INTEGRAL CALCULUS

CHAPTER I

DEFINITION AND FUNDAMENTAL PROPERTIES

1.1. Two view-points of Integration.

There are two distinct view-points from which the process of integration can be considered. We may consider *integration as the inverse of differentiation* and make this as our starting point; or else, we may start with defining *integration as a certain summation* and then proceed to show that the result is identical with the reversal of a differentiation.

The establishment of an identity of the two view-points is referred to as the *Fundamental Theorem* of Integral Calculus. We shall consider both the points of view, starting first of all with the former, reserving the consideration of the latter for a subsequent chapter.

1.2. Integration, the inverse process of differentiation.

If $f(x)$ be a given function of x and if another function $F(x)$ be obtained such that its differential coefficient with respect to x is equal to $f(x)$, then $F(x)$ is defined as an *integral*, or more properly *an indefinite integral of $f(x)$ with respect to x.*

The process of finding an integral of a function of x is called *Integration* and the operation is indicated by writing the *integral sign** $\int$ before the given function and dx after the given function, the differential dx indicating that x is *the variable of integration*. The function to be integrated, viz., $f(x)$ is called the *Integrand*.

Symbolically, if $\frac{d}{dx} F(x) = f(x)$,

then $$\int f(x)\,dx = F(x),$$

* Historically this sign is elongated S, the initial letter of the word 'sum', since integration was originally studied as a process of summation. (See *Chapter VI.*)

where $\int f(x)\,dx$ is called an indefinite integral of $f(x)$ with respect to x.

Thus, considered as symbols of operation,

$$\frac{d}{dx}(\ ) \text{ and } \int (\ )\,dx \text{ are inverse to each other.}$$

Alternatively, using differentials, the above result may be written as follows :

Since $\quad d(F(x)) = f(x)\,dx,$

$\therefore \quad \int f(x)\,dx = F(x).$

[Thus, since $d(\sin x) = \cos x\,dx$, $\therefore \int \cos x\,dx = \sin x$.]

Again, if $f(x)$ be a given function, $F(x)$ an integral of $f(x)$, and $x = a$ and $x = b$ be two given values of x, the change in the value of the integral function $F(x)$ as x changes from a to b, i.e., the quantity $F(b) - F(a)$ is defined as the **definite integral*** of $f(x)$ between the 'limits' a and b, which is denoted by the symbol

$$\int_a^b f(x)\,dx.$$

In other words, if $\frac{d}{dx}F(x) = f(x)$, for all values of x between a and b,

then $F(b) - F(a) = \int_a^b f(x)\,dx$,

which is called *the definite integral of $f(x)$ from a to b*, and 'b' is called the *upper limit* and 'a' the *lower limit* of the definite integral.

Cor. $\int f'(x)\,dx = f(x)$ and $\int_a^b f'(x)\,dx = f(b) - f(a)$.

* Provided $f(x)$ satisfies certain general conditions which will be discussed later. (*See Chapter VI*).

1.3. Constant of integration.

It may be noted that if $\frac{d}{dx} F(x) = f(x)$, then we also have $\frac{d}{dx}\{F(x) + C\} = f(x)$, where C is an arbitrary constant. Thus, if $\int f(x)\,dx = F(x)$, a general value of the indefinite integral $\int f(x)\,dx = F(x) + C$.

In other words, in finding the indefinite integral of a function $f(x)$, *an arbitrary constant* is to be added to the result to make it general. This is the reason why the integral is referred to as an indefinite integral. The arbitrary constant is usually referred to as the *constant of integration*.

It is easily seen, however, that in evaluating a definite integral this constant of integration cancels out and its value is thus definite.

Again, we may see that *two functions having the same derivative differ only by a constant*. For, if $\frac{d}{dx}\phi(x) = \frac{d}{dx}\psi(x)$, and if $y = \phi(x) - \psi(x)$, then we get $\frac{dy}{dx} = 0$ always, and thus the rate of change of y with respect to x is zero everywhere and hence y is constant*. Thus, it is possible to get *the indefinite integral of the same function in different forms* by different processes, but *ultimately these forms can at most differ from each other by constant quantities only*. Hence, an arbitrary constant added to the indefinite integral of a given function obtained by any process makes the result perfectly general.

In the following pages, we shall first of all deal with indefinite integrals. For the sake of convenience *the arbitrary constant of integration has generally been omitted* but it is always understood to be present in every case, *and should be supplied by the students in the result*.

It should also be noted that in case an indefinite integral consists of a sum or difference of two or more integrals, the addition of one arbitrary constant for each integral is equivalent to the addition of a single arbitrary constant, denoting their algebraic sum, in the final result. For illustrations, see Illustrative Examples in Art. 1.7.

* For analytical proof see Authors' *Differential Calculus*, Art. 6.7, Ex. 1.

1.4. General laws satisfied by integrals.

(i) *The Integral of the sum or difference of any finite number of functions is equal to the sum or difference of the integrals of the functions taken separately.*

This follows immediately from the known results of the Differential Calculus. For we know that

$$\frac{d}{dx}\{f_1(x)+f_2(x)-f_3(x)\ldots\ldots\}$$
$$=f_1'(x)+f_2'(x)-f_3'(x)+\ldots$$
$$\therefore\quad \int\{f_1'(x)+f_2'(x)-f_3'(x)+\ldots\}\,dx$$
$$=f_1(x)+f_2(x)-f_3(x)+\ldots$$
$$=\int f_1'(x)\,dx+\int f_2'(x)\,dx-\int f_3'(x)\,dx+\ldots$$

[since $\frac{d}{dx}f(x)=f'(x)$, $\therefore f(x)=\int f'(x)\,dx$, etc.].

(ii) *The operation of integration is commutative with regard to a constant, i.e.,* a factor of the integrand which is constant with regard to the variable of integration can be taken outside the sign of integration.

Symbolically, $\int A\,f(x)\,dx = A\int f(x)\,dx$.

This follows immediately from the fact that

$$\frac{d}{dx}\left\{AF(x)\right\}=A\frac{d}{dx}\left\{F(x)\right\}=Af(x),\text{ say,}$$

so that $\int Af(x)\,dx = AF(x) = A\int f(x)\,dx$,

since, by our supposition, $\frac{d}{dx}F(x)=f(x)$.

(iii) Combining the above two results, we can write

$$\int\{Af_1(x)\pm Bf_2(x)\pm Cf_3(x)+\ldots\}\,dx$$
$$=A\int f_1(x)\,dx\pm B\int f_2(x)\,dx\pm C\int f_3(x)\,dx+\ldots$$

1.5. Fundamental Integrals.

A slight acquaintance with the Differential Calculus will at once suggest the integrals in many elementary cases. As the first step towards the facility in integration, the student must be thoroughly familiar with the following fundamental integrals :

FUNDAMENTAL INTEGRALS

(i) $\int x^n \, dx = \frac{x^{n+1}}{n+1} \; (n \neq -1).$

Cor. $\int \frac{dx}{x^n} = -\frac{1}{(n-1)x^{n-1}} \; (n \neq -1).$

$\int \frac{dx}{\sqrt{x}} = 2\sqrt{x}.$

$\int dx = x.$

(ii) $\int \frac{dx}{x} = \log|x|.$

(iii) $\int e^{mx} \, dx = \frac{e^{mx}}{m}.$

Cor. $\int e^x \, dx = e^x.$

$\int a^x \, dx = \frac{a^x}{\log_e a} \; (a > 0, \; a \neq 1).$

(iv) $\int \sin mx \, dx = -\frac{\cos mx}{m}.$

Cor. $\int \sin x \, dx = -\cos x.$

(v) $\int \cos mx \, dx = \frac{\sin mx}{m}.$

Cor. $\int \cos x \, dx = \sin x.$

(vi) $\int \sec^2 mx \, dx = \frac{\tan mx}{m}.$

Cor. $\int \sec^2 x \, dx = \tan x.$

(vii) $\int \operatorname{cosec}^2 mx \, dx = -\frac{\cot mx}{m}.$

Cor. $\int \operatorname{cosec}^2 x \, dx = -\cot x.$

(viii) $\int \sec mx \tan mx \, dx = \frac{\sec mx}{m}.$

Cor. $\int \sec x \tan x \, dx = \sec x.$

(ix) $\int \operatorname{cosec} mx \cot mx \, dx = -\frac{\operatorname{cosec} mx}{m}.$

Cor. $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x.$

(x) $\int \sinh mx \, dx = \frac{\cosh mx}{m}.$

Cor. $\int \sinh x \, dx = \cosh x.$

(xi) $\int \cosh mx \, dx = \frac{\sinh mx}{m}.$

Cor. $\int \text{Cosh } x \, dx = \sinh x.$

The results can be easily verified by differentiating the right side in each case.

Other standard results of integration which are also very useful for application will be found in the subsequent chapters.

Note 1. Since the integral (i) is frequently used, for the sake of convenience, we give here a concise verbal statement of the result, *viz*., "*Increase the index by one and divide it by the increased index*".

Note 2. Since log x is real when $x > 0$ and $\frac{d}{dx}(\log x) = \frac{1}{x}$,

so $\int \frac{1}{x} dx = \log x$ is defined for $x > 0$. When $x < 0$, *i.e.*, $-x > 0$, $\frac{d}{dx}\log(-x) = \frac{-1}{-x} = \frac{1}{x}$. Therefore when $x < 0$, $\int \frac{1}{x} dx = \log(-x)$.

Hence, both these results will be included if we write $\int \frac{1}{x} dx = \log|x|$

In the formula and examples where integrals of this type occurs, *i.e.*, where the value of an integral involves the logarithm of a function and the function may become negative for some values of the variable of the function, the absolute value sign | | enclosing the function should be given, but it has generally been omitted, though it is always understood to be present and it should be supplied by the students.

Note 3. Different algebraical symbols $a, b, m, n,$ etc. occurring in integrands are generally supposed to be different unless otherwise stated.

Note 4. In the above integrals (iii), (iv), (v), (x), (xi) it is tacitly assumed that m is a non-zero constant.

1.6. Standard methods of integration.

The different modes of integration all aim at reducing a given integral to one of the Fundamental or known integrals. As a matter of fact, there are two principal processes :

(i) *The method of substitution, i.e.,* a change of the independent variable.

(ii) *Integration by parts.*

In some cases, when the integrand is a rational fraction it may be broken into *partial fractions* by the rules of Algebra, and then each part may be integrated by one of the above methods. (See *Chapter V*)

In some cases of irrational functions, the method of *Integration by rationalization* is adopted, which is a special case of (i) above.

In some cases, integration by the method of *Successive Reduction* is resorted to, which really falls under case (ii).(*See Chapter IX*).

It may be noted that the classes of integrals which are reducible to one or other of the fundamental forms by the above processes are very limited, and that the large majority of the expressions, under proper restrictions, can only be integrated by the aid of *infinite series*, and in some cases when the integrand involves expressions under a radical sign containing powers of x beyond the second, the investigation of such integrals has necessitated the introduction of higher classes of transcendental function such as elliptic functions, etc.

In fact, integration is, on the whole, a more difficult operation than differentiation. The Differential Calculus gives general rules for differentiation, but Integral Calculus gives no such corresponding general rules for performing the inverse operation. Integration is essentially a tentative process. In fact, so simple an integral in appearance as

$$\int \sqrt{x} \cos x \, dx, \text{ or } \int \frac{\sin x}{x} \, dx$$

can not be worked out; that is, there is no *elementary function* whose derivative is $\sqrt{x} \cos x$, or $(\sin x)/x$, though the integrals exist. There is quite a large number of integrals of these types.

1.7. Illustrative Examples.

Ex. 1. *Integrate* $\int \sin^2 x \, dx$.

$$I = \int \tfrac{1}{2}(1 - \cos 2x) \, dx = \int \tfrac{1}{2} \, dx - \int \tfrac{1}{2} \cos 2x \, dx$$
$$= \tfrac{1}{2}\int dx - \tfrac{1}{2} \int \cos 2x \, dx = \tfrac{1}{2} x - \tfrac{1}{2} \frac{\sin 2x}{2} + C$$
$$= \tfrac{1}{2} x - \tfrac{1}{4} \sin 2x + C.$$

Ex. 2. *Integrate* $\int \tan^2 x \, dx$.

$$I = (\sec^2 x - 1) \, dx = \int \sec^2 x \, dx - \int dx$$
$$= \tan x - x + C.$$

Ex. 3. *Integrate* $\int \frac{5(x-3)^2}{x\sqrt{x}} \, dx$.

$$I = \int \frac{5x^2 - 30x + 45}{x\sqrt{x}} \, dx = 5 \int x^{\frac{1}{2}} \, dx - 30 \int \frac{dx}{\sqrt{x}} + 45 \int x^{-\frac{3}{2}} \, dx$$

$$= 5.\tfrac{2}{3} x^{\frac{3}{2}} - 30.2\sqrt{x} + 45(-2x^{-\frac{1}{2}}) + C$$

$$= \tfrac{10}{3} x^{\frac{3}{2}} - 60\sqrt{x} - 90 x^{-\frac{1}{2}} + C.$$

Ex. 4. *Integrate* $\int \sin 3x \cos 2x \, dx$.

$$\sin 3x \cos 2x = \tfrac{1}{2} . 2 \sin 3x \cos 2x = \tfrac{1}{2} (\sin 5x + \sin x).$$

$$\therefore \quad I = \tfrac{1}{2}\left[\int \sin 5x\, dx + \int \sin x\, dx\right]$$
$$= \tfrac{1}{2}\left[-\tfrac{1}{5}\cos 5x - \cos x\right] + C.$$

Note. Henceforth the arbitrary constant of integration will be omitted in the Illustrative examples, as also in the answers to the set of examples.

EXAMPLES I

Integrate the following :-

1. (i) $\displaystyle\int \frac{(1+x)^3}{x}\, dx.$ (ii) $\displaystyle\int \sqrt{x}\left(x^5 + \frac{3}{x}\right) dx.$

2. (i) $\displaystyle\int \cos^2 x\, dx.$ (ii) $\displaystyle\int \frac{\tan x}{\cot x}\, dx.$

(iii) $\displaystyle\int \frac{1-\tan^2 x}{1+\tan^2 x}\, dx.$ (iv) $\displaystyle\int \frac{1+\tan^2 x}{1+\cot^2 x}\, dx.$

3. $\displaystyle\int \sec x\,(\sec x - \tan x)\, dx.$

4. (i) $\displaystyle\int \cos^2 ax\, dx.$ (ii) $\displaystyle\int \cot^2 x\, dx.$

5. (i) $\displaystyle\int \frac{2e^{2x} + 3e^{4x} + 4}{e^{3x}}\, dx.$ (ii) $\displaystyle\int \frac{e^{3x} + e^{5x}}{e^{x} + e^{-x}}\, dx.$

(iii) $\displaystyle\int \frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}\, dx.$

6. $\displaystyle\int (e^{a\log x} + e^{x\log a})\, dx.$

7. (i) $\displaystyle\int \left(x\sqrt{x} - \tfrac{1}{3}\sqrt{x} + \frac{11}{\sqrt{x}}\right) dx.$ (ii) $\displaystyle\int \frac{1-x^8}{1-x}\, dx.$

8. (i) $\displaystyle\int \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^3 dx$ (ii) $\displaystyle\int (x+2)(x+3)^2\, dx.$

9. (i) $\displaystyle\int \frac{a^x + a^{2x} + a^{3x}}{a^{4x}}\, dx.$ (ii) $\displaystyle\int \frac{8^{1+x} + 4^{1-x}}{2^x}\, dx.$

10. (i) $\displaystyle\int \frac{(1-2x^2)^2}{x\sqrt[3]{x}}\, dx.$ (ii) $\displaystyle\int \frac{a\sin^3 x + b\cos^3 x}{\sin^2 x\cos^2 x}\, dx.$

(iii) $\int \frac{\cos^3 x + \sin^3 x}{\cos x + \sin x}\, dx$.

11. (i) $\int \frac{\sin x + \operatorname{cosec} x}{\tan x}\, dx$. (ii) $\int \cos x^{\circ}\, dx$.

12. $\int \frac{\operatorname{cosec} x + \tan^2 x + \sin^2 x}{\sin x}\, dx$.

13. (i) $\int \frac{x^3 - 4x^2 + 5x - 2}{x^2 - 2x + 1}\, dx$. (ii) $\int \frac{x^3 - 6x + 9}{x + 3}\, dx$.

14. (i) $\int \frac{\sin x + \cos x}{\sqrt{(1 + \sin 2x)}}\, dx$. (ii) $\int \frac{\cos x - \cos 2x}{1 - \cos x}\, dx$

(iii) $\int \frac{\sec 2x - 1}{\sec 2x + 1}\, dx$. (iv) $\int \frac{dx}{\cosh x + \sinh x}$.

15. (i) $\int \sqrt{(1 + \sin x)}\, dx$. (ii) $\int \sqrt{(1 - \sin x\, dx)}$.

[$1 \pm \sin x = (\sin \frac{1}{2}x \pm \cos \frac{1}{2}x)^2$]

16. $\int \frac{\cos x - \sin x}{\cos x + \sin x}(2 + 2\sin 2x)\, dx$.

17. (i) $\int \sqrt{(1 + \cos x\, dx)}$. (ii) $\int \sqrt{(1 - \cos x\, dx)}$.

(iii) $\int (3\sin x\cos^2 x - \sin^3 x)\, dx$.

18. (i) $\int \frac{dx}{1 + \sin x}$. (ii) $\int \frac{dx}{1 + \cos x}$.

[*(i) Multiply numerator and denominator by* $1 - \sin x$.]

19. (i) $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}\, dx$. (ii) $\int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x}\, dx$.

[*(ii) Multiply numerator and denominator by* $\sin 3x$.]

20. (i) $\int \frac{dx}{\sin^2 x \cos^2 x}$. [*H. S.'78, '85, J. E. '81*]

(ii) $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} \, dx$.

[(i) *Put* $\sin^2 x + \cos^2 x$ in the numerator .]

(iii) $\int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} \, dx$. (iv) $\int \frac{\cos^4 x - \sin^4 x}{\sqrt{(1 + \cos 4x)}} \, dx$.

21. $\int \frac{\cos x}{\sin^2 x} (1 - 3\cos^3 x) \, dx$.

22. (i) $\int \cos^3 x \, dx$. (ii) $\int \sin^4 x \, dx$. [*H. S. '85*]

[(i) *Write* $\cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x)$.
(*For another method see Art.* 4.3.)]

23. (i) $\int \sin mx \sin nx \, dx$. (ii) $\int \cos 2x \cos 3x \, dx$.

24. (i) $\int \sin^2 x \cos^2 x \, dx$. (ii) $\int \sin^2 x \cos 2x \, dx$.

25. $\int \sin x \sin 2x \sin 3x \, dx$.

ANSWERS

1. (i) $\log x + 3x + \frac{3}{2}x^2 + \frac{1}{3}x^3$ (ii) $\frac{2}{13}x^{13/2} + 6x^{1/2}$.
2. (i) $\frac{1}{2}x + \frac{1}{4}\sin 2x$. (ii) $\tan x - x$. (iii) $\frac{1}{2}\sin 2x$. (iv) $\tan x - x$.
3. $\tan x - \sec x$ 4. (i) $\frac{1}{2}x + \frac{\sin 2ax}{4a}$ (ii) $-\cot x - x$.
5. (i) $-2e^{-x} + 3e^x - \frac{4}{3}e^{-x}$. (ii) $\frac{1}{4}e^{4x}$. (iii) $\frac{1}{3}x^3$.
6. $\frac{x^{a+1}}{a+1} + \frac{a^x}{\log a}$. 7. (i) $\frac{2}{5}x^{5/2} - \frac{2}{8}x^{3/2} + 22x^{1/2}$.

(ii) $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 + \frac{1}{6}x^6 + \frac{1}{7}x^7 + \frac{1}{8}x^8$.

8. (i) $a^2 x - \frac{9}{5}a^{4/3}x^{5/3} + \frac{9}{7}a^{2/3}x^{7/3} - \frac{1}{3}x^3$.

(ii) $\frac{1}{12}x(3x^3 + 32x^2 + 126x + 216)$.

9. (i) $-\dfrac{1}{\log a}\left[\dfrac{a^{-3x}}{3}+\dfrac{a^{-2x}}{2}+\dfrac{a^{-x}}{1}\right]$.

(ii) $\dfrac{4}{\log 2}\left[2^{2x}-\frac{1}{3}.2^{-3x}\right]$.

10. (i) $-3x^{-1/3}-\frac{12}{5}x^{5/3}+\frac{12}{11}x^{-11/3}$. (ii) $a\sec x-b\operatorname{cosec}x$.

(iii) $x+\frac{1}{4}\cos 2x$. 11. (i) $\sin x-\operatorname{cosec}x$. (ii) $\frac{180}{\pi}\sin x^{\circ}$.

12. $-\cot x+\sec x-\cos x$. 13. (i) $\frac{1}{2}x^2-2x$. (ii) $\frac{1}{3}x^3-\frac{3}{2}x^2+3x$

14. (i) x. (ii) $x+2\sin x$. (iii) $\tan x-x$. (iv) $\sinh x-\cosh x$.

15. (i) $2(\sin\frac{1}{2}x-\cos\frac{1}{2}x)$ or $(2\sqrt{1-\sin x})$.

(ii) $2\sqrt{1+\sin x}$. 16. $\sin 2x$. 17. (i) $2\sqrt{2}\sin\frac{1}{2}x$.

(ii) $-2\sqrt{2}\cos\frac{1}{2}x$. (iii) $-\frac{1}{3}\cos 3x$.

18. (i) $\tan x-\sec x$. (ii) $-\cot x+\operatorname{cosec}x$.

19. (i) $2(\sin x+x\cos\alpha)$.

(ii) $-(\sin x+\frac{1}{2}\sin 2x)$. 20. (i) $\tan x-\cot x$.

(ii) $\tan x-\cot x-3x$. (iii) $-\frac{1}{2}\sin 2x$. (iv) $\frac{1}{\sqrt{2}}x$.

21. $-\operatorname{cosec}x+3\cot x+\frac{9}{2}x+\frac{3}{4}\sin 2x$.

22. (i) $\frac{1}{12}\sin 3x+\frac{3}{4}\sin x$. (ii) $\frac{1}{32}\sin 4x-\frac{1}{4}\sin 2x+\frac{3}{8}x$.

23. (i) $\dfrac{\sin(m-n)x}{2(m-n)}-\dfrac{\sin(m+n)x}{2(m+n)}$.

(ii) $\frac{1}{10}\sin 5x+\frac{1}{2}\sin x$. 24. (i) $\frac{1}{8}x-\frac{1}{32}\sin 4x$.

(ii) $\frac{1}{4}(\sin 2x-x-\frac{1}{4}\sin 4x)$.

25. $-\frac{1}{8}\cos 2x-\frac{1}{16}\cos 4x+\frac{1}{24}\cos 6x$.

CHAPTER II

METHOD OF SUBSTITUTION

2.1. Change of variable.

Let $I = \int f(x)\, dx$, and let $x = \phi(z)$.

Then, by definition, $\frac{dI}{dx} = f(x)$ and $\frac{dx}{dz} = \phi'(z)$.

Now, $\frac{dI}{dz} = \frac{dI}{dx}\frac{dx}{dz} = f(x).\phi'(z) = f\{\phi(z)\}\,\phi'(z)$.

$\therefore$ by definition, $I = \int f\{\phi(z)\}\,\phi'(z)\, dz$.

Note 1. Thus, if in the integral $\int f(x)\,dx$ we put $x = \phi(z)$ we are to replace x by $\phi(z)$ in the expression $f(x)$ and also we are to replace dx by $\phi'(z)\, dz$ and then we have to proceed with the integration with z as the new variable. After evaluating the integral we are to replace z by the equivalent expression in x.

Note that though from $x = \phi(z)$ we can write $\frac{dx}{dz} = \phi'(z)$ in making our substitution in the given integral, we generally use it in the differential form $dx = \phi'(z)\, dz$. It really means that when x and z are connected by the relation $x = \phi(z)$, I being the funetion of x whose differential coefficient with respect to x is $f(x)$, it is, when expressed in terms of z, identical with the function whose differential coefficient with respect to z is $f\{\phi(z)\}\,\phi'(z)$ which later, by a proper choice of $\phi(z)$, may possibly be of a standard form, and therefore easy to find out.

Note 2. Sometimes it is found convenient to make the substitution in the form $\psi(x) = z$ where corresponding differential form will be $\psi'(x)\,dx = dz$; by means of these two relations, $f(x)\,dx$ is transformed into the form $F(z)\,dz$.

2.2. Illustrative Examples.

Ex. 1. *Integrate* $\int (a + bx)^n\, dx$.

Put $a + bx = z$. $\quad\therefore b\,dx = dz$. $\quad\therefore dx = (1/b)\,dz$.

$\therefore I = \int z^n \frac{1}{b} dz = \frac{1}{b}\int z^n\, dz = \frac{1}{b}\,\frac{z^{n+1}}{n+1} = \frac{1}{(n+1)b}\,.\,(a + bx)^{n+1}$.

Ex. 2. *Integrate* $\int \frac{dx}{x\sqrt{(x^2-a^2)}}$.

Put $x = a \sec\theta$. $\therefore\ dx = a \sec\theta \tan\theta \, d\theta$.

$$\therefore\ I = \int \frac{a \sec\theta \tan\theta \, d\theta}{a \sec\theta . \, a \tan\theta} = \frac{1}{a}\int d\theta = \frac{1}{a}\theta = \frac{1}{a} \sec^{-1}\frac{x}{a}.$$

Cor. $$\int \frac{dx}{x\sqrt{(x^2-1)}} = \sec^{-1} x.$$

Ex. 3. *Integrate* $\frac{\sin^{-1} x}{\sqrt{(1-x^2)}}dx$.

Put $\sin^{-1} x = z$. $\therefore\ \frac{1}{\sqrt{(1-x^2)}}dx = dz$.

$$\therefore\ I = \int z\, dz = \tfrac{1}{2} z^2 = \tfrac{1}{2}(\sin^{-1} x)^2.$$

Ex. 4. *Show that*

(i) $\int \tan x \, dx = \log|\sec x|$.

(ii) $\int \cot x \, dx = \log|\sin x|$.

(i) Put $\cos x = z$; then $-\sin x\, dx = dz$.

$$\therefore\ I = \int \frac{\sin x}{\cos x}dx = -\int \frac{dz}{z} = -\log z$$

$$= -\log\cos x = \log\frac{1}{\cos x} = \log|\sec x|.$$

(ii) Similarly, by substituting $\sin x = z$, this result follows.

Otherwise :

(i) $\int \tan x \, dx = \int \frac{\sec x \tan x}{\sec x}dx = \log|\sec x|$.

(ii) $\int \cot x \, dx = \int \frac{\cos x}{\sin x}dx = \log|\sin x|$ [*See Ex. 5 below*]

Ex. 5. *Show that*

$$\int \frac{f'(x)}{f(x)}\,dx = \log|\,f(x)\,|\,.$$

Put $f(x) = z$ $\quad\therefore\quad f'(x)\,dx = dz$.

$$\therefore \quad I = \int \frac{dz}{z} = \log|\,z\,| = \log|f(x)|\,.$$

Hence,

If the integrand be a fraction such that its numerator is the differenial coefficient of the denominator, then the integral is equal to log | (denominator) | .

$$\text{Thus}, \int \frac{\cos x - \sin x}{\sin x + \cos x}\,dx = \log|\,(\sin x + \cos x)\,|\,.$$

$$\int \frac{2ax + b}{ax^2 + bx + c}\,dx = \log|\,(ax^2 + bx + c)\,|.$$

The principle is also illustrated in Ex. 4 above.

Ex. 6. *Integrate* $\displaystyle\int \frac{2\sin x}{5 + 3\cos x}\,dx$.

I can be written as $\displaystyle -\frac{2}{3}\int \frac{-3\sin x}{5 + 3\cos x}\,dx$.

Now, since the numerator of the integrand is the differential coefficient of the denominator,

$\therefore\ I = -\frac{2}{3}\log|\,(5 + 3\cos x)\,|\,.$

Ex. 7. *Integrate* $\displaystyle\int \frac{dx}{\sqrt{(x + a)} + \sqrt{(x + b)}}$.

Multiplying the numerator and denominator by $\sqrt{x + a} - \sqrt{x + b}$, we have

$$I = \int \frac{\sqrt{x + a} - \sqrt{x + b}}{a - b}\,dx = \frac{1}{a - b}\left[\int \sqrt{x + a}\,dx - \int \sqrt{x + b}\,dx\right].$$

Putting $x + a = z$, so that $dx = z$.

$\int \sqrt{x+a}\,dx = \int \sqrt{z}\,dz = \frac{2}{3} z^{3/2} = \frac{2}{3}(x+a)^{3/2}$.

Similarly, $\int \sqrt{x+b}\,dx = \frac{2}{3}(x+b)^{3/2}$.

$$\therefore\ I = \frac{2}{3}\,\frac{1}{a-b}\left[(x+a)^{3/2} - (x+b)^{3/2}\right].$$

Ex. 8. *Integrate* $\displaystyle\int \frac{(a+bx)^2}{(a'+b'x)^3}\,dx$.

Put $a' + b'x = z$, or, $x = \dfrac{z-a'}{b'}$. $\quad\therefore\ dx = \dfrac{1}{b'}\,dz$.

Now the given integral becomes

$$\int \frac{\left\{a + \frac{b}{b'}(z-a')\right\}^2}{z^3}\,\frac{dz}{b'} = \frac{1}{b'^3}\int \frac{(bz + ab' - a'b)^2}{z^3}\,dz$$

$$= \frac{1}{b'^3}\left[b^2 \int \frac{dz}{z} + 2b\,(ab'-a'b)\int \frac{dz}{z^2} + (ab'-a'b)^2 \int \frac{dz}{z^3}\right]$$

$$= \frac{b^2}{b'^3}\log z - \frac{2b\,(ab'-a'b)}{b'^3}\,\frac{1}{z} - \frac{(ab'-a'b)^2}{2b'^3}\,\frac{1}{z^2}$$

$$= \frac{b^2}{b'^3}\log(a'+b'x) - \frac{2b\,(ab'-a'b)}{b'^3\,(a'+b'x)} - \frac{(ab'-a'b)^2}{2b'^3(a'+b'x)^2}.$$

Note. By the same process we can integrate $\displaystyle\int \frac{(a+bx)^m}{(a'+b'x)^n}\,dx$,

where m is *a positive integer*, n being a rational number . [*Cf.* § 9.13]

Ex. 9. *Integrate* $\displaystyle\int \frac{dx}{x^3(a+bx)^2}$.

Put $a + bx = zx$, or, $\dfrac{a}{x} + b = z$. Then $-\dfrac{a}{x^2}\,dx = dz$.

The given integral then

$$= -\frac{1}{a}\int \frac{dz}{x.z^2x^2} = -\frac{1}{a}\int \frac{dz}{z^2}\left(\frac{z-b}{a}\right)^3$$

$$= -\frac{1}{a^4}\int\left(z - 3b + \frac{3b^2}{z} - \frac{\ ^3}{\ }\right)dz$$

$$= -\frac{1}{a^4}\left[\frac{z^2}{2} - 3bz + 3b^2\log z + \frac{b^3}{z}\right]$$

$$= -\frac{1}{a^4}\left[\frac{1}{2}\left(\frac{a+bx}{x}\right)^2 - 3b\left(\frac{a+bx}{x}\right) + 3b^2\log\frac{a+bx}{x} + b^3\left(\frac{x}{a+bx}\right)\right].$$

Note. By the same substitution the integral $\int \frac{dx}{x^m(a+bx)^n}$ can be obtained where m and n are positive integers, or even when they are fractions such that $m + n$ is a positive integer greater than unity. For another method see § 9.13.

EXAMPLES II(A)

Integrate the following :-

1. (i) $\int e^{\tan^{-1}x}\frac{1}{1+x^2}\,dx$. (ii) $\int e^{a\sin^{-1}x}\frac{1}{\sqrt{(1-x^2)}}\,dx$

(iii) $\int \frac{\cos(\log x)}{x}\,dx$. (iv) $\int \frac{\cos^2 x}{\sin^4 x}\,dx$.

(v) $\int \frac{\cos 4x - \cos 2x}{\sin 4x - \sin 2x}\,dx$. (vi) $\int \frac{dx}{\operatorname{cosec} 2x - \cot 2x}$.

2. (i) $\int x\sqrt{x^2+1}\,dx$. (ii) $\int x^2\sqrt{a^3+x^3}\,dx$.

3. (i) $\int \frac{1}{\sqrt{x}}\cos\sqrt{x}\,dx$. (ii) $\int \frac{dx}{1+\cos x}$.

4. (i) $\int \frac{\sqrt{(\tan x)}}{\sin x\cos x}\,dx$. (ii) $\int \sqrt{\frac{\sin x}{\cos^5 x}}\,dx$.

5. (i) $\int \frac{1+\cos x}{\sqrt[3]{(x+\sin x)}}\,dx$. (ii) $\int \frac{1+\cos x}{x+\sin x}\,dx$.

6. (i) $\int \frac{\tan(\log x)}{x}\,dx$. (ii) $\int \frac{dx}{x\log x}$.

7. (i) $\int \frac{\cos x\,dx}{\sqrt{(1+\sin x)}}$. (ii) $\int \frac{\cos x\,dx}{(a+b\sin x)^2}$.

8. (i) $\int \frac{dx}{x^2\sqrt{(1-x^2)}}$. (ii) $\int \frac{dx}{x^2\sqrt{(1+x^2)}}$.

[*Put* $x = \sin\theta$.]

(iii) $\int \frac{dx}{(1-x^2)\sqrt{(1-x^2)}}$ (iv) $\int \frac{dx}{(1+x^2)\sqrt{(1+x^2)}}$

9 (i) $\int \frac{e^x-1}{e^x+1}\,dx$. (ii) $\int \frac{dx}{e^x+1}$. [*H. S. '85*]

[*Multiply the numerator and denominator of* (i) *by* $e^{-x/2}$ *, and that of* (ii) *by* e^{-x} *.*]

10. (i) $\int \frac{\tan x}{\log\cos x}\,dx$. (ii) $\int \frac{\cot x}{\log\sin x}\,dx$.

(iii) $\int \frac{\sec x\,\mathrm{cosec}\,x}{\log\tan x}\,dx$. (iv) $\int \frac{\sec x\,dx}{\log(\sec x+\tan x)}$

11. (i) $\int \frac{\sin 2x\,dx}{a\sin^2 x+b\cos^2 x}$. (ii) $\int \frac{\tan x\,dx}{a+b\tan^2 x}$.

(iii) $\int \frac{\sin 2x\,dx}{(a^2\cos^2 x+b^2\sin^2 x)^2}$.

(iv) $\int \frac{\tan x\sec^2 x}{(a^2+b^2\tan^2 x)^2}\,dx$.

12. (i) $\int x\sin x^2\,dx$. (ii) $\int \frac{dx}{\sin x\cos x}$.

13. (i) $\int \frac{3x-1}{\sqrt{(3x^2-2x+7)}}\,dx$. (ii) $\int \frac{x\,dx}{\sqrt{(x^2-a^2)}}$.

14. (i) $\int \frac{dx}{(1+x^2)\sqrt{(\tan^{-1}x+3)}}$. (ii) $\int \frac{\sec^4 x\,dx}{\sqrt{(\tan x)}}$.

15. (i) $\int \frac{e^{2x}}{e^x+1}\,dx$. (ii) $\int \frac{dx}{(e^x-1)^2}$.

$\int \frac{dx}{\sqrt{(e^x-1)}}$. [*H. S. '87*]

16. $\int \frac{e^x(1+x)}{\cos^2(xe^x)}\,dx$. [Put $xe^x = z$.]

17. (i) $\int \frac{dx}{\sqrt{(x+1)} - \sqrt{(x-1)}}$.

(ii) $\int \frac{dx}{\sqrt{(2x+5)} + \sqrt{(2x-3)}}$.

(iii) $\int \frac{dx}{\{(x-3)+(x-4)\}\sqrt{\{(x-3)(x-4)\}}}$.

18. (i) $\int \frac{dx}{\sqrt{x} + x}$. (ii) $\int \frac{x\,dx}{(2x+1)^3}$.

[Put $\sqrt{x} = z$.]

19. (i) $\int \frac{dx}{\sqrt{x} - 1}$. (ii) $\int \frac{x}{\sqrt{x} + 1}\,dx$.

20. (i) $\int (3x+2)\sqrt{2x+1}\,dx$. (ii) $\int x\,\sqrt[3]{(x+a)}\,dx$.

21. (i) $\int \frac{1+x}{1-x}\,dx$. (ii) $\int \frac{x^6}{x-1}\,dx$.

22. (i) $\int \frac{2x+3}{3x+4}\,dx$. (ii) $\int \frac{x}{a+bx}\,dx$.

23. (i) $\int \frac{2x+1}{\sqrt{(3x+2)}}\,dx$. (ii) $\int \frac{x}{\sqrt[3]{(a+bx)}}\,dx$.

24. $\int \sqrt{\frac{a+x}{a-x}}\,dx$. [Put $x = a\cos 2\theta$.]

25. $\int \frac{2x^3+3x^2+4x+5}{2x+1}\,dx$.

26. (i) $\int \frac{\sqrt{x}}{\sqrt{(a^3-x^3)}}\,dx$. (ii) $\int \frac{x^2}{\sqrt{(a^6-x^6)}}\,dx$.

[Put $x^3 = a^3\sin^2\theta$ in (i) and $a^3\sin\theta$ in (ii).]

(iii) $\int \frac{dx}{x^3\sqrt{(x^2-1)}}$. (iv) $\int \frac{x^3\,dx}{\sqrt{(1-x^2)}}$.

27. $\int \sqrt{\frac{x}{a-x}}\,dx$. [Put $x = a\sin^2\theta$.]

28. (i) $\int \frac{dx}{(a^2 - x^2)^{\frac{3}{2}}}$. (ii) $\int \frac{dx}{(1-x)\sqrt{(1-x^2)}}$.

29. (i) $\int \left(1 - \frac{1}{x^2}\right) e^{x + \frac{1}{x}}\,dx$. (ii) $\int \frac{x^2+1}{(x^2-1)^2}\,dx$.

30. $\int \frac{a\cos x - b\sin x}{a\sin x + b\cos x + c}\,dx$.

31. (i) $\int \frac{dx}{x(a + b\log x)}$. (ii) $\int \frac{(\log\sec x)^2}{\cot x}\,dx$.

32. $\int \frac{x^2+1}{\sqrt[3]{(x^3 + 3x + 6)}}\,dx$.

33. (i) $\int \cos x\cos(\sin x)\,dx$. (ii) $\int \sin x\cot^3 x\,dx$.

(iii) $\int \tan x\tan 2x\tan 3x\,dx$.

34. $\int \cos\left(2\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right)dx$. [Put $x = \cos\theta$.]

35. (i) $\int \frac{\cos x}{\cos(x+\alpha)}\,dx$. (ii) $\int \frac{\cot\alpha - \cot x}{\cot\alpha + \cot x}\,dx$.

36. (i) $\int \frac{dx}{x^2(a-bx)^2}$ (ii) $\int \frac{x^7\,dx}{(1-x^4)^2}$.

37. (i) $\int \frac{x^{\frac{1}{2}}}{1 + x^{3/4}}\,dx$. [Put $x = z^4$.] (ii) $\int \frac{\sqrt{(1+x^2)}}{x^4}\,dx$.

38. (i) $\int \frac{dx}{x\sqrt{(x^4-1)}}$. (ii) $\int \frac{2x\,ax}{(1-x^2)\sqrt{(x^4-1)}}$

[Put $x^2 = \sec\theta$.]

39. $\int \{f(x)\,\phi'(x) + \phi(x)\,f'(x)\}\,dx$.

40. Integrate $\frac{1}{2}f'(x)$ with respect to x^4 where
$$f(x) = \tan^{-1}x + \log\sqrt{1+x} - \log\sqrt{1-x}.$$

ANSWERS

1. (i) $e^{\tan^{-1}x}$. (ii) $a^{-1}e^{a\sin^{-1}x}$. (iii) $\sin(\log x)$.
(iv) $-\frac{1}{3}\cot^3 x$. (v) $\frac{1}{3}\log\cos 3x$. (vi) $\log$ si·

2. (i) $\frac{1}{3}(x^2+1)^{3/2}$. (ii) $\frac{2}{9}(a^3+x^3)^{3/2}$.

3. (i) $2\sin\sqrt{x}$. (ii) $\tan\frac{1}{2}x$.

4. (i) $2\sqrt{\tan x}$. (ii) $\frac{2}{3}\tan^{3/2}x$.

5. (i) $\frac{3}{2}(x+\sin x)^{2/3}$. (ii) $\log(x+\sin x)$.

6. (i) $\log\sec(\log x)$. (ii) $\log(\log x)$. **7.** (i) $2\sqrt{1+\sin x}$.

(ii) $\dfrac{-1}{b(a+b\sin x)}$. **8.** (i) $-\dfrac{\sqrt{1-x^2}}{x}$. (ii) $-\dfrac{\sqrt{1+x^2}}{x}$.

(iii) $x/\sqrt{1-x^2}$. (iv) $x/\sqrt{x^2+1}$.

9. (i) $2\log(e^{x/2}+e^{-x/2})$. (ii) $-\log(1+e^{-x})$.

10. (i) $-\log(\log\cos x)$. (ii) $\log(\log\sin x)$.
(iii) $\log(\log\tan x)$. (iv) $\log\{\log(\sec x+\tan x)\}$.

11. (i) $\dfrac{1}{a-b}\log(a\sin^2 x+b\cos^2 x)$.

(ii) $\dfrac{1}{2(b-a)}\log(a\cos^2 x+b\sin^2 x)$.

(iii) $\dfrac{1}{(a^2-b^2)}\left\{\dfrac{1}{a^2\cos^2 x+b^2\sin^2 x}\right\}$.

(iv) $-\dfrac{1}{2b^2}\left\{\dfrac{1}{a^2+b^2\tan^2 x}\right\}$.

12. (i) $-\frac{1}{2}\cos x^2$. (ii) $\log\tan x$. **13.** (i) $\sqrt{3x^2-2x+7}$.

(ii) $\sqrt{x^2-a^2}$. **14.** (i) $2\sqrt{3+\tan^{-1}x}$. (ii) $2\tan^{1/2}x+\frac{2}{5}\tan^{5/2}x$.

15. (i) $e^x - \log(e^x + 1)$. (ii) $\tan^{-1}(e^x)$.

(iii) $x - \log(e^x - 1) - (e^x - 1)^{-1}$. (iv) $2\tan^{-1}\{\sqrt{(e^x - 1)}\}$

16. $\tan(xe^x)$.

17. (i) $\frac{1}{3}\{(x+1)^{3/2} + (x-1)^{3/2}\}$.

(ii) $\frac{1}{24}\{(2x+5)^{3/2} - (2x-3)^{3/2}\}$.

(iii) $\sec^{-1}(2x-7)$. 18. (i) $2\log(1+\sqrt{x})$.

(ii) $-(4x+1)/\{8(2x+1)^2\}$. 19. (i) $2\sqrt{x} + 2\log(\sqrt{x} - 1)$.

(ii) $\frac{2}{3}x^{3/2} - x + 2\sqrt{x} - 2\log(\sqrt{x} + 1)$.

20. (i) $\frac{3}{10}(2x+1)^{5/2} + \frac{1}{6}(2x+1)^{3/2}$.

(ii) $\frac{3}{7}(x+a)^{7/3} - \frac{3}{4}a(x+a)^{4/3}$.

21. (i) $-x - 2\log(1-x)$.

(ii) $\frac{1}{6}x^6 + \frac{1}{5}x^5 + \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + \log(x-1)$.

22. (i) $\frac{2}{3}x + \frac{1}{9}\log(3x+4)$. (ii) $b^{-2}[(a+bx) - a\log(a+bx)]$.

23. (i) $\frac{4}{27}(3x+2)^{3/2} - \frac{2}{9}(3x+2)^{1/2}$.

(ii) $\frac{3}{5}b^{-2}(a+bx)^{5/3} - \frac{3}{2}ab^{-2}(a+bx)^{2/3}$.

24. $-a\cos^{-1}(x/a) - \sqrt{a^2 - x^2}$.

25. $\frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{3}{2}x + \frac{7}{4}\log(2x+1)$.

26. (i) $\frac{2}{3}\sin^{-1}(x/a)^{\frac{3}{2}}$. (ii) $\frac{1}{3}\sin^{-1}(x/a)^3$.

(iii) $\frac{1}{2}\left\{\sec^{-1}x + \frac{1}{x^2}\sqrt{x^2 - 1}\right\}$.

(iv) $-\sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{3/2}$.

27. $a\sin^{-1}\left(\frac{x}{a}\right)^{1/2} - \sqrt{x(a-x)}$.

28. (i) $\frac{x}{a^2\sqrt{(a^2-x^2)}}$ (ii) $\sqrt{\frac{1+x}{1-x}}$.

29. (i) $e^{x+\frac{1}{x}}$. (ii) $-x/(x^2-1)$.

30. $\log(a\sin x+b\cos x+c)$.

31. (i) $b^{-1}\log(a+b\log x)$. (ii) $\frac{1}{3}(\log\sec x)^3$.

32. $\frac{1}{2}(x^3+3x+6)^{2/3}$.

33. (i) $\sin(\sin x)$. (ii) $-(\sin x+\operatorname{cosec} x)$.

(iii) $\frac{1}{3}\log\sec 3x-\frac{1}{2}\log\sec 2x-\log\sec x$. 34. $-\frac{1}{2}x^2$.

35.. (i) $x\cos\alpha-\sin\alpha\log\cos(x+\alpha)$.

(ii) $x\cos 2\alpha-\sin 2\alpha\log\sin(x+\alpha)$.

36. (i) $\frac{2b}{a^2}\log\frac{x}{a-bx}-\frac{(a-2bx)}{a^2x(a-bx)}$. (ii) $\frac{1}{4}\left\{\log(1-x^4)+\frac{1}{1-x^4}\right\}$.

37. (i) $\frac{4}{3}\{x^{3/4}-\log(1+x^{3/4})\}$ (ii) $-\sqrt{(1+x^2)^3}/3x^3$

38. (i) $\frac{1}{2}\sec^{-1}x^2$. (ii) $\sqrt{(x^2+1)}/\sqrt{(x^2-1)}$. 39. $f(x)\phi(x)$.

40. $-\log(1-x^4)$.

2.3. Standard Integrals.

(A) $$\int\frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}\ .(a\neq 0)$$

Proof. Put $x=a\tan\theta$; then $dx=a\sec^2\theta\,d\theta$.

$$\therefore\ I=\int\frac{a\sec^2\theta\,d\theta}{a^2\sec^2\theta}=\frac{1}{a}\int d\theta=\frac{1}{a}\theta=\frac{1}{a}\tan^{-1}\frac{x}{a}.$$

Cor. $$\int\frac{dx}{1+x^2}=\tan^{-1}x.$$

Note. Putting $x=a\cot\theta$, the above integral takes up the form $-(1/a)\cot^{-1}(x/a)$, which evidently differs from the previous form by a constant. Usually

$$\int-\frac{dx}{a^2+x^2}\text{ is written in the form }\frac{1}{a}\cot^{-1}\frac{x}{a}.$$

(B) $\displaystyle\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| \quad (|x| > |a|).$

Proof. $\displaystyle\int \frac{dx}{x^2 - a^2} = \int \frac{1}{2a} \left\{ \frac{1}{x-a} - \frac{1}{x+a} \right\} dx$

$$= \frac{1}{2a} \left\{ \int \frac{dx}{x-a} - \int \frac{dx}{x+a} \right\}$$

$$= \frac{1}{2a} \left\{ \log|(x-a)| - \log|(x+a)| \right\},$$

since the numerator is the differential coefficient of the denominator in each case, [*See* Ex. 5, *Art.* 2.2.],

$$= \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right|.$$

Note. The above is an example of integration by breaking up the integrand into fractions. [*See Chapter V.*]

(C) $\displaystyle\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| \quad (|x| < |a|).$

The proof is as before, noticing that

$$\frac{1}{a^2 - x^2} = \frac{1}{2a} \left(\frac{1}{a+x} + \frac{1}{a-x} \right).$$

(D) $\displaystyle\int \frac{dx}{\sqrt{(x^2 \pm a^2)}} = \log |(x + \sqrt{x^2 \pm a^2})|.$

Proof. Put $\sqrt{(x^2 \pm a^2)} = z - x$, or, $z = x + \sqrt{(x^2 \pm a^2)}$.

$$\therefore \quad dz = \left(1 + \frac{2x}{2\sqrt{(x^2 \pm a^2)}} \right) dx = \frac{z}{\sqrt{(x^2 \pm a^2)}} dx.$$

$$\therefore \quad \int \frac{dx}{\sqrt{(x^2 \pm a^2)}} = \int \frac{dz}{z} = \log z$$

$$= \log |(x + \sqrt{(x^2 \pm a^2)})|.$$

Note. Students acquainted with hyperbolic functions may work out the integrals (*D*) by substitution $x = a \sinh z$, or $x = a \cosh z$ according as the denominator is $\sqrt{(x^2 + a^2)}$, or $\sqrt{(x^2 - a^2)}$.

Thus, putting $x = a \sinh z$, we have $dx = a \cosh z\, dz$.

(1) Hence, $\displaystyle\int\frac{dx}{\sqrt{(x^2+a^2)}} = \int\frac{a\cosh z}{a\sqrt{(1+\sinh^2 z)}}\,dz = \int dz = z = \sinh^{-1}\frac{x}{a}$

and this is shown in Trigonometry* $= \log\left|\dfrac{x+\sqrt{(x^2+a^2)}}{a}\right|$

$$= \log\left|(x+\sqrt{(x^2+a^2)}\right| - \log a.$$

This form differs from the result given above by a constant [*Cf Remarks in Art. 1.3*]. In the result, $-\log a$ being a constant may be dropped, since it may be supposed to be included in the constant of integration.

In that case the forms given in (*D*) and here are the same. Similar remark applies to the results of (2) and (3) below.

Similarly, by putting $x = a\cosh z$, we have

(2) $\displaystyle\int\frac{dx}{\sqrt{(x^2-a^2)}} = \cosh^{-1}\frac{x}{a} = \log\left|\left(\frac{x+\sqrt{(x^2-a^2}}{a}\right)\right|$.

In many text-books, for the sake of brevity, inverse hyperbolic forms are used in preference to the logarithmic forms.

The first of the integrals (*D*) can also be evaluated by putting $x = a\tan\theta$, so that $dx = a\sec^2\theta\, d\theta$. Thus,

(3) $\displaystyle\int\frac{dx}{\sqrt{(x^2+a^2)}} = \int\frac{a\sec^2\theta\, d\theta}{a\sec\theta} = \int\sec\theta\, d\theta = \int\frac{\sec\theta(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta}d\theta$

$= \log(\sec\theta + \tan\theta)$ [*by Ex. 5, Art. 2.2*]

$= \log(\sqrt{(1+\tan^2\theta)} + \tan\theta)$

$$= \log\left(\sqrt{1+\frac{x^2}{a^2}} + \frac{x}{a}\right) = \log\left|\frac{x+\sqrt{(x^2+a^2)}}{a}\right|.$$

Similarly, put $x = a\sec\theta$, in the other integral.

Similarly the integrals of (*B*) and (C) can be obtained by hyperbolic substitution.

(B) Thus, putting $x = a\coth\theta$, $dx = -a\,\text{cosech}^2\theta\, d\theta$ and
$(x^2-a^2) = a^2(\coth^2\theta - 1) = a^2\,\text{cosech}^2\theta$.

* See *Das & Mukherjees' Higher Trigonomnetry, Art, 12.9.*

$$\therefore\ I = \int \frac{-a\,\text{cosech}^2\theta}{a^2\,\text{cosech}^2\theta}\,d\theta = -\frac{1}{a}\int d\theta = -\frac{1}{a}\,\theta = -\frac{1}{a}\coth^{-1}\frac{x}{a}.$$

(C) Similarly, putting $x = a\tanh\theta$,

$$I = \int \frac{a\,\text{sech}^2\theta\,d\theta}{a^2\,\text{sech}^2\theta} = \frac{1}{a}\int d\theta = \frac{1}{a}\,\theta = \frac{1}{a}\tanh^{-1}\frac{x}{a}.$$

(E) $\displaystyle\int \frac{dx}{\sqrt{(a^2 - x^2)}} = \sin^{-1}\frac{x}{a}$. ($|x| < |a|$)

Put $x = a\sin\theta$; then $dx = a\cos\theta\,d\theta$.

$$\therefore\ I = \int \frac{a\cos\theta\,d\theta}{a\cos\theta} = \int d\theta = \theta = \sin^{-1}\frac{x}{a}.$$

Cor. $\displaystyle\int \frac{dx}{\sqrt{(1 - x^2)}} = \sin^{-1}x$.

Note 1. Putting $x = a\cos\theta$, the integral $-\displaystyle\int \frac{dx}{\sqrt{(a^2 - x^2)}}$ can be put in the form $\cos^{-1}\dfrac{x}{a}$ instead of $-\sin^{-1}\dfrac{x}{a}$.

Note 2. In (A) to (E), it is tacitly assumed that $a \neq 0$.

2.4. $\displaystyle\int \frac{dx}{ax^2 + bx + c}$. ($a \neq 0$)

The above integral can be written as

$$\frac{1}{a}\int \frac{dx}{x^2 + \frac{b}{a}x + \frac{c}{a}} = \frac{1}{a}\int \frac{dx}{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}}$$

and putting $x + \dfrac{b}{2a} = z$, this reduces to the form $\dfrac{1}{a}\displaystyle\int \frac{dz}{z^2 \pm k^2}$ according as $4ac - b^2$ is positive or negative, and this is of the form (A) or (B) of Art. 2.3.

If a be negative, we may take $-(1/a)$, which is positive, outside the integral sign and it then reduces to the form (C).

2.5. $\displaystyle\int \frac{px + q}{ax^2 + bx + c}\,dx$. ($a \neq 0,\ p \neq 0$)

Here, noting that the differential of $ax^2 + bx + c = (2ax + b)dx$, the given integral can be written as

$$\frac{p}{2a}\int \frac{2ax + \frac{2aq}{p}}{ax^2 + bx + c}\,dx = \frac{p}{2a}\int \frac{(2ax + b) + \frac{2aq}{p} - b}{ax^2 + bx + c}\,dx$$

$$= \frac{p}{2a}\left\{\int \frac{(2ax + b)}{ax^2 + bx + c}\,dx + \frac{2aq - pb}{p}\int \frac{dx}{ax^2 + bx + c}\right\}$$

The first integral $= \log(ax^2 + bx + c)$, since the numerator of the integral is equal to the differential coefficient of the denominator. The second integral is evaluated as in the previous article.

Note 1. To express $px + q$ as the sum of two terms we might also proceed thus :-

Let $px + q = l$ (differential coefficient of the denominator) $+ m$, where the constants l, m are to be determined by comparing the coefficients.

Note 2. In Arts. 2.4 and 2.5, if $ax^2 + bx + c$ breaks up into two real linear factors, then instead of proceeding as above, we may break up the integrand into the sum of partial fractions, and then integrate each separately. [*See Chapter V.*]

2.6. $\displaystyle\int \frac{dx}{\sqrt{ax^2 + bx + c}}$. $(a \neq 0)$.

If *a be positive,* we can write the integral as

$$\frac{1}{\sqrt{a}}\int \frac{dx}{\sqrt{\left\{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right\}}}$$

and substituting z for $x + \frac{b}{2a}$, this reduces to the form

$$\frac{1}{\sqrt{a}}\int \frac{dz}{\sqrt{(z^2 \pm k^2)}},$$

according as $4ac - b^2$ is positive or negative, and this is of the form (D) of Art. 2.3.

If *a be negative,* say, $= -a'$, we can write the integral as

$$\frac{1}{\sqrt{a'}} \int \frac{dx}{\sqrt{\left\{\frac{4a'c + b^2}{4a'^2} - \left(x - \frac{b}{2a'}\right)^2\right\}}}$$

and putting $x - \frac{b}{2a'} = z$, this reduces to the form

$$\frac{1}{\sqrt{a'}} \int \frac{dz}{\sqrt{(k^2 - z^2)}}$$

which is of the form (E) of Art. 2.3.

Note. *If the quadratic under the radical* in the above case *breaks up into two real linear factors*, we may, instead of proceeding as above, *substitute* z^2 *for one of the factors* and then proceed. The method is illustrated in Exs. 2 and 3 of Art. 2.9.

2.7. $\int \frac{px + q}{\sqrt{(ax^2 + bx + c)}}\, dx. \quad (a \neq 0,\ p \neq 0)$

As in Art. 2.5, the above integral can be written as

$$\frac{p}{2a} \int \frac{(2ax + b) + \frac{2aq}{p} - b}{\sqrt{(ax^2 + bx + c)}}\, dx$$

$$= \frac{p}{2a} \left\{ \int \frac{2ax + b}{\sqrt{(ax^2 + bx + c)}}\, dx + \frac{2aq - pb}{p} \int \frac{dx}{\sqrt{(ax^2 + bx + c)}} \right.$$

The first integral, on putting $ax^2 + bx + c = z$, * reduces to

$$\int \frac{dz}{\sqrt{z}} = 2\sqrt{z} = 2\sqrt{ax^2 + bx + c}.$$

The second integral is obtained as in Art. 2.6.

2.8. (A) $\int \frac{dx}{(ax + b)\sqrt{(cx + d)}} \quad (a \neq 0,\ c \neq 0)$

Put $cx + d = z^2$. $\therefore \quad c\,dx = 2z\,dz$.

* We may also put $ax^2 + bx + c = z^2$

The integral then reduces to

$$\frac{2}{c}\int \frac{z\,dz}{\left(a\,\frac{z^2-d}{c}+b\right).z} = 2\int \frac{dz}{az^2+(bc-ad)}$$

which reduces to the form (A), (B) or (C) of Art. 2.3.

(B) $$\int \frac{dx}{(px+q)\sqrt{(ax^2+bx+c)}}\ ,\ (a \neq 0,\ p \neq 0)$$

Here, put $px + q = \frac{1}{z}$, so that

$$p\,dx = -\frac{dz}{z^2} \text{ and } x = \frac{1}{p}\left(\frac{1}{z}-q\right).$$

The given integral then reduces to

$$-\frac{1}{p}\int \frac{dz}{z^2\,\frac{1}{z}\sqrt{\left\{\frac{a}{p^2}\left(\frac{1}{z}-q\right)^2+\frac{b}{p}\left(\frac{1}{z}-q\right)+c\right\}}}$$

$$= -\frac{1}{p}\int \frac{dz}{\sqrt{\left\{\frac{a}{p^2}(1-qz)^2+\frac{bz}{p}(1-qz)+cz^2\right\}}}$$

which when simplified takes up the form

$$-\int \frac{dz}{\sqrt{(Az^2+Bz+C)}}$$

when we proceed as in Art. 2.6.

2.9. Illustrative Examples.

Ex.1. *Integrate* $\int \frac{7x-9}{x^2-2x+35}\,dx$.

$$I = \int \frac{\frac{7}{2}(2x-2)-2}{x^2-2x+35}\,dx$$

$$= \frac{7}{2}\int \frac{2x-2}{x^2-2x+35}\,dx - 2\int \frac{dx}{x^2-2x+35}$$

$$= \frac{7}{2}\log(x^2 - 2x + 35) - 2\int \frac{dx}{(x-1)^2 + 34}$$

$$= \frac{7}{2}\log(x^2 - 2x + 35) - \frac{2}{\sqrt{34}}\tan^{-1}\left(\frac{x-1}{\sqrt{34}}\right).$$

Ex. 2. *Integrate* $\int \frac{dx}{\sqrt{(2 + 3x - 2x^2)}}$

Here, $I = \frac{dx}{\sqrt{\{(1 + 2x)(2 - x)\}}}$.

Put $2 - x = z^2$. $\therefore\ -dx = 2z\,dz$

and $1 + 2x = 1 + 2(2 - z^2) = 5 - 2z^2$.

The integral reduces to

$$-\int \frac{2z\,dz}{\sqrt{\{(5 - 2z^2)z^2\}}} = -\sqrt{2}\int \frac{dz}{\sqrt{(\frac{5}{2} - z^2)}}$$

$= \sqrt{2}\cos^{-1}(\sqrt{\tfrac{2}{5}}z)$ [See (E) Note, Art. 2.3.]

$$= \sqrt{2}\cos^{-1}\sqrt{\frac{2(2-x)}{5}}.$$

Ex. 3. *Integrate* $\int \frac{dx}{\sqrt{\{(x-\alpha)(\beta - x)\}}}$ $(\beta > \alpha)$.

Put $x - \alpha = z^2$. $\therefore\ dx = 2z\,dz$; and $\beta - x = \beta - \alpha - z^2$.

$$\therefore\ I = \int \frac{2z\,dz}{\sqrt{\{z^2(\beta - \alpha - z^2)\}}} = 2\int \frac{dz}{\sqrt{(k^2 - z^2)}},\ \text{where } k^2 = \beta - \alpha,$$

$$= 2\sin^{-1}\frac{z}{k} = 2\sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}}$$

Note. This integral can also be evaluated by putting

$$x = \alpha\cos^2\theta + \beta\sin^2\theta.$$

Ex. 4. *Integrate* $\int \frac{dx}{(2x+3)\sqrt{(x^2 + 3x + 2)}}$.

The integral is of the form (B) of Art. 2.8.

Put $2x + 3 = \frac{1}{z}$. $\therefore\ 2\,dx = -\frac{1}{z^2}\,dz$. $\therefore\ dx = -\frac{1}{2}\frac{dz}{z^2}$.

And $x = \frac{1}{2}\left(\frac{1}{z} - 3\right)$; $z = \frac{1}{2x+3}$.

$$\therefore \quad I = -\tfrac{1}{2}\int \frac{dz}{z^2 \frac{1}{z}\sqrt{\left\{\frac{1}{2^2}\left(\frac{1}{z}-3\right)^2 + \frac{3}{2}\left(\frac{1}{z}-3\right)+2\right\}}}$$

$$= -\int \frac{dz}{\sqrt{(1-z^2)}} = -\sin^{-1} z = -\sin^{-1}\left(\frac{1}{2x+3}\right).$$

Alternatively :

$$I = \int \frac{dx}{(2x+3)\sqrt{\{\frac{1}{4}(4x^2+12x+8)\}}} = \int \frac{dx}{(2x+3)\frac{1}{2}\sqrt{\{(2x+3)^2-1\}}}.$$

Put $2x + 3 = z$. $\therefore$ $2dx = dz \,.\, dx = \frac{1}{2}dz$.

$$\therefore \qquad I = \frac{dz}{z\sqrt{(z^2-1)}} = \sec^{-1} z = \sec^{-1}(2x-3).$$

Although apparently the forms of the two results are different, it can be easily shown (by using the properties of inverse circular functions) that one differs from the other by a constant.

Note. It can be easily seen that the linear expression is the sum of the two linear factors of the quadratic expression under the radical ; also the linear expression is the derivative of the quadratic expression.

Ex. 5. *Integrate* $\int \sqrt{\frac{1+x}{1-x}}\, dx$. [*H. S. '81*]

Rationalizing the numerator, we have

$$I = \int \frac{1+x}{\sqrt{(1-x^2)}}\, dx = \int \frac{dx}{\sqrt{(1-x^2)}} + \int \frac{x\,dx}{\sqrt{(1-x^2)}}.$$

Put $1 - x^2 = z^2$ in the 2nd integral, so that $-2x\,dx = 2z\,dz$.

$\therefore$ 2nd integral then $= -\int dz = -z = -\sqrt{(1-x^2)}$.

$\therefore \; I = \sin^{-1} x - \sqrt{(1-x^2)}$.

Note. Integrals of the type $\int \sqrt{\frac{ax+b}{cx+d}}\, dx$ $(a \neq 0,\, c \neq 0)$ of which the above is a particular case can be evaluated exactly in the same way.

Ex. 6. *Integrate* $\int \frac{x^2+4}{x^2+2x+3}\, dx$.

Here, $$I = \int \left(1 - \frac{2x - 1}{x^2 + 2x + 3} \right) dx$$

$$= \int dx - \int \frac{2x - 1}{x^2 + 2x + 3} dx .$$

2nd integral $$= \int \frac{(2x + 2) - 3}{x^2 + 2x + 3} dx$$

$$= \int \frac{2x + 2}{x^2 + 2x + 3} dx - 3 \int \frac{dx}{(x + 1)^2 + (\sqrt{2})^2}$$

$$= \log (x^2 + 2x + 3) - \frac{3}{\sqrt{2}} \tan^{-1} \left(\frac{x + 1}{\sqrt{2}} \right)$$

$\therefore$ $$I = x - \log (x^2 + 2x + 3) + \frac{3}{\sqrt{2}} \tan^{-1} \left(\frac{x + 1}{\sqrt{2}} \right)$$

Note. In order to evaluate integrals of the type $\int \frac{f(x)}{ax^2 + bx + c} dx$, (of which the above is a particular case), where $f(x)$ is a rational function of x of second or of higher degree, divide the numerator by the denominator till the numerator is of the first degree ; then the integral reduces to the sum of integrals of the type $\int A_m x^m dx$ ($m >$ or $= 0$) and of the type

$$\int \frac{px + q}{ax^2 + bx + c} dx .$$

EXAMPLES II (B)

Integrate :-

1. $\int \frac{3x^2}{1 + x^6} dx .$

2. (i) $\int \frac{x\,dx}{x^4 + 1}$. (ii) $\int \frac{x\,dx}{x^4 - 1}$.

3. (i) $\int \frac{dx}{e^x + e^{-x}}$. [*H. S. '78, 86*] (ii) $\int \frac{x^3\,dx}{\sqrt{(a^8 - x^8)}}$.

   [*Put* $e^x = z$.] [*Put* $x^4 = z$.]

4. (i) $\int \frac{x^2 + \sin^2 x}{1 + x^2} \sec^2 x\, dx$. (ii) $\int \frac{\sin x\, dx}{3 + \sin^2 x}$.

5. (i) $\int \frac{x\,dx}{\sqrt{(a^4 + x^4)}}$ (ii) $\int \frac{x^2 - 1}{x\sqrt{(1 + x^4)}}\,dx$.

[(ii) *Put* $x + x^{-1} = z$.]

6. $\int \frac{x\,dx}{\sqrt{\{(x^2 - a^2)(b^2 - x^2)\}}}\ (b^2 > a^2)$

[*Put* $x^2 - a^2 = z^2$.]

7. (i) $\int \frac{dx}{1 + x + x^2}$. [*H. S. '80*] (ii) $\int \frac{dx}{4x^2 + 4x + 5}$

8. (i) $\int \frac{dx}{1 + x - x^2}$. (ii) $\int \frac{dx}{6x^2 + 7x + 2}$

9. $\int \frac{x\,dx}{x^4 + 2x^2 + 2}$.

10. $\int \frac{\cos x\,dx}{\sin^2 x + 4\sin x + 3}$

11. $\int \frac{e^x\,dx}{e^{2x} + 2e^x + 5}$

12. $\int \frac{dx}{\sqrt{(1 - x^2)}\{1 + (\sin^{-1} x)^2\}}$.

13. $\int \frac{x^2\,dx}{x^6 - 6x^3 + 5}$.

14. $\int \frac{dx}{x\{10 + 7\log x + (\log x)^2\}}$.

15. (i) $\int \frac{x\,dx}{x^2 + 2x + 1}$. (ii) $\int \frac{x + 1}{3 + 2x - x^2}\,dx$.

16. (i) $\int \frac{x + 1}{x^2 + 4x + 5}\,dx$. (ii) $\int \frac{2x + 3}{4x^2 + 1}\,dx$.

17. (i) $\int \frac{(4x + 3)\,dx}{3x^2 + 3x + 1}$. (ii) $\int \frac{x\,dx}{2 - 6x - x^2}$

18. $\int \frac{x^2}{x^2 - 4}\,dx$.

19. (i) $\int \frac{x^2 + 2x}{x^2 + 2x + 2}\, dx$. (ii) $\int \frac{x^2 - x + 1}{x^2 + x + 1}\, dx$.

20. $\int \frac{x^3 + x^2 + 2x + 1}{x^2 - x + 1}\, dx$.

21. $\int \frac{dx}{\sqrt{(x^2 + x - 2)}}$.

22. (i) $\int \frac{dx}{\sqrt{(1 - x - x^2)}}$. (ii) $\int \frac{dx}{\sqrt{(3 + 3x + x^2)}}$.

23. $\int \frac{dx}{\sqrt{(2x^2 + 3x + 4)}}$.

24. $\int \frac{dx}{\sqrt{(x^2 - 7x + 12)}}$. [*Put* $x - 4 = z^2$.]

25. $\int \frac{dx}{\sqrt{(6 + 11x - 10x^2)}}$.

26. $\int \frac{\cos x\, dx}{\sqrt{(5 \sin^2 x - 12 \sin x + 4)}}$.

27. $\int \frac{dx}{\sqrt{\{(x - \alpha)(x - \beta)\}}}$.

28. (i) $\int \frac{dx}{\sqrt{(2ax - x^2)}}$. (ii) $\int \frac{dx}{\sqrt{(2ax + x^2)}}$

29. (i) $\int \frac{x + b}{\sqrt{(x^2 + a^2)}}\, dx$. [*J. E.* '83] (ii) $\int \frac{2x + 3}{\sqrt{(x^2 + x + 1)}}\, dx$.

30. $\int \frac{x - 2}{\sqrt{(2x^2 - 8x + 5)}}\, dx$.

31. (i) $\int \frac{(x + 1)}{\sqrt{(4 + 8x - 5x^2)}}\, dx$. (ii) $\int \frac{(2x - 1)\, dx}{\sqrt{(4x^2 + 4x + 2)}}\, dx$.

32. (i) $\int \frac{dx}{(2 + x)\sqrt{(1 + x)}}$. (ii) $\int \frac{dx}{(2x + 1)\sqrt{(4x + 3)}}$.

33. $\int \frac{dx}{\sqrt{(\frac{2}{3}x^3 - x^2 + \frac{1}{3})}}$

34. (i) $\int \sqrt{\left(\frac{x-3}{x-4}\right)}\, dx$. (ii) $\int \sqrt{\left(\frac{2x+1}{3x+2}\right)}\, dx$.

35. (i) $\int \frac{dx}{(1-x)\sqrt{x}}$. (ii) $\int \frac{\sqrt{x}\, dx}{x-1}$.

[Put $x = z^2$.]

36. (i) $\int \frac{dx}{x\sqrt{(x^2 \pm a^2)}}$. (ii) $\int \frac{dx}{(1+x)\sqrt{(1-x^2)}}$.

(iii) $\int \frac{dx}{x\sqrt{(9x^2+4x+1)}}$. (iv) $\int \frac{dx}{(1+x)\sqrt{(1+2x-x^2)}}$ [C. P. '86]

(v) $\int \frac{dx}{x\sqrt{(x^2+2x-1)}}$. (vi) $\int \frac{dx}{(1+x)\sqrt{(1+x-x^2)}}$

(vii) $\int \frac{dx}{(x-3)\sqrt{(x^2-6x+8)}}$.

37. (i) $\int \frac{\sqrt{(a^2-x^2)}}{x}\, dx$. (ii) $\int \frac{dx}{x+\sqrt{(x-1)}}$.

38. $\int \frac{dx}{x\sqrt{(1+x^3)}}$. [Put $1 + x^3 = z^2$.] [C. P. '81]

39. (i) $\int \sqrt{\frac{a+x}{x}}\, dx$. (ii) $\int \frac{\sqrt{x-a}}{x}\, dx$.

40. If $a < x < b$, show that

$$\int \frac{dx}{(x-a)\sqrt{\{(x-a)(b-x)\}}} = \frac{2}{a-b}\sqrt{\frac{b-x}{x-a}}.$$

ANSWERS

1. $\tan^{-1}(x^3)$. 2. (i) $\frac{1}{2}\tan^{-1}(x^2)$. (ii) $\frac{1}{4}\log\frac{x^2-1}{x^2+1}$.

3. (i) $\tan^{-1}(e^x)$. (ii) $\frac{1}{4}\sin^{-1}\left(\frac{x}{a}\right)^4$.

4. (i) $\tan x - \tan^{-1} x$. (ii) $\frac{1}{4}\log\frac{2-\cos x}{2+\cos x}$.

5. (i) $\frac{1}{2}\log(x^2+\sqrt{x^4+a^4})$. (ii) $\log\{(1+x^2+\sqrt{1+x^4})/x\}$.

6. $\sin^{-1}\sqrt{\frac{x^2-a^2}{b^2-a^2}}$. 7. (i) $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$. (ii) $\frac{1}{4}\tan^{-1}(x+\frac{1}{2})$.

8. (i) $\frac{1}{\sqrt{5}}\log\frac{\sqrt{5}+2x-1}{\sqrt{5}-2x+1}$. (ii) $\log\frac{2x+1}{3x+2}$. 9. $\frac{1}{2}\tan^{-1}(x^2+1)$.

10. $\frac{1}{2}\log\frac{1+\sin x}{3+\sin x}$. 11. $\frac{1}{2}\tan^{-1}\{\frac{1}{2}(e^x+1)\}$. 12. $\tan^{-1}(\sin^{-1}x)$

13. $\frac{1}{12}\log\frac{x^3-5}{x^3-1}$. 14. $\frac{1}{3}\log\frac{2+\log x}{5+\log x}$. 15. (i) $\log(x+1)+\frac{1}{x+1}$.

(ii) $-\log(x-3)$. 16. (i) $\frac{1}{2}\log(x^2+4x+5)-\tan^{-1}(x+2)$

(ii) $\frac{1}{4}\log(4x^2+1)+\frac{3}{2}\tan^{-1}(2x)$.

17. (i) $\frac{2}{3}\log(3x^2+3x+1)+\frac{2}{\sqrt{3}}\tan^{-1}\{\sqrt{3}(2x+1)\}$.

(ii) $\frac{-3}{2\sqrt{(11)}}\log\frac{\sqrt{11}+3+x}{\sqrt{(11)}-3-x}-\frac{1}{2}\log(2-6x-x^2)$.

18. $x+\log\frac{x-2}{x+2}$. 19. (i) $x-2\tan^{-1}(x+1)$.

(ii) $x-\log(x^2+x+1)+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$.

20. $\frac{1}{2}x^2+2x+\frac{3}{2}\log(x^2-x+1)+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)$.

21. $2\log(\sqrt{x+2}+\sqrt{x-1})$. 22. (i) $\sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right)$.

(ii) $\log(2x+3+2\sqrt{3+3x+x^2})$.

23. $\frac{1}{\sqrt{2}}\log(x+\frac{3}{4}+\sqrt{x^2+\frac{3}{2}x+2})$.

24. $2\log(\sqrt{x-3}+\sqrt{x-4})$. 25. $\sqrt{\frac{2}{5}}\sin^{-1}\sqrt{\frac{10x+4}{19}}$

26. $-\frac{2}{\sqrt{5}}\log\{\sqrt{2-5\sin x}+\sqrt{5(2-\sin x)}\}$.

27. $2\log(\sqrt{x-\alpha}+\sqrt{x-\beta})$. 28. (i) $\sin^{-1}\left(\frac{x-a}{a}\right)$

(ii) $\log(x+a+\sqrt{x^2+2ax})$.

29. (i) $\sqrt{x^2 + a^2} + b \log (x + \sqrt{x^2 + a^2})$.

(ii) $2\sqrt{x^2 + x + 1} + 2 \log (x + \frac{1}{2} + \sqrt{x^2 + x + 1})$.

30. $\frac{1}{2}\sqrt{2x^2 - 8x + 5}$. 31. (i) $\frac{9}{5\sqrt{5}} \sin^{-1}\left(\frac{5x-4}{6}\right) - \frac{1}{5}\sqrt{4 + 8x - 5x^2}$.

(ii) $\frac{1}{2}\sqrt{4x^2 + 4x + 2} - \log (2x + 1 + \sqrt{4x^2 + 4x + 2})$.

32. (i) $2 \tan^{-1}\left(\sqrt{1 + x}\right)$. (ii) $\frac{1}{2} \log \left(\frac{\sqrt{4x + 3 - 1}}{\sqrt{(4x + 3)} + 1}\right)$

33. $\log \frac{\sqrt{2x + 1} - \sqrt{3}}{\sqrt{(2x + 1)} + \sqrt{3}}$.

34. (i) $\sqrt{(x - 3)(x - 4)} + \log (\sqrt{x - 3} + \sqrt{x - 4})$.

(ii) $\frac{1}{6}[2\sqrt{(2x + 1)(3x + 2)} - \sqrt{\frac{2}{3}} \log (\sqrt{3}\sqrt{2x + 1} + \sqrt{2}\sqrt{3x + 2})$

35. (i) $\log \frac{1 + \sqrt{x}}{1 - \sqrt{x}}$. (ii) $2\sqrt{x} + \log \frac{\sqrt{x} - 1}{\sqrt{x} + 1}$.

36. (i) $\frac{1}{2a} \log \frac{\sqrt{x^2 + a^2} - a}{\sqrt{(x^2 + a^2)} + a}$; $\frac{1}{a} \sec^{-1} \frac{x}{a}$. (ii) $-\sqrt{\frac{1 - x}{1 + x}}$

(iii) $\log x - \log (1 + 2x + \sqrt{9x^4 + 4x + 1})$.

(iv) $\frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{x\sqrt{2}}{1 + x}\right)$. (v) $\sin^{-1}\left(\frac{x - 1}{x\sqrt{2}}\right)$.

(vi) $\sin^{-1}\left(\frac{3x + 1}{(1 + x)\sqrt{5}}\right)$. (vii) $\sec^{-1}(x - 3)$.

37. (i) $\sqrt{a^2 - x^2} + a \log \frac{a - \sqrt{a^2 - x^2}}{x}$.

(ii) $\log (x + \sqrt{x - 1}) - \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2\sqrt{x - 1} + 1}{\sqrt{3}}\right)$

38. $\frac{2}{3} \log (\sqrt{1 + x^3} - 1) - \log x$.

39. (i) $a \log (\sqrt{x} + \sqrt{x + a}) + \sqrt{x(x + a)}$.

(ii) $2\sqrt{x - a} - 2\sqrt{a} \tan^{-1}\left(\sqrt{\frac{x - a}{a}}\right)$.

CHAPTER III

INTEGRATION BY PARTS

3.1. Integration of a product 'by parts'.

We know from Differential Calculus that, if u and v_1 are two differentiable functions of x,

$$\frac{d}{dx}(uv_1) = \frac{du}{dx}v_1 + u\frac{dv_1}{dx}.$$

$\therefore$ integrating both sides with respect to x, we have

$$uv_1 = \int\left(\frac{du}{dx}v_1\right)dx + \int\left(u\frac{dv_1}{dx}\right)dx,$$

$$\text{or, } \int\left(u\frac{dv_1}{dx}\right)dx = uv_1 - \int\left(\frac{du}{dx}v_1\right)dx.$$

Suppose $\frac{dv_1}{dx} = v$, then $v_1 = \int v\,dx$.

Hence, the above result can be written as

$$\int(uv)dx = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx.$$

The above formula for the integration of a product of two functions is referred to as *integration by parts*.

It states that

the integral of the product of two functions
= *1st function (unchanged)* × *integral of 2nd*
− *integral of* [differential coefficient of 1st × *integral of 2nd*].

3.2. Illustrative Examples.

Ex. 1. *Integrate* $\int xe^x\,dx$.

$$I = x\int e^x\,dx - \int\left\{\frac{dx}{dx}\int e^x\,dx\right\}dx,$$

$$= xe^x - \int 1.e^x\,dx, \quad = xe^x - e^x.$$

Note. In the above integral, instead of taking x as the first function and e^x as the second, if we take e^x as the first function and x as the second, then applying the rule for integration by parts we get

$$\int (e^x . x)\, dx = e^x . \tfrac{1}{2}x^2 - \int e^x . \tfrac{1}{2}x^2\, dx .$$

The integral $\frac{1}{2}\int e^x x^2\, dx$ on the right side is more complicated than the one we started with, for it involves x^2 instead of x.

Thus, *while applying the rule for integration by parts to the product of two functions, care should be taken to choose properly the first function, i.e., the function not to be integrated.*

A little practice and experience will enable the student to make the right choice.

Ex. 2. *Integrate* $\int \log x\, dx$.

$$I = \int \log x . 1\, dx,$$

$$= \log x \int dx - \int \left\{ \frac{d}{dx}(\log x) . \int dx \right\} dx,$$

$$= \log x . x - \int \frac{1}{x} . x\, dx,$$

$$= x \log x - \int dx,$$

$$= x \log x - x .$$

Ex. 3. *Integrate* $\int \tan^{-1} x\, dx$.

$$I = \int \tan^{-1} x . 1\, dx,$$

$$= \tan^{-1} x . \int dx - \int \left\{ \frac{d}{dx}\left(\tan^{-1} x\right) \int dx \right\} dx,$$

$$= \tan^{-1} x . x - \int \frac{1}{1 + x^2} . x\, dx,$$

$$= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1 + x^2}\, dx,$$

$$= x \tan^{-1} x - \tfrac{1}{2} \log (1 + x^2) . \quad [\text{By Ex. 5, Art. 2.2}]$$

Note. Very often *an integral involving a single logarithmic function or a single inverse circular function can be evaluated by the application of the rule for integration by parts*, by considering the integral as the product of the given function and unity, and taking the given function as the first function and unity as the second.

This principle is illustrated in Exs. 2 and 3 above and Ex. 4 below.

Ex. 4. *Integrate* $\int \log(x + \sqrt{x^2 + a^2})\, dx$.

$$I = \int \log(x + \sqrt{x^2 + a^2}) . 1\, dx,$$

$$= \log(x + \sqrt{x^2 + a^2}) \int dx - \int \left[\frac{d}{dx} \left\{ \log\left(x + \sqrt{x^2 + a^2} \right) \right\} . \int dx \right] dx,$$

$$= \log(x + \sqrt{x^2 + a^2}) . x - \int \frac{1}{\sqrt{(x^2 + a^2)}} . x\, dx,$$

$$= x \log(x + \sqrt{x^2 + a^2}) - \int \frac{x\, dx}{\sqrt{(x^2 + a^2)}}.$$

To evaluate $\int \frac{x\, dx}{\sqrt{(x^2 + a^2)}}$, put $x^2 + a^2 = z^2$, so that $x\, dx = z\, dz$.

$$\therefore \int \frac{x\, dx}{\sqrt{(x^2 + a^2)}} = \int \frac{z\, dz}{z} = \int dz = z = \sqrt{x^2 + a^2}.$$

$$\therefore I = x \log(x + \sqrt{x^2 + a^2}) - \sqrt{x^2 + a^2}.$$

Ex. 5. *Integrate* $\int x^3 e^x\, dx$.

$I = x^3 e^x - 3 \int x^2 e^x\, dx$, integrating by parts

$= x^3 e^x - 3(x^2 e^x - 2 \int x e^x\, dx)$, integrating by parts again

$= x^3 e^x - 3[x^2 e^x - 2\{x e^x - \int e^x\, dx\}]$,

$= x^3 e^x - 3[x^2 e^x - 2(x e^x - e^x)]$,

$= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x$,

$= (x^3 - 3x^2 + 6x - 6) e^x$.

3.3 Standard Integrals.

(A) $\int e^{ax} \cos bx\, dx = \dfrac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}$

$$= \frac{e^{ax}}{\sqrt{(a^2 + b^2)}} \cos\left(bx - \tan^{-1}\frac{b}{a}\right).$$

(B) $\int e^{ax} \sin bx\, dx = \dfrac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}$

$$= \frac{e^{ax}}{\sqrt{(a^2 + b^2)}} \sin\left(bx - \tan^{-1}\frac{b}{a}\right).$$

(Here $a \neq 0$)

Proof. (A) Integrating by parts,

$$\int e^{ax} \cos bx\, dx = e^{ax}.\frac{\sin bx}{b} - \int \left(ae^{ax}.\frac{\sin bx}{b}\right) dx,$$

$$= \frac{e^{ax}\sin bx}{b} - \frac{a}{b}\int e^{ax} \sin bx\, dx.$$

Now, integrating by parts, the right side of this integral

$$= \frac{e^{ax}\sin bx}{b} - \frac{a}{b}\left\{ e^{ax}\frac{-\cos bx}{b} - \int ae^{ax}\left(\frac{-\cos bx}{b}\right)dx\right\},$$

$$= \frac{e^{ax}\sin bx}{b} + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}\int e^{ax} \cos bx\, dx.$$

transposing,

$$\left(1 + \frac{a^2}{b^2}\right)\int e^{ax} \cos bx\, dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{b^2}.$$

Now, dividing both sides by $1 + \dfrac{a^2}{b^2}$, i.e., $\dfrac{a^2 + b^2}{b^2}$,

we get $\int e^{ax} \cos bx\, dx = \dfrac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}$.

Again, putting $a = r\cos\alpha$, $b = r\sin\alpha$, so that $r = \sqrt{(a^2 + b^2)}$ and $\alpha = \tan^{-1}(b/a)$ on the right side of this integral, we have the right side

$$= \frac{e^{ax} r \cos(bx - \alpha)}{a^2 + b^2} = \frac{e^{ax}}{\sqrt{(a^2 + b^2)}} \cos\left(bx - \tan^{-1}\frac{b}{a}\right).$$

Integral (B) can be evaluated exactly in the same way.

3.4. Alternative method

Let $\quad P = \int e^{ax} \cos bx \, dx$

and $\quad Q = \int e^{ax} \sin bx \, dx$.

$$\therefore \; P + iQ = \int e^{ax} (\cos bx + i \sin bx) \, dx = \int e^{ax} e^{ibx} \, dx$$

$$= \int e^{(a+ib)x} \, dx = \frac{e^{(a+ib)x}}{a + ib} = \frac{a - ib}{a^2 + b^2} e^{ax} e^{ibx}$$

$$= \frac{e^{ax}}{a^2 + b^2} (a - ib)(\cos bx + i \sin bx).$$

Equating real and imaginary parts we get the values of P and Q.

Note 1. The above integrals can also be obtained thus :

Denoting the integrals (A) and (B) by I_1 and I_2 and integrating each by parts, we shall get

$$bI_1 + aI_2 = e^{ax} \sin bx$$

and $\quad aI_1 - bI_2 = e^{ax} \cos bx$,

from which I_1 and I_2 can easily be determined.

Note 2. Exactly in the same way the integrals $\int e^{ax} \cos(bx + c)\, dx$ and $\int e^{ax} \sin(bx + c)\, dx$ can be evaluated.

3.5. Standard Intergals.

(C) $\displaystyle\int \sqrt{x^2 + a^2} \, dx = \frac{x\sqrt{x^2 + a^2}}{2} + \frac{a^2}{2} \log|(x + \sqrt{x^2 + a^2})|$

or $\displaystyle\quad = \frac{x\sqrt{x^2 + a^2}}{2} + \frac{a^2}{2} \sinh^{-1}\frac{x}{a}$.

(D) $\int \sqrt{x^2 - a^2}\, dx = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \log|(x + \sqrt{x^2 - a^2})|$

or $\qquad = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \cosh^{-1} \frac{x}{a}$.

(E) $\int \sqrt{a^2 - x^2}\, dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$

Proof. (C) Integrating by parts,

$$\int \sqrt{x^2 + a^2}\, dx = \sqrt{x^2 + a^2} \,.\, x - \int \frac{2x}{2\sqrt{(x^2 + a^2)}} \,.\, x\, dx,$$

$$= x\sqrt{x^2 + a^2} - \int \frac{x^2}{\sqrt{(x^2 + a^2)}}\, dx. \qquad \ldots (1)$$

Also, $\int \sqrt{x^2 + a^2}\, dx = \int \frac{x^2 + a^2}{\sqrt{(x^2 + a^2)}}\, dx$,

$$= \int \frac{x^2}{\sqrt{(x^2 + a^2)}}\, dx + a^2 \int \frac{dx}{\sqrt{(x^2 + a^2)}}. \qquad \ldots (2)$$

Adding (1) and (2) and dividing by 2,

$$\int \sqrt{x^2 + a^2}\, dx = \frac{x\sqrt{x^2 + a^2}}{2} + \frac{a^2}{2} \int \frac{dx}{\sqrt{(x^2 + a^2)}}$$

$$= \frac{x\sqrt{x^2 + a^2}}{2} + \frac{a^2}{2} \log|(x + \sqrt{x^2 + a^2})|.$$

[By *Art*. 2.3 (D)]

Proof. (D) $\int \sqrt{x^2 - a^2}\, dx$

$$= \sqrt{x^2 - a^2} \,.\, x - \int \frac{2x}{2\sqrt{(x^2 - a^2)}} \,.\, x\, dx,$$

$$= x\sqrt{x^2 - a^2} - \int \frac{x^2}{\sqrt{(x^2 - a^2)}}\, dx,$$

$$= x\sqrt{x^2 - a^2} - \int \frac{(x^2 - a^2) + a^2}{\sqrt{(x^2 - a^2)}}\, dx,$$

$$= x\sqrt{x^2 - a^2} - \int \frac{x^2 - a^2}{\sqrt{(x^2 - a^2)}}\, dx - a^2 \int \frac{dx}{\sqrt{(x^2 - a^2)}},$$

$$= x\sqrt{x^2 - a^2} - \int \sqrt{x^2 - a^2}\, dx - a^2 \int \frac{dx}{\sqrt{(x^2 - a^2)}}.$$

Now transposing $\int \sqrt{(x^2 - a^2)}\, dx$ to the left side and dividing by 2,

$$\int \sqrt{x^2 - a^2}\, dx = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \log |\, (x + \sqrt{x^2 - a^2})\,| .$$

[By *Art.* 2.3 (*D*)]

Note. The integral (C) can be evaluated by the method of evaluating the integral (D), and the integral (D) can also be evaluated by the method of evaluating the integral (C).

(E) Although this integral can be easily evaluated by either of the methods employed in evaluating the integrals (C) and (D) above, yet another method, the method of substitution, may be adopted in evaluating this integral.

Putting $x = a \sin\theta$, so that $dx = a \cos\theta\, d\theta$, we get

$$\int \sqrt{a^2 - x^2}\, dx = a^2 \int \cos^2\theta\, d\theta ,$$

$$= a^2 . \tfrac{1}{2} \int (1 + \cos 2\theta)\, d\theta,$$

$$= \tfrac{1}{2} a^2 [\int \cos 2\theta\, d\theta + \int d\theta] ,$$

$$= \tfrac{1}{2} a^2 [\tfrac{1}{2} \sin 2\theta + \theta],$$

$$= \tfrac{1}{2} a^2 . \sin\theta \cos\theta + \tfrac{1}{2} a^2 \theta ,$$

$$= \tfrac{1}{2} a^2 \frac{x}{a} \sqrt{1 - \frac{x^2}{a^2}} + \tfrac{1}{2} a^2 \sin^{-1} \frac{x}{a},$$

$$= \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} .$$

Note. The integrals (C) and (D) can also be evaluated by putting $x = a \sinh z$ and $x = a \cosh z$ respectively.

3.6. $\int \sqrt{ax^2 + bx + c}\, dx.\quad (a \neq 0)$

To integrate this, express $ax^2 + bx + c$ as the sum or difference of two squares, as the case may be ; that is, express $ax^2 + bx + c$ in either of the forms $a\{(x + l)^2 \pm m^2\}$ or $a'\{m^2 - (x + l)^2\}$ and then substitute z for $x + l$. Now the integral reduces to one of the forms (C), (D) or (E) discussed above. This is illustrated in Ex. 3 of Art 3.10.

3.7. $\int (px + q)\sqrt{ax^2 + bx + c}\, dx.\quad (a \neq 0)$

To integrate this, put $px + q = \frac{p}{2a}(2ax + b) + \left(q - \frac{bp}{2a}\right)$;

then the integral reduces to the sum of two integrals, the first of which can be immediately integrated by putting $z = ax^2 + bx + c$, and the second is of the form of the previous article. This is illustrated in Ex. 4 of Art. 3.10.

3.8. $\int e^x\{f(x) + f'(x)\}\, dx.$

Integrating by parts $\int e^x f(x)\, dx$, we have

$\int e^x f(x)\, dx = \int f(x) e^x dx = f(x) e^x - \int f'(x) e^x dx.$

$\therefore$ transposing, $\int e^x\{f(x) + f'(x)\}\, dx = e^x f(x).$

Alternatively, we may integrate by parts $\int e^x f'(x)\, dx$, and derive the same result.

Note. $\int e^x \phi(x)\, dx$, when $\phi(x)$ can be broken up as *the sum of two functions of x such that one is the differential coefficient of the other, can be easily integrated as above.*

3.9. Generalized rule for Integration by parts.

Let u and v be two differentiable functions of x (differentiable n times), and let us denote

$$\frac{d}{dx}(u) \text{ by } u' \text{ and } \int v\, dx \text{ by } v_1,$$

$$\frac{d}{dx}(u') \text{ by } u'' \text{ and } \int v_1\,dx \text{ by } v_2, \text{ etc.}$$

Now, $\int uv\,dx = uv_1 - \int u'v_1\,dx$ (by integrating by parts) ... (1)

Again, $\int u'v_1\,dx = u'v_2 - \int u''v_2\,dx,$ ... (2)

$$\int u''v_2\,dx = u''v_3 - \int u'''v_3\,dx, \quad \dots (3)$$

$$\int u'''v_3\,dx = u'''v_4\,dx - \int u^4 v_4\,dx, \quad \dots (4)$$

where u^4 denotes u''''.

Combining (1), (2), (3) and (4) we get

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + (-1)^4 \int u^4 v_4\,dx \quad \dots (5)$$

And generally

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \dots + (-1)^{n-1} u^{n-1} v + (-1)^n \int u^n v_n\,dx \quad \dots (6)$$

where u^n denotes u with n dashes.

Illustration.

Integrate $\int x^4 \cos x\,dx.$

$I = x^4 \sin x - 4x^3(-\cos x) + 12x^2(-\sin x) - 24x(\cos x) + 24 \sin x$

$= x^4 \sin x + 4x^3 \cos x - 12x^2 \sin x - 24x \cos x + 24 \sin x$

3.10. Illustrative Examples.

Ex. 1. *Integrate* $\int e^{2x} \sin 3x \cos x\,dx.$

$I = \frac{1}{2}\int e^{2x} . 2 \sin 3x \cos x\,dx,$

$= \frac{1}{2}\int e^{2x}(\sin 4x + \sin 2x)\,dx,$

$= \frac{1}{2}[\int e^{2x} \sin 4x\,dx + \int e^{2x} \sin 2x\,dx],$

$= \frac{1}{2}\left[\frac{e^{2x}}{\sqrt{(20)}}\sin(4x - \tan^{-1} 2) + \frac{e^{2x}}{\sqrt{8}}\sin(2x - \tan^{-1} 1)\right],$

$$= \frac{e^{2x}}{2}\left[\frac{1}{\sqrt{(20)}}\sin(4x - \tan^{-1}2) + \frac{1}{\sqrt{8}}\sin\left(2x - \frac{\pi}{4}\right)\right].$$

Ex. 2. *Integrate* $\int \frac{\cos^3 x}{e^{3x}}\,dx$.

$$I = \int e^{-3x}\cos^3 x\,dx = \tfrac{1}{4}\int e^{-3x}(\cos 3x + 3\cos x)\,dx,$$

$$= \tfrac{1}{4}[\int e^{-3x}\cos 3x\,dx + 3\int e^{-3x}\cos x\,dx],$$

$$= \frac{1}{4}\left[\frac{e^{-3x}}{18}(-3\cos 3x + 3\sin 3x) + 3\frac{e^{-3x}}{10}(-3\cos x + \sin x)\right],$$

$$= \frac{e^{-3x}}{8}\left\{\frac{1}{3}(\sin 3x - \text{co}(\quad)) + \frac{3}{5}(\sin x - 3\cos x)\right\}.$$

Ex. 3. *Integrate* $\int \sqrt{4 + 8x - 5x^2}\,dx$.

$$I = \int \sqrt{5(\tfrac{4}{5} + \tfrac{8}{5}x - x^2)}\,dx,$$

$$= \sqrt{5}\int \sqrt{\tfrac{36}{25} - (\tfrac{16}{25} - \tfrac{8}{5}x + x^2)}\,dx,$$

$$= \sqrt{5}\int \sqrt{(\tfrac{6}{5})^2 - (x - \tfrac{4}{5})^2}\,dx,$$

$$= \sqrt{5}\int \sqrt{a^2 - z^2}\,dz, \text{ (putting } z = x - \tfrac{4}{5} \text{ and } a = \tfrac{6}{5})$$

$$= \sqrt{5}\left[\frac{z\sqrt{a^2 - z^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{z}{a}\right], \qquad [\textit{By Art. } 3.4\,(E)]$$

$$= \sqrt{5}\left[\frac{(5x-4)\sqrt{4 + 8x - 5x^2}}{10\sqrt{5}} + \frac{18}{25}\sin^{-1}\left(\frac{5x-4}{6}\right)\right],$$

on restoring the values of a and z and simplifying

$$= \frac{1}{10}(5x - 4)\sqrt{4 + 8x - 5x^2} + \frac{18}{5\sqrt{5}}\sin^{-1}\left(\frac{5x-4}{6}\right).$$

Ex. 4. *Integrate* $\int (3x - 2)\sqrt{x^2 - x + 1}\,dx$.

Since $3x - 2 = \frac{3}{2}(2x - 1) - \frac{1}{2}$,

$$\therefore I = \tfrac{3}{2}\int (2x - 1)\sqrt{x^2 - x + 1}\,dx - \tfrac{1}{2}\int \sqrt{x^2 - x + 1}\,dx.$$

To evaluate the 1st integral,

$$\text{put } z = x^2 - x + 1 . \therefore \quad dz = (2x - 1)\,dx .$$

$$\therefore \text{ 1st integral } = \int \sqrt{z}\,dz = \tfrac{2}{3} z^{\frac{3}{2}} = \tfrac{2}{3}(x^2 - x + 1)^{\frac{3}{2}};$$

$$\text{2nd integral } = \int \sqrt{(x - \tfrac{1}{2})^2 + \tfrac{3}{4}}\,dx .$$

$$= \int \sqrt{(z^2 + a^2)}\,dz, \text{ putting } z = x - \tfrac{1}{2} \text{ and } a^2 = \tfrac{3}{4}$$

$$= \frac{z\sqrt{(z^2 + a^2)}}{2} + \frac{a^2}{2}\log(z + \sqrt{z^2 + a^2}),$$

$$= \tfrac{1}{4}(2x - 1)\sqrt{x^2 - x + 1} + \tfrac{3}{8}\log(x - \tfrac{1}{2} + \sqrt{x^2 - x + 1}) .$$

$$\therefore I = (x^2 - x + 1)^{\frac{3}{2}} - \tfrac{1}{8}(2x - 1)\sqrt{x^2 - x + 1},$$

$$\tfrac{3}{16}\log|\,(x - \tfrac{1}{2} + \sqrt{x^2 - x + 1})\,| .$$

Ex. 5. *Integrate* $\displaystyle\int \frac{x^2 + x + 1}{\sqrt{(x^2 + 2x + 3)}}\,dx$.

$$I = \int \frac{(x^2 + 2x + 3) - (x + 2)}{\sqrt{(x^2 + 2x + 3)}}\,dx,$$

$$= \int \frac{x^2 + 2x + 3}{\sqrt{(x^2 + 2x + 3)}}\,dx - \int \frac{x + 2}{\sqrt{(x^2 + 2x + 3)}}\,dx,$$

$$= \int \sqrt{x^2 + 2x + 3}\,dx - \int \frac{1/2\,(2x + 2) + 1}{\sqrt{(x^2 + 2x + 3)}}\,dx,$$

$$= \int \sqrt{(x + 1)^2 + 2}\,dx - \frac{1}{2}\int \frac{(2x + 2)\,dx}{\sqrt{(x^2 + 2x + 3)}} - \int \frac{dx}{\sqrt{\{(x + 1)^2 + 2\}}} .$$

Denoting the right-side integrals by I_1, I_2, I_3,

$$I_1 - I_3 = \int \sqrt{z^2 + a^2}\,dz - \int \frac{dz}{\sqrt{(z^2 + a^2)}}, \text{ (where } z = x + 1,\ a^2 = 2)$$

$$= \tfrac{1}{2} z\sqrt{z^2 + a^2} + \tfrac{1}{2} a^2 \log(z + \sqrt{z^2 + a^2}) - \log(z + \sqrt{z^2 + a^2})$$

$= \frac{1}{2}(x+1)\sqrt{x^2+2x+3}$, on restoring the value of z and a^2.

Putting $x^2+2x+3 = z$, so that $(2x+2)\,dx = dz$,

$$I_2 = \int \frac{dz}{\sqrt{z}} = 2\sqrt{z} = 2\sqrt{x^2+2x+3}.$$

$$\therefore\ I = \tfrac{1}{2}(x+1)\sqrt{x^2+2x+3} - \sqrt{x^2+2x+3},$$

$$= \tfrac{1}{2}(x-1)\sqrt{x^2+2x+3}.$$

Ex. 6. *Integrate* $\int \frac{xe^x}{(x+1)^2}\,dx$ [*J. E. '80*]

$$I = \int \frac{(x+1)e^x - e^x}{(x+1)^2}\,dx = \int \frac{e^x}{x+1}\,dx - \int \frac{e^x}{(x+1)^2}\,dx.$$

Integrating by parts the first integral

$$\int \frac{1}{x+1}e^x\,dx = \frac{1}{x+1}e^x + \int \frac{1}{(x+1)^2}e^x dx.$$

$$\therefore\quad I = \frac{e^x}{x+1}.$$

Ex. 7. *Prove that (if $a \neq b$),*

(i) $\int e^{ax}\sinh bx\,dx = \frac{e^{ax}}{a^2-b^2}(a\sinh bx - b\cosh bx).$

(ii) $\int e^{ax}\cosh bx\,dx = \frac{e^{ax}}{a^2-b^2}(a\cosh bx - b\sinh bx).$

(i) Integrating by parts,

$$I = \frac{e^{ax}\cosh bx}{b} - \int ae^{ax}\frac{\cosh bx}{b}\,dx,$$

$$= \frac{e^{ax}\cosh bx}{b} - \frac{a}{b}\int e^{ax}\cosh bx\,dx. \quad \ldots (1)$$

Again integrating by parts,

$$\int e^{ax}\cosh bx\,dx = \frac{e^{ax}\sinh bx}{b} - \frac{a}{b}\int e^{ax}\sinh bx\,dx,$$

$$= \frac{e^{ax}\sinh bx}{b} - \frac{a}{b}I. \quad \ldots (2)$$

From (1) and (2),

$$I = \frac{e^{ax}\cosh bx}{b} - \frac{a}{b^2}e^{ax}\sinh bx + \frac{a^2}{b^2}I.$$

Transposing,

$$\left(1 - \frac{a^2}{b^2}\right)I = \frac{e^{ax}}{b^2}(b\cosh bx - a\sinh bx).$$

$$\therefore\ I = \frac{e^{ax}}{a^2 - b^2}(a\sinh bx - b\cosh bx).$$

(ii) This integral can be evaluated in the same way.

Alternatively, we can use the exponential values of sinh x and cosh x to evaluate these integrals.

Thus, $\int e^{ax}\sinh bx\,dx = \int e^{ax}\frac{1}{2}(e^{bx} - e^{-bx})\,dx$,

$$= \tfrac{1}{2}\int\{e^{(a+b)x} - e^{(a-b)x}\}\,dx,$$

$$= \frac{1}{2}\left\{\frac{e^{(a+b)x}}{a+b} - \frac{e^{(a-b)x}}{a-b}\right\},$$

$$= \frac{1}{2}e^{ax}\left\{\frac{e^{bx}}{a+b} - \frac{e^{-bx}}{a-b}\right\},$$

$$= \frac{1}{2}e^{ax}\left\{\frac{(a-b)e^{bx} - (a+b)e^{-bx}}{a^2 - b^2}\right\},$$

$$= \frac{e^{ax}}{a^2 - b^2}[a\tfrac{1}{2}(e^{bx} - e^{-bx}) - b\tfrac{1}{2}(e^{bx} + e^{-bx})],$$

$$= \frac{e^{ax}}{a^2 - b^2}[a\sinh bx - b\cosh bx].$$

EXAMPLES III

1. Integrate the following with respect to x :-

(i) $x\sin x$. (ii) $x^2\cos x$. (iii) xe^{ax}.

(iv) $x^n\log x$. (v) x^2e^x. [*H. S.* '83] (vi) $x\sec^2 x$.

(vii) $\sin^{-1}x$. (viii) $\cos^{-1}x$. [*H. S.* '80] (ix) $\text{cosec}^{-1}x$.

(x) $\sec^{-1}x$. (xi) $\cot^{-1}x$. (xii) $\cos^{-1}(1/x)$.

(xiii) $x\sin^{-1}x$. (xiv) $x^2\tan^{-1}x$. (xv) $x\cos nx$.

(xvi) $(\log x)^2$. (xvii) $x \log x$.[H. S. '79, '86] (xviii) $\sin^{-1}\sqrt{x}$.

(xix) $\log(1 + x)^{1+x}$. (xx) $\dfrac{\log(x + 1)}{(x + 1)^2}$.

(xxi) $\log(1 + 2x^2 + x^4)$. (xxii) $\log(x^2 + 5x + 6)$.

(xxiii) $x^3 \cos 2x$. (xxiv) $x^3(\log x)^2$.

Integrate :-

2. (i) $\int x \sin^2 x \, dx$. (ii) $\int x \sin x \cos x \, dx$.

3. (i) $\int \log(x - \sqrt{x^2 - 1}) \, dx$. (ii) $\int \log(x^2 - x + 1) \, dx$.

4. (i) $\int \left\{ \dfrac{1}{\log x} - \dfrac{1}{(\log x)^2} \right\} dx$. (ii) $\int \dfrac{x}{1 + \cos x} \, dx$.

5 (i) $\int \sin x \log(\sec x + \tan x) \, dx$. [*H. S. '86*]

(ii) $\int \cos x \log(\operatorname{cosec} x + \cot x) \, dx$.

6. (i) $\int \cos 2x \log(1 + \tan x) \, dx$.

(ii) $\int \operatorname{cosec}^2 x \log \sec x \, dx$.

7. (i) $\int \sin^{-1}(3x - 4x^3) \, dx$. (ii) $\int (\sin^{-1} x)^3 \, dx$.

8. (i) $\int \cos^{-1} \dfrac{1 - x^2}{1 + x^2} \, dx$. (ii) $\int \tan^{-1} \dfrac{2x}{1 - x^2} \, dx$.

9. (i) $\int \sin^{-1} \dfrac{2x}{1 + x^2} \, dx$. (ii) $\int \tan^{-1} \dfrac{3x - x^3}{1 - 3x^2} \, dx$.

10. (i) $\int \dfrac{\cos^{-1} x}{x^3} \, dx$. (ii) $\int \tan^{-1} \sqrt{\dfrac{1 - x}{1 + x}} \, dx$.

11. (i) $\int \dfrac{x \sin^{-1} x}{\sqrt{(1 - x^2)}} \, dx$. (ii) $\int \dfrac{\sin^{-1} x}{(1 - x^2)^{3/2}} \, dx$.

12. (i) $\int e^x \sin x \, dx$. (ii) $\int e^x \cos x \, dx$.

(iii) $\int 2^x \sin x \, dx$. (iv) $\int 3^x \cos 3x \, dx$.

(v) $\int e^x \sinh x\, dx$. (vi) $\int e^x \cosh x\, dx$.

13. (i) $\int e^x \sin^2 x\, dx$. (ii) $\int e^x \sin x \sin 2x\, dx$.

14. $\int \frac{e^{m \tan^{-1} x}}{(1 + x^2)^2}\, dx$. [*Put* $\tan^{-1} x = z$.]

15. (i) $\int \frac{x + \sin x}{1 + \cos x} dx$. (ii) $\int \frac{\log (x + 1)}{\sqrt{(x + 1)}} dx$.

16. $\int \tan^{-1} \left(\frac{1 + \cos x}{\sin x} \right) dx$. [*C. P. '88*]

17. $\int \sin^{-1} \sqrt{\frac{x}{x + a}}\, dx$. [*Put* $x = a \tan^2 \theta$] [*J. E. E '79, '86*]

18. $\int e^x (\cos x + \sin x)\, dx$. [*C. P. '84*]

19. (i) $\int \frac{e^x}{x} (1 + x \log x)\, dx$. [*C. P. '82, H. S.'87*] (ii) $\int \frac{\log x\, dx}{(1 + \log x)^2}$.

20. (i) $\int e^x (\tan x - \log \cos x)\, dx$.

(ii) $\int e^x \sec x . (1 + \tan x)\, dx$.

(iii) $\int e^x \{ \log (\sec x + \tan x) + \sec x \}\, dx$.

21. (i) $\int e^x \frac{x^2 + 1}{(x + 1)^2}\, dx$. (ii) $\int e^x \frac{(1 - x)^2}{(1 + x^2)^2} dx$.

(iii) $\int e^x \frac{x - 1}{(x + 1)^3}\, dx$.

22. (i) $\int e^x \frac{1 + \sin x}{1 + \cos x}\, dx$. (ii) $\int e^x \frac{1 - \sin x}{1 - \cos x}\, dx$.

(iii) $\int e^x \frac{2 - \sin 2x}{1 - \cos 2x} dx$. (iv) $\int e^x \frac{2 + \sin 2x}{1 + \cos 2x} dx$.

23. $\int \sqrt{25 - 9x^2}\, dx$.

24. (i) $\int \sqrt{5 - 2x + x^2}\, dx$. (ii) $\int \sqrt{10 - 4x + 4x^2}\, dx$.

25. (i) $\int \sqrt{18x - 65 - x^2}\, dx$. (ii) $\int \sqrt{4 - 3x - 2x^2}\, dx$.

26. $\int \sqrt{5x^2 + 8x + 4}\, dx$.

27. $\int \frac{dx}{x + \sqrt{(x^2 - 1)}}$.

28. $\int \sqrt{2ax - x^2}\, dx$.

29. $\int \sqrt{(x - \alpha)(\beta - x)}\, dx$. [Put $x = \alpha \cos^2\theta + \beta \sin^2\theta$.]

30. (i) $\int (x - 1)\sqrt{x^2 - 1}\, dx$. (ii) $\int (x + b)\sqrt{x^2 + a^2}\, dx$.

31. (i) $\int (x - 1)\sqrt{x^2 - x + 1}\, dx$.

(ii) $\int (x + 2)\sqrt{2x^2 + 2x + 1}\, dx$.

32. (i) $\int \frac{x^2 + x + 1}{\sqrt{(1 - x^2)}}\, dx$. (ii) $\int \frac{x^2 + 2x + 3}{\sqrt{(x^2 + x + 1)}}\, dx$.

33. $\int \frac{x^3 + 2x^2 + x - 7}{\sqrt{(x^2 + 2x + 3)}}\, dx$.

34. (i) $\int \sqrt{\frac{a + x}{a - x}}\, dx$. [C. P. '85] (ii) $\int x\sqrt{\frac{a - x}{a + x}}\, dx$.

35. $\int \frac{(x + 1)\sqrt{x + 2}}{\sqrt{(x - 2)}}\, dx$.

36. If $u = \int e^{ax} \cos bx\, dx$, $v = \int e^{ax} \sin bx\, dx$,

prove that

(i) $\tan^{-1}\frac{v}{u} + \tan^{-1}\frac{b}{a} = bx$.

(ii) $(a^2 + b^2)(u^2 + v^2) = e^{2ax}$.

ANSWERS

1. (i) $-x\cos x+\sin x$. (ii) $(x^2-2)\sin x+2x\cos x$.

(iii) $\frac{e^{ax}}{a^2}(ax-1)$. (iv) $\frac{x^{n+1}}{n+1}\left[\log x-\frac{1}{n+1}\right]$.

(v) $e^x(x^2-2x+2)$. (vi) $x\tan x+\log\cos x$.

(vii) $x\sin^{-1}x+\sqrt{1-x^2}$. (viii) $x\cos^{-1}x-\sqrt{1-x^2}$.

(ix) $x\operatorname{cosec}^{-1}x+\log(x+\sqrt{x^2-1})$.

(x) $x\sec^{-1}x-\log(x+\sqrt{x^2-1})$.

(xi) $x\cot^{-1}x+\frac{1}{2}\log(1+x^2)$.

(xii) $x\sec^{-1}x-\log(x+\sqrt{x^2-1})$.

(xiii) $\frac{1}{2}x^2\sin^{-1}x-\frac{1}{4}\sin^{-1}x+\frac{1}{4}x\sqrt{1-x^2}$.

(xiv) $\frac{1}{3}x^3\tan^{-1}x-\frac{1}{6}x^2+\frac{1}{6}\log(1+x^2)$.

(xv) $\frac{x\sin nx}{n}+\frac{\cos nx}{n^2}$. (xvi) $x(\log x)^2-2x\log x+2x$.

(xvii) $\frac{1}{4}x^2(2\log x-1)$. (xviii) $(x-\frac{1}{2})\sin^{-1}\sqrt{x}+\frac{1}{2}\sqrt{x(1-x)}$

(xix) $\frac{1}{2}(1+x)^2\log(1+x)-\frac{1}{4}x(x+2)$.

(xx) $-(1+x)^{-1}[\log(1+x)+1]$.

(xxi) $2\{x\log(1+x^2)-2x+2\tan^{-1}x\}$.

(xxii) $(x+2)\log(x+2)+(x+3)\log(x+3)-2x$.

(xxiii) $\frac{1}{4}x(2x^2-3)\sin 2x+\frac{3}{8}(2x^2-1)\cos 2x$.

(xxiv) $\frac{1}{4}x^4\{(\log x)^2-\frac{1}{2}\log x+\frac{1}{8}\}$.

2. (i) $\frac{1}{8}(2x^2-2x\sin 2x-\cos 2x)$. (ii) $-\frac{1}{4}x\cos 2x+\frac{1}{8}\sin 2x$.

3. (i) $x\log(x-\sqrt{x^2-1})+\sqrt{x^2-1}$.

(ii) $(x-\frac{1}{2})\log(x^2-x+1)-2x+\sqrt{3}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)$.

4. (i) $x(\log x)^{-1}$. (ii) $x\tan\frac{1}{2}x+2\log\cos\frac{1}{2}x$.

5. (i) $x-\cos x\log(\sec x+\tan x)$.

(ii) $\sin x\log(\operatorname{cosec}x+\cot x)+x$.

6. (i) $\sin x \cos x \log(1 + \tan x) - \frac{1}{2}x + \frac{1}{2}\log(\sin x + \cos x)$.

(ii) $-\cot x \log(\sec x) + x$.

7. (i) $3(x \sin^{-1} x + \sqrt{1 - x^2})$.

(ii) $x(\sin^{-1} x)^3 + 3\sqrt{1 - x^2}(\sin^{-1} x)^2 - 6(x \sin^{-1} x + \sqrt{1 - x^2})$.

8. (i) $2x \tan^{-1} x - \log(1 + x^2)$. (ii) Same as (i).

9. (i) Same as 8 (i). (ii) $3x \tan^{-1} x - \frac{3}{2}\log(1 + x^2)$.

10. (i) $\dfrac{x\sqrt{1 - x^2} - \cos^{-1} x}{2x^2}$. (ii) $\frac{1}{2}[x \cos^{-1} x - \sqrt{(1 - x^2)}]$.

11. (i) $x - \sqrt{1 - x^2}\sin^{-1} x$. (ii) $\dfrac{x \sin^{-1} x}{\sqrt{(1 - x^2)}} + \frac{1}{2}\log(1 - x^2)$.

12. (i) $\frac{1}{2}e^x(\sin x - \cos x)$. (ii) $\frac{1}{2}e^x(\sin x + \cos x)$.

(iii) $\dfrac{2^x \sin\{x - \cot^{-1}(\log 2)\}}{\sqrt{\{1 + (\log 2)^2\}}}$.

(iv) $\dfrac{3^x\{3\sin 3x + (\log 3)\cos 3x\}}{9 + (\log 3)^2}$.

(v) $\frac{1}{4}(\cosh 2x + \sinh 2x) - \frac{1}{2}x$. (vi) $\frac{1}{4}(\cosh 2x + \sinh 2x) + \frac{1}{2}x$.

13. (i) $\frac{1}{2}e^x\{1 - \frac{1}{5}(\cos 2x + 2\sin 2x)\}$.

(ii) $\frac{1}{4}e^x\{(\cos x + \sin x) - \frac{1}{5}(\cos 3x + 3\sin 3x)\}$.

14. $\dfrac{e^{m\tan^{-1} x}}{2}\left[\dfrac{1}{m} + \dfrac{1}{m^2 + 4}\left\{m\dfrac{1 - x^2}{1 + x^2} + \dfrac{4x}{1 + x^2}\right\}\right]$.

15. (i) $x \tan \frac{1}{2}x$. (ii) $2\sqrt{x + 1}\log(x + 1) - 4\sqrt{x + 1}$.

16. $\frac{1}{2}\pi x - \frac{1}{4}x^2$. 17. $(x + a)\tan^{-1}\left(\dfrac{x}{a}\right)^{1/2} - \sqrt{ax}$.

18. $e^x \sin x$. 19. (i) $e^x \log x$. (ii) $x/(1 + \log x)$.

20. (i) $e^x \log \sec x$. (ii) $e^x \sec x$. (iii) $e^x \log(\sec x + \tan x)$.

21. (i) $e^x \dfrac{x - 1}{x + 1}$. (ii) $\dfrac{e^x}{1 + x}$. (iii) $\dfrac{e^x}{(1 + x)^2}$.

22. (i) $e^x \tan \frac{1}{2}x$. (ii) $-e^x \cot \frac{1}{2}x$. (iii) $-e^x \cot x$.

(iv) $e^x \tan x$ 23. $\dfrac{x}{2}\sqrt{25 - 9x^2} + \dfrac{25}{6}\sin^{-1}\dfrac{3x}{5}$.

24. (i) $\frac{1}{2}(x-1)\sqrt{5-2x+x^2} + 2\log(x-1+\sqrt{5-2x+x^2})$.

(ii) $\frac{1}{4}(2x-1)\sqrt{10-4x+4x^2}$

$+\frac{9}{4}\log(2x-1+\sqrt{10-4x+4x^2})$.

25. (i) $\frac{1}{2}(x-9)\sqrt{18x-65-x^2} + 8\sin^{-1}\frac{1}{4}(x-9)$.

(ii) $\frac{1}{8}(4x+3)\sqrt{4-3x-2x^2} + \frac{41\sqrt{2}}{32}\sin^{-1}\frac{4x+3}{\sqrt{41}}$.

26. $\frac{1}{10}(5x+4)\sqrt{5x^2+8x+4}$

$+\frac{2}{5\sqrt{5}}\log\{(5x+4)+\sqrt{5(5x^2+8x+4)}\}$.

27. $\frac{1}{2}\{x(x-\sqrt{x^2-1}) + \log(x+\sqrt{x^2-1})\}$.

28. $\frac{1}{2}(x-a)\sqrt{2ax-x^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x-a}{a}\right)$.

29. $\frac{1}{4}\left[(2x-\alpha-\beta)\sqrt{(x-\alpha)(\beta-x)} + (\beta-\alpha)^2\sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}}\right]$.

30. (i) $\frac{1}{3}(x^2-1)^{3/2} - \frac{1}{2}x\sqrt{x^2-1} + \frac{1}{2}\log(x+\sqrt{x^2-1})$.

(ii) $\frac{1}{3}(x^2+a^2)^{\frac{3}{2}} + \frac{1}{2}bx\sqrt{x^2+a^2}$

$+\frac{1}{2}a^2b\log(x+\sqrt{x^2+a^2})$.

31. (i) $\frac{1}{3}(x^2-x+1)^{\frac{3}{2}} - \frac{1}{8}(2x-1)\sqrt{x^2-x+1}$

$-\frac{3}{16}\log(x-\frac{1}{2}+\sqrt{x^2-x+1})$.

(ii) $\frac{1}{6}(2x^2+2x+1)^{\frac{3}{2}} + \frac{3}{8}(2x+1)\sqrt{2x^2+2x+1}$

$+\frac{[illegible]}{8\sqrt{2}}\log\{(2x+1)+\sqrt{2}(2x^2+2x+1)\}$.

32. (i) $\frac{3}{2}\sin^{-1}x - \frac{1}{2}(x+2)\sqrt{1-x^2}$.

(ii) $\frac{1}{4}(2x+5)\sqrt{x^2+x+1} + \frac{15}{8}\log\{(x+\frac{1}{2})+\sqrt{x^2+x+1}\}$.

33. $\frac{1}{3}(x^2+2x+3)^{\frac{3}{2}} - \frac{1}{2}(x+5)\sqrt{x^2+2x+3}$
$- 6\log(x+1+\sqrt{x^2+2x+3})$.

34. (i) $a\sin^{-1}\frac{x}{a} - \sqrt{a^2-x^2}$. (ii) $(\frac{1}{2}x-a)\sqrt{a^2-x^2} - \frac{1}{2}a^2\sin^{-1}\frac{x}{a}$.

35. $\frac{1}{2}(x+6)\sqrt{x^2-4} + 4\log(x+\sqrt{x^2-4})$.

CHAPTER IV

SPECIAL TRIGONOMETRIC FUNCTIONS

4.1. Standard Integrals.

(A) $\int \operatorname{cosec} x \, dx = \log \left| \tan \frac{x}{2} \right|$.

Proof. $$\int \operatorname{cosec} x \, dx = \int \frac{dx}{\sin x} = \int \frac{dx}{2 \sin \frac{1}{2}x \cos \frac{1}{2}x}$$

$$= \int \frac{\frac{1}{2} \sec^2 \frac{1}{2}x}{\tan \frac{1}{2}x} dx$$

(on multiplying the numerator and denominator by $\sec^2 \frac{1}{2}x$)

$$= \log \left| \tan \tfrac{1}{2}x \right|,$$

since the numerator is the differential coefficient of the denominator.

(B) $\int \sec x \, dx = \log \left| \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \right|$.

$$= \log \left| (\sec x + \tan x) \right|.$$

Proof. $$\int \sec x \, dx = \int \frac{dx}{\cos x} = \int \frac{dx}{\sin (\frac{1}{2}\pi + x)}$$

$$= \int \frac{dx}{2 \sin (\frac{1}{4}\pi + \frac{1}{2}x) \cos (\frac{1}{4}\pi + \frac{1}{2}x)}$$

$$= \int \frac{\frac{1}{2} \sec^2 (\frac{1}{4}\pi + \frac{1}{2}x) \, dx}{\tan (\frac{1}{4}\pi + \frac{1}{2}x)}$$

$$= \log \left| \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}x) \right| \text{ as in (A)}.$$

Note. *Alternative Methods* :

$$\int \operatorname{cosec} x \, dx = \int \frac{\operatorname{cosec} x \, (\operatorname{cosec} x - \cot x)}{\operatorname{cosec} x - \cot x} dx = \log \left| (\operatorname{cosec} x - \cot x) \right|$$

$$\int \operatorname{cosec} x \, dx = \int \frac{dx}{\sin x} = \int \frac{\sin x}{\sin^2 x} dx$$

$$= -\int \frac{d(\cos x)}{1-\cos^2 x} = -\int \frac{dz}{1-z^2}, \text{ where } z = \cos x$$

$$= \frac{1}{2}\log\frac{1-z}{1+z} = \frac{1}{2}\log\left|\frac{1-\cos x}{1+\cos x}\right| .$$

$$\int \sec x\,dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\,dx = \log|(\sec x + \tan x)| ,$$

since the numerator is the derivative of the denominator.

$$\int \sec x\,dx = \int \frac{\cos x}{\cos^2 x}\,dx = \int \frac{d(\sin x)}{1-\sin^2 x}$$

$$= \int \frac{dz}{1-z^2} = \frac{1}{2}\log\frac{1+z}{1-z}, \text{ where } z = \sin x$$

$$= \frac{1}{2}\log\left|\frac{1+\sin x}{1-\sin x}\right| .$$

$$\int \sec x\,dx = \int \frac{dx}{\cos x} = \int \frac{dx}{\cos^2 \frac{1}{2}x - \sin^2 \frac{1}{2}x}$$

$$= \int \frac{\sec^2 \frac{1}{2}x\,dx}{1-\tan^2 \frac{1}{2}x} = 2\int \frac{dz}{1-z^2}, \text{ where } z = \tan\tfrac{1}{2}x$$

$$= \log\frac{1+z}{1-z} = \log\left|\frac{1+\tan\frac{1}{2}x}{1-\tan\frac{1}{2}x}\right| .$$

It should be noted that the different forms in which the integrals cosec x and of sec x are obtained by different methods can be easily showr to be identical by elementary trigonometry.

Thus,

$$\frac{1}{2}\log\left|\frac{1-\cos x}{1+\cos x}\right| = \frac{1}{2}\log\left|\frac{2\sin^2\frac{1}{2}x}{2\cos^2\frac{1}{2}x}\right| = \frac{1}{2}\log\left|\tan^2\tfrac{1}{2}x\right|$$

$$= \log\left|\tan\tfrac{1}{2}x\right| \text{ ; etc.}$$

4.2. $\displaystyle\int \frac{dx}{a + b\cos x}$.

The given integral

$$= \int \frac{dx}{a(\cos^2\frac{1}{2}x + \sin^2\frac{1}{2}x) + b(\cos^2\frac{1}{2}x - \sin^2\frac{1}{2}x)}$$

$$= \int \frac{\sec^2 \frac{1}{2}x \, dx}{(a+b) + (a-b)\tan^2 \frac{1}{2}x}$$

(on multiplying the numerator and denominator by $\sec^2 \frac{1}{2}x$).

Case I. $a > b$.

Put $\sqrt{a-b}\,\tan \frac{1}{2}x = z$. $\therefore \frac{1}{2}\sqrt{a-b}\,\sec^2 \frac{1}{2}x\,dx = dz$.

The given integral now becomes

$$\frac{2}{\sqrt{(a-b)}}\int \frac{dz}{(a+b)+z^2}$$

$$= \frac{2}{\sqrt{(a^2-b^2)}} \tan^{-1} \frac{z}{\sqrt{(a+b)}} \quad \text{[See (A), Art. 2.3]}$$

$$= \frac{2}{\sqrt{(a^2-b^2)}} \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}} \tan \frac{x}{2}\right),$$

i.e., $$\frac{1}{\sqrt{(a^2-b^2)}} \cos^{-1}\left(\frac{b + a\cos x}{a + b\cos x}\right).$$

Case II. $a < b$.

Put $\sqrt{b-a}\,\tan \frac{1}{2}x = z$; $\therefore \frac{1}{2}\sqrt{b-a}\,\sec^2 \frac{1}{2}x\,dx = dz$.

As before, the required integral becomes

$$\frac{2}{\sqrt{(b-a)}}\int \frac{dz}{(a+b)-z^2}$$

$$= \frac{2}{\sqrt{(b-a)}} \cdot \frac{1}{2\sqrt{(b+a)}} \log \left| \left\{ \frac{\sqrt{b+a}+z}{\sqrt{(b+a)}-z} \right\} \right|$$

[*See* (C), *Art.* **2.3**.]

$$= \frac{1}{\sqrt{(b^2-a^2)}} \log \left| \left\{ \frac{\sqrt{b+a} + \sqrt{b-a}\,\tan \frac{1}{2}x}{\sqrt{(b+a)} - \sqrt{(b-a)}\,\tan \frac{1}{2}x} \right\} \right|$$

Note 1. Here it is assumed that $a > 0$, $b > 0$, if $a < 0$, $b > 0$ or $a > 0$, $b < 0$, or $a < 0$, $b < 0$, then the integral can be evaluated exactly in the same way.

Note 2. (i) If $b = a$, the integrand reduces to $(1/2a)\sec^2 \frac{1}{2}x$, the integral of which is $(1/a)\tan \frac{1}{2}x$.

(ii) If $b = -a$, the integrand reduces to $(1/2a)\operatorname{cosec}^2 \frac{x}{2}$, the integral of which is $-(1/a)\cot \frac{1}{2}x$.

Note 3. By an exactly similar process, the integral $\int \frac{dx}{a + b \sin x}$ or more generally $\int \frac{dx}{a + b\cos x + c\sin x}$ can be evaluated by breaking $\sin x$ and $\cos x$ in terms of $\frac{1}{2}x$ and then multiplying the numerator and the denominator of the integrand by $\sec^2 \frac{1}{2}x$ and substituting z for $\tan \frac{x}{2}$. This is illustrated in Examples 3 and 4 of Art. 4.8 below.

In fact, *any rational function of sin x, cos x* can be easily integrated by expressing $\sin x$ and $\cos x$ in terms of $\tan \frac{1}{2}x$, *i.e.*, by writing

$$\sin x = \frac{2\tan \frac{1}{2}x}{1 + \tan^2 \frac{1}{2}x} \text{ and } \cos x = \frac{1 - \tan^2 \frac{1}{2}x}{1 + \tan^2 \frac{1}{2}x}$$

and then putting $\tan \frac{1}{2}x = z$.

Similar integrals involving *hyperbolic functions* can be evaluated by an exactly similar process.

4.3. Positive integral powers of sine and cosine.

(A) *Odd positive index.*

Any odd positive power of sines and cosines can be integrated immediately by substituting $\cos x = z$ and $\sin x = z$ respectively as shown below.

Ex. (i). $\int \sin^3 x\, dx = \int \sin^2 x \sin x\, dx = -\int (1 - \cos^2 x)\, d(\cos x)$

$= -\int (1 - z^2)\, dz$ [putting z for $\cos x$]

$= -(z - \frac{1}{3}z^3) = -(\cos x - \frac{1}{3}\cos^3 x)$.

Ex. (ii). $\int \cos^5 x\, dx = \int \cos^4 x \cos x\, dx = \int (1 - \sin^2 x)^2\, d(\sin x)$

[*H. S.* '81]

$= \int (1 - z^2)^2\, dx$ [putting z for $\sin x$]

$= \int (1 - 2z^2 + z^4)\, dz = z - \frac{2}{3}z^3 + \frac{1}{5}z^5$

$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x$.

(B) *Even positive index.*

In order to integrate any even positive power of sine and cosine, we should first express it in terms of multiple angles by means of trigonometry and then integrate it.

Ex. (iii). *Integrate* $\int \cos^4 x \, dx$.

$$\cos^4 x = \{\tfrac{1}{2}(1 + \cos 2x)\}^2 = \tfrac{1}{4}\{1 + 2\cos 2x + \cos^2 2x\}$$
$$= \tfrac{1}{4}[1 + 2\cos 2x + \tfrac{1}{2}(1 + \cos 4x)]$$
$$= \tfrac{3}{8} + \tfrac{1}{2}\cos 2x + \tfrac{1}{8}\cos 4x.$$
$$\therefore \int \cos^4 x \, dx = \int (\tfrac{3}{8} + \tfrac{1}{2}\cos 2x + \tfrac{1}{8}\cos 4x)\, dx$$
$$= \tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x.$$

Note 1. It should be noted that when the index is large, it would be more convenient to express the powers of sines or cosines of angles in terms of multiple angles by the use of De Moivre's theorem, as shown below.

Ex. (iv). *Integrate* $\int \sin^8 x \, dx$.

Let $\cos x + i \sin x = y$, then $\cos x - i\sin x = \dfrac{1}{y}$. $\quad \therefore \cos nx + i \sin nx = y^n$, $\quad \cos nx - i \sin nx = \dfrac{1}{y^n}$.

$$\therefore \quad y + \frac{1}{y} = 2\cos x \quad y^n + \frac{1}{y^n} = 2\cos nx$$
$$y - \frac{1}{y} = 2i\sin x \quad y^n - \frac{1}{y^n} = 2i\sin nx.$$

$\therefore \; 2^8 i^8 \sin^8 x$

$$= \left(y - \frac{1}{y}\right)^8$$
$$= \left(y^8 + \frac{1}{y^8}\right) - 8\left(y^6 + \frac{1}{y^6}\right) + 28\left(y^4 - \frac{1}{y^4}\right) - 56\left(y^2 - \frac{1}{y^2}\right) + 70$$
$$= 2\cos 8x - 8.2\cos 6x + 28.2\cos 4x - 56.2\cos 2x + 70.$$
$$\therefore \sin^8 x = 2^{-7}(\cos 8x - 8\cos 6x + 28\cos 4x - 56\cos 2x + 35).$$
$$\therefore \int \sin^8 x \, dx = 2^{-7}\int(\cos 8x - 8\cos 6x + 28\cos 4x - 56\cos 2x + 35)\, dx$$

$$= \frac{1}{2^7}\left[\frac{\sin 8x}{8} - \frac{8\sin 6x}{6} + 28\frac{\sin 4x}{4} - 56\frac{\sin 2x}{2} + 35x\right]$$

$$= \frac{1}{2^7}\left[\tfrac{1}{8}\sin 8x - \tfrac{4}{3}\sin 6x + 7\sin 4x - 28\sin 2x + 35x\right].$$

Note 2. When the index is a large odd positive integer, then also we can first express the function in terms of multiple angles as above and then integrate it; but in this case, it is better to adopt the method shown above in (A), when the index is small.

Thus $\int \sin^3 x\, dx = \int \frac{1}{4}(3\sin x - \sin 3x)\, dx = -\frac{3}{4}\cos + \frac{1}{12}\cos 3x.$

4.4. Products of positive integral powers of sine and cosine.

Any product of the form $\sin^p x \cos^q x$ admits of immediate integration as in Sec. A, Art. 4.3, whenever *either p or q is a positive odd integer*, whatever the other may be. But when *both p and q are positive even indices*, we may first express the function as the sum of a series of sines or cosines of multiples of x as in Sec. B, Art. 4.3, and then integrate it.

Ex. (i). *Integrate* $\int \sin^2 x \cos^5 x\, dx$.

$$I = \int \sin^2 x \cos^4 x \cos x\, dx$$
$$= \int \sin^2 x (1 - \sin^2 x)^2\, d(\sin x)$$
$$= \int z^2 (1 - z^2)^2\, dz, \text{ [putting } z = \sin x]$$
$$= \int (z^2 - 2z^4 + z^6)\, dz$$
$$= \tfrac{1}{3}z^3 - \tfrac{2}{5}z^5 + \tfrac{1}{7}z^7$$
$$= \tfrac{1}{3}\sin^3 x - \tfrac{2}{5}\sin^5 x + \tfrac{1}{7}\sin^7 x.$$

Ex. (ii). *Integrate* $\int \sin^4 x \cos^2 x\, dx$.

Let $\cos x + i\sin x = y$, then $\cos x - i\sin x = \dfrac{1}{y}$ $\quad \therefore \cos nx + i\sin nx = y^n$, $\cos nx - i\sin nx = \dfrac{1}{y^n}$.

$$\therefore\ y + \frac{1}{y} = 2\cos x \qquad y^n + \frac{1}{y^n} = 2\cos nx,$$

$$y - \frac{1}{y} = 2i\sin x \qquad y^n - \frac{1}{y^n} = 2i\sin nx.$$

$\therefore\ 2^6 i^4 \sin^4 x \cos^2 x$

$$= \left(y-\frac{1}{y}\right)^4 \left(y+\frac{1}{y}\right)^2 = \left(y-\frac{1}{y}\right)^2 \left(y^2-\frac{1}{y^2}\right)^2$$

$$= \left(y^2 - 2 + \frac{1}{y^2}\right)\left(y^4 - 2 + \frac{1}{y^4}\right)$$

$$= \left(y^6 + \frac{1}{y^6}\right) - 2\left(y^4 + \frac{1}{y^4}\right) - \left(y^2 + \frac{1}{y^2}\right) + 4$$

$$= 2\cos 6x - 2.2\cos 4x - 2\cos 2x + 4.$$

$\therefore\quad \sin^4 x \cos^2 x = 2^{-5}[\cos 6x - 2\cos 4x - \cos 2x + 2].$

$\therefore\quad \int \sin^4 x \cos^2 x\, dx = 2^{-5}\int(\cos 6x - 2\cos 4x - \cos 2x + 2)\, dx$

$$= \frac{1}{2^5}\left[\frac{\sin 6x}{6} - \frac{2\sin 4x}{4} - \frac{\sin 2x}{2} + 2x\right].$$

Note. *The expression* $\sin^p x\ \cos^q x$ *also admits of immediate integration* in terms of $\tan x$ or $\cot x$ if $p + q$ *be a negative even integer*, whatever p and q may be. In this case, the best substitution is $\tan x$ or $\cot x = z$. For other case of $\sin^p x \cos^q x$, a reduction formula is generally required. (See § 8.14 — 8.17.)

Ex. (iii). *Integrate* $\int \frac{\sin^2 x}{\cos^6 x}\, dx$.

Here, $p + q = 2 - 6 = -4$. $\therefore$ put $\tan x = z$, then $\sec^2 x\, dx = dz$.

Now, $I = \int \tan^2 x . \sec^4 x\, dx$

$$= \int z^2(1 + z^2)\, dz = \tfrac{1}{3}z^3 + \tfrac{1}{5}z^5$$

$$= \tfrac{1}{3}\tan^3 x + \tfrac{1}{5}\tan^5 x.$$

Ex. (iv). *Integrate* $\int \frac{dx}{\sin^{1/2} x \cos^{7/2} x}$.

Here, $p + q = -\frac{1}{2} - \frac{7}{2} = -4$. $\therefore$ put $\tan x = z$, then $\sec^2 x\, dx = dz$.

Now, $\quad I = \int \frac{\sec^4 x\, dx}{\tan^{1/2} x} = \int \frac{1 + z^2}{z^{1/2}}\, dz$

$$= \int(z^{-1/2} + z^{3/2})\, dz = 2z^{1/2} + \tfrac{2}{5}z^{5/2}$$

$$= 2\tan^{1/2} x + \tfrac{2}{5}\tan^{5/2} x.$$

4.5. Integral powers of tangent and cotangent.

Any integral power of tangent and cotangent can be readily integrated. Thus,

(i) $\int \tan^3 x\, dx = \int \tan x . \tan^2 x\, dx = \int \tan x (\sec^2 x - 1)\, dx$

$$= \int \tan x\, d (\tan x) - \int \tan x\, dx = \tfrac{1}{2} \tan^2 x - \log \sec x .$$

(ii) $\int \cot^4 x\, dx = \int \cot^2 x (\operatorname{cosec}^2 x - 1)\, dx$

$$= \int \cot^2 x \operatorname{cosec}^2 z\, dx - \int \cot^2 x\, dx$$

$$= - \int \cot^2 x\, d (\cot x) - \int (\operatorname{cosec}^2 x - 1)\, dx$$

$$= - \tfrac{1}{3} \cot^3 x + \cot x + x .$$

4.6. Positive integral powers of secant and cosecant.

(A) *Even positive index.*

Even positive powers of secant or cosecant admit of immediate integration in terms of $\tan x$ or $\cot x$. Thus,

(i) $\int \sec^4 x\, dx = \int (1 + \tan^2 x) \sec^2 x\, dx$

$$= \int \sec^2 x\, dx + \int \tan^2 x\, d (\tan x)$$

$$= \tan x + \tfrac{1}{3} \tan^3 x.$$

(ii) $\int \operatorname{cosec}^6 x\, dx = \int \operatorname{cosec}^4 x . \operatorname{cosec}^2 x\, dx$

$$= \int (1 + \cot^2 x)^2 \operatorname{cosec}^2 x\, dx$$

$$= - \int (1 + 2 \cot^2 x + \cot^4 x)\, d (\cot x)$$

$$= - \cot x - \tfrac{2}{3} \cot^3 x - \tfrac{1}{5} \cot^5 x .$$

(B) *Odd positive index.*

Odd positive powers of secant and cosecant are to be integrated by the application of the rule of integration by parts.

(iii) $\int \sec^3 x\, dx = \int \sec x . \sec^2 x\, dx = \sec x \tan x - \int \sec x \tan^2 x\, dx$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1)\, dx$$

$$= \sec x \tan x + \int \sec x\, dx - \int \sec^3 x\, dx .$$

$\therefore$ transposing $\int \sec^3 x\, dx$ to the left side, writing the value of $\int \sec x\, dx$, and dividing by 2, we get

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right).$$

(iv) $\int \sec^5 x\,dx = \int \sec^3 x \sec^2 x\,dx$

$$= \sec^3 x \tan x - \int 3\sec^3 x \tan^2 x\,dx$$

$$= \sec^3 x \tan x - 3\int \sec^3 x(\sec^2 x - 1)\,dx$$

$$= \sec^3 x \tan x + 3\int \sec^3 x\,dx - 3\int \sec^5 x\,dx.$$

Now, transposing $3\int \sec^5 x\,dx$ and writing the value of $\int \sec^3 x\,dx$ we get ultimately

$$\int \sec^5 x\,dx = \frac{\tan x \sec^3 x}{4} + \frac{3}{4}\,\frac{\tan x \sec x}{2} + \frac{3}{4}\cdot\frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right).$$

(v) $\int \operatorname{cosec}^3 x\,dx = \int \operatorname{cosec} x \operatorname{cosec}^2 x\,dx$

$$= -\operatorname{cosec} x \cot x - \int \operatorname{cosec} x \cot^2 x\,dx$$

$$= -\operatorname{cosec} x \cot x - \int \operatorname{cosec} x(\operatorname{cosec}^2 x - 1)\,dx$$

$$= -\operatorname{cosec} x \cot x + \int \operatorname{cosec} x\,dx - \int \operatorname{cosec}^3 x\,dx.$$

$\therefore$ transposing $\int \operatorname{cosec}^3 x\,dx$ and writing the value of $\int \operatorname{cosec} x\,dx$,

$$\int \operatorname{cosec}^3 x\,dx = -\tfrac{1}{2}\operatorname{cosec} x \cot x + \tfrac{1}{2}\log\tan\tfrac{1}{2}x.$$

4.7. Hyperbolic Functions.

(i) $\int \sinh x\,dx = \int \frac{1}{2}(e^x - e^{-x})\,dx = \frac{1}{2}(e^x + e^{-x}) = \cosh x.$

(ii) $\int \cosh x\,dx = \int \frac{1}{2}(e^x + e^{-x})\,dx = \frac{1}{2}(e^x - e^{-x}) = \sinh x.$

(iii) $\int \tanh x\,dx = \int \frac{\sinh x}{\cosh x}\,dx = \log|(\cosh x)|.$

(iv) $\int \coth x\,dx = \int \frac{\cosh x}{\sinh x}\,dx = \log|(\sinh x)|.$

(v) $\int \operatorname{cosech} x\,dx = \int \frac{dx}{\sinh x} = 2\int \frac{dx}{e^x - e^{-x}}$

$$= 2\int \frac{e^x\,dx}{e^{2x} - 1}$$

$$= \int \left(\frac{1}{e^x - 1} - \frac{1}{e^x + 1} \right) dx\,(e^x)$$

$$= \log \left| \frac{e^x - 1}{e^x + 1} \right|$$

$$= \log \left| \tanh \tfrac{1}{2} x \right|$$

(on dividing the numerator and denominator by $e^{x/2}$).

(vi) sech x dx $= \int \frac{dx}{\cosh x} = 2\int \frac{e^x}{1 + e^{2x}}\,dx$

$$= 2\int \frac{d\ (e^x)}{1 + e^{2x}} = 2 \tan^{-1} (e^x)$$

$$= 2 \tan^{-1} (\cosh x + \sinh x).$$

Otherwise :

$$\int \operatorname{sech} x\,dx = \int \frac{dx}{\cosh x} = \int \frac{dx}{\cosh^2 \frac{1}{2}x + \sinh^2 \frac{1}{2}x}$$

$$= 2 \int \frac{\frac{1}{2} \operatorname{sech}^2 \frac{1}{2}x}{1 + \tanh^2 \frac{1}{2}x}\,dx$$

$$= \int \frac{dz}{1 + z^2} \quad [\text{ on putting } z = \tanh \tfrac{1}{2}x]$$

$$= 2 \tan^{-1} z = 2 \tan^{-1} (\tanh \tfrac{1}{2}x).$$

(vii) $\int \operatorname{sech}^2 x\,dx = \tanh x$.

(viii) $\int \operatorname{cosech}^2 x\,dx = - \coth x$.

(ix) $\int \operatorname{sech} x \tanh x\,dx = - \operatorname{sech} x$.

(x) $\int \operatorname{cosech} x \coth x\,dx = - \operatorname{cosech} x$.

4.8. Illustrative Examples.

Ex. 1. *Integrate* $\int \frac{dx}{\sin x + \cos x}$.

$$I = \int \frac{dx}{2 \sin \frac{1}{2}x \cos \frac{1}{2}x + \cos^2 \frac{1}{2}x - \sin^2 \frac{1}{2}x}$$

$$= \int \frac{\sec^2 \frac{1}{2} x \, dx}{2 \tan \frac{1}{2}x + 1 - \tan^2 \frac{1}{2}x}$$

(*on multiplying the numerator and denominator by* $\sec^2 \frac{1}{2}x$)

$$= \int \frac{2dz}{2z + 1 - z^2} \text{ (putting } \tan \tfrac{1}{2}x = z\text{)}$$

$$\int \frac{2dz}{2 - (z^2 - 2z + 1)} = 2 \int \frac{dz}{(\sqrt{2})^2 - (z - 1)^2}$$

$$= 2 \int \frac{dy}{a^2 - y^2} \text{ where } a = \sqrt{2}, y = z - 1$$

$$= 2 \frac{1}{2a} \log \frac{a + y}{a - y} = \frac{1}{\sqrt{2}} \log \frac{\sqrt{2} + (\tan \frac{1}{2}x - 1)}{\sqrt{2} - (\tan \frac{1}{2}x - 1)}.$$

Ex. 2. *Integrate* $\int \frac{dx}{a \sin x + b \cos x}$.

Put $a = r \cos \theta$, $b = r \sin \theta$, then $a \sin x + b \cos x = r \sin (x + \theta)$.

Here, $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1} \frac{b}{a}$;

$$I = \int \frac{dx}{r \sin (x + \theta)} = \frac{1}{r} \int \operatorname{cosec} (x + \theta) \, dx$$

$$= \frac{1}{r} \int \operatorname{cosec} z \, dz, \text{ where } z = x + \theta$$

$$= \frac{1}{r} \log \left| \tan \frac{z}{2} \right|$$

$$= \frac{1}{\sqrt{(a^2 + b^2)}} \log \left| \tan \left(\frac{x}{2} + \frac{1}{2} \tan^{-1} \frac{b}{a} \right) \right|.$$

Note. Since, as above, $\sin x + \cos x = \sqrt{2} \sin \left(x + \frac{\pi}{4} \right)$.

$$\therefore \int \frac{dx}{\sin x + \cos x} = \frac{1}{\sqrt{2}} \int \text{cosec}\left(x + \frac{\pi}{4}\right) dx$$

$$= \frac{1}{\sqrt{2}} \log \left| \tan\left(\frac{x}{2} + \frac{\pi}{8}\right) \right| .$$

Ex. 3. *Integrate* $\int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x}$.

Multiply the numerator and denominator by $\sec^2 x$ and put $\tan x = z$

$$\therefore\ I = \int \frac{dz}{a^2 z^2 + b^2} = \frac{1}{a^2} \int \frac{dz}{z^2 + k^2}, \text{ where } k = \frac{b}{a}$$

$$= \frac{1}{a^2} \frac{1}{k} \tan^{-1} \frac{z}{k} = \frac{1}{ab} \tan^{-1}\left(\frac{a}{b} \tan x\right).$$

Ex. 4. *Integrate* $\int \frac{dx}{5 - 13 \sin x}$.

$$I = \frac{dx}{5(\sin^2 \tfrac{1}{2}x + \cos^2 \tfrac{1}{2}x) - 13.2 \sin \tfrac{1}{2}x \cos \tfrac{1}{2}x} .$$

Multiplying the numerator and denominator by $\sec^2 \frac{1}{2}x$, this

$$= \int \frac{\sec^2 \tfrac{1}{2}x\, dx}{5(\tan^2 \tfrac{1}{2}x + 1) - 26 \tan \tfrac{1}{2}x}$$

$$= \int \frac{2\, dz}{5z^2 - 26z + 5}, \text{ [putting } \tan \tfrac{1}{2}x = z]$$

$$= \frac{2}{5} \int \frac{dz}{(z - \frac{13}{5})^2 - (\frac{12}{5})^2}$$

$$= \frac{2}{5} \int \frac{du}{u^2 - a^2}, \text{ where } u = z - \frac{13}{5} \text{ and } a = \frac{12}{5}$$

$$= \frac{2}{5} \frac{1}{2a} \log \frac{u - a}{u + a} = \frac{1}{12} \log \frac{z - 5}{z - \frac{1}{5}}$$

$$= \frac{1}{12} \log \left| \frac{5 \tan \tfrac{1}{2}x - 25}{5 \tan \tfrac{1}{2}x - 1} \right|$$

on restoring the value of z.

Ex. 5. *Integrate* $\int \frac{dx}{13 + 3 \cos x + 4 \sin x}$.

$$I = \int \frac{dx}{13(\sin^2 \frac{1}{2}x + \cos^2 \frac{1}{2}x) + 3(\cos^2 \frac{1}{2}x - \sin^2 \frac{1}{2}x) + 4.2 \sin \frac{1}{2}x \cos \frac{1}{2}x}$$

Multiplying the numerator and denominator by $\sec^2 \frac{1}{2} x$, this

$$= \int \frac{\sec^2 \frac{1}{2} x \, dx}{10 \tan^2 \frac{1}{2}x + 8 \tan \frac{1}{2}x + 16}$$

$$= \int \frac{2\, dz}{10 z^2 + 8 z + 16}, \quad [\text{ putting } z = \tan \tfrac{1}{2} x]$$

$$= \frac{1}{5} \int \frac{dz}{(z + \frac{2}{5})^2 + (\frac{6}{5})^2} = \frac{1}{5} \int \frac{du}{u^2 + a^2},$$

where $u = z + \frac{2}{5}$, $a = \frac{6}{5}$

$$= \frac{1}{5} \frac{1}{a} \tan^{-1} \frac{u}{a} = \frac{1}{6} \tan^{-1} \frac{5z + 2}{6} = \frac{1}{6} \tan^{-1} \frac{5 \tan \frac{1}{2} x + 2}{6}$$

Ex. 6. *Integrate* $\int \frac{2 \sin x + 3 \cos x}{3 \sin x + 4 \cos x} dx$.

Let $2 \sin x + 3 \cos x$

$= l$ (denominator) + m (differential coefficient of denominator)

$$= l (3 \sin x + 4 \cos x) + m (3 \cos x - 4 \sin x)$$

$$= (3l - 4m) \sin x + (4l + 3m) \cos x .$$

Now comparing the coefficients of $\sin x$ and $\cos x$ of both sides, we get $3l - 4m = 2$ and $4l + 3m = 3$, whence $l = \frac{18}{25}$, $m = \frac{1}{25}$.

$\therefore\ 2 \sin x + 3 \cos x = \frac{18}{25} (3 \sin x + 4 \cos x) + \frac{1}{25} (3 \cos x - 4 \sin x)$.

$$\therefore\ I = \frac{18}{25} \int dx + \frac{1}{25} \int \frac{3 \cos x - 4 \sin x}{3 \sin x + 4 \cos x} dx$$

$$= \frac{18}{25} x + \frac{1}{25} \log | (3 \sin x + 4 \cos x) | .$$

Note. Generally $\int \frac{a \sin x + b \cos x}{c \sin x + d \cos x} dx$ can be treated in the same way.

Ex. 7. *Integrate* $\int \frac{1}{\sin (x - a) \sin (x - b)} dx$.

$$\frac{1}{\sin(x-a)\sin(x-b)} = \frac{1}{\sin(a-b)} \frac{\sin\{(x-b)-(x-a)\}}{\sin(x-a)\sin(x-b)}$$

$$= \frac{1}{\sin(a-b)}\left[\frac{\cos(x-a)}{\sin(x-a)} - \frac{\cos(x-b)}{\sin(x-b)}\right].$$

$$\therefore\ I = \frac{1}{\sin(a-b)}\left[\int \frac{\cos(x-a)}{\sin(x-a)}\,dx - \int \frac{\cos(x-b)}{\sin(x-b)}\,dx\right]$$

$$= \frac{1}{\sin(a-b)}\{\log\sin(x-a) - \log\sin(x-b)\}$$

$$= \frac{1}{\sin(a-b)}\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right|.$$

Ex. 8. *Integrate* $\int \frac{\tan x}{\sqrt{(a+b\tan^2 x)}}\,dx,\ b>a.$

$$I = \frac{\sin x\,dx}{\sqrt{(a\cos^2 x + b\sin^2 x}} = \int \frac{\sin x\,dx}{\sqrt{\{b-(b-a)\cos^2 x}}$$

$$= \frac{1}{\sqrt{(b-a)}}\int \frac{\sin x\,dx}{\sqrt{\left(\frac{b}{b-a}-\cos^2 x\right)}} = -\frac{1}{\sqrt{(b-a)}}\int \frac{dz}{\sqrt{(k^2-z^2)}},$$

$$\left[\text{putting } z = \cos x \text{ and } k^2 = \frac{b}{(b-a)}\right]$$

$$= \frac{1}{\sqrt{(b-a)}}\cos^{-1}\frac{z}{k} = \frac{1}{\sqrt{(b-a)}}\cos^{-1}\left[\sqrt{\frac{b-a}{b}}\cos x\right].$$

[See *Art.* 2.3 (E) *Note.*]

Ex. 9. *Integrate* $\int \frac{dx}{3+4\cosh x}$.

$$I = \int \frac{dx}{3(\cosh^2\frac{1}{2}x - \sinh^2\frac{1}{2}x) + 4(\cosh^2\frac{1}{2}x + \sinh^2\frac{1}{2}x)}$$

$$= \int \frac{dx}{7\cosh^2\frac{1}{2}x + \sinh^2\frac{1}{2}x} = \int \frac{\operatorname{sech}^2\frac{1}{2}x}{7+\tanh^2\frac{1}{2}x}\,dx$$

(on multiplying the numerator and denominator by $\operatorname{sech}^2\frac{1}{2}x$).

Put $\tanh\frac{1}{2}x = z$; then $\frac{1}{2}\operatorname{sech}^2\frac{1}{2}x\,dx = dz$.

$$\therefore\ I = 2\int \frac{dz}{7+z^2} = \frac{2}{\sqrt{7}} \tan^{-1}\frac{z}{\sqrt{7}} = \frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{1}{\sqrt{7}}\tanh\tfrac{1}{2}x\right).$$

EXAMPLES IV

Integrate with respect to x the following functions :

1. (i) $\operatorname{cosec} 2x$. (ii) $\cos^3 x$. (iii) $\sin^4 x$.

 (iv) $\sin^5 x$. (v) $\sin^2 x \cos^2 x$.

 (vi) $\sin^3 x \cos^3$. (vii) $\sin^4 x \cos^4 x$.

 (viii) $\sin^2 x \cos^3 x$. (ix) $\cos^2 x \sin^3 x$.

 (x) $\sin 2x \cos^3 x$. (xi) $\sin 3x \cos^3 x$.

 (xii) $\sec^2 x \operatorname{cosec}^2 x$. (xiii) $\sin^5 x \sec^6 x$.

2. (i) $\cot^3 x$. (ii) $\tan^4 x$. (iii) $\sec^6 x$.

 (iv) $\operatorname{cosec}^4 x$. (v) $\operatorname{cosec}^5 x$. (vi) $\tan^2 x \sec^4 x$

Evaluate the following integrals :

3. (i) $\int \frac{\cos 2x}{\sin x}\,dx$. (ii) $\int \frac{\cos 2x}{\cos x}\,dx$.

 (iii) $\int \frac{\sin x}{\sin 2x}\,dx$. (iv) $\int \frac{\cos x}{\cos 2x}\,dx$.

 (v) $\int \left(\frac{\tan x}{\cos x}\right)^4 dx$. (vi) $\int \frac{x\cos x}{\sin^2 x}\,dx$.

4. (i) $\int \sqrt{\sin x}\,\cos^3 x\,dx$. (ii) $\int \frac{dx}{\sqrt{(\sin^5 x \cos^7 x)}}$.

5. (i) $\frac{dx}{(\sin x+\cos x)^2}$. (ii) $\int \frac{dx}{1+\sin 2x}$.

6. $\int \frac{dx}{3\sin x - 4\cos x}$.

7. $\int \frac{dx}{(3\sin x + 4\cos x)^2}$.

8. (i) $\int \frac{\sin x}{\cos 2x}\,dx$. (ii) $\int \frac{\sin x}{\sin 3x}\,dx$.

9. (i) $\int \frac{dx}{\cos^2 x - \sin^2 x}$. (ii) $\int \frac{dx}{1 - \sin^4 x}$.

10. $\int \frac{\cot^2 x + 1}{\cot^2 x - 1}\, dx$.

11. $\int \frac{dx}{4\cos^3 x - 3\cos x}$.

12. (i) $\int \frac{\sin^3 x}{\cos^{2/5} x}\, dx$. (ii) $\int \frac{\sin^5 x}{\cos^2 x}\, dx$.

13. (i) $\int \frac{dx}{\sin x \cos^2 x}$. (ii) $\int \frac{dx}{\sin x \cos^3 x}$.

(iii) $\int \frac{\sin 2x\, dx}{\sin 5x \sin 3x}$. (iv) $\int \frac{dx}{\cos 3x - \cos x}$.

[*Put* $\sin^2 x + \cos^2 x$ *in the numerator of* (i) *and* (ii).]

14. (i) $\int \frac{dx}{\sin^4 x \cos^2 x}$. (ii) $\int \frac{dx}{\sin^4 x \cos^4 x}$.

[*Put* $\tan x = z$ *in* (i) *and* (ii).]

15. (i) $\int \frac{\sqrt{\tan x}}{\sin x \cos x}\, dx$. [*J. E. E.* '81]

(ii) $\int \frac{\cos x - \sin x}{\sqrt{(\sin 2x)}}\, dx$.

16. (i) $\int \frac{dx}{4 - 5\sin^2 x}$. (ii) $\int \frac{dx}{1 + \cos^2 x}$.

17. (i) $\int \frac{\sin 2x}{\sin^4 x + \cos^4 x}\, dx$. [*H. S.* '86]

(ii) $\int \frac{dx}{\sin^4 x + \cos^4 x}$.

[(ii) *Write* $\sin^4 x + \cos^4 x = \cos^2 2x + \frac{1}{2}\sin^2 2x$.]

18. (i) $\int \frac{\sin x}{\sqrt{(1 + \sin x)}}\, dx$. (ii) $\int \frac{\sin 2x\, dx}{(\sin x + \cos x)^2}$.

19. $\displaystyle\int \frac{\sin^2 x}{(1 + \cos x)^2}\, dx.$

20. (i) $\displaystyle\int \frac{dx}{1 + \tan x}.$ (ii) $\displaystyle\int \frac{dx}{1 + \cos\alpha \cos x}.$

21. (i) $\displaystyle\int \frac{\cos x}{\sin x + \cos x}\, dx.$

(ii) $\displaystyle\int \frac{\cos x\, dx}{2 \sin x + 3 \cos x}.$ [*C. P. '87*]

[*Numerator* $= \frac{1}{2}\{(\sin x + \cos x) + (\cos x - \sin x)\}$.]

(iii) $\displaystyle\int \sqrt{\left(\frac{\operatorname{cosec} x - \cot x}{\operatorname{cosec} x + \cot x}\right)} \frac{\sec x}{\sqrt{(1 + 2 \sec x)}}\, dx.$

22. $\displaystyle\int \frac{\sec x}{a + b \tan x}\, dx.$

23. $\displaystyle\int \frac{dx}{a + b \tan x} dx.$

24. $\displaystyle\int \frac{dx}{a + b \sin x}\, dx.$

25. (i) $\displaystyle\int \frac{dx}{5 + 4 \sin x}.$ (ii) $\displaystyle\int \frac{dx}{4 + 5 \sin x}.$

(iii) $\displaystyle\int \frac{dx}{4 + 3 \sinh x}.$ (iv) $\displaystyle\int \frac{dx}{4 + 3 \cosh x}.$

26. (i) $\displaystyle\int \frac{dx}{5 + 4 \cos x}.$ [*C. P. '86, '88*] (ii) $\displaystyle\int \frac{dx}{3 + 5 \cos x}.$

27. (i) $\displaystyle\int \frac{dx}{\cos\alpha + \cos x}.$ [*J. E. E '89*] (ii) $\displaystyle\int \frac{\cos x\, dx}{5 - 3 \cos x}.$

28. $\displaystyle\int \frac{dx}{a^2 - b^2 \cos^2 x}.$

29. $\displaystyle\int \frac{\sin x\, dx}{\sqrt{(a^2 \cos^2 x + b^2 \sin^2 x)}}.$

30. $\int \frac{\sin 2x\, dx}{(a + b\cos x)^2}$.

31. $\int \frac{11\cos x - 16\sin x}{2\cos x + 5\sin x}\, dx$.

32. (i) $\int \frac{dx}{1 - \cos x + \sin x}$. (ii) $\int \frac{dx}{3 + 2\sin x + \cos x}$.

33. $\int \frac{6 + 3\sin x + 14\cos x}{3 + 4\sin x + 5\cos x}\, dx$.

34. $\int \sqrt{1 + \sec x\, dx}$. [*Put* $\sqrt{2}\sin\frac{1}{2}x = z$.]

35. (i) $\int \frac{1}{\sec x + \operatorname{cosec} x}\, dx$. [C. P. '85]

(ii) $\int \frac{dx}{\sin x + \tan x}$. [C. P. '89]

36. $\left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx$.

[*Put* $\sin x - \cos x = z$ *and note* $2\sin x\cos x = 1 - (\sin x - \cos x)^2$.]

37. $\int \sqrt{\left\{\frac{\sin(x - \alpha)}{\sin(x + \alpha)}\right\}}\, dx$.

38. $\frac{x^2\, dx}{(x\sin x + \cos x)^2}$.

ANSWERS

1. (i) $\frac{1}{2}\log\tan x$. (ii) $\sin x - \frac{1}{3}\sin^3 x$. (iii) $\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x$

(iv) $\frac{2}{3}\cos^3 x - \cos x - \frac{1}{5}\cos^5 x$. (v) $\frac{1}{8}x - \frac{1}{32}\sin 4x$.

(vi) $\frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x$. (vii) $\frac{1}{128}[3x - \sin 4x + \frac{1}{8}\sin 8x]$

(viii) $\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x$. (ix) $\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x$. (x) $-\frac{2}{5}\cos^5 x$

(xi) $\frac{3}{2}\sin^2 x - \frac{7}{4}\sin^4 x + \frac{2}{3}\sin^6 x$. (xii) $\tan x - \cot x$.

(xiii) $\sec x - \frac{2}{3}\sec^3 x + \frac{1}{5}\sec^5 x$.

2. (i) $-\frac{1}{2}\cot^2 x - \log \sin x$. (ii) $\frac{1}{3}\tan^3 x - \tan x + x$.

(iii) $\tan x \left(1 + \frac{2}{3}\tan^2 x + \frac{1}{5}\tan^4 x\right)$. (iv) $-\cot x - \frac{1}{3}\cot^3 x$.

(v) $-\frac{1}{4}\cot x \operatorname{cosec}^3 x - \frac{3}{8}\cot x \operatorname{cosec} x + \frac{3}{8}\log \tan \frac{1}{2}x$.

(vi) $\tan^3 x \left(\frac{1}{3} + \frac{1}{5}\tan^2 x\right)$. 3. (i) $\log \tan \frac{1}{2}x + 2\cos x$.

(ii) $2\sin x - \log(\sec x + \tan x)$. (iii) $\frac{1}{2}\log(\sec x + \tan x)$.

(iv) $\dfrac{1}{2\sqrt{2}} \log \dfrac{1 + \sqrt{2}\sin x}{1 - \sqrt{2}\sin x}$. (v) $\frac{1}{5}\tan^5 x + \frac{1}{7}\tan^7 x$.

(vi) $\log \tan \frac{1}{2}x - x \operatorname{cosec} x$. 4. (i) $\frac{2}{21}\sqrt{\sin x}\,(7\sin x - 3\sin^3 x)$.

(ii) $2\cot^{\frac{3}{2}} x \left(\frac{1}{5}\tan^4 x + 2\tan^2 x - \frac{1}{3}\right)$. 5. (i) & (ii) $-\dfrac{1}{1 + \tan x}$.

6. $\frac{1}{5}\log \tan \left(\frac{1}{2}x - \frac{1}{2}\tan^{-1}\frac{4}{3}\right)$. 7. $-\dfrac{1}{3(3\tan x + 4)}$.

8. (i) $\dfrac{1}{2\sqrt{2}} \log \dfrac{1 + \sqrt{2}\cos x}{1 - \sqrt{2}\cos x}$. (ii) $\dfrac{1}{2\sqrt{3}} \log \dfrac{\sqrt{3} + \tan x}{\sqrt{3} - \tan x}$.

9. (i) $\frac{1}{2}\log \tan \left(\frac{1}{4}\pi + x\right)$. (ii) $\frac{1}{2}\tan x + \dfrac{1}{2\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x)$.

10. $\frac{1}{2}\log \tan \left(\frac{1}{4}\pi + x\right)$. 11. $\frac{1}{3}\log \tan \left(\frac{1}{4}\pi + \frac{3}{2}x\right)$.

12. (i) $-\frac{5}{3}\cos^{3/5} x + \frac{5}{13}\cos^{13/5} x$. (ii) $\sec x + 2\cos x - \frac{1}{3}\cos^3 x$.

13. (i) $\sec x + \log \tan \frac{1}{2}x$. (ii) $\frac{1}{2}\tan^2 x + \log \tan x$.

(iii) $\frac{1}{3}\log \sin 3x - \frac{1}{5}\log \sin 5x$.

(iv) $\frac{1}{4}[\operatorname{cosec} x - \log(\sec x + \tan x)]$.

14. (i) $\tan x - 2\cot x - \frac{1}{3}\cot^3 x$.

(ii) $\frac{1}{3}(\tan^3 x - \cot^3 x) + 3(\tan x - \cot x)$.

15. (i) $2\sqrt{\tan x}$. (ii) $\log(\cos x + \sin x + \sqrt{\sin 2x})$.

16. (i) $\dfrac{1}{4}\log \dfrac{2 + \tan x}{2 - \tan x}$. (ii) $\dfrac{1}{\sqrt{2}}\tan^{-1}\left(\dfrac{\tan x}{\sqrt{2}}\right)$.

17. (i) $\tan^{-1}(\tan^2 x)$. (ii) $\dfrac{1}{\sqrt{2}}\tan^{-1}\left(\dfrac{1}{\sqrt{2}}\tan 2x\right)$.

18. (i) $2\sqrt{1 - \sin x} - \sqrt{2}\log \tan\left(\frac{1}{4}x + \frac{1}{8}\pi\right)$.

(ii) $x + \dfrac{1}{1 + \tan x}$. 19. $2\tan \frac{1}{2}x - x$.

20. (i) $\frac{1}{2}\{x + \log(\sin x + \cos x)\}$.

(ii) $2 \operatorname{cosec} \alpha \tan^{-1}(\tan \frac{1}{2}\alpha \tan \frac{1}{2}x)$.

21. (i) $\frac{1}{2}\{x + \log(\sin x + \cos x)\}$.

(ii) $\frac{3}{13}x + \frac{2}{13}\log(2\sin x + 3\cos x)$.

(iii) $\sin^{-1}(\frac{1}{2}\sec^2 \frac{1}{2}x)$.

22. $\frac{1}{\sqrt{(a^2 + b^2)}} \log \tan\left(\frac{x}{2} + \frac{1}{2}\tan^{-1}\frac{a}{b}\right)$.

23. $\frac{a}{a^2 + b^2}x + \frac{b}{a^2 + b^2}\log(a\cos x + b\sin x)$.

24. $\frac{2}{\sqrt{(a^2 - b^2)}}\tan^{-1}\left\{\frac{a\tan\frac{1}{2}x + b}{\sqrt{(a^2 - b^2)}}\right\}$, if $a > b$;

$\frac{1}{\sqrt{(b^2 - a^2)}}\log\left\{\frac{a\tan\frac{1}{2}x + b - \sqrt{(b^2 - a^2)}}{a\tan\frac{1}{2}x + b + \sqrt{(b^2 - a^2)}}\right\}$, if $a < b$.

25. (i) $\frac{2}{3}\tan^{-1}\frac{1}{3}(5\tan\frac{1}{2}x + 4)$. (ii) $\frac{1}{3}\log\frac{2\tan\frac{1}{2}x + 1}{2\tan\frac{1}{2}x + 4}$.

(iii) $\frac{1}{5}\log\frac{1 + 2\tanh\frac{x}{2}}{4 - 2\tanh\frac{x}{2}}$. (iv) $\frac{1}{\sqrt{7}}\log\frac{\sqrt{7} + \tanh\frac{1}{2}x}{\sqrt{7} - \tanh\frac{1}{2}x}$.

26. (i) $\frac{2}{3}\tan^{-1}(\frac{1}{3}\tan\frac{1}{2}x)$. (ii) $\frac{1}{4}\log\left(\frac{2 + \tan\frac{1}{2}x}{2 - \tan\frac{1}{2}x}\right)$.

27. (i) $\frac{1}{\sin\alpha}\log\frac{\cos\frac{1}{2}(x - \alpha)}{\cos\frac{1}{2}(x + \alpha)}$. (ii) $-\frac{1}{3}x + \frac{5}{6}\tan^{-1}(2\tan\frac{1}{2}x)$.

28. $\frac{1}{a\sqrt{(a^2 - b^2)}}\tan^{-1}\left(\frac{a}{\sqrt{(a^2 - b^2)}}\tan x\right)$, if $a > b$;

$\frac{1}{2a\sqrt{(b^2 - a^2)}}\log\frac{a\tan x - \sqrt{b^2 - a^2}}{a\tan x + \sqrt{(b^2 - a^2)}}$, if $a < b$.

29. $\frac{-1}{\sqrt{(a^2 - b^2)}}\log(\sqrt{a^2 - b^2}\cos x + \sqrt{a^2\cos^2 x + b^2\sin^2 x})$,

if $a > b$; $\frac{-1}{\sqrt{(b^2 - a^2)}}\sin^{-1}\left(\frac{\sqrt{b^2 - a^2}}{b}\cos x\right)$, if $a < b$.

30. $-2b^{-2}\log(a + b\cos x) - 2ab^{-2}(a + b\cos x)^{-1}$.

31. $3 \log (2 \cos x + 5 \sin x) - 2x$. **32.** (i) $-\log (1 + \cot \frac{1}{2}x)$

(ii) $\tan^{-1} (1 + \tan \frac{1}{2}x)$. **33.** $2x + \log (3 + 4 \sin x + 5 \cos x)$.

34. $2 \sin^{-1} (\sqrt{2} \sin \frac{1}{2}x)$.

35. (i) $\frac{1}{2}[\sin x - \cos x - \frac{1}{\sqrt{2}} \log \tan (\frac{1}{2}x + \frac{1}{8}\pi)]$.

(ii) $\frac{1}{2} \log \tan \frac{1}{2}x - \frac{1}{4} \tan^2 \frac{1}{2}x$. **36.** $\sqrt{2} \sin^{-1} (\sin x - \cos x)$

37. $\cos \alpha \cos^{-1} (\cos x \sec \alpha) - \sin \alpha \log (\sin x + \sqrt{\sin^2 x - \sin^2 \alpha})$.

38. $\dfrac{\sin x - \cos x}{x \sin x + \cos x}$.

CHAPTER V

RATIONAL FRACTIONS

[*Method of breaking up into partial fractions*]

5.1. Integration of Rational Fractions.

When we have to integrate a rational fraction, say, $f(x)/\phi(x)$*, if $f(x)$ be not of a lower degree than $\phi(x)$, we shall first express $f(x)/\phi(x)$ by ordinary division in the form

$$C_p x^p + C_{p-1} x^{p-1} + \ldots + C_0 + \frac{\psi(x)}{\phi(x)},$$

where $C_p x^p + \ldots\ldots + C_0$ is the quotient, and $\psi(x)$ is the remainder and hence of a lower degree than $\phi(x)$.

$$\text{Then } \int \frac{f(x)}{\phi(x)}\,dx = C_p \frac{x^{p+1}}{p+1} + \ldots\ldots + \int \frac{\psi(x)}{\phi(x)}\,dx.$$

So we shall now consider how to integrate that rational fraction $\psi(x)/\phi(x)$ in which *the numerator is of a lower degree than the denominator.* The best way of effecting the integration is first to decompose the fraction into a number of partial fractions and then to integrate each term separately.

We shall not enter here into a detailed discussion of the theory of partial fractions for which the student is referred to treaties on Higher Algebra, but we shall briefly indicate the different methods adopted in breaking up a fraction into partial fractions according to the nature of the factors of the denominator of the fraction.

We know from the Theory of Equations that $\phi(x)$ can always be broken up into real factors which may be linear or quadratic and some of which may be repeated.

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* When $f(x)$ and $\phi(x)$ are algebraic expressions, containing terms involving positive integral powers of x only, of the form

$$a_0 + a_0 x + a_0 x^2 + \ldots + a_n x^2.$$

Thus, the general form of $\phi(x)$ is

$$A(x-\alpha)(x-\beta)(x-\gamma)^p(x+\delta)^q \ldots \{(x-l)^2+m^2\}\ldots\{(x-l')^2+m'^2\}^r.$$

Case I. *When the denominator contains factors real linear, but none repeated.*

To each non-repeated linear factor of the denominator, such as $x-a$, there corresponds a partial fraction of the from $A/(x-a)$, where A is a constant. The given fraction can be expressed as a sum of fractions of this type and the unknown constant A's can be determined easily as shown by the following examples.

Ex.1. *Integrate* $\int \frac{x^2+x-1}{x^3+x^2-6x}\,dx$. [*P. P. 1981*]

$$x^3+x^2-6x = x(x^2+x-6) = x(x+3)(x-2).$$

Let $$\frac{x^2+x-1}{x(x+3)(x-2)} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-2}.$$

Multiplying both sides by $x(x+3)(x-2)$, we get

$$x^2+x-1 = A(x+3)(x-2) + Bx(x-2) + Cx(x+3).$$

Putting $x = 0, -3, 2$ successively on both sides, we get

$$A = \tfrac{1}{6},\ B = \tfrac{1}{3}, C = \tfrac{1}{2}.$$

$\therefore$ the given integral is

$$\frac{1}{6}\int \frac{dx}{x} + \frac{1}{3}\int \frac{dx}{x+3} + \frac{1}{2}\int \frac{dx}{x-2}$$

$$= \tfrac{1}{6}\log x + \tfrac{1}{3}\log(x+3) + \tfrac{1}{2}\log(x-2).$$

Ex. 2. *Integrate* $\int \frac{x^3}{(x-a)(x-b)(x-c)}\,dx$.

Here the numerator is of the same degree as the denominator and if the numerator be divided by the denominator the fraction would be of the form

$1 + \frac{P}{Q}$, where $Q = (x-a)(x-b)(x-c)$ and P is of a lower degree than Q.

Hence, we can write

$$\frac{x^3}{(x-a)(x-b)(x-c)} = 1 + \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}. \quad \ldots \quad (1)$$

$$\therefore \quad x^3 = (x-a)(x-b)(x-c) + A(x-b)(x-c) + B(x-c)(x-a) + C(x-a)(x-b) \ldots . \quad (2)$$

Putting $x = a, b, c$ successively on both sides of the above identity (2), we get

$$A = \frac{a^3}{(a-b)(a-c)}, \; B = \frac{b^3}{(b-c)(b-a)}, \; C = \frac{c^3}{(c-a)(c-b)}.$$

$\therefore$ from (1), it follows that the given integral

$$= \int dx + \frac{a^3}{(a-b)(a-c)}\int \frac{dx}{x-a} + \frac{b^3}{(b-c)(b-a)}\int \frac{dx}{x-b} + \frac{c^3}{(c-a)(c-b)}\int \frac{dx}{x-c}$$

$$= x + \frac{a^3}{(a-b)(a-c)} \log(x-a) + \frac{b^3}{(b-c)(b-a)} \log(x-b) + \frac{c^3}{(c-a)(c-b)} \log(x-c).$$

Case II. *When the denominator contains factors, real, linear, but some repeated.*

To each p-fold linear factor, such as $(x-a)^p$, there will correspond the sum of p partial fractions of the form

$$\frac{A_p}{(x-a)^p} + \frac{A_{p-1}}{(x-a)^{p-1}} + \ldots\ldots\ldots + \frac{A_1}{(x-a)},$$

where the constants $A_p, A_{p-1}, \ldots, A_1$ can be evaluate easily.

Ex. 3. *Integrate* $\displaystyle\int \frac{x^2}{(x+1)^2(x+2)}\,dx.$

Let $$\frac{x^2}{(x+1)^2(x+2)} = \frac{A}{(x+1)^2} + \frac{B}{(x+1)} + \frac{C}{(x+2)}.$$

Multiplying both sides by $(x+1)^2(x+2)$, we get

$$x^2 = A(x+2) + B(x+1)(x+2) + C(x+1)^2.$$

Putting $x = -1, -2$ successively, we get $A = 1$, $C = 4$.

Again, equating the coefficients of x^2 on both sides,

$$B + C = 1; \qquad \therefore B = -3, \text{ since } C = 4.$$

$$\therefore \text{ the given integral } = \int \frac{dx}{(x+1)^2} - 3\int \frac{dx}{x+1} + 4\int \frac{dx}{x+2}$$

$$= -\frac{1}{x+1} - 3\log(x+1) + 4\log(x+2).$$

Note. The partial fractions in the above case can also be obtained in the following way. Denote the first power of the repeated factor, *i.e.*, $x + 1$ by z, then the fraction $= \frac{1}{z^2}\frac{(z-1)^2}{z+1}$. Now, divide the Numerator by the Denominator of the second fraction, after writing them in ascending powers of z, till the highest power of the repeated factor, *viz.*, z^2, appears in the remainder. Thus the fraction $= \frac{1}{z^2}\left(1 - 3z + \frac{4z^2}{1+z}\right) = \frac{1}{z^2} - \frac{3}{z} + \frac{4}{1+z}$.

Now replace z by $x + 1$, and the required partial fractions are obtained.

Case III. *When the denominator contains factors, real, quadratic, but none repeated.*

To each non-repeated quadratic factor, such as $x^2 + px + q$, (or, $x^2 + q, q \neq 0$) there corresponds a partial fraction of the form $(Ax + B)/(x^2 + px + q)$, the method of integration of which is explained in Art. 2.5.

Ex. 4. *Integrate* $\dfrac{x}{(x-1)(x^2+4)}\,dx$.

Let $$\frac{x}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx + C}{x^2+4}.$$

$$\therefore \quad x = A(x^2 + 4) + (Bx + C)(x - 1).$$

Putting $x = 1$ on both sides, we get $A = \frac{1}{5}$.

Equating the coefficients of x^2 and x on both sides, we get

$A + B = 0$ and $C - B = 1$; hence, $B = -\frac{1}{5}$, $C = \frac{4}{5}$.

$\therefore$ the given integral becomes

$$\frac{1}{5}\int\frac{dx}{x-1}-\frac{1}{5}\int\frac{x-4}{x^2+4}\,dx=\frac{1}{5}\int\frac{dx}{x-1}-\frac{1}{10}\int\frac{2x\,dx}{x^2+4}+\frac{4}{5}\int\frac{dx}{x^2+4}$$

$$=\frac{1}{5}\log(x-1)-\frac{1}{10}\log(x^2+4)+\frac{2}{5}\tan^{-1}\frac{x}{2}\ .$$

Ex. 5. *Integrate* $\displaystyle\int\frac{dx}{(x^2+a^2)(x^2+b^2)}$.

$$\frac{1}{(x^2+a^2)(x^2+b^2)}=\frac{1}{a^2-b^2}\left[\frac{1}{x^2+b^2}-\frac{1}{x^2+a^2}\right].$$

$\therefore$ the given integral $\displaystyle=\frac{1}{a^2-b^2}\left[\int\frac{dx}{x^2+b^2}-\int\frac{dx}{x^2+a^2}\right]$

$$=\frac{1}{a^2-b^2}\left[\frac{1}{b}\tan^{-1}\frac{x}{b}-\frac{1}{c}\tan^{-1}\frac{x}{a}\right].$$

Ex. 6. *Integrate* $\displaystyle\int\frac{dx}{x^3+1}$

Since $x^3+1=(x+1)(x^2-x+1)$,

let us assume $\displaystyle\frac{1}{x^3+1}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}$.

$\therefore\ 1=A(x^2-x+1)+(Bx+C)(x+1)$.

Putting $x=-1$, we get $A=\frac{1}{3}$.

Equating the coefficients of x^2 and the constant terms, we have

$$A+B=0 \text{ and } A+C=1.\quad \therefore\ B=-\tfrac{1}{3},\ C=\tfrac{2}{3}.$$

$\therefore$ the given integral becomes

$$\frac{1}{3}\int\frac{dx}{x+1}-\frac{1}{3}\int\frac{x-2}{x^2-x+1}\,dx$$

$$=\frac{1}{3}\int\frac{dx}{x+1}-\frac{1}{6}\int\frac{(2x-1)-3}{x^2-x+1}\,dx$$

$$=\frac{1}{3}\int\frac{dx}{x+1}-\frac{1}{6}\int\frac{2x-1}{x^2-x+1}\,dx+\frac{1}{2}\int\frac{dx}{x^2-x+1}$$

$$=\frac{1}{3}\log(x+1)-\frac{1}{6}\log(x^2-x+1)+\frac{1}{2}\int\frac{dx}{(x-\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{3}\log(x+1) - \frac{1}{6}\log(x^2 - x + 1) + \frac{1}{2}\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)$$

$$= \frac{1}{3}\log(x+1) - \frac{1}{6}\log(x^2 - x + 1) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)$$

Case IV. *When the denominator contains factors, real, quadratic, but some repeated.*

In this case we shall require the use of Reduction Formula to perform the integration, for the general discussion of which see Chapter IX.

Ex. 7. *Integrate* $\int \frac{dx}{(1+x^2)^2}$.

Although this case comes under Case (IV), it can be treated more simply as follows : Put $x = \tan\theta$.

$$\therefore I = \int \frac{\sec^2\theta\, d\theta}{\sec^4\theta} = \int \cos^2\theta\, d\theta$$

$$= \int \tfrac{1}{2}(1 + \cos 2\theta)\, d\theta = \tfrac{1}{2}(\theta + \tfrac{1}{2}\sin 2\theta)$$

$$= \frac{1}{2}\left\{\theta + \frac{1}{2}\frac{2\tan\theta}{1+\tan^2\theta}\right\} = \frac{1}{2}\left\{\tan^{-1}x + \frac{x}{1+x^2}\right\}.$$

5.2. Two Special Cases.

(A) In many cases, if *the numerator and the denominator* of a given fraction *contain even powers of* x *only*, we can first write the fraction in a simpler form by putting z for x^2, and then break it up nto partial fractions involving z, i.e., x^2, and then integrate it.

Ex. 8. *Integrate* $\int \frac{x^2}{x^4 + x^2 - 2}\, dx$.

Putting $x^2 = z$, we have

$$\frac{x^2}{x^4 + x^2 - 2} = \frac{z}{z^2 + z - 2} = \frac{z}{(z+2)(z-1)} = \frac{A}{z+2} + \frac{B}{z-1}, \text{ say}.$$

$$\therefore \quad z = A(z-1) + B(z+2).$$

Puting $z = -2$ and 1, we get respectively $A = \frac{2}{3}$, $B = \frac{1}{3}$.

$$\therefore \quad \frac{x^2}{x^4+x^2-2} = \frac{2}{3}\,\frac{1}{x^2+2} + \frac{1}{3}\,\frac{1}{x^2-1}.$$

$$\therefore \quad I = \frac{2}{3}\int\frac{dx}{x^2+2} + \frac{1}{3}\int\frac{dx}{x^2-1}$$

$$= \frac{2}{3}\,\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + \frac{1}{3}\,\frac{1}{2}\log\frac{x-1}{x+1}.$$

(B) If, in a fraction, *the numerator contains only odd powers of* x and the denominator only even powers, then it is found more convenient to change the variable first by putting $x^2 = z$ and then break it up into partial fractions as usual.

Ex. 9. *Integrate* $\displaystyle\int\frac{x^3\,dx}{x^4+3x^2+2}$.

Put $x^2 = z$. $\therefore$ $2x\,dx = dz$. $\therefore$ $x^3\,dx = \frac{1}{2}z\,dz$.

$$\therefore \; I = \frac{1}{2}\int\frac{z\,dz}{z^2+3z+2}.$$

Now, $\displaystyle\frac{z}{z^2+3z+2} = \frac{z}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2}$, say.

We determine as usual, $A = -1$, $B = 2$.

$$\therefore \; I = \frac{1}{2}\left[2\int\frac{dz}{z+2} - \int\frac{dz}{z+1}\right] = \frac{1}{2}\left[2\log(z+2) - \log(z+1)\right]$$

$$= \log(x^2+2) - \tfrac{1}{2}\log(x^2+1).$$

5.3. *Integral of the form*

$$\int\frac{dx}{(x-a)^m(x-b)^n},$$

where m *and* n *are positive integers and* a *and* b *are unequal, positive or negative,* can be evaluated by putting $x - a = z(x-b)$.

Ex. 10. *Integrate* $\displaystyle\int\frac{dx}{(x-1)^2(x-2)^3}$.

Put $x - 1 = z(x-2)$.

$$\therefore \quad x = \frac{1-2z}{1-z}. \qquad \therefore \; dx = -\frac{dz}{(1-z)^2}.$$

Hence, the integral transforms into

$$\int \frac{(1-z)^3}{z^2}\,dz = \int \frac{1-3z+3z^2-z^3}{z^2}\,dz$$

$$= -\frac{1}{z} - 3\log z + 3z - \tfrac{1}{2}z^2$$

$$= -\left(\frac{x-2}{x-1}\right) - 3\log\left(\frac{x-1}{x-2}\right) + 3\left(\frac{x-1}{x-2}\right) - \frac{1}{2}\left(\frac{x-1}{x-2}\right)^2.$$

EXAMPLES V

Integrate the following :

1. $\displaystyle\int \frac{(x-1)\,dx}{(x-2)(x-3)}$.

2. $\displaystyle\int \frac{x\,dx}{(x-a)(x-b)}$.

3. $\displaystyle\int \frac{(x-1)\,dx}{(x+2)(x-3)}$.

4. (i) $\displaystyle\int \frac{x\,dx}{x^2-12x+35}$. (ii) $\displaystyle\int \frac{3x\,dx}{x^2-x-2}$.

5. $\displaystyle\int \frac{x^2\,dx}{(x-1)(x-2)(x-3)}$.

6. (i) $\displaystyle\int \frac{(2x+3)\,dx}{x^3+x^2-2x}$. (ii) $\displaystyle\int \frac{(x^2+1)\,dx}{x(x^2-1)}$.

7. (i) $\displaystyle\int \frac{x^3\,dx}{x^2+7x+12}$. (ii) $\displaystyle\int \frac{(x-1)(x-5)}{(x-2)(x-4)}\,dx$.

8. (i) $\displaystyle\int \frac{1-3x^2}{3x-x^3}\,dx$. (ii) $\displaystyle\int \frac{x\,dx}{(3-x)(3+2x)}$.

9. (i) $\displaystyle\int \frac{x^2\,dx}{(x-a)(x-b)(x-c)}$. (ii) $\displaystyle\int \frac{dx}{(x-a)^2(x-b)}$.
[C. P. '81]

(iii) $\displaystyle\int \frac{dx}{(x-2)^2(x-1)^3}$. (iv) $\displaystyle\int \frac{dx}{(x+1)^2(x+2)^3}$.

10. (i) $\int \frac{x^2\,dx}{(x+1)(x+2)^2}$. (ii) $\int \frac{(3-x)}{x^2+x^3}\,dx$.

11. (i) $\int \frac{dx}{x^3-x^2-x+1}$. (ii) $\int \frac{dx}{x(x+1)^2}$.

12. (i) $\int \frac{dx}{(x^2-1)^2}$. (ii) $\int \frac{(x+1)\,dx}{(x-1)^2(x+2)^2}$

13. (i) $\int \frac{dx}{(x-1)^3(x+1)}$. (ii) $\int \frac{(3x+2)\,dx}{x(x+1)^3}$

14. (i) $\int \frac{dx}{1-x^3}$. (ii) $\int \frac{2+x^2}{1-x^3}\,dx$.

15. (i) $\int \frac{x}{x^4-1}\,dx.$ (ii) $\int \frac{dx}{x(x^4-1)}$

16. (i) $\int \frac{x^2}{1-x^4}\,dx$. (ii) $\int \frac{dx}{x^4-1}$.

17. (i) $\int \frac{x\,dx}{(x^2+a^2)(x^2+b^2)}$. (ii) $\int \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)}$.

18. (i) $\int \frac{x^3\,dx}{(x^2+a^2)(x^2+b^2)}$. (ii) $\int \frac{x^4\,dx}{(x^2+a^2)(x^2+b^2)}$.

19. $\int \frac{dx}{(x^2+a^2)(x+b)}$.

20. (i) $\int \frac{x\,dx}{(1+x)(1+x^2)}$. (ii) $\int \frac{x}{x^3-1}\,dx$.

21. $\int \frac{(x^2-1)}{x^4+x^2+1}\,dx$.

22. (i) $\int \frac{x^2\,dx}{x^4-x^2-12}$. (ii) $\int \frac{x\,dx}{x^4-x^2-2}$.

23. $\int \frac{dx}{(x^2+4x+5)^2}$.

$\int \frac{x^2\,dx}{(x^2+1)(2x^2+1)}$.

25. $\displaystyle\int \frac{dx}{x(1+x+x^2+x^3)}$.

26. $\displaystyle\int \frac{dx}{x^4+x^2+1}$.

$[x^4+x^2+1=(x^2+x+1)(x^2-x+1)]$

27. (i) $\displaystyle\int \frac{dx}{x^4+1}$. (ii) $\displaystyle\int \frac{x^2+1}{x^4+1}\,dx$.

$[x^4+1=(x^2+x\sqrt{2}+1)(x^2-x\sqrt{2}+1)]$

28. (i) $\displaystyle\int \frac{dx}{\cos x(5+3\cos x)}$. (ii) $\displaystyle\int \frac{dx}{\sin 2x-\sin x}$.

29. (i) $\displaystyle\int \frac{dx}{1+3e^x+2e^{2x}}$. (ii) $\displaystyle\int \frac{e^x\,dx}{e^x-3e^{-x}+2}$

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30. $\displaystyle\int \frac{dx}{\sin x(3+2\cos x)}$. [Put $\cos x = z$.]

31. Show that $\displaystyle\int \frac{dx}{\phi(x)}$

$$=\frac{1}{n!}\left[\log x+\sum_{r=1}^{n}(-1)^r\binom{n}{r}\log(x+r)\right],$$

where $\phi(x)=\prod_{r=0}^{n}(x+r)$.

32. Show that $\displaystyle\int \frac{x^{n-1}}{f(x)}\,dx$

$$=\sum_{r=1}^{n}\frac{(a_r)^{n-1}}{f'(a_r)}\log(x-a_r),$$

where $f(x)=\prod_{r=1}^{n}(x-a_r)$, $[a_i\neq a_k$ if $i\neq k]$.

ANSWERS

1. $2\log(x-3)-\log(x-2)$.

2. $\dfrac{1}{a-b}\{a\log(x-a)-b\log(x-b)\}$.

3. $\frac{1}{5}\{3\log(x+2)+2\log(x-3)\}$.

4. (i) $\frac{1}{2}\{7\log(x-7)-5\log(x-5)\}$. (ii) $\log\{(x-2)^2(x+1)\}$

5. $\frac{1}{2}\log(x-1)-4\log(x-2)+\frac{9}{2}\log(x-3)$.

6. (i) $\frac{5}{3}\log(x-1)-\frac{3}{2}\log x-\frac{1}{6}\log(x+2)$. (ii) $\log(x^2-1)-\log x$.

7. (i) $\frac{1}{2}x^2-7x-27\log(x+3)+64\log(x+4)$. (ii) $x-\dfrac{3}{2}\log\dfrac{x-4}{x-2}$.

8. (i) $\frac{1}{3}\log\{x(x^2-3)^4\}$. (ii) $-\frac{1}{3}\log(3-x)-\frac{1}{6}\log(3+2x)$.

9. (i) $\dfrac{a^2}{(a-b)(a-c)}\log(x-a)+\dfrac{b^2}{(b-c)(b-a)}\log(x-b)$

$$+\frac{c^2}{(c-a)(c-b)}\log(x-c).$$

(ii) $\dfrac{1}{(b-a)(x-a)}+\dfrac{1}{(b-a)^2}\log\dfrac{x-b}{x-a}$.

iii) $-\dfrac{x-1}{x-2}-3\log\dfrac{x-2}{x-1}+3\dfrac{x-2}{x-1}-\dfrac{1}{2}\left(\dfrac{x-2}{x-1}\right)^2$.

(iv) $-\dfrac{x+2}{x+1}-3\log\dfrac{x+1}{x+2}+3\dfrac{x+1}{x+2}-\dfrac{1}{2}\left(\dfrac{x+1}{x+2}\right)^2$.

10. (i) $\dfrac{4}{x+2}+\log(x+1)$. (ii) $-\dfrac{3}{x}-4\log x+4\log(x+1)$.

11. (i) $-\dfrac{1}{2(x-1)}+\dfrac{1}{4}\log\dfrac{x+1}{x-1}$. (ii) $\dfrac{1}{x+1}+\log\dfrac{x}{x+1}$.

12. (i) $\dfrac{1}{4}\log\dfrac{x+1}{x-1}-\dfrac{1}{2}\dfrac{x}{x^2-1}$.

(ii) $\dfrac{1}{9}\left(\dfrac{1}{x+2}-\dfrac{2}{x-1}-\dfrac{1}{3}\log\dfrac{x-1}{x+2}\right)$.

13. (i) $\dfrac{1}{4}\left\{\dfrac{x-2}{(x-1)^2}+\dfrac{1}{2}\log\dfrac{x-1}{x+1}\right\}$. (ii) $2\log\dfrac{x}{x+1}+\dfrac{4x+3}{2(x+1)^2}$.

14. (i) $\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} - \frac{1}{3}\log\frac{1-x}{\sqrt{(1+x+x^2)}}$.

(ii) $-\log(1-x) + \frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}$.

15. (i) $\frac{1}{4}\{\log(x^2-1) - \log(x^2+1)\}$. (ii) $\frac{1}{4}\log(x^4-1) - \log x$.

16. (i) $\frac{1}{4}\{\log(1+x) - \log(1-x)\} - \frac{1}{2}\tan^{-1}x$.

(ii) $\frac{1}{4}\log\frac{x-1}{x+1} - \frac{1}{2}\tan^{-1}x$.

17. (i) $\frac{1}{2(a^2-b^2)}\log\frac{x^2+b^2}{x^2+a^2}$. (ii) $\frac{1}{a^2-b^2}\left\{a\tan^{-1}\frac{x}{a} - b\tan^{-1}\frac{x}{b}\right\}$.

18. (i) $\frac{1}{2(a^2-b^2)}\{a^2\log(x^2+a^2) - b^2\log(x^2+b^2)\}$.

(ii) $x + \frac{a^3}{b^2-a^2}\tan^{-1}\frac{x}{a} + \frac{b^3}{a^2-b^2}\tan^{-1}\frac{x}{b}$.

19. $\frac{1}{a^2+b^2}\left\{\log\frac{x+b}{\sqrt{(x^2+a^2)}} + \frac{b}{a}\tan^{-1}\frac{x}{a}\right\}$.

20. (i) $-\frac{1}{2}\log(1+x) + \frac{1}{4}\log(1+x^2) + \frac{1}{2}\tan^{-1}x$.

(ii) $\frac{1}{3}\log\frac{x-1}{\sqrt{(x^2+x+1)}} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}$.

21. $\frac{1}{2}[\log(x^2-x+1) - \log(x^2+x+1)]$.

22. (i) $\frac{1}{7}\log\frac{x-2}{x+2} + \frac{\sqrt{3}}{7}\tan^{-1}\frac{x}{\sqrt{3}}$. (ii) $\frac{1}{6}\{\log(x^2-2) - \log(x^2+1)\}$.

23. $\frac{1}{2}\left\{\tan^{-1}(x+2) + \frac{x+2}{x^2+4x+5}\right\}$. 24. $\tan^{-1}x - \frac{1}{\sqrt{2}}\tan^{-1}(x\sqrt{2})$.

25. $\log x - \frac{1}{2}\log(1+x) - \frac{1}{4}\log(1+x^2) - \frac{1}{2}\tan^{-1}x$.

26. $\frac{1}{4}\log\frac{1+x+x^2}{1-x+x^2} + \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x\sqrt{3}}{1-x^2}\right)$.

27. (i) $\frac{1}{4\sqrt{2}}\log\frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2} + \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x\sqrt{2}}{1-x^2}\right)$.

(ii) $\frac{1}{\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{1-x^2}$

28. (i) $\frac{1}{5}\log\tan\left(\frac{1}{4}\pi + \frac{1}{2}x\right) - \frac{3}{10}\tan^{-1}\left(\frac{1}{2}\tan\frac{1}{2}x\right)$.

(ii) $\frac{1}{6}\log(1 + \cos x) + \frac{1}{2}\log(1 - \cos x) - \frac{2}{3}\log(1 - 2\cos x)$.

29. (i) $x + \log(1 + e^x) - 2\log(1 + 2e^x)$. (ii) $\frac{1}{4}\log\{(e^x - 1)(e^x + 3)^3\}$.

30. $-\frac{1}{2}\log(1 + \cos x) + \frac{1}{10}\log(1 - \cos x) + \frac{2}{5}\log(3 + 2\cos x)$.