cos\theta,\quad C_2 = -\frac{1}{k}v_0\cos\theta;\quad K_1 = v_0\sin\theta,\quad K_2 = -\frac{1}{k}v_0\sin\theta - \frac{1}{k^2}g.$$

Thus, $x = \dfrac{1}{k}(v_0\cos\theta)(1 - e^{-kt})$, $y = \dfrac{1}{k}\{(\dfrac{g}{k} + v_0\sin\theta)(1 - e^{-kt}) - gt\}$.

6. Two masses, m_1 and m_2, are separated by a spring for which $k = k_2\ \text{Nm}^{-1}$ and m_1 is attached to a support by a spring for which $k = k_1\ \text{Nm}^{-1}$ as in the figure. After the system is brought to rest, the masses are displaced a metres downward and released. Discuss their motion.

Let positive direction be downward and let x_1 and x_2 denote the displacement of the masses at time t from their respective positions at rest. The elongation of the upper spring is then x_1 and that of the lower spring is $x_2 - x_1$. The corresponding restoring forces in the springs are

$-k_1x_1 + k_2(x_2 - x_1)$ acting on m_1

and $-k_2(x_2 - x_1)$ acting on m_2.

The equations of motion are

$$m_1\frac{d^2x_1}{dt^2} = -k_1x_1 + k_2(x_2 - x_1) \quad\text{and}\quad m_2\frac{d^2x_2}{dt^2} = -k_2(x_2 - x_1)$$

or 1) $[m_1D^2 + (k_1 + k_2)]x_1 - k_2x_2 = 0$ and 2) $(m_2D^2 + k_2)x_2 - k_2x_1 = 0$.

Operating on 1) with $(m_2D^2 + k_2)$ and substituting from 2),

$$(m_2D^2 + k_2)(m_1D^2 + k_1 + k_2)x_1 - k_2(m_2D^2 + k_2)x_2 = (m_2D^2 + k_2)(m_1D^2 + k_1 + k_2)x_1 - k_2^2x_1 = 0$$

or

$$(D^4 + (\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2})D^2 + \frac{k_1k_2}{m_1m_2})x_1 = 0.$$

Denoting the roots of the characteristic equation by $\pm i\alpha$, $\pm i\beta$, where

$$\alpha^2,\ \beta^2 = \frac{1}{2}\left| -(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}) \pm \sqrt{(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2})^2 - 4\frac{k_1k_2}{m_1m_2}} \right|$$

$x_1 = C_1e^{i\alpha t} + C_2e^{-i\alpha t} + C_3e^{i\beta t} + C_4e^{-i\beta t}$ and

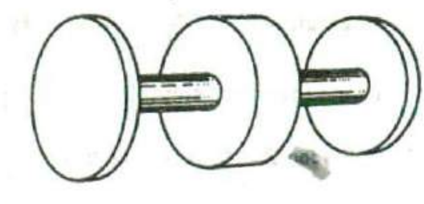
$$x_2 = \frac{1}{k_2}(m_1D^2 + k_1 + k_2)x_1 = \frac{k_1 + k_2 - m_1\alpha^2}{k_2}(C_1e^{i\alpha t} + C_2e^{-i\alpha t}) + \frac{k_1 + k_2 - m_1\beta^2}{k_2}(C_3e^{i\beta t} + C_4e^{-i\beta t})$$

$$= \mu(C_1e^{i\alpha t} + C_2e^{-i\alpha t}) + \nu(C_3e^{i\beta t} + C_4e^{-i\beta t}).$$

Using the initial conditions $x_1 = x_2 = a$, $\dfrac{dx_1}{dt} = \dfrac{dx_2}{dt} = 0$ when $t = 0$,

$$C_1 = C_2 = \frac{a}{2}(\frac{\nu - 1}{\nu - \mu}) = \frac{a}{2m_1}(\frac{k_1 - m_1\beta^2}{\alpha^2 - \beta^2}) \quad\text{and}\quad C_3 = C_4 = -\frac{a}{2m_1}(\frac{k_1 - m_1\alpha^2}{\alpha^2 - \beta^2}).$$

7. A uniform shaft carries three disks as in the adjoining figure. The polar moment of inertia of the disk at either end is I, and that of the disk at the middle is $4I$. The torsional stiffness constant of the shaft between two disks (the torque required to produce an angular displacement difference of one radian between successive disks) is k. Find the motion of the disks if a torque $2T_0\sin\omega t$ is applied to the middle disk, assuming that at $t=0$ the disks are at rest and there is no twist in the shaft.



At time t, let the angular displacement of the disk at either end be θ_1 and that of the disk at the middle be θ_2. The differences of the angular twists of the ends of the two pieces of shaft, from left to right, are $\theta_2-\theta_1$ and $\theta_1-\theta_2$. The restoring torques acting on the disks are $k(\theta_2-\theta_1)$, $k(\theta_1-\theta_2)-k(\theta_2-\theta_1)$ and $-k(\theta_1-\theta_2)$ respectively. The net torque acting on a mass when rotating is equal to the product of the polar moment of inertia of the mass about the axis of rotation and its angular acceleration; hence the equation of motion of the middle disk is

1) $$4I\frac{d^2\theta_2}{dt^2} = k(\theta_1-\theta_2) - k(\theta_2-\theta_1) + 2T_0\sin\omega t \quad\text{or}\quad (2ID^2+k)\theta_2 = k\theta_1 + T_0\sin\omega t$$

and that of either end disk is

2) $$I\frac{d^2\theta_1}{dt^2} = k(\theta_2-\theta_1) \qquad\text{or}\qquad (ID^2+k)\theta_1 = k\theta_2.$$

Operating on 2) with $(2ID^2+k)$ and substituting from 1),

$$(2ID^2+k)(ID^2+k)\theta_1 = k(2ID^2+k)\theta_2 = k^2\theta_1 + T_0k\sin\omega t, \qquad\text{or}$$

3) $$D^2(2I^2D^2+3kI)\theta_1 = T_0k\sin\omega t.$$

The characteristic roots are $0, 0, ai, -ai$, where $a^2 = 3k/2I$, and

4) $$\theta_1 = C_1 + C_2t + C_3\cos at + C_4\sin at + \frac{T_0k\sin\omega t}{I\omega^2(2I\omega^2-3k)}$$

$$= C_1 + C_2t + C_3\cos at + C_4\sin at + \frac{T_0k}{2I^2\omega^2(\omega^2-a^2)}\sin\omega t.$$

From 2), $\theta_2 = (\frac{I}{k}D^2+1)\theta_1$ and

5) $$\theta_2 = C_1 + C_2t + C_3(1-\frac{I}{k}a^2)\cos at + C_4(1-\frac{I}{k}a^2)\sin at + \frac{T_0k - T_0\omega^2I}{2I^2\omega^2(\omega^2-a^2)}\sin\omega t.$$

From 4) and 5), we obtain by differentiation,

4′) $$\frac{d\theta_1}{dt} = C_2 - C_3a\sin at + C_4a\cos at + \frac{T_0k}{2I^2\omega(\omega^2-a^2)}\cos\omega t, \qquad\text{and}$$

5′) $$\frac{d\theta_2}{dt} = C_2 - C_3a(1-\frac{I}{k}a^2)\sin at + C_4a(1-\frac{I}{k}a^2)\cos at + \frac{T_0k - T_0\omega^2I}{2I^2\omega(\omega^2-a^2)}\cos\omega t.$$

Using the initial conditions $\theta_1 = \theta_2 = 0$, $\frac{d\theta_1}{dt} = \frac{d\theta_2}{dt} = 0$ when $t=0$, we have $C_1 + C_3 = 0$,

$C_1 + C_3(1-\frac{I}{k}a^2) = 0,\quad C_2 + C_4 a + \frac{T_0 k}{2I^2\omega(\omega^2-a^2)} = 0,$ and $C_2 + C_4 a(1-\frac{I}{k}a^2) + \frac{T_0 k - T_0\omega^2 I}{2I^2\omega(\omega^2-a^2)} = 0.$

Then $C_1 = C_3 = 0,\quad C_4 = -\frac{T_0\omega}{3Ia(\omega^2-a^2)},\quad C_2 = \frac{T_0}{3I\omega}.$

$$\theta_1 = \frac{T_0}{3I}\left(\frac{t}{\omega} + \frac{a^2 \sin \omega t}{\omega^2(\omega^2-a^2)} - \frac{\omega \sin at}{a(\omega^2-a^2)}\right) = \frac{T_0}{3I}\left(\frac{t}{\omega} + \frac{a^3 \sin \omega t - \omega^3 \sin at}{a\omega^2(\omega^2-a^2)}\right),\quad \text{and}$$

$$\theta_2 = \theta_1 - \frac{T_0(a \sin \omega t - \omega \sin at)}{2Ia(\omega^2-a^2)}.$$

8. The fundamental equations of a transformer are

$$1)\quad M\frac{di_1}{dt} + L_2\frac{di_2}{dt} + R_2 i_2 = 0, \qquad 2)\quad M\frac{di_2}{dt} + L_1\frac{di_1}{dt} + R_1 i_1 = E(t),$$

where $i_1(t)$ and $i_2(t)$ denote the currents, while $M,\ L_1,\ L_2,\ R_1,\ R_2$ are constants.
Assuming $M^2 < L_1L_2$, show that

$$A)\quad (L_1L_2-M^2)\frac{d^2 i_1}{dt^2} + (R_1L_2+R_2L_1)\frac{di_1}{dt} + R_1R_2 i_1 = R_2E(t) + L_2E'(t).$$

$$B)\quad (L_1L_2-M^2)\frac{d^2 i_2}{dt^2} + (R_1L_2+R_2L_1)\frac{di_2}{dt} + R_1R_2 i_2 = -ME'(t).$$

Solve the system when $E(t) = E_0$, a constant.

Differentiating 1) and 2) with respect to t,

$$3)\quad M\frac{d^2 i_1}{dt^2} + L_2\frac{d^2 i_2}{dt^2} + R_2\frac{di_2}{dt} = 0, \qquad 4)\quad M\frac{d^2 i_2}{dt^2} + L_1\frac{d^2 i_1}{dt^2} + R_1\frac{di_1}{dt} = E'(t).$$

Multiplying 3) by M and 4) by L_2, and subtracting,

$$(L_1L_2-M^2)\frac{d^2 i_1}{dt^2} + R_1L_2\frac{di_1}{dt} - MR_2\frac{di_2}{dt} = L_2E'(t).$$

Subtituting for $\frac{di_2}{dt}$ from 2), we obtain *A)*.

Multiplying 3) by L_1 and 4) by M, and subtracting,

$$(L_1L_2-M^2)\frac{d^2 i_2}{dt^2} + R_2L_1\frac{di_2}{dt} - R_1M\frac{di_1}{dt} = -ME'(t).$$

Subtituting for $\frac{di_1}{dt}$ from 1), we obtain *B)*.

When $E(t) = E_0$, equation *A)* is $(L_1L_2-M^2)\frac{d^2 i_1}{dt^2} + (R_1L_2+R_2L_1)\frac{di_1}{dt} + R_1R_2 i_1 = R_2E_0.$

Let $\alpha,\beta = \frac{1}{2}\,\frac{-(R_1L_2+R_2L_1) \pm \sqrt{(R_1L_2-R_2L_1)^2 + 4M^2R_1R_2}}{L_1L_2 - M^2}$ denote the characteristic roots.

Then $$i_1 = C_1e^{\alpha t} + C_2e^{\beta t} + \frac{E_0}{R_1}.$$

To find i_2, multiply 1) by M and 2) by L_2, and subtract to obtain

$$MR_2i_2 = (L_1L_2-M^2)\frac{di_1}{dt} + L_2R_1i_1 - L_2E_0.$$

Then $$i_2 = \frac{1}{MR_2}[(L_1L_2-M^2)(\alpha C_1e^{\alpha t}+\beta C_2e^{\beta t}) + L_2R_1(C_1e^{\alpha t}+C_2e^{\beta t})].$$

Note that since $M^2 < L_1L_2$, both α and β are negative. Then after a time, the primary current becomes approximately constant $= E_0/R_1$ and the secondary current i_2 becomes negligible.

9. A moving particle of mass m is attracted to a fixed point O by a central force which varies inversely as the square of the distance of the particle from O. Show that the equation of its path is a conic having the fixed point as focus.

Using polar coordinates with O as pole, the equations of motion are

1) $m[\frac{d^2\rho}{dt^2} - \rho(\frac{d\theta}{dt})^2] = -\frac{K}{\rho^2} = -\frac{mk^2}{\rho^2}$ or $\frac{d^2\rho}{dt^2} - \rho(\frac{d\theta}{dt})^2 = -\frac{k^2}{\rho^2}$,

2) $m[2\frac{d\rho}{dt}\frac{d\theta}{dt} + \rho\frac{d^2\theta}{dt^2}] = 0$ or $2\frac{d\rho}{dt}\frac{d\theta}{dt} + \rho\frac{d^2\theta}{dt^2} = 0$.

From 2), $\frac{d}{dt}(\rho^2\frac{d\theta}{dt}) = 0$ and $\rho^2\frac{d\theta}{dt} = C_1$.

Let $\sigma = \frac{1}{\rho}$. Then $\frac{d\theta}{dt} = \frac{C_1}{\rho^2} = C_1\sigma^2$, $\frac{d\rho}{dt} = \frac{d\rho}{d\sigma}\frac{d\sigma}{dt} = -\frac{1}{\sigma^2}\frac{d\sigma}{d\theta}\frac{d\theta}{dt} = -C_1\frac{d\sigma}{d\theta}$, and

$\frac{d^2\rho}{dt^2} = \frac{d}{dt}(-C_1\frac{d\sigma}{d\theta}) = -C_1\frac{d^2\sigma}{d\theta^2}\frac{d\theta}{dt} = -C_1^2\sigma^2\frac{d^2\sigma}{d\theta^2}$. Substituting in 1) and simplifying, we have

1′) $\frac{d^2\sigma}{d\theta^2} + \sigma = \frac{k^2}{C_1^2}$, a linear equation with constant coefficients. Solving,

$$\sigma = C_2\cos(\theta+C_3) + \frac{k^2}{C_1^2} \quad \text{or} \quad \rho = \frac{1}{\frac{k^2}{C_1^2} + C_2\cos(\theta+C_3)} = \frac{C_1^2/k^2}{1+\frac{C_2C_1^2}{k^2}\cos(\theta+C_3)}.$$

Writing $C_1^2/k^2 = l$, $|C_2C_1^2/k^2| = e$, $C_3 = \alpha$, this becomes $\rho = \frac{l}{1 \pm e\cos(\theta+\alpha)}$, the equation of a conic having O as focus.

SUPPLEMENTARY PROBLEMS

10. Find the family of curves orthogonal to the family of surfaces $x^2 + y^2 + 2z^2 = C$.

Ans. $y = Ax,\ z = By^2$

11. Find the family of surfaces orthogonal to the family of curves $y = C_1 x,\ x^2 + y^2 + 2z^2 = C_2$.

Ans. $z = C(x^2 + y^2)$

12. A particle of mass m is attracted to the origin O by a force varying directly as its distance from O. If it starts at $(a,0)$ with velocity v_0 in a direction making an angle θ with the horizontal, find the position at time t.

Ans. $x = a\cos kt + \dfrac{v_0\cos\theta}{k}\sin kt,\quad y = \dfrac{v_0\sin\theta}{k}\sin kt$

13. The currents $i_1,\ i_2,\ i = i_1 + i_2$ in a certain network satisfy the equations

$$20i + 0.1\frac{di_2}{dt} = 5, \qquad 4i + i_1 + 1000q_1 = 1.$$

Determine the currents subject to the initial conditions $i = i_1 = i_2 = 0$ when $t = 0$.

Hint: Use $i_1 = \dfrac{dq_1}{dt}$ to obtain $\dfrac{d^2q_1}{dt^2} + 240\dfrac{dq_1}{dt} + 40{,}000q_1 = 0$.

Ans. $i_1 = -\frac{1}{4}e^{-120t}\sin 160t,\quad i_2 = \frac{1}{4}(1 - e^{-120t}\cos 160t) + \frac{1}{8}e^{-120t}\sin 160t$

14. Initially tank *I* contains 400 l of brine with 100 kg of salt, and tank *II* contains 200 l of fresh water. Brine from tank *I* runs into tank *II* at 12 l/min, and from tank *II* into tank *I* at 8 l/min, If each tank is kept well stirred, how much will tank *I* contain after 50 minutes?

Hint: $q_1 + q_2 = 100,\ \dfrac{dq_1}{dt} = \dfrac{2q_2}{50+t} - \dfrac{3q_1}{100-t}$. *Ans.* 34·375 kg.

CHAPTER 24

Numerical Approximations to Solutions

IN MANY APPLICATIONS it is required to find the value $\bar{y}$ of y corresponding to $x = x_0 + h$ from the particular solution of a given differential equation

1) $$y' = f(x,y)$$

satisfying the initial conditions $y = y_0$ when $x = x_0$. Such problems have been solved by first finding the primitive

2) $$y = F(x) + C$$

of 1), then selecting the particular solution

3) $$y = g(x)$$

through (x_0, y_0) and finally computing the required value $\bar{y} = g(x_0 + h)$

When no method is available for finding the primitive, it is necessary to use some procedure for approximating the desired value. Integrating 1) between the limits $x = x_0, y = y_0$ and $x = x, y = y$ we obtain

4) $$y = y_0 + \int_{x_0}^{x} f(x,y)\,dx.$$

The value of y when $x = x_0 + h$ is then

5) $$\bar{y} = y_0 + \int_{x_0}^{x_0+h} f(x,y)\,dx.$$

The methods of this chapter will consist of procedures for approximating 4) or 5).

PICARD'S METHOD. For values of x near $x = x_0$ the corresponding value of $y = g(x)$ is near $y_0 = g(x_0)$. Thus, a first approximation y_1 of $y = g(x)$ is obtained by replacing y by y_0 in the right member of 4), that is,

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx.$$

A second approximation, y_2 is then obtained by replacing y by y_1 in the right member of 4), that is,

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx.$$

Continuing this procedure, a succession of functions of x

$$y_0,\ y_1,\ y_2,\ y_3,\ \cdots\cdots$$

is obtained, each giving a better approximation of the required solution than the preceeding one.

See Problems 1-2.

Picard's method is of considerable theoretical value. In general, it is unsatisfactory as a practical means of approximation because of difficulties which arise in performing the necessary integrations.

TAYLOR SERIES. The Taylor expansion of $y = g(x)$ near (x_0, y_0) is

6) $$y = g(x_0) + (x-x_0)\,g'(x_0) + \frac{1}{2}(x-x_0)^2\,g''(x_0) + \frac{1}{6}(x-x_0)^3\,g'''(x_0) + \cdots\cdots$$

From 1), $y' = g'(x) = f(x, y)$; hence, by repeated differentiation,

$$y'' = g''(x) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y},$$

7) $$y''' = g'''(x) = \frac{d}{dx}\left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right) = \left(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\right)\left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + 2f\frac{\partial^2 f}{\partial x\,\partial y} + f\left(\frac{\partial f}{\partial y}\right)^2 + f^2\frac{\partial^2 f}{\partial y^2}, \quad \text{etc.}$$

For convenience, write $p = \frac{\partial f}{\partial x}$, $q = \frac{\partial f}{\partial y}$, $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x\,\partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$ and

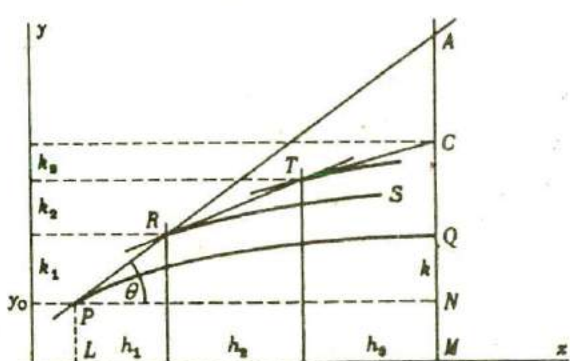
Let $f_0, p_0, q_0, \cdots$ denote the values of $f, p, q, \cdots$ at (x_0, y_0). Substituting in 6) the results of 7) and evaluating for $x = x_0 + h$, we obtain

8) $$\bar{y} = y_0 + h\cdot f_0 + \frac{1}{2}h^2(p_0 + f_0\cdot q_0) + \frac{1}{6}h^3(r_0 + p_0\cdot q_0 + 2f_0\cdot s_0 + f_0\cdot q_0^2 + f_0^2\cdot t_0)$$
$$+ \cdots\cdots\cdots$$

This series may be used to compute $\bar{y}$; it is evident, however, that additional terms will be increasingly complex.

See Problems 3–4.

FIRST DERIVATIVE METHOD. A procedure involving only first derivatives, that is, using only the first two terms of Taylor series, follows.



As a first approximation of $\bar{y}$, take the first two terms of 8)

$$\bar{y} \approx y_0 + h\,f(x_0, y_0).$$

To interpret this approximation geometrically, let PQ be the integral curve of 1) through $P(x_0, y_0)$ and and let Q be the point on the curve corresponding to $x = x_0 + h$. Then $\bar{y} = MQ = y_0 + k$. If θ is the angle of inclination of the tangent at P, then from 1) $\tan\theta = f(x_0, y_0)$ and the approximation

$$y_0 + h\,f(x_0, y_0) = LP + h\tan\theta = MN + NA = MA.$$

To obtain a better approximation, let the interval LM of width h be divided into n subintervals of widths $h_1, h_2, \ldots, h_n$. (In the figure, $n = 3$.) Let the line $x = x_0 + h_1$ meet PA in $R(x_0+h_1, y_0+k_1) = (x_1, y_1)$. Then

$$y_1 = y_0 + k_1 = y_0 + h_1 f(x_0, y_0).$$

Let RS be the integral curve of 1) through R, and on its tangent at R take T having coordinates $(x_1 + h_2, y_1 + k_2) = (x_2, y_2)$. Then

$$y_2 = y_1 + k_2 = y_1 + h_2 f(x_1, y_1) = y_1 + h_2 f(x_0+h_1,\ y_0+h_1 f_0).$$

After a sufficient number of repetitions, we reach finally an approximation MC of MQ. It is clear from the figure that the accuracy will increase as the number of subintervals is increased in such a manner that the widths of the subintervals decrease.

See Problems 5–6.

RUNGE'S METHOD. From 5) and 8) we obtain

$$9)\qquad k = \bar{y} - y_0 = \int_{x_0}^{x_0+h} f(x,y)\,dx$$

$$= h f_0 + \frac{1}{2}h^2 (p_0 + f_0 q_0) + \frac{1}{6}h^3 (r_0 + p_0 q_0 + 2f_0 s_0 + f_0 q_0^2 + f_0^2 t_0) + \cdots\cdots$$

Assume for the moment that the values y_0, y_1, y_2 of $y = g(x)$ corresponding to $x_0,\ x_1 = x_0 + \frac{1}{2}h,\ x_2 = x_0 + h$ are known. Then by Simpson's Rule,

$$10)\qquad k = \int_{x_0}^{x_0+h} f(x,y)\,dx \approx \frac{h}{6}[f(x_0\ y_0) + 4f(x_0 + \tfrac{1}{2}h,\ y_1) + f(x_0+h,\ y_2)].$$

Actually, only y_0 is known. Runge's Method is based on certain approximations y_1 and y_2,

$$y_1 \approx y_0 + \tfrac{1}{2}hf(x_0, y_0) = y_0 + \tfrac{1}{2}hf_0,$$
$$y_2 \approx y_0 + hf(x_0+h,\ y_0+hf_0),$$

chosen so that when k, found by 10, is expanded as a power series in h the first three terms coincide with those of the right member of 9). Thus 10) becomes

$$11)\qquad k \approx \frac{h}{6}\{f_0 + 4f(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}hf_0) + f[x_0+h,\ y_0+hf(x_0+h,\ y_0+hf_0)]\}.$$

These calculations are best made as follows:

$$k_1 = hf_0,\quad k_2 = hf(x_0+h, y_0+k_1),\quad k_3 = hf(x_0+h, y_0+k_2),\quad k_4 = hf(x_0+\tfrac{1}{2}h, y_0+\tfrac{1}{2}k_1),$$

$$k \approx \frac{1}{6}(k_1 + 4k_4 + k_3).$$

Note. Since the approximation of k obtained here differs from the value as given by 8) in the terms containing powers of h greater than 3, the approximation may be poor if $f_0 > 1$.

See Problems 7–11.

KUTTA-SIMPSON METHOD. Various modifications of the Runge Method have been made by Kutta. One of these, known as Kutta's Simpson's Rule uses the following calculations:

$$k_1 = hf_0,\quad k_2 = hf(x_0+\tfrac{1}{2}h, y_0+\tfrac{1}{2}k_1),\quad k_3 = hf(x_0+\tfrac{1}{2}h, y_0+\tfrac{1}{2}k_2),\quad k_4 = hf(x_0+h, y_0+k_3),$$

$$k \approx \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

See Problem 12.

SIMULTANEOUS FIRST ORDER DIFFERENTIAL EQUATIONS. Approximations to that solution of the simultaneous differential equations

$$\frac{dy}{dx} = f(x,y,z), \qquad \frac{dz}{dx} = g(x,y,z)$$

for which $y = y_0$ and $z = z_0$ when $x = x_0$, may be obtained by the use of Picard's Method, Taylor Series, Runge's Method, or Kutta-Simpson Method. The necessary modifications of the formulas given above are made in Solved Problems 13–14. Further extensions to three or more simultaneous first order equations may be readily made.

DIFFERENTIAL EQUATIONS OF ORDER n. The differential equation

$$\frac{d^n y}{dx^n} = f(x, y, y', y'', \cdots\cdots, y^{n-1})$$

where $y' = \dfrac{dy}{dx}$, $y'' = \dfrac{d^2y}{dx^2}$, ..., may be reduced to the system of simultaneous first order equations

$$\frac{dy}{dx} = y_1, \quad \frac{dy_1}{dx} = y_2, \quad \cdots\cdots, \quad \frac{dy_{n-2}}{dx} = y_{n-1}, \quad \frac{dy_{n-1}}{dx} = f(x, y, y_1, y_2, \cdots\cdots, y_{n-1}).$$

When initial conditions $x = x_0$, $y = y_0$, $y' = (y_1)_0$, $y'' = (y_2)_0$, $\cdots$, $y^{n-1} = (y_{n-1})_0$ are given, the methods of the preceding paragraph apply.

EXAMPLE. The second order differential equation $\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 4y = 0$ is equivalent to the system of simultaneous first order differential equations

$$\frac{dy}{dx} = z, \quad \frac{dz}{dx} = 4y - 2xz.$$

See Problems 15–16.

SOLVED PROBLEMS

1. Use Picard's Method to approximate y when $x = 0.2$, given that $y = 1$ when $x = 0$, and $dy/dx = x - y$.

Here $f(x,y) = x - y$, $x_0 = 0$, $y_0 = 1$. Then

$$y_1 = y_0 + \int_0^x f(x,y_0)dx = 1 + \int_0^x (x-1)dx = \frac{1}{2}x^2 - x + 1,$$

$$y_2 = y_0 + \int_0^x f(x,y_1)dx = 1 + \int_0^x (-\frac{1}{2}x^2 + 2x - 1)dx = -\frac{1}{6}x^3 + x^2 - x + 1,$$

$$y_3 = y_0 + \int_0^x f(x,y_2)dx = 1 + \int_0^x (\frac{1}{6}x^3 - x^2 + 2x - 1)dx = \frac{1}{24}x^4 - \frac{1}{3}x^3 + x^2 - x + 1,$$

$$y_4 = y_0 + \int_0^x f(x,y_3)dx = 1 + \int_0^x (-\frac{1}{24}x^4 + \frac{1}{3}x^3 - x^2 + 2x - 1)dx = -\frac{x^5}{120} + \frac{x^4}{12} - \frac{x^3}{3} + x^2 - x + 1,$$

$$y_5 = \frac{1}{720}x^6 - \frac{1}{60}x^5 + \frac{1}{12}x^4 - \frac{1}{3}x^3 + x^2 - x + 1, \quad \cdots\cdots\cdots$$

When $x = 0.2$, $y_0 = 1$, $y_1 = 0.82$, $y_2 = 0.83867$, $y_3 = 0.83740$, $y_4 = 0.83746$, $y_5 = 0.83746$. Thus, to five decimal places, $\bar{y} = 0.83746$.

Note. The primitive of the given differential equation is $y = x - 1 + Ce^{-x}$. The particular solution satisfying the initial conditions $x = 0, y = 1$ is $y = x - 1 + 2e^{-x}$. Replacing e^{-x} by its MacLaurin series, we have $y = 1 - x + x^2 - \frac{1}{3}x^3 + \frac{1}{12}x^4 - \frac{1}{60}x^5 + \frac{1}{360}x^6 + \cdots\cdots\cdots$. Upon comparing this with the successive approximations obtained above, it seems reasonable to suppose that the sequence of approximations given by Picard's Method tends to the exact solution as a limit.

2. Use Picard's Method to approximate the value of y when $x = 0.1$, given that $y = 1$ when $x = 0$, and $dy/dx = 3x + y^2$.

Here $f(x,y) = 3x + y^2$, $x_0 = 0$, $y_0 = 1$. Then

$$y_1 = y_0 + \int_0^x (3x + y_0^2)\,dx = 1 + \int_0^x (3x + 1)\,dx = \frac{3}{2}x^2 + x + 1,$$

$$y_2 = y_0 + \int_0^x (3x + y_1^2)\,dx = 1 + \int_0^x (\frac{9}{4}x^4 + 3x^3 + 4x^2 + 5x + 1)\,dx = \frac{9}{20}x^5 + \frac{3}{4}x^4 + \frac{4}{3}x^3 + \frac{5}{2}x^2 + x + 1,$$

$$y_3 = 1 + \int_0^x (\frac{81}{400}x^{10} + \frac{27}{40}x^9 + \frac{141}{80}x^8 + \frac{17}{4}x^7 + \frac{1157}{180}x^6 + \frac{136}{15}x^5 + \frac{125}{12}x^4 + \frac{23}{3}x^3 + 6x^2 + 5x + 1)\,dx$$

$$= \frac{81}{4400}x^{11} + \frac{27}{400}x^{10} + \frac{47}{240}x^9 + \frac{17}{32}x^8 + \frac{1157}{1260}x^7 + \frac{68}{45}x^6 + \frac{25}{12}x^5 + \frac{23}{12}x^4 + 2x^3 + \frac{5}{2}x^2 + x + 1.$$

When $x = 0.1$, $y_0 = 1$, $y_1 = 1.115$, $y_2 = 1.1264$, $y_3 = 1.12721$.

3. If $\frac{dy}{dx} = x - y$, use the Taylor Series Method to approximate y when:

a) $x = 0.2$, given that $y = 1$ when $x = 0$.
b) $x = 1.6$, given that $y = 0.4$ when $x = 1$.

a) Here

$$\begin{aligned} &y = g(x), & &g(x_0) = 1, & &y''' = g'''(x) = -y'', & &g'''(x_0) = -2,\\ &y' = g'(x) = x - y, & &g'(x_0) = -1, & &y^{iv} = g^{iv}(x) = -y''', & &g^{iv}(x_0) = 2,\\ &y'' = g''(x) = 1 - y', & &g''(x_0) = 2, & &y^{v} = g^{v}(x) = -y^{iv}, & &g^{v}(x_0) = -2, \text{ etc.} \end{aligned}$$

and equation 6) becomes $y = 1 - x + x^2 - \frac{1}{3}x^3 + \frac{1}{12}x^4 - \frac{1}{60}x^5 + \cdots\cdots\cdots$. Then

$$\bar{y} = 1 - 0.2 + 0.04 - \frac{1}{3}(0.008) + \frac{1}{12}(0.0016) - \frac{1}{60}(0.00032) + \cdots \approx 0.83746. \text{ (See Problem 1.)}$$

b) Here $g(x_0) = 0.4$, $g'(x_0) = 0.6$, $g''(x_0) = 0.4$, $g'''(x_0) = -0.4$, $g^{iv}(x_0) = 0.4$, etc., and equation 6) becomes

$$y = 0.4 + 0.6h + 0.4\frac{h^2}{2} - 0.4\frac{h^3}{6} + 0.4\frac{h^4}{24} - 0.4\frac{h^5}{120} + 0.4\frac{h^6}{720} + \cdots\cdots\cdots, \text{ where } h = x - x_0.$$

When $x = 1.6$, $h = 0.6$ and

$$\bar{y} = 0.4 + 0.6(0.6) + 0.4(0.18) - 0.4(0.036) + 0.4(0.0054) - 0.4(0.000648) + 0.4(0.0000648) + \cdots\cdots\cdots$$

$$\approx 0.81953.$$

4. If $\frac{dy}{dx} = 3x + y^2$, use the Taylor Series Method to approximate y when:

a) $x = 0.1$, given that $y = 1$ when $x = 0$.
b) $x = 1.1$, given that $y = 1.2$ when $x = 1$.

a) Here $(x_0, y_0) = (0, 1)$, $g(x_0) = 1$,

$$\begin{array}{lllll} y' & = g'(x) & = 3x + y^2, & g'(x_0) & = 1, \\ y'' & = g''(x) & = 3 + 2yy', & g''(x_0) & = 5, \\ y''' & = g'''(x) & = 2(y')^2 + 2yy'', & g'''(x_0) & = 12, \\ y^{iv} & = g^{iv}(x) & = 6y'y'' + 2yy''', & g^{iv}(x_0) & = 54, \\ y^{v} & = g^{v}(x) & = 6(y'')^2 + 8y'y''' + 2yy^{iv}, & g^{v}(x_0) & = 354, \end{array}$$

and 6) becomes

$y = 1 + x + \frac{5}{2}x^2 + 2x^3 + \frac{9}{4}x^4 + \frac{177}{60}x^5 + \cdots\cdots\cdots$. When $x = 0.1$,

$\bar{y} = 1 + 0.1 + 0.025 + 0.002 + 0.00022 + 0.00003 + \cdots\cdots \approx 1.12725$. (See Problem 2.)

b) Here $(x_0, y_0) = (1, 1.2)$, $g(x_0) = 1.2$, $g'(x_0) = 4.44$, $g''(x_0) = 13.656$, $g'''(x_0) = 72.202$, $g^{iv}(x_0) = 537.078$, $g^{v}(x_0) = 4973$, $\cdots\cdots\cdots$, and 6) becomes

$$y = 1.2 + 4.44h + 13.656\frac{h^2}{2} + 72.202\frac{h^3}{6} + 537.078\frac{h^4}{24} + 4973\frac{h^5}{120} + \cdots\cdots\cdots,$$

where $h = x - x_0$. When $x = 1.1$, $h = 0.1$ and

$\bar{y} = 1.2 + 0.1(4.44) + 0.01(6.828) + 0.001(12.03) + 0.0001(22.4) + 0.00001(41) + \cdots \approx 1.7270$.

5. Use the First Derivative Method, with $n = 4$, to approximate y when $x = 1.1$, given that $y = 1.2$ when $x = 1$ and $dy/dx = 3x + y^2$. See Problem 4b.

Here $h = 0.1$ and we take $h_1 = h_2 = h_3 = h_4 = 0.025$. We seek $y_0 + k_1 + k_2 + k_3 + k_4 = y_3 + k_4$.

a) $(x_0, y_0) = (1, 1.2)$, $h_1 = 0.025$, $f(x_0, y_0) = 4.44$, $k_1 = h_1 f(x_0, y_0) = 0.111$;
$y_1 = y_0 + k_1 = 1.311$.

b) $(x_1, y_1) = (1.025, 1.311)$, $h_2 = 0.025$, $f(x_1, y_1) = 4.7937$, $k_2 = h_2 f(x_1, y_1) = 0.1198$;
$y_2 = y_1 + k_2 = 1.4308$.

c) $(x_2, y_2) = (1.05, 1.4308)$, $h_3 = 0.025$, $f(x_2, y_2) = 5.1972$, $k_3 = h_3 f(x_2, y_2) = 0.1299$;
$y_3 = y_2 + k_3 = 1.5607$.

d) $(x_3, y_3) = (1.075, 1.5607)$, $h_4 = 0.025$, $f(x_3, y_3) = 5.6608$, $k_4 = h_4 f(x_3, y_3) = 0.1415$;
$\bar{y} \approx y_3 + k_4 = 1.7022$.

6. Use the First Derivative Method, with $n = 4$, to approximate y when $x = 1.4$, given that $y = 0.2$ when $x = 1$ and $\frac{dy}{dx} = (x^2 + 2y)^{1/2}$.

Here $h = 0.4$ and we take $h_1 = h_2 = h_3 = h_4 = 0.1$.

a) $(x_0, y_0) = (1, 0.2)$, $h_1 = 0.1$, $f(x_0, y_0) = \sqrt{1.4} = 1.183$, $k_1 = h_1 f(x_0, y_0) = 0.1183$;
$y_1 = y_0 + k_1 = 0.3183$.

b) $(x_1, y_1) = (1.1, 0.3183)$, $h_2 = 0.1$, $f(x_1, y_1) = 1.359$, $k_2 = h_2 f(x_1, y_1) = 0.1359$;
$y_2 = y_1 + k_2 = 0.4542$.

c) $(x_2, y_2) = (1.2, 0.4542)$, $h_3 = 0.1$, $f(x_2, y_2) = 1.532$, $k_3 = h_3 f(x_2, y_2) = 0.1532$;
$y_3 = y_2 + k_3 = 0.6074$.

d) $(x_3, y_3) = (1.3, 0.6074)$, $h_4 = 0.1$, $f(x_3, y_3) = 1.704$, $k_4 = h_4 f(x_3, y_3) = 0.1704$;
$$\bar{y} \approx y_3 + k_4 = 0.7778.$$

7. Use Runge's Method to approximate y when $x = 1.6$, given that $y = 0.4$ when $x = 1$ and $dy/dx = x - y$. (See problem 3*b*.)

Here $(x_0, y_0) = (1, 0.4)$, $h = 0.6$, $f_0 = 1 - 0.4 = 0.6$. Then

$$k_1 = hf_0 = 0.36,$$
$$k_2 = hf(x_0+h,\ y_0+k_1) = 0.6[(1+0.6) - (0.4+0.36)] = 0.504,$$
$$k_3 = hf(x_0+h,\ y_0+k_2) = 0.6[(1+0.6) - (0.4+0.504)] = 0.4176,$$
$$k_4 = hf(x_0+\tfrac{1}{2}h,\ y_0+\tfrac{1}{2}k_1) = 0.6[(1+0.3) - (0.4+0.18)] = 0.432,$$
$$k \approx \frac{1}{6}(k_1 + 4k_4 + k_3) = \frac{1}{6}[0.36 + 4(0.432) + 0.4176] = 0.4176, \quad \text{and} \quad \bar{y} = y_0 + k \approx 0.8176.$$

The difference between this approximation and that found in Problem 3*b* arises from the fact that $h = 0.6$. In finding the value of y when $x = 1.1$, (that is, $h = 0.1$), the Taylor series gives $\bar{y} = 0.4 + 0.6(0.1) + 0.4(0.005) - 0.4(0.00017) + 0.4(0.000004) - \cdots\cdots \approx 0.46193$, while by Runge's Method

$$k_1 = 0.1(0.6) = 0.06, \quad k_2 = 0.1(1.1 - 0.46) = 0.064, \quad k_3 = 0.1(1.1 - 0.464) = 0.0636,$$
$$k_4 = 0.1(1.05 - 0.43) = 0.062, \quad k \approx \frac{1}{6}(k_1 + 4k_4 + k_3) = 0.06193, \quad \text{and} \quad \bar{y} \approx 0.46193.$$

8. Use Runge's Method to approximate y when $x = 0.1$, given that $y = 1$ when $x = 0$ and $dy/dx = 3x + y^2$.

Here $(x_0, y_0) = (0, 1)$, $h = 0.1$, $f_0 = 1$. Then

$$k_1 = hf_0 = 0.1,$$
$$k_2 = hf(x_0+h,\ y_0+k_1) = 0.1[3(0+0.1) + (1+0.1)^2] = 0.151,$$
$$k_3 = hf(x_0+h,\ y_0+k_2) = 0.1[3(0+0.1) + (1+0.151)^2] = 0.16248,$$
$$k_4 = hf(x_0+\tfrac{1}{2}h,\ y_0+\tfrac{1}{2}k_1) = 0.1[3(0+0.05) + (1+0.05)^2] = 0.12525,$$
$$k \approx \frac{1}{6}(k_1 + 4k_4 + k_3) = \frac{1}{6}[0.1 + 4(0.12525) + 0.16248] = 0.12725, \quad \text{and} \quad \bar{y} = y_0 + k \approx 1.12725.$$

(See Problems 2 and 4*a*.)

9. Use Runge's Method to approximate y when $x = 1.1$, given that $y = 1.2$ when $x = 1$ and $dy/dx = 3x + y^2$.

Here $(x_0, y_0) = (1, 1.2)$, $h = 0.1$, $f_0 = 4.44$. Then

$$k_1 = hf_0 = 0.444,$$
$$k_2 = hf(x_0+h,\ y_0+k_1) = 0.1[3(1+0.1) + (1.2+0.444)^2] = 0.600274,$$
$$k_3 = hf(x_0+h,\ y_0+k_2) = 0.1[3(1+0.1) + (1.2+0.60027)^2] = 0.654097,$$
$$k_4 = hf(x_0+\tfrac{1}{2}h,\ y_0+\tfrac{1}{2}k_1) = 0.1[3(1+0.05) + (1.2+0.222)^2] = 0.517208,$$
$$k \approx \frac{1}{6}(k_1 + 4k_4 + k_3) = \frac{1}{6}[0.444 + 4(0.517208) + 0.654097] = 0.527822, \quad \text{and}$$
$$\bar{y} = y_0 + k \approx 1.727822.$$

Comparing this result with that obtained in Problem 4*b*, it is to be noted that the approximation is better than might have been expected in view of the value $f_0 = 4.44$.

10. Use Runge's Method to approximate y when $x = 0.8$ for that particular solution of $dy/dx = \sqrt{x+y}$ satisfying $y = 0.41$ when $x = 0.4$.

Here $(x_0, y_0) = (0.4,\ 0.41)$, $h = 0.4$, $f_0 = \sqrt{0.81} = 0.9$. Then

$$k_1 = hf_0 = 0.36,$$
$$k_2 = hf(x_0+h,\ y_0+k_1) = 0.4\sqrt{1.57} = 0.50120,$$
$$k_3 = hf(x_0+h,\ y_0+k_2) = 0.4\sqrt{1.7112} = 0.52325,$$
$$k_4 = hf(x_0+\tfrac{1}{2}h,\ y_0+\tfrac{1}{2}k_1) = 0.4\sqrt{1.19} = 0.43635,$$
$$k \approx \frac{1}{6}(k_1 + 4k_4 + k_3) = 0.43811, \quad \text{and} \quad \bar{y} = y_0 + k \approx 0.84811.$$

11. Solve Problem 10, first approximating y when $x = 0.6$ and then, using this pair of values as (x_0, y_0), approximate the required value of y.

First, $(x_0, y_0) = (0.4,\ 0.41)$, $h = 0.2$, $f_0 = \sqrt{0.81} = 0.9$. Then

$$k_1 = hf_0 = 0.18,$$
$$k_2 = hf(x_0+h,\ y_0+k_1) = 0.2\sqrt{1.19} = 0.21817,$$
$$k_3 = hf(x_0+h,\ y_0+k_2) = 0.2\sqrt{1.22817} = 0.22165,$$
$$k_4 = hf(x_0+\tfrac{1}{2}h,\ y_0+\tfrac{1}{2}k_1) = 0.2,$$
$$k \approx \frac{1}{6}(k_1 + 4k_4 + k_3) = 0.20028, \quad \text{and} \quad \bar{y} = y_0 + k \approx 0.61028.$$

Next, take $(x_0, y_0) = (0.6,\ 0.61028)$, $h = 0.2$. Then $f_0 = \sqrt{1.21028} = 1.1001$,

$$k_1 = hf_0 = 0.22002,$$
$$k_2 = hf(x_0+h,\ y_0+k_1) = 0.2\sqrt{1.63030} = 0.25537,$$
$$k_3 = hf(x_0+h,\ y_0+k_2) = 0.2\sqrt{1.66565} = 0.25812,$$
$$k_4 = hf(x_0+\tfrac{1}{2}h,\ y_0+\tfrac{1}{2}k_1) = 0.2\sqrt{1.42029} = 0.23836,$$
$$k \approx \frac{1}{6}(k_1 + 4k_4 + k_3) = 0.23860, \quad \text{and} \quad \bar{y} = y_0 + k \approx 0.84888.$$

12. Solve Problem 10, using the Kutta-Simpson Method.

Here $(x_0, y_0) = (0.4,\ 0.41)$, $h = 0.4$, $f_0 = \sqrt{0.81} = 0.9$. Then

$$k_1 = hf_0 = 0.36,$$
$$k_2 = hf(x_0+\tfrac{1}{2}h,\ y_0+\tfrac{1}{2}k_1) = 0.4\sqrt{1.19} = 0.43635,$$
$$k_3 = hf(x_0+\tfrac{1}{2}h,\ y_0+\tfrac{1}{2}k_2) = 0.4\sqrt{1.22817} = 0.44329,$$
$$k_4 = hf(x_0+h,\ y_0+k_3) = 0.4\sqrt{1.65329} = 0.51432,$$
$$k \approx \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.43893, \quad \text{and} \quad \bar{y} = y_0 + k \approx 0.84893.$$

13. Use Picard's Method to approximate y and z corresponding to $x = 0.1$ for that particular solution of

$$\frac{dy}{dx} = f(x,y,z) = x + z, \qquad \frac{dz}{dx} = g(x,y,z) = x - y^2$$

satisfying $y = 2$, $z = 1$ when $x = 0$.

For the first approximations,

$$y_1 = y_0 + \int_0^x f(x,y_0,z_0)dx = 2 + \int_0^x (1+x)dx = 2 + x + \tfrac{1}{2}x^2,$$
$$z_1 = z_0 + \int_0^x g(x,y_0,z_0)dx = 1 + \int_0^x (-4+x)dx = 1 - 4x + \tfrac{1}{2}x^2.$$

For the second approximations,

$$y_2 = y_0 + \int_0^x f(x,y_1,z_1)dx = 2 + \int_0^x (1-3x+\tfrac{1}{2}x^2)dx = 2 + x - \frac{3}{2}x^2 + \frac{1}{6}x^3,$$
$$z_2 = z_0 + \int_0^x g(x,y_1,z_1)dx = 1 + \int_0^x (-4-3x-3x^2-x^3-\tfrac{1}{4}x^4)dx$$
$$= 1 - 4x - \frac{3}{2}x^2 - x^3 - \frac{1}{4}x^4 - \frac{1}{20}x^5.$$

For the third approximations,

$$y_3 = y_0 + \int_0^x f(x,y_2,z_2)dx = 2 + \int_0^x (1 - 3x - \frac{3}{2}x^2 - x^3 - \frac{1}{4}x^4 - \frac{1}{20}x^5)dx$$
$$= 2 + x - \frac{3}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{4}x^4 - \frac{1}{20}x^5 - \frac{1}{120}x^6,$$
$$z_3 = z_0 + \int_0^x g(x,y_2,z_2)dx = 1 + \int_0^x (-4 - 3x + 5x^2 + \frac{7}{3}x^3 - \frac{31}{12}x^4 + \frac{1}{2}x^5 - \frac{1}{36}x^6)dx$$
$$= 1 - 4x - \frac{3}{2}x^2 + \frac{5}{3}x^3 + \frac{7}{12}x^4 - \frac{31}{60}x^5 + \frac{1}{12}x^6 - \frac{1}{252}x^7,$$

and so on.

When $x = 0.1$: $y_1 = 2.105$, $z_1 = 0.605$

$y_2 = 2.08517$, $z_2 = 0.58397$

$y_3 = 2.08447$, $z_3 = 0.58672$.

14. Use Runge's Method to approximate y and z when $x = 0.3$ for that particular solution of the system $\frac{dy}{dx} = x + \sqrt{z} = f(x,y,z)$, $\frac{dz}{dx} = y - \sqrt{z} = g(x,y,z)$ satisfying $y = 0.5$, $z = 0$ when $x = 0.2$.

Here $(x_0,y_0,z_0) = (0.2, 0.5, 0)$, $h = 0.1$, $f_0 = 0.2$, $g_0 = 0.5$. Then

$$k_1 = hf_0 = 0.02,$$
$$l_1 = hg_0 = 0.05,$$
$$k_2 = hf(x_0+h,\ y_0+k_1,\ z_0+l_1) = 0.1(0.3 + \sqrt{0.05}) = 0.05236,$$
$$l_2 = hg(x_0+h,\ y_0+k_1,\ z_0+l_1) = 0.1(0.52 - \sqrt{0.05}) = 0.02964,$$
$$k_3 = hf(x_0+h,\ y_0+k_2,\ z_0+l_2) = 0.1(0.3 + \sqrt{0.02964}) = 0.047216,$$
$$l_3 = hg(x_0+h,\ y_0+k_2,\ z_0+l_2) = 0.1(0.52 - \sqrt{0.02964}) = 0.034784,$$
$$k_4 = hf(x_0+\tfrac{1}{2}h,\ y_0+\tfrac{1}{2}k_1,\ z_0+\tfrac{1}{2}l_1) = 0.1(0.25 + \sqrt{0.025}) = 0.040811,$$
$$l_4 = hg(x_0+\tfrac{1}{2}h,\ y_0+\tfrac{1}{2}k_1,\ z_0+\tfrac{1}{2}l_1) = 0.1(0.51 - \sqrt{0.025}) = 0.035189,$$

$$k \approx \frac{1}{6}(k_1 + 4k_4 + k_3) = 0.03841, \qquad l \approx \frac{1}{6}(l_1 + 4l_4 + l_3) = 0.03759,$$

and $\bar{y} = y_0 + k \approx 0.53841$, $\bar{z} = z_0 + l \approx 0.03759$.

15. Use the Taylor Series Method to approximate the value of θ corresponding to $t = 0.05$ for that particular solution of $\frac{d^2\theta}{dt^2} = -8\sin\theta$ satisfying $\theta = \pi/4$, $\frac{d\theta}{dt} = 1$ when $t = 0$.

The given differential equation is equivalent to the system

$$\frac{d\theta}{dt} = \phi = f(t,\theta,\phi), \qquad \frac{d\phi}{dt} = -8\sin\theta = g(t,\theta,\phi)$$

with initial conditions $t = 0,\ \theta = \pi/4,\ \phi = 1.$ Then

$$\begin{array}{llll} \frac{d\theta}{dt} = \theta' = \phi & \theta_0' = 1 & \phi' = -8\sin\theta & \phi_0' = -4\sqrt{2} \\ \theta'' = \phi' & \theta_0'' = -4\sqrt{2} & \phi'' = -8\theta'\cos\theta & \phi_0'' = -4\sqrt{2} \\ \theta''' = \phi'' & \theta_0''' = -4\sqrt{2} & \phi''' = 8(\theta')^2\sin\theta - 8\theta''\cos\theta & \\ \theta^{iv} = \phi''' & \theta_0^{iv} = 4\sqrt{2} + 32 & & \phi_0''' = 4\sqrt{2}(1+4\sqrt{2}) \end{array}$$

and $\quad \theta = \pi/4 + t - 4\sqrt{2}\,\frac{t^2}{2} - 4\sqrt{2}\,\frac{t^3}{6} + 4(8+\sqrt{2})\,\frac{t^4}{24} + \cdots\cdots\cdots = 0.82821.$

16. Use the Kutta-Simpson Method to approximate y corresponding to $x = 0.1$ for that particular solution of $\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 4y = 0$ satisfying $y = 0.2,\ \frac{dy}{dx} = 0.5$ when $x = 0.$

The given equation with initial conditions is equivalent to the system

$$\frac{dy}{dx} = z = f(x,y,z), \qquad \frac{dz}{dx} = 4y - 2xz = g(x,y,z)$$

with initial conditions $x = 0,\ y = 0.2,\ z = 0.5.$

Here $(x_0, y_0, z_0) = (0,\ 0.2,\ 0.5),\ h = 0.1,\ f_0 = 0.5,\ g_0 = 0.8.$ Then

$$\begin{aligned} k_1 &= hf_0 = 0.05, \\ l_1 &= hg_0 = 0.08, \\ k_2 &= hf(x_0+\tfrac{1}{2}h,\ y_0+\tfrac{1}{2}k_1,\ z_0+\tfrac{1}{2}l_1) = 0.1(0.54) = 0.054, \\ l_2 &= hg(x_0+\tfrac{1}{2}h,\ y_0+\tfrac{1}{2}k_1,\ z_0+\tfrac{1}{2}l_1) = 0.1(0.846) = 0.0846, \\ k_3 &= hf(x_0+\tfrac{1}{2}h,\ y_0+\tfrac{1}{2}k_2,\ z_0+\tfrac{1}{2}l_2) = 0.1(0.5423) = 0.05423, \\ l_3 &= hg(x_0+\tfrac{1}{2}h,\ y_0+\tfrac{1}{2}k_2,\ z_0+\tfrac{1}{2}l_2) = 0.1(0.85377) = 0.085377, \\ k_4 &= hf(x_0+h,\ y_0+k_3,\ z_0+l_3) = 0.1(0.585377) = 0.0585377, \end{aligned}$$

$k \approx \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.0541663,$ and $\quad \bar{y} = y_0 + k \approx 0.25417.$

SUPPLEMENTARY PROBLEMS

17. Approximate y when $x = 0.2$ if $dy/dx = x + y^2$ and $y = 1$ when $x = 0$, using *a)* Picard's method, *b)* Taylor series, and *c)* the First Derivative method with $n = 4$.

Ans. *a)* $y_1 = 1.22$, $y_2 = 1.2657$, $y_3 = 1.2727$; *b)* 1.2735; *c)* 1.2503

18. Approximate y when $x = 0.1$ if $dy/dx = x - y^2$ and $y = 1$ when $x = 0$, using *a)* Picard's method, *b)* Taylor series, and *c)* the First Derivative method with $n = 4$.

Ans. *a)* $y_1 = 0.905$, $y_2 = 0.9143$, $y_3 = 0.9138$; *b)* 0.9138; *c)* 0.9107

19. Use Runge's method to approximate y when $x = 0.025$ if $dy/dx = x + y$ and $y = 1$ when $x = 0$.

Ans. 1.0256

20. Use Runge's method to approximate y when $x = 2.2$ if $dy/dx = 1 + y/x$ and $y = 2$ when $x = 2$.

Ans. 2.4096

21. Use Runge's method to approximate y when $x = 0.5$ if $dy/dx = \sqrt{x + 2y}$ and $y = 0.17$ when $x = 0.3$.

Ans. 0.3607

22. Solve Problem 21 using the Kutta-Simpson method. *Ans.* 0.3611

23. Use Runge's method to approximate y and z when $x = 0.2$ for the particular solution of the system $dy/dx = y + z$, $dz/dx = x^2 + y$ satisfying $y = 0.4$, $z = 0.1$ when $x = 0.1$.

Ans. $y \approx 0.4548$, $z \approx 0.1450$

24. Use the Kutta-Simpson method to approximate y when $x = 0.2$ for that particular solution of $\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = 0$ satisfying $y = 0.1$, $\frac{dy}{dx} = 0.2$ when $x = 0.1$. *Ans.* 0.1191

CHAPTER 25

Integration in Series

EQUATIONS OF ORDER ONE. The existence theorem of Chapter 2 for a differential equation of the form

1) $$\frac{dy}{dx} = f(x,y)$$

gives a sufficient condition for a solution. In the proof using power series, y is found in the form of a Taylor series

2) $$y = A_0 + A_1(x-x_0) + A_2(x-x_0)^2 + \cdots\cdots + A_n(x-x_0)^n + \cdots\cdots,$$

where for convenience y_0 has been replaced by A_0. This series *i*) satisfies the differential equation 1), *ii*) has the value $y = y_0$ when $x = x_0$, and *iii*) is convergent for all values of x sufficiently near $x = x_0$.

A. To obtain the solution of 1) satisfying the condition $y = y_0$ when $x = 0$:
a) Assume the solution to be of the form

$$y = A_0 + A_1x + A_2x^2 + A_3x^3 + \cdots\cdots + A_nx^n + \cdots\cdots$$

in which $A_0 = y_0$ and the remaining A's are constants to be determined.

b) Substitute the assumed series in the differential equation and proceed as in the Method of Undetermined Coefficients of Chapter 15.

EXAMPLE 1. Solve $y' = x^2 + y$ in series satisfying the condition $y = y_0$ when $x = 0$.

Since $f(x,y) = x^2 + y$ is single valued and continuous while $\partial f/\partial y = 1$ is continuous over any rectangle of values (x,y) enclosing $(0,y_0)$, the conditions of the Existence Theorem are satisfied and we assume the solution

$$y = A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \cdots\cdots + A_nx^n + \cdots\cdots$$

Now, within the region of convergence, this series may be differentiated term by term yielding a series which converges to the derivative y'. Hence,

$$y' = A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 + \cdots\cdots + nA_nx^{n-1} + \cdots\cdots$$

and

$$y' - x^2 - y = (A_1 - A_0) + (2A_2 - A_1)x + (3A_3 - A_2 - 1)x^2 + (4A_4 - A_3)x^3 + \cdots\cdots\cdots + (nA_n - A_{n-1})x^{n-1} + \cdots\cdots\cdots = 0.$$

In order that this series vanish for all values of x in some region surrounding $x = 0$, it is necessary and sufficient that the coefficients of each power of x vanish. Thus,

$A_1 - A_0 = 0$ and $A_1 = A_0 = y_0$. $\qquad 3A_3 - A_2 - 1 = 0$ and $A_3 = \frac{1}{3} + \frac{1}{6}y_0$.

$2A_2 - A_1 = 0$ and $A_2 = \frac{1}{2}A_1 = \frac{1}{2}A_0 = \frac{1}{2}y_0$. $\qquad 4A_4 - A_3 = 0$ and $A_4 = \frac{1}{12} + \frac{1}{24}y_0$.

....

$$nA_n - A_{n-1} = 0 \quad \text{and} \quad A_n = \frac{1}{n}A_{n-1}, \quad n \geq 4.$$

This latter relation, called a *recursion formula*, may be used to compute additional coefficients; thus,

$$A_5 = \frac{1}{5}A_4 = \frac{1}{60} + \frac{1}{120}y_0, \qquad A_6 = \frac{1}{6}A_5 = \frac{1}{360} + \frac{1}{720}y_0, \quad \ldots\ldots\ldots$$

It is also possible to obtain the coefficients as follows:

Since $A_n = \frac{1}{n}A_{n-1}$ and $A_{n-1} = \frac{1}{n-1}A_{n-2}$, $A_n = \frac{1}{n(n-1)}A_{n-2}$. But $A_{n-2} = \frac{1}{n-2}A_{n-3}$, $\ldots$;
hence, $A_n = \frac{1}{n(n-1)(n-2)\cdot\ldots\cdot 4}A_3 = \frac{1}{n(n-1)(n-2)\cdot\ldots\cdot 4\cdot 3}(1+\frac{1}{2}A_0) = \frac{1}{n!}(2+y_0)$, $n \geqq 3$.

When the values of the A's are substituted in the assumed series, we have

$$y = y_0 + y_0 x + \frac{1}{2}y_0 x^2 + (\frac{1}{3} + \frac{1}{6}y_0)x^3 + (\frac{1}{12} + \frac{1}{24}y_0)x^4 + \ldots\ldots + \frac{1}{n!}(2+y_0)x^n + \ldots\ldots$$

$$= (y_0 + 2)(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots\ldots + \frac{1}{n!}x^n + \ldots\ldots) - x^2 - 2x - 2$$

$$= (y_0 + 2)e^x - x^2 - 2x - 2.$$

The given differential equation may be solved using the integrating factor e^{-x}; thus,

$ye^{-x} = \int x^2 e^{-x}\,dx = (-x^2 - 2x - 2)e^{-x} + C$ and $y = Ce^x - x^2 - 2x - 2$.

Using the initial condition, $y = y_0$ when $x = 0$, $C = y_0 + 2$, and $y = (y_0 + 2)e^x - x^2 - 2x - 2$, as before.

B. To obtain the solution of 1) satisfying the condition $y = y_0$ when $x = x_0$:

a) Make the substitution $x - x_0 = v$, that is,

$$x = v + x_0, \quad \frac{dy}{dx} = \frac{dy}{dv}$$

resulting in $dy/dv = F(y, y)$.

b) Use the procedure of *A* to obtain the solution of this equation satisfying the condition $y = y_0$ when $v = 0$.

c) Make the substitution $v = x - x_0$ in the solution.

Example 2. Solve $y' = x^2 - 4x + y + 1$ satisfying the condition $y = 3$ when $x = 2$.

First make the substitution $x = v + 2$ and obtain $\frac{dy}{dv} = v^2 + y - 3$. We seek the solution satisfying $y = 3$ when $v = 0$; hence, we assume the series solution

$$y = 3 + A_1 v + A_2 v^2 + A_3 v^3 + A_4 v^4 + \ldots\ldots\ldots + A_n v^n + \ldots\ldots\ldots$$

Then $\frac{dy}{dv} = A_1 + 2A_2 v + 3A_3 v^2 + 4A_4 v^3 + \ldots\ldots\ldots + nA_n v^{n-1} + \ldots\ldots\ldots$

and

$$\frac{dy}{dv} - v^2 - y + 3 = A_1 + (2A_2 - A_1)v + (3A_3 - A_2 - 1)v^2 + (4A_4 - A_3)v^3 + \ldots\ldots\ldots\ldots$$
$$+ (nA_n - A_{n-1})v^{n-1} + \ldots\ldots\ldots = 0.$$

Equating the coefficients to zero, we have: $A_1 = 0$, $2A_2 - A_1 = 0$ and $A_2 = 0$, $3A_3 - A_2 - 1 = 0$ and $A_3 = 1/3$, $4A_4 - A_3 = 0$ and $A_4 = 1/12$, $\ldots\ldots\ldots$

The recursion formula $A_n = \frac{1}{n}A_{n-1}$ yields

$$A_n = \frac{1}{n}A_{n-1} = \frac{1}{n(n-1)}A_{n-2} = \cdots\cdots = \frac{1}{n(n-1)(n-2)\cdots\cdots 4}A_3 = \frac{2}{n!}, \quad n \geq 3.$$

Thus,
$$y = 3 + \frac{1}{3}v^3 + \frac{1}{12}v^4 + \cdots\cdots + \frac{2}{n!}v^n + \cdots\cdots$$
$$= 3 + \frac{2}{3!}(x-2)^3 + \frac{2}{4!}(x-2)^4 + \cdots\cdots + \frac{2}{n!}(x-2)^n + \cdots\cdots$$

See also Problems 1-4.

LINEAR EQUATIONS OF ORDER TWO. Consider the equation

3) $$P_0(x)\,y'' + P_1(x)\,y' + P_2(x)\,y = 0$$

where the P's are polynomials in x. We shall call $x = a$ an *ordinary point* of 3) if $P_0(a) \neq 0$; otherwise, a *singular point*.

If $x=0$ is an ordinary point, 3) may be solved in series about $x=0$ as

4) $$y = A\{\text{series in } x\} + B\{\text{series in } x\},$$

in which A and B are arbitrary constants. The two series are linearly independent and both are convergent in a region surrounding $x=0$. The procedure for equations of order one in the section above may be used to obtain 4).

See Problem 5-7.

SOLVED PROBLEMS

EQUATIONS OF ORDER ONE.

1. Solve $\frac{dy}{dx} = \frac{2x-y}{1-x}$ in series satisfying the condition $y = y_0$ when $x = 0$.

Assume the series to be $y = A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \cdots\cdots + A_nx^n + \cdots\cdots,$

where $A_0 = y_0$. Then $y' = A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 + \cdots\cdots + nA_nx^{n-1} + \cdots\cdots.$

Substituting in the given differential equation $(1-x)y' - 2x + y = 0$, we have

$$(1-x)(A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 + \cdots\cdots + nA_nx^{n-1} + \cdots\cdots)$$
$$- 2x + (A_0 + A_1x + A_2x^2 + A_3x^3 + \cdots\cdots + A_nx^n + \cdots\cdots) = 0,$$

or

$$(A_1 + A_0) + (2A_2 - 2)x + (3A_3 - A_2)x^2 + (4A_4 - 2A_3)x^3 + \cdots\cdots + [(n+1)A_{n+1} - (n-1)A_n]x^n + \cdots\cdots = 0.$$

(Note. In finding the general term in the line immediately above, we may write a number of terms on either side of the general term of the assumed series for y, differentiate each in getting y', carry out the required multiplications, and pick out the terms in x^n OR learn to write the required term using the general term of the assumed series and its derivative. In the present problem we wish the term in x^n when the substitutions are made in $y' - xy' - 2x + y = 0$. First, we need the term in x^n of y' when we have the term in x^{n-1}. We simply replace n by $(n+1)$ in nA_nx^{n-1} and obtain $(n+1)A_{n+1}x^n$. The remaining terms $-nA_nx^n + A_nx^n$ are obvious.)

Equating the coefficients of distinct powers of x to zero yields

$A_1 + A_0 = 0$ and $A_1 = -A_0$, $\quad 3A_3 - A_2 = 0$ and $A_3 = \frac{1}{3}A_2 = \frac{1}{3}$,

$2A_2 - 2 = 0$ and $A_2 = 1$, $\quad 4A_4 - 2A_3 = 0$ and $A_4 = \frac{1}{2}A_3 = \frac{1}{6}$,

....

$$(n+1)A_{n+1} - (n-1)A_n = 0 \quad \text{and} \quad A_{n+1} = \frac{n-1}{n+1}A_n, \quad (n \geq 2).$$

Now $A_n = \frac{n-2}{n}A_{n-1} = \frac{(n-2)(n-3)}{n(n-1)}A_{n-2} = \frac{(n-2)(n-3)(n-4)}{n(n-1)(n-2)}A_{n-3} = \cdots\cdots$

$$= \frac{(n-2)(n-3)(n-4)\cdots\cdots 2\cdot 1}{n(n-1)(n-2)\cdots\cdots 4\cdot 3}A_2 = \frac{2}{n(n-1)}, \quad n \geq 2.$$

Thus, $y = y_0(1-x) + x^2 + \frac{1}{3}x^3 + \frac{1}{6}x^4 + \frac{1}{10}x^5 + \cdots\cdots + \frac{2}{n(n-1)}x^n + \cdots\cdots$

$$= y_0(1-x) + \sum_{n=2}^{\infty} \frac{2}{n(n-1)}x^n.$$

Using the ratio test, $\lim_{n\to\infty}\left|\frac{A_{n+1}x^{n+1}}{A_n x^n}\right| = |x| \lim_{n\to\infty}\frac{n-1}{n+1} = |x|$.

The series converges for $|x| < 1$.

Note. By means of the integrating factor $1/(1-x)$ the solution of the differential equation is $y = 2(1-x)\ln(1-x) + 2x + C(1-x)$. The particular integral required is

$$y = y_0(1-x) + 2(1-x)\ln(1-x) + 2x.$$

2. Solve $(1-xy)y' - y = 0$ in powers of x.

Assume the series to be $y = A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \cdots\cdots + A_nx^n + \cdots\cdots$. Then

$$y' = A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 + \cdots\cdots + nA_nx^{n-1} + \cdots\cdots \quad \text{and}$$

$$(1-xy)y' - y$$

$$= (1 - A_0x - A_1x^2 - A_2x^3 - A_3x^4 - \cdots\cdots - A_nx^{n+1} - \cdots\cdots)(A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 + \cdots\cdots + nA_nx^{n-1} + \cdots\cdots) - (A_0 + A_1x + A_2x^2 + A_3x^3 + \cdots\cdots + A_nx^n + \cdots\cdots)$$

$$= (A_1 - A_0) + (2A_2 - A_0A_1 - A_1)x + (3A_3 - 2A_0A_2 - A_1^2 - A_2)x^2 + (4A_4 - 3A_0A_3 - 3A_1A_2 - A_3)x^3 + \cdots\cdots = 0.$$

Equating to zero the coefficients of distinct powers of x,

$A_1 - A_0 = 0$ and $A_1 = A_0$,

$2A_2 - A_0A_1 - A_1 = 0$ and $A_2 = \frac{1}{2}A_1(1 + A_0) = \frac{1}{2}A_0(1 + A_0)$,

$3A_3 - 2A_0A_2 - A_1^2 - A_2 = 0$ and $A_3 = \frac{1}{3}(2A_0A_2 + A_1^2 + A_2) = \frac{1}{6}A_0(1 + 5A_0 + 2A_0^2)$,

$4A_4 - 3A_0A_3 - 3A_1A_2 - A_3 = 0$ and $A_4 = \frac{1}{24}A_0(1 + 17A_0 + 26A_0^2 + 6A_0^3)$,

....

Thus, $y = A_0[1 + x + \frac{1}{2!}(1+A_0)x^2 + \frac{1}{3!}(1+5A_0+2A_0^2)x^3 + \frac{1}{4!}(1+17A_0+26A_0^2+6A_0^3)x^4 + \cdots\cdots\cdot]$.

We shall not attempt to obtain a recursion formula here nor to test for convergence.

3. Solve $xy' - y - x - 1 = 0$ in powers of $(x-1)$.

Setting $x = z+1$, the equation becomes $(z+1)\frac{dy}{dz} - y - z - 2 = 0$. Since we seek its solution in powers of z, assume the series to be

$$y = A_0 + A_1 z + A_2 z^2 + A_3 z^3 + A_4 z^4 + \cdots\cdots\cdots + A_n z^n + \cdots\cdots\cdots. \quad \text{Then}$$

$$\frac{dy}{dz} = A_1 + 2A_2 z + 3A_3 z^2 + 4A_4 z^3 + \cdots\cdots\cdots + nA_n z^{n-1} + \cdots\cdots\cdots \quad \text{and}$$

$$(z+1)\frac{dy}{dz} - y - z - 2$$

$$= (z+1)(A_1 + 2A_2 z + 3A_3 z^2 + 4A_4 z^3 + \cdots\cdots + nA_n z^{n-1} + \cdots\cdots) - z - 2 - (A_0 + A_1 z + A_2 z^2 + A_3 z^3 + \cdots\cdots + A_n z^n + \cdots\cdots)$$

$$= (A_1 - 2 - A_0) + (2A_2 - 1)z + (3A_3 + A_2)z^2 + (4A_4 + 2A_3)z^3 + \cdots\cdots\cdots + [(n+1)A_{n+1} + (n-1)A_n]z^n + \cdots\cdots\cdots = 0.$$

Equating to zero the coefficients of the distinct powers of z,

$A_1 - 2 - A_0 = 0$ and $A_1 = 2 + A_0$, $\qquad 3A_3 + A_2 = 0$ and $A_3 = -\frac{1}{3}A_2 = -\frac{1}{6}$,

$2A_2 - 1 = 0$ and $A_2 = \frac{1}{2}$, $\qquad 4A_4 + 2A_3 = 0$ and $A_4 = -\frac{1}{2}A_3 = \frac{1}{12}$,

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$$(n+1)A_{n+1} + (n-1)A_n = 0 \quad \text{and} \quad A_{n+1} = -\frac{n-1}{n+1}A_n, \quad n \geq 2.$$

From Problem 1, $A_n = (-1)^n \frac{(n-2)(n-3)\cdots\cdots 2\cdot 1}{n(n-1)\cdots\cdots\cdots\cdots 4\cdot 3} A_2 = (-1)^n \frac{1}{n(n-1)}$, $n \geq 2$,

and $$y = A_0 + (2+A_0)z + \frac{1}{2}z^2 - \frac{1}{6}z^3 + \frac{1}{12}z^4 - \cdots\cdots + (-1)^n \frac{1}{n(n-1)} z^n + \cdots\cdots.$$

Replacing z by $(x-1)$, we have

$$y = A_0 x + 2(x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3 + \frac{1}{12}(x-1)^4 - \cdots\cdots\cdots$$

$$= A_0 x + 2(x-1) + \sum_{n=2}^{\infty} (-1)^n \frac{1}{n(n-1)}(x-1)^n.$$

Using the ratio test, $\lim_{n\to\infty} \left| \frac{A_{n+1}z^{n+1}}{A_n z^n} \right| = |z| \lim_{n\to\infty} \frac{n-1}{n+1} = |z| = |x-1|$.

The series converges for $|x-1| < 1$.

4. Solve $y' - x^2 - e^y = 0$ satisfying the condition $y = 0$ when $x = 0$.

In view of the initial condition, assume the series to be

$$y = A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \dots\dots\dots.$$

Then $y' = A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 + 5A_5x^4 + \dots\dots\dots.$

Also, $e^y = 1 + y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3 + \frac{1}{4!}y^4 + \dots\dots\dots$

$$= 1 + (A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \dots) + \frac{1}{2!}[A_1^2x^2 + 2A_1A_2x^3 + (A_2^2 + 2A_1A_3)x^4 + \dots\dots]$$

$$+ \frac{1}{3!}(A_1^3x^3 + 3A_1^2A_2x^4 + \dots\dots) + \frac{1}{4!}(A_1^4x^4 + \dots\dots) + \dots\dots\dots$$

$$= 1 + A_1x + (A_2 + \frac{1}{2}A_1^2)x^2 + (A_3 + A_1A_2 + \frac{1}{6}A_1^3)x^3$$

$$+ (A_4 + \frac{1}{2}A_2^2 + A_1A_3 + \frac{1}{2}A_1^2A_2 + \frac{1}{24}A_1^4)x^4 + \dots\dots\dots.$$

Substituting in the differential equation,

$$(A_1 - 1) + (2A_2 - A_1)x + (3A_3 - 1 - A_2 - \frac{1}{2}A_1^2)x^2 + (4A_4 - A_3 - A_1A_2 - \frac{1}{6}A_1^3)x^3$$

$$+ (5A_5 - A_4 - \frac{1}{2}A_2^2 - A_1A_3 - \frac{1}{2}A_1^2A_2 - \frac{1}{24}A_1^4)x^4 + \dots\dots\dots = 0.$$

Equating coefficients of distinct powers of x to zero,

$$A_1 - 1 = 0 \text{ and } A_1 = 1, \qquad 2A_2 - A_1 = 0 \text{ and } A_2 = \frac{1}{2}A_1 = \frac{1}{2}.$$

$$3A_3 - 1 - A_2 - \frac{1}{2}A_1^2 = 0 \text{ and } A_3 = \frac{1}{3}(1 + A_2 + \frac{1}{2}A_1^2) = \frac{2}{3}.$$

$$4A_4 - A_3 - A_1A_2 - \frac{1}{6}A_1^3 = 0 \text{ and } A_4 = \frac{1}{4}(A_3 + A_1A_2 + \frac{1}{6}A_1^3) = \frac{1}{3}.$$

$$5A_5 - A_4 - \frac{1}{2}A_2^2 - A_1A_3 - \frac{1}{2}A_1^2A_2 - \frac{1}{24}A_1^4 = 0 \text{ and } A_5 = \frac{17}{60}, \dots\dots\dots$$

and $y = x + \frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{1}{3}x^4 + \frac{17}{60}x^5 + \dots\dots\dots.$

LINEAR EQUATIONS OF ORDER TWO.

5. Solve $(1 + x^2)y'' + xy' - y = 0$ in powers of x.

Here $P_0(x) = 1 + x^2$, $P_0(0) \neq 0$ and $x = 0$ is an ordinary point.

We assume the series

$$y = A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \dots\dots + A_nx^n + \dots\dots.$$

Then $y' = A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 + \dots\dots + nA_nx^{n-1} + \dots\dots,$

and $y'' = 2A_2 + 6A_3x + 12A_4x^2 + \dots\dots + n(n-1)A_nx^{n-2} + \dots\dots.$

Substituting in the given differential equation,

$$(1+x^2)[2A_2 + 6A_3x + 12A_4x^2 + \cdots + n(n-1)A_nx^{n-2} + \cdots] + x(A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 + \cdots + nA_nx^{n-1} + \cdots) - (A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \cdots + A_nx^n + \cdots) = 0,$$

or $(2A_2 - A_0) + 6A_3x + (12A_4 + 3A_2)x^2 + \cdots + [(n+2)(n+1)A_{n+2} + (n^2-1)A_n]x^n + \cdots = 0.$

Equating to zero the coefficients of the distinct powers of x,

$2A_2 - A_0 = 0$ and $A_2 = \frac{1}{2}A_0$, $6A_3 = 0$ and $A_3 = 0$, $12A_4 + 3A_2 = 0$ and $A_4 = -\frac{1}{8}A_0$, ...

$$(n+2)(n+1)A_{n+2} + (n^2-1)A_n = 0 \quad \text{and} \quad A_{n+2} = -\frac{n-1}{n+2}A_n.$$

From the latter relation it is clear that $A_3 = A_5 = A_7 = \cdots = 0$, this is, $A_{n+2} = 0$ if n is odd. If n is even, $(n = 2k)$, then

$$A_{2k} = -\frac{2k-3}{2k}A_{2k-2} = \frac{(2k-3)(2k-5)}{2k(2k-2)}A_{2k-4} = \cdots = (-1)^{k+1}\frac{1\cdot3\cdot5\cdot\cdots\cdot(2k-3)}{2^k k!}A_0.$$

Thus, the complete solution is

$$y = A_0(1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 - \frac{5}{128}x^8 + \cdots) + A_1x$$

$$= A_0[1 + \frac{1}{2}x^2 + \sum_{k=2}^{\infty}(-1)^{k+1}\frac{1\cdot3\cdot5\cdot\cdots\cdot(2k-3)}{2^k k!}x^{2k}] + A_1x$$

$$= A_0[1 + \frac{1}{2}x^2 - \sum_{k=2}^{\infty}(-1)^{k}\frac{1\cdot3\cdot5\cdot\cdots\cdot(2k-3)}{2^k k!}x^{2k}] + A_1x.$$

Here $\lim_{n\to\infty}\left|\frac{A_{n+2}x^{n+2}}{A_nx^n}\right| = x^2 \lim_{n\to\infty}\frac{n-1}{n+2} = x^2$, and the series converges for $|x| < 1$.

6. Solve $y'' - x^2y' - y = 0$ in powers of x.

Here $P_0(x) = 1$ and $x = 0$ is an ordinary point. We assume the series

$$y = A_0 + A_1x + A_2x^2 + A_3x^3 + \cdots + A_nx^n + \cdots. \qquad \text{Then}$$

$$y' = A_1 + 2A_2x + 3A_3x^2 + \cdots + nA_nx^{n-1} + \cdots,$$

$$y'' = 2A_2 + 6A_3x + 12A_4x^2 + 20A_5x^3 + \cdots + n(n-1)A_nx^{n-2} + \cdots, \quad \text{and}$$

$$y'' - x^2y' - y = (2A_2 - A_0) + (6A_3 - A_1)x + (12A_4 - A_1 - A_2)x^2 + (20A_5 - 2A_2 - A_3)x^3 + \cdots + [(n+2)(n+1)A_{n+2} - (n-1)A_{n-1} - A_n]x^n + \cdots = 0.$$

Equating to zero the coefficients of the distinct powers of x,

$2A_2 - A_0 = 0$ and $A_2 = \frac{1}{2}A_0$, $6A_3 - A_1 = 0$ and $A_3 = \frac{1}{6}A_1$, $12A_4 - A_1 - A_2 = 0$ and $A_4 = \frac{1}{24}A_0 + \frac{1}{12}A_1$,

....

$$(n+2)(n+1)A_{n+2} - (n-1)A_{n-1} - A_n = 0 \quad \text{and} \quad A_{n+2} = \frac{(n-1)A_{n-1} + A_n}{(n+1)(n+2)}, \quad n \geq 1.$$

The complete solution is $y = A_0(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{20}x^5 + \frac{1}{720}x^6 + \frac{13}{2520}x^7 + \cdots\cdots)$

$$+ A_1(x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5 + \frac{7}{360}x^6 + \frac{41}{5040}x^7 + \cdots\cdots).$$

7. Solve $y'' - 2x^2y' + 4xy = x^2 + 2x + 2$ in powers of x.

Assume the series to be

$$y = A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \cdots\cdots + A_nx^n + \cdots\cdots. \quad \text{Then}$$
$$y' = A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 + 5A_5x^4 + \cdots\cdots + nA_nx^{n-1} + \cdots\cdots,$$
$$y'' = 2A_2 + 6A_3x + 12A_4x^2 + 20A_5x^3 + \cdots\cdots + n(n-1)A_nx^{n-2} + \cdots\cdots, \quad \text{and}$$

$$y'' - 2x^2y' + 4xy - x^2 - 2x - 2 = (2A_2 - 2) + (6A_3 + 4A_0 - 2)x + (12A_4 + 2A_1 - 1)x^2 + 20A_5x^3 + \cdots\cdots$$
$$+ [(n+2)(n+1)A_{n+2} - 2(n-1)A_{n-1} + 4A_{n-1}]x^n + \cdots\cdots = 0.$$

Equating the coefficients to zero, we obtain

$2A_2 - 2 = 0$ and $A_2 = 1$, $6A_3 + 4A_0 - 2 = 0$ and $A_3 = \frac{1}{3} - \frac{2}{3}A_0$, $A_4 = \frac{1}{12} - \frac{1}{6}A_1$, $A_5 = 0$,

$\cdots\cdots\cdots\cdots$

$$(n+2)(n+1)A_{n+2} - 2(n-3)A_{n-1} = 0 \quad \text{and} \quad A_{n+2} = \frac{2(n-3)}{(n+1)(n+2)}A_{n-1}, \quad n \geq 3.$$

The complete solution is

$$y = A_0(1 - \frac{2}{3}x^3 - \frac{2}{45}x^6 - \frac{2}{405}x^9 - \cdots\cdots) + A_1(x - \frac{1}{6}x^4 - \frac{1}{63}x^7 - \frac{1}{567}x^{10} - \cdots\cdots)$$
$$+ x^2 + \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \frac{1}{126}x^7 + \frac{1}{405}x^9 + \frac{1}{1134}x^{10} + \cdots\cdots$$

8. Solve $y'' + (x-1)y' + y = 0$ in powers of $x - 2$.

Put $x = v + 2$ in the given equation and obtain $\frac{d^2y}{dv^2} + (v+1)\frac{dy}{dv} + y = 0$ which is to be integrated in powers of v. Assume the series

$$y = A_0 + A_1v + A_2v^2 + A_3v^3 + A_4v^4 + \cdots\cdots + A_nv^n + \cdots\cdots. \quad \text{Then}$$
$$\frac{dy}{dv} = A_1 + 2A_2v + 3A_3v^2 + 4A_4v^3 + \cdots\cdots + nA_nv^{n-1} + \cdots\cdots,$$
$$\frac{d^2y}{dv^2} = 2A_2 + 6A_3v + 12A_4v^2 + \cdots\cdots + n(n-1)A_nv^{n-2} + \cdots\cdots, \quad \text{and}$$

$$\frac{d^2y}{dv^2} + (v+1)\frac{dy}{dv} + y = (2A_2 + A_1 + A_0) + (6A_3 + 2A_1 + 2A_2)v + (12A_4 + 3A_2 + 3A_3)v^2 + \cdots\cdots$$
$$+ [(n+2)(n+1)A_{n+2} + (n+1)A_n + (n+1)A_{n+1}]v^n + \cdots\cdots = 0.$$

Equating the coefficients of powers of v to zero, we obtain

$$A_2 = -\frac{1}{2}(A_0 + A_1), \quad A_3 = -\frac{1}{3}(A_1 + A_2) = \frac{1}{6}(A_0 - A_1), \quad A_4 = -\frac{1}{4}(A_2 + A_3) = \frac{1}{12}(A_0 + 2A_1), \quad \cdots\cdots$$

$$(n+2)(n+1)A_{n+2} + (n+1)A_n + (n+1)A_{n+1} = 0 \quad \text{and} \quad A_{n+2} = -\frac{1}{n+2}(A_n + A_{n+1}).$$

Thus, noting that $v = x-2$, the complete solution is

$$y = A_0[1 - \frac{1}{2}(x-2)^2 + \frac{1}{6}(x-2)^3 + \frac{1}{12}(x-2)^4 - \frac{1}{20}(x-2)^5 - \frac{1}{180}(x-2)^6 + \cdots\cdots]$$
$$+ A_1[(x-2) - \frac{1}{2}(x-2)^2 - \frac{1}{6}(x-2)^3 + \frac{1}{6}(x-2)^4 - \frac{1}{36}(x-2)^6 + \cdots\cdots].$$

SUPPLEMENTARY PROBLEMS

9. Solve $(1-x)y' = x^2 - y$ in powers of x.

Ans. $y = A_0(1-x) + x^3(\frac{1}{3} + \frac{1}{6}x + \frac{1}{10}x^2 + \cdots\cdots + \frac{1\cdot 2}{(n+2)(n+3)}x^n + \cdots\cdots)$

10. Solve $xy' = 1 - x + 2y$ in powers of $x-1$. Also integrate directly.
Hint: Let $x - 1 = z$ and solve $(z+1)\frac{dy}{dz} = -z + 2y$ in powers of z.

Ans. $y = A_0[1 + 2(x-1) + (x-1)^2] + \frac{1}{2} + (x-1)$

11. Solve $y' = 2x^2 + 3y$ in powers of x.

Ans. $y = A_0[1 + 3x + 9x^2/2 + 9x^3/2 + 27x^4/8 + \cdots\cdots] + (2x^3/3 + x^4/2 + \cdots\cdots)$

12. Solve $(x+1)y' = x^2 - 2x + y$ in powers of x.

Ans. $y = A_0(1+x) - x^2 + 2x^3/3 - x^4/3 + x^5/5 - 2x^6/15 + \cdots\cdots$

13. Solve $y'' + xy = 0$ in powers of x.

R.F. $A_n = -\frac{1}{n(n-1)}A_{n-3}$, $n \geq 3$; convergent for all x.

Ans. $y = A_0(1 - x^3/6 + x^6/180 - \cdots\cdots) + A_1(x - x^4/12 + x^7/504 - \cdots\cdots)$

14. Solve $y'' + 2x^2y = 0$ in powers of x.

R.F. $A_n = -\frac{2}{n(n-1)}A_{n-4}$; convergent for all x.

Ans. $y = A_0(1 - x^4/6 + x^8/168 - \cdots\cdots) + A_1(x - x^5/10 + x^9/360 - \cdots\cdots)$

15. Solve $y'' - xy' + x^2y = 0$ in powers of x.

R.F. $n(n-1)A_n - (n-2)A_{n-2} + A_{n-4} = 0$, $n \geq 4$.

Ans. $y = A_0(1 - x^4/12 - x^6/90 + x^8/3360 + \cdots) + A_1(x + x^3/6 - x^5/40 - x^7/144 - \cdots)$

16. Solve $(1-x^2)y'' - 2xy' + p(p+1)y = 0$, where p is a constant, in powers of x. (Legendre Equation)

R.F $A_n = \frac{(n-2-p)(n+p-1)}{n(n-1)}A_{n-2}$; convergent for $|x| < 1$.

Ans. $y = A_0(1 - \frac{p(p+1)}{2!}x^2 + \frac{(p-2)p(p+1)(p+3)}{4!}x^4 - \cdots\cdots\cdots\cdots)$
$+ A_1(x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-3)(p-1)(p+2)(p+4)}{5!}x^5 - \cdots\cdots\cdots\cdots)$

17. Solve $y'' + x^2y = 1 + x + x^2$ in powers of x. R.F. $A_n = -\frac{1}{n(n-1)}A_{n-4}$; convergent for all x.

Ans. $y = A_0(1 - x^4/12 + x^8/672 - \cdots\cdots) + A_1(x - x^5/20 + x^9/1440 - \cdots\cdots)$
$+ x^2/2 + x^3/6 + x^4/12 - x^6/60 - x^7/252 - x^8/672 + \cdots\cdots$

CHAPTER 26

Integration in Series

WHEN $x = a$ IS A SINGULAR POINT OF THE DIFFERENTIAL EQUATION

1) $$P_0(x)\, y'' + P_1(x)\, y' + P_2(x)\, y = 0,$$

in which $P_i(x)$ are polynomials, the procedure of the preceding chapter will not yield a complete solution in series about $x = a$.

EXAMPLE 1. For the equation $x^2y'' + (x^2 - x)y' + 2y = 0$, $x = 0$ is a singular point since $P_0(0) = 0$. If we assume a solution of the form

(i) $$y = A_0 + A_1x + A_2x^2 + A_3x^3 + \cdots\cdots\cdots$$

and substitute in the given equation, we obtain

$$2A_0 + A_1x + (2A_2 + A_1)x^2 + (5A_3 + 2A_2)x^3 + \cdots\cdots\cdots = 0.$$

In order that this relation be satisfied identically, it is necessary that $A_0 = 0,\ A_1 = 0, A_2 = 0,\ A_3 = 0, \cdots\cdot$; hence, there is no series of the form (i) satisfying the given equation.

A SINGULAR POINT $x = a$ OF 1) IS CALLED *REGULAR* IF, when 1) is put in the form

1′) $$y'' + \frac{R_1(x)}{x-a}y' + \frac{R_2(x)}{(x-a)^2}y = 0,$$

$R_1(x)$ and $R_2(x)$ can be expanded in Taylor series about $x = a$.

EXAMPLE 2. For the equation $(1 + x)y'' + 2xy' - 3y = 0$, $x = -1$ is a singular point since $P_0(-1) = 1 + (-1) = 0$. When the equation is put in the form

$$y'' + \frac{R_1(x)}{x+1}y' + \frac{R_2(x)}{(x+1)^2}y = y'' + \frac{2x}{x+1}y' + \frac{-3(x+1)}{(x+1)^2}y = 0,$$

the Taylor expansions about $x = -1$ of $R_1(x)$ and $R_2(x)$ are

$$R_1(x) = 2x = 2(x+1) - 2 \quad \text{and} \quad R_2(x) = -3(x+1).$$

Thus, $x = -1$ is a regular singular point.

EXAMPLE 3. For the equation $x^3y'' + x^2y' + y = 0$, $x = 0$ is a singular point. Writing the equation in the form

$$y'' + \frac{1}{x}y' + \frac{1/x}{x^2}y = 0,$$

it is seen that $R_2(x) = 1/x$ cannot be expanded in a Taylor series about $x = 0$. Thus, $x = 0$ is not a regular singular point.

WHEN $x = 0$ IS A REGULAR SINGULAR POINT OF 1), there always exists a series solution of the form

2) $$y = x^m \sum_{n=0}^{\infty} A_n x^n = A_0 x^m + A_1 x^{m+1} + A_2 x^{m+2} + \cdots + A_n x^{m+n} + \cdots,$$

with $A_0 \neq 0$, and we shall proceed to determine m and the A's so that 2) satisfies 1).

EXAMPLE 4. Solve in series $2xy'' + (x+1)y' + 3y = 0$.

Here, $x = 0$ is a regular singular point, Substituting

$$y = A_0 x^m + A_1 x^{m+1} + A_2 x^{m+2} + \cdots + A_n x^{m+n} + \cdots,$$

$$y' = mA_0 x^{m-1} + (m+1)A_1 x^m + (m+2)A_2 x^{m+1} + \cdots + (m+n)A_n x^{m+n-1} + \cdots,$$

$$y'' = (m-1)mA_0 x^{m-2} + m(m+1)A_1 x^{m-1} + (m+1)(m+2)A_2 x^m + \cdots + (m+n-1)(m+n)A_n x^{m+n-2} + \cdots$$

in the given differential equation, we have

(i) $$m(2m-1)A_0 x^{m-1} + [(m+1)(2m+1)A_1 + (m+3)A_0]x^m + [(m+2)(2m+3)A_2 + (m+4)A_1]x^{m+1} + \cdots + [(m+n)(2m+2n-1)A_n + (m+n+2)A_{n-1}]x^{m+n-1} + \cdots = 0.$$

Since $A_0 \neq 0$, the coefficient of the first term will vanish provided $m(2m-1) = 0$, that is, provided $m = 0$ or $m = ½$. However, without regrard to m, all terms after the first will vanish provided the A's satisfy the recursion formula

$$A_n = -\frac{m+n+2}{(m+n)(2m+2n-1)} A_{n-1}, \quad n \geq 1.$$

Thus, the series

2′) $$\bar{y} = A_0 x^m \Big[1 - \frac{m+3}{(m+1)(2m+1)}x + \frac{(m+3)(m+4)}{(m+1)(m+2)(2m+1)(2m+3)}x^2 - \frac{(m+4)(m+5)}{(m+1)(m+2)(2m+1)(2m+3)(2m+5)}x^3 + \cdots\Big]$$

satisfies the equation

(ii) $$2x\bar{y}'' + (x+1)\bar{y}' + 3\bar{y} = m(2m-1)A_0 x^{m-1}.$$

The right hand member of (ii) will be zero when $m = 0$ or $m = ½$. When $m = 0$, we have from 2′) with $A_0 = 1$, the particular solution

$$y_1 = 1 - 3x + 2x^2 - 2x^3/3 + \cdots,$$

and when $m = ½$ with $A_0 = 1$, the particular solution

$$y_2 = \sqrt{x}(1 - 7x/6 + 21x^2/40 - 11x^3/80 + \cdots).$$

The complete solution is then

$$y = Ay_1 + By_2$$
$$= A(1 - 3x + 2x^2 - 2x^3/3 + \cdots) + B\sqrt{x}(1 - 7x/6 + 21x^2/40 - 11x^3/80 + \cdots).$$

The coefficient of the lowest power of x in (i), (also, the coefficient in the right hand member of (ii), has the form $f(m)A_0$. The equation $f(m) = 0$ is called the *indicial equation*. The linearly independent solutions y_1 and y_2 above correspond to the distinct roots $m = 0$ and $m = \frac{1}{2}$ of this equation.

In the Solved Problems below, the roots of the indicial equation will be:

a) distinct and do not differ by an integer.
b) equal, or
c) distinct and differ by an integer.

The first case is illustrated in the example above and also in Problems 1–2.

When the roots m_1 and m_2 of the indicial equation are equal, the solutions corresponding will be identical. The complete solution is then obtained as

$$y = A\bar{y}\Big|_{m=m_1} + B\frac{\partial\bar{y}}{\partial m}\Big|_{m=m_1}$$ See Problems 3-4.

When the two roots $m_1 < m_2$ of the indicial equation differ by an integer, the greater of the roots m_2 will always yield a solution while the smaller root m_1 may or may not. In the latter case, we set $A_0 = B_0(m - m_1)$ and obtain the complete solution as

$$y = A\bar{y}\Big|_{m=m_1} + B\frac{\partial\bar{y}}{\partial m}\Big|_{m=m_1}$$ See Problems 5–7.

The series, expanded about $x = 0$, which appear in these complete solutions converge *always* in the region of the complex plane bounded by two circles centred at $x = 0$. The radius of one of the circles is arbitrarily small while that of the other extends to the finite singular point of the differential equation nearest $x = 0$. It is clear that the series obtained in Example 4 converge also at $x = 0$; moreover, since the differential equation has but one singular point $x = 0$, these series converge for all finite values of x.

THE COMPLETE SOLUTION OF

3) $$P_0(x)\,y'' + P_1(x)\,y' + P_2(x)\,y = Q$$

consists of the sum of the complementary function (complete solution of 1)), and any particular integral of 3). A procedure for obtaining a particular integral when Q is a sum of positive and negative powers of x is illustrated in Problem 8.

LARGE VALUES OF x. It is at times necessary to solve a differential equation 1) for large values of x. In such instances the series thus far obtained, even when valid for all finite values of x, are impractical.

To solve an equation in series convergent for large values of x or "about the point at infinity", we transform the given equation by means of the substitution

$$x = 1/z$$

and solve, if possible, the resulting equation in series near $z = 0$.

See Problems 9–10.

SOLVED PROBLEMS

1. Solve in series $2x^2y'' - xy' + (x^2+1)y = 0.$

Substituting

$$y = A_0x^m + A_1x^{m+1} + A_2x^{m+2} + \cdots\cdots + A_nx^{m+n} + \cdots\cdots$$

$$y' = mA_0x^{m-1} + (m+1)A_1x^m + (m+2)A_2x^{m+1} + \cdots\cdots + (m+n)A_nx^{m+n-1} + \cdots\cdots$$

$$y'' = (m-1)mA_0x^{m-2} + (m+1)mA_1x^{m-1} + (m+1)(m+2)A_2x^m + \cdots\cdots + (m+n-1)(m+n)A_nx^{m+n-2} + \cdots$$

in the given differential equation, we obtain

$$(m-1)(2m-1)A_0x^m + m(2m+1)A_1x^{m+1} + \{[(m+2)(2m+1)+1]A_2 + A_0\}x^{m+2} + \cdots\cdots$$
$$+ \{[(m+n)(2m+2n-3)+1]A_n + A_{n-2}\}x^{m+n} + \cdots\cdots = 0.$$

Now all terms except the first two will vanish if $A_2, A_3, \cdots\cdots$ satisfy the recursion formula

i) $$A_n = -\frac{1}{(m+n)(2m+2n-3)+1}A_{n-2}, \quad n \geq 2.$$

The roots of the indicial equation, $(m-1)(2m-1) = 0$, are $m = \frac{1}{2}, 1$, and for either value the first term will vanish. Since, however, neither of these values of m will cause the second term to vanish, we take $A_1 = 0$. Using 1), it follows that $A_1 = A_3 = A_5 = \cdots\cdots = 0$. Thus,

$$\bar{y} = A_0x^m\left(1 - \frac{1}{(m+2)(2m+1)+1}x^2 + \frac{1}{[(m+2)(2m+1)+1][(m+4)(2m+5)+1]}x^4 - \cdots\cdots\right)$$

satisfies $$2x^2\bar{y}'' - x\bar{y}' + (x^2+1)\bar{y} = (m-1)(2m-1)A_0x^m$$

and the right hand member will be 0 when $m = \frac{1}{2}$ or $m = 1$.

When $m = \frac{1}{2}$ and $A_0 = 1$, we have $y_1 = \sqrt{x}(1 - x^2/6 + x^4/168 - x^6/11088 + \cdots\cdots)$

and when $m = 1$, with $A_0 = 1$, we have $y_2 = x(1 - x^2/10 + x^4/360 - x^6/28080 + \cdots\cdots)$.

The complete solution is then

$$y = Ay_1 + By_2$$
$$= A\sqrt{x}(1 - x^2/6 + x^4/168 - x^6/11088 + \cdots) + Bx(1 - x^2/10 + x^4/360 - x^6/28080 + \cdots).$$

Since $x = 0$ is the only finite singular point, the series converge for all finite values of x.

2. Solve in series $3xy'' + 2y' + x^2y = 0.$

Substituting for y, y', and y'' as in the problem above, we have

$$m(3m-1)A_0x^{m-1} + (m+1)(3m+2)A_1x^m + (m+2)(3m+5)A_2x^{m+1} + [(m+3)(3m+8)A_3 + A_0]x^{m+2}$$
$$+ \cdots\cdots + [(m+n)(3m+3n-1)A_n + A_{n-3}]x^{m+n-1} + \cdots\cdots = 0.$$

All terms after the third will vanish if $A_3, A_4, \cdots\cdots$ satisfy the recursion formula

$$A_n = -\frac{1}{(m+n)(3m+3n-1)}A_{n-3}, \quad n \geq 3.$$

The roots of the indicial equation $m(3m-1)=0$ are $m = 0,\ 1/3$. Since neither will cause the second and third terms to vanish, we take $A_1 = A_2 = 0$. Then, using the recursion formula, $A_1 = A_4 = A_7 = \cdots = 0$ and $A_2 = A_5 = A_8 = \cdots = 0$. Thus the series

1) $$\bar{y} = A_0 x^m \left(1 - \frac{1}{(m+3)(3m+8)}x^3 + \frac{1}{(m+3)(m+6)(3m+8)(3m+17)}x^6 - \cdots\cdots\right)$$

satisfies $$3x\bar{y}'' + 2\bar{y}' + x^2\bar{y} = m(3m-1)A_0 x^{m-1}.$$

For $m = 0$, with $A_0 = 1$, we obtain from 1) $y_1 = 1 - x^3/24 + x^6/2448 - \cdots\cdots$ and for $m = 1/3$, with $A_0 = 1$, we obtain $y_2 = x^{1/3}(1 - x^3/30 + x^6/3420 - \cdots\cdots)$.

The complete solution is

$$y = Ay_1 + By_2 = A(1 - x^3/24 + x^6/2448 - \cdots\cdots) + Bx^{1/3}(1 - x^3/30 + x^6/3420 - \cdots\cdots).$$

The series converge for all finite values of x.

ROOTS OF INDICIAL EQUATION EQUAL.

3. Solve in series $xy'' + y' - y = 0$.

Substituting for y, y', and y'' as in Problems 1 and 2 above, we obtain

$$m^2A_0x^{m-1} + [(m+1)^2A_1 - A_0]x^m + [(m+2)^2A_2 - A_1]x^{m+1} + \cdots\cdots + [(m+n)^2A_n - A_{n-1}]x^{m+n-1} + \cdots\cdots = 0.$$

All terms except the first will vanish if $A_1, A_2, \cdots\cdots$ satisfy the re[illegible]form[illegible]

1) $$A_n = \frac{1}{(m+n)^2}A_{n-1}, \quad n \geq [illegible]$$

Thus,

$$\bar{y} = A_0x^m\left(1 + \frac{1}{(m+1)^2}x + \frac{1}{(m+1)^2(m+2)^2}x^2 + \frac{1}{(m+1)^2(m+2)^2(m+3)^2}x^3 + \cdots\cdots\right)$$

satisfies

2) $$x\bar{y}'' + \bar{y}' - \bar{y} = m^2A_0x^{m-1}.$$

The roots of the indicial equation are $m = 0,0$. Hence there corresponds but one series solution satisfying 2) with $m = 0$. However, regarding $\bar{y}$ as a function of the independent variables x and m,

$$\frac{\partial\bar{y}'}{\partial m} = \frac{\partial}{\partial m}\left(\frac{\partial\bar{y}}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial\bar{y}}{\partial m}\right) = \left(\frac{\partial\bar{y}}{\partial m}\right)'$$

and $$\frac{\partial\bar{y}''}{\partial m} = \frac{\partial}{\partial m}\frac{\partial}{\partial x}\left(\frac{\partial\bar{y}}{\partial x}\right) = \frac{\partial}{\partial x}\frac{\partial}{\partial m}\left(\frac{\partial\bar{y}}{\partial x}\right) = \frac{\partial}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial\bar{y}}{\partial m}\right) = \left(\frac{\partial\bar{y}}{\partial m}\right)'',$$

and we have by differentiating 2) partially with respect to m,

3) $$x\left(\frac{\partial\bar{y}}{\partial m}\right)'' + \left(\frac{\partial\bar{y}}{\partial m}\right)' - \left(\frac{\partial\bar{y}}{\partial m}\right) = 2mA_0x^{m-1} + m^2A_0x^{m-1}\ln x.$$

From 2) and 3) it follows that $y_1 = \bar{y}\big|_{m=0}$ and $y_2 = \frac{\partial\bar{y}}{\partial m}\Big|_{m=0}$ are solutions of the given differential equation. Taking $A_0 = 1$, we find

$$\frac{\partial \bar{y}}{\partial m} = x^m \ln x\,[1 + \frac{1}{(m+1)^2}x + \frac{1}{(m+1)^2(m+2)^2}x^2 + \frac{1}{(m+1)^2(m+2)^2(m+3)^2}x^3 + \cdots\cdots\cdots]$$

$$+ x^m[-\frac{2}{(m+1)^3}x - (\frac{2}{(m+1)^3(m+2)^2} + \frac{2}{(m+1)^2(m+2)^3})x^2 - (\frac{2}{(m+1)^3(m+2)^2(m+3)^2}$$

$$+ \frac{2}{(m+1)^2(m+2)^3(m+3)^2} + \frac{2}{(m+1)^2(m+2)^2(m+3)^3})x^3 - \cdots\cdots\cdots]$$

$$= \bar{y}\ln x - 2x^m[\frac{1}{(m+1)^3}x + (\frac{1}{(m+1)^3(m+2)^2} + \frac{1}{(m+1)^2(m+2)^3})x^2$$

$$+ (\frac{1}{(m+1)^3(m+2)^2(m+3)^2} + \frac{1}{(m+1)^2(m+2)^3(m+3)^2} + \frac{1}{(m+1)^2(m+2)^2(m+3)^3})x^3 + \cdots].$$

Then $y_1 = \bar{y}\big|_{m=0} = 1 + x + \frac{x^2}{(2!)^2} + \frac{x^3}{(3!)^2} + \cdots\cdots\cdots,$

$$y_2 = \frac{\partial \bar{y}}{\partial m}\bigg|_{m=0} = y_1 \ln x - 2[x + \frac{1}{(2!)^2}(1+\frac{1}{2})x^2 + \frac{1}{(3!)^2}(1+\frac{1}{2}+\frac{1}{3})x^3 + \cdots\cdots\cdots],$$

and the complete solution is

$$y = Ay_1 + By_2 = (A + B\ln x)[1 + x + \frac{1}{(2!)^2}x^2 + \frac{1}{(3!)^2}x^3 + \cdots\cdots]$$

$$- 2B[x + \frac{1}{(2!)^2}(1+\frac{1}{2})x^2 + \frac{1}{(3!)^2}(1+\frac{1}{2}+\frac{1}{3})x^3 + \cdots\cdots].$$

The series converge for all finite values of $x \neq 0$.

4. Solve in series $xy'' + y' + x^2y = 0$.

Substituting for y, y', and y'', we obtain

$$m^2A_0x^{m-1} + (m+1)^2A_1x^m + (m+2)^2A_2x^{m+1} + [(m+3)^2A_3 + A_0]x^{m+2} + \cdots\cdots\cdots$$
$$+ [(m+n)^2A_n + A_{n-3}]x^{m+n-1} + \cdots\cdots\cdots = 0.$$

The two roots of the indicial equation are equal. We take $A_0 = 1$, $A_1 = A_2 = 0$, and the remaining A's satisfying the recursion formula $A_n = -\frac{1}{(m+n)^2}A_{n-3}$.

Then $A_1 = A_4 = A_7 = \cdots = 0$, $A_2 = A_5 = A_8 = \cdots = 0$,

$$\bar{y} = x^m(1 - \frac{1}{(m+3)^2}x^3 + \frac{1}{(m+3)^2(m+6)^2}x^6 - \frac{1}{(m+3)^2(m+6)^2(m+9)^2}x^9 + \cdots\cdots\cdots)$$

and, following the procedure of Problem 3 above,

$$\frac{\partial \bar{y}}{\partial m} = \bar{y}\ln x + 2x^m[\frac{1}{(m+3)^3}x^3 - (\frac{1}{(m+3)^3(m+6)^2} + \frac{1}{(m+3)^2(m+6)^3})x^6 +$$

$$(\frac{1}{(m+3)^3(m+6)^2(m+9)^2} + \frac{1}{(m+3)^2(m+6)^3(m+9)^2} + \frac{1}{(m+3)^2(m+6)^2(m+9)^3})x^9 - \cdots].$$

Using the root $m = 0$ of the indicial equation,

$$y_1 = \bar{y}\big|_{m=0} = 1 - \frac{1}{3^2}x^3 + \frac{1}{3^4(2!)^2}x^6 - \frac{1}{3^6(3!)^2}x^9 + \cdots\cdots\cdots$$

and $y_2 = \left.\dfrac{\partial\bar{y}}{\partial m}\right|_{m=0} = y_1 \ln x + 2\left[\dfrac{1}{3^3}x^3 - \dfrac{1}{3^5(2!)^2}(1+\dfrac{1}{2})x^6 + \dfrac{1}{3^7(3!)^2}(1+\dfrac{1}{2}+\dfrac{1}{3})x^9 - \cdots\cdots\right].$

The complete solution is

$$y = Ay_1 + By_2 = (A + B\ln x)\left[1 - \frac{1}{3^2}x^3 + \frac{1}{3^4(2!)^2}x^6 - \frac{1}{3^6(3!)^2}x^9 + \cdots\cdots\right]$$
$$+ 2B\left[\frac{1}{3^3}x^3 - \frac{1}{3^5(2!)^2}(1+\frac{1}{2})x^6 + \frac{1}{3^7(3!)^2}(1+\frac{1}{2}+\frac{1}{3})x^9 - \cdots\cdots\cdots\right].$$

The series converge for all finite values of $x \neq 0$.

ROOTS OF INDICIAL EQUATION DIFFERING BY AN INTEGER.

5. Solve in series $xy'' - 3y' + xy = 0$.

Substituting for y, y', and y'', we obtain

$$(m-4)mA_0x^{m-1} + (m-3)(m+1)A_1x^m + [(m-2)(m+2)A_2 + A_0]x^{m+1} + \cdots\cdots$$
$$+ [(m+n-4)(m+n)A_n + A_{n-2}]x^{m+n-1} + \cdots\cdots = 0.$$

The roots of the indicial equation are $m = 0, 4$, and we have the second special case mentioned above since the difference of the two roots is an integer. We take $A_1 = 0$ and choose the remaining A's to satisfy the recursion formula

$$A_n = -\frac{1}{(m+n-4)(m+n)}A_{n-2}, \quad n \geq 2.$$

It is clear that this relation yields finite values when $m = 4$, the larger of the roots, but when $m = 0$, $A_4 \to \infty$. Since the root $m = 0$ gives difficulty, we replace A_0 by $B_0(m-0) = B_0m$ and note that the series

$$\bar{y} = A_0x^m\left[1 - \frac{1}{(m-2)(m+2)}x^2 + \frac{1}{m(m-2)(m+2)(m+4)}x^4 - \frac{1}{m(m-2)(m+2)^2(m+4)(m+6)}x^6\right.$$
$$\left.+ \frac{1}{m(m-2)(m+2)^2(m+4)^2(m+6)(m+8)}x^8 - \cdots\cdots\cdots\right]$$

$$= B_0x^m\left[m - \frac{m}{(m-2)(m+2)}x^2 + \frac{1}{(m-2)(m+2)(m+4)}x^4 - \frac{1}{(m-2)(m+2)^2(m+4)(m+6)}x^6\right.$$
$$\left.+ \frac{1}{(m-2)(m+2)^2(m+4)^2(m+6)(m+8)}x^8 - \cdots\cdots\cdots\right]$$

satisfies the equation

$$x\bar{y}'' - 3\bar{y}' + x\bar{y} = (m-4)mA_0x^{m-1} = (m-4)m^2B_0x^{m-1}.$$

Since the right hand member contains the factor m^2, it follows by the argument made in Problem 3 that $\bar{y}$ and $\frac{\partial \bar{y}}{\partial m}$, with $m = 0$, are solutions of the given differential equation. We find

$$\frac{\partial \bar{y}}{\partial m} = \bar{y}\ln x + B_0 x^m \Big[1 + \frac{m^2+4}{[(m-2)(m+2)]^2}x^2 - \frac{1}{(m-2)(m+2)(m+4)}\Big(\frac{1}{m-2} + \frac{1}{m+2} + \frac{1}{m+4}\Big)x^4$$
$$+ \frac{1}{(m-2)(m+2)^2(m+4)(m+6)}\Big(\frac{1}{m-2} + \frac{2}{m+2} + \frac{1}{m+4} + \frac{1}{m+6}\Big)x^6$$
$$- \frac{1}{(m-2)(m+2)^2(m+4)^2(m+6)(m+8)}\Big(\frac{1}{m-2} + \frac{2}{m+2} + \frac{2}{m+4} + \frac{1}{m+6} + \frac{1}{m+8}\Big)x^8 + \cdots\cdots\Big].$$

Using the root $m = 0$, with $B_0 = 1$, we obtain

$$y_1 = \bar{y}\Big|_{m=0} = -\frac{1}{2\cdot 2\cdot 4}x^4 + \frac{1}{2\cdot 2^2\cdot 4\cdot 6}x^6 - \frac{1}{2\cdot 2^2\cdot 4^2\cdot 6\cdot 8}x^8 + \cdots\cdots\cdots\cdots$$

and

$$y_2 = \frac{\partial \bar{y}}{\partial m}\Big|_{m=0} = y_1 \ln x + 1 + \frac{1}{2^2}x^2 + \frac{1}{2^5\, 2!}x^4 - \frac{1}{2^6\, 3!\, 1!}\Big(1 + \frac{1}{2} + \frac{1}{3}\Big)x^6$$
$$+ \frac{1}{2^8\, 4!\, 2!}\Big[\Big(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\Big) + \frac{1}{2}\Big]x^8 - \frac{1}{2^{10}\, 5!\, 3!}\Big[\Big(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\Big) + \Big(\frac{1}{2}+\frac{1}{3}\Big)\Big]x^{10} + \cdots,$$

The complete solution is

$$y = Ay_1 + By_2$$
$$= (A + B\ln x)\Big\{-\frac{1}{2^5\, 2!}x^4 + \frac{1}{2^5\, 3!\, 1!}x^6 - \frac{1}{2^7\, 4!\, 2!}x^8 + \cdots\cdots\Big\}$$
$$+ B\Big\{1 + \frac{1}{2^2}x^2 + \frac{1}{2^5\, 2!}x^4 - \frac{1}{2^6\, 3!\, 1!}\Big(1+\frac{1}{2}+\frac{1}{3}\Big)x^6 + \frac{1}{2^8\, 4!\, 2!}\Big[\Big(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\Big) + \frac{1}{2}\Big]x^8$$
$$- \frac{1}{2^{10}\, 5!\, 3!}\Big[\Big(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\Big) + \Big(\frac{1}{2}+\frac{1}{3}\Big)\Big]x^{10} + \cdots\cdots\Big].$$

The series converge for all finite values of $x \neq 0$.

6. Solve in series $(x - x^2)y'' - 3y' + 2y = 0$.

Substituting for y, y', and y'', we obtain

$$(m-4)mA_0x^{m-1} + [(m-3)(m+1)A_1 - (m-2)(m+1)A_0]x^m + [(m-2)(m+2)A_2 - (m-1)(m+2)A_1]x^{m+1}$$
$$+ \cdots\cdots + [(m+n-4)(m+n)A_n - (m+n-3)(m+n)A_{n-1}]x^{m+n-1} + \cdots\cdots = 0.$$

The recursion formula is $A_n = \frac{m+n-3}{m+n-4}A_{n-1}$ so that

$$1)\quad \bar{y} = A_0x^m\Big[1 + \frac{m-2}{m-3}x + \frac{m-1}{m-3}x^2 + \frac{m}{m-3}x^3 + \frac{m+1}{m-3}x^4 + \frac{m+2}{m-3}x^5 + \frac{m+3}{m-3}x^6 + \cdots\cdots\Big]$$

satisfies the differential equation

$$(x - x^2)\bar{y}'' - 3\bar{y}' + 2\bar{y} = (m-4)mA_0x^{m-1}.$$

The roots $m = 0, 4$ of the indicial equation differ by an integer. However, when $m = 0$ the expected vanishing of the denominator in the coefficient of x^4 does not occur since the factor m appears in both numerator and denominator and thus cancels out. Note that the coefficient of x^3 is zero when $m = 0$.

Thus, with $A_0 = 1$,

$$y_1 = \bar{y}\big|_{m=0} = 1 + 2x/3 + x^2/3 + 0 - x^4/3 - 2x^5/3 - 3x^6/3 - 4x^7/3 - \cdots\cdots$$

and

$$y_2 = \bar{y}\big|_{m=4} = x^4(1 + 2x + 3x^2 + 4x^3 + \cdots\cdots)$$

so that $y_1 = (1 + 2x/3 + x^2/3) - y_2/3$.

The complete solution is

$$y = C_1y_1 + C_2y_2 = C_1(1 + 2x/3 + x^2/3) + (C_2 - C_1/3)y_2$$

$$= A(x^2 + 2x + 3) + Bx^4(1 + 2x + 3x^2 + 4x^3 + \cdots\cdots)$$

$$= A(x^2 + 2x + 3) + B\frac{x^4}{(1-x)^2}.$$

There are finite singular points at $x = 0$ and $x = 1$. The series converge for $|x| < 1$.

7. Solve in series $xy'' + (x-1)y' - y = 0$.

Substituting for y, y', and y'', we obtain

$$(m-2)mA_0x^{m-1} + [(m-1)(m+1)A_1 + (m-1)A_0]x^m + [m(m+2)A_2 + mA_1]x^{m+1} + \cdots\cdots$$

$$+ [(m+n-2)(m+n)A_n + (m+n-2)A_{n-1}]x^{m+n-1} + \cdots\cdots = 0.$$

The roots of the indicial equation are $m = 0, 2$ which differ by an integer. We choose the A's to satisfy the recursion formula

$$A_n = -\frac{m+n-2}{(m+n-2)(m+n)}A_{n-1} = -\frac{1}{m+n}A_{n-1}.$$

At this point we see that no $A_i \to \infty$ for $m = 0$, the smaller root, as in Problem 5. This is due, of course, to the fact that the factor $m+n-2$ cancels out. Thus, since

$$\bar{y} = A_0x^m\left[1 - \frac{1}{m+1}x + \frac{1}{(m+1)(m+2)}x^2 - \frac{1}{(m+1)(m+2)(m+3)}x^3 + \cdots\cdots\right]$$

satisfies

$$x\bar{y}'' + (x-1)\bar{y}' - \bar{y} = (m-2)mA_0x^{m-1},$$

we obtain, with $A_0 = 1$ and $m = 0$, $m = 2$ respectively,

$$y_1 = \bar{y}\big|_{m=0} = 1 - x + x^2/2! - x^3/3! + \cdots\cdots = e^{-x}$$

and

$$y_2 = \bar{y}\big|_{m=2} = x^2 - 2x^3/3! + 2x^4/4! - 2x^5/5! + \cdots\cdots = 2(e^{-x} + x - 1).$$

The complete solution is $y = C_1e^{-x} + C_2[2(e^{-x} + x - 1)] = Ae^{-x} + B(1-x)$, convergent for all finite values of x.

PARTICULAR INTEGRAL.

8. Solve $(x^2-x)y'' + 3y' - 2y = x + 3/x^2$ near $x = 0$.

Substituting for y, y', and y'' as in Problem 6, we obtain the condition

1) $$m(4-m)A_0x^{m-1} + [(m+1)(3-m)A_1 + (m+1)(m-2)A_0]x^m + \cdots\cdots + [(m+n)(4-m-n)A_n + (m+n)(m+n-3)A_{n-1}]x^{m+n-1} + \cdots\cdots = x + 3/x^2.$$

To find the complementary function, we set the left member of 1) equal to zero and proceed as before.

The recursion formula is $A_n = \dfrac{m+n-3}{m+n-4}A_{n-1}$, and thus

$$\bar{y} = A_0x^m\left(1 + \frac{m-2}{m-3}x + \frac{m-1}{m-3}x^2 + \frac{m}{m-3}x^3 + \frac{m+1}{m-3}x^4 + \cdots\cdots\right)$$

satisfies

2) $$(x^2-x)\bar{y}'' + 3\bar{y}' - 2\bar{y} = m(4-m)A_0x^{m-1}.$$

The right hand member of 2) will be 0 when $m = 0, 4$. $m = 0$ with $A_0 = 1$, we have

$$y_1 = 1 + 2x/3 + x^2/3 - x^4/3 - 2x^5/3 - 3x^6/3 - 4x^7/3 - \cdots\cdots$$

and for $m = 4$ with $A_0 = 1$, we have

$$y_2 = x^4(1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots\cdots).$$

Then $y_1 = (1 + 2x/3 + x^2/3) - y_2/3$ and (See Problem 6) the complementary function is

$$y = A(x^2 + 2x + 3) + Bx^4/(1-x)^2.$$

In finding a particular integral, we consider each of the terms of the right member of the given differential equation separately. Setting the right member of 2) equal to x, that is,

$$m(4-m)A_0x^{m-1} = x, \text{ identically},$$

we have $m = 2$ and $A_0 = \frac{1}{4}$. For $m = 2$, the recursion formula is $A_n = \dfrac{n-1}{n-2}A_{n-1}$; thus, $A_1 = A_2 = A_3 = \cdots\cdots = 0$. The particular integral corresponding to the term x is $x^2/4$.

Again, setting the right member of 2) equal to $3/x^2$, that is,

$$m(4-m)A_0x^{m-1} = 3/x^2, \text{ identically},$$

we have $m = -1$ and $A_0 = -3/5$. For $m = -1$, $A_n = \dfrac{n-4}{n-5}A_{n-1}$; thus, $A_1 = \frac{3}{4}A_0$, $A_2 = \frac{1}{2}A_0$, $A_3 = \frac{1}{4}A_0$, $A_4 = A_5 = A_6 = \cdots\cdots = 0$. The particular integral corresponding to the term $3/x^2$ is $-\frac{3}{5}x^{-1}(1 + \frac{3}{4}x + \frac{1}{2}x^2 + \frac{1}{4}x^3)$. The required complete solution is

$$y = A(x^2 + 2x + 3) + \frac{Bx^4}{(1-x)^2} - \frac{3}{5x} - \frac{9}{20} - \frac{3}{10}x + \frac{1}{10}x^2$$

$$= C(x^2 + 2x + 3) + \frac{Bx^4}{(1-x)^2} + \frac{1}{4}x^2 - \frac{3}{5x}.$$

Note. A partial check of the work is obtained by showing that the particular integral $y = x^2/4 - 3/5x$ satisfies the differential equation.

Since $x = 1$ is the only other finite singular point, the series converge in the annular region bounded by a circle of arbitrarily small radius and a circle of radius one, both centred at $x = 0$.

EXPANSION FOR LARGE VALUES OF x

9. Solve $2x^2(x-1)y'' + x(3x+1)y' - 2y = 0$ in series convergent near $x = \infty$.

The substitution

$$x = \frac{1}{z}, \qquad y' = \frac{dy}{dz}\frac{dz}{dx} = -\frac{1}{x^2}\frac{dy}{dz} = -z^2\frac{dy}{dz}, \qquad y'' = \frac{2}{z^3}\frac{dy}{dz} + \frac{1}{z^4}\frac{d^2y}{dz^2} = z^4\frac{d^2y}{dz^2} + 2z^3\frac{dy}{dz}$$

transforms the given equation into

$$2(z-z^2)\frac{d^2y}{dz^2} + (1-5z)\frac{dy}{dz} - 2y = 0$$

for which $z = 0$, the transform of $x = \infty$, is a regular singular point. We next assume the series solution

$$y = A_0 z^m + A_1 z^{m+1} + A_2 z^{m+2} + \cdots\cdots + A_n z^{m+n} + \cdots\cdots$$

and obtain the condition

$$m(2m-1)A_0 z^{m-1} + \{(m+1)(2m+1)A_1 - (2m^2+3m+2)A_0\}z^m + \cdots\cdots$$
$$+ \{(m+n)(2m+2n-1)A_n - [2(m+n)^2 - (m+n) + 1]A_{n-1}\}z^{m+n-1} + \cdots = 0.$$

The recursion formula is $A_n = \dfrac{2(m+n)^2 - (m+n) + 1}{(m+n)(2m+2n-1)} A_{n-1}$, and thus the series

$$\bar{y} = A_0 z^m\left(1 + \frac{2m^2+3m+2}{(m+1)(2m+1)}z + \frac{2m^2+3m+2}{(m+1)(2m+1)}\cdot\frac{2m^2+7m+7}{(m+2)(2m+3)}z^2 + \cdots\cdots\right)$$

satisfies

$$2(z-z^2)\frac{d^2\bar{y}}{dz^2} + (1-5z)\frac{d\bar{y}}{dz} - 2\bar{y} = m(2m-1)A_0 z^{m-1}.$$

For $m = 0$, with $A_0 = 1$, we have

$$y_1 = 1 + 2z + 7z^2/3 + 112z^3/45 + \cdots\cdots = 1 + \frac{2}{x} + \frac{7}{3x^2} + \frac{112}{45x^3} + \cdots\cdots,$$

and for $m = \frac{1}{2}$, with $A_0 = 1$, we have

$$y_2 = z^{\frac{1}{2}}(1 + 4z/3 + 22z^2/15 + 484z^3/315 + \cdots\cdots) = x^{-\frac{1}{2}}\left(1 + \frac{4}{3x} + \frac{22}{15x^2} + \frac{484}{315x^3} + \cdots\cdots\right).$$

The complete solution is

$$y = Ay_1 + By_2 = A\left(1 + \frac{2}{x} + \frac{7}{3x^2} + \frac{112}{45x^3} + \cdots\right) + Bx^{-\frac{1}{2}}\left(1 + \frac{4}{3x} + \frac{22}{15x^2} + \frac{484}{315x^3} + \cdots\right).$$

The series in z converge for $|z| < 1$, that is, for all z inside a circle of radius 1, centred at $z = 0$.

The series in x converge for $|x| > 1$, that is, for all x outside a circle of radius 1, centred at $x = 0$.

10. Solve $x^3y'' + x(1-x)y' + y = 0$ in series convergent near $x = \infty$.

Making the substitution $x = 1/z$ as in Problem 9, we obtain

$$1)\qquad z\frac{d^2y}{dz^2} + (3-z)\frac{dy}{dz} + y = 0$$

for which $z = 0$ is a regular singular point. We next assume the series solution

$$y = A_0z^m + A_1z^{m+1} + A_2z^{m+2} + \cdots\cdots + A_nz^{m+n} + \cdots\cdots,$$

substitute in 1), and obtain

$$m(m+2)A_0z^{m-1} + [(m+1)(m+3)A_1 - (m-1)A_0]z^m + [(m+2)(m+4)A_2 - mA_1]z^{m+1} + \cdots\cdots$$
$$+ [(m+n)(m+n+2)A_n - (m+n-2)A_{n-1}]z^{m+n-1} + \cdots\cdots = 0.$$

The roots of the indicial equation are $m = 0, -2$ and differ by an integer. From the recursion formula $A_n = \dfrac{m+n-2}{(m+n)(m+n+2)}A_{n-1}$ it is seen that $A_2 \to \infty$ when $m = -2$. We replace A_0 by $B_0(m+2)$ and note that the series

$$\bar{y} = B_0z^m\Big[(m+2) + \frac{(m-1)(m+2)}{(m+1)(m+3)}z + \frac{(m-1)m}{(m+1)(m+3)(m+4)}z^2 + \frac{(m-1)m}{(m+3)^2(m+4)(m+5)}z^3$$
$$+ \frac{(m-1)m(m+2)}{(m+3)^2(m+4)^2(m+5)(m+6)}z^4 + \cdots\cdots\Big]$$

satisfies the equation
$$z\frac{d^2\bar{y}}{dz^2} + (3-z)\frac{d\bar{y}}{dz} + \bar{y} = B_0m(m+2)^2z^{m-1}.$$

Hence,

$$\frac{\partial\bar{y}}{\partial m} = \bar{y}\ln z + B_0z^m\Big\{1 + \Big[\frac{2m+1}{(m+1)(m+3)} - \frac{(m-1)(m+2)}{(m+1)(m+3)}\Big(\frac{1}{m+1} + \frac{1}{m+3}\Big)\Big]z +$$
$$\Big[\frac{2m-1}{(m+1)(m+3)(m+4)} - \frac{(m-1)m}{(m+1)(m+3)(m+4)}\Big(\frac{1}{m+1} + \frac{1}{m+3} + \frac{1}{m+4}\Big)\Big]z^2 +$$
$$\Big[\frac{2m-1}{(m+3)^2(m+4)(m+5)} - \frac{(m-1)m}{(m+3)^2(m+4)(m+5)}\Big(\frac{2}{m+3} + \frac{1}{m+4} + \frac{1}{m+5}\Big)\Big]z^3 +$$
$$\Big[\frac{3m^2+2m-2}{(m+3)^2(m+4)^2(m+5)(m+6)} - \frac{(m-1)m(m+2)}{(m+3)^2(m+4)^2(m+5)(m+6)}\Big(\frac{2}{m+3} + \frac{2}{m+4} + \frac{1}{m+5} + \frac{1}{m+6}\Big)\Big]z^4$$
$$+ \cdots\cdots\Big\}$$

also satisfies this equation.

Using $m = -2$ with $B_0 = 1$, we find

$$y_1 = \bar{y}\Big|_{m=-2} = z^{-2}(-3z^2 + z^3) = \frac{1}{x} - 3 \quad\text{and}$$

$$y_2 = \frac{\partial\bar{y}}{\partial m}\Big|_{m=-2} = y_1\ln z + z^{-2}(1 + 3z + 4z^2 - 11z^3/3 + z^4/8 + \cdots)$$
$$= y_1\ln\frac{1}{x} + x^2 + 3x + 4 - 11/3x + 1/8x^2 + \cdots\cdots.$$

The complete solution is

$$y = Ay_1 + By_2 = \Big(A + B\ln\frac{1}{x}\Big)(1/x - 3) + B(x^2 + 3x + 4 - 11/3x + 1/8x^2 + \cdots\cdots).$$

The series converge for all values of $x \neq 0$.

SUPPLEMENTARY PROBLEMS

Solve in series near $x = 0$.

11. $2(x^2 + x^3)y'' - (x - 3x^2)y' + y = 0.$

R.F. $A_n = -A_{n-1}$

Ans. $y = (A\sqrt{x} + Bx)(1 - x + x^2 - x^3 + \cdots\cdots).$ Converges for $|x| < 1.$

12. $4xy'' + 2(1-x)y' - y = 0.$

R.F. $A_n = \dfrac{1}{2(m+n)} A_{n-1}$

Ans. $y = A(1 + \dfrac{x}{2\cdot 1!} + \dfrac{x^2}{2^2\cdot 2!} + \dfrac{x^3}{2^3\cdot 3!} + \cdots\cdots) + B\sqrt{x}(1 + \dfrac{x}{1\cdot 3} + \dfrac{x^2}{1\cdot 3\cdot 5} + \dfrac{x^3}{1\cdot 3\cdot 5\cdot 7} + \cdots\cdots).$

Converges for all finite values of x.

13. $2x^2y'' - xy' + (1 - x^2)y = 0.$

R.F. $A_n = \dfrac{1}{(m+n-1)(2m+2n-1)} A_{n-2}$, n even; $A_n = 0$, n odd.

Ans. $y = Ax(1 + \dfrac{x^2}{2\cdot 5} + \dfrac{x^4}{2\cdot 4\cdot 5\cdot 9} + \dfrac{x^6}{2\cdot 4\cdot 6\cdot 5\cdot 9\cdot 13} + \cdots\cdots\cdots\cdots)$

$+ B\sqrt{x}(1 + \dfrac{x^2}{2\cdot 3} + \dfrac{x^4}{2\cdot 4\cdot 3\cdot 7} + \dfrac{x^6}{2\cdot 4\cdot 6\cdot 3\cdot 7\cdot 11} + \cdots\cdots\cdots\cdots).$

Converges for all finite values of x.

14. $xy'' + y' + xy = 0.$

R.F. $A_n = -\dfrac{1}{(m+n)^2} A_{n-2}$, n even; $A_n = 0$, n odd.

Ans. $y = (A + B\ln x)(1 - \dfrac{x^2}{2^2} + \dfrac{x^4}{2^2\cdot 4^2} - \dfrac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots\cdots\cdots)$

$+ B[\dfrac{x^2}{2^2} - \dfrac{x^4}{2^2\cdot 4^2}(1 + \dfrac{1}{2}) + \dfrac{x^6}{2^2\cdot 4^2\cdot 6^2}(1 + \dfrac{1}{2} + \dfrac{1}{3}) - \cdots\cdots\cdots].$

Converges for all finite values of $x \neq 0$.

15. $x^2y'' - xy' + (x^2 + 1)y = 0.$ R.F. $A_n = -\dfrac{1}{(m+n-1)^2} A_{n-2}$, n even; $A_n = 0$, n odd.

Ans. $y = (A + B\ln x)x(1 - \dfrac{x^2}{2^2} + \dfrac{x^4}{2^4(2!)^2} - \dfrac{x^6}{2^6(3!)^2} + \cdots\cdots\cdots)$

$+ Bx[\dfrac{x^2}{2^2} - \dfrac{x^4}{2^4(2!)^2}(1 + \dfrac{1}{2}) + \dfrac{x^6}{2^6(3!)^2}(1 + \dfrac{1}{2} + \dfrac{1}{3}) + \cdots\cdots\cdots].$

Converges for all finite values of $x \neq 0$.

16. $xy'' - 2y' + y = 0.$ R.F. $A_n = -\dfrac{1}{(m+n-3)(m+n)}A_{n-1}$

Ans. $y = (A + B\ln x)\left(-\dfrac{x^3}{12} + \dfrac{x^4}{48} - \dfrac{x^5}{480} + \dots\right) + B\left(1 + \dfrac{x}{2} + \dfrac{x^2}{4} + \dfrac{x^3}{36} - \dfrac{19x^4}{576} + \dfrac{137x^5}{28800} - \dots\right).$

Converges for all finite values of $x \neq 0$.

17. $xy'' + 2y' + xy = 0.$ R.F. $A_n = -\dfrac{1}{(m+n)(m+n+1)}A_{n-2}$, n even; $A_n = 0$, n odd.

Ans. $y = Ax^{-1}\left(1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dots\dots\right) + B\left(1 - \dfrac{x^2}{3!} + \dfrac{x^4}{5!} - \dots\dots\right).$

Converges for all finite values of $x \neq 0$.

18. $x^2(x+1)y'' + x(x+1)y' - y = 0.$

Singular points: $x = 0, -1$. R.F. $A_n = -\dfrac{m+n-1}{m+n+1}A_{n-1}$.

Ans. $y = Ax(1 - x/3 + x^2/6 - x^3/10 + \dots\dots) + Bx^{-1}(1 + x).$

Converges in the annular region bounded by a circle of arbitrarily small radius and a circle of radius one, both centred at $x = 0$.

19. $2xy'' + y' - y = x + 1.$ R.F. $A_n = \dfrac{1}{(m+n)(2m+2n-1)}A_{n-1}$

Ans. $y = A(1 + x + x^2/6 + x^3/90 + \dots\dots) + B\sqrt{x}(1 + x/3 + x^2/30 + x^3/630 + \dots\dots)$

$+ \dfrac{1}{6}x^2(1 + x/15 + x^2/420 + x^3/18900 + \dots\dots) - 1.$

Converges for all finite values of x.

Solve in series near $x = \infty$.

20. $2x^3y'' + x^2y' + y = 0.$ R.F. $A_n = -\dfrac{1}{(m+n)(2m+2n+1)}A_{n-1}$

Ans. $y = A\left(1 - \dfrac{1}{3x} + \dfrac{1}{30x^2} - \dfrac{1}{630x^3} + \dots\dots\right) + B\sqrt{x}\left(1 - \dfrac{1}{x} + \dfrac{1}{6x^2} - \dfrac{1}{90x^3} + \dots\dots\right).$

Converges for all finite values of $x \neq 0$.

21. $x^3y'' + (x^2 + x)y' - y = 0.$ R.F. $A_n = \dfrac{1}{m+n}A_{n-1}$

Ans. $y = \left(A + B\ln\dfrac{1}{x}\right)\left(1 + \dfrac{1}{x} + \dfrac{1}{2x^2} + \dfrac{1}{6x^3} + \dots\right) + B\left[\dfrac{1}{x} + \dfrac{1}{2x^2}\left(1 + \dfrac{1}{2}\right) + \dfrac{1}{6x^3}\left(1 + \dfrac{1}{2} + \dfrac{1}{3}\right) + \dots\right].$

Converges for all finite values of $x \neq 0$.

CHAPTER 27

The Legendre, Bessel, and Gauss Equations

THE THREE DIFFERENTIAL EQUATIONS to be considered here are solved by the methods of the preceding chapter. The first two have important applications in mathematical physics. The solutions of all three have many interesting properties.

THE LEGENDRE EQUATION

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0.$$

A solution of this equation in series convergent near $x = 0$, an ordinary point, was called for in Problem 16, Chapter 25. Under certain conditions on p which will be stated later, we shall obtain here the solution convergent near $x = \infty$. Using the substitution $x = 1/z$ (see Chapter 26) the equation becomes

$$(z^4 - z^2)\frac{d^2y}{dz^2} + 2z^3\frac{dy}{dz} + p(p+1)y = 0$$

for which $z = 0$ is a regular singular point.

Putting $y = A_0z^m + A_1z^{m+1} + A_2z^{m+2} + \dots\dots + A_nz^{m+n} + \dots\dots,$ we have

$$\{-m(m-1)+p(p+1)\}A_0z^m + \{-m(m+1)+p(p+1)\}A_1z^{m+1} + \{[-(m+1)(m+2)+p(p+1)]A_2$$
$$+ m(m+1)A_0\}z^{m+2} + \dots\dots + \{[-(m+n)(m+n-1)+p(p+1)]A_n + (m+n-2)(m+n-1)A_{n-2}\}z^{m+n}$$
$$+ \dots\dots = 0.$$

We take $A_1 = 0$ and $A_n = \dfrac{(m+n-2)(m+n-1)}{(m+n)(m+n-1)-p(p+1)}A_{n-2}$, and see that

$$\bar{y} = A_0z^m\Big[1 + \frac{m(m+1)}{(m+1)(m+2)-p(p+1)}z^2 + \frac{m(m+1)(m+2)(m+3)}{[(m+1)(m+2)-p(p+1)][(m+3)(m+4)-p(p+1)]}z^4$$
$$+ \frac{m(m+1)(m+2)(m+3)(m+4)(m+5)}{[(m+1)(m+2)-p(p+1)][(m+3)(m+4)-p(p+1)][(m+5)(m+6)-p(p+1)]}z^6 + \dots\dots\Big]$$

satisfies the equation

$$(z^4-z^2)\frac{d^2\bar{y}}{dz^2} + 2z^3\frac{d\bar{y}}{dz} + p(p+1)\bar{y} = [-m(m-1)+p(p+1)]A_0z^m = (m+p)(-m+p+1)A_0z^m.$$

For $m = -p$ with $A_0 = 1$, we obtain

1) $$y_1 = z^{-p}\Big[1 - \frac{p(p-1)}{2(2p-1)}z^2 + \frac{p(p-1)(p-2)(p-3)}{2\cdot4(2p-1)(2p-3)}z^4 - \frac{p(p-1)(p-2)(p-3)(p-4)(p-5)}{2\cdot4\cdot6(2p-1)(2p-3)(2p-5)}z^6 + \dots\dots\dots\dots\Big]$$

$$= x^p\Big[1 - \frac{p(p-1)}{2(2p-1)}x^{-2} + \frac{p(p-1)(p-2)(p-3)}{2\cdot4(2p-1)(2p-3)}x^{-4} - \frac{p(p-1)(p-2)(p-3)(p-4)(p-5)}{2\cdot4\cdot6(2p-1)(2p-3)(2p-5)}x^{-6} + \dots\dots\dots\dots\Big].$$

For $n = p+1$ with $A_0 = 1$, we obtain

2) $$y_2 = z^{p+1}\left[1 + \frac{(p+1)(p+2)}{2(2p+3)}z^2 + \frac{(p+1)(p+2)(p+3)(p+4)}{2\cdot 4(2p+3)(2p+5)}z^4 + \frac{(p+1)(p+2)(p+3)(p+4)(p+5)(p+6)}{2\cdot 4\cdot 6(2p+3)(2p+5)(2p+7)}z^6 + \cdots\cdots\right]$$

$$= x^{-p-1}\left[1 + \frac{(p+1)(p+2)}{2(2p+3)}x^{-2} + \frac{(p+1)(p+2)(p+3)(p+4)}{2\cdot 4(2p+3)(2p+5)}x^{-4} + \frac{(p+1)(p+2)(p+3)(p+4)(p+5)(p+6)}{2\cdot 4\cdot 6(2p+3)(2p+5)(2p+7)}x^{-6} + \cdots\cdots\right]$$

Thus, $$y = Ay_1 + By_2$$

is the complete solution, convergent for $|x| > 1$, provided that $p \neq 1/2, 3/2, 5/2, \cdots\cdots$ or $p \neq -3/2, -5/2, \cdots\cdots$.

Suppose p is a positive integer including 0 and consider the solution y_1 which is a polynomial, say $u_p(x)$. Putting $p = 0, 1, 2, 3, \cdots\cdots$ in 1), we have

$$u_0(x) = 1, \quad u_1(x) = x, \quad u_2(x) = x^2 - 1/3, \quad u_3(x) = x^3 - 3x/5, \quad \cdots\cdots\cdots,$$

$$u_k(x) = \sum_{n=0}^{[\frac{1}{2}k]} (-1)^n \frac{k(k-1)\cdots\cdots(k-2n+1)}{2^n\, n!\,(2k-1)\cdots\cdots(2k-2n+1)} x^{k-2n}, \quad \cdots\cdots,$$

Where $[\frac{1}{2}k]$ denotes the greatest integer $\leq \frac{1}{2}k$ (i.e., if $[\frac{1}{2}k] = 3$ if $k = 7$, $[\frac{1}{2}k] = 4$ if $k = 8$).

The polynomials defined by

3) $$P_p(x) = \frac{(2p)!}{2^p(p!)^2}u_p(x) = \frac{1\cdot 3\cdot 5\cdots\cdot(2p-1)}{p!}u_p(x), \quad p = 0, 1, 2, \cdots\cdots,$$

are called Legendre polynomials. The first few of these are:

$$P_0(x) = u_0(x) = 1,$$

$$P_1(x) = u_1(x) = x,$$

$$P_2(x) = \frac{1\cdot 3}{2!}u_2(x) = \frac{3}{2}x^2 - \frac{1}{2},$$

$$P_3(x) = \frac{1\cdot 3\cdot 5}{3!}u_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x,$$

$$P_4(x) = \frac{1\cdot 3\cdot 5\cdot 7}{4!}u_4(x) = \frac{5\cdot 7}{2\cdot 4}x^4 - 2\frac{3\cdot 5}{2\cdot 4}x^2 + \frac{1\cdot 3}{2\cdot 4},$$

$$P_5(x) = \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{5!}u_5(x) = \frac{7\cdot 9}{2\cdot 4}x^5 - 2\frac{5\cdot 7}{2\cdot 4}x^3 + \frac{3\cdot 5}{2\cdot 4}x,$$

$$P_6(x) = \frac{1\cdot 3\cdot\cdots\cdot 11}{6!}u_6(x) = \frac{7\cdot 9\cdot 11}{2\cdot 4\cdot 6}x^6 - 3\frac{5\cdot 7\cdot 9}{2\cdot 4\cdot 6}x^4 + 3\frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}x^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6},$$

$$P_7(x) = \frac{1\cdot 3\cdot\cdots\cdot 13}{7!}u_7(x) = \frac{9\cdot 11\cdot 13}{2\cdot 4\cdot 6}x^7 - 3\frac{7\cdot 9\cdot 11}{2\cdot 4\cdot 6}x^5 + 3\frac{5\cdot 7\cdot 9}{2\cdot 4\cdot 6}x^3 - \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}x, \quad \text{etc.}$$

It is clear from 3) that $P_p(x)$ is a particular solution of the Legendre equation $(1-x^2)y'' - 2xy' + p(p+1) = 0$.

See Problems 1-6.

THE BESSEL EQUATION

$$x^2y'' + xy' + (x^2 - k^2)y = 0.$$

It is evident that $x = 0$ is a regular singular point. To obtain the solution in series, convergent near $x = 0$, we substitute

$$y = A_0x^m + A_1x^{m+1} + A_2x^{m+2} + \cdots\cdots + A_nx^{m+n} + \cdots\cdots$$

and obtain

$$(m^2 - k^2)A_0x^m + \{(m+1)^2 - k^2\}A_1x^{m+1} + \{[(m+2)^2 - k^2]A_2 + A_0\}x^{m+2} + \cdots\cdots$$
$$+ \{[(m+n)^2 - k^2]A_n + A_{n-2}\}x^{m+n} + \cdots\cdots = 0.$$

We take $A_1 = 0$ and $A_n = -\dfrac{1}{(m+n)^2 - k^2}A_{n-2}$ and see that

$$\bar{y} = A_0x^m\{1 - \frac{1}{(m+2)^2 - k^2}x^2 + \frac{1}{[(m+2)^2 - k^2][(m+4)^2 - k^2]}x^4$$
$$- \frac{1}{[(m+2)^2 - k^2][(m+4)^2 - k^2][(m+6)^2 - k^2]}x^6 + \cdots\cdots\}$$

satisfies the equation $\qquad x^2\bar{y}'' + x\bar{y}' + (x^2 - k^2)\bar{y} = (m^2 - k^2)A_0x^m.$

For $m = k$ with $A_0 = 1$, we obtain

$$y_1 = x^k\{1 - \frac{1}{4(k+1)}x^2 + \frac{1}{4^2\cdot 2!(k+1)(k+2)}x^4 - \frac{1}{4^3\cdot 3!(k+1)(k+2)(k+3)}x^6 + \cdots\cdots\}$$

and for $m = -k$ with $A_0 = 1$, we obtain

$$y_2 = x^{-k}\{1 - \frac{1}{4(1-k)}x^2 + \frac{1}{4^2\cdot 2!(1-k)(2-k)}x^4 - \frac{1}{4^3\cdot 3!(1-k)(2-k)(3-k)}x^6 + \cdots\cdots\}.$$

Note that $y_2 = y_1$ if $k = 0$, y_1 is meaningless if k is a negative integer, and y_2 is meaningless if k is a positive integer. Except for these cases, the complete solution of the Bessel equation is $y = Ay_1 + By_2$, convergent for all $x \neq 0$.

The Bessel functions of the *first kind* are defined by

$$J_k(x) = \frac{1}{2^k\cdot k!}y_1 = (\frac{x}{2})^k\{\frac{1}{k!} - \frac{1}{1!(k+1)!}(\frac{x}{2})^2 + \frac{1}{2!(k+2)!}(\frac{x}{2})^4 - \frac{1}{3!(k+3)!}(\frac{x}{2})^6 + \cdots\},$$

$J_{-k}(x) = (-1)^kJ_k(x)$, where k is a positive integer including 0.

Of these, $\qquad J_0(x) = 1 - \dfrac{1}{(1!)^2}(\dfrac{x}{2})^2 + \dfrac{1}{(2!)^2}(\dfrac{x}{2})^4 - \dfrac{1}{(3!)^2}(\dfrac{x}{2})^6 + \cdots\cdots$

and $\qquad J_1(x) = (\dfrac{x}{2})\{1 - \dfrac{1}{1!2!}(\dfrac{x}{2})^2 + \dfrac{1}{2!3!}(\dfrac{x}{2})^4 - \dfrac{1}{3!4!}(\dfrac{x}{2})^6 + \cdots\cdots\}$

are more frequently used.

See Problems 7-10.

THE GAUSS EQUATION

$$(x - x^2)y'' + [\gamma - (\alpha + \beta + 1)x]y' - \alpha\beta y = 0.$$

To obtain the solution in series, convergent near $x = 0$, substitute

$$y = A_0x^m + A_1x^{m+1} + A_2x^{m+2} + \cdots\cdots + A_nx^{m+n} + \cdots\cdots$$

and obtain

$$m(m+\gamma-1)A_0x^{m-1} + \{(m+1)(m+\gamma)A_1 - [m(m+\alpha+\beta)+\alpha\beta]A_0\}x^m + \cdots\cdots$$
$$+ \{(m+n)(m+n+\gamma-1)A_n - [(m+n-1)(m+n+\alpha+\beta-1)+\alpha\beta]A_{n-1}\}x^{m+n-1} + \cdots\cdots = 0.$$

We take $A_n = \dfrac{(m+n-1)(m+n+\alpha+\beta-1)+\alpha\beta}{(m+n)(m+n+\gamma-1)}A_{n-1}$ and see that

$$\bar{y} = A_0x^m\Big[1 + \frac{m(m+\alpha+\beta)+\alpha\beta}{(m+1)(m+\gamma)}x + \frac{m(m+\alpha+\beta)+\alpha\beta}{(m+1)(m+\gamma)}\cdot\frac{(m+1)(m+\alpha+\beta+1)+\alpha\beta}{(m+2)(m+\gamma+1)}x^2$$
$$+ \frac{m(m+\alpha+\beta)+\alpha\beta}{(m+1)(m+\gamma)}\cdot\frac{(m+1)(m+\alpha+\beta+1)+\alpha\beta}{(m+2)(m+\gamma+1)}\cdot\frac{(m+2)(m+\alpha+\beta+2)+\alpha\beta}{(m+3)(m+\gamma+2)}x^3 + \cdots\cdots\Big]$$

satisfies the equation

$$(x-x^2)\bar{y}'' + [\gamma-(\alpha+\beta+1)x]\bar{y}' - \alpha\beta\bar{y} = m(m+\gamma-1)A_0x^{m-1}.$$

For $m = 0$, with $A_0 = 1$, we obtain

$$y_1 = 1 + \frac{\alpha\cdot\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\cdot\gamma(\gamma+1)(\gamma+2)}x^3 + \cdots\cdots,$$

and for $m = 1-\gamma$, $\gamma \neq 1$, with $A_0 = 1$, we obtain

$$y_2 = x^{1-\gamma}\Big[1 + \frac{(\alpha-\gamma+1)(\beta-\gamma+1)}{1(2-\gamma)}x + \frac{(\alpha-\gamma+1)(\alpha-\gamma+2)(\beta-\gamma+1)(\beta-\gamma+2)}{1\cdot2(2-\gamma)(3-\gamma)}x^2$$
$$+ \frac{(\alpha-\gamma+1)(\alpha-\gamma+2)(\alpha-\gamma+3)(\beta-\gamma+1)(\beta-\gamma+2)(\beta-\gamma+3)}{1\cdot2\cdot3(2-\gamma)(3-\gamma)(4-\gamma)}x^3 + \cdots\cdots\Big].$$

The series y_1, known as the *hypergeometric series*, is convergent for $|x| < 1$ and is represented by

$$y_1 = F(\alpha, \beta, \gamma, x).$$

Note that $$y_2 = x^{1-\gamma}F(\alpha-\gamma+1,\ \beta-\gamma+1,\ 2-\gamma,\ x)$$

is of the same type. Thus, if γ is non-integral (including 0), the general solution is

$$y = Ay_1 + By_2 = AF(\alpha, \beta, \gamma, x) + Bx^{1-\gamma}F(\alpha-\gamma+1,\ \beta-\gamma+1,\ 2-\gamma,\ x).$$

There are numerous special cases, depending upon the values of α, β, and γ. Some of these will be treated in the Solved Problems.

SOLVED PROBLEMS

THE LEGENDRE EQUATION.

1. Verify that $2^p p!\, P_p(x) = \dfrac{d^p}{dx^p}(x^2-1)^p$. (Rodrigues' Formula)

By the binomial theorem, $(x^2-1)^p = \displaystyle\sum_{n=0}^{p} (-1)^n \frac{p!}{n!\,(p-n)!}\, x^{2p-2n}$ Then

$$\frac{d^p}{dx^p}(x^2-1)^p = \sum_{n=0}^{[\frac{1}{2}p]} (-1)^n \frac{p!}{n!\,(p-n)!}(2p-2n)(2p-2n-1)\cdots\cdots(p-2n+1)\,x^{p-2n}$$

$$= \sum_{n=0}^{[\frac{1}{2}p]} (-1)^n \frac{2p(2p-1)\cdots(2p-2n+1)}{2p(2p-1)\cdots(2p-2n+1)}(2p-2n)(2p-2n-1)\cdots(p-2n+1)\frac{(p-2n)(p-2n-1)\cdots1}{(p-2n)(p-2n-1)\cdots1}\cdot\frac{p!}{n!\,(p-n)!}\,x^{p-2n}.$$

Now (in the denominator) $2p(2p-1)\cdots(2p-2n+1) = 2^n[p(p-1)\cdots(p-n+1)][(2p-1)(2p-3)\cdots(2p-2n+1)]$ and when multiplied by $(p-n)!$ yields $2^n p!\,[(2p-1)(2p-3)\cdots(2p-2n+1)]$. Hence,

$$\frac{d^p}{dx^p}(x^2-1)^p = \sum_{n=0}^{[\frac{1}{2}p]} (-1)^n \frac{(2p)!\,p!}{2^n p!\,[(2p-1)(2p-3)\cdots(2p-2n+1)]\,(p-2n)!\,n!}\,x^{p-2n}$$

$$= \sum_{n=0}^{[\frac{1}{2}p]} (-1)^n \frac{(2p)!}{2^n\,n!\,p!}\cdot\frac{p(p-1)\cdots(p-2n+1)}{(2p-1)(2p-3)\cdots(2p-2n+1)}\,x^{p-2n}$$

$$= \frac{(2p)!}{p!}\,u_p(x) = 2^p\cdot p!\,P_p(x).$$

2. Show that $P_p(x) = \displaystyle\sum_{n=0}^{[\frac{1}{2}p]} (-1)^n \frac{(2p-2n)!}{2^p\,n!\,(p-n)!\,(p-2n)!}\,x^{p-2n}$. From Problem 1 above,

$$\frac{d^p}{dx^p}(x^2-1)^p = \sum_{n=0}^{[\frac{1}{2}p]} (-1)^n \frac{p!}{n!\,(p-n)!}(2p-2n)(2p-2n-1)\cdots(p-2n+1)\,x^{p-2n}$$

$$= \sum_{n=0}^{[\frac{1}{2}p]} (-1)^n (2p-2n)(2p-2n-1)\cdots(p-2n+1)\frac{(p-2n)!}{(p-2n)!}\cdot\frac{p!}{n!\,(p-n)!}\,x^{p-2n}$$

$$= \sum_{n=0}^{[\frac{1}{2}p]} (-1)^n \frac{(2p-2n)!\,p!}{n!\,(p-n)!\,(p-2n)!}\,x^{p-2n}.$$

Hence,
$$P_p(x) = \frac{1}{2^p\,p!}\cdot\frac{d^p}{dx^p}(x^2-1)^p = \sum_{n=0}^{[\frac{1}{2}p]} (-1)^n \frac{(2p-2n)!}{2^p\,n!\,(p-n)!\,(p-2n)!}\,x^{p-2n}.$$

3. Evaluate $\int_{-1}^{1} P_r(x)\, P_s(x)\, dx$.

Using Rodrigues' formula (Problem 1),

$$\int_{-1}^{1} P_r(x)\, P_s(x)\, dx = \frac{1}{2^{r+s}\, r!\, s!} \int_{-1}^{1} \frac{d^r}{dx^r}(x^2-1)^r \cdot \frac{d^s}{dx^s}(x^2-1)^s\, dx.$$

Let $u = \frac{d^r}{dx^r}(x^2-1)^r$ and $dv = \frac{d^s}{dx^s}(x^2-1)^s dx$. Then $du = \frac{d^{r+1}}{dx^{r+1}}(x^2-1)^r\, dx$, $v = \frac{d^{s-1}}{dx^{s-1}}(x^2-1)^s$,

and
$$\int_{x=-1}^{x=1} u\, dv = uv\Big]_{x=-1}^{x=1} - \int_{x=-1}^{x=1} v\, du$$

$$= \frac{d^r}{dx^r}(x^2-1)^r \cdot \frac{d^{s-1}}{dx^{s-1}}(x^2-1)^s\Big]_{-1}^{1} - \int_{-1}^{1} \frac{d^{r+1}}{dx^{r+1}}(x^2-1)^r \cdot \frac{d^{s-1}}{dx^{s-1}}(x^2-1)^s\, dx.$$

Now $\frac{d^{s-j}}{dx^{s-j}}(x^2-1)^s\Big]_{-1}^{1} = 0$, for $j = 1, 2, \cdots, s-1$; hence, after one integration by parts,

$$\int_{-1}^{1} P_r(x)\, P_s(x)\, dx = -\frac{1}{2^{r+s}\, r!\, s!} \int_{-1}^{1} \frac{d^{r+1}}{dx^{r+1}}(x^2-1)^r \cdot \frac{d^{s-1}}{dx^{s-1}}(x^2-1)^s\, dx.$$

A second integration by parts yields

$$\int_{-1}^{1} P_r(x)\, P_s(x)\, dx = \frac{1}{2^{r+s}\, r!\, s!} \int_{-1}^{1} \frac{d^{r+2}}{dx^{r+2}}(x^2-1)^r \cdot \frac{d^{s-2}}{dx^{s-2}}(x^2-1)^s\, dx$$

and, after s integrations by parts, we have formally

$$(A) \quad \int_{-1}^{1} P_r(x)\, P_s(x)\, dx = \frac{(-1)^s}{2^{r+s}\, r!\, s!} \int_{-1}^{1} (x^2-1)^s \cdot \frac{d^{r+s}}{dx^{r+s}}(x^2-1)^r\, dx.$$

Suppose $s > r$. Then, since $(x^2-1)^r = x^{2r} - rx^{2r-2} + \cdots\cdots + (-1)^r$, $\frac{d^{r+s}}{dx^{r+s}}(x^2-1)^r = 0$

and $\int_{-1}^{1} P_r(x)\, P_s(x)\, dx = 0$. Since r and s enter symmetrically, this relation holds also when $r > s$.

Thus, it holds when $r \neq s$.

Suppose $s = r$. Then (A) becomes

$$(B) \quad \int_{-1}^{1} P_r^2(x)\, dx = \frac{(-1)^r}{2^{2r}\,(r!)^2} \int_{-1}^{1} (x^2-1)^r \cdot \frac{d^{2r}}{dx^{2r}}(x^2-1)^r\, dx.$$

Now $\frac{d^{2r}}{dx^{2r}}(x^2-1)^r = (2r)!$. Hence, $\int_{-1}^{1} P_r^2(x)\, dx = \frac{(-1)^r(2r)!}{2^{2r}(r!)^2} \int_{-1}^{1} (x^2-1)^r\, dx$

$$= \frac{(-1)^r(2r)!}{2^{2r}(r!)^2} \cdot (-1)^r \cdot 2 \int_0^{\frac{1}{2}\pi} \sin^{2r+1}\theta\, d\theta = \frac{(2r)!}{2^{2r}(r!)^2} \cdot \frac{2^{r+1}\, r!}{1\cdot 3\cdots(2r+1)} = \frac{2}{2r+1},$$ using the

substitution $x = \cos\theta$ and Wallis' Formula $\cdot \int_0^{\frac{1}{2}\pi} \sin^{2n+1}\theta\, d\theta = \dfrac{2^n n!}{1\cdot 3\cdots(2n+1)}$.

4. Express $f(x) = x^4 + 2x^3 + 2x^2 - x - 3$ in terms of Legendre polynomials.

Since $P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$, then $x^4 = \frac{8}{35}P_4(x) + \frac{6}{7}x^2 - \frac{3}{35}$ and

$$f(x) = (\frac{8}{35}P_4(x) + \frac{6}{7}x^2 - \frac{3}{35}) + 2x^3 + 2x^2 - x - 3 = \frac{8}{35}P_4(x) + 2x^3 + \frac{20}{7}x^2 - x - \frac{108}{35}.$$

Now $x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}x$ and $f(x) = \frac{8}{35}P_4(x) + \frac{4}{5}P_3(x) + \frac{20}{7}x^2 + \frac{1}{5}x - \frac{108}{35}$;

$x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}$ and

$$f(x) = \frac{8}{35}P_4(x) + \frac{4}{5}P_3(x) + \frac{40}{21}P_2(x) + \frac{1}{5}x - \frac{224}{105}$$

$$= \frac{8}{35}P_4(x) + \frac{4}{5}P_3(x) + \frac{40}{21}P_2(x) + \frac{1}{5}P_1(x) - \frac{224}{105}P_0(x).$$

5. Show that $(1-2xt+t^2)^{-\frac{1}{2}} = P_0(x) + P_1(x)\,t + P_2(x)\,t^2 + \cdots\cdots + P_k(x)\,t^k + \cdots\cdots$.

Now
$$(1-2xt+t^2)^{-\frac{1}{2}} = [1-(2xt-t^2)]^{-\frac{1}{2}} = 1 + \frac{1}{2}(2xt-t^2) + \frac{(1/2)(3/2)}{2}(2xt-t^2)^2 + \cdots\cdots$$
$$+ \frac{1\cdot 3\cdots(2k-5)}{2^{k-2}(k-2)!}(2xt-t^2)^{k-2} + \frac{1\cdot 3\cdots(2k-3)}{2^{k-1}(k-1)!}(2xt-t^2)^{k-1} + \frac{1\cdot 3\cdots(2k-1)}{2^k k!}(2xt-t^2)^k + \cdots\cdots.$$

But
$$(2xt-t^2)^k = (2x)^k t^k - \cdots\cdots,$$
$$(2xt-t^2)^{k-1} = (2x)^{k-1}t^{k-1} - (k-1)(2x)^{k-2}t^k + \cdots\cdots,$$
$$(2xt-t^2)^{k-2} = (2x)^{k-2}t^{k-2} - (k-2)(2x)^{k-3}t^{k-1} + \frac{(k-2)(k-3)}{2!}(2x)^{k-4}t^k - \cdots\cdots, \text{ etc.}$$

Hence,
$$(1-2xt+t^2)^{-\frac{1}{2}} = 1 + xt + (\frac{3}{2}x^2 - \frac{1}{2})t^2 + \cdots\cdots\cdots\cdots + [\frac{1\cdot 3\cdots(2k-1)}{2^k k!}2^k x^k$$
$$- \frac{1\cdot 3\cdots(2k-3)}{2^{k-1}(k-1)!}(k-1)2^{k-2}x^{k-2} + \frac{1\cdot 3\cdots(2k-5)}{2^{k-2}(k-2)!}\frac{(k-2)(k-3)}{2!}2^{k-4}x^{k-4} + \cdots\cdots]t^k + \cdots\cdots\cdots$$

$$= 1 + xt + (\frac{3}{2}x^2 - \frac{1}{2})t^2 + \cdots\cdots\cdots\cdots + \frac{1\cdot 3\cdots(2k-1)}{k!}[x^k$$
$$- \frac{k(k-1)}{2(2k-1)}x^{k-2} + \frac{k(k-1)(k-2)(k-3)}{2\cdot 4(2k-1)(2k-3)}x^{k-4} + \cdots\cdots]t^k + \cdots\cdots$$
$$= P_0(x) + P_1(x)\,t + P_2(x)\,t^2 + \cdots\cdots + P_k(x)\,t^k + \cdots\cdots.$$

6. Show that $P_p(1) = 1,\ p = 0,1,2,3,\cdots\cdots\cdots$.

Put $x = 1$ in the identity established in Problem 5. Then
$$(1-2t+t^2)^{-\frac{1}{2}} = (1-t)^{-1} = 1 + t + t^2 + \cdots\cdots + t^p + \cdots\cdots$$
$$= P_0(1) + P_1(1)\,t + P_2(1)\,t^2 + \cdots\cdots + P_p(1)\,t^p + \cdots\cdots, \text{ identically.}$$

Hence, $P_0(1) = P_1(1) = \cdots\cdots = P_p(1) = \cdots\cdots = 1.$

THE BESSEL EQUATION.

7. Prove $\frac{d}{dx} J_0(x) = -J_1(x)$.

$$J_0(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n!)^2} (\frac{x}{2})^{2n}$$

$$= 1 - (\frac{x}{2})^2 + \frac{1}{(2!)^2}(\frac{x}{2})^4 - \frac{1}{(3!)^2}(\frac{x}{2})^6 + \cdots + (-1)^{n+1} \frac{1}{[(n+1)!]^2}(\frac{x}{2})^{2n+2} + \cdots\cdots$$

and

$$\frac{d}{dx} J_0(x) = -(\frac{x}{2}) + \frac{1}{1!\,2!}(\frac{x}{2})^3 - \frac{1}{2!\,3!}(\frac{x}{2})^5 + \cdots\cdots + (-1)^{n+1} \frac{1}{n!\,(n+1)!}(\frac{x}{2})^{2n+1} + \cdots\cdots$$

$$= -[\frac{x}{2} - \frac{1}{1!\,2!}(\frac{x}{2})^3 + \cdots\cdots + (-1)^n \frac{1}{n!\,(n+1)!}(\frac{x}{2})^{2n+1} + \cdots\cdots]$$

$$= -\sum_{n=0}^{\infty} (-1)^n \frac{1}{n!\,(n+1)!}(\frac{x}{2})^{2n+1} = -J_1(x).$$

More briefly,

$$\frac{d}{dx} J_0(x) = \frac{d}{dx} \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n!)^2}(\frac{x}{2})^{2n} = \frac{d}{dx}[1 + \sum_{n=1}^{\infty} (-1)^n \frac{1}{(n!)^2}(\frac{x}{2})^{2n}]$$

$$= \frac{d}{dx}[1 - \sum_{n=0}^{\infty} (-1)^n \frac{1}{[(n+1)!]^2}(\frac{x}{2})^{2n+2}] = -\sum_{n=0}^{\infty} (-1)^n \frac{1}{n!\,(n+1)!}(\frac{x}{2})^{2n+1} = -J_1(x).$$

8. Prove *a)* $\frac{d}{dx} x^k \cdot J_k(x) = x^k J_{k-1}(x)$, *b)* $\frac{d}{dx} x^{-k} \cdot J_k(x) = -x^{-k} J_{k+1}(x)$,

where k is a positive integer.

a) $$\frac{d}{dx} x^k \cdot J_k(x) = \frac{d}{dx} \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{k+2n}\, n!\,(k+n)!} x^{2k+2n}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{2k+2n}{2^{k+2n}\, n!\,(k+n)!} x^{2k+2n-1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{k+2n-1}\, n!\,(k+n-1)!} x^{2k+2n-1}$$

$$= x^k \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!\,(k+n-1)!} (\frac{x}{2})^{k+2n-1} = x^k J_{k-1}(x).$$

b) $$\frac{d}{dx} x^{-k} \cdot J_k(x) = \frac{d}{dx} \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{k+2n}\, n!\,(k+n)!} x^{2n}$$

$$= \frac{d}{dx}[\frac{1}{2^k\, k!} - \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{k+2n+2}\,(n+1)!\,(k+n+1)!} x^{2n+2}]$$

$$= -\sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{k+2n+1}\, n!\,(k+n+1)!} x^{2n+1} = -x^{-k} \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!\,(k+n+1)!}(\frac{x}{2})^{k+2n+1} = -x^{-k} J_{k+1}(x).$$

9. Prove $\quad a)\ \ J_{k-1}(x) - J_{k+1}(x) = 2\dfrac{d}{dx}J_k(x), \qquad b)\ \ J_{k-1}(x) + J_{k+1}(x) = \dfrac{2k}{x}J_k(x),$

where k is a positive integer.

From Problem 8,

$$A)\quad \frac{d}{dx}x^k\cdot J_k(x) = x^k\frac{d}{dx}J_k(x) + kx^{k-1}J_k(x) = x^k J_{k-1}(x) \qquad \text{and}$$

$$B)\quad \frac{d}{dx}x^{-k}\cdot J_k(x) = x^{-k}\frac{d}{dx}J_k(x) - kx^{-k-1}J_k(x) = -x^{-k}J_{k+1}(x).$$

Then from $A)$,

$$1)\quad \frac{d}{dx}J_k(x) + \frac{k}{x}J_k(x) = J_{k-1}(x);$$

and from $B)$,

$$2)\quad \frac{d}{dx}J_k(x) - \frac{k}{x}J_k(x) = -J_{k+1}(x).$$

When 1) and 2) are added, we have $a)$; when 2) is subtracted from 1), we have $b)$.

Note that when $b)$ is subtracted from $a)$, we have

$$2\frac{d}{dx}J_k(x) - \frac{2k}{x}J_k(x) = -2J_{k+1}(x) \qquad \text{or} \qquad \frac{d}{dx}J_k(x) = \frac{k}{x}J_k(x) - J_{k+1}(x).$$

Note also that $b)$ is a recursion formula for Bessel functions.

10. Show that

$$e^{\frac{1}{2}x(t-1/t)} = J_0(x) + tJ_1(x) + \cdots\cdots + t^k J_k(x) + \cdots\cdots + \frac{1}{t}J_{-1}(x) + \cdots\cdots + \frac{1}{t^k}J_{-k}(x) + \cdots\cdots\cdots = \sum_{n=-\infty}^{+\infty} t^n J_n(x).$$

$$\begin{aligned} e^{\frac{1}{2}x(t-1/t)} &= e^{\frac{1}{2}xt}\cdot e^{-x/2t} \\ &= \left[1 + \frac{xt}{2} + \frac{x^2t^2}{2^2\,2!} + \frac{x^3t^3}{2^3\,3!} + \cdots\cdots + \frac{x^nt^n}{2^n\,n!} + \cdots\cdots\right]\left[1 - \frac{x}{2t} + \frac{x^2}{2^2\,2!\,t^2}\right. \\ &\qquad \left. - \frac{x^3}{2^3\,3!\,t^3} + \cdots\cdots + (-1)^n\frac{x^n}{2^n\,n!\,t^n} + \cdots\cdots\right]. \end{aligned}$$

In this product, the terms free of t are

$$1 - \left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \cdots\cdots + (-1)^n\frac{1}{(n!)^2}\left(\frac{x}{2}\right)^{2n} + \cdots\cdots = J_0(x),$$

the coefficient of t^k is

$$\begin{aligned} &\frac{x^k}{2^k\,k!} - \frac{x^{k+1}}{2^{k+1}\,(k+1)!}\cdot\frac{x}{2} + \frac{x^{k+2}}{2^{k+2}\,(k+2)!}\cdot\frac{x^2}{2^2\,2!} - \frac{x^{k+3}}{2^{k+3}\,(k+3)!}\cdot\frac{x^3}{2^3\,3!} + \cdots\cdots\cdots \\ &= \frac{1}{k!}\left(\frac{x}{2}\right)^k - \frac{1}{1!\,(k+1)!}\left(\frac{x}{2}\right)^{k+2} + \frac{1}{2!\,(k+2)!}\left(\frac{x}{2}\right)^{k+4} - \frac{1}{3!\,(k+3)!}\left(\frac{x}{2}\right)^{k+6} + \cdots\cdots\cdots \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{1}{n!\,(k+n)!}\left(\frac{x}{2}\right)^{k+2n} = J_k(x), \end{aligned}$$

and the coefficient of $\dfrac{1}{t^k}$ is

$$(-1)^k\left[\frac{x^k}{2^k\,k!} - \frac{x^{k+1}}{2^{k+1}\,(k+1)!}\cdot\frac{x}{2} + \frac{x^{k+2}}{2^{k+2}\,(k+2)!}\cdot\frac{x^2}{2^2\,2!} - \frac{x^{k+3}}{2^{k+3}\,(k+3)!}\cdot\frac{x^3}{2^3\,3!} + \cdots\right]$$

$$= (-1)^k\left[\frac{1}{k!}\left(\frac{x}{2}\right)^k - \frac{1}{1!\,(k+1)!}\left(\frac{x}{2}\right)^{k+2} + \frac{1}{2!\,(k+2)!}\left(\frac{x}{2}\right)^{k+4} - \frac{1}{3!\,(k+3)!}\left(\frac{x}{2}\right)^{k+6} + \cdots\right]$$

$$= (-1)^k J_k(x) = J_{-k}(x).$$

THE GAUSS EQUATION.

11. Solve in series $(x - x^2)y'' + (\frac{3}{2} - 2x)y' - \frac{1}{4}y = 0.$

Here $\alpha+\beta+1 = 2,\ \gamma = 3/2,\ \alpha\beta = 1/4$; thus $\alpha = \beta = 1/2,$ and $\gamma = 3/2.$

Then $y_1 = F(\alpha,\beta,\gamma,x) = F(\frac{1}{2},\frac{1}{2},\frac{3}{2},x) = 1 + \frac{x}{6} + \frac{3x^2}{40} + \frac{5x^3}{112} + \cdots\cdots$

and $\quad y_2 = x^{1-\gamma}F(\alpha-\gamma+1,\ \beta-\gamma+1,\ 2-\gamma,\ x) = x^{-\frac{1}{2}}F(0,0,\frac{1}{2},x) = 1/\sqrt{x}\,,$

and the complete solution is $\quad y = A\,F(\frac{1}{2},\frac{1}{2},\frac{3}{2},x) + B/\sqrt{x}\,.$

12. Solve in series $(x - x^2)y'' + 4(1-x)y' - 2y = 0.$

Here $\alpha+\beta+1 = 4,\ \gamma = 4,\ \alpha\beta = 2$; then $\alpha = 1,\ \beta = 2,\ \gamma = 4$ or $\alpha = 2,\ \beta = 1,\ \gamma = 4.$

For either choice, $y_1 = F(1,2,4,x) = F(2,1,4,x)$

$$= 1 + \frac{x}{2} + \frac{3x^2}{10} + \frac{x^3}{5} + \frac{x^4}{7} + \frac{3x^5}{28} + \cdots\cdots$$

Since $\gamma = 4$, the fourth term in y_2 has zero for denominator. However, one of $\alpha-\gamma+2$, $\beta-\gamma+2$ in the third term is zero so that

$$y_2 = x^{-3}F(-2,-1,-2,x) = x^{-3}F(-1,-2,-2,x) = x^{-3}(1-x)$$

and the complete solution is

$$y = A\,F(1,2,4,x) + B\,\frac{1-x}{x^3}\,.$$

13. Show that $\quad a)\ F(\alpha,\beta,\beta,x) = (1-x)^{-\alpha}, \qquad b)\ xF(1,1,2,-x) = \ln(1+x).$

a) $$F(\alpha,\beta,\beta,x) = 1 + \frac{\alpha\cdot\beta}{1\cdot\beta}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\beta\,(\beta+1)}x^2 + \cdots\cdots$$

$$= 1 + \alpha x + \frac{\alpha(\alpha+1)}{2!}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)}{3!}x^3 + \cdots\cdots = (1-x)^{-\alpha}.$$

b) $$xF(1,1,2,-x) = x\left[1 + \frac{1\cdot1}{1\cdot2}(-x) + \frac{1\cdot2\cdot1\cdot2}{1\cdot2\cdot2\cdot3}(-x)^2 + \frac{1\cdot2\cdot3\cdot1\cdot2\cdot3}{1\cdot2\cdot3\cdot2\cdot3\cdot4}(-x)^3 + \cdots\cdots\right]$$

$$= x(1 - \frac{1}{2}x + \frac{1}{3}x^2 - \frac{1}{4}x^3 + \cdots\cdots) = \ln(1+x).$$

SUPPLEMENTARY PROBLEMS

14. Compute *a)* $P_4(2) = 55.3750$, *b)* $J_0(1) = 0.7652$, *c)* $J_1(1) = 0.4401$, *d)* $F(1,1,10,-1) = 0.9147$.

15. Verify each of the following by using the series expansion of $P_p(x)$.

a) $(x^2-1)P_p'(x) = (p+1)[P_{p+1}(x) - xP_p(x)] = p[x\,P_p(x) - P_{p-1}(x)]$.
b) $P_{p+1}'(x) = xP_p'(x) + (p+1)P_p(x)$.
c) $(2p+1)P_p(x) = P_{p+1}'(x) - P_{p-1}'(x) = \frac{1}{x}[(p+1)P_{p+1}(x) + pP_{p-1}(x)]$.

16. If $P_6(2) = a$ and $P_7(2) = b$, show that
a) $P_6'(2) = \frac{7}{3}(b-2a)$, *b)* $P_7'(2) = \frac{7}{3}(2b-a)$, *c)* $P_8(2) = \frac{1}{8}(30b-7a)$, *d)* $P_8'(2) = \frac{1}{3}(52b-14a)$.

17. If $J_0(2) = a$ and $J_1(2) = b$, show that *a)* $J_2(2) = b-a$, *b)* $J_1'(2) = a - \frac{1}{2}b$, *c)* $J_2'(2) = a$.

18. Show that the change of independent variable $x^2 = t$ reduces the Legendre equation to a Gauss equation.

19. *a)* Show that the change of dependent variable $y = x^{1/2}z$ transforms $y'' + y = 0$ into a Bessel equation.

b) Write the solution of the Bessel equation as $y = C_1x^{1/2}J_{1/2}(x) + C_2x^{1/2}J_{-1/2}(x)$ and show that $J_{1/2}(x)$ and $J_{-1/2}(x)$ may be defined as $ax^{-1/2}\sin x$ and $bx^{-1/2}\cos x$ respectively.

c) Show that if the relations of Problem 8 are to hold for $k = \pm\frac{1}{2}$, then $a = b$.

Note. These functions are defined with $a = \sqrt{2/\pi}$.

20. Use the substitution $y = x^{1/2}z$ and then $x = (3t/2)^{2/3}$ to show that $y'' + xy = 0$ is a special case of the Bessel equation, and solve.

Hint: $z'' + tz' + (t^2 - 1/9)z = 0$.

Ans.
$$y = Ax\left[1 - \frac{x^3}{2^2\,3} + \frac{x^6}{2!\,2^2\,3^2\,7} - \frac{x^9}{3!\,2^2\,3^3\,7\cdot10} + \cdots\cdots\right]$$
$$+ B\left[1 - \frac{x^3}{3\cdot2} + \frac{x^6}{2!\,3^2\,2\cdot5} - \frac{x^9}{3!\,3^3\,2\cdot5\cdot8} + \cdots\cdots\cdots\cdots\right].$$

21. Solve $(x^2-3x+2)y'' + 4xy' + 2y = 0$ after reducing it to a Gauss equation by a substitution of the form $x = \xi z + \eta$.

Hint: $y = AF(1,2,-4,x-1) + B(x-1)^5F(6,7,6,x-1)$ is not a complete solution since the sixth term of $F(1,2,-4,x-1)$ becomes infinite.

Ans. $y = A\,F(1,2,8,2-x) + B(2-x)^{-7}F(-6,-5,-6,2-x)$

22. Express each of the following as Gauss functions.

a) $\frac{1}{1-x} = F(1,\beta,\beta,x)$

b) $\arcsin x = xF(\frac{1}{2},\frac{1}{2},\frac{3}{2},x^2)$

c) $\arctan x = x\,F(1,\frac{1}{2},\frac{3}{2},-x^2)$

d) $e^x = \lim_{\alpha\to\infty} F(\alpha,1,1,x/\alpha)$

e) $\sin x = \lim_{\substack{\alpha\to\infty\\ \beta\to\infty}} x\,F(\alpha,\beta,\frac{3}{2},-\frac{x^2}{4\alpha\beta})$.

CHAPTER 28

Partial Differential Equations

PARTIAL DIFFERENTIAL EQUATIONS are those which contain one or more partial derivatives. They must, therefore, involve at least two independent variables. The *order* of a partial differential equation is that of the derivative of highest order in the equation. For example, considering z as dependent variable and x, y as independent variables,

1) $$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z \qquad \text{or} \qquad 1') \ xp + yq = z$$

is of order one and

2) $$\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x\,\partial y} + \frac{\partial^2 z}{\partial y^2} = 0 \qquad \text{or} \qquad 2') \ r + 3s + t = 0$$

is of order two. In writing 1′) and 2′), use has been made of the standard notation:

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x\,\partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}.$$

Partial differential equations may be derived by the elimination of arbitrary constants from a given relation between the variables and by the elimination of arbitrary functions of the variables. They also may arise in connection with geometrical and physical problems.

ELIMINATION OF ARBITRARY CONSTANTS. Consider z to be a function of two independent variables x and y defined by

3) $$g(x, y, z, a, b) = 0,$$

in which a and b are two arbitrary constants. By differentiating 3) partially with respect to x and y, we obtain

4) $$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial g}{\partial x} + p\frac{\partial g}{\partial z} = 0$$

and

5) $$\frac{\partial g}{\partial y} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial y} = \frac{\partial g}{\partial y} + q\frac{\partial g}{\partial z} = 0.$$

In general, the arbitrary constants may be eliminated from 3), 4), 5) yielding a partial differential equation of order one

6) $$f(x, y, z, p, q) = 0.$$

EXAMPLE 1. Eliminate the arbitrary constants a and b from $z = ax^2 + by^2 + ab$.

Differentiating partially with respect to x and y, we have

$$\frac{\partial z}{\partial x} = p = 2ax \qquad \text{and} \qquad \frac{\partial z}{\partial y} = q = 2by.$$

Solving for a and b from these equations and substituting in the given relation, we obtain

$$z = (\tfrac{1}{2}\tfrac{p}{x})x^2 + (\tfrac{1}{2}\tfrac{q}{y})y^2 + (\tfrac{1}{2}\tfrac{p}{x})(\tfrac{1}{2}\tfrac{q}{y}) \qquad \text{or} \qquad pq + 2px^2y + 2qxy^2 = 4xyz,$$

a partial differential equation of order one.

If z is a function of x and y defined by a relation involving but one arbitrary constant, it is usually possible to obtain two distinct partial differential equations of order one by eliminating the constant.

EXAMPLE 2. Eliminate a from $z = a(x+y)$.

Differentiating with respect to x gives $p = a$, so that the partial differential equation $z = p(x+y)$ is obtained. Similarly, differentiation with respect to y gives $q = a$ and the equation $z = q(x+y)$.

If the number of arbitrary constants to be eliminated exceeds the number of independent variables, the resulting partial differential equation (or equations) is usually of order higher than the first.

EXAMPLE 3. Eliminate a, b, c from $z = ax + by + cxy$.

Differentiating partially with respect to x and y, we have

(i) $p = a + cy$ and (ii) $q = b + cx$.

These, together with the given relation, are not sufficient for the elimination of three constants. Differentiating (i) partially with respect to x, we have

$$\frac{\partial}{\partial x}p = \frac{\partial^2 z}{\partial x^2} = r = 0,$$

a partial differential equation of order two. Differentiating (ii) partially with respect to y, we have

$$\frac{\partial}{\partial y}q = \frac{\partial^2 z}{\partial y^2} = t = 0, \qquad \text{of order two.}$$

Differentiating (i) partially with respect to y or (ii) with respect to x, we obtain

$$\frac{\partial}{\partial y}p = \frac{\partial}{\partial x}q = \frac{\partial^2 z}{\partial x\,\partial y} = s = c.$$

From (i), $p = a + sy$ and $a = p - sy$; from (ii), $b = q - sx$.

Substituting for a, b, c in the given relation, we obtain

$$z = (p - sy)x + (q - sx)y + sxy = px + qy - sxy,$$

or order two.

Thus, we have three partial differential equations $r = 0$, $t = 0$, $z = px + qy - sxy$ of the same (minimum) order associated with the given relation.

See also Problems 1–4.

ELIMINATION OF ARBITRARY FUNCTIONS. Let $u = u(x,y,z)$ and $v = v(x,y,z)$ be independent functions of the variables x, y, z, and let

7) $$\phi(u,v) = 0$$

be an arbitrary relation between them. Regarding z as the dependent variable and differentiating partially with respect to x and y, we obtain

8) $$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0 \qquad \text{and}$$

9) $$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0.$$

Eliminating $\frac{\partial\phi}{\partial u}$ and $\frac{\partial\phi}{\partial v}$ from 8) and 9), we have

$$\begin{vmatrix} \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} & \frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} & \frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z} \end{vmatrix} = \left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) - \left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right)$$

$$= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + p\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right) + q\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial x}\right) = 0.$$

Writing $\lambda P = \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}$, $\lambda Q = \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}$, $\lambda R = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$,

this takes the form

$$Pp + Qq = R,$$

a partial differential equation linear in p and q and free of the arbitrary function $\phi(u, v)$.

EXAMPLE 4. Find the differential equation arising from $\phi(z/x^3, y/x) = 0$, where ϕ is an arbitrary function of the arguments.

We write the functional relation in the form $\phi(u,v) = 0$ with $u = z/x^3$ and $v = y/x$. Differentiating partially with respect to x and y, we have

$$\frac{\partial\phi}{\partial u}\left(\frac{p}{x^3} - \frac{3z}{x^4}\right) + \frac{\partial\phi}{\partial v}\left(-\frac{y}{x^2}\right) = 0, \qquad \frac{\partial\phi}{\partial u}\left(\frac{q}{x^3}\right) + \frac{\partial\phi}{\partial v}\left(\frac{1}{x}\right) = 0.$$

The elimination of $\frac{\partial\phi}{\partial u}$ and $\frac{\partial\phi}{\partial v}$ yields

$$\begin{vmatrix} p/x^3 - 3z/x^4 & -y/x^2 \\ q/x^3 & 1/x \end{vmatrix} = p/x^4 - 3z/x^5 + qy/x^5 = 0 \quad \text{or} \quad px + qy = 3z.$$

The arbitrary functional relation may also be given by $\frac{z}{x^3} = f(\frac{y}{x})$ or $z = x^3 f(\frac{y}{x})$ where f is an arbitrary function of its argument. Using $v = y/x$ and differentiating $z = x^3 f(v)$ with respect to x and y yields

$$p = 3x^2 f(v) + x^3\frac{df}{dv}\frac{\partial v}{\partial x} = 3x^2 f(v) + x^3\left(\frac{df}{dv}\right)\left(-\frac{y}{x^2}\right) = 3x^2 f(v) - xy\,f'(v),$$

$$q = x^3\frac{df}{dv}\frac{\partial v}{\partial y} = x^3\left(\frac{df}{dv}\right)\left(\frac{1}{x}\right) = x^2 f'(v).$$

When $f'(v)$ is eliminated from these, we have

$$px + qy = 3x^3 f(v) = 3z$$

as before. See also Problems 5–8.

SOLVED PROBLEMS

1. Eliminate a and b from $z = (x^2+a)(y^2+b)$.

Differentiating partially with respect to x and y, $p = 2x(y^2+b)$ and $q = 2y(x^2+a)$. Then $y^2+b = \frac{p}{2x}$, $x^2+a = \frac{q}{2y}$, and $z = (x^2+a)(y^2+b) = (\frac{q}{2y})(\frac{p}{2x})$ or $pq = 4xyz$.

We could also eliminate a and b as follows: $pq = 4xy(y^2+b)(x^2+a) = 4xyz$.

2. Find the differential equation of the family of spheres of radius 5 with centres on the plane $x = y$.

The equation of the family of spheres is 1) $(x-a)^2+(y-a)^2+(z-b)^2 = 25$, a and b being arbitrary constants. Differentiating partially with respect to x and y, and dividing by 2, we have

$$(x-a) + (z-b)p = 0 \quad \text{and} \quad (y-a) + (z-b)q = 0.$$

Let $z-b = -m$; then $x-a = pm$ and $y-a = qm$. Making these replacements in 1), we get

$$m^2(p^2+q^2+1) = 25.$$

Now $x-y = (p-q)m$. Then $m = \frac{x-y}{p-q}$, $m^2(p^2+q^2+1) = \frac{(x-y)^2}{(p-q)^2}(p^2+q^2+1) = 25$, and the required differential equation is $(x-y)^2(p^2+q^2+1) = 25(p-q)^2$.

3. Show that the partial differential equation obtained by eliminating the arbitrary constants a, c from $z = ax + h(a)y + c$, where $h(a)$ is an arbitrary function of a, is free of the variables x, y, z.

Differentiating $z = ax + h(a)y + c$ partially with respect to x and y, we obtain $p = a$ and $q = h(a)$. The differential equation resulting from the elimination of a is $q = h(p)$ or $f(p,q) = 0$, where f is an arbitrary function of its arguments. This equation contains p and q but none of the variables x, y, z.

4. Show that the partial differential equation obtained by eliminating the arbitrary constants a and b from

$$z = ax + by + f(a,b),$$

the *extended Clairaut equation*, is

$$z = px + qy + f(p,q).$$

Differentiating $z = ax + by + f(a,b)$ with respect to x and y yields $p = a$ and $q = b$, and the required differential equation follows immediately.

5. Find the differential equation arising from $\phi(x+y+z,\ x^2+y^2-z^2) = 0$.

Let $u = x+y+z$, $v = x^2+y^2-z^2$ so that the given relation is $\phi(u,v) = 0$.

Differentiation with respect to x and y yields

$$\frac{\partial\phi}{\partial u}(1+p) + \frac{\partial\phi}{\partial v}(2x-2zp) = 0 \qquad \frac{\partial\phi}{\partial u}(1+q) + \frac{\partial\phi}{\partial v}(2y-2zq) = 0.$$ Eliminating $\frac{\partial\phi}{\partial u}$ and $\frac{\partial\phi}{\partial v}$, we have

$$\begin{vmatrix} 1+p & 2x-2zp \\ 1+q & 2y-2zq \end{vmatrix} = 2(y-x) + 2p(y+z) - 2q(z+x) = 0 \quad \text{or} \quad (y+z)p - (x+z)q = x-y.$$

6. Eliminate the arbitrary function $\phi(x+y)$ from $z = \phi(x+y)$.

Let $x+y = u$ so that the given relation is $z = \phi(u)$.

Differentiating with respect to x and y yields $p = \frac{d\phi}{du} = \phi'(u)$ and $q = \phi'(u)$.

Thus, $p = q$ is the resulting differential equation.

7. The equation of any cone with vertex at $P_0(x_0, y_0, z_0)$ is of the form $\phi(\frac{x-x_0}{z-z_0}, \frac{y-y_0}{z-z_0}) = 0$.
Find the differential equation.

Let $\frac{x-x_0}{z-z_0} = u$, $\frac{y-y_0}{z-z_0} = v$ so that the given relation is $\phi(u,v) = 0$.

Differentiating with respect to x and y, we have

$$\frac{\partial\phi}{\partial u}\left(\frac{1}{z-z_0} - p\frac{x-x_0}{(z-z_0)^2}\right) + \frac{\partial\phi}{\partial v}\left(-p\frac{y-y_0}{(z-z_0)^2}\right) = 0$$

$$\frac{\partial\phi}{\partial u}\left(-q\frac{x-x_0}{(z-z_0)^2}\right) + \frac{\partial\phi}{\partial v}\left(\frac{1}{z-z_0} - q\frac{y-y_0}{(z-z_0)^2}\right) = 0.$$

Eliminating $\frac{\partial\phi}{\partial u}$ and $\frac{\partial\phi}{\partial v}$ we obtain $p(x-x_0) + q(y-y_0) = z-z_0$.

8. Eliminate the arbitrary functions $f(x)$ and $g(y)$ from $z = yf(x) + xg(y)$.

Differentiating partially with respect to x and y, we have

1) $p = yf'(x) + g(y)$ and 2) $q = f(x) + xg'(y)$.

Since it is not possible to eliminate f, g, f', g' from these relations and the given one, we find the second partial derivatives

3) $r = yf''(x)$, $s = f'(x) + g'(y)$, $t = xg''(y)$.

From 1) and 2) we find $f'(x) = \frac{1}{y}[p - g(y)]$ and $g'(y) = \frac{1}{x}[q - f(x)]$. Hence,

$$s = f'(x) + g'(y) = \frac{1}{y}[p - g(y)] + \frac{1}{x}[q - f(x)].$$

Thus, $xys = x[p-g(y)] + y[q-f(x)] = px + qy - [yf(x) + xg(y)] = px + qy - z$ is the resulting partial differential equation.

Note that the differential equation is of order two although, in general, a higher order is expected. However, since one of the relations 3) involves only the first derivatives of f and g, it is possible to eliminate f, g, f', g' between this relation, 1), 2), and the given relation.

9. Find the differential equation of all surfaces cutting the family of cones $x^2 + y^2 - a^2z^2 = 0$ orthogonally.

Let $z = f(x,y)$ be the equation of the required surfaces. At a point $P(x,y,z)$ on the surface, a set of direction numbers of the normal to the surface is $[p, q, -1]$. Likewise, at P a set of direction numbers of the normal to the cone through P is $[x, y, -a^2z]$. Since these directions are orthogonal,

$$px + qy + a^2z = 0.$$

The elimination of a^2 between this and the given equation yields the required differential equation

$$z(px + qy) + x^2 + y^2 = 0.$$

10. A surface which is the envelope of a one-parameter family of planes is called a developable surface. (Such a surface can be deformed (developed) into a plane without stretching or tearing.) Obtain the differential equation of developable surfaces.

Let $z = f(x,y)$ be the equation of a developable surface.

The tangent plane at a point (x_0, y_0, z_0) of the surface has equation

$$1)\quad F = (x - x_0)p + (y - y_0)q - (z - z_0) = 0.$$

Now when p and q satisfy a relation $\phi(p,q) = 0$, 1) is a one-parameter family of planes having $z = f(x,y)$ as envelope. Thus $\phi(p,q) = 0$ or $q = \lambda(p)$ is the required differential equation.

The cone of Problem 9 is a developable surface since $p = \dfrac{x}{a^2 z}$, $q = \dfrac{y}{a^2 z}$ satisfies $\phi(p,q) = a^2(p^2 + q^2) - 1 = 0$.

11. Eliminate the arbitrary functions ϕ_1 and ϕ_2 from

$$z = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) = \phi_1(u) + \phi_2(v)$$

in which $m_1 \neq m_2$ are fixed constants.

Differentiating partially, we obtain

$$r = m_1^2 \frac{d^2\phi_1}{du^2} + m_2^2 \frac{d^2\phi_2}{dv^2}, \qquad s = m_1 \frac{d^2\phi_1}{du^2} + m_2 \frac{d^2\phi_2}{dv^2}, \qquad t = \frac{d^2\phi_1}{du^2} + \frac{d^2\phi_2}{dv^2}.$$

Eliminating $\dfrac{d^2\phi_1}{du^2}, \dfrac{d^2\phi_2}{dv^2}$ we have $\begin{vmatrix} m_1^2 & m_2^2 & r \\ m_1 & m_2 & s \\ 1 & 1 & t \end{vmatrix} = (m_1 - m_2)r - (m_1^2 - m_2^2)s + (m_1^2 m_2 - m_1 m_2^2)t = 0$

or, since $m_1 \neq m_2$, $\quad r - (m_1 + m_2)s + m_1 m_2 t = 0.$

12. Show that *(a)* $z = ax^3 + by^3$ and *(b)* $z = ax^3 + bx^2y + cxy^2 + dy^4/x$ give rise to the same differential equation.

a) Differentiating $z = ax^3 + by^3$ partially with respect to x and y, we have

$$p = 3ax^2 \text{ and } q = 3by^2.$$

Thus, $px + qy = 3(ax^3 + by^3) = 3z$ is the resulting differential equation.

b) Differentiating $z = ax^3 + bx^2y + cxy^2 + dy^4/x$ partially with respect to x and y, we have

$$p = 3ax^2 + 2bxy + cy^2 - dy^4/x^2 \quad\text{and}\quad q = bx^2 + 2cxy + 4dy^3/x.$$

Thus, $px + qy = 3(ax^3 + bx^2y + cxy^2 + dy^4/x) = 3z$ as before.

The fact that these two equations, one with two arbitrary constants and the other with four, give rise to the same differential equation will indicate the subordinate role which the arbitrary constant will play here. In its place we will have arbitrary functions. Since *(a)* may be written as

$$z = ax^3 + by^3 = x^3[a + b(y/x)^3] = x^3 \cdot g(y/x),$$

while *(b)* may be written as

$$z = x^3[a + b(y/x) + c(y/x)^2 + d(y/x)^4] = x^3 \cdot h(y/x),$$

each is a particular case of $z = x^3 \cdot f(y/x)$ considered in Example 4.

SUPPLEMENTARY PROBLEMS

Eliminate the arbitrary constants a, b, c from each of the following equations.

13. $z = (x-a)^2 + (y-b)^2$ *Ans.* $4z = p^2 + q^2$

14. $z = axy + b$ $xp - yq = 0$

15. $ax + by + cz = 1$ $r = 0, \ s = 0,$ or $t = 0$

16. $z = axe^y + \frac{1}{2}a^2e^{2y} + b$ $q = xp + p^2$

17. $z = xy + y\sqrt{x^2 - a^2} + b$ $pq = xp + yq$

18. $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ $xzr + xp^2 - zp = 0, \ yzt + yq^2 - zq = 0,$ or $zs + pq = 0$

Eliminate the arbitrary constants a, b and the arbitrary functions ϕ, f, g.

19. $z = x^2\phi(x-y)$ or $\psi(z/x^2, x-y) = 0$ *Ans.* $2z = xp + xq$

20. $xyz = \phi(x+y+z)$ $x(y-z)p + y(z-x)q = z(x-y)$

21. $z = (x+y)\,\phi(x^2 - y^2)$ $yp + xq = z$

22. $z = f(x) + e^y g(x)$ $t - q = 0$

23. $x = f(z) + g(y)$ $ps - qr = 0$

24. $z = f(xy) + g(x+y)$ *Ans.* $x(y-x)r - (y^2 - x^2)s + y(y-x)t + (p-q)(x+y) = 0$

25. $z = f(x+z) + g(x+y)$ *Ans.* $qr - (1+p+q)s + (1+p)t = 0$

26. $z = ax^2 + g(y)$ $p - xr = 0$ or $s = 0$

27. $z = \frac{1}{2}(a^2+2)x^2 + axy + bx + \phi(y + ax)$ $r - 2t + rt - s^2 = 2$

28. Find the differential equation of all spheres of radius 2 having their centres in the xOy plane.
Hint: Eliminate a and b from $(x-a)^2 + (y-b)^2 + z^2 = 4$. *Ans.* $z^2(p^2 + q^2 + 1) = 4$

29. Find the differential equation of planes having equal x- and y- intercepts. *Ans.* $p - q = 0$

30. Find the differential equation of all surfaces of revolution having the z- axis as axis of rotation.

Hint: Eliminate ϕ from $z = \phi(\sqrt{x^2 + y^2}) = \psi(x^2 + y^2)$. *Ans.* $yp - xq = 0$

CHAPTER 29

Linear Partial Differential Equations of Order One

THE PARTIAL DIFFERENTIAL EQUATIONS of order one

$1_1)$ $px + qy = 3z$ and $1_2)$ $px^2 + qy = z^3$

are called *linear* to indicate that they are of the first degree in p and q. Note that, unlike linear ordinary differential equations, there is no restriction on the degree of the dependent variable z.

All partial differential equations of order one which are not linear, as

$2_1)$ $p^2 + q^2 = 1$ and $2_2)$ $p + \ln q = 2z^3$,

are called *non-linear*.

LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF ORDER ONE. Equation $1_1)$ was obtained in Chapter 28, Example 4, from the arbitrary functional relation

3) $$\phi(z/x^3,\ y/x) = 0$$

or its equivalent $z/x^3 = f(y/x)$. This solution, involving an arbitrary function, is called the *general solution* of $1_1)$.

The differential equation was also obtained (Chapter 28, Problem 12) by eliminating the arbitrary constants from

$4_1)$ $$z = ax^3 + by^3$$

and from

$4_2)$ $$z = ax^3 + bx^2y + cxy^2 + dy^4/x.$$

A study of the problems of that chapter indicates that relations involving two arbitrary constants usually yield non-linear partial differential equations of order one, while those involving more than two arbitrary constants yield equations of order higher than one. However, as was pointed out in Chapter 28, Problem 12, both of these relations are particular cases of the arbitrary functional relation 3). It is clear then that the general solution of 1) yields a much greater variety of solutions than that obtained (in the case of ordinary differential equations) through the appearance of arbitrary constants; for example,

$$z/x^3 = A\sin(y/x)^2 + B\cos(y/x) + C\ln(y/x) + De^{y/x} + E(y/x)^{12}$$

is included in the general solution 3).

THE GENERAL SOLUTION. A linear partial differential equation of order one, involving a dependent variable z and two independent variables x and y, is of the form

5) $$Pp + Qq = R$$

where P, Q, R are functions of x, y, z.

If $P = 0$ or $Q = 0$, 5) may be solved easily. Thus, the equation $\frac{\partial z}{\partial x} = 2x + 3y$ has as solution $z = x^2 + 3xy + \phi(y)$, where ϕ is an arbitrary function.

Lagrange reduced the problem of finding the general solution of 5) to that of solving an auxiliary system (called the Lagrange system) of *ordinary* differential equations

6) $$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

by showing (see Problem 7) that

7) $$\phi(u, v) = 0, \qquad (\phi, \text{ arbitrary})$$

is the general solution of 5) provided $u = u(x, y, z) = a$ and $v = v(x, y, z) = b$ are two independent solutions of 6). Here, a and b are arbitrary constants and at least one of u, v must contain z.

EXAMPLE 1. Find the general solution of

1) $$px + qy = 3z.$$

The auxiliary system is $\dfrac{dx}{x} = \dfrac{dy}{y} = \dfrac{dz}{3z}$.

From $\dfrac{dx}{x} = \dfrac{dz}{3z}$, we obtain $u = z/x^3 = a$; and from $\dfrac{dx}{x} = \dfrac{dy}{y}$, we obtain $v = y/x = b$.

Thus, the general solution is $\phi(z/x^3, y/x) = 0$, where ϕ is arbitrary.

Of course, from $\dfrac{dy}{y} = \dfrac{dz}{3z}$, we obtain $z/y^3 = c$, and we may write

$$\psi(z/x^3, z/y^3) = 0 \qquad \text{or} \qquad \lambda(z/y^3, y/x) = 0,$$

where ψ and λ are arbitrary. However these are all equivalent and we shall call any one of them *the* general solution.

The above procedure may be extended readily to solve linear first order differential equations involving more than two independent variables.

EXAMPLE 2. Find the general solution of

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + t\frac{\partial z}{\partial t} = xyt,$$

z being the dependent variable.

The auxilary system is $\dfrac{dx}{x} = \dfrac{dy}{y} = \dfrac{dt}{t} = \dfrac{dz}{xyt}$.

We obtain readily $u = x/y = a$, $v = t/y = b$.

A third independent solution may be found by using the multipliers yt, xt, xy, -3. Since

$$x(yt) + y(xt) + t(xy) + (xyt)(-3) = 0,$$

$$yt\,dx + xt\,dy + xy\,dt - 3\,dz = 0$$

and $$xyt - 3z = c.$$

Thus, the general solution is $\phi(x/y,\ t/y,\ xyt - 3z) = 0$.

COMPLETE SOLUTIONS. If $u = a$ and $v = b$ are two independent solutions of 6) and if α, β are arbitrary constants,

8) $$u = \alpha v + \beta$$

is called *a complete solution* of 5). Thus, for the equation of Example 1.

$$z/x^3 = \alpha(y/x) + \beta$$

is a complete solution.

A complete solution 8) represents a two-parameter family of surfaces which does not have an envelope, since the arbitrary constants enter linearly. It is possible, however, to select one-parameter families of surfaces from among 8) which have envelopes. As shown in Problem 8, these envelopes (surfaces) are merely particular surfaces of the general solution.

SOLVED PROBLEMS

1. Find the general solution of $2p + 3q = 1$.

The auxiliary system is $\dfrac{dx}{2} = \dfrac{dy}{3} = \dfrac{dz}{1}$.

From $\dfrac{dx}{2} = \dfrac{dz}{1}$, we have $x - 2z = a$; and from $\dfrac{dx}{2} = \dfrac{dy}{3}$, we have $3x - 2y = b$. Thus, the general solution is

$$\phi(x - 2z,\ 3x - 2y) = 0.$$

The complete solution $x - 2z = \alpha(3x - 2y) + \beta$ is a two-parameter family of planes. The one-parameter family determined by taking $\beta = \alpha^2$ has equation

$$A)\quad x - 2z = \alpha(3x - 2y) + \alpha^2.$$

Differentiating $A)$ with respect to α yields $0 = 3x - 2y + 2\alpha$ or $\alpha = -\frac{1}{2}(3x - 2y)$.

Substituting for α in $A)$, we obtain the envelope, a parabolic cylinder, $x - 2z = -\frac{1}{4}(3x - 2y)^2$. This cylinder is clearly a part of the general solution.

2. Find the general solution of $y^2zp - x^2zq = x^2y$.

The auxiliary equations are $\dfrac{dx}{y^2z} = \dfrac{dy}{-x^2z} = \dfrac{dz}{x^2y}$.

From $\dfrac{dz}{x^2y} = \dfrac{dy}{-x^2z}$ or $z\,dz + y\,dy = 0$, we have $y^2 + z^2 = a$; from $\dfrac{dx}{y^2z} = \dfrac{dy}{-x^2z}$, we have $x^3 + y^3 = b$.

Thus, the general solution is $\phi(y^2 + z^2, x^3 + y^3) = 0$.

3. Find the general solution of $(y - z)p + (z - y)q = z - x$.

The auxiliary system is $\dfrac{dx}{y - z} = \dfrac{dy}{x - y} = \dfrac{dz}{z - x}$.

Since $(y - z) + (x - y) + (z - x) = 0$, $dx + dy + dz = 0$ and $x + y + z = a$.
Since $x(y - z) + z(x - y) + y(z - x) = 0$, $x\,dx + z\,dy + y\,dz = 0$ and $x^2 + 2yz = b$.

Thus, the general solution is $\phi(x^2 + 2yz,\ x+y+z) = 0$.

The complete solution $x^2 + 2yz = \alpha(x + y + z) + \beta$ represents a family of hyperboloids.

4. Find the general solution of $(x^2-y^2-z^2)p + 2xyq = 2xz$.

The auxiliary system is $\dfrac{dx}{x^2-y^2-z^2} = \dfrac{dy}{2xy} = \dfrac{dz}{2xz}$.

From $\dfrac{dy}{2xy} = \dfrac{dz}{2xz}$, we obtain $y/z = a$.

From $\dfrac{dz}{2xz} = \dfrac{x\,dx + y\,dy + z\,dz}{x(x^2-y^2-z^2) + y(2xy) + z(2xz)} = \dfrac{x\,dx + y\,dy + z\,dz}{x(x^2+y^2+z^2)}$ or $\dfrac{dz}{z} = \dfrac{2(x\,dx + y\,dy + z\,dz)}{x^2+y^2+z^2}$,

we obtain $\dfrac{x^2+y^2+z^2}{z} = b$.

Thus, the general solution is $\phi(\frac{y}{z}, \frac{x^2+y^2+z^2}{z}) = 0$.

The complete solution $x^2+y^2+z^2 = \alpha y + \beta z$ consists of the spheres through the origin with centres on the plane yOz.

5. Solve $ap + bq + cz = 0$.

The auxiliary system is $\dfrac{dx}{a} = \dfrac{dy}{b} = \dfrac{dz}{-cz}$. From $\dfrac{dx}{a} = \dfrac{dy}{b}$, we obtain $ay - bx = A$.

If $a \neq 0$, $\dfrac{dz}{-cz} = \dfrac{dx}{a}$ yields $\ln z = -\frac{c}{a}x + \ln B$ or $z = Be^{-cx/a}$, and the general solution may be written as $z = e^{-cx/a}\phi(ay-bx)$. If $b \neq 0$, $\dfrac{dz}{-cz} = \dfrac{dy}{b}$ yields $z = Ce^{-cy/b}$, and the general solution may be written as $z = e^{-cy/b}\psi(ay-bx)$.

6. Solve 1) $2p+q+z=0$, 2) $p-3q+2z=0$, 3) $2p+3q+5z=0$, 4) $q+2z=0$.

1) Comparing with Problem 5 above, $a=2$, $b=1$, $c=1$.
The general solution is $z = e^{-x/2}\phi(2y-x)$ or $z = e^{-y}\psi(2y-x)$.

2) Here, $a=1$, $b=-3$, $c=2$. The general solution is $z = e^{-2x}\phi(y+3x)$ or $z = e^{2y/3}\psi(y+3x)$.

3) The general solution is $z = e^{-5x/2}\phi(2y-3x)$ or $z = e^{-5y/3}\psi(2y-3x)$.

4) The general solution is $z = e^{-2y}\phi(-x) = e^{-2y}\psi(x)$.

7. Show that if $u = u(x,y,z) = a$ and $v = v(x,y,z) = b$ are two independent solutions of $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$, where P,Q,R are functions of x,y,z, then $\phi(u,v) = 0$, with ϕ arbitrary, is the general solution of $Pp + Qq = R$.

Taking the differentials of $u = a$ and $v = b$, we have

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0, \qquad \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz = 0.$$

Since u and v are independent functions, we may solve for the ratios

$$dx : dy : dz = \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right) : \left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right) : \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) = P : Q : R.$$

But these are the relations (see Chapter 28) defining P,Q,R in the equation $Pp + Qq = R$ whose general solution is $\phi(u,v) = 0$.

8. Let $u = \alpha v + \beta$ be a complete solution of $Pp + Qq = R$. From this two-parameter family of surfaces, select a one-parameter family by setting $\beta = h(\alpha)$, where h is a given function of α, and obtain the envelope.

The envelope of the family

$$1)\quad u = \alpha v + h(\alpha)$$

is obtained by eliminating α between 1) and

$$2)\quad 0 = v + h'(\alpha).$$

Solving 2) for $\alpha = \mu(v)$ and substituting in 1), we have

$$3)\quad u = v\cdot\mu(v) + h[\mu(v)] = \lambda(v).$$

Now 3) is a part of the general solution $\phi(u,v) = 0$. Thus, unlike the case of ordinary differential equations, the envelope is not a new locus.

If $h(\alpha)$ is taken as an arbitrary function of α, $\lambda(v)$ is an arbitrary function of v, and 3) *is* the general solution. Thus, the general solution of a linear partial differential equation of order one is the totality of envelopes of all one-parameter families 1) obtained from a complete solution. It is to be noted that when $h(\alpha)$ is arbitrary, the elimination of α between 1) and 2) is not possible; thus, the general solution cannot be obtained from the complete solution.

9. Show that the conditions for exactness of the ordinary differential equation

$$\mu(x,y)\,M(x,y)\,dx + \mu(x,y)\,N(x,y)\,dy = 0$$

is a linear partial differential equation of order one. Thus, show how to find an integrating factor of $M\,dx + N\,dy = 0$. (See Chapter 4.)

If

$$\mu M\,dx + \mu N\,dy = 0$$

is exact, then $\dfrac{\partial}{\partial y}(\mu M) = \dfrac{\partial}{\partial x}(\mu N)$ or $M\dfrac{\partial\mu}{\partial y} - N\dfrac{\partial\mu}{\partial x} = \mu\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)$.

This is a linear partial differential equation of order one for which the auxiliary system is

$$1)\quad \frac{dx}{-N} = \frac{dy}{M} = \frac{d\mu}{\mu\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)}.$$

Any solution, involving μ, of this system is an integrating factor of $M\,dx + N\,dy = 0$.

Writing 1) in the form

$$2)\quad \frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{-N}\,dx = \frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}\,dy = \frac{d\mu}{\mu},$$

it is evident that if

$\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{-N} = f(x)$, then $\mu = e^{\int f(x)\,dx}$ is an integrating factor; or if $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = g(y)$, $\mu = e^{\int g(y)\,dy}$ is an integrating factor. Moreover, if the equation is linear (that is, $y' + Py = Q$), then $M = Py - Q$, $N = 1$ and 2) becomes $P\,dx = \dfrac{-P}{Py-Q}\,dy = \dfrac{d\mu}{\mu}$ and $\mu = e^{\int P\,dx}$ is an integrating factor.

10. Find an integrating factor for $(2x^3y - y^2)dx - (2x^4 + xy)dy = 0$. (See Problem 9 above.)

Here $M = 2x^3y - y^2$, $N = -(2x^4 + xy)$, $\dfrac{\partial M}{\partial y} = 2x^3 - 2y$, $\dfrac{\partial N}{\partial x} = -(8x^3 + y)$.

We seek a solution involving μ of $\dfrac{dx}{2x^4+xy} = \dfrac{dy}{2x^3y-y^2} = \dfrac{d\mu}{\mu(y-10x^3)}$.

From $\dfrac{d\mu}{\mu(y-10x^3)} = \dfrac{-2y\,dx-3x\,dy}{-2y(2x^4+xy)-3x(2x^3y-y^2)} = \dfrac{-2y\,dx-3x\,dy}{xy(y-10x^3)}$ or $\dfrac{d\mu}{\mu} = \dfrac{-2y\,dx-3x\,dy}{xy}$

we obtain $\ln\mu = -2\ln x - 3\ln y$. Thus, $\mu = x^{-2}y^{-3}$ is an integrating factor.

11. Find the integral surface of $x^2p+y^2q+z^2=0$ which passes the hyperbola $xy = x+y,\ z=1$.

The auxiliary system is $\dfrac{dx}{x^2} = \dfrac{dy}{y^2} = \dfrac{dz}{-z^2}$.

From $\dfrac{dx}{x^2} = \dfrac{dz}{-z^2}$ we obtain $u = \dfrac{x+z}{xz} = a$, and from $\dfrac{dy}{y^2} = \dfrac{dz}{-z^2}$ we obtain $v = \dfrac{y+z}{yz} = b$.

We first eliminate x_0, y_0, z_0 between $x_0y_0 = x_0+y_0$, $z_0 = 1$ and $u = \dfrac{x_0+z_0}{x_0z_0} = \dfrac{x_0+1}{x_0} = a$ and $v = \dfrac{y_0+z_0}{y_0z_0} = \dfrac{y_0+1}{y_0} = b$. Solving the latter for $x_0 = \dfrac{1}{a-1}$, $y_0 = \dfrac{1}{b-1}$ and substituting in $x_0y_0 = x_0+y_0$, we obtain $\dfrac{1}{(a-1)(b-1)} = \dfrac{1}{a-1}+\dfrac{1}{b-1}$ or $a+b=3$ as the relation which must exist between a and b. Then the equation of the required surface is

$$a+b = u+v = \frac{x+z}{xz}+\frac{y+z}{yz} = 3 \quad\text{or}\quad 2xy+z(x+y) = 3xyz.$$

SUPPLEMENTARY PROBLEMS

Find the general solution of each of the following equations.

12. $p+q=z$ *Ans.* $z = e^y\phi(x-y)$

13. $3p+4q=2$ $3y-4x = f(3z-2x)$ or $\phi(3y-4x,\ 3z-2x)=0$

14. $yq-xp=z$ $\phi(xy,\ xz)=0$

15. $xzp+yzq=xy$ $y = x\phi(xy-z^2)$

16. $x^2p+y^2q=z^2$ $x-y = xy\phi(1/x-1/z)$

17. $yp-xq+x^2-y^2=0$ $\phi(x^2+y^2,\ xy-z)=0$

18. $yzp-xzq=xy$ $\phi(x^2+y^2,\ y^2+z^2)=0$

19. $zp+yq=x$ $x+z = y\phi(x^2-z^2)$

20. $x(y-z)p+y(z-x)q=z(x-y)$ $\phi(xyz,\ x+y+z)=0$

21. $x(y^2-z^2)p+y(z^2-x^2)q=z(x^2-y^2)$ $\phi(xyz,\ x^2+y^2+z^2)=0$

22. Find the equation of all thesurfaces whose tangent planes pass through the point $(0,0,1)$.
Hint: Solve $xp+yq=z-1$. *Ans.* $z = 1+x\phi(y/x)$

23. Find the equation of the surface satisfying $4yzp+q+2y=0$ and passing through $y^2+z^2=1,\ x+z=2$.
Ans $y^2+z^2+x+z=3$

CHAPTER 30

Non-linear Partial Differential Equations of Order One

COMPLETE AND SINGULAR SOLUTIONS. Let the non-linear partial differential equation of order one

1) $$f(x,y,z,p,q) = 0$$

be derived from

2) $$g(x,y,z,a,b) = 0$$

by eliminating the arbitrary constants a and b. Then 2) is called a (or the) *complete solution* of 1).

This complete solution represents a two-parameter family of surfaces which may or may not have an envelope. To find the envelope (if one exists) we eliminate a and b from

$$g = 0, \quad \frac{\partial g}{\partial a} = 0, \quad \frac{\partial g}{\partial b} = 0.$$

If the eliminant

3) $$\lambda(x,y,z) = 0$$

satisfies 1), it is called the *singular solution* of 1); if

$$\lambda(x,y,z) = \xi(x\ y\ z)\cdot\eta(x,y,z)$$

and if $\xi = 0$ satisfies 1) while $\eta = 0$ does not, $\xi = 0$ is the singular solution. As in the case of ordinary differential equations (Chapter 10), the singular solution may be obtained from the partial differential equation by eliminating p and q from

$$f = 0, \quad \frac{\partial f}{\partial p} = 0, \quad \frac{\partial f}{\partial q} = 0.$$

EXAMPLE 1. It is readily verified that $z = ax + by - (a^2 + b^2)$ is a complete solution of $z = px + qy - (p^2 + q^2)$. Eliminating a and b from

$$g = z - ax - by + a^2 + b^2 = 0, \quad \frac{\partial g}{\partial a} = -x + 2a = 0, \quad \frac{\partial g}{\partial b} = -y + 2b = 0,$$

we have $z = \frac{1}{2}x^2 + \frac{1}{2}y^2 - \frac{1}{4}(x^2+y^2) = \frac{1}{4}(x^2+y^2)$. This satisfies the differential equation and is the singular solution. The complete solution represents a two-parameter family of planes which envelope the paraboloid $x^2 + y^2 = 4z$.

GENERAL SOLUTION. If, in the complete solution 2), one of the constants, say b, is replaced by a known function of the other, say $b = \phi(a)$, then

$$g(x,y,z,a,\phi(a)) = 0$$

is a one-parameter family of the surfaces of 1). If this family has an envelope, its equation may be found as usual by eliminating a from

$$g(x,y,z,a,\phi(a)) = 0 \quad \text{and} \quad \frac{\partial}{\partial a}\, g(x,y,z,a,\phi(a)) = 0$$

and determining that part of the result which satisfies 1).

EXAMPLE 2. Set $b = \phi(a) = a$ in the complete solution of Example 1. The result of eliminating a from $g = z - a(x+y) + 2a^2 = 0$ and $\frac{\partial g}{\partial a} = -(x+y) + 4a = 0$ is $z = \frac{1}{8}(x+y)^2$ which can be readily shown to satisfy the differential equation of Example 1. This is a parabolic cylinder with its elements parallel to the xOy plane.

The totality of solutions obtained by varying $\phi(a)$ is called the *general solution* of the differential equation. Thus, from Example 2, $8z = (x+y)^2$ is included in the general solution of the differential equation of Example 1.

When $b = \phi(a)$, ϕ arbitrary, is used, the elimination of a between

$$g = 0 \quad \text{and} \quad \frac{\partial g}{\partial a} = 0$$

is not possible; hence, we are unable to express the general solution as a single equation, involving an arbitrary function, as we were in the case of the linear equation.

SOLUTIONS. Before considering a general method for obtaining a complete solution of 1), we give special procedures for handling four types of equations.

TYPE I: $f(p,q) = 0$. Example: $p^2 - q^2 = 1$.

From Problem 3, Chapter 28, it follows that a complete solution is

4) $$z = ax + h(a)y + c,$$

where $f(a, h(a)) = 0$, and a and c are arbitrary constants.

The equations for determining the singular solution are

$$z = ax + h(a)y + c, \qquad 0 = x + h'(a)y, \qquad 0 = 1.$$

Thus, there is no singular solution.

The general solution is obtained by putting $c = \phi(a)$, ϕ arbitrary, and eliminating a between

5) $$z = ax + h(a)y + \phi(a) \quad \text{and} \quad 0 = x + h'(a)y + \phi'(a).$$

The first equation of 5) for a stipulated function $\phi(a)$ represents a one-parameter family of planes and its envelope (a part of the general solution) is a developable surface. (See Problem 10, Chapter 28.)

EXAMPLE 3. Solve $p^2 - q^2 = 1$.

Here $f(p,q) = p^2 - q^2 - 1 = 0$, $f(a,h(a)) = a^2 - [h(a)]^2 - 1 = 0$ and $h(a) = (a^2-1)^{1/2}$.

A complete solution is $z = ax + (a^2-1)^{1/2}y + c$.

A neater form is obtained by putting $a = \sec\alpha$; then $h(a) = \tan\alpha$ and we have

$$z = x\sec\alpha + y\tan\alpha + c.$$

If we set $c = \phi(\alpha) = 0$, the result of eliminating α from

$$z = x\sec\alpha + y\tan\alpha, \quad 0 = x\tan\alpha + y\sec\alpha \quad \text{or} \quad 0 = x\sin\alpha + y$$

is

$$z^2 = x^2 - y^2.$$

This developable surface (cone) is a part of the general solution of the given differential equation.

Note that we might have taken $h(a) = -(a^2-1)^{1/2}$ and obtained as a complete solution

$$z = ax - (a^2-1)^{1/2}y + c.$$

See also Problems 1-2.

TYPE II: $z = px + qy + f(p,q)$. Example: $z = px + qy + 3p^{1/3}q^{1/3}$.

From Problem 4, Chapter 28, it follows that a complete solution is

6) $$z = ax + by + f(a,b).$$

This is known as the extended Clairaut type, for obvious reasons. This complete solution consists of a two-parameter family of planes. The singular solution (if one exists) is a surface having the complete solution as its tangent planes.

EXAMPLE 4. Solve $z = px + qy + 3p^{1/3}q^{1/3}$.

A complete solution is $z = ax + by + 3a^{1/3}b^{1/3}$.

The derivatives with respect to a and b are $x + a^{-2/3}b^{1/3} = 0$ and $y + a^{1/3}b^{-2/3} = 0$.

Then $$ax + by = -2a^{1/3}b^{1/3}, \quad xy = a^{-1/3}b^{-1/3},$$

and, substituting in the complete solution, we obtain the singular solution

$$z = a^{1/3}b^{1/3} = 1/xy \quad \text{or} \quad xyz = 1.$$

See also Problems 3-4

TYPE III: $f(z,p,q) = 0$. Example: $z = p^2 + q^2$.

Assume $z = F(x+ay) = F(u)$, where a is the arbitrary constant. Then

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} = \frac{dz}{du} \quad \text{and} \quad q = \frac{dz}{du}\frac{\partial u}{\partial y} = a\frac{dz}{du}.$$

When these are substituted in the given differential equation, we obtain an ordinary differential equation of order one

$$f(z, \frac{dz}{du}, a\frac{dz}{du}) = 0$$

whose solution is the required complete solution.

EXAMPLE 5. Solve $z = p^2 + q^2$.

Put $z = F(x+ay) = F(u)$. Then $p = dz/du$, $q = a\,dz/du$, and the given equation may be reduced to $z = (\frac{dz}{du})^2 + a^2(\frac{dz}{du})^2$.

Solving $\frac{dz}{du} = \frac{\sqrt{z}}{\sqrt{1+a^2}}$ or $\frac{dz}{\sqrt{z}} = \frac{1}{\sqrt{1+a^2}}du$, we obtain $2\sqrt{z} = \frac{1}{\sqrt{1+a^2}}u + k = \frac{1}{\sqrt{1+a^2}}(u+b)$.

Thus, a complete solution is $4(1+a^2)z = (x+ay+b)^2$, a family of parabolic cylinders.

Taking the derivatives with respect to a and b, we have

$$8az - 2(x+ay+b)y = 0, \quad x + ay + b = 0.$$

The singular solution is $z = 0$.

See also Problems 5-7.

TYPE IV: $f_1(x,p) = f_2(y,q)$. Example: $p - x^2 = q + y^2$.

Set $f_1(x,p) = a$, $f_2(y,q) = a$, where a is an arbitrary constant, and solve to obtain

$$p = F_1(x,a) \quad \text{and} \quad q = F_2(y,a).$$

Since z is a function of x and y, $dz = p\,dx + q\,dy = F_1(x,a)\,dx + F_2(y,a)\,dy$. Thus,

7) $$z = \int F_1(x, a)\, dx + \int F_2(y, a)\, dy + b,$$

containing two arbitrary constants, is the required complete solution.

EXAMPLE 6. Solve $p - q = x^2 + y^2$ or $p - x^2 = q + y^2$.

Setting $p - x^2 = a$, $q + y^2 = a$, we obtain $p = a + x^2$, $q = a - y^2$.

Integrating $dz = p\,dx + q\,dy = (a + x^2)dx + (a - y^2)dy$, the required complete solution is $z = ax + x^3/3 + ay - y^3/3 + b$. There is no singular solution.

See also Problems 8-9.

TRANSFORMATIONS. As in the case of ordinary differential equations, it is possible at times to find a transformation of the variables which will reduce a given equation to one of the above four types.

The combination px, for example, suggests the transformation $X = \ln x$, since then

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{dX}{dx} = \frac{1}{x}\frac{\partial z}{\partial X} \qquad \text{and} \qquad px = \frac{\partial z}{\partial X}.$$

Thus, $q = px + p^2x^2$ becomes $\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial X} + (\dfrac{\partial z}{\partial X})^2$, of Type I.

Similarly, the combination qy suggests the transformation $Y = \ln y$.

The appearance of $\frac{p}{z}, \frac{q}{z}$ in an equation suggests the transformation $Z = \ln z$ since then

$$p = \frac{\partial z}{\partial x} = \frac{dz}{dZ}\frac{\partial Z}{\partial x} = z\frac{\partial Z}{\partial x} \qquad \text{and} \qquad \frac{p}{z} = \frac{\partial Z}{\partial x}; \quad \text{similarly,} \quad \frac{q}{z} = \frac{\partial Z}{\partial y}.$$

Thus, $\frac{q}{z} = (\frac{p}{z})^2$ becomes $\dfrac{\partial Z}{\partial y} = (\dfrac{\partial Z}{\partial x})^2$, of Type I.

See also Problems 10-14.

COMPLETE SOLUTION. CHARPIT'S METHOD. Consider the non-linear partial differential equation

1) $$f(x, y, z, p, q) = 0.$$

Since z is a function of x and y, it follows that

8) $$dz = p\,dx + q\,dy.$$

Let us assume $p = u(x, y, z, a)$, where a is an arbitrary constant, substitute in 1) and solve to obtain $q = v(x, y, z, a)$. For these values of p and q, 8) becomes

8_1) $$dz = u\,dx + v\,dy.$$

Now if 8_1) can be integrated, yielding

9) $$g(x, y, z, a, b) = 0,$$

this is a complete solution of 1).

EXAMPLE 7. Solve $pq + qx = y$.

Take $p = a - x$, substitute in $pq + qx = y$, and solve for $q = y/a$.

Substituting in $dz = p\,dx + q\,dy$, we have $dz = (a - x)dx + (y/a)dy$, an integrable equation, with solution

$$z = ax - \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2/a + k \qquad \text{or} \qquad 2az = 2a^2x - ax^2 + y^2 + b.$$

Since the success of the above procedure depends upon making a fortunate choice for p, it cannot be suggested as a standard procedure. We turn now to a general method for solving 1). This consists in finding an equation

10) $$F(x,y,z,p,q) = 0$$

such that 1) and 10) may be solved for $p = P(x,y,z)$ and $q = Q(x,y,z)$, (that is, such that

11) $$\Delta = \begin{vmatrix} \frac{\partial f}{\partial p} & \frac{\partial f}{\partial q} \\ \frac{\partial F}{\partial p} & \frac{\partial F}{\partial q} \end{vmatrix} \neq 0, \quad \text{identically}),$$

and such that for these values of p and q the total differential equation

8) $$dz = p\,dx + q\,dy = P(x,y,z)\,dx + Q(x,y,z)\,dy$$

is integrable, that is, $P\frac{\partial Q}{\partial z} - Q\frac{\partial P}{\partial z} - \frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} = \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} = 0.$

Differentiating 1) and 10) partially with respect to x and y, we find

12) $$\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = 0,$$

13) $$\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial y} = 0,$$

14) $$\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} = 0,$$

15) $$\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial y} = 0.$$

Multiplying 12) by $\frac{\partial F}{\partial p}$, 13) by $\frac{\partial F}{\partial q}$, 14) by $-\frac{\partial f}{\partial p}$, 15) by $-\frac{\partial f}{\partial q}$, and adding, we obtain (noting that $\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$)

$$\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial p} + \left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial q} - \frac{\partial f}{\partial p}\frac{\partial F}{\partial x} - \frac{\partial f}{\partial q}\frac{\partial F}{\partial y} - \left(p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial z} = 0.$$

This is a linear partial differential equation in F, considered as a function of the independent variables x, y, z, p, q. The auxiliary system is

16) $$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-(p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q})} = \frac{dF}{0}.$$

Thus, we may take for 10) any solution of this system which involves p or q, or both, which contains an arbitrary constant, and for which 11) holds.

EXAMPLE 8. Solve $q = -xp + p^2$.

Here $f = p^2 - xp - q$ so that $\frac{\partial f}{\partial x} = -p$, $\frac{\partial f}{\partial y} = 0$, $\frac{\partial f}{\partial z} = 0$, $\frac{\partial f}{\partial p} = 2p - x$, $\frac{\partial f}{\partial q} = -1$, and

$\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} = -p$, $\quad \frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} = 0$, $\quad -(p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q}) = -2p^2 + xp + q$.

The auxiliary system (16) is $\frac{dp}{-p} = \frac{dq}{0} = \frac{dx}{-2p+x} = \frac{dy}{1} = \frac{dz}{-2p^2+xp+q}$.

From $\frac{dp}{-p} = \frac{dy}{1}$, we have $\ln p = -y + \ln a$ or $p = ae^{-y}$.

Using the given differential equation, $q = -xp + p^2 = -axe^{-y} + a^2e^{-2y}$.

Then $dz = p\,dx + q\,dy$ becomes $dz = ae^{-y}dx + (-axe^{-y} + a^2e^{-2y})dy$. Integrating,

$$z = axe^{-y} - \tfrac{1}{2}a^2e^{-2y} + b.$$

There is no singular solution.

See also Problem 15.

SOLVED PROBLEMS

(In these solutions, the equations leading to the general solution will not be given.)

TYPE I: $f(p,q) = 0$.

1. Solve $p^2 + q^2 = 9$.

A complete solution is $z = ax + by + c$, where $a^2 + b^2 = 9$.

The equations for determining the singular solution are

$z = ax + \sqrt{9-a^2}\,y + c$, $\quad 0 = x - \frac{a}{\sqrt{9-a^2}}y$, $\quad 0 = 1$. Thus, there is no singular solution.

2. Solve $pq + p + q = 0$.

A complete solution is $z = ax + by + c$, where $ab + a + b = 0$, or $z = ax - \frac{a}{a+1}y + c$.

There is no singular solution.

TYPE II: $z = px + qy + f(p,q)$.

3. Solve $z = px + qy + p^2 + pq + q^2$.

A complete solution is $z = ax + by + a^2 + ab + b^2$.

Differentiating the complete solution with respect to a and b, we have

$$0 = x + 2a + b, \qquad 0 = y + a + 2b.$$

Solving to obtain $a = (y - 2x)/3$, $b = (x - 2y)/3$ and substituting in the complete solution, the singular solution is $3z = xy - x^2 - y^2$.

4. Solve $z = px + qy + p^2q^2$.

A complete solution is $z = ax + by + a^2b^2$. The equations obtained by differentiating with respect to a and b are $0 = x + 2ab^2$ and $0 = y + 2a^2 b$. Then $a = -\sqrt[3]{\frac{y^2}{2x}}$, $b = -\sqrt[3]{\frac{x^2}{2y}}$ and the singular solution is $z = -x\sqrt[3]{\frac{y^2}{2x}} - y\sqrt[3]{\frac{x^2}{2y}} + \sqrt[3]{\frac{x^2y^2}{16}} = -\frac{3}{4}\sqrt[3]{4}\, x^{2/3}y^{2/3}$.

TYPE III: $f(z,p,q) = 0$.

5. Solve $4(1+z^3) = 9z^4pq$.

Assume $z = F(x+ay) = F(u)$. Then $p = \frac{dz}{du}$, $q = a\frac{dz}{du}$, and the given equation becomes

$$4(1+z^3) = 9az^4(\frac{dz}{du})^2 \quad \text{or} \quad \frac{3\sqrt{a}\; z^2}{\sqrt{1+z^3}}\,dz = 2\,du.$$

Integrating, $\sqrt{a(1+z^3)} = u+b$, and a complete solution is $a(1+z^3) = (x+ay+b)^2$.

Using the results of differentiating this with respect to a and b,

$$1 + z^3 = 2(x+ay+b)y \quad \text{and} \quad 0 = 2(x+ay+b),$$

the singular solution is $z^3 + 1 = 0$.

6. Solve $p(1-q^2) = q(1-z)$.

Assume $z = F(x+ay) = F(u)$. Then $p = \frac{dz}{du}$, $q = a\frac{dz}{du}$, and the given equation becomes

$$(\frac{dz}{du})[1-a^2(\frac{dz}{du})^2] = a\frac{dz}{du}(1-z) \quad \text{or} \quad (\frac{dz}{du})[1-a+az-a^2(\frac{dz}{du})^2] = 0.$$

Then $\frac{dz}{du} = 0$ and $z = c$; or $1-a+az-a^2(\frac{dz}{du})^2 = 0$, $\frac{a\,dz}{\sqrt{1-a+az}} = du$ and

$$2\sqrt{1-a+az} = u+b = x+ay+b \quad \text{or} \quad 4(1-a+az) = (x+ay+b)^2.$$

Each of $z = c$ and $4(1-a+az) = (x+ay+b)^2$ is a solution; the latter is a complete solution. Using it, the equations for obtaining the singular solution are

$g = 4(1-a+az) - (x+ay+b)^2 = 0$, $\frac{\partial g}{\partial a} = 4(-1+z) - 2y(x+ay+b) = 0$, $\frac{\partial g}{\partial b} = -2(x+ay+b) = 0$; there is no singular solution.

7. Solve $1 + p^2 = qz$.

Assume $z = F(x+ay) = F(u)$. Then $p = \frac{dz}{du}$, $q = a\frac{d}{d}$ d the given equation becomes

$$(\frac{dz}{du})^2 - az\frac{dz}{du} + 1 = 0 \quad \text{or} \quad \frac{dz}{az - \sqrt{a^2z^2-4}} = \tfrac{1}{2}\,du.$$

Rationalizing the left member of the latter equation, we obtain $(az + \sqrt{a^2z^2-4})dz = 2\,du$ whose solution is $\frac{1}{2}az^2 + \frac{1}{a}[\frac{az}{2}\sqrt{a^2z^2-4} - 2\ln(az+\sqrt{a^2z^2-4})] = 2(u+b)$.

A complete solution is then $a^2z^2 + az\sqrt{a^2z^2-4} - 4\ln(az+\sqrt{a^2z^2-4}) = 4a(x+ay+b)$.

Note that $a^2z^2 - az\sqrt{a^2z^2-4} + 4\ln(az+\sqrt{a^2z^2-4}) = 4a(x+ay+b)$, obtained from $\frac{dz}{az+\sqrt{a^2z^2-4}} = \frac{1}{2}\,du$, is also a complete solution.

TYPE IV: $f_1(x,p) = f_2(y,q)$.

8. Solve $\sqrt{p} - \sqrt{q} + 3x = 0$ or $\sqrt{p} + 3x = \sqrt{q}$.

Set $\sqrt{p} + 3x = a$ and $\sqrt{q} = a$. Then $p = (a-3x)^2$ and $q = a^2$. A complete solution is

$$z = \int p\,dx + \int q\,dy + b = \int (a-3x)^2\,dx + a^2\int dy + b \quad \text{or} \quad z = -\frac{1}{9}(a-3x)^3 + a^2y + b.$$

There is no singular solution.

9. Solve $q = -px + p^2$.

Set $p^2 - px = a$ and $q = a$. Then $p = \frac{1}{2}(x + \sqrt{x^2+4a})$.

A complete solution is $z = \frac{1}{2}\int (x + \sqrt{x^2+4a})\,dx + a\int dy + b$

$$\text{or} \quad z = \tfrac{1}{4}(x^2 + x\sqrt{x^2+4a}) + a\ln(x + \sqrt{x^2+4a}) + ay + b.$$

Another complete solution is obtained by the method of Charpit in Example 8.
There is no singular solution.

USE OF TRANSFORMATIONS.

10. Solve $pq = x^m y^n z^{2l}$ or $\dfrac{pz^{-l}}{x^m}\cdot\dfrac{qz^{-l}}{y^n} = 1$.

The transformation

$$Z = \frac{z^{1-l}}{1-l},\quad X = \frac{x^{m+1}}{m+1},\quad Y = \frac{y^{n+1}}{n+1},\quad \frac{\partial Z}{\partial X} = \frac{\partial Z}{\partial x}\frac{dx}{dX} = z^{-l}p\frac{1}{x^m},\quad \frac{\partial Z}{\partial Y} = \frac{\partial Z}{\partial y}\frac{dy}{dY} = z^{-l}q\frac{1}{y^n}$$

reduces the given differential equation to $\dfrac{\partial Z}{\partial X}\cdot\dfrac{\partial Z}{\partial Y} = 1$.

This equation is of Type I and its solution is $Z = aX + \frac{1}{a}Y + c$.

A complete solution of the given equation is $\dfrac{z^{1-l}}{1-l} = a\dfrac{x^{m+1}}{m+1} + \dfrac{y^{n+1}}{a(n+1)} + c$.

There is no singular solution.

11. Solve $x^2p^2 + y^2q^2 = z$.

1) The transformation

$$X = \ln x,\quad Y = \ln y,\quad Z = 2z^{\frac{1}{2}},\quad \frac{\partial Z}{\partial X} = \frac{\partial Z}{\partial x}\frac{dx}{dX} = pxz^{-\frac{1}{2}},\quad \frac{\partial Z}{\partial Y} = \frac{\partial Z}{\partial y}\frac{dy}{dY} = qyz^{-\frac{1}{2}}$$

reduces the given equation to $z(\frac{\partial Z}{\partial X})^2 + z(\frac{\partial Z}{\partial Y})^2 = z$ or $(\frac{\partial Z}{\partial X})^2 + (\frac{\partial Z}{\partial Y})^2 = 1$, of Type I.

A complete solution is $Z = aX + bY + c$ or $4z = (a\ln x + b\ln y + c)^2$, where $a^2 + b^2 = 1$. The singular solution is $z = 0$.

2) The transformation $X = \ln x$, $Y = \ln y$, $p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial X}\dfrac{dX}{dx} = \dfrac{1}{x}\dfrac{\partial z}{\partial X}$, $q = \dfrac{1}{y}\dfrac{\partial z}{\partial Y}$

reduces the given differential equation to $(\frac{\partial z}{\partial X})^2 + (\frac{\partial z}{\partial Y})^2 = z$, of Type III.

We set $z = F(X + aY) = F(u)$. Then $\frac{\partial z}{\partial X} = \frac{dz}{du}\frac{\partial u}{\partial X} = \frac{dz}{du}$, $\frac{dz}{dY} = a\frac{dz}{du}$, and

$$(\frac{dz}{du})^2 + a^2(\frac{dz}{du})^2 = z \quad \text{or} \quad \sqrt{1+a^2}\,\frac{dz}{\sqrt{z}} = du.$$

Integrating. $2\sqrt{1+a^2}\, z^{\frac{1}{2}} = u + b = X + aY + b = \ln x + a \ln y + b.$

A complete solution is $4(1+a^2)z = (\ln x + a\ln y + b)^2$. The singular solution is $z = 0$.

12. Solve $4xyz = pq + 2px^2y + 2qxy^2$.

Let $x = X^{\frac{1}{2}}$, $y = Y^{\frac{1}{2}}$. Then $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{dX}{dx} = 2X^{\frac{1}{2}}\frac{\partial z}{\partial X}$ and $q = \frac{\partial z}{\partial Y}\frac{dY}{dy} = 2Y^{\frac{1}{2}}\frac{\partial z}{\partial Y}$.

Substituting in the given equation, we have $z = X\frac{\partial z}{\partial X} + Y\frac{\partial z}{\partial Y} + \frac{\partial z}{\partial X}\frac{\partial z}{\partial Y}$ of Type II.

A complete solution is $z = aX + bY + ab$ or $z = ax^2 + by^2 + ab$.

Eliminating a and b from this and $0 = x^2 + b$, $0 = y^2 + a$, obtained by differentiating it with respect to a and b, the singular solution is found to be $z + x^2y^2 = 0$.

13. Solve $p^2x^2 = z(z - qy)$.

The transformation $Y = \ln y$, $X = \ln x$, $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{dX}{dx} = \frac{1}{x}\frac{\partial z}{\partial X}$, $q = \frac{1}{y}\frac{\partial z}{\partial Y}$ reduces the given equation to $A)$ $(\frac{\partial z}{\partial X})^2 = z(z - \frac{\partial z}{\partial Y})$, of Type III.

We set $z = F(X + aY) = F(u)$. Then $\frac{\partial z}{\partial X} = \frac{dz}{du}$, $\frac{\partial z}{\partial Y} = a\frac{dz}{du}$, and $A)$ becomes $(\frac{dz}{du})^2 = z^2 - az\frac{dz}{du}$.

Then $\frac{dz}{du} = \frac{1}{2}z(\sqrt{a^2+4} - a)$, $2\frac{dz}{z} = (\sqrt{a^2+4} - a)du$, and $\ln z^2 = (\sqrt{a^2+4} - a)(u + b)$.

A complete solution is $\ln z^2 = (\sqrt{a^2+4} - a)(\ln x + a\ln y + b)$.
There is no singular solution.

14. Solve $p^2 + q^2 = z^2(x+y)$ or $(\frac{p}{z})^2 + (\frac{q}{z})^2 = x + y$.

The transformation $Z = \ln z$, $p = z\frac{\partial Z}{\partial x}$, $q = z\frac{\partial Z}{\partial y}$ reduces the given equation to

$$(\frac{\partial Z}{\partial x})^2 + (\frac{\partial Z}{\partial y})^2 = x + y \quad \text{or} \quad (\frac{\partial Z}{\partial x})^2 - x = y - (\frac{\partial Z}{\partial y})^2, \quad \text{of Type IV.}$$

Set $(\frac{\partial Z}{\partial x})^2 - x = a = y - (\frac{\partial Z}{\partial y})^2$. Then $\frac{\partial Z}{\partial x} = (a+x)^{\frac{1}{2}}$ and $\frac{\partial Z}{\partial y} = (y-a)^{\frac{1}{2}}$.

A complete solution is $Z = \int(a+x)^{\frac{1}{2}}dx + \int(y-a)^{\frac{1}{2}}dy + b$

$$\text{or} \quad \ln z = \frac{2}{3}(a+x)^{3/2} + \frac{2}{3}(y-a)^{3/2} + b.$$

CHARPIT'S METHOD.

15. Solve $16p^2z^2 + 9q^2z^2 + 4z^2 - 4 = 0$.

Let $f(x,y,z,p,q) = 16p^2z^2 + 9q^2z^2 + 4z^2 - 4$.

Then $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z} = 32p^2z + 18q^2z + 8z$, $\frac{\partial f}{\partial p} = 32pz^2$, $\frac{\partial f}{\partial q} = 18qz^2$, and the auxiliary system

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-(p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q})} \quad \text{is}$$

$$\frac{dp}{32p^3z + 18pq^2z + 8pz} = \frac{dq}{32p^2qz + 18q^3z + 8qz} = \frac{dx}{-32pz^2} = \frac{dy}{-18qz^2} = \frac{dz}{-32p^2z^2 - 18q^2z^2}.$$

Using the multipliers $4z, 0, 1, 0, 4p$, we find

$$4z(32p^3z + 18pq^2z + 8pz) + 1(-32pz^2) + 4p(-32p^2z^2 - 18q^2z^2) = 0$$

and so
$$dx + 4p\,dz + 4z\,dp = 0.$$

Then $x + 4pz = a$ and $p = -\frac{x-a}{4z}$. Substituting for p in the given differential equation, we find $(x-a)^2 + 9q^2z^2 + 4z^2 - 4 = 0$. Using the root $q = \frac{2}{3z}\sqrt{1 - z^2 - \frac{1}{4}(x-a)^2}$,

$$dz = p\,dx + q\,dy = -\frac{x-a}{4z}dx + \frac{2}{3z}\sqrt{1-z^2-\tfrac{1}{4}(x-a)^2}\,dy \quad \text{or} \quad dy = \frac{3[z\,dz + \frac{1}{4}(x-a)dx]}{2\sqrt{1-z^2-\frac{1}{4}(x-a)^2}}.$$

Then $y - b = -\frac{3}{2}\sqrt{1-z^2-\frac{1}{4}(x-a)^2}$ or $\frac{(x-a)^2}{4} + \frac{(y-b)^2}{9/4} + z^2 = 1$ is a complete solution.

This is a family of ellipsoids with centres on the xOy plane. The semi-axes of the ellipsoids are 2 units parallel to the x-axis, 3/2 units parallel to the y-axis, and 1 unit parallel to the z-axis. The singular solution consists of the parallel planes $z = \pm 1$.

Another complete solution may be found by noting that the equation is of Type III. Using $F(x + ay) = F(u)$ and setting $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$, the given equation becomes

$$16z^2\left(\frac{dz}{du}\right)^2 + 9a^2z^2\left(\frac{dz}{du}\right)^2 + 4z^2 - 4 = 0 \quad \text{or} \quad \frac{z\,dz}{\sqrt{1-z^2}} = \frac{2}{\sqrt{16+9a^2}}du. \quad \text{Then}$$

$$-\sqrt{1-z^2} = \frac{2}{\sqrt{16+9a^2}}(u+b) = \frac{2}{\sqrt{16+9a^2}}(x+ay+b).$$

This complete solution $(16+9a^2)(1-z^2) = 4(x+ay+b)^2$ represents a family of elliptic cylinders with elements parallel to the xOy plane. The major axis of a cross section lies in the xOy plane and the minor axis is 2 units parallel to the z-axis.

SUPPLEMENTARY PROBLEMS

Find a complete solution and the singular solution (if any).

16. $p = q^2$ *Ans.* $z = b^2x + by + c$

17. $p^2 + p = q^2$ $z = ax + by + c$ where $b^2 = a^2 + a$

18. $pq = p + q$ $(b-1)z = bx + b(b-1)y + c$

19. $z = px + qy + pq$ $z = ax + by + ab$; s.s., $z = -xy$

20. $p^2 + q^2 = 4z$ $z(1+a^2) = (x + ay + b)^2$; s.s., $z = 0$

21. $pz = 1 + q^2$ $z^2 - z\sqrt{z^2 - 4a^2} + 4a^2 \ln(z + \sqrt{z^2 - 4a^2}) = 4(x + ay + b)$

22. $z^2(p^2 + q^2 + 1) = 1$ $(1 + a^2)(1 - z^2) = (x + ay + b)^2$; s.s., $z^2 - 1 = 0$

23. $p^2 + pq = 4z$ $(1 + a)z = (x + ay + b)^2$; s.s., $z = 0$

24. $p^2 - x = q^2 - y$ $3(z - b) = 2(x + a)^{3/2} + 2(y + a)^{3/2}$

25. $yp - x^2q^2 = x^2y$ $4(a-1)y^3 = (3z - ax^3 - b)^2$

26. $(1 - y^2)xq^2 + y^2p = 0$ $(2z - ax^2 + b)^2 = 4a(y^2 - 1)$

27. $x^4p^2 - yzq - z^2 = 0$ $x \ln z = a + (a^2 - 1)x \ln y + bx$
Hint: Use $X = 1/x$, $Y = \ln y$, $Z = \ln z$.

28. $x^4p^2 + y^2zq - 2z^2 = 0$ $xy \ln z = ay + (a^2 - 2)x + bxy$
Hint: Use $X = 1/x$, $Y = 1/y$, $Z = \ln z$.

29. $x^4p^2 + y^2q = 0$ $x^2(zy + a + by)^2 + ay^2 = 0$

30. $2py^2 - q^2z = 0$ $z^2 = a^2x + ay^2 + b$

31. $q = xp + p^2$ $z = 2axe^y + 2a^2e^{2y} + b$

32. $zp^2 - y^2p + y^2q = 0$ $yz^2 = 2(axy + ay^2 + a^2 + by)$

Hint: $\dfrac{dp}{p^3} = \dfrac{dz}{-p^2z}$; $pz = a$ and $q = \dfrac{a}{z}\left(1 - \dfrac{a}{y^2}\right)$.

33. $pq + 2x(y+1)p + y(y+2)q - 2(y+1)z = 0$

Ans. $z = ax + b(y^2 + 2y + a)$; s.s. $z + x(y^2 + 2y) = 0$

CHAPTER 31

Homogeneous Partial Differential Equations of Higher Order with Constant Coefficients

AN EQUATION SUCH AS

1) $$(x^2+y^2)\frac{\partial^3 z}{\partial x^3} + 2x\frac{\partial^3 z}{\partial x\,\partial y^2} + \frac{\partial^3 z}{\partial y^3} - \frac{\partial^2 z}{\partial x^2} + 5xy\frac{\partial^2 z}{\partial x\,\partial y} + x^3\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} + yz = e^{x+y}$$

which is linear in the dependent variable z and its partial derivatives is called a *linear partial differential equation*. The order of 1) is three, being the order of the highest ordered derivative.

A linear partial differential equation such as

2) $$x^2\frac{\partial^3 z}{\partial x^3} + xy\frac{\partial^3 z}{\partial x^2\,\partial y} + 2\frac{\partial^3 z}{\partial x\,\partial y^2} + \frac{\partial^3 z}{\partial y^3} = x^2 + y^3,$$

in which the derivatives involved are all of the same order, will be called homogeneous, although there is no agreement among authors in the use of this term.

HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS.

Consider

3) $$A\frac{\partial z}{\partial x} + B\frac{\partial z}{\partial y} = 0,$$

4) $$A\frac{\partial^2 z}{\partial x^2} + B\frac{\partial^2 z}{\partial x\,\partial y} + C\frac{\partial^2 z}{\partial y^2} = 0,$$

5) $$A\frac{\partial^2 z}{\partial x^2} + B\frac{\partial^2 z}{\partial x\,\partial y} + C\frac{\partial^2 z}{\partial y^2} = x + 2y,$$

in which the numbers A, B, C are real constants.

It will be seen as we proceed that the methods for solving equations 3)–5) parallel those used in solving the linear ordinary differential equation

$$f(D)y = Q(x) \quad \text{where} \quad D = \frac{d}{dx}.$$

We shall employ two operators, $D_x = \frac{\partial}{\partial x}$ and $D_y = \frac{\partial}{\partial y}$, so that equations 3)–5) may be written as

3′) $$f(D_x, D_y)z = (AD_x + BD_y)z = 0,$$

4′) $$f(D_x, D_y)z = (AD_x^2 + BD_xD_y + CD_y^2)z = 0,$$

5′) $$f(D_x, D_y)z = (AD_x^2 + BD_xD_y + CD_y^2)z = x + 2y.$$

Equation 3′) is of order one and the general solution (Chapter 29) is $z = \phi(y - \frac{B}{A}x)$, ϕ arbitrary.

Suppose now that $z = \phi(y + mx) = \phi(u)$, ϕ arbitrary, is a solution of 4′); then substituting

$$D_x z = \frac{\partial z}{\partial x} = \frac{d\phi}{du}\frac{\partial u}{\partial x} = m\frac{d\phi}{du}, \qquad D_y z = \frac{\partial z}{\partial y} = \frac{d\phi}{du}\frac{\partial u}{\partial y} = \frac{d\phi}{du}$$

in 4′) we obtain

$$\frac{d^2\phi}{du^2}(Am^2 + Bm + C) = 0.$$

Since ϕ is arbitrary, $d^2\phi/du^2$ is not zero identically: hence, m is one of the roots $m = m_1, m_2$ of $Am^2 + Bm + C = 0$. If $m_1 \neq m_2$, $z = \phi_1(y + m_1x)$ and $z = \phi_2(y + m_2x)$ are distinct solution of 4′). Clearly,

$$z = \phi_1(y + m_1x) + \phi_2(y + m_2x)$$

is also a solution; it contains two arbitrary functions and is the *general solution*.

More generally, if

6) $$f(D_x, D_y)z = (D_x - m_1D_y)(D_x - m_2D_y)\cdots\cdots(D_x - m_nD_y)z = 0$$

and if $m_1 \neq m_2 \neq \cdots\cdots \neq m_n$, then

7) $$z = \phi_1(y + m_1x) + \phi_2(y + m_2x) + \cdots\cdots + \phi_n(y + m_nx)$$

is the general solution of $f(D_x, D_y)z = 0$.

EXAMPLE 1. Solve $(D_x^2 - D_xD_y - 6D_y^2)z = (D_x + 2D_y)(D_x - 3D_y)z = 0$.

Here, $m_1 = -2$, $m_2 = 3$, and the general solution is $y = \phi_1(y - 2x) + \phi_2(y + 3x)$.

See also Problems 1–2.

If $m_1 = m_2 = \cdots\cdots = m_k \neq m_{k+1} \neq \cdots\cdots \neq m_n$, so that 6) becomes

6′) $$f(D_x, D_y)z = (D_x - m_1D_y)^k(D_x - m_{k+1}D_y)\cdots\cdots(D_x - m_nD_y)z = 0,$$

the part of the general solution given by the corresponding k equal factors is

$$\phi_1(y + m_1x) + x\phi_2(y + m_1x) + x^2\phi_3(y + m_1x) + \cdots\cdots + x^{k-1}\phi_k(y + m_1x),$$

and the general solution of 6′) is

$$z = \phi_1(y + m_1x) + x\phi_2(y + m_1x) + \cdots\cdots + x^{k-1}\phi_k(y + m_1x) + \phi_{k+1}(y + m_{k+1}x) + \cdots\cdots + \phi_n(y + m_nx),$$

where $\phi_1, \phi_2, \cdots\cdots\phi_n$ are arbitrary functions.

EXAMPLE 2. Solve $(D_x^3 - D_x^2D_y - 8D_xD_y^2 + 12D_y^3)z = (D_x - 2D_y)^2(D_x + 3D_y)z = 0$.

Here, $m_1 = m_2 = 2$, $m_3 = -3$ and the general solution is $z = \phi_1(y + 2x) + x\phi_2(y + 2x) + \phi_3(y - 3x)$.

See also Problems 3-4.

If one of the numbers, say m_1, of 6) is imaginary then another, say m_2, is the conjugate of m_1. Let $m_1 = a + bi$ and $m_2 = a - bi$ so that 6) becomes

6'') $f(D_x, D_y)z = [D_x - (a+bi)D_y][D_x - (a-bi)D_y](D_x - m_3D_y) \cdots\cdots (D_x - m_nD_y)z = 0.$

The part of the general solution given by the first two factors is

$$\phi_1(y + ax + ibx) + \phi_1(y + ax - ibx) + i[\phi_2(y + ax + ibx) - \phi_2(y + ax - ibx)],$$

(ϕ_1, ϕ_2 arbitrary, real functions), and the general solution of 6'') is

$$z = \phi_1(y + ax + ibx) + \phi_1(y + ax - ibx) + i[\phi_2(y + ax + ibx) - \phi_2(y + ax - ibx)] + \phi_3(y + m_3x) + \cdots\cdots + \phi_n(y + m_nx).$$

EXAMPLE 3. Solve $(D_x^4 - D_x^3D_y + 2D_x^2D_y^2 - 5D_xD_y^3 + 3D_y^4)z$

$$= (D_x - D_y)^2[D_x + \tfrac{1}{2}(1 + i\sqrt{11})D_y][D_x + \tfrac{1}{2}(1 - i\sqrt{11})D_y]z = 0.$$

Here, $m_1 = m_2 = 1$, $m_3 = -\frac{1}{2}(1 + i\sqrt{11})$, $m_4 = -\frac{1}{2}(1 - i\sqrt{11})$, and the general solution is

$$z = \phi_1(y+x) + x\phi_2(y+x) + \phi_3[y - \tfrac{1}{2}(1 + i\sqrt{11})x] + \phi_3[y - \tfrac{1}{2}(1 - i\sqrt{11})x] + i[\phi_4\{y - \tfrac{1}{2}(1 + i\sqrt{11})x\} - \phi_4\{y - \tfrac{1}{2}(1 - i\sqrt{11})x\}].$$

See also Problem 5.

The general solution of

5') $$f(D_x, D_y)z = (AD_x^2 + BD_xD_y + CD_y^2)z = x + 2y$$

consists of the general solution of the reduced equation

4') $$f(D_x, D_y)z = (AD_x^2 + BD_xD_y + CD_y^2)z = 0$$

plus any particular integral of 5'). We shall speak of the general solution of 4') as the *complementary function* of 5').

In setting up procedures for obtaining a particular integral of

8) $$f(D_x, D_y)z = (D_x - m_1D_y)(D_x - m_2D_y)\cdots\cdots(D_x - m_nD_y)z = F(x, y),$$

we define the operator $\dfrac{1}{f(D_x, D_y)}$ by the identity

$$f(D_x, D_y)\frac{1}{f(D_x, D_y)}F(x, y) = F(x, y).$$

The particular integral, denoted by

$$z = \frac{1}{f(D_x, D_y)}F(x,y) = \frac{1}{(D_x - m_1D_y)(D_x - m_2D_y)\cdots\cdots(D_x - m_nD_y)}F(x,y),$$

may be found, as in Chapter 13, by solving n equations of the first order

9) $$u_1 = \frac{1}{D_x - m_nD_y}F(x, y), \quad u_2 = \frac{1}{D_x - m_{n-1}D_y}u_1, \quad \cdots\cdots, \quad z = u_n = \frac{1}{D_x - m_1D_y}u_{n-1}.$$

Note that each of the equations of 9) is of the form

10) $$p - mq = g(x,y)$$

and that we need only a solution .the simpler the better. In Problem 6 below, the following rule is established for obtaining one such solution of 10): Evaluate $z = \int g(x, a-mx)\,dx$, omitting the arbitrary constant of integration, and then replace a by $y+mx$.

EXAMPLE 4. Solve $(D_x^2 - D_xD_y - 6D_y^2)z = (D_x + 2D_y)(D_x - 3D_y)z = x + y$.

From Example 1, the complementary function is $z = \phi_1(y-2x) + \phi_2(y+3x)$.

To obtain the particular integral denoted by $z = \dfrac{1}{D_x + 2D_y}\left[\dfrac{1}{D_x - 3D_y}(x+y)\right]$:

a) Set $u = \dfrac{1}{D_x - 3D_y}(x+y)$ and obtain a particular integral of $(D_x - 3D_y)u = x + y$.

Using the procedure of Problem 6, we have $u = \int(x + a - 3x)\,dx = ax - x^2$ and, replacing a by $y+3x$, $u = xy + 2x^2$.

b) Set $z = \dfrac{1}{D_x + 2D_y}u = \dfrac{1}{D_x + 2D_y}(xy + 2x^2)$ and obtain a particular integral of

$$(D_x + 2D_y)z = xy + 2x^2.$$

Then $z = \int[x(a+2x) + 2x^2]\,dx = \frac{1}{2}ax^2 + \frac{4}{3}x^3$ and, replacing a by $y-2x$, $z = \frac{1}{2}x^2y + \frac{1}{3}x^3$.

Thus the general solution is $z = \phi_1(y-2x) + \phi_2(y+3x) + \frac{1}{2}x^2y + \frac{1}{3}x^3$.

See also Problems 8–9.

The method of undetermined coefficients may be used if $F(x,y)$ involves $\sin(ax+by)$ or $\cos(ax+by)$.

EXAMPLE 5. Solve

$$(D_x^2 + 5D_xD_y + 5D_y^2)z = [D_x - \tfrac{1}{2}(-5+\sqrt{5})D_y][D_x - \tfrac{1}{2}(-5-\sqrt{5})D_y]z = x\sin(3x-2y).$$

The complementary function is $z = \phi_1[y + \frac{1}{2}(-5+\sqrt{5})x] + \phi_2[y + \frac{1}{2}(-5-\sqrt{5})x]$.
Take as a particular integral

$$z = Ax\sin(3x-2y) + Bx\cos(3x-2y) + C\sin(3x-2y) + D\cos(3x-2y).$$ Then

$$D_x^2z = (6A - 9D)\cos(3x-2y) - (6B+9C)\sin(3x-2y) - 9Ax\sin(3x-2y) - 9Bx\cos(3x-2y),$$

$$D_xD_yz = (-2A+6D)\cos(3x-2y) + (2B+6C)\sin(3x-2y) + 6Ax\sin(3x-2y) + 6Bx\cos(3x-2y),$$

$$D_y^2z = -4D\cos(3x-2y) - 4C\sin(3x-2y) - 4Ax\sin(3x-2y) - 4Bx\cos(3x-2y),$$

and $$(D_x^2 + 5D_xD_y + 5D_y^2)z = Ax\sin(3x-2y) + Bx\cos(3x-2y) + (C+4B)\sin(3x-2y) + (D-4A)\cos(3x-2y) = x\sin(3x-2y).$$

Then $A = 1,\ B = C = 0,\ D = 4$ and the particular integral is

$$z = x\sin(3x-2y) + 4\cos(3x-2y).$$ The general solution is

$$z = \phi_1[y + \tfrac{1}{2}(-5+\sqrt{5})x] + \phi_2[y + \tfrac{1}{2}(-5-\sqrt{5})x] + x\sin(3x-2y) + 4\cos(3x-2y).$$

See also Problem 10.

Short methods for obtaining particular integrals, analogous to those of Chapter 16, may be used.

a) $\dfrac{1}{f(D_x, D_y)}e^{ax+by} = \dfrac{1}{f(a,b)}e^{ax+by}$, provided $f(a,b) \neq 0$.

If $f(a,b) = 0$, write $f(D_x, D_y) = (D_x - \frac{a}{b}D_y)^r g(D_x, D_y)$, where $g(a,b) \neq 0$; then

$$\frac{1}{(D_x - \frac{a}{b}D_y)^r}\,\frac{1}{g(D_x, D_y)}e^{ax+by} = \frac{1}{g(a,b)}\,\frac{1}{(D_x - \frac{a}{b}D_y)^r}e^{ax+by} = \frac{1}{g(a,b)}\,\frac{x^r}{r!}e^{ax+by}.$$

b) $$\frac{1}{f(D_x^2, D_xD_y, D_y^2)}\sin(ax+by) = \frac{1}{f(-a^2, -ab, -b^2)}\sin(ax+by) \quad \text{and}$$

$$\frac{1}{f(D_x^2, D_xD_y, D_y^2)}\cos(ax+by) = \frac{1}{f(-a^2, -ab, -b^2)}\cos(ax+by),$$

provided $f(-a^2, -ab, -b^2) \neq 0$.

EXAMPLE 6. Solve $(D_x^2 - 3D_xD_y + 2D_y^2)z = (D_x - D_y)(D_x - 2D_y)z = e^{2x+3y} + e^{x+y} + \sin(x-2y)$.

The complementary function is $z = \phi_1(y+x) + \phi_2(y+2x)$.

Now $\dfrac{1}{D_x^2 - 3D_xD_y + 2D_y^2}e^{2x+3y} = \dfrac{1}{2^2 - 3\cdot2\cdot3 + 2\cdot3^2}e^{2x+3y} = \dfrac{1}{4}e^{2x+3y}$ is one term of the particular integral. Since $\phi_1(y+x)$ includes e^{x+y}, we write

$$\frac{1}{D_x^2 - 3D_xD_y + 2D_y^2}e^{x+y} = \frac{1}{D_x - D_y}\left(\frac{1}{D_x - 2D_y}e^{x+y}\right) = \frac{1}{D_x - D_y}\left(\frac{1}{1-2\cdot1}e^{x+y}\right) = -\frac{1}{D_x - D_y}e^{x+y} = -xe^{x+y}.$$

Also, $\dfrac{1}{D_x^2 - 3D_xD_y + 2D_y^2}\sin(x-2y) = \dfrac{1}{-1-3(2)+2(-1)(-2)^2}\sin(x-2y) = -\dfrac{1}{15}\sin(x-2y)$.

Thus, the general solution is $z = \phi_1(y+x) + \phi_2(y+2x) + \frac{1}{4}e^{2x+3y} - xe^{x+y} - \frac{1}{15}\sin(x-2y)$.

c) If $F(x,y)$ is a polynomial, that is $F(x,y) = \Sigma p_{ij}x^iy^j$, where i, j are positive integers or zero and p_{ij} are constants, the procedure illustrated below may be used.

EXAMPLE 7. Solve $(D_x^2 - D_xD_y - 6D_y^2)z = x + y$. (Example 4.)

For a particular integral, write

$$\frac{1}{D_x^2 - D_xD_y - 6D_y^2}(x+y) = \frac{1}{D_x^2}\,\frac{1}{1 - \frac{D_y}{D_x} - 6\frac{D_y^2}{D_x^2}}(x+y) = \frac{1}{D_x^2}\{[1 + \frac{D_y}{D_x} + \cdots](x+y)\} = \frac{1}{D_x^2}(x + y + \frac{1}{D_x})$$

$$= \frac{1}{D_x^2}(x+y+x) = \frac{1}{D_x^2}(2x+y) = \frac{1}{3}x^3 + \frac{1}{2}x^2y.$$ Note that $D_y(x+y) = 1$ and $\frac{1}{D_x} = \int dx$.

See also Problems 11–13.

SOLVED PROBLEMS

1. Solve $(D_x^3 + 2D_x^2D_y - D_xD_y^2 - 2D_y^3)z = (D_x - D_y)(D_x + D_y)(D_x + 2D_y)z = 0.$

Here $m_1 = 1,\ m_2 = -1,\ m_3 = -2$ and the general solution is

$$z = \phi_1(y+x) + \phi_2(y-x) + \phi_3(y-2x).$$

2. Solve $(D_x^3 - 5D_x^2D_y + 5D_xD_y^2 + 3D_y^3)z = (D_x - 3D_y)[D_x - (1+\sqrt{2})D_y][D_x - (1-\sqrt{2})D_y]z = 0.$

Here $m_1 = 3,\ m_2 = 1+\sqrt{2},\ m_3 = 1-\sqrt{2}$ and the general solution is

$$z = \phi_1(y+3x) + \phi_2[y+(1+\sqrt{2})x] + \phi_3[y+(1-\sqrt{2})x].$$

3. Solve $(D_x^3 + 3D_x^2D_y - 4D_y^3)z = (D_x - D_y)(D_x + 2D_y)^2 z = 0.$

Since $m_1 = 1,\ m_2 = m_3 = -2,$ the general solution is

$$z = \phi_1(y+x) + \phi_2(y-2x) + x\,\phi_3(y-2x).$$

Another form of the general solution is

$$z = \phi_1(y+x) + \phi_2(y-2x) + y\,\phi_3(y-2x).$$

4. Solve $(D_x^4 - 2D_x^2D_y^2 + D_y^4)z = (D_x - D_y)^2(D_x + D_y)^2 z = 0.$

Here $m_1 = m_2 = 1,\ m_3 = m_4 = -1$ and the general solution is

$$z = \phi_1(y+x) + x\,\phi_2(y+x) + \phi_3(y-x) + x\,\phi_4(y-x).$$

5. Solve $(D_x^2 - 2D_xD_y + 5D_y^2)z = [D_x - (1+2i)D_y][D_x - (1-2i)D_y]z = 0.$

Since $m_1 = 1+2i,\ m_2 = 1-2i,$ the general solution is

$$z = \phi_1(y+x+2ix) + \phi_1(y+x-2ix) + i[\phi_2(y+x+2ix) - \phi_2(y+x-2ix)],$$

where ϕ_1, ϕ_2 are real functions.

If we take $\phi_1(u) = \cos u$ and $\phi_2(u) = e^u$, then since

$$e^{ibx} = \cos bx + i\sin bx, \qquad \sin bx = \frac{1}{2i}(e^{ibx} - e^{-ibx}),$$

$$e^{-ibx} = \cos bx - i\sin bx, \qquad \cos bx = \frac{1}{2}(e^{ibx} + e^{-ibx}),$$

$$\phi_1(y+x+2ix) = \cos(y+x)\cos(2ix) - \sin(y+x)\sin(2ix)$$
$$= \cos(y+x)\cosh 2x - i\sin(y+x)\sinh 2x,$$

$$\phi_1(y+x-2ix) = \cos(y+x)\cos(2ix) + \sin(y+x)\sin(2ix)$$
$$= \cos(y+x)\cosh 2x + i\sin(y+x)\sinh 2x,$$

$$\phi_2(y+x+2ix) - \phi_2(y+x-2ix) = e^{y+x+2ix} - e^{y+x-2ix} = e^{y+x}(e^{2ix} - e^{-2ix}) = 2ie^{y+x}\sin 2x.$$

Thus, we obtain as a particular integral

$$z = [\cos(y+x)\cosh 2x - i\sin(y+x)\sinh 2x] + [\cos(y+x)\cosh 2x + i\sin(y+x)\sinh 2x]$$
$$+ i(2ie^{y+x}\sin 2x) = 2\cos(y+x)\cosh 2x - 2e^{y+x}\sin 2x.$$

Note that z is a real function of x and y.

6. Show that a particular integral of $p - mq = g(x,y)$ may be found by integrating $dz = g(x, a-mx)dx$, omitting the arbitrary constant of integration, and then replacing a by $y + mx$.

The auxiliary system is $\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{g(x,y)}$. Integrating the equation formed with the first two terms, we have $y + mx = a$. Using this relation, the equation

$$\frac{dx}{1} = \frac{dz}{g(x,y)} \qquad \text{becomes} \qquad \frac{dx}{1} = \frac{dz}{g(x,a-mx)}.$$

Then $z = \int g(x,a-mx)dx$ and, in order that no arbitrary constants be involved, we replace a by $y + mx$ in the solution.

7. Using the procedure of Problem 6, find particular integrals of

$$a)\ \ p + 3q = \cos(2x + y), \qquad b)\ \ p - 2q = (y + 1)e^{3x}.$$

a) Here $m = -3$ and $g(x,y) = \cos(2x + y)$.

Then $z = \int g(x,a-mx)dx = \int \cos(2x + a + 3x)dx = \frac{1}{5}\sin(5x + a)$ and, replacing a by $y - 3x$, the required particular integral is $z = \frac{1}{5}\sin(2x + y)$.

b) $z = \int g(x,a-mx)dx = \int (a - 2x + 1)e^{3x}\,dx = \frac{1}{3}(a + 1)e^{3x} - \frac{2}{3}xe^{3x} + \frac{2}{9}e^{3x}$.

Replacing a by $y + 2x$, we have $z = \frac{1}{3}(y + 2x + 1)e^{3x} - \frac{2}{3}xe^{3x} + \frac{2}{9}e^{3x} = \frac{1}{3}(y + \frac{5}{3})e^{3x}$.

8. Solve $(D_x^2 + 2D_xD_y - 8D_y^2)z = (D_x - 2D_y)(D_x + 4D_y)z = \sqrt{2x + 3y}$.

The complementary function is $z = \phi_1(y + 2x) + \phi_2(y - 4x)$.

To obtain the particular integral denoted by $\frac{1}{(D_x - 2D_y)(D_x + 4D_y)}\sqrt{2x + 3y}$, we obtain from $(D_x + 4D_y)u = \sqrt{2x + 3y}$ the solution

$$u = \int [2x + 3(a - mx)]^{1/2}\,dx = \int [2x + 3(a + 4x)]^{1/2}\,dx$$
$$= \int (14x + 3a)^{1/2}\,dx = \frac{1}{21}(14x + 3a)^{3/2} = \frac{1}{21}(2x + 3y)^{3/2}$$

and from $(D_x - 2D_y)z = u = \frac{1}{21}(2x + 3y)^{3/2}$, the solution

$$z = \frac{1}{21}\int [(2x + 3(a - 2x)]^{3/2}\,dx = -\frac{1}{210}(3a - 4x)^{5/2} = -\frac{1}{210}(2x + 3y)^{5/2}.$$

The general solution is $z = \phi_1(y + 2x) + \phi_2(y - 4x) - \frac{1}{210}(2x + 3y)^{5/2}$.

9. Solve $(D_x - 2D_y)^2(D_x + 3D_y)z = e^{2x+y}$.

The complementary function is $z = \phi_1(y + 2x) + x\phi_2(y + 2x) + \phi_3(y - 3x)$.

To obtain the particular integral denoted by $\frac{1}{(D_x - 2D_y)(D_x - 2D_y)(D_x + 3D_y)}e^{2x+y}$, we obtain from $(D_x + 3D_y)u = e^{2x+y}$ the solution $u = \int e^{2x + (a + 3x)}dx = \int e^{5x+a}\,dx = \frac{1}{5}e^{5x+a} = \frac{1}{5}e^{2x+y}$

from $(D_x - 2D_y)v = u = \frac{1}{5}e^{2x+y}$ the solution $v = \frac{1}{5}\int e^{2x+(a-2x)}\,dx = \frac{1}{5}xe^{a} = \frac{1}{5}xe^{2x+y}$;

and from $(D_x - 2D_y)z = v = \frac{1}{5}xe^{2x+y}$ the solution $z = \frac{1}{5}\int xe^{a}\,dx = \frac{1}{10}x^2e^{a} = \frac{1}{10}x^2e^{2x+y}$.

The general solution is $z = \phi_1(y+2x) + x\,\phi_2(y+2x) + \phi_3(y-3x) + \frac{1}{10}x^2e^{2x+y}$.

10. Solve $(D_x^3 + D_x^2D_y - D_xD_y^2 - D_y^3)z = (D_x + D_y)^2(D_x - D_y)z = e^x \cos 2y$.

The complementary function $z = \phi_1(y-x) + x\phi_2(y-x) + \phi_3(y+x)$.

Take as a particular integral $z = Ae^x \cos 2y + Be^x \sin 2y$. Then

$D_x^3 z = Ae^x \cos 2y + Be^x \sin 2y$, $\quad D_xD_y^2 z = -4Ae^x \cos 2y - 4Be^x \sin 2y$,

$D_x^2D_y z = -2Ae^x \sin 2y + 2Be^x \cos 2y$, $\quad D_y^3 z = 8Ae^x \sin 2y - 8Be^x \cos 2y$.

Substituting in the given differential equation, we have

$(5A + 10B)e^x \cos 2y + (5B - 10A)e^x \sin 2y = e^x \cos 2y$, so that $A = 1/25$ and $B = 2/25$.

The particular integral is $z = \frac{1}{25}e^x \cos 2y + \frac{2}{25}e^x \sin 2y$, and the general solution is

$$z = \phi_1(y-x) + x\phi_2(y-x) + \phi_3(y+x) + \frac{1}{25}e^x \cos 2y + \frac{2}{25}e^x \sin 2y.$$

11. Solve $(D_x^2 - 2D_xD_y)z = D_x(D_x - 2D_y)z = e^{2x} + x^3y$.

The complementary function is $z = \phi_1(y) + \phi_2(y+2x)$.

A particular integral is given by $\frac{1}{D_x^2 - 2D_xD_y}e^{2x} + \frac{1}{D_x^2 - 2D_xD_y}x^3y$. The first term yields $\frac{1}{(2)^2 - 2(2)(0)}e^{2x} = \frac{1}{4}e^{2x}$. Writing the second term

$$\frac{1}{D_x^2}\frac{1}{1 - 2\frac{D_y}{D_x}}x^3y = \frac{1}{D_x^2}\left(1 + 2\frac{D_y}{D_x} + \cdots\right)x^3y = \frac{1}{D_x^2}\left(x^3y + \frac{2}{D_x}x^3\right) = \frac{1}{D_x^2}\left(x^3y + \frac{1}{2}x^4\right),$$

we obtain $\frac{x^5y}{20} + \frac{x^6}{60}$. The general solution is $z = \phi_1(y) + \phi_2(y+2x) + \frac{1}{4}e^{2x} + \frac{x^5y}{20} + \frac{x^6}{60}$.

12. Solve $(D_x^3 - 7D_xD_y^2 - 6D_y^3)z = (D_x + D_y)(D_x + 2D_y)(D_x - 3D_y)z = \sin(x+2y) + e^{3x+y}$.

The complementary function is $z = \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x)$. A particular integral is given by

$$\frac{1}{(D_x + D_y)(D_x^2 - D_xD_y - 6D_y^2)}\sin(x+2y) + \frac{1}{(D_x - 3D_y)(D_x^2 + 3D_xD_y + 2D_y^2)}e^{3x+y}.$$

(Note. The separation in the first term is one of convenience, i.e., we could have written $\frac{1}{(D_x + 2D_y)(D_x^2 - 2D_xD_y - 3D_y^2)}\sin(x+2y)$. The separation in the second term is necessary, however, since e^{3x+y} is part of the term $\phi_3(y+3x)$ of the complementary function.)

For the first term: $\dfrac{1}{(D_x+D_y)(D_x^2-D_xD_y-6D_y^2)}\sin(x+2y) = \dfrac{1}{D_x+D_y}\,\dfrac{1}{-1+2+24}\sin(x+2y)$

$= \dfrac{1}{25}\,\dfrac{D_x-D_y}{D_x^2-D_y^2}\sin(x+2y) = \dfrac{1}{25(3)}(D_x-D_y)\sin(x+2y) = -\dfrac{1}{75}\cos(x+2y).$

For the second term: $\dfrac{1}{(D_x-3D_y)(D_x^2+3D_xD_y+2D_y^2)}e^{3x+y} = \dfrac{1}{D_x-3D_y}\,\dfrac{e^{3x+y}}{9+9+2}$

$= \dfrac{1}{20}\,\dfrac{1}{D_x-3D_y}e^{3x+y} = \dfrac{1}{20}xe^{3x+y}.$

The general solution is $z = \phi_1(y-x)+\phi_2(y-2x)+\phi_3(y+3x)-\dfrac{1}{75}\cos(x+2y)+\dfrac{1}{20}xe^{3x+y}.$

13. Solve $(D_x^3-7D_xD_y^2-6D_y^3)z = \cos(x-y)+x^2+xy^2+y^3.$

The reduced equation is that of Problem 12. A particular integral is given by

$$\frac{1}{(D_x+D_y)(D_x^2-D_xD_y-6D_y^2)}\cos(x-y) + \frac{1}{D_x^3-7D_xD_y^2-6D_y^3}(x^2+xy^2+y^3).$$

(Note that $\cos(x-y)$ is part of the complementary function; hence, the corresponding factor (D_x+D_y) must be treated separately.)

For the first term: $\dfrac{1}{(D_x+D_y)(D_x^2-D_xD_y-6D_y^2)}\cos(x-y) = \dfrac{1}{4}\,\dfrac{1}{D_x+D_y}\cos(x-y).$ We must solve

$(D_x+D_y)u = \frac{1}{4}\cos(x-y)$, obtaining $u = \frac{1}{4}\int\cos[x-(a+x)]\,dx = \frac{1}{4}\int\cos(-a)\,dx$

$= \frac{1}{4}x\cos(-a) = \frac{1}{4}x\cos(x-y).$

For the second term: $\dfrac{1}{D_x^3-7D_xD_y^2-6D_y^3}(x^2+xy^2+y^3) = \dfrac{1}{D_x^3\left(1-7\dfrac{D_y^2}{D_x^2}-6\dfrac{D_y^3}{D_x^3}\right)}(x^2+xy^2+y^3)$

$= \dfrac{1}{D_x^3}\left(1+7\dfrac{D_y^2}{D_x^2}+6\dfrac{D_y^3}{D_x^3}\right)(x^2+xy^2+y^3) = \dfrac{1}{D_x^3}\left[x^2+xy^2+y^3+\dfrac{7}{D_x^2}(2x+6y)+\dfrac{6}{D_x^3}(6)\right]$

$= \dfrac{1}{D_x^3}(x^2+xy^2+y^3)+\dfrac{7}{D_x^5}(2x+6y)+\dfrac{36}{D_x^6} = \dfrac{5}{72}x^6+\dfrac{1}{60}x^5(1+21y)+\dfrac{1}{24}x^4y^2+\dfrac{1}{6}x^3y^3.$

The general solution is

$$z = \phi_1(y-x)+\phi_2(y-2x)+\phi_3(y+3x)+\frac{1}{4}x\cos(x-y)+\frac{5}{72}x^6+\frac{1}{60}x^5(1+21y) + \frac{1}{24}x^4y^2+\frac{1}{6}x^3y^3.$$

SUPPLEMENTARY PROBLEMS

Solve each of the following equations.

14. $(D_x^2 - 8D_xD_y + 15D_y^2)z = 0.$ *Ans.* $z = \phi_1(y+3x) + \phi_2(y+5x)$

15. $(D_x^2 - 2D_xD_y - D_y^2)z = 0.$ *Ans.* $z = \phi_1[y + x(1+\sqrt{2})] + \phi_2[y + x(1-\sqrt{2})]$

16. $(D_x^2 - 4D_xD_y + 4D_y^2)z = 0.$ *Ans.* $z = \phi_1(y+2x) + x\phi_2(y+2x)$

17. $(D_x^3 + 2D_x^2D_y - D_xD_y^2 - 2D_y^3)z = 0.$ *Ans.* $z = \phi_1(y+x) + \phi_2(y-x) + \phi_3(y-2x)$

18. $(D_x^3D_y^2 + D_x^2D_y^3)z = 0.$ *Ans.* $z = \phi_1(y) + x\phi_2(y) + \phi_3(x) + y\phi_4(x) + \phi_5(y-x)$

19. $(D_x^2 + 5D_xD_y + 6D_y^2)z = e^{x-y}.$ *Ans.* $z = \phi_1(y-2x) + \phi_2(y-3x) + \frac{1}{2}e^{x-y}$

20. $(D_x^2 + D_y^2)z = x^2y^2.$

Ans. $z = \phi_1(y+ix) + \phi_1(y-ix) + i[\phi_2(y+ix) - \phi_2(y-ix)] + \frac{1}{180}(15x^4y - x^6)$

21. $(D_x^3 - 3D_x^2D_y + 4D_y^3)z = e^{y+2x}.$ *Ans.* $z = \phi_1(y-x) + \phi_2(y+2x) + x\phi_3(y+2x) + \frac{1}{6}x^2e^{y+2x}$

22. $(D_x^3 + 2D_x^2D_y - D_xD_y^2 - 2D_y^3)z = (y+2)e^x.$ *Ans.* $z = \phi_1(y+x) + \phi_2(y-x) + \phi_3(y-2x) + ye^x$

23. $(D_x^3 - 3D_x^2D_y - 4D_xD_y^2 + 12D_y^3)z = \sin(y+2x).$

Ans. $z = \phi_1(y-2x) + \phi_2(y+2x) + \phi_3(y+3x) + \frac{1}{4}x\sin(y+2x)$

24. $(D_x^3 - 3D_xD_y^2 + 2D_y^3)z = \sqrt{x+2y}.$ *Ans.* $z = \phi_1(y+x) + x\phi_2(y+x) + \phi_3(y-2x) + \frac{8}{525}(x+2y)^{7/2}$

25. $(D_x^3 + D_x^2D_y - 6D_xD_y^2)z = x^2 + y^2.$

Ans. $z = \phi_1(y) + \phi_2(y+2x) + \phi_3(y-3x) + \frac{2}{15}x^5 - \frac{1}{12}x^4y + \frac{1}{6}x^3y^2$

26. $(D_x^3 - 4D_x^2D_y + 5D_xD_y^2 - 2D_y^3)z = e^{y+x} + e^{y-2x} + e^{y+2x}.$

Ans. $z = \phi_1(y+x) + x\phi_2(y+x) + \phi_3(y+2x) - \frac{1}{2}x^2e^{y+x} - \frac{1}{36}e^{y-2x} + xe^{y+2x}$

27. $(D_x^3 - 2D_x^2D_y)z = 2e^{2x} + 3x^2y.$

Ans. $z = \phi_1(y) + x\phi_2(y) + \phi_3(y+2x) + \frac{1}{4}e^{2x} + \frac{1}{20}x^5y + \frac{1}{60}x^6$

28. $(D_x^3 - 3D_xD_y^2 - 2D_y^3)z = \cos(x+2y) - e^y(3+2x).$

Ans. $z = \phi_1(y-x) + x\phi_2(y-x) + \phi_3(y+2x) + \frac{1}{27}\sin(x+2y) + xe^y$

CHAPTER 32

Non-homogenous Linear Equations with Constant Coefficients

A NON-HOMOGENEOUS LINEAR partial differential equation with constant coefficients such as

$$f(D_x, D_y)z = (D_x^2 - D_y^2 + 3D_x + D_y + 2)z = (D_x + D_y + 1)(D_x - D_y + 2)z = x^2 + xy$$

is called *reducible*, since the left member can be resolved into factors each of which is of the first degree in D_x, D_y, while

$$f(D_x, D_y)z = (D_x D_y + 2D_y^3)z = D_y(D_x + 2D_y^2)z = \cos(x - 2y),$$

which cannot be so resolved, is called *irreducible*.

REDUCIBLE NON-HOMOGENEOUS EQUATIONS. Consider the reducible non-homogeneous equation

1) $f(D_x, D_y)z = (a_1 D_x + b_1 D_y + c_1)(a_2 D_x + b_2 D_y + c_2)\cdots\cdots(a_n D_x + b_n D_y + c_n)z = 0,$

where the a_i, b_i, c_i are constants. Any solution of

2) $$(a_i D_x + b_i D_y + c_i)z = 0$$

is a solution of 1). From Problem 5, Chapter 29, the general solution of 2) is

3) $$z = e^{-c_i x/a_i}\,\phi(a_i y - b_i x), \quad a_i \neq 0,$$

or

3′) $$z = e^{-c_i y/b_i}\,\psi(a_i y - b_i x), \quad b_i \neq 0,$$

with ϕ and ψ arbitrary functions of their argument. Thus, if no two factors of 1) are linearly dependent (that is, if no factor is a mere multiple of another), the general solution of 1) consists of the sum of n arbitrary functions of the types 3) and 3′).

EXAMPLE 1. Solve $(2D_x + D_y + 1)(D_x - 3D_y + 2)z = 0.$

The general solution is $z = e^{-y}\phi_1(2y - x) + e^{-2x}\phi_2(y + 3x)$. Note that the first term on the right may be replaced by $e^{-x/2}\psi_1(2y - x)$ and the second by $e^{2y/3}\psi_2(y + 3x)$.

EXAMPLE 2. Solve $(2D_x + 3D_y - 5)(D_x + 2D_y)(D_x - 2)(D_y + 2)z = 0.$

The general solution is $z = e^{5x/2}\phi_1(2y - 3x) + \phi_2(y - 2x) + e^{2x}\phi_3(y) + e^{-2y}\phi_4(x).$

See also Problems 1–2.

If

4) $f(D_x, D_y)z = (a_1 D_x + b_1 D_y + c_1)^k (a_{k+1} D_x + b_{k+1} D_y + c_{k+1})\cdots(a_n D_x + b_n D_y + c_n)z = 0,$

where no two of the n factors are linearly dependent except as indicated, the part of the general solution corresponding to the k repeated factors is

$$e^{-c_1 x/a_1}[\phi_1(a_1 y - b_1 x) + x\phi_2(a_1 y - b_1 x) + \cdots + x^{k-1}\phi_k(a_1 y - b_1 x)].$$

EXAMPLE 3. Solve $(2D_x + D_y + 5)(D_x - 2D_y + 1)^2 z = 0.$

The general solution is $z = e^{-5y}\phi_1(2y - x) + e^{-x}[\phi_2(y + 2x) + x\phi_3(y + 2x)].$

See also Problem 3

THE GENERAL SOLUTION OF

5) $$f(D_x, D_y)z = (a_1D_x + b_1D_y + c_1)(a_2D_x + b_2D_y + c_2)\cdots(a_nD_x + b_nD_y + c_n)z = F(x,y)$$

is the sum of the general solution of 1), (now called the complementary function of 5), and a particular integral of 5),

6) $$z = \frac{1}{f(D_x, D_y)} F(x,y).$$

The general procedure for evaluating 6) as well as short methods applicable to particular forms of $F(x,y)$ are those of the previous chapter.

EXAMPLE 4. Solve $f(D_x,D_y)z = (D_x^2 - D_xD_y - 2D_y^2 + 2D_x - 4D_y)z$
$= (D_x - 2D_y)(D_x + D_y + 2)z = ye^x + 3xe^{-y}.$

The complementary function is $z = \phi_1(y + 2x) + e^{-2x}\phi_2(y - x).$

To evaluate $\frac{1}{f(D_x,D_y)} ye^x = \frac{1}{(D_x - 2D_y)(D_x + D_y + 2)} ye^x$, we first solve $(D_x + D_y + 2)u = ye^x$

whose auxiliary system is $\frac{dx}{1} = \frac{dy}{1} = \frac{du}{ye^x - 2u}$. We obtain $y = x + a$ readily and the equation

$\frac{du}{ye^x - 2u} = \frac{dx}{1}$ or $\frac{du}{dx} + 2u = ye^x = (x + a)e^x$. This linear equation has e^{2x} as integrating factor; hence,

$ue^{2x} = \int (x + a)e^{3x}dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + \frac{1}{3}ae^{3x} = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + \frac{1}{3}(y - x)e^{3x}$ and

$u = \frac{1}{3}ye^x - \frac{1}{9}e^x.$

We then solve $(D_x - 2D_y)z = u = \frac{1}{3}ye^x - \frac{1}{9}e^x$ obtaining the required particular integral (see Problem 6, Chapter 31)

$$z = \int [\tfrac{1}{3}(a - 2x)e^x - \tfrac{1}{9}e^x]dx = \frac{1}{3}ae^x - \frac{2}{3}xe^x + \frac{2}{3}e^x - \frac{1}{9}e^x$$

$$= \frac{1}{3}(y + 2x)e^x - \frac{2}{3}xe^x + \frac{5}{9}e^x = \frac{1}{3}(y + \frac{5}{3})e^x.$$

To evaluate $\frac{1}{(D_x - 2D_y)(D_x + D_y + 2)}(3xe^{-y})$, we solve $(D_x + D_y + 2)u = 3xe^{-y}$ whose auxiliary system

is $\frac{dx}{1} = \frac{dy}{1} = \frac{du}{3xe^{-y} - 2u}$. Then $y = x + a$, and from $\frac{du}{3xe^{-y} - 2u} = \frac{dy}{1}$ or

$\dfrac{du}{dy} + 2u = 3xe^{-y} = 3(y-a)e^{-y}$, $ue^{2y} = 3\int(y-a)e^{y}\,dy = 3(y-1-a)e^{y} = 3(x-1)e^{y}$ and

$u = 3(x-1)e^{-y}$. Solving in turn $(D_x - 2D_y)z = u = 3(x-1)e^{-y}$, the required particular integral is

$z = 3\int(x-1)e^{-a+2x}\,dx = \frac{3}{2}(xe^{-a+2x} - \frac{3}{2}e^{-a+2x}) = \frac{3}{2}(x-\frac{3}{2})e^{-y}$.

The general solution is $z = \phi_1(y+2x) + e^{-2x}\phi_2(y-x) + \frac{1}{3}(y+\frac{5}{3})e^{x} + \frac{3}{2}(x-\frac{3}{2})e^{-y}$.

EXAMPLE 5. Solve $f(D_x,D_y)z = (D_x^2 - D_xD_y - 2D_y^2 + 6D_x - 9D_y + 5)z$

$= (D_x + D_y + 5)(D_x - 2D_y + 1)z = e^{2x+y} + e^{x+y}$.

The complementary function is $z = e^{-5x}\phi_1(y-x) + e^{-x}\phi_2(y+2x)$.

For the particular integral corresponding to the first term of $F(x,y)$, we use

$$\frac{1}{f(D_x,D_y)}e^{ax+by} = \frac{1}{f(a,b)}e^{ax+by}, \quad f(a,b) \neq 0,$$

and obtain $$\frac{1}{D_x^2 - D_xD_y - 2D_y^2 + 6D_x - 9D_y + 5}e^{2x+y} = \frac{1}{4-2-2+12-9+5}e^{2x+y} = \frac{1}{8}e^{2x+y}.$$

In evaluating $\dfrac{1}{f(D_x,D_y)}e^{x+y}$, we note that $f(1,1) = 0$. This means that e^{x+y} is a part of the complementary function. (To see this, take $\phi_2(y+2x) = e^{y+2x} + \psi_2(y+2x)$; then

$e^{-x}\phi_2(y+2x) = e^{-x}[e^{y+2x} + \psi_2(y+2x)] = e^{y+x} + e^{-x}\psi_2(y+2x)$.) We write

$$\frac{1}{f(D_x,D_y)}e^{x+y} = \frac{1}{D_x - 2D_y + 1}\,\frac{1}{D_x + D_y + 5}e^{x+y} = \frac{1}{7}\,\frac{1}{D_x - 2D_y + 1}e^{x+y} = \frac{1}{7}xe^{x+y}.$$

The general solution is $z = e^{-5x}\phi_1(y-x) + e^{-x}\phi_2(y+2x) + \frac{1}{8}e^{2x+y} + \frac{1}{7}xe^{x+y}$.

See also Problems 4–5.

The use of the formula

7) $$\frac{1}{f(D_x,D_y)}Ve^{ax+by} = e^{ax+by}\frac{1}{f(D_x+a,D_y+b)}V, \quad V = V(x,y),$$

is illustrated below.

EXAMPLE 6. Solve $(D_x^3 + 3D_x^2D_y - 2D_x^2)z = D_x^2(D_x + 3D_y - 2)z = (x^2+2y)e^{2x+y}$.

The complementary function is $z = \phi_1(y) + x\phi_2(y) + e^{2x}\phi_3(y-3x)$. A particular integral is

$$z = \frac{1}{D_x^2(D_x + 3D_y - 2)}(x^2+2y)e^{2x+y} = e^{2x+y}\frac{1}{(D_x+2)^2(D_x + 3D_y + 3)}(x^2+2y).$$

Setting $(D_x + 3D_y + 3)u = x^2 + 2y$, the auxiliary system is $\dfrac{dx}{1} = \dfrac{dy}{3} = \dfrac{du}{x^2+2y-3u}$.

Then $y = 3x + a$, and from $\frac{du}{x^2 + 2y - 3u} = \frac{dx}{1}$ or $\frac{du}{dx} + 3u = x^2 + 2y$, we have

$$ue^{3x} = \int (x^2 + 6x + 2a)e^{3x}\,dx = e^{3x}(\tfrac{1}{3}x^2 + \tfrac{16}{9}x - \tfrac{16}{27} + \tfrac{2}{3}a) \text{ and } u = \tfrac{1}{3}x^2 - \tfrac{2}{9}x - \tfrac{16}{27} + \tfrac{2}{3}y.$$

Next, setting $(D_x + 2)v = u$ and making use of the integrating factor e^{2x}, y being regarded as a constant

$$ve^{2x} = \int e^{2x}(\tfrac{1}{3}x^2 - \tfrac{2}{9}x - \tfrac{16}{27} + \tfrac{2}{3}y)dx = (\tfrac{1}{6}x^2 - \tfrac{5}{18}x - \tfrac{17}{108} + \tfrac{1}{3}y)e^{2x} \text{ and } v = \tfrac{1}{6}x^2 - \tfrac{5}{18}x - \tfrac{17}{108} + \tfrac{1}{3}y.$$

Finally, setting $(D_x + 2)w = v$, we have

$$we^{2x} = \int e^{2x}(\tfrac{1}{6}x^2 - \tfrac{5}{18}x - \tfrac{17}{108} + \tfrac{1}{3}y)dx = (\tfrac{1}{12}x^2 - \tfrac{2}{9}x + \tfrac{7}{216} + \tfrac{1}{6}y)e^{2x}$$

and $\quad w = \frac{1}{12}x^2 - \frac{2}{9}x + \frac{7}{216} + \frac{1}{6}y.$

Then $z = we^{2x+y}$ and the general solution is

$$z = \phi_1(y) + x\phi_2(y) + e^{2x}\phi_3(3y - x) + (\tfrac{1}{12}x^2 - \tfrac{2}{9}x + \tfrac{7}{216} + \tfrac{1}{6}y)e^{2x+y}.$$

See also Problems 6–7.

IRREDUCIBLE EQUATIONS WITH CONSTANT COEFFICIENTS. Consider the linear equation with constant coefficients

8) $$f(D_x, D_y)z = 0.$$

Since $D_x^r D_y^s(ce^{ax+by}) = ca^r b^s e^{ax+by}$, where a, b, c are constants, the result of substituting

9) $$z = ce^{ax+by}$$

in 8) is $c\,f(a, b)e^{ax+by} = 0$. Thus, 9) is a solution of 8) provided

10) $$f(a, b) = 0,$$

with c arbitrary. Now for any chosen value of a (or b) one or more values of b (or a) are obtained by means of 10). Thus, there exist infinitely many pairs of numbers (a_i, b_i) satisfying 10). Moreover,

11) $$z = \sum_{i=1}^{\infty} c_i e^{a_i x + b_i y}, \quad \text{where} \quad f(a_i, b_i) = 0,$$

is a solution of 8).

If $$f(D_x, D_y)z = (D_x + hD_y + k)\,g(D_x, D_y)z,$$

then any pair (a, b) for which $a + hb + k = 0$ satisfies 10). Consider all such pairs $(a_i, b_i) = (-hb_i - k, b_i)$. By 11),

$$z = \sum_{i=1}^{\infty} c_i e^{-(hb_i + k)x + b_i y} = e^{-kx}\sum_{i=1}^{\infty} c_i e^{b_i(y - hx)}$$

is a solution of 8) corresponding to the linear factor $(D_x + hD_y + k)$ of $f(D_x, D_y)$.

This is, of course, $e^{-kx}\phi(y-hx)$, ϕ arbitrary, used above. Thus, if $f(D_x, D_y)$ has no linear factor, 11) will be called the solution of 8); however, if $f(D_x, D_y)$ has $m < n$ linear factors, we shall write part of the solution involving arbitrary functions (corresponding to the linear factors) and the remainder involving arbitrary constants.

EXAMPLE 7. Solve $f(D_x, D_y)z = (D_x^2 + D_x + D_y)z = 0$.

The equation is irreducible. Here $f(a,b) = a^2 + a + b = 0$ so that for any $a = a_i$, $b_i = -a_i(a_i+1)$. Thus the solution is

$$z = \sum_{i=1}^{\infty} c_i e^{a_i x + b_i y} = \sum_{i=1}^{\infty} c_i e^{a_i x - a_i(a_i+1)y},$$ with c_i and a_i arbitrary constants.

EXAMPLE 8. Solve $(D_x + 2D_y)(D_x - 2D_y + 1)(D_x - D_y^2)z = 0$.

Corresponding to the linear factors we have $\phi_1(y-2x)$ and $e^{-x}\phi_2(y+2x)$ respectively.

For the irre[illegible]ible factor $D_x - D_y^2$ we have $a - b^2 = 0$ or $a = b^2$.

The required solution is

$$z = \phi_1(y-2x) + e^{-x}\phi_2(y+2x) + \sum_{i=1}^{\infty} c_i e^{b_i^2 x + b_i y},$$ with c_i and b_i arbitrary constants.

In obtaining a particular integral of $f(D_x, D_y)z = F(x,y)$, all procedures used heretofore are available.

EXAMPLE 9. Solve $f(D_x, D_y)z = (D_x - D_y^2)z = e^{2x+3y}$.

Fro[illegible] 8, the complementary function is $z = \sum_{i=1}^{\infty} c_i e^{b_i^2 x + b_i y}$.

For the particular integral: $\dfrac{1}{D_x - D_y^2} e^{2x+3y} = \dfrac{1}{2-(3)^2} e^{2x+3y} = -\dfrac{1}{7} e^{2x+3y}$.

The required solution is $z = \sum_{i=1}^{\infty} c_i e^{b_i^2 x + b_i y} - \frac{1}{7} e^{2x+3y}$

See also Problems 8-11.

THE CAUCHY (ORDINARY) DIFFERENTIAL EQUATION $f(xD)y = F(x)$ is transformed into a linear equation with constant coefficients by means of the substitution $x = e^z$ (see Chapter 17). The analogue in partial differential equations is an equation of the form

$$f(xD_x, yD_y)z = \sum_{r,s} c_{rs} x^r y^s D_x^r D_y^s z = F(x,y), \quad c_{rs} = \text{constant},$$

which is reduced to a linear partial differential equation with constant coefficients by the substitution

$$x = e^u, \quad y = e^v.$$

EXAMPLE 10. Solve $(x^2D_x^2 + 2xyD_xD_y - xD_x)z = x^3/y^2$.

The substitution $x = e^u$, $y = e^v$, $xD_x z = D_u z$, $yD_y z = D_v z$, $x^2D_x^2 z = D_u(D_u - 1)z$, $xyD_xD_y = D_uD_v z$, $y^2D_y^2 z = D_v(D_v - 1)z$ transforms the given equation into

$$[D_u(D_u - 1) + 2D_uD_v - D_u]z = D_u(D_u + 2D_v - 2)z = e^{3u-2v}$$

whose solution is $z = \phi_1(v) + e^{2u}\phi_2(v - 2u) - \frac{1}{9}e^{3u-2v}$.

Thus, the general solution (expressed in the original variables) is

$$z = \phi_1(\ln y) + x^2\phi_2(\ln \frac{y}{x^2}) - \frac{1}{9}\frac{x^3}{y^2} \quad \text{or} \quad z = \psi_1(y) + x^2\psi_2(\frac{y}{x^2}) - \frac{1}{9}\frac{x^3}{y^2}.$$

See also Problems 12-13

SOLVED PROBLEMS

REDU[illegible]

1. Solve $(D_x^2 - D_y^2 + 3D_x - 3D_y)z = (D_x - D_y)(D_x + D_y + 3)z = 0.$

   The general solution is $z = \phi_1(y + x) + e^{-3x}\phi_2(y - x)$.

2. Solve $D_x(2D_x - D_y + 1)(D_x + 2D_y - 1)z = 0.$

   T[illegible] $z = \phi_1(y) + e^y\phi_2(2y + x) + e^x\phi_3(y - 2x)$.

3. Solve $(2D_x + 3D_y - 1)^2$ [illegible]^3 [illegible] 0. The general solution is

$$z = e^{\frac{1}{2}x}[\phi_1(2y - 3x) + x\phi_2(2y - 3x)] + e^y[\phi_3(y + 3x) + y\phi_4(y + 3x) + y^2\phi_5(y + 3x)].$$

4. Solve $(2D_xD_y + D_y^2 - 3D_y)z = D_y(2D_x + D_y - 3)z = 3\cos(3x - 2y).$

   The complementary function is $z = \phi_1(x) + e^{3y}\phi_2(2y - x)$. A particular integral is

$$\frac{1}{2D_xD_y + D_y^2 - 3D_y}3\cos(3x - 2y) = \frac{3}{2(6) - 4 - 3D_y}\cos(3x - 2y) = \frac{3}{8 - 3D_y}\cos(3x - 2y)$$

$$= \frac{3(8 + 3D_y)}{64 - 9D_y^2}\cos(3x - 2y) = \frac{3}{100}(8 + 3D_y)\cos(3x - 2y) = \frac{3}{50}[4\cos(3x - 2y) + 3\sin(3x - 2y)].$$

   The general solution is $z = \phi_1(x) + e^{3y}\phi_2(2y - x) + \frac{3}{50}[4\cos(3x - 2y) + 3\sin(3x - 2y)]$.

5. Solve $D_x(D_x + D_y - 1)(D_x + 3D_y - 2)z = x^2 - 4xy + 2y^2.$

   The complementary function is $z = \phi_1(y) + e^x\phi_2(y - x) + e^{2x}\phi_3(y - 3x)$.

   A particular integral is denoted by $z = \dfrac{1}{D_x(D_x + D_y - 1)(D_x + 3D_y - 2)}(x^2 - 4xy + 2y^2)$.

To evaluate it, consider $\dfrac{1}{D_x+3D_y-2}(x^2-4xy+2y^2) = \frac{1}{2}\dfrac{1}{-1+\frac{1}{2}(D_x+3D_y)}(x^2-4xy+2y^2)$

$= \frac{1}{2}[-1-\frac{1}{2}(D_x+3D_y)-\frac{1}{4}(D_x+3D_y)^2-\cdots](x^2-4xy+2y^2)$

$= \frac{1}{2}[-(x^2-4xy+2y^2)-(-5x+4y)-7/2] = -\frac{1}{2}(x^2-4xy+2y^2-5x+4y+7/2).$

Consider next $\dfrac{-\frac{1}{2}}{D_x+D_y-1}(x^2-4xy+2y^2-5x+4y+7/2) = \frac{1}{2}\dfrac{1}{1-(D_x+D_y)}(x^2-4xy+2y^2-5x+4y+7/2)$

$= \frac{1}{2}[1+(D_x+D_y)+(D_x+D_y)^2+\cdots](x^2-4xy+2y^2-5x+4y+7/2) = \frac{1}{2}(x^2-4xy+2y^2-7x+4y+\frac{1}{2}).$

Finally, $z = \dfrac{\frac{1}{2}}{D_x}(x^2-4xy+2y^2-7x+4y+\frac{1}{2}) = \frac{1}{2}(x^3/3-2x^2y+2xy^2-7x^2/2+4xy+x/2).$

The general solution is

$$z = \phi_1(y) + e^x\phi_2(y-x) + e^{2x}\phi_3(y-3x) + \frac{1}{12}(2x^3-12x^2y+12xy^2-21x^2+24xy+3x).$$

TYPE: $\dfrac{1}{f(D_x,D_y)}e^{ax+by}V(x,y).$

6. Solve $(D_x+D_y-1)(D_x+D_y-3)(D_x+D_y)z = e^{x+y+2}\cos(2x-y).$

The complementary function is $z = e^x\phi_1(y-x) + e^{3x}\phi_2(y-x) + \phi_3(y-x).$

For the particular integral, $\dfrac{1}{(D_x+D_y-1)(D_x+D_y-3)(D_x+D_y)}e^{x+y+2}\cos(2x-y)$

$= e^{x+y}\dfrac{1}{(D_x+D_y+1)(D_x+D_y-1)(D_x+D_y+2)}e^2\cos(2x-y)$

$= e^{x+y+2}\dfrac{1}{(D_x^2+2D_xD_y+D_y^2-1)(D_x+D_y+2)}\cos(2x-y) = -\frac{1}{2}e^{x+y+2}\dfrac{1}{D_x+D_y+2}\cos(2x-y)$

$= -\dfrac{1}{2}e^{x+y+2}\dfrac{D_x+D_y-2}{D_x^2+2D_xD_y+D_y^2-4}\cos(2x-y) = \dfrac{1}{10}e^{x+y+2}(D_x+D_y-2)\cos(2x-y)$

$= -\dfrac{1}{10}e^{x+y+2}[\sin(2x-y)+2\cos(2x-y)].$ The general solution is

$$z = e^x\phi_1(y-x) + e^{3x}\phi_2(y-x) + \phi_3(y-x) - \frac{1}{10}e^{x+y+2}[\sin(2x-y)+2\cos(2x-y)].$$

7. Solve $D_x(D_x-2D_y)(D_x+D_y)z = e^{x+2y}(x^2+4y^2).$

The complementary function is $z = \phi_1(y) + \phi_2(y+2x) + \phi_3(y-x).$

For the particular integral

$\dfrac{1}{D_x(D_x-2D_y)(D_x+D_y)}e^{x+2y}(x^2+4y^2) = e^{x+2y}\dfrac{1}{(D_x+1)(D_x-2D_y-3)(D_x+D_y+3)}(x^2+4y^2),$ we first find $u = \frac{1}{D_x + D_y + 3}(x^2 + 4y^2) = \frac{1}{3}\frac{1}{1 + \frac{1}{3}(D_x + D_y)}(x^2 + 4y^2)$

$= \frac{1}{3}[1 - \frac{1}{3}(D_x + D_y) + \frac{1}{9}(D_x + D_y)^2 + \cdots\cdots](x^2 + 4y^2)$

$= \frac{1}{3}[x^2 + 4y^2 - \frac{2}{3}(x + 4y) + \frac{10}{9}] = \frac{1}{27}(9x^2 + 36y^2 - 6x - 24y + 10),$

then $v = \frac{1}{D_x - 2D_y - 3}u = -\frac{1}{3}\frac{1}{1 + \frac{1}{3}(2D_y - D_x)}u = -\frac{1}{3}[1 - \frac{1}{3}(2D_y - D_x) + \frac{1}{9}(2D_y - D_x)^2 - \cdots]u$

$= -\frac{1}{81}(9x^2 + 36y^2 - 72y + 58),$

and finally, $z = \frac{1}{D_x + 1}v = (1 - D_x + D_x^2 + \cdots)v = -\frac{1}{81}(9x^2 + 36y^2 - 18x - 72y + 76).$

The general solution is

$$z = \phi_1(y) + \phi_2(y + 2x) + \phi_3(y - x) - \frac{1}{81}(9x^2 + 36y^2 - 18x - 72y + 76)e^{x+2y}.$$

TYPE: IRREDUCIBLE EQUATIONS.

8. Solve $f(D_x, D_y)z = (D_x - D_y^2)z = e^{x+y}$.

The complementary function is $z = \sum_{i=1}^{\infty} c_i e^{b_i^2 x + b_i y}$ from Example 9.

The short method for evaluating the particular integral $\frac{1}{f(D_x, D_y)}e^{x+y}$ cannot be used, since $f(a,b) = f(1,1) = 0$. We shall use the method of undetermined coefficients, assuming the particular integral to be of the form $z = Axe^{x+y} + Bye^{x+y}$.

Now $D_x z = (A + Ax + By)e^{x+y}$, $D_y^2 z = (Ax + 2B + By)e^{x+y}$ and $(D_x - D_y^2)z = (A - 2B)e^{x+y} = e^{x+y}$; hence $A - 2B = 1$. Taking $A = 1$, $B = 0$, we have as particular integral $z = xe^{x+y}$; taking $A = 0$, $B = -\frac{1}{2}$, we have $z = -\frac{1}{2}ye^{x+y}$; and so on. Choosing the first, the required solution is

$$z = \sum_{i=1}^{\infty} c_i e^{b_i^2 x + b_i y} + xe^{x+y}.$$

9. Solve $(2D_x^2 - D_y^2 + D_x)z = x^2 - y$.

The complementary function is $z = \sum_{i=1}^{\infty} c_i e^{a_i x + b_i y}$, $2a_i^2 - b_i^2 + a_i = 0$.

The particular integral $\frac{1}{2D_x^2 - D_y^2 + D_x}(x^2 - y) = -\frac{1}{D_y^2}\frac{1}{1 - \frac{D_x + 2D_x^2}{D_y^2}}(x^2 - y)$

$= -\frac{1}{D_y^2}[1 + \frac{D_x + 2D_x^2}{D_y^2} + \frac{(D_x + 2D_x^2)^2}{D_y^4} + \cdots](x^2 - y) = -\frac{1}{D_y^2}[x^2 - y + \frac{2x + 4}{D_y^2} + \frac{2}{D_y^4}]$

$$= -\frac{1}{D_y^2}(x^2 - y + xy^2 + 2y^2 + y^4/12) = -\frac{1}{2}x^2y^2 + \frac{1}{6}y^3 - \frac{1}{12}xy^4 - \frac{1}{6}y^4 - \frac{1}{360}y^6.$$

The required solution is $z = \sum_{i=1}^{\infty} c_i e^{a_i x \pm \sqrt{2a_i^2 + a_i}\, y} - \frac{1}{2}x^2y^2 + \frac{1}{6}y^3 - \frac{1}{12}xy^4 - \frac{1}{6}y^4 - \frac{1}{360}y^6.$

10. Find a particular integral of $(D_x^2 + D_y)(D_x - D_y - D_y^2)z = \sin(2x + y).$

A particular integral is given by

$$\frac{1}{(D_x^2 + D_y)(D_x - D_y - D_y^2)}\sin(2x + y) = \frac{1}{(-4 + D_y)(D_x - D_y + 1)}\sin(2x + y)$$

$$= \frac{1}{D_xD_y - D_y^2 - 4D_x + 5D_y - 4}\sin(2x + y) = \frac{1}{5D_y - 4D_x - 5}\sin(2x + y)$$

$$= \frac{5D_y - 4D_x + 5}{25D_y^2 - 40D_xD_y + 16D_x^2 - 25}\sin(2x + y) = -\frac{1}{34}[5\sin(2x + y) - 3\cos(2x + y)].$$

The method of undetermined coefficients with $z = A\sin(2x + y) + B\cos(2x + y)$ may also be used here.

11. Find a particular integral of $(D_x - 2D_y + 5)(D_x^2 + D_y + 3)z = e^{3x + 4y}\sin(x - 2y).$

A particular integral is $\dfrac{1}{(D_x - 2D_y + 5)(D_x^2 + D_y + 3)}e^{3x+4y}\sin(x - 2y)$

$$= e^{3x+4y}\frac{1}{(D_x - 2D_y)(D_x^2 + 6D_x + D_y + 16)}\sin(x - 2y) = e^{3x+4y}\frac{1}{(D_x - 2D_y)(6D_x + D_y + 15)}\sin(x - 2y)$$

$$= e^{3x+4y}\frac{1}{6D_x^2 - 11D_xD_y - 2D_y^2 + 15D_x - 30D_y}\sin(x - 2y) = \frac{1}{5}e^{3x+4y}\frac{1}{3D_x - 6D_y - 4}\sin(x - 2y)$$

$$= \frac{1}{5}e^{3x+4y}\frac{3D_x - 6D_y + 4}{9D_x^2 - 36D_xD_y + 36D_y^2 - 16}\sin(x - 2y) = -\frac{1}{1205}e^{3x+4y}(3D_x - 6D_y + 4)\sin(x - 2y)$$

$$= -\frac{1}{1205}e^{3x+4y}[15\cos(x - 2y) + 4\sin(x - 2y)].$$

TYPE: $f(xD_x, yD_y)z = 0.$

12. Solve $(xD_x^3D_y^2 - yD_x^2D_y^3)z = 0$ or $(x^3y^2D_x^3D_y^2 - x^2y^3D_x^2D_y^3)z = 0.$

The substitution $x = e^u,\quad y = e^v,\quad x^3y^2D_x^3D_y^2 z = D_u(D_u - 1)(D_u - 2)D_v(D_v - 1)z,$
$x^2y^3D_x^2D_y^3 z = D_u(D_u - 1)D_v(D_v - 1)(D_v - 2)z$ transforms the given equation into

$$D_uD_v(D_u - 1)(D_v - 1)(D_u - D_v)z = 0.$$

The required solution is

$z = \phi_1(v) + \phi_2(u) + e^u\phi_3(v) + e^v\phi_4(u) + \phi_5(v + u)$ or, in the original variables,

$$z = \phi_1(\ln y) + \phi_2(\ln x) + x\phi_3(\ln y) + y\phi_4(\ln x) + \phi_5(\ln xy)$$
$$= \psi_1(y) + \psi_2(x) + x\psi_3(y) + y\psi_4(x) + \psi_5(xy).$$

13. Solve $(x^2D_x^2 - 4y^2D_y^2 - 4yD_y - 1)z = x^2y^3 \ln y$.

The substitution $x = e^u$, $y = e^v$ transforms the given equation into

$$[D_u(D_u - 1) - 4D_v(D_v - 1) - 4D_v - 1]z = (D_u^2 - 4D_v^2 - D_u - 1)z = ve^{2u+3v}.$$

A particular integral of this equation is given by $\dfrac{1}{D_u^2 - 4D_v^2 - D_u - 1}ve^{2u+3v}$

$$= e^{2u+3v}\frac{1}{(D_u+2)^2 - 4(D_v+3)^2 - (D_u+2) - 1}v = e^{2u+3v}\frac{1}{D_u^2 - 4D_v^2 + 3D_u - 24D_v - 35}v.$$

By inspection, a solution of $(D_u^2 - 4D_v^2 + 3D_u - 24D_v - 35)w = v$ is found to be $w = -\dfrac{1}{35}v + \dfrac{24}{(35)^2}$.

Hence, the particular integral is $z = -\dfrac{1}{(35)^2}e^{2u+3v}(35v - 24)$.

The required solution of the given differential equation is

$$z = \sum_{i=1}^{\infty} c_i e^{a_i u + b_i v} - \frac{1}{1225}e^{2u+3v}(35v - 24) \quad \text{or, in the original variables,}$$

$$z = \sum_{i=1}^{\infty} c_i x^{a_i} y^{b_i} - \frac{1}{1225}x^2y^3(35 \ln y - 24), \quad a_i^2 - 4b_i^2 - a_i - 1 = 0.$$

SUPPLEMENTARY PROBLEMS

Solve each of the following equations.

14. $(D_x + D_y + 1)(D_x - 2D_y - 1)z = 0$. *Ans.* $z = e^{-x}\phi_1(y - x) + e^x\phi_2(y + 2x)$

15. $(D_x + 2D_y - 3)(D_x + D_y - 1)z = 0$. *Ans.* $z = e^{3x}\phi_1(y - 2x) + e^x\phi_2(y - x)$

16. $(2D_x + D_y + 1)(D_x^2 + 3D_xD_y - 3D_x)z = 0$. *Ans.* $z = \phi_1(y) + e^{-y}\phi_2(2y - x) + e^y\phi_3(y - 3x)$

17. $(D_xD_y + D_y^2)(D_x - D_y - 2)z = 0$. *Ans.* $z = \phi_1(x) + \phi_2(y - x) + e^{2x}\phi_3(y + x)$

18. $(D_x + 2D_y)(D_x + 2D_y + 1)(D_x + 2D_y + 2)^2 z = 0$.

Ans. $z = \phi_1(y - 2x) + e^{-x}\phi_2(y - 2x) + e^{-y}[\phi_3(y - 2x) + y\phi_4(y - 2x)]$

19. $(D_x + D_y)(D_x + D_y - 2)z = \sin(x + 2y)$.

Ans. $z = \phi_1(y - x) + e^{2x}\phi_2(y - x) + \dfrac{1}{117}[6\cos(x + 2y) - 9\sin(x + 2y)]$

20. $(D_x + D_y - 1)(D_x + 2D_y + 2)z = e^{3x+4y} + y(1 - 2x)$.

Ans. $z = e^x\phi_1(y - x) + e^{-y}\phi_2(y - 2x) + xy + \dfrac{3}{2} + \dfrac{1}{78}e^{3x+4y}$

21. $(D_x^2 + D_xD_y + D_y - 1)z = e^x + e^{-x}$. *Ans.* $z = e^{-x}\phi_1(y) + e^x\phi_2(y - x) + \dfrac{1}{2}xe^x - \dfrac{1}{2}xe^{-x}$

22. $(D_x^3 - D_xD_y^2 - D_x^2 + D_xD_y)z = (x+2)/x^3$. *Ans.* $z = \phi_1(y) + \phi_2(y+x) + e^x\phi_3(y-x) + \ln x$

23. $(3D_xD_y - 2D_y^2 - D_y)z = \cos(3y+2x)$. *Ans.* $z = \phi_1(x) + e^{\frac{1}{2}y}\phi_2(3y+2x) - \frac{1}{3}\sin(3y+2x)$

24. $(D_x^2 + D_xD_y - D_y^2 + D_x - D_y)z = e^{2x-3y}$. *Ans.* $z = \sum c_i e^{a_ix+b_iy} - \frac{1}{6}e^{2x-3y}$, $a_i^2 + a_ib_i - b_i^2 + a_i - b_i = 0$

25. $(3D_x^2 - 2D_y^2 + D_x - 1)z = 3e^{x+y}\sin(x+y)$.

Ans. $z = \sum c_i e^{a_ix+b_iy} - e^{x+y}\cos(x+y)$, $3a_i^2 - 2b_i^2 + a_i - 1 = 0$

26. $(D_x^2 + 2D_xD_y^2 - 2D_y + 3)z = e^{x+y}\cos(x+2y)$.

Ans. $z = \sum c_i e^{a_ix+b_iy} - \frac{1}{13}e^{x+y}\cos(x+2y)$, $a_i^2 + 2a_ib_i - 2b_i + 3 = 0$

27. $(D_x^2 + D_xD_y + D_x + D_y + 1)z = e^{-2x}(x^2 + 2y^2)$.

Ans. $z = \sum c_i e^{a_ix+b_iy} + \frac{1}{27}e^{-2x}(9x^2 + 18y^2 + 18x + 12y + 16)$, $a_i^2 + a_ib_i + a_i + b_i + 1 = 0$

28. $(D_x^2D_y + D_y^2 - 2)z = e^{2y}\cos 3x + e^x\sin 2y$.

Ans. $z = \sum c_i e^{a_ix+b_iy} - \frac{1}{16}e^{2y}\cos 3x - \frac{1}{20}e^x(\cos 2y + 3\sin 2y)$, $a_i^2b_i + b_i^2 - 2 = 0$

29. $(xyD_xD_y - y^2D_y^2 - 3xD_x + 2yD_y)z = 0$. *Ans.* $z = \phi_1(\ln xy) + y^3\phi_2(\ln x) = \psi_1(xy) + y^3\psi_2(x)$

30. $(x^2D_x^2 - 2xyD_xD_y - 3y^2D_y^2 + xD_x - 3yD_y) = x^2y\sin(\ln x^2)$.

Ans. $z = \phi_1(x^3y) + \phi_2(y/x) - \frac{1}{65}x^2y[4\cos(\ln x^2) + 7\sin(\ln x^2)]$

31. $(x^2D_x^2 + xyD_xD_y - 2y^2D_y^2 - xD_x - 6yD_y)z = 0$. *Ans.* $z = \phi_1(y/x^2) + x^2\phi_2(xy)$

32. $(x^2D_x^2 - xyD_xD_y - 2y^2D_y^2 + xD_x - 2yD_y)z = \ln(y/x) - 1/2$.

Ans. $z = \phi_1(x^2y) + \phi_2(y/x) + \frac{1}{2}(\ln x)^2\ln y + \frac{1}{2}\ln x\ln y$

33. $(x^2yD_x^2D_y - xy^2D_xD_y^2 - x^2D_x^2 + y^2D_y^2)z = \dfrac{x^3+y^3}{xy}$.

Ans. $z = x\phi_1(y) + y\phi_2(x) + \phi_3(xy) - \frac{1}{6}(\frac{x^3-y^3}{xy})$

CHAPTER 33

Partial Differential Equations of Order Two with Variable Cofficients

THE MOST GENERAL LINEAR PARTIAL DIFFERENTIAL EQUATION of order two in two independent variables has the form

1) $$Rr + Ss + Tt + Pp + Qq + Zz = F$$

where R, S, T, P, Q, Z, F are functions of x and y only and not all R, S, T are zero.

Before considering the general equation, a number of special types will be treated.

TYPE I.

2a) $$r = \frac{\partial^2 z}{\partial x^2} = F/R = F_1(x,y)$$

2b) $$s = \frac{\partial^2 z}{\partial x\, \partial y} = F/S = F_2(x,y)$$

2c) $$t = \frac{\partial^2 z}{\partial y^2} = F/T = F_3(x,y).$$

These are reducible equations with constant coefficients (Chapter 32), but a more direct method of solving will be used here.

EXAMPLE 1. Solve $s = x - y$.

Integrating $s = \frac{\partial^2 z}{\partial x\, \partial y} = x - y$ with respect to y, $p = \frac{\partial z}{\partial x} = xy - \frac{1}{2}y^2 + \psi(x)$, ψ arbitrary.

Integrating this relation with respect to x, $z = \frac{1}{2}x^2 y - \frac{1}{2}xy^2 + \phi_1(x) + \phi_2(y)$,

where $\frac{d}{dx}\phi_1(x) = \psi(x)$ and $\phi_2(y)$ are arbitrary functions.

TYPE II.

3a) $$Rr + Pp = R\frac{\partial p}{\partial x} + Pp = F$$

3b) $$Ss + Pp = S\frac{\partial p}{\partial y} + Pp = F$$

3c) $$Ss + Qq = S\frac{\partial q}{\partial x} + Qq = F$$

3d) $$Tt + Qq = T\frac{\partial q}{\partial y} + Qq = F.$$

These are essentially linear *ordinary* differential equations of order one in which p (or q) is the dependent variable.

EXAMPLE 2. Solve $xr + 2p = (9x+6)e^{3x+2y}$.

Considering p as the dependent variable, x as the independent variable, and y as constant, the equation is $x\dfrac{\partial p}{\partial x} + 2p = (9x+6)e^{3x+2y}$ for which x is an integrating factor.

Integrating $x^2\dfrac{\partial p}{\partial x} + 2xp = (9x^2+6x)e^{3x+2y}$, we have

$$x^2 p = \frac{1}{D_x}(9x^2+6x)e^{3x+2y} = \frac{1}{3}e^{3x+2y}(1 - \frac{D_x}{3} + \frac{D_x^2}{9} - \dots)(9x^2+6x)$$

$$= 3x^2 e^{3x+2y} + \phi_1(y) \quad \text{or} \quad p = \frac{\partial z}{\partial x} = 3e^{3x+2y} + \frac{1}{x^2}\phi_1(y).$$

Then $z = e^{3x+2y} - \frac{1}{x}\phi_1(y) + \phi_2(y)$ is the required solution.

TYPE III.

4a) $$Rr + Ss + Pp = F \quad \text{or} \quad R\frac{\partial p}{\partial x} + S\frac{\partial p}{\partial y} = F - Pp$$

4b) $$Ss + Tt + Qq = F \quad \text{or} \quad S\frac{\partial q}{\partial x} + T\frac{\partial q}{\partial y} = F - Qq.$$

These are linear partial differential equations of order one with p (or q) as dependent variable and x, y as independent variables.

EXAMPLE 3. Solve $2xr - ys + 2p = 4xy^2$ or $2x\dfrac{\partial p}{\partial x} - y\dfrac{\partial p}{\partial y} = 4xy^2 - 2p$.

Using the method of Lagrange (Chapter 29), the auxiliary system is $\dfrac{dx}{2x} = \dfrac{dy}{-y} = \dfrac{dp}{4xy^2 - 2p}$.

From the first two ratios, we obtain readily $xy^2 = a$.

By inspection, $2y^4(2x) + 2py(-y) - y^2(4xy^2 - 2p) = 0$. Thus,

$$2y^4\,dx + 2py\,dy - y^2\,dp = 0 \quad \text{or} \quad 2\,dx - \frac{y^2\,dp - 2py\,dy}{y^4} = 0, \quad \text{and} \quad \frac{p}{y^2} - 2x = b.$$

The general solution is $p/y^2 - 2x = \psi(xy^2)$. Then

$$p = \frac{\partial z}{\partial x} = 2xy^2 + y^2\psi(xy^2) \quad \text{and} \quad z = x^2y^2 + \phi_1(xy^2) + \phi_2(y), \quad \text{where} \quad \frac{\partial}{\partial x}\phi_1(xy^2) = y^2\,\psi(xy^2).$$

TYPE IV.

5a) $$Rr + Pp + Zz = F \quad \text{or} \quad R\frac{\partial^2 z}{\partial x^2} + P\frac{\partial z}{\partial x} + Zz = F$$

5b) $$Tt + Qq + Zz = F \quad \text{or} \quad T\frac{\partial^2 z}{\partial y^2} + Q\frac{\partial z}{\partial y} + Zz = F.$$

These are essentially linear ordinary differential equations of order two with x as independent variable in 5*a*) and y as independent variable in 5*b*).

EXAMPLE 4. Solve $t - 2xq + x^2 z = (x-2)e^{3x+2y}$.

The equation may be written as $(D_y^2 - 2xD_y + x^2)z = (D_y - x)^2 z = (x-2)e^{3x+2y}$.

The complementary function is $z = e^{xy}\phi_1(x) + xe^{xy}\phi_2(x)$ and a particular integral is

$$\frac{1}{(D_y - x)^2}(x-2)e^{3x+2y} = \frac{x-2}{(2-x)^2}e^{3x+2y} = \frac{e^{3x+2y}}{x-2}.$$

The required solution is $z = e^{xy}\phi_1(x) + xe^{xy}\phi_2(x) + \dfrac{e^{3x+2y}}{x-2}$.

See also Problems 1-8.

LAPLACE'S TRANSFORMATION. This transformation on

1) $$Rr + Ss + Tt + Pp + Qq + Zz = G(u,v)$$

consists of changing from the independent variables x, y to a new set u, v, where

6) $$u = u(x,y), \quad v = v(x,y)$$

are to be chosen so that the resulting equation is simpler than 1). By means of 6), we obtain

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = z_u u_x + z_v v_x, \qquad q = \frac{\partial z}{\partial y} = z_u u_y + z_v v_y,$$

$$r = \frac{\partial p}{\partial x} = z_u u_{xx} + (z_{uu}u_x + z_{uv}v_x)u_x + z_v v_{xx} + (z_{uv}u_x + z_{vv}v_x)v_x$$

$$= z_{uu}(u_x)^2 + 2z_{uv}u_x v_x + z_{vv}(v_x)^2 + z_u u_{xx} + z_v v_{xx}.$$

$$s = \frac{\partial p}{\partial y} = z_u u_{xy} + (z_{uu}u_y + z_{uv}v_y)u_x + z_v v_{xy} + (z_{uv}u_y + z_{vv}v_y)v_x$$

$$= z_{uu}u_x u_y + z_{uv}(u_x v_y + u_y v_x) + z_{vv}v_x v_y + z_u u_{xy} + z_v v_{xy}.$$

$$t = \frac{\partial q}{\partial y} = z_{uu}(u_y)^2 + 2z_{uv}u_y v_y + z_{vv}(v_y)^2 + z_u u_{yy} + z_v v_{yy}.$$

Let

1') $$R'z_{uu} + S'z_{uv} + T'z_{vv} + P'z_u + Q'z_v + Zz = F$$

be obtained by making the above replacements in 1) and rearranging. We shall need only the coefficients

$$R' = R(u_x)^2 + Su_x u_y + T(u_y)^2 \quad \text{and} \quad T' = R(v_x)^2 + Sv_x v_y + T(v_y)^2.$$

We note that both are of the form

7) $$R(\xi_x)^2 + S\xi_x\xi_y + T(\xi_y)^2 = (a\xi_x + b\xi_y)(e\xi_x + f\xi_y).$$

i) Suppose $b/a \neq f/e$; then, if for u we take any solution of $a\xi_x + b\xi_y = 0$ and for v any solution of $e\xi_x + f\xi_y = 0$, 1) is transformed into 1') with $R' = T' = 0$.

EXAMPLE 5. Solve *a)* $x^2(y-1)r - x(y^2-1)s + y(y-1)t + xyp - q = 0$,

b) $y(x+y)(r-s) - xp - yq - z = 0$.

a) Here 7) is $x^2(y-1)(\xi_x)^2 - x(y^2-1)\xi_x\xi_y + y(y-1)(\xi_y)^2 = 0$

or $x^2(\xi_x)^2 - x(y+1)\xi_x\xi_y + y(\xi_y)^2 = (x\xi_x - y\xi_y)(x\xi_x - \xi_y) = 0$.

Now $x\xi_x - y\xi_y = 0$ is satisfied by $\xi = u = xy$ and $x\xi_x - \xi_y = 0$ is satisfied by $\xi = v = xe^y$. Moreover, it is easily shown that these solutions also satisfy the given differential equation. Hence, the required solution is

$$z = \phi_1(xy) + \phi_2(xe^y).$$

b) Here 7) is $y(x+y)[(\xi_x)^2 - \xi_x\xi_y] = 0$ or $(\xi_x - \xi_y)\xi_x = 0$.

Now $\xi_x - \xi_y = 0$ is satisfied by $\xi = x+y$ and $\xi_x = 0$ by $\xi = y$. However, neither of these solutions will satisfy the given differential equation.

We take $u = x+y$ and $v = y$. Then $p = z_u$, $q = z_u + z_v$, $r = z_{uu}$, $s = z_{uu} + z_{uv}$, and the given differential equation becomes

$$-y(x+y)z_{uv} - xz_u - yz_u - yz_v - z = 0 \quad \text{or} \quad uvz_{uv} + uz_u + vz_v + z = 0.$$

This may be written as

$$z_{uv} + \frac{1}{v}z_u + \frac{1}{u}z_v + \frac{1}{uv}z = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v} + \frac{1}{v}z\right) + \frac{1}{u}\left(\frac{\partial z}{\partial v} + \frac{1}{v}z\right) = \left(\frac{\partial}{\partial u} + \frac{1}{u}\right)\left(\frac{\partial z}{\partial v} + \frac{1}{v}z\right) = 0.$$

Let $\frac{\partial z}{\partial v} + \frac{1}{v}z = w$; then $\frac{\partial w}{\partial u} + \frac{1}{u}w = 0$ and $wu = \psi(v)$. Now

$$\frac{\partial z}{\partial v} + \frac{1}{v}z = w = \frac{1}{u}\psi(v), \quad zv = \frac{1}{u}\lambda(v) + \phi_2(u), \quad \text{and} \quad z = \frac{1}{u}\phi_1(v) + \frac{1}{v}\phi_2(u),$$

where $\frac{d}{dv}\lambda(v) = v\cdot\psi(v)$ and $\phi_1(v) = \frac{1}{v}\lambda(v)$. The required solution is $z = \frac{\phi_1(y)}{x+y} + \frac{\phi_2(x+y)}{y}$.

EXAMPLE 6. Solve $x^2r - y^2t + px - qy = x^2$.

Here 7) is $x^2(\xi_x)^2 - y^2(\xi_y)^2 = (x\xi_x - y\xi_y)(x\xi_x + y\xi_y) = 0$.

Now $x\xi_x - y\xi_y = 0$ is satisfied by $\xi = xy$ and $x\xi_x + y\xi_y = 0$ by $\xi = x/y$. It is found readily that these solutions satisfy the reduced equation $x^2r - y^2t + px - qy = 0$; hence, the complementary function is $z = \phi_1(x/y) + \phi_2(xy)$. However, this complementary function may be obtained along with the particular integral as follows. Take $u = xy$ and $v = x/y$; then

$$p = yz_u + \frac{1}{y}z_v, \quad q = xz_u - \frac{x}{y^2}z_v, \quad r = y^2z_{uu} + 2z_{uv} + \frac{1}{y^2}z_{vv}, \quad t = x^2z_{uu} - 2\frac{x^2}{y^2}z_{uv} + \frac{x^2}{y^4}z_{vv} + \frac{2x}{y^3}z_v,$$

and the given equation becomes $4x^2z_{uv} = x^2$ or $z_{uv} = \frac{1}{4}$.

Integrating first with respect to u, $\quad z_v = \psi(v) + \frac{1}{4}u$,

and then with respect to v, $\quad z = \phi_1(v) + \phi_2(u) + \frac{1}{4}uv = \phi_1(x/y) + \phi_2(xy) + \frac{1}{4}x^2$,

where $\frac{d}{dv}\phi_1(v) = \psi(v)$.

See Problems 9-10.

ii) Suppose $b/a = f/e$; then $R(\xi_x)^2 + S\xi_x\xi_y + T(\xi_y)^2 = m(a\xi_x + b\xi_y)^2$. This case is treated in Problem 11.

NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF ORDER TWO. One possible method for solving a given non-linear partial differential equation of order two

8) $$F(x,y,z,p,q,r,s,t) = 0$$

is suggested by several of the examples of linear equations above. In each of Examples 1-3, the first step consisted in finding a relation of the form

9) $$u = \psi(v), \quad \psi \text{ arbitrary},$$

where $u = u(x,y,z,p,q)$ and $v = v(x,y,z,p,q)$, from which the given differential equation could be derived by eliminating the arbitrary function. Such a relation 9) is called an *intermediate integral* of 8). For example, $p - xy + \frac{1}{2}y^2 = \psi(x)$ is an intermediate integral of $s = x - y$, (Example 1).

It can be shown that the most general partial differential equation having

$$u = \psi(v), \quad \psi \text{ arbitrary},$$

where $u = u(x,y,z,p,q)$ and $v = v(x,y,z,p,q)$, as intermediate integral has the form

10) $$Rr + Ss + Tt + U(rt - s^2) = V,$$

where R, S, T, U, V are functions of x, y, z, p, q. However, it is evident from the definitions of $R, S, \cdots, V$ that not every equation of the form 10) has an intermediate integral. The discussion below concerns Monge's method for determining an intermediate integral of 10), assuming that one exists.

TYPE: $Rr + Ss + Tt = V$. Consider the equation

11) $$Rr + Ss + Tt = V,$$

that is, 10) with U identically zero. Since we seek z as a function of x and y, we have always

12_1) $$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = p\,dx + q\,dy,$$

12_2) $$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = r\,dx + s\,dy,$$

12_3) $$dq = \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy = s\,dx + t\,dy.$$

Solving the latter two for $r = \dfrac{dp - s\,dy}{dx}$, $t = \dfrac{dq - s\,dx}{dy}$ and substituting in 11), we obtain

$$R\,\frac{dp - s\,dy}{dx} + Ss + T\,\frac{dq - s\,dx}{dy} = V \quad \text{or}$$

13) $$s[R(dy)^2 - S\,dx\,dy + T(dx)^2] = R\,dy\,dp + T\,dx\,dq - V\,dx\,dy.$$

The equations

$14_1)$ $$R(dy)^2 - S\,dx\,dy + T(dx)^2 = 0$$

$14_2)$ $$R\,dy\,dp + T\,dx\,dq - V\,dx\,dy = 0$$

are called *Monge's equations*.

Suppose $R(dy)^2 - S\,dx\,dy + T(dx)^2 = (A\,dy + B\,dx)^2 = 0$. If now $u = u(x,y,z,p,q) = a$, $v = v(x,y,z,p,q) = b$ satisfy the system

$$\begin{cases} A\,dy + B\,dx = 0 \\ R\,dy\,dp + T\,dx\,dq - V\,dx\,dy = 0, \end{cases}$$

then $$u = \psi(v)$$

is an intermediate integral of 11) since $u = a$, $v = b$ satisfy 13) and, hence, 11).

Suppose $R(dy)^2 - S\,dx\,dy + T(dx)^2 = (A_1\,dy + B_1\,dx)(A_2\,dy + B_2\,dx) = 0$, where $A_1B_2 - A_2B_1 \neq 0$ indentically. We now have two systems

$$\begin{cases} A_1\,dy + B_1\,dx = 0 \\ R\,dy\,dp + T\,dx\,dq - V\,dx\,dy = 0 \end{cases} \quad \text{and} \quad \begin{cases} A_2\,dy + B_2\,dx = 0 \\ R\,dy\,dp + T\,dx\,dq - V\,dx\,dy = 0. \end{cases}$$

If either system is integrable, we are led to an intermediate integral of 11); if both are integrable, we have two intermediate integrals at our disposal. Procedures for finding a solution of a given equation for which intermediate integrals have been obtained will be discussed in the examples and solved problems.

EXAMPLE 7. Solve $q(yq+z)r - p(2yq+z)s + yp^2t + p^2q = 0$.

Here $R = q(yq+z)$, $S = -p(2yq+z)$, $T = yp^2$, $V = -p^2q$; Monge s equations are

$$\begin{aligned} R(dy)^2 - S\,dx\,dy + T(dx)^2 &= q(yq+z)(dy)^2 + p(2yq+z)dx\,dy + yp^2(dx)^2 \\ &= (q\,dy + p\,dx)[(yq+z)dy + yp\,dx] = 0 \end{aligned}$$

and $$R\,dy\,dp + T\,dx\,dq - V\,dx\,dy = q(yq+z)dy\,dp + yp^2\,dx\,dq + p^2q\,dx\,dy = 0.$$

We seek a solution of the system $\begin{cases} q\,dy + p\,dx = 0 \\ q(yq+z)dy\,dp + yp^2\,dx\,dq + p^2q\,dx\,dy = 0. \end{cases}$

Combining the first equation and $12_1)$, we have $dz = 0$ and $z = a$. Substituting in the second equation $dy = -p\,dx/q$, obtained from the first, we obtain

$$(yq+z)dp - p(y\,dq + q\,dy) = 0.$$

We add $-p\,dz = 0$ to this, obtaining

$$(yq+z)dp - p(y\,dq + q\,dy + dz) = 0 \quad \text{or} \quad \frac{dp}{p} = \frac{y\,dq + q\,dy + dz}{yq+z}$$

with solution $\frac{yq+z}{p} = b$. Then $yq + z = p\cdot f(z)$ is an intermediate integral. The Lagrange system for this first order equation is $\frac{dx}{f(z)} = \frac{dy}{-y} = \frac{dz}{z}$. From $\frac{dy}{-y} = \frac{dz}{z}$ we obtain $yz = a$, and from $\frac{dx}{f(z)} = \frac{dz}{z}$ we obtain $x = \int f(z)\frac{dz}{z} = \phi_1(z) + b$. Thus, the required solution is

$$x = \phi_1(z) + \phi_2(yz).$$

Consider next the second system $\left[\begin{array}{l} (yq+z)dy + yp\,dx = 0 \\ q(yq+z)dy\,dp + yp^2 dx\,dq + p^2 q\,dx\,dy = 0. \end{array}\right.$

From the first equation, $p\,dx + q\,dy = -z\,dy/y$; then $dz = -z\,dy/y$ and $yz = a$. Substituting from the first equation, the second becomes

$$qy\,dp - py\,dq - pq\,dy = 0 \qquad \text{or} \qquad \frac{dp}{p} - \frac{dq}{q} - \frac{dy}{y} = 0$$

with solution $qy/p = b$. Then $qy = p\cdot g(yz)$ is an intermediate integral. The Lagrange system is $\frac{dx}{g(yz)} = \frac{dy}{-y}$, $dz = 0$. Then $z = a$ and the first equation $\frac{dx}{g(ya)} = \frac{dy}{-y}$ has solution $x = -\int g(ya)\frac{dy}{y} = \phi_2(ya) + b$. We thus obtain $x = \phi_1(z) + \phi_2(yz)$ as before.

The solution may also be obtained by using the two intermediate integrals simultaneously. Upon solving them for $p = \frac{z}{f(z) - g(yz)}$, $q = \frac{z\cdot g(yz)}{y[f(z) - g(yz)]}$

and substituting in $p\,dx + q\,dy = dz$, we have $yz\,dx + zg(yz)dy = yf(z)dz - yg(yz)dz$.
Writing $f(z) = zf_1(z)$ and $g(yz) = -yzg_1(yz)$, this equation becomes

$$dx = f_1(z)dz + g_1(yz)[z\,dy + y\,dz]$$

and, integrating,

$$x = \phi_1(z) + \phi_2(yz).$$

See also Problems 12-16.

TYPE: $Rr + Ss + Tt + U(rt - s^2) = V$. Consider equation 10) with $U \neq 0$. By substituting $r = \frac{dp - s\,dy}{dx}$, $t = \frac{dq - s\,dx}{dy}$ as in the preceding type, we obtain

$$s[R(dy)^2 - S\,dx\,dy + T(dx)^2 + U(dx\,dp + dy\,dq)] = R\,dy\,dp + T\,dx\,dq + U\,dp\,dq - V\,dx\,dy.$$

The equations

15$_1$) $$R(dy)^2 - S\,dx\,dy + T(dx)^2 + U(dx\,dp + dy\,dq) = 0$$

15$_2$) $$R\,dy\,dp + T\,dx\,dq + U\,dp\,dq - V\,dx\,dy = 0$$

are called *Monge's equations*. Note that when $U = 0$, these equations are 14$_1$) and 14$_2$). However, unlike 14$_1$) and 14$_2$), neither can be factored.

We shall attempt to choose $\lambda = \lambda(x, y, z, p, q)$ so as to obtain a factorable combination

16 $$\lambda[R(dy)^2 - S\,dx\,dy + T(dx)^2 + U(dx\,dp + dy\,dq)] + R\,dy\,dp + T\,dx\,dq + U\,dp\,dq - V\,dx\,dy$$
$$= (a\,dy + b\,dx + c\,dp)(\alpha\,dy + \beta\,dx + \gamma\,dq)$$
$$= a\alpha(dy)^2 + (a\beta + b\alpha)dx\,dy + b\beta(dx)^2 + c\beta\,dx\,dp + a\gamma\,dy\,dq + c\alpha\,dy\,dp + b\gamma\,dx\,dq + c\gamma\,dp\,dq = 0.$$

Comparing coefficients, we have

$$a\alpha = R\lambda, \quad a\beta + b\alpha = -S\lambda - V, \quad b\beta = T\lambda, \quad c\beta = U\lambda = a\gamma, \quad c\alpha = R, \quad b\gamma = T, \quad c\gamma = U.$$

The first relation will be satisfied by taking $a = \lambda$ and $\alpha = R$; this choice determines $b = T/U$, $\beta = \lambda U$, $c = 1$, $\gamma = U$. The remaining relation $a\beta + b\alpha = -S\lambda - V$

takes the form $$U\lambda^2 + \frac{TR}{U} = -S\lambda - V$$ or

17) $$U^2\lambda^2 + SU\lambda + TR + UV = 0.$$

In general 17) will have two distinct roots $\lambda = \lambda_1$, $\lambda = \lambda_2$; thus, 16) can be factored as

18_1) $$(\lambda_1 U dy + T dx + U dp)(R dy + \lambda_1 U dx + U dq) = 0$$ and

18_2) $$(\lambda_2 U dy + T dx + U dp)(R dy + \lambda_2 U dx + U dq) = 0.$$

There are four systems to be considered. The system $\lambda_1 U dy + T dx + U dp = 0$, $\lambda_2 U dy + T dx + U dp = 0$ implies $(\lambda_1 - \lambda_2) U dy = 0$ and, hence, unless $\lambda_1 = \lambda_2$, $U dy = 0$ identically. Similarly, the system, $R dy + \lambda_1 U dx + U dq = 0$, $R dy + \lambda_2 U dx + U dq = 0$ implies $U dx = 0$ identically. We therefore shall use only the systems

19) $$\begin{cases} \lambda_1 U dy + T dx + U dp = 0 \\ R dy + \lambda_2 U dx + U dq = 0 \end{cases} \quad \text{and} \quad \begin{cases} \lambda_2 U dy + T dx + U dp = 0 \\ R dy + \lambda_1 U dx + U dq = 0. \end{cases}$$

Each system, if integrable, yields an intermediate integral of 10).

EXAMPLE 8. Solve $3s - 2(rt - s^2) = 2$.

Here, $R = 0$, $S = 3$, $T = 0$, $U = -2$, $V = 2$. Then $U^2\lambda^2 + SU\lambda + TR + UV = 4\lambda^2 - 6\lambda - 4 = 0$, $\lambda_1 = -\frac{1}{2}$ and $\lambda_2 = 2$. We seek solutions of the systems

$$\begin{cases} \lambda_1 U dy + T dx + U dp = dy - 2dp = 0 \\ R dy + \lambda_2 U dx + U dq = -4dx - 2dq = 0 \end{cases} \quad \text{and} \quad \begin{cases} \lambda_2 U dy + T dx + U dp = -4dy - 2dp = 0 \\ R dy + \lambda_1 U dx + U dq = dx - 2dq = 0. \end{cases}$$

From the first system, $y - 2p = a$ and $2x + q = b$; then (i) $y - 2p = f(2x + q)$ is an intermediate integral. From the second system, $2y + p = a$ and $x - 2q = b$; then (ii) $2y + p = g(x - 2q)$ is an intermediate integral. Since q appears in the argument of both f and g, it is no longer possible to obtain a solution of the given equation involving two arbitrary functions by solving for p and q and substituting in $dz = p\,dx + q\,dy$.

We shall attempt to find a solution involving arbitrary constants from the intermediate integral $y - 2p = f(2x + q)$. To obtain an integrable equation, take $f(2x + q) = \alpha(2x + q) + \beta$, where α and β are arbitrary constants. The Lagrange system for

$$y - 2p = \alpha(2x + q) + \beta \quad \text{or} \quad 2p + \alpha q = y - 2\alpha x - \beta$$

is

$$\frac{dx}{2} = \frac{dy}{\alpha} = \frac{dz}{y - 2\alpha x - \beta}.$$

From the first two members, $\alpha x = 2y + \xi$. Substituting for αx, the last two members become

$$\frac{dy}{\alpha} = \frac{dz}{-3y - 2\xi - \beta}$$

or $\alpha\,dz = (-3y - 2\xi - \beta)dy$ and $\alpha z = -\frac{3}{2}y^2 - 2\xi y - \beta y + \eta$.

Thus, $\alpha z = \frac{5}{2}y^2 - (2\alpha x + \beta)y + \phi_1(\alpha x - 2y)$ is a solution of the given equation involving one arbitrary function and two arbitrary constants.

Treating the second intermediate integral similarly, we take $2y+p = \gamma(x-2q)+\delta$ or $p+2\gamma q = \gamma x - 2y+\delta$, where γ and δ are arbitrary constants. The corresponding Lagrange system is $\dfrac{dx}{1} = \dfrac{dy}{2\gamma} = \dfrac{dz}{\gamma x - 2y + \delta}$.

From the first two members, $y = 2\gamma x + \xi$. Now the first and third members become $\dfrac{dx}{1} = \dfrac{dz}{-3\gamma x - 2\xi + \delta}$ and $z = -\frac{3}{2}\gamma x^2 - 2\xi x + \delta x + \eta$. Thus, $z = \frac{5}{2}\gamma x^2 - (2y-\delta)x + \phi_2(y-2\gamma x)$ is also a solution involving one arbitrary function and two arbitrary constants.

A solution involving two arbitrary functions of parameters λ and μ will next be found. Set $2x+q = \lambda$ and $x-2q = \mu$ so that $x = (2\lambda+\mu)/5$. Then (i) and (ii) become $y-2p = f(\lambda)$ and $2y+p = g(\mu)$, and $y = [f(\lambda)+2g(\mu)]/5$. Now

(iii) $$p = \tfrac{1}{2}[y-f(\lambda)] = -2y+g(\mu)$$ and

(iv) $$q = \lambda - 2x = \tfrac{1}{2}(x-\mu).$$

Substituting the second value of p and the first value of q in $dz = p\,dx + q\,dy$, we have

$$\begin{aligned} dz &= [-2y+g(\mu)]dx + (\lambda-2x)dy \\ &= -2(y\,dx + x\,dy) + \frac{1}{5}g(\mu)[2\,d\lambda + d\mu] + \frac{1}{5}\lambda[f'(\lambda)d\lambda + 2g'(\mu)d\mu] \\ &= -2(y\,dx+x\,dy) + \frac{2}{5}[\lambda g'(\mu)d\mu + g(\mu)d\lambda] + \frac{1}{5}[\lambda f'(\lambda)+f(\lambda)]d\lambda - \frac{1}{5}f(\lambda)d\lambda + \frac{1}{5}g(\mu)d\mu \end{aligned}$$

and

$$\begin{aligned} z &= -2xy + \frac{2}{5}\lambda g(\mu) + \frac{1}{5}\lambda f(\lambda) - \phi_1(\lambda) + \phi_2(\mu) \\ &= -2xy + \lambda y - \phi_1(\lambda) + \phi_2(\mu). \end{aligned}$$

This solution may have been obtained by using the first value of p in (iii) and the second value of q in (iv).
See also Problems 17-18.

SOLVED PROBLEMS

1. Solve $r = x^2e^{-y}$ or $\dfrac{\partial^2 z}{\partial x^2} = x^2e^{-y}$.

One integration with respect to x yields $p = \dfrac{\partial z}{\partial x} = \dfrac{x^3}{3}e^{-y} + \phi_1(y)$, and the second integration with respect to x yields $z = \dfrac{x^4}{12}e^{-y} + x\phi_1(y) + \phi_2(y)$.

2. Solve $xy^2s = 1 - 4x^2y$.

Integrating $\dfrac{\partial^2 z}{\partial x\,\partial y} = x^{-1}y^{-2} - 4xy^{-1}$ with respect to y, $\dfrac{\partial z}{\partial x} = -x^{-1}y^{-1} - 4x\ln y + \psi(x)$.

Integrating this with respect to x, $z = -\dfrac{1}{y}\ln x - 2x^2\ln y + \phi_1(x) + \phi_2(y)$,

where $\dfrac{d}{dx}\phi_1(x) = \psi(x)$.

3. Solve $xys - px = y^2$.

Integrating $\dfrac{y\dfrac{\partial p}{\partial y} - p}{y^2} = \dfrac{1}{x}$ with respect to y, we get $\dfrac{p}{y} = \dfrac{y}{x} + \psi(x)$ or $\dfrac{\partial z}{\partial x} = \dfrac{y^2}{x} + y\psi(x)$.

Integrating with respect to x, we get $z = y^2 \ln x + y\phi_1(x) + \phi_2(y)$, where $\dfrac{d}{dx}\phi_1(x) = \psi(x)$.

4. Solve $t - xq = -\sin y - x\cos y$.

Integrating $\dfrac{\partial q}{\partial y} - xq = -(\sin y + x\cos y)$, using the integrating factor e^{-xy}, we obtain

$e^{-xy}q = -\int e^{-xy}(\sin y + x\cos y)dy = e^{-xy}\cos y + \psi(x)$ or $q = \dfrac{\partial z}{\partial y} = \cos y + e^{xy}\psi(x)$.

A second integration, with respect to y, yields $z = \sin y + e^{xy}\phi_1(x) + \phi_2(x)$, where $\phi_1(x) = \psi(x)/x$.

5. Solve $sy - 2xr - 2p = 6xy$.

The auxiliary system for the equation $2x\dfrac{\partial p}{\partial x} - y\dfrac{\partial p}{\partial y} = -6xy - 2p$ is $\dfrac{dx}{2x} = \dfrac{dy}{-y} = \dfrac{dp}{-6xy-2p}$.

From the first and second ratios, we find $xy^2 = a$. By inspection,

$$2y^3(2x) - (2yp + 2xy^2)(-y) + y^2(-6xy - 2p) = 0$$

so that
$$2y^3dx - (2yp + 2xy^2)dy + y^2dp = 0,$$

or $\dfrac{y^2(dp + 2x\,dy + 2y\,dx) - 2y(p + 2xy)dy}{y^4} = 0$, and $\dfrac{p + 2xy}{y^2} = b$.

Thus, we obtain as solution $p + 2xy = y^2\psi(xy^2)$. Then

$\dfrac{\partial z}{\partial x} = -2xy + y^2\psi(xy^2)$ and $z = -x^2y + \phi_1(xy^2) + \phi_2(y)$, where $\dfrac{\partial}{\partial x}\phi_1(xy^2) = y^2\psi(xy^2)$.

6. Solve $xs + yt + q = 10x^3y$.

The auxiliary system for the equation $x\dfrac{\partial q}{\partial x} + y\dfrac{\partial q}{\partial y} = 10x^3y - q$ is $\dfrac{dx}{x} = \dfrac{dy}{y} = \dfrac{dq}{10x^3y-q}$.

From the first two ratios, $x/y = a$. By inspection,

$$(q - 8x^3y)x - 2x^4(y) + x(10x^3y - q) = 0$$

so that $(q - 8x^3y)dx - 2x^4dy + x\,dq = 0$, or $x\,dq + q\,dx = 8x^3y\,dx + 2x^4dy$,

and
$$qx = 2x^4y + b.$$

The general solution is $qx = 2x^4y + \psi(y/x)$. Thus,

$\dfrac{\partial z}{\partial y} = 2x^3y + \dfrac{1}{x}\psi(\dfrac{y}{x})$ and $z = x^3y^2 + \phi_1(\dfrac{y}{x}) + \phi_2(x)$, where $\dfrac{\partial}{\partial y}\phi_1(\dfrac{y}{x}) = \dfrac{1}{x}\psi(\dfrac{y}{x})$.

7. Solve $t - q - \frac{1}{x}(\frac{1}{x} - 1)z = xy^2 - x^2y^2 + 2x^3y - 2x^3$.

The equation may be written as $[D_y^2 - D_y - \frac{1}{x}(\frac{1}{x} - 1)]z = xy^2 - x^2y^2 + 2x^3y - 2x^3$.

The complementary function is $z = e^{y/x}\phi_1(x) + e^{y - y/x}\phi_2(x)$.

For a particular integral we try $z = Ay^2 + By + C$, where A, B, C are functions of x or constants. Then $[D_y^2 - D_y - \frac{1}{x}(\frac{1}{x} - 1)]z = 2A - 2Ay - B - (\frac{1}{x^2} - \frac{1}{x})(Ay^2 + By + C) = xy^2 - x^2y^2 + 2x^3y - 2x^3$, identically. Equating coefficients of the several powers of y, we have

$$-(\frac{1}{x^2} - \frac{1}{x})A = x(1 - x), \quad -2A - (\frac{1}{x^2} - \frac{1}{x})B = 2x^3, \quad 2A - B - (\frac{1}{x^2} - \frac{1}{x})C = -2x^3.$$

Then $A = -x^3$, $B = C = 0$ and the required solution is $z = e^{y/x}\phi_1(x) + e^{y - y/x}\phi_2(x) - x^3y^2$.

8. Solve $ys + p - yq - z = (1 - x)(1 + \ln y)$.

This equation is solved readily by noting that it may be put in the form

$$\frac{\partial^2 z}{\partial x\,\partial y} + \frac{1}{y}\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} - \frac{z}{y} = \frac{\partial}{\partial x}(\frac{\partial z}{\partial y} + \frac{1}{y}z) - (\frac{\partial z}{\partial y} + \frac{1}{y}z) = \frac{1 - x}{y}(1 + \ln y).$$

Setting $w = \frac{\partial z}{\partial y} + \frac{1}{y}z$, the equation becomes $\frac{\partial w}{\partial x} - w = \frac{1 - x}{y}(1 + \ln y)$ for which e^{-x} is an integrating factor. Then

$$e^{-x}w = \frac{1 + \ln y}{y}\int^x (e^{-x} - xe^{-x})dx = \frac{1 + \ln y}{y}(xe^{-x}) + \psi(y) \quad \text{and} \quad w = x\,\frac{1 + \ln y}{y} + e^x\psi(y).$$

In turn, integrating $\frac{\partial z}{\partial y} + \frac{1}{y}z = x\frac{1 + \ln y}{y} + e^x\psi(y)$, using the integrating factor y, we find

$$yz = x\int^y (1 + \ln y)dy + e^x\int^y y\psi(y)\,dy = xy\ln y + e^x\phi_1(y) + \phi_2(x).$$

LAPLACE'S TRANSFORMATION.

9. Solve $t - s + p - q(1 + 1/x) + z/x = 0$.

Setting $(\xi_y)^2 - \xi_x\xi_y = 0$ and solving, we have $\xi = x$ and $\xi = x + y$.

For the choice $u = x$ and $v = x + y$, $p = z_u + z_v$, $q = z_v$, $s = z_{uv} + z_{vv}$, and $t = z_{vv}$. Substituting in the given equation, we have $z_{uv} - z_u + \frac{1}{x}(z_v - z) = \frac{\partial}{\partial u}(\frac{\partial z}{\partial v} - z) + \frac{1}{x}(\frac{\partial z}{\partial v} - z) = 0$.

Let $\frac{\partial z}{\partial v} - z = w$; then $\frac{\partial w}{\partial u} + \frac{w}{u} = 0$ and $uw = u(\frac{\partial z}{\partial v} - z) = \psi(v)$.

Integrating $\frac{\partial z}{\partial v} - z = \frac{1}{u}\psi(v)$, we have $e^{-v}z = \frac{1}{u}\phi_1(v) + \phi(u)$ or $z = \frac{e^v}{u}\phi_1(v) + e^v\phi(u)$.

In the original variables, $z = \frac{e^{x+y}}{x}\phi_1(x+y) + e^{x+y}\phi(x) = \frac{1}{x}f(x+y) + e^y g(x)$, where $f(x+y) = e^{x+y}\phi_1(x+y)$ and $g(x) = e^x\phi(x)$.

10. Solve $xys - x^2r - px - qy + z = -2x^2y$.

From $xy\xi_x\xi_y - x^2(\xi_x)^2 = x\xi_x(y\xi_y - x\xi_x) = 0$, we obtain $\xi = y$ and $\xi = xy$.

Using $u = xy$, $v = y$, $p = yz_u$, $q = xz_u + z_v$, $r = y^2z_{uu}$, $s = z_u + xyz_{uu} + yz_{uv}$, the given differential equation becomes

$$z_{uv} - \frac{1}{v}z_u - \frac{1}{u}z_v + \frac{1}{uv}z = -\frac{2u}{v^2} \quad \text{or} \quad \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v} - \frac{1}{v}z\right) - \frac{1}{u}\left(\frac{\partial z}{\partial v} - \frac{1}{v}z\right) = -\frac{2u}{v^2}.$$

Let $\frac{\partial z}{\partial v} - \frac{1}{v}z = w$; then $\frac{\partial w}{\partial u} - \frac{w}{u} = -\frac{2u}{v^2}$, and $\frac{w}{u} = -\frac{2u}{v^2} + \psi(v)$ or $w = -\frac{2u^2}{v^2} + u\,\psi(v)$.

Integrating $w = \frac{\partial z}{\partial v} - \frac{1}{v}z = -\frac{2u^2}{v^2} + u\,\psi(v)$, we have $\frac{z}{v} = \frac{u^2}{v^2} + u\,\psi_1(v) + \phi_2(u)$ or

$$z = \frac{u^2}{v} + uv\,\psi_1(v) + v\,\phi_2(u) = \frac{u^2}{v} + u\,\lambda_1(v) + v\,\phi_2(u).$$

In the original variables, $z = xy\,\lambda_1(y) + y\,\phi_2(xy) + x^2y = x\phi_1(y) + y\phi_2(xy) + x^2y$.

11. Solve $x^2r - 2xys + y^2t - xp + 3yq = 8y/x$.

Here $x^2(\xi_x)^2 - 2xy\xi_x\xi_y + y^2(\xi_y)^2 = (x\xi_x - y\xi_y)^2 = 0$, and since the factors are not distinct we obtain only $\xi = xy$.

We set $u = xy$ and take $v = y$; then $p = yz_u$, $q = xz_u + z_v$, $r = y^2z_{uu}$, $s = z_u + xyz_{uu} + yz_{uv}$, $t = x^2z_{uu} + 2xz_{uv} + z_{vv}$ and the given differential equation becomes

$$y^2z_{vv} + 3yz_v = 8y/x \quad \text{or} \quad v^2z_{vv} + 3vz_v = 8v^2/u,$$

an equation of the Cauchy type. However, it is seen that v is an integrating factor; hence

$$v^3z_{vv} + 3v^2z_v = 8v^3/u \quad \text{and} \quad v^3z_v = 2v^4/u + \phi(u).$$

Then $z_v = \frac{2v}{u} + \frac{1}{v^3}\phi(u)$ and

$$z = \frac{v^2}{u} - \frac{1}{2v^2}\phi(u) + \phi_1(u)$$

$$= \frac{v^2}{u} + \frac{1}{v^2}\psi(u) + \phi_1(u)$$

$$= \phi_1(xy) + \frac{1}{y^2}\psi(xy) + \frac{y}{x}$$

or $z = \phi_1(xy) + x^2\,\phi_2(xy) + \frac{y}{x}$, where $\psi(xy) = x^2y^2\,\phi_2(xy)$.

MONGE'S METHOD.

12. Solve $qs - pt = q^3$.

The Monge equations are $q\,dx\,dy + p(dx)^2 = 0$ and $p\,dx\,dq + q^3\,dx\,dy = 0$.

From the first equation, $q\,dy + p\,dx = 0$; then $dz = p\,dx + q\,dy = 0$ and $z = a$.

Substituting $q\,dy = -p\,dx$ in the second equation yields $dq - q^2 dx = 0$; thus $1/q + x = b$ and $1/q + x = f(z)$ or $[x - f(z)]q = -1$ is an intermediate integral.

The required solution is obtained by solving this first order equation; thus

$$xz - \int f(z)dz = -y + \phi_2(x) \quad \text{or} \quad y + xz = \phi_1(z) + \phi_2(x), \quad \text{where} \quad \phi_1'(z) = f(z).$$

13. Solve $q^2 r - 2pqs + p^2 t = pq^2$.

The Monge equations are $(q\,dy + p\,dx)^2 = 0$ and $q^2\,dy\,dp + p^2\,dx\,dq - pq^2\,dx\,dy = 0$.

From the first equation, $q\,dy + p\,dx = 0$; then $dz = p\,dx + q\,dy = 0$ and $z = a$.

Substituting $q\,dy = -p\,dx$ in the second yields $-q\,dp + p\,dq + pq\,dx = 0$ or $-\dfrac{dp}{p} + \dfrac{dq}{q} + dx = 0$ and $e^x q/p = b$. Thus $e^x q - p\,f(z) = 0$ is an intermediate integral. The Lagrange system for this equation is $\dfrac{dx}{f(z)} = \dfrac{dy}{-e^x}$, $dz = 0$.

From the second equation, $z = c$. Then the first becomes $\dfrac{dx}{f(c)} = \dfrac{dy}{-e^x}$ with solution $e^x/f(c) + y = d$. As required solution, we find

$$y = -e^x/f(z) + \phi_2(z) = e^x\,\phi_1(z) + \phi_2(z), \quad \text{where} \quad \phi_1(z) = -1/f(z).$$

14. Solve $x(r + 2xs + x^2 t) = p + 2x^3$.

The Monge equations are $(dy)^2 - 2x\,dx\,dy + x^2(dx)^2 = (dy - x\,dx)^2 = 0$

$$\text{and} \quad x\,dy\,dp + x^3\,dx\,dq - (p + 2x^3)dx\,dy = 0.$$

We seek a solution of the system $dy - x\,dx = 0$, $x\,dy\,dp + x^3\,dx\,dq - (p + 2x^3)dx\,dy = 0$.

From the first equation, $x^2 - 2y = a$. Substituting $dy = x\,dx$ in the second, we get

$$x\,dp + x^2 dq - (p + 2x^3)dx = 0.$$

Using the integrating factor $1/x^2$, we obtain the intermediate integral $p + xq = x^3 + x\,f(x^2 - 2y)$.

The Lagrange system is $\dfrac{dx}{1} = \dfrac{dy}{x} = \dfrac{dz}{x^3 + x\,f(x^2 - 2y)}$. The first two members yield $x^2 - 2y = c$ and then

the first and third become $\dfrac{dx}{1} = \dfrac{dz}{x^3 + x\,f(c)}$. Solving,

$$z = \tfrac{1}{4}x^4 + \tfrac{1}{2}x^2 f(c) + \phi(c) \quad \text{or} \quad z = \tfrac{1}{4}x^4 + \tfrac{1}{2}x^2 f(x^2 - 2y) + \phi(x^2 - 2y).$$

15. Solve $q(1+q)r - (1+2q)(1+p)s + (1+p)^2 t = 0$.

The Monge equations are

$$q(1+q)(dy)^2 + (1+2q)(1+p)dx\,dy + (1+p)^2(dx)^2 = [q\,dy + (1+p)dx][(1+q)dy + (1+p)dx] = 0$$

and $$q(1+q)dy\,dp + (1+p)^2\,dx\,dq = 0.$$

Consider first the system

$$q\,dy + (1+p)dx = 0$$
$$q(1+q)dy\,dp + (1+p)^2 dx\,dq = 0.$$

From the first equation, $p\,dx + q\,dy = -dx$; then $dz = -dx$ and $x+z=a$. The substitution of $q\,dy = -(1+p)dx$ in the second yields

$$-(1+q)dp + (1+p)dq = 0$$

from which we obtain $\frac{1+p}{1+q} = b$. Thus, $\frac{1+p}{1+q} = f(x+z)$ is an intermediate integral.

Consider next the system

$$(1+q)dy + (1+p)dx = 0$$
$$q(1+q)dy\,dp + (1+p)^2 dx\,dq = 0.$$

From the first, $p\,dx + q\,dy = -(dx+dy)$ so that $dz = -(dx+dy)$ and $x+y+z=a$. The substitution of $(1+q)dy = -(1+p)dx$ in the second gives $-q\,dp + (1+p)dq = 0$ which is satisfied by $\frac{1+p}{q} = b$. Thus, $\frac{1+p}{q} = g(x+y+z)$ is an intermediate integral.

Solving the two intermediate integrals for $p = \frac{fg+f-g}{g-f}$, $q = \frac{f}{g-f}$ and substituting in the relation $p\,dx + q\,dy = dz$, we have

$$(fg+f-g)dx + f\,dy = (g-f)dz, \qquad fg\,dx = -f(dx+dy+dz) + g(dx+dz),$$

$$dx = -\frac{dx+dy+dz}{g(x+y+z)} + \frac{dx+dz}{f(x+z)}, \quad \text{and} \quad x = \phi_1(x+y+z) + \phi_2(x+z).$$

16. Solve $(x-z)[xq^2r - q(x+z+2px)s + (z+px+pz+p^2x)t] = (1+p)q^2(x+z)$.

Monge's equations are

$$xq^2(dy)^2 + q(x+z+2px)dx\,dy + (1+p)(z+px)(dx)^2 = [q\,dy + (1+p)dx][xq\,dy + (z+px)dx] = 0$$

and $$(x-z)[xq^2dy\,dp + (1+p)(z+px)dx\,dq] - (1+p)q^2(x+z)dx\,dy = 0.$$

Consider first the system

$$q\,dy + (1+p)dx = 0$$
$$(x-z)xq^2dy\,dp + (1+p)(z+px)(x-z)dx\,dq - (1+p)q^2(x+z)dx\,dy = 0.$$

From the first equation, $p\,dx + q\,dy = -dx$; then $dz = -dx$ and $x+z=a$. Substituting $q\,dy = -(1+p)dx$, $a-x$ in the second, we have

i) $$-(2x-a)xq\,dp + (2x-a)(a-x+px)dq + (1+p)qa\,dx = 0.$$

To solve this equation, consider x as a constant so that $dx = 0$. Then i) becomes

$$-(2x-a)xq\,dp + (2x-a)(a-x+px)dq = 0 \quad \text{or} \quad x(q\,dp - p\,dq) - (a-x)dq = 0$$

and $\frac{xp+a-x}{q} = \psi(x)$. To determine we take the differential of this relation,

$$q(x\,dp + p\,dx - dx) - (xp+a-x)dq = q^2d\psi$$

and obtain $$xq\,dp - xp\,dq = q^2d\psi - pq\,dx + q\,dx + a\,dq - x\,dq.$$

From i), $$xq\,dp - xp\,dq = \frac{(2x-a)(a-x)dq + (1+p)qa\,dx}{2x-a} = (a-x)dq + \frac{(1+p)qa\,dx}{2x-a};$$

then $$q^2 d\psi - pq\,dx + q\,dx + a\,dq - x\,dq = (a-x)dq + \frac{(1+p)qa\,dx}{2x-a},$$

$$d\psi = \frac{2(px+a-x)}{q(2x-a)}dx = \frac{2\psi}{2x-a}dx \quad \text{and} \quad \frac{\psi}{2x-a} = b = f(x+z).$$

Thus, $\dfrac{xp+a-x}{q(2x-a)} = \dfrac{xp+z}{q(x-z)} = f(x+z)$ is an intermediate integral.

Consider next the system

$$xq\,dy + (z+px)dx = 0$$
$$(x-z)xq^2 dy\,dp + (1+p)(z+px)(x-z)dx\,dq - (1+p)q^2(x+z)dx\,dy = 0.$$

From the first equation, $p\,dx + q\,dy = -z\,dx/x$; then $dz = -z\,dx/x$ and $xz = a$. Substituting $xq\,dy = -(z+px)dx$, $z = a/x$ in the second, we have

ii) $$-xq(x^2-a)dp + x(1+p)(x^2-a)dq + (1+p)q(x^2+a)dx = 0.$$

Considering x as a constant, this becomes $q\,dp - (1+p)dq = 0$ and we have $\dfrac{1+p}{q} = \psi(x)$. From this relation we find $q\,dp - (1+p)dq = q^2 d\psi$, while from ii) $q\,dp - (1+p)dq = \dfrac{(1+p)q(x^2+a)}{x(x^2-a)}dx$.

Then $d\psi = \dfrac{(1+p)q(x^2+a)}{q^2x(x^2-a)}dx = \dfrac{\psi(x^2+a)}{x(x^2-a)}dx = (-\dfrac{dx}{x} + \dfrac{2x\,dx}{x^2-a})\psi$, $\ln\psi = -\ln x + \ln(x^2-a) + \ln b$,

and $\psi = \dfrac{b(x^2-a)}{x} = \dfrac{1+p}{q}$. Thus, $\dfrac{1+p}{q(x-z)} = g(xz)$ is an intermediate integral.

Solving the two intermediate integrals, we find $p = \dfrac{f-zg}{xg-f}$ and $q = \dfrac{1}{xg-f}$; then

$$dz = p\,dx + q\,dy = \frac{f-zg}{xg-f}dx + \frac{1}{xg-f}dy \quad \text{or} \quad f(x+z)(dx+dz) + dy = zg(xz)dx + xg(xz)dz.$$

Thus, $y + \phi_1(x+z) = \phi_2(xz)$ is the required solution.

17. Solve $3r + s + t + (rt - s^2) = -9$.

Here, $R = 3$, $S = T = U = 1$, $V = -9$; then

$$U^2\lambda^2 + SU\lambda + TR + UV = \lambda^2 + \lambda - 6 = 0 \quad \text{and} \quad \lambda_1 = 2,\ \lambda_2 = -3.$$

We seek solutions of the system (see equations 19)

$\lambda_1 U\,dy + T\,dx + U\,dp = 2\,dy + dx + dp = 0$, $R\,dy \quad dx + U\,dq = 3\,dy - 3\,dx + dq = 0$

and $\lambda_2 U\,dy + T\,dx + U\,dp = -3\,dy + dx + dp = 0$, $R\,dy \quad dx + U\,dq = 3\,dy + 2\,dx + dq = 0$.

From the first system, we have $2y + x + p = a$, $3y - 3x + q = b$; thus, $p + 2y + x = f(q + 3y - 3x)$ is an intermediate integral. From the second system, we have $-3y + x + p = c$, $3y + 2x + q = d$; thus, $p - 3y + x = g(q + 3y + 2x)$ is an intermediate integral. Since q appears in the argument of both f and g, it will not be possible to solve for p and q as before, and it will not be possible to find a solution involving two arbitrary functions. We give two solutions involving arbitrary constants.

Replacing the arbitrary function f of the first intermediate integral by $\alpha(q + 3y - 3x) + \beta$, we obtain

$$p + 2y + x = \alpha(q + 3y - 3x) + \beta \quad \text{or} \quad p - \alpha q = (3\alpha - 2)y - (3\alpha + 1)x + \beta$$

for which the Langrange system is $\frac{dx}{1} = \frac{dy}{-a} = \frac{dz}{(3a-2)y-(3a+1)x+\beta}$. From $\frac{dx}{1} = \frac{dy}{-a}$, we find $y + ax = \xi$;

then $$\frac{dx}{1} = \frac{dz}{(3a-2)y-(3a+1)x+\beta} = \frac{dz}{-(3a^2+a+1)x + 3a\xi - 2\xi + \beta}$$ and

$$z = -\tfrac{1}{2}(3a^2+a+1)x^2 + (3a\xi - 2\xi + \beta)x + \eta = -\tfrac{1}{2}(3a^2+a+1)x^2 + (3ay+3a^2x-2y-2ax+\beta)x+\eta.$$

Thus, $z = \frac{1}{2}(3a^2-5a-1)x^2 + (3a-2)xy + \beta x + \phi_1(y+ax)$ is a solution involving one arbitrary function and two arbitrary constants.

Replacing the arbitrary function $g(q+3y+2x)$ of the second intermediate integral by the linear function $\gamma(q+3y+2x)+\delta$, we obtain

$$p - 3y + x = \gamma(q+3y+2x)+\delta \quad \text{or} \quad p - \gamma q = 3(\gamma+1)y + (2\gamma-1)x + \delta$$

for which the Lagrange system is $\frac{dx}{1} = \frac{dy}{-\gamma} = \frac{dz}{3(\gamma+1)y+(2\gamma-1)x+\delta}$. From $\frac{dx}{1} = \frac{dy}{-\gamma}$, we get

$y + \gamma x = \xi$; then $$\frac{dx}{1} = \frac{dz}{3(\gamma+1)y + (2\gamma-1)x + \delta} = \frac{dz}{-(3\gamma^2+\gamma+1)x + 3\gamma\xi + 3\xi + \delta}$$ and

$$z = -\tfrac{1}{2}(3\gamma^2+\gamma+1)x^2 + (3\gamma\xi+3\xi+\delta)x + \eta.$$

Thus, $z = \frac{1}{2}(3\gamma^2+5\gamma-1)x^2 + 3(\gamma+1)xy + \delta x + \phi_2(y+\gamma x)$ is also a solution.

18. Solve $xqr + (p+q)s + ypt + (xy-1)(rt-s^2) + pq = 0$.

Here, $R = xq$, $S = p+q$, $T = yp$, $U = xy-1$, $V = -pq$; then

$$U^2\lambda^2 + SU\lambda + TR + UV = (xy-1)^2\lambda^2 + (p+q)(xy-1)\lambda + pq = 0 \quad \text{and} \quad \lambda_1 = \frac{-p}{xy-1}, \ \lambda_2 = \frac{-q}{xy-1}.$$

Consider first the system $\begin{cases} -p\,dy + yp\,dx + (xy-1)dp = 0 \\ xq\,dy - q\,dx + (xy-1)dq = 0 \end{cases}$. The system is not integrable since neither equation is integrable.

Consider next the system $-q\,dy + yp\,dx + (xy-1)dp = 0, \quad xq\,dy - p\,dx + (xy-1)dq = 0.$

We multiply the second equation by y, add the first, and divide by $xy-1$ to obtain $q\,dy + dp + y\,dq = 0$ and thus $p + yq = a$. Again, we multiply the first equation by x, add the second, and divide by $xy-1$ to obtain $p\,dx + x\,dp + dq = 0$ and thus $xp + q = b$. However, the form of the resulting intermediate integral $xp + q = f(yq+p)$ or $yq + p = g(xp+q)$ does not permit a solution involving two arbitrary functions.

To obtain a solution, involving one arbitrary function and two arbitrary constants, we replace $f(yq+p)$ by the linear function $a(yq+p)+\beta$ in the first form of the intermediate integral above and have

$$(x-a)p + (1-ay)q = \beta.$$

The corresponding Lagrange system is $\frac{dx}{x-a} = \frac{dy}{1-ay} = \frac{dz}{\beta}$. From the first two members we obtain

$a\ln(x-a) + \ln(1-ay) = \ln\xi$ or $(x-a)^a(1-ay) = \xi$, and from the first and third members we get $z = \beta\ln(x-a) + \eta$. Thus, the solution is

$$z = \beta\ln(x-a) + \phi[(x-a)^a(1-ay)].$$

SUPPLEMENTARY PROBLEMS

Solve.

19. $r = xy$ *Ans.* $z = x\,\phi_1(y) + \phi_2(y) + \frac{1}{6}x^3y$

20. $s = x^2 + y^2$ $z = \phi_1(x) + \phi_2(y) + \frac{1}{3}(x^3y + xy^3)$

21. $t = -x^2\sin(xy)$ $z = y\,\phi_1(x) + \phi_2(x) + \sin(xy)$

22. $xr - p = 0$ $z = x^2\,\phi_1(y) + \phi_2(y)$

23. $xr + p = 1/x^2$ $z = \phi_1(y)\ln x + \phi_2(y) + 1/x$

24. $yt - q = 2x^2y$ $z = y^2\,\phi_1(x) + \phi_2(x) + x^2y^2\ln y$

25. $ys - p = xy^2\sin(xy)$ $z = y\,\phi_1(x) + \phi_2(y) - \sin(xy)$

26. $t + q = xe^{-y}$ $z = e^{-y}\,\phi_1(x) + \phi_2(x) - xye^{-y}$

27. $r + s = 3y^2$ $z = \phi_1(x-y) + \phi_2(y) + xy^3$

28. $xyr + x^2s - yp = x^3e^y$ $z = \phi_1(x^2-y^2) + \phi_2(y) + \frac{1}{2}x^2e^y$

29. $2yt - xs + 2q = x^2y$ $z = \phi_1(x^2y) + \phi_2(x) + \frac{1}{4}x^2y^2$

30. $xr + ys + p = 8xy^2 + 9x^2$ $z = \phi_1(x/y) + \phi_2(y) + x^2y^2 + x^3$

LAPLACE'S TRANSFORMATION.

31. $6r - s - t = 18y - 4x$ *Ans.* $z = \phi_1(x-3y) + \phi_2(x+2y) + y(2x^2+y^2)$

32. $x(xy-1)r - (x^2y^2-1)s + y(xy-1)t + (x-1)p + (y-1)q = 0$ *Ans.* $z = \phi_1(xe^y) + \phi_2(ye^x)$

33. $x(y-x)r - (y^2-x^2)s + y(y-x)t + (y+x)(p-q) = 2(x+y+1)$
    Hint: Let $x+y = u$, $xy = v$. *Ans.* $z = \phi_1(x+y) + \phi_2(xy) + x - y + \ln x$

34. $(y-1)r - (y^2-1)s + y(y-1)t + p - q = 2ye^{2x}(1-y)^3$
    Ans. $z = \phi_1(x+y) + \phi_2(ye^x) + (x+y)y^2e^{2x}$

35. $xyr - (x^2-y^2)s - xyt + py - qx = 2(x^2-y^2)$ *Ans.* $z = \phi_1(x^2+y^2) + \phi_2(y/x) - xy$

36. $r - 2s + t + p - q = e^x(2y-3) - e^y$ *Ans.* $z = \phi_1(x+y) + e^y\,\phi_2(x+y) + xe^y + ye^x$
    Hint: Let $x+y = u$, $y = v$.

37. $y^2(r-2s+t) - y(p-q) - z = y^2$ *Ans.* $z = y\,\phi_1(x+y) + \frac{1}{y}\,\phi_2(x+y) + \frac{1}{3}y^2$

MONGE'S METHOD.

38. $(e^x-1)(qr-ps) = pqe^x$ I.I. $p = \psi(z)$. G.S.: $x = \phi_1(z) + \phi_2(y) + e^x$

39. $r - 3s - 10t = -3$

I.I: $p + 2q = \psi_1(y + 5x), \quad p - 5q = \psi_2(y - 2x)$

G.S.: $z = \phi_1(y + 5x) + \phi_2(y - 2x) + xy$

40. $q^2 r - 2pqs + p^2 t = 0$

I.I.: $p = q\,\psi(z)$. G.S.: $x\phi_1(z) + y = \phi_2(z)$.

41. $qr - (1 + p + q)s + (1 + p)t = 0$

I.I.: $p - q = \psi_1(x + z), \quad p + 1 = q\,\psi_2(x + y)$

G.S.: $z = f(x + z) + g(x + y)$

42. $(1 - q)^2 r - 2(2 - p - 2q + pq)s + (2 - p)^2 t = 0$

I.I.: $\dfrac{1 - q}{2 - p} = \psi(y + 2x - z)$

G.S.: $x + y\,\phi_1(y + 2x - z) = \phi_2(y + 2x - z)$

43. $5r - 10s + 4t - (rt - s^2) = -1$

I.I.: $3y + 4x - p = f(5y + 7x - q), \quad 7y + 4x - p = g(5y + 3x - q)$

$z = 2x^2 + 3xy + \frac{5}{2}y^2 - 2\alpha x^2 - \beta x + \phi_1(y + \alpha x)$ or $z = 2x^2 + 7xy + \frac{5}{2}y^2 + 2\gamma x^2 - \delta x + \phi_2(y + \gamma x)$

44. $2r - 6s + 2t + (rt - s^2) = 4$

I.I.: $2y + 2x + p = f(2y + 4x + q), \quad 4y + 2x + p = g(2y + 2x + q)$

Sol. $z = \alpha x^2 + \beta x - (x + y)^2 + \phi_1(y + \alpha x)$ or $z = -\gamma x^2 + \delta x - x^2 - 4xy - y^2 + \phi_2(y + \gamma x)$

45. $3r - 6s + 4t - (rt - s^2) = 3$

I.I.: $3y + 4x - p = f(3y + 3x - q)$. Sol.: $z = 2x^2 + 3xy + \frac{3}{2}y^2 + \beta x + \phi(y + \alpha x)$.

46. $yr - ps + t + y(rt - s^2) = -1$

I.I.: $yp + x = f(q + y)$. Sol. $6\alpha^2 z = 2y^3 - 3\alpha^2 y^2 + 6\alpha xy + 6\beta y + \phi(\alpha x + \frac{1}{2}y^2)$.

47. $xqr - (x + y)s + ypt + xy(rt - s^2) = 1 - pq$

I.I.: $xp + y = f(yq + x)$. Sol.: $z = \alpha x + y/\alpha + \beta \ln x + \phi(x^{\alpha} y)$.