Chapter 10

Logic and Propositional Calculus

10.1 INTRODUCTION

Many proofs in mathematics and many algorithms in computer science use logical expressions such as

"IF p THEN q" or "IF p_1 AND p_2, THEN q_1 OR q_2"

It is therefore necessary to know the cases in which these expressions are either TRUE or FALSE: what we refer to as the truth values of such expressions. We discuss these issues in this chapter.

We also investigate the truth value of quantified statements, which are statements which use the logical quantifiers "for every" and "there exists".

10.2 PROPOSITIONS AND COMPOUND PROPOSITIONS

A *proposition* (or *statement*) is a declarative sentence which is true or false, but not both. Consider, for example, the following eight sentences:

(i) Paris is in France.

(ii) $1 + 1 = 2$.

(iii) $2 + 2 = 3$.

(iv) London is in Denmark.

(v) $9 < 6$.

(vi) $x = 2$ is a solution of $x^2 = 4$.

(vii) Where are you going?

(viii) Do your homework.

All of them are propositions except (vii) and (viii). Moreover, (i), (ii), and (vi) are true, whereas, (iii), (iv), and (v) are false.

Compound Propositions

Many propositions are *composite*, that is, composed of *subpropositions* and various connectives discussed subsequently. Such composite propositions are called *compound propositions*. A proposition is said to be *primitive* if it cannot be broken down into simpler propositions, that is, if it is not composite.

EXAMPLE 10.1

(*a*) "Roses are red and violets are blue" is a compound proposition with subpropositions "Roses are red" and "Violets are blue".

(*b*) "John is intelligent or studies every night" is a compound proposition with subpropositions "John is intelligent" and "John studies every night".

(*c*) The above propositions (i) through (vi) are all primitive propositions; they cannot be broken down into simpler propositions.

> The fundamental property of a compound proposition is that its truth value is completely determined by the truth values of its subpropositions together with the way in which they are connected to form the compound propositions.

The next section studies some of these connectives.

10.3 BASIC LOGICAL OPERATIONS

This section discusses the three basic logical operations of conjunction, disjunction, and negation which correspond, respectively, to the English words "and", "or", and "not".

Conjunction $p \wedge q$

Any two propositions can be combined by the word "and" to form a compound proposition called the *conjunction* of the original propositions. Symbolically,

$$p \wedge q$$

read "p and q", denotes the conjunction of p and q. Since $p \wedge q$ is a proposition it has a truth value, and this truth value depends only on the truth values of p and q. Specifically:

Definition 10.1: If p and q are true, then $p \wedge q$ is true; otherwise $p \wedge q$ is false.

The truth value of $p \wedge q$ may be defined equivalently by the table in Fig. 10-1(a). Here, the first line is a short way of saying that if p is true and q is true, then $p \wedge q$ is true. The second line says that if p is true and q is false, then $p \wedge q$ is false. And so on. Observe that there are four lines corresponding to the four possible combinations of T and F for the two subpropositions p and q. Note that $q \wedge q$ is true only when both p and q are true.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

(a) "p and q"

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

(b) "p or q"

p	$\neg p$
T	F
F	T

(c) "not p"

Fig. 10-1

EXAMPLE 10.2 Consider the following four statements:

(i) Paris is in France and $2 + 2 = 4$.
(ii) Paris is in France and $2 + 2 = 5$.
(iii) Paris is in England and $2 + 2 = 4$.
(iv) Paris is in England and $2 + 2 = 5$.

Only the first statement is true. Each of the other statements is false since at least one of its substatements is false.

Disjunction, $p \vee q$

Any two propositions can be combined by the word "or" to form a compound proposition called the *disjunction* of the original propositions. Symbolically,

$$p \vee q$$

read "p or q", denotes the disjunction of p and q. The truth value of $p \vee q$ depends only on the truth values of p and q as follows.

Definition 10.2: If p and q are false, then $p \vee q$ is false; otherwise $p \vee q$ is true.

The truth value of $p \vee q$ may be defined equivalently by the table in Fig. 10-1(b). Observe that $p \vee q$ is false only in the fourth case when both p and q are false.

EXAMPLE 10.3 Consider the following four statements:

(i) Paris is in France or $2+2=4$. (iii) Paris is in England or $2+2=4$.
(ii) Paris is in France or $2+2=5$. (iv) Paris is in England or $2+2=5$.

Only the last statement (iv) is false. Each of the other statements is true since at least one of its substatements is true.

Remark: The English word "or" is commonly used in two distinct ways. Sometimes it is used in the sense of "p or q or both", i.e., at least one of the two alternatives occurs, as above, and sometimes it is used in the sense of "p or q but not both", i.e., exactly one of the two alternatives occurs. For example, the sentence "He will go to Harvard or to Yale" uses "or" in the latter sense, called the *exclusive disjunction*. Unless otherwise stated, "or" shall be used in the former sense. This discussion points out the precision we gain from our symbolic language: $p \vee q$ is defined by its truth table and *always* means "p and/or q".

Negation, $\neg p$

Given any proposition p, another proposition, called the *negation* of p, can be formed by writing "It is not the case that . . ." or "It is false that . . ." before p or, if possible, by inserting in p the word "not". Symbolically,

$$\neg p$$

read "not p", denotes the negation of p. The truth value of $\neg p$ depends on the truth value of p as follows.

Definition 10.3: If p is true, then $\neg p$ is false; and if p is false, then $\neg p$ is true.

The truth value of $\neg p$ may be defined equivalently by the table in Fig. 10-3(c). Thus the truth value of the negation of p is always the opposite of the truth value of p.

EXAMPLE 10.4 Consider the following six statements.

(a_1) Paris is in France. (b_1) $2+2=5$.
(a_2) It is not the case that Paris is in France. (b_2) It is not the case that $2+2=5$.
(a_3) Paris is not in France. (b_3) $2+2 \neq 5$.

Then (a_2) and (a_3) are each the negation of (a_1); and (b_2) and (b_3) are each the negation of (b_1). Since (a_1) is true, (a_2) and (a_3) are false; and since (b_1) is false, (b_2) and (b_3) are true.

Remark: The logical notation for the connectives "and", "or", and "not" are not completely standard. For example, some texts use:

$$\begin{array}{rl} p \,\&\, q,\ p \cdot q \text{ or } pq & \text{for } p \wedge q \\ p+q & \text{for } p \vee q \\ p',\ \bar{p} \text{ or } \sim p & \text{for } \neg p \end{array}$$

10.4 PROPOSITIONS AND TRUTH TABLES

Let $P(p, q, \ldots)$ denote an expression constructed from logical variables $p, q, \ldots$, which take on the value TRUE (T) or FALSE (F), and the logical connectives $\wedge$, $\vee$, and $\neg$ (and others discussed subsequently). Such an expression $P(p, q, \ldots)$ will be called a *proposition*.

The main property of a proposition $P(p, q, \ldots)$ is that its truth value depends exclusively upon the truth values of its variables, that is, the truth value of a proposition is known once the truth value of each of its variables is known. A simple concise way to show this relationship is through a *truth table*. We describe a way to obtain such a truth table below.

Consider, for example, the proposition $\neg(p \wedge \neg q)$. Figure 10-2(a) indicates how the truth table of $\neg(p \wedge \neg q)$ is constructed. Observe that the first columns of the table are for the variables $p, q, \ldots$ and and that there are enough rows in the table to allow for all possible combinations of T and F for these variables. (For 2 variables, 4 rows are necessary; for 3 variables, 8 rows are necessary; and, in general, for n variables 2^n rows are required.) There is then a column for each "elementary" stage of the construction of the proposition, the truth table at each step being determined from the previous stages by the definitions of the connectives $\wedge$, $\vee$, $\neg$. Finally we obtain the truth value of the proposition, which appears in the last column.

The actual truth table of the proposition $\neg(p \wedge \neg q)$ is shown in Fig. 10-2(b). It consists precisely of the columns in Fig. 10-2(a) which appear under the variables and under the proposition; the other columns were merely used in the construction of the truth table.

p	q	$\neg q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

(a)

p	q	$\neg(p \wedge \neg q)$
T	T	T
T	F	F
F	T	T
F	F	T

(b)

Fig. 10-2

Remark: In order to avoid an excessive number of parentheses, we sometimes adopt an order of precedence for the logical connectives. Specifically:

$\neg$ has precedence over $\wedge$ which has precedence over $\vee$.

For example, $\neg p \wedge q$ means $(\neg p) \wedge q$ and not $\neg(p \wedge q)$.

Alternative Method for Constructing a Truth Table

Another way to construct the truth table for $\neg(p \wedge \neg q)$ follows:

(a) First we construct the truth table shown in Fig. 10-3. That is, first we list all the variables and the combinations of their truth values. Then the proposition is written on the top row to the right of its variables with sufficient space so that there is a column under each variable and each connective in the proposition. Also there is a final row labeled "Step".

p	q	$\neg$	$(p$	$\wedge$	$\neg$	$q)$
T	T					
T	F					
F	T					
F	F					
Step						

Fig. 10-3

(*b*) Next, additional truth values are entered into the truth table in various steps as shown in Fig. 10-4. That is, first the truth values of the variables are entered under the variables in the proposition, and then there is a column of truth values entered under each logcial operation. We also indicate the step in which each column of truth values is entered in the table.

The truth table of the proposition then consists of the original columns under the variables and the last step, that is, the last column entered into the table.

p	q	$\neg$	$(p$	$\wedge$	$\neg$	$q)$
T	T		T			T
T	F		T			F
F	T		F			T
F	F		F			F
Step			1			1

(*a*)

p	q	$\neg$	$(p$	$\wedge$	$\neg$	$q)$
T	T		T		F	T
T	F		T		T	F
F	T		F		F	T
F	F		F		T	F
Step			1		2	1

(*b*)

p	q	$\neg$	$(p$	$\wedge$	$\neg$	$q)$
T	T		T	F	F	T
T	F		T	T	T	F
F	T		F	F	F	T
F	F		F	F	T	F
Step			1	3	2	1

(*c*)

p	q	$\neg$	$(p$	$\wedge$	$\neg$	$q)$
T	T	T	T	F	F	T
F	T	F	T	T	T	F
F	F	T	F	F	F	T
F	F	T	F	F	T	F
Step		4	1	3	2	1

(*d*)

Fig. 10-4

10.5 TAUTOLOGIES AND CONTRADICTIONS

Some propositions $P(p, q, \ldots)$ contain only T in the last column of their truth tables or, in other words, they are true for any truth values of their variables. Such propositions are called *tautologies*. Analogously, a proposition $P(p, q, \ldots)$ is called a *contradiction* if it contains only F in the last column of its truth table or, in other words, if it is false for any truth values of its variables. For example, the proposition "p or not p", that is, $p \vee \neg p$, is a tautology, and the proposition "p and not p", that is, $p \wedge \neg p$, is a contradiction. This is verified by looking at their truth tables in Fig. 10-5. (The truth tables have only two rows since each proposition has only the one variable p.)

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

(*a*) $p \vee \neg p$

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

(*b*) $p \wedge \neg p$

Fig. 10-5

Note that the negation of a tautology is a contradiction since it is always false, and the negation of a contradiction is a tautology since it is always true.

Now let $P(p, q, \ldots)$ be a tautology, and let $P_1(p, q, \ldots)$, $P_2(p, q, \ldots) \ldots$ be any propositions. Since $P(p, q, \ldots)$ does not depend upon the particular truth values of its variables $p, q, \ldots$, we can substitute P_1 for p, P_2 for $q, \ldots$ in the tautology $P(p, q, \ldots)$ and still have a tautology. We state this result formally.

Theorem 10.1 (Principle of Substitution): If $P(p, q, \ldots)$ is a tautology, then $P(P_1, P_2, \ldots)$ is a tautology for any propositions $P_1, P_2, \ldots$.

10.6 LOGICAL EQUIVALENCE

Two propositions $P(p, q, \ldots)$ and $Q(p, q, \ldots)$ are said to be *logically equivalent*, or simply *equivalent* or *equal*, denoted by

$$P(p, q, \ldots) \equiv Q(p, q, \ldots)$$

if they have identical truth tables. Consider, for example, the truth tables of $\neg(p \wedge q)$ and $\neg p \vee \neg q$ appearing in Fig. 10-6. Observe that both truth tables are the same, that is, both propositions are false in the first case and true in the other three cases. Accordingly, we can write

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

In other words, the propositions are logically equivalent.

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

(*a*) $\neg(p \wedge q)$

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

(*b*) $\neg p \vee \neg q$

Fig. 10-6

Remark: Consider the statement

"It is not the case that roses are red and violets are blue"

This statement can be written in the form $\neg(p \wedge q)$, where

p is "roses are red" and q is "violets are blue"

However, as noted above, $\neg(p \wedge q) \equiv \neg p \vee \neg q$. Thus the statement

"Roses are not red, or violets are not blue"

has the same meaning as the given statement.

10.7 ALGEBRA OF PROPOSITIONS

Propositions satisfy various laws which are listed in Table 10-1. (In this table, T and F are restricted to the truth values "true" and "false" respectively.) We state this result formally.

Theorem 10.2: Propositions satisfy the laws of Table 10-1.

Table 10-1 Laws of the Algebra of Propositions

Idempotent laws	
(1a) $p \vee p \equiv p$	(1b) $p \wedge p \equiv p$
Associative laws	
(2a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$	(2b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Commutative laws	
(3a) $p \vee q \equiv q \vee p$	(3b) $p \wedge q \equiv q \wedge p$
Distributive laws	
(4a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	(4b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity laws	
(5a) $p \vee \mathrm{T} \equiv p$	(5b) $p \wedge \mathrm{F} \equiv p$
(6a) $p \vee \mathrm{T} \equiv \mathrm{T}$	(6b) $p \wedge \mathrm{F} \equiv \mathrm{F}$
Complement laws	
(7a) $p \vee \neg p \equiv \mathrm{T}$	(8a) $\neg \mathrm{T} \equiv \mathrm{F}$
(7b) $p \wedge \neg p \equiv \mathrm{F}$	(8b) $\neg \mathrm{F} \equiv \mathrm{T}$
Involution law	
(9) $\neg\neg p \equiv p$	
DeMorgan's laws	
(10a) $\neg(p \vee q) \equiv \neg p \wedge \neg q$	(10b) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

10.8 CONDITIONAL AND BICONDITIONAL STATEMENTS

Many statements, particularly in mathematics, are of the form "If p then q". Such statements are called *conditional* statements, and are denoted by

$$p \rightarrow q$$

The conditional $p \rightarrow q$ is frequently read "p implies q" or "p only if q".

Another common statement is of the form "p if and only if q". Such statements are called *biconditional* statements, and are denoted by

$$p \leftrightarrow q$$

The truth values of $p \rightarrow q$ and $p \leftrightarrow q$ are defined by the tables in Fig. 10-7. Observe that:

(*a*) The conditional $p \rightarrow q$ is false only when the first part p is true and the second part q is false. Accordingly, when p is false, the conditional $p \rightarrow q$ is true regardless of the truth value of q.

(*b*) The biconditional $p \leftrightarrow q$ is true whenever p and q have the same truth values and false otherwise.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

(*a*) $p \rightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

(*b*) $p \leftrightarrow q$

Fig. 10-7

The truth table of the proposition $\neg p \vee q$ appears in Fig. 10-8. Observe that the truth tables of $\neg p \vee q$ and $p \rightarrow q$ are identical, that is, they are both false only in the second case. Accordingly, $p \rightarrow q$ is logically equivalent to $\neg p \vee q$; that is,

$$p \rightarrow q \equiv \neg p \vee q$$

In other words, the conditional statement "If p then q" is logically equivalent to the statement "Not p or q" which only involves the connectives $\vee$ and $\neg$ and thus was already a part of our language. We may regard $p \rightarrow q$ as an abbreviation for an oft-recurring statement.

p	q	$\neg p$	$\neg p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

$\neg p \vee q$

Fig. 10-8

10.9 ARGUMENTS

An *argument* is an assertion that a given set of propositions $P_1, P_2, \ldots, P_n$, called *premises*, yields (has as a consequence) another proposition Q, called the *conclusion*. Such an argument is denoted by

$$P_1, P_2, \ldots, P_n \vdash Q$$

The notion of a "logical argument" or "valid argument" is formalized as follows.

Definition 10.4: An argument $P_1, P_2, \ldots, P_n \vdash Q$ is said to be *valid* if Q is true whenever all the premises $P_1, P_2, \ldots, P_n$ are true. An argument which is not valid is called a *fallacy*.

EXAMPLE 10.5

(*a*) The following argument is valid:

$$p, p \rightarrow q \vdash q \text{ (\textbf{\textit{Law of Detachment}})}$$

The proof of this rule follows from the truth table in Fig. 10-9. Specifically, p and $p \rightarrow q$ are true simultaneously only in Case (row) 1, and in this case q is true.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Fig. 10-9

(*b*) The following argument is a fallacy:

$$p \rightarrow q,\ q \vdash p$$

For $p \rightarrow q$ and q are both true in Case (row) 3 in the truth table in Fig. 10-9, but in this case p is false.

Now the propositions $P_1, P_2, \ldots, P_n$ are true simultaneously if and only if the proposition $P_1 \wedge P_2 \wedge \cdots \wedge P_n$ is true. Thus the argument $P_1, P_2, \ldots, P_n \vdash Q$ is valid if and only if Q is true whenever $P_1 \wedge P_2 \wedge \cdots \wedge P_n$ is true or, equivalently, if the proposition $(P_1 \wedge P_2 \wedge \cdots \wedge P_n) \rightarrow Q$ is a tautology. We state this result formally.

Theorem 10.3: The argument $P_1, P_2, \ldots, P_n \vdash Q$ is valid if and only if the proposition $(P_1 \wedge P_2 \wedge \cdots \wedge P_n) \rightarrow Q$ is a tautology.

We apply this theorem in the next example.

EXAMPLE 10.6 A fundamental principle of logical reasoning states:

"If p implies q and q implies r, then p implies r"

That is, the following argument is valid:

$$p \rightarrow q,\ q \rightarrow r \ \vdash\ p \rightarrow r \quad (\textit{Law of Syllogism})$$

This fact is verified by the truth table in Fig. 10-10, which shows that the following proposition is a tautology:

$$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$$

Equivalently, the argument is valid since the premises $p \rightarrow q$ and $q \rightarrow r$ are true simultaneously only in Cases (rows) 1, 5, 7, 8 and in these cases the conclusion $p \rightarrow r$ is also true. (Observe that the truth table required $2^3 = 8$ lines since there are three variables, p, q, r.)

p	q	r	$[(p$	$\rightarrow$	$q)$	$\wedge$	$(q$	$\rightarrow$	$r)]$	$\rightarrow$	$(p$	$\rightarrow$	$r)$
T	T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T	F	F	T	T	F	F
T	F	T	T	F	F	F	F	T	T	T	T	T	T
T	F	F	T	F	F	F	F	T	F	T	T	F	F
F	T	T	F	T	T	T	T	T	T	T	F	T	T
F	T	F	F	T	T	F	T	F	F	T	F	T	F
F	F	T	F	T	F	T	F	T	T	T	F	T	T
F	F	F	F	T	F	T	F	T	F	T	F	T	F
Step			1	2	1	3	1	2	1	4	1	2	1

Fig. 10-10

We now apply the above theory to arguments involving specific statements. We emphasize that the validity of an argument does not depend upon the truth values nor the content of the statements appearing in the argument, but upon the particular form of the argument. This is illustrated in the following example.

EXAMPLE 10.7 Consider the following argument:

S_1: If a man is a bachelor, he is unhappy.
S_2: If a man is unhappy, he dies young.

....

S: Bachelors die young.

Here the statement S below the line denotes the conclusion of the argument, and the statements S_1 and S_2 above the line denote the premises. We claim that the argument $S_1, S_2 \vdash S$ is valid. For the argument is of the form

$$p \rightarrow q,\ q \rightarrow r \ \vdash\ p \rightarrow r$$

where p is "He is a bachelor", q is "He is unhappy" and r is "He dies young"; and by Example 10.6 this argument (law of syllogism) is valid.

10.10 LOGICAL IMPLICATION

A proposition $P(p, q, \ldots)$ is said to *logically imply* a proposition $Q(p, q, \ldots)$, written

$$P(p, q, \ldots) \Rightarrow Q(p, q, \ldots)$$

if $Q(p, q, \ldots)$ is true whenever $P(p, q, \ldots)$ is true.

EXAMPLE 10.8 We claim that p logically implies $p \vee q$. For consider the truth table in Fig. 10-11. Observe that p is true in Cases (rows) 1 and 2, and in these cases $p \vee q$ is also true. Thus $p \Rightarrow p \vee q$.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Fig. 10-11

Now if $Q(p, q, \ldots)$ is true whenever $P(p, q, \ldots)$ is true, then the argument

$$P(p, q, \ldots) \vdash Q(p, q, \ldots)$$

is valid; and conversely. Furthermore, the argument $P \vdash Q$ is valid if and only if the conditional statement $P \rightarrow Q$ is always true, i.e., a tautology. We state this result formally.

Theorem 10.4: For any propositions $P(p, q, \ldots)$ and $Q(p, q, \ldots)$ the following three statements are equivalent:

(i) $P(p, q, \ldots)$ logically implies $Q(p, q, \ldots)$.

(ii) The argument $P(p, q, \ldots) \vdash Q(p, q, \ldots)$ is valid.

(iii) The proposition $P(p, q, \ldots) \rightarrow Q(p, q, \ldots)$ is a tautology.

We note that some logicians and many texts use the word "implies" in the same sense as we use "logically implies", and so they distinguish between "implies" and "if . . . then". These two distinct concepts are, of course, intimately related as seen in the above theorem.

10.11 PROPOSITIONAL FUNCTIONS, QUANTIFIERS

Let A be a given set. A *propositional function* (or an *open sentence* or *condition*) defined on A is an expression

$$p(x)$$

which has the property that $p(a)$ is true or false for each $a \in A$. That is, $p(x)$ becomes a statement (with a truth value) whenever any element $a \in A$ is substituted for the variable x. The set A is called the *domain* of $p(x)$, and the set T_p of all elements of A for which $p(a)$ is true is called the *truth set* of $p(x)$. In other words,

$$T_p = \{x : x \in A,\ p(x) \text{ is true}\} \qquad \text{or} \qquad T_p = \{x : p(x)\}$$

Frequently, when A is some set of numbers, the condition $p(x)$ has the form of an equation or inequality involving the variable x.

EXAMPLE 10.9 Find the truth set T_p of each propositional function $p(x)$ defined on the set $\mathbf{P} = \{1, 2, 3, \ldots\}$.

(*a*) Let $p(x)$ be "$x + 2 > 7$". Then

$$T_p = \{x : x \in \mathbf{P},\ x + 2 > 7\} = \{6, 7, 8, \ldots\}$$

consisting of all integers greater than 5.

(*b*) Let $p(x)$ be "$x + 5 < 3$". Then

$$T_p = \{x : x \in \mathbf{P},\ x + 5 < 3\} = \varnothing$$

the empty set. In other words, $p(x)$ is not true for any positive integer in $\mathbf{P}$.

(*c*) Let $p(x)$ be "$x + 5 > 1$". Then

$$T_p = \{x : x \in \mathbf{P},\ x + 5 > 1\} = \mathbf{P}$$

Thus $p(x)$ is true for every element in $\mathbf{P}$.

Remark: The above example shows that if $p(x)$ is a propositional function defined on a set A then $p(x)$ could be true for all $x \in A$, for some $x \in A$, or for no $x \in A$. The next two subsections discusses quantifiers related to such propositional functions.

Universal Quantifier

Let $p(x)$ be a propositional function defined on a set A. Consider the expression

$$(\forall x \in A)p(x) \qquad \text{or} \qquad \forall x\, p(x) \tag{10.1}$$

which reads "For every x in A, $p(x)$ is a true statement" or, simply, "For all x, $p(x)$". The symbol

$$\forall$$

which reads "for all" or "for every" is called the *universal quantifier*. The statement (*10.1*) is equivalent to the statement

$$T_p = \{x : x \in A,\ p(x)\} = A \tag{10.2}$$

that is, that the truth set of $p(x)$ is the entire set A.

The expression $p(x)$ by itself is an open sentence or condition and therefore has no truth value. However, $\forall x, p(x)$ that is, $p(x)$ preceded by the quantifier $\forall$, does have a truth value which follows from the equivalence of (*10.1*) and (*10.2*). Specifically:

Q_1 : If $\{x : x \in A,\ p(x)\} = A$ then $\forall x,\ p(x)$ is true; otherwise, $\forall x,\ p(x)$ is false.

EXAMPLE 10.10

(*a*) The proposition $(\forall n \in \mathbf{P})\ (n + 4 > 3)$ is true since

$$\{n : n + 4 > 3\} = \{1, 2, 3, \ldots\} = \mathbf{P}$$

(*b*) The proposition $(\forall n \in \mathbf{P})\ (n + 2 > 8)$ is false since

$$\{n : n + 2 > 8\} = \{7, 8, \ldots\} \neq \mathbf{P}$$

(*c*) The symbol $\forall$ can be used to define the intersection of an indexed collection $\{A_i : i \in I\}$ of sets A_i as follows:

$$\bigcap(A_i : i \in I) = \{x : \forall i \in I,\ x \in A_i\}$$

Existential Quantifier

Let $p(x)$ be a propositional function defined on a set A. Consider the expression

$$(\exists x \in A)p(x) \qquad \text{or} \qquad \exists x,\ p(x) \tag{10.3}$$

which reads "There exists an x in A such that $p(x)$ is a true statement" or, simply, "For some x, $p(x)$". The symbol

$$\exists$$

which reads "there exists" or "for some" or "for at least one" is called the *existential quantifier*. Statement (*10.3*) is equivalent to the statement

$$T_p = \{x : x \in A,\ p(x)\} \neq \varnothing \tag{10.4}$$

i.e., that the truth set of $p(x)$ is not empty. Accordingly, $\exists x, p(x)$, that is, $p(x)$ preceded by the existential quantifier $\exists$ does have a truth value. Specifically,

> Q_2: If $\{x : p(x)\} \neq \varnothing$ then $\exists x, p(x)$ is true; otherwise, $\exists x, p(x)$ is false.

EXAMPLE 10.11

(*a*) The proposition $(\exists n \in \mathbf{P})\ (n + 4 < 7)$ is true since

$$\{n : n + 4 < 7\} = \{1, 2\} \neq \varnothing$$

(*b*) The proposition $(\exists n \in \mathbf{P})\ (n + 6 < 4)$ is false since

$$\{n : n + 6 < 4\} = \varnothing$$

(*c*) The symbol $\exists$ can be used to define the union of an indexed collection $\{A_i : i \in I\}$ of sets A_i as follows:

$$\bigcup(A_i : i \in I) = \{x : \exists i \in I,\ x \in A_i\}$$

Notation

Let $A = \{2, 3, 5\}$ and let $p(x)$ be the sentence "x is a prime number" or, simply "x is prime". Then the proposition

"Two is prime and three is prime and five is prime" (*)

can be denoted by

$$p(2) \wedge p(3) \wedge p(5) \qquad \text{or} \qquad \wedge\,(a \in A,\ p(a))$$

which is equivalent to the statement

"Every number in A is prime" or $\forall a \in A,\ p(a)$ (**)

Similarly, the proposition

"Two is prime or three is prime or five is prime"

can be denoted by

$$p(2) \vee p(3) \vee p(5) \qquad \text{or} \qquad \vee\,(a \in A,\ p(a))$$

which is equivalent to the statement

"At least one number in A is prime" or $\exists a \in A,\ p(a)$

Alternatively, we can write

$$\wedge(a \in A,\ p(a)) \equiv \forall a \in A,\ p(a) \qquad \text{and} \qquad \vee\,(a \in A,\ p(a)) \equiv \exists a \in a,\ p(a)$$

where the symbols $\wedge$ and $\vee$ are used instead of $\forall$ and $\exists$.

Remark: If A were an infinite set, then a statement of the form $(*)$ could be made since the sentence would not end; but a statement of the form $(**)$ can always be made, even when A is infinite.

10.12 NEGATION OF QUANTIFIED STATEMENTS

Consider the statement: "All math majors are male". Its negation is either of the following equivalent statements:

"It is not the case that all math majors are male"

"There exists at least one math major who is a female (not male)"

Symbolically, using M to denoted the set of math majors, the above can be written as

$$\neg(\forall x \in M)\ (x \text{ is male}) \equiv (\exists x \in M)\ (x \text{ is not male})$$

or, when $p(x)$ denotes "x is male",

$$\neg(\forall x \in M)p(x) \equiv (\exists x \in M)\neg p(x) \qquad \text{or} \qquad \neg\forall x, p(x) \equiv \exists x \neg p(x)$$

The above is true for any proposition $p(x)$. That is:

Theorem 10.5 (DeMorgan): $\neg(\forall x \in A)p(x) \equiv (\exists x \in A)\neg p(x)$.

In other words, the following two statements are equivalent:

(1) It is not true that, for all $a \in A$, $p(a)$ is true.

(2) There exists an $a \in A$ such that $p(a)$ is false.

There is an analogous theorem for the negation of a proposition which contains the existential quantifier.

Theorem 10.6 (DeMorgan): $\neg(\exists x \in A)p(x) \equiv (\forall x \in A)\neg p(x)$.

That is, the following two statements are equivalent:

(1) It is not true that for some $a \in A$, $p(a)$ is true.

(2) For all $a \in A$, $p(a)$ is false.

EXAMPLE 10.12

(*a*) The following statements are negatives of each other:

"For all positive integers n we have $n + 2 > 8$"

"There exists a positive integer n such that $n + 2 \not> 8$"

(*b*) The following statements are also negatives of each other:

"There exists a college student who is 60 years old"

"Every college student is not 60 years old"

Remark: The expression $\neg p(x)$ has the obvious meaning; that is:

"The statement $\neg p(a)$ is true when $p(a)$ is false, and vice versa"

Previously, $\neg$ was used as an operation on statements; here $\neg$ is used as an operation on propositional functions. Similarly, $p(x) \wedge q(x)$, read "$p(x)$ and $q(x)$", is defined by:

"The statement $p(a) \wedge q(a)$ is true when $p(a)$ and $q(a)$ are true"

Similarly, $p(x) \vee q(x)$, read "$p(x)$ or $q(x)$", is defined by:

"The statement $p(a) \vee q(a)$ is true when $p(a)$ or $q(a)$ is true"

Thus in terms of truth sets:

(i) $\neg p(x)$ is the complement of $p(x)$.

(ii) $p(x) \wedge q(x)$ is the intersection of $p(x)$ and $q(x)$.

(iii) $p(x) \vee q(x)$ is the union of $p(x)$ and $q(x)$.

One can also show that the laws for propositions also hold for propositional functions. For example, we have DeMorgan's laws:

$$\neg (p(x) \wedge q(x)) \equiv \neg p(x) \vee \neg q(x) \quad \text{and} \quad \neg (p(x) \vee q(x)) \equiv \neg p(x) \wedge \neg q(x)$$

Counterexample

Theorem 10.6 tells us that to show that a statement $\forall x, p(x)$ is false, it is equivalent to show that $\exists x \neg p(x)$ is true or, in other words, that there is an element x_0 with the property that $p(x_0)$ is false. Such an element x_0 is called a *counterexample* to the statement $\forall x, p(x)$.

EXAMPLE 10.13

(*a*) Consider the statement $\forall x \in \mathbf{R}, |x| \neq 0$. The statement is false since 0 is a counterexample, that is, $|0| \neq 0$ is not true.

(*b*) Consider the statement $\forall x \in \mathbf{R}, x^2 \geq x$. The statement is not true since, for example, 1/2 is a counterexample. Specifically, $(1/2)^2 \geq 1/2$ is not true, that is, $(1/2)^2 < 1/2$.

(*c*) Consider the statement $\forall x \in \mathbf{P}, x^2 \geq x$. This statement is true where $\mathbf{P}$ is the set of positive integers. In other words, there does not exist a positive integer n for which $n^2 < n$.

Propositional Functions with More than One Variable

A propositional function (of n variables) defined over a product set $A = A_1 \times \cdots \times A_n$ is an expression

$$p(x_1, x_2, \ldots, x_n)$$

which has the property that $p(a_1, a_2, \ldots, a_n)$ is true or false for any n-tuple $(a_1, \ldots, a_n)$ in A. For example,

$$x + 2y + 3z < 18$$

is a propositional function on $\mathbf{P}^3 = \mathbf{P} \times \mathbf{P} \times \mathbf{P}$. Such a propositional function has no truth value. However, we do have the following:

Basic Principle: A propositional function preceded by a quantifier for each variable, for example,

$$\forall x\, \exists y,\ p(x, y) \qquad \text{or} \qquad \exists x\, \forall y\, \exists z,\ p(x, y, z)$$

denotes a statement and has a truth value.

EXAMPLE 10.14 Let $B = \{1, 2, 3, \ldots, 9\}$ and let $p(x, y)$ denote "$x + y = 10$". Then $p(x, y)$ is a propositional function on $A = B^2 = B \times B$.

(*a*) The following is a statement since there is a quantifier for each variable:

$$\forall x\, \exists y,\ p(x, y) \qquad \text{that is,} \qquad \text{"For every } x\text{, there exists a } y \text{ such that } x + y = 10\text{"}$$

This statement is true. For example, if $x = 1$, let $y = 9$; if $x = 2$, let $y = 8$, and so on.

(*b*) The following is also a statement:

$\exists y\, \forall x,\ p(x, y)$, that is, "There exists a y such that, for every x, we have $x + y = 10$"

No such y exists; hence this statement is false.

> **Warning!** Observe that the only difference between (*a*) and (*b*) in the above Example 10.14 is the order of the quantifiers. Thus a different ordering of the quantifiers may yield a different statement.

We note that, when translating quantified statements into English, the expression "such that" frequently follows "there exists".

Negating Quantified Statements with More than One Variable

Quantified statements with more than one variable may be negated by successively applying Theorems 10.5 and 10.6. Thus each $\forall$ is changed to $\exists$, and each $\exists$ is changed to $\forall$ as the negation symbol $\neg$ passes through the statement from left to right. For example

$$\begin{aligned}\neg[\forall x \exists y \exists z,\ p(x, y, z)] &\equiv \exists x \neg[\exists y \exists z,\ p(x, y, z)] \equiv \exists x \forall y[\neg \exists z,\ p(x, y, z)\\ &\equiv \exists x\, \forall y\, \forall z,\ \neg p(x, y, z)\end{aligned}$$

Naturally, we do not put in all the steps when negating such quantified statements.

EXAMPLE 10.15

(*a*) Consider the quantified statement:

"Every student has at least one course where the lecturer is a teaching assistant"

Its negation is the statement:

"There is a student such that in every course the lecturer is not a teaching assistant"

(*b*) The formal definition that L is the limit of a sequence $a_1, a_2, \ldots$ follows:

$$\forall \varepsilon > 0,\ \exists n_0 \in \mathbf{P},\ \forall n > n_0,\ |a_n - L| < \varepsilon$$

Thus L is not the limit of the sequence $a_1,\ a_2, \ldots$ when

$$\exists \varepsilon > 0,\ \forall n_0 \in \mathbf{P},\ \exists n > n_0,\ |a_n - L| \geq \varepsilon$$

Solved Problems

PROPOSITIONS AND LOGICAL OPERATIONS

10.1. Let p be "It is cold" and let q be "It is raining". Give a simple verbal sentence which describes each of the following statements: (*a*) $\neg p$; (*b*) $p \wedge q$; (*c*) $p \vee q$; (*d*) $q \vee \neg p$.

In each case, translate $\wedge$, $\vee$ and $\sim$ to read "and", "or", and "It is false that" or "not", respectively, and then simplify the English sentence.

(*a*) It is not cold.
(*b*) It is cold and raining.
(*c*) It is cold or it is raining.
(*d*) It is raining or it is not cold.

10.2. Let p be "Erik reads *Newsweek*", let q be "Erik reads *The New Yorker*", and let r be "Erik reads *Time*". Write each of the following in symbolic form:

(*a*) Erik reads *Newsweek* or *The New Yorker*, but not *Time*.

(*b*) Erik reads *Newsweek* and *The New Yorker*, or he does not read *Newsweek* and *Time*.

(*c*) It is not true that Erik reads *Newsweek* but not *Time*.

(*d*) It is not true that Erik reads *Time* or *The New Yorker* but not *Newsweek*.

Use $\vee$ for "or", $\wedge$ for "and" (or, its logical equivalent, "but"), and $\neg$ for "not" (negation).

(*a*) $(p \vee q) \wedge \neg r$; (*b*) $(p \wedge q) \vee \neg (p \wedge r)$; (*c*) $\neg (p \wedge \neg r)$; (*d*) $\neg [(r \vee q) \wedge \neg p]$.

TRUTH VALUES AND TRUTH TABLES

10.3. Determine the truth value of each of the following statements:

(*a*) $4 + 2 = 5$ and $6 + 3 = 9$. (*c*) $4 + 5 = 9$ and $1 + 2 = 4$.

(*b*) $3 + 2 = 5$ and $6 + 1 = 7$. (*d*) $3 + 2 = 5$ and $4 + 7 = 11$.

The statement "p and q" is true only when both substatements are true. Thus:

(*a*) false, (*b*) true; (*c*) false; (*d*) true.

10.4. Find the truth table of $\neg p \wedge q$.

See Fig. 10.12, which gives both methods for constructing the truth table.

p	q	$\neg p$	$\neg p \wedge q$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

(*a*) Method 1

p	q	$\neg$	p	$\wedge$	q
T	T	F	T	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	F	F	F
	Step	2	1	3	1

(*b*) Method 2

Fig. 10-12

10.5. Verify that the proposition $p \vee \neg (p \wedge q)$ is a tautology.

Construct the truth table of $p \vee \neg (p \wedge q)$ as shown in Fig. 10.13. Since the truth value of $p \vee \neg (p \wedge q)$ is T for all values of p and q, the proposition is a tautology.

p	q	$p \wedge q$	$\neg (p \wedge q)$	$p \vee \neg (p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Fig. 10-13

10.6. Show that the propositions $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are logically equivalent.

Construct the truth tables for $\neg(p \wedge q)$ and $\neg p \vee \neg q$ as in Fig. 10.14. Since the truth tables are the same (both propositions are false in the first case and true in the other three cases), the propositions $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are logically equivalent and we can write

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

(a) $\neg(p \wedge q)$

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

(b) $\neg p \vee \neg q$

Fig. 10-14

10.7. Using the laws in Table 10-1 to show that $\neg(p \vee q) \vee (\neg p \wedge q) \equiv \neg p$.

Statement		**Reason**
(1) $\neg(p \vee q) \vee (\neg p \wedge q)$	$\equiv (\neg p \wedge \neg q) \vee (\neg p \wedge q)$	DeMorgan's law
(2)	$\equiv \neg p \wedge (\neg q \vee q)$	Distributive law
(3)	$\equiv \neg p \wedge t$	Complement law
(4)	$\equiv \neg p$	Identity law

CONDITIONAL STATEMENTS

10.8. Rewrite the following statements without using the conditional:

(a) If it is cold, he wears a hat.

(b) If productivity increases, then wages rise.

Recall that "If p then q" is equivalent to "Not p or q"; that is, $p \rightarrow q \equiv \neg p \vee q$. Hence,

(a) It is not cold or he wears a hat.

(b) Productivity does not increase or wages rise.

10.9. Determine the contrapositive of each statement:

(a) If John is a poet, then he is poor.

(b) Only if Marc studies will he pass the test.

(a) The contrapositive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$. Hence the contrapositive of the given statement is

"If John is not poor, then he is not a poet"

(b) The given statement is equivalent to "If Marc passes the test, then he studied". Hence its contrapositive is

"If Marc does not study, then he will not pass the test"

10.10. Write the negation of each statement as simply as possible.

(*a*) If she works, she will earn money.

(*b*) He swims if and only if the water is warm.

(*c*) If it snows, then they do not drive the car.

(*a*) Note that $\neg(p \to q) \equiv p \wedge \neg q$; hence the negation of the statement follows:

"She works or she will not earn money"

(*b*) Note that $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q \equiv \neg p \leftrightarrow q$; hence the negation of the statement is either of the following:

"He swims if and only if the water is not warm"

"He does not swim if and only if the water is warm"

(*c*) Note that $\neg(p \to \neg q) \equiv p \wedge \neg\neg q \equiv p \wedge q$. Hence the negation of the statement follows:

"It snows and they drive the car"

ARGUMENTS

10.11. Show that the following argument is a fallacy: $p \to q, \neg p \vdash \neg q$.

Construct the truth table for $[(p \to q) \wedge \neg p] \to \neg q$ as in Fig. 10.15. Since the proposition $[(p \to q)(\wedge \neg p] \to \neg q$ is not a tautology, the argument is a fallacy. Equivalently, the argument is a fallacy since in third line of the truth table $p \to q$ and $\neg p$ are true but $\neg q$ is false.

p	q	$p \to q$	$\neg p$	$(p \to q) \wedge \neg p$	$\neg q$	$[(p \to q) \wedge \neg p] \to \neg q$
T	T	T	F	F	F	T
T	F	F	F	F	T	T
F	T	T	T	T	F	F
F	F	T	T	T	T	T

Fig. 10-15

10.12. Determine the validity of the following argument: $p \to q, \neg q \vdash \neg p$.

Construct the truth table for $[(p \to q) \wedge \neg q] \to \neg p$ as in Fig. 10.16. Since the proposition $[(p \to q) \vee \neg q] \to \neg p$ is a tautology, the argument is valid.

p	q	$[(p$	$\to$	$q)$	$\wedge$	$\neg$	$q]$	$\to$	$\neg$	p
T	T	T	T	T	F	F	T	T	F	T
T	F	T	F	F	F	T	F	T	F	T
F	T	F	T	T	F	F	T	T	T	F
F	F	F	T	F	T	T	F	T	T	F
Step		1	2	1	3	2	1	4	2	1

Fig. 10-16

10.13. Prove that the following argument is valid: $p \to \neg q, r \to q, r \vdash \neg p$.

Construct the truth tables of the premises and conclusion as in Fig. 10.17. Now, $p \to \neg q$, $r \to q$, and r are true simultaneously only in the fifth line of the table, where $\neg p$ is also true. Hence, the argument is valid.

	p	q	r	$p \to \neg q$	$r \to q$	$\neg q$
1	T	T	T	F	T	F
2	T	T	F	F	T	F
3	T	F	T	T	F	F
4	T	F	F	T	T	F
5	F	T	T	T	T	T
6	F	T	F	T	T	T
7	F	F	T	T	F	T
8	F	F	F	T	T	T

Fig. 10-17

10.14. Test the validity of the following argument:

If two sides of a triangle are equal, then the opposite angles are equal.
Two sides of a triangle are not equal.

The opposite angles are not equal.

First translate the argument into the symbolic form $p \to q, \neg p \vdash \neg q$, where p is "Two sides of a triangle are equal" and q is "The opposite angles are equal". By problem 10.11, this argument is a fallacy.

Remark: Although the conclusion *does* follow from the second premise and axioms of Euclidean geometry, the above argument does not constitute such a proof since the argument is a fallacy.

10.15. Determine the validity of the following argument:

If 7 is less than 4, then 7 is not a prime number.
7 is not less than 4.

---

7 is a prime number.

First translate the argument into symbolic form. Let p be "7 is less than 4" and q be "7 is a prime number". Then the argument is of the form

$$p \to \neg q, \neg p \vdash q$$

Now, we construct a truth table as shown in Fig. 10.18. The above argument is shown to be a fallacy since, in the fourth line of the truth table, the premises $p \to \neg q$ and $\neg p$ are true, but the conclusion q is false.

Remark: The fact that the conclusion of the argument happens to be a true statement is irrelevant to the fact that the argument presented is a fallacy

p	q	$\neg q$	$p \to \neg q$	$\neg p$
T	T	F	F	F
T	F	T	T	F
F	T	F	T	T
F	F	T	T	T

Fig. 10-18

10.16. Show that $p \wedge q$ logically implies $p \leftrightarrow q$.

Consider the truth tables of $p \wedge q$ and $p \leftrightarrow q$ shown in Fig. 10.19. Now $p \wedge q$ is true only in the first line of the table and, in this case, the proposition $p \leftrightarrow q$ is also true. Thus $p \wedge q$ logically implies $p \leftrightarrow q$.

p	q	$p \wedge q$	$p \leftrightarrow q$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	T

Fig. 10-19

QUANTIFIERS AND PROPOSITIONAL FUNCTIONS

10.17. Let $A = \{1, 2, 3, 4, 5\}$. Determine the truth value of each of the following statements:

(*a*) $(\exists x \in A)(x + 3 = 10)$ (*c*) $(\exists x \in A)(x + 3 < 5)$

(*b*) $(\forall x \in A)(x + 3 < 10)$ (*d*) $(\forall x \in A)(x + 3 \leq 7)$

(*a*) False. For no number in A is a solution to $x + 3 = 10$.

(*b*) True. For every number in A satisfies $x + 3 < 10$.

(*c*) True. For if $x_0 = 1$, then $x_0 + 3 < 5$, i.e., 1 is a solution.

(*d*) False. For if $x_0 = 5$, then $x_0 + 3$ is not less than or equal 7. In other words, 5 is not a solution to the given condition.

10.18. Determine the truth value of each of the following statements where $\mathbf{U} = \{1, 2, 3\}$ is the universal set:

(*a*) $\exists x \, \forall y, \ x^2 < y + 1$; (*b*) $\exists x \, \forall y, \ x^2 + y^2 < 12$; (*c*) $\forall x \, \forall y, \ x^2 + y^2 < 12$.

(*a*) True. For if $x = 1$, then 1, 2, and 3 are all solutions to $1 < y + 1$.

(*b*) True. For each x_0, let $y = 1$; then $x_0^2 + 1 < 12$ is a true statement.

(*c*) False. For if $x_0 = 2$ and $y_0 = 3$, then $x_0^2 + y_0^2 < 12$ is not a true statement.

10.19. Negate each of the following statements:

(*a*) $\exists x \, \forall y, \ p(x, y)$; (*b*) $\forall x \, \forall y, \ p(x, y)$; (*c*) $\exists y \, \exists x \, \forall z, \ p(x, y, z)$.

Use $\neg \forall x p(x) \equiv \exists x \neg p(x)$ and $\neg \exists x p(x) \equiv \forall x \neg p(x)$:

(*a*) $\neg(\exists x \, \forall y, \ p(x, y)) \equiv \forall x \, \exists y \neg p(x, y)$.

(*b*) $\neg(\forall x \, \forall y, \ p(x, y)) \equiv \exists x \, \exists y \neg p(x, y)$.

(*c*) $\neg(\exists y \, \exists x \, \forall z, \ p(x, y, z)) \equiv \forall y \, \forall x \, \exists z \neg p(x, y, z)$.

10.20. Let $p(x)$ denote the sentence "$x + 2 > 5$". State whether or not $p(x)$ is a propositional function on each of the following sets: (*a*) $\mathbf{P}$, the set of positive integers; (*b*) $M = \{-1, -2, -3, \ldots\}$; (*c*) $\mathbf{C}$, the set of complex numbers.

(*a*) Yes.

(*b*) Although $p(x)$ is false for every element in M, $p(x)$ is still a propositional function on M.

(*c*) No. Note that $2i + 2 > 5$ does not have any meaning. In other words, inequalities are not defined for complex numbers.

10.21. Negate each of the following statements: (*a*) All students live in the dormitories. (*b*) All mathematics majors are males. (*c*) Some students are 25 (years) or older.

Use Theorem 4.5 to negate the quantifiers.

(*a*) At least one student does not live in the dormitories. (Some students do not live in the dormitories.)

(*b*) At least one mathematics major is female. (Some mathematics majors are female.)

(*c*) None of the students is 25 or older. (All the students are under 25.)

Supplementary Problems

PROPOSITION AND LOGICAL OPERATIONS

10.22. Let p be "Audrey speaks French" and let q be "Audrey speaks Danish". Give a simple verbal sentence which describes each of the following:

(*a*) $p \vee q$; (*b*) $p \wedge q$; (*c*) $p \wedge \neg q$; (*d*) $\neg p \vee \neg q$; (*e*) $\neg\neg p$; (*f*) $\neg(\neg p \wedge \neg q)$.

10.23. Let p denote "He is rich" and let q denote "He is happy". Write each statement in symbolic form using p and q. Note that "He is poor" and "He is unhappy" are equivalent to $\neg p$ and $\neg q$, respectively.

(*a*) If he is rich, then he is unhappy.

(*b*) He is neither rich nor happy.

(*c*) It is necessary to be poor in order to be happy.

(*d*) To be poor is to be unhappy.

10.24. Find the truth table for: (*a*) $p \vee \neg q$; (*b*) $\neg p \wedge \neg q$.

10.25. Verify that the proposition $(p \wedge q) \wedge \neg(p \vee q)$ is a contradiction.

ARGUMENTS

10.26. Test the validity of each argument:

(*a*) If it rains, Erik will be sick.
It did not rain.
____
Erik was not sick.

(*b*) If it rains, Erik will be sick.
Erik was not sick.
____
It did not rain.

10.27. Test the validity of the following argument:

If I study, then I will not fail mathematics.
If I do not play basketball, then I will study.
But I failed mathematics.
____
Therefore I must have played basketball.

10.28. Show that $p \leftrightarrow \neg q$ does not logically imply $p \rightarrow q$.

QUANTIFIERS

10.29. Let $A = \{1, 2, \ldots, 9, 10\}$. Consider each of the following sentences. If it is a statement, then determine its truth value. If it is a propositional function, determine its truth set.

(a) $(\forall x \in A)(\exists y \in A)(x + y < 14)$ (c) $(\forall x \in A)(\forall y \in A)(x + y < 14)$

(b) $(\forall y \in A)(x + y < 14)$ (d) $(\exists y \in A)(x + y < 14)$

10.30. Negate each of the following statements:

(a) If the teacher is absent, then some students do not complete their homework.

(b) All the students completed their homework and the teacher is present.

(c) Some of the students did not complete their homework or the teacher is absent.

10.31. Negate each of the statements in Problem 10.17.

10.32. Find a counterexample for each statement where $\mathbf{U} = \{3, 5, 7, 9\}$ is the universal set:

(a) $\forall x,\ x + 3 \geq 7$; (b) $\forall x$, x is odd; (c) $\forall x$, x is prime; (d) $\forall x, |x| = x$.

Answers to Supplementary Problems

10.22. In each case, translate $\wedge$, $\vee$, and $\neg$ to read "and", "or", and "It is false that" or "not", respectively; and then simplify the English sentence.

10.23. (a) $p \rightarrow \neg q$; (b) $\neg p \wedge \neg q$; (c) $q \rightarrow \neg p$; (d) $\neg p \leftrightarrow \neg q$

10.24. The truth tables appear in Fig. 10-20.

p	q	$\neg q$	$p \vee \neg q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

(a)

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

(b)

Fig. 10-20

10.25. It is a contradiction since its truth table in Fig. 10-21 is false for all values of p and q.

p	q	$p \wedge q$	$p \vee q$	$\neg(p \vee q)$	$(p \wedge q) \wedge \neg(p \vee q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

Fig. 10-21

10.26. First translate the arguments into symbolic form: (a) $p \rightarrow q, \neg p \vdash \neg q$, (b) $p \rightarrow q, \neg q \vdash \neg p$. By Problem 10.11, argument (a) is a fallacy. By Problem 10.12, argument (b) is valid.

10.27. Translate the argument into the following symbolic form where p is "I study", q is "I fail mathematics", and r is "I play basketball":

$$p \to \neg q,\ \neg r \to p,\ q \vdash r$$

Construct the truth tables as in Fig. 10.22 where the premises $p \to \neg q$, $\neg r \to p$, and q are true simultaneously only in the fifth row of the table, and in that case the conclusion r is also true. Hence the argument is valid.

p	q	r	$\neg q$	$p \to \neg q$	$\neg r$	$\neg r \to p$
T	T	T	F	F	F	T
T	T	F	F	F	T	T
T	F	T	T	T	F	T
T	F	F	T	T	T	T
F	T	T	F	T	F	T
F	T	F	F	T	T	F
F	F	T	T	T	F	T
F	F	F	T	T	T	F

Fig. 10-22

p	q	$\neg q$	$p \leftrightarrow \neg q$	$p \to q$
T	T	F	F	T
T	F	T	T	F
F	T	F	T	T
F	F	T	F	T

Fig. 10-23

10.28. Method 1. Construct the truth tables of $p \leftrightarrow \neg q$ and $p \to q$ as in Fig. 10.23. Note that $p \leftrightarrow \neg q$ is true in line 2 of the truth table whereas $p \to q$ is false.

Method 2. Construct the truth table of the proposition $(p \leftrightarrow \neg q) \to (p \to q)$. It will not be a tautology; hence, by Theorem 10.4, $p \leftrightarrow \neg q$ does not logically imply $p \to q$.

10.29. (*a*) The open sentence in two variables is preceded by two quantifiers; hence it is a statement. Moreover, the statement is true.

(*b*) The open sentence is preceded by one quantifier; hence it is a propositional function of the other variable. Note that for every $y \in A$, $x_0 + y < 14$ if and only if $x_0 = 1, 2,$ or 3. Hence the truth set is $\{1, 2, 3\}$.

(*c*) It is a statement and it is false: if $x_0 = 8$ and $y_0 = 9$, then $x_0 + y_0 < 14$ is not true.

(*d*) It is an open sentence in x. The truth set is A itself.

10.30. (*a*) The teacher is absent and all the students completed their homework.

(*b*) Some of the students did not complete their homework or the teacher is absent.

(*c*) All the students completed their homework and the teacher is present.

10.31. (*a*) $(\forall x \in A)(x + 3 \neq 10)$ (*c*) $(\forall x \in A)(x + 3 \geq 5)$

(*b*) $(\exists x \in A)(x + 3 \geq 10)$ (*d*) $(\exists x \in A)(x + 3 > 7)$

10.32. (*a*) Here 5, 7, and 9 are counterexamples.

(*b*) The statement is true; hence no counterexample exists.

(*c*) Here 9 is the only counterexample.

(*d*) The statement is true; hence there is no counterexample.

Chapter 11

Boolean Algebra

11.1 INTRODUCTION

Both sets and propositions satisfy similar laws which are listed in Tables 1-1 and 10-1 (appearing in Chapters 1 and 10, respectively). These laws are used to define an abstract mathematical structure called a *Boolean algebra*, which is named after the mathematician George Boole (1813–1864).

11.2 BASIC DEFINITIONS

Let B be a nonempty set with two binary operations $+$ and $*$, a unary operation $'$, and two distinct elements 0 and 1. Then B is called a *Boolean algebra* if the following axioms hold where a, b, c are any elements in B:

$[\mathbf{B}_1]$ ***Commutative laws***:
(1a) $a + b = b + a$ (1b) $a * b = b * a$

$[\mathbf{B}_2]$ ***Distributive laws***:
(2a) $a + (b * c) = (a + b) * (a + c)$ (2b) $a * (b + c) = (a * b) + (a * c)$

$[\mathbf{B}_3]$ ***Identity laws***:
(3a) $a + 0 = a$ (3b) $a * 1 = a$

$[\mathbf{B}_4]$ ***Complement laws***:
(4a) $a + a' = 1$ (4b) $a * a' = 0$

We will sometimes designate a Boolean algebra by $\langle B, +, *, ', 0, 1\rangle$ when we want to emphasize its six parts. We say 0 is the *zero* element, 1 is the *unit* element and a' is the *complement* of a. We will usually drop the symbol $*$ and use juxtaposition instead. Then (2b) is written $a(b + c) = ab + ac$ which is the familiar algebraic identity of rings and fields. However, (2a) becomes $a + bc = (a + b)(a + c)$, which is certainly not a usual identity in algebra.

The operations $+$, $*$ and $'$ are called sum, product, and complement respectively. We adopt the usual convention that, unless we are guided by parentheses, $'$ has precedence over $*$, and $*$ has precedence over $+$. For example,

$$a + b * c \text{ means } a + (b * c) \text{ and not } (a + b) * c \qquad a * b' \text{ means } a * (b') \text{ and not } (a * b)'$$

Of course when $a + b * c$ is written $a + bc$ then the meaning is clear.

EXAMPLE 11.1

(a) Let $\mathbf{B} = \{0, 1\}$, the set of *bits* (binary digits), with the binary operations of $+$ and $*$ and the unary operation defined by Fig. 11-1. Then $\mathbf{B}$ is a Boolean algebra. (Note $'$ simply changes the bit, i.e., $1' = 0$ and $0' = 1$.)

+	1	0
1	1	1
0	1	0

*	1	0
1	1	0
0	0	0

′	1	0
	0	1

Fig. 11-1

(*b*) Let $\mathbf{B}^n = \mathbf{B} \times \mathbf{B} \times \cdots \times \mathbf{B}$ (n factors) where the operations of $+$, $*$ and $'$ are defined componentwise using Fig. 11-1. For notational convenience, we write the elements of $\mathbf{B}^n$ as n-bit sequences without commas, e.g., $x = 110011$ and $y = 111000$ belong to $\mathbf{B}^6$. Hence

$$x + y = 111011, \qquad x * y = 110000, \qquad x' = 001100$$

Then $\mathbf{B}^n$ is a Boolean algebra. Here $0 = 000 \cdots 0$ is the zero element, and $1 = 111 \cdots 1$ is the unit element. We note that $\mathbf{B}^n$ has 2^n elements.

(*c*) Let $\mathbf{D}_{70} = \{1, 2, 5, 7, 10, 14, 35, 79\}$, the divisors of 70. Define $+$, $*$ and $'$ by

$$a + b = \mathrm{lcm}(a, b), \qquad a * b = \gcd(a, b), \qquad a' = \frac{70}{a}$$

Then $\mathbf{D}_{70}$ is a Boolean algebra with 1 the zero element and 70 the unit element.

(*d*) Let $\mathscr{C}$ be a collection of sets closed under the set operations of union, intersection, and complement. Then $\mathscr{C}$ is a Boolean algebra with the empty set $\varnothing$ as the zero element and the universal set $\mathbf{U}$ as the unit element.

Subalgebras, Isomorphic Boolean Algebras

Suppose C is a nonempty subset of a Boolean algebra B. We say C is a *subalgebra* of B if C itself is a Boolean algebra (with respect to the operations of B). We note that C is a subalgebra of B if and only if C is closed under the three operations of B, i.e., $+$, $*$, and $'$. For example, $\{1, 2, 35, 70\}$ is a subalgebra of D_{70} in Example 11.1(*c*).

Two Boolean algebras B and B' are said to be *isomorphic* if there is a one-to-one correspondence $f: B \to B'$ which preserves the three operations, i.e., such that

$$f(a + b) = f(a) + f(b), \qquad f(a * b) = f(a) * f(b) \qquad \text{and} \qquad f(a') = f(a)'$$

for any elements a, b in B.

11.3 DUALITY

The *dual* of any statement in a Boolean algebra B is the statement obtained by interchanging the operations $+$ and $*$, and interchanging their identity elements 0 and 1 in the original statement. For example, the dual of

$$(1 + a) * (b + 0) = b \qquad \text{is} \qquad (0 * a) + (b * 1) = b$$

Observe the symmetry in the axioms of a Boolean algebra B. That is, the dual of the set of axioms of B is the same as the original set of axioms. Accordingly, the important principle of duality holds in B. Namely,

Theorem 11.1 (Principle of Duality): The dual of any theorem in a Boolean algebra is also a theorem.

In other words, if any statement is a consequence of the axioms of a Boolean algebra, then the dual is also a consequence of those axioms since the dual statement can be proven by using the dual of each step of the proof of the original statement.

11.4 BASIC THEOREMS

Using the axioms $[\mathbf{B}_1]$ through $[\mathbf{B}_4]$, we prove (Problem 11.5) the following theorem.

Theorem 11.2: Let a, b, c be any elements in a Boolean algebra B.

(i) ***Idempotent laws***:

(5a) $a + a = a$ (5b) $a * a = a$

(ii) ***Boundedness laws***:

(6a) $a + 1 = 1$ (6b) $a * 0 = 0$

(iii) ***Absorption laws***:

(7a) $a + (a * b) = a$ (7b) $a * (a + b) = a$

(iv) ***Associative laws***:

(8a) $(a + b) + c = a + (b + c)$ (8b) $(a * b) * c = a * (b * c)$

Theorem 11.2 and our axioms still do not contain all the properties of sets listed in Table 1-1. The next two theorems (proved in Problems 11.6 and 11.7) give us the remaining properties.

Theorem 11.3: Let a be any element of a Boolean algebra B.

(i) (Uniqueness of Complement)

If $a + x = 1$ and $a * x = 0$, then $x = a'$.

(ii) (Involution law) $(a')' = a$

(iii) (9a) $0' = 1$, (9b) $1' = 0$

Theorem 11.4 (DeMorgan's laws): (10a) $(a + b)' = a' * b'$. (10b) $(a * b)' = a' + b'$.

11.5 BOOLEAN ALGEBRAS AS LATTICES

By Theorem 11.2 and axiom $[\mathbf{B}_1]$, every Boolean algebra B satisfies the associative, commutative, and absorption laws and hence is a lattice where $+$ and $*$ are the join and meet operations, respectively. With respect to this lattice, $a + 1 = 1$ implies $a \leq 1$ and $a * 0 = 0$ implies $0 \leq a$, for any element $a \in B$. Thus B is a bounded lattice. Furthermore, axioms $[\mathbf{B}_2]$ and $[\mathbf{B}_4]$ show that B is also distributive and complemented. Conversely, every bounded, distributive, and complemented lattice L satisfies the axioms $[\mathbf{B}_1]$ through $[\mathbf{B}_4]$. Accordingly, we have the following

Alternate Definition: A Boolean algebra B is a bounded, distributive, and complemented lattice.

Since a Boolean algebra B is a lattice, it has a natural partial ordering (and so its diagram can be drawn). Recall (Chapter 7) that we define $a \leq b$ when the equivalent conditions $a + b = b$ and $a * b = a$ hold. Since we are in a Boolean algebra, we can actually say much more. Specifically, the following theorem (proved in Problem 11.8) applies.

Theorem 11.5: The following are equivalent in a Boolean algebra:

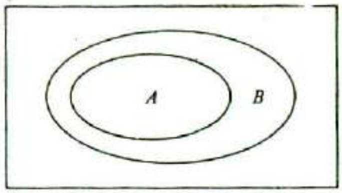
$$(1)\ a + b = b, \qquad (2)\ a * b = a, \qquad (3)\ a' + b = 1, \qquad (4)\ a * b' = 0.$$

Thus in a Boolean algebra we can write $a \leq b$ whenever any of the above four conditions is known to be true.

EXAMPLE 11.2

(a) Consider a Boolean algebra of sets. Then set A precedes set B if A is a subset of B. Theorem 11.4 states that if $A \subseteq B$, as illustrated in the Venn diagram in Fig. 11-2, then the following conditions hold:

$$(1)\ A \cup B = B, \qquad (2)\ A \cap B = A, \qquad (3)\ A^c \cup B = U, \qquad (4)\ A \cap B^c = \varnothing.$$



A is a subset of B

Fig. 11-2

(*b*) Consider the Boolean algebra of the proposition calculus. Then the proposition P precedes the proposition Q if P logically implies Q, i.e., if $P \Rightarrow Q$.

11.6 REPRESENTATION THEOREM

Let B be a finite Boolean algebra. Recall (Section 7.9) that an element a in B is an atom if a immediately succeeds 0, that is if $0 << a$. Let A be the set of atoms of B and let $P(A)$ be the Boolean algebra of all subsets of the set A of atoms. By Theorem 7.15, each $x \neq 0$ in B can be expressed uniquely (except for order) as the sum (join) of atoms, i.e. elements of A Say,

$$x = a_1 + a_2 + \cdots + a_r$$

is such a representation. Consider the function $f: B \rightarrow P(A)$ defined by

$$f(x) = \{a_1, a_2, \ldots, a_r\}$$

The mapping is well-defined since the representation is unique.

Theorem 11.6: The above mapping $f: B \rightarrow \mathscr{P}(A)$ is an isomorphism.

Thus we see the intimate relationship between set theory and abstract Boolean algebras in the sense that every finite Boolean algebra is structurally the same as a Boolean algebra of sets.

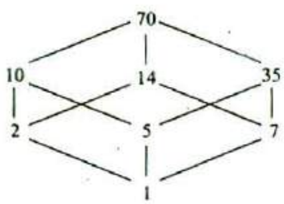
If a set A has n elements, then its power set $\mathscr{P}(A)$ has 2^n elements. Thus the above theorem gives us our next result.

Corollary 11.7: A finite Boolean algebra has 2^n elements for some positive integer n.

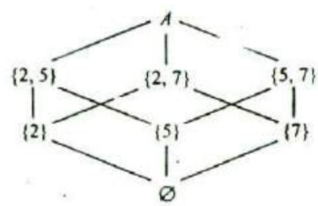
EXAMPLE 11.3 Consider the Boolean algebra $\mathbf{D}_{70} = \{1, 2, 5, \ldots, 70\}$ whose diagram is given in Fig. 11-3(*a*). Note that $A = \{2, 5, 7\}$ is the set of atoms of $\mathbf{D}_{70}$. The following is the unique representation of each non-atom by atoms:

$$10 = 2 \vee 5, \qquad 14 = 2 \vee 7, \qquad 35 = 5 \vee 7, \qquad 70 = 2 \vee 5 \vee 7$$

Figure 11-3(*b*) gives the diagram of the Boolean algebra of the power set $\mathscr{P}(A)$ of the set A of atoms. Observe that the two diagrams are structurally the same.



(*a*) $\mathbf{D}_{70}$



(*b*) $\mathscr{P}(A)$

Fig. 11-3

11.7 SUM-OF-PRODUCTS FORM FOR SETS

This section motivates the concept of the sum-of-products form in Boolean algebra by an example of set theory. Consider the Venn diagram in Fig. 11-4 of three sets A, B, C. Observe that these sets partition the rectangle (universal set) into eight numbered sets which can be represented as follows:

$$\begin{array}{llll} (1)\ A \cap B \cap C & (3)\ A \cap B^c \cap C & (5)\ A \cap B^c \cap C^c & (7)\ A^c \cap B^c \cap C \\ (2)\ A \cap B \cap C^c & (4)\ A^c \cap B \cap C & (6)\ A^c \cap B \cap C^c & (8)\ A^c \cap B^c \cap C^c \end{array}$$

Each of these eight sets is of the form $A^* \cap B^* \cap C^*$, where

$$A^* = A \text{ or } A^c, \qquad B^* = B \text{ or } B^c, \qquad C^* = C \text{ or } C^c$$

Consider any nonempty set expression E involving the sets A, B, and C, say,

$$E = [(A \cap B^c)^c \cup (A^c \cap C^c)] \cap [B^c \cup C)^c \cap (A \cup C^c)]$$

Then E will represent some area in Fig. 11-4 and hence will uniquely equal the union of one or more of the eight sets.

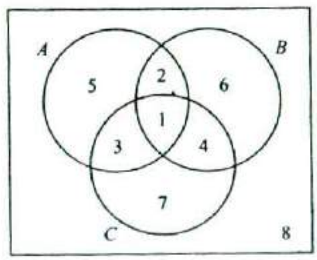


Fig. 11-4

Suppose we now interpret a union as a sum and an intersection as a product. Then the above eight sets are products, and the unique representation of E will be a sum (union) of products. This unique representation of E is the same as the complete sum-of-products expansion in Boolean algebras which we discuss below.

11.8 SUM-OF-PRODUCTS FORM FOR BOOLEAN ALGEBRAS

Consider a set of variables (or letters or symbols), say, $x_1, x_2, \ldots, x_n$. A *Boolean expression E* in these variables, sometimes written $E(x_1, \ldots, x_n)$, is any variable or any expression built up from the variables using the Boolean operations $+$, $*$ and $'$. (Naturally, the expression E must be *well-formed*, that is, where $+$ and $*$ are used as binary operations, and $'$ is used as a unary operation.) For example,

$$E_1 = (x + y'z)' + (xyz' + x'y)' \qquad \text{and} \qquad E_2 = ((xy'z' + y)' + x'z)'$$

are Boolean expressions in x, y, and z.

A *literal* is a variable or complemented variable, such as x, x', y, y', and so on. A *fundamental product* is a literal or a product of two or more literals in which no two literals involve the same variable. Thus

$$xz', \quad xy'z, \quad x, \quad y', \quad x'yz$$

are fundamental products, but $xyx'z$ and $xyzy$ are not. Note that any product of literals can be reduced to either 0 or a fundamental product, e.g., $xyx'z = 0$ since $xx' = 0$ (complement law), and $xyzy = xyz$ since $yy = y$ (idempotent law).

A fundamental product P_1 is said to be *contained in* (or *included in*) another fundamental product P_2 if the literals of P_1 are also literals of P_2. For example, $x'z$ is contained in $x'yz$, but $x'z$ is not contained in $xy'z$ since x' is not a literal of $xy'z$. Observe that if P_1 is contained in P_2, say $P_2 = P_1 * Q$, then, by the absorption law,

$$P_1 + P_2 = P_1 + P_1 * Q = P_1$$

Thus, for instance, $x'z + x'yz = x'z$.

Definition: A Boolean expression E is called a *sum-of-products* expression if E is a fundamental product or the sum of two or more fundamental products none of which is contained in another.

Definition: Let E be any Boolean expression. A *sum-of-products form* of E is an equivalent Boolean sum-of-products expression.

EXAMPLE 11.4 Consider the expressions

$$E_1 = xz' + y'z + xyz' \qquad \text{and} \qquad E_2 = xz' + x'yz' + xy'z$$

Although the first expression E_1 is a sum of products, it is not a sum-of-products expression. Specifically, the product xz' is contained in the product xyz'. However, by the absorption law, E_1 can be expressed as

$$E_1 = xz' + y'z + xyz' = xz' + xyz' + y'z = xz' + y'z$$

This yields a sum-of-products form for E_1. The second expression E_2 is already a sum-of-products expression.

Algorithm for Finding Sum-of-Products Forms

The following four-step algorithm uses the Boolean algebra laws to transform any Boolean expression E into an equivalent sum-of-products expression:

Algorithm 11.8A: The input is a Boolean expression E. The output is a sum-of-products expression equivalent to E.

Step 1. Use DeMorgan's laws and involution to move the complement operation into any parenthesis until finally the complement operation only applies to variables. Then E will consist only of sums and products of literals.

Step 2. Use the distributive operation to next transform E into a sum of products.

Step 3. Use the commutative, idempotent, and complement laws to transform each product in E into 0 or a fundamental product.

Step 4. Use the absorption and identity laws to finally transform E into a sum-of-products expression.

EXAMPLE 11.5 Suppose Algorithm 11.8A is applied to the following Boolean expression:

$$E = ((xy)'z)'((x' + z)(y' + z'))'$$

Step 1. Using DeMorgan's laws and involution, we obtain

$$E = ((xy)'' + z')((x' + z)' + (y' + z')') = (xy + z')(xz' + yz)$$

E now consists only of sums and products of literals.

Step 2. Using the distributive laws, we obtain

$$E = xyxz' + xyyz + xz'z' + yzz'$$

E now is a sum of products.

Step 3. Using the commutative, idempotent, and complement laws, we obtain

$$E = xyz' + xyz + xz' + 0$$

Each term in E is a fundamental product or 0.

Step 4. The product ac' is contained in abc'; hence, by the absorption law,

$$xz' + (xz' * y) = xz'$$

Thus we may delete abc' from the sum. Also, by the identity law for 0, we may delete 0 from the sum. Accordingly,

$$E = xyz + xz'$$

E is now represented by a sum-of-products expression.

Complete Sum-of-Products Forms

A Boolean expression $E = E(x_1, x_2, \ldots, x_n)$ is said to be a *complete sum-of-products* expression if E is a sum-of-products expression where each product P involves all the n variables. Such a fundamental product P which involves all the variables is called a *minterm*, and there is a maximum of 2^n such products for n variables. The following theorem applies.

Theorem 11.8: Every nonzero Boolean expression $E = E(x_1, x_2, \ldots, x_n)$ is equivalent to a complete sum-of-products expression and such a representation is unique.

The above unique representation of E is called the *complete sum-of-products form* of E. Recall that Algorithm 11.8A tells us how to transform E into a sum-of-products form. The following algorithm shows how to transform a sum-of-products form into a complete sum-of-products form.

Algorithm 11.8B: The input is a Boolean sum-of-products expression $E = E(x_1, x_2, \ldots, x_n)$. The output is a complete sum-of-products expression equivalent to E.

Step 1. Find a product P in E which does not involve the variable x_i, and then multiply P by $x_i + x_i'$, deleting any repeated products. (This is possible since $x_i + x_i' = 1$, and $P + P = P$.)

Step 2. Repeat Step 1 until every product P in E is a minterm, i.e., every product P involves all the variables.

EXAMPLE 11.6 Express $E(x, y, z) = x(y'z)'$ in its complete sum-of-products form.

(*a*) Apply Algorithm 11.8A to E to obtain

$$E = x(y'z)' = x(y + z') = xy + xz'$$

Now E is represented by a sum-of-products expression.

(*b*) Apply Algorithm 11.8B to obtain

$$\begin{aligned} E &= xy(z + z') + xz'(y + y') = xyz + xyz' + xyz' + xy'z' \\ &= xyz + xyz' + xy'z' \end{aligned}$$

Now E is represented by its complete sum-of-products form.

Warning: The terminology in this section has not been standardized. The sum-of-products form for a Boolean expression E is also called the *disjunctive normal form* or DNF of E. The complete sum-of-products form for E is also called the *full disjunctive normal form*, or the *disjunctive canonical form*, or the *minterm canonical form* of E.

11.9 MINIMAL BOOLEAN EXPRESSIONS, PRIME IMPLICANTS

There are many ways of representing the same Boolean expression E. Here we define and investigate a minimal sum-of-products form for E. We must also define and investigate prime implicants of E since the minimal sum-of-products involves such prime implicants. Other minimal forms exist, but their investigation lies beyond the scope of this text.

Minimal Sum-of-Products

Consider a Boolean sum-of-products expression E. Let E_L denote the number of literals in E (counted according to multiplicity), and let E_S denote the number of summands in E. For instance, suppose

$$E = xyz' + x'y't + xy'z't + x'yzt$$

Then $E_L = 3 + 3 + 4 + 4 = 14$ and $E_S = 4$.

Suppose E and F are equivalent Boolean sum-of-products expressions. We say E is *simpler* than F if

$$\text{(i) } E_L < F_L \text{ and } E_S \leq F_L, \qquad \text{or} \qquad \text{(ii) } E_L \leq F_L \text{ and } E_S < F_L$$

We say E is *minimal* if there is no equivalent sum-of-products expression which is simpler than E. We note that there can be more than one equivalent minimal sum-of-products expression.

Prime Implicants

A fundamental product P is called a *prime implicant* of a Boolean expression E if

$$P + E = E$$

but no other fundamental product contained in P has this property. For instance, suppose

$$E = xy' + xyz' + x'yz'$$

One can show (Problem 11.15) that

$$xz' + E = E \qquad \text{but} \qquad x + E \neq E \qquad \text{and} \qquad z' + E \neq E$$

Thus xz' is a prime implicant of E.

The following theorem applies.

Theorem 11.9: A minimal sum-of-products form for a Boolean expression E is a sum of prime implicants of E.

The following subsections give a method for finding the prime implicants of E based on the notion of the consensus of fundamental products. This method can then be used to find a minimal sum-of-products form for E. Section 11.10 gives a geometric method for finding such prime implicants.

Consensus of Fundamental Products

Let P_1 and P_2 be fundamental products such that exactly one variable, say x_k, appears uncomplemented in one of P_1 and P_2 and complemented in the other. Then the *consensus* of P_1 and P_2 is the product (without repetitions) of the literals of P_1 and the literals of P_2 after x_k and x_k' deleted. (We do not define the consensus of $P_1 = x$ and $P_2 = x'$.)

The following lemma (proved in Problem 11.19) applies.

Lemma 11.10: Suppose Q is the consensus of P_1 and P_2. Then $P_1 + P_2 + Q = P_1 + P_2$.

EXAMPLE 11.7 Find the consensus Q of P_1 and P_2 where:

(*a*) $P_1 = xyz's$ and $P_2 = xy't$.

Delete y and y' and then multiply the literals of P_1 and P_2 (without repetition) to obtain $Q = xz'st$.

(*b*) $P_1 = xy'$ and $P_2 = y$.

Deleting y and y' yields $Q = x$.

(*c*) $P_1 = x'yz$ and $P_2 = x'yt$.

No variable appears uncomplemented in one of the products and complemented in the other. Hence P_1 and P_2 have no consensus.

(*d*) $P_1 = x'yz$ and $P_2 = xyz'$.

Each of x and z appear complemented in one of the products and uncomplemented in the other. Hence P_1 and P_2 have no consensus.

Consensus Method for Finding Prime Implicants

The following algorithm, called the *consensus method*, is used to find the prime implicants of a Boolean expression.

Algorithm 11.9A (Consensus Method): The input is a Boolean expression

$$E = P_1 + P_2 + \cdots + P_m$$

where the P's are fundamental products. The output expresses E as a sum of its prime implicants (Theorem 11.11).

Step 1. Delete any fundamental product P_i which includes any other fundamental product P_j. (Permissible by the absorption law.)

Step 2. Add the consensus of any P_i and P_j providing Q does not include any of the P's. (Permissible by Lemma 11.10.)

Step 3. Repeat Step 1 and/or Step 2 until neither can be applied.

The following theorem gives the basic property of the above algorithm.

Theorem 11.11: The consensus method will eventually stop, and then E will be the sum of its prime implicants.

EXAMPLE 11.8 Let $E = xyz + x'z' + xyz' + x'y'z + x'yz'$. Then:

$$\begin{aligned}
E &= xyz + x'z' + xyz' + x'y'z && (x'yz' \text{ includes } x'z')\\
&= xyz + x'y' + xyz' + x'y'z + xy && (\text{Consensus of } xyz \text{ and } xyz')\\
&= x'z' + x'y'z + xy && (xyz \text{ and } xyz' \text{ include } xy)\\
&= x'z' + x'y'z + xy + x'y' && (\text{Consensus of } x'z' \text{ and } x'y'z)\\
&= x'z' + xy + x'y' && (x'y'z \text{ includes } x'y')\\
&= x'z' + xy + x'y' + yz' && (\text{Consensus of } x'z' \text{ and } xy)
\end{aligned}$$

Now neither step in the consensus method will change E. Thus E is the sum of its prime implicants, which are $x'z'$, xy, $x'y'$, and yz'.

Finding a Minimal Sum-of-Products Form

The consensus method (Algorithm 11.9A) can be used to express a Boolean expression E as a sum of all its prime implicants. Using such a sum, one may find a minimal sum-of-products form for E as follows.

Algorithm 11.9B: The input is a Boolean expression $E = P_1 + P_2 + \cdots + P_m$ where the P's are all the prime implicants of E. The output expresses E as a minimal sum-of-products.

Step 1. Express each prime implicant P as a complete sum-of-products.

Step 2. Delete one by one those prime implicants whose summands appear among the summands of the remaining prime implicants.

EXAMPLE 11.9 We apply Algorithm 11.9B to

$$E = x'z' + xy + x'y' + yz'$$

(By Example 11.8, E is now expressed as the sum of all its prime implicants.)

Step-1. Express each prime implicant of E as a complete sum-of-products to obtain:

$$\begin{aligned} x'z' &= x'z'(y + y') = x'yz' + x'y'z' \\ xy &= xy(z + z') = xyz + xyz' \\ x'y' &= x'y'(z + z') = x'y'z + x'y'z' \\ yz' &= yz'(x + x') = xyz' + x'yz' \end{aligned}$$

Step 2. The summands of $x'z'$ are $x'yz$ and $x'y'z'$ which appear among the other summands. Thus delete $x'z'$ to obtain

$$E = xy + x'y' + yz'$$

The summands of no other prime implicant appear among the summands of the remaining prime implicants, and hence this is a minimal sum-of-products form for E. In other words, none of the remaining prime implicants is *superfluous*, that is, none can be deleted without changing E.

11.10 KARNAUGH MAPS

Karnaugh maps, where minterms involving the same variables are represented by squares, are pictorial devices for finding prime implicants and minimal forms for Boolean expressions involving at most six variables. We will only treat the cases of two, three, and four variables. In the context of Karnaugh maps, we will sometimes use the terms "squares" and "minterm" interchangeably. Recall that a minterm is a fundamental product which involves all the variables, and that a complete sum-of-products expression is a sum of minterms.

First we need to define the notion of adjacent products. Two fundamental products P_1 and P_2 are said to be *adjacent* if P_1 and P_2 have the same variables and if they differ in exactly one literal. Thus there must be an uncomplemented variable in one product and complemented in the other. In particular, the sum of two such adjacent products will be a fundamental product with one less literal (Problem 11.51).

EXAMPLE 11.10 Find the sum of the following adjacent products P_1 and P_2:

(*a*) $P_1 = xyz'$ and $P_2 = xy'z'$.

$$P_1 + P_2 = xyz' + xy'z' = xz'(y + y') = xz'(1) = xz'$$

(*b*) $P_1 = x'yzt$ and $P_2 = x'yz't$.

$$P_1 + P_2 = x'yzt + x'yz't = x'yt(z + z') = x'yt(1) = x'yt$$

(c) $P_1 = x'yzt$ and $P_2 = xyz't$.

Here P_1 and P_2 are not adjacent since they differ in two literals. In particular,

$$P_1 + P_2 = x'yzt + xyz't = (x' + x)y(z + z')t = (1)y(1)t = yt$$

(d) $P_1 = xyz'$ and $P_2 = xyzt$.

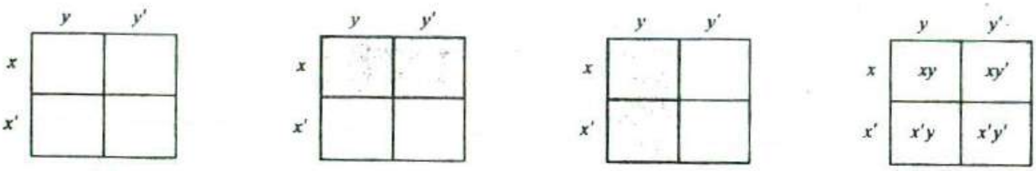
Here P_1 and P_2 are not adjacent since they have different variables. Thus, in particular, they will not appear as squares in the same Karnaugh map.

Case of Two Variables

The Karnaugh map corresponding to Boolean expressions $E = E(x, y)$ with two variables x and y is shown in Fig. 11-5(a). The Karnaugh map may be viewed as a Venn diagram where x is represented by the points in the upper half of the map, shaded in Fig. 11-5(b), and y is represented by the points in the left half of the map, shaded in Fig. 11-5(c). Thus x' is represented by the points in the lower half of the map, and y' is represented by the points in the right half of the map. Accordingly, the four possible minterms with two literals,

$$xy, \quad xy', \quad x'y, \quad x'y'$$

are represented by the four squares in the map, as labeled in Fig. 11-5(d). Note that two such squares are adjacent, as defined above, if and only if the squares are geometrically adjacent (have a side in common).



(a) (b) x shaded (c) y shaded (d)

Fig. 11-5

Any complete sum-of-products Boolean expression $E(x, y)$ is a sum of minterms and hence can be represented in the Karnaugh map by placing checks in the appropriate squares. A prime implicant of $E(x, y)$ will be either a pair of adjacent squares in E or an *isolated* square, i.e., a square which is not adjacent to any other square of $E(x, y)$. A minimal sum-of-products form for $E(x, y)$ will consist of a minimal number of prime implicants which cover all the squares of $E(x, y)$ as illustrated in the next example.

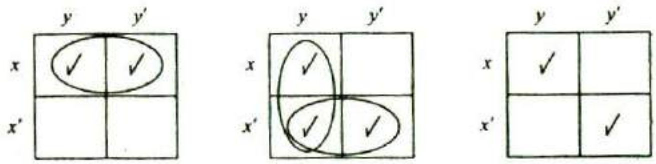
EXAMPLE 11.11 Find the prime implicants and a minimal sum-of-products form for each of the following complete sum-of-products Boolean expressions:

(a) $E_1 = xy + xy'$; (b) $E_1 = xy + x'y + x'y'$; (c) $E_1 = xy + x'y'$.

This can be solved by using Karnaugh maps as follows:

(a) Check the squares corresponding to xy and xy' as in Fig. 11-6(a). Note that E_1 consists of one prime implicant, the two adjacent squares designated by the loop in Fig. 11-6(a). This pair of adjacent squares represents the variable x, so x is a (the only) prime implicant of E_1. Consequently, $E_1 = x$ is its minimal sum.

(b) Check the squares corresponding to xy, $x'y$, and $x'y'$ as in Fig. 11-6(b). Note that E_2 contains two pairs of adjacent squares (designated by the two loops) which include all the squares of E_2. The vertical pair represents y and the horizontal pair represents x'; hence y and x' are the prime implicants of E_2. Thus $E_2 = x' + y$ is its minimal sum.



(a) E_1 (b) E_2 (c) E_3

Fig. 11-6

(c) Check the squares corresponding to xy and $x'y'$ as in Fig. 11-6(c). Note that E_3 consists of two isolated squares which represent xy and $x'y'$; hence xy and $x'y'$ are the prime implicants of E_3 and $E_3 = xy + x'y'$ is its minimal sum.

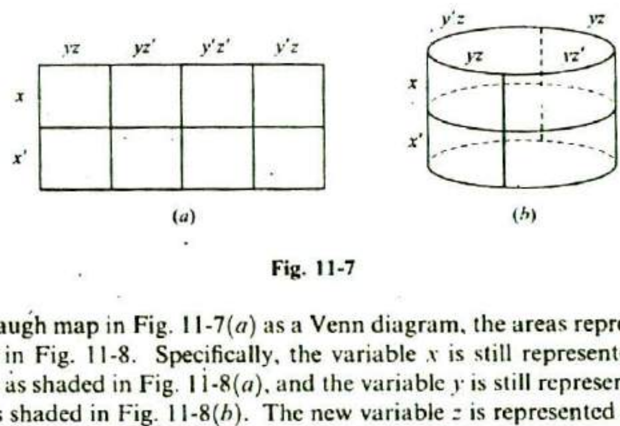
Case of Three Variables

The Karnaugh map corresponding to Boolean expressions $E = E(x, y, z)$ with three variables x, y, z is shown in Fig. 11-7(a). Recall that there are exactly eight minterms with three variables:

$$xyz, \quad xyz', \quad xy'z', \quad xy'z, \quad x'yz, \quad x'yz', \quad x'y'z', \quad x'y'z$$

These minterms are listed so that they correspond to the eight squares in the Karnaugh map in the obvious way.

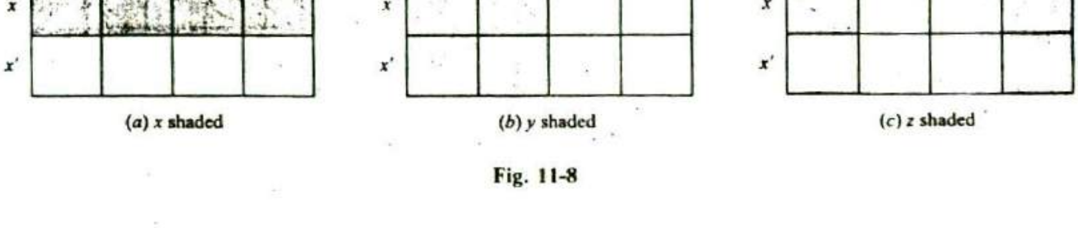
Furthermore, in order that every pair of adjacent products in Fig. 11-7(a) are geometrically adjacent, the right and left edges of the map must be identified. This is equivalent to cutting out, bending, and gluing the map along the identified edges to obtain the cylinder pictured in Fig. 11-7(b), where adjacent products are now represented by squares with one edge in common.



(a) (b)

Fig. 11-7

Viewing the Karnaugh map in Fig. 11-7(a) as a Venn diagram, the areas represented by the variables x, y, and z are shown in Fig. 11-8. Specifically, the variable x is still represented by the points in the upper half of the map, as shaded in Fig. 11-8(a), and the variable y is still represented by the points in the left half of the map, as shaded in Fig. 11-8(b). The new variable z is represented by the points in the left and right quarters of the map, as shaded in Fig. 11-8(c). Thus x', y', and z' are represented, respectively, by points in the lower half, right half, and middle two quarters of the map.



(a) x shaded (b) y shaded (c) z shaded

Fig. 11-8

By a *basic rectangle* in the Karnaugh map with three variables, we mean a square, two adjacent squares, or four squares which form a one-by-four or a two-by-two rectangle. These basic rectangles correspond to fundamental products of three, two, and one literal, respectively. Moreover, the fundamental product represented by a basic rectangle is the product of just those literals that appear in every square of the rectangle.

Suppose a complete sum-of-products Boolean expression $E = E(x, y, z)$ is represented in the Karnaugh map by placing checks in the appropriate squares. A prime implicant of E will be a *maximal basic rectangle* of E, i.e., a basic rectangle contained in E which is not contained in any larger basic rectangle in E. A minimal sum-of-products form for E will consist of a *minimal cover* of E, that is, a minimal number of maximal basic rectangles of E which together include all the squares of E.

EXAMPLE 11.12 Find the prime implicants and a minimal sum-of-products form for each of the following complete sum-of-products Boolean expressions:

(*a*) $E_1 = xyz + xyz' + x'yz' + x'y'z$

(*b*) $E_2 = xyz + xyz' + xy'z + x'yz + x'y'z$

(*c*) $E_3 = xyz + xyz' + x'yz' + x'y'z' + x'y'z$

This can be solved by using Karnaugh maps as follows:

(*a*) Check the squares corresponding to the four summands as in Fig. 11-9(*a*). Observe that E_1 has three prime implicants (maximal basic rectangles), which are circled; these are xy, yz', and $x'y'z$. All three are needed to cover E_1, hence the minimal sum for E_1 is

$$E_1 = xy + yz' + x'y'z$$

(*b*) Check the squares corresponding to the five summands as in Fig. 11-9(*b*). Note that E_2 has two prime implicants, which are circled. One is the two adjacent squares which represents xy, and the other is the two-by-two square (spanning the identified edges) which represents z. Both are needed to cover E_2, so the minimal sum for E_2 is

$$E_2 = xy + z$$

(*c*) Check the squares corresponding to the five summands as in Fig. 11-9(*c*). As indicated by the loops, E_3 has four prime implicants, xy, yz', $x'z'$, and $x'y'$. However, only one of the two dashed ones, i.e., one of yz' or $x'z'$, is needed in a minimal cover of E_3. Thus E_3 has two minimal sums:

$$E_3 = xy + yz' + x'y' = xy + x'z' + x'y'.$$

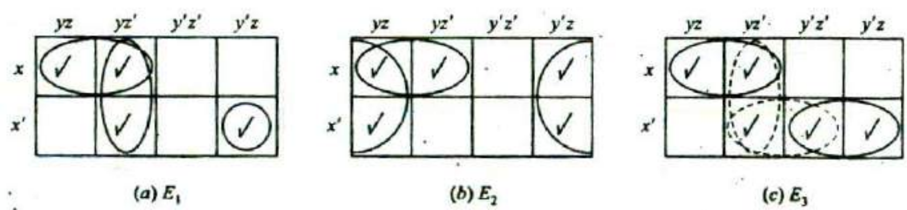


Fig. 11-9

Case of Four Variables

The Karnaugh map corresponding to Boolean expressions $E = E(x, y, z, t)$ with four variables x, y, z, t is shown in Fig. 11-10. Each of the 16 squares corresponds to one of the 16 minterms with four variables,

$$xyzt, \quad xyzt', \quad xyz't', \quad xyz't, \quad \ldots, \quad x'yz't$$

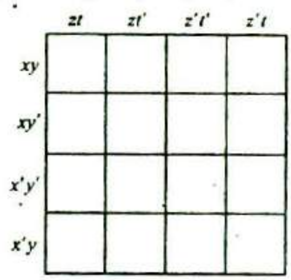


Fig. 11-10

This is indicated by the labels of the row and column of the square. Observe that the top line and the left side are labeled so that adjacent products differ in precisely one literal. Again, we must identify the left edge with the right edge (as we did with three variables) but we must also identify the top edge with the bottom edge. (These identifications give rise to a donut-shaped surface called a *torus*, and we may view our map as really being a torus.)

A *basic rectangle* in a four-variable Karnaugh map is a square, two adjacent squares, four squares which form a one-by-four or two-by-two rectangle, or eight squares which form a two-by-four rectangle. These basic rectangles correspond to fundamental products of four, three, two, and one literal, respectively. Again, maximal basic rectangles are the prime implicants. The minimizing technique for a Boolean expression $E(x, y, z, t)$ is the same as before.

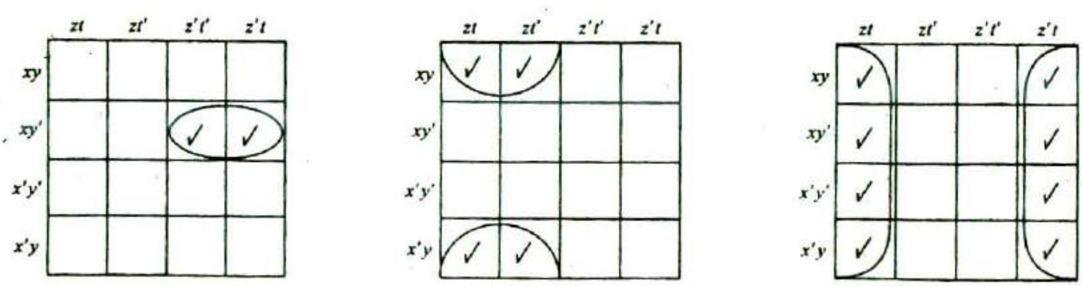
EXAMPLE 11.13 Find the fundamental product P represented by the basic rectangle in the Karnaugh maps shown in Fig. 11-11.

In each case, find the literals which appear in all the squares of the basic rectangle; P is the product of such literals.

(*a*) x, y, and z' appear in both squares; hence $P = xy'z'$.

(*b*) Only y and z appear in all four squares; hence $P = yz$.

(*c*) Only t appears in all eight squares; hence $P = t$.



(*a*) (*b*) (*c*)

Fig. 11-11

EXAMPLE 11.14 Find a minimal sum-of-products form for $E = xy' + xyz + x'y'z' + x'yzt'$.

Check all the squares representing each fundamental product. That is, check all four squares representing xy', the two squares representing xyz, the two squares representing $x'y'z'$ and the one square representing $x'yzt'$, as in Fig. 11-12. A minimal cover of the map consists of the three designated basic rectangles. The two-by-two squares represent the fundamental products xz and $y'z'$, and the two adjacent squares (on top and bottom) represents yzt'. Hence the following is a minimal sum for E:

$$E = xz + y'z' + yzt'$$

	zt	zt'	$z't'$	$z't$
xy	✓	✓		
xy'	✓	✓	✓	✓
$x'y'$			✓	✓
$x'y$		✓		

Fig. 11-12

Solved Problems

BOOLEAN ALGEBRAS

11.1. Write the dual of each Boolean equation: (*a*) $(a * 1) * (0 + a') = 0$;. (*b*) $a + a'b = a + b$.

(*a*) To obtain the dual equation, interchange $+$ and $*$, and interchange 0 and 1. This yields

$$(a + 0) + (1 * a') = 1$$

(*b*) First write the equation using $*$: $a + (a' * b) = a + b$. Then the dual is $a * (a' + b) = a * b$, which can we written as

$$a(a' + b) = ab$$

11.2. Recall (Chapter 7) that the set $\mathbf{D}_m$ of divisors of m is a bounded, distributive lattice with $a + b = a \vee b = \operatorname{lcm}(a, b)$ and $a * b = a \wedge b = \gcd(a, b)$. (*a*) Show that $\mathbf{D}_m$ is a Boolean algebra if m is square free, i.e., if m is a product of distinct primes. (*b*) Find the atoms of $\mathbf{D}_m$.

(*a*) We need only show that $\mathbf{D}_m$ is complemented. Let x be in $\mathbf{D}_m$ and let $x' = m/x$. Since m is a product of distinct primes, x and x' have different divisors. Hence $x * x' = \gcd(x, x') = 1$ and $x + x' = \operatorname{lcm}(x, x') = m$. Recall that 1 is the zero element (lower bound) of $\mathbf{D}_m$, and that m is the identity element (upper bound) of $\mathbf{D}_m$. Thus x' is a complement of x, and so $\mathbf{D}_m$ is a Boolean algebra.

(*b*) The atoms of $\mathbf{D}_m$ are the prime divisors of m.

11.3. Consider the Boolean algebra $\mathbf{D}_{210}$.

(*a*) List its elements and draw its diagram.

(*b*) Find the set A of atoms.

(*c*) Find two subalgebras with eight elements.

(*d*) Is $X = \{1, 2, 6, 210\}$ a sublattice of $\mathbf{D}_{210}$? A subalgebra?

(*e*) Is $Y = \{1, 2, 3, 6\}$ a sublattice of $\mathbf{D}_{210}$? A subalgebra?

(a) The divisors of 210 are 1, 2, 3, 5, 6, 7, 10, 14, 15, 21, 30, 35, 42, 70, 105 and 210. The diagram of $\mathbf{D}_{210}$ appears in Fig. 11-13.

(b) $A = \{2, 3, 5, 7\}$, the set of prime divisors is 210.

(c) $B = \{1, 2, 3, 35, 6, 70, 105, 210\}$ and $C = \{1, 5, 6, 7, 30, 35, 42, 210\}$ are subalgebras of $\mathbf{D}_{210}$.

(d) X is a sublattice since it is linearly ordered. However, X is not a subalgebra since 35 is the complement of 2 in $\mathbf{D}_{210}$ but 35 does not belong to X. (In fact, no Boolean algebra with more than two elements is linearly ordered.)

(e) Y is a sublattice of $\mathbf{D}_{210}$ since it is closed under $+$ and $*$. However, Y is not a subalgebra of $\mathbf{D}_{210}$ since it is not closed under complements in $\mathbf{D}_{210}$, e.g., $35 = 2'$ does not belong to Y. (We note that Y itself is a Boolean algebra, in fact, $Y = \mathbf{D}_6$.)

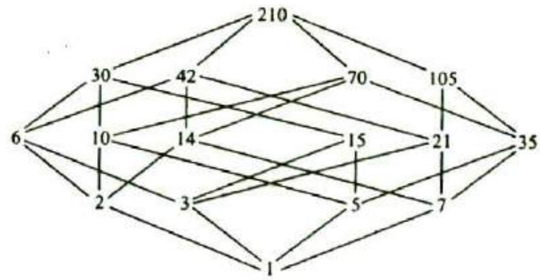


Fig. 11-13

11.4. Find the number of subalgebras of $\mathbf{D}_{210}$.

A subalgebra of $\mathbf{D}_{210}$ must contain two, four, eight or sixteen elements.

(i) There can be only one two-element subalgebra which consists of the upper and lower bounds, i.e., $\{1, 210\}$.

(ii) Since $\mathbf{D}_{210}$ contains sixteen elements, the only sixteen-element subalgebra is $\mathbf{D}_{210}$ itself.

(iii) Any four-element subalgebra is of the form $\{1, x, x', 210\}$, i.e., consists of the upper and lower bounds and a nonbound element and its complement. There are fourteen nonbound elements in $\mathbf{D}_{210}$ and so there are $14/2 = 7$ pairs $\{x, x'\}$. Thus $\mathbf{D}_{210}$ has seven 4-element subalgebras.

(iv) Any eight-element subalgebra S will itself contain three atoms s_1, s_2, s_3. We can choose s_1 and s_2 to be any two of the four atoms of $\mathbf{D}_{210}$ and then s_3 must be the product of the other two atoms, e.g., we can let $s_1 = 2$, $s_2 = 3$, $s_3 = 5 \cdot 7 = 35$ (which determines the subalgebra B above), or we can let $s_1 = 5$, $s_2 = 7$, $s_3 = 2 \cdot 3 = 6$ (which determines the subalgebra C above). There are $\binom{4}{2} = 6$ ways to choose s_1 and s_2 from the four atoms of $\mathbf{D}_{210}$ and so $\mathbf{D}_{210}$ has six 8-element subalgebras.

Accordingly, $\mathbf{D}_{210}$ has $1 + 1 + 7 + 6 = 15$ subalgebras.

11.5. Prove Theorem 11.2: Let a, b, c be any elements in a Boolean algebra B.

(i) Idempotent laws:

(5a) $a + a = a$ (5b) $a * a = a$

(ii) Boundedness laws:

(6a) $a + 1 = 1$ (6b) $a * 0 = 0$

(iii) Absorption laws:

(7a) $a + (a * b) = a$ (7b) $a * (a + b) = a$

(iv) Associative laws:

(8a) $(a + b) + c = a + (b + c)$ (8b) $(a * b) * c = a * (b * c)$

The proofs follow:

(5*b*) $a = a * 1 = a * (a + a') = (a * a) + (a * a') = (a * a) + 0 = a * a$

(5*a*) Follows from (5*b*) and duality.

(6*b*) $a * 0 = (a * 0) + 0 = (a * 0) + (a * a') = a * (0 + a') = a * (a' + 0) = a * a' = 0$

(6*a*) Follows from (6*b*) and duality.

(7*b*) $a * (a + b) = (a + 0) * (a + b) = a + (0 * b) = a + (b * 0) = a + 0 = a$

(7*a*) Follows from (7*b*) and duality.

(8*b*) Let $L = (a * b) * c$ and $R = a * (b * c)$. We need to prove that $L = R$. We first prove that $a + L = a + R$. Using the absorption laws in the last two steps,

$$a + L = a + ((a * b) * c) = (a + (a * b)) * (a + c) = a * (a + c) = a$$

Also, using the absorption law in the last step,

$$a + R = a + (a * (b * c)) = (a + a) * (a + (b * c)) = a * (a + (b * c)) = a$$

Thus $a + L = a + R$. Next we show that $a' + L = a' + R$. We have

$$\begin{aligned} a' + L &= a' + ((a * b) * c) = (a' + (a * b)) * (a' + c) \\ &= ((a' + a) * (a' + b)) * (a' + c) = (1 * (a' + b)) * (a' + c) \\ &= (a' + b) * (a' + c) = a' + (b * c) \ . \end{aligned}$$

Also,

$$\begin{aligned} a' + R &= a' + (a * (b * c)) = (a' + a) * (a' + (b * c)) \\ &= 1 * (a' + (b * c)) = a' + (b * c) \end{aligned}$$

Thus $a' + L = a' + R$. Consequently

$$\begin{aligned} L &= 0 + L = (a * a') + L = (a + L) * (a' + L) = (a + R) * (a' + R) \\ &= (a * a') + R = 0 + R = R \end{aligned}$$

(8*a*) Follows from (8*b*) and duality.

11.6. Prove Theorem 11.3: Let a be any element of a Boolean Algebra B.

(i) (Uniqueness of complement) If $a + x = 1$ and $a * x = 0$, then $x = a'$.

(ii) (Involution law) $(a')' = a$

(iii) (9*a*) $0' = 1$. (9*b*) $1' = 0$.

(i) We have

$$a' = a' + 0 = a' + (a * x) = (a' + a) * (a' + x) = 1 * (a' + x) = a' + x$$

Also,

$$x = x + 0 = x + (a * a') = (x + a) * (x + a') = 1 * (x + a') = x + a'$$

Hence $x = x + a' = a' + x = a'$.

(ii) By definition of complement, $a + a' = 1$ and $a * a' = 0$. By commutativity, $a' + a = 1$ and $a' * a = 0$. By uniqueness of complement, a is the complement of a', that is, $a = (a')'$.

(iii) By boundedness law (6*a*), $0 + 1 = 1$, and by identity axiom (3*b*), $0 * 1 = 0$. By uniqueness of complement, 1 is the complement of 0, that is, $1 = 0'$. By duality, $0 = 1'$.

11.7. Prove Theorem 11.4 (DeMorgan's laws):

(10a) $(a+b)' = a' * b'$. (10b) $(a*b)' = a' + b'$

(10a) We need to show that $(a+b)+(a'*b') = 1$ and $(a+b)*(a'*b') = 0$; then by uniqueness of complement, $a'*b' = (a+b)'$. We have

$$(a+b)+(a'*b') = b+a+(a'*b') = b+(a+a')*(a+b')$$
$$= b+1*(a+b') = b+a+b' = b+b'+a = 1+a = 1$$

Also,

$$(a+b)*(a'*b') = ((a+b)*a')*b'$$
$$= ((a*a')+(b*a'))*b' = (0+(b*a'))*b'$$
$$= (b*a')*b' = (b*b')*a' = 0*a' = 0$$

Thus $a'*b' = (a+b)'$.

(10b) Principle of duality (Theorem 11.1).

11.8. Prove Theorem 11.5: The following are equivalent in a Boolean algebra:

(1) $a+b=b$, (2) $a*b=a$, (3) $a'+b=1$, (4) $a*b'=0$.

By Theorem 7.8, (1) and (2) are equivalent. We show that (1) and (3) are equivalent. Suppose (1) holds. Then

$$a'+b = a'+(a+b) = (a'+a)+b = 1+b = 1$$

Now suppose (3) holds. Then

$$a+b = 1*(a+b) = (a'+b)*(a+b) = (a'*a)+b = 0+b = b$$

Thus (1) and (3) are equivalent.

We next show that (3) and (4) are equivalent. Suppose (3) holds. By DeMorgan's law and involution,

$$0 = 1' - (a'+b)' = a''*b' = a*b'$$

Conversely, if (4) holds then

$$1 = 0' = (a*b')' = a'+b'' = a'+b$$

Thus (3) and (4) are equivalent. Accordingly, all four are equivalent.

11.9. Prove Theorem 11.6: The mapping $f: B \to \mathscr{P}(A)$ is an isomorphism where B is a Boolean algebra, $\mathscr{P}(A)$ is the power set of the set A of atoms, and

$$f(x) = \{a_1, a_2, \ldots, a_n\}$$

where $x = a_1 + \cdots + a_n$ is the unique representation of a as a sum of atoms.

Recall (Chapter 7) that if the a's are atoms then $a_i^2 = a_i$ but $a_i a_j = 0$ for $a_i \neq a_j$. Suppose x, y are in B and suppose

$$x = a_1 + \cdots + a_r + b_1 + \cdots + b_s$$
$$y = b_1 + \cdots + b_s + c_1 + \cdots + c_t$$

where

$$A = \{a_1, \ldots, a_r, b_1, \ldots, b_s, c_1, \ldots, c_t, d_1, \ldots, d_k\}$$

is the set of atoms of B. Then

$$x+y = a_1 + \cdots + a_r + b_1 + \cdots + b_s + c_1 + \cdots + c_t$$
$$xy = b_1 + \cdots + b_s$$

Hence

$$\begin{aligned} f(x+y) &= \{a_1,\ldots,a_r,b_1,\ldots,b_s,c_1,\ldots,c_t\} \\ &= \{a_1,\ldots,a_r,b_1,\ldots,b_s\} \cup \{b_1,\ldots,b_s,c_1,\ldots,c_t\} \\ &= f(x) \cup f(y) \\ f(xy) &= \{b_1,\ldots,b_s\} \\ &= \{a_1,\ldots,a_r,b_1,\ldots,b_s\} \cap \{b_1,\ldots,b_s,c_1,\ldots,c_t\} \\ &= f(x) \cap f(y) \end{aligned}$$

Let $y = c_1 + \cdots + c_t + d_1 + \cdots + d_k$. Then $x + y = 1$ and $xy = 0$, and so $y = x'$. Thus

$$f(x') = \{c_1,\ldots,c_t,d_1,\ldots,d_k\} = \{a_1,\ldots,a_r,b_1,\ldots,b_2\}^c = (f(x))^c$$

Since the representation is unique, f is one-to-one and onto. Hence f is a Boolean algebra isomorphism.

BOOLEAN EXPRESSIONS

11.10. Reduce the following Boolean products to either 0 or a fundamental product:

(a) $xyx'z$; (b) $xyzy$; (c) $xyz'yx$; (d) $xyz'yx'z'$.

Use the commutative law $x * y = y * x$, the complement law $x * x' = 0$, and the idempotent law $x * x = x$:

(a) $zyx'z = xx'yz = 0yz = 0$

(b) $xyzy = xyyz = xyz$

(c) $xyz'yx = xxyyz' = xyz'$

(d) $xyz'yx'z' = xx'yyz'z' = 0yz' = 0$

11.11. Express each Boolean expression $E(x, y, z)$ as a sum-of-products and then in its complete sum-of-products form:

(a) $E = x(xy' + x'y + y'z)$; (b) $E = z(x' + y) + y'$.

First use Algorithm 11.8A to express E as a sum-of-products, and then use Algorithm 11.8B to express E as a complete sum-of-products.

(a) First we have $E = xxy' + xx'y + xy'z = xy' + xy'z$. Then

$$E = xy'(z + z') + xy'z = xy'z + xy'z' + xy'z = xy'z + xy'z'$$

(b) First we have

$$E = z(x' + y) + y' = x'z + yz + y'$$

Then

$$\begin{aligned} E = x'z + yz + y' &= x'z(y + y') + yz(x + x') + y'(x + x')(z + z') \\ &= x'yz + x'y'z + xyz + x'yz + xy'z + xy'z' + x'y'z + x'y'z' \\ &= xyz + xy'z + xy'z' + x'yz + x'y'z + x'y'z' \end{aligned}$$

11.12. Express $E(x, y, z) = (x' + y)' + x'y$ in its complete sum-of-products form.

We have $E = (x' + y)' + x'y = +xy' + x'y$, which would be the complete sum-of-products form of E if E were a Boolean expression in x and y. However, it is specified that E is a Boolean expression in the three variables x, y, and z. Hence,

$$E = xy' + x'y = xy'(z + z') + x'y(z + z') = xy'z + xy'z' + x'yz + x'yz'$$

is the complete sum-of-products form of E.

11.13. Express each Boolean expression $E(x, y, z)$ as a sum-of-products and then in its complete sum-of-products form:

(*a*) $E = y(x + yz)'$; (*b*) $E = x(xy + y' + x'y)$.

(*a*) $E = y(x'(yz)') = yx'(y' + z') = yx'y' + x'yz' = x'yz'$
which already is in its complete sum-of-products form.

(*b*) First we have $E = xxy + xy' + xx'y = xy + xy'$. Then

$$E = xy(z + z') + xy'(z + z') = xyz + xyz' + xy'z + xy'z'$$

11.14. Express each set expression $E(A, B, C)$ involving sets A, B, C as a union of intersections:

(*a*) $E = (A \cup B)^c \cap (C^c \cup B)$; (*b*) $E = (B \cap C)^c \cap (A^c \cup C)^c$

Use Boolean notation, $'$ for complement, $+$ for union and $*$ (or juxtaposition) for intersection, and then express E as a sum of products (union of intersections).

(*a*) $E = (A + B)'(C' + B) = A'B'(C' + B) = A'B'C' + A'B'B = A'B'C'$ or $E = A^c \cap B^c \cap C^c$

(*b*) $E = (BC)'(A' + C)' = (B' + C')(AC') = AB'C' + AC'$ or $E = (A \cap B^c \cap C^c) \cup (A \cap C^c)$

11.15. Let $E = xy' + xyz' + x'yz'$. Prove that (*a*) $xz' + E = E$; (*b*) $x + E \neq E$, (*c*) $z' + E \neq E$.

Since the complete sum-of-products form is unique, $A + E = E$, where $A \neq 0$, if and only if the summands in the complete sum-of-products form for A are among the summands in the complete sum-of-products form for E. Hence, first find the complete sum-of-products form for E:

$$E = xy'(z + z') + xyz' + x'yz' = xy'z + xy'z' + xyz' + x'yz'$$

(*a*) Express xz' in complete sum-of-products form:

$$xz' = xz'(y + y') = xyz' + xy'z'$$

Since the summands of xz' are among those of E, we have $xz' + E = E$.

(*b*) Express x in complete sum-of-products form:

$$x = x(y + y')(z + z') = xyz + xyz' + xy'z + xy'z'$$

The summand xyz of x is not a summand of E; hence $x + E \neq E$.

(*c*) Express z' in complete sum-of-products form:

$$z' = z'(x + x')(y + y') = xyz' + xy'z' + x'yz' + x'y'z'$$

The summand $x'y'z'$ of z' is not a summand of E; hence $z' + E \neq E$.

MINIMAL BOOLEAN EXPRESSIONS, PRIME IMPLICANTS

11.16. For any Boolean sum-of-products expression E, we let E_L denote the number of literals in E (counting multiplicity) and E_S denote the number of summands in E. Find E_L and E_S for each of the following:

(*a*) $E = xy'z + x'z' + yz' + x$ (*c*) $E = xyt' + x'y'zt + xz't$

(*b*) $E = x'y'z + xyz + y + yz' + x'z$ (*d*) $E = (xy' + z)' + xy'$

Simply add up the number of literals and the number of summands in each expression:

(*a*) $E_L = 3 + 2 + 2 + 1 = 8$, $E_S = 4$.

(*b*) $E_L = 3 + 3 + 1 + 2 + 2 = 11$, $E_S = 5$.

(*c*) $E_L = 3 + 4 + 3 = 10$, $E_S = 3$.

(*d*) Because E is not written as a sum of products, E_L and E_S are not defined.

11.17. Given E and F are equivalent Boolean sum-of-products, define:

(*a*) E is simpler than F; (*b*) E is minimal.

(*a*) E is simpler than F if $E_L < F_L$ and $E_S \leq F_S$, or if $E_L \leq F_L$ and $E_S < F_S$.

(*b*) E is minimal if there is no equivalent sum-of-products expression which is simpler than E.

11.18. Find the consensus Q of the fundamental products P_1 and P_2 where:

(*a*) $P_1 = xy'z'$, $P_2 = xyt$ (*c*) $P_1 = xy'z'$, $P_2 = x'y'zt$

(*b*) $P_1 = xyz't$, $P_2 = xzt$ (*d*) $P_1 = xyz'$, $P_2 = xz't$

The consensus Q of P_1 and P_2 exists if there is exactly one variable, say x_k, which is complemented in one of P_1 and P_2 and uncomplemented in the other. Then Q is the product (without repetition) of the literals in P_1 and P_2 after x_k and x_k' have been deleted.

(*a*) Delete y' and y and then multiply the literals of P_1 and P_2 (without repetition) to obtain $Q = xz't$.

(*b*) Deleting z' and z yields $Q = xyt$.

(*c*) They have no consensus since both x and z appear complemented in one of the products and uncomplemented in the other.

(*d*) They have no consensus since no variable appears complemented in one of the products and uncomplemented in the other.

11.19. Suppose Q is the consensus of P_1 and P_2. Prove that $P_1 + P_2 + Q = P_1 + P_2$.

Since the literals commute, we can assume without loss of generality that

$$P_1 = a_1a_2\cdots a_rt, \qquad P_2 = b_1b_2\cdots b_st', \qquad Q = a_1a_2\cdots a_rb_1b_2\cdots b_s$$

Now, $Q = Q(t + t') = Qt + Qt'$. Because Qt contains P_1, $P_1 + Qt = P_1$; and because Qt' contains P_2, $P_2 + Qt' = P_2$. Hence

$$P_1 + P_2 + Q = P_1 + P_2 + Qt + Qt' = (P_1 + Qt) + (P_2 + Qt') = P_1 + P_2$$

11.20. Let $E = xy' + xyz' + x'yz'$. Find: (*a*) the prime implicants of E; (*b*) a minimal sum for E.

(*a*) Apply Algorithm 11.9A (consensus method) as follows:

$$\begin{aligned} E &= xy' + xyz' + x'yz' + xz' && \text{(Consensus of } xy' \text{ and } xyz'\text{)} \\ &= xy' + x'yz' + xz' && (xyz' \text{ includes } xz') \\ &= xy' + x'yz' + xz' + yz' && \text{(Consensus of } x'yz' \text{ and } xz'\text{)} \\ &= xy' + xz' + yz' && (x'yz' \text{ includes } yz') \end{aligned}$$

Neither step in the consensus method can now be applied. Hence xy', xz', and yz' are the prime implicants of E.

(*b*) Apply Algorithm 11.9B. Write each prime implicant of E in complete sum-of-products form obtaining:

$$\begin{aligned} xy' &= xy'(z + z') = xy'z + xy'z' \\ xz' &= xz'(y + y') = xyz' + xy'z' \\ yz' &= yz'(x + x') = xyz' + x'yz' \end{aligned}$$

Only the summands xyz' and $xy'z'$ of xz' appear among the other summands and hence xz' can be eliminated as superfluous. Thus $E = xy' + yz'$ is a minimal sum for E.

11.21. Let $E = xy + y't + x'yz' + xy'zt'$. Find: (*a*) the prime implicants of E; (*b*) a minimal sum for E.

(*a*) Apply Algorithm 11.9A (consensus method) as follows:

$$\begin{aligned}
E &= xy + y't + x'yz' + xy'zt' + xzt' && \text{(Consensus of } xy \text{ and } xy'zt'\text{)}\\
&= xy + y't + x'yz' + xzt' && (xy'zt' \text{ includes } xzt')\\
&= xy + y't + x'yz' + xzt' + yz' && \text{(Consensus of } xy \text{ and } x'yz'\text{)}\\
&= xy + y't + xzt' + yz' && (x'yz' \text{ includes } yz')\\
&= xy + y't + xzt' + yz' + xt && \text{(Consensus of } xy \text{ and } y't\text{)}\\
&= xy + y't + xzt' + yz' + xt + xz && \text{(Consensus of } xzt' \text{ and } xt\text{)}\\
&= xy + y't + yz' + xt + xz && (xzt' \text{ includes } xz)\\
&= xy + y't + yz' + xt + xz + z't && \text{(Consensus of } y't \text{ and } yz'\text{)}
\end{aligned}$$

Neither step in the consensus method can now be applied. Hence the prime implicants of E are $xy, y't, yz', xt, xz$, and $z't$.

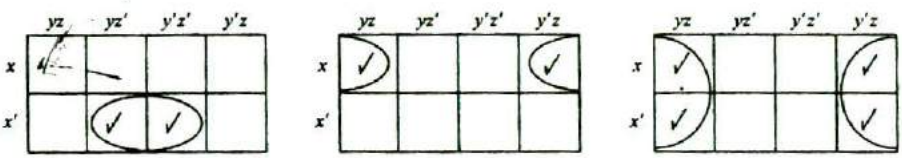
(*b*) Apply Algorithm 11.9B. Write each prime implicant in complete sum-of-products form and then delete one by one those which are superfluous, i.e. those whose summands appear among the other summands. This finally yields

$$E = y't + xz + yz'$$

as a minimal sum for E.

KARNAUGH MAPS

11.22. Find the fundamental product P represented by each basic rectangle in the Karnaugh map in Fig. 11-14.



(*a*) (*b*) (*c*)

Fig. 11-14

In each case find those literals which appear in all the squares of the basic rectangle; then P is the product of such literals.

(*a*) x' and z' appear in both squares; hence $P = x'z'$.

(*b*) x and z appear in both squares; hence $P = xz$.

(*c*) Only z appears in all four squares; hence $P = z$.

11.23. Let R be a basic rectangle in a Karnaugh map for four variables x, y, z, t. State the number of literals in the fundamental product P corresponding to R in terms of the number of squares in R.

P will have 1, 2, 3, or 4 literals according as R has 8, 4, 2, or 1 squares.

11.24. Find the fundamental product P represented by each basic rectangle R in the Karnaugh map in Fig. 11-15.

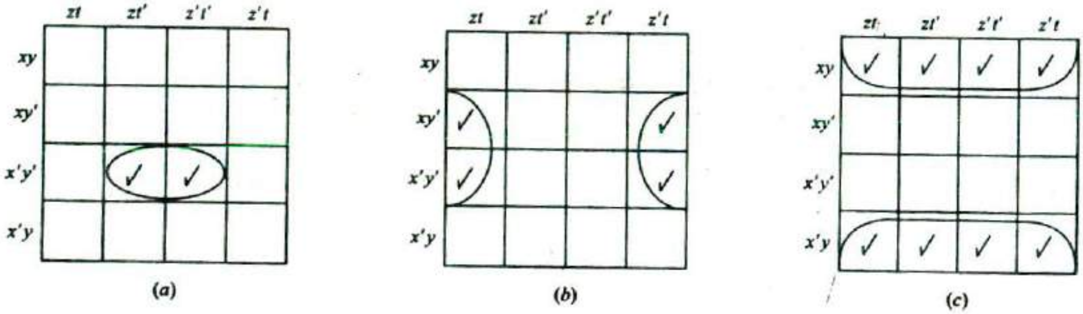


Fig. 11-15

In each case find those literals which appear in all the squares of the basic rectangle; then P is the product of such literals. (Problem 11.23 indicates the number of such literals in P.)

(*a*) There are two squares in R, so P has three literals. Specifically, x', y', t' appear in both squares; hence $P = x'y't'$.

(*b*) There are four squares in R, so P has two literals. Specifically, only y' and t appear in all four squares; hence $P = y't$.

(*c*) There are eight squares in R, so P has only one literal. Specifically, only y appears in all eight squares; hence $P = y$.

11.25. Let E be the Boolean expression given in the Karnaugh map in Fig. 11-16.

(*a*) Write E in its complete sum-of-products form. (*b*) Find a minimal form for E.

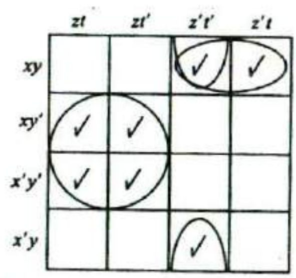


Fig. 11-16

(*a*) List the seven fundamental products checked to obtain

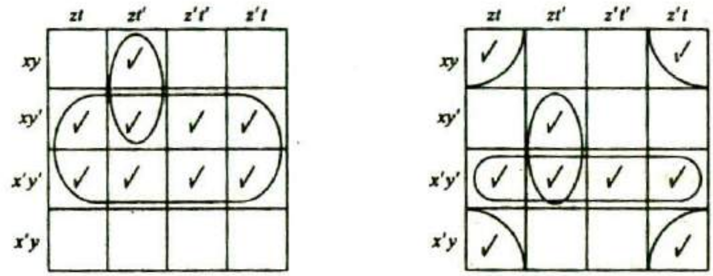
$$E = xyz't' + xyz't + xy'zt + xy'zt' + x'y'zt + x'y'zt' + x'yz't'$$

(*b*) The two-by-two maximal basic rectangle represents $y'z$ since only y' and z appear in all four squares. The horizontal pair of adjacent squares represents xyz', and the adjacent squares overlapping the top and bottom edges represent $yz't'$. As all three rectangles are needed for a minimal cover,

$$E = y'z + xyz' + yz't'$$

is the minimal sum for E.

11.26. Consider the Boolean expressions E_1 and E_2 in variables x, y, z, t which are given by the Karnaugh maps in Fig. 11-17. Find a minimal sum for (*a*) E_1; (*b*) E_2.



(*a*) E_1 (*b*) E_2

Fig. 11-17

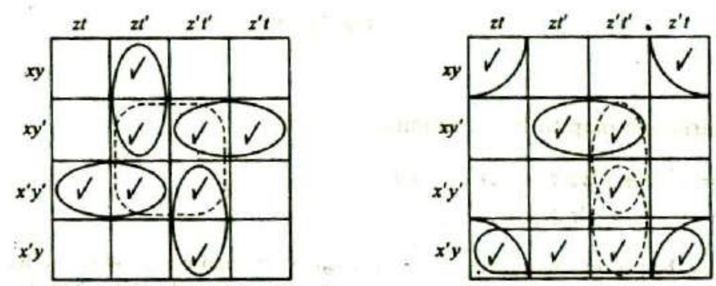
(*a*) Only y' appears in all eight squares of the two-by-four maximal basic rectangle, and the designated pair of adjacent squares represents xzt'. As both rectangles are needed for a minimal cover, thus the following is the minimal sum for E_1:

$$E_1 = y' + xzt'$$

(*b*) The four corner squares form a two-by-two maximal basic rectangle which represents yt, since only y and t appear in all the four squares. The four-by-one maximal basic rectangle represents $x'y'$, and the two adjacent squares represent $y'zt'$. As all three rectangles are needed for a minimal cover, hence the following is the minimal sum for E_2:

$$E_2 = yt + x'y' + y'zt'$$

11.27. Consider the Boolean expressions E_1 and E_2 in variables x, y, z, t which are given by the Karnaugh maps in Fig. 11-18. Find a minimal sum for: (*a*) E_1; (*b*) E_2.



(*a*) E_1 (*b*) E_2

Fig. 11-18

(*a*) There are five prime implicants, designated by the four loops and the dashed circle. However, the dashed circle is not needed to cover all the squares, whereas the four loops are required. Thus the four loops give the minimal sum for E_1; that is,

$$E_1 = xzt' + xy'z' + x'y'z + x'z't'$$

(*b*) There are five prime implicants, designated by the five loops of which two are dashed. Only one of the two dashed loops is needed to cover the square $x'y'z't'$. Thus there are two minimal sums for E_2 as follows:

$$E_2 = x'y + yt + xy't' + y'z't' = x'y + yt + xy't' + x'z't'$$

11.28. Use a Karnaugh map to find a minimal sum for:

(*a*) $E_1 = x'y'z' + x'yz' + xy'z + xyz'$.

(*b*) $E_2 = x'yz' + x'yz + xy'z + xyz' + xyz$.

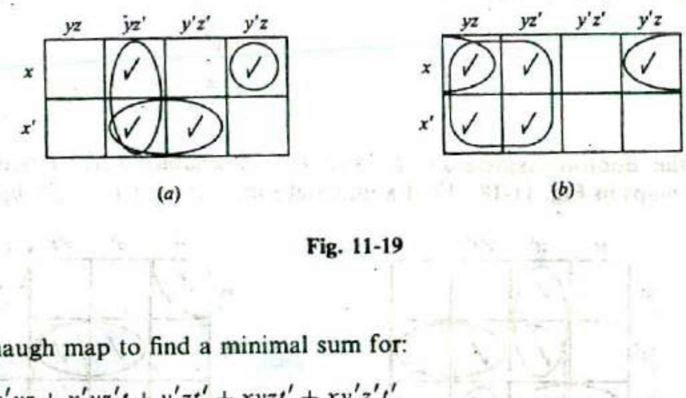
Each term in E_1 and E_2 contains the three variables x, y, z, and hence it corresponds to a square in the Karnaugh map (with three variables).

(*a*) Checking the appropriate squares gives the Karnaugh map in Fig. 11-19(*a*). There are three prime implicants, as indicated by the three loops, which form a minimal cover of E_1. Thus a minimal form for E_1 follows:

$$E_1 = yz' + x'z' + xy'z$$

(*b*) The Karnaugh map appears in Fig. 11-19(*b*). There are two prime implicants, as indicated by the two loops, which form a minimal cover of E_2. Thus a minimal form for E_2 follows:

$$E_2 = xz + y$$



(*a*) (*b*)

Fig. 11-19

11.29. Use a Karnaugh map to find a minimal sum for:

(*a*) $E_1 = x'yz + x'yz't + y'zt' + xyzt' + xy'z't'$.

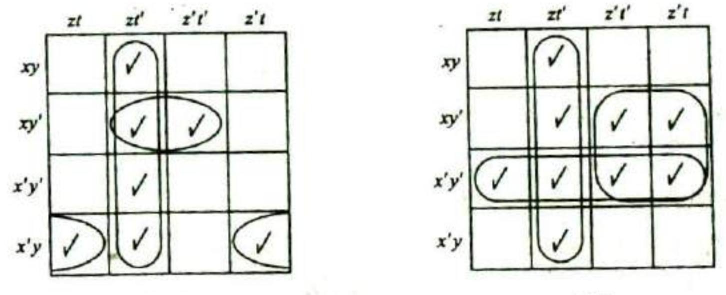
(*b*) $E_2 = y't' + y'z't + x'y'zt + yzt'$.

(*a*) Check the two squares corresponding to each of $x'yz$ and $y'zt'$, and check the square corresponding to each of $x'yz't$, $xyzt'$, and $xy'z't'$. This gives the Karnaugh map in Fig. 11-20(*a*). A minimal cover consists of the three designated loops. Thus a minimal sum for E_1 follows:

$$E_1 = zt' + xy't' + x'yt$$

(*b*) Check the four squares corresponding to zt', check the two squares corresponding to each of $y'z't$ and yzt', and check the square corresponding to $x'y'zt$. This gives the Karnaugh map in Fig. 11-20(*b*). A minimal cover consists of the three designated maximal basic rectangles. Thus a minimal sum for E_2 follows:

$$E_2 = zt' + xy't' + x'yt$$



(a) E_1 (b) E_2

Fig. 11-20

Supplementary Problems

BOOLEAN ALGEBRAS

11.30. Write the dual of each Boolean expression:

(a) $a(a' + b) = ab$; (b) $(a + 1)(a + 0) = a$; (c) $(a + b)(b + c) = ac + b$.

11.31. Consider the lattices $\mathbf{D}_m$ of divisors of m (where $m > 1$).

(a) Show that $\mathbf{D}_m$ is a Boolean algebra if and only if m is *square-free*, that is, m is a product of distinct primes.

(b) If $\mathbf{D}_m$ is a Boolean algebra, show that the atoms are the distinct prime divisors of m.

11.32. Consider the following lattices: (a) $\mathbf{D}_{20}$; (b) $\mathbf{D}_{55}$; (c) $\mathbf{D}_{99}$; (d) $\mathbf{D}_{130}$. Which of them are Boolean algebras, and what are their atoms?

11.33. Consider the Boolean algebra $\mathbf{D}_{110}$.

(a) List its elements and draw its diagram. (b) Find all its subalgebras.

(c) Find the number of sublattices with four elements. (d) Find the set A of atoms of $\mathbf{D}_{110}$.

(e) Give the isomorphic mapping $f: \mathbf{D}_{110} \to \mathscr{P}(A)$ as defined in Theorem 11.6.

11.34. Let B be a Boolean algebra. Show that: (a) For any x in B, $0 \le x \le 1$. (b) $a < b$ if and only if $b' < a'$.

11.35. An element x in a Boolean algebra is called a *maxterm* if the identity 1 is its only successor. Find the maxterms in the Boolean algebra $\mathbf{D}_{210}$ pictured in Fig. 11-13.

11.36. Let B be a Boolean algebra. (a) Show that complements of the atoms of B are the maxterms. (b) Show that any element x in B can be expressed uniquely as a product of maxterms.

11.37. Let B be a 16-element Boolean algebra and let S be an 8-element subalgebra of B. Show that two of the atoms of S must be atoms of B.

11.38. Let $B = (B, +, *, ', 0, 1)$ be a Boolean algebra. Define an operation $\triangle$ on B (called the symmetric difference) by

$$x \triangle y = (x * y') + (x' * y)$$

Prove that $R = (B, \triangle, *)$ is a commutative Boolean ring.

11.39. Let $R = (R, +, \cdot)$ be a Boolean ring with identity $1 \neq 0$. Define

$$x' = 1 + x, \quad x + y = x + y + x \cdot y, \quad x * y = x \cdot y$$

Prove that $B = (R, +, *, ', 0, 1)$ is a Boolean algebra.

BOOLEAN EXPRESSIONS, PRIME IMPLICANTS

11.40. Reduce the following Boolean products to either 0 or a fundamental product:

(*a*) $xy'zxy'$; (*b*) $xyz'sy'ts$; (*c*) $xy'xz'ty'$; (*d*) $xyz'ty't$

11.41. Express each Boolean expression $E(x, y, z)$ as a sum-of-products and then in its complete sum-of-products form:

(*a*) $E = x(xy' + x'y + y'z)$. (*b*) $E = (x + y'z)(y + z')$. (*c*) $E = (x' + y)' + y'z$.

11.42. Express each Boolean expression $E(x, y, z)$ as a sum-of-products and then in its complete sum-of-products form:

(*a*) $E = (x'y)'(x' + xyz')$. (*b*) $(x + y)'(xy')'$. (*c*) $E = y(x + yz)'$.

11.43. Find the consensus Q of the fundamental products P_1 and P_2 where:

(*a*) $P_1 = xy'z$, $P_2 = xyt$ (*c*) $P_1 = xy'zt$, $P_2 = xyz'$
(*b*) $P_1 = xyz't'$, $P_2 = xzt'$ (*d*) $P_1 = xy't$, $P_2 = xzt$

11.44. For any Boolean sum-of-products expression E, we let E_L denote the number of literals in E (counting multiplicity) and E_S denote the number of summands in E. Find E_L and E_S for each of the following:

(*a*) $E = xyz't + x'yt + xy'zt$. (*b*) $E = xyzt + xt' + x'y't + yt$

11.45. Apply the consensus method (Algorithm 11.9A) to find the prime implicants of each Boolean expression:

(*a*) $E_1 = xy'z' + x'y + x'y'z' + x'yz$.
(*b*) $E_2 = xy' + x'z't + xyzt' + x'y'zt'$.
(*c*) $E_3 = xyzt + xyz't' + xz't' + x'y'z' + x'yz't$.

11.46. Find a minimal sum-of-products form for each of the Boolean expressions in Problem 11.45.

KARNAUGH MAPS

11.47. Find all possible minimal sums for each Boolean expression E given by the Karnaugh maps in Fig. 11-21.

	yz	yz'	$y'z'$	$y'z$
x	✓		✓	✓
x'	✓	✓		

(*a*)

	yz	yz'	$y'z'$	$y'z$
x	✓		✓	✓
x'	✓	✓		✓

(*b*)

	yz	yz'	$y'z'$	$y'z$
x	✓			✓
x'	✓	✓	✓	✓

(*c*)

Fig. 11-21

11.48. Find all possible minimal sums for each Boolean expression E given by the Karnaugh maps in Fig. 11-22.

	zt	zt'	$z't'$	$z't$
xy		✓		✓
xy'	✓	✓		✓
$x'y'$		✓		
$x'y$	✓	✓	✓	✓

(*a*)

	zt	zt'	$z't'$	$z't$
xy	✓	✓	✓	
xy'		✓	✓	✓
$x'y'$		✓		
$x'y$	✓	✓	✓	

(*b*)

	zt	zt'	$z't'$	$z't$
xy	✓			✓
xy'		✓	✓	
$x'y'$		✓		
$x'y$	✓	✓	✓	✓

(*c*)

Fig. 11-22

11.49. Use a Karnaugh map to find a minimal sum for each Boolean expression:

(*a*) $E = xy + x'y + x'y'$. (*b*) $E = x + x'yz + xy'z'$.

11.50. Use a Karnaugh map to find a minimal sum for each Boolean expression:

(*a*) $E = y'z + y'z't' + z't$. (*b*) $E = y'zt + xzt' + xy'z'$.

11.51. Show that the sum of two adjacent products will be a fundamental product with one fewer literal.

Answers to Supplementary Problems

11.30. (*a*) $a + a'b = a + b$; (*b*) $a \cdot 0 + a \cdot 1 = a$; (*c*) $ab + bc = (a + c)b$

11.32. (*b*) $\mathbf{D}_{55}$; atoms 5 and 11; (*d*) $\mathbf{D}_{130}$; atoms 2, 5 and 13

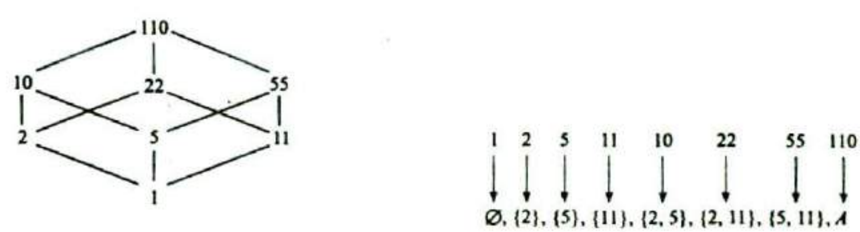
11.33. (*a*) There are eight elements 1, 2, 5, 10, 11, 22, 55, 110. See Fig. 11-23(*a*).

(*b*) There are five subalgebras: $\{1, 110\}$, $\{1, 2, 55, 110\}$, $\{1, 5, 22, 110\}$, $\{1, 10, 11, 110\}$, $\mathbf{D}_{110}$.

(*c*) There are fifteen sublattices which include the above four three subalgebras.

(*d*) $A = \{2, 5, 11\}$

(*e*) See Fig. 11-23(*b*).



(*a*) $\mathbf{D}_{110}$ (*b*) $f: \mathbf{D}_{110} \longrightarrow P(A)$

Fig. 11-23

11.35. Maxterms: 30, 42, 70, 105

11.36. (*b*) *Hint*: Use duality.

11.40. (*a*) $xy'z$; (*b*) 0; (*c*) $xy'z't$; (*d*) 0

11.41. (*a*) $E = xy' + xy'z = xy'z' + xy'z$
(*b*) $E = xy + xz' = xyz + xyz' + xy'z'$
(*c*) $E = xy' + y'z = xy'z + xy'z' + x'y'z$

11.42. (*a*) $E = xyz' + x'y' = xyz' + x'y'z + x'y'z'$
(*b*) $E = x'y' = x'y'z + x'y'z'$
(*c*) $E = x'yz'$

11.43. (*a*) $Q = xzt$; (*b*) $Q = xyt'$; (*c*) and (*d*) Does not exist.

11.44. (*a*) $E_L = 11$, $E_S = 3$; (*b*) $E_L = 11$, $E_S = 4$

11.45. (*a*) $x'y$, $x'z'$, $y'z'$; (*b*) xy', xzt', $y'zt'$, $x'z't$, $y'z't$; (*c*) $xyzt$, $xz't'$, $y'z't'$, $x'y'z'$, $x'z't$

11.46. (*a*) $E = x'y + x'z'$
(*b*) $E = xy' + xzt' + x'z't + y'z't$
(*c*) $E = xyzt + xz't' + x'y'z' + x'z't$

11.47. (*a*) $E = xy' + x'y + yz = xy' + x'y + xz'$
(*b*) $E = xy' + x'y + z$
(*c*) $E = x' + z$

11.48. (*a*) $E = x'y + zt' + xz't + xy'z = x'y + zt' + xz't + xy't$
(*b*) $E = yz + yt' + zt' + xy'z'$
(*c*) $E = x'y + yt + xy't' + x'zt = x'y + yt + xy't' + y'zt$

11.49. (*a*) $E = x' + y$; (*b*) $E = xz' + yz$

11.50. (*a*) $E = y' + z't$; (*b*) $E = xy' + zt' + y'zt$