INTEGRAL CALCULUS

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PREFACE TO THE FIRST EDITION

THIS Book is prepared with a view to be used as a text-book for the B.A. and B.Sc. students of the Indian Universities. We have tried to make the exposition of the fundamental principles clear as well as concise without going into unnecessary details ; and at the same time an attempt has been made to make the treatement as much rigorous and up-to-date as is possible within the scope of this elementary work.

We have devoted a separate chapter for the discussion of infinite (or improper) integrals and the integration of infinite series in order to emphasise their peculiarity upon the students. Important formulae and results of Differential Calculus as also of this book are given in the beginning for ready reference. A good number of typical examples have been worked out by way of illustration.

Examples for exercises have been selected very carefully and include many which have been set in the Pass and Honours Examinations of different Universities. University questions of recent years have been added at the end to give the students an idea of the standard of the examination.

Our thanks are due to several friends for their helpful suggestions in the preparation of the work and especially to our pupil, Prof. H. K. Ganguly, M.A. for verifying the answers of all the examples of the book.

Corrections and suggestions will be thankfully received.

CALCUTTA
January, 1938

B. C. D.
B. N. M.

PREFACE TO THE NINETEENTH EDITION

In this edition a few examples have been added here and there and a set of Miscellaneous Examples has been added at the end. A few misprints that had crept into the previous edition have been corrected. The generalization of the Rule of Integration by Parts and alternative proofs evaluating two important integrals

$\int e^{ax} \begin{matrix}\cos \\ \sin\end{matrix} bx\, dx$ and $\int \frac{dx}{\sin x + \cos x}$ have been given in the Section C of the Appendix.

Calcutta
July, 1969

B. C. D.
B. N. M.

PREFACE OF THE TWENTIETH EDITION

THIS edition is practically a reprint of the previous edition ; only a method of finding the C. G. of the Volume and Surface of Revolution has been given in the section E of the Appendix and some additional examples of various types on C. G. and Moment of Inertia have been given in the Miscellaneous Examples II of the Appendix.

Calcutta
July, 1971

B. C. D.
B. N. M.

PREFACE TO THE FORTY-FIRST EDITION

In this edition some rearrangement of the matters have been made so as to enable the students to understand the subject more easily. Mistakes and misprints have been corrected as far as possible.

We thank Sri B. Mahalanabis, M.A. and Sri Malay Chatterjee B.E. (Cal) for their help in identification and rectification of mistakes. Our thanks are due to the authorities and staff of Messrs U. N. Dhur & Sons (P) Ltd. for helping us in bringing out the book within such a short time. We thank the authorities and staff of Messrs Micromeg (India) Private Limited for the help extended by them in organising computerised typesetting of the book. We also thank the authorities and staff of Messrs Varnakshar for helping in printing the book on time.

Calcutta
September, 1994

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The University of North Bengal

Indefinite Integral. Definite Integral as the limit of a sum and its geometric interpretation. Fundamental theorem of Integral Calculus. Elementary properties of Definite Integrals. Evaluation of Definite Integrals. Reduction formula for

$$\int \sin^n \theta \, d\theta, \int \cos^n \theta \, d\theta \text{ and } \int \sin^m \theta \cos^n \theta \, d\theta.$$

Rectification and Quadrature. Calculation of volume and surface of solids of Revolution.

Differential equations : Genesis of Differential Equation. Family of curves represented by $\frac{dy}{dx} = f(x, y)$. Solution of first order Differential Equation. Solution of Higher Order Linear Differential Equation with constant Coefficients. Simple applications in Geometry and Mechanics.

The University of Burdwan

Integral Calculus : Indefinite integral, standard forms. Rules of Integration. Method of substitution, Integration by parts, partial fractions. Definite Integral as the limit of a sum, its geometrical interpretation. Elementary properties of definite integrals. Fundamental theorem connecting definite and indefinite integrals. Certain definite integrals, viz.

$$\int_0^{\pi/2} \sin^n x \, dx, \int_0^{\pi/2} \cos^n x \, dx, \int_0^{\pi/2} \sin^m x \cos^n x \, dx.$$

(m, n positive integers).

Simpson's one-third rule for numerical evaluation of definite integrals. Idea of improper integrals.

Rectification of a plane curve, Quadrature, Volumes and Surfaces of solids of revolution. Centre of gravity of simple bodies.

Differential Equation : Genesis of a differential equation. Family of curves represented by $\frac{dy}{dx} = f(x, y)$. Solution of first order differential equations. Evaluation of special solutions passing through a given point. Linear differential equations with constant cofficients, both homogeneous and non-homogeneous. Evaluation of special solutions for given x_0, y_0, y'_0 of second order linear differential equations with constant coefficient. Simple applications.

Syllabus for two-year pass Degree Course
Integral Calculus
The University of Calcutta

Integral Calculus (30 Marks) : Integrations of the form $\int \frac{dx}{a + p \cos x}$, $\int \frac{l \sin x + m \cos x}{n \sin x + p \cos x} dx$ and integration of Rational functions. Evalua-tion of definite integrals.

Integration as the limit of a sum (with equally-spaced as well as unequal intervals).

Reduction formulae of $\int \sin^m x \cos^n x \, dx$, $\int \frac{\sin^m x}{\cos^n x} dx$, $\int \tan^n x dx$ and associated problems (*m* and *n* are non-negative integer).

Definition of Improper Integrals : Use of Beta and Gamma functions (Convergence and important relations being assumed).

Working knowledge of Double Integral.

Applications : Rectification, Volume and surface areas of solids formed by revolution of plane curves and areas (by x-axis and y-axis). [Problems only]

Differential Equations (20 Marks) : Order, degree and Solution of an ordinary differential Equation. (ODE) presence of arbitrary constants, formation of ODE.

First order equations : (i) variables separable. (ii) Homogeneous equations and equations reducible to Homogeneous forms. (iii) Euler's and Bernoulli's Equations (Linear). (iv) Exact Equations and those reducible to such equations. (v) Clairaut's equation : General and Singular solutions.

Second order linear Equations : Second order linear differential Equations with constant co-efficients. Euler's Homogeneous equations.

Simple applications : Orthogonal Trajectories

CONTENTS

INTEGRAL CALCULUS

DIFFERENTIAL EQUATIONS

ABBREVIATIONS USED IN THE BOOK

1. ***I.*** stands for "the Integral".
2. ***C. P.*** stands for "set in the B. A. and B. Sc. Pass Examinations of the Calcutta University".
3. ***C. H.*** stands for "set in the B. A. and B. Sc. Honours Examinations of the Calcutta University".
4. ***P. P.*** Examinations of the Patna University.

IMPORTANT FORMULAE AND RESULTS
of
(A) TRIGONOMETRY

I. Fundamental relations.

(i) $\sin^2\theta + \cos^2\theta = 1$
(ii) $\sec^2\theta = 1 + \tan^2\theta$
(iii) $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$

(iv) $\sin(-\theta) = -\sin\theta$
(v) $\cos(-\theta) = \cos\theta$
(vi) $\tan(-\theta) = -\tan\theta$

II. Multiple angles.

(i) $\sin 2A = 2 \sin A \cos A$.

(ii) $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$.

(iii) $\sin 2A = \dfrac{2\tan A}{1 + \tan^2 A}$; (iv) $\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$

(v) $1 - \cos 2A = 2\sin^2 A$
(vi) $1 + \cos 2A = 2\cos^2 A$

(vii) $\tan^2 A = \dfrac{1 - \cos 2A}{1 + \cos 2A}$

(viii) $1 + \sin 2A = (\sin A + \cos A)^2$.

(ix) $1 - \sin 2A = (\sin A - \cos A)^2$.

(x) $\sin 3A = 3\sin A - 4\sin^3 A$.

(xi) $\cos 3A = 4\cos^3 A - 3\cos A$.

(xii) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$.

(xiii) $\cos 3A = \dfrac{\cot^3 A - 3\cot A}{3\cot^2 A - 1}$.

III. Special angles.

$\sin 0° = 0$
$\cos 0° = 1$
$\tan 0° = 0$

$\operatorname{cosec} 0° = \infty$
$\sec 0° = 1$
$\cot 0° = \infty$

$\sin 90° = 1$
$\cos 90° = 0$
$\tan 90° = \infty$

$\operatorname{cosec} 90° = 1$
$\sec 90° = \infty$
$\cot 90° = 0$

$\sin 30° = \frac{1}{2}$
$\cos 30° = \frac{1}{2}\sqrt{3}$
$\tan 30° = 1/\sqrt{3}$

$\sin 60° = \frac{1}{2}\sqrt{3}$
$\cos 60° = \frac{1}{2}$
$\tan 60° = \sqrt{3}$

$$\left.\begin{array}{l}\sin 45^\circ = 1/\sqrt{2}\\ \cos 45^\circ = 1/\sqrt{2}\\ \tan 45^\circ = 1\end{array}\right\} \qquad \left.\begin{array}{l}\sin 180^\circ = 0\\ \cos 180^\circ = -1\\ \tan 180^\circ = 0\end{array}\right\}$$

$\sin 120^\circ = \frac{1}{2}\sqrt{3}$, $\qquad \cos 120^\circ = -\frac{1}{2}$.

$\sin 15^\circ = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$; $\qquad \cos 15^\circ = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$;

$\tan 15^\circ = 2 - \sqrt{3}$.

$\sin 75^\circ = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$; $\qquad \cos 75^\circ \dfrac{\sqrt{3}-1}{2\sqrt{2}}$;

$\tan 75^\circ = 2 + \sqrt{3}$.

$\sin 18^\circ = \frac{1}{4}(\sqrt{5}-1)$; $\qquad \cos 36^\circ = \frac{1}{4}(\sqrt{5}+1)$.

$\sin 22\frac{1}{2}^\circ = \frac{1}{2}\sqrt{(2-\sqrt{2})}$; $\cos 22\frac{1}{2}^\circ = \frac{1}{2}\sqrt{(2+\sqrt{2})}$.

IV. Inverse Trigonometric functions.

(i) $\operatorname{cosec}^{-1} x = \sin^{-1} \dfrac{1}{x}$; $\cot^{-1} x = \tan^{-1} \dfrac{1}{x}$; $\sec^{-1} x = \cos^{-1} \dfrac{1}{x}$.

(ii) $\sin^{-1} x + \cos^{-1} x = \frac{1}{2}\pi$.

(iii) $\tan^{-1} x + \cot^{-1} x = \frac{1}{2}\pi$ $\qquad$ (iv) $\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{1}{2}\pi$.

(v) $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \dfrac{x+y}{1-xy}$.

(vi) $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \dfrac{x-y}{1+xy}$

(vii) $3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$.

(viii) $3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$.

(ix) $3\tan^{-1} x = \tan^{-1} \dfrac{3x - x^3}{1 - 3x^2}$.

(x) $2\tan^{-1} x = \sin^{-1} \dfrac{2x}{1+x^2} = \cos^{-1} \dfrac{1-x^2}{1+x^2} = \tan^{-1} \dfrac{2x}{1-x^2}$

V. Complex Arguments.

(i) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

(ii) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \ldots + (-1)^n \dfrac{x^{2n}}{(2n)!} + \ldots$ to ∞

(iii) $\sin x = x - \frac{x^3}{3!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots$ to ∞.

(iv) $\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \ldots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)}$

$+ \ldots$ to ∞, $-1 \le x \le 1$.

(v) $e^{ix} = \cos x + i \sin x$; $e^{-ix} = \cos x - i \sin x$.

(vi) $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$; $\sin x = \frac{1}{2i}(e^{ix} + e^{-ix})$.

(vii) $x^n + \frac{1}{x^n} = 2\cos n\theta$; $x^n - \frac{1}{x^n} = 2i \sin n\theta$.

(viii) $2^{n-1} \cos^n \theta = \cos n\theta + n \cos (n-2)\theta$

$$+ \frac{n(n-1)}{2!} \cos (n-4)\theta + \ldots$$

(n being a positive integer)

(ix) $(-1)^{n/2} 2^{n-1} \sin^n \theta$

$= \cos n\theta - n \cos (n-2)\theta + \frac{n(n-1)}{2!} \cos (n-4)\theta -$

(n being an even positive integer).

(x) $(-1)^{(n-1)/2} 2^{n-1} \sin^n \theta$

$= \sin n\theta - n \sin (n-2)\theta + \frac{n(n-1)}{2!} \sin (n-4)\theta -$

(n being an odd positive integer).

VI. Hyperbolic Functions.

(i) $\cosh x = \frac{1}{2}(e^x + e^{-x})$; $\sinh x = \frac{1}{2}(e^x - e^{-x})$.

(ii) $e^x = \cosh x + \sinh x$; $e^{-x} = \cosh x - \sinh x$.

(iii) $\cosh^2 x - \sinh^2 x = 1$.

(iv) $\text{sech}^2 x + \tanh^2 x = 1$.

(v) $\coth^2 x - \text{cosech}^2 x = 1$.

(vi) $\sinh 2x = 2 \sinh x \cosh x$.

(vii) $\cosh 2x = \cosh^2 x + \sinh^2 x$

$= 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$.

(viii) $\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$.

(ix) $\sinh(-x) = -\sinh x$; $\cosh(-x) = \cosh x$.

(x) $\sinh 0 = 0$; $\cosh 0 = 1$; $\tanh 0 = 0$.

(xi) $\sinh x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \ldots + \dfrac{x^{2n+1}}{(2n+1)!} + \ldots$ to ∞.

(xii) $\cosh x = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \ldots \dfrac{x^{2n}}{(2n)!} + \ldots$ to ∞.

(xiii) $\sinh^{-1} x = \log \{x + \sqrt{(x^2+1)}\}$ for all x.

(xiv) $\cosh^{-1} x = \log \{x + \sqrt{(x^2-1)}\}$ $(x \geq 1)$.

(xv) $\tanh^{-1} x = \dfrac{1}{2} \log \dfrac{1+x}{1-x}$ $(x^2 < 1)$.

(xvi) $\operatorname{cosech}^{-1} x = \log \dfrac{1+\sqrt{(1+x^2)}}{x}$ $(x \neq 0)$.

(xvii) $\operatorname{sech}^{-1} x = \log \dfrac{1+\sqrt{(1-x^2)}}{x}$ $(0 < x \leq 1)$.

(xv) $\coth^{-1} x = \frac{1}{2} \log \dfrac{x+1}{x-1}$ $(x^2 > 1)$.

VII. Special series.

(i) $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots$ to $\infty = \dfrac{\pi^2}{6}$.

(ii) $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots$ to $\infty = \dfrac{\pi^2}{8}$.

(iii) $\dfrac{1}{1^4} + \dfrac{1}{2^4} + \dfrac{1}{3^4} + \ldots$ to $\infty = \dfrac{\pi^4}{90}$.

(iv) $\dfrac{1}{1^4} + \dfrac{1}{3^4} + \dfrac{1}{5^4} + \ldots$ to $\infty = \dfrac{\pi^4}{96}$.

VIII. Logarithm.

$\log_a m = \log_b m / \log_b a = \log_b m \times \log_a b$.

(B) DIFFERENTIAL CALCULUS

1. Fundamental properties.

(i) $\frac{d}{dx}\{u \pm v \pm w \pm \ldots\ldots \text{ to } n \text{ terms}\}$

$$= \frac{du}{dx} \pm \frac{dv}{dx} \pm \frac{dw}{dx} \pm \ldots\ldots \text{ to } n \text{ terms.}$$

(ii) $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

(iii) $\frac{d}{dx}(cu) = c\frac{du}{dx}$.

(iv) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

(v) $\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}$ {where $x = f(z)$ and $z = \phi(x)$ }

II. Standard differential coefficients.

(i) $\frac{d}{dx}(c) = 0$.

(ii) $\frac{d}{dx}(x^n) = nx^{n-1}$.

(iii) $\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{n}{x^{n+1}}$.

(iv) $\frac{d}{dx}(a^x) = a^x \log_e a$

(v) $\frac{d}{dx}(e^x) = e^x$.

(vi) $\frac{d}{dx}(\log. x) = \frac{1}{x}\log. e$

(vii) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

(viii) $\frac{d}{dx}(\sin x) = \cos x$

(ix) $\frac{d}{dx}(\cos x) = -\sin x$.

(x) $\frac{d}{dx}(\tan x) = \sec^2 x$

(xi) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(xii) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

(xiii) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(xiv) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{(1-x^2)}}$. $(-1 < x < 1)$

(xv) $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{(1-x^2)}}, (-1 < x < 1)$

(xvi) $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$.

(xvii) $\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$.

(xviii) $\frac{d}{dx}(\operatorname{cosec}^{-1}x) = -\frac{1}{x\sqrt{(x^2-1)}}, (|x| > 1)$

(xix) $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{(x^2-1)}}, (|x| > 1)$

(xx) $\frac{d}{dx}(\sinh x) = \cosh x.$

(xxi) $\frac{d}{dx}(\cosh x) = \sinh x.$

(xxii) $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x.$

(xxiii) $\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x.$

(xxiv) $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tan x.$

(xxv) $\frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x.$

(xxvi) $\frac{d}{dx} = \sinh^{-1}x) = \frac{1}{\sqrt{(1+x^2)}}$.

(xxvii) $\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{(x^2-1)}}$. $(x>1)$

(xxviii) $\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2}$. $(x<1)$

(xxix) $\frac{d}{dx}(\coth^{-1})\,x = -\frac{1}{x^2-1}$. $(x>1)$

(xxx) $\frac{d}{dx}(\operatorname{cosech}^{-1}x) = -\frac{1}{x\sqrt{(x^2+1)}}$.

(xxxi) $\frac{d}{dx}(\operatorname{sech}^{-1}x) = -\frac{1}{x\sqrt{(1-x^2)}}$. $(x<1)$

III. Important results associated with curves.

(i) Cartesian subtangent $= \dfrac{y}{y_1}$.

(ii) Cartesian subnormal $= yy_1$.

(iii) Cartesian normal $= y\sqrt{(1 + y_1^2)}$.

(iv) Cartesian tangent $= \dfrac{y}{y_1} \cdot \sqrt{1 + y_1^2}$.

(v) Polar subtangent $= r^2 \dfrac{d\theta}{dr} = -\dfrac{d\theta}{du}$; $\left(u = \dfrac{1}{r}\right)$.

(vi) Polar subnormal $= \dfrac{dr}{d\theta} = -\dfrac{1}{u^2}\dfrac{du}{d\theta}$; $\left(u = \dfrac{1}{r}\right)$.

(vii) $\tan\psi = \dfrac{dy}{dx}$; $\cos\psi = \dfrac{dx}{ds}$; $\sin\psi = \dfrac{dy}{ds}$.

(viii) $\tan\phi = r\dfrac{d\theta}{dr}$; $\cos\phi = \dfrac{dr}{ds}$; $\sin\phi = r\dfrac{d\theta}{ds}$.

(ix) $ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2$.

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2;\ \left(\frac{ds}{dy}\right)^2 = 1 + \left(\frac{dx}{dy}\right)^2.$$

$$\left(\frac{ds}{dr}\right)^2 = 1 + r^2\left(\frac{d\theta}{dr}\right)^2;\ \left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2.$$

(x) $p = r\sin\phi$; $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}$; $\left(\dfrac{dr}{d\theta}\right)^2 = u^2 + \left(\dfrac{du}{d\theta}\right)^2$.

(xi) $\rho = \dfrac{ds}{d\psi} = \dfrac{(1 + y_1^2)^{3/2}}{y_2} = \dfrac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2} = r\dfrac{dr}{dp} = p + \dfrac{d^2p}{d\psi^2}$

(C) INTEGRAL CALCULUS

I. Fundamental Properties.

(i) $\int \{f_1(x) \pm f_2(x) \pm f_3(x) \ldots \text{to } n \text{ terms}\}\, dx$

$$= \int f_1(x)\,dx \pm \int f_2(x)\,dx \pm \int f_3(x)\,dx + \ldots \text{to } n \text{ terms}.$$

(ii) $\int cf(x)\,dx = c\int f(x)\,dx.$

II. Standard Integrals.

(i) $\int x^n\,dx = \frac{x^{n+1}}{n+1} (n \neq -1).$

(ii) $\int \frac{dx}{x^n} = -\frac{1}{(n-1)x^{n-1}} (n \neq 1).$

(iii) $\int dx = x.$ (iv) $\int \frac{dx}{\sqrt{x}} = 2\sqrt{x}.$

(v) $\int \frac{dx}{x} = \log|x|.$ (vi) $\int e^{mx}\,dx = \frac{e^{mx}}{m}.$

(vii) $\int e^x\,dx = e^x.$ (viii) $\int a^x\,dx = \frac{a^x}{\log_e a} (a > 0).$

(ix) $\int \sin mx\,dx = -\frac{\cos mx}{m}.$

(x) $\int \sin x\,dx = -\cos x.$

(xi) $\int \cos mx\,dx = \frac{\sin mx}{m}.$

(xii) $\int \cos x\,dx = \sin x.$

(xiii) $\int \sec^2 x\,dx = \tan x.$

(xiv) $\int \operatorname{cosec}^2 x\,dx = -\cot x.$

(xv) $\int \sec x \tan x \, dx = \sec x.$

(xvi) $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x.$

(xvii) $\int \sinh x \, dx = \cosh x.$

(xviii) $\int \cosh x \, dx = \sinh x.$

(xix) $\int \tanh x \, dx = \log |(\cosh x)|.$

(xx) $\int \coth x \, dx = \log |(\sinh x)|.$

(xxi) $\int \operatorname{cosech} x \, dx = \log |(\tanh \tfrac{1}{2} x)|$

(xxii) $\int \operatorname{sech} x \, dx = 2 \tan^{-1}(e^x).$

(xxiii) $\int \operatorname{sech}^2 x \, dx = \tanh x.$

(xxiv) $\int \operatorname{cosech}^2 x \, dx = -\coth x.$

III. Standard Integrals.

(i) $\int \frac{f'(x)}{f(x)} dx = \log |f(x)|.$

(ii) $\int \tan x \, dx = \log |\sec x|.$

(iii) $\int \cot x \, dx = \log |\sin x|.$

(iv) $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} \; (a \neq 0).$

(v) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x - a}{x + a} \right| \quad (|x| > |a|).$

(vi) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| \quad (|x| < |a|).$

(vii) $\int \frac{dx}{\sqrt{(x^2 + a^2)}} = \log|x + \sqrt{x^2 + a^2}| = \sinh^{-1} \frac{x}{a}$

(viii) $\int \frac{dx}{\sqrt{(x^2 - a^2)}} = \log|x + \sqrt{x^2 - a^2}| = \cosh^{-1} \frac{x}{a}$.

(ix) $\int \frac{dx}{\sqrt{(a^2 - x^2)}} = \sin^{-1} \frac{x}{a} \left(|x| < |a|\right)$

(x) $\int \frac{dx}{x\sqrt{(x^2 - a^2)}} = \frac{1}{a} \sec^{-1} \left(\frac{x}{a}\right)$.

(xi) $\int \frac{dx}{x\sqrt{(x^2 - 1)}} = \sec^{-1} x.$

(xii) $\int (uv)\, dx = u \int v\, dx - \int \left(\frac{du}{dx} \cdot \int v\, dx \right) dx.$

Integral of the product of two functions
= 1st function × integral of 2nd
− integral of [Differential coefficient of 1st × integral of 2nd].

(xiii) $\int e^{ax} \cos bx\, dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}$

$$= \frac{e^{ax}}{\sqrt{(a^2 + b^2)}} \cos\left(bx - \tan^{-1} \frac{b}{a}\right).$$

(xiv) $\int e^{ax} \sin bx\, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}$

$$= \frac{e^{ax}}{\sqrt{(a^2 + b^2)}} \sin\left(bx - \tan^{-1} \frac{b}{a}\right).$$

(xv) $\int \sqrt{x^2 + a^2}\, dx = \frac{x\sqrt{x^2 + a^2}}{2} + \frac{a^2}{2} \log|(x + \sqrt{x^2 + a^2})|$

$$= \frac{x\sqrt{x^2 + a^2}}{2} + \frac{a^2}{2} \sinh^{-1} \frac{x}{a}.$$

(xvi) $\int \sqrt{x^2 - a^2}\, dx = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \log|(x + \sqrt{x^2 - a^2})|$

$= \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \cosh^{-1} \frac{x}{a}$.

(xvii) $\int \sqrt{a^2 - x^2}\, dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$

(xviii) $\int \operatorname{cosec} x\, dx = \log|\tan \tfrac{1}{2}x|$

$= \log|\operatorname{cosec} x - \cot x|$.

(xix) $\int \sec x\, dx = \log|\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}x)|$

$= \log|\sec x + \tan x|$.

IV. Definite Integrals.

(A) *Definition.*

(i) $\underset{h \to 0}{Lt}\, h \sum_{r=0}^{n-1} f(a + rh)$ or, $\underset{h \to 0}{Lt}\, h \sum_{r=0}^{n} f(a + rh)$

$= \int_a^b f(x)\, dx$, where $nh = b - a$.

(ii) $\underset{n \to \infty}{Lt} \frac{b-a}{n} \sum f\left(a + \frac{r}{n}(b-a)\right) = \int_a^b f(x)\, dx$

(iii) $\underset{n \to \infty}{Lt} \frac{1}{n} \sum f\left(\frac{r}{n}\right) = \int_0^1 f(x)\, dx$.

(B) *Properties.*

(i) $\int_a^b f(x)\, dx = F(b) - F(a)$, if $\frac{d}{dx} F(x) = f(x)$.

(ii) $\int_a^b f(x)\, dx = \int_a^b f(z)\, dz$.

(iii) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$

(iv) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \ (a < c < b).$

(v) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$

(vi) $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx,$ if $f(a+x) = f(x).$

(vii) $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx,$ if $f(2a-x) = f(x),$

$= 0,$ if $f(2a-x) = -f(x).$

(*a*) $\int_0^{\pi} \sin x\,dx = 2\int_0^{\frac{1}{2}\pi} \sin x\,dx.$

(*b*) $\int_0^{\pi} \cos x\,dx = 0$

(viii) $\int_{-a}^{+a} f(x)\,dx = 2\int_0^a f(x)\,dx,$ if $f(-x) = f(x),$

$= 0,$ if $f(-x) = -f(x).$

(ix) $\int_0^{\frac{1}{2}\pi} \sin^n x\,dx = \int_0^{\frac{1}{2}\pi} \cos^n x\,dx.$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\ \cdots\ \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$\text{or } \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\ \cdots\ \frac{4}{5}\cdot\frac{2}{3}\cdot 1$$

according as n *is an even or odd integer.*

V. Beta and Gamma Functions : Infinite integrals.

(i) $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^\infty \frac{x^{m-1}\,dx}{(1+x)^{m+n}}$

$$= \int_0^\infty \frac{x^{n-1}dx}{(1+x)^{m+n}} \quad (m > 0, n > 0).$$

(ii) $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx \quad (n > 0)$.

(iii) $B(m, n) = B(n, m) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

(iv) $\Gamma(n+1) = n\Gamma(n), \Gamma(n+1) = n!$, if n is a *positive* integer.

(v) $\Gamma(1) = 1, \Gamma(\frac{1}{2}) = \sqrt{\pi}, \Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin m\pi}$
$(0 < m < 1)$.

(vi) $\int_0^{\pi/2} \sin^p\theta \cos^q\theta\,d\theta$

$$= \frac{1}{2}\cdot\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)} \quad \begin{bmatrix} p > -1 \\ q > -1 \end{bmatrix}$$

$$= \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right).$$

(vii) $\int_0^{\pi/2} \sin^p\theta\,d\theta = \int_0^{\pi/2} \cos^p\theta\,d\theta = \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)}$
$(p > -1)$.

(viii) $\int_0^\infty e^{-kx} x^{n-1}\,dx = \frac{\Gamma(n)}{k^n} \quad (k < 0)$.

(ix) $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

(x) $\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$ $(a > 0)$.

(xi) $\int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$ $(a > 0)$.

(xii) $\int_0^\infty \frac{\sin bx}{x}\,dx = \frac{1}{2}\pi$ or $-\frac{1}{2}\pi$ according as $b >$ or < 0.

(xiii) $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

VI. Areas of Plane curves.

1. *Cartesian co-ordinates* :

(*a*) If the curve be $y = f(x)$, area $= \int y\,dx$.

(*b*) If the curve be $x = f(y)$, area $= \int x\,dy$.

(*c*) If the curve be $x = \phi(t)$, $y = \psi(t)$ (*Parametric form*), the formula for the area is either (*a*) or *b*.

2. *Polar co-ordinates* :

If the curve be $r = f(\theta)$, area $= \frac{1}{2}\int r^2\,d\theta$.

3. *Important areas* :

(*a*) Area of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is πab.

(*b*) Area of the circle $x^2 + y^2 = a^2$ is πa^2.

VII. Lengths of Plane curves.

1. *Cartesian co-ordinates* :

Arc length $= \int ds = \int \frac{ds}{dx}\,dx$

$$= \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

if the equation of the curve be $y = f(x)$;

$$= \int \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dx,$$

if the equation of the curve be $x = f(y)$.

$$\text{Arc length} = \int ds = \int \frac{ds}{dy}\, dy.$$

$$\text{Arc length} = \int ds = \int \frac{ds}{dt}\, dt$$

$$= \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt,$$

if the equation of the curve be $x = \phi(t), y = \psi(t)$.

2. *Polar co-ordinates* :

(*a*) $\text{Arc length} = \int ds = \int \frac{ds}{d\theta}\, d\theta$

$$= \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta,$$

if the equation of the curve be $r = f(\theta)$.

(*a*) $\text{Arc length} = \int ds = \int \frac{ds}{dr}\, dr$

$$= \int \sqrt{r^2 + \left(\frac{d\theta}{dr}\right)^2}\, dr,$$

if the equation of the curve be $r = f(\theta), \frac{d\theta}{dr}$, *i.e.*, $1 \Big/ \frac{dr}{d\theta}$ being expressed in terms of r.

3. *Important lengths* :

Perimeter of the circle $x^2 + y^2 = a^2$ is $2\pi a$.

Perimeter of the cycloid, $x = a(\theta + \sin\theta)$,
$y = a(1 - \cos\theta)$ is $8a$.

For the cycloid, $x = a(\theta + \sin\theta)$,
$y = a(1 - \cos\theta)$, $s^2 = 8ay$.

Perimeter of the cardioide $r = a(1 - \cos\theta)$ is $8a$.

VIII. Volumes of the solids of revolution

1. *Cartesian co-ordinates :*

(*a*) Volume $V = \pi \int y^2\, dx$,
if the axis of revolution is the x-axis.

(*b*) Volume $V = \pi \int x^2\, dy$,
if the axis of revolution is the y-axis.

2. *Polar co-ordinates :*

Put $x = r\cos\theta$, $y = r\sin\theta$ in (*a*) and (*b*).

3. *Important volumes :*

(*a*) Volume of the solid generated by the revolution of the circle $x^2 + y^2 = a^2$, *i.e.*, the volume of the sphere of radius $a = \frac{4}{3}\pi a^3$.

(*b*) Volume of the ellipsoid formed by the revolution of the ellipse $x^2/a^2 + y^2/b^2 = 1$,

(i) round the major axis $= \frac{4}{3}\pi ab^2$;
(ii) round the minor axis $= \frac{4}{3}\pi a^2 b$;

(*c*) Volume of the solid generated by the revolution of the straight line $y = x\tan\alpha$, about the x-axis,

i.e., volume of a right circular cone of height h, radius a and semi-vertical angle $\alpha = \frac{1}{3}\pi h^3 \tan^2\alpha$
$= \frac{1}{3}\pi a^2 h$.

(*d*) Volume of a right circular cylinder of height h, and base of radius a is $= \pi a^2 h$.

IX. Surface areas of the solids of revolution.

1. *Cartesian co-ordinates* :

(*a*) Surface area $S = 2\pi \int y\, ds = 2\pi \int y \frac{ds}{dx}\, dx$

$$= 2\pi \int y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

if the axis of revolution is the x-axis.

(*b*) Surface area $S = 2\pi \int x\, ds = 2\pi \int x \frac{ds}{dy}\, dy$

$$= 2\pi \int x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy,$$

if the axis of revolution is the y-axis.

2. *Polar co-ordinates* :

Put $x = r\cos\theta, y = r\sin\theta$ in (*a*) and (*b*).

3. *Important surface area* :

Surface area of a sphere of radius a is $4\pi a^2$.

X. Symbolical Operators.

(i) $\frac{1}{f(D)} e^{ax} = \frac{1}{f(x)} e^{ax}$, if $f(a) \neq 0$.

(ii) $\frac{1}{f(D)} e^{ax}\ V = e^{ax} \frac{1}{f(D + a)} V,$

where V is any function of x.

(iii) $\frac{1}{\phi(D^2)} \sin(ax + b) = \frac{1}{\phi(-a^2)} \sin(ax + b),$

if $\phi(-a^2) \neq 0$.

(iv) $$\frac{1}{\phi(D)} \cos(ax+b) = \frac{1}{\phi(-a^2)} \cos(ax+b),$$

if $\phi(-a^2) \neq 0$.

(v) $$\frac{D}{f(D)}\, x \quad \left\{ x - \frac{1}{f(D)} f'(D) \right\} \frac{1}{f(D)} V,$$

where V is any function of x.