

SET THEORY AND RELATED TOPICS

Second Edition

SEYMOUR LIPSCHUTZ

Use this book with any text or as a self-study guide

Teaches effective problem-solving

530 problems and worked solutions

Covers functions, relations,
cardinal and ordinal numbers,
transfinite induction, Zorn's lemma,
algebra of propositions,
Boolean algebra

**MORE THAN
30 MILLION
SCHAUM'S
OUTLINES
SOLD**

Use with these courses: Finite Mathematics Modern Mathematics Advanced Calculus
 Probability Real Variables Discrete Mathematics

SCHAUM'S OUTLINE OF

THEORY AND PROBLEMS

OF

SET THEORY

and Related Topics

Second Edition

SEYMOUR LIPSCHUTZ, Ph.D.

*Professor of Mathematics
Temple University*

NEW EDITION - 2005 - 2006

SCHAUM'S OUTLINE SERIES

McGRAW-HILL

*New York San Francisco Washington, D.C. Auckland Bogotá Caracas Lisbon
London Madrid Mexico City Milan Montreal New Delhi
San Juan Singapore Sydney Tokyo Toronto*

SEYMOUR LIPSCHUTZ, who is presently on the mathematics faculty of Temple University, formerly taught at the Polytechnic Institute of Brooklyn and was visiting professor in the Computer Science Department of Brooklyn College. He received his Ph.D. in 1960 at the Courant Institute of Mathematical Sciences of New York University. Some of his other books in the Schaum's Outline Series are *Beginning Linear Algebra*; *Discrete Mathematics, 2nd ed.*; *Probability*; and *Linear Algebra, 2nd ed.*

Schaum's Outline of Theory and Problems of
SET THEORY
International Editions 1999
Revised Edition - 2005

Exclusive rights by McGraw-Hill Book Co - Singapore, for manufacture and export. This book cannot be re-exported from the country to which it is consigned by McGraw-Hill.

Copyright © 1998, 1964 by The McGraw-Hill Companies, Inc. All rights reserved. Except as permitted under the United States Copyright Act of 1976, no part of this publication may be reproduced or distributed in any form or by any means, or stored in a data base or retrieval system, without the prior written permission of the publisher.

3 4 5 6 7 8 9 10 KKP-PMP-20

Library of Congress Cataloging-in-Publication Data

When ordering this title, use ISBN 0-07-116494-4

Printed in Singapore

McGraw-Hill

A Division of The McGraw-Hill Companies



Preface

The theory of sets lies at the foundations of mathematics. Concepts in set theory, such as functions and relations, appear explicitly or implicitly in every branch of mathematics. These concepts also appear in many related fields such as computer science, the physical sciences, and engineering. This text is an informal, nonaxiomatic treatment of the theory of sets.

The material is divided into three Parts, since the logical development is thereby not disturbed while the usefulness as a text and reference book on any of several levels is increased. Part I contains an introduction to the elementary operations of sets and a detailed discussion of the concepts of relation and function. Part II develops the theory of cardinal and ordinal numbers in the classical approach of Cantor. It also considers partially ordered sets, and the Axiom of Choice and its equivalents including Zorn's lemma. Part III treats those topics which are usually associated with the elementary theory of sets, that is, logic and Boolean algebra.

This second edition of *Set Theory* covers more material than the first edition. In particular, it includes a deeper discussion of the real numbers \mathbf{R} and a more complete discussion of the integers \mathbf{Z} . Furthermore, it includes a discussion of algorithms and their complexity in the chapter on functions, and it includes new material, including Karnaugh maps, in the chapter on Boolean algebra.

Each chapter begins with clear statements of pertinent definitions, principles, and theorems together with illustrative and other descriptive material. This is followed by graded sets of solved and supplementary problems. The solved problems serve to illustrate and amplify the theory, bring into sharp focus those fine points without which the student continually feels himself on unsafe ground, and provide the repetition of basic principles so vital to effective learning. Numerous proofs of theorems and derivations of basic results are included among the solved problems. The supplementary problems serve as a complete review of each chapter.

Finally, the author wishes to thank the staff of the McGraw-Hill Schaum's Outline Series, especially Barbara Gilson, Mary Loebig Giles, and Maureen Walker, for their excellent cooperation.

SEYMOUR LIPSCHUTZ

Temple University

Contents

PART I *Elementary Theory of Sets*

Chapter 1	SETS AND BASIC OPERATIONS ON SETS	1
	1.1 Introduction. 1.2 Sets and Elements. 1.3 Universal Set, Empty Set. 1.4 Subsets. 1.5 Venn Diagrams. 1.6 Set Operations. 1.7 Algebra of Sets, Duality. 1.8 Finite Sets, Counting Principles. 1.9 Classes of Sets, Power Sets. 1.10 Arguments and Venn Diagrams. 1.11 Mathematical Induction. 1.12 Axiomatic Development of Set Theory.	
Chapter 2	SETS AND ELEMENTARY PROPERTIES OF THE REAL NUMBERS	34
	2.1 Introduction. 2.2 Real Number System \mathbf{R} . 2.3 Order and Inequalities. 2.4 Absolute Value, Distance. 2.5 Intervals. 2.6 Bounded Sets, Completion Property. 2.7 Integers \mathbf{Z} (Optional Material). 2.8 Greatest Common Divisor, Euclidean Algorithm. 2.9 Fundamental Theorem of Arithmetic.	
Chapter 3	RELATIONS	64
	3.1 Introduction. 3.2 Product Sets. 3.3 Relations. 3.4 Pictorial Representations of Relations. 3.5 Composition of Relations. 3.6 Types of Relations. 3.7 Closure Properties. 3.8 Partitions. 3.9 Equivalence Relations. 3.10 Partial Ordering Relations. 3.11 n -Ary Relations.	
Chapter 4	FUNCTIONS	94
	4.1 Introduction. 4.2 Functions. 4.3 Composition of Functions. 4.4 One-to-One, Onto, and Invertible Functions. 4.5 Mathematical Functions, Exponential and Logarithmic Functions. 4.6 Recursively Defined Functions.	
Chapter 5	FURTHER THEORY OF SETS AND FUNCTIONS	117
	5.1 Introduction. 5.2 Operations on Collections of Sets. 5.3 Indexed Collections of Sets. 5.4 Sequences, Summation Symbol. 5.5 Fundamental Products. 5.6 Functions and Diagrams. 5.7 Special Kinds of Functions, Fundamental Factorization. 5.8 Associated Set Functions. 5.9 Choice Functions. 5.10 Algorithms and Functions. 5.11 Complexity of Algorithms.	

PART II *Cardinals, Ordinals, Transfinite Induction*

Chapter 6	CARDINAL NUMBERS	141
	6.1 Introduction. 6.2 One-to-One Correspondence, Equipotent Sets. 6.3 Denumerable and Countable Sets. 6.4 Real Numbers \mathbf{R} and the Power of the Continuum. 6.5 Cardinal Numbers. 6.6 Ordering of Cardinal Numbers. 6.7 Cardinal Arithmetic.	

Chapter 7	ORDERED SETS AND LATTICES	166
	7.1 Introduction. 7.2 Ordered Sets. 7.3 Set Constructions and Order. 7.4 Partially Ordered Sets and Hasse Diagrams. 7.5 Minimal and Maximal Elements, First and Last Elements. 7.6 Consistent Enumeration. 7.7 Supremum and Infimum. 7.8 Isomorphic (Similar) Ordered Sets. 7.9 Order Types of Linearly Ordered Sets. 7.10 Lattices. 7.11 Bounded, Distributive, Complemented Lattices.	
Chapter 8	ORDINAL NUMBERS.	204
	8.1 Introduction. 8.2 Well-Ordered Sets. 8.3 Transfinite Induction. 8.4 Limit Elements. 8.5 Initial Segments. 8.6 Similarity Between a Well-Ordered Set and Its Subsets. 8.7 Comparison of Well-Ordered Sets. 8.8 Ordinal Numbers. 8.9 Inequalities and Ordinal Numbers. 8.10 Ordinal Addition. 8.11 Ordinal Multiplication. 8.12 Structure of Ordinal Numbers. 8.13 Auxiliary Construction of Ordinal Numbers.	
Chapter 9	AXIOM OF CHOICE, ZORN'S LEMMA, WELL-ORDERING THEOREM	219
	9.1 Introduction. 9.2 Cartesian Products and Choice Functions. 9.3 Axiom of Choice. 9.4 Well-Ordering Theorem, Zorn's Lemma. 9.5 Cardinal and Ordinal Numbers. 9.6 Alephs. 9.7 Paradoxes in Set Theory.	
PART III <i>Related Topics</i>		
Chapter 10	LOGIC AND PROPOSITIONAL CALCULUS	229
	10.1 Introduction. 10.2 Propositions and Compound Propositions. 10.3 Basic Logical Operations. 10.4 Propositions and Truth Tables. 10.5 Tautologies and Contradictions. 10.6 Logical Equivalence. 10.7 Algebra of Propositions. 10.8 Conditional and Biconditional Statements. 10.9 Arguments. 10.10 Logical Implication. 10.11 Propositional Functions, Quantifiers. 10.12 Negation of Quantified Statements.	
Chapter 11	BOOLEAN ALGEBRA.	252
	11.1 Introduction. 11.2 Basic Definitions. 11.3 Duality. 11.4 Basic Theorems. 11.5 Boolean Algebras as Lattices. 11.6 Representation Theorem. 11.7 Sum-of-Products Form for Sets. 11.8 Sum-of-Products Form for Boolean Algebras. 11.9 Minimal Boolean Expressions, Prime Implicants. 11.10 Karnaugh Maps.	
	INDEX	281