CHAPTER I

ELECTROSTATIC AND ELECTROMAGNETIC THEORY

ELECTROSTATICS

Coulomb's Law. The earliest recorded facts in connection with the subject of electricity were obtained as a result of experiments carried out by the ancient Greek philosopher Thales of Miletus, about 600 B.C., and related to the forces of repulsion and attraction between bodies charged with static electricity. Those facts were qualitative only, and it was left for Coulomb, many centuries later, to state them in a quantitative form by his Inverse Square Law, which is the most fundamental law of electrostatics—

$$F \propto \frac{Q_1 Q_2}{\varepsilon r^2} \qquad . \quad . \quad . \quad . \quad (1.1)$$

where F is the force between two small bodies charged respectively with Q_1 and Q_2 units of electricity, their centres being a distance r apart, and ε is a constant depending upon the medium in which the bodies are situated, and is called the "permittivity" of the medium.

In the rationalized M.K.S. system of units this expression is written as

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon r^2} \text{ newtons} . \qquad . \quad . \quad (1.2)$$

where Q_1 and Q_2 are the charges in coulombs, r is in metres, and $\varepsilon = \varepsilon_0 \varepsilon_r$ where ε_0 is the primary electric constant, having a value $\frac{1}{36\pi \times 10^9}$, and ε_r is the permittivity of the medium relative to that of a vacuum. $\varepsilon_r = 1$ for a vacuum, and air may be considered to have the same value. In this system two infinitely small bodies each having unit charge and being 1 metre apart in air experience a force of 9×10^9 newtons.

Electric Field Round Charged Conductors. If unit positive charge of electricity be placed in the neighbourhood of a charged body it will experience a force of attraction or repulsion according as the charged body is negatively or positively charged. If this unit charge be allowed to move freely, it will trace out a "line of electric force." For all points on this line the resultant force on the unit charge will be in a direction tangential to the line at the given point. The electric field in the neighbourhood of any charged conductor or system of charged conductors can be represented by such lines of force, arrow heads placed upon them giving the direction in which unit positive charge would move along the line.

The magnitude of the force upon unit positive charge, placed at any point, is a measure of the "electric force" or "field strength" at that point, it being assumed that the introduction of the unit charge does not affect the distribution of charge upon the conductors to which the field is due.

The Electric Field. In the preceding paragraph lines of force are spoken of as giving the direction of the field at any point. If we have two adjacent charges of opposite polarity and there are no

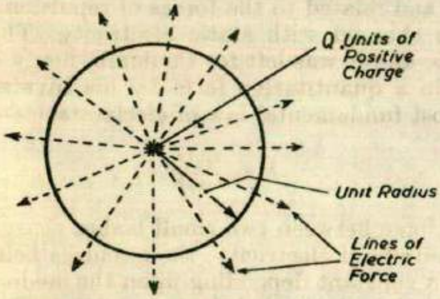

FIG. 1.1. LINES OF ELECTRIC FORCE ROUND A POSITIVE CHARGE

other charges in the vicinity, we can visualize lines of force emanating from one charge and terminating on the other. These lines of force can be looked upon as representing an electric flux between the charges. The unit of electric flux is now defined as being the total flux originating from a unit charge. It follows that

$$\psi = Q \text{ coulombs}$$

where ψ is the electric flux radiating from a charge of Q coulombs.

If a charge Q is situated at the centre of a sphere of radius 1 metre then the electric flux density D at the surface of the sphere is given by

$$D = \frac{Q}{4\pi} \text{ coulombs per sq. metre}$$

The electric field strength E at any point is defined as the force on a unit charge situated at that point and it follows from Equation (1.2) that E, at the surface of a sphere of unit radius, is

$$E = \frac{Q}{4\pi\varepsilon} = \frac{D}{\varepsilon} \quad . \quad . \quad . \quad . \quad (1.3)^*$$

In air, $\varepsilon_r = 1$, so that $\varepsilon = \varepsilon_0$ and $E = \dfrac{D}{\varepsilon_0}$.

* Note that this law is similar to the magnetic law—
$H = B/\mu$ where B = magnetic flux density,
H = magnetic field strength,
μ = magnetic permeability of the medium.

E is also termed the electric force and, since E is the force in newtons on a charge of 1 coulomb, it will be shown subsequently that the unit of E is 1 volt per metre.

Tubes of Flux. Fig. 1.2 represents a number of lines of electric

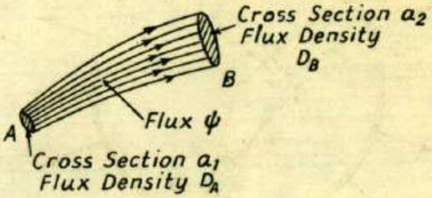

FIG. 1.2. TUBE OF FLUX

field strength forming a "tube of flux." If A and B are two points in an electric field such that the field strength at A is greater than that at B, then, from Equation (1.3), the field strength at A is

$$E_A = \frac{D_A}{\varepsilon_0}$$

where D_A = lines per unit surface of cross-section of the tube at A.

Similarly, at B

$$E_B = \frac{D_B}{\varepsilon_0}$$

If ψ is the electric flux in the tube, and a_1 and a_2 are the areas of cross-section of the tube at A and B, these areas being measured perpendicular to the direction of the field at the points, then

$$E_A = \frac{D_A}{\varepsilon_0} = \frac{\psi}{a_1\varepsilon_0}$$

$$E_B = \frac{D_B}{\varepsilon_0} = \frac{\psi}{a_2\varepsilon_0}$$

Electric Field Inside a Charged Spherical Conductor. Imagine a hollow sphere of conducting material which has been given a charge of Q positive units of electricity. If its area of surface be S the density of charge on the surface (which will be uniform) is $\frac{Q}{S}$ per unit of surface. The electric field at the surface will be at all points normal to the surface, since the sphere is of conducting material. This follows from a consideration of the fact that, if it were not so, the field would have a tangential component which would produce a movement of charge until the direction of the field became normal. Consider a point P inside the sphere (Fig. 1.3)

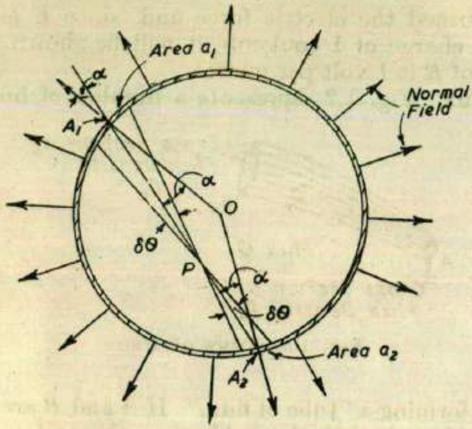

FIG. 1.3. FIELD INSIDE A SPHERICAL CONDUCTOR

at which small areas of surface a_1 and a_2 subtend a solid angle $\delta\theta$ as shown. Points A_1 and A_2 are mid-points of the areas a_1 and a_2. Angle OA_1P = angle $OA_2P = a$.

Let $A_1P = d_1$, $A_2P = d_2$

ε = permittivity of the medium inside the sphere.

Then Charge on area $a_1 = \frac{Q}{S} \cdot a_1$

" " " $a_2 = \frac{Q}{S} \cdot a_2$

Since the field is everywhere normal to the surface, the field strength at P due to charge on a_1 is

$$\frac{Qa_1}{S} \frac{\cos a}{4\pi\varepsilon\, d_1^2} \text{ in direction } A_1P$$

Similarly, the field strength at P due to charge on a_2 is

$$\frac{Qa_2}{S} \cdot \frac{\cos a}{4\pi\varepsilon\, d_2^2} \text{ in direction } A_2P$$

directly opposite to direction A_1P.

Now, the solid angle subtended at the centre of a sphere of radius R by any area A on its surface is $\frac{A}{R^2}$.

Hence,

$$\text{Solid angle } \delta\theta = \frac{a_1 \cos \alpha}{d_1^2} = \frac{a_2 \cos \alpha}{d_2^2}$$

Thus, the field strengths at P due to charges on a_1 and a_2 are opposite and are each equal to $\frac{Q}{4\pi\varepsilon S} \cdot \delta\theta$, giving a resultant field strength due to these two charges of zero.

As the same is true for all similar pairs of areas such as a_1 and a_2, the *total field strength at any point inside a charged spherical conductor is zero.*

Field in the Neighbourhood of a Charged Straight Conductor. Fig. 1.4 (a) represents a long, thin, straight conductor which carries a

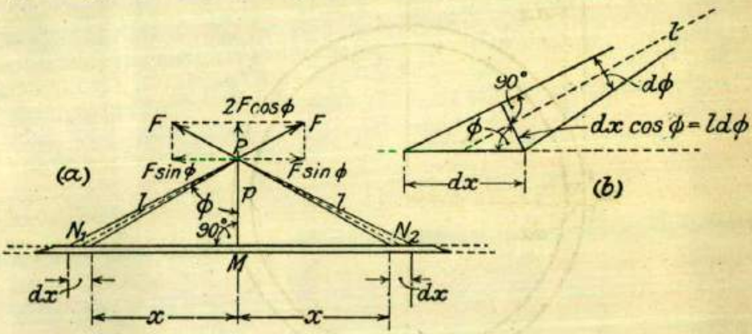

Fig. 1.4. Electrostatic Field Near a Straight Conductor

uniform charge of Q per unit length. P is a point whose perpendicular distance from the conductor is p, and p is small compared with the length of the wire. Consider two elements of the conductor each of length dx, as shown at N_1 and N_2, the elements being equidistant from P.

Let $$N_1P = N_2P = l$$

Then, if the elements dx are so small that the charges on them can be considered as concentrated at N_1 and N_2, the forces (F) upon unit positive charge placed at P will be each equal to $\frac{Qdx}{4\pi\varepsilon l^2}$, from Equation (1.2), where ε is the permittivity of the medium.

The directions of these forces will, as shown, each make an angle of $(90 - \phi)$ with the direction of the conductor, and will together be equivalent to one force of $2F \cos \phi$ in direction MP, the horizontal components neutralizing one another.

The same applies to all such pairs of elements as those shown, so that the total force upon unit charge at P—i.e. the field strength at

P—due to the whole length of the wire, will be in the direction MP, and is given by

$$E_p = \int_{x=0}^{x=\infty} \frac{2Q \cos \phi}{4\pi\varepsilon l^2}\, dx$$

where E_p = total field strength at P, if the distance p is small compared with the length of the wire.

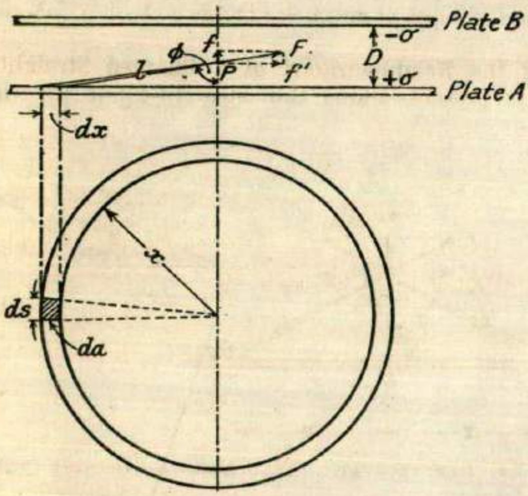

FIG. 1.5. ELECTROSTATIC FIELD BETWEEN TWO CHARGED PLATES

From Fig. 1.4 (b), it can be seen that, if dx is very small, then

$$ld\phi = dx \cos \phi = dx \cdot \frac{p}{l}$$

$\therefore \; \frac{dx}{l^2} = \frac{d\phi}{p}$ where $d\phi$ is the angle subtended at P by dx

$$\therefore E_p = \int_{\phi=0}^{\phi=+\frac{\pi}{2}} \frac{2Q\cos\phi}{4\pi\varepsilon p}\, d\phi = \frac{2Q}{4\pi\varepsilon p} \text{ in the direction } MP \qquad (1.4)$$

Field in the Space Between Two Charged Parallel Conducting Plates. Fig. 1.5 represents the two conducting plates, which are close together. Their extent is supposed to be so great as compared with their distance apart that the electrostatic field on or near their common axis is unaffected by the fringing field at the edges of the plates. Let plate A be charged positively and B charged negatively. Neglecting the edge effects the distribution of charge will be uniform.

Let charge density on $A = +\sigma$ units per unit of surface
" " $B = -\sigma$ units per unit of surface

P is a point between the plates on or near their common axis. Let their distance apart be D.

A similar method to that set out in the preceding paragraph can be followed, except that the field strengths at P due to *elemental rings* must now be considered instead of elements of length of conductor as previously considered.

Since the charge on area da is $+\sigma da = +\sigma ds . dx$, the force ($F$) upon a unit positive charge at P due to this area is $+\dfrac{\sigma . ds . dx}{4\pi\varepsilon . l^2}$, ε being the permittivity of the medium between the plates. This force F may be split up into two components, f and f', perpendicular to and parallel with the plates as shown. Since the components of all such forces as F due to the whole of the elemental ring in a direction parallel to the plates will neutralize one another, the total force at P due to the whole of the elemental ring will be the sum of all components such as f perpendicular to the plates. Calling this total perpendicular force due to the ring f_r, we have

$$f_r = \sum_{s=0}^{s=2\pi x} f$$

$$\text{Now } f = F\cos\phi = \frac{\sigma . ds . dx}{4\pi\varepsilon . l^2}\cos\phi$$

$$\therefore\ f_r = \int_{s=0}^{s=2\pi x} \frac{\sigma . dx}{4\pi\varepsilon . l^2}\cos\phi . ds = \frac{\sigma . dx}{4\pi\varepsilon . l^2}\cos\phi \times 2\pi x$$

If P = total force at P in a direction perpendicular to the plates due to all such elemental rings, then

$$P = \int_{x=0}^{x=\infty} \frac{\sigma\cos\phi}{4\pi\varepsilon . l^2} \times 2\pi x . dx$$

As in the previous section,

$$ld\phi = dx . \cos\phi$$

i.e.
$$\cos\phi = \frac{ld\phi}{dx}$$

$$\therefore\ P = \int_{x=0}^{x=\infty} \frac{\sigma l}{4\pi\varepsilon l^2} 2\pi x\, dx . \frac{d\phi}{dx}$$

$$= \int_{\phi=0}^{\phi=\frac{\pi}{2}} \frac{\sigma 2\pi}{4\pi\varepsilon l} x \, . \, d\phi$$

$$= \int_{\phi=0}^{\phi=\frac{\pi}{2}} \frac{\sigma \, . \, 2\pi}{4\pi\varepsilon} \sin\phi \, . \, d\phi$$

$$= \frac{2\pi\sigma}{4\pi\varepsilon} = \frac{\sigma}{2\varepsilon}$$

This force will be one of repulsion if unit positive charge is placed at P. There will also be an equal force attracting the unit charge to

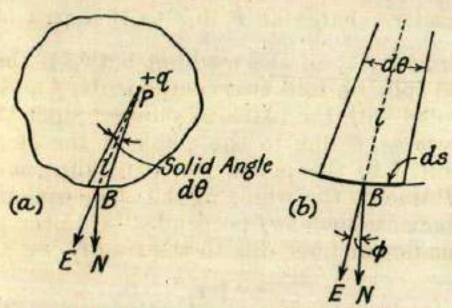

FIG. 1.6. ILLUSTRATING GAUSS'S THEOREM

plate B. Thus the total force on unit positive charge at P—i.e. the field strength at P—is

$$E_p = \frac{\sigma}{\varepsilon} \qquad . \quad . \quad . \quad . \quad (1.5)$$

Gauss's Theorem. Briefly this theorem states that the total electric flux traversing a surface which completely encloses a charge of Q units is Q. This holds true whatever the shape of the surrounding surface, and for any dielectric. Consider a small element of surface ds upon the surface surrounding a charge of $+Q$ (Fig. 1.6 (*a*)). Let this element subtend a solid angle $d\theta$ at P, and let the angle between field strength E at B, due to the charge, and the normal N at B be ϕ. Let the distance $PB = l$.

Then, electric flux crossing element ds is $D \cos\phi \, . \, ds$

$$= E\varepsilon \, . \cos\phi \, . \, ds$$

$$= \frac{Q}{4\pi\varepsilon l^2} \, . \, \varepsilon \cos\phi \, . \, ds$$

$$= \frac{Q}{4\pi l^2} \cos\phi \, . \, ds$$

Thus the total flux crossing the whole surface is

$$\psi = \Sigma \frac{Q}{4\pi l^2} \cos \phi \, . \, ds$$

or, since $\frac{ds \, . \cos \phi}{l^2}$ = solid angle $d\theta$,

$$\psi = \frac{\Sigma Q d\theta}{4\pi} = Q \quad . \quad . \quad . \quad . \quad . \quad (1.6)$$

If there are a number of charges inside the surface, some positive and some negative, then, if the charges are Q_1, Q_2, etc.,

$$\psi = (Q_1 \pm Q_2 \pm Q_3 \pm \ldots) \quad . \quad . \quad . \quad (1.7)$$

the flux in the outward direction being considered positive.

Coulomb's Theorem. This theorem states, in effect, that *the electric field strength at the surface of a conductor, charged to a surface density of σ units per unit of surface, is $\frac{\sigma}{\varepsilon}$*, where ε is the permittivity of the medium outside the conductor.

This follows from Gauss's Theorem. Consider an element of surface of the conductor ds. This element carries a charge of σds units. From Gauss's Theorem the flux radiating from this charge is $\sigma \, . \, ds$, and, since no flux exists inside the conductor, the whole of this flux passes outwards normally.

Thus, electric flux density at the surface $D = \frac{\sigma \, . \, ds}{ds} = \sigma$.

Hence the field strength is

$$E = \frac{D}{\varepsilon} = \frac{\sigma}{\varepsilon} \quad . \quad . \quad . \quad . \quad (1.8)$$

its direction being normal to the surface.

Potential. If unit positive charge is moved towards a positively charged body, work is done in overcoming the force of repulsion acting on the charge. If this movement of the unit charge is from a point P_1 to some point P_2 nearer to the positively charged body, then the point P_2 is said to be at a higher electric potential than point P_1 and the difference of potential between the two points is defined as the quantity of work required to move unit positive charge from the point at the lower potential to the point at the higher potential.

In general, the potential of any point in the vicinity of a system of charged bodies is defined as the work required to move unit positive charge from an infinite distance to the point considered, assuming that the distribution of the charges on the bodies is unaffected by the approach of the unit charge.

The *unit of potential difference* is the volt and is defined as the potential difference between two points such that 1 joule of work is done in moving unit positive charge (as defined on p. 1) from the point at the lower potential to the point at the higher potential, the potential being assumed unaltered by the presence of the unit charge.

If two points at a very small distance ds apart have a difference of potential dV units, then the work done in moving unit positive charge from one point to the other up the gradient of potential is Eds, where E is the average force on the unit charge during the movement.

Thus $$dV = -Eds$$

or $$E = -\frac{dV}{ds} \quad . \quad . \quad . \quad (1.9)$$

where E is the field strength at any point in an electric field, $\frac{dV}{ds}$ being the potential gradient at the point, the positive direction of s being *down* the gradient of potential. Again, the potential difference V between any two points A and B is given by

$$V_{AB} = \int_A^B E \,.\, ds \quad . \quad . \quad . \quad (1.10)$$

Potential at a Point Due to a Number of Charges. The potential at any point P distant d from a single charge of Q units is equal to the work done in bringing unit charge from an infinite distance up to the point P, i.e.

$$\text{Potential at } P, \quad V_P = \int_d^\infty E \,.\, ds = \int_d^\infty \frac{Q}{4\pi\varepsilon s^2}\, ds$$

$$\therefore V_P = \frac{Q}{4\pi\varepsilon d} \text{ volts} \quad . \quad . \quad . \quad . \quad (1.11)$$

Similarly, the potential at a point P due to a number of charges Q_1, Q_2, etc., distant d_1, d_2, etc., respectively, from P is given by

$$V_P = \left[\frac{Q_1}{d_1} + \frac{Q_2}{d_2} + \frac{Q_3}{d_3} + \text{etc.}\right]\frac{1}{4\pi\varepsilon} \quad . \quad . \quad (1.12)$$

ε being the permittivity of the medium.

If a number of conductors have charges $Q_1, Q_2 \ldots Q_n$, then their potentials $V_1, V_2, \ldots V_n$ will be given by the expressions

$$V_1 = P_{11}Q_1 + P_{21}Q_2 + \ldots P_{n1}Q_n, \text{ etc.}$$

If all but (say) the second conductor are uncharged and it has unit charge, then by putting $Q_1 = Q_3 \ldots Q = Q_4 = {}_n = 0$, and $Q_2 = 1$, we find that P_{21}, for example, is the potential of the first body produced by unit charge on the second. The value of P_{21} is evidently obtained by considering all the contributions $\frac{\delta Q}{4\pi\varepsilon d}$ which the elementary charges on the second conductor produce at the first. The coefficients P are called the "potential coefficients" of the conductors and depend only on the geometry of the system.

Green's Reciprocity Theorem enables us to assert that $P_{12} = P_{21}$, etc., so that in general $P_{mn} = P_{nm}$. Thus, the potential produced at conductor n by unit charge on conductor m (all other charges being zero) is equal to the potential of m produced by unit charge on n.

Equipotential Surfaces. An equipotential surface is a surface such that all points on it are at the same potential. Obviously the potential gradient $\frac{dV}{ds}$ for such a surface is zero, and from Equation (1.9) it follows that the field strength along such a surface is also zero. Thus the lines of electric field strength of the field in which the equipotential surface is situated have no component along the surface, i.e. they cut such a surface at right angles.

Capacitance. Consider, again, a number of conductors having charges Q_1, Q_2, Q_3, etc., and potentials V_1, V_2, V_3, etc. If the series of equations

$$V_1 = P_{11}Q_1 + P_{21}Q_2 + \dots + P_{n1}Q_n$$
$$V_2 = P_{12}Q_1 + P_{22}Q_2 + \dots + P_{n2}Q_n, \text{ etc.,}$$

is solved for the values of Q, we obtain

$$Q_1 = c_{11}V_1 + c_{21}V_2 + \dots + c_{n1}V_n$$
$$Q_2 = c_{12}V_1 + c_{22}V_2 + \dots + c_{n2}V_n, \text{ etc.}$$

Thus, from the theory of linear simultaneous equations,

$$c_{21} = \frac{-\begin{vmatrix} P_{21} & P_{31} \dots P_{n1} \\ P_{22} & P_{33} \dots P_{n3} \\ \dots & \dots \\ P_{2n} & P_{3n} \dots P_{nn} \end{vmatrix}}{\begin{vmatrix} P_{11} & P_{21} \dots P_{n1} \\ P_{12} & P_{22} \dots P_{n2} \\ \dots & \dots \\ P_{1n} & P_{2n} \dots P_{nn} \end{vmatrix}}$$

and the other c values can be found similarly. From the solution $P_{mn} = P_{nm}$ it readily follows that $c_{mn} = c_{nm}$.

The coefficients c_{11}, c_{22} are termed "coefficients of capacitance."

By putting $V_1 = 1, V_2 = V_3 = \ldots = V_n = 0$ it is easily seen that c_{11} is the charge required to produce unit potential on the first conductor, all the other conductors having zero potential. c_{12} (or c_{21}) is the coefficient of induction between the first and second conductors and is always negative or zero. By putting $V_1 = V_3 = V_4 = \ldots = V_n = 0$ and $V_2 = 1$ it is seen that c_{21} is the charge produced on the first conductor by unit potential on the second and must be negative for positive potential. Similarly c_{12}

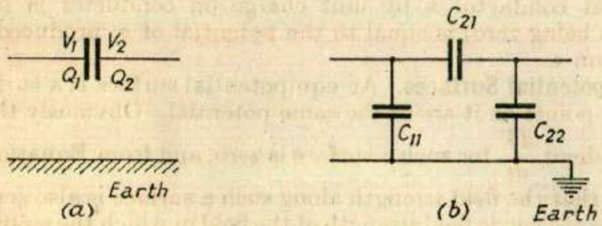

FIG. 1.7

is the charge produced on the second conductor by unit potential on the first and, as shown above, this is equal to c_{21}.

In the case of a capacitor consisting of two metal plates separated by an insulator, or "dielectric," as in Fig. 1.7 (*a*), the equations will reduce to

$$Q_1 = c_{11}V_1 + c_{21}V_2$$
$$Q_2 = c_{12}V_1 + c_{22}V_2$$

where $c_{21} = c_{12}$.

The equations may be written in the form

$$Q_1 = (c_{11} + c_{21})V_1 - c_{21}(V_1 - V_2)$$
$$Q_2 = -c_{12}(V_2 - V_1) + (c_{22} + c_{12})V_2$$

or $$Q_1 = C_{11}V_1 + C_{21}(V_1 - V_2)$$

and $$Q_2 = C_{12}(V_2 - V_1) + C_{22}V_2$$

where $C_{11} = (c_{11} + c_{21})$, $C_{21} = -c_{21} = -c_{12} = C_{12}$ and $C_{22} = (c_{22} + c_{12})$.

C_{11} and C_{22} are the earth capacitances of the plates and C_{21} is usually referred to as *the* capacitance of the capacitor. These capacitances are shown in Fig. 1.7(*b*).

In the conventional diagram (*b*) the conductors of all the capacitors are understood to have no earth capacitance. Obviously the second circuit gives exactly the same result as the first. Generally C_{21} is much greater than C_{11} or C_{22}.

If a single conductor is very remote from all other conductors, the coefficient C_{11} takes a value depending only on the geometry of the conductor. The value of C_{11} under these conditions is known as the "self-capacitance" of the conductor.

The coefficients of capacitance are calculated, in any practical case, by finding the electric field strength at each point due to a given distribution of charges on the conductors and integrating along lines between the conductors to find the potential differences. This gives the potential coefficients P, and by solving the equations, the coefficients of capacitance can be found.

In the simple case of a capacitor formed by two conductors with negligible earth capacitances the capacitance C is given by

$$C = \frac{Q}{V} \qquad \qquad (1.13)$$

If Q is the charge, in coulombs, producing a potential difference of V volts between the conductors, then C will be given in farads. This last unit is defined as the capacitance of a field such that a charge of one coulomb causes a potential difference between the conductors, between which the field exists, of one volt.

A capacitor is most commonly thought of as an arrangement of two conductors placed comparatively close together so that a strong electric field exists between them.

Energy Stored in an Electric Field. If two conductors X and Y are charged so as to have a potential difference of V volts, then an electrostatic field will exist between them, and this field will represent a quantity of stored energy, since, from the definition of potential, work must be done to produce a potential difference between two points. If the charges on the two conductors X and Y are $+Q$ and $-Q$ units respectively, and the conductors be considered as originally uncharged, then the potential difference V may be considered as produced by the transference of Q units of charge from Y to X.

Since the potential difference V is, from Equation (1.13), proportional to the charge at any time, the average potential difference due to transference of the Q units is $\frac{1}{2}V$, and the work done during the transference is, from the definition of potential, $\frac{1}{2}QV$. This is obviously equal to the energy stored. Hence—

Energy stored in the field between the conductors

$$= \tfrac{1}{2}QV = \tfrac{1}{2}(VC)V$$

$$= \tfrac{1}{2}CV^2 \text{ joules} \qquad \qquad (1.14)$$

where C is the capacitance of the field.

Energy Stored per Unit Volume of Dielectric in an Electric Field. It can be shown experimentally that the energy stored in a capacitor is actually stored in the dielectric between the conductors bounding the field. The energy stored per unit volume of dielectric

$$= \frac{\varepsilon . E^2}{2} \quad . \quad . \quad . \quad . \quad (1.15)$$

as shown below, if the field strength is E and the permittivity is ε.

Consider two charged conductors having surface densities of charge σ and σ_1, whose potentials are V and V_1, the charges on the conductors

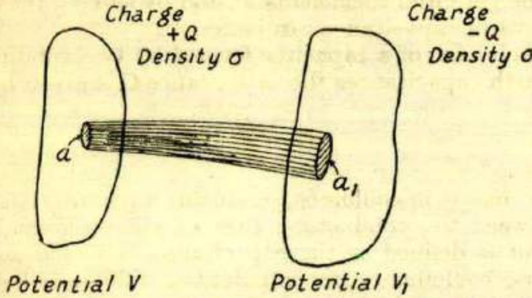

FIG. 1.8. TUBE OF FLUX BETWEEN TWO CHARGED CONDUCTORS

being $+Q$ and $-Q$ (Fig. 1.8). The tube of flux shown starts on area a, and finishes on a_1. It contains a flux ψ, where

$$\psi = \sigma a = \sigma_1 a_1$$

If ε is the permittivity of the medium between the conductors, the field strength at any point P in the tube of flux, distant x from end a, is given by

$$E_x = \frac{\psi}{a_x \varepsilon} \quad . \quad . \quad . \quad . \quad (1.16)$$

where a_x is the area of cross-section of the tube at P.

Now, on the assumption that the energy stored per unit volume of the dielectric is $\frac{\varepsilon E^2}{2}$, then the energy stored in an element of the tube of length dx at P is

$$\frac{\varepsilon . E_x^2}{2} a_x . dx$$

and the total energy stored in the tube

$$= \int_0^l \frac{\varepsilon . E_x^2}{2} . a_x . dx$$

$$= \frac{\varepsilon}{2} \int_0^l E_x^2 a_x dx$$

where l is the length of the tube,

$$= \frac{\varepsilon E_x}{2} a_x \int_0^l E_x \, . \, dx$$

$$= \frac{\psi}{2} \int_0^l E_x \, . \, dx \quad \text{from (1.16)}$$

$$= \frac{\sigma a}{2} \int_0^l E_x \, . \, dx$$

$$= \frac{\sigma a}{2} (V - V_1), \text{ since } \int_0^l E_x \, . \, dx \text{ is the work}$$

done in moving unit charge from a_1 to a, i.e. the potential difference $(V - V_1)$.

But, since the charge on a is σa the expression $\frac{\sigma a}{2}(V - V_1)$ gives the energy stored in the small capacitor, whose plates are a and a_1 from (1.14), which is, of course, the same as the energy of the tube of flux considered.

Thus, the assumption that the energy stored per unit volume of dielectric is $\frac{\varepsilon \, . \, E^2}{2}$ is correct for this tube of flux and, since the same reasoning applies to all such tubes of flux, the assumption is true generally.

In general, the energy stored in a dielectric the field strength in which is a variable quantity, as above, is given by

$$\iiint \frac{\varepsilon \, . \, E^2}{2} \, dv$$

where E is the field strength at any point and dv is an element of volume of the field at this point.*

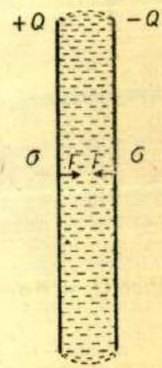

FIG. 1.9. ELECTROSTATIC FIELD BETWEEN TWO CHARGED PLATES

Force of Attraction Between Oppositely Charged Parallel Conducting Surfaces. Fig. 1.9 represents two parallel conducting surfaces of equal

* Note that the expression obtained for the energy stored per unit volume of dielectric, viz. $\frac{\varepsilon E^2}{2}$, is similar to the expression for energy stored in a magnetic field, i.e. $\frac{\mu H^2}{2}$ per unit volume, where

H = the magnetic field strength, and
μ = the permeability of the medium (see page 44)

area and close together, possessing charges of $+Q$ and $-Q$ units. Since their areas are equal, the surface density of charge (σ) is the same on both surfaces and the field strength in between them is at all points $\frac{\sigma}{\varepsilon}$, where ε is the permittivity of the medium between them. It is assumed that the effect of the fringing field at the edges of the surfaces is negligible.

Let F be the force of attraction between the surfaces. Then, if one surface is moved away from the other by an infinitely small distance δx, the work done is $F\delta x$, it being assumed that the movement is so small that the field strength is unaffected by the movement. If the area of each surface is A, then the increase in the energy stored, owing to the increase in volume of the dielectric, is $\frac{A\delta x \,.\, \varepsilon E^2}{2}$, which is, of course, equal to the work done.

$$\therefore F\delta x = \frac{A\delta x\varepsilon E^2}{2}$$

$$F = \frac{A\varepsilon E^2}{2}$$

$$= \frac{A\varepsilon}{2}\left(\frac{\sigma}{\varepsilon}\right)^2$$

$$= \frac{A\sigma^2}{2\varepsilon}$$

$$\therefore F = \frac{\sigma Q}{2\varepsilon} \quad . \quad . \quad . \quad . \quad (1.17)$$

since $Q = A\sigma$.

Magnetism

It is convenient, in magnetic theory, to use the concept of a magnetic pole, i.e. a point at which the magnetic field originates or terminates. Such a magnetic pole has no physical existence but derives from the fact that the field due to a magnet can be represented by two such poles, of opposite polarity, a fixed distance apart.

This concept provides a very useful means for deriving the equations of the magnetic field. An excellent exposition of modern field theory will be found in Ref. (24).

Coulomb's Law. Coulomb's Inverse Square Law is true for magnetic quantities as well as for electrostatic quantities. The force F, in newtons, between two magnetic poles of pole strength m_1 and m_2 webers, distant r metres apart, is

$$F = \frac{m_1 m_2}{4\pi\mu_0\mu_r r^2} \quad . \quad . \quad . \quad . \quad (1.18)$$

where μ_0 is the primary magnetic constant and has a value $4\pi \times 10^{-7}$, μ_r is the permeability of the medium, separating the poles, relative to that of a vacuum (for which μ_r is unity). It is customary to write μ for $\mu_0 . \mu_r$. A unit magnetic pole is defined as giving unit magnetic flux, the unit of flux being the "weber."

The force between the poles is one of repulsion or attraction, according to whether they are of like or unlike polarity respectively,

Magnetic poles are considered as being concentrated at a point.

Lines of Force and Magnetic Field Strength. The magnetic field existing in the neighbourhood of a magnetic pole can be represented by lines of force similar to the lines of electric force which are used to represent an electrostatic field, the arrowheads upon the lines of force indicating the direction in which a unit *north* pole would move if placed upon the line of force. The "strength" of a magnetic field at any point is expressed by the "force which would be exerted on a north pole of unit strength placed at the point," it being assumed that the introduction of the unit pole does not affect the field. Thus, the magnetic field strength at a point, distant r from a pole of strength m units, in air, is given by

$$H = \frac{m}{4\pi\mu_0 r^2} \quad . \quad . \quad . \quad . \quad (1.19)$$

The Magnetic Field. A number of lines of force are spoken of collectively as "magnetic flux" and the number of lines per unit area of cross-section as the "flux density." The symbols used to represent these quantities are Φ and B respectively. If a flux Φ crosses an area A square metres in a direction perpendicular to the area, then

$$\Phi = B . A \quad . \quad . \quad . \quad . \quad (1.20)$$

The units of Φ and B are the weber and the weber per square metre respectively.

Gauss's theorem can be applied to the magnetic field in a manner similar to that used in applying it to the electric field; in this case it states that the total flux which traverses, in a normal direction, a surface completely surrounding a pole of strength m webers is m webers. It follows that the flux density at a distance r metres from a pole of strength m is given by

$$B = \frac{m}{4\pi r^2}$$

By comparing this result with Equation (1.19) it is clear that

$$B = \mu_0 H \text{ in air}$$

or, in any medium,

$$B = \mu H \quad . \quad . \quad . \quad (1.21)$$

The distinction between H and B is that of cause and effect: H is the magnetic field strength or magnetizing force and B the magnetic flux density which results from this force. It can be seen from Equation (1.19) that H is the force in newtons on a unit pole of strength 1 weber, but the practical unit of H is the "ampere per metre." This is not apparent from the previous equations but arises from the concept that all magnetic fields are due to electric currents flowing in loops. The electrons revolving around the atomic nuclei are, in effect, tiny current loops which give rise to the field we associate with magnets. The derivation of the practical unit of H will follow when we consider the field due to an electric current.

Magnetic Moment and Intensity of Magnetization of a Magnet. A magnet having poles of strength m units distant l apart is said

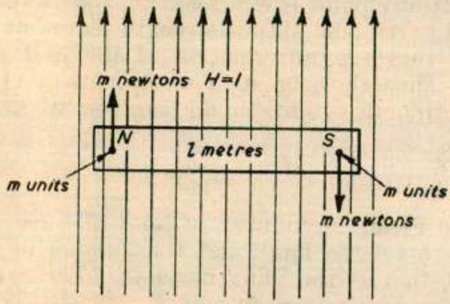

FIG 1.10. BAR MAGNET SITUATED IN A MAGNETIC FIELD

to have a "magnetic moment" of ml units. This term arises from the fact that, if the magnet were placed in a magnetic field of unit strength so that its magnetic axis (i.e. the line joining its poles) was perpendicular to the direction of the field, it would be acted on by two forces each of m units, as shown in Fig. 1.10, these forces forming a couple whose turning moment is ml. The symbol M is used to express magnetic moment; hence

$$M = ml \qquad . \quad . \quad . \quad . \quad (1.22)$$

The intensity of magnetization J of a magnet is expressed by the ratio $\frac{\text{pole strength}}{\text{cross-sectional area}}$ the magnet being assumed to be uniformly magnetized.

Thus
$$J = \frac{m}{A} \qquad . \quad . \quad . \quad . \quad (1.23)$$

In the case of a bar magnet, of uniform cross-section A, and of length l, the poles being considered at the extreme ends, the magnetic moment is ml and the volume of the magnet V is lA.

Then
$$J = \frac{m}{A} = \frac{ml}{Al} = \frac{M}{V}$$

and thus the intensity of magnetization may be expressed as the magnetic moment per unit volume, the magnetization being uniform.

If the magnetization is not uniform, J varies at different points in the magnet, and is expressed by $\frac{\delta M}{\delta V}$, where δM is the magnetic moment of an element of volume δV taken at the point considered.

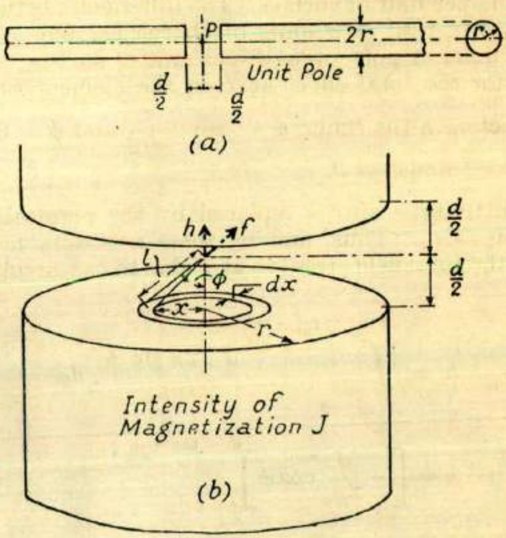

Fig. 1.11. Magnetizing Force in the Air Gap of a Magnetized Iron Bar

Relations Between Intensity of Magnetization, Flux Density, and Magnetizing Force. In order to determine the connection between the intensity of magnetization in a magnetized body and the flux density in the body, consider a long, thin rod of some magnetic substance (say iron) which has been uniformly magnetized. The demagnetizing effect of the ends of the rod (see Chap. IX) may be neglected if it is very long and thin. Now imagine a very narrow air gap of length d in the rod, with a unit magnetic pole placed in the air gap, on the axis of the rod, and equidistant from the bounding faces of the air gap, as shown in Fig. 1.11(a). This unit pole will be repelled from one face and attracted to the other with equal forces, since it is equidistant from them, and the direction of these forces will, of course, be along the axis of the rod.

Let the rod be of circular cross-section, radius r, and let its intensity of magnetization be J. Then, as shown in Fig. 1.11(b), the magnetizing force in the air gap at P, due to the intensity of magnetization J in the iron rod, can be considered as being produced by a number of elemental rings of radius x and width dx, having a pole strength per unit area of J (i.e. intensity of magnetization of J).

If Fig. 1.11 is compared with Fig. 1.5, it will be observed that the present determination of magnetizing force at P is almost exactly similar to the determination of the electric force at a point P in between two oppositely-charged surfaces with surface densities of charge σ units per unit of surface. The differences between the two cases are that instead of σ units of charge per unit of surface we now have J units of pole strength per unit of surface, and that the integration for the total effect at P of the element rings must be performed between the limits $\phi = \tan^{-1}\frac{r}{d/2}$ and $\phi = 0$, instead of the limits $\phi = \frac{\pi}{2}$ and $\phi = 0$.

The permittivity ε also is replaced by the permeability, which is here taken as μ_0. Thus, making these necessary modifications, we have for the total field strength at P due to *one* circular bounding surface

$$h = \int_{\phi=0}^{\phi=\tan^{-1}\frac{r}{d/2}} \frac{J\,.\,2\pi\sin\phi}{4\pi\mu_0}\,.\,d\phi$$

i.e.

$$h = \left[-\frac{J}{2\mu_0}\cos\phi\right]_{\phi=0}^{\phi=\tan^{-1}\frac{r}{d/2}}$$

$$= -\frac{J}{2\mu_0}\left[\frac{\frac{d}{2}}{\sqrt{\left(\frac{d}{2}\right)^2 + r^2}} - 1\right]$$

If the air gap is very small, so that $\frac{d}{2}$ is negligible compared with the radius r, then

$$h = \frac{J}{2\mu_0} \quad . \quad . \quad . \quad . \quad (1.24)$$

This is the force in the gap due to *one* bounding surface only. If this is a force of repulsion upon the unit pole at P, then the other surface, which is of opposite polarity, will attract the unit pole at P with an equal force. Thus, the total field strength at P is J/μ_0 and, since the introduction of the very narrow air gap does not affect the intensity of magnetization in the iron, the same would be true in the iron if the air gap did not exist. The flux density, also, in the air gap is J, which, again, is the flux density which will exist in the iron due to the intensity of magnetization J.

If this iron rod is placed in a magnetic field of strength H, the direction of this field being along the axis of the rod and in the same direction as the intensity J, then the total flux density B in the iron is $\mu_0 H + J'$ where J' is the intensity of magnetization in the iron when the rod is situated in the field of intensity H and will, of course, be different from the intensity of magnetization J, existing before the rod was placed in the magnetizing field.

If the iron rod has no magnetization before being placed in the magnetizing field of strength H, and if J is the intensity of magnetization *produced by the field*, then

$$B = \mu_0 H + J \qquad . \quad . \quad . \quad (1.25)$$

The term J, which is the flux density which exists due to the magnetization of the iron itself, is sometimes called the "ferric induction." "Ferric induction" is, therefore, the flux density which exists in excess of that produced in air by the same magnetizing force, and the intensity of magnetization is $B - \mu_0 H$.

Equation (1.25) can be written

$$B = \mu H = \mu_0 \mu_r H \qquad . \quad . \quad . \quad (1.26)$$

where μ is the "magnetic permeability" of the iron, and μ_r is the relative permeability of the iron as compared with that of a vacuum.

It follows that

$$\mu_r = \frac{B}{\mu_0 H}$$

If the iron has some residual magnetism before being placed in the magnetic field, then the ratio $\frac{B}{\mu_0 H}$, B being the total flux density in the iron, is not the correct value of the permeability.

If J were directly proportional to H ($= kH$ say) for all values of H, then Equation (1.25) could be written

$$B = H(\mu_0 + k) = \mu_0 \mu_r H$$

and the relative permeability μ_r would be a constant factor. This, however, is not so, J having a maximum value depending upon the iron or other magnetic material considered, and μ_r is thus a variable factor depending upon the value of H (or B).

When, during the magnetization of a specimen of magnetic material, the maximum value of J is attained, further increase in the magnetizing force H only increases the flux density B by increasing H (Equation (1.25)), and the material is said to be "saturated."

Fig. 1.12 shows a typical magnetization (B–H) curve for cast steel. It can be seen that, above a value of B of about 1·6 Wb/m² the increase in flux density as H is increased is very small, which means that the intensity of magnetization J of the iron is being increased only very slightly by increase of H.

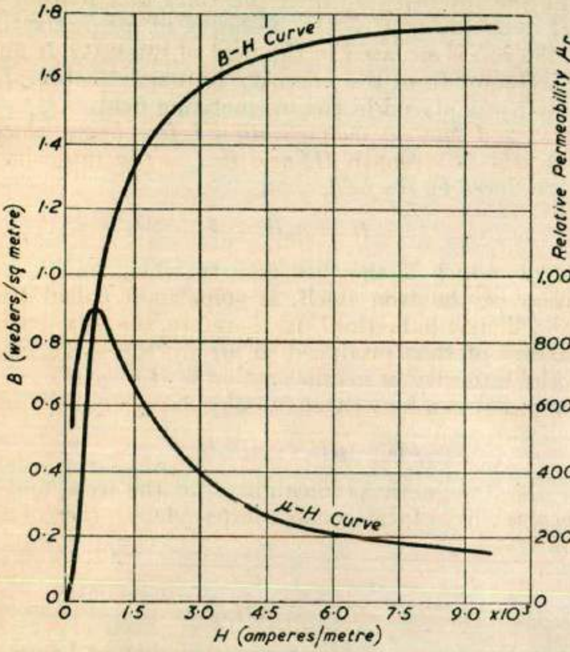

FIG. 1.12. TYPICAL B–H AND μ–H CURVES FOR CAST STEEL

The second curve in Fig. 1.12 shows the variation of relative permeability with magnetizing force H.

Magnetic Susceptibility. The ratio

$$\frac{\text{intensity of magnetization}}{\text{magnetizing force producing this intensity of magnetization}}$$

is called the "magnetic susceptibility" of the material and is given by

$$\text{Magnetic susceptibility} = \frac{J}{H} \quad . \quad . \quad . \quad . \quad (1.27)$$

$$= \frac{B - \mu_0 H}{H} \text{ from equation (1.25)}$$

$$\therefore \text{ Magnetic susceptibility} = \mu_0(\mu_r - 1) \quad . \quad . \quad . \quad . \quad (1.28)$$

Obviously, since $\mu_r = 1$ for non-magnetic substances, the magnetic susceptibility of such substances is equal to μ_0. The magnetic susceptibility is alternatively defined as $\frac{J}{\mu_0 H}$, when it becomes $\mu_r - 1$.

Magnetic Potential. This can be defined in a similar way to that used in the definition of electrostatic potential. The work required to move unit north pole from an infinite distance to a given point is considered to be the magnetic potential of the point.

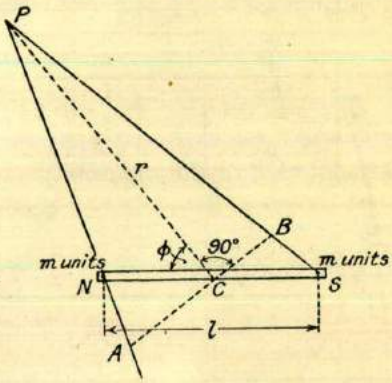

FIG. 1.13. POTENTIAL NEAR A BAR MAGNET

Thus, in the M.K.S. system, if H_x is the field strength at a distance x metres from a pole of strength m units, the force upon unit pole situated at this distance away is H_x newtons, and the work done in moving the unit pole from infinite distance to a point in the neighbourhood of the pole is $\int_x^\infty H_x \, . \, dx$ joules. Also $H_x = \frac{m}{4\pi\mu_0 x^2}$, so that the potential at a point distant r metres from a pole of strength m units, in air, is given by

$$V = \int_r^\infty \frac{m}{4\pi\mu_0 x^2} \, dx = \frac{m}{4\pi\mu_0 r} \qquad . \qquad . \qquad (1.29)$$

Potential Near a Short Bar Magnet. Fig. 1.13 represents a bar magnet whose length l is small compared with the distance r from its centre to the point (P) whose magnetic potential is being considered.

Let the magnet have a pole strength of m units and centre C as shown, and let the line ACB, perpendicular to PC, cut PS and PN produced, in B and A respectively. Since PC is great compared with the length of the magnet, $PA = PB = r$ very nearly, and $N\hat{A}C = S\hat{B}C = 90°$ very nearly.

Thus
$$PN = PA - AN = r - \frac{l}{2}\cos\phi$$
$$PS = PB + BS = r + \frac{l}{2}\cos\phi$$

$\therefore$ Potential at P is

$$V_p = \left[\frac{m}{r - \frac{l}{2}\cos\phi} - \frac{m}{r + \frac{l}{2}\cos\phi}\right]\frac{1}{4\pi\mu_0}$$

the negative sign being due to the opposite polarities of the poles of strength m units.

$$V_p = \left[\frac{m\left(r + \frac{l}{2}\cos\phi\right) - m\left(r - \frac{l}{2}\cos\phi\right)}{r^2 - \frac{l^2}{4}\cos^2\phi}\right]\frac{1}{4\pi\mu_0}$$

$$= \frac{ml\cos\phi}{4\pi\mu_0 r^2} \text{ if } r \text{ is great compared with } l$$

Thus,
$$V_p = \frac{M\cos\phi}{4\pi\mu_0 r^2} \qquad . \quad . \quad . \quad . \quad . \quad (1.30)$$

where M is the magnetic moment of the bar magnet.

Force of Attraction Between Oppositely-magnetized Surfaces. Fig. 1.14 represents the ends of two magnetized bars, each of cross-section A, the ends having opposite polarity. Let B be the flux density (considered uniform) between them. If μ is the permeability of the medium between the poles, the value of H in the field is $\frac{B}{\mu}$. Let F be the force of attraction existing between them. Then if one of the bars is moved an infinitesimal distance dx farther away from the other, the work done is Fdx. Now, the energy stored per unit volume of a magnetic field is $\frac{\mu H^2}{2}$ (see page 45). Thus, assuming that the value of H is unaltered by the infinitesimal movement, the increase in the energy stored in the field

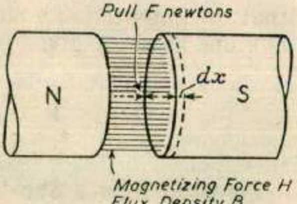

FIG. 1.14. ATTRACTION BETWEEN MAGNETIZED SURFACES

is $Adx \frac{\mu H^2}{2}$, which is, of course, equal to the work done in overcoming the force of attraction F.

$$\therefore Fdx = Adx \frac{\mu H^2}{2}$$

$$F = \mu \frac{AH^2}{2}$$

$$= \frac{AB^2}{2\mu}$$

or, if the field is in air,

$$F = \frac{AB^2}{2\mu_0} \quad . \quad . \quad . \quad . \quad (1.31)$$

This force is in newtons if A is in sq. m and B in webers per sq. m.

Magnetic Shells. A thin iron sheet, magnetized as in Fig. 1.15, may be thought of as consisting of an infinite number of small bar magnets, with all the north poles on one side of the sheet and all the

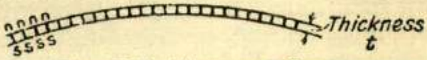

FIG. 1.15. MAGNETIC SHELL

south poles on the other. This constitutes what is known as a "magnetic" shell, and the idea is useful in considering various problems in electromagnetism. The "strength of the shell" is defined as "magnetic moment per unit area." Thus, if m is the pole strength per unit area and the thickness of the shell is t, the "strength of the shell" is mt.

ELECTROMAGNETISM

The study of electromagnetism originated with Oersted's discovery that a pivoted magnetic needle in the neighbourhood of a conducting wire is deflected when a current of electricity flows in the wire. This means that a magnetic field exists around a wire which carries current, and this leads to the present definition of the unit of current (the ampere). The ampere is defined as that current which, if maintained in two straight parallel conductors of infinite length and negligible cross-section, spaced 1 metre apart in vacuum, will produce a force between the conductors of 2×10^{-7} newtons per metre length. The relationship on which this definition is based will be derived in due course, but for the moment we must accept the unit of current as being the ampere.

Ampere's Theorem. Ampere showed that the magnetic effect of a current i units, flowing in a small closed circuit, is the same as that of a small bar magnet placed with its axis perpendicular to the plane of the circuit, provided that the magnetic moment of such a magnet is equal to $\mu_0 i dA$, where dA is the area of the small circuit.

By considering a number of such small circuits placed together as shown in Fig. 1.16(a), with currents of i units flowing in each, it can be shown that, *for any closed circuit* carrying a current of i units, the magnetic effect is the same as that of a magnetic shell occupying

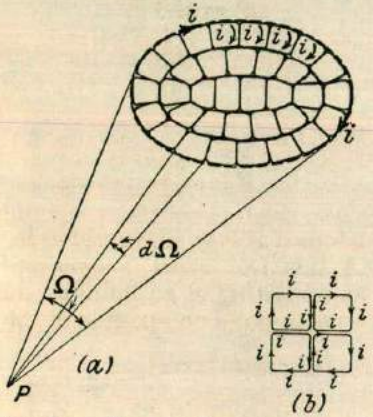

FIG. 1.16. POTENTIAL DUE TO CURRENT IN A CLOSED CIRCUIT

the space enclosed by the circuit, provided the "strength" of such a shell is equal to the current i. This constitutes what is known as Ampere's Theorem.

Weber showed experimentally that the potential at any point P distant r from a small closed circuit of area dA carrying i units of current is

$$V_p = \frac{dA \cdot i \cos\phi}{4\pi r^2} \qquad (1.32)$$

where ϕ is the angle between the normal to the plane (whose dimensions are small compared with the distance r) and the distance r.

From Equation (1.30), the potential at a distance r from a short bar magnet is

$$V_p = \frac{M \cos\phi}{4\pi\mu_0 r^2}$$

Thus, if the small magnet which is equivalent to the closed circuit has magnetic moment M we can write

$$\frac{dA \cdot i \cdot \cos\phi}{4\pi r^2} = \frac{M \cos\phi}{4\pi\mu_0 r^2}$$

$$\therefore \mu_0 dA \cdot i = M$$

Again, from Equation (1.32),

$$V_p = \frac{i}{4\pi} \cdot d\Omega$$

where $d\Omega$ is the solid angle subtended at P by the closed circuit.

The potential at P due to a large number of small circuits (Fig.1.16) is

$$V_p = \sum \frac{id\Omega}{4\pi} = \frac{i\Omega}{4\pi} \quad . \quad . \quad . \quad (1.33)$$

where Ω is the total solid angle subtended at P by all the small circuits.

It can be seen also that the currents i in the small circuits neutralize one another at all parts except the outside (as in Fig. 1.16 (*b*)), so that the whole is equivalent to one circuit lying along the perimeter of the group of small circuits and carrying a current i. Thus it is shown that, for any circuit carrying i units of current, the potential at a point P is $i\Omega/4\pi$, where Ω is the solid angle subtended at the point by the circuit. Again, if each of the small circuits of Fig. 1.16 (*a*), each of area dA, be replaced by a small magnet, of moment M, then

$$\mu_0 i = \frac{M}{dA}$$

Now, obviously, $\frac{M}{dA}$ is the strength of the magnetic shell formed by such a replacement of the small circuits by bar magnets of moment M, and thus the circuit is equivalent to a magnetic shell having a strength equal to $\mu_0 i$.

Potential Energy of a Current and a Magnetic Flux. From the preceding section it follows that the potential energy of a magnetic pole of strength m units, at a point P in the neighbourhood of a closed circuit in which a current i flows, is $\frac{mi\Omega}{4\pi}$, where Ω is the solid angle subtended at P by the circuit. This expression represents energy, since, from the definition of potential difference as the work done in moving unit pole from one point to another, the work done in moving m units is $mV_p = \frac{mi\Omega}{4\pi}$.

Now, a magnetic pole of strength m radiates m lines of force distributed uniformly in all directions. Thus the magnetic flux threading the current-carrying circuit is $\frac{\Omega}{4\pi} \times m = \frac{\Omega m}{4\pi}$.

Hence, the potential energy of the current and magnetic flux

$$= i\Phi \quad . \quad . \quad . \quad . \quad (1.34)$$

where Φ is the flux threading through the current-carrying circuit.

[Note that, if the flux Φ changes with time t, then $i\frac{d\Phi}{dt}$ represents rate of change of energy, i.e. power.]

Forces Due to and Acting upon Current in a Long Straight Conductor. Biot and Savart were the first to examine, by means of a compass needle, the magnetic field strength at different distances from a straight conductor which is carrying current. They showed that the magnetic field strength was "inversely proportional to the distance from the conductor in which the current flows." This is known as the Biot–Savart Law.

Fig. 1.17(a) shows a small element of a conductor of length dl carrying a current i and situated at a distance r from a magnetic

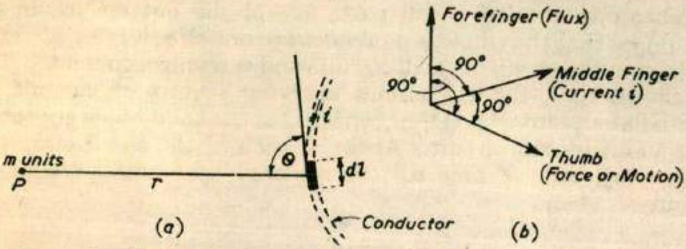

FIG. 1.17. MAGNETIC FIELD DUE TO CURRENT IN A CONDUCTOR

pole of strength m units in a direction making an angle of θ with the element of conductor.

Laplace established the equation, related to the above case, that the force upon the element of conductor is

$$f = \frac{m \,.\, i \,.\, dl \sin \theta}{4\pi r^2} \quad . \quad . \quad . \quad (1.35)$$

Now, the flux density at the conductor due to the pole is, by Equation (1.19), $\frac{m}{4\pi r^2}$. Hence, the force is

$$f = Bidl \sin \theta$$

If the conductor is straight and is situated in a uniform field of flux density B whose direction is perpendicular to the conductor then

$$F = Bil \quad . \quad . \quad . \quad . \quad . \quad (1.36)$$

where F is the total force on the conductor. F is in newtons if l is in metres and i in amperes.

If in Fig. 1.17(a) the pole m has north polarity, the force f is, by the *left-hand rule* (Fig. 1.17(b)), perpendicular to the plane of the paper *outwards*.

The *Left-hand Rule* states that if the thumb, forefinger, and middle finger of the left hand are placed mutually at right angles, then the corresponding directions of magnetic field, current and force (or motion) are given as shown in Fig. 1.17(*b*).

The force upon the pole, due to the current in the conductor, is equal in magnitude, but opposite in direction, to that of the pole upon the conductor; force upon the pole is $\frac{midl}{4\pi r^2}\sin\theta$ in a direction perpendicular to the plane of the paper *inwards.*

The force upon *unit* pole at P (i.e. the magnetizing force at P) due to the current is

$$dH = \frac{idl\sin\theta}{4\pi r^2} \quad . \quad . \quad . \quad . \quad (1.37)$$

Now, if the conductor is straight and carries i units of current, the value of H at a point P, which is perpendicularly distant D from

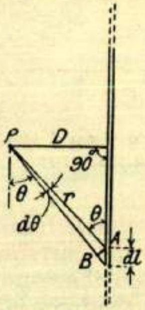

FIG. 1.18. MAGNETIC FIELD NEAR A STRAIGHT CONDUCTOR

the conductor, can be found as follows: Consider an element of conductor dl as in Fig. 1.18. The magnetizing force due to it is, as above, $dH = \frac{idl\sin\theta}{4\pi r^2}$. The length $AB = dl\sin\theta = rd\theta$, and $\sin\theta = \frac{D}{r}$.

Thus $$dH = i\frac{\sin\theta}{4\pi D}d\theta$$

and the total magnetizing force at P is

$$H = 2\int_{\theta=0}^{\theta=+\frac{\pi}{2}} i\frac{\sin\theta}{4\pi D}\,d\theta$$

if D is small compared with the length of the conductor.

$$\therefore H = \frac{i}{2\pi D} \quad . \quad . \quad . \quad . \quad (1.38)$$

It follows from this equation that the work done in carrying unit north pole through a circular path surrounding the conductor is

$$H \times 2\pi D = \frac{i}{2\pi D} \times 2\pi D$$

$$\therefore \text{ Work done} = i \quad . \quad . \quad . \quad . \quad (1.39)$$

Force Between Two Parallel Current-carrying Conductors. The forces of attraction or repulsion between two parallel conductors A, B carrying currents i_1 and i_2 respectively and distant D apart,

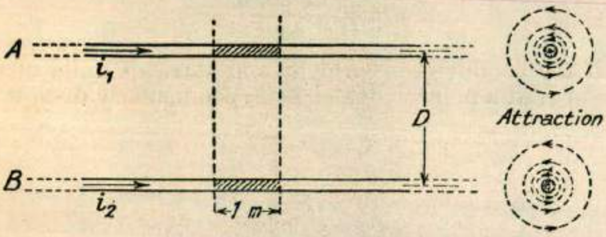

FIG. 1.19. FORCE BETWEEN TWO PARALLEL CURRENT-CARRYING CONDUCTORS

as in Fig. 1.19, can be calculated from the equations obtained in the previous paragraph. If the currents are in the same direction (as shown), the force between the conductors is one of attraction, and if in opposite directions the force is one of repulsion.

Consider the attractive force of unit length of conductor A upon unit length of conductor B adjacent and parallel to it as shown. Conductor B is situated in a magnetic field, due to A, of flux density $\frac{\mu_0 i_1}{2\pi D}$ (from Equation (1.38)). Hence, from Equation (1.36), the force per unit length of conductor is

$$F = \frac{\mu_0 i_1 i_2}{2\pi D}$$

The force of attraction on conductor A, due to conductor B, is, by similar reasoning, the same.

Thus, the force per unit length between the conductors is

$$F = \frac{\mu_0 i_1 i_2}{2\pi D}$$

or, for a length l,

$$F = \frac{\mu_0 i_1 i_2}{2\pi D} l \quad . \quad . \quad . \quad . \quad (1.40)$$

The force is in newtons if i_1 and i_2 are in amperes and D is in metres. It is assumed that the distance D is small compared with the lengths of the conductors. The expression forms the basis of the definition of the ampere given on p. 60.

The dots, placed on the conductor sections to the right of Fig. 1.19, indicate that the current in them flows in an outward direction. A cross so placed indicates inward direction.

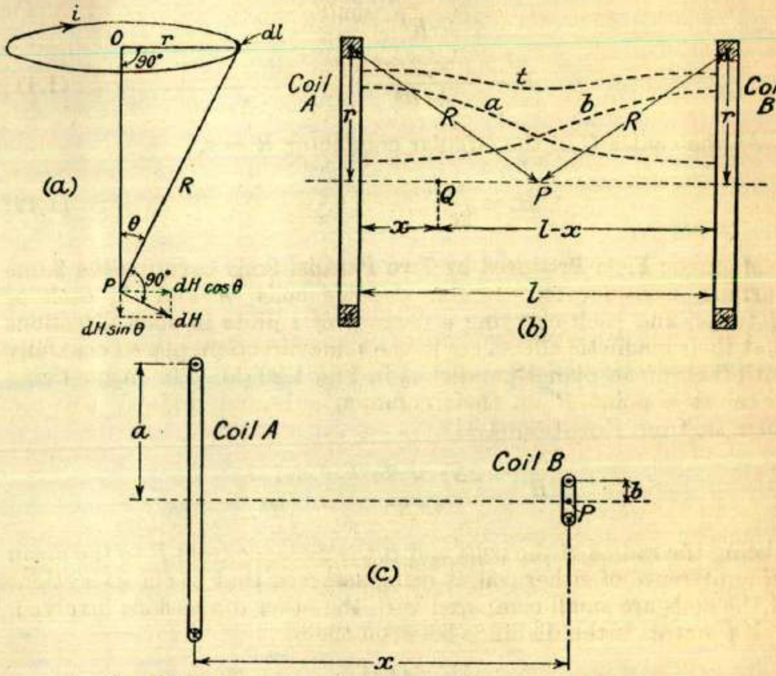

FIG. 1.20. MAGNETIC FIELD NEAR CIRCULAR, CURRENT-CARRYING COILS

Magnetic Field Due to Current in a Circular Conductor. Fig. 1.20 (*a*) shows a circular conductor carrying i units of current. Consider the magnetizing force (dH) at a point P, on the axis of the circle, due to the current in an element dl of the conductor. Then

$$dH = \frac{idl}{4\pi R^2}$$

in a direction perpendicular to the line joining the element to P.

This force may be split up into two components, one in direction OP produced, namely $dH \sin \theta$, and one perpendicular to OP, namely $dH \cos \theta$. Considering all such elements of the circular conductor it is seen that the components perpendicular to OP neutralize one another, leaving, as the force at P, only the sum of all components in a direction OP produced. Thus the total magnetizing force at P due to the current is in direction OP, and is

$$H = \Sigma dH \sin\theta = \Sigma \frac{idl}{4\pi R^2} \sin\theta$$

$$= \frac{i \times 2\pi r}{4\pi R^2} \sin\theta$$

$$H = \frac{i \times 2\pi r^2}{4\pi R^3} \quad . \quad . \quad . \quad . \quad (1.41)$$

At the centre O of the circular conductor $R = r$,

$$\therefore H = \frac{i}{2r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.42)$$

Magnetic Field Produced by Two Parallel Coils Carrying the Same Current. Consider two similar circular coils A and B, each of N turns, and each carrying a current of i units in such direction that their magnetic effects are in the same direction, placed coaxially with their mean planes parallel as in Fig. 1.20 (b). The magnetizing force at a point P on their common axis and midway between them is, from Equation (1.41),

$$H = \frac{2Ni \times 2\pi r^2}{4\pi R^3} = \frac{Ni\, r^2}{R^3}$$

r being the radius of the coils and R the distance from P to the mean circumference of either coil, it being assumed that the cross-sections of the coils are small compared with the other dimensions involved.

If l metres is the distance between them,

$$R^2 = r^2 + \left(\frac{l}{2}\right)^2 = r^2 + \frac{l^2}{4}$$

$$\text{Thus } H \text{ at } P = \frac{Ni\,.\,r^2}{\left(r^2 + \frac{l^2}{4}\right)^{\frac{3}{2}}}$$

For a point Q on the axis distant x from coil A,

$$H \text{ at } Q \text{ due to coil } A = \frac{Ni\,.\,r^2}{2(r^2 + x^2)^{\frac{3}{2}}}$$

$$H \text{ at } Q \text{ due to coil } B = \frac{Ni\,.\,r^2}{2[r^2 + (l - x)^2]^{\frac{3}{2}}}$$

Thus, resultant H at Q

$$= \frac{Ni \,.\, r^2}{2(r^2 + x^2)^{\frac{3}{2}}} + \frac{Ni \,.\, r^2}{2[r^2 + (l - x)^2]^{\frac{3}{2}}}$$

$$= \frac{r^2 Ni}{2}\left[\frac{1}{(r^2 + x^2)^{\frac{3}{2}}} + \frac{1}{(r^2 + (l - x)^2)^{\frac{3}{2}}}\right] \qquad (1.43)$$

If the ordinates representing the values of H at various points along the axis are drawn, the result gives curves as shown in Fig. 1.20 (b). The curve marked (a) gives H due to coil A, and that marked (b) the value of H due to coil B. The curve marked (t) gives the resultant value of H, which is seen to fall slightly towards the midway position P.

If the distance l between the coils is made equal to the radius r of the coils, Equation (1.43) becomes

$$H_Q = \frac{r^2 Ni}{2}\left[\frac{1}{(r^2 + x^2)^{\frac{3}{2}}} + \frac{1}{[r^2 + (r - x)^2]^{\frac{3}{2}}}\right]$$

Now $\qquad [r^2 + (r - x)^2]^{\frac{3}{2}} = [r^2 + x^2 + r^2 - 2rx]^{\frac{3}{2}}$

Expanding we have

$$[r^2 + x^2 + r^2 - 2rx]^{\frac{3}{2}} = (r^2 + x^2)^{\frac{3}{2}} + \frac{3}{2} \cdot (r^2 + x^2)^{\frac{1}{2}}\,(r^2 - 2rx)$$
$$+ \frac{3}{2}\;\frac{1}{2}\,\frac{(r^2 + x^2)^{-\frac{1}{2}}\;(r^2 - 2rx)^2}{2\,.\,1} + \quad \ldots$$

If $x = \frac{r}{2}$ all terms but the first disappear. If x has any value which is of the same order as $\frac{r}{2}$ the succeeding terms are small compared with the first. Thus, the resultant magnetizing force is

$$H_Q \doteqdot \frac{r^2 Ni}{2}\left[\frac{2}{(r^2 + x^2)^{\frac{3}{2}}}\right]$$

This arrangement of the two coils produces a field of almost uniform H-value between the two coils for a considerable distance on either side of the mid-point P, and was used by von Helmholtz in a special form of tangent galvanometer for use in the absolute measurement of current.*

Force Between Two Parallel Coaxial Circles in which Currents are Flowing. Consider the two circles A and B shown in Fig. 1.20(c). They are coaxial, and their planes are parallel. Let circle A have radius a, and carry a current i_A, while circle B has radius b, and carries a current i_B. Assume also that the radius b is very small.

* A diagram of the field between two coils so placed is given in Maxwell's *Electricity and Magnetism*, Vol. II, and the complete theory is given in Gray's *Absolute Measurements in Electricity and Magnetism*, Vol. II, Part I.

From Equation (1.41) the value of H at the centre P of circle B due to the current in circle A is $\frac{i_A a^2}{2(a^2 + x^2)^{\frac{3}{2}}}$ where x is the distance between the circles. The flux density (considered uniform, since the radius b is very small) inside circle B is therefore $\frac{\mu_0}{2} \cdot \frac{i_A a^2}{(a^2 + x^2)^{\frac{3}{2}}}$ and thus the total flux threading through circle B is $\frac{\mu_0}{2} \cdot \frac{\pi b^2 \,.\, i_A a^2}{(a^2 + x^2)^{\frac{3}{2}}}$.

From Equation (1.34) the potential energy of the coils is

$$\frac{\mu_0}{2} \cdot \frac{\pi b^2 \,.\, i_A a^2}{(a^2 + x^2)^{\frac{3}{2}}} \times i_B = \frac{\mu_0}{2} \cdot \frac{\pi i_A i_B \,.\, a^2 b^2}{(a^2 + x^2)^{\frac{3}{2}}} = \text{P.E.}$$

The force between the circles (of attraction or repulsion, depending upon the directions of the currents i_A and i_B) is from the law $F = \frac{dV}{dx}$ equal to $\frac{d(\text{P.E.})}{dx}$;

i.e. $$F = \frac{\frac{3}{2}\mu_0 \pi i_A i_B \,.\, a^2 b^2 x}{(a^2 + x^2)^{\frac{5}{2}}} \quad . \quad . \quad . \quad (1.44)$$

The force is one of attraction if the currents are in the same direction, and of repulsion if the currents are in opposite directions. It can be shown by differentiation that F is maximum when $x = \frac{a}{2}$

The complete theory in connection with the force between two parallel current-carrying coaxial coils, when the radius of neither may be considered very small, and when they each have a number of turns giving a finite cross-section, is necessary for the purpose of measuring current in absolute units by means of a current balance, and has been given by Gray (Ref. (10)) and by J. V. Jones (Ref. (11)).

Magnetic Field of a Solenoid. Consider the solenoid shown in Fig. 1.21 to be made up of a large number of circular conductors such as that considered above.

Let N = total number of turns on solenoid
i = current in the solenoid in amperes
l = length of solenoid in metres
r = radius of solenoid in metres
c = number of turns per metre length $= \frac{N}{l}$

Then, magnetizing force at the centre O of the solenoid due to a length dl of the solenoid is

$$dH = c.i.dl \times \frac{2\pi r^2}{4\pi R^3} \text{ (from Equation (1.41))}$$

in the direction of the axis of the solenoid.

The length $AB = Rd\theta = dl \sin\theta$

$$\therefore \quad dl = \frac{Rd\theta}{\sin\theta}$$

$$dH = ci \times \frac{r^2}{2R^3} \cdot \frac{Rd\theta}{\sin\theta}$$

Also $$\sin\theta = \frac{r}{R}$$

Substituting for $\frac{r}{R}$ in the equation for dH,

$$dH = \frac{ci}{2} \sin\theta \, d\theta$$

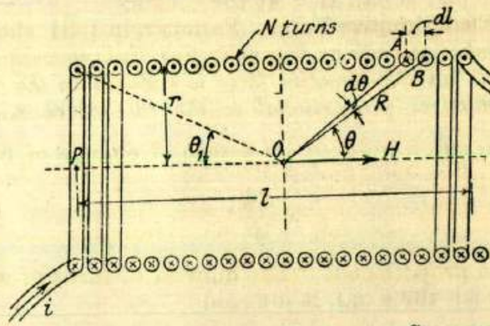

FIG. 1.21. MAGNETIC FIELD OF A SOLENOID

The expression for the total value of H at O due to the whole solenoid is, therefore,

$$H = \int_{\theta_1}^{\pi - \theta_1} \frac{ci}{2} \sin\theta \, d\theta$$

where θ_1 is the limiting value of θ and is $\tan^{-1} \frac{2r}{l}$.

$$\therefore \quad H = ci \cos\theta_1$$

$$H = \frac{Ni}{l} \cos\theta_1 \quad . \quad . \quad . \quad . \quad . \quad (1.45)$$

or, if the solenoid is very long compared with its radius r,

$$H = \frac{Ni}{l} \text{ since } \cos\theta_1 \doteqdot 1$$

This expression gives the unit of H, the ampere-turn per metre. Strictly speaking the unit is the ampere per metre since the product

Ni is dimensionally amperes, but use of the term ampere-turn helps to emphasize that H is due to a current loop.

The value of H at one end of the solenoid (at P) is obtained by integrating between the limits $\pi/2$ and 0 (if the assumption is made that l is very great compared with r)

Thus, at one end

$$H_P = \int_0^{\frac{\pi}{2}} \frac{ci}{2} \sin \theta d\theta$$

giving
$$H_P = \frac{N}{2l} i \quad . \quad . \quad . \quad . \quad . \quad (1.46)$$

i.e. the H at one end is half that at the centre.

Induction of Electromotive Force. Faraday, in 1831, showed that, *whenever the number of lines of magnetic flux linking with an electric circuit is changed, an electromotive force is induced in the circuit, the magnitude of which is proportional to the rate of change of flux.* Thus, if e is the e.m.f. induced by a rate of change of flux of $\frac{d\Phi}{dt}$,

$$e \propto \frac{d\Phi}{dt}$$

The e.m.f. is also proportional to the number of turns of wire, N, in the circuit in which the e.m.f. is induced.

Thus
$$e \propto N \frac{d\Phi}{dt}$$

If e is expressed in volts and Φ in webers, then

$$e = N \frac{d\Phi}{dt}$$

The unit of magnetic flux is defined by this relationship. The weber is the magnetic flux which, linking a circuit of 1 turn, produces in it an e.m.f. of 1 volt as the flux is reduced to zero at a uniform rate, in one second.

Lenz's Law states that "*the direction of the induced e.m.f. is such as to tend to oppose the change in the inducing flux.*" The mathematical expression of this law, in conjunction with Faraday's Law, introduces a negative sign, and gives

$$e = -N \frac{d\Phi}{dt} \quad . \quad . \quad . \quad . \quad (1.47)$$

Statically-induced E.M.F.s. An e.m.f. induced in a stationary electric circuit, by a change in the magnetic flux linking with it, is referred to as a "statically induced" e.m.f. as distinct from a "dynamically induced" e.m.f., which occurs when an electric conductor cuts through a stationary flux.

The simplest method of producing a statically-induced e.m.f. is by inserting one pole of a bar magnet in the space enclosed by a coil of wire. If the coil forms a closed electric circuit, then, upon inserting the magnetic pole, the statically induced e.m.f. produces a current in such a direction that its magnetic effect *opposes* the magnetic field due to the pole. Upon withdrawing the pole, the e.m.f. is in the opposite direction. The current produced by it then has a magnetic effect in the *same direction* as that of the magnetic pole. In each case, therefore, the direction of induced e.m.f. is such as to oppose the change in the interlinking flux.

Statically induced e.m.f.s are more often produced in a circuit by alternating current which is flowing in an adjacent circuit, the

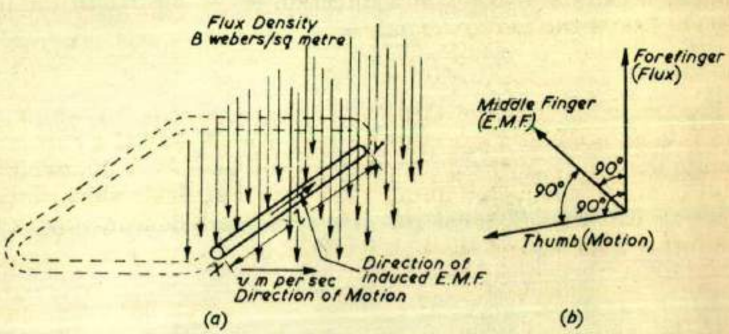

FIG. 1.22. DYNAMICALLY INDUCED E.M.F.

latter being so placed that some of the flux produced by the current in it threads through the other circuit. The magnitude of the induced e.m.f. at any instant depends upon the rate of change of current in the inducing circuit, and upon the relative positions of the two circuits.

Dynamically-induced E.M.F.s. Whenever an electric conductor *cuts through* a magnetic flux an e.m.f. is "dynamically induced" in the conductor. This e.m.f. is proportional to "the rate of cutting flux." There is no difference, as regards the phenomena involved, between statically and dynamically induced e.m.f.s. It can be seen that, if the conductor in Fig. 1.22 (*a*) forms part of a closed circuit (shown dotted), and moves left to right, the direction of the induced e.m.f. is from X to Y, since the flux linking with the circuit is increased. If the conductor moves right to left the direction of e.m.f. is Y to X, since the interlinking flux is then reduced.

The direction of the e.m.f. is given quite simply by the right-hand rule. By this rule, the corresponding directions of motion, flux, and induced e.m.f., are given by the thumb, forefinger, and middle finger respectively of the right hand, these being so placed as to be mutually perpendicular (see Fig. 1.22 (*b*)).

The magnitude of the induced e.m.f. in a conductor which moves in a plane *perpendicular to the flux*, as in Fig. 1.22 (*a*), is

$$e = \text{flux cut per second}$$

$$= B \times \text{area swept out by the conductor per second}$$

$$\therefore \quad e = Blv \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.48)$$

If l is the length of the conductor XY in metres, and v its velocity perpendicular to the field in metres per second, e is expressed in volts.

If a conductor cuts through magnetic flux whilst moving in a direction making an angle θ with that of the lines of force, then the component of its velocity in a direction perpendicular to the field is $v \sin \theta$, and the induced e.m.f. is

$$e = Blv \sin \theta \quad . \quad . \quad . \quad . \quad (1.49)$$

Energy in an Electric Circuit. If the conductor in which the e.m.f. is induced forms part of a closed circuit, and if a current of i units flows in this circuit, then there will be a force opposing the motion of the conductor through the magnetic field, which force is given by Equation (1.36) as Bil. Thus, the work done in moving the conductor a distance x through the field

$$= Bilx$$

If the conductor takes a time t sec to pass through distance x when moving with velocity v, then

$$\text{work done} = Bil \,.\, vt$$

$$= eit$$

$$= \text{energy given to the electric circuit}$$

$$\therefore \text{Energy given to the electric circuit} = eit \quad . \quad . \quad . \quad (1.50)$$

Magnetic Hysteresis. This phenomenon is observed when the current flowing in a solenoid, for the purpose of magnetizing a bar or ring of iron, or other magnetic material upon which the solenoid is wound, is reduced. It is found that the flux density in the iron corresponding to this value of the current is higher than the flux density, corresponding to the *same value of the current*, which was produced when the current was being increased from zero (say) to some maximum value—i.e. the magnetism lags behind the magnetizing force producing it. This effect is known as the "hysteresis effect."

Consider an iron ring upon which is wound, uniformly, a magnetizing winding through which a current can be passed in either direction, as in Fig. 1.23. Suppose the ring has, initially, no magnetism. If the current in the magnetizing winding is increased from zero to some reasonably high value, the magnetizing force acting upon the ring is also increased, since $H = \frac{Ni}{l}$ N being the number of turns on the magnetizing winding, and l metres the length of the magnetic path in the ring.

If the flux density in the ring is measured (by means of a search

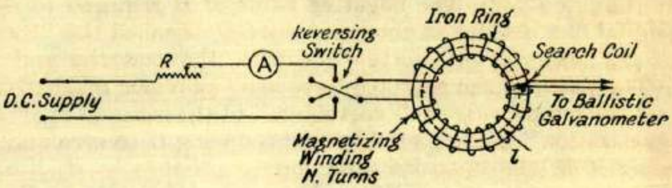

FIG. 1.23. MAGNETIZATION OF AN IRON RING

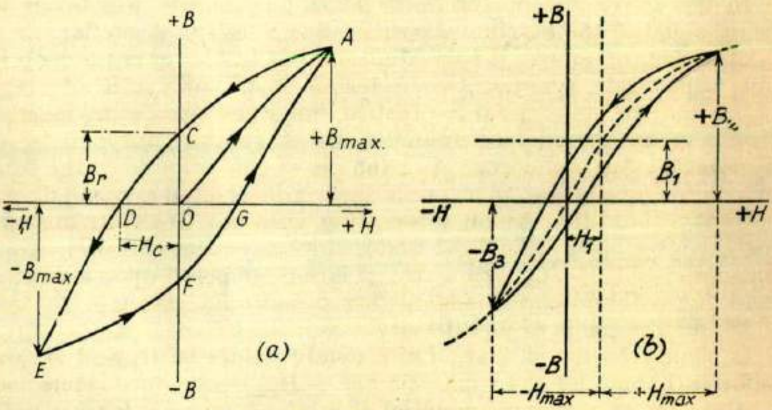

FIG. 1.24. HYSTERESIS LOOPS

coil and ballistic galvanometer as described in Chapter IX) for various values of H, and the B-H curve plotted, it will be found that its shape is as shown by the portion OA in Fig. 1.24 (a). If the current is reduced gradually to zero again, after some flux density B_{max} has been attained, the curve for descending values of H takes the form AC, this curve, owing to the hysteresis effect, being above the ascending curve OA. The flux density B_r remaining in the iron when H is again zero is referred to as the "residual magnetism." This value depends upon the magnitude of B_{max} and upon the material of the specimen, being high for permanent magnet steels, and very low for such material as silicon sheet steel. Since H is now zero, $B_r = J_r$, where J_r is the intensity of magnetization left in the iron when the magnetizing force is removed.

The residual flux density, after magnetization up to saturation point, is referred to as the "remanence" of the iron or steel concerned.

If the direction of the magnetizing current is now reversed, and gradually increased in this reverse direction until a flux density B_{max} is again obtained (but in the opposite direction), the curve follows the line *CDE*. The negative value of H required to reduce the residual flux density to zero, namely *OD*, is called the "coercive force," H_c, and will of course vary with the material and with B_{max}. The value of the coercive force after previous magnetization up to saturation point is the "coercivity" of the iron. If the "cycle of magnetization" is completed by first reducing H to zero and then increasing it in the opposite direction to produce $+ B_{max}$ again, the curve follows the line *EFGA*. The loop so formed is called the "hysteresis loop."

In the above, the maximum negative flux density was taken as being equal to the maximum positive flux density. A similar effect is observed, of course, if this is not so, but in this case the loop is unsymmetrical. Symmetrical hysteresis loops, or cycles of magnetization, are most usual in practice, but when alternating magnetism is superimposed upon unidirectional magnetism, unsymmetrical hysteresis loops will occur. An example of this is found in the core of a radio transformer, when one of the windings often carries a direct current in addition to an alternating current. Such an unsymmetrical loop is shown in Fig. 1.24 (*b*), where a symmetrical alternating magnetic field ($+ H_{max}$ to $- H_{max}$) is superimposed upon a steady field H_1, with its corresponding flux density B_1, giving $+ B_2$ and $- B_3$ for the limits of flux density.

It should be noted that, if the steady values of H_1 and B_1 are sufficiently high for the application of $+ H_{max}$ to produce saturation in the iron or magnetic material then the hysteresis loop is considerably distorted as shown. In such a case $B_2 - B_1$ is not equal to $B_1 + B_3$, although these quantities would be approximately equal if the saturation point of the material is not approached, as when $+ H_{max}$ and $- H_{max}$ are small. In this latter case the hysteresis loop would be very nearly the same in shape as the normal symmetrical loop but would, of course, be displaced relative to the axes of coordinates by H_1 and B_1. The space available will not permit further consideration of this question, but references to works on the subject are given in the bibliography (Refs. (4), (7), (8)).

Incremental Permeability. The term "incremental permeability," introduced by Dr. Thomas Spooner,* refers to a type of permeability which has become increasingly important with the advance of radio communication, since it relates to the case of superimposed

* *Trans. A.I.E.E.*, Vol. XLII, p. 42.

direct and alternating magnetizations. Spooner defines it as "the ratio of ΔB to ΔH for any position on a magnetization curve, or hysteresis loop, where ΔB and ΔH may be of any magnitude, but ΔH must be in the reverse direction from the immediately preceding change."

Referring to Fig. 1.25, the incremental permeability at various points on the major hysteresis loop is given by $\Delta B_1/\Delta H_1$, $\Delta B_2/\Delta H_2$, etc., these being the slopes of the minor hysteresis loops corresponding to increments of H as shown.

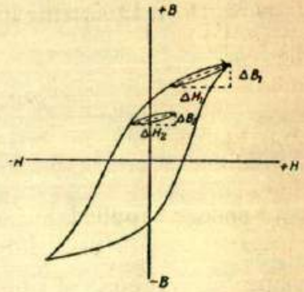

FIG. 1.25.

L. G. A. Sims, discussed and summarized this question in a valuable paper before the British Association in September, 1937 (Ref. (13)).

Hysteresis Loss. Although no energy is required merely to maintain a magnetic field, it is found that energy is required to bring about a cycle of magnetization in a magnetic material. Energy is required to build up a magnetic field owing to the opposing e.m.f., which is induced in the magnetizing circuit when the flux in the magnetic material is increased. This energy is stored in the magnetic field, but it is found that the quantity of energy returned to the magnetizing circuit, when the current is reduced, is less than the quantity supplied when the field was built up. The difference is due to "molecular magnetic friction"—as it has been called by Steinmetz—and the energy absorbed is converted into heat. The energy absorbed when a magnetic material is passed through one cycle of magnetization can be shown to be proportional to the area of the hysteresis loop as below.

Relation between Hysteresis Loss and the Area of the Hysteresis Loop. Consider the magnetization of a ring specimen of iron by means of a magnetizing winding as shown in Fig. 1.23.

If the length of magnetic path in the ring is l metres, and its cross-section is a sq. m, if the number of magnetizing turns is N, and the current in the magnetizing circuit at any instant is i, then the induced e.m.f. (in volts) at any instant is

$$e = \frac{Nd\,(Ba)}{dt}$$

where B = flux density in the ring in webers per square metre.

Thus the power supplied at any instant to overcome this back e.m.f. (i.e. to build up the magnetic field in the ring) is

$$ei = i \, . \, N \frac{d\,(Ba)}{dt}$$

and the energy supplied in order to build up the magnetic field in time t sec is

$$\int_0^t eidt = \int_0^t aiN \cdot \frac{dB}{dt} \cdot dt = a \int_{-B_r}^{B_{max}} iN \cdot dB$$

since, when $t = 0$, $B = B_r$ as shown in Fig. 1.26 in which $OF = OE = B_r$.

Now, the magnetizing force acting upon the ring is, at any instant, given by

$$H = \frac{Ni}{l}$$

from which $$Ni = lH$$

and energy supplied $$= la \int_{-B_r}^{B_{max}} HdB$$

Since la = volume of ring in cubic metres, it follows that the energy in joules supplied per cubic metre to build up the field

$$= \int_{-B_r}^{B_{max}} HdB$$

This energy is stored in the magnetic field, and is represented in Fig. 1.26 by the area $FACDF$.

Upon reducing the current (and hence the flux) the induced e.m.f. is in the *same direction* as the applied e.m.f., so that energy is now returned to the magnetizing circuit as the flux is reduced. From the above reasoning the energy returned during the reduction of the magnetizing force from H_{max} to 0 is $\int_{B_r}^{B_{max}} HdB$ joules per cubic metre, where B_r is the residual flux density.

This energy is represented in the figure by the area ECD. Thus the energy *absorbed by the specimen due to hysteresis* is the difference between energy put in and energy returned to the magnetizing circuit, and is represented by the shaded area $FACEF$.

If the current is now reversed and the above process repeated, H being finally brought back to the starting point A of the cycle of magnetization, the energy in joules absorbed per cubic metre per cycle due to hysteresis will be $\int_0^0 HdB$

= area of hysteresis loop

The area of the loop is, of course, measured to scale, so that the energy absorbed per cubic metre per cycle in joules

$$= \text{area of loop in square centimetres} \times bh \qquad (1.51)$$

where b = flux density in webers per sq. m represented by 1 cm on B axis

h = number of ampere-turns per m, represented by 1 cm on H axis

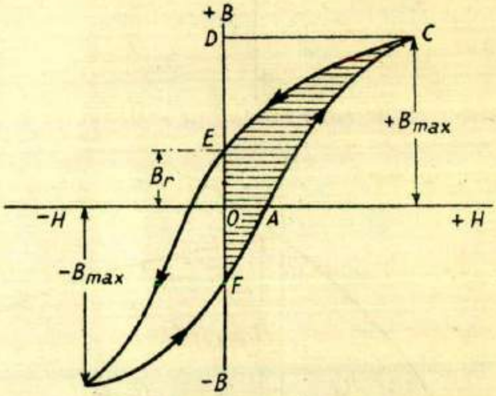

FIG. 1.26. HYSTERESIS LOSS

Vertical Scale: 1 cm = b webers per sq. metre
Horizontal Scale: 1 cm = h ampere-turns per metre

This expression for the energy loss in terms of the area of the loop obviously applies whether the loop is symmetrical or not.

Steinmetz Hysteresis Law. Steinmetz has shown that the empirical law

$$W_h = k \, . \, B_{max}^{1 \cdot 6} \qquad . \quad . \quad . \quad (1.52)$$

gives the hysteresis loss for iron with sufficient accuracy for most practical purposes, provided the maximum flux density B_{max} lies between 0·1 and 1·2 webers per sq. metre

W_h = energy loss in joules per cubic metre per cycle

k = the hysteresis coefficient of the material, and is constant for any given material

The magnitude of k varies with the material. Its value for annealed sheet steel lies between 250 and 500, and for silicon steel is about 210.

For values of B_{max} above 1·2, the energy loss increases at a higher rate than the 1·6th power of B_{max}, this rate increasing with increasing values of B_{max}. For values of B_{max} below 0·1 also, the loss varies as some power of B_{max} greater than 1·6.

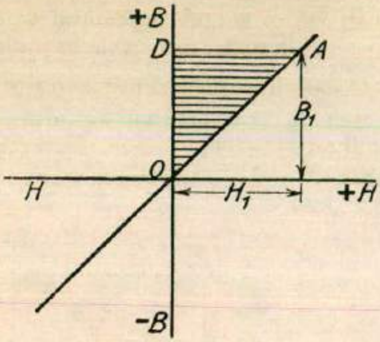

FIG. 1.27. ENERGY STORED IN A MAGNETIC FIELD

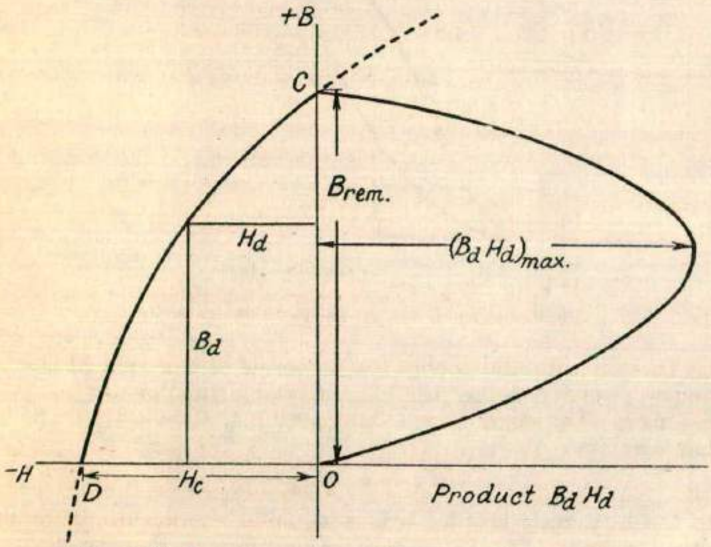

FIG. 1.28. DEMAGNETIZATION CURVE OF MAGNET STEEL

Steinmetz (Ref. (3)) has shown that for silicon steel the law

$$W_h = k' \,.\, B_{max}^2 \quad . \quad . \quad . \quad . \quad (1.53)$$

is more nearly correct, k' being about 457.

With regard to the hysteresis loss in the case of an unsymmetrical cycle, Ball has shown (Ref. (7)) that this obeys the law

$$W_h = k'' B_{max}^{1.6}$$

but that in this case k'' is not constant for any given material but varies with the average value of the flux density.

Thus if B_2 and B_3 are the limiting values of flux density as in Fig. 1.24 (*b*),

$$B_{max} = \frac{B_2 - B_3}{2}$$

Also, Ball found that if the average value of the flux density is

$$B_{av} = \frac{B_2 + B_3}{2}$$

$$k'' = k + \alpha B_{av}^{1.9}$$

where k is the normal hysteresis coefficient of the material.

Thus, in general, energy loss in joules per cubic metre per cycle is

$$W_h = (k + aB_{av}^{1.9})B_{max}^{1.6}$$

$$W_h = \left[k + a\left(\frac{B_2 + B_3}{2}\right)^{1.9}\right] \left(\frac{B_2 - B_3}{2}\right)^{1.6} \qquad (1.54)$$

where B_3 is the lower value of the flux density and may, or may not, be negative.

Values of a are: for ordinary annealed sheet steel $a = 340$. for annealed silicon sheet steel $a = 320$. Obviously if $B_2 = -B_3$, as in a symmetrical cycle, Equation (1.54) reduces to $W_h = kB_2^{1.6}$.

Energy Stored per Unit Volume of Magnetic Field in Air. In the case of a magnetic field in air or other non-magnetic material, the hysteresis loop reduces to a straight line, i.e. the hysteresis loss is zero. If the same scales are used for both B and H, this line makes an angle of 45° with either axis, as in Fig. 1.27.

The energy stored in the field per cubic metre when the flux density is B_1 is $\int_0^{B_1} HdB$

$$= \text{area } OAD \text{ (to scale)}$$

$$= \frac{1}{2} \cdot B_1H_1$$

But $B_1 = \mu_0H_1$, since the field is in air.

$$\therefore \text{ Energy stored in joules per cu.m} = \frac{B_1^2}{2\mu_0} = \frac{\mu_0H_1^2}{2} \qquad (1.55)$$

Demagnetization Curve (Permanent-magnet Design). In the design of permanent magnets, in which the flux density in the air gap is required to be as large as possible, while the magnetism must be resistant to demagnetizing forces, the remanence, coercivity, and the demagnetization curve (portion CD of Fig. 1.24 (*a*)) of the steel to be used are of great importance. For good magnet steel the product $B_r \times H_c$ should be large.

Fig. 1.28 shows the demagnetization curve of magnet steel, this being a portion of the hysteresis loop for the steel in which the maximum magnetization has been up to saturation point, so that OC represents the remanence and OD the coercivity.

As the demagnetizing force H_d is increased, the flux density falls as shown, and at any point, when the flux density in the steel is B_d there is, remaining in the steel, an m.m.f. per metre length of path in it of H_d. This m.m.f. is that which would drive the flux across an air gap in the magnetic circuit of the magnet. The products $B_d \cdot H_d$ are plotted. They exhibit a maximum value $(B_d \cdot H_d)_{max}$ which is a criterion of the value of the steel for permanent-magnet purposes. The most economical design of magnet is that for which B_d and H_d are such that their product has this maximum value. (See Refs. (8), (12), (15), (16), and (17).) The properties of some of the permanent magnet materials are given in Chap. XVII.

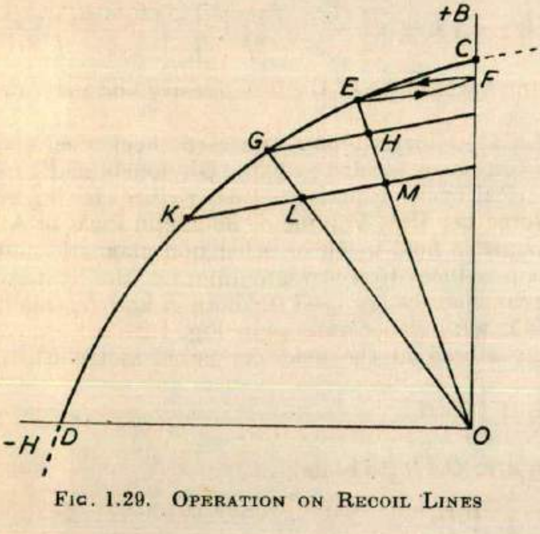

FIG. 1.29. OPERATION ON RECOIL LINES

Suppose the dimensions of the magnet are: length L_m, cross-section A_m; and of its air gap: length L_g, cross-section A_g; and let the respective flux densities be B_d and B_g. Then

m.m.f. across the air gap = m.m.f. in the magnet

i.e.
$$H_g \cdot L_g = H_d \cdot L_m$$

Now, the energy stored in the air gap is

$$(B_g^2/2\mu_0)\, A_g \cdot L_g \text{ joules}$$

$$= \frac{B_g A_g \cdot B_g L_g}{2\mu_0} = \frac{\text{Flux across gap} \times \text{m.m.f. across gap}}{2}$$

If the flux in the gap is equal to the flux in the magnet, the energy stored in the gap is

$$\frac{A_m \,.\, B_d \times H_d \,.\, L_m}{2} = \frac{B_d \,.\, H_d}{2} \times A_m \,.\, L_m$$

Since the volume of steel is $A_m \,.\, L_m$, the energy in the air gap is

$$\frac{B_d \,.\, H_d}{2} \text{ per cubic metre}$$

This is obviously maximum when the product $B_d \,.\, H_d$ is maximum, and, for minimum volume of magnet steel, the magnet must operate at the values of B_d and H_d corresponding to this point. The flux density B_g required in the air gap, and the dimensions of the air gap, are usually known, so the dimensions of the magnet can be obtained from the following expressions:

$$A_m = \frac{B_g \,.\, A_g}{B_d}, \qquad L_m = \frac{H_g \,.\, L_g}{H_d}$$

In practice, due to leakage, the flux in the magnet is greater than that in the gap and we must write

$$K_g \,.\, B_g \,.\, A_g = B_d \,.\, A_m$$

K_g is a constant dependent upon the magnet circuit configuration and can have any value between 1·8 and 9·0.

Where a magnet is provided with soft-iron pole pieces, a part F of the m.m.f. delivered by the magnet is expended in overcoming the reluctance of the joints and pole pieces, and in this case,

$$F + H_g \,.\, L_g = H_d \,.\, L_m$$

This can be written in an alternative form,

$$K_l \,.\, H_g \,.\, L_g = H_d \,.\, L_m$$

where K_l is a constant, usually 1·3 and 1·5.

Combining these two expressions,

$$\frac{B_d}{H_d} = \frac{K_g \,.\, B_g \,.\, A_g \,.\, L_m}{K_l \,.\, H_g \,.\, L_g \,.\, A_m} = \mu_0 \frac{K_g \,.\, A_g \,.\, L_m}{K_l \,.\, L_g \,.\, A_m}$$

The right-hand term is a constant for any given system and is called the "unit permeance." It is the slope of a line from the origin of co-ordinates to the magnet operating point.

It is clear from these considerations that, to achieve a satisfactory choice of magnet dimensions, it is necessary to have some knowledge of the factors K_g and K_l and these can only be determined by trial on a similar system.

Recoil Loops. Referring to Fig. 1.29, suppose the magnet is operating at E and providing flux in an air gap, and the gap is then completely closed. The magnet operating point will then run along the lower line to the point F: it will not return to C. If, now, the gap is restored, the magnet will return along the upper line to E. A minor hysteresis loop has been traced out which is known as a recoil loop. The recoil loop is very narrow and it is a sufficient approximation to replace it by a straight line called a recoil line.

Stabilization. If the magnet is initially operating at point E on the characteristic, the permeance of the system is given by the line OE. Suppose that the magnet is now subjected to an external disturbing field which has a demagnetizing effect and is equivalent to an increase in air gap; then the operating point moves to G. When the disturbing field is removed, the operating point must lie on the original permeance line, but the magnetization now moves along a recoil line to the point H. There is therefore a permanent change in the operating point and a change in the gap flux. This is clearly intolerable when the magnet is used in a permanent-magnet moving-coil instrument, because momentary exposure to an external field may cause a permanent change in calibration.

Stabilization is carried out by applying a fairly large disturbing field which forces the magnetization down to point K, and the recoil is then to M, which is now taken as the normal operating point. If the magnet is subsequently subjected to a disturbing field it will move to point L, say, but when this field is removed it will return to M and there is no permanent change in the operating point.

Modern permanent-magnet materials are often in the form of short blocks of material, and if these are removed from their pole pieces, effectively introducing a very large gap, they will then operate subsequently on a very low recoil line and their magnetic properties are virtually lost. Because of this, magnets are usually supplied unmagnetized and are magnetized in the system of which they form part.

TABLE I

UNRATIONALIZED C.G.S. FORMS OF THE NUMBERED FORMULAE IN CHAPTER I

Formula Number	C.G.S. Form*
(1.1)	$F \propto \dfrac{Q_1Q_2}{\varepsilon r^2}$
(1.2)	$F = \dfrac{Q_1Q_2}{\varepsilon r^2}$ dynes

* It should be borne in mind that all formulae in Table I, even when unchanged in form, are in C.G.S. units which differ from the units used in Chapter I. The primary constants ε_0 and μ_0 are both unity in this system, so that ε and μ correspond to the relative permittivity and relative permeability respectively. The electrical quantities Q and V are in electrostatic units, the current i is in electromagnetic units, all dimensions are in centimetres, and energy is in ergs (see Chapter II and Table II).

(*Note*. In Equations (1.38) and (1.40) D is a distance.)

TABLE I—(*contd.*)

Formula Number	C.G.S Form*
(1.3)	$E = \frac{D}{\varepsilon}$
(1.4)	$E_p = \frac{2Q}{\varepsilon p}$
(1.5)	$E_p = \frac{4\pi\sigma}{\varepsilon}$
(1.6)	$\Psi = 4\pi Q$
(1.7)	$\Psi = 4\pi(Q_1 \pm Q_2 \pm Q_3 \ldots)$
(1.8)	$E = \frac{4\pi\sigma}{\varepsilon}$
(1.9)	Unchanged
(1.10)	Unchanged
(1.11)	$V_p = \frac{Q}{\varepsilon d}$
(1.12)	$V_p = \frac{1}{\varepsilon}\left[\frac{Q_1}{d_1} + \frac{Q_2}{d_2} + \frac{Q_3}{d_3} + \ldots\right]$
(1.13)	Unchanged (cm)
(1.14)	Unchanged (ergs)
(1.15)	Energy per unit volume $= \frac{\varepsilon E^2}{8\pi}$ ergs
(1.16)	$E_x = \frac{\Psi}{a_x \varepsilon}$
(1.17)	$F = \frac{2\pi\sigma . Q}{\varepsilon}$
(1.18)	$F = \frac{m_1 m_2}{r^2}$ dynes
(1.19)	$H = \frac{m}{r^2}$ oersteds
(1.20)	Unchanged (maxwells)
(1.21)	Unchanged (gauss)
(1.22)	Unchanged
(1.23)	Unchanged
(1.24)	$h = 2\pi J$
(1.25)	$B = H + 4\pi J$
(1.26)	$B = \mu H$ (gauss)
(1.27)	Unchanged
(1.28)	Magnetic susceptibility $= \frac{-1}{\pi}$

* See footnote on page

TABLE I—(*contd.*)

Formula Number	C.G.S. Form*
(1.29)	$V = \frac{m}{r}$
(1.30)	$V_p = \frac{M \cos \phi}{r^2}$
(1.31)	$F = \frac{AB^2}{8\pi}$
(1.32)	$V_p = \frac{dA \,.\, i \cos \phi}{r^2}$
(1.33)	$V_p = i\Omega$
(1.34)	Unchanged
(1.35)	$f = \frac{m \,.\, i \,.\, dl \sin \theta}{r^2}$
(1.36)	$F = Hil$ dynes
(1.37)	$dH = \frac{i \,.\, dl \sin \theta}{r^2}$
(1.38)	$H = \frac{2i}{D}$ oersteds
(1.39)	Work done $= 4\pi i$
(1.40)	$F = \frac{2i_1 i_2 l}{D}$
(1.41)	$H = \frac{i \times 2\pi r^2}{R^3}$ oersteds
(1.42)	$H = \frac{2\pi i}{r}$ oersteds
(1.43)	H at $Q = 2\pi r^2 Ni \left[\frac{1}{(r^2 + x^2)^{\frac{3}{2}}} + \frac{1}{(r^2 + (l - x)^2)^{\frac{3}{2}}}\right]$
(1.44)	$F = -\frac{6\pi^2 i_A i_B \,.\, a^2 b^2 x}{(a^2 + x^2)^{\frac{5}{2}}}$
(1.45)	$H = \frac{4\pi Ni}{l} \cos \theta_1$ oersteds
(1.46)	$H_p = \frac{2\pi Ni}{l}$ oersteds
(1.47)	Unchanged
(1.48)	Unchanged
(1.49)	Unchanged
(1.50)	Unchanged
(1.51)	Energy per cm³ in ergs $= \frac{1}{4\pi} \times$ area of loop in sq. cm $\times$ bh

* See footnote on page 48.

TABLE I—(*contd.*)

Formula Number	C.G.S. Form*
(1.52)	Unchanged
(1.53)	Unchanged
(1.54)	Unchanged
(1.55)	Energy stored in ergs per $\text{cm}^3 = \frac{B_1^2}{8\pi} = \frac{H_1^2}{8\pi}$

* See footnote on page 48.

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CHAPTER II

UNITS, DIMENSIONS, AND STANDARDS

Absolute Units. An absolute system of units may be defined as a system in which the various units are all expressed in terms of a small number of fundamental units. The word "absolute" in this sense does not imply supreme accuracy: it is used as opposed to "relative." Absolute measurements do not compare the measured quantity with arbitrary units of the same kind, but are made in terms of appropriate fundamental units.

The Committee of the British Association on Electrical Units and Standards, in formulating the absolute system of units in 1863, had the idea "that the units should not be defined by a series of master standards, each defining one quantity in the way in which the units of length and mass are defined, but that each electrical unit should be defined by some natural law which expresses the relation between the quantity concerned and the fundamental quantities of length, mass and time, for which internationally accepted standards have already been established" (Ref. (66)). Electrical and magnetic units involve, in addition, the properties of the media in which the electrical or magnetic actions take place, i.e. the permittivity in the case of electrostatic forces and the permeability in the case of magnetic forces.

The British Association Committee on Practical Standards for Electrical Measurements adopted as the fundamental units of length, mass and time, the centimetre, gramme, and second respectively, and thus brought into existence the C.G.S. system of units.

Two systems of C.G.S. units exist: one involving only the permittivity ε of the medium as well as units of length, mass, and time; the other involving permeability as well as units of length, mass, and time. The first is known as the electrostatic C.G.S. system of units (e.s.c.g.s. or e.s.u. system), and the second as the electromagnetic C.G.S. system (e.m.c.g.s. or e.m.u. system).

The electromagnetic system is the more convenient from the point of view of most electrical measurements, and is, therefore, much more generally used than the electrostatic system. If a quantity is expressed in "C.G.S. units" without the additional designation "electromagnetic" or "electrostatic," it may be taken that the electromagnetic system is indicated.

In the electrostatic C.G.S. system the permittivity is, for purposes of definition, taken as unity, as is the permeability in the electromagnetic system.

Dimensions of Velocity, Acceleration, and Force. Since velocity $= \frac{\text{length}}{\text{time}}$, this can be expressed in the dimensional notation as

$$[v] = \frac{[L]}{[T]} = [LT^{-1}] \quad . \quad . \quad . \quad . \quad . \quad (2.1)$$

the square brackets indicating that the equality is dimensional only, and does not refer to numerical values.

Thus, velocity has the "dimensions" $[LT^{-1}]$.

Similarly,

$$\text{Acceleration} = \frac{\text{velocity}}{\text{time}} = \frac{\text{length}}{\text{time} \times \text{time}}$$

or, dimensionally,

$$[a] = \frac{[L]}{[T^2]} = [LT^{-2}] \, . \quad (2.2)$$

$$\text{Force} = \text{mass} \times \text{acceleration}$$

thus, representing the dimension of mass by $[M]$,

$$[F] = [M]\,[LT^{-2}] = [MLT^{-2}] \quad . \quad . \quad (2.3)$$

Dimensions in Electrostatic and Electromagnetic Systems. From Coulomb's Inverse Square Law we have, using the C.G.S. system,*

$$F = \frac{Q_1 Q_2}{\varepsilon r^2}$$

i.e. $$\text{Force} = \frac{(\text{quantity of electricity})^2}{\varepsilon \times \text{length}^2}$$

where ε is the permittivity of the medium.

Dimensionally,

$$[F] = \frac{[Q]^2}{[\varepsilon \, . \, L^2]}$$

i.e $$[MLT^{-2}] = \frac{[Q]^2}{[\varepsilon L^2]}$$

From which

$$[Q] = [\varepsilon^{\frac{1}{2}} L^{\frac{3}{2}} M^{\frac{1}{2}} T^{-1}] \quad . \quad . \quad . \quad . \quad (2.4)$$

which gives the dimensions of Q in the electrostatic system.

Also, in magnetism, the force between two magnetic poles of pole

* Dimensions can be derived in a similar manner using the M.K.S. relationships. The results are the same except that ε_0 and μ_0 are included in ε and μ, but see p. 59.

strengths m_1 and m_2, distant r apart, in a medium of permeability μ is

$$F = \frac{m_1 m_2}{\mu r^2}$$

$$\text{Force} = \frac{\text{pole strength} \times \text{pole strength}}{\mu \times \text{length}^2}$$

Dimensionally,

$$[MLT^{-2}] = \frac{[m]^2}{[\mu L^2]}$$

From which

$$[m] = \mu^{\frac{1}{2}} L^{\frac{3}{2}} M^{\frac{1}{2}} T^{-1} \quad . \quad . \quad . \quad . \quad . \quad (2.5)$$

in the electromagnetic system.

Thus, it is seen that the dimensions of these two quantities, one electrostatic and one magnetic, involve the dimensions of either ε or μ as well as those of length, mass, and time. It will be seen later that the same holds for all such quantities.

Instead of using either μ or ε as the necessary fourth fundamental dimension, any one of the electrical or magnetic magnitudes could be used. G Giorgi (see Ref. (67)) suggested that quantity of electricity Q might be used and that this would eliminate fractional exponents in dimensional expressions. Thus, for example, the product QV (where V = potential) has the dimensions of work so that

$$[QV] = [ML^2T^{-2}]$$

from which

$$[V] = [ML^2T^{-2}Q^{-1}]$$

Dimensions of Permeability (μ) **and** (ε). If they are taken as fundamental dimensions, the dimensions of these quantities cannot be expressed in terms of length, mass, and time, but a relationship between them can be found.

As seen in the preceding paragraph, the dimensions of a quantity of electricity can be expressed in terms of ε, etc., as

$$[Q] = [\varepsilon^{\frac{1}{2}} L^{\frac{3}{2}} M^{\frac{1}{2}} T^{-1}]$$

Now, the force exerted upon a magnetic pole, of strength m units placed at the centre of a circular wire of radius r due to a current of i flowing in an arc of the circle of length l is given by

$$F = \frac{mil}{r^2}$$

$$\therefore i = \frac{Fr^2}{ml}$$

Quantity of electricity flowing in time t is

$$Q = it = \frac{Fr^2 t}{ml}$$

Dimensionally,

$$[Q] = \frac{[MLT^{-2}][L^2][T]}{[\mu^{\frac{1}{2}}L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}][L]}$$

substituting the expression for m from the previous paragraph.

$$\therefore [Q] = [M^{\frac{1}{2}}L^{\frac{1}{2}}\mu^{-\frac{1}{2}}] \quad . \quad . \quad . \quad . \quad (2.6)$$

This gives the dimensions of Q in the electromagnetic system. Since Q must have the same dimensions in either system, we have

$$[\varepsilon^{\frac{1}{2}}L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}] = [M^{\frac{1}{2}}L^{\frac{1}{2}}\mu^{-\frac{1}{2}}]$$

$$\therefore [\varepsilon^{\frac{1}{2}}LT^{-1}] = [\mu^{-\frac{1}{2}}]$$

or

$$[LT^{-1}] = [\mu^{-\frac{1}{2}}\varepsilon^{-\frac{1}{2}}] \quad . \quad . \quad . \quad . \quad (2.7)$$

Now, the dimensions $[LT^{-1}]$ are those of a velocity.

$$\therefore \frac{1}{\sqrt{\mu\varepsilon}} = \text{a velocity}$$

In any system of units the "permeability of free space" μ_0 and the "permittivity of free space" ε_0 are related by the equation

$$\mu_0\varepsilon_0 = \frac{1}{c^2}$$

Where c is the velocity of light in the system of units considered. From this relationship the dimensions of any electrical quantity can be converted from those of the electrostatic system to those of the electromagnetic system, and *vice versa*.

Dimensions of Electrical and Magnetic Quantities. The dimensions of the various quantities can be derived from the known relationships between them, as shown below.

1. *Electric Current.*

$$\text{Current} = \frac{\text{quantity}}{\text{time}}$$

$$[I] = \frac{[\varepsilon^{\frac{1}{2}}L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}]}{[T]} = [\varepsilon^{\frac{1}{2}}L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}]$$

in the E.S. system.

To convert these dimensions to those of the E.M. system—involving μ instead of ε—substitute $\mu^{-\frac{1}{2}} . L^{-1}T$ for $\varepsilon^{\frac{1}{2}}$ from Equation (2.7).

Thus, in the E.M. system

$$[I] = [\mu^{-\frac{1}{2}}L^{-1}TL^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}] = [\mu^{-\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}]$$

2. *Electric Potential.* By definition,

$$\text{Potential} = \frac{\text{work}}{\text{quantity of electricity}}$$

Thus, representing the dimensions of potential by $[V]$,

$$[V] = \frac{[MLT^{-2}][L]}{[\varepsilon^{\frac{1}{2}}L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}]} = [\varepsilon^{-\frac{1}{2}}L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}]$$

in the E.S. system.

Converting to the E.M. system, we have

$$[V] = [\mu^{\frac{1}{2}} . LT^{-1} . L^{\frac{1}{2}} . M^{\frac{1}{2}}T^{-1}] = [\mu^{\frac{1}{2}} . L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}]$$

in the E.M. system.

3. *Magnetic Flux.*

$$\text{e.m.f.} = \text{rate of change of flux}$$
$$= \frac{\text{flux}}{\text{time}}$$
$$\therefore \text{Flux} = \text{e.m.f.} \times \text{time}$$

Dimensionally $[\Phi] = [\varepsilon^{-\frac{1}{2}}L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}]\,[T] = [\varepsilon^{-\frac{1}{2}}L^{\frac{1}{2}}M^{\frac{1}{2}}]$

in the E.S. system, or

$$[\Phi] = [\mu^{\frac{1}{2}}L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}] \text{ in the E.M. system.}$$

The dimensions of the most important electrical and magnetic quantities in both systems, together with the relationships from which they are derived, are given in Table II.

Practical and C.G.S. Units. Some of the electromagnetic C.G.S. units are too small for practical purposes, while others are too large. The British Association Committee fixed the practical unit of current and resistance as $\frac{1}{10}$ and 10^9 electromagnetic units of current and resistance respectively. The magnitudes of the practical units of other quantitites, in terms of the absolute electromagnetic units, can be found from the known relationships connecting the various quantities and are given in the table.

For example,

$$\text{e.m.f.} = \text{current} \times \text{resistance}$$

$\therefore$ The practical unit of e.m.f. = the practical unit of current $\times$ the practical unit of resistance

$= \frac{1}{10}$ of the e.m. unit of current $\times$ 10^9 e.m. units of resistance

$= 10^8$ e.m. units of e.m.f.

The International Conference on Electrical Units and Standards in London, 1908, agreed that the magnitudes of the fundamental

TABLE II

Dimensions of Electrical and Magnetic Quantities

Quantity	Symbol	Equation from which the Dimensions are Derived	Dimensions		Practical Unit	Number of Electro-magnetic Units in One Practical Unit	Ratio— 1 E.S. unit / 1 E.M. unit (neglecting L, M, and T)	Number of Electrostatic Units in One Practical Unit
			Electromagnetic System	Electrostatic System				
Quantity of electricity	Q, q	$F = \frac{Q_1Q_2}{\varepsilon r^2}$	$L^{\frac{1}{2}}M^{\frac{1}{2}}\mu^{-\frac{1}{2}}$	$L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}\varepsilon^{\frac{1}{2}}$	Coulomb Ampere-hour	10^{-1} 360	$\varepsilon^{\frac{1}{2}}\mu^{\frac{1}{2}} = \frac{1}{3 \times 10^{10}}$	3×10^{9} $10{\cdot}8 \times 10^{12}$
Current	I, i	$I = \frac{Q}{t}$	$L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}$	$L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}\varepsilon^{\frac{1}{2}}$	Ampere	10^{-1}	$\varepsilon^{\frac{1}{2}}\mu^{\frac{1}{2}} = \frac{1}{3 \times 10^{10}}$	3×10^{9}
E.m.f. or potential	V, e, E	$E = \frac{\text{Work}}{\text{Quantity}}$	$L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}\mu^{\frac{1}{2}}$	$L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\varepsilon^{-\frac{1}{2}}$	Volt	10^{8}	$\varepsilon^{-\frac{1}{2}}\mu^{-\frac{1}{2}} = 3 \times 10^{10}$	$\frac{1}{3 \times 10^{2}}$
Resistance	R	$R = \frac{E}{I}$	$LT^{-1}\mu$	$L^{-1}T\varepsilon^{-1}$	Ohm	10^{9}	$\varepsilon^{-1}\mu^{-1} = 9 \times 10^{20}$	$\frac{1}{9 \times 10^{11}}$
Capacitance	C	$C = \frac{Q}{E}$	$L^{-1}T^{2}\mu^{-1}$	$L\varepsilon$	Farad	10^{-9}	$\varepsilon\mu = \frac{1}{9 \times 10^{20}}$	9×10^{11}
Inductance	L	$e = L\frac{di}{dt}$	$L\mu$	$L^{-1}T^{2}\varepsilon^{-1}$	Henry	10^{9}	$\varepsilon^{-1}\mu^{-1} = 9 \times 10^{20}$	$\frac{1}{9 \times 10^{11}}$
Impedance	Z	$Z = \frac{E}{I}$	$LT^{-1}u$	$L^{-1}T\varepsilon^{-1}$	Ohm	10^{9}	$\varepsilon^{-1}\mu^{-1} = 9 \times 10^{20}$	$\frac{1}{9 \times 10^{11}}$
Pole strength	m	$F = \frac{m_1m_2}{\mu r^2}$	$L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}$	$L^{\frac{1}{2}}M^{\frac{1}{2}}\varepsilon^{-\frac{1}{2}}$				
Magnetic field intensity	H	$H = \frac{B}{\mu}$	$L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}$	$L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-2}\varepsilon^{\frac{1}{2}}$				
Magnetic flux	Φ	$e = N\frac{d\phi}{dt}$	$L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}$	$L^{\frac{1}{2}}M^{\frac{1}{2}}\varepsilon^{-\frac{1}{2}}$				
Magnetic flux density	B	$B = \frac{\phi}{a}$	$L^{-\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{\frac{1}{2}}$	$L^{-\frac{3}{2}}M^{\frac{1}{2}}\varepsilon^{-\frac{1}{2}}$				
Magnetomotive force	F	$F = H \, . \, l$	$L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}$	$L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}\varepsilon^{\frac{1}{2}}$				
Permeability	μ		μ	$L^{-2}T^{2}\varepsilon^{-1}$				
Reluctance	S	$S = \frac{l}{a} \times \frac{1}{\mu}$	$L^{-1}\mu^{-1}$	$LT^{-2}\varepsilon$				
Magnetic potential	V	$V = \frac{\text{Work}}{\text{Pole strength}}$	$L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}$	$L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}\varepsilon^{\frac{1}{2}}$				
Electric power	P	$P = \frac{\text{Work}}{\text{Time}} = EI$	$L^{2}MT^{-3}$	$L^{2}MT^{-3}$	Watt	10^{7}	1	10^{7}
Electric energy	W	$W = \text{Force} \times \text{distance} = EIt$	$L^{2}MT^{-2}$	$L^{2}MT^{-2}$	Joule Kilowatt-hour	10^{7} $3{\cdot}6 \times 10^{13}$	1 1	10^{7} $3{\cdot}6 \times 10^{13}$

electrical units should be determined on the E.M.C.G.S. system and adopted as these fundamentals (i) the Ohm = 10^9 e.m.u., (ii) the Ampere = 0·1 e.m.u., (iii) the Volt = 10^8 e.m.u. and (iv) the Watt = 10^7 e.m.u.

The *electromagnetic C.G.S. unit of resistance* is defined as the resistance of a conductor such that 1 erg of energy is expended per second when unit current passes through it.

The *electromagnetic C.G.S. unit of current* is defined as the current which, flowing in the arc of a circle 1 cm in length and 1 cm in radius, produces a force of 1 dyne on a unit magnetic pole placed at its centre.

The determination of the number of electrostatic C.G.S. units in one practical unit can best be illustrated by an example. In the case of e.m.f., from the corresponding dimensions given in Table II,

$$\left[\frac{1 \text{ e.s.u.}}{1 \text{ e.m.u.}}\right] = \left[\frac{L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\varepsilon^{-\frac{1}{2}}}{L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2} \,.\, \mu^{\frac{1}{2}}}\right]$$

Now, if the dimensions of L, M, and T are neglected, this ratio is equal to $\varepsilon^{-\frac{1}{2}}\mu^{-\frac{1}{2}}$. But it has been shown above that

$$\varepsilon^{-\frac{1}{2}}\mu^{-\frac{1}{2}} = \text{a velocity}$$
$$= 2{\cdot}998 \times 10^{10} \text{ cm/sec in the c.g.s system}$$

$$\therefore \frac{1 \text{ e.s.u.}}{1 \text{ e.m.u.}} = 2{\cdot}998 \times 10^{10}$$

1 e.s.u. of e.m.f. or potential = $2{\cdot}998 \times 10^{10}$ e.m.u. units of e.m.f. or potential

Note. In the following conversions the approximation $2{\cdot}998 \doteqdot 3$ has been used for the sake of simplification.

Thus, if the volt is equivalent to 10^8 e.m.u. it will be equivalent to $\dfrac{10^8}{3 \times 10^{10}} = \dfrac{1}{3 \times 10^2}$ e.s.u.

It must be noted that the above ratio of $\dfrac{1 \text{ e.s.u.}}{1 \text{ e.m.u.}}$ does not in all cases equal 3×10^{10}, but depends upon the dimensions. Thus, in the case of capacitance, the ratio $\dfrac{1 \text{ e.s.u.}}{1 \text{ e.m.u.}}$ equals, neglecting dimensions of L, M, and T, the product

$$\mu\varepsilon = \frac{1}{(3 \times 10^{10})^2} = \frac{1}{9 \times 10^{20}}$$

from which

1 e.s.u. of capacitance = $\dfrac{1}{9 \times 10^{20}}$ of 1 e.m.u. of capacitance

The other practical units can be determined from the four fundamental units, and are as follows—

Quantity. The quantity of electricity passed through a circuit by 1 amp in 1 sec is 1 *Coulomb*

$$= 10^{-1} \text{ e.m.u. of quantity}$$

Work or Energy. The unit of work or energy is 1 *Joule* (or 1 watt-second), and is the energy expended by 1 watt in 1 sec

$$1 \text{ joule} = 10^7 \text{ ergs (C.G.S. units of work)}$$

Capacitance. A capacitor has unit capacitance—1 *Farad*—when 1 coulomb of electricity raises the potential difference between its plates 1 volt.

$$1 \text{ farad} = \frac{1 \text{ coulomb}}{1 \text{ volt}} = \frac{10^{-1}}{10^8} = 10^{-9} \text{ e.m.u.}$$

Inductance. The practical unit of inductance is 1 *Henry.* It is the inductance of a circuit such that a rate of change of current in the circuit of 1 amp per sec induces an e.m.f. of 1 volt.

$$1 \text{ henry} = \frac{1 \text{ volt}}{1 \text{ amp per sec}} = \frac{10^8}{10^{-1}} = 10^9 \text{ e.m.u.}$$

The C.G.S. unit of inductance is 1 cm.

Thus $$1 \text{ henry} = 10^9 \text{ cm}$$

Since the farad is too large a unit for many practical cases, the microfarad (represented symbolically as "μF") or the micromicrofarad (represented by "$\mu\mu$F" symbolically) are used as more convenient units.

In the same way, the millihenry and microhenry are often used as more convenient units of inductance than the henry.

In the electrostatic system, capacitance has the dimensions $L\epsilon$, which gives the electrostatic C.G.S. unit as 1 cm.

Additional names have been given to several of the C.G.S. units by the Symbols, Units, and Nomenclature (S.U.N.) Committee of the International Union of Pure and Applied Physics.* The most important of these are—

The Maxwell (the C.G.S. unit of magnetic flux),
The Gauss (the C.G.S. unit of magnetic flux density),
The Oersted (the C.G.S. unit of intensity of magnetizing field),
The Gilbert (the C.G.S. unit of magnetomotive force).

The name *Weber* has been given to the practical unit of magnetic flux (1 weber = 10^8 maxwells), with 1 weber per square metre as the corresponding unit of flux density.

* Report published October, 1934.

The M.K.S. (or Giorgi) System of Units. This system, which uses the metre, kilogramme and second as the units of length, mass and time instead of the centimetre, gramme and second, as in the C.G.S. systems, was originally suggested by Prof. G. Giorgi in 1901. After international discussions on the matter (Refs. (65), (67)) the system was finally adopted by the International Electrotechnical Commission (I.E.C.) at its meeting in 1938 at Torquay, when the connecting link between the electrical and mechanical units recommended was the "permeability of free space with the value of $\mu_0 = 10^{-7}$ in the unrationalized system or $\mu_0 = 4\pi 10^{-7}$ in the rationalized system." (The question of rationalization which does not form an essential part of the M.K.S. system but is actually a separate consideration, will be discussed a little later.)

The M.K.S. system is an absolute system of units, based on its own definitions. It is independent of the C.G.S. systems, which it can be used to replace although the long-established C.G.S. systems are not likely to be entirely discarded for some time to come.

Its great advantage is that its units are identical with the practical units and are the same whether built up from the electromagnetic or electrostatic theory. The rather cumbersome conversions necessary to relate the units of the electromagnetic and electrostatic C.G.S. systems to those of the practical system—given in Table II—are thus avoided.

It is important to remember that, in the M.K.S. system, the constants μ_0 and ε_0, the "permeability and permittivity of free space," must be used in their proper places in the expressions for the particular units being considered. They cannot be omitted, because they are not unity, as in the C.G.S. systems. From the relationship

$$\frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c = \text{velocity of light}$$

(see also p. 55) it follows that if, in the unrationalized system, $\mu_0 = 10^{-7}$ then $\varepsilon_0 = 1{\cdot}113 \times 10^{-10}$. L. H. A. Carr (Ref. 65) suggests that μ_0 should be given dimensions $L^{-2}T^2$ and that ε_0 is a pure numeric. If this is accepted, then the dimensions of the quantities given in Table II become the same in the two systems—electromagnetic and electrostatic—and the fourth and fifth columns of the table could then be replaced by one column (substituting everywhere these dimensions for μ and taking ε as dimensionless). Thus, the first three lines of this replacing column would read

$$L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}$$

$$L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}$$

$$L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}$$

Some of the principal derived units, mechanical and electrical, in the M.K.S. system are given below; some of these units have, necessarily, been introduced already, in Chapter I, but are included again here for completeness in the treatment.

Area . . .	The square metre
Volume . .	The cubic metre
Velocity . .	The metre per second
Acceleration .	The metre per second per second
Force . . .	The Newton, which is the force which produces, in a mass of 1 kilogramme, an acceleration of 1 metre per sec^2 (1 newton $= 10^5$ dynes).
Work or Energy .	The Joule, or Newton-metre
Power . . .	The Watt, or Newton-metre per second
Electric Current .	The Ampere, defined as the constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular sections, and placed 1 metre apart in a vacuum, will produce between these conductors a force equal to 2×10^{-7} newton per metre of length.

The M.K.S. units of quantity of electricity, e.m.f., resistance, inductance and capacitance are respectively the Coulomb, Volt, Ohm, Henry and Farad. The unit of magnetic flux is the Weber, which is equivalent to 10^8 maxwells (or lines of force) so that a rate of change of 1 weber per second, in the flux linking one turn, produces unit e.m.f. Flux density is measured in Webers per square metre.

The use of the constant $\mu_0 = 10^{-7}$, in defining the (unrationalized) M.K.S. units needs some explanation.

Consider the expression for the force between parallel, current-carrying conductors. In the electromagnetic C.G.S. system, this is often written

$$F = 2\,\frac{i_1 i_2 l}{d}$$

where F is in dynes, i_1 and i_2 are currents in e.m. units and l and d are in centimetres.

Strictly the expression should be written

$$F = 2\,\frac{i_1 i_2 l}{d} \times 1$$

the figure 1 representing the permeability of air or vacuum.

Now, if F is to be in newtons and the currents i_1 and i_2 in M.K.S. units (i.e. amperes) the expression becomes

$$F = 2\,\frac{i_1 i_2}{10^7}\,\frac{l}{d} = 2\,\frac{i_1 i_2 l}{d} \times 10^{-7}$$

But, if $\mu_0 (= 10^{-7})$ is included, as it must be in the M.K.S. system, we have

$$F = 2\frac{i_1 i_2 l}{d}\mu_0$$

which is in strict accordance with the M.K.S. definition of the unit of current when i_1, i_2, l and d are all made equal to unity.

It should be understood clearly that if, in any calculation, some medium is involved which has a permeability μ_r times that of air, this value must be included in addition to μ_0 so that we would then have a factor $\mu_r \mu_0$.

Rationalized Systems of Units. The basis of a "rationalized" system of units is the conception of *unit* flux issuing from unit magnetic pole, or from unit charge, instead of a flux of 4π. Clearly "rationalization" is not specially connected with the M.K.S. system: it could be used with any system founded on the same basic electric or magnetic definitions. But, if we are to adopt the M.K.S. system, there is a good argument for adopting it in rationalized form and so performing two stages of adoption at the same time. The question was considered at the I.E.C., Torquay, meeting in 1938 but it was agreed to postpone formal endorsement until some later date.

In the rationalized system the 4π disappears in some relationships only to appear in another place in others. The advantage of rationalization must then be judged by its effect upon the relationships which one considers to be most important. In the I.E.E. papers mentioned in Ref. (65) strong arguments for rationalization are offered as well as some reasons against it. Limitations of space prevent the statement of these arguments here; the reader specially interested in the question must be referred to these papers.

In one of the papers, by H. Marriott and A. L. Cullen, a distinction is made between rationalization of theory and that of the units themselves. Table III, compiled from this paper, shows the differences between the unrationalized and rationalized forms of some of the most important formulae. In the table μ and ε are the absolute permeability and permittivity of the medium (as distinct from the values for free space).

In the rationalized system of units the permeability of free space is $\mu_0 = 4\pi 10^{-7}$ (instead of $\mu_0 = 10^{-7}$ in the unrationalized system). Again, the permittivity of free space is $\varepsilon_0 = \dfrac{1{\cdot}113 \times 10^{-10}}{4\pi}$ $= 8{\cdot}854 \times 10^{-12}$ instead of $\varepsilon_0 = 1{\cdot}113 \times 10^{-10}$;

E. Bradshaw (Ref. (65)) gives the relative sizes of the units in the C.G.S. and rationalized and unrationalized M.K.S. systems, and Table IV reproduces some of the relationships for the more commonly used quantities.

Relationships Between the Mechanical, Electrical, and Thermal Practical Units. The relationships connecting the practical units

TABLE III

Rationalized and Unrationalized Forms of Electrical and Magnetic Equations or Formulae

Equation or Formula	Rationalized	Unrationalized
Force F between isolated electric charges Q_1 and Q_2 distant x	$F = \frac{1}{\varepsilon}\frac{Q_1Q_2}{4\pi x^2}$	$F = \frac{1}{\varepsilon}\frac{Q_1Q_2}{x^2}$
Electric flux density D near a uniformly charged plane surface of charge density σ	$D = \sigma$	$D = 4\pi\sigma$
Electric field strength E near a uniformly charged plane surface of charge density σ	$E = \frac{\sigma}{\varepsilon}$	$E = \frac{4\pi\sigma}{\varepsilon}$
Electric flux density D at distance r from a point charge Q	$D = \frac{Q}{4\pi r^2}$	$D = \frac{Q}{r^2}$
Electric field E at distance r from a point charge Q	$E = \frac{1}{\varepsilon}\frac{Q}{4\pi r^2}$	$E = \frac{Q}{\varepsilon r^2}$
Electric potential V at distance r from a point charge Q	$V = -\frac{1}{\varepsilon}\frac{Q}{4\pi r}$	$V = -\frac{Q}{\varepsilon r}$
Electric energy density	$\frac{1}{2}\varepsilon E^2$	$\frac{\varepsilon E^2}{8\pi}$
Capacitance of parallel-plate capacitor, of plate area A and separation d	$C = \frac{\varepsilon A}{d}$	$C = \frac{\varepsilon A}{4\pi d}$
Capacitance of concentric capacitor of radii a, b and length l	$C = \frac{\varepsilon \cdot 2\pi l}{\log_e \frac{b}{a}}$	$C = \frac{\varepsilon l}{2 \log_e \frac{b}{a}}$
Capacitance of concentric sphere capacitor of radii a and b	$C = \varepsilon \frac{4\pi ab}{b - a}$	$C = \frac{\varepsilon ab}{b - a}$
Force F between small isolated magnetic poles m_1 and m_2 distance x	$F = \frac{1}{\mu}\frac{m_1m_2}{4\pi x^2}$	$F = \frac{1}{\mu}\frac{m_1m_2}{x^2}$
Magnetic flux density B near a uniformly magnetized plane surface of pole strength surface density m'	$B = m'$	$B = 4\pi m'$
Magnetic field strength H near a uniformly maghetized plane surface of pole strength surface density m'	$H = \frac{m'}{\mu}$	$H = \frac{4\pi m'}{\mu}$
Magnetic flux density B at distance r from a point magnetic pole m	$B = \frac{m}{4\pi r^2}$	$B = \frac{m}{r^2}$
Magnetic energy density	$\frac{1}{2}\mu H^2$	$\frac{\mu H^2}{8\pi}$
Force F between parallel isolated current elements $I_1\delta l_1$ and $I_2\delta l_2$. The perpendicular distance between the two directions of the currents I_1 and I_2 is x	$F = \mu \frac{I_1\delta l_1 I_2\delta l_2}{4\pi x^2}$	$F = \mu \frac{I_1\delta l_1 I_2\delta l_2}{x^2}$

TABLE III—(*contd.*)

Equation or Formula	Rationalized	Unrationalized
Magnetic field δH at distance x from current element $I\delta l$. θ = angle between directions $I\delta l$ and x	$\delta H = \frac{I\delta l}{4\pi x^2} \sin\theta$	$\delta H = \frac{I\delta l}{x^2} \sin\theta$
Force δF on current element $I\delta l$ situated in magnetic flux density B. ϕ = angle between the directions of B and $I\delta l$	$\delta F = BI\delta l \sin\phi$	$\delta F = BI\delta l \sin\phi$
Magnetic field strength H inside a long solenoid of n turns per unit length carrying current I	$H = nI$	$H = 4\pi nI$
Magnetic field H at a distance r from a long straight wire carrying a current I	$H = \frac{I}{2\pi r}$	$H = \frac{2I}{r}$
Ampere-turns for a closed magnetic path	$NI = \oint H \,.\, dl$	$NI = \frac{1}{4\pi}\oint H \,.\, dl$
E.m.f. induced in conductor δl moving transversely at velocity v in magnetic flux density B. ϕ = angle between the direction of B and the plane $(v, \delta l)$	$\delta E = Bv\delta l \sin\phi$	$\delta E = Bv\delta l \sin\phi$
E.m.f. induced by changing magnetic flux linkage	$E = -\frac{d\phi}{dt}$	$E = -\frac{d\phi}{dt}$
Inductance L of single-turn solenoid of cross-sectional area A and length d	$L = \frac{\mu A}{d}$	$L = \frac{4\pi\mu A}{d}$

for the measurement of mechanical power, energy, and heat, with those for the measurement of electrical power and energy are so important that a consideration of them here is, perhaps, not misplaced.

Power.

$$1 \text{ h.p.} = 33{,}000 \text{ ft-lb per min}$$
$$= 550 \text{ ft-lb per sec}$$
$$= 550 \times 12 \times 2{\cdot}54 \times 453{\cdot}6 \times 981 \text{ cm-dynes (or ergs) per sec}$$
$$= 746 \times 10^7 \text{ ergs per sec}$$

∴ Since $1 \text{ watt} = 10^7 \text{ ergs per sec}$

$$1 \text{ h.p.} = 746 \text{ watts}$$

$$1 \text{ ft-lb per sec} = \frac{746}{550} = 1{\cdot}357 \text{ watts}$$

TABLE IV

K_1 = number of M.K.S. unrationalized units in 1 M.K.S. rationalized unit

K_2 = number of C.G.S. electromagnetic units in 1 M.K.S. unrationalized unit

K_3 = number of C.G.S. electrostatic units in 1 C.G.S. electromagnetic unit

K_1K_2 = number of C.G.S. electromagnetic units in 1 M.K.S. rationalized unit

$K_1K_2K_3$ = number of C.G.S. electrostatic units in 1 M.K.S. rationalized unit

K_2K_3 = number of C.G.S. electrostatic units in 1 M.K.S. unrationalized unit

c = free space velocity of propagation = $2{\cdot}998 \times 10^8$ metres/sec

Quantity	Symbol	M.K.S. Rationalized Unit	K_1	K_2	K_3
Force	F	Newton	1	10^5	1
Energy	W	Newton-metre : Joule	1	10^7	1
Power	P	Joule/sec : Watt	1	10^7	1
Current	I	Ampere	1	10^{-1}	$100c$
Charge	Q	Coulomb	1	10^{-1}	$100c$
Charge surface density	σ	Coulomb/metre²	1	10^{-5}	$100c$
Potential difference	V	Joule/coulomb : Volt	1	10^8	$\frac{1}{(100c)}$
Electric field strength	E	Volt/metre	1	10^6	$\frac{1}{(100c)}$
Resistance	R	Volt/ampere : Ohm	1	10^9	$\frac{1}{(100c)^2}$
Resistivity	ρ	Ohm-metre	1	10^{11}	$\frac{1}{(100c)^2}$
Electric flux	ψ	Coulomb	4π	10^{-1}	$100c$
Electric flux density	D	Coulomb/metre²	4π	10^{-5}	$100c$
Capacitance	C	Coulomb/volt : Farad	1	10^{-9}	$(100c)^2$
Permittivity	ε	Farad/metre	4π	10^{-11}	$(100c)^2$
Magnetic flux	Φ	Volt-sec : Weber	1	10^8	$\frac{1}{(100c)}$
Magnetic flux density	B	Weber/metre²	1	10^4	$\frac{1}{(100c)}$
Magneto motive force	F	Ampere	4π	10^{-1}	$100c$
Magnetic field strength	H	Ampere/metre	4π	10^{-3}	$100c$
Inductance	L	Weber/ampere : Henry	1	10^9	$\frac{1}{(100c)^2}$
Permeability	μ	Henry/metre	$1/4\pi$	10^7	$\frac{1}{(100c)^2}$
Magnetic pole strength	m	Weber	$1/4\pi$	10^8	$\frac{1}{(100c)}$
Resistance of free space	R_0	Ohm	$1/4\pi$	10^9	$\frac{1}{(100c)^2}$

Energy.

$$
\begin{aligned}
1 \text{ kilowatt-hour (kWh)} &= 1{,}000 \text{ watt-hours} \\
1 \text{ h.p.-hr} &= 746 \text{ watt-hours} \\
&= 0{\cdot}746 \text{ kWh} \\
1 \text{ ft-lb} &= 12 \times 2{\cdot}54 \times 453{\cdot}6 \times 981 \text{ cm-dynes (or ergs)} \\
&= 1{\cdot}357 \times 10^7 \text{ ergs} \\
&= 1{\cdot}357 \text{ joules (or watt-sec)} \\
&= \frac{1{\cdot}357}{60 \times 60 \times 1{,}000} \text{ kWh}
\end{aligned}
$$

or
$$1 \text{ ft-lb} = 0{\cdot}000000377 \text{ kWh}$$

Thermal Units.

$$
\begin{aligned}
1 \text{ gramme-caloric} &= 4{\cdot}18 \times 10^7 \text{ ergs} \\
&= 4{\cdot}18 \text{ joules (or watt-seconds)}
\end{aligned}
$$

1 B.Th.U. (i.e. the heat required to raise the temperature of 1 lb water 1° F)

$$
\begin{aligned}
&= 778 \text{ ft-lb} \\
&= 778 \times 0{\cdot}000000377 \text{ kWh of electrical energy} \\
&= 0{\cdot}000293 \text{ kWh}
\end{aligned}
$$

1 Centigrade heat unit (i.e. the heat required to raise the temperature of 1 lb of water 1° C)

$$
\begin{aligned}
&= \frac{9}{5} \times 778 \text{ ft-lb} \\
&= 0{\cdot}000528 \text{ kWh}
\end{aligned}
$$

Example 1. Calculate the number of kWh of electrical energy obtained per hour from a generating plant whose overall efficiency is 18 per cent, given,

Number of pounds of coal burnt per hour	= 7,000 lb
Calorific value of the coal	= 12,000 B.Th.U. per lb
Number of B.Th.U. input per hour	$= 7{,}000 \times 12{,}000$
Output in B.Th.U. per hour	$= 0{\cdot}18 \times 84 \times 10^6$
Output in kWh. per hour	$= 0{\cdot}18 \times 84 \times 10^6 \times 0{\cdot}000293$
	= 4,420 kWh per hour

(The power output is thus 4,420 kilowatts.)

Example 2. Calculate the number of kWh expended in pulling a train of weight 250 tons, ½ mile up an incline of 1 in 75, at a steady speed of 20 miles an hour, by means of an electric locomotive, if 70 per cent of the energy input is usefully employed. Frictional resistance to motion may be taken as 16 lb per ton.

Calculate also the current taken by the motors if the supply voltage is 500 volts.

$$
\begin{aligned}
\text{Tractive effort} &= \frac{250 \times 2{,}240}{75} + 250 \times 16 \text{ lb} \\
&= 11{,}467 \text{ lb} \\
\text{Work done} &= 11{,}467 \times 2{,}640 \\
&= 30{,}250{,}000 \text{ ft-lb} \\
\therefore \text{Energy input} &= \frac{30{,}250{,}000}{0{\cdot}7} \times 0{\cdot}000000377 \text{ kWh} \\
&= 16{\cdot}3 \text{ kWh}
\end{aligned}
$$

Time taken to travel ½ mile $= \frac{1}{40}$ hr

$$\therefore \text{Power input} = \frac{16 \cdot 3 \times 1{,}000}{\frac{1}{40}}$$

$$= 652{,}000 \text{ watts}$$

$$\therefore \text{Current} = \frac{652{,}000}{500} = 1{,}304 \text{ amp}$$

Dimensional Equations. If a certain physical quantity y is proportional to the product of two or more other physical quantities, each of which is raised to some power which is unknown, e.g. $y \propto x^m z^n w^p$, then the unknown indices m, n, and p can be determined by substituting their dimensions for the quantities y, x, z, and w, and equating the corresponding indices of L, M, T, μ, and ε, as in the following example.

Example. The electrical power in a circuit is proportional to the voltage, and to the resistance of the circuit, each raised to some power. Determine these powers by the use of the dimensions of the quantities involved.

Let $$P \propto E^m R^n$$

or $$P = k \, . \, E^m R^n$$

where k is a number which has no dimensions.

Then, substituting the dimensions of the quantities from Table II, we have using the electromagnetic system,

$$L^2 M T^{-3} = k\,[(L^{\frac{3}{2}} M^{\frac{1}{2}} T^{-2} \mu^{\frac{1}{2}})^m \times (L T^{-1} \mu)^n]$$

Equating corresponding indices, we have

For L—

$$2 = \frac{3}{2} m + n$$

For M—

$$1 = \frac{1}{2} m, \text{ i.e. } m = 2,\ n = -1$$

Also, *for T*—

$$-3 = -2m - n$$

which is satisfied also by $m = 2$, $n = -1$

For μ—

$$0 = \frac{1}{2} m + n$$

which is again satisfied by $m = 2$, $n = -1$,

$$\therefore P \propto \frac{E^2}{R}$$

The dimensions of the physical quantities involved can also be used to check, or detect, possible errors in equations which have been derived, perhaps, from somewhat complicated theory. This use is illustrated in the following example.

Example. It is suspected that an error has been made in the derivation of the expression

$$I = \frac{E\omega M}{\sqrt{(\omega^2 M^2 + R_1 R_2)^2 + \omega^2 L_1 R_1^2}}$$

for the current in a circuit, in terms of the voltage E, angular velocity ω, mutual inductance M, self-inductance L_1 and resistances R_1 and R_2. Ascertain if this is so and, if necessary, make a correction to ensure that the equation is dimensionally correct.

ω, being an angular velocity, has the dimension T^{-1}, and M has, of course, the dimensions of inductance, i.e. $L\mu$ in the E.M. system. Then, substituting the dimensions of the various quantities in the electromagnetic system, we have—

Left-hand Side	*Right-hand Side*
$L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}$	$\dfrac{(L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-2}\mu^{\frac{1}{2}})(T^{-1})(L\mu)}{[(T^{-2}L^2\mu^2 + L^2T^{-2}\mu^2)^2 + T^{-2}\,.\,L\mu\,.\,L^2T^{-2}\mu^2]^{\frac{1}{2}}}$
	$= \dfrac{L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-3}\mu^{\frac{3}{2}}}{[(T^{-2}L^2\mu^2 + L^2T^{-2}\mu^2)^2 + T^{-4}L^3\mu^3]^{\frac{1}{2}}}$

Since the sum of terms which have the same dimensions as one another must have the same dimensions as its constituent terms, the right-hand side can be written

$$\frac{L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-3}\mu^{\frac{3}{2}}}{[T^{-4}L^4\mu^4 + T^{-4}L^3\mu^3]^{\frac{1}{2}}}$$

In order that the dimensions of this expression shall be the same as those of the left-hand side, the second term in the denominator should have dimensions $T^{-4}L^4\mu^4$, so that the dimensions of the denominator as a whole would be

$$[T^{-4}L^4\mu^4]^{\frac{1}{2}} = T^{-2}L^2\mu^2$$

which would give $L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}$ for the dimensions of the right-hand side, and would thus make the whole equation dimensionally correct.

Thus, the dimensions $L\mu$ are missing from the last term in the denominator of the right-hand side. Since these are the dimensions of inductance, the original equation, to be dimensionally correct, should have read

$$I = \frac{E\omega M}{\sqrt{(\omega^2 M^2 + R_1 R_2)^2 + \omega^2 L_1 R_1^2 \mathrm{L}}}.$$

L being an inductance, either self or mutual.

Determination of "c" (i.e. the Ratio of the Electromagnetic to the Electrostatic Unit of Electricity). It has already been stated that

$$\mu^{-\frac{1}{2}}\varepsilon^{-\frac{1}{2}} = \text{a velocity} = c$$

and that this velocity can be shown, experimentally, to be that of light, i.e. $2{\cdot}998 \times 10^{10}$ or 3×10^{10} cm per sec, very nearly.

To determine this velocity, the ratio of the electromagnetic and electrostatic values of some electrical quantity must be measured. This ratio can be measured for any of the four quantities—capacitance, resistance, e.m.f., and quantity of electricity. Of these, the first is perhaps the best, and will be described.

The method used necessitates the calculation of the capacitance of some simple form of capacitor in electrostatic C.G.S. units, and also the measurement of its capacitance in electromagnetic C.G.S. units in terms of a resistance whose value in electromagnetic C.G.S. units is known.

The value of the velocity c can be obtained from the ratio of the calculated electrostatic value to the measured electromagnetic value as below.

Let C_{ES} be the calculated value of the capacitance in e.s.u.
,, C_{EM} be the measured value in e.m.u.

From Table II—

$$\frac{1 \text{ e.s.u. of capacitance}}{1 \text{ e.m.u. of capacitance}} = \left[\frac{\varepsilon . L}{L^{-1}T^{2}\mu^{-1}}\right] = \varepsilon\mu \text{ neglecting the dimensions of } L \text{ and } T$$

But $\varepsilon^{-\frac{1}{2}}\mu^{-\frac{1}{2}} = c$, $\quad \therefore \varepsilon\mu = \frac{1}{c^2}$

$$\therefore \frac{1 \text{ e.s.u. of capacitance}}{1 \text{ e.m.u. of capacitance}} = \frac{1}{c^2}$$

Thus, 1 e.s.u. of capacitance $= \frac{1}{c^2}$ of 1 e.m.u. of capacitance, or the number of e.s.u. of capacitance in 1 e.m.u $= c^2$.

Now, if a certain length is expressed as L' ft or L'' in.,

$$\frac{L''}{L'} = \frac{12}{1} = \text{No. of inches in 1 ft}$$

By analogy, $\frac{C_{ES}}{C_{EM}}$ = No. of e.s.u.s of capacitance in 1 e.m.u. $= c^2$

$$\therefore \quad \sqrt{\frac{C_{ES}}{C_{EM}}} = c \qquad (2.8)$$

Procedure. The capacitance C_{ES} having been calculated from the dimensions of the capacitor, C_{EM} must be measured. There are several ways of carrying out this measurement of capacitance in electromagnetic C.G.S. units, the best of which is Maxwell's bridge method, described by him in his *Electricity and Magnetism*, Article 775.

In this method, the capacitor C to be measured is connected in one arm of a bridge network, as shown in Fig. 2.1. A commutator is also connected in this arm, by means of which the capacitor is alternately charged and discharged. The commutator is driven by a small motor, supplied from a steady source, and whose speed can be varied as required.

P, Q, and R are non-inductive resistors whose values in absolute electromagnetic units are known. G is a sensitive galvanometer, of resistance g, and b is the resistance of the battery circuit.

The resistances of the leads in the capacitor branch are made negligibly small.

When the commutator is in such a position that the moving contact 3 of Fig. 2.1 is on contact 2, the capacitor C is discharged and the currents flowing in the various arms, including the galvanometer branch, are steady currents from the battery. When contact 3 is on contact 1, the capacitor is charged to the potential of the battery

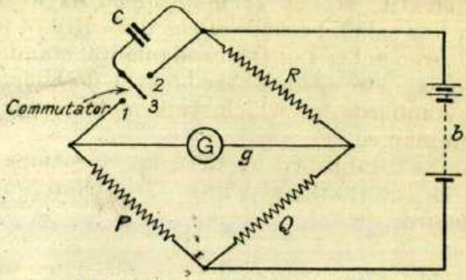


FIG. 2.1. CIRCUIT FOR THE DETERMINATION OF c

and the galvanometer current is altered owing to the varying current taken by the capacitor while it is being charged. When the capacitor is fully charged, no current is taken by it, and the galvanometer current again takes up its steady value.

The galvanometer current can be made zero by suitable adjustment of the resistances P, Q, and R, and of the speed of the commutator.

The expression* for the capacitance of capacitor C is

$$C_{EM} = \frac{Q}{nPR}\left[\frac{1-\left(\dfrac{Q^2}{(P+Q+g)\,(Q+b+R)}\right)}{\left\{1+\dfrac{Qb}{P(Q+b+R)}\right\}\left\{1+\dfrac{Qg}{R(Q+P+g)}\right\}}\right] . \quad (2.9)$$

n is the frequency of the commutator. To a close approximation $C_{EM} = \dfrac{Q}{nPR}$ provided $P + R$ is large compared with Q.

The resistances must be expressed in electromagnetic C.G.S. units of resistance. If expressed in ohms, the value of C will be in farads. Many investigators have measured the ratio c by the above and other methods. From their results it appears that

$$c = 2{\cdot}998 \times 10^{10} \text{ cm per sec}$$

while the average value obtained by many investigators for the velocity of light is $2{\cdot}9986 \times 10^{10}$ cm per sec. These figures are taken from the *Dictionary of Applied Physics*, Vol. II, p. 960, where a record of much work carried out on the subject is given.

* J. J. Thomson first gave this equation, and the theory from which it is derived is given also in Laws's *Electrical Measurements*, p. 364.

International and Absolute Units. Although the British Association Committee on Electrical Measurements adopted the absolute system of units in 1863, and this was confirmed at an International Conference on Electrical Units in London in 1908, this conference decided to specify material standards to be calibrated in absolute units and thereafter set up or maintained as working standards. This decision was taken because of the difficulty of making accurate absolute measurements; but the fundamental standards determined on the electromagnetic system based on the centimetre, gramme and second—the standards for which were more permanent than electrical ones—remained.

The four units established by these specifications (defined below) were known as International Units. The Ohm was chosen as the primary standard.

Definitions of International Units. The **International Ohm** is the resistance offered to the passage of an unvarying electric current by a column of mercury at the temperature of melting ice, of mass 14·4521 g, of uniform cross-sectional area and of length 106·300 cm.

Although unnecessary for the purpose of definition, the cross-section of such a column is very nearly 1 sq. mm.

The **International Ampere** is the unvarying electric current which, when passed through a solution of silver nitrate in water, "in accordance with Specification II attached to these resolutions," deposits silver at the rate of 0·00111800 g per sec.

The **International Volt** is the steady electric pressure which, applied to a conductor of resistance 1 international ohm, produces a current of 1 international ampere.

The **International Watt** is the electrical energy per second expended when an unvarying electric current of 1 international ampere flows under a pressure of 1 international volt.

The centimetre, gramme, and second were selected as the units of length, mass, and time, by an International Electrical Congress at Paris in 1881, and were defined by them

One of the main reasons for adopting an absolute system of units originally was the difficulty of constructing standards which did not vary appreciably with time. Since then, however, wire resistance standards have been developed which are sufficiently permanent for their use to act as a better method of maintaining the international ohm than by occasionally setting up and measuring the "mercury" ohm. Thus it has become the practice of the national standardizing laboratories to maintain the international ohm by wire resistance standards.

As also, by 1930, it was clear that the absolute ohm and ampere could be determined as accurately as the international units, a decision was finally taken in 1946 to abandon the international units, reverting to the fundamental units defined in 1908. The date of the change-over was 1st January, 1948.

Following a number of determinations of the absolute ohm and ampere by various national standards laboratories during the period 1934–1942 the National Physical Laboratory has adopted the following conversion factors—

	1 international	ohm = 1·00049	absolute	ohms
	1 ,,	ampere = 0·99985	,,	ampere
from which,	1 ,,	volt = 1·00034	,,	volts
	1 ,,	watt = 1·00019	,,	watts
	1 ,,	henry = 1·00049	,,	henrys
	1 ,,	farad = 0·99951	,,	farad

An appendix to Reference 66 gives "Definitions of the Units recommended by the International Committee on Weights and Measures for Legal and Similar Purposes."

Legal Standards. For legal purposes it is necessary to lay down some simpler, if less accurate, standards than those referred to above.

It was laid down by an Order in Council (London Gazette, 1949, No. 38683, p. 3810) that the legal standards shall be—

Electrical Resistance. A standard of electrical resistance denominated one Ohm, agreeing in value within one hundredth part of 1 per cent with that of the fundamental unit, and being the resistance between the copper terminals of the instrument marked "Board of Trade Ohm Standard verified, 1894 and 1909," to the passage of an unvarying electrical current when the coil of insulated wire forming part of the aforesaid instrument is in all parts at a temperature of 14·9° C.

Electrical Current. A standard of electrical current denominated one Ampere, agreeing in value within one tenth of 1 per cent with the fundamental unit, and being the current which is passing in and through the coils of wire forming part of the instrument marked "Board of Trade Ampere Standard verified, 1894 and 1909," when on reversing the current in the fixed coils the change in the forces acting upon the suspended coil in its sighted position is exactly balanced by the force exerted by gravity in Teddington upon the iridioplatinum weight marked *A* and forming part of the said instrument.

Electrical Pressure. A standard of electrical pressure denominated one Volt, agreeing in value within one tenth of 1 per cent with the fundamental unit and being the pressure which when applied between the terminals forming part of the instrument marked "Board of Trade Volt Standard verified, 1894 and 1909 and 1948," causes that rotation of the suspended portion of the instrument which is exactly measured by the coincidence of the sighting wire with the image of the fiducial mark *A* before and after application of the pressure and with that of the fiducial mark *B* during the application of the pressure, these images being produced by the suspended mirror and observed by means of the eyepiece.

The legal standards are maintained at the National Physical Laboratory.

Absolute Measurements. 1. MEASUREMENT OF RESISTANCE. In Table II resistance has the dimensions $LT^{-1}\mu$ in the electromagnetic system. The dimensions are those of a velocity, and thus absolute measurements of resistance involve the measurement of either a velocity, or of length and time, which determine a velocity. Such measurements often involve the measurement of inductance and time, since inductance has the dimensions of length in the electromagnetic system.

At least eight different methods have been used, but only one* can be given here.

Lorenz Method. This method, originally used by Lorenz in 1873, has since been used for the absolute measurement of resistance, sometimes in a modified form, by a number of investigators.

At the National Physical Laboratory the latest determination of the ohm, by this method, was made in the years 1933–1936 (*see*

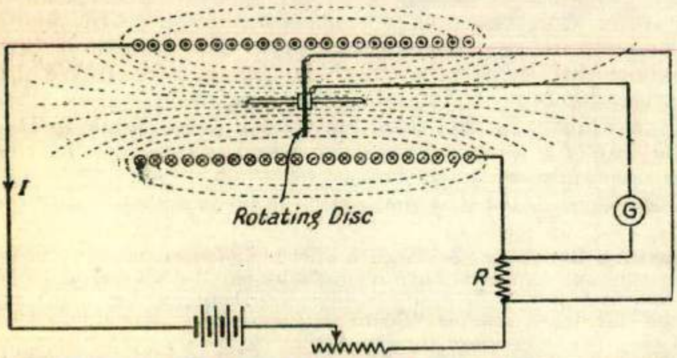


FIG. 2.2. LORENZ METHOD FOR THE ABSOLUTE MEASUREMENT OF RESISTANCE

P. Vigoureux, *N.P.L. Collected Researches*, 1938, Vol. 24, p. 277). For descriptions of American work on the subject see H. L. Curtis, "A Review of the Methods for the Absolute Determination of the Ohm," *Journal of the Washington Academy of Sciences*, 1942, Vol. 32, p. 40, H. L. Curtis, C. Moon and C. M. Sparks, "A Determination of the Absolute Ohm using an Improved Self Inductor," *Journal of Research of the National Bureau of Standards*, 1938, Vol. 21, p. 375 and H. L. Curtis, "Review of Recent Absolute Determinations of the Ohm and Ampere," *Journal of Research of the National Bureau of Standards*, 1944, Vol. 33, p. 235.

In the original experiments a circular metal disc, mounted concentrically inside a solenoid, was driven at a uniform speed of rotation.

A steady current was passed through the solenoid, in series with which was a low resistance R, from the terminals of which leads were taken to two small brushes, one pressing on the edge of the rotating disc and another making contact with the disc near its centre. A sensitive galvanometer was included in one of these leads as shown in Fig. 2.2.

* Other methods are given in the *Dictionary of Applied Physics*, Vol. II, in the section on "Electrical Measurements."

As the disc rotates e.m.f.s are induced in it, since it is placed at right angles to the field of the solenoid. The connections from the brushes on the disc to the terminals of R are so made that the induced e.m.f. in the disc is opposed by the voltage drop due to the solenoid current I in the resistor R. Thus, when the induced e.m.f. is exactly equal to the voltage drop, IR, no current passes through the galvanometer, which therefore gives no deflection.

Let M be the mutual inductance between the disc and the solenoid.

i.e. M = the magnetic flux passing perpendicularly through the disc surface when 1 ampere flows in the solenoid.

Thus, the flux cutting the disc when I amperes flow through the solenoid $= MI$ webers. The speed of rotation of the disc (together with the current I and resistance R, if necessary) can be adjusted until no current flows through the galvanometer.

Let N rev per sec be the speed of rotation for zero galvanometer deflection.

Then e.m.f. induced in the disc $= MIN$ volts

Voltage drop in the resistor $= IR$ volts,

R being expressed in ohms.

Thus $$MIN = IR$$

or $$R = MN \text{ ohms} \qquad . \quad . \quad . \quad (2.10)$$

The value of the mutual inductance M is calculated from the dimensions of the solenoid and disc, and from their relative positions, using methods such as those described in Chapter V.

As a check upon this expression from the point of view of the dimensions of the quantities involved, consider the dimensions of the product MN.

$$[M] = \frac{\text{Magnetic flux}}{\text{Current}} = \frac{[L^{\frac{3}{2}} M^{\frac{1}{2}} T^{-1} \mu^{\frac{1}{2}}]}{[L^{\frac{1}{2}} M^{\frac{1}{2}} T^{-1} \mu^{-\frac{1}{2}}]} = [L\mu]$$

$$[N] = \frac{\text{Revolutions}}{\text{Seconds}} = [T^{-1}]$$

$\therefore [MN] = [LT^{-1}\mu]$, which are the dimensions of resistance

If the resistance R is that of a column of mercury of known dimensions, the resistivity of mercury can thus be obtained in absolute measure, from which the resistance of the international unit of resistance in absolute units can be calculated. Otherwise the resistance R may be some resistance whose magnitude, in terms of the standard ohm, is known to a high degree of precision.

The e.m.f.s in the disc may be thought of as existing in an infinite number of radial elements, each cutting through a field of flux density equal to $\frac{MI}{\text{area of the disc}} = \frac{MI}{\pi r^2}$, where r is the radius of the disc in metres. The e.m.f. across the brushes is thus that induced in a radial element of length r, moving with a mean linear velocity of $\pi r N$ metres per sec, through a field of flux density $\frac{MI}{\pi r^2}$. Thus, from Equation (1.48), the e.m.f. induced in this element (i.e. the e.m.f. across the brushes) is

$$\frac{MI}{\pi r^2} \cdot r \cdot \pi r N = MIN \text{ volts}$$

which is the same as the expression given above.

Precautions Necessary to Ensure Accuracy of Measurement. To obtain an accuracy of measurement of the resistance of 1 part in 10,000, both M and N must be determined with an accuracy of a few parts in 100,000.

The speed N may be determined by stroboscopic methods (see Chapter XXII) or by a directly driven chronograph, the latter being F. E. Smith's method (see Refs. (1), (5)). He also incorporated a fly-wheel to ensure uniformity of speed.

To obtain the necessary accuracy in the value of M, both the disc and solenoid must be carefully constructed and their dimensions accurately measured. The former of the solenoid is usually a marble cylinder, very carefully machined, the dimensions being obtained by the use of precision measuring apparatus. The winding is of bare copper wire, wound in grooves cut in the cylindrical surface.

Since the effective dimensions of the disc, when rotating, cannot be obtained with the same accuracy, the value of M is made as little dependent upon these dimensions as possible by suitably choosing the dimensions of the solenoid relative to the disc diameter. The disc is usually of phosphor-bronze.

The effect of the earth's magnetic field upon the e.m.f. induced in the disc is made small by arranging the plane of the latter in the magnetic meridian. Two measurements are made—one with the current I reversed—to eliminate this effect.

To reduce the effects of thermo-electric e.m.f.s at the brush contacts, F. E. Smith used two phosphor-bronze discs of special construction, and two solenoids.

The accuracy of such absolute measurements depends upon the precision with which the component apparatus can be made and upon that of physical measurements such as those of length and time. It also depends on the variation of the dimensions with time.

Length can now be measured, under the experimental conditions applying to such work, to within 0·0001 mm, giving an accuracy of about one part in a million for the components used in these measurements. Time, or frequency, can be measured to about one part in ten million.

The overall uncertainty in the values determined for the ohm is probably not more than 20 parts in a million, and the most recent comparisons of the results of determinations at various national laboratories show a total spread of only 12 parts per million for the ohm and 4 parts per million for the ampere.

2. Measurement of Current. The dimensions of current, in the electromagnetic system, being $L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}\mu^{-\frac{1}{2}}$, the dimensions of (current)^2 are LMT^{-2} if μ is regarded as non-dimensional. But these

are the dimensions of force, so that absolute measurements of current involve the measurement of force.

This force may be exerted in two ways—

(*a*) By the current in a solenoid upon a suspended magnetic needle—as in a tangent or sine galvanometer.

(*b*) By the current in one part of a circuit upon another part of the circuit in series with it, and carrying the same current—as in an electrodynamometer or current balance.

Galvanometer methods suffer from the disadvantages that there is always some uncertainty about the exact position of the poles of the magnetic needle used, and also that the horizontal component of the earth's magnetic field must be separately determined with great accuracy before the results of current measurements can be interpreted.

Electrodynamometers measure current in terms of the torsion of a suspension wire or of a bifilar suspension, and this is not very satisfactory. Methods of measurement which utilize some form of current balance are therefore probably the most satisfactory, and are most commonly used.

Tangent Galvanometer Method. If a current of I amperes flows in the coil of a tangent galvanometer, it can easily be shown that the steady deflection θ (see Fig. 2.3) is such that

$$I = \frac{2Hr}{N} \tan \theta \qquad . \quad . \quad . \quad (2.11)$$

where r = the mean radius of the galvanometer coil in metres,
N = number of turns on this coil,
and H = the horizontal component of the earth's magnetic field in amperes per metre.

Obviously the current can be obtained from this expression, in terms of the deflection, the dimensions of the coil, and H.

The following assumptions are made in deriving this expression—

(*a*) That the plane of the galvanometer coil lies exactly in the magnetic meridian, and is exactly vertical.

(*b*) That the magnetic needle is infinitesimally short.

(*c*) That the needle is suspended at the *exact* centre of the coil.

(*d*) That the axis of the needle is horizontal.

These assumptions are obviously not all justifiable in practice. Again, unless the galvanometer coil has only a single layer, and is exactly circular, the value of r may be somewhat uncertain. The accuracy of the measurement depends, also, directly upon the accuracy with which H is known for the particular place at which the measurement is being made. This last is a great disadvantage of the method, since it usually necessitates a separate—and highly accurate—determination of H, and this is about as difficult a measurement as that of the current itself. Kohlrausch devised a method of measuring H and the current simultaneously (see *Philosophical Magazine*, Vol. XXXIX), but the method does not appear to have been adopted generally.

Corrections can be applied to allow for some of the divergencies between practice and theory. Two of these—given by F. E. Smith in the *Dictionary of Applied Physics*, Vol. II, p. 231—are as follows—

(*a*) To allow for the fact that all the turns on the galvanometer

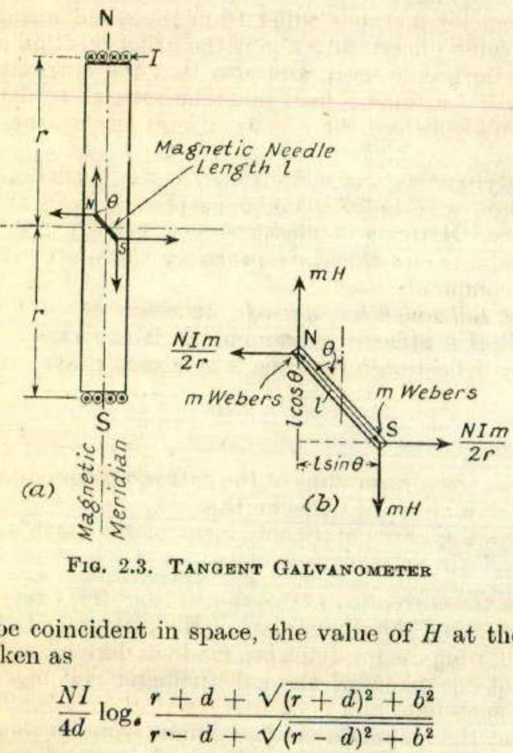


FIG. 2.3. TANGENT GALVANOMETER

coil cannot be coincident in space, the value of H at the centre of the coil is taken as

$$\frac{NI}{4d}\log_e \frac{r + d + \sqrt{(r + d)^2 + b^2}}{r - d + \sqrt{(r - d)^2 + b^2}}$$

instead of $\frac{NI}{2r}$ as assumed in the elementary theory of the galvanometer. In this expression $2b$ = axial length of the coil, $2d$ = radial depth of the coil, both in metres. This expression is due to A. Gray, and is given in his *Absolute Measurements*.

(*b*) To allow for the fact that the centre of the needle is not exactly at the centre of the coil, the correction factor to be applied to the value of H due to the current is

$$1 + \frac{3}{2} \cdot \frac{\delta y^2 + \delta z^2 - 2\delta x^2}{r^2}$$

where δx, δy, and δz are the displacements of the centre of the needle relative to the centre of the coil. These displacements are measured, of course, in three, mutually perpendicular, directions, δx being measured along the axis of the coil.

If, however, corrections are to be applied to allow for all departures from the theoretical assumptions, the method becomes very cumbersome.

Helmholtz modified the tangent galvanometer by adding a second coil and placing the needle midway between the two coils in the uniform field produced by this arrangement (see Chapter I). The correction for axial displacement of the centre of the needle from the centre of a coil is thus rendered unnecessary.*

Rayleigh Current Balance. The principle of this instrument will first of all be discussed. If a current-carrying coil is placed with its plane parallel to that of another current-carrying coil and in such a position that their axes are coincident, a force—either of attraction or repulsion—will exist between the coils, depending upon the current directions. This force is proportional to the product of the two currents in the coils. If the coils are connected in series, so that the same current flows through both, the force between them is proportional to the square of the current passing. This force can be measured if one of the coils is movable, and is suspended from one arm of a balance, the force thus being "weighed"; hence the name "current weigher" given to such instruments. Lord Rayleigh and Mrs. Sidgwick, in their experiments for the determination of the electrochemical equivalent of silver, used two parallel coaxial fixed coils with a moving coil suspended between them, the three coils being so arranged relative to one another that the force upon the moving coil was maximum. This arrangement is shown in Fig. 2.4.

The force acting on the moving coil, and measured by the balance, is given by

$$\mathrm{F} = I^2 \frac{dM}{dx} \text{ newtons} \quad . \quad . \qquad (2.12)$$

where I is the current in amperes in the three coils in series; M is the mutual inductance of the coils, and depends upon their numbers of turns, and upon their dimensions and relative positions; dx is an element of length along the axis of the three coils. The value of M can be calculated from the dimensions of the coils by means of formulae given by Gray (Ref. (7)) or by J. V. Jones (Ref. (8)).

In the above apparatus, if the three coils are so placed that the moving coil is at a distance of half their radius from each of the fixed coils, the value of $\frac{dM}{dx}$ becomes dependent only on the ratio

* The theory of this galvanometer is given in Gray's *Absolute Measurements in Electricity and Magnetism*, Vol. II, Part 1.

$\frac{\text{radius of fixed coil}}{\text{radius of moving coil}}$. Under these circumstances, also, very little error is introduced by a slight inaccuracy in the axial position of the moving coil (see Chapter I).

Bosscha (Ref. (9)) introduced an electrical method of measuring the ratio of the coil radii which does away with the necessity for measuring the mean radii of the coils themselves—somewhat uncertain measurements in the case of multi-layer coils.

The measurement of current in absolute units, by means of the Rayleigh balance, thus becomes little more than a careful weighing, very accurate measurements of dimensions being avoided. This is perhaps the greatest advantage of this form of current balance.

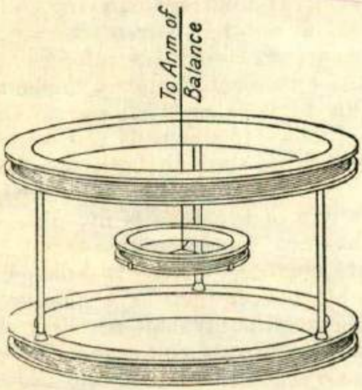


Fig. 2.4. Arrangement of Coils in the Rayleigh Current Balance

Other advantages, common to all forms of current balance, are that neither measurement of the horizontal component of the earth's magnetic field nor a determination of the torsion constants of a suspension are required.

The weighing is usually carried out by observing the change in the weights necessary to balance the moving coil when the current in it is reversed, this having the effect, of course, of reversing the force upon the moving coil. It should be noted that the expression for the force, given above, is in newtons; and, since the weights used in the weighing will be grammes, the value of g—the acceleration due to gravity—must be known. In some forms of current balance the accuracy with which g is known determines the accuracy of the current measurement.

Many forms of current balance have been constructed on this principle and have been used for the determination of the ampere in absolute units.* At the National Physical Laboratory the most recent determination of the ampere was made during the years 1930–35 (see P. Vigoureux, *N.P.L. Collected Researches*, 1938, Vol. 24, p. 173).

In all cases a precision balance of special form is used. Great care is necessary in the construction to ensure that the flexible leads for the purpose of leading current into the moving coil or coils exert no appreciable torque upon the moving system. Other important points are the selection of truly non-magnetic material for the bobbins of the coils. Marble is perhaps the best material from this point of view. Brass is suitable if selected with care. The cooling of the coils, also, is very important, water jackets being used for the

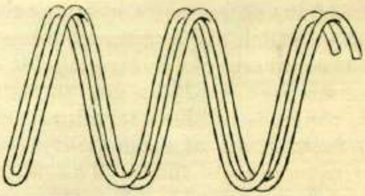


FIG. 2.5. BIFILAR WINDING

fixed coils and a water-cooled chamber being provided for the moving coil.

3. DETERMINATION OF THE VOLT. The value of the volt is obtained by Ohm's Law using the values of the ohm and ampere determined by the methods previously described. In practice the e.m.f. of a standard cell is determined and the standard cell then forms, with the ohm, the second available standard.

Standard Resistors. The standard resistor known as the "legal ohm," as representing, for general commercial purposes, the unit of resistance, has already been referred to. Since it is considerably easier to compare resistances than to determine their value in absolute measure, it is convenient to have available standard resistors which can be used as reference standards. For general purposes, measurements of resistance can be made with sufficient accuracy by comparison with such standards. The values of sub-standard resistors can be determined by comparison with these, such sub-standards being used in the calibration of laboratory standards of resistance.

One form of standard resistor consists of a coil of platinum silver wire non-inductively wound on a metal bobbin The wire is wound as shown in Fig. 2.5. In this bifilar method of winding, the

* Detailed descriptions of these pieces of apparatus are given in the *Dictionary of Applied Physics*, Vol. II, pp. 235, etc., and in Laws's *Electrical Measurements*, p. 90.

wire is doubled back on itself before winding. This gives the effect of two wires, side by side, carrying currents in opposite directions. The magnetic fields due to the two currents neutralize one another, giving a very small inductance.

The coil is insulated from the metal bobbin by a layer of shellacked silk which is baked before the wire is wound on. The wire is laid in one layer in order that the cooling shall be as efficient as possible, it being essential that the coil shall not be appreciably heated during use. After winding, the coil is usually shellacked and baked at a temperature of about 140° C. This serves the double purpose of drying out the coil and of annealing the wire, the latter being necessary in order to remove conditions of strain, due to bending, from the wire, and so ensure greater permanence of the resistance of the coil. The coil is fixed inside an outer cylindrical metal case, which has an ebonite top to which the coil and bobbin are attached, and the space between the coil and the outer cylinder is filled with paraffin wax. The terminals consist of long copper rods, hard-soldered to the resistance coil, the ends of these terminals being amalgamated. In use, the coil is maintained at a constant temperature for some hours before measurements are made. This is done by immersing the major portion of it in water.

The Board of Trade ohm is of this form.

More recent designs for standard resistance coils differ mainly in the new materials used for the coil former; e.g. Barber, Gridley and Hall (Ref. (73)) have described the construction at N.P.L. of strain-free coils which lie in grooves on Perspex discs.

Many other forms of standard resistor have been constructed, the most important being those designed and constructed by the Standards Laboratories of different countries, such as the German Physikalisch-Technische Reichsanstalt, the American Bureau of Standards, and the National Physical Laboratory of this country.*

After experiments with platinum wire and a gold-chromium alloy it has been concluded that forms of manganin are the best materials for standard resistance coils. By suitable heat treatments and mountings, freeing the coils from strain, and by enclosing them in hermetically sealed containers, it is possible to obtain a constancy over a year within one part in ten million.

The reference standard of the N.P.L. consists of a group of five or more coils of nominal value 1 ohm which have shown the greatest relative constancy over preceding years. The average value of the group is assumed to have remained constant, and values with an accuracy of one part in a million are assigned to the reference standards for the purpose of international comparisons.

* Several forms are described in the *Dictionary of Applied Physics*, Vol. II, p. 700, etc., and in the publications mentioned in Refs. (2), (10), (11), (12), (66), (69), (70), (71), (72), at the end of this chapter.

Requirements of Standard Resistors. The most important properties of resistors which are to be used as standards of reference are—

1. Permanence. The necessity for this property is obvious. In order to avoid variation, with time, of the resistance value of the finished standard, annealing during manufacture is essential. Thorough drying out by baking after covering the wire insulation with shellac is also necessary, and if the coil or strip is immersed in oil for cooling purposes, care must be taken to ensure that the oil is free from acid and water, in order to avoid corrosion of the resistance alloy.

2. Robust and strain-free construction.

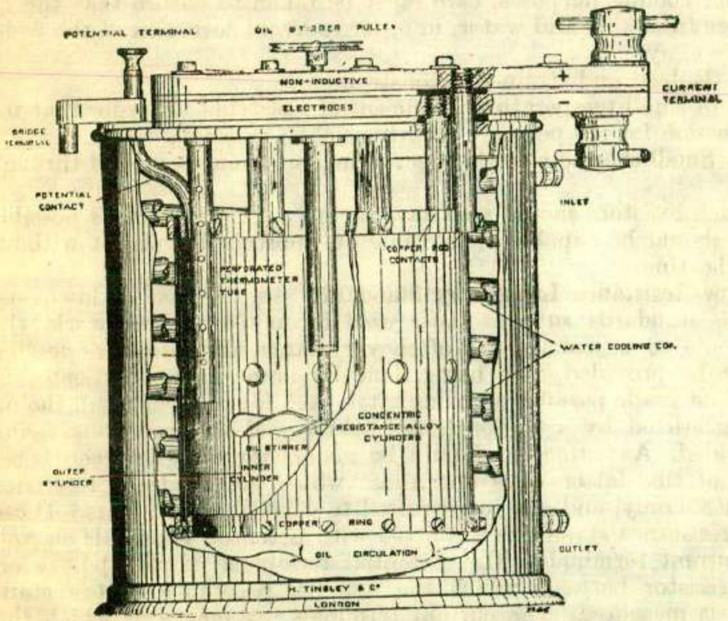
3. A small temperature coefficient of resistance, in order that the correction for temperature variations shall be small.

4. Small thermo-electric effects when a current is passed through it.

Such resistors should also have as low an inductance as possible and should be capable of carrying an appreciable current without overheating.

Low-Resistance Laboratory Standards. In the case of low-resistance standards such as those used for potentiometer work, the currents to be carried are often very large, and adequate cooling must be provided, this being done by immersing the resistor in oil (first-grade paraffin oil being often used for this purpose), the oil being stirred by a motor-driven stirrer and water cooling being provided. A distinction should be made, however, between resistors of this latter class and those which are used for reference purposes only, and are not required to carry large currents. These low-resistance standards are fitted with potential terminals as well as current terminals. The potential terminals fix the points on the resistor between which the nominal resistance of the standard is measured. The current terminals, by means of which the resistor is connected to the supply circuit, should be at an appreciable distance from the tapping points of the potential leads in order that the current distribution shall be uniform throughout the cross-section of the resistance material, before the tapping points are reached (*see* Ref. (13)).

Fig. 2.6 shows a low-resistance standard of the Drysdale-Tinsley non-inductive type, designed to carry heavy currents such as may be required in potentiometer and other work. In addition to the ordinary current and potential terminals, it has, fitted to the potential terminals, mercury contacts for use in a standardizing bridge (see Chap. VII). This type of resistor is manufactured by Messrs. H. Tinsley & Co., and is designed for use with either direct or alternating currents (up to 1,000 c/s). The resistance material used is manganin, silver-soldered to copper rings, which are screwed to heavy copper lugs to which they are also soldered with tin-lead solder. The manganin resistance strips are in the form of concentric cylinders, through which the current passes axially in opposite directions, thus giving a very low inductance. A range of resistors of the type shown in the figure is manufactured, having resistance values from 0·02 ohm down to 0·0001 ohm. The watts dissipated are 200 for resistors from 0·02 ohm down to 0·005 ohm, and 500 from 0·001 ohm down to 0·0001 ohm.



(*H. Tinsley & Co., Ltd.*)

FIG. 2.6. DRYSDALE-TINSLEY NON-INDUCTIVE LOW RESISTANCE STANDARD (500 Watt Type)

Resistance Materials. It is desirable that a material to be used in the construction of standard resistors should possess the following properties—

(*a*) High resistivity, in order that the standard resistor, when constructed, may be reasonably compact.

(*b*) Permanence. There should be as little variation in resistance with time as possible.

(*c*) It should have a low thermo-electric force with copper.

(*d*) Low temperature coefficient, in order that the correction for temperature variation may be small.

(*e*) It should not easily oxidize, and should be unaffected by moisture, acids, etc.

In addition, it should, if possible, be easily worked and jointed.

From intercomparison, over a long period of years, by various investigators, of a number of standard coils made up in 1864 by Mathiessen and Hockin, on behalf of the British Association, it appeared that platinum was the best material from the point of view of permanence, though later work has shown the superiority of manganin. Platinum has the disadvantage of a high temperature coefficient—about 0·4 per cent per 1° C. Many alloys, such as platinum-silver, platinum-iridium, German silver, manganin, etc., have been used as resistance materials, and much research, beginning with the work of Mathiessen (Ref. (14)) has been carried out upon the subject.

Manganin. Weston, in 1889, discovered that alloys of copper, manganese, and nickel, have a very small temperature coefficient. Manganin* is an alloy of this type. Lindeck (Ref. (12)), Bash, and others, have since further investigated the properties of such alloys, and it has been found that the composition —84 per cent copper, 12 per cent manganese, 3·5 per cent nickel, and 0·5 per cent iron—has an extremely low temperature coefficient and is most suitable for resistance purposes.

TABLE V: Properties of Other Resistance Materials

Material	Composition (approx.)	Resistivity (microhm-cm)	Temperature Coefficient (% per ° C)	Thermo-electric e.m.f. against Copper (micro-volts)	Remarks
Therlo	Copper 71% Aluminium 16·5% Manganese 10·5% Iron 2%	47 (at 20° C)	0·0005	Very low	Comparatively new material. Properties similar to manganin.
Platinum-silver	1 part platinum, 2 parts silver	31·6	0·03	Small	High temperature coefficient.
Constantan	Copper and nickel	50 (at 20° C approx.)	− 0·001	40	Cheap. Easy to work. High thermo-electric e.m.f. is a disadvantage.
Eureka	Copper 60% Nickel 40%	As	for Constan	tan.	
German silver	Copper 63% Zinc 22% Nickel 15%	30 (at 20° C)	0·03	35	The presence of zinc in alloys produces unstable properties.
Platinoid	German silver with addition of about 1% tungsten.	34 to 40	0·02 to 0·03	20	Tungsten improves the permanence.
Nichrome		95 (at 20° C) approx.	0·04		Used for resistors of rougher class, especially at high temperatures. Is non-corrosive.
Platinum		11	0·36		Used in resistance thermometry.
Iron		12	0·4		Used for resistors when its magnetic properties and high temperature coefficient are unimportant.
Karma	Nickel 75% Chromium 20% Balance iron and aluminium.	133	0·002	2	High-value precision resistors.

* The name "Manganin" is a registered trade mark belonging to Isabellen-Huette, Heusler K.-G. (Germany), who first produced this alloy commercially in 1889.

Copper-manganese-nickel alloys similar in characteristic to manganin are made by a number of manufacturers under their own trade names, a typical example being the alloy *Minalpha* made by Johnson Matthey & Co., Ltd. Fig. 2.7 shows a temperature-resistance curve for *Minalpha*, and it can be seen that the temperature coefficient of resistance is practically zero over a limited range of temperature at 26°C.

The resistivity of *Minalpha* is 41·5 microhm-centimetres at 20°C, and the temperature coefficient is about + 0·0004 per cent per 1°C at 20°C, as indicated by the curve. The thermo-electric e.m.f. against copper is − 0·5 microvolts per 1°C.

Resistance alloys show a change in electrical resistance when they are subjected to mechanical strain, and this is possibly the primary

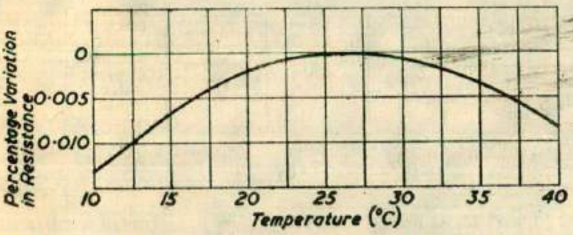


FIG. 2.7. THE TEMPERATURE–RESISTANCE CURVE OF MINALPHA
(Johnson, Matthey & Co., Ltd.)

cause of drift in the value of resistors with time. In order to achieve the highest stability a strain-free construction must be adopted, i.e. the resistance wire should be properly annealed after winding and mounted in such a manner that it is free from mechanical constraints.

The manganin alloys have a strain sensitivity, expressed as the ratio of percentage resistance change to percentage strain, of about 0·4, which is quite low—a further factor in their favour.

Annealing of manganin to remove initial strains is best done by heating the material to a temperature of 550°C. If this is done in air there will be some oxidation of the surface, and the wire must be pickled in chromic acid and then washed in distilled water to remove the oxide layer. Annealing in an inert gas prevents oxidation, but in practice some pickling is still necessary to achieve the utmost stability. This process cannot be applied to insulated wires, and in these circumstances the wound resistor is annealed by heating at about 140°C for at least 10 hours. Resistance standards having values of 1 and 10 ohms are made with bare wire which has been subjected to a high-temperature anneal. The resistance coil is virtually self-supporting and can be mounted with a minimum of mechanical constraint, thus achieving a virtually strain-free construction.

These conditions are not readily realized with higher-value resistors, which are normally wound with insulated wire on a supporting bobbin; as a result of this, the long-term stability of such resistors is not comparable with that of the 1 ohm standards. The notable feature of the new resistors due to Barber, Gridley and Hall (Ref. (73)), previously referred to, is the attainment of a true strain-free construction with 1,000 ohm coils.

It is a common practice to coat insulated wires after winding with a protective coating of shellac; the shellac absorbs moisture from the atmosphere which causes it to swell and stress the wire, giving rise to small variations of resistance with time.

Current Standards. It is obviously impossible to set up a standard of current in the same sense that a standard of resistance can be set up, and in practice the standard cell is maintained as a second working standard with the ohm. Any voltage can be accurately measured by comparison with a standard cell, using a precision potentiometer, and current is therefore measured with the standard ohm and a standard cell, using a potentiometer.

Kelvin Current Balance. The current balance as used for the determination of the absolute value of the ampere has already been described. Lord Kelvin designed an instrument the action of which depends upon the same principle and which has been used for the accurate measurement of current. The instrument mentioned in the definition of the legal ampere is a special form of the Kelvin balance.

The Kelvin balance is now described as a matter of historical interest only, because a far higher degree of accuracy in direct-current measurement can be achieved with a standard resistor and potentiometer, and in alternating-current measurement by using a transfer device such as an electrostatic voltmeter or a vacuo-thermo-junction. These particular devices are described in subsequent chapters.

The Kelvin balance has been used in the past for the measurement of currents from 0·1 to 10 amperes. The instrument consists of six coils, four fixed and two moving, the latter being carried on a beam which can rotate in a vertical plane, like the beam of a chemical balance. Instead of a knife edge, as the means of pivoting this beam, it is suspended at its centre by two flexible copper ribbons, each consisting of a large number of fine wires. These ribbons also act as leads to the moving coils. The latter are situated between the two pairs of fixed coils as shown in Fig. 2.8, all six coils being connected in series, the connections being such that the currents flow as shown. Under these conditions the top fixed coil on the right attracts the adjacent moving coil A, while the bottom fixed coil repels A. On the left the top fixed coil repels the adjacent moving coil B, while the bottom fixed coil attracts B. The total effect is thus to cause an anti-clockwise movement of the beam carrying the moving coils. This anti-clockwise torque is balanced by means of weights carried by a small carriage which runs on a graduated bar attached to the moving beam. This carriage is moved by means of cords which pass through holes in the case of the instrument. To ensure that the weights shall always be placed in the same position on the carriage, the latter is fitted with two small conical pins, which fit into holes in the weights.

To use the instrument, a known weight, whose value is suitable for use with the current to be measured, is placed on the carriage, and a counterpoise of the same value is placed in the aluminium V-shaped trough attached to the right-hand moving coil. The

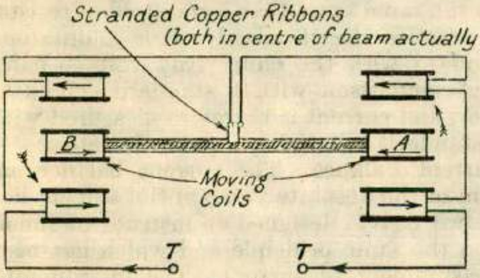


FIG 2.8. ARRANGEMENT OF COILS IN KELVIN CURRENT BALANCE

carriage is then moved to zero at the left-hand end of the graduated scale, and the clamping device, for removing the weight of the moving system from the copper ligaments when the instrument is not in use, is freed. The moving system should then be balanced as indicated by the pointers—one at each end of the beam—which move over small vertical scales attached to the base. A means of adjustment is provided to obtain complete balance with zero current if this condition should not be obtained without.

When current flows through the instrument, the anti-clockwise torque produced by it is balanced by moving the carriage, with its weights, along the scale to the right. If $2l$ cm is the length of the scale, and balance is obtained with a movement of the weight of x cm from zero, then, the moving system being suspended at its centre, the total turning moment due to the weights, each of weight W grammes (say), is $Wl - W(l - x) = Wx$ g-cm. At balance this turning moment is equal to that due to the current, which latter is proportional to the square of the current. The current I is calculated from the equation

$$I = K2\sqrt{D} \quad . \quad . \quad . \quad . \quad (2.13)$$

where D is the displacement of the moving weight in scale divisions for balance, and K is a constant for the instrument which depends also upon the weight used. A fixed inspectional scale for approximate readings is fitted behind the moving scale, the former being graduated in terms of $2\sqrt{D}$. Four sliding weights and four counterpoise weights are supplied with the instrument, in order to obtain different ranges, the carriage constituting the smallest of the sliding weights. These four weights are in the ratio 1, 4, 16, 64. The first being the weight of the carriage, the last three are 3, 15, and 63 times the weight of the carriage respectively.

Kelvin balances can be used with alternating current as well as direct, since, the currents in all the coils being the same (because they are in series), all the magnetic fields of the coils reverse direction together, thus producing a turning moment which is always in the same direction.

Voltage Standards—Standard Cells. The Weston Standard Cell is now used exclusively as the standard of e.m.f.; it was patented

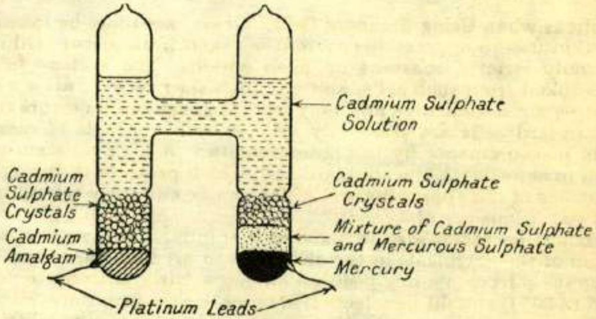


FIG. 2.9. WESTON STANDARD CELL

in its original form by Weston in 1892. It has completely supplanted the earlier Clark cell (Ref. (19)). The present form of Weston cell is illustrated in Fig. 2.9. The positive element is mercury covered by a depolarizer of mercurous sulphate, and the negative element is an amalgam of 1 part of cadmium and 7 parts of mercury. The electrolyte is a saturated solution of cadmium sulphate, and to ensure saturation cadmium sulphate crystals are added to it. Lord Rayleigh suggested the H-form shown, the two limbs being hermetically sealed. The connections to an external circuit are made by platinum wires sealed into the glass.

The present cells are usually of the acid type in which the cadmium sulphate is dissolved in 0·1 N sulphuric acid. The acid cell is less subject to irregular variations and has less temperature hysteresis than the neutral cell.

The e.m.f. of the acid cell is 1·01859 volts at 20° C: it falls by 40 microvolts per 1° C rise in temperature.

Each limb of the standard cell has a comparatively high temperature coefficient and the overall value is the difference between the values for the two limbs. Because of this it is important to ensure that the whole cell is at the same temperature and protected from draughts. Standard cells must be totally enclosed and preferably oil-immersed to ensure evenness of temperature.

The temperature coefficient of the standard cell is related to the solubility of the cadmium sulphate, and unsaturated cells, i.e. with no crystals undissolved, have a negligible temperature coefficient. Such cells have a lower stability than the saturated cells but are often used in industrial applications where wide temperature variations are encountered.

The internal resistance of the acid-saturated cell is about 1,000 ohms and rises slowly with age. The e.m.f. of these cells does not change by more than 1 or 2 parts in 10^5 over several years.

Precautions when Using Standard Cells. Great care must be taken to ensure that when in use no appreciable current is taken from a standard cell, as the e.m.f. is only strictly constant on open circuit. The voltage falls when a current is taken from such cells, and although they recover after a time, such disturbances are undesirable and may lead to considerable errors in measurement. Standard cells are thus only used in null methods of measurement, such as in measurements by the potentiometer. A high resistance should be connected in series with the standard cell, which protects it during the initial manipulations of the apparatus and which can be cut out when approximately balanced conditions are obtained.

Care should be taken also in moving a standard cell, as any appreciable shaking up of the chemicals in the cell tends to produce variations of e.m.f.

For storage purposes a dry position having a fairly uniform temperature of about 15° to 20° C should be selected. in order to avoid troubles from hysteresis effects due to temperature variations, and to avoid any possibility of leakage currents due to moisture on the insulating material between the terminals of the cell.

The use of a group of standard cells as a working standard in international comparisons of voltage is described in Refs. (66) and (69).

Standards of Mutual and Self-Inductance. It has been seen previously (Table II) that the dimensions of inductance in the electromagnetic system are those of length. Thus standards of inductance, both self and mutual, depend for their value upon their dimensions, together with the number of turns of wire in them, this latter being a mere number which has no dimensions.

The self-inductance of a coil, or the mutual inductance of a system of coils, can be calculated from the dimensions of the coils by the use of formulae which have been given by many workers on this subject.*

In the construction of a primary standard of inductance, whether mutual or self, some form must be adopted for which a rigidly accurate formula exists for calculation purposes. The design should be such as to facilitate the accurate measurement of the dimensions

* References to publications giving such formulae are given at the end of the chapter.

of the standard, for, if the formula used is rigidly correct, the errors in the calculated value, as compared with the actual value of the inductance, will depend very largely upon the accuracy of such measurements. There should, also, be no doubt about what lengths should be taken as the effective dimensions of the standard. For this reason the coils are usually single layer, and are often wound with bare wire laid in a screw thread cut in a marble cylinder. Other factors influencing the design are that the dimensions should be subject to as little variation as possible with time in order to ensure permanence of the inductance of the standard, and also that the bobbins used for the coils should be absolutely non-magnetic.

It has been found that marble is the best material for the purpose, its advantages being: (*a*) it does not warp and is unaffected by moisture and atmospheric conditions; (*b*) its electrical resistance is very high, so that it serves as an insulator when bare wire is wound on it; (*c*) its relative permeability (which would be exactly unity for a completely non-magnetic material) is 0·999988, as given by Coffin (Ref. (25)); (*d*) its coefficient of expansion is only about 0·000004 per degree Centigrade; (*e*) it is comparatively cheap and easy to work, so that any desired shape can be obtained.

It is essential that metal shall be avoided as far as possible in the construction of such coils, as eddy currents set up in metal parts may appreciably affect the value of the inductance of the standard. For the same reason, standards constructed for use with heavy currents, when the conductors must be of large section, employ stranded wire to reduce the eddy current effect. Capacitance effects should also be avoided as far as possible, and the resistance of the windings should be low compared with the inductance.

Measurements of the dimensions of coils to be used as primary standards are carried out by means of a precision measuring apparatus, one form of which is described by Coffin (Ref. (25)).

Primary Standards of Mutual Inductance. Such standards are always fixed standards—i.e. they are of single value. Variable standards of inductance will be described in a later chapter. The general form of such primary standards is a single-layer coil, uniformly wound and of circular cross-section, its axial length being large compared with its cross-sectional diameter, which forms the primary circuit, with a coil of small axial length placed at its centre, the latter forming the secondary circuit. The secondary coil may be wound on top of the primary coil, so that their cross-sections are as nearly coincident as possible, or it may be wound on a separate bobbin and placed inside the primary coil, the first form being the better from the point of view of ease of construction and measurement.

The flux density at the centre of the primary coil is

$$\frac{\mu_0 N i}{l} \cos \theta_1 = B \text{ webers per square metre}$$

where N is the number of turns on the coil, l its axial length in metres, and i the current, in amperes, flowing in it. θ_1 is the angle between the axis of the coil and a line drawn from the centre point of the axis to a point on the circumference of an end turn of the coil (see Fig. 1.21).

If the secondary coil, placed at the centre of the primary, has n turns, and is of cross-section a sq. m, the flux linkages with this secondary coil per unit current in the primary (which is the mutual inductance) is

$$\frac{nBa}{i} = \frac{\mu_0 Nna}{l} \cos \theta_1$$

or

$$M = \frac{\mu_0 Nna \cos \theta_1}{l} \text{ henrys} \quad . \quad . \quad . \quad (2.14)$$

It should be noted, however, that in the derivation of Equation (2.14) assumptions are involved which are not quite justifiable for the purpose of calculation of mutual inductance for standards purposes, and that more exact formulae are applied in practice. The above equation gives a fairly close approximation.

Campbell Primary Standard of Mutual Inductance. The Campbell type of primary standard (Ref. (26)) consists of a primary coil of bare copper wire wound under tension in a screw thread cut in a marble cylinder. It is a single-layer coil and is divided into two equal parts connected in series and displaced from one another by a distance equal to three times the axial length of one of them. The secondary coil, consisting of a number of layers of wire wound in a channel cut in the circumference of a marble ring, is placed so that it is concentric and coaxial with the primary coil cylinder. This coil is situated midway between the two portions of the primary coil, and a means of adjustment is provided to enable the coil to be brought into the correct position relative to the primary coil.

With this construction the magnitude of the mutual inductance obtainable is much greater than is possible if both primary and secondary coils are single-layered, whilst the difficulty of accurately measuring the effective radius of the multi-layered secondary is overcome by arranging its dimensions so that small variations of radius or of axial position have a negligible effect upon the mutual inductance. With the relative positions of the secondary and the two portions of the primary coil as stated above, maximum mutual inductance is obtained by making the mean radius of the secondary coil about 1·46 times that of the primary coil. This means that the circumference of the secondary coil is situated in the position of zero magnetic field when current flows in the primary coil (see Fig. 2.10). Thus the mutual inductance will not be appreciably affected by small errors in measurement of the secondary coil radius, or by a small departure from the true midway position between the two halves of the primary winding.

The mutual inductance is calculated by J. V. Jones's formula, mentioned previously.

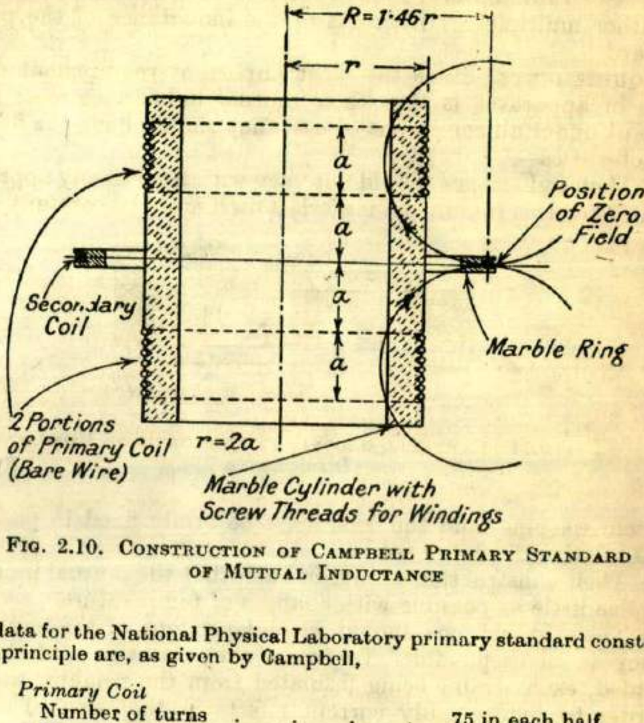


FIG. 2.10. CONSTRUCTION OF CAMPBELL PRIMARY STANDARD OF MUTUAL INDUCTANCE

The data for the National Physical Laboratory primary standard constructed on this principle are, as given by Campbell,

Primary Coil	
Number of turns	75 in each half
Diameter	30 cm
Axial length of each half. . .	15 cm
Distance between inner ends of the two halves	15 cm
Secondary Coil	
Number of turns	485
Mean diameter	$43{\cdot}73_7$ cm
Axial depth	1·00 cm
Radial depth	0·86 cm

The mutual inductance of this standard is given as 10·0178 millihenrys.

Secondary Standards of Mutual Inductance. Such secondary standards are used as standards of mutual inductance for general laboratory purposes. Since they are not absolute standards it is not essential that their dimensions shall be determined with great accuracy, it being merely essential that they shall have a mutual inductance which is as near as possible to the nominal value for which they are designed. When constructed they are compared with a primary standard, and their mutual inductance is adjusted, if necessary, until it is within, say, 1 part in 10,000 of their nominal value. Such standards are constructed to have nominal values which are either multiples or fractions of the inductance of the primary standard.

Requirements. Since the most important requirement of such pieces of apparatus is that their mutual inductance shall remain constant under all conditions of use, they should have the following characteristics—

(*a*) Their inductance should not vary with time to any appreciable extent. For this reason the materials used must be carefully chosen

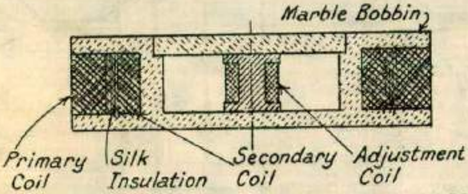


Fig. 2.11. Campbell Secondary Standard of Mutual Inductance

to avoid warping, and the coils must be firmly fixed in position to avoid relative displacement.

(*b*) Their construction should be such that the mutual inductance varies as little as possible with changes of temperature.

(*c*) Their inductance should be independent of the supply frequency as far as possible. To ensure this, the wire used should be stranded, each strand being insulated from the neighbouring ones, in order to reduce eddy current effects in the wire. The inter-capacitance of the windings should be small, also, and the insulation should be as perfect as possible.

Secondary standards usually consist of two coils wound on a bobbin of marble or hard, paraffined wood, the coils being separated by a flange. The wire is stranded copper, with double silk coverings. After winding, the coils and bobbin are immersed in hot paraffin wax. When withdrawn and allowed to cool, the wax firmly fixes the wires in the coils in position.

Adjustment to the value of mutual inductance required is done by carrying one end of one of the coils through a further arc of a circle in order to give the effect of a fraction of a turn. Campbell (Ref. (27)) gives a method of adjustment utilizing a third coil, of small diameter, concentric and coaxial with the other two, and connected in series with the secondary (Fig. 2.11). Adjustment is by alteration of the number of turns on the small coil, a variation of one in the number of turns on the small coil having the effect of a variation of a fraction of a turn on the larger coil.

Primary Standards of Self-inductance. Although mutual inductances are more generally regarded as the primary standards of inductance, owing to the greater accuracy with which their values can be calculated from their dimensions, standards of self-inductance have been constructed at several of the national laboratories already mentioned. Their magnitudes are calculated from formulae previously referred to, and permanence is ensured by winding the coil of bare hard-drawn copper wire under tension in a screw thread cut in a marble cylinder which has been very carefully ground so as to be as nearly truly cy'indrical as possible. Deseriptions of such

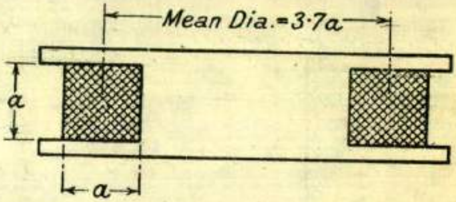


FIG. 2.12. MAXWELL'S DIMENSIONS FOR SELF-INDUCTANCE STANDARD

standards at the Bureau of Standards and at the Physikalisch Technische Reichsanstalt have been given by J. G. Coffin (Ref. (25)), and by Gruneisen and Giebe (Ref. (28)), respectively. These two coils are of inductances 216·24 mH and about 10 mH respectively.

Secondary Standards of Self-inductance. As in the case of secondary standards of mutual inductance, self-inductance secondary standards are constructed to have a nominal value which is usually a simple fraction of 1 henry. Such standards are compared with a primary standard of inductance and are used as reference standards for general laboratory work.

For the purpose of obtaining the largest possible time constant (i.e. ratio $\left.\frac{\text{inductance}}{\text{resistance}}\right)$ when winding an inductance coil, Maxwell recommended the use of the relative dimensions given in Fig. 2.12. An approximate formula for the inductance of a coil having these relative dimensions is

$$L \doteqdot 6\pi N^2 r \times 10^{-9} \text{ henrys}$$

where N = number of turns on coil

r = mean radius of the coil in centimetres

Since $r = 1{\cdot}85a$

$$L \doteqdot 11{\cdot}1\pi N^2 a \times 10^{-9} \text{ henrys} \quad . \quad . \quad . \quad . \quad (2.15)$$

Later work by Shawcross and Wells (Ref. (26)) on this subject showed that Maxwell did not consider enough terms in the formula which he used in this calculation, and that a coil of shape somewhat similar to that of

Fig. 2.12 but having a mean diameter $3a$ (instead of $3{\cdot}7a$) gives a slightly greater time constant (0·5 per cent greater).

The formula for the inductance of such a coil (dimensions in centimetres) is

$$L = 16{\cdot}83\ N^2r \times 10^{-9} \text{ henrys}$$

or

$$L = 25{\cdot}24\ N^2a \times 10^{-9} \text{ henrys} \quad . \quad . \quad . \quad (2.16)$$

Actually the maximum time constant is obtained by making the mean diameter $2{\cdot}95a$, but $3a$ is more convenient and is a sufficiently close approximation.

Coils for use as secondary standards are wound of silk-covered stranded copper wire on bobbins of marble, or of mahogany impregnated with paraffin. After winding, the coils are immersed in molten

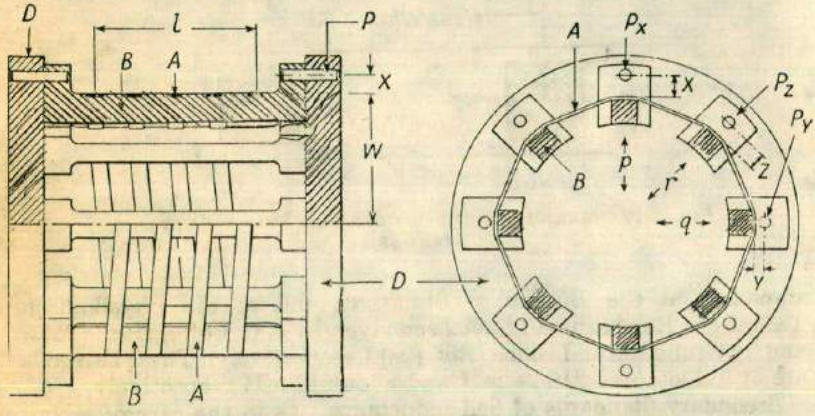


FIG. 2.13. CONSTRUCTION OF SULLIVAN-GRIFFITHS TEMPERATURE-COMPENSATED SELF-INDUCTANCE STANDARD

paraffin wax for some time. Upon being removed and cooled, the paraffin wax solidifies and rigidly fixes the coil wires in position.

The chief cause of non-cyclic temperature coefficient and general instability of inductance is always the non-equality of expansion of the conductor and the radial dimension (W) of the former. A design for stable inductances with sensibly zero temperature coefficient, due to W. H. F. Griffiths, is illustrated in Fig. 2.13. Members (B) of Keramot are supported rigidly by the end cheeks (D) of special laminated Bakelite. The conductor (A), of thin strip copper or Litzendraht wire, is wound in grooves. The pins (P) form the *chief* locating means for the relative positioning of the B and D members. The temperature coefficient of linear expansion of the D members varies with the direction of the radius owing to its laminated manufacture.

In order that the *resultant* temperature coefficient of the dimension

W shall exactly equal that of the linear expansion coefficient of the conductor A, the following condition must be satisfied—

$$\delta'(W + X) - \beta X = \delta''(W + Y) - \beta Y$$
$$= \delta'''(W + Z) - \beta Z = \alpha W$$

where δ', δ'', and δ''' are the measured temperature coefficients of expansion of the D members in the radial directions p, q and r respectively, and the expansion coefficients of the conductor and B members are α and β respectively, the latter being uniform in all directions.

From the above expression it is seen that the radial dimensions of the pins P_x, P_y, and P_z are given respectively by

$$X = \frac{\alpha - \delta'}{\delta' - \beta} \cdot W$$

$$Y = \frac{\alpha - \delta''}{\delta'' - \beta} \cdot W$$

$$Z = \frac{\alpha - \delta'''}{\delta''' - \beta} \cdot W$$

Having thus, by the above determination of the correct dimensions $X + W$, $Y + W$, and $Z + W$, ensured that the temperature coefficient δ of the former may be safely replaced by that, α, of the conductor, it can be shown that the resultant temperature coefficient of inductance is

$$\frac{\Delta L}{L} = 2\alpha + \gamma(\alpha - \beta) - \beta$$

where γ is a variable, depending upon the ratio of length to diameter, which has been enumerated for all possible shapes of coil by Griffiths.

The principle can be applied to multi-layered coils up to 1 henry with equal success. The temperature coefficients of *true* inductance are always less than 5×10^{-6} per deg C and can by careful design be reduced to 10^{-6}

Griffiths* shows how the temperature coefficient of these inductances may be affected by frequency within certain bands and discusses the effects of humidity, current and self-capacitance upon their ultimate stability.

Primary Standards of Capacitance. Such standards are capacitors whose capacitance can be accurately calculated, by means of an exact formula, from their dimensions. Capacitance in the electrostatic system of units has the dimensions of length (Table II). The

* "Recent Improvements in Air Cored Inductances," *Wireless Engineer*, Vol. XIX, No. 220, pp. 8–19 and No. 220, pp. 56–63. (See also H. W. Sullivan, 1954 catalogue.)

capacitance of absolute capacitors can thus be expressed in terms of lengths—i.e. of their dimensions—and it is therefore of prime importance that such dimensions shall be very accurately known and also that these dimensions shall not vary once the capacitor has been constructed. Owing to the fact that air is the only dielectric whose permittivity is definitely known and which is free from absorption and dielectric loss (see Chap. IV), it is always used as the dielectric in primary standard capacitors. Three types have been used as primary standards, viz. the concentric-spheres type, the concentric-cylinders type with "guard rings" (see Chap. IV), and the parallel-plate type with guard plates. Of these, the last is perhaps the least satisfactory, as it requires very careful adjustment if the calculated value of capacitance is to be accurately realized. The necessity for the guard rings and the formulae for the calculation of the capacitance of these types will be considered in Chapter IV.

The disadvantages of air as a dielectric in such capacitors are as follows—

(a) Its "dielectric strength" (see Chap. IV) is low, which necessitates a comparatively long gap between plates in order to withstand breakdown of the air when a voltage is applied.

(b) Its permittivity is low compared with solid dielectrics, which fact, combined with the long gap referred to above, means that an air capacitor is very bulky if the capacitance is to be other than very small.

(c) Dust particles, settling in the gap between the plates, cause leakage troubles unless precautions, such as thorough drying of the air in the capacitor, are taken to avoid this. The minimum distance between plates to ensure freedom from dust troubles should be 2 to 3 mm.

(d) Since there is no solid dielectric between the plates to act as a spacer, the plates must be rigidly fixed in position by supports of some solid dielectric. Very few of such insulating materials are satisfactory for this purpose, owing to their tendency to warp and cause displacement of the plates from their original position. Fused quartz and amberite are used for such purposes.

Absolute standards of capacitance were originally developed in connection with the measurement of *c*—the ratio of the electromagnetic to the electrostatic C.G.S. unit of quantity—as described earlier in the chapter. Rosa and Dorsey (Ref. (30)) have described fully several types of absolute standards of capacitance constructed by them for this purpose.

Standard Air Capacitors for High-voltage Testing. The development of methods of measuring the dielectric loss and power factor of capacitors at high voltages (see Chap. IV) has led to the construction of several types of standard air capacitors for use in making such measurements.

These are either of the parallel plate or concentric-cylinder type, guard rings being employed in each case in order to shield the capacitor from external electrostatic influences and to render more definite the effective area of the electrodes, so that the area to be used in calculating the capacitance from the dimensions shall be subject to no uncertainty. The air gap between the plates must be large in order to withstand the applied voltage, and the edges of the plates must be rounded in order to avoid brush discharges, which would produce a loss of power due to ionization of the air at such edges. For the same reason the surfaces of the plates must be free from irregularities, which necessitates a very careful grinding of these surfaces during the construction. High-voltage capacitors of the parallel-plate type have been used by various investigators, including Shanklin (Ref. (31)), who used a high-tension plate suspended from the ceiling by insulating cord, with two low-tension plates, one on either side, the latter being provided with earthed guard rings. This was used up to 60,000 volts. Rayner, Standring, Davis, and Bowdler (Ref. (32)), at the National Physical Laboratory, employed a somewhat similar construction, the high-tension plate in this case having a rounded edge of 3 in. radius. A full description of the capacitor is given by them in the paper referred to. Dunsheath (Ref. (33)) has described a parallel-plate capacitor used by him for the same purpose.

The concentric-cylinder type of capacitor, developed by Petersen (Ref. (34)), has been more generally adopted, and is more satisfactory than the parallel-plate type owing to the difficulty of efficiently screening the latter. Petersen's form consists of a cylindrical low-tension electrode with a guard cylinder of the same diameter at each end. This is surrounded by the high-tension cylinder, which is concentric with the inner one and which projects beyond the ends of the low-tension cylinder by a considerable length at each end. The ends of this high-tension cylinder are bell-shaped. Freedom from brush discharge is thus obtained, whilst the screening is efficient and the capacitance of the arrangement is easily calculable, within fairly narrow limits, from the formula

$$C = \frac{100l}{1{\cdot}8 \log_e \frac{D}{d}} \text{ micromicrofarads} \quad . \quad . \quad (2.17)$$

where l is the active length in metres of the low-tension electrode, d being its diameter and D the internal diameter of the outer electrode.

Rayner (Ref. (35)), Semm (Ref. (36)), Churcher and Dannatt (Refs. (37) and (50)), and others have used capacitors of this type. Fig. 2.14 shows the construction of a standard capacitor designed by Churcher and Dannatt for use at 300 kV (r.m.s.). The electrodes are of machined cast iron having a specially smooth finish to avoid surface irregularities which cause premature breakdown of the capacitor when the voltage is applied. The high-tension cylinder is suspended inside the low-tension cylinder from separate supports, and is insulated from the latter by a Micarta tube.

An accuracy of 0·2 per cent in the calculated capacitance of the capacitor was aimed at in the design. The average breakdown voltage is 310 kV (r.m.s.).

The original paper (Ref. (50)) should be referred to for details. Fig. 2.14 is drawn to scale, but much detail is omitted in order to show the main features more clearly.

Compressed-gas Capacitors. Fig. 2.15 shows the construction of the compressed-gas capacitor for a maximum working voltage of 250 kV (r.m.s.) made by Associated Electrical Industries. The

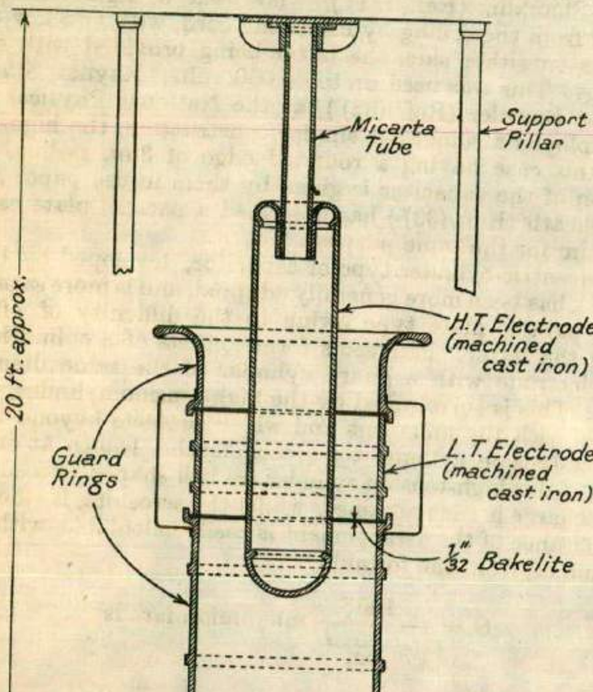
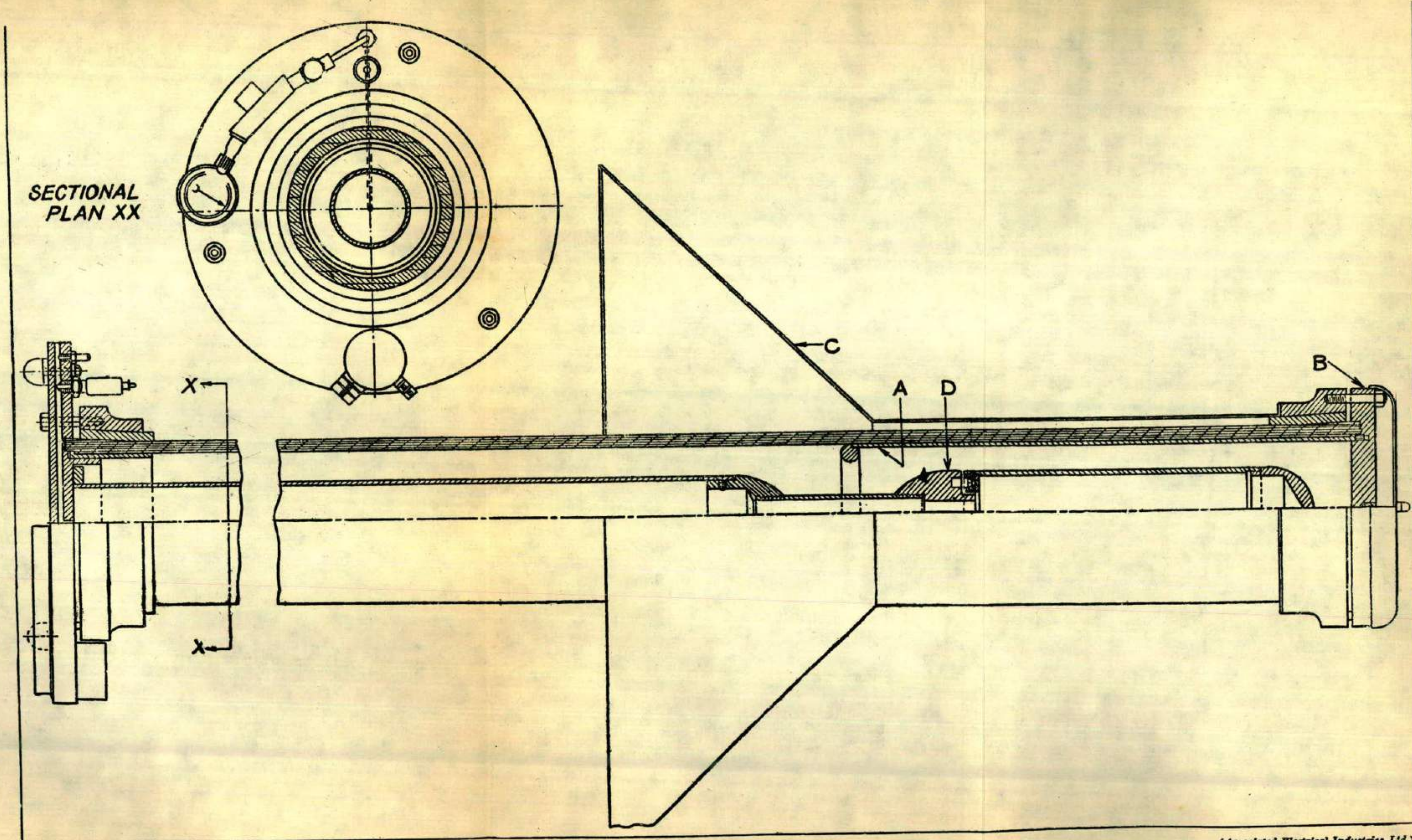


FIG. 2.14. CHURCHER AND DANNATT STANDARD AIR CAPACITOR FOR 300 kV

main features are the high-tension electrode *A*, consisting of a steel tube fitting tightly inside a Micarta tube, the latter being long enough to give the necessary insulation to ground for the working voltage. Connection from the electrode to the top plate *B* is made through a spring contact. The high-tension terminal, with domed head, is fitted to the centre of a stress distributor *C* (used for voltages above 150 kV), which is secured to the top plate by small screws.

The low-tension electrode is supported on a central post terminating in the cap *D*, which acts as a guard ring, this being insulated



(Associated Electrical Industries Ltd.)

FIG. 2.15. A 250-KV COMPRESSED-GAS CAPACITOR

from the effective part of the electrode by an insulating collar. The lead from the electrode, which is screened throughout its length, is brought down to a screened terminal box.

The capacitance of the capacitor is $50\mu\mu F$, and its loss angle is less than 0·00001 degree. The gas used may be either air or nitrogen and it must be clean and dry. The working gas pressure is 150 lb per sq. in. gauge reading.

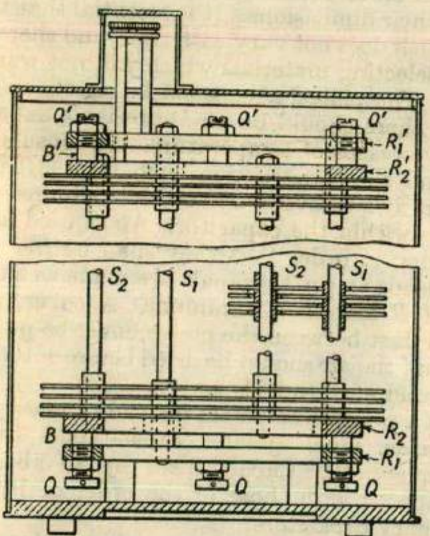
Secondary Standard Capacitor. Capacitors for use as secondary standards are calibrated capacitors whose dimensions need not be accurately known since the magnitude of their capacitance is not calculated from their dimensions. It is essential that they shall have a capacitance which does not vary with time, and therefore care must be exercised in selecting materials which will not warp and so alter the dimensions. The plates, also, must be rigidly fixed in position, and, if possible, there should be no appreciable expansion of them with moderate increases of temperature. The insulation should be very efficient and the construction such that leakage is avoided. Sharp edges must also be avoided in order to eliminate troubles from brush discharges within the capacitor. Air is used as the dielectric in such capacitors, in order that they shall be free from dielectric losses, and the leads to the terminals are made as short as possible, to reduce the I^2R loss to a minimum. A cover, to prevent the accumulation of dust between the plates, must be provided, and it is desirable also that the air should be dried before entering the interior, as moisture is conducive to leakage.

By the use of a number of plates instead of merely two, as in the primary standards, much greater capacitances can be obtained without excessive bulk. Capacitances up to about 0·02 μF are obtainable compared with those of the order of 100 to $200\mu\mu F$ in the case of primary capacitors.

Glazebrook and Muirhead (Ref. (38)) designed a secondary standard air capacitor for the committee of the British Association in 1890. It consisted of twenty-four concentric brass tubes, the thickness of whose walls was about $\frac{1}{32}$ in. Twelve of these tubes were supported in a vertical position by a conical brass casting, the outside surface of which formed a series of twelve steps over which the tubes fitted and to which they were screwed. This casting, with its tubes attached, was carried by three ebonite pillars about 3 in. high. The other twelve tubes were fitted to a similar stepped brass casting, which was carried by the outside case so that these tubes hung downwards in the air spaces between the first twelve cylinders. The terminal of the insulated cylinders was in the form of a brass rod passing through a central hole in the upper brass casting, and insulated from it by an ebonite plug, this rod being screwed into the bottom brass casting. The internal air was dried by a small dish of sulphuric acid placed inside the case. The capacitance of this capacitor was about 0·021 μF.

Giebe (Ref. (47)) in 1909 described a modified form of the above capacitor constructed by him, and also a plate type which he found to be superior to the former and which is shown in Fig. 2.16. It consists of a large number of thin, circular plates of magnalium—a magnesium-aluminium alloy—with a space of about 2 mm between successive plates. In one form there are 71 plates in all—35 connected to one terminal and 36 to the other. Hague* gives a full description of this capacitor.

Messrs. H. W. Sullivan, Ltd., manufacture a range of standard



(From *Alternating Current Bridge Methods.* Hague)

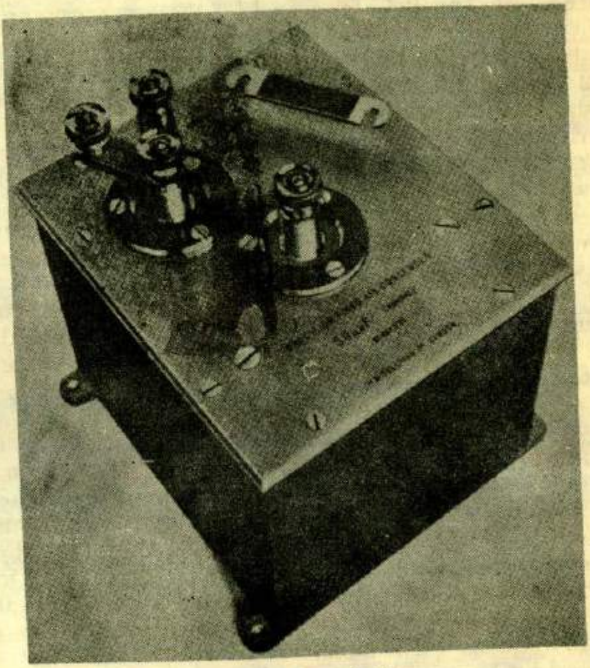
FIG. 2.16. GIEBE'S PLATE AIR CAPACITOR

air capacitors in which the insulation between the two conducting systems consists of small pieces of silica-quartz, a material having very low dielectric loss. Their long-period permanence is 1 part in 20,000 and the temperature coefficient less than 1 part per 100,000 per degree centigrade. This extraordinarily low temperature coefficient is brought about by a method of compensation described fully by W. H. F. Griffiths (Ref. (63)). These fixed standards of capacitance have a special terminal system to eliminate errors of stray lead capacitance. The two main terminals of the capacitor remain connected to the testing apparatus, and the exactly standardized capacitance is inserted by means of a strap (see Fig. 2.17). Power factors $\leqslant 0{\cdot}00001$ are obtained at 1,000 c/s and maintained

* *A.C. Bridge Methods*, 2nd Edition, p. 122.

up to radio frequencies. The three-terminal types have inter-terminal electrostatic shielding.

The Capacitance Increment Standard due to W. H. F. Griffiths (H. W. Sullivan, Ltd.) introduces into a calibrating circuit, by a rotary motion of a variable air capacitor, four very accurately known capacitances of, say, 100, 200, 300, and 500 $\mu\mu$F or 10, 20, 30, and 50 $\mu\mu$F. These capacitances are inserted strictly as circuit



(*H. W. Sullivan, Ltd.*)
FIG. 2.17. SULLIVAN-GRIFFITHS FIXED AIR CAPACITOR

increments and may thus be used as a very accurate means of calibrating variable capacitors by either direct or substitution bridge methods.

Recent improvements in precision variable air capacitors (silica insulated) have been responsible for stabilities which are now measured in parts in a million. In the Sullivan and Griffiths instrument the capacitance accuracy is 1 part in 10^4, and it is made to be unaffected by variable stray electrostatic fields in the vicinity of the terminals and connecting leads by a special screened lead device. Power factors of 0·00001 or less are maintained up to frequencies of

10^5 or 10^6 c/s, and the temperature coefficient is compensated by means of a bi-metallic method also due to W. H. F. Griffiths (Ref. (63)). Also due to Griffiths is the recently invented Decade Variable Air Capacitor Standard which is capable of very great accuracy and is described on p. 265.

Laboratory Standards of Capacitance. The secondary standards described above are unsuitable for general laboratory purposes. As laboratory standards, capacitors having a solid dielectric instead of air are used. The dielectrics used for these purposes are mica and paraffined paper, the former being the better. Both of these materials have a permittivity greater than that of air (mica 3 to 8, paper 2 (about)), and therefore give a greater capacitance for a given size than when air is the dielectric. They have also high resistivity and dielectric strength, both of which characteristics are necessary for the purpose for which they are used. W. H. F. Griffiths (H. W. Sullivan, Ltd.) has designed fixed-value standards from 0·01 to 1·0 μF and single and multi-decade standards up to 5 μF continuously subdivisible down to 1 $\mu\mu$F having a *direct reading* accuracy of 0·01 per cent. Extraordinarily low power factors of 0·00005 are obtained on the higher capacitances. Temperature coefficients as low as 0·001 per cent per °C are obtained, and the internal inductance and series resistance are reduced to such low values that frequency corrections of both capacitance and power factor are unnecessary up to quite high frequencies even for an accuracy of 0·01 per cent (15 kc/s for 1 μF, 200 kc/s for 0·01 μF).

Paraffined-paper capacitors are not so reliable as the mica type, and are not suitable as standards for precision work. They have a greater dielectric loss (and therefore power factor) than mica capacitors, the power factor as stated by Grover (Ref. (40)) for a range of them varying from 0·0017 to 0·017. Grover also found the frequency variation to be of the order of 4 parts in 1,000 for a frequency range of 50 to 1,000 c/s, the capacitance decreasing with increase of frequency. The manufacture of paper and other capacitors is described by Mansbridge (Ref. (41)).

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CHAPTER III

CIRCUIT ANALYSIS

In alternating current circuits generally, and especially in networks, the symbolic (j) notation is of great use in simplifying the calculation of the various quantities involved. For this reason it will be considered here before proceeding to work in which such calculations are necessary.

Fig. 3.1 shows a vector representing (say) a voltage, which, expressed in the usual trigonometrical notation, is given by

$$v = V_{max} \sin (\omega t + a)$$

This vector could otherwise be defined by stating its resolved com-

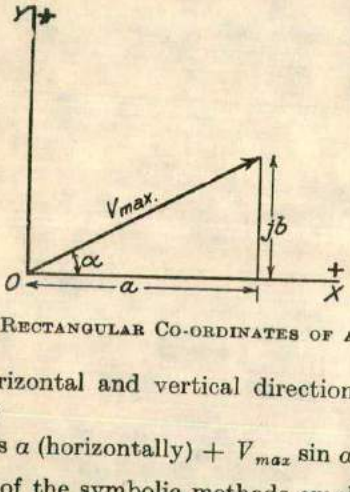


Fig. 3.1 Rectangular Co-ordinates of a Vector

ponents in the horizontal and vertical directions—i.e. along axes OX and OY. Thus

$$v = V_{max} \cos \alpha \text{ (horizontally)} + V_{max} \sin \alpha \text{ (vertically)}$$

The commonest of the symbolic methods employs this means of expression, the horizontal component being written simply as $V_{max} \cos \alpha$ and the vertical component being distinguished by placing a letter j in front of it. Thus, symbolically, the vector is expressed as

$$[V] = V_{max} \cos \alpha + jV_{max} \sin \alpha = V_{max} [\cos \alpha + j \sin \alpha]$$

or $$[V] = a + jb$$

where a and b are its horizontal and vertical components, the brackets [ ] indicating that the notation is symbolic.*

* The fact that a quantity is expressed symbolically may be indicated also by a dot placed under the symbol, thus— $\underset{\cdot}{E}$, $\underset{\cdot}{I}$, etc.

In the same way, the vectors shown in Fig. 3.2 can be represented symbolically as

$$[V_1] = a + jb$$
$$[V_2] = -c + jd$$
$$[V_3] = -e - jf$$
$$[V_4] = k - jh$$

respectively, the directions OX and OY being positive and directions OX' and OY' negative.

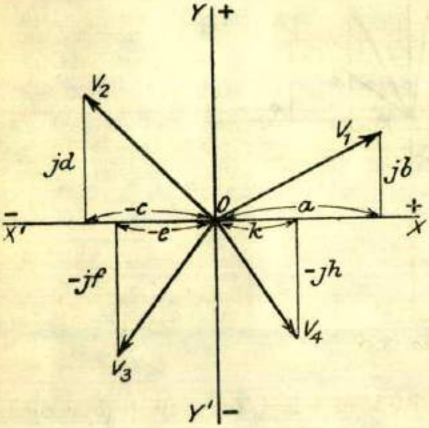


FIG. 3.2 SYMBOLIC REPRESENTATION

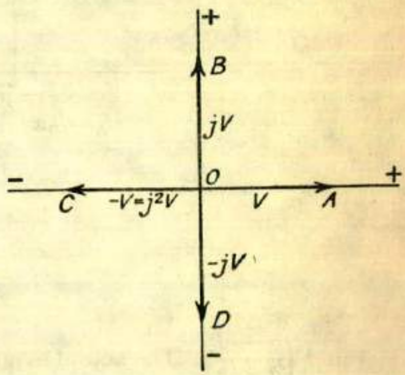


FIG. 3.3

Actual Value of the Operator "j". In Fig. 3.3, the vectors OA, OB, OC, and OD are all of the same magnitude V. Expressing them symbolically, we have

$$OA = V$$
$$OB = jV$$
$$OC = -V$$
$$OD = -jV$$

From this it appears that the multiplication of a vector V, such as OA, by j means that it is rotated through 90° in an anti-clockwise direction. Then, multiplying OB by j, we rotate it through another 90° to OC. Thus,

$$OA \times j \times j = OC$$

or
$$j^2 V = -V$$

i.e.
$$j^2 = -1$$
$$j = \sqrt{-1}$$

The operator j is thus an imaginary quantity, and can be treated as having the value $\sqrt{-1}$ in all calculations in which it occurs.

Addition of Vectors. Vectors are added by adding together their horizontal and vertical components separately. Thus the sum of vectors $[V_1] = a + jb$ and $[V_2] = c + jd$ is $[V] = (a + c) + j(b + d)$ and its phase angle relative to the horizontal axis OX is

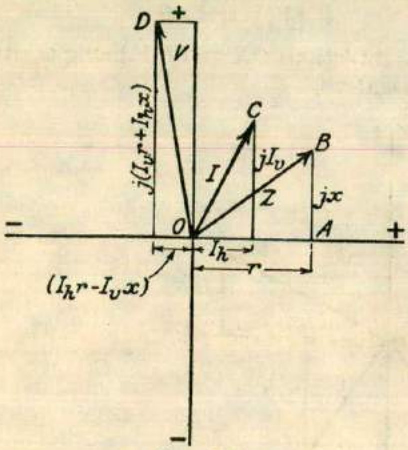


FIG. 3.4

$\tan^{-1}\left(\frac{b + d}{a + c}\right)$. The actual magnitude of V is $\sqrt{(a + c)^2 + (b + d)^2}$.

Subtraction of Vectors. If a vector $[V_1] = a + jb$ is to be subtracted from a vector $[V_2] = c + jd$, then the resultant is

$$[V] = [V_2] - [V_1] = c + jd - (a + jb)$$
$$= c - a + j(d - b)$$

its actual value being

$$\sqrt{(c - a)^2 + (d - b)^2}$$

and its phase angle

$$\tan^{-1}\left(\frac{d - b}{c - a}\right)$$

Multiplication of Complex Quantities. The product of two complex quantities $[V_1] = a + jb$ and $[V_2] = c + jd$ is

$$[V] = (a + jb)(c + jd) = ac + j(bc + ad) + j^2bd$$
$$= ac - bd + j(bc + ad)$$
$$= A + jB$$

where A and B are the horizontal and vertical resolved components of the product $[V]$. The numerical value of V is obviously $\sqrt{A^2 + B^2}$.

As an example, an impedance expressed as $[Z] = r + jx$, when multiplied by a current $[I] = I_h + jI_v$ gives a voltage

$$[V] = (I_h + jI_v)(r + jx)$$
$$= (I_h r - I_v x) + j(I_v r + I_h x)$$

To find the numerical value of V,

$$V^2 = (I_h r - I_v x)^2 + (I_v r + I_h x)^2$$
$$= (I_h^2 + I_v^2)(r^2 + x^2)$$

Thus $V = (\sqrt{I_h^2 + I_v^2})(\sqrt{r^2 + x^2})$

and since $\sqrt{I_h^2 + I_v^2} = I$, we have

$$V = I\sqrt{r^2 + x^2}$$

which is, of course, the result which would be obtained by trigonometrical methods. Fig. 3.4 illustrates this example. The triangle OAB is the "impedance triangle," giving the symbolic expression $r + jx$ for the impedance, while OC and OD are the current and voltage vectors respectively.

Division of a Vector Quantity by a Complex Quantity. If a vector quantity $a + jb$ is to be divided by a complex quantity $c + jd$ the quotient is $\frac{a + jb}{c + jd}$.

Rationalizing the denominator, we have

$$[V] = \frac{(a + jb)(c - jd)}{(c + jd)(c - jd)} = \frac{ac + j(bc - ad) - j^2bd}{c^2 - j^2d^2}$$
$$= \frac{(ac + bd) + j(bc - ad)}{c^2 + d^2}$$
$$= \frac{ac + bd}{c^2 + d^2} + j\frac{(bc - ad)}{c^2 + d^2}$$
$$= C + jD$$

where C and D are the resolved parts of the resultant vector.

The numerical value of V is, as before, given by $V = \sqrt{C^2 + D^2}$.

Other Forms of Representation. Exponential Form. This is really an extension of the trigonometrical form of expression. It was seen that a vector quantity could be expressed in the form

$$[V] = V(\cos\alpha + j\sin\alpha)$$

If the angle α is in radians, $\sin\alpha$ and $\cos\alpha$ can be expanded below—

$$\sin\alpha = \alpha - \frac{\alpha^3}{\underline{|3}} + \frac{\alpha^5}{\underline{|5}} - \frac{\alpha^7}{\underline{|7}} + \dots$$

$$\cos\alpha = 1 - \frac{\alpha^2}{\underline{|2}} + \frac{\alpha^4}{\underline{|4}} - \frac{\alpha^6}{\underline{|6}} + \dots$$

$$\therefore [V] = V\left[\left(1 - \frac{a^2}{|\underline{2}} + \frac{a^4}{|\underline{4}} - \frac{a^6}{|\underline{6}} + \ldots\right) + j\left(a - \frac{a^3}{|\underline{3}} + \frac{a^5}{|\underline{5}} - \frac{a^7}{|\underline{7}} + \ldots\right)\right]$$

$$= V\left[1 + ja - \frac{a^2}{|\underline{2}} - \frac{ja^3}{|\underline{3}} + \frac{a^4}{|\underline{4}} + \frac{ja^5}{|\underline{5}} - \frac{a^6}{|\underline{6}} - \frac{ja^7}{|\underline{7}} + \frac{a^8}{|\underline{8}} + \cdots\right]$$

Substituting j^2 for -1 in the above we have

$$[V] = V\left[1 + ja + \frac{j^2a^2}{|\underline{2}} + \frac{j^3a^3}{|\underline{3}} + \frac{j^4a^4}{|\underline{4}} + \frac{j^5a^5}{|\underline{5}} + \frac{j^6a^6}{|\underline{6}} + \frac{j^7a^7}{|\underline{7}} + \frac{j^8a^8}{|\underline{8}} + \ldots\right]$$

or $[V] = Ve^{ja}$,

since the series is the expansion of e^{ja}, where e is the base of natural logarithms. Thus, if a current I is given by $\frac{V}{Z}$ where the voltage $V] = Ve^{ja}$ and the impedance $[Z] = Ze^{j\beta}$, then

$$[I] = \frac{[V]}{[Z]} = \frac{Ve^{ja}}{Ze^{j\beta}} = \frac{V}{Z}e^{j(a-\beta)}$$

This is illustrated in Fig. 3.5.

Polar Form (1). This form of representation, suggested by Prof. Diamant (*Trans. Am. I.E.E.*, Vol. XXXV, p. 957), has not been very generally applied, but is nevertheless useful in some types of problems.

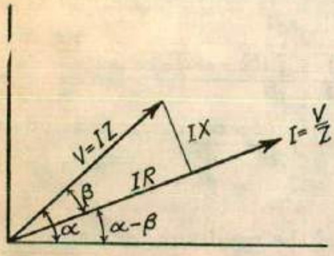


Fig. 3.5. Exponential Form of Representation

In this method, the vector quantity $[V] = V(\cos a + j \sin a)$ is expressed as VJ^m, where J represents an operator which, when applied to a vector, rotates it through an angle of 90°. In this respect it is similar to j. The index m is the ratio of the angle which the vector makes with the horizontal axis to one right angle. Thus, in the vector V mentioned above, $m = \frac{a}{\pi/2}$, expressing the angles in circular measure. If m is positive, the rotation is anti-clockwise, and if negative the rotation is clockwise. The three-phase voltage vectors shown in Fig. 3.14 could be expressed in this form as

$$[E_1] = EJ^0$$
$$[E_2] = EJ^{+\frac{4}{3}}$$
$$[E_3] = EJ^{-\frac{4}{3}}$$

Since $m = \frac{a}{\pi/2} \therefore a = m\frac{\pi}{2}$ and the vector V can be written

$$[V] = V\left(\cos m\frac{\pi}{2} + j\sin m\frac{\pi}{2}\right)$$

$$= V\left(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2}\right)^m \text{ from De Moivre's Theorem}$$

$$= V(0 + j\,.\,1)^m$$

$$= V(j^m) = VJ^m$$

Thus j and J have the same meaning.

Polar Form (2). Another form of representation which is fairly frequently used is $V/\underline{\alpha}$ meaning that the vector is of length V and is rotated in an anti-clockwise direction, so that it makes an angle of a with the horizontal. This form is merely conventional. Using this mode of expression, the three-phase vectors referred to in the previous paragraph can be expressed as

$$[E_1] = E/\underline{0}$$

$$[E_2] = E\,/\underline{120^\circ} \quad \text{or } E\,/\underline{\tfrac{2\pi}{3}}$$

$$[E_3] = E\,/\underline{-120^\circ} \text{ or } E\,/\underline{\tfrac{-2\pi}{3}}$$

The product of two vectors $[E_1] = E_1/\underline{\alpha}$ and $[E_2] = E_2/\underline{\beta}$ may be expressed as

$$E = E_1E_2/\underline{\alpha + \beta}$$

and the quotient of two such vectors as

$$E = \frac{E_1}{E_2}/\underline{\alpha - \beta}$$

Application of the Symbolic Method to Alternating Current Problems. The application of the rectangular form of representation to problems in a.c. circuits can be illustrated by means of examples. Several different types of circuits and problems are given below.

Example 1 (Simple Series Circuit). A sinusoidal voltage of r.m.s. value 100 volts and frequency 50 cycles per second is applied to the circuit shown in Fig. 3.6. Calculate the current in the circuit and find its phase relative to that of the applied voltage.

The impedances in the circuit can be expressed symbolically as—

Impedance of $R_1 = 3$

" $L_1 = j\omega L_1 = j \times 314 \times 0{\cdot}0159 = 5j$

" $R_2 = 4$

" $L_2 = j\omega L_2 = j \times 314 \times 0{\cdot}0477 = 15j$

" $C = -j \times \frac{1}{\omega C} = \frac{-j10^6}{314 \times 318} = -10j$

[NOTE. The negative sign in the capacitor impedance is explained by considering the current as a horizontal vector, when the voltage drop across the capacitor will be vertically downwards (since it lags 90° in phase behind

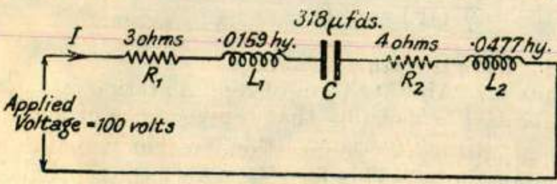


FIG. 3.6

the current). Thus, if the current is expressed symbolically as I, the voltage drop (given by current × impedance of capacitor) is $-j\frac{I}{\omega C}$, i.e. the impedance is $\frac{-j}{\omega C}$.]

The total impedance of the circuit is the sum of these symbolic expressions. Thus

$$[Z] = 3 + 4 + 5j + 15j - 10j$$
$$= 7 + 10j$$

Then $Z^2 = 7^2 + 10^2 = 149$

$$Z = \sqrt{149} = 12{\cdot}2$$

and the current is $\frac{100}{12{\cdot}2} = 8{\cdot}22$ amp

If the current vector is horizontal, so that

$$[I] = 8{\cdot}22 + j\,.\,0$$

the voltage is

$$[V] = [I][Z] = [8{\cdot}22 + j\,.\,0][7 + 10j] = 57{\cdot}54 + 82{\cdot}2j$$

The voltage thus leads the current by an angle ϕ such that $\tan\phi = \frac{82{\cdot}2}{57{\cdot}54} = \frac{10}{7}$. It is, of course, the angle of the impedance triangle.

Example 2 (Series-parallel Circuit). A sinusoidal voltage, of r.m.s. value 100 volts and frequency 50 cycles per second, is applied to the circuit shown in Fig. 3.7. Calculate the current in the main circuit and the currents in the two branch circuits. Take the voltage vector as horizontal, so that $[V] = 100 + j\,.\,0$.

Impedance of $R_1 = 8$

" $L_1 = j\omega L_1 = 314 \times 0{\cdot}0477j = 15j$

" $C_1 = \frac{-j}{\omega C_1} = \frac{-10^6 j}{314 \times 159} = -20j$

Total impedance $= 8 - 5j$

$$\left.\begin{aligned}\text{Impedance of } R_1 &= 10\\ \text{,, } L_1 &= j\omega L_1 = 314 \times 0.0636j = 20j\end{aligned}\right\} \begin{aligned}&\text{Total impedance}\\ &= 10 + 20j\end{aligned}$$

$$\left.\begin{aligned}\text{Impedance of } R_3 &= 7\\ \text{,, } C_2 &= \frac{-j}{\omega C_2} = \frac{-j10^6}{314 \times 318} = -10j\end{aligned}\right\} \begin{aligned}&\text{Total impedance}\\ &= 7 - 10j\end{aligned}$$

Admittance of branch I $= \frac{1}{10 + 20j} = \frac{10 - 20j}{10^2 + 20^2}$

$= 0.02 - 0.04j = [Y_1]$

Admittance of branch II $= \frac{1}{7 - 10j} = \frac{7 + 10j}{7^2 + 10^2}$

$= 0.047 + 0.067j = [Y_2]$

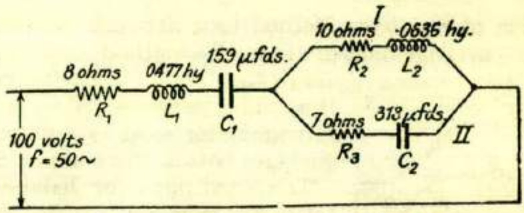


FIG. 3.7. SERIES-PARALLEL CIRCUIT

Total admittance of the two branches in parallel $= [Y_1] + [Y_2] = 0.067 + 0.027j$

Total impedance of the two branches in parallel $= \frac{1}{0.067 + 0.027j} = \frac{0.067 - 0.027j}{0.067^2 + 0.027^2}$

$= 12.8 - 5.16j$

Thus the total impedance of the complete circuit is

$$8 - 5j + 12.8 - 5.16j = 20.8 - 10.16j$$

and the current I in the main circuit is given by

$$I = \frac{100}{20.8 - 10.16j} = \frac{100\,(20.8 + 10.16j)}{20.8^2 + 10.16^2}$$

$$= 3.88 + 1.9j$$

Its numerical value is therefore $\sqrt{3.88^2 + 1.9^2}$

$= 4.32$ amp

and its phase relative to the applied voltage is $\tan^{-1}\frac{1.9}{3.88}$ leading (since the imaginary term in the expression for the current is positive).

The voltage drop across the two parallel branches is

$$(3.88 + 1.9j)(12.8 - 5.16j) = 59.46 + 4.3j$$

Current in branch I $= \frac{59.46 + 4.3j}{10 + 20j}$

$$= \frac{(59.46 + 4.3j)(10 - 20j)}{10^2 + 20^2} = 1.37 - 2.29j$$

Its numerical value is $\sqrt{1{\cdot}37^2 + 2{\cdot}29^2} = 2{\cdot}67$ amp and it lags behind the applied voltage V by an angle $\tan^{-1} \frac{2{\cdot}29}{1{\cdot}36}$.

$$\text{Current in branch II} = \frac{59{\cdot}46 + 4{\cdot}3j}{7 - 10j}$$

$$= \frac{(59{\cdot}46 + 4{\cdot}3j)(7 + 10j)}{7^2 + 10^2} = 2{\cdot}51 + 4{\cdot}2j$$

Its numerical value is $\sqrt{2{\cdot}51^2 + 4{\cdot}2^2} = 4{\cdot}88$ amp and it leads the applied voltage by a phase angle $\tan^{-1} \frac{4{\cdot}2}{2{\cdot}5}$.

Adding the two branch currents gives $3{\cdot}88 + 1{\cdot}9j$ which is the original expression for the current in the main circuit.

Application of Symbolic Method to a Network Problem. Fig. 3.8 shows Wien's arrangement of Maxwell's method of comparing a self-inductance with a capacitance by means of a bridge network. V.G. is a vibration galvanometer used as a detector for frequencies within the commercial range.

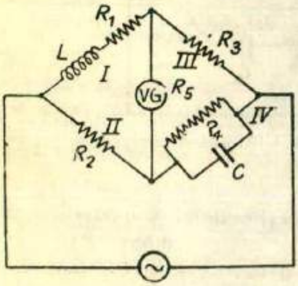


FIG. 3.8. WIEN BRIDGE NETWORK

The conditions for balance with this network are that

$$R_1R_4 = R_2R_3 = \frac{L}{C}$$

as can be seen from the following.

At balance, when no current flows through the galvanometer circuit,

$$\frac{Z_1}{Z_3} = \frac{Z_2}{Z_4}$$

where Z_1, Z_2, Z_3, and Z_4 are the impedances of branches I, II, III, and IV respectively.

$$[Z_1] = R_1 + j\omega L$$
$$[Z_2] = R_2$$
$$[Z_3] = R_3$$

Total admittance of branch IV is

$$[Y_4] = \frac{1}{R_4} + \frac{1}{\frac{-j}{\omega C}}$$

$$= \frac{1}{R_4} + \frac{j\omega C}{-j^2} = \frac{1}{R_4} + j\omega C$$

$\therefore$ Total impedance of branch IV is

$$\frac{1}{[Y_4]} = \frac{R_4}{1 + j\omega CR_4} = [Z_4]$$

$$\therefore \frac{R_1 + j\omega L}{R_3} = \frac{R_2}{\left(\frac{R_4}{1 + j\omega CR_4}\right)} = \frac{R_2}{R_4}(1 + j\omega CR_4)$$

Cross-multiplying,

$$R_1R_4 + j\omega LR_4 = R_2R_3 + jR_2R_3\omega CR_4$$

Equating real and imaginary terms, we have

$$R_1R_4 = R_2R_3$$

and $$j\omega LR_4 = jR_2R_3\omega CR_4$$

from which $$L = R_2R_3C$$

Thus $$R_1R_4 = R_2R_3 = \frac{L}{C} \quad . \quad . \quad . \quad . \quad (3.1)$$

The symbolic method can be used, also, to calculate the current in the galvanometer circuit when the bridge network is out of balance. In Fig. 3.9 the network is represented simply by impedances, and the cyclic currents (used by Maxwell to simplify network calculations) X, $X + Y$, and A, are assumed to flow in the three meshes as shown.

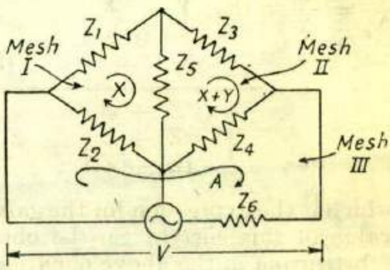


Fig. 3.9

Z_5 is the impedance of the galvanometer circuit, Z_6 that of the alternator branch, and V the alternator voltage.

Then the current in the galvanometer circuit is $(X + Y - X) = Y$. Using Kirchhoff's second law—that the algebraic sum of the potential differences in any closed circuit is zero— we have

Mesh I.

$$Z_1X + Z_5(-Y) + Z_2(X - A) = 0$$

or $$X(Z_1 + Z_2) - Z_5Y - Z_2A = 0$$

Mesh II.

$$Z_3(X + Y) + Z_4(X + Y - A) + Z_5Y = 0$$

or $$X(Z_3 + Z_4) + Y(Z_3 + Z_4 + Z_5) - AZ_4 = 0$$

Mesh III.

$$Z_2(A - X) + Z_4(A - X - Y) + Z_6A = V$$

or $$-X(Z_2 + Z_4) - YZ_4 + A(Z_2 + Z_4 + Z_6) = V$$

Thus the three equations, from which Y is to be obtained, are

$$X(Z_1 + Z_2) - YZ_5 - AZ_2 - 0 = 0 \quad . \quad \text{(i)}$$
$$X(Z_3 + Z_4) + Y(Z_3 + Z_4 + Z_5) - AZ_4 - 0 = 0 \quad . \quad \text{(ii)}$$
$$-X(Z_2 + Z_4) - YZ_4 + A(Z_2 + Z_4 + Z_6) - V = 0 \quad . \quad \text{(iii)}$$

expressing the equations in the form most suited to their solution by the method of determinants. Then, from the algebraic theory of determinants, we have

$$\frac{Y}{\begin{vmatrix} Z_1 + Z_2 & -Z_2 & 0 \\ Z_3 + Z_4 & -Z_4 & 0 \\ -(Z_2 + Z_4) & Z_2 + Z_4 + Z_6 & -V \end{vmatrix}} = \frac{1}{\begin{vmatrix} Z_1 + Z_2 & -Z_5 & -Z_2 \\ Z_3 + Z_4 & Z_3 + Z_4 + Z_5 & -Z_4 \\ -(Z_2 + Z_4) & -Z_4 & Z_2 + Z_4 + Z_6 \end{vmatrix}}$$

or, expressing the determinants by Δ_v and Δ respectively, we have

$$\frac{Y}{\Delta_v} = \frac{1}{\Delta} \text{ so that } Y = \frac{\Delta_v}{\Delta}$$

$$= \frac{V(Z_1Z_4 - Z_2Z_3)}{\begin{vmatrix} Z_1 + Z_2 & -Z_2 & -Z_2 \\ Z_3 + Z_4 & Z_3 + Z_4 + Z_5 & -Z_4 \\ -(Z_2 + Z_4) & -Z_4 & Z_2 + Z_4 + Z_6 \end{vmatrix}}$$

which is the expression for the galvanometer current. The numerical value of this current can be obtained, in any particular case, by substituting in the above equation the symbolic expressions for the impedances of the various branches.

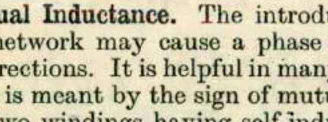


FIG. 3.10.

The Sign of Mutual Inductance. The introduction of a mutual inductance into a network may cause a phase reversal depending upon the winding directions. It is helpful in many network problems to understand what is meant by the sign of mutual inductance.

Fig. 3.10 shows two windings having self-inductances L_1 and L_2 and a mutual inductance M between them. If the windings are connected togethei so that the same current passing through both windings gives a flux in the same direction, then the total self-inductance of the two windings in series is $L_1 + L_2 + 2M$ (see also p. 182). The mutual inductance between the windings is then positive in sign. Reversal of the direction of one winding gives opposing flux directions; the overall self-inductance is then

$L_1 + L_2 - 2M$ and the mutual inductance is negative in sign. The directions of current are fixed. Fig. 3.11 shows the conventional current directions when the two windings in Fig. 3.10 are separated, and Fig. 3.12 shows the more usual convention in which the windings are presumed to circulate common flux in the same direction for the positive connection.

The self- and mutually-induced voltages are conventionally assumed to act in the same direction as the current flow but having

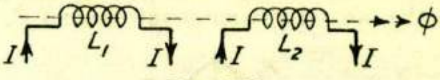


FIG. 3.11.

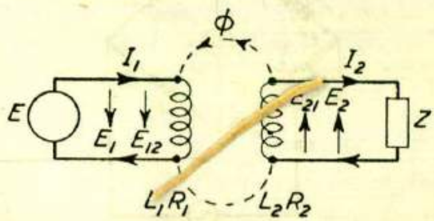


FIG. 3.12.

negative sign. Hence, for the network shown in Fig. 3.12, if M is positive the network equations are

$$E + E_1 + E_{12} = R_1 I_1$$

or

$$E - j\omega L I_1 - j\omega M I_2 = R_1 I_1$$

and

$$E_2 + E_{21} = I_2 R_2 + I_2 Z$$

or

$$-j\omega L I_2 - j\omega M I_1 = I_2 R_2 + I_2 Z$$

A negative connection of M simply necessitates a change in sign of the terms in M.

The adoption of this convention is not of great importance in networks containing a single mutual inductance, because it usually becomes quite apparent during the course of analysis if an error has been made in the sign. It is essential, however, to adhere to a convention in the solution of networks containing a number of mutual inductances.

Network Containing a Mutual Inductance. Fig. 3.13 shows the connections of Heaviside's mutual inductance bridge for the determination of a self-inductance in terms of a mutual inductance. R_1, R_2, R_3, and R_4 are non-inductive resistances. while L_1 and L_2 are self-inductances, there being, in addition, mutual inductance M between the alternator circuit and branch IV. The inductance L_2 forms the secondary of this mutual inductance.

The treatment of a problem when mutual inductance is present is somewhat different from that used in the previous bridge network.

Since, at balance, the voltage across the detector branch is zero, the voltage drops across branches I and IV are equal. Thus

$$i'(R_1 + j\omega L_1) = (R_4 + j\omega L_2)i'' + j\omega Mi \quad . \quad . \quad \text{(i)}$$

The mutual inductance term $j\omega Mi$ represents the voltage induced in arm IV by a current of i in the alternator branch. The convention is employed that this voltage is in the opposite direction to the current i, i.e. in the same direction as the current i''.

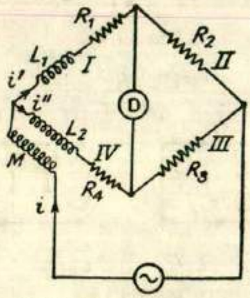


FIG. 3.13. HEAVISIDE BRIDGE

We have also that

$$i = i' + i''$$

and, since no current flows in the detector branch, the current in arm II is also i' and in arm III is i''. Thus

$$R_2 i' = R_3 i''$$

Substituting for i in equation (i),

$$i'(R_1 + j\omega L_1) = (R_4 + j\omega L_2)i'' + j\omega M(i' + i'')$$

or $$i'(R_1 + j\omega L_1 - j\omega M) = (R_4 + j\omega L_2 + j\omega M)i''$$

Substituting $\dfrac{R_2}{R_3} i'$ for i'',

$$i'(R_1 + j\omega L_1 - j\omega M) = (R_4 + j\omega L_2 + j\omega M)\frac{R_2}{R_3} i'$$

$$R_1 + j\omega L_1 - j\omega M = \frac{R_4 R_2}{R_3} + j\omega L_2 \frac{R_2}{R_3} + j\omega M \frac{R_2}{R_3}$$

Equating real and imaginary terms we have

$$R_1 = \frac{R_4 R_2}{R_3} \text{ or } R_1 R_3 = R_2 R_4 \quad . \quad . \quad . \quad (3.2)$$

Also

$$j\omega L_1 - j\omega M = j\omega L_2 \frac{R_2}{R_3} + j\omega M \frac{R_2}{R_3}$$

from which

$$R_3(L_1 - M) = R_2(L_2 + M) \quad . \quad . \quad . \quad (3.3)$$

Application of the Symbolic Method to Polyphase Circuit Calculations. The full consideration of such problems is both outside the scope of this work and too lengthy for inclusion here, but the application of the symbolic notation to one three-phase circuit problem will be considered.

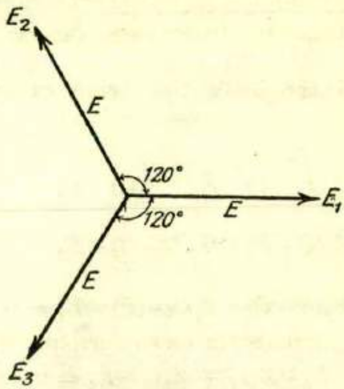


FIG. 3.14. THREE-PHASE VOLTAGE VECTORS

Fig. 3.14 shows a balanced system of three-phase voltages E_1, E_2, and E_3. When in the phase positions shown, these can be expressed symbolically as

$$[E_1] = [E + j0]$$
$$[E_2] = [-E \cos 60° + jE \sin 60°]$$
$$= E[-0{\cdot}5 + 0{\cdot}866j]$$
$$[E_3] = [-E \cos 60° - jE \sin 60°]$$
$$= E[-0{\cdot}5 - 0{\cdot}866j]$$

The symbolic sum of the voltages is, of course, zero.

Fig. 3.15 shows a three-phase network with alternator phase voltages E_1, E_2, and E_3, having positive directions as shown. I_1, I_2, and I_3 represent the phase (and line) currents, and z_1, z_2, and z_3 the line impedances. P, Q, and S are the mesh currents, and Z_1, Z_2, Z_3 are the phase impedances, including the impedances of both alternator and load phases and the line impedances.

Then, whether the system is balanced or not, the following

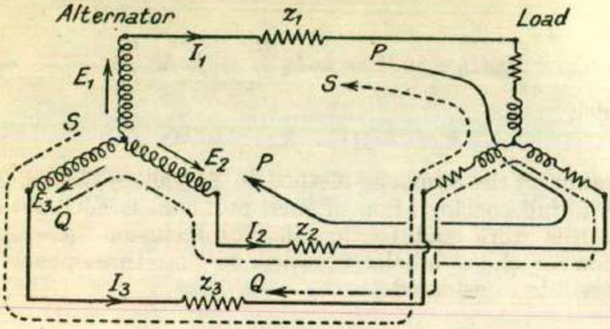


FIG. 3.15. THREE-PHASE CIRCUIT

equations hold, all the quantities being expressed in symbolic notation.

Mesh P.

$$E_1 - Z_1(P - S) - Z_2(P - Q) - E_2 = 0$$

Mesh Q.

$$E_2 - Z_2(Q - P) - Z_3(Q - S) - E_3 = 0$$

Mesh S.

$$E_3 - Z_3(S - Q) - Z_1(S - P) - E_1 = 0$$

Substituting line currents for mesh currents, we have

$$E_1 - Z_1I_1 + Z_2I_2 - E_2 = 0$$
$$E_2 - Z_2I_2 + Z_3I_3 - E_3 = 0$$
$$E_3 - Z_3I_3 + Z_1I_1 - E_1 = 0$$

or

$$E_1 - E_2 = Z_1I_1 - Z_2I_2$$
$$E_2 - E_3 = Z_2I_2 - Z_3I_3$$
$$E_3 - E_1 = Z_3I_3 - Z_1I_1$$

Eliminating I_2 and I_3 by the use of the relationship $I_1 + I_2 + I_3 = 0$, we have

$$\frac{E_1 - E_2}{Z_2} + \frac{E_1 - E_3}{Z_3} = I_1 \left(1 + \frac{Z_1}{Z} + \frac{Z_1}{Z_3}\right)$$

$$\therefore I_1 = \frac{E_1 - E_2}{Z_2Z_1\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}\right)} + \frac{E_1 - E_3}{Z_3Z_1\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}\right)}$$

$$= \frac{E_1 - E_2}{Z_1 + Z_2 + \frac{Z_1Z_2}{Z_3}} + \frac{E_1 - E_3}{Z_1 + Z_3 + \frac{Z_1Z_3}{Z_2}} \quad . \quad . \quad (3.4)$$

Expressions can be found for I_2 and I_3 in the same way.

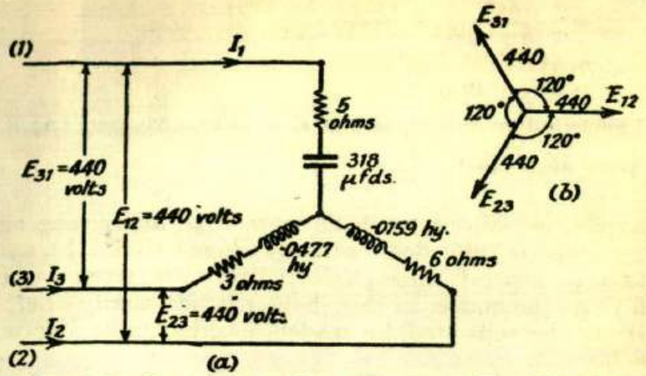


FIG. 3.16. THREE-PHASE STAR-CONNECTED LOAD CIRCUIT

Example. The three-phase star-connected load shown in Fig. 3.16 (*a*) is connected to a three-phase system having a line voltage of 440 volts. Assuming the line voltages to be unaffected by the unbalanced load, calculate the current flowing in the branch containing the capacitor. The supply frequency is 50 cycles per sec.

Fig. 3.16 (*b*) shows the phase relationships of the three line voltages.

Then $E_{12} = E_1 - E_2,\ E_{23} = E_2 - E_3,\ E_{31} = E_3 - E_1$

where E_1, E_2, and E_3 are the phase voltages of the supply as used in the expression for I_1 in the above paragraph.

From the figure,

$$[E_{12}] = 440 + j\,.\,0$$
$$[E_{23}] = -\,440 \cos 60° - j\,.\,440 \sin 60°$$
$$= -\,440\,(0{\cdot}5 + 0{\cdot}866j)$$
$$[E_{31}] = -\,440 \cos 60° + j\,.\,440 \sin 60°$$
$$= -\,440\,(0{\cdot}5 - 0{\cdot}866j)$$

The impedances of the load branches are

$$[Z_1] = 5 - \frac{j10^6}{314 \times 318} = 5 - 10j$$
$$[Z_2] = 6 + 314 \times 0{\cdot}0159j = 6 + 5j$$
$$[Z_3] = 3 + 314 \times 0{\cdot}0477j = 3 + 15j$$

Then
$$[I_1] = \frac{E_1 - E_2}{Z_1 + Z_2 + \dfrac{Z_1 Z_2}{Z_3}} + \frac{E_1 - E_3}{Z_1 + Z_3 + \dfrac{Z_1 Z_3}{Z_2}}$$

$$[I_1] = \frac{440}{5 - 10j + 6 + 5j + \dfrac{(5 - 10j)(6 + 5j)}{3 + 15j}}$$
$$+ \frac{440\,(0{\cdot}5 - 0{\cdot}866j)}{5 - 10j + 3 + 15j + \dfrac{(5 - 10j)(3 + 15j)}{6 + 5j}}$$

$$= \frac{440}{9{\cdot}78 - 10{\cdot}57j} + \frac{440\,(0{\cdot}5 - 0{\cdot}866j)}{28 - 4{\cdot}1j}$$
$$= 440\,[0{\cdot}069 + 0{\cdot}023j]$$
$$= 30{\cdot}4 + 10{\cdot}1j$$

Thus the numerical value of I_1 is $\sqrt{30{\cdot}4^2 + 10{\cdot}1^2} = 32{\cdot}1$ amp and it leads E_{12} by a phase angle $\tan^{-1} \frac{10{\cdot}1}{30{\cdot}4}$.

The application of the symbolic method to alternating current bridge networks is fully dealt with by Hague (Ref. (1)), and the application to general three-phase networks is given by Dover (Ref. (2)). As the matter in this chapter is necessarily brief, these works should be consulted by readers desiring fuller information on the subject.

Symmetrical Components. The method of calculation referred to as that of "Symmetrical Components" involves, and is an extension of, the symbolic methods already described. It is especially applicable to the solution of problems in connection with unbalanced polyphase networks, and simplifies the calculation when it would be very difficult, if not impossible, by other methods. The most usual polyphase system is, of course, the three-phase, and the symmetrical components method will be discussed here with reference to such a system.

The method, which was largely developed by C. L. Fortescue (Ref. (8)), involves the analysis of an unbalanced system of three-phase vectors into three systems which are each balanced but which have different phase sequences. These are referred to as the *positive-sequence*, *negative-sequence*, and *zero-sequence* systems respectively and constitute the symmetrical components of the three original unbalanced vectors. In other words, each of these vectors is split up into three components, each of which forms part of a balanced system.

Fig. 3.17 shows positive-, negative-, and zero-sequence systems of vectors in diagrams (i), (ii), and (iii). It must be clearly understood that all vectors are assumed to rotate in an anti-clockwise direction in accordance with the standard convention. The sequence is determined, not by any differences in direction of rotation, but by the order in which the vectors pass any fixed position. Thus, in the positive sequence this order is a, b, c, whereas in the negative sequence it is a, c, b. The three zero-sequence vectors are in phase with one another so that they pass any fixed position together. All three systems of vectors are balanced, so that E_{a1}, E_{b1} and E_{c1} are all equal in magnitude as are E_{a2}, E_{b2} and E_{c2}, and E_{a0}, E_{b0}, E_{c0}.*

In diagram (iv) of Fig. 3.17 the three vectors E_{a1}, E_{a2}, and E_{a0} of diagrams (i), (ii), and (iii) are added vectorially to give vector E_a,

* The notation used here is that commonly adopted in published work on the subject of symmetrical components.

as are E_{b1}, E_{b2}, E_{b0} and E_{c1}, E_{c2}, E_{c0} to give E_b and E_c respectively. It will be noticed that the result is an unbalanced system of vectors E_a, E_b, and E_c. Conversely, the unbalanced system E_a, E_b, and E_c can be split up into, or replaced by, the three balanced systems shown in diagrams (i), (ii), and (iii), these three diagrams being supposed to

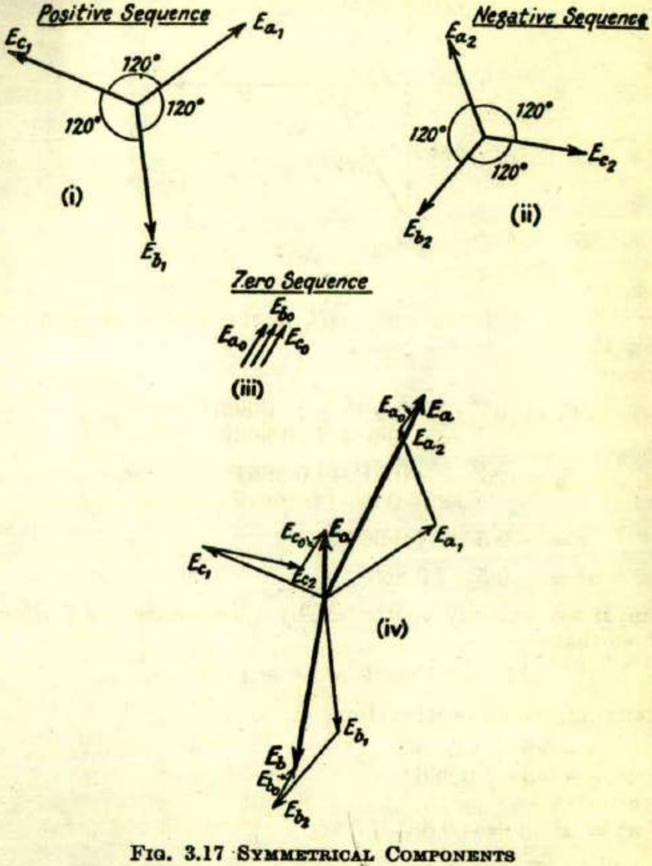


FIG. 3.17 SYMMETRICAL COMPONENTS

correspond to the same instant of time, so that they may be superposed in one diagram to show the correct phase relationships between the nine vectors.

We must now consider the mathematical treatment of symmetrical components. For this purpose we introduce a vector or operator a which is comparable with the operator j already used, except that the multiplication of a vector by a rotates it through an angle of 120° in an anti-clockwise direction instead of by 90° as does multiplication by j. Referring to Fig. 3.18, OP represents a

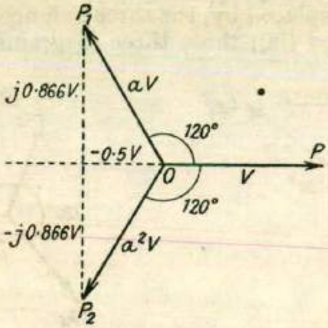


FIG. 3.18

vector V (or $V + j \,.\, 0$), while OP_1 is the vector aV and OP_2 the vector a^2V.

Obviously

$$OP_1 = aV = -0{\cdot}5\,V + j\,.\,0{\cdot}866V = (-0{\cdot}5 + j\,.\,0{\cdot}866)V$$

and

$$OP_2 = a^2V = -0{\cdot}5V - j\,0{\cdot}866V = (-0{\cdot}5 - j\,0{\cdot}866)V$$

so that $a = -0{\cdot}5 + j\,0{\cdot}866$

and $a^2 = -0{\cdot}5 - j\,0{\cdot}866$

Again, if we multiply vector OP_2 by a we rotate it a further 120° to OP, so that

$$a^3V = OP = V \text{ or } a^3 = 1^*$$

Summarizing, we have, therefore,

$$a = -0{\cdot}5 + j\,0{\cdot}866$$
$$a^2 = -0{\cdot}5 - j\,0{\cdot}866$$
$$a^3 = 1$$
$$a^4 = a^3\ a = -0{\cdot}5 + j\,0{\cdot}866$$

and so on.

It is important to note, also, that

$$1 + a + a^2 = 1 - 0{\cdot}5 + j\,0{\cdot}866 - 0{\cdot}5 - j\,0{\cdot}866 = 0$$

Adopting the exponential notation of page 110, a is the unit vector $1.e^{j\frac{2\pi}{3}}$ or $1.e^{j120°}$ so that any vector $Me^{j\theta}$, when multiplied by a, becomes $Me^{j(\theta+120°)}$. The vector OP may be written in this notation as Ve^{j0}, OP_1 as $Ve^{j120°}$, and OP_2 as $Ve^{j240°}$.

* 1, a and a^2 are the cube roots of unity.

Using the operator a in relation to the symmetrical components of Fig. 3.17 we have, for example, $E_{c1} = aE_{a1}$ and $E_{b1} = a^2 E_{a1}$, so that we obtain the following equations—

$$\left.\begin{aligned} E_{a1} &= E_{a1} \\ E_{b1} &= a^2 E_{a1} \\ E_{c1} &= a\, E_{a1} \end{aligned}\right\} \quad \text{Positive-sequence system}$$

$$\left.\begin{aligned} E_{a2} &= E_{a2} \\ E_{b2} &= a\, E_{a2} \\ E_{c2} &= a^2\, E_{a2} \end{aligned}\right\} \quad \text{Negative-sequence system}$$

$$E_{a0} = E_{b0} = E_{c0} \quad \text{Zero-sequence system}$$

Hence the three unbalanced vectors E_a, E_b, and E_c may be expressed by the symbolic expressions

$$E_a = E_{a0} + E_{a1} + E_{a2} = E_{a0} + E_{a1} + E_{a2} \quad . \quad . \quad . \quad \text{(i)}$$

$$E_b = E_{b0} + E_{b1} + E_{b2} = E_{a0} + a^2E_{a1} + aE_{a2} \quad . \quad . \quad \text{(ii)}$$

$$E_c = E_{c0} + E_{c1} + E_{c2} = E_{a0} + aE_{a1} + a^2E_{a2} \quad . \quad . \quad \text{(iii)}$$

The symmetrical components E_{a0}, E_{a1}, and E_{a2} may be derived, in terms of the three unbalanced vectors E_a, E_b, and E_c, from these equations as shown below.

Positive-sequence Components. Utilizing the values obtained in equations (i), (ii), and (iii) we have, for the sum of E_a, aE_b, and a^2E_c,

$$\begin{aligned} E_a + aE_b + a^2E_c &= E_{a0}\,(1 + a + a^2) \\ &\quad + E_{a1}\,(1 + a^3 + a^3) + E_{a2}\,(1 + a^2 + a^4) \\ &= 3E_{a1} \qquad \text{since } a^4 = a,\ a^3 = 1 \\ &\qquad\qquad\quad \text{and } 1 + a + a^2 = 0 \end{aligned}$$

Hence $$E_{a1} = \frac{E_a + aE_b + a^2E_c}{3}$$

Negative-sequence Components. Again, from (i), (ii) and (iii),

$$\begin{aligned} E_a + a^2E_b + aE_c &= E_{a0}\,(1 + a + a^2) \\ &\quad + E_{a1}\,(1 + a^4 + a^2) \\ &\quad + E_{a2}\,(1 + a^3 + a^3) \\ &= 3E_{a2} \end{aligned}$$

Hence $$E_{a2} = \frac{E_a + a^2E_b + aE_c}{3}$$

Zero-sequence Components. Adding (i), (ii) and (iii) we have

$$\begin{aligned} E_a + E_b + E_c &= 3E_{a0} + E_{a1}\,(1 + a^2 + a) \\ &\qquad + E_{a2}\,(1 + a + a^2) \\ &= 3E_{a0} \end{aligned}$$

Hence

$$E_{a0} = E_{b0} = E_{c0} = \frac{E_a + E_b + E_c}{3}$$

The zero-sequence components are thus each equal to one-third of the vector sum of the three unbalanced vectors.

These statements and equations may, from the use of the symbol E throughout, be taken to apply to voltage vectors, but they apply in exactly the same way to current or impedance vectors.

From the above, two important facts are immediately apparent. First, if a system is balanced, the zero-sequence and negative-sequence components are both zero, since we may write $E_b = a^2E_a$ and $E_c = aE_a$. from which

$$E_{a0} = \frac{E_a + a_a{}^2E + aE_a}{3} = \frac{E_a(1 + a + a^2)}{3} = 0$$

$$E_{a2} = \frac{E_a + a^4E_a + a^2E_a}{3} = \frac{E_a(1 + a + a^2)}{3} = 0$$

The positive-sequence components are then equal to the balanced vectors themselves, since

$$E_{a1} = \frac{E_a + a^3E_a + a^3E_a}{3} = \frac{3E_a}{3} = E$$

Again, although three vectors may not constitute a balanced system, yet if their resultant is zero (i.e. if the vector sum $E_a + E_b + E_c$ is zero) the zero-sequence components must be zero since $E_{a0} = (E_a + E_b + E_c)/3 = 0$. It follows, therefore, that in a mesh-connected system there are no zero-sequence components of voltage, and in a star-connected three-wire system with an insulated neutral point there are no zero-sequence components of current.

It must be realized that in the few pages of space here available no more than a brief introduction to the method of symmetrical components can be given. For fuller treatment, including the application of the method to the calculation of networks upon which there are faults, for which purpose it is particularly suited, the reader should refer to the works mentioned at the end of the chapter.

Before concluding, however, the application of the method will be illustrated by an alternative solution of the problem given on page 121.

Example. Let E_a, E_b, and E_c be the voltages, line to neutral, applied to the impedances 1, 2, and 3 in Fig. 3.16 (a).

Then

$$E_a = (5 - 10j)\,I_1 = (5 - 10j)\,[I_{a0} + I_{a1} + I_{a2}]$$
$$E_b = (6 + 5j)\,I_2 = (6 + 5j)\,[I_{b0} + I_{b1} + I_{b2}]$$
$$E_c = (3 + 15j)\,I_3 = (3 + 15j)\,[I_{c0} + I_{c1} + I_{c2}]$$

where I_1, I_2, and I_3 are the three line currents, expressed in symbolic notation, and I_{a0}, I_{a1}, I_{a2}, etc., their symmetrical components. Since there is no fourth wire the zero-sequence components I_{a0}, I_{b0}, and I_{c0} are zero, so that we have

$$E_a = (5 - 10j) \ [I_{a1} + I_{a2}] \quad . \quad . \quad . \quad . \quad . \quad . \quad (x)$$
$$E_b = (6 + 5j) \ [I_{b1} + I_{b2}] = (6 + 5j) \ [a^2 I_{a1} + a I_{a2}] \quad . \quad . \quad . \quad (y)$$
$$E_c = (3 + 15j)[I_{c1} + I_{c2}] = (3 + 15j)[a I_{a1} + a^2 I_{a2}] \quad . \quad . \quad . \quad (z)$$

Now, from the vector diagram of Fig. 3.16 (*b*),

$$E_a - E_b = 440; \ E_b - E_c = a^2 . 440; \ E_c - E_a = a . 440$$

From (x) and (y), by subtraction, giving a and a^2 their known values,

$$E_a - E_, = I_{a1}[3{\cdot}66 - 2{\cdot}3j] + I_{a2}[12{\cdot}34 - 12{\cdot}7j] = 440$$

Again, from (y) and (z),

$$E_b - E_c = I_{a1}[15{\cdot}84 - 2{\cdot}8j] + I_{a2}[- 18{\cdot}84 + 12{\cdot}8j]$$
$$= a^2 . 440 = - 220 - 381j$$

And, from (z) and (x),

$$E_c - E_a = I_{a1}[- 19{\cdot}5 + 5{\cdot}1j] + I_{a2}[6{\cdot}5 - 0{\cdot}1j]$$
$$= a . 440 = - 220 + 381j$$

Only the first two of these equations are needed to evaluate I_{a1} and I_{a2}. First eliminating I_{a1} by multiplying the first equation by $[15{\cdot}84 - 2{\cdot}8j]$ and the second by $[3{\cdot}66 - 2{\cdot}3j]$ and subtracting, we have for I_{a2}

$$I_{a2} = 12{\cdot}6 + 18{\cdot}8j$$

$$\text{and} \quad I_{a1} = \frac{440 - [12{\cdot}34 - 12{\cdot}7j]I_{a2}}{3{\cdot}66 - 2{\cdot}3j} = 17{\cdot}8 - 8{\cdot}5j$$

Then

$$I_1 = I_{a1} + I_{a2} = 30{\cdot}4 + 10{\cdot}1j$$

which is the same result as previously obtained on page 121.

We may proceed to find I_2 and I_3 as follows—

$$I_2 = a^2 Ia_1 + a I_{a2}$$
$$= - 38{\cdot}86 - 9{\cdot}55j$$
$$I_3 = a I_{a1} + a^2 I_{a2}$$
$$= 8{\cdot}46 - 0{\cdot}55j$$

Graphical methods of determining the symmetrical components of three unbalanced voltages or currents are discussed fully in Chapter XIII of the book by Wagner and Evans mentioned in Ref. (7). The measurement of such quantities is also dealt with in Chapter XIV of the same book. Specially constructed meters for the analysis of unbalanced voltages and currents into their symmetrical components are described in a paper by T. A. Rich (Ref. (11)).

Some Important Network Theorems

Although it is possible to solve the majority of linear network problems with the aid of Kirchhoff's laws, in many cases the working can be simplified by the use of certain network theorems which will now be stated.

The Superposition Theorem. *The resultant current in any element of a linear network, due to the simultaneous action of a number of generators, may be found by considering one generator at a time and adding the currents due to the individual generators.*

The application of the theorem may best be illustrated by means of an example. The circuit shown in Fig. 3.19 (*a*) consists of two alternators in parallel supplying current to a load impedance Z. The alternators have open-circuit e.m.f.s. E_1 and E_2 and internal impedances Z_1 and Z_2 respectively. This circuit is resolved into the two circuits in Figs. 3.19 (*b*) and (*c*).

Fig. 3.19 (*b*) shows the circuit with the generator E_2 removed and

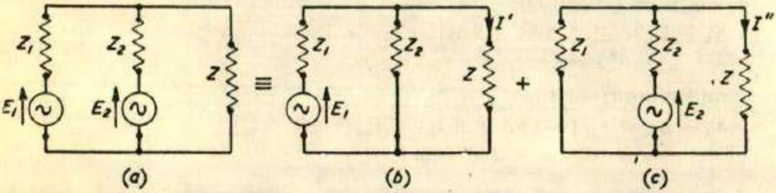


FIG. 3.19. APPLICATION OF THE SUPERPOSITION THEOREM

replaced by a passive Z_2 equal to its internal impedance; the partial load current in this case is

$$I' = \frac{E_1}{Z_1 + \frac{ZZ_2}{Z + Z_2}} \cdot \frac{Z_2}{Z + Z_2} = E_1 \cdot \frac{Z_2}{ZZ_1 + Z_1Z_2 + ZZ_2}$$

In Fig. 3.19 (*c*), E_1 is removed and replaced by a passive impedance Z_1 equal to its internal impedance, and the partial load current is

$$I'' = E_2 \cdot \frac{Z_1}{ZZ_1 + Z_1Z_2 + ZZ_2}$$

Applying the Superposition Theorem, the load current I is given by

$$I = I' + I'' = \frac{E_1Z_2 + E_2Z_1}{ZZ_1 + Z_1Z_2 + ZZ_2}$$

The current contributed by each generator can be found in a similar manner.

This theorem can be verified, in general terms, by taking a network having a number of meshes and generators, and deriving, by means of Kirchhoff's laws, an expression for the current in one mesh: the theory of determinants may usefully be employed for this purpose. It will be found that the current I_n is expressed in the form

$$I_n = k_1E_1 + k_2E_2 + k_3E_3 + \ldots k_nE_n$$

where E_1, E_2, etc., are the generator voltages, and k_1, k_2, etc., are constant coefficients expressed in terms of the network impedances. It is clear from the form of this result that each generator makes an independent contribution to the current I_n.

Thevenin's Theorem. This is a very useful theorem which often simplifies calculations when it is required to find the current in one branch of a network. It is really an extension of the Superposition Theorem.

The current flowing through an impedance, when this is connected across any two points A and B of an active network, is found by dividing the voltage appearing across the two points before the impedance is inserted by the sum of this impedance and the impedance of the network as seen from the points A and B. In determining the network impedance all

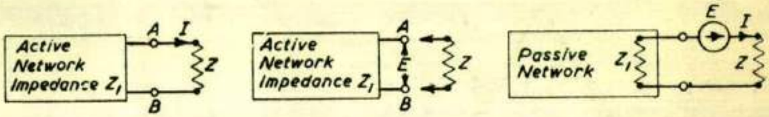


FIG. 3.20. ILLUSTRATING THÉVENIN'S THEOREM

the voltage sources must be replaced by passive impedances equal to their internal impedances.

Referring to Fig. 3.20, the current in the branch Z, when connected across A and B, is given by $\frac{E}{Z_1 + Z}$, where E is the open-circuit potential difference appearing across A and B when Z is removed, and Z_1 is the internal impedance of the active network, measured between A and B.

A number of different rigorous proofs have been developed for this theorem but it can be simply justified in the following manner. Suppose a generator of zero internal impedance and e.m.f. E, equal but opposite in sign to that appearing across AB, is introduced into Z, and this branch is then closed across AB; then no current will flow in Z. Now, if the generator in Z is removed, it follows from the Superposition Theorem that the change of current in Z is $\frac{E}{Z_1 + Z}$, and this is the actual current in Z when closed on AB.

The Star–Delta Transformation. This transformation can be used to simplify complicated networks. It is of considerable value in the solution of some bridge networks and examples of its use will be found in subsequent chapters.

Consider the delta and star arrangement of impedances shown in Fig. 3.21. If the two circuits are to be equivalent, then the impedance between corresponding terminals must be the same in both cases.

The impedance between terminals A and B in the delta network is

$$Z_{AB} = \frac{Z_1(Z_2 + Z_3)}{Z_1 + Z_2 + Z_3}$$

and in the star network,

$$Z_{AB} = Z_a + Z_b$$

Similar expressions may be derived for the other pairs of terminals giving the following equations for equivalence—

Terminals AB $$Z_a + Z_b = \frac{Z_1(Z_2 + Z_3)}{Z_1 + Z_2 + Z_3} \quad . \quad . \quad . \quad \text{(i)}$$

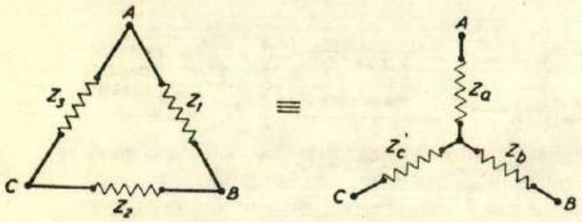


FIG. 3.21. STAR–DELTA TRANSFORMATION

Terminals BC $$Z_b + Z_c = \frac{Z_2(Z_1 + Z_3)}{Z_1 + Z_2 + Z_3} \quad . \quad . \quad . \quad \text{(ii)}$$

Terminals CA $$Z_c + Z_a = \frac{Z_3(Z_1 + Z_2)}{Z_1 + Z_2 + Z_3} \quad . \quad . \quad . \quad \text{(iii)}$$

Subtract (ii) from (i), obtaining

$$Z_a - Z_c = \frac{Z_3(Z_1 - Z_2)}{Z_1 + Z_2 + Z_3} \quad . \quad . \quad . \quad \text{(iv)}$$

Add (iii) to (iv) and we obtain the result

$$Z_a = \frac{Z_1 Z_3}{Z_1 + Z_2 + Z_3}$$

By symmetry,

$$Z_b = \frac{Z_1 Z_2}{Z_1 + Z_2 + Z_3} \text{ and } Z_c = \frac{Z_2 Z_3}{Z_1 + Z_2 + Z_3}$$

The inverse transformation may be deduced from these results as follows—

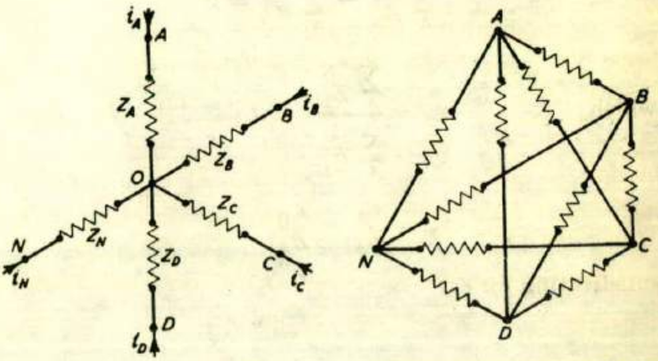
$$Z_1 + Z_2 + Z_3 = \frac{Z_1 Z_3}{Z_a} = \frac{Z_1 Z_2}{Z_b} = \frac{Z_2 Z_3}{Z_c}$$

from which

$$\frac{Z_1}{Z_3} = \frac{Z_b}{Z_c}, \quad \frac{Z_2}{Z_3} = \frac{Z_b}{Z_a}$$

Now $$Z_a = \frac{Z_1 Z_3}{Z_1 + Z_2 + Z_3} = \frac{Z_1}{\frac{Z_1}{Z_3} + \frac{Z_2}{Z_3} + 1}$$

$$= \frac{Z_1}{\frac{Z_b}{Z_c} + \frac{Z_b}{Z_a} + 1} = \frac{Z_1 Z_c Z_a}{Z_a Z_b + Z_b Z_c + Z_c Z_a}$$



Star System of Impedances *Equivalent Pair-connected System*

FIG. 3.22. GENERAL STAR–MESH TRANSFORMATION

Hence $$Z_1 = \frac{Z_a Z_b + Z_b Z_c + Z_c Z_a}{Z_c}$$

Similarly $$Z_2 = \frac{Z_a Z_b + Z_b Z_c + Z_c Z_a}{Z_a}$$

and $$Z_3 = \frac{Z_a Z_b + Z_b Z_c + Z_c Z_a}{Z_b}$$

The General Star–Mesh Transformation. The star–delta transformation is a particular case of a more general transformation (Rosen's Theorem) which enables a star of impedances having n arms to be converted to a pair-connected system having $\frac{n(n-1)}{2}$ arms.

Consider the star of n impedances shown in Fig. 3.22. Let e_A, e_B, e_C, . . . e_N be the potentials of the points $A, B, C,$ . . . . N. Let e_O be the potential of the star point O. For an equivalent pair-connected system the currents entering the network at A, B, C, etc., must be the same in both systems.

In the star network

$$i_A = \frac{e_A - e_O}{Z_A},\ i_B = \frac{e_B - e_O}{Z_B},\ i_N = \frac{e_N - e_O}{Z_N}$$

and
$$i_A + i_B + i_C \ldots + i_N = 0$$

i.e.
$$\sum_A^N \frac{e_N - e_O}{Z_N} = 0$$

or
$$\sum_A^N \frac{e_N}{Z_N} - \sum_A^N \frac{e_O}{Z_N} = 0$$

from which
$$e_O = \frac{\sum_A^N \frac{e_N}{Z_N}}{\sum_A^N \frac{1}{Z_N}}$$

Now,
$$i_A = \frac{e_A}{Z_A} - \frac{e_O}{Z_A}$$

and; substituting for e_O,

$$i_A = \frac{e_A}{Z_A} - \frac{1}{Z_A}\frac{\sum_A^N \frac{e_N}{Z_N}}{\sum_A^N \frac{1}{Z_N}}$$

or
$$i_A = \frac{1}{Z_A \sum_A^N \frac{1}{Z_N}}\left[\sum_A^N \frac{e_A}{Z_N} - \frac{e_A}{Z_A} - \frac{e_B}{Z_B} - \frac{e_C}{Z_C} \ldots - \frac{e_N}{Z_N}\right]$$

which simplifies to

$$i_A = \frac{1}{Z_A \sum_A^N \frac{1}{Z_N}}\left[\frac{e_A - e_B}{Z_B} + \frac{e_A - e_C}{Z_C} \ldots + \frac{e_A - e_N}{Z_N}\right]$$

or

$$i_A = \frac{e_A - e_B}{Z_A Z_B \sum_A^N \frac{1}{Z_N}} + \frac{e_A - e_C}{Z_A Z_C \sum_A^N \frac{1}{Z_N}} \ldots + \frac{e_A - e_N}{Z_A Z_N \sum_A^N \frac{1}{Z_N}}$$

It is apparent that

$$\frac{e_A - e_B}{Z_A Z_B \sum_A^N \frac{1}{Z_N}}$$

is the current which would flow in an impedance

$$Z_A Z_B \sum_A^N \frac{1}{Z_N}$$

placed between A and B, and

$$\frac{e_A - e_N}{Z_A Z_N \sum_A^N \frac{1}{Z_N}}$$

is the current in an impedance

$$Z_A Z_N \sum_A^N \frac{1}{Z_N}$$

between A and N.

Thus the equivalent pair-connected system is obtained by taking two terminals at a time and placing between them an impedance of the form

$$Z_A Z_N \sum_A^N \frac{1}{Z_N}$$

The pair-connected system is constructed by starting with terminal A and joining it to every other terminal with the appropriate impedance, then repeating the procedure from terminal B with impedances of the form

$$Z_B Z_N \sum_A^N \frac{1}{Z_N}$$

and so on until every pair is connected.

This theorem is of considerable assistance in the study of earth-capacitance effects in a.c. bridges. See also Ref. (1).

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CHAPTER IV

CAPACITORS, CAPACITANCE, AND DIELECTRICS

General Considerations. In Chapter I capacitance was defined with reference to a number of conductors having different charges and being at different potentials. Self- and earth-capacitances were also discussed. Before proceeding to develop formulae for the capacitances of various common arrangements of conductors encountered in practice it may be well to give these matters a little further consideration.

Fig. 4.1 shows a general system of conductors in air, situated at various distances from earth and from one another. If all these

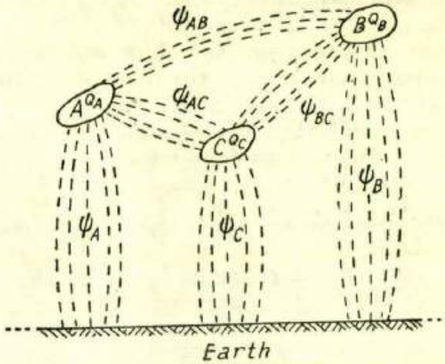


FIG. 4.1. SYSTEM OF CHARGED CONDUCTORS NEAR TO EARTH

conductors are at the same potential above earth, varying quantities of electric flux will pass from them to earth, these fluxes depending, in each case, upon the size and shape of the conductor, and upon its position relative to earth—i.e. upon the "earth capacitance" of each conductor. No flux will pass from one conductor to another, since they are all at the same potential above earth. The quantities of positive electricity existing upon the various conductors will be different, since their earth capacitances are different and their potentials the same.

Suppose the capacitances of the various conductors to earth are given by C_A, C_B, C_C, etc. Suppose now that the conductors are charged to different potentials V_A, V_B, V_C, etc., above earth. In this case, not only will some flux pass from each conductor to earth, but, in addition, flux will pass between any one conductor and each of the others in the system Each of these inter-conductor fluxes will be proportional to the difference of potential of the conductors between which it exists, and its direction will, of course, depend upon which of the two conductors concerned is at the higher potential. If conductor A is at a higher potential than any of the other conductors, fluxes will flow from it, which may be represented by ψ_{AB}, ψ_{AC}, ψ_{AD}, and so on, the second suffix letter indicating, in each case, the conductor to which the particular flux radiated from A flows. If B is at the second highest potential, the fluxes radiating from it are $-\psi_{BA}$, ψ_{BC}, ψ_{BD}, etc., and for conductor C, $-\psi_{CA}$, $-\psi_{CB}$, ψ_{CD}, etc., assuming it to be the third highest in potential. As stated above there will be, in each case, an earth flux which may be represented by ψ_A, ψ_B, ψ_C, etc.

It may be supposed that a portion of the total charge of each conductor is associated with each of the fluxes radiating from that conductor. These portions of charge will, of course, be proportional to the corresponding fluxes, and therefore will be proportional to the differences in potential between the pairs of conductors. Representing these portions of charge, in the case of A by Q_{AB}, Q_{AC}, Q_{AD}, etc., and in the case of B by Q_{BA}, Q_{BC}, Q_{BD}, etc., and so on, we have for the total charges on the various conductors

$$
\begin{aligned}
Q_A &= C_A V_A + C_{AB}(V_A - V_B) + C_{AC}(V_A - V_C) + C_{AD}(V_A - V_D) + \dots \\
Q_B &= C_B V_B + C_{AB}(V_B - V_A) + C_{BC}(V_B - V_C) + C_{BD}(V_B - V_D) + \dots \\
Q_C &= C_C V_C + C_{AC}(V_C - V_A) + C_{BC}(V_C - V_B) + C_{CD}(V_C - V_D) + \dots
\end{aligned}
\qquad (4.1)
$$

Thus, if there are n capacitors, each one has n component capacitances, including its earth capacitance.

In most cases in practice we are concerned with two (or it may be three or four) conductors, which are so near together. compared with their distances from other conductors and from earth, that the capacitances due to the latter can be neglected. Thus, in the case of a capacitor having two plates, A and B, near together, it is only the capacitance C_{AB} which is considered, and this is spoken of as the capacitance of the capacitor. In the cases considered in the following pages, earth capacitances, and inter-capacitances with conductors other than those forming the arrangement under consideration, will be neglected unless otherwise stated. The earth capacitance, and inter-capacitance with other conductors, may, however, be of considerable importance if the capacitor is of small capacitance and large dimensions. In the case of capacitors of capacitance $\frac{1}{10}$ microfarad and over, earth capacitances are usually negligible.

Capacitance of Various Systems of Conductors. 1. CAPACITANCE OF AN ISOLATED SPHERICAL CONDUCTOR. Suppose the spherical conductor to be perfectly insulated and at an infinite distance from all other conductors. Let its radius be R metres and let the medium surrounding it have permittivity ε, where $\varepsilon = \varepsilon_0 \varepsilon_r$. ε_0 is the permittivity of a vacuum, and ε_r the relative permittivity of the surrounding medium. For air, $\varepsilon_r = 1$.

If a charge of Q coulombs be given to the sphere, the electric field strength at any point outside it is the same as it would be if the charge were concentrated at the centre of the sphere. Thus, the field strength at any point P, distant x metres from the centre of the sphere, is, from Equation (1.2),

$$E = \frac{Q}{4\pi\varepsilon x^2}$$

and the potential of the sphere is given by

$$V = \int_R^\infty \frac{Q}{4\pi\varepsilon x^2}\, dx = \frac{Q}{4\pi\varepsilon R}$$

$\therefore$ The capacitance of the isolated sphere $= \dfrac{Q}{Q/4\pi\varepsilon R} = 4\pi\varepsilon R$ (4.2)

If the sphere is in air, its capacitance is

$$C = \frac{R}{9 \times 10^9} \text{ farads}$$

2. Capacitance of a Spherical Conductor Inside a Concentric Hollow Conducting Sphere. Let the radii of the inner and outer spheres be R_1 and R_2 metres respectively, the latter being the radius of the inner spherical surface of the outer sphere. Let ε be the permittivity of the medium between them.

If a charge of $+Q$ coulombs be given to the inner sphere a charge of $-Q$ coulombs will be induced on the inner surface of the outer sphere. Since, as shown in Chapter I, the field strength at any point inside a hollow charged conductor is zero, the field strength at any point between the two spheres will be that due to the inner sphere only. Taking any point P, distant x metres from the centre of the inner sphere, and, as before, considering the charge on this sphere to be concentrated at its centre, we have, for the field strength at P,

$$E_P = \frac{Q}{4\pi\varepsilon x^2}$$

The potential differenc- ,etween the spheres is given by

$$V = \int_{R_1}^{R_2} \frac{Q}{4\pi\varepsilon x^2}\, dx = \left[-\frac{Q}{4\pi\varepsilon x}\right]_{R_1}^{R_2}$$

$$\therefore V = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

Hence, the capacitance of the arrangement is

$$C = \frac{Q}{V} = \frac{Q}{\frac{Q}{4\pi\varepsilon}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)} = \frac{4\pi\varepsilon R_1 R_2}{R_2 - R_1} \quad . \quad . \quad (4.3)$$

3. Capacitance Between Two Spheres at a Relatively Great Distance Apart. In this case each sphere will have its own self-capacitance, and also a mutual capacitance with the other sphere. Suppose that the two spheres have equal and opposite charges, and are at a relatively great distance apart, and infinitely distant from all other bodies.*

Under these conditions, if the charges upon spheres A and B are $+Q$ and $-Q$ coulombs, and their potentials V_1 and V_2, then the capacitance between the spheres is

$$C = \frac{Q}{V_1 - V_2}$$

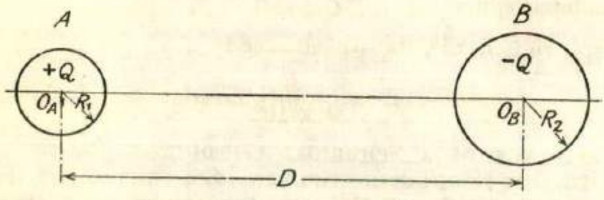


Fig. 4.2. Two Charged Spheres

Let the spheres have radii R_1 and R_2 metres respectively, and let their distance apart be D metres in air (see Fig. 4.2). Then the potential at the centre O_A of sphere A due to its own charge is $\frac{Q}{4\pi\varepsilon_0 R_1}$. If the second sphere is distant from sphere A, the potential at O_A due to the charge on B is $-\frac{Q}{4\pi\varepsilon_0 D}$.

$$\therefore V_1 = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{R_1} - \frac{Q}{D}\right]$$

By similar reasoning,

$$V_2 = \frac{1}{4\pi\varepsilon_0}\left[-\frac{Q}{R_2} + \frac{Q}{D}\right]$$

* The capacitance of a system of two charged spheres in the general case has been fully investigated by Russell (Ref. (12)).

Thus the capacitance between the spheres

$$C = \frac{Q}{V_1 - V_2} = \frac{4\pi\varepsilon_0 Q}{\frac{Q}{R_1} - \frac{Q}{D} - \left(\frac{-Q}{R_2} + \frac{Q}{D}\right)} = \frac{4\pi\varepsilon_0}{\frac{1}{R_1} - \frac{2}{D} + \frac{1}{R_2}}$$

or

$$C = \frac{4\pi\varepsilon_0 R_1 R_2 D}{D(R_1 + R_2) - 2R_1R_2}$$

If the medium is not air but has a permittivity ε, then

$$C = \frac{4\pi\varepsilon R_1 R_2 D}{D(R_1 + R_2) - 2R_1R_2} \quad . \quad . \quad . \quad . \quad . \quad (4.4)$$

If the spheres are equal,

$$C = \frac{4\pi\varepsilon RD}{2\ (D - R)}$$

where R is the common radius.

Russell (*loc. cit.*) gives the *capacitance of the two spheres in parallel*, i.e. when connected by a thin wire so that they are at the same potential, as

$$C_P = 4\pi\varepsilon_0 \left(R_1 + R_2 - \frac{2R_1R_2}{D}\right) \frac{D^2}{D^2 - R_1R_2} \quad . \quad . \quad . \quad (4.5)$$

(in air), using the symbols as above. If the spheres have equal radii R, then

$$C_P = \frac{8\pi\varepsilon_0 RD}{D + R} \text{ or } C_P = \frac{8\pi\varepsilon RD}{D + R}$$

when in a medium of permittivity ε.

For *two equal spheres close together*, the capacitance between the spheres is given approximately by

$$C = 4\pi\varepsilon_0 \ \frac{R}{2}\left(1 + \frac{x}{6R}\right)\left(1{\cdot}2704 + \frac{1}{2}\log_e \frac{R}{x} + \frac{x}{18R}\right) \quad . \quad (4.6)$$

farads in air, where R is the common radius and x the nearest distance between them ($= D - 2R$).

4. Capacitance Between Two Conducting Plates. Consider two equal conducting plates, placed parallel to one another, and at a distance D metres apart, this being small compared with the dimensions of the plates, so that the fringing effect at the edges of the plates can be neglected. Let the area of each plate (one side only) be A sq. m, and let the charges on the plates be $+ Q$ and $- Q$ coulombs.

From Chapter I the field strength at a point between the plates is $\frac{\sigma}{\varepsilon}$, where σ is the density of the charge and equals $\frac{Q}{A}$. Then the potential difference between the plates is

$$V = \int_0^D \frac{Q}{\varepsilon A}\, dx = \frac{QD}{\varepsilon A}$$

Thus, $$C = \frac{Q}{V} = \frac{Q}{QD/\varepsilon A} = \frac{\varepsilon A}{D} \quad . \quad . \quad . \quad (4.7)$$

Suppose that, instead of there being only one dielectric between the plates, there are several parallel layers of dielectrics of thicknesses D_1, D_2, D_3, etc., and having permittivities ε_1, ε_2, ε_3, etc., respectively, as in Fig. 4.3.

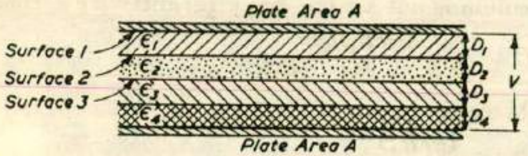


FIG. 4.3. DIELECTRICS IN SERIES IN A PLATE CAPACITOR

Then potential difference between surfaces 1 and 2 is

$$V_{12} = \int_0^{D_1} \frac{Q}{\varepsilon_1 A}\, dx = \frac{Q}{\varepsilon_1 A} D_1$$

while that between surfaces 2 and 3 is

$$V_{23} = \frac{Q}{\varepsilon_2 A} D_2$$

and so on. Thus, the total potential difference V between the parallel conducting plates is

$$V = V_{12} + V_{23} + V_{34} + \ldots$$
$$= \frac{Q}{A}\left(\frac{D_1}{\varepsilon_1} + \frac{D_2}{\varepsilon_2} + \frac{D_3}{\varepsilon_3} + \ldots\right)$$

and the capacitance between the plates is therefore

$$C = \frac{Q}{V} = \frac{A}{\left(\frac{D_1}{\varepsilon_1} + \frac{D_2}{\varepsilon_2} + \frac{D_3}{\varepsilon_3} + \ldots\right)} \quad . \quad . \quad (4.8)$$

Effect of Additional Plates. If two more similar plates are added, one of which is connected to each of the existing plates (Fig. 4.4), and the same dielectric placed between them, then the effective area for the whole capacitor thus formed is $3A$, and the capacitance is thus increased to $\frac{3\varepsilon A}{D}$.

In general, since the use of N plates creates $N-1$ spaces (each of width D) the capacitance of such a capacitor with N plates is

$$C = \frac{\varepsilon \, . \, (N-1) \, A}{D} \quad . \quad . \quad . \quad (4.9)$$

By this means the capacitance of a plate capacitor can be made large whilst using plates with only a comparatively small surface area.

Although these formulae must be considered as approximations, if the plates are close together they are sufficiently accurate for most

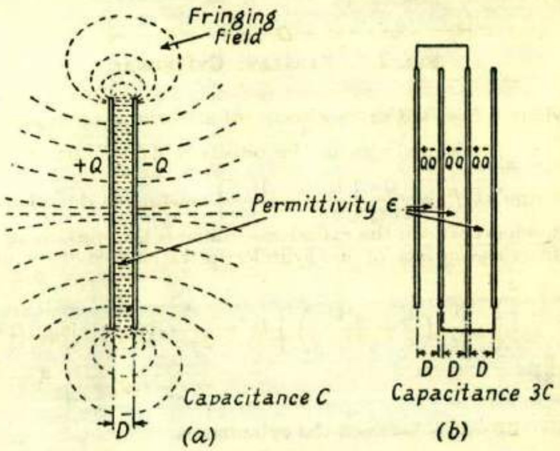


FIG. 4.4.. PLATE CAPACITOR

practical purposes, even though the capacitor may be in the vicinity of other conductors.

5. CAPACITANCE BETWEEN TWO LONG, PARALLEL CONDUCTING CYLINDERS. This problem can be resolved into two separate cases, namely: (*a*) when the cylinders are at a distance apart which is great compared with their diameters; (*b*) when they are comparatively close together.

In the former case it is considerably easier to calculate the capacitance between them than in the latter. This case will be considered first.

Case (*a*).

Fig. 4.5 represents two long parallel conducting cylinders, perpendicular to the plane of the paper, each of diameter d metres placed at a distance D metres apart in air, D being great compared with d and the cylinders being at a great distance from all other conductors.

Let $+Q$ and $-Q$ coulombs be the charges per metre axial length on A and

B respectively. In this case it may be assumed that the charges are concentrated at the axes of the cylinders.

From Equation (1.4), the field strength at P, distant x from cylinder A, due to this cylinder is $\frac{Q}{2\pi\varepsilon_0 x}$, which is the force (in newtons, if Q is in coulombs and x in metres) upon unit charge placed at P. This force is in the direction AB.

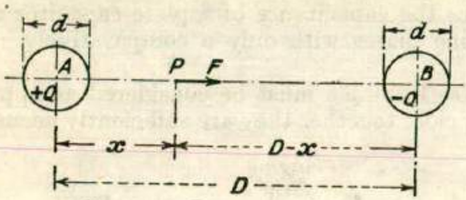


Fig. 4.5. Parallel Cylinders

Similarly, cylinder B would exert a force (of attraction) upon unit charge at P of $\frac{Q}{2\pi\varepsilon_0(D-x)}$ newtons, also in the direction AB. Thus the total force upon unit charge at P is $\frac{Q}{2\pi\varepsilon_0}\left[\frac{1}{x}+\frac{1}{D-x}\right]$ newtons in direction AB. The potential difference between the cylinders—which is the work done in moving unit charge from the surface of one cylinder to the surface of the other—is

$$\int_{x=\frac{d}{2}}^{x=D-\frac{d}{2}}\left[\frac{Q}{2\pi\varepsilon_0}\left(\frac{1}{x}+\frac{1}{D-x}\right)\right]dx = \frac{Q}{2\pi\varepsilon_0}\left[\log_e x - \log_e (D-x)\right]$$

$$= \frac{Q}{\pi\varepsilon_0}\log_e \frac{2D-d}{d}$$

i.e. potential difference between the cylinders is

$$V = \frac{Q}{\pi\varepsilon_0}\log_e \frac{2D-d}{d} \quad . \quad . \quad . \quad . \quad (4.10)$$

$\therefore$ The capacitance between the cylinders *per metre axial length*

$$= \frac{Q}{V} = \frac{\pi\varepsilon_0}{\log_e \frac{2D-d}{d}}$$

If the relative permittivity of the medium between the cylinders is ε_r, then, of course,

$$C = \frac{1{\cdot}21\,\varepsilon_r}{10^{11}\log_{10}\left(\frac{2D}{d}-1\right)} \text{ farads per metre length}$$

or, the capacitance per mile of two such parallel cylinders in air is

$$\frac{1{\cdot}95}{10^{8}\log_{10}\left(\frac{2D}{d}-1\right)} \text{ farads} \quad . \quad . \quad . \quad . \quad (4.11)$$

If D is great compared with d,

$$C \doteqdot \frac{1{\cdot}95}{10^8 \log_{10} \frac{2D}{d}} \text{ farads per mile}$$

Case (*b*). When the cylinders are comparatively close together the treatment of the problem differs from that of Case (*a*), owing to the fact that the charges of $+Q$ and $-Q$ cannot now be assumed to be concentrated at the axes of the cylinders. The charges must now be taken as concentrated along other axes, parallel to and in the same plane as the axes of the cylinders, but displaced so that the distance apart of the axes along which the charges are assumed to be concentrated is now less than the distance D. To derive an expression for the capacitance in this case the distribution of the electrostatic field between the cylinders must first be considered.

When the cylinders are at a great distance apart, as in Case (*a*), the lines of force of the electrostatic field radiate from the cylinders uniformly in all directions, each line cutting the surfaces of the cylinders perpendicularly. Since the potential of a point along any one line of force decreases as the distance of the point from cylinder A is increased, a number of equipotential surfaces exist which are in the form of cylinders concentric with the cylindrical conductors, the lines of force cutting all of these cylinders perpendicularly.

If the cylindrical conductors are comparatively close together these equipotential surfaces are still cylinders, but they are not concentric with the surfaces of the cylindrical conductors whose capacitance is to be determined, nor are they concentric with one another.

It can be shown* that the equations of the traces of these cylindrical equipotential surfaces in the plane of the paper are $r_1 = Mr$, where r_1 and r are the distances of any point on one of the circular traces from the traces X and Y of the axes along which the charges $+Q$ and $-Q$ may be assumed to be concentrated and from which the lines of electrostatic force radiate (these lines of force being circles, as in Fig. 4.6), and M is a constant which differs for different traces. By giving M different values a series of circular traces is obtained, as shown in the figure. When $M = 1$ the trace is a straight line, this being the trace of a plane the potential of all points on which is zero.

Now, since the surfaces of the cylindrical conductors are equipotential surfaces, the equations of whose traces in the plane of the paper are given by the above relationship ($r_1 = Mr$), it follows that the traces X and Y are not coincident with the axes of the conducting cylinders, but are displaced as shown in Fig. 4.6.

* See T. F. Wall's *Electrical Engineering*, p. 46.

Calculation of Capacitance. To calculate the positions of the axes whose traces are X and Y, proceed as below.

Let the points X and Y be displaced inwards from the centres of the two circles which are the traces of the cylindrical conductors A and B by a distance m in each case, and let their distance apart be l. Then $l = D - 2m$.

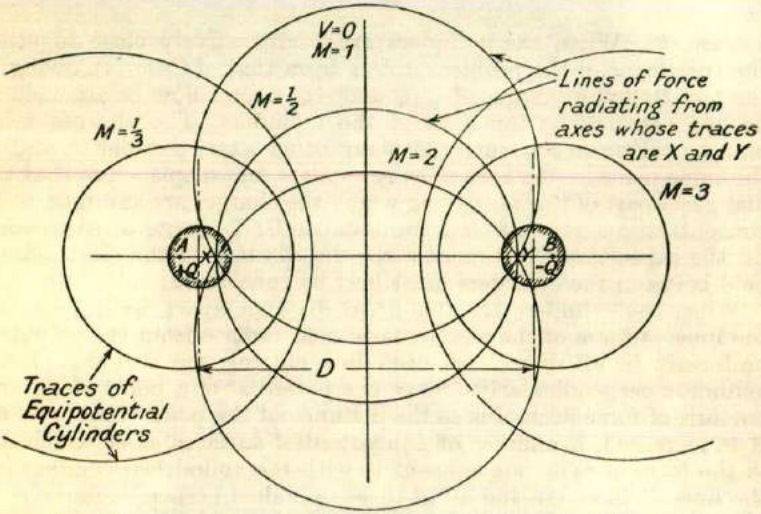


FIG. 4.6. ELECTROSTATIC FIELD BETWEEN CHARGED PARALLEL CYLINDERS WHICH ARE NEAR TOGETHER

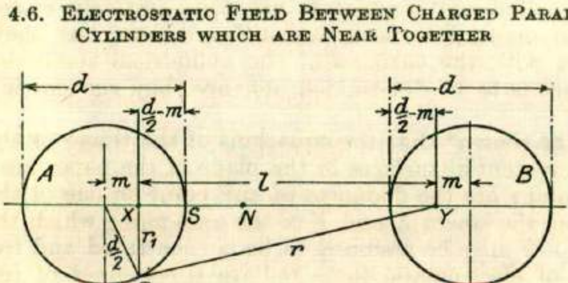


FIG. 4.7.

Since the surfaces of the cylindrical conductors are equipotential surfaces, the equation $r_1 = Mr$ holds for their traces. Consider the point P (Fig. 4.7) on the trace of cylinder A on a line through X perpendicular to the line XY.

Then $\left(\frac{d}{2}\right)^2 = m^2 + XP^2$, and $XP = M(PY)$, since for point P, $r_1 = XP$ and $r = PY$.

Also $\quad l^2 + XP^2 = PY^2$

and $$m = \frac{D - l}{2}$$

For the point S,

$$r_1 = XS = \frac{d}{2} - m \text{ and } r = SY = l - \left(\frac{d}{2} - m\right)$$

Since for all points on the circular trace of A

$$r_1 = Mr$$

we have $$\frac{d}{2} - m = M\left[l - \left(\frac{d}{2} - m\right)\right] \text{ for point } S$$

$$\therefore M = \frac{\frac{d}{2} - m}{l - \left(\frac{d}{2} - m\right)}$$

Now $$l^2 + XP^2 = PY^2 = \left(\frac{XP}{M}\right)^2 = \frac{XP^2}{\left[\dfrac{\frac{d}{2} - m}{l - \left(\frac{d}{2} - m\right)}\right]^2}$$

and $$\left(\frac{d}{2}\right)^2 = m^2 + XP^2$$

$$\therefore l^2 + \left(\frac{d}{2}\right)^2 - m^2 = \frac{\left(\frac{d}{2}\right)^2 - m^2}{\left[\dfrac{\frac{d}{2} - m}{l - \frac{d}{2} + m}\right]^2}$$

Substituting $m = \dfrac{D - l}{2}$ and solving for l we have the solution,

$$l = \sqrt{D^2 - d^2}$$

If d is small compared with D, we have $l = D$, as in Case (a).

Thus, to calculate the capacitance between the cylinders, the treatment is exactly the same as that of Case (a), except that the charges $+Q$ and $-Q$ per metre axial length are considered concentrated along parallel axes whose distance apart is now l instead of D.

We have then, for the field strength at a point such as N (Fig. 4.7) distant x from X,

$$E = \frac{Q}{2\pi\varepsilon_0 x} + \frac{Q}{2\pi\varepsilon_0 (l - x)}$$

and for the potential difference between the cylinders,

$$V = \int_{x = \frac{d}{2} - m}^{x = l - \left(\frac{d}{2} - m\right)} \frac{Q}{2\pi\varepsilon_0}\left(\frac{1}{x} + \frac{1}{l - x}\right) dx$$

$$= \frac{Q}{\pi\varepsilon_0} \log_e \frac{l - \left(\frac{d}{2} - m\right)}{\frac{d}{2} - m}$$

or, since $l = \sqrt{D^2 - d^2}$ and $2m = D - l$

$$V = \frac{Q}{\pi\varepsilon_0}\log_e\left[\frac{\sqrt{D^2-d^2} - (d-D)}{\sqrt{D^2-d^2} + (d-D)}\right] \quad . \quad . \quad . \quad (4.12)$$

Thus capacitance per metre axial length is

$$C = \frac{Q}{V} = \frac{\pi\varepsilon_0}{\log_e\left[\frac{\sqrt{D^2-d^2} - (d-D)}{\sqrt{D^2-d^2} + (d-D)}\right]}$$

Rationalizing and simplifying,* we have

$$C = \frac{\pi\varepsilon_0}{\log_e\left[\frac{\sqrt{D^2-d^2}}{d}\right]}$$

If the permittivity of the medium between the cylinders is ε, we have

$$C = \frac{\pi\varepsilon}{\log_e \frac{(D + \sqrt{D^2-d^2}}{d}} \quad . \quad . \quad . \quad . \quad (4.13)$$

or $$C = \frac{1\cdot 95}{10^8 \log_{10}\left(\frac{D + \sqrt{D^2-d^2}}{d}\right)} \text{ farads per mile of double conductor in air}$$

These capacitances are given in farads *per mile*, since the arrangement of two long parallel conducting cylinders is chiefly met with in overhead transmission lines where the most useful unit of length is the mile. Formulae for the general case of two parallel cylindrical conductors have been given by Russell (Ref. (13)).

6. Capacitance between Two Coaxial Cylinders. An important case of this arrangement in practice is, of course, a concentric cable.

Consider two long conducting concentric cylinders, the diameter of the inner one being d metres and the inner diameter of the outer one being D metres. Let $+Q$ and $-Q$ coulombs be their charges per metre axial length. The lines of force of the electrostatic field will be radial, and the equipotential surfaces will be cylindrical and coaxial with the two conducting cylinders. The field strength at some point at a radial distance of x metres from the common axis of the cylinders will be $\frac{Q}{2\pi\varepsilon_0 x}$ if the dielectric separating the cylinders is air.

Thus the potential difference between the cylinders is

$$\int_{\frac{d}{2}}^{\frac{D}{2}} \frac{Q}{2\pi\varepsilon_0 x}\,dx = \frac{Q}{2\pi\varepsilon_0}\left[\log_e \frac{D}{2} - \log_e \frac{d}{2}\right]$$

$$V = \frac{Q}{2\pi\varepsilon_0}\log_e \frac{D}{d}$$

The capacitance per metre length is

$$C = \frac{Q}{V} = \frac{2\pi\varepsilon_0}{\log_e \frac{D}{d}}$$

The general expression for a length l metres, the dielectric having a permittivity ε, is

$$C = \frac{2\pi\varepsilon l}{\log_e \frac{D}{d}} \quad . \quad . \quad . \quad . \quad (4.14)$$

or

$$C = \frac{3{\cdot}89\varepsilon_r}{10^8 \log_{10} \frac{D}{d}} \text{ farads per mile}$$

7. Capacitance of a Single Straight Conductor Parallel to Earth. *Method of Electric Images.* This method is based upon the concept of an "image" of a conductor placed above the earth's surface, this image being of the same size and shape as the conductor considered and lying as far beneath the surface of the earth as the conductor considered is above the surface. The earth's surface is thus in the plane of zero potential for these two conductors—considering the image as being in actual fact a conductor placed at a distance $2H$ from the original one, H being the height of this original conductor above the earth.

Since the earth's surface is at zero potential, the electrostatic field from the charged conductor above the earth, to the surface of the earth, has the same distribution as the field which would exist between the conductor and the zero potential plane, in the case of two conductors placed at a distance of $2H$ apart.

Fig. 4.8 shows the trace of a cylindrical conductor A lying parallel to the earth's surface, and at a height H metres above the earth; A' is its image. If conductor A has a charge of $+Q$ coulombs per metre axial length, then the potential difference between it and conductor A', which is supposed to have $-Q$ coulombs per metre axial length, is, from Equation (4.10),

$$2V = \frac{Q}{\pi\varepsilon_0}\log_e \frac{4H - d}{d}$$

where d is the diameter of the conductors and is assumed small compared with H; V is the potential of A above that of the earth, and is also the potential of A' below earth potential.

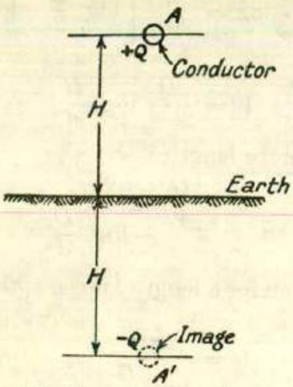


FIG. 4.8. CYLINDRICAL CONDUCTOR PARALLEL TO EARTH

Thus $$V = \frac{Q}{2\pi\varepsilon_0} \log_e \frac{4H - d}{d}$$

and the capacitance per metre length of one conductor to earth is

$$C = \frac{2\pi\varepsilon}{\log_e \frac{4H - d}{d}} \quad . \quad . \quad . \quad . \quad . \quad (4.15)$$

where the dielectric has permittivity ε.

The capacitance per mile of one conductor to earth in air is therefore

$$C = \frac{3{\cdot}89}{10^8 \log_{10} \frac{4H - d}{d}} \text{ farads per mile}$$

If d is small compared with H (as is usually the case when an overhead line is considered) then

$$C = \frac{3{\cdot}89}{10^8 \log_{10} \frac{4H}{d}} = \frac{3{\cdot}89}{10^8 \log_{10} \frac{2H}{r}} \text{ farads per mile}$$

where r is the radius of the conductor in metres.

If d is not small compared with H, the calculation of capacitance must be based upon Equation (4.12) instead of Equation (4.10) as above.

8. CAPACITANCE BETWEEN TWO LONG, STRAIGHT CONDUCTORS, PARALLEL TO THE EARTH AND TO ONE ANOTHER. Consider two long

cylindrical conductors M and N parallel to earth and to one another, their diameters being d, their distance apart D, and their height above earth H, metres; and let M' and N' be their images (Fig. 4.9). Suppose d small compared with H.

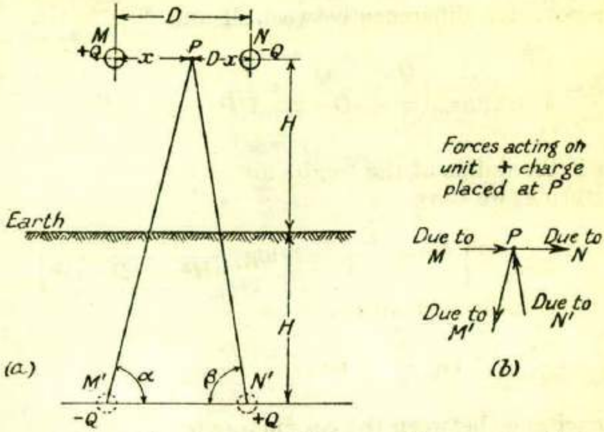


FIG. 4.9. TWO CHARGED PARALLEL CONDUCTORS NEAR TO EARTH

Let M and N have charges of $+Q$ and $-Q$ per metre axial length respectively and M' and N' charges of $-Q$ and $+Q$ units per metre length respectively.

Consider a point P on the horizontal line joining the centres of M and N and distant x metres from M. The field strength at P is due to all four conductors M, N, M', and N'. Thus field strength at P *in the direction MN* is—

$$\text{Due to } M. \ \frac{Q}{2\pi\varepsilon_0 x}$$

$$\text{Due to } N. \ \frac{Q}{2\pi\varepsilon_0(D-x)}$$

$$\text{Due to } M'. \ \frac{-Q}{2\pi\varepsilon_0 PM'}\cos\alpha = \frac{-Q}{2\pi\varepsilon_0\sqrt{4H^2+x^2}}\cdot\frac{x}{\sqrt{4H^2+x^2}}$$

$$= \frac{-Qx}{2\pi\varepsilon_0(4H^2+x^2)}$$

$$\text{Due to } N'. \ \frac{-Q}{2\pi\varepsilon\sqrt{(D-x)^2+4H^2}}\cos\beta$$

$$= \frac{-Q(D-x)}{2\pi\varepsilon_0[(D-x)^2+4H^2]}$$

Resultant field strength at P is

$$\frac{1}{2\pi\varepsilon_0}\left[\frac{Q}{x}+\frac{Q}{D-x}-\frac{Qx}{4H^2+x^2}-\frac{Q(D-x)}{(D-x)^2+4H^2}\right]$$

and the potential difference between M and N is

$$V=\int_r^{D-r}\frac{1}{2\pi\varepsilon_0}\left(\frac{Q}{x}+\frac{Q}{D-x}-\frac{Qx}{4H^2+x^2}-\frac{Q(D-x)}{(D-x)^2+4H^2}\right)dx$$

where r is the radius of the conductors.

Integrating, we have

$$V=\frac{1}{2\pi\varepsilon_0}Q\left[2\log_e\frac{D-r}{r}+\log_e\frac{4H^2+r^2}{4H^2+(D-r)^2}\right] \qquad (4.16)$$

If D is great compared with r,

$$V=\frac{Q}{2\pi\varepsilon_0}\left[2\log_e\frac{D}{r}+\log_e\frac{4H^2}{4H^2+D^2}\right]$$

The capacitance between the conductors is

$$C=\frac{Q}{V}=\frac{2\pi\varepsilon_0}{2\log_e\frac{D}{r}+\log_e\frac{4H^2}{4H^2+D^2}} \quad . \quad . \quad . \quad (4.17)$$

$$=\frac{\pi\varepsilon_0}{\log_e\frac{D}{r}\left(\frac{2H}{\sqrt{4H^2+D^2}}\right)} \text{ per metre length in air}$$

or $$C=\frac{1{\cdot}95}{10^8\log_{10}\frac{D}{r}\left(\frac{2H}{\sqrt{4H^2+D^2}}\right)} \text{ farads per mile}$$

The capacitance of two parallel cylinders which are at a great distance from earth was previously found to be

$$\frac{1{\cdot}95}{10^8\log_{10}\frac{2D}{d}}=\frac{1{\cdot}95}{10^8\log_{10}\frac{D}{r}} \text{ farads per mile}$$

D being great compared with r.

Thus the proximity of the earth introduces the term $\frac{2H}{\sqrt{4H^2+D^2}}$ in the denominator, as shown above.

The capacitance of a system of three or more conductors, parallel and near to the earth, can be found by similar methods (Refs. (1), (5), (8)).

Capacitors in Series and Parallel. (*a*) SERIES. If a number of capacitors are connected in series, as in Fig. 4.10 (*a*), a potential difference of V being applied between the outer terminals, there will be potential differences v_1, v_2, v_3, etc., between the different pairs of plates.

Let the capacitances of the capacitors (neglecting earth capacitances) be C_1, C_2, C_3, etc. If a quantity of electricity Q is given to the system of capacitors by means of a current which flows for a

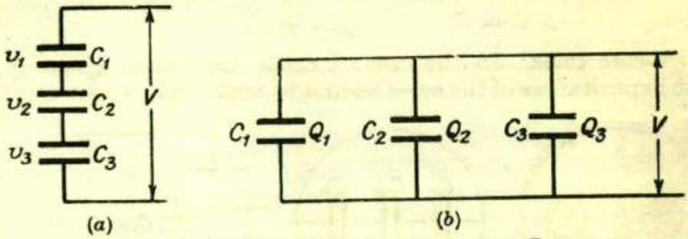


FIG. 4.10. CAPACITORS IN SERIES AND IN PARALLEL

short time through them until they are charged to the total potential difference V, then

$$v_1 = \frac{Q}{C_1} \qquad v_2 = \frac{Q}{C_2} \qquad v_3 = \frac{Q}{C_3}$$

and so on.

If C is the capacitance of the whole system, the potential difference for which is V, then

$$C = \frac{Q}{V} \text{ or } V = \frac{Q}{C}$$

Thus, since, $V = v_1 + v_2 + v_3$,

$$\frac{Q}{C} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3} + \ldots$$

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots \qquad (4.18)$$

(*b*) PARALLEL. If a potential difference V is applied to a number of capacitors connected in parallel (Fig. 4.10 (*b*)), then the potential difference across the plates of such capacitors is, in each case, V, but the quantities of electricity given to the capacitors are now different for the different capacitors. If these quantities are Q_1, Q_2, Q_3, etc., then

$$v_1 = \frac{Q_1}{C_1} \text{ or } Q_1 = v_1 C_1$$

$$v_2 = \frac{Q_2}{C_2} \text{ or } Q_2 = v_2 C_2$$

and so on.

But $$v_1 = v_2 = v_3 = \ldots = V$$

and the total quantity of electricity is

$$Q = Q_1 + Q_2 + Q_3 + \ldots = CV$$

where C is the total capacitance.

Thus
$$\begin{aligned} CV &= C_1v_1 + C_2v_2 + C_3v_3 + \ldots \\ &= V(C_1 + C_2 + C_3 + \ldots) \\ \therefore C &= C_1 + C_2 + C_3 + \ldots \end{aligned} \qquad (4.19)$$

Two-core Cable. In the case of multi-core cables generally, the earth capacitances of the cores cannot be neglected. A two-core cable

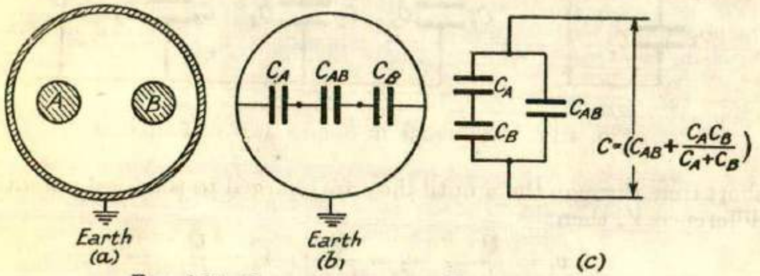


FIG. 4.11. CAPACITANCE OF A TWO-CORE CABLE

consists essentially of two long parallel conductors embedded in some insulating material, the whole being enclosed by an earthed, conducting cylinder, as in Fig. 4.11 (*a*).

This arrangement is equivalent to the system of capacitors shown in Fig. 4.11(*b*). If the cores are represented by A and B, then C_{AB} is the capacitance between cores, and C_A and C_B are the earth capacitances of the two conductors. We thus have C_A and C_B in series with one another, this series circuit being in parallel with C_{AB}, the equivalent arrangement being represented in Fig. 4.11 (*c*). The capacitance of C_A and C_B in series is $\frac{C_A C_B}{C_A + C_B}$, and when this is connected in parallel with C_{AB}, the total, or working, capacitance is $C_{AB} + \frac{C_A C_B}{C_A + C_B}$.

Three-core Cable. The capacitances which exist in the case of a three-core cable are shown in Fig. 4.12 (*a*), in which C_1 is the inter-core capacitance, and C_0 the earth capacitance. Diagram (*b*) shows the equivalent circuit of such a cable when used on a three-phase system of line voltage E.

To facilitate calculations of the charging current per line it is usual to resolve the system shown in diagram (*b*) into either an *equivalent mesh* system, as in diagram (*c*), or an *equivalent star* system as in diagram (*d*). In the first case, the three capacitances C_0 are replaced by three imaginary capacitances C_m, connected in mesh, in parallel with the inter-core capacitances C_1, and having such values

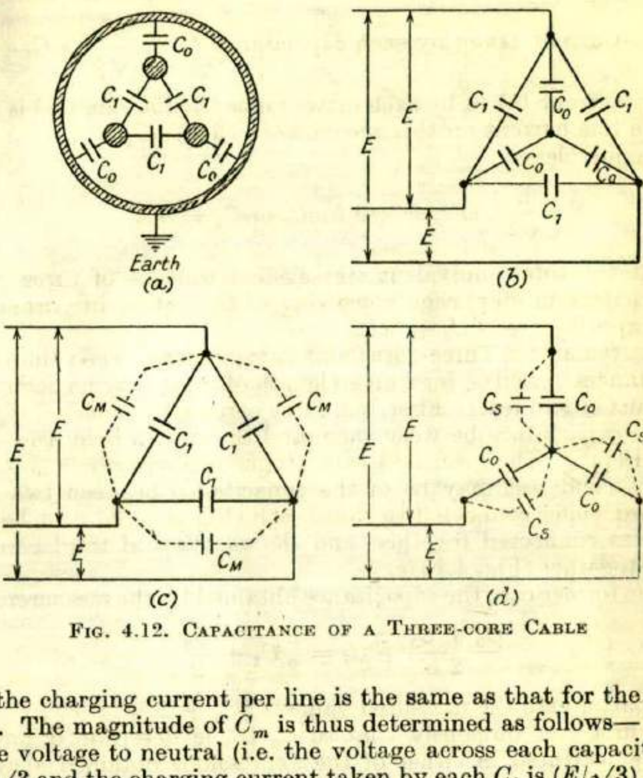


FIG. 4.12. CAPACITANCE OF A THREE-CORE CABLE

that the charging current per line is the same as that for the actual cable. The magnitude of C_m is thus determined as follows—

The voltage to neutral (i.e. the voltage across each capacitor C_0) is $E/\sqrt{3}$ and the charging current taken by each C_0 is $(E/\sqrt{3})\,.\,\omega\,C_0$.

In diagram (*c*) the current taken by each capacitor C_m is $E\,.\,\omega\,C_m$ and the *line* current on this account is thus $\sqrt{3}.\ E\,.\,\omega\,C_m$.

For equivalence this line current must be equal to $(E/\sqrt{3})\,.\,\omega\,C_0$.

Thus,
$$\sqrt{3}\,.\,E\,.\,\omega C_m = \frac{E}{\sqrt{3}}\,.\,\omega C_0$$

or
$$C_m = \frac{C_0}{3}$$

Hence the total equivalent mesh system consists of three groups of C_1 each in parallel with C_m, i.e. three capacitances $C_1 + \dfrac{C_0}{3}$ connected in mesh.

Diagram (*d*) shows the equivalent star system, in which the capacitors C_1 are replaced by three capacitors C_s, each in parallel with C_0 and of such values that the line currents are the same as for the actual cable.

To determine the value of C_s—

$$\text{Current taken by each capacitance } C_s = \frac{E}{\sqrt{3}} \cdot \omega C_s$$

Now, current taken by each capacitance C_1 (diagram (*b*)) is $E\omega C_1$, and the line current on this account $= \sqrt{3} \,.\, E\omega C_1$.

For equivalence

$$\frac{E}{\sqrt{3}} \cdot \omega C_s = \sqrt{3}\, E\omega C_1 \text{ or } C_s = 3C_1$$

So that the total equivalent star system consists of three groups of capacitors in star, each consisting of C_0 and C_s in parallel, i.e. three capacitances of $C_0 + 3C_1$.

Measurements of Three-core Cable Capacitances. The values of the capacitances C_0 and C_1 for a given length of cable may be determined by means of two tests. First, the three cores are connected together and the capacitance between them and the sheath is measured (see Fig. 4.13 (*a*)). The measured capacitance is obviously $3C_0$.

The second test may be of the capacitance between two cores, the third being connected to the sheath (Fig. 4.13 (*b*)); or between two cores connected together and the sheath and third core connected together (Fig. 4.13 (*c*)).

In the former case the capacitance obtained by the measurement is

$$\frac{C_0 + C_1}{2} + C_1 = \frac{3}{2}C_1 + \frac{C_0}{2}$$

In the latter case the measured value is $2C_0 + 2C_1$.

The first test obviously enables C_0 to be determined, and this value, substituted in either of the expressions obtained above for the two alternative methods of carrying out the second test, renders C_1 calculable.

Distributed Capacitance. In the foregoing paragraphs it has been assumed in all cases that the surfaces of the conductors considered are equipotential surfaces.

There are many important cases in practice when this is not so, and in these cases the calculation of capacitance cannot be carried out by the simple methods used above. In wire-wound solenoids we have capacitance between adjacent turns, and layers, and all the conductors in one layer are obviously not at the same potential. The earth capacitances of the turns in the coil also are not all the same. In such coils we have what is referred to as "distributed capacitance."

The effect of such distributed capacitance is, in many cases, small for low-frequency work, and an equivalent circuit, which represents such a coil sufficiently accurately for most purposes, can then be obtained by assuming the coil itself to be free from capacitance but as having a simple capacitor connected in parallel with it, and also having simple capacitors connected between parts of the coil and earth. The latter represent the distributed earth capacitance, while the former represents the distributed inter-turn capacitance.

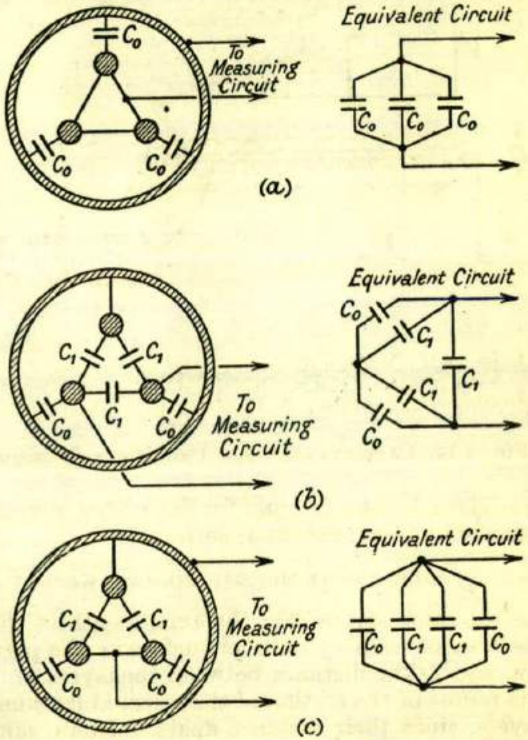


FIG. 4.13. CABLE CAPACITANCE MEASUREMENTS

If such a coil is to be used for very high frequency work, e.g. radio frequency work, such approximate methods of representation are not justifiable, since the distributed capacitance of the coil may, at such frequencies, become of more importance than its inductance.

Capacitance of a Two-layer Solenoid. Fig. 4.14 represents a solenoid of circular section, having two layers of insulated wire wound continuously so that, in effect, the layers are connected together at one end as shown. If a steady potential difference V is applied to the terminals a, a' of the coil, then the potential difference between layers will vary from V at the left-hand end of the coil to zero at the right-hand end, and the electrostatic field between adjacent turns will thus decrease from a maximum to zero, moving from left to right. Morecroft (*Principles of Radio Communication*, Chap. II) calculates the internal capacitance of such a coil by treating it as, essentially, two coaxial conducting cylinders, whose capacitance, if the layers of wire are close together compared with the diameter

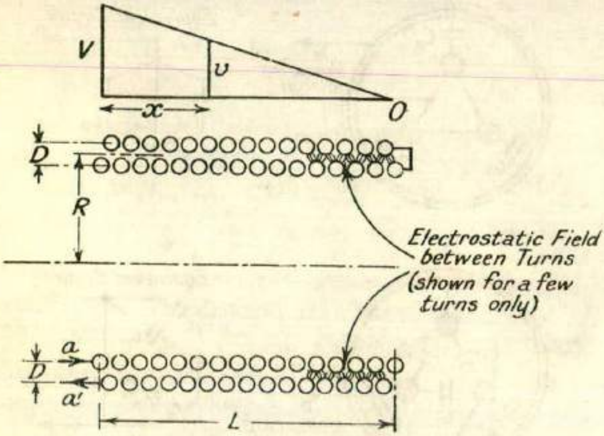


FIG. 4.14. CAPACITANCE OF A TWO-LAYER SOLENOID

of the coil, is given by the formula for flat plates, assuming at first that the cylinders are equipotential surfaces.

Thus $C = \frac{\varepsilon A}{D}$, where C is the capacitance when the potential difference is the same throughout the axial length of the cylinders, A being the area of each cylindrical surface, ε the permittivity of the medium, and D the distance between the layers.

If R is the radius of the section of the solenoid (assumed the same for both layers, since their distance apart is small) and L is their axial length, then $A = 2\pi RL$ and $C = \frac{2\pi\varepsilon RL}{D}$, or capacitance per metre axial length is $\frac{2\pi\varepsilon R}{D}$.

Actually the potential difference between layers varies along the axial length from V to zero. Assuming this variation to be according to a straight-line law, we have

Energy stored in axial length dx,

$$dW = \frac{c \cdot v^2}{2} = \frac{2\pi\varepsilon R}{D} \cdot \frac{v^2}{2} \cdot dx$$

where v is the potential difference between layers at any point of axial distance x from the left-hand end (Fig 4.14). Since $\frac{V}{L} = \frac{v}{L-x}$ we have $v = \left(1 - \frac{x}{L}\right) V$ and

$$dW = \frac{2\pi\varepsilon R}{D}\frac{V^2}{2}\left(1 - \frac{x}{L}\right)^2 dx$$

∴ Total energy stored is

$$W = \int_0^L \frac{\pi\varepsilon R V^2}{D}\left(1 - \frac{x}{L}\right)^2 dx$$

$$W = \frac{\pi\varepsilon R V^2}{D}\left[-\frac{L}{3}\left(1 - \frac{x}{L}\right)^3\right]_0^L = \frac{\pi\varepsilon R V^2 L}{3D}$$

Thus, if C' is the distributed capacitance to be calculated,

$$W = C'\frac{V^2}{2}$$

$$\therefore C'\frac{V^2}{2} = \frac{\pi\varepsilon R V^2 L}{3D}$$

or

$$C' = \frac{2\pi\varepsilon R L}{3D} \qquad . \quad . \quad . \quad . \quad . \quad (4.20)$$

Morecroft (*loc. cit.*) gives the distributed capacitance for a solenoid of N layers as

$$C' = C_0 \times \frac{4}{3}\left(\frac{N-1}{N}\right)^2 \qquad . \quad . \quad . \quad . \quad (4.21)$$

where C_0 is the capacitance between the outermost and innermost layers.

Breit (*Physical Review*, XVIII, p. 133 (1921)) gives the capacitance for a short single-layer solenoid in air as approximately $\frac{0{\cdot}07l}{9 \times 10^9}$ farads, where l is the length (in metres) of one turn of wire on the solenoid.

Shielding and Guard Rings. In making measurements involving the use of capacitors it is often desirable—and in some cases absolutely necessary—to shield pieces of apparatus from the effect of electrostatic fields which are external to the apparatus itself. This is done by surrounding the apparatus by an earthed metal screen which may be of thin aluminium or copper sheet, or in the form of a wire mesh. Charges which may be induced in this screen pass to earth and have no effect upon the apparatus inside.

Guard rings are used in order to overcome the difficulty of calculating accurately the capacitance of a capacitor which has a fringing electrostatic field at its edges. The distribution of such fringing fields is somewhat uncertain and this renders exact calculations of capacitance difficult.

In calculating the capacitance of a parallel-plate capacitor in a previous paragraph it was assumed that the effect of the field at the edge of the plate could be neglected. The simple formula obtained is rendered much more accurate by the use of a guard ring as shown in Fig. 4.15. The guard ring consists of a metal plate of the same thickness as the plate A which it surrounds, and from which it is

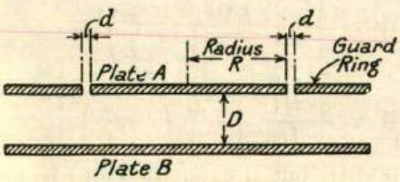


Fig. 4.15. Guard Ring

separated by a narrow and uniform air gap. This ring is usually of the same outside dimensions as the opposing plate B of the capacitor, and is, in use, at the same potential as the plate A which it surrounds. Under these conditions the electrostatic field between the plates is perpendicular to the plates even up to the extreme edge of plate A, the fringing field being now transferred to the edges of the guard ring. The effective area of the plates to be used in the capacitance formula is now taken, of course, as the area of plate A.

A formula which corrects for the width of the air gap between plate A and the guard ring (which gap should be of zero length if no correction is to be used) has been given by Maxwell and is

$$C \doteqdot \left[\frac{R^2}{4D} + \frac{1}{4}\cdot\frac{Rd}{D + 0{\cdot}22d}\left(1 + \frac{d}{2R}\right)\right] \times \frac{1}{9 \times 10^9} \text{ farads} \qquad (4.22)$$

where the plate A (assumed circular) has a radius R, D is the distance between the plates, and d the width of the air gap, all in metres, the dielectric being air.

When no guard ring is used, the edge effect can be taken into account in the calculation of capacitance by a formula due to Kirchhoff. This formula is

$$C = \left[\frac{R^2}{4D} + \frac{R}{4\pi D}\left[D\left\{\log_e \frac{16\pi R(D + t)}{D^2} - 1\right\} + t \log_e\left(1 + \frac{D}{t}\right)\right]\right] \times \frac{1}{9 \times 10^9} \text{ farads} \quad . \quad . \qquad (4.23)$$

where R is the radius of the circular plates of the capacitor, t being the thickness of the plates and D the distance between them, the dielectric being air.

In cylindrical capacitors the guard ring takes the form of two cylinders, of the same diameter as the cylindrical electrode to which they are adjacent, and placed one at each end of, and coaxial with, this electrode. They are connected together and are, in use, charged to the same potential as the electrode between them. Their use was described in Chapter II in connection with high-voltage air capacitors.

Dielectrics. The broadest definition of a dielectric is, simply, "an insulator." More precisely, a dielectric is some medium in which a constant electrostatic field can be maintained without involving the supply of any appreciable amount of energy from outside sources. The term "dielectric" is applied when an insulating material is used to separate two neighbouring conductors such as the plates of a capacitor. As will be seen later, dielectrics increase the capacitance of a system of conductors as compared with the capacitance of the same system of conductors existing in vacuo. No dielectrics are at present known which, when placed between two conductors, decrease the capacitance between them.

Three very important quantities in connection with any dielectric are—

(*a*) Its "dielectric strength."
(*b*) Its "permittivity" or "dielectric constant."
(*c*) Its "dielectric loss angle" or power factor.

(*a*) Dielectric Strength. This may be defined as the ability of a dielectric to withstand breakdown when a voltage is applied to it. All insulating materials should, of course, have a very high resistivity, so that only an extremely small current flows through them when a voltage is applied. This is, however, an entirely different property from dielectric strength. If a gradually increasing voltage is applied between, say, the opposite faces of a slab of an insulating material, the material becomes electrically strained, the electrostatic field in it increasing in intensity with increasing voltage. Eventually a value of the field strength is reached at which the material "breaks down," i.e. the material is punctured and is rendered useless for insulation purposes. This effect is observed in the case of all insulating materials, although the magnitude of the field strength, or "potential gradient," for which it occurs differs for different materials. In liquid or gaseous dielectrics the breakdown is only temporary.

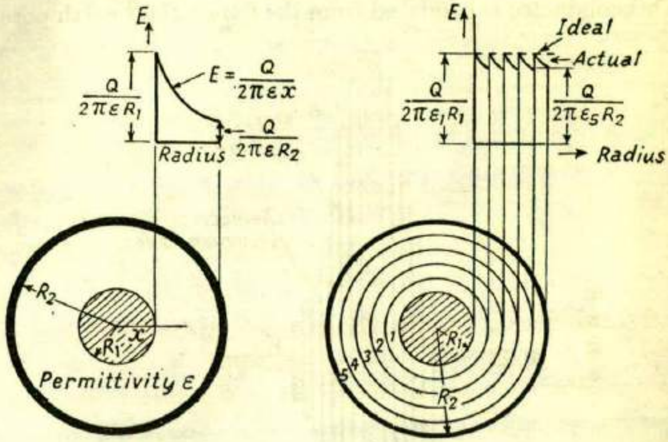
The dielectric strength is expressed in volts per millimetre or per centimetre, or in kilovolts per centimetre, etc.

The true or intrinsic dielectric strength of solid materials can be measured only if all discharges in the ambient medium are eliminated and if the heating effect of the applied field is negligible. Such intrinsic strengths are difficult to measure, but have been obtained for a few good dielectrics and lie in the region of 5×10^6 V/cm. When the dielectric strength is measured in the conventional manner between disc or sphere electrodes the breakdown is due to intense local concentration of stress at the end of ionic discharges outside the material, and values from 5 to 50 times lower than the intrinsic value are obtained. It is these lower values which are quoted in Table VII (p. 165). The dielectric strength so measured depends on the geometry of the electrodes, on the nature of the ambient medium (air or oil) and on the thickness of the specimen, but no exact laws can be quoted. If the time of a test is prolonged to days or weeks in order to represent the useful life of the material, still lower values of breakdown strength are obtained which depend either on the erosion of microscopic holes through the material by ionic bombardment or on electrochemical changes in the structure of the insulation. In low-grade materials, failure may be due to thermal instability, resulting from the heat liberated by dielectric losses.

When the applied voltage is alternating, the frequency of the supply affects the dielectric strength; and also, since the maximum value of the voltage is responsible for the breakdown, the wave-form of the voltage, as well as its r.m.s. value, is important. The shape of the electrodes by means of which the voltage is applied is important, since the distribution of the electrostatic field depends upon this shape, which therefore affects the dielectric strength. The true dielectric strength is the strength at breakdown when the electrostatic field is uniform.

Potential Gradient. In practice the potential gradient is an important matter. Consider the case of a single-core cable with a conducting outer sheath. From page 6 we have for the field strength at a point between two coaxial cylinders, and at a distance x from their common axis, $E = \frac{Q}{2\pi\varepsilon x}$, where Q is the charge on the inner conductor per metre axial length. Since the potential between two points is given by $\int E dx$, E is the potential gradient at any point. If R_1 is the radius of the core, and R_2 the internal radius of the sheath, the potential gradient at the surface of the core is $\frac{Q}{2\pi\varepsilon R_1}$, and at the internal surface of the sheath, $\frac{Q}{2\pi\varepsilon R_2}$; the gradient in between these points varying as shown in Fig. 4.16 (a). Now, if the dielectric between the core and sheath consists of only one material, of permittivity ε, which is capable of withstanding, without breakdown, the maximum stress $\frac{Q}{2\pi\varepsilon R_1}$ at the core surface, then the outer layers of dielectric, approaching the sheath, will not be economically used.

Graded Cables. To effect a more economical utilization of the dielectric between the core and sheath, several different dielectrics, of permittivities ε_1, ε_2, ε_3, etc., are used, these being arranged so that their permittivities are in descending order as the radius increases. Cables insulated in this way are referred to as "graded" cables. Obviously, if the dielectric used could be varied continuously so that ε varied inversely as the radius x, an absolutely uniform potential gradient could be obtained, between core and sheath, as shown in the dotted line in Fig. 4.16 (*b*). Actually the potential gradient varies in the manner shown in the full-line curve.



(*a*) (*b*)

Figs. 4.16. Potential Gradient in Single-core Cable

In the previous work the potential difference between two coaxial cylinders of radii R_1 and R_2 was found to be

$$V = \frac{Q}{2\pi\varepsilon} \log_e \frac{R_2}{R_1}$$

from which

$$Q = \frac{2\pi\varepsilon V}{\log_e \dfrac{R_2}{R_1}} \quad . \quad . \quad . \quad . \quad (4.24)$$

Substituting this value for Q, we have for the potential gradient at any radius x

$$E = \frac{V}{x \log_e \dfrac{R_2}{R_1}} \quad . \quad . \quad . \quad . \quad (4.25)$$

when only one dielectric, of permittivity ε, is used.

Another method of obtaining a uniform potential gradient between two coaxial cylinders is by the interposition of metal intersheaths (consisting of cylindrical sheets of metal foil coaxial with the two conductors) in the dielectric, between the charged con ductors. As an example of the use of such intersheaths, a "condenser bushing" will be considered.

Condenser Bushing. This is a type of bushing which is commonly used for the terminals of high-voltage transformers and switchgear. Fig. 4.17 shows a conductor A which is charged to some high voltage V This conductor is insulated from the flange B (at earth potential,

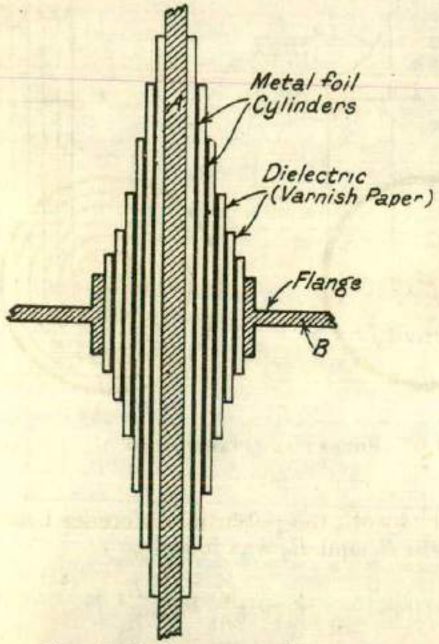


FIG. 4.17. CONDENSER BUSHING

say), by a condenser bushing consisting of some dielectric material with metal-foil cylindrical sheaths of different lengths and radii embedded in it, thus splitting up what is essentially a capacitor, having the high-tension conductor and flange as its plates, into a number of capacitors in series. The capacitances of the capacitors formed by the metal-foil cylinders are given by the equation

$$C = \frac{2\pi\varepsilon l}{\log_e \frac{R_2}{R_1}}$$

l being the axial length of the capacitor and R_1 and R_2 the radii of its cylindrical plates (assumed to be of negligible thickness in the case of the metal foil). If these capacitors all have the same capacitance, since Q is the same for all (being the charge per metre axial length of the high-voltage conductor), the potential differences between their plates will be equal. They can be made to have the same capacitance by suitably choosing the axial lengths of successive sheets of foil together with the ratios of their radii $\frac{R_2}{R_1}$. If the radial spaces between successive sheets of foil are made equal and the lengths adjusted to make the capacitances equal, the potential gradient in the dielectric is uniform, but the edges of foil sheets lie on a curve, thus giving unequal surfaces of dielectric between the edges of successive sheets. This is undesirable from the point of view of

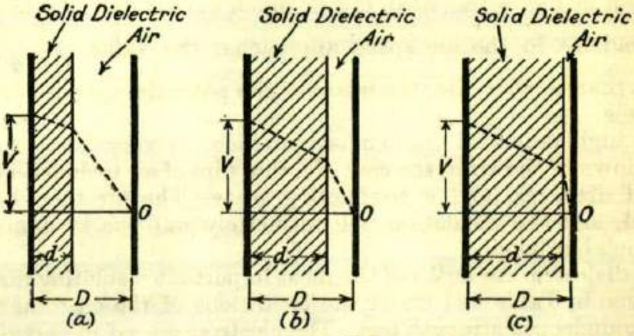


FIG. 4.18. EFFECT OF DIELECTRIC THICKNESS UPON POTENTIAL GRADIENT IN A PLATE CAPACITOR

flashover by "creeping" along the surface. If the differences between the lengths of successive sheets are made equal, the radial potential gradient is not uniform. A compromise between the two conditions is usually adopted.

Effect of Varying Thicknesses of Solid Dielectric upon the Potential Gradient Between Parallel Plates. Fig. 4.18 shows the effect upon the potential gradient of varying the thickness of a slab of solid dielectric which is situated between the plates of a parallel-plate capacitor, one plate being charged to a potential V volts and the other being at earth potential. The remaining space is air.

If σ is the surface density of charge on the plates and ε the permittivity of the solid dielectric, we have—

$$\text{Potential gradient in solid dielectric} = \frac{\sigma}{\varepsilon} = E_D$$

$$\text{Potential gradient in air space} = \frac{\sigma}{\varepsilon_0} = E_A$$

Thus $$\varepsilon E_D = \varepsilon_0 E_A$$

Also, if d is the thickness of solid dielectric,

$$E_D d + E_A(D - d) = V$$

Substituting for E_D we have, since $\varepsilon = \varepsilon_0 \varepsilon_r$,

$$\frac{E_A}{\varepsilon_r} d + E_A(D - d) = V$$

or

$$E_A = \frac{V}{D - d\left(1 - \frac{1}{\varepsilon_r}\right)} \quad . \quad . \quad . \quad (4.26)$$

Thus, increase of d increases the potential gradient in the air space, as is shown in Fig. 4.18. Also, if ε_r is much greater than 1. the potential gradient in the air space approaches the value $\frac{V}{D - d}$, which means that, in this case, the whole of the potential drop is across the air space.

The high potential gradient so produced is very likely to cause breakdown of the air in the case of a thin film of air included between a solid dielectric and a conducting plate. The air then becomes ionized, and the insulation will ultimately fail due to damage by ionic bombardment.

The dielectric strengths of the most important insulating materials are given in Table VII under the conditions of the customary one-minute dielectric strength test. The electrodes used in carrying out tests of dielectric strength are usually flat plates with rounded edges or smooth spheres of large diameter. In either case a fairly uniform electrostatic field is obtained.

(*b*) Relative Permittivity. This quantity is defined as the ratio

$$\frac{\text{The capacitance of a capacitor having the material considered as its dielectric}}{\text{The capacitance of the same capacitor with air as the dielectric}} = \varepsilon_r$$

Strictly, the capacitance in the denominator should be that when a vacuum exists between the plates, since the relative permittivity of a vacuum is unity, while that of air is about 1·0006. Most gaseous dielectrics have permittivity of the same order as that of air, while solid and liquid dielectrics have values of ε_r varying from about 2 upwards, as shown in Table VII.

(*c*) Dielectric Loss and Power Factor. If a steady voltage V is applied to the plates of a perfect capacitor a "charging current" flows from the supply for a short time and gives to the capacitor a certain quantity Q of electricity, which is sufficient to produce a potential difference between the capacitor plates of V volts. When this potential difference has been attained, the current ceases to

TABLE VII

PROPERTIES OF DIELECTRICS

Dielectric	Approx. Dielectric Strength Volts/mm	Relative Permittivity ϵ_r	Power Factor (f = 50 c/s except where noted)
Bakelite	20,000–25,000	5–6	
Bitumen (vulcanized)	14,000	4·5	
Cotton cloth (varnished)	3,000–4,000	4·5–5·5	0·2
Ebonite	10,000–40,000	2·8	0·01
Empire cloth	10,000–20,000	2	
Fibre	5,000	4–6	
Glass (plate)	5,000–12,000	6–7	0·006 (f = 800–1,000)
Guttapercha	10,000–20,000	3–5	
Hard rubber (loaded)	10,000–25,000	3·5–4·5	0·016
Marble	6,000	8	
Mica (Muscovite)	40,000–150,000	4·5–7	0·0003
Mycalex		6–7	0·002–0·005
Paper (dry)	4,000–10,000	1·9–2·9	0·005
Paraffin wax	8,000	2·2	0·0003 (f = 800–1,000)
Polystyrene		2·5–2·7	0·0002
Polythene		2·3	0·0001
Porcelain	9,000–20,000	5·5–6·5	0·005–0·01
Shellac	5,000–20,000	2·3–3·8	0·008
Silica (fused transparent)		3·8	0·0001–0·0003
Slate	3,000	6–7·5	
Steatite		4·1–6·5	0·002
Mineral insulating oil	25,000–30,000	2–2·5	0·0002
Water	—	40–90 (decreases with increase of temperature)	

NOTE. Owing to the different qualities of the various materials and to the variations in results according to the conditions of the test (e.g. frequency, and temperature), the above figures must be regarded as approximations only. The properties of dielectrics, including many of the recently introduced plastic materials, are given in Refs. (39) to (44).

flow, the quantity of electricity Q, which has been supplied, being given by $Q = CV$, where C is the capacitance and is, of course, dependent upon the permittivity of the dielectric. In a perfect capacitor, therefore, the dielectric has only one electrical property, namely that of permittivity. It is found that with all practical dielectrics the current does not cease after a short time but dies away gradually over a long period of time as shown in Fig. 4.19. This means that dielectrics have other properties beyond that of permittivity.

A very small "conduction" current will, of course flow through the dielectric because of the fact that the resistance of the dielectric, though very high, is not infinite. This does not explain, however, the phenomena observed in most dielectrics, since the current is at first larger than that due to plain conduction and also it is not a constant current, but dies away gradually.

This second phenomenon is referred to as "absorption" and dielectrics in which it occurs are said to be "absorptive." All dielectrics are absorptive to some degree. If an absorptive capacitor, after being charged, is discharged, the discharging connection being removed after a short time, it is found that the potential difference between the plates gradually rises again, i.e. the capacitor charges itself. This is known as the "residual" effect. Absorption is explained by assuming that there is a viscous movement of the molecules or

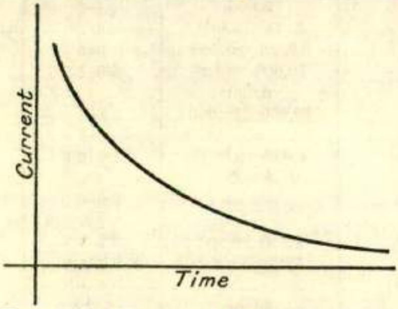


FIG. 4.19. CHARGING CURRENT IN AN IMPERFECT CAPACITOR

ions of a dielectric when the plates between which it is situated are charged. In charging such a capacitor there are rapid electronic and molecular movements which correspond to the initial charging current. Thereafter there are slower molecular and ionic movements which correspond to the absorption current. Finally, there is a steady flow of ions which corresponds to the true conduction current.

The capacitance of a capacitor may thus be divided into two components, viz. the "geometric capacitance" and the "absorptive capacitance." In measuring the capacitance of a capacitor on direct current, the time of charging is thus very important. The shorter the charging time (provided this is long enough to charge the capacitor to the potential difference applied), the nearer the measured capacitance approaches the "geometric" capacitance. Fig. 4.20 shows the variation of the quantity of charge with time in an absorptive capacitor. The measurement of resistance of dielectrics must also be carried out having regard to the time of application of the p.d., since the current for a given applied voltage varies with time as shown above.

Dunsheath (Ref. (10)) represents an absorptive capacitor symbolically, as in Fig. 4.21. The capacitor C_1 represents the geometric capacitance, the resistance R_1 represents the pure conduction effect, and C_2 and R_2 in series represent the absorption effect. In real materials the behaviour can rarely be represented by a single circuit R_2C_2. There is instead a whole spectrum of similar RC circuits in parallel, with different values of RC.

With alternating currents the absorption of the dielectric is intimately connected with the loss of power in the dielectric. In

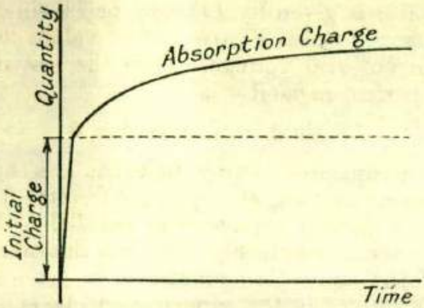
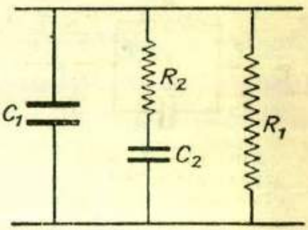
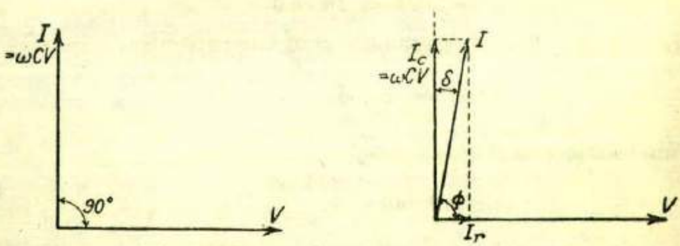


FIG. 4.20. ABSORPTION IN AN ABSORPTIVE CAPACITOR



(From "*High Voltage Cables*," Dunsheath.)

FIG. 4.21. SYMBOLIC COMBINATION TO IMITATE PRACTICAL DIELECTRIC



(*a*) (*b*)

FIGS. 4.22. CAPACITOR VECTOR DIAGRAMS

the case of air and most other gases, the losses are very small, and such dielectrics may be regarded as almost perfect.

If a sinusoidal voltage is applied to a perfect capacitor, the current which flows into the capacitor leads the voltage in phase by 90°, as shown in the vector diagram in Fig. 4.22 (*a*). If the voltage is

$$v = V_{max} \sin \omega t$$

the current in a perfect capacitor of capacitance C farads is

$$i = \omega C \, . \, V_{max} \cos \omega t$$

Its r.m.s. value is ωCV amp, where V is the r.m.s. value of the applied voltage. Owing to the dielectric loss, the current in capacitors used in practice leads the voltage by some angle which is slightly less than 90°, as in Fig. 4.22 (*b*). The angle ϕ is the "phase angle" of the capacitor, the power factor being $\cos \phi$. The angle δ, which equals $90 - \phi$, is called the "loss angle." Obviously the power factor may also be expressed as $\sin \delta$.

In a perfect capacitor $\phi = 90°$, and therefore $\delta = 0$. The dielectric loss in an imperfect capacitor is given by $IV \cos \phi$ or $IV \sin \delta$, where I and V are r.m.s. values of current and voltage. Thus the loss in a perfect capacitor is

$$IV \sin \delta = 0, \text{ since } \delta = 0$$

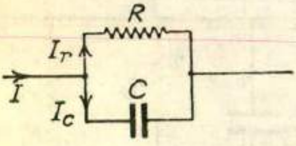


FIG. 4.23. SYMBOLIC REPRESENTATION OF AN IMPERFECT CAPACITOR

A capacitor having dielectric loss can be represented, at any single frequency, by a perfect capacitor in parallel with a resistance as in Fig. 4.23, but the value of the equivalent resistance in general varies with frequency. The current I in the capacitor can be split up into a current I_r in the resistance branch, in phase with the voltage, and a current I_c in the capacitor branch, leading the voltage by 90°. These components are shown in Fig. 4.22 (*b*). Then

$$I_c = \omega CV = I \cos \delta$$

where C is the effective capacitance of the capacitor,

$$\therefore C = \frac{I}{\omega V} \cos \delta$$

The dielectric loss $P = IV \sin \delta$

$$= V \sin \delta \times \frac{V\omega C}{\cos \delta}$$

$$= V^2\omega C \tan \delta \quad \text{watts} \quad . \quad . \quad (4.27)$$

if C is in farads and V in volts.

The works referred to at the end of the chapter should be consulted by those who wish to carry the study of dielectric loss further. Refs. (15), (16), and (40) give the effect of frequency and of temperature upon dielectric loss. W. H. F. Griffiths* has investigated the question of losses in variable air capacitors.

Measurement of Dielectric Loss and Power Factor. The two groups of methods of measuring dielectric losses which have been used are—

(*a*) Wattmeter methods,
(*b*) Bridge methods.

* *Experimental Wireless and The Wireless Engineer*, Vol. VIII, No. 90 March, 1931.

The cathode-ray oscillograph has also been applied to such measurements and is still used to investigate the dielectric properties of non-linear materials which would give no balance in a bridge circuit. One example of such a material is barium titanate.

(*a*) WATTMETER METHODS. These are now very seldom used and will be described here only briefly.

Fig. 4.24 shows the connection diagram for a dynamometer wattmeter when used for this purpose. Owing to the very small power loss and low power factor (usually less than 0·01) the wattmeter must be very sensitive. A "null" method of use is preferable, the wattmeter reading being made zero by adjustment of the variable

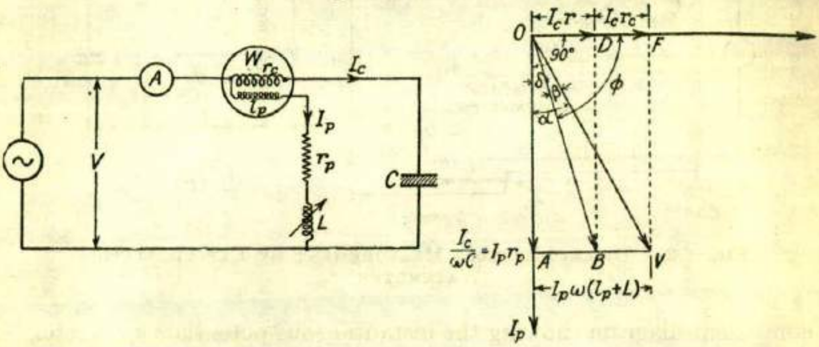


FIG. 4.24. WATTMETER METHOD OF MEASURING DIELECTRIC LOSS AND POWER FACTOR

inductance L in the voltage-coil circuit; this brings about a 90° phase difference between I_c and I_p.

Since the loss angle δ, of the capacitor C under test, is very small, as is also the angle β, we may write

$$\tan\beta = \frac{BV}{OB} = \frac{BV}{OA} \text{ approx.}$$

$$= \frac{I_c r_c}{I_c/\omega C} = \omega C r_c \text{ approx.}$$

Thus $$\beta = \tan^{-1}\omega C r_c$$

Again $$\alpha = \tan^{-1}\frac{\omega(l_p + L)}{r_p}, \text{ so that}$$

$$\phi = 90 - \alpha + \beta$$

$$= 90 - \tan^{-1}\frac{\omega(l_p + L)}{r_p} + \tan^{-1}\omega C r_c$$

The power factor of the capacitor is $\cos\phi$ and its loss angle $\delta = \alpha - \beta$.

Obviously, in addition to the value of the variable inductance L, the values of the inductance l_p and resistance r_p of the voltage-coil circuit, of the resistance r_c of the current coil and C and ω must be known.

Rosa (Ref. (18)) has described several null methods of measurement of dielectric loss using wattmeters.

Electrostatic Wattmeter Method. This method has been used by many investigators. Fig. 4.25 (*a*) shows the connections for the method as used by Rayner (Ref. (20)). Fig. 4.25 (*b*) gives the

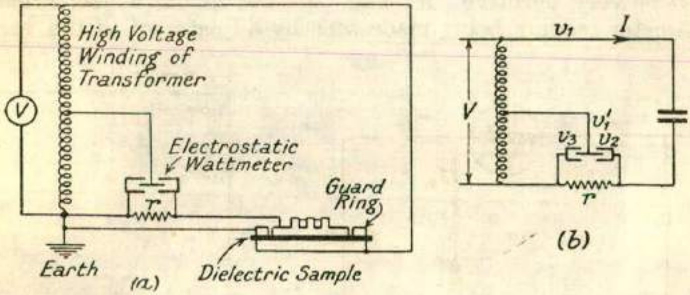


FIG. 4.25. DIELECTRIC LOSS MEASUREMENT BY ELECTROSTATIC WATTMETER

equivalent diagram showing the instantaneous potentials v_1, v_2, etc., at various points; r is a non-inductive resistance.

The moving vane of the electrostatic instrument is connected to a tapping point on the high-voltage winding of a transformer from which the supply is obtained.

The sample of insulating material whose dielectric loss is to be measured is connected as shown and is provided with a guard ring which is earthed.

From the theory of the electrostatic wattmeter given in Chap. XX, it can be shown that the mean torque of the wattmeter is proportional to

$$\frac{r}{n}(P + rI^2) - \frac{r^2I^2}{2}$$

where P = dielectric power loss

I = r.m.s. value of the current

Then, if K is the constant of the instrument and D is the deflection, we have

$$\frac{r}{n}(P + rI^2) - \frac{r^2I^2}{2} = KD$$

from which $$P = \frac{nKD}{r} + \frac{n-2}{2}rI^2 \quad . \quad . \quad . \quad (4.28)$$

If the tapping point on the transformer winding is adjusted so that $n = 2$, the second term becomes zero, and we have

$$P = \frac{2KD}{r}$$

This avoids the correction for the power loss in the resistance r.

The voltage used by Rayner in his measurements was 10,000 volts.

(*b*) BRIDGE METHODS. The Schering bridge method is now the most widely used of all methods of measuring dielectric loss and power factor. All bridge methods consist essentially of a Wheatstone bridge network, the battery supply being replaced by an a.c. supply at either power frequency or some higher frequency. The detector used depends upon the frequency, a vibration galvanometer being used for power frequency work and telephones for work at higher frequencies, the latter being often of the order of 800 to 1,000 cycles per second.

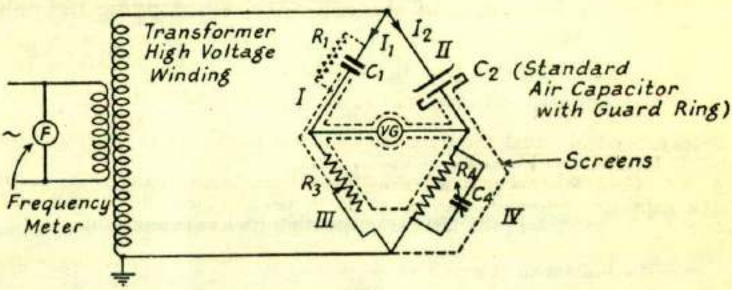


FIG. 4.26. CONNECTIONS OF SCHERING BRIDGE

Fig. 4.26 gives the connections of the Schering bridge, which can be used with high or low voltages. C_1 is the capacitor whose power factor is to be measured, R_1 being an imaginary resistance representing its dielectric loss component. C_2 is a standard air capacitor of the type described in Chapter II. R_3 and R_4 are non-inductive resistors, the former being variable. C_4 is a variable capacitor. Earthed screens are provided in order to avoid errors due to intercapacitance between the high- and low-voltage arms of the bridge. Instead of earthing one point on the network as shown in the figure, the earth capacitance effect on the galvanometer and leads is eliminated by means of a "Wagner earth" device (Ref. (22)), which will be described in a later chapter. *V.G.* is a vibration galvanometer of a special design suited to the purpose. This must have a high current sensitivity, since the impedances of arms 1 and 2 of the bridge are usually very high. For the same reason, this method of measurement involves only a small power loss. Since the impedances of branches 3 and 4 are usually small compared with those of arms I and 2, the galvanometer and the resistances are at a potential of only a few volts above earth even when a high-voltage supply (of the order of 100 kilovolts) is used, except in the case of breakdown of one of the capacitor arms I and II.

In use, the bridge is balanced by successive variation of R_3 and C_4 until the vibration galvanometer indicates zero deflection. Then, at balance,

$$C_1 = C_2 \cdot \frac{R_4}{R_3} \cos^2 \delta = C_2 \cdot \frac{R_4}{R_3} \text{ approx.} \qquad (4.29)$$

since δ is small, and

$$\tan \delta = R_4 \omega \cdot C_4 \qquad (4.30)$$

where $\omega = 2\pi \times$ frequency

$\delta =$ the loss angle of the capacitor, $\sin \delta$ giving the power factor

$C_1 =$ the effective parallel capacitance of the test capacitor

$C_2 =$ the capacitance of the standard capacitor

Theory. Consider first the impedances of the four arms of the bridge numbered I, II, III, and IV in Fig. 4.26.

Arm I. Consider this arm as consisting of the effective parallel capacitance of the capacitor whose power factor is to be obtained, in parallel with a resistance R_1, as shown, the latter representing its loss component.

$$\text{Total admittance of arm I} = \frac{1}{R_1} + \frac{1}{\frac{-j}{\omega C_1}}$$

$$= \frac{1}{R_1} + j\omega C_1$$

$$\therefore \text{Impedance of arm I} = \frac{1}{\frac{1}{R_1} + j\omega C} = \frac{R_1}{1 + j\omega C_1 R_1} = z$$

Arm II. Impedance $= \dfrac{-j}{\omega C_2} = z_2$

Arm III. Impedance $= R_3 = z_3$

Arm IV. Impedance $= \dfrac{R_4}{1 + j\omega C_4 R_4} = z$

Under balance conditions,

$$\frac{z_1}{z_3} = \frac{z_2}{z_4}$$

i.e.

$$\frac{R_1}{R_3(1 + j\omega C_1 R_1)} = \frac{\frac{-j}{\omega C_2}}{\frac{R_4}{1 + j\omega C_4 R_4}} = \frac{-j}{\omega C_2 R_4}(1 + j\omega C_4 R_4)$$

Rationalizing, we have

$$\frac{R_1(1 - j\omega C_1 R_1)}{R_3(1 + \omega^2 C_1^2 R_1^2)} = \frac{-j}{\omega C_2 R_4}(1 + j\omega C_4 R_4)$$

Equating real terms,

$$\frac{R_1}{1 + \omega^2 C_1^2 R_1^2} = \frac{C_4 R_3}{C_2}$$

Now, from Fig. 4.27, which shows the vector diagram for the capacitor (C_1 and R_1 in parallel) when a voltage E is applied to it,

$$\cos\delta = \frac{E\omega C_1}{E\sqrt{\frac{1}{R_1^2} + \omega^2 C_1^2}} = \frac{\omega C_1 R_1}{\sqrt{1 + \omega^2 C_1^2 R_1^2}}$$

or

$$\cos^2\delta = \frac{\omega^2 C_1^2 R_1^2}{1 + \omega^2 C_1^2 R_1^2}$$

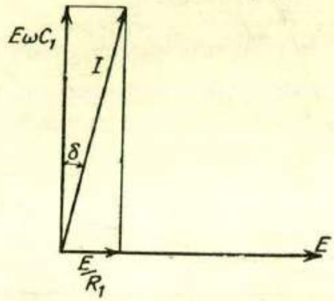


FIG. 4.27. VECTOR DIAGRAM FOR C_1 AND R_1 IN PARALLEL

Substituting $\cos^2\delta$ in the equation of real terms obtained above, we have

$$\frac{\cos^2\delta}{\omega^2 C_1^2 R_1} = \frac{C_4 R_3}{C_2}$$

$$\therefore C_1 = \frac{C_2 \cos^2\delta}{\omega^2 C_1 C_4 R_1 R_3}$$

From Fig. 4.28, showing the complete vector diagram for the bridge net work under balance conditions,

$$\tan\delta = \frac{\omega C_4}{\frac{1}{R_4}} = \omega C_4 R_4$$

(which is the expression previously stated), and also

$$\tan\delta = \frac{\frac{1}{R_1}}{\omega C_1} = \frac{1}{\omega C_1 R_1}$$

$$\therefore \omega C_4 R_4 = \frac{1}{\omega C_1 R_1}$$

or

$$R_4 = \frac{1}{\omega^2 C_1 C_4 R_1}$$

Substituting R_4 in the expression for C_1 gives

$$C_1 = \frac{C_2 R_4}{R_3} \cos^2\delta$$

as previously stated.

Imperfect Capacitor as a Series Circuit. An alternative method to that of representing an imperfect capacitor diagrammatically as a perfect capacitor C_1 in parallel with a resistance R_1, is to represent it as a perfect capacitor C_s in series with a resistance R_s. It is much simpler to derive the bridge balance equations using the series representation.

The impedances in the two cases are

$$\frac{R_1}{1 + j\omega C_1 R_1} = \frac{R_1(1 - j\omega C_1 R_1)}{1 + \omega^2 C_1^2 R_1^2} \text{ in the parallel representation}$$

and

$$R_s - \frac{j}{\omega C_s} \text{ in the series representation}$$

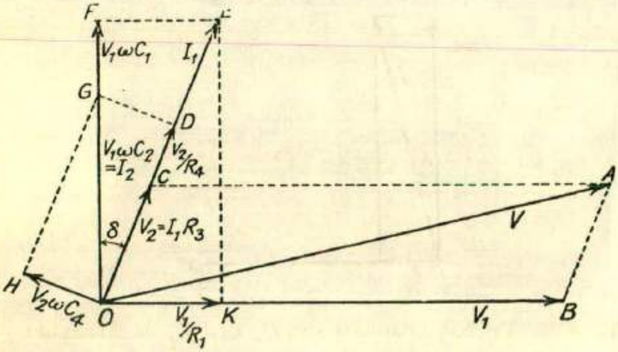


FIG. 4.28. VECTOR DIAGRAM FOR SCHERING BRIDGE UNDER BALANCE CONDITIONS

By equating the real and imaginary terms in the two impedances we obtain the relationships

$$R_s = \frac{R_1}{1 + \omega^2 C_1^2 R_1^2}$$

and

$$C_s = \frac{1 + \omega^2 C_1^2 R_1^2}{\omega^2 C_1 R_1^2}$$

The veetor diagram of Fig. 4.28 needs, perhaps, some explanation. Vector OA represents the voltage applied to the bridge from the supply transformer. OB is the voltage drop V_1 across arm II which, when no current flows in the vibration galvanometer branch (i.e. under balance conditions), is equal in magnitude and phase to the voltage drop across arm I. Vector OC is the drop V_2 across arm III, which is equal in magnitude and phase to that across arm IV. The vector sum of OB and OC obviously gives the total bridge voltage OA. The current I_1 flowing in arms I and III is represented by vector OE, while OG represents the current I_2 flowing in branches II and IV. OF and OK represent the component parts of current I_1 when split up between the capacitance C_1 and resistance R_1. In the same way OD and OH represent the components of the current I_2 when split up between R_4 and C_4.

The magnitudes of some of the vectors, e.g. *OC*, are exaggerated for the sake of clearness. V_2 will, in reality, be very small compared with V_1 and V.

A direct-reading Schering bridge for the measurement of permittivity and power factor of solid dielectrics at 1,600 cycles per sec and voltages of 100–200 is manufactured by Messrs. H. W. Sullivan, Ltd. This covers a range of capacitance up to 1,000 $\mu\mu$F.

The Cambridge Instrument Co. manufacture both low- and high-voltage Schering bridges.

Muirhead and Co., Ltd. make a Schering bridge, with a Wagner earth attachment (see p. 247) which is intended for the measurement of power factor and permittivity of insulating materials in accordance with the recommendations of *British Standard Specification* No. 234.

A portable high-voltage Schering bridge made by H. Tinsley and Co. has with it a screened, loss-free air capacitor of 100 $\mu\mu$F (within $\pm \frac{1}{2}$ per cent) and is for use at 11 kV. It may be used up to 150 kV with a compressed-air capacitor of nominal capacitance 100 $\mu\mu$F having a power factor $\not>$ 0·0001 at 50 cycles per sec. This requires an air pressure of 250 to 300 lb per sq. in.

L. Hartshorn (Ref. (45)) adapted the Schering bridge to the measurement of very small capacitances (below 1 $\mu\mu$F), and the Hartshorn form of the bridge is the best method of measuring the permittivity and dielectric loss of sheet materials. B.S. 234 and B.S. 903 give detailed specifications for its use for this purpose.

A very full discussion of the Schering bridge, in its various forms, is given in Hague's *Alternating Current Bridge Methods*.

Dielectric Loss Measurement by Cathode-ray Oscillograph. The construction of the cathode-ray oscillograph is dealt with in Chap. XV. For the present purpose it is sufficient to know that it consists of a vacuum tube having, at one end, a filament which gives off a stream of electrons in a thin beam, or pencil, when the tube is in use. This beam passes two pairs of parallel plates, set at right angles to one another, and is deflected by potential differences applied to these pairs. A continuous path will be traced out by the beam on the fluorescent screen of the tube if the p.d.s are alternating. This path will be a straight line if the p.d.s are sinusoidal and are in phase but will be an ellipse if they are not in phase. The area of this ellipse is maximum—for any given maximum values of the two potential differences—when they are 90° out of phase with one another. Under these conditions, the semi-axes of the ellipse give the maximum values of the two potential differences to scale. The electron beam, having negligible inertia, can immediately take up a deflected position which is proportional, at any given time, to the deflecting force.

When used for dielectric loss measurements, a potential difference proportional to the applied voltage is applied to one pair of plates, and one proportional to the integral of current through the dielectric to the other pair. This is obtained in the form of the p.d. across a relatively larger capacitor in series with the sample.

It will be shown below that the area of the ellipse traced out by the electron beam is then proportional to the power loss in the dielectric. If there is no power loss—as in the case of an air capacitor—the p.d.s applied to the plates are in phase with one another and the path traced out is a straight line.

A record of the ellipse traced out in power loss measurements can be obtained photographically.

J. P. Minton (Ref. (23)) used a cathode-ray oscillograph for

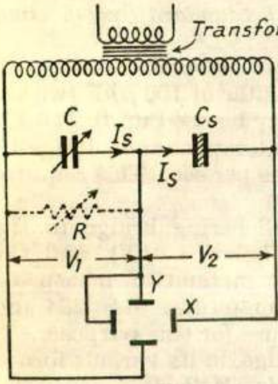


FIG. 4.29

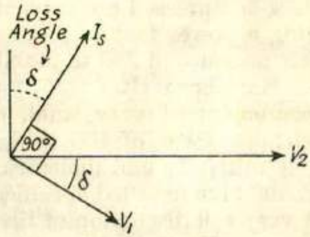


Fig. 4.30

MEASUREMENT OF DIELECTRIC LOSS BY C.R. OSCILLOGRAPH

dielectric loss and power factor measurements. The full circuit arrangements are given by Hartshorn (Ref. (16)).

Fig. 4.29 shows a simpler arrangement than that of Minton which, nevertheless, will serve to illustrate this method of dielectric loss measurement.

C_s is the dielectric sample and C a loss-free capacitor of much greater capacitance than C_s. The resistor R, shown dotted, may be used, if desired, for compensating the loss angle of C_s. The c.r. oscillograph plates X and Y are connected as shown. (If the voltage on C_s is low, an amplifier will be needed between C and the Y plates since the method is inaccurate unless the voltage on C is much less than that on C_s.)

In the theory of the method which is given below it is assumed that the resistor R is omitted. The vector diagram may then be drawn as in Fig. 4.30, which shows the voltage V_2 across C_s, the current I_s through both C_s and C, and the voltage V_1 across the latter.

The power loss in C_s is $V_2 I_s \sin \delta$. From the vector diagram, if $v_2 = V_{2\,max} \sin \omega t$, then $v_1 = V_{1max} \sin (\omega t - \delta)$.

The deflection produced by the Y plates is proportional to v_1 and we may write

$$y \text{ deflection} = \alpha \, . \, V_{1\,max} \sin (\omega t - \delta)$$
$$= \alpha \, . \, \frac{I_{s\,max}}{\omega C} \sin (\omega t - \delta)$$

Also, $\quad x \text{ deflection} = \beta \, . \, V_{2\,max} \sin \omega t$

where α and β are proportionality constants.

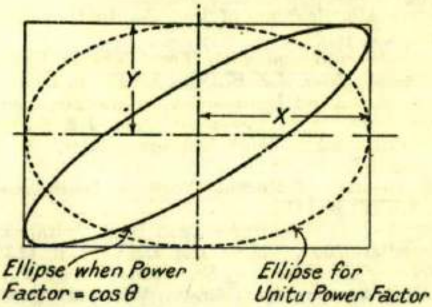


FIG. 4.31. FORMS OF OSCILLOGRAMS OBTAINED WITH CATHODE-RAY OSCILLOGRAPH

Now, the area of the ellipse traced out on the oscillograph screen is

$$A = \int y \, . \, dx$$
$$= \int_0^T \alpha \, . \, \frac{I_{s\,max}}{\omega C} \, . \sin (\omega t - \delta) \, . \, \beta \, . \, V_{2\,max} \omega \cos \omega t \, . \, dt$$

(where T is the periodic time)

$$= \frac{\alpha\beta \, . \, I_{s\,max} V_{2\,max}}{C} \int_0^T \sin (\omega t - \delta) \cos \omega t \, . \, dt$$
$$= \frac{\alpha\beta}{C} \, . \, \frac{I_{s\,max} V_{2\,max}}{2} \int_0^T [\sin (2\omega t - \delta) + \sin \delta] \, dt$$

i.e. $\quad A = \frac{\alpha\beta}{C} \, . \, I_s V_2 \, . \sin \delta \, . \, \frac{2\pi}{\omega}$

since $\quad T = \frac{2\pi}{\omega}$

Thus the area of the ellipse is proportional to $V_2 I_s \sin \delta$, which is the dielectric loss.

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