CHAPTER VI

DEFINITE INTEGRALS

6.1. Thus far we have defined integration as *the inverse of differentiation*. Now, we shall define integration as a *process of summation*. In fact, the integral calculus was invented in the attempt to calculate the area bounded by curves by supposing the given area to be divided into an infinite number of infinitesimal parts called elements, the sum of all these elements being the area required. Historically the integral sign is merely the elongated S used by early writers to denote the sum.

This new definition, as explained in the next article, is of fundamental importance, because it is used in most of the applications of the integral calculus to practical problems.

6.2. Integration as the limit of a sum.

The generalized definition is given in Note 2 below. We first start with a special case of that definition which is advantageous for application in most cases.

Let $f(x)$ be a *bounded *single-valued continuous function defined in the interval* (a, b), a and b *being both finite quantities* and $b > a$; and let the interval (a, b) be divided into n equal sub-intervals, each of length h, by the points

$$a + h,\ a + 2h,\ \ldots,\ a + (n-1)h, \text{ so that } nh = b - a;$$

then $\underset{h \to 0}{\text{Lt}}\ h\,[f(a) + f(a+h) + f(a+2h) + \ldots + f(a + \overline{n-1}\,h)]$

$$\text{i.e., shortly } \underset{h \to 0}{Lt}\ h \sum_{r=0}^{n-1} f(a + rh), \ (nh = b - a),$$

$$\text{or, } \underset{n \to \infty}{Lt}\ \frac{b-a}{n} \Sigma f\left(a + (b-a)\frac{r}{n}\right)$$

(since $n \to \infty$ when $h \to 0$)

* *i.e.*, which does not become infinite at any point. See Authors' *Differential Calculus, Art. 1.6.*

is defined as *the definite integral of* $f(x)$ *with respect to* x *between the limits* a *and* b, *and is denoted by the symbol*

$$\int_a^b f(x)\,dx,$$

where 'a' is called the *lower* or *inferior limit*, and 'b' is called the *upper* or *superior limit*.

Cor. Putting $a = 0$, we get

$$\int_0^b f(x)\,dx = \underset{h \to 0}{Lt}\ h \sum_{r=0}^{n-1} f(rh), \text{ where } nh = b.$$

Note 1. $\int_a^b f(x)\,dx$ is also sometimes defined as

$$\underset{h \to 0}{Lt}\ h \sum_{r=0}^{n-1} f(a + rh), \text{ or } \underset{h \to 0}{Lt}\ h \sum_{r=1}^{n} f(a + rh);$$

these definitions differ from one another only in the inclusion or exclusion of the terms $hf(a)$ and $hf(a + nh)$, *i.e.*, $hf(b)$ which ultimately vanish.

It should be carefully noted that whichever of these slightly different forms of the definition we use, we always arrive at the same result. Sometimes, for the sake of simplicity, we use one or the other of these definitions.

Supposing the interval (a, b) to be divided into n equal parts each of length Δx by the points $x_0 (= a), x_1, x_2, \ldots\ldots\ldots, x_n (= b)$, the definite integral

$$\int_a^b f(x)\,dx \text{ may also be defined as } \underset{n \to \infty}{Lt} \sum_{r=0}^{n-1} f(x_r)\,\Delta x.$$

Note 2. The above definition of a definite integral is a special case of the more generalized definition as given below.

Let $f(x)$ be a bounded function defined in the interval (a, b); and let the interval (a, b) be divided *in any manner into* n sub-intervals (equal or unequal) of lengths $\delta_1, \delta_2, \ldots\ldots, \delta_n$. In each sub-interval choose a perfectly arbitrary point (which may be within or at either end-point of the interval) : and let these points be $x = \zeta_1, \zeta_2, \ldots\ldots\ldots, \zeta_n$.

Let $$S_n = \sum_{r=1}^{n} \delta_r f(\zeta_r).$$

Now, let n increase indefinitely in such a way that the greatest of the lengths $\delta_1, \delta_2, \ldots, \delta_n$ tends to zero. If, in this case, S_n tends to a definite limit *which is independent of the way in which the interval* (a, b) *is sub-divided and the intermediate points* $\zeta_1, \zeta_2, \ldots, \zeta_n$ *are chosen*, then *this limit, when it exists*, is called the *definite integral of* $f(x)$ *from* a *to* b.

It can be shown that, when $f(x)$ *is a continuous function, the above limit always exists.*

In the present volume, however, in Art. 6.4 we prove that if, in addition to $f(x)$ being continuous in the interval, there exists a function of which it is the differential coefficient then the above limit exists.

In the definition of the Article above, for the sake of simplicity, $f(x)$ is taken to be a continuous function, the intervals are taken to be of equal lengths, and $\zeta_1, \zeta_2, \ldots, \zeta_n$ are taken as the end-points of the successive sub-intervals.

The method of unequal sub-divisions of the interval is illustrated in Ex. 5 below.

6.3. Illustrative Examples.

Ex. 1. *Evaluate from first principle* $\int_a^b e^x\,dx$.

From the definition,

$$\int_a^b e^x\,dx = \underset{h \to 0}{Lt}\; h \sum_{r=0}^{n-1} e^{a+rh}, \text{ where } nh = b - a,$$

$$= \underset{h \to 0}{Lt}\; h\,[e^a + e^{a+h} + \ldots + e^{a+(n-1)h}]$$

$$= \underset{h \to 0}{Lt}\; h \cdot e^a [1 + e^h + e^{2h} + \ldots + e^{(n-1)h}]$$

$$= \underset{h \to 0}{Lt}\; h \cdot e^a \frac{e^{nh} - 1}{e^h - 1}$$

$$= e^a(e^{b-a} - 1) \cdot \underset{h \to 0}{Lt}\; \frac{h}{e^h - 1}, \text{ since } nh = b - a,$$

$$= e^b - e^a, \qquad \left[\text{since, } \underset{h \to 0}{Lt}\; \frac{h}{e^h - 1} = 1\right].$$

Ex. 2. *Find from the definition, the value of*

$$\int_0^1 x^2\,dx .$$

From the definition,

$$\int_0^1 x^2\,dx = \underset{h\to 0}{Lt}\; h \sum_{r=1}^{n} (rh)^2, \text{ where } nh = 1,$$

$$= \underset{h\to 0}{Lt}\; h\,[1^2h^2 + 2^2h^2 + \ldots + n^2h^2]$$

$$= \underset{h\to 0}{Lt}\; [h^3(1^2 + 2^2 + \ldots + n^2)]$$

$$= \underset{h\to 0}{Lt}\; h^3\,\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{1}{6}\cdot \underset{h\to 0}{Lt}\; (2n^3h^3 + 3n^2h^2.h + nh.h^2)$$

$$= \frac{1}{6}\cdot \underset{h\to 0}{Lt}\; (2 + 3h + h^2), \text{ since } nh = 1,$$

$$= \frac{1}{6}.2 = \frac{1}{3}\ .$$

Ex. 3. *Prove ab initio* $\displaystyle\int_a^b \frac{1}{x^2}\,dx = \frac{1}{a} - \frac{1}{b}$

Here by the definition,

$$\int_a^b \frac{1}{x^2}\,dx = \underset{h\to 0}{Lt}\; h\left[\frac{1}{a^2} + \frac{1}{(a+h)^2} + \frac{1}{(a+2h)^2} + \ldots + \frac{1}{(a+(n-1)h)^2}\right]$$

$$(\text{where } nh = b - a).$$

Denoting the right-hand series by S, since, obviously,

$$\frac{1}{(a+rh)^2} \text{ is} > \frac{1}{(a+rh)(a+(r+1)h)} \text{ and} < \frac{1}{(a+(r-1)h)(a+rh)},$$

we get $S > h\left[\dfrac{1}{a(a+h)} + \dfrac{1}{(a+h)(a+2h)} + \ldots + \dfrac{1}{(a+(n-1)h)(a+nh)}\right]$,

i.e., $> \left[\left(\dfrac{1}{a} - \dfrac{1}{a+h}\right) + \left(\dfrac{1}{a+h} - \dfrac{1}{a+2h}\right) + \ldots + \left(\dfrac{1}{a+(n-1)h} - \dfrac{1}{a+nh}\right)\right]$, i.e., $\left(\frac{1}{a}-\frac{1}{a+nh}\right)$, i.e., $>\frac{1}{a}-\frac{1}{b}$; $\left[\text{since } nh=b-a\right]$

Also, $S<h\left[\frac{1}{(a-h)a}+\frac{1}{a(a+h)}+\ldots+\frac{1}{\{a+(n-2)h\}\{a+(n-1)h\}}\right]$,

i.e., $<\left[\left(\frac{1}{a-h}-\frac{1}{a}\right)+\left(\frac{1}{a}-\frac{1}{a+h}\right)+\ldots+\left(\frac{1}{a+(n-2)h}-\frac{1}{a+(n-1)h}\right)\right]$

i.e., $<\left(\frac{1}{a-h}-\frac{1}{a+(n-1)h}\right)$, i.e., $<\left(\frac{1}{a-h}-\frac{1}{b-h}\right)$.

Hence, $\left(\frac{1}{a}-\frac{1}{b}\right)<S<\left(\frac{1}{a-h}-\frac{1}{b-h}\right)$

and this being true for all values of h, proceeding to the limit when $h\to 0$, $\left(\frac{1}{a-h}-\frac{1}{b-h}\right)$ clearly tends to $\left(\frac{1}{a}-\frac{1}{b}\right)$ and S by definition becomes $\int_a^b \frac{dx}{x^2}$, and hence $\int_a^b \frac{dx}{x^2}=\frac{1}{a}-\frac{1}{b}$.

For an alternative method, see Ex. 5 ; here $m=-2$.

Ex. 4. *Prove by summation,* $\int_a^b \sin x\,dx=\cos a-\cos b$.

$$\int_a^b \sin x\,dx=\underset{h\to 0}{Lt}\, h\sum_{r=0}^{n-1}\sin(a+rh), \text{ where } nh=b-a,$$

$$=\underset{h\to 0}{Lt}\, h\,[\sin a+\sin(a+h)+\sin(a+2h)+\ldots \text{to } n \text{ terms}]$$

$$=\underset{h\to 0}{Lt}\, h.\sin\left\{a+(n-1)\frac{h}{2}\right\}\frac{\sin\frac{1}{2}nh}{\sin\frac{1}{2}h}$$

$$=\underset{h\to 0}{Lt}\,\frac{\frac{1}{2}h}{\sin\frac{1}{2}h}\,2\sin\tfrac{1}{2}nh.\sin\left\{a+(n-1)\frac{h}{2}\right\}$$

$$=\underset{h\to 0}{Lt}\,\frac{\frac{1}{2}h}{\sin\frac{1}{2}h}\left[\cos\left(a-\frac{1}{2}h\right)-\cos\left\{a+(2n-1)\frac{h}{2}\right\}\right]$$

$$=\underset{h\to 0}{Lt}\,[\cos(a-\tfrac{1}{2}h)-\cos(a+nh-\tfrac{1}{2}h)], \text{ since } \underset{\theta\to 0}{Lt}\,\frac{\theta}{\sin\theta}=1,$$

$$= \underset{h \to 0}{Lt} \left[\cos\left(a - \tfrac{1}{2}h\right) - \cos\left(b - \tfrac{1}{2}h\right)\right], \text{ since } a + nh = b,$$

$$= \cos a - \cos b.$$

Ex. 5. *Evaluate* $\int_a^b x^m \, dx$ *where* m *is any number, positive or negative, integer or fraction, but* $\neq -1$ $(0 < a < b)$.

Let us divide the interval (a, b) into n parts by the points of division $a, ar, ar^2, \ldots, ar^{n-1}, ar^n$, where $ar^n = b$, *i.e.*, $r = (b/a)^{1/n}$.

Evidently as $n \to \infty$, $r = (b/a)^{1/n} \to 1$, so that each of the intervals $a(r-1), ar(r-1), \ldots, ar^{n-1}(r-1) \to 0$. Now, by the generalized definition, as given in Note 2, Art. 6.2,

$$\int_a^b x^m \, dx = \underset{n \to \infty}{Lt} \left[a^m . a(r-1) + (ar)^m . ar(r-1) + (ar^2)^m . (ar^2)(r-1) + \ldots \text{ to } n \text{ terms} \right]$$

$$= \underset{r \to 1}{Lt}\, a^{m+1}(r-1)\left[1 + r^{m+1} + r^{2(m+1)} + \ldots \text{ to } n \text{ terms}\right]$$

$$= \underset{r \to 1}{Lt}\, a^{m+1}(r-1)\frac{(r^{m+1})^n - 1}{r^{m+1} - 1}, \; [m + 1 \neq 0]$$

$$= \underset{r \to 1}{Lt}\, a^{m+1}\frac{r-1}{r^{m+1}-1}\left\{(r^n)^{m+1} - 1\right\}$$

$$= \underset{r \to 1}{Lt}\, \frac{r-1}{r^{m+1}-1}\, a^{m+1}\left\{\left(\frac{b}{a}\right)^{m+1} - 1\right\}$$

$$= \underset{r \to 1}{Lt}\, \frac{r-1}{r^{m+1}-1}\left(b^{m+1} - a^{m+1}\right)$$

$$= \frac{b^{m+1} - a^{m+1}}{m+1}. \quad [m \neq -1]$$

$$\left[\text{Since } \underset{r \to 1}{Lt}\, \frac{r-1}{r^{m+1}-1}, \text{ being of the form } \frac{0}{0} = \underset{r \to 1}{Lt}\, \frac{1}{(m+1)r^m} = \frac{1}{m+1}. \right]$$

Note 1. x^m being continuous in (a, b) is integrable in (a, b), a unique limit of the summation S_n as given in Note 2, Art. 6.2, exists. So it is immaterial in what mode we calculate it. The same remark holds for the next example.

Note. 2. In evaluating $\int_0^b x^m dx \; [m \neq -1, b > 0]$ we may first evaluate $\int_a^b x^m dx \; [0 < a < b]$ as above, and then make $a \to 0+$.

Ex. 6. *Show from the definition*

$$\int_a^b \frac{1}{x}\, dx = \log \frac{b}{a} \;.\; (0 < a < b)$$

As in Ex. 5, divide the interval (a, b) into n parts by the points of division $a, ar, ar^2, \ldots\ldots\ldots\ldots, ar^{n-1}, ar^n$, where $ar^n = b$, *i.e.*, $r = (b/a)^{1/n}$. Evidently as $n \to \infty$, $r = (b/a)^{1/n} \to 1$, so that each of the intervals $a(r-1), ar(r-1) \ldots\ldots \to 0$. Now, by the generalized definition,

$$\int_a^b \frac{1}{x}\, dx = \underset{n \to \infty}{Lt} \sum_{k=1}^{n} \frac{1}{ar^{k-1}} (ar^k - ar^{k-1})$$

$$= \underset{n \to \infty}{Lt} \Sigma (r-1) = \underset{n \to \infty}{Lt} n(r-1)$$

$$= \underset{n \to \infty}{Lt} n[(b/a)^{1/n} - 1]$$

$$= \underset{h \to 0}{Lt} \left[\frac{e^h - 1}{h} \log \frac{b}{a}\right], \text{ where } h = \frac{1}{n} \log \frac{b}{a}$$

$$= \log \frac{b}{a} \left[\text{since } \underset{h \to 0}{Lt} \frac{e^h - 1}{h} = 1\right].$$

Ex. 7. *Find ab initio the value of* $\int_0^{\frac{1}{4}\pi} \sec^2 x\, dx$.

By definition, the required integral

$$I = \underset{h \to 0}{Lt}\; h \sum_{r=1}^{n} \sec^2 rh, \text{ where } nh = \tfrac{1}{4}\pi.$$

Now, $\sec(r-1)h \cdot \sec rh < \sec^2 rh < \sec rh \cdot \sec(r+1)h$, since $\sec x$ increases with x in $0 < x < \frac{1}{4}\pi$.

$$\text{Also, } \sec rh \cdot \sec(r+1)h = \frac{1}{\sin h} \frac{\sin\{(r+1)h - rh\}}{\cos rh \cdot \cos(r+1)h}$$

$$= \frac{1}{\sin h}\{\tan(r+1)h - \tan rh\}.$$

Similarly, $\sec(r-1)h \cdot \sec rh = \dfrac{1}{\sin h}\{\tan rh - \tan(r-1)h\}$.

Thus, I lies between $\underset{h \to 0}{Lt} \dfrac{h}{\sin h} \displaystyle\sum_{r=1}^{n} \{\tan rh - \tan(r-1)h\}$

and $\underset{h \to 0}{Lt} \dfrac{h}{\sin h} \displaystyle\sum_{r=1}^{n} \{\tan(r+1)h - \tan rh\}$,

i.e., $\underset{h \to 0}{Lt} \dfrac{h}{\sin h}\{\tan nh - \tan 0\}$ and $\underset{h \to 0}{Lt} \dfrac{h}{\sin h}\{\tan(n+1)h - \tan h\}$.

Since $nh = \frac{1}{4}\pi$, and $Lt\,(h/\sin h) = 1$ as $h \to 0$, both the above limits tend to $\tan \frac{1}{4}\pi$, *i.e.*, 1.

Hence, I has the value 1.

6.4. Definition of definite integral based on the notion of bounds.

We have *two* methods of defining definite integrals : one based on the *notion of limits*, the other based on the *notion of bounds*.

The first method based on the notion of limits is given in Note 2, Art. 6.2.

The second method based on the notion of bounds is given below.

Let the interval (a, b) be divided in any manner into a number (say n) of sub-intervals by taking interme-diate points

$$a = x_0 < x_1 < x_2 \;\ldots\ldots < x_{n-1} < x_n = b.$$

Let M_r and m_r be the upper and lower bounds of $f(x)$ in the r-th sub-interval (x_{r-1}, x_r) and let δ_r denote the length of this sub-interval. The lower bound (denoted by J) of the aggregate of the sums $S = \Sigma M_r \delta_r$ (obtained by considering all possible modes of sub-division) is called the *Upper Integral* and is denoted by

$$\int_a^b f(x)\,dx,$$

and the upper bound (denoted by j) of the aggregate of the sums $s = \Sigma m_r \delta_r$ is called the *Lower Integral* and is denoted by

$$\int_a^b f(x)\,dx.$$

When the lower and upper integrals are equal, *i.e.*, when $j = J$, then $f(x)$ is said to be *integrable* and the common value is said to be the integral of $f(x)$ in (a, b) and is denoted by

$$\int_a^b f(x)\,dx.$$

It can be shown by what is known as *Darboux's theorem* that both the definitions are equivalent when $f(x)$ is integrable.

Note. The integral defined above, when it exists, is called a Riemann integral, as it was first obtained by the great mathematician Riemann.

6.5. Necessary and sufficient condition for integrability.

We give below, without proof, the *necessary and sufficient condition* for the integrability of a bounded function $f(x)$.

If there be at least one pair of sums S, s of $f(x)$ for a sub-division of the interval (a, b) such that

$$S - s < \varepsilon,$$

where ε is any arbitrarily small positive number, then $f(x)$ is integrable.

Note. It can be easily shown that the sum or difference of two or more functions integrable in (a, b) is also integrable in (b, a).

6.6. Integrable functions.

(i) Functions continuous in a closed interval (a, b) are integrable in that interval.

(ii) Functions with only a *finite* number of *finite* discontinuities in a closed interval (a , b) are integrable in that interval.

(iii) Functions *monotonic and bounded* in an interval (a , b) are integrable in that interval.

6.7. Important Theorems.

I. *If* $f(x)$ *is integrable in the closed interval* (a , b) *and if* $f(x) \geq 0$ *for all* x *in* (a, b), then

$$\int_a^b f(x)\,dx \geq 0 \; (b > a).$$

Since $f(x) \geq 0$ in (a , b), it follows that in the interval (x_{r-1} , x_r) the lower bound $m_r \geq 0$ and therefore

$$s = \Sigma m_r \delta_r \geq 0 .$$

$\therefore$ j , which is the upper bound of the set of numbers s, ≥ 0 .

Since $f(x)$ is integrable, $j = \int_a^b f(x)\,dx$

$$\text{and hence } \int_a^b f(x)\,dx \geq 0 .$$

Alternatively.

Since $f(x)$ is integrable in (a , b),

$$\therefore \qquad \int_a^b f(x)\,dx = \underset{n \to \infty}{Lt} \Sigma f(\zeta_r)\,\delta_r .$$

Since $f(x) \geq 0$ in (a , b), $\therefore$ $f(\zeta_r) \geq 0$ in (a , b).

$$\therefore \quad \underset{n \to \infty}{Lt} \Sigma f(\zeta_r)\,\delta_r \geq 0 \text{ in } (a , b).$$

$$\therefore \qquad \int_a^b f(x)\,dx \geq 0 \text{ in } (a , b).$$

Note. It can be shown similarly that if $f(x) \leq 0$ in (a, b) then

$$\int_a^b f(x)\,dx \leq 0.$$

II. *If* $f(x)$ *and* $g(x)$ *are integrable in* (a, b) *and* $f(x) \geq g(x)$ *in* (a, b), then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx \quad (b > a).$$

Consider the function $\psi(x) = f(x) - g(x)$.

Then $\psi(x)$ is integrable in (a, b) and $\psi(x) \geq 0$ in (a, b).

$\therefore$ by (I), $\int_a^b \psi(x)\,dx \geq 0$ in (a, b),

i.e., $\int_a^b [f(x) - g(x)]\,dx \geq 0$ in (a, b),

i.e., $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

III. *If* M *and* m *are the upper and lower bounds of the integrable function* $f(x)$ *in* (a, b), $b > a$, then

$$m(b - a) \leq \int_a^b f(x)\,dx \leq M(b - a).$$

Since $m \leq f(x) \leq M$ in (a, b),

$\therefore$ $\{f(x) - m\} \geq 0$ in (a, b),

$\therefore$ $\int_a^b \{f(x) - m\}\,dx \geq 0$.

$\therefore$ $\int_a^b f(x)\,dx \geq m \int_a^b dx$, i.e., $\geq m(b - a)$.

Similarly, since $M - f(x) \geq 0$, we can show that

$$M(b-a) \geq \int_a^b f(x)\,dx.$$

Hence the result.

This is known as the *First Mean Value Theorem of Integral Calculus.*

Cor. The above theorem can be written in the form

$$\int_a^b f(x)\,dx = (b-a)\mu, \text{ when } m \leq \mu \leq M;$$

and if further $f(x)$ is *continuous* in (a, b) then $f(x)$ attains the value μ for some value ζ of x such that $a \leq \zeta \leq b$* , *and so*

$$\int_a^b f(x)\,dx = (b-a) f(\zeta).$$

IV. *If $f(x)$ and $g(x)$ are integrable in (a, b) and if $g(x)$ maintains the same sign throughout (a, b)*, then

$$\int_a^b f(x)g(x)\,dx = \mu \int_a^b g(x)\,dx, \text{ where } m \leq \mu \leq M,$$

m and M being the lower and upper bounds of $f(x)$ in (a, b).

Let us assume, for the sake definiteness, that $g(x)$ is always positive in (a, b).

Now, $\quad m \leq f(x) \leq M$ in (a, b).

Since $\quad g(x)$ is positive,

$\therefore \quad mg(x) \leq f(x)g(x) \leq Mg(x)$,

$\therefore \quad f(x)g(x) - mg(x) \geq 0$,

$$\therefore \quad \int_a^b \{f(x)g(x) - mg(x)\}\,dx \geq 0.$$

* See Authors' Differential Calculus.

$\therefore$ $$\int_a^b f(x)\,g(x)\,dx \geq m\int_a^b g(x)\,dx$$

and $$f(x)\,g(x) - Mg(x) \leq 0,$$

$\therefore$ $$\int_a^b \{f(x)\,g(x)\,dx - Mg(x)\}\,dx \leq 0,$$

i.e., $$\int_a^b f(x)\,g(x)\,dx \leq M\int_a^b g(x)\,dx.$$

$\therefore$ $$m\int_a^b g(x)\,dx \leq \int_a^b f(x)\,g(x)\,dx \leq M\int_a^b g(x)\,dx,$$

$\therefore$ $$\int_a^b f(x)\,g(x)\,dx = \mu\int_a^b g(x)\,dx,\text{ where } m \leq \mu \leq M.$$

Cor. If further $f(x)$ is *continuous*, then $f(x)$ attains the value μ for some value ζ of x, where $a \leq \zeta \leq b$, *i.e.*, $f(\zeta) = \mu$.

$\therefore$ when $f(x)$ is *continuous*

$$\int_a^b f(x)\,g(x)\,dx = f(\zeta)\int_a^b g(x)\,dx.$$

Note. This is the *generalized form of the First Mean Value Theorem*. The theorem III can be obtained from this by putting $g(x) = 1$.

V. *If $f(t)$ is bounded and integrable in the closed interval (a, b) and if*

$$F(x) = \int_a^x f(t)\,dt, \textit{ where } x \textit{ is any point in } (a, b), \textit{ then}$$

(1) *$F(x)$ is a continuous function of x in (a, b).*

(2) *If $f(x)$ is continuous throughout (a, b), then the derivative of $F(x)$ exists at every point of (a, b) and $= f(x)$.*

(3) *If $f(x)$ is continuous throughout (a, b) and if $\phi(x)$ be a function of x such that $\phi'(x) = f(x)$ throughout (a, b), then*

$$F(x) = \int_a^x f(t)\,dt = \phi(x) - \phi(a).$$

(1) Let us consider a point $x + h$ in the neighbourhood of x in (a, b).

Then, $$F(x+h) = \int_a^{x+h} f(t)\,dt,$$

$$F(x+h) - F(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$$

$$= \int_x^{x+h} f(t)\,dt = \mu h,$$

by Cor. of (III), where μ lies between the upper and lower bounds of $f(t)$ in the interval $(x, x+h)$. Since $f(t)$ is integrable, m and M are finite and so is μ.

$$\therefore \quad \underset{h \to 0}{Lt} \; [F(x+h) - F(x)] = \underset{h \to 0}{Lt} \; \mu h = 0.$$

$$\therefore \quad \underset{h \to 0}{Lt} \; F(x+h) = F(x).$$

Thus, $F(x)$ is a continuous function of x in (a, b).

(2) We have $$F(x+h) - F(x) = \int_x^{x+h} f(t)\,dt$$

$$= hf(\zeta), \text{ where } x \le \zeta \le x + h,$$

since $f(t)$ is continuous. [See *Cor. of (III)*]

$$\therefore \quad \frac{F(x+h) - F(x)}{h} = f(\zeta), \text{ for } h \ne 0.$$

When $h \to 0$, $\zeta \to x$ and $f(\zeta) \to f(x)$, since $f(t)$ is continuous.

$\therefore \quad \because \lim_{h \to 0} \dfrac{F(x+h)-F(x)}{h}$ exists, and $= f(x)$,

i.e., $\quad F'(x) = f(x)$.

(3) Since $f(x)$ is continuous throughout (a, b), as proved above,

$$F'(x) = f(x), \text{ i.e., } \quad F'(x) = \phi'(x),$$

$\therefore \quad F'(x) - \phi'(x) = 0$.

Let $\quad \psi(x) \equiv F(x) - \phi(x)$.

$\therefore \quad \psi'(x) = 0$ everywhere in (a, b).

Hence, $\psi(x) \equiv F(x) - \phi(x) =$ a constant c in (a, x). (1)

[See *Differential Calculus, Art. 6.7, Ex. 1.*]

When $x = a, F(a) = \displaystyle\int_a^a f(t)\,dt = 0$.

Since, from (1), $F(a) - \phi(a) = c, \quad \therefore \quad -\phi(a) = c$.

Consequently, from (1), $F(x) = \phi(x) + c = \phi(x) - \phi(a)$,

i.e., $$\int_a^x f(t)\,dt = \phi(x) - \phi(a).$$

In particular,

$$\int_a^b f(t)\,dt = \phi(b) - \phi(a).$$

Note. The relation given in (3) is known as the *Fundamental theorem of Integral Calculus*. [For *an alternative proof*, See *Art. 6.12.*]

6.8. Change of variable in an integral.

To change the variable in the integral $\displaystyle\int_a^b f(x)\,dx$

by the substitution $x = \phi(t)$, it is necessary that

(i) $\phi(t)$ possesses a derivative at every point of the interval $\alpha \leq t \leq \beta$, where $\phi(\alpha) = a$ and $\phi(\beta) = b$, and $\phi'(t) \neq 0$ for any value t in (α, β).

(ii) $f[\phi(t)]$ and $\phi'(t)$ are bounded and integrable in (α, β). When the above conditions hold good, then and then only we have

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f[\phi(t)]\,\phi'(t)\,dt.$$

Illustration :

Let $$I = \int_{-1}^{+1} \frac{dx}{1 + x^2}.$$

Putting $x = \tan\theta$, we get $I = \int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} d\theta = \frac{1}{2}\pi$.

Putting $x = 1/t$, we get

$$I = -\int_{-1}^{1} \frac{dt}{1 + t^2} = -\tfrac{1}{2}\pi.$$

The reason for the discrepancy lies in the fact that $1/t$ does not possess a derivative at $t = 0$, an interior point of $(-1, 1)$; in fact, the function itself is undefined when $t = 0$.

6.9. Primitives and Integrals.

If $\phi'(x) = f(x)$, then $\phi(x)$ is the *primitive* of $f(x)$. The *integral* of $f(x)$, on the other hand, is

$\underset{n \to \infty}{Lt} \Sigma f(\zeta_r)\,\delta_r$, or symbolically

$\int_a^b f(x)\,dx$, *i.e.,* the analytical substitute for an area

in case $f(x)$ has a continuous graph.

The distinction between the two is that while integrals can be *calculated,* primitives cannot be calculated.

The question as to whether a primitive exists and the question of the existence of an integral of $f(x)$ in (a, b) are entirely inde pendent questions. It is only in the case of continuous functions that they are the same.

Indefinite integrals can properly be described as the *Calculus of primitives.*

The connection between primitives and integrals is represe[nted] by the Fundamental Theorem of Integral Calculus, *viz.*,

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

Illustration :

(i) $f(x) = x \cdot \sin \dfrac{1}{x^2} - \dfrac{1}{x}\cos\dfrac{1}{x^2}$, $(x \neq 0)$

$\quad = 0 \; (x = 0).$

Here, $\dfrac{d}{dx}\left\{\dfrac{1}{2}x^2 \sin\dfrac{1}{x^2}\right\} = f(x)$ for $x \neq 0$ and $= 0$ for $x = 0$,

so that primitive exists, but $\displaystyle\int_{-1}^{+1} f(x)\,dx$ does not exist.

(ii) $f(x) = 0 \; (x \neq 0)$, $= 1 \; (x = 0)$; here in $(0, 1)$ $\displaystyle\int_0^1 f(x)\,dx$ exists and $= 0$, but no primitive exists.

6.10. Illustrative Examples.

Ex. 1. *Show that* $\dfrac{1}{2} < \displaystyle\int_0^1 \frac{dx}{\sqrt{(4 - x^2 + x^5)}} < \frac{\pi}{6}$.

We have $\quad 4 > 4 - (x^2 - x^5)$ in $(0, 1)$,

or, $\quad \sqrt{4} > \sqrt{(4 - x^2 + x^5)}$.

$\therefore \quad \dfrac{1}{\sqrt{4}}$, *i.e.*, $\dfrac{1}{2} < \dfrac{1}{\sqrt{(4 - x^2 + x^5)}}$.

$\therefore \quad \displaystyle\int_0^1 \frac{1}{2}\,dx$, *i.e.*, $\dfrac{1}{2} < \displaystyle\int_0^1 \frac{dx}{\sqrt{(4 - x^2 + x^5)}}$.

Again, $4 - x^2 < 4 - x^2 + x^5$ in $(0, 1)$.

$$\therefore \quad \frac{1}{\sqrt{(4 - x^2)}} > \frac{1}{\sqrt{(4 - x^2 + x^5)}} .$$

$$\therefore \quad \int_0^1 \frac{dx}{\sqrt{(4 - x^2)}} > \int_0^1 \frac{dx}{\sqrt{(4 - x^2 + x^5)}} .$$

$$\therefore \quad \left[\sin^{-1} \frac{1}{2} x \right]_0^1 \text{ i.e. } \sin^{-1} \frac{1}{2} \text{ i.e. } \frac{\pi}{6} > \int_0^1 \frac{dx}{\sqrt{(4 - x^2 + x^5)}}$$

Hence the result.

Ex. 2. *If* $\int_a^b f(x)\, dx$ *exists, show that* $\left| \int_a^b f(x)\, dx \right| \le \int_a^b | f(x) |\, dx$.

We have $| f(\zeta_1)\delta_1 + f(\zeta_2)\delta_2 + \ldots + f(\zeta_n)\delta_n |$

$$\le | f(\zeta_1) | | \delta_1 | + | f(\zeta_2) | | \delta_2 | + \ldots\ldots | f(\zeta_n) | | \delta_n |,$$

i.e., $| \Sigma f(\zeta_r)\delta_r | \le \Sigma | f(\zeta_r) | | \delta_r |$.

$\therefore \quad Lt\, | \Sigma f(\zeta_r)\delta_r | \le \Sigma\, Lt\, | f(\zeta_r) | | \delta_r |$.

$$\therefore \; \left| \int_a^b f(x)\, dx \right| \le \int_a^b f(x) |\, dx .$$

Otherwise :

Since $\int_a^b f(x)\, dx$ exists, $\quad \therefore \int_a^b | f(x) |\, dx$ exists.

We have $-| f(x) | \le f(x) \le | f(x) |$.

$$\therefore \quad -\int_a^b | f(x) |\, dx \le \int_a^b f(x)\, dx \le \int_a^b | f(x) |\, dx$$

$$\text{i.e.,} \quad \left| \int_a^b f(x)\, dx \right| \le \int_a^b | f(x) |\, dx .$$

6.11. Geometrical Interpretation of $\int_a^b f(x)\,dx$.

Let the function $f(x)$, which we suppose to be finite and continuous in the interval (a, b) [$b > a$], be represented graphically and let $y = f(x)$ be the equation of the *continuous curve* PQ, and let $\overline{AC}$, $\overline{BD}$ be two ordinates corresponding to the points $x = a$, $x = b$, meeting the curve at finite points.

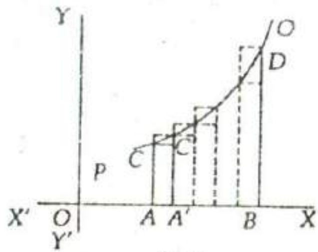

Fig.1

We have $OA = a$, $OB = b$ and $\therefore$ $AB = b - a$.

Let $\overline{AB}$ be divided into n equal parts each of length h.

$\therefore$ $nh = b - a$, or, $a + nh = b$.

Let the ordinates be erected thro-ugh the points whose abscissæ are $a + h$, $a + 2h$, ..., $a + (n - 1)h$ to meet the curve at finite points.

Let us complete the set of inner rectangles $ACC'A'$, ... and also the set of outer rectangles.

Let S denote the area enclosed between the curve $y = f(x)$, two ordinates $x = a$, $x = b$, and the x-axis.

Let S_1 denote the sum of the inner rectangles.

$\therefore$ $S_1 < S$. [$f(x)$ monotone increasing]

Now, $S_1 = hf(a) + hf(a + h) + \ldots\ldots + hf(a + \overline{n-1}\,h)$

$$= h \sum_{r=0}^{n-1} f(a + rh).$$

Let S_2 denote the sum of the outer rectangles. $\therefore$ $S_2 > S$.

Now, $S_2 = hf(a+h) + hf(a+2h) + \ldots + hf(a+nh)$

$$= h \sum_{r=0}^{n-1} f(a+rh) - hf(a) + hf(b) \quad [\text{since } a + nh = b].$$

We have, $S_1 < S < S_2$.

Now, let the number of sub-division increase indefinitely, and consequently the length of each of the sub-intervals diminishes indefinitely.

Thus, as $n \to \infty$, $h \to 0$.

$\therefore$ both $hf(a)$ and $hf(b) \to 0$, since $f(a)$ and $f(b)$ are finite.

$$\therefore \quad S_1 \to \underset{h \to 0}{Lt}\ h \sum_{r=0}^{n-1} f(a+rh) = \int_a^b f(x)\,dx,$$

$$\text{and } S_2 \to \underset{h \to 0}{Lt}\ h \sum_{r=0}^{n-1} f(a+rh) = \int_a^b f(x)\,dx.$$

Since we have always $S_1 < S < S_2$,

$$\therefore \quad S = \int_a^b f(x)\,dx.$$

Thus, $\int_a^b f(x)\,dx$ *geometrically represents the area of the space enclosed by the curve* $y = f(x)$, *the ordinates* $x = a$, $x = b$, *and the x-axis.*

Note. The arguments here postulate a concave curve. Similar arguments apply for a convex curve, or even for a curve which alternately rises and falls in the interval.

6.12. Fundamental Theorem of Integral Calculus.

If $f(x)$ is integrable in (a, b) $[a < b]$, and if there exists a function $\phi(x)$ such that $\phi'(x) = f(x)$ in (a, b), then

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a).$$

Divide the interval (a, b) into n parts by taking intermediate points

$$a = x_0 < x_1 < x_2 < \ldots\ldots < x_n = b.$$

Then we have, by the Mean Value Theorem of Differential Calculus,

$$\phi(x_r) - \phi(x_{r-1}) = (x_r - x_{r-1})\,\phi'(\zeta_r),\ [x_{r-1} < \zeta_r < x_r]$$

$$\therefore \quad \sum_{r=1}^{n} \phi'(\zeta_r)\,\delta_r = \sum_{r=1}^{n} [\phi(x_r) - \phi(x_{r-1})]$$

$$[\text{ where } \delta_r = x_r - x_{r-1}]$$

$$= [\phi(x_1) - \phi(x_0)] + [\phi(x_2) - \phi(x_1)] + \ldots\ldots$$

$$\ldots\ldots + [\phi(x_{n-1}) - \phi(x_{n-2})] + [\phi(x_n) - \phi(x_{n-1})]$$

$$= \phi(x_n) - \phi(x_0) = \phi(b) - \phi(a).$$

$\therefore \underset{\delta \to 0}{Lt} \Sigma\, \phi'(\zeta_r)\,\delta_r = \phi(b) - \phi(a)$, where δ is the greatest of the sub-intervals δ_r. Since $f(x)$, and hence $\phi'(x)$, is integrable in (a, b), therefore

$$\underset{\delta \to 0}{Lt}\ \Sigma\ \phi'(\zeta_r)\,\delta_r = \int_a^b \phi'(x)\,dx = \int_a^b f(x)\,dx.$$

$$\therefore \quad \int_a^b f(x)\,dx = \phi(b) - \phi(a)$$

Note 1. The above theorem establishes a *connection between the integration as a particular kind of summation, and the integration as an operation inverse* *to differentiation*. This also establishes the existence of the limit of the sum referred to in Art. 6.2, Note 2.

Note 2. From the above theorem it is clear that *the definite integral is a function of its upper and lower limits* and not of the independent variable x.

Note 3. It should be noted that if the upper limit is the independent variable, the integral is not a definite integral but simply another form of the indefinite integral. Thus, suppose $\int f(x)\,dx = \phi(x)$; then

$$\int_a^x f(x)\,dx = \phi(x) - \phi(a) = \phi(x) + \text{a constant} = \int f(x)\,dx.$$

6.13. Evaluation of the Definite Integral.

By the help of the above theorem, the value of a definite integral can be obtained much more easily than by the tedious process of summation. The success in the evaluation of a definite integral by this method mainly depends upon the success in the evaluation of the corresponding indefinite integral, as will be seen from the following illustrative examples. The application of the above theorem in the evaluation of the definite integral is very simple.

Suppose we require to evaluate $\int_a^b f(x)\,dx$.

First evaluate the indefinite integral $\int f(x)\,dx$ *by the usual methods, and suppose the result is* $\phi(x)$.

Next substitute for x *in* $\phi(x)$ *first the upper limit and then the lower limit, and subtract the last result from the first.*

Thus, $\int_a^b f(x)\,dx = \phi(b) - \phi(a)$.

Now, $\phi(b) - \phi(a)$ is very often shortly written as $\Big[\phi(x)\Big]_a^b$

It should be carefully noted that *in a definite integral the arbitrary constant of integration does not appear*.

For, if we write $\int f(x)\,dx = \phi(x) + c = \psi(x)$, say, then $\int_a^b f(x)\,dx = \psi(b) - \psi(a) = [\phi(b) + c] - [\phi(a) + c]$

$$= \phi(b) - \phi(a).$$

Thus, *while evaluating a definite integral, arbitrary constant need not be added in the value of the corresponding indefinite integral.*

6.14. Illustrative Examples.

Ex. 1. *Evaluate* $\int_a^b x^n\,dx.$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

$$\therefore \quad \int_a^b x^n\,dx = \left[\frac{x^{n+1}}{n+1}\right]_a^b = \frac{1}{n+1}\left[b^{n+1} - a^{n+1}\right];\ n+1 \neq 0.$$

Ex. 2. *Evaluate* $\int_0^{\pi/2} \cos^2 x\,dx.$

$$\int \cos^2 x\,dx = \frac{1}{2}\int 2\cos^2 x\,dx = \frac{1}{2}\int (1 + \cos 2x)\,dx$$

$$= \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x.$$

$$\therefore \quad \int_0^{\pi/2} \cos^2 x\,dx = \left[\frac{1}{2}x + \frac{1}{4}\sin 2x\right]_0^{\pi/2}$$

$$= \frac{1}{4}\pi + \frac{1}{4}\sin\pi = \frac{1}{4}\pi.$$

Ex. 3. *Evaluate* $\int_0^1 \frac{1-x}{1+x}\,dx.$

$$\int \frac{1-x}{1+x}\,dx = \int\left(\frac{2}{1+x} - 1\right)dx$$

$$= 2\int \frac{1}{1+x}\,dx - \int dx = 2\log(1+x) - x.$$

$$\therefore I = \left[2 \log(1+x) - x\right]_0^1 = 2 \log 2 - 1 - 2 \log 1 = 2 \log 2 - 1.$$

Ex. 4. *Evaluate* $\int_0^a \frac{dx}{a^2 + x^2}$.

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}.$$

$$\therefore I = \left[\frac{1}{a} \tan^{-1} \frac{x}{a}\right]_0^a = \frac{1}{a} \tan^{-1} 1 - \frac{1}{a} \tan^{-1} 0$$

$$= \frac{1}{a} \cdot \frac{\pi}{4} - \frac{1}{a} \cdot 0 = \frac{\pi}{4a}.$$

Note. Two points should be noted when evaluating a definite integral for which the indefinite integral involves an inverse trigonometrical function.

(i) The result must never be expressed in degrees ; for the ordinary rules for the differentiation and integration of trigonometrical functions hold only when the angles are measured in radians.

(iii) In substituting the limits in the inverse functions, care should be taken to choose the right values of the expressions obtained. Unless otherwise mentioned, usually the principal values are used.

6.15. Substitution in a Definite Integral.

While integrating an indefinite integral by the substitution of a new variable, it is sometimes rather troublesome to transform the result back into the original variable. In all such cases, while integrating the corresponding intergal between limits (*i.e.*, corresponding definite integral), we can avoid the tedious process of restoring the original variable, *by changing the limits of the definite integral to correspond with the change in the variable.*

Therefore in a definite integral the substitution should be effected in three places (i) in the integrand, (ii) in the differential and (iii) in the limits.

The following illustrative examples show the procedure to be employed.

6.16. Illustrative Examples.

Ex. 1. *Evaluate* $\int_0^1 \frac{\sin^{-1} x}{\sqrt{(1-x^2)}}\,dx$.

Put $\sin^{-1} x = \theta$. $\therefore d\theta = \frac{1}{\sqrt{(1-x^2)}}\,dx$.

0 and 1 are the limits of x; the corresponding limits of θ where $\theta = \sin^{-1} x$ are found as follows :

When $x = 0$, $\theta = \sin^{-1} 0 = 0$.

When $x = 1$, $\theta = \sin^{-1} 1 = \frac{1}{2}\pi$.

$$\therefore \quad I = \int_0^{\pi/2} \theta\, d\theta = \left[\frac{1}{2}\theta^2\right]_0^{\pi/2} = \frac{1}{8}\pi^2.$$

Note . Of course this example can be worked out by first finding the indefinite integral in terms of x and then substituting the limits.

Ex. 2. *Evaluate* $\int_0^a \sqrt{a^2 - x^2}\,dx$.

Put $x = a \sin\theta$. $\therefore dx = a\cos\theta\, d\theta$.

Also, when $x = 0$, $\theta = 0$, and when $x = a$, $\theta = \frac{1}{2}\pi$.

$$\therefore I = \int_0^{\frac{1}{2}\pi} a^2 \cos^2\theta\, d\theta = a^2 \int_0^{\frac{1}{2}\pi} \cos^2\theta\, d\theta.$$

Now, $\int \cos^2\theta\, d\theta = \frac{1}{2}\int (1 + \cos 2\theta)\, d\theta = \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]$.

$$\therefore I = a^2\, \frac{1}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\frac{1}{2}\pi} = \tfrac{1}{4}\pi a^2.$$

Ex. 3. *Evaluate* $\int_\alpha^\beta \sqrt{(x-\alpha)(\beta - x)}\,dx$.

Put $x = \alpha\cos^2\theta + \beta\sin^2\theta$. $\therefore\ dx = 2(\beta-\alpha)\sin\theta\cos\theta\, d\theta$;
also, $x - \alpha = \beta\sin^2\theta - \alpha(1-\cos^2\theta) = (\beta-\alpha)\sin^2\theta$,

$$\beta - x = \beta(1-\sin^2\theta) - \alpha\cos^2\theta = (\beta-\alpha)\cos^2\theta.$$

$\therefore$ when $x = \alpha$, $(\beta-\alpha)\sin^2\theta = 0$.

$\therefore\ \sin\theta = 0$ since $\beta \neq \alpha$. $\therefore\ \theta = 0$.

Similarly, when $x = \beta$, $(\beta-\alpha)\cos^2\theta = 0$.

$\therefore\ \cos\theta = 0$. $\therefore\ \theta = \frac{1}{2}\pi$.

$$\therefore\ I = 2(\beta-\alpha)^2\int_0^{\pi/2}\sin^2\theta\cos^2\theta\, d\theta.$$

Now, $\sin^2\theta\cos^2\theta = \frac{1}{4}\cdot 4\sin^2\theta\cos^2\theta = \frac{1}{4}\sin^2 2\theta = \frac{1}{8}(1-\cos 4\theta)$.

Also, $\int(1-\cos 4\theta)\, d\theta = \theta - \frac{1}{4}\sin 4\theta$.

$$\therefore\ I = 2(\beta-\alpha)^2\frac{1}{8}\int_0^{\pi/2}(1-\cos 4\theta)\, d\theta = \frac{1}{4}(\beta-\alpha)^2\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2}$$

$$= \frac{1}{4}(\beta-\alpha)^2\left[\frac{1}{2}\pi - \frac{1}{4}\sin 2\pi\right] = \frac{1}{8}\pi(\beta-\alpha)^2.$$

Ex. 4. *Evaluate* $\displaystyle\int_\alpha^\beta \frac{dx}{\sqrt{\{(x-\alpha)(\beta-x)\}}}$ $(\beta > \alpha)$. *[J. E. E. '79]*

As in Ex. 3, put $x = \alpha\cos^2\theta + \beta\sin^2\theta$.

$$\therefore\quad I = \int_0^{\frac{1}{2}\pi} 2\, d\theta = 2\cdot\frac{1}{2}\pi = \pi.$$

Ex. 5. *Show that* $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{(1-2x^2)\sqrt{(1-x^2)}} = \frac{1}{2}\log(2+\sqrt{3})$.

Put $x = \sin\theta$. Then $dx = \cos\theta\, d\theta$; also when $x = 0$, $\theta = 0$, and when $x = \frac{1}{2}$, $\theta = \frac{1}{6}\pi$.

$$\therefore\ I = \int_0^{\frac{1}{6}\pi}\frac{\cos\theta\, d\theta}{\cos 2\theta\cos\theta} = \int_0^{\frac{1}{6}\pi}\sec 2\theta\, d\theta$$

$$= \left[\tfrac{1}{2}\log\tan\left(\tfrac{1}{4}\pi + \theta\right) \right]_0^{\pi/6}$$

$$= \tfrac{1}{2}\left[\log\tan\tfrac{5}{12}\pi - \log\tan\tfrac{1}{4}\pi \right] = \tfrac{1}{2}\log(2 + \sqrt{3}).$$

Ex. 6. *Show that* $\int_0^{\frac{1}{2}\pi} \sin^6\theta\cos^3\theta\, d\theta = \frac{2}{63}$.

Let $\sin\theta = x$. $\therefore \cos\theta\, d\theta = dx$;

also when $\theta = 0$, $x = 0$ and when $\theta = \frac{1}{2}\pi$, $x = 1$.

$$\therefore I = \int_0^{\frac{1}{2}\pi} \sin^6\theta(1 - \sin^2\theta).\cos\theta\, d\theta = \int_0^1 x^6(1 - x^2)\, dx$$

$$= \int_0^1 x^6\, dx - \int_0^1 x^8\, dx = \left[\frac{x^7}{7}\right]_0^1 - \left[\frac{x^9}{9}\right]_0^1 = \frac{1}{7} - \frac{1}{9} = \frac{2}{63}.$$

6.17. Series represented by Definite Integrals.

The definition of a definite integral as the limit of a sum enables us to evaluate easily the limits of the sums of certain series, when the number of terms tends to infinity by identifying them with some definite integrals. This is illustrated in the following examples.

In identifying a series with a definite integral, it should be noted that the definite integral

$$\int_a^b f(x)\,dx = \underset{h \to 0}{Lt}\; h\,\Sigma f(a + rh), \text{ when } nh = b - a,$$

may be expressed as

$$\underset{n \to \infty}{Lt}\; \frac{b-a}{n}\,\Sigma f\left(a + r\frac{b-a}{n}\right) = \int_a^b f(x)\,dx.$$

In the special case when $a = 0$, $b = 1$, we have $h = 1/n$.

Hence, in this case, we have

$$\underset{n \to \infty}{Lt} \; \frac{1}{n} \Sigma f\left(\frac{r}{n}\right) = \int_0^1 f(x)\, dx,$$

[As if we write x for r/n and dx for $1/n$.]

or, puttting $h = 1/n$, $\underset{h \to 0}{Lt} \; h \Sigma f(rh) = \int_0^1 f(x)\, dx$.

[As if we write x for rh and dx for h.]

6.18. Illustrative Examples.

Ex. 1. Evaluate $\underset{n \to \infty}{Lt} \left\{ \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n} \right\}$. [*H. S. '88*]

Dividing the numerator and denominator of each term of the above series by n, the given series becomes

$$= \underset{n \to \infty}{Lt} \left\{ \frac{\frac{1}{n}}{1 + \frac{1}{n}} + \frac{\frac{1}{n}}{1 + \frac{2}{n}} + \dots\dots + \frac{\frac{1}{n}}{1 + \frac{n}{n}} \right\}$$

$$= \underset{n \to \infty}{Lt} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{1 + \frac{r}{n}} = \underset{h \to 0}{Lt} \; h \sum_{r=1}^{n} \frac{1}{1+rh} \quad \left[\text{putting } h = \frac{1}{n} \right]$$

$$= \int_0^1 \frac{1}{1+x}\, dx = \Big[\log(1+x) \Big]_0^1 = \log 2.$$

Ex. 2. *Evaluate*

$$\underset{n \to \infty}{Lt} \left\{ \left(1 + \frac{1}{n^2}\right)^{\frac{2}{n^2}} \left(1 + \frac{2^2}{n^2}\right)^{\frac{4}{n^2}} \left(1 + \frac{3^2}{n^2}\right)^{\frac{6}{n^2}} \cdots \left(1 + \frac{n^2}{n^2}\right)^{\frac{2n}{n^2}} \right\}.$$

Let A denote the given expression ; then

$$\log A = \sum_{r=1}^{n} \frac{2r}{n^2} \log\left(1 + \frac{r^2}{n^2}\right).$$

$$\therefore \underset{n \to \infty}{Lt} \log A = \underset{n \to \infty}{Lt} \frac{1}{n} \sum_{r=1}^{n} 2 \frac{r}{n} \log \left(1 + \frac{r^2}{n^2}\right)$$

$$= \int_0^1 2x \log(1 + x^2)\, dx$$

$$= \int_1^2 \log z\, dz, \text{ [putting } 1 + x^2 = z \text{]}$$

$$= \Big[z \log z - z \Big]_1^2 = 2 \log 2 - 1 = \log \frac{4}{e}.$$

Since $\log \underset{n \to \infty}{Lt} A = \underset{n \to \infty}{Lt} \log A = \log \frac{4}{e}$,

$\therefore \underset{n \to \infty}{Lt} A$, *i.e.*, the limit $= \frac{4}{e}$.

Ex. 3. *Prove that* $\underset{n \to \infty}{Lt} \frac{1^m + 2^m + 3^m + \ldots\ldots + n^m}{n^{m+1}} = \frac{1}{m+1}$ [$m > -1$].

Left side

$$= \underset{n \to \infty}{Lt} \frac{1}{n} \left[\left(\frac{1}{n}\right)^m + \left(\frac{2}{n}\right)^m + \ldots\ldots + \left(\frac{n}{n}\right)^m \right]$$

$$= \underset{n \to \infty}{Lt} \frac{1}{n} \sum_{r=1}^{n} \left(\frac{r}{n}\right)^m = \underset{h \to 0}{Lt}\, h \sum_{r=1}^{n} (rh)^m \left[\text{where } h = \frac{1}{n}\right]$$

$$= \int_0^1 x^m dx = \left[\frac{x^{m+1}}{m+1}\right]_0^1 = \frac{1}{m+1}.$$

EXAMPLES VI(A)

1. Find by the method of summation the values of :-

(i) $\int_a^b e^{-x}\, dx$. (ii) $\int_a^b e^{kx}\, dx$.

(iii) $\int_0^1 x^3\, dx$. (iv) $\int_0^1 (ax + b)\, dx$.

(v) $\int_0^{\frac{1}{2}\pi} \sin x\, dx$. (vi) $\int_a^b \cos\theta\, d\theta$.

(vii) $\int_0^1 \sqrt{x}\, dx$. (viii) $\int_1^4 \frac{1}{\sqrt{x}}\, dx$.

(ix) $\int_0^a \sin\ nx\, dx$. (x) $\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \operatorname{cosec}^2 x\, dx$.

Evaluate the following integrals (*Ex.* 2 *to Ex.* 12) :-

2. (i) $\int_0^1 x^3 \sqrt{1 + 3x^4}\, dx$. (ii) $\int_0^{2a} \sqrt{2ax - x^2}\, dx$.

(iii) $\int_1^{e^2} \frac{dx}{x(1 + \log x)^2}$. [*H. S.* '85] (iv) $\int_0^1 \frac{dx}{(x^2 + 1)^2}$

3. $\int_0^1 xe^x\, dx$. [*H. S.* '80]

4. (i) $\int_0^1 \sin^{-1} x\, dx$. (ii) $\int_0^1 \tan^{-1} x\, dx$.

(iii) $\int_0^1 (\cos^{-1} x)^2\, dx$. (iv) $\int_0^1 x \log(1 + 2x)\, dx$.

(v) $\int_0^1 x(\tan^{-1} x)^2\, dx$. (vi) $\int_0^1 x^2 \sqrt{(4 - x^2)}\, dx$.

5. (i) $\int_0^\pi \sin mx\ \sin\ nx\, dx$. [*J. E E.*'82]

(ii) $\int_0^\pi \cos\ mx\ \cos nx\, dx$. (*m, n being integers*)

(iii) $\int_0^{\pi/2} \sin x \sin 2x \, dx$.

6. (i) $\int_0^{\pi} \sin^2 nx \, dx$. (ii) $\int_0^{\pi} \cos^2 nx \, dx$.

(n being an integer)

7. (i) $\int_0^1 \frac{x \, dx}{\sqrt{(1 + x^2)}}$. (ii) $\int_0^a \frac{dx}{(a^2 + x^2)^{3/2}}$.

(iii) $\int_0^a \frac{dx}{\sqrt{(ax - x^2)}}$. (iv) $\int_2^3 \frac{dx}{\sqrt{\{(x-1)(5-x)\}}}$.

8. (i) $\int_0^{\frac{1}{2}\pi} x \sin x \, dx$. (ii) $\int_0^{\frac{1}{4}\pi} \sec x \, dx$.

(iii) $\int_0^{\frac{1}{2}\pi} (\sec\theta - \tan\theta) \, d\theta$.

9. (i) $\int_0^{\frac{1}{4}\pi} \tan x \, dx$. (ii) $\int_0^{\frac{1}{4}\pi} \tan^2 x \, dx$.

10. (i) $\int_0^{\frac{1}{2}\pi} \cos 2x \cos 3x \, dx$. (ii) $\int_0^{\frac{1}{2}\pi} \sin^2 x \cos^2 x \, dx$. [*H. S. '82*]

(iii) $\int_0^{\frac{1}{4}\pi} x \cos x \cos 3x \, dx$. (iv) $\int_0^{\frac{1}{4}\pi} \sec^4\theta \, d\theta$.

11. (i) $\int_1^{\sqrt{e}} x \log x \, dx$. (ii) $\int_0^{\frac{1}{2}\pi} x^2 \sin x \, dx$. [*H. S. '81*]

(iii) $\int_0^{\frac{1}{2}\pi} \sin\phi \cos\phi \sqrt{(a^2 \sin^2\phi + b^2 \cos^2\phi)} \, d\phi$.

12. (i) $\int_0^a \frac{dx}{a^2 + x^2}$. (ii) $\int_1^2 \frac{dx}{x(1 + 2x)^2}$

(iii) $\int_0^{\frac{1}{2}\pi} \frac{dx}{a + b\cos x}$ $(a > b > 0)$.

(iv) $\int_0^{\pi} \frac{dx}{1 - 2a\cos x + a^2}$ $(0 < a < 1)$.

Show that (*Ex. 13 to Ex. 28 (ii)*]:-

13. $\int_0^{\log 2} \frac{e^x}{1 + e^x}\,dx = \log\frac{3}{2}$.

14. $\int_a^b \frac{\log x}{x}\,dx = \frac{1}{2}\log\left(\frac{b}{a}\right)\log(ab)$.

15. $\int_0^a \sin^{-1}\frac{2t}{1 + t^2}\,dt = 2a\tan^{-1}a - \log(1 + a^2)$. [*H. S. '85*]

16. (i) $\int_1^2 \sqrt{(x - 1)(2 - x)}\,dx = \frac{1}{8}\pi$.

(ii) $\int_8^{15} \frac{dx}{(x - 3)\sqrt{(x + 1)}} = \frac{1}{2}\log\frac{5}{3}$. [*H. S. '85*]

17. $\int_0^a \frac{a^2 - x^2}{(a^2 + x^2)^2}\,dx = \frac{1}{2a}$.

18. $\int_0^{\frac{3}{4}\pi} \frac{\sin x\,dx}{1 + \cos^2 x} = \frac{\pi}{4} + \tan^{-1}\frac{1}{\sqrt{2}}$.

19. $\int_0^{\frac{1}{2}\pi} \cos^3 x \,.\, \sqrt[4]{\sin x\,dx} = \frac{32}{65}$.

20. (i) $\int_0^{\frac{1}{2}\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi}{2ab}$. [$a, b > 0$]

(ii) $\int_0^{\frac{1}{4}\pi} \frac{\sin^2 x\cos^2 x}{(\sin^3 x + \cos^3 x)^2}\, dx = \frac{1}{6}$.

21. (i) $\int_0^{\frac{1}{2}\pi} \frac{dx}{4 + 5\sin x} = \frac{1}{3}\log 2$.

(ii) $\int_0^{\frac{1}{2}\pi} \frac{dx}{5 + 4\sin x} = \frac{2}{3}\tan^{-1}\frac{1}{3}$.

22. (i) $\int_0^{\frac{1}{2}\pi} \frac{dx}{5 + 3\cos x} = \frac{1}{2}\tan^{-1}\frac{1}{2}$.

(ii) $\int_0^{\frac{1}{2}\pi} \frac{dx}{3 + 5\cos x} = \frac{1}{4}\log 3$.

(iii) $\int_0^{\frac{1}{2}\pi} \frac{dx}{1 + 4\cot^2 x} = \frac{\pi}{6}$.

23. $\int_0^{\frac{1}{2}\pi} \frac{dx}{1 + \cos\theta\cos x} = \frac{\theta}{\sin\theta}$.

24. $\int_0^{\frac{1}{2}\pi} \frac{\cos x\, dx}{(1 + \sin x)(2 + \sin x)} = \log\frac{4}{3}$.

25. $\int_0^{\frac{1}{4}\pi} \frac{\sin 2x}{\sin^4 x + \cos^4 x}\, dx = \frac{\pi}{4}$.

26. (i) $\int_0^{\frac{1}{2}\pi} \frac{dx}{(a^2\cos^2 x + b^2\sin^2 x)^2} = \frac{\pi}{4}\,\frac{a^2 + b^2}{a^3 b^3}$. [$a, b > 0$]

[*J. E. E. '88*]

[*Multiply numerator and denominator by* $\sec^4 x$; *then put* $b\tan x = a\tan\theta$]

(ii) $\int_0^{\frac{1}{2}\pi} \frac{x \sin x \cos x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2}\, dx = \frac{\pi}{4ab^2(a+b)}$. $[a, b > 0]$

27. $\int_2^e \left\{ \frac{1}{\log x} - \frac{1}{(\log x)^2} \right\} dx = e - \frac{2}{\log 2}$

28. (i) $\int_2^3 \frac{dx}{(x-1)\sqrt{(x^2-2x)}} = \frac{\pi}{3}$.

(ii) $\int_0^1 \frac{dx}{(1+x)\sqrt{(1+2x-x^2)}} = \frac{\pi}{4\sqrt{2}}$.

29. Evaluate the following :-

(i) $\underset{n \to \infty}{Lt} \left[\frac{1}{n+m} + \frac{2}{n+2m} + \ldots\ldots + \frac{1}{n+nm} \right]$.

(ii) $\underset{n \to \infty}{Lt} \left[\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \ldots\ldots + \frac{n}{n^2+n^2} \right]$.

[H. S. '86]

(iii) $\underset{n \to \infty}{Lt} \left[\frac{1}{\sqrt{(n^2-1^2)}} + \frac{1}{\sqrt{(n^2-2^2)}} + \ldots + \frac{1}{\sqrt{\{n^2-(n-1)^2\}}} \right]$.

(iv) $\underset{n \to \infty}{Lt} \left[\frac{1}{\sqrt{(2n-1^2)}} + \frac{1}{\sqrt{(4n-2^2)}} + \ldots + \frac{1}{n} \right]$.

[*Write* $n = \sqrt{(2n^2 - n^2)}$ *in the last term.*]

(v) $\underset{n \to \infty}{Lt} \left[\frac{1^2}{n^3+1^3} + \frac{2^2}{n^3+2^3} + \ldots\ldots + \frac{n^2}{2n^3} \right]$.

[C. P. '84]

(vi) $\underset{n \to \infty}{Lt} \left[\frac{1}{n} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \ldots\ldots + \frac{1}{8n} \right]$.

(vii) $\underset{n \to \infty}{Lt} \left[\frac{1}{n} + \frac{\sqrt{n^2-1^2}}{n^2} + \ldots + \frac{\sqrt{n^2-(n-1)^2}}{n^2} \right]$.

(viii) $\underset{n \to \infty}{Lt} \sum_{r=1}^{n-1} \frac{1}{n} \sqrt{\left(\frac{n+r}{n-r} \right)}$.

(ix) $\underset{n \to \infty}{Lt} \left\{ \left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right) \ldots\ldots \left(1 + \frac{n}{n}\right) \right\}^{1/n}$. [J. E. E. '86]

(x) $\underset{n \to \infty}{Lt} \left\{ \left(1 + \frac{1^2}{n^2}\right)\left(1 + \frac{2^2}{n^2}\right) \ldots \left(1 + \frac{n^2}{n^2}\right) \right\}^{1/n}$.

(xi) $\underset{n \to \infty}{Lt} \sum_{r=1}^{n} \frac{n + r}{n^2 + r^2}$.

(xii) $\underset{n \to \infty}{Lt} \sum_{r=1}^{n} \frac{n}{(n + r)\sqrt{\{r(2n + r)\}}}$.

(xiii) $\underset{n \to \infty}{Lt} \left[\frac{1}{n} + \frac{1}{n + 1} + \frac{1}{n + 2} + \ldots\ldots + \frac{1}{3n} \right]$. [H. S. '83, '88]

(xiv) $\underset{n \to \infty}{Lt} \frac{1}{\sqrt{n}} \left[1 + \frac{1}{\sqrt{2}} + \ldots\ldots + \frac{1}{\sqrt{n}} \right]$. [J. E. E. '86]

(xv) $\underset{n \to \infty}{Lt} \left[\frac{\sqrt{(n + 1)} + \sqrt{(n + 2)} + \ldots + \sqrt{2n}}{n\sqrt{n}} \right]$.

(xvi) $\underset{n \to \infty}{Lt} \left[\frac{n!}{n^n} \right]^{1/n}$.

30. If $\int_0^a \frac{dx}{\sqrt{(x + a)} + \sqrt{x}} = \int_0^{\frac{1}{4}\pi} \frac{\sin\theta \, d\theta}{\cos^2\theta}$

find the value of a.

31. If a be positive and the positive value of the square root is taken, show that

$$\int_{-1}^{+1} \frac{dx}{\sqrt{(1 - 2ax + a^2)}} = 2 \text{ if } a < 1;$$

$$= \frac{2}{a} \text{ if } a > 1.$$

32. If m and n are *positive integers*, show that

(i) $\int_{-\pi}^{+\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n. \end{cases}$

(ii) $\int_{-\pi}^{+\pi} \sin mx \cos nx \, dx = 0$.

(iii) $\int_{-\pi}^{+\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n. \end{cases}$

ANSWERS

1. (i) $(e^{-a} - e^{-b})$. (ii) $(e^{kb} - e^{ka})/k$. (iii) $\frac{1}{4}$.
   (iv) $\frac{1}{2}a + b$. (v) 1. (vi) $\sin b - \sin a$. (vii) $\frac{2}{3}$. (viii) 2.
   (ix) $(1 - \cos na)/n$. (x) 1.

2. (i) $\frac{7}{18}$. (ii) $\frac{1}{2}\pi a^2$. (iii) $\frac{2}{3}$. (iv) $\frac{1}{8}(\pi + 2)$.

3. 1. 4. (i) $\frac{1}{2}\pi - 1$. (ii) $\frac{1}{4}\pi - \frac{1}{2}\log 2$. (iii) $\pi - 2$. (iv) $\frac{3}{8}\log 3$.
   (v) $\frac{1}{4}\pi(\frac{1}{4}\pi - 1) + \frac{1}{2}\log 2$ (vi) $\frac{1}{3}\pi - \frac{1}{4}\sqrt{3}$.

5. (i) $\dfrac{\sin(m-n)\pi}{2(m-n)} - \dfrac{\sin(m+n)\pi}{2(m+n)}$. (ii) 0. (iii) $\frac{2}{3}$.

6. (i) $\frac{1}{2}\pi$. (ii) $\frac{1}{2}\pi$. 7. (i) $\sqrt{2} - 1$. (ii) $\dfrac{1}{a^2\sqrt{2}}$. (iii) π.
   (iv) $\frac{1}{6}\pi$. 8. (i) 1. (ii) $\log(\sqrt{2} + 1)$. (iii) $\log 2$.

9. (i) $\frac{1}{2}\log 2$. (ii) $1 - \frac{1}{4}\pi$. 10. (i) $\frac{3}{5}$. (ii) $\frac{2}{15}$. (iii) $\frac{1}{16}(\pi - 3)$.
   (iv) $\frac{4}{3}$. 11. (i) $\frac{1}{4}$. (ii) $\pi - 2$. (iii) $\frac{1}{3}\;\dfrac{a^2 + ab + b^2}{a + b}$.

12. (i) $\dfrac{\pi}{4a}$. (ii) $\log\frac{6}{5} - \frac{2}{15}$. (iii) $\dfrac{1}{\sqrt{(a^2 - b^2)}}\cos^{-1}\left(\dfrac{b}{a}\right)$.
   (iv) $\dfrac{\pi}{1 - a^2}$. 29. (i) $\dfrac{1}{m}\log(1 + m)$. (ii) $\frac{1}{4}\pi$. (iii) $\frac{1}{2}\pi$. (iv) $\frac{1}{2}\pi$.
   (v) $\frac{1}{3}\log 2$. (vi) $\frac{3}{8}$. (vii) $\frac{1}{4}\pi$. (viii) $\frac{1}{2}\pi + 1$. (ix) $4/e$.
   (x) $2e^{1/2(\pi - 4)}$. (xi) $\dfrac{\pi}{4} + \dfrac{1}{2}\log 2$. (xii) $\frac{1}{3}\pi$. (xiii) $\log 3$.
   (xiv) 2. (xv) $\frac{4}{3}\sqrt{2} - \frac{2}{3}$. (xvi) e^{-1}. 30. $\frac{9}{16}$.

—

6.19. General Properties of Definite Integrals.

(i) $\int_a^b f(x)\,dx = \int_a^b f(z)\,dz$.

Let $\int f(x)\,dx = \phi(x)$; $\therefore \int_a^b f(x)\,dx = \phi(b) - \phi(a)$;

then, $\int f(z)\,dz = \phi(z)$; $\therefore \int_a^b f(z)\,dz = \phi(b) - \phi(a)$.

(ii) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

Let $\int f(x)\,dx = \phi(x)$; $\therefore \int_a^b f(x)\,dx = \phi(b) - \phi(a)$;

and $-\int_b^a f(x)\,dx = -[\phi(a) - \phi(b)] = \phi(b) - \phi(a)$.

Thus, *an interchange of the limits changes the sign of the integral.*

(iii) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$. $(a < c < b)$.

Let $\int f(x)\,dx = \phi(x)$; $\therefore \int_a^b f(x)\,dx = \phi(b) - \phi(a)$.

Right side $= \{\phi(c) - \phi(a)\} + \{\phi(b) - \phi(c)\} = \phi(b) - \phi(a)$.

Generalization.

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \ldots$$

$$+ \int_{c_{n-1}}^{c_n} f(x)\,dx + \int_{c_n}^{b} f(x)\,dx,$$

when $a < c_1 < c_2 < \ldots\ldots < c_n < b$.

(iv) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

Proof. Put $a - x = z$, $\therefore$ $dx = -dz$;

also when $x = 0$, $z = a$, and when $x = a$, $z = 0$.

$\therefore$ right side $= -\int_a^0 f(z)\,dz = \int_0^a f(z)\,dz = \int_0^a f(x)\,dx$.

Illustration : $\int_0^{\pi/2} \sin x\,dx = \int_0^{\pi/2} \sin\left(\frac{\pi}{2} - x\right)dx = \int_0^{\pi/2} \cos x$

(v) $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$, if $f(x) = f(a+x)$

[C. P. '86]

Proof .

$$\int_0^{na} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx + \ldots + \int_{(n-1)a}^{na} f(x)$$

Put $z + a = x$, then $dx = dz$,

also when $x = a$, $z = 0$, and when $x = 2a$, $z = a$;

$$\therefore \int_a^{2a} f(x)\,dx = \int_0^a f(z+a)\,dz = \int_0^a f(a+x)\,dx$$

$$= \int_0^a f(x)\,dx.$$

Similarly, it can be shown that

$$\int_{2a}^{3a} f(x)\,dx = \int_a^{2a} f(x)\,dx = \int_0^a f(x)\,dx;$$

and so on. Thus, each of the integrals on the right side can be shown to be equal to $\int_0^a f(x)\,dx$. Hence the result.

Illustration :

Since $\sin^8 x = \sin^8(\pi + x)$, $\therefore \int_0^{4\pi} \sin^8 x\,dx = 4\int_0^{\pi} \sin^8 x\,dx$.

(vi) $\displaystyle\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a - x)\,dx$.

Proof. $\displaystyle\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_a^{2a} f(x)\,dx$.

[*By* (*iii*)]

Put $x = 2a - z$ in the 2nd integral ; then $dx = -dz$, also when $x = a$, $z = a$; and when $x = 2a$, $z = 0$.

$\therefore$ the second integral on the right side, *viz.*,

$$\int_a^{2a} f(x)\,dx = -\int_a^{0} f(2a - z)\,dz = \int_0^{a} f(2a - z)\,dz.$$

[*By* (*ii*)]

$$= \int_0^{a} f(2a - x)\,dx. \qquad [\text{By } (i)]$$

Hence the result.

(vii) $\displaystyle\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$, if $f(2a - x) = f(x)$,

and $\displaystyle\int_0^{2a} f(x)\,dx = 0$, if $f(2a - x) = -f(x)$.

These two results follow immediately from (vi).

Illustration :

Since $\sin(\pi - x) = \sin x$, and $\cos(\pi - x) = -\cos x$,

$$\therefore \int_0^{\pi} \sin x \, dx = 2\int_0^{\pi/2} \sin x \, dx \text{ ; and } \int_0^{\pi} \cos x \, dx = 0,$$

and generally, $\int_0^{\pi} f(\sin x)\, dx = 2\int_0^{\pi/2} f(\sin x)\, dx$,

and $\int_0^{\pi} f(\cos x)\, dx = 0$, if $f(\cos x)$ is an *odd* function of $\cos x$.

$$\text{(viii)} \int_{-a}^{+a} f(x)\, dx = \int_0^{a} \{f(x) + f(-x)\}\, dx.$$

Proof. $\int_{-a}^{+a} f(x)\, dx = \int_{-a}^{0} f(x)\, dx + \int_0^{+a} f(x)\, dx.$

Now, putting $x = -z$,

$$\int_{-a}^{0} f(x)\, dx = -\int_{a}^{0} f(-z)\, dz = \int_0^{a} f(-z)\, dz$$

$$= \int_0^{a} f(-x)\, dx.$$

Hence, the result follows.

Cor. If *$f(x)$ is an odd function of x*, i.e., $f(-x) = -f(x)$,

$$\int_{-a}^{+a} f(x)\, dx = 0,$$

and if *$f(x)$ is an even function of x*, i.e., $f(-x) = f(x)$,

$$\int_{-a}^{+a} f(x)\, dx = 2\int_0^{a} f(x)\, dx.$$

Illustration :

$$\int_{-\pi/2}^{+\pi/2} \sin^5 x \, dx = 0, \text{ and}$$

$$\int_{-\pi/2}^{+\pi/2} \sin^6 x \, dx = 2 \int_0^{\pi/2} \sin^6 x \, dx.$$

6.20. Illustrative Examples.

By the help of the above properties of definite integrals we can evaluate many definite integrals without evaluating the corresponding indefinite integrals, as shown in the following examples.

Ex. 1. *Show that* $\int_0^{\pi/2} \log \tan x \, dx = 0$. [*H. S. '85*]

$$I = \int_0^{\pi/2} \log \tan \left(\frac{\pi}{2} - x \right) dx \qquad [\text{ By (iv), Art. 6.8 }]$$

$$= \int_0^{\pi/2} \log \cot x \, dx = - \int_0^{\pi/2} \log \tan x \, dx = - I.$$

$\therefore \quad 2I = 0; \qquad \therefore I = 0.$

Ex. 2. *Show that* $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{(\sin x)} + \sqrt{(\cos x)}} dx = \frac{\pi}{4}$. [*C. P. '88*]

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\left\{\sin\left(\frac{\pi}{2} - x\right)\right\}} + \sqrt{\left\{\cos\left(\frac{\pi}{2} - x\right)\right\}}} dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{(\cos x)} + \sqrt{(\sin x)}} dx.$$

$$\therefore \quad 2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{(\sin x)} + \sqrt{(\cos x)}} dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{(\cos x)} + \sqrt{(\sin x)}} dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{(\sin x)} + \sqrt{(\cos x)}}\, dx = \int_0^{\pi/2} dx = \Big[x\Big]_0^{\pi/2} = \tfrac{1}{2}\pi .$$

$\therefore\ I = \frac{1}{4}\pi .$

Ex. 3. *Show that* $\displaystyle\int_0^{\pi/2} \log \sin x\, dx = \int_0^{\pi/2} \log \cos x\, dx = \frac{\pi}{2} \log \frac{1}{2}$.

[C. H. '86]

$$\int_0^{\pi/2} \log \sin x\, dx = \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} - x\right) dx$$

$$= \int_0^{\pi/2} \log \cos x\, dx .$$

[By Art. 6.19, (iv)]

$$\therefore\ 2I = \int_0^{\pi/2} \log \sin x\, dx + \int_0^{\pi/2} \log \cos x\, dx$$

$$= \int_0^{\pi/2} (\log \sin x + \log \cos x)\, dx = \int_0^{\pi/2} \log(\sin x \cos x)\, dx$$

$$= \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right) dx = \int_0^{\pi/2} (\log \sin 2x - \log 2)\, dx$$

$$= \int_0^{\pi/2} \log \sin 2x\, dx - \frac{\pi}{2} \log 2 .$$

Put $2x = z$; $\quad\therefore\ dx = \frac{1}{2} dz$.

$$\therefore\quad \int_0^{\pi/2} \log \sin 2x\, dx = \frac{1}{2}\int_0^{\pi} \log \sin z\, dz$$

$$= \frac{1}{2}\int_0^{\pi} \log \sin x\, dx = \int_0^{\pi/2} \log \sin x\, dx = I$$ [By (vii), Art. 6.19,]

$\therefore\ 2I = I - \frac{\pi}{2} \log 2$; $\quad\therefore\ I = -\frac{\pi}{2} \log 2 = \frac{\pi}{2} \log \frac{1}{2}$

Ex. 4. *Show that* $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\log 2$. [C. H '61 & C.P. '85]

Put $x = \tan\theta$; $\therefore dx = \sec^2\theta\, d\theta$; also when $x = 0$, $\theta = 0$; and when $x = 1$, $\theta = \frac{1}{4}\pi$.

$$\therefore I = \int_0^{\frac{1}{4}\pi} \log(1+\tan\theta)\,d\theta = \int_0^{\frac{1}{4}\pi} \log\{1+\tan(\tfrac{1}{4}\pi - \theta)\}\,d\theta.$$

[*By Art. 6.19, (iv)*]

Now, $1 + \tan\left(\frac{\pi}{4} - \theta\right) = 1 + \frac{1-\tan\theta}{1+\tan\theta} = \frac{2}{1+\tan\theta}$;

$$\therefore I = \int_0^{\frac{1}{4}\pi} \log\frac{2}{1+\tan\theta}\,d\theta = \int_0^{\frac{1}{4}\pi} \{\log 2 - \log(1+\tan\theta)\}\,d\theta$$

$$= \int_0^{\frac{1}{4}\pi} \log 2\,d\theta - \int_0^{\frac{1}{4}\pi} \log(1+\tan\theta)\,d\theta = \tfrac{1}{4}\pi \,.\log 2 - I.$$

$\therefore 2I = \frac{1}{4}\pi \,.\log 2$; $\quad \therefore I = \frac{\pi}{8}\log 2$.

Ex. 5. *Show that* $\int_{-a}^{+a} \frac{xe^{x^2}}{1+x^2}\,dx = 0$.

$$I = \int_{-a}^{0} \frac{xe^{x^2}}{1+x^2}\,dx + \int_0^a \frac{xe^{x^2}}{1+x^2}\,dx = I_1 + I_2 \text{ say}.$$

Putting $x = -z$ in the first integral,

$$I_1 = \int_a^0 \frac{ze^{z^2}}{1+z^2}\,dz = -\int_0^a \frac{ze^{z^2}}{1+z^2}\,dz = -\int_0^a \frac{xe^{x^2}}{1+x^2}\,dx = -I_2.$$

Hence the result.

6.21. Logarithmic and Exponential Functions.

The fundamental concepts of Calculus furnish a more adequate theory of logarithmic and exponential functions than the met-hods adopted in elementary books. There an exponential function is first introduced, and then logarithm is defined as the inverse function;

but in the treatment of these functions by the principles of Calculus, logarithm is first defined by means of a definite integral, and then exponential function is introduced as the inverse of logari-thm. From the stand-point of these new definitions, certain important inequalities and limits can be obtained more easily and satisfactorily.

A. Logarithmic Function.

The *natural* logarithm $\log x$ is defined as

$$\log x = \int_1^x \frac{dt}{t}, \qquad \ldots (1)$$

where x is any *positive number, i.e.,* $x > 0$.

Thus, $\log x$ denotes the area under the curve $y = 1/t$ from $t = 1$ to $t = x$.

From the definition it follows that $\log 1 = 0$, and [since $1/t$ is continuous for $t > 0$] from the fundamental theorem of Integral Calculus it follows that $\log x$ is a continuous function and has a derivative given by

$$\frac{d}{dx}(\log x) = \frac{1}{x}. \qquad \ldots (2)$$

Since the derivative is always positive, $\log x$ increases steadily with x (*i.e.*, $\log x$ is a monotone increasing function).

Putting $t = 1/u$ in the integral for x, we get

$$\log x = \int_1^x \frac{dt}{t} = -\int_1^{1/x} \frac{du}{u} = -\log \frac{1}{x}. \qquad \ldots (3)$$

Putting $t = yu$ [y = a fixed number > 0] in the integral for $\log(xy)$, we get

$$\log(xy) = \int_1^{xy} \frac{dt}{t} = \int_{1/y}^{x} \frac{du}{u} = \int_1^x \frac{du}{u} - \int_1^{1/y} \frac{du}{u}$$

$$= \log x - \log(1/y) = \log x + \log y. \qquad \ldots (4)$$

n this way, other well-known properties of logarithms can be oped.

nce $\log x$ is a continuous monotone function of x, having lue 0 for $x = 1$, and tending to infinity as x increases, there be some number greater than 1 such that for this value of x ve $\log x = 1$, and this number is called e. Thus e is defined equation

$$\log e = 1, \text{ i.e., } \int_1^e \frac{dt}{t} = 1. \quad \ldots (5)$$

xponential Function.

$y = \log x$, then we write $x = e^y$ ... (6)

n this way the exponential e^y is defined for all real values In particular $e^0 = 1$, since $\log 1 = 0$. As y is a continuous ion of x, x is a continuous function of y.

$x = e^y$, so that $y = \log x$, and so

$$\frac{dy}{dx} = \frac{1}{x}; \quad \therefore \quad \frac{dx}{dy} = 1 \bigg/ \frac{dy}{dx} = x = e^y,$$

$$\text{i.e., } \frac{d}{dy}(e^y) = e^y. \quad \ldots (7)$$

More generally, $\frac{d}{dy}(e^{ay}) = ae^{ay}$.

$a > 0$) is defined as $e^{x \log a}$, so that $\log a^x = x \log a$.

Thus, $10^x = e^{x \log 10}$.

The inverse function of a^y is called the *logarithm to the base a*.

Thus, if $x = a^y$, $y = \log_a x$.

. Some Inequalities and Limits.

(i) *To prove* $2 < e < 3$.

$$\log 2 = \int_1^2 \frac{dt}{t}, \; 1 < t < 2; \quad \therefore \; \tfrac{1}{2} < 1/t < 1.$$

$$\therefore \quad \int_1^2 \frac{dt}{t} < \int_1^2 dt, \text{ i.e., } < 1, \quad \text{i.e., } < \int_1^e \frac{dt}{t} \;.\; \therefore 2 < e.$$

$$\int_1^3 \frac{dt}{t} = \int_1^2 \frac{dt}{t} + \int_2^3 \frac{dt}{t} = \int_0^1 \frac{du}{2-u} + \int_0^1 \frac{du}{2+u}$$

(by putting $t = 2 - u$ and $t = 2 + u$)

$$= 4\int_0^1 \frac{du}{4-u^2} > 4\int_0^1 \frac{du}{4}, \text{ i.e., } > 1, \text{ i.e., } > \int_1^e \frac{dt}{t},$$

$\therefore 3 > e, \quad$ i.e., $e < 3$.

(*ii*) *To prove* $\dfrac{x}{1+x} < \log(1+x) < x \; (x > 0)$.

From definition, $\log(1+x) = \displaystyle\int_1^{1+x} \frac{dt}{t}$.

$\therefore 1 < t < 1 + x. \qquad \therefore 1/(1+x) < 1/t < 1.$

$$\therefore \quad \frac{1}{1+x}\int_1^{1+x} dt < \int_1^{1+x} \frac{dt}{t} < \int_1^{1+x} dt,$$

i.e, $\quad \dfrac{x}{1+x} < \log(1+x) < x.$

(*iii*) *To prove* $\displaystyle \mathop{Lt}_{x \to 0} \frac{1}{x} \log(1+x) = 1$.

From (ii), $\dfrac{1}{1+x} < \dfrac{\log(1+x)}{x} < 1$, and since $1/(1+x)$ and 1 both tend to 1 as $x \to 0$, the required limit $= 1$.

(*iv*) *To prove* $\displaystyle \mathop{Lt}_{x \to 0} \frac{a^x - 1}{x} = \log a$.

Since the derivative of a^x is $a^x \log a$, and that for $x = 0$ is $\log a$, it follows, from the definition of the derivative for $x = 0$, that

$$\mathop{Lt}_{h \to 0} \frac{a^h - a^0}{h}, \text{ i.e., } \mathop{Lt}_{h \to 0} \frac{a^h - 1}{h} = \log a.$$

Putting x for h, the required result follows.

When $a = e$, we get $\underset{x \to 0}{Lt} \dfrac{e^x - 1}{x} = 1$.

(*v*) *To prove* $\underset{n \to \infty}{Lt} \left\{ 1 + \dfrac{x}{n} \right\}^n = e^x$.

Since $\dfrac{d}{dt} \log (1 + xt) = \dfrac{x}{1 + xt}$, it follows that the derivative of log $(1 + xt)$ for $t = 0$ is x. Hence, from the definition of the derivative for $x = 0$, we get

$$\underset{h \to 0}{Lt} \frac{\log (1 + xh)}{h} = x.$$

Putting $h = 1 / \zeta$, we see that

$$\underset{\zeta \to \infty}{Lt} \zeta \log \left(1 + \frac{x}{\zeta} \right), \text{ i.e., } \underset{\zeta \to \infty}{Lt} \log \left(1 + \frac{x}{\zeta} \right)^{\zeta} = x.$$

Since the exponential function is continuous, it follows

$$\underset{\zeta \to \infty}{Lt} \left(1 + \frac{x}{\zeta} \right)^{\zeta} = e^x.$$

If we suppose $\zeta \to \infty$ through positive integral values only, the required result follows.

Putting $x = 1$, we get $\underset{n \to \infty}{Lt} \left(1 + \dfrac{1}{n} \right)^n = e$.

(*vi*) *To prove* $\underset{n \to \infty}{Lt} n (\sqrt[n]{x} - 1) = \log x$

Since the derivative of $e^y = e^y$, and that for $y = 0$ is 1, we have, from the definition of the derivative for $y = 0$,

$$\underset{h \to 0}{Lt} \frac{e^h - e^0}{h}, \text{ i.e. } \underset{h \to 0}{Lt} \frac{e^h - 1}{h},$$

Putting z / n for h where z is any arbitrary number, and n ranges over the sequence of positive integers, we get

$$\underset{n \to \infty}{Lt} \left\{ n \frac{e^{z/n} - 1}{z} \right\} = 1, \text{ i.e., } \underset{n \to \infty}{Lt} n (\sqrt[n]{(e^z)} - 1) = z.$$

Putting $z = \log x$, so that $e^z = x$, the required result follows.

(*vii*) *To prove* $\underset{x \to \infty}{Lt} \dfrac{\log x}{x^{\alpha}} = 0$, *when* $\alpha > 0$.

If $t > 1$ and $\beta > 0$, then $t^{-1} < t^{\beta - 1}$.

$$\therefore \log x = \int_1^x \frac{dt}{t} < \int_1^x t^{\beta-1}\, dt, \text{ i.e., } < \frac{x^{\beta}-1}{\beta}, \text{ i.e., } < \frac{x^{\beta}}{\beta} \text{ for } x > 1.$$

Suppose $\alpha > \beta$.

$$\therefore 0 < \frac{\log x}{x^{\alpha}} < \frac{x^{\beta}}{\beta x^{\alpha}}, \text{ i.e., } < \frac{1}{\beta}\,\frac{1}{x^{\alpha-\beta}} \text{ for } x > 1.$$

But $(1/x^{\alpha-\beta}) \to 0$, as $x \to \infty$, since $\alpha > \beta$.

Hence the result.

Note. Replacing x by n, where n is a positive integer,

$$\underset{n\to\infty}{Lt} \frac{\log n}{n^{\alpha}} = 0, \text{ when } \alpha > 0 \text{ (} n \to \infty \text{ through positive integral values).}$$

(*viii*) *To prove* $\underset{y\to\infty}{Lt} \frac{y^{\alpha}}{e^{y}} = 0$, *for all values of* n, *however great.*

From (vii), $x^{-\beta} \log x \to 0$, when $x \to \infty$, for $\beta > 0$.

Putting $\alpha = 1/\beta$ in the left side and raising it to the power α, we get $x^{-1}(\log x)^{\alpha} \to 0$, as $x \to \infty$. Now putting $x = e^{y}$, so that $\log x = y$, the required result follows.

6.23. Two Important Definite Integrals.

A. *If* n *be a positive integer,**

$$\int_0^{\frac{1}{2}\pi} \sin^n x\, dx = \int_0^{\frac{1}{2}\pi} \cos^n x\, dx \qquad [\text{ C. P. '82 }]$$

$$= \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4} \cdots\cdots \frac{3}{4}\,\frac{1}{2}\,\frac{\pi}{2}, \qquad [\text{ C. P. '84 }]$$

or,
$$= \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4} \cdots\cdots \frac{4}{5}\,\frac{2}{3}.1,$$

according as n *is even or odd.*

Proof. $\int \sin^n x\, dx = \int \sin^{n-1} x . \sin x\, dx$

$$= \sin^{n-1} x . (-\cos x) + (n-1) \int \sin^{n-2} x \cos x\, dx$$

(integrating by parts)

* For other forms of these integrals see § 9.3 .

$$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,(1-\sin^2 x)\,dx$$

$$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^n dx.$$

$\therefore$ transposing $-(n-1)\int \sin^n x\,dx$ to the left side and dividing by n, we have

$$\int \sin^n x\,dx = -\frac{\sin^{n-1}x\cos x}{n} + \frac{(n-1)}{n}\int \sin^{n-2}x\,dx \quad .(1)$$

$$\therefore \int_0^{\frac{1}{2}\pi} \sin^n x\,dx = \left[-\frac{\sin^{n-1}x\cos x}{n}\right]_0^{\frac{1}{2}\pi} + \frac{n-1}{n}\int_0^{\frac{1}{2}\pi} \sin^{n-2}x\,dx$$

$$= \frac{n-1}{n}\int_0^{\frac{1}{2}\pi} \sin^{n-2}x\,dx.$$

Hence, denoting $\int_0^{\frac{1}{2}\pi} \sin^n x\,dx$ by I_n, we have

$$I_n = \frac{n-1}{n} I_{n-2}. \quad \ldots (2)$$

Changing n into $n-2$, $n-4$, etc. successively, we have, from (2),

$$I_{n-2} = \frac{n-3}{n-2} I_{n-4}\,;\; I_{n-4} = \frac{n-5}{n-4} I_{n-6}, \text{ etc.}$$

$$\therefore I_n = \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4}\cdots\frac{3}{4}\,\frac{1}{2}\,I_0,$$

$$\therefore I_n = \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4}\cdots\frac{4}{5}\,\frac{2}{3}\,I_0,$$

according as n is even or odd.

But $I_0 = \int_0^{\frac{1}{2}\pi} dx = \frac{1}{2}\pi$

and $I_1 = \int_0^{\frac{1}{2}\pi} \sin x\,dx = \left[-\cos x\right]_0^{\frac{1}{2}\pi} = 1.$

Thus, we get the required value of $\int_0^{\frac{1}{2}\pi} \sin^n x\, dx$.

Exactly in the same way it can be shown that $\int_0^{\frac{1}{2}\pi} \cos^n x\, dx$ has precisely the same value as the above integral in either case, n being even or odd.

Otherwise, it can be shown thus :

$$\int_0^{\frac{1}{2}\pi} \cos^n x\, dx = \int_0^{\frac{1}{2}\pi} \cos^n (\tfrac{1}{2}\pi - x)\, dx = \int_0^{\frac{1}{2}\pi} \sin^n x\, dx.$$

Note. The student can easily detect the law of formation of the factors in the above formulæ, noting that when the index is even, an additional factor $\frac{1}{2}\pi$ is written at the end but when the index is odd, no factor involving π is introduced. The formula (1) and (2) above are called **Reduction Formula.** [See Chapter IX.]

B. $\int_0^{\frac{1}{2}\pi} \sin^m x \cos^n x\, dx$, *m , n being positive integers.**

[*C. P. '88*]

$$\int \sin^m x \cos^n x\, dx = \int \cos^{n-1} x \,.\, (\sin^m x \cos x)\, dx$$

$$= \cos^{n-1} x \, \frac{\sin^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \cos^{n-2} x \sin x \sin^{m+1} x\, dx$$

$\left[\text{integrating by parts and noting } \int \sin^m x \cos x\, dx = \frac{\sin^{m+1} x}{m+1}\right]$

$$= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} \int \sin^m x\,(1 - \cos^2 x) \cos^{n-2} x\, dx$$

$$= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} \int \sin^m x \cos^{n-2} x\, dx$$

$$- \frac{n-1}{m+1} \int \sin^m x \cos^n x\, dx .$$

* See Chapter IX, Art. 9.15.

Hence, transposing and dividing by $\dfrac{m+n}{m+1}$, we have $\int \sin^m x \cos^n x \, dx$

$$= \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} \int \sin^m x \cos^{n-2} x \, dx \; . \; . \; . \quad (1)$$

$$\therefore \int_0^{\frac{1}{2}\pi} \sin^m x \cos^n x \, dx = \left[\frac{\sin^{m+1} x \cos^{n-1} x}{m+n} \right]_0^{\frac{1}{2}\pi}$$

$$+ \frac{n-1}{m+n} \int_0^{\frac{1}{2}\pi} \sin^m x \cos^{n-2} x \, dx$$

$$= \frac{n-1}{m+n} \int_0^{\frac{1}{2}\pi} \sin^m x \cos^{n-2} x \, dx . \qquad . \; . \; . \quad (2)$$

Again, writing $\int \sin^m x \cos^n x \, dx = \int \sin^{m-1} x \, (\cos^n x \sin x) \, dx$ and integrating by parts and proceeding as above, we get

$$\int \sin^m x \cos^n x \, dx = - \frac{\sin^{m-1} x \cos^{n+1} x}{m+n}$$

$$+ \frac{m-1}{m+n} \int \sin^{m-2} x \cos^n x \, dx$$

and hence taking it between the limits 0 and $\frac{1}{2}\pi$, we get

$$\int_0^{\frac{1}{2}\pi} \sin^m x \cos^n x \, dx = \frac{m-1}{m+n} \int_0^{\frac{1}{2}\pi} \sin^{m-2} x \cos^n x \, dx \; . \; . \; . \; . \quad (3)$$

Thus, denoting $\int_0^{\frac{1}{2}\pi} \sin^m x \cos^n x \, dx$ by $I_{m,n}$ we have from (2) and (3)

$$\left. \begin{aligned} I_{m,n} &= \frac{n-1}{m+n} I_{m,n-2} \\ &= \frac{m-1}{m+n} I_{m-2,n} \end{aligned} \right\} . \qquad . \; . \; . \quad (4)$$

Again, since $\int_0^{\frac{1}{2}\pi} \sin^m x \cos^n x\, dx$

$$= \int_0^{\frac{1}{2}\pi} \sin^m (\tfrac{1}{2}\pi - x) \cos^n (\tfrac{1}{2}\pi - x)\, dx$$

$$= \int_0^{\frac{1}{2}\pi} \sin^n x \cos^m x\, dx,$$

$$\therefore \quad I_{m,n} = I_{n,m}. \qquad \ldots (5)$$

By means of the formulæ (2) and (3), either index can be reduced by 2, and by repetitions of this process we can, since m and n are positive integers, make the original integral, *viz.*, $I_{m,n}$ depend upon one in which the indices are 1 or 0. The result, therefore, finally involves one or other of the following integrals :

$$\left.\begin{aligned} &\int_0^{\frac{1}{2}\pi} \sin x \cos x\, dx = \tfrac{1}{2}; \int_0^{\frac{1}{2}\pi} dx = \frac{\pi}{2}; \\ &\int_0^{\frac{1}{2}\pi} \sin x\, dx = 1; \int_0^{\frac{1}{2}\pi} \cos x\, dx = 1 \end{aligned}\right\} \qquad \ldots (6)$$

Thus, finally we have

$$\int_0^{\frac{1}{2}\pi} \sin^m x \cos^n x\, dx = \int_0^{\frac{1}{2}\pi} \cos^m x \sin^n x\, dx$$

$$= \frac{1.3.5\ldots(m-1).1.3.5\ldots(n-1)}{2.4.6.\ldots(m+n)} \frac{\pi}{2},$$

when *both* m *and* n *are even integers* ; and

$$= \frac{2.4.6.\ldots(m-1)}{(n+1)(n+3)\ldots(n+m)},$$

when one of the two indices, say m, *is an odd integer*.

By (iv) of Art 6.19,

$$I_{m,n} = \int_0^{\frac{1}{2}\pi} \sin^m \left(\tfrac{1}{2}\pi - x\right) \cos^n \left(\tfrac{1}{2}\pi - x\right) dx$$

$$= \int_0^{\frac{1}{2}\pi} \sin^n x \cos^m x \, dx$$

$$= I_{n,m} . \qquad \ldots (1)$$

From (2) of Art. 6.10(B), we get

$$\left.\begin{aligned} I_{m,n} &= \frac{n-1}{m+n} \cdot I_{m,n-2} \\ &= \frac{m-1}{m+n} I_{m-2,n}, \text{ by (1)} \end{aligned}\right\} \qquad \ldots (2)$$

If n is an even integer we can deduce from the first result of (2) by integration

$$\begin{aligned} I_{m,n} &= \frac{n-1}{m+n} \frac{n-3}{m+n-2} \cdot I_{m,n-4} = \ldots \\ &= \frac{(n-1)(n-3)\ldots 3.1}{(m+n)(m+n-2)\ldots(m+2)} \cdot I_{m,0} \\ &= \frac{1.3.5\ldots(n-1)}{(m+n)(m+n-2)\ldots(m+2)} \int_0^{\frac{1}{2}\pi} \sin^m x \, dx . \end{aligned}$$

The result now follows from (A) of § 6.23.

Note 1. The above definite integrals are of great use in the application of Integral Calculus to practical problems ; *e.g.*, in the determination of centre of gravity, in the calculation of area, etc. ; and also many elementary definite integrals on suitable substitution reduce to one or other of the above forms, as shown in the following examples.

6.24. Illustrative Examples.

Ex. 1. *Evaluate* $\int_0^1 x^6 \sqrt{(1 - x^2)}\, dx$.

Put $x = \sin\theta$; $\quad \therefore dx = \cos\theta \, d\theta$ and $1 - x^2 = \cos^2\theta$;

also when $x = 0$, $\theta = 0$, and when $x = 1$, $\theta = \frac{1}{2}\pi$.

The given integral then reduces to

$$\int_0^{\frac{1}{2}\pi} \sin^6\theta \cos^2\theta \, d\theta = \frac{1.3.5.1}{2.4.6.8} \frac{\pi}{2} = \frac{5\pi}{256} .$$

Ex. 2. *Evaluate* $\int_0^1 x^2 (1-x)^{\frac{3}{2}} dx$.

Put $x = \sin^2\theta$; $\therefore dx = 2\sin\theta\cos\theta \, d\theta$

and when $x = 0, 1$, we have $\theta = 0, \frac{1}{2}\pi$ respectively.

$$\therefore \quad I = 2\int_0^{\frac{1}{2}\pi} \sin^5\theta \cos^4\theta \, d\theta = 2\,\frac{2.4}{5.7.9} = \frac{16}{315} .$$

Ex. 3. *Evaluate* $\int_0^{\pi} \cos^n x \, dx$.

Since $\cos^n x = -\cos^n(\pi - x)$ when n is odd,
and $= \cos^n(\pi - x)$ when n is *even*,

$\therefore$ by Art. 6.8 (vii), it follows that $I = 0$ when n is odd,

and $$I = 2\int_0^{\frac{1}{2}\pi} \cos^n x \, dx, \text{ when } n \text{ is even}$$

$$= 2\,\frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4} \cdots \frac{3}{4}\,\frac{1}{2}\,\frac{\pi}{2} .$$ [*By Art. 6.23(A)*]

EXAMPLES VI(B)

Show that :-

1. (i) $\int_a^b f(a+b-x)\,dx = \int_a^b f(x)\,dx$.

(ii) $\int_{a-c}^{b-c} f(x+c)\,dx = \int_a^b f(x)\,dx$.

(iii) $\int_a^b f(nx)\,dx = \frac{1}{n}\int_{na}^{nb} f(x)\,dx$.

2. $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{\sin x}{\sin x + \cos x}\, dx = \frac{\pi}{4}$. [C. P. '86]

3. $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{\cos x - \sin x}{1 + \sin x \cos x}\, dx = 0$.

4. $\displaystyle\int_0^{\frac{1}{2}\pi} (a \cos^2 x + b \sin^2 x)\, dx = \tfrac{1}{4}\pi (a + b)$. [C. P. '85]

5. $\displaystyle\int_0^{\frac{1}{2}\pi} \sin 2x \log \tan x\, dx = 0$.

6. $\displaystyle\int_0^{\pi} x f(\sin x)\, dx = \tfrac{1}{2}\pi \int_0^{\pi} f(\sin x)\, dx$.

7. $\displaystyle\int_0^{\pi} x \log \sin x\, dx = \tfrac{1}{2}\pi^2 \log \tfrac{1}{2}$. [C. P. '75]

8. $\displaystyle\int_0^{\pi} x \sin x \cos^2 x\, dx = \frac{\pi}{3}$.

9. $\displaystyle\int_0^{\pi} x \sin^2 x\, dx = \frac{\pi^2}{4}$.

10. $\displaystyle\int_0^{\pi} \frac{\sin 4x}{\sin x}\, dx = 0$.

11. $\displaystyle\int_{-a}^{+a} x \sqrt{a^2 - x^2}\, dx = 0$.

12. $\displaystyle\int_0^{2\pi} \sin^4 \tfrac{1}{2}x \cos^5 \tfrac{1}{2}x\, dx = 0$.

13. $\int_0^1 \log \sin(\frac{1}{2}\pi\theta)\, d\theta = \log \frac{1}{2}$. [*Put* $\frac{1}{2}\pi\theta = x$]

14. $\int_0^1 \frac{\log x}{\sqrt{(1-x^2)}}\, dx = \frac{\pi}{2} \log \frac{1}{2}$. [*Put* $x = \sin\theta$]

15. $\int_0^{\frac{1}{4}\pi} \log(1 + \tan\theta)\, d\theta = \frac{\pi}{8} \log 2$. [*C. P.* '76, '83]

16. $\int_0^{\pi} x \cos^4 x\, dx = \frac{3}{16}\pi^2$.

17. (i) $\int_0^{\frac{1}{2}\pi} \cos^6 x\, dx = \frac{5}{32}\pi$. (ii) $\int_0^{\frac{1}{2}\pi} \sin^9 x\, dx = \frac{128}{315}$.
[*C. P.* '82]

(iii) $\int_0^{\frac{1}{2}\pi} \sin^4\theta \cos^8\theta\, d\theta = \frac{7\pi}{2048}$.

(iv) $\int_0^{\frac{1}{2}\pi} \sin^4 x \cos^5 x\, dx = \frac{8}{315}$.

(v) $\int_0^{\pi} (1 + \cos x)^3\, dx = \frac{5\pi}{2}$.

(vi) $\int_0^{\pi} \sin^3 x \cos^3 x\, dx = 0$. [*H. S.* '80]

(vii) $\int_0^{\pi} \cos^7\theta\, d\theta = 0$. (viii) $\int_{-\pi/2}^{+\pi/2} \sin^7 x\, dx = 0$.

18. (i) $\int_0^1 x^3 (1 - x)^3\, dx = \frac{1}{140}$.

(ii) $\int_0^1 x^3 (1-x^2)^{\frac{5}{2}} dx = \frac{2}{63}$.

(iii) $\int_0^a \frac{x^4}{\sqrt{(a^2-x^2)}} dx = \frac{3\pi}{16} a^4$.

(iv) $\int_0^1 \frac{x^6 dx}{\sqrt{(1-x^2)}} = \frac{5}{32} \pi$.

19. $\int_0^1 \frac{dx}{(x^2-2x+2)^3} = \frac{3\pi+8}{32}$. [*Put* $x = 1 + \tan\theta$]

20. (i) $\int_0^\pi \frac{x \sin x}{1+\cos^2 x} dx = \frac{\pi^2}{4}$. [*C. H. '75, J. E. E. '89*]

(ii) $\int_0^{\frac{1}{2}\pi} \frac{\sin^2 x}{\sin x + \cos x} dx = \frac{1}{\sqrt{2}} \log(\sqrt{2}+1)$.

(iii) $\int_0^\pi \frac{x \tan x}{\sec x + \tan x} dx = \frac{1}{2}\pi(\pi - 2)$. [*C. H. 1964*]

(iv) $\int_0^{\frac{1}{2}\pi} \frac{x\, dx}{\sin x + \cos x} = \frac{\pi}{2\sqrt{2}} \log(\sqrt{2}+1)$. [*C. P. 1977*]

(v) $\int_0^{\frac{1}{2}\pi} \frac{x\, dx}{\sec x + \operatorname{cosec} x} = \frac{\pi}{4} [1 + \frac{1}{\sqrt{2}} \log(\sqrt{2}-1)]$.

(vi) $\int_0^1 \cot^{-1}(1 - x + x^2) dx = \frac{1}{2}\pi - \log 2$.

(vii) $\int_0^\pi \frac{x\, dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \frac{\pi^2}{2ab}$, $(a, b > 0)$.

(viii) $\displaystyle\int_0^{\frac{1}{4}\pi} \frac{x\,dx}{1 + \cos 2x + \sin 2x} = \frac{\pi}{16} \log 2 .$

(ix) $\displaystyle\int_0^{a} \frac{a\,(x - \sqrt{a^2 - x^2}\,)^2}{(2x^2 - a^2)^2}\,dx = \frac{1}{\sqrt{2}} \log (\sqrt{2} + 1) .$

(x) $\displaystyle\int_0^{\pi} \frac{x\,dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} = \frac{\pi^2 (a^2 + b^2)}{4a^3 b^3} .$

[*J. E. E. '88*]

21. If $I_n = \displaystyle\int_0^{\frac{1}{4}\pi} \tan^n \theta \, d\theta$, show that $I_n = \dfrac{1}{n-1} - I_{n-2}$

Hence find the value of $\displaystyle\int_0^{\frac{1}{4}\pi} \tan^6 x \, dx$.

22. Show that, *if* m and n are positive and m is an integer,

$$\int_0^1 x^{n-1} (1 - x)^{m-1}\, dx = \int_0^1 x^{m-1} (1 - x)^{n-1}\, dx$$

$$= \frac{1.2.3 \,.\,.\,.\,.\,.\, (m - 1)}{n\,(n + 1) \,.\,.\,.\,.\, (n + m - 1)} .$$

ANSWERS

21. $\frac{13}{15} - \frac{1}{4}\pi$.

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CHAPTER VII

INFINITE (OR IMPROPER) INTEGRALS AND INTEGRATION OF INFINITE SERIES

7.1. Infinite integrals.

In discussing definite integrals we have hitherto supposed that the range of integration is finite and the integrand is continuous in the range. If in an integral either the range is infinite or the integrand has an infinite discontinuty in the range (*i.e.*, the integrand tends to infinity at some points of the range), the integral is usually called an *Infinite Integral*, and by some writers an *Improper Integral*. Simple cases of infinite integrals occur in elementary problems; for example, in the problem of finding the area between a plane curve and its asymptote. We give below the definitions of infinite integrals in different cases.

(A) Infinite range.

(i) $\int_a^{\infty} f(x)\,dx$ *is defined as* $\underset{\varepsilon \to \infty}{Lt} \int_a^{\varepsilon} f(x)\,dx$,

provided $f(x)$ is integrable in (a, ε), and this limit exists.

(ii) $\int_{-\infty}^{b} f(x)\,dx$ *is defined as* $\underset{\varepsilon \to -\infty}{Lt} \int_{\varepsilon}^{b} f(x)\,dx$,

provided $f(x)$ is integrable in (ε, b), and this limit exists.

(iii) If the infinite integrals $\int_{-\infty}^{a} f(x)\,dx$ and $\int_a^{\infty} f(x)\,dx$

both exist, we say that $\int_{-\infty}^{+\infty} f(x)\,dx$ exists, and

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx .$$

Note. In the above cases, when the limit tends to a finite number, the integral is said to be **convergent**, when it tends to infinity with a fixed sign, it is said to be **divergent**, and when it does not tend to any fixed limit, finite or infinite, it is said to be **oscillatory**. When an integral is divergent or oscillatory, some writers say that *the integral does not exist or the integral has no meaning*. [See Ex. 2, § 7.2]

(B) Integrand infinitely discontinuous at a point.

(i) *If $f(x)$ is infinitely discontinuous only at the end point a, i.e.*, if $f(x) \to \infty$ as $x \to a$, then

$$\int_a^b f(x)\,dx \text{ is defined as } \underset{\varepsilon \to 0}{Lt} \int_{a+\varepsilon}^b f(x)\,dx,\ \varepsilon > 0,$$

provided $f(x)$ be integrable in $(a+\varepsilon, b)$ and this limit exists.

(ii) *If $f(x)$ is infinitely discontinuous only at the end point b, i.e.*, if $f(x) \to \infty$ as $x \to b$, then

$$\int_a^b f(x)\,dx \text{ is defined as } \underset{\varepsilon \to 0}{Lt} \int_a^{b-\varepsilon} f(x)\,dx,\ \varepsilon > 0,$$

provided $f(x)$ be integrable in $(a, b-\varepsilon)$ and this limit exists.

(iii) *If $f(x)$ is infinitely discontinuous only at an internal point c* $(a < c < b)$, *i.e.*, if $f(x) \to \infty$ as $x \to c$, then

$$\int_a^b f(x)\,dx = \underset{\varepsilon \to 0}{Lt} \int_a^{c-\varepsilon} f(x)\,dx + \underset{\varepsilon' \to 0}{Lt} \int_{c+\varepsilon'}^b f(x)$$

when $\varepsilon \to 0$ and $\varepsilon' \to 0$ *independently*.

Note. It sometimes happens that no definite limit exists when ε and ε' tend to zero *independently*, but that a limit exists when $\varepsilon = \varepsilon'$. [See *Ex.* 7, *Art.* 7.2.] When $\varepsilon = \varepsilon'$, the value of the limit on the right side, when it exists, is called the *principal value* of the improper integral and is very often

denoted by $P\int_a^b f(x)\,dx$.

(iv) *If a and b are both points of infinite discontinuity,*

then $\int_a^b f(x)\,dx$ *is defined as* $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ when these two integrals exist, as defined above, c being a point between a and b.

7.2. Illustrative Examples.

Ex. 1. *Evaluate* $\int_0^\infty e^{-x}\,dx$.

$$I = \underset{\varepsilon \to \infty}{Lt} \int_0^\varepsilon e^{-x}\,dx = \underset{\varepsilon \to \infty}{Lt} (1 - e^{-\varepsilon}) = 1.$$

Ex. 2. *Evaluate* $\int_0^\infty \cos tx\,dx$.

$$I = \underset{\varepsilon \to \infty}{Lt} \int_0^\varepsilon \cos tx\,dx = \underset{\varepsilon \to \infty}{Lt} \frac{\sin t\varepsilon}{t}$$; but this limit does not exist.

Hence *the integral does not exist.**

Ex. 3. *Evaluate* $\int_{-\infty}^{+\infty} \frac{dx}{1 + x^2}$.

$$I = \int_{-\infty}^{a} \frac{dx}{1 + x^2} + \int_a^\infty \frac{dx}{1 + x^2}.$$

$$\int_{-\infty}^{a} \frac{dx}{1 + x^2} = \underset{\varepsilon \to -\infty}{Lt} \int_\varepsilon^a \frac{dx}{1 + x^2} = \underset{\varepsilon \to -\infty}{Lt} (\tan^{-1} a - \tan^{-1}\varepsilon)$$
$$= \tan^{-1} a + \tfrac{1}{2}\pi;$$

$$\int_a^\infty \frac{dx}{1 + x^2} = \underset{\varepsilon' \to \infty}{Lt} \int_a^{\varepsilon'} \frac{dx}{1 + x^2} = \underset{\varepsilon' \to \infty}{Lt} (\tan^{-1}\varepsilon' - \tan^{-1} a)$$
$$= \tfrac{1}{2}\pi - \tan^{-1} a.$$

* Although this integral does not exist in the manner defined above, it is expressed in terms of Dirac's delta function [$\delta(t)$] in modern mathematics. Detailed discussion is outside the scope of this book.

$\therefore \quad I = (\tan^{-1} a + \frac{1}{2}\pi) + (\frac{1}{2}\pi - \tan^{-1} a) = \pi .$

Ex. 4. *Evaluate* $\int_0^1 \frac{dx}{x^{2/3}}$.

Here $\frac{1}{x^{2/3}}$ tends to ∞ as x tends to $+0$.

$$\therefore \int_0^1 \frac{dx}{x^{2/3}} = \underset{\varepsilon \to 0}{Lt} \int_\varepsilon^1 \frac{dx}{x^{2/3}} = \underset{\varepsilon \to 0}{Lt} \; 3(1 - \varepsilon^{1/3}) = 3 .$$

Ex. 5. *Evaluate* $\int_{-1}^{+1} \frac{dx}{x^2}$.

Here $\frac{1}{x^2} \to \infty$ as $x \to 0$, an interior point of the interval $(-1, 1)$.

$$\therefore \quad I = \int_{-1}^{0} \frac{dx}{x^2} + \int_0^1 \frac{dx}{x^2} .$$

Now, $\int_0^1 \frac{dx}{x^2} = \underset{\varepsilon \to 0}{Lt} \int_\varepsilon^1 \frac{dx}{x^2} = \underset{\varepsilon \to 0}{Lt} \left(\frac{1}{\varepsilon} - 1 \right);$

this limit does not exist. So $\int_0^1 \frac{dx}{x^2}$ does not exist .

Similarly, $\int_{-1}^{0} \frac{dx}{x^2}$ does not exist.

Note. In examples of this type usually a mistake is committed in this way :

Since $\int \frac{1}{x^2}\, dx = -\frac{1}{x}$, $\therefore \int_{-1}^{+1} \frac{1}{x^2}\, dx = \left[-\frac{1}{x} \right]_{-1}^{+1} = -2$,

which is wrong.

In this connection, it should be carefully noted that the relation

$\int_a^b f(x)\, dx = F(b) - F(a)$ cannot be used without special examination unless $F'(x) = f(x)$ for all values of x from a to b, both inclusive.

Here, since the relation $\frac{d}{dx}\left(-\frac{1}{x}\right) = \frac{1}{x^2}$ fails to have any meaning when $x = 0$, and 0 is a value between -1 and $+1$, we cannot directly apply the Fundamental Theorem of Integral Calculus to evaluate this definite integral.

Ex. 6. *Show that* $\int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2}$, $a > 0$.

$$\int_0^\varepsilon e^{-ax} \cos bx \, dx = \left[\frac{e^{-ax}(-a\cos bx + b\sin bx)}{a^2 + b^2}\right]_0^\varepsilon \quad [\,Art.\ 3.3\,]$$

$$= \frac{1}{a^2 + b^2}\{e^{-a\varepsilon}(-a\cos b\varepsilon + b\sin b\varepsilon) - (-a)\}.$$

$$\therefore \int_0^\infty e^{-ax} \cos bx \, dx = \underset{\varepsilon \to \infty}{Lt} \int_0^\varepsilon e^{-ax} \cos bx \, dx$$

$$= \underset{\varepsilon \to \infty}{Lt} \left[\frac{1}{a^2 + b^2}\{e^{-a\varepsilon}(-a\cos b\varepsilon + b\sin b\varepsilon) + a\}\right].$$

Now, $\underset{\varepsilon \to \infty}{Lt}\ e^{-a\varepsilon}(-a\cos b\varepsilon + \sin b\varepsilon) = 0$.

[Since $e^{-a\varepsilon} \to 0$ and $\cos b\varepsilon$ and $\sin b\varepsilon$ are bounded.]

$$\therefore \int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2}.$$

Ex. 7. *Evaluate* $\int_{-1}^{+1} \frac{dx}{x}$.

The itegrand here is undefined for $x = 0$.

$$\therefore \int_{-1}^{+1} \frac{1}{x}\, dx = \underset{\varepsilon \to 0}{Lt} \int_{-1}^{-\varepsilon} \frac{dx}{x} + \underset{\varepsilon' \to 0}{Lt} \int_{\varepsilon'}^{1} \frac{dx}{x}$$

$$= Lt \left[\log(-x)\right]_{-1}^{-\varepsilon} + Lt \left[\log x\right]_{\varepsilon'}^{1}$$

$$= Lt \log \varepsilon - Lt \log \varepsilon' = Lt \log \frac{\varepsilon}{\varepsilon'}$$

as ε and ε' tend to zero independently.

But this limit is not definite, since it depends upon the ratio $\varepsilon : \varepsilon'$, which may be anything, ε and ε' being both arbitrary positive numbers.

But if we put $\varepsilon = \varepsilon'$, we get $\displaystyle\int_{-1}^{+1} \frac{dx}{x} = \underset{\varepsilon \to 0}{Lt} \log 1 = 0$.

Thus, although the general value of the integral does not exist, its principal value exists.

Note. $\displaystyle\int_{-1}^{-\varepsilon} \frac{dx}{x}$, when the range of integration is such that x is negative throughout, may be written, by putting $z = -x$, as $\displaystyle\int_{1}^{\varepsilon} \frac{dz}{z}$

$$= \Big[\log z \Big]_1^{\varepsilon} = \Big[\log(-x) \Big]_{-1}^{-\varepsilon}, \text{ for } \log x \text{ is imaginary here.}$$

Ex. 8. *Evaluate* $\displaystyle\int_0^{\infty} \frac{x^2}{(1+x^2)^2}\, dx$.

Put $x = \tan\theta$. $\therefore$ $dx = \sec^2\theta\, d\theta$; as x increases from 0 to ∞, θ increases from 0 to $\frac{1}{2}\pi$.

$$I = \int_0^{\frac{1}{2}\pi} \frac{\tan^2\theta \sec^2\theta\, d\theta}{\sec^4\theta} = \int_0^{\frac{1}{2}\pi} \sin^2\theta\, d\theta = \tfrac{1}{4}\pi.$$

Note. Thus, sometimes an *infinite integral can be transformed into an ordinary definite integal by a suitable substitution*. But whenever a substitution is used to evaluate an infinite integral, we must see that the transformation is legitimate.

Ex. 9. *Show that* $\displaystyle\int_0^{\infty} e^{-x} x^n\, dx = n!$, *n being a positive integer.*

Let I_n denote the given integral.

$$I_n = \underset{\varepsilon \to \infty}{Lt} \int_0^{\varepsilon} e^{-x} x^n\, dx$$

$$= \underset{\varepsilon \to \infty}{Lt} \left\{ \left[-e^{-x} x^n \right]_0^{\varepsilon} + n \int_0^{\varepsilon} e^{-x} x^{n-1}\, dx \right\}$$

[integrating by parts]

$$= n \underset{\varepsilon \to \infty}{Lt} \int_0^{\varepsilon} e^{-x} x^{n-1}\, dx, \text{ since } \underset{\varepsilon \to \infty}{Lt}\ e^{-\varepsilon}.\varepsilon^n = 0.$$

[See Das & Mukherjees' *Differential Calculus*, *Chapter on Indeterminate Forms, sum no. 2 (iii).*]

$$= n I_{n-1} = n(n-1) I_{n-2} \text{ (as before)}$$

$$= n(n-1)(n-2) I_{n-3} = \text{etc.}$$

$$= n(n-1)(n-2) \ldots 2.1 \int_0^{\infty} e^{-x}\, dx$$

$$= n\,!, \text{ since } \int_0^{\infty} e^{-x}\, dx = 1. \quad \text{[See Ex. 1 above]}$$

7.3. The integral $\int_0^{\infty} e^{-x^2} dx$.

Since e^{-x^2} ($= 1/e^{x^2}$) is positive and $< \dfrac{1}{1+x^2}$, (for $x > 0$)

it follows that $\int_0^X e^{-x^2} dx$ increases monotonically with X,

and $\int_0^X e^{-x^2} dx < \int_0^X \dfrac{dx}{1+x^2}$, *i.e.*, $< \tan^{-1} X$,

[See § *6.7.*]

This being true for all positive values of X, however large, and as $\tan^{-1} X$ increases with X and $\to \frac{1}{2}\pi$ as $X \to \infty$, it follows

that $\int_0^X e^{-x^2} dx$ monotonically increases with X, and is bounded above.

Thus, the infinite integral $\int_0^{\infty} e^{-x^2}\,dx$ is *convergent.*

Denote it by I.

Now, a being any positive number, replace x by ax.

Then, $I = \int_0^{\infty} ae^{-a^2x^2}\,dx$.

$$\therefore \quad Ie^{-a^2} = \int_0^{\infty} ae^{-a^2(1+x^2)}\,dx.$$

Since $ae^{-a^2(1+x^2)}$ is a continuous function for all positive values of x and a (which are independent), assuming the validity of integration under an integral sign in this case

$$I\int_0^{\infty} e^{-a^2}da = \int_0^{\infty} \left\{ \int_0^{\infty} ae^{-a^2(1+x^2)}\,da \right\} dx. \qquad \ldots (1)$$

Also for any particular value of x, $\int_0^{\varepsilon} ae^{-a^2(1+x^2)}\,da$

$$= \left[-\frac{1}{2}\,\frac{1}{1+x^2}\,e^{-a^2(1+x^2)} \right]_0^{\varepsilon} = \frac{1}{2(1+x^2)}\left[1 - e^{-\varepsilon^2(1+x^2)}\right]$$

$$\to \frac{1}{2(1+x^2)} \text{ as } \varepsilon \to \infty.$$

Hence from (1), $I^2 = \int_0^{\infty} \frac{1}{2}\,\frac{1}{1+x^2}.dx$

$$= \frac{1}{2}\,\frac{\pi}{2}, \text{ or, } \quad I = \tfrac{1}{2}\sqrt{\pi},$$

i.e., $\int_0^{\infty} e^{-x^2}dx = \frac{1}{2}\sqrt{\pi}.$*

* For an alternative proof see *Chapter IX, Art. 8.21.*

7.4. The integral $\int_0^\infty \frac{\sin bx}{x}\, dx$.

Let $u = \int_0^\infty \frac{e^{-ax} \sin bx}{x}\, dx,\ a > 0$.

Assuming the validity of differentiation under the integral sign, we have

$$\frac{du}{db} = \int_0^\infty e^{-ax} \cos bx\, dx$$

$$= \frac{a}{a^2 + b^2},\ a > 0. \quad \text{[See Ex. 6 Art. 7.2.]}$$

Now, integrating with respect to b,

$$u = a \int \frac{db}{a^2 + b^2} = a \frac{1}{a} \tan^{-1} \frac{b}{a} + C = \tan^{-1} \frac{b}{a} + C \quad \ldots(1)$$

where C is the constant of integration.

From the given integral, we see that when $b = 0,\ u = 0$.

$\therefore$ from (1), we deduce $C = 0$.

$$\therefore \int_0^\infty \frac{e^{-ax} \sin bx}{x}\, dx = \tan^{-1} \frac{b}{a}. \quad \ldots (2)$$

Assuming u a continuous function of a, we deduce from (2), when $a \to 0$,

$$\int_0^\infty \frac{\sin bx}{x}\, dx = \frac{\pi}{2},\ \text{or}\ -\frac{\pi}{2} \quad \ldots (3)$$

according as $b >$ or < 0.

Cor. When $b = 1$ we have

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}. \quad \ldots (4)$$

Note. There are other methods of obtaining the result. Students may consult any text-book on Mathematical Analysis.

7.5. Integration of Infinite Series.

We have proved in Art. 1.4 that the integral of the sum of a finite number of terms is equal to the sum of the integrals of these terms. Now, the question arises whether this principle can be extended to the case when the number of terms is *not finite*. In other words, is it always permissible to integrate an infinite series term by term ? It is beyond the scope of an elementary treatise like this to investigate the conditions under which an infinite series can properly be integrated term by term. We should merely state the theorem that applies to most of the series that are ordinarily met with in elementary mathematics. For a fuller discussion, students may consult any text-book on Mathematical Analysis.

Theorem *A power series can be integrated term by term throughout any interval of convergence, but not necessarily extending to the end-points of the interval.*

Thus, if $f(x)$ can be expanded in a convergent infinite power series for all values of x in a certain continuous range, *viz.*,

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots \text{ to } \infty,$$

then $$\int_a^b f(x)\,dx = \int_a^b (a_0 + a_1 x + a_2 x^2 + \ldots\ldots)\,dx$$

$$= \Sigma \int_a^b a_r x^r\,dx,$$

or, $$\int_a^x f(x)\,dx = \int_a^x (a_0 + a_1 x + a_2 x^2 + \ldots\ldots)\,dx$$

$$= \Sigma \int_a^x a_r x^r\,dx,$$

provided the intervals (a, b) and (a, x) lie within the interval of convergence of the power series.

Ex. *Find by integration the series for* $\tan^{-1} x$.

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots \text{ to } \infty, \text{ if } x^2 < 1.$$

$\therefore$ integrating both sides between the limits 0 and x,

$$\int_0^x \frac{dx}{1+x^2} = \int_0^x (1 - x^2 + x^4 - x^6 + \ldots)\, dx.$$

$$\therefore \quad \tan^{-1} x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \ldots, \; -1 < x < 1.$$

EXAMPLES VII

Evaluate, when possible, the following integrals :

1. (i) $\displaystyle\int_0^\infty \frac{dx}{1+x^2}$. (ii) $\displaystyle\int_0^\infty \frac{x\,dx}{x^2+4}$

2. (i) $\displaystyle\int_2^\infty \frac{dx}{x^2-1}$. (ii) $\displaystyle\int_0^\infty xe^{-x^2}\,dx$.

3. (i) $\displaystyle\int_{-1}^{+1} \frac{dx}{x^3}$. (ii) $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{x^3}$.

4. (i) $\displaystyle\int_0^\pi \frac{\sin x\,dx}{\cos^2 x}$. (ii) $\displaystyle\int_0^\pi \frac{dx}{1+\cos x}$.

5. (i) $\displaystyle\int_1^\infty \frac{dx}{x(1+x)}$. (ii) $\displaystyle\int_0^2 \frac{dx}{2-x}$.

6. (i) $\displaystyle\int_{-1}^{+1} \sqrt{\frac{1+x}{1-x}}\,dx$. (ii) $\displaystyle\int_0^\infty \frac{dx}{(1+x^2)^4}$.

7. (i) $\displaystyle\int_1^{+\infty} \frac{x\,dx}{(1+x^2)^2}$. (ii) $\displaystyle\int_{-\infty}^{+\infty} \frac{x\,dx}{x^4+1}$.

8. (i) $\int_0^2 \frac{dx}{(1-x)^2}$. (ii) $\int_0^\infty \frac{dx}{(x+1)(x+2)}$

Show that : (Ex.9 to Ex.22)

9. $$\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)} \cdot [a, b > 0]$$

10. $$\int_0^\infty \frac{x\,dx.}{(x^2+a^2)(x^2+b^2)} = \frac{1}{a^2-b^2}\log\frac{a}{b} \cdot [a, b > 0]$$

11. $$\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2(a+b)} \cdot \quad [a, b > 0]$$

12. (i) $\int_0^\infty e^{-x}(\cos x - \sin x)\,dx = 0$. (ii) $\int_0^\infty \frac{\log x}{1+x^2}\,dx = 0$

[*(ii) Divide the range* $(0, \infty)$ *into two parts* $(0, 1)$ *and* $(1, \infty)$.]

13. $$\int_{-\infty}^{+\infty} \frac{dx}{x^2+2x+2} = \pi.$$

14. $$\int_0^1 x^n \log x\,dx = -\frac{1}{(n+1)^2}\ (n > -1).$$

15. $$\int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}\ (a > 0).$$

16. (i) $$\int_0^\infty \frac{dx}{x^2+2x\cos\theta+1} = 2\int_0^1 \frac{dx}{x^2+2x\cos\theta+1}$$

(ii) $$\int_0^\infty \frac{dx}{\{x+\sqrt{(1+x^2)}\}^n} = \frac{n}{n^2-1},$$

where n is an integer greater than one.

(iii) $\displaystyle\int_0^\infty \frac{x\,dx}{(1+x)(1+x^2)} = \frac{\pi}{4}$.

17. (i) $\displaystyle\int_0^\infty \frac{\sin ax \cos bx}{x}\,dx = \frac{1}{2}\pi,\ 0,\ \text{or}\ \frac{1}{4}\pi$

according as $a >, <,$ or $= b$ (a and b being supposed positive).

(ii) $\displaystyle\int_0^\infty \frac{(\sin 2x + \cos 2x)^2 - (\sin x + \cos x)^2}{x}\,dx = 0$.

18. $\displaystyle\int_0^\infty \frac{\sin^3 x}{x}\,dx = \frac{\pi}{4}$.

19. $\displaystyle\int_0^\infty \frac{\sin^5 x}{x}\,dx = \frac{3}{16}\pi$.

20. $\displaystyle\int_0^\infty \frac{\sin^2 mx}{x^2}\,dx = \frac{\pi}{2}m,\ \text{or}\ -\frac{\pi}{2}m$

according as $m >$, or < 0.

21. $\displaystyle\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$.

22. $\displaystyle\int_0^\infty \left(\frac{\sin x}{x}\right)^3 dx = \frac{3\pi}{8}$.

23. Find by integration the power series for the following :

(i) $\log(1+x)$; (ii) $\log(1-x)$; (iii) $\sin^{-1} x$.

Show that :- (Ex.24 to Ex.26)

24. (i) $\displaystyle\int \frac{dx}{\sqrt{(\sin x)}} = 2\sqrt{\sin x}\left[1 + \frac{1}{2}\,\frac{\sin^2 x}{5} + \frac{1.3}{2.4}\,\frac{\sin^4 x}{9} + \dots\right]$.

(ii) $\displaystyle\int \frac{dx}{\sqrt{(1+x^4)}} = \frac{x}{1} - \frac{1}{2}\,\frac{x^5}{5} + \frac{1.3}{2.4}\,\frac{x^9}{9} - \dots;\ [x^2 < 1]$.

(iii) $\int_0^x \frac{\sin x}{x}\,dx = x - \frac{x^3}{3.3!} + \frac{x^5}{5.5!} - \dots$

(iv) $\int_a^b \frac{e^x}{x}\,dx = \log\frac{b}{a} + (b-a) + \frac{b^2-a^2}{2.2!} + \frac{b^3-a^3}{3.3!} + .$

(v) $\int_0^{\frac{1}{2}\pi} \sqrt{1 - e^2\sin^2\phi}\,d\phi$, where $e^2 < 1$,

$$= \frac{\pi}{2}\left\{1 - \left(\frac{1}{2}\right)^2 \frac{e^2}{1} - \left(\frac{1.3}{2.4}\right)^2 \frac{e^4}{3} - \dots\right\}.$$

(vi) $\int_0^{\frac{1}{2}\pi} \frac{dx}{\sqrt{(1 - k^2\sin^2 x)}}$, where $k^2 < 1$,

$$= \frac{\pi}{2}\left\{1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1.3}{2.4}\right)^2 . k^4 + \dots\right\}$$

(vii) $\int_0^1 \frac{dx}{1 + x^2} = 1 - \frac{1}{3} + \frac{1}{5} - \dots$

(viii) $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1 - x^2)}} = \frac{1}{2} + \frac{1}{2.3.2^3} + \frac{1.3}{2.4.5.2^5} + \dots$

25. $\int_0^1 \frac{\tan^{-1}x}{x}\,dx = \sum_0^{\infty} (-1)^n \frac{1}{(2n+1)^2}$

26. (i) $\int_0^1 \frac{\log x}{1+x}\,dx = -\frac{\pi^2}{12}$. $\left[\textit{Use}\ \Sigma\frac{1}{n^2} = \frac{\pi^2}{6}.\right]$

(ii) $\int_0^1 \frac{\log(1-x)}{x}\,dx = -\frac{\pi^2}{6}$

27. (i) Show that if, $a > 0$,

$$\int_0^1 \frac{x^{a-1}}{1+x}\,dx = \frac{1}{a} - \frac{1}{a+1} + \frac{1}{a+2} - \frac{1}{a+3} + \dots$$

Hence deduce the value of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots\dots$

(ii) Show that, if $a > 0, b \; 0$,

$$\int_0^1 \frac{x^{a-1}}{1+x^b}\,dx = \frac{1}{a} - \frac{1}{a+b} + \frac{1}{a+2b} - \frac{1}{a+3b} + \dots$$

28. Show that

$$\int_0^1 x^{2p-1} \log(1+x)\,dx = \frac{1}{2p}\left[\frac{1}{1.2} + \frac{1}{3.4} + \dots + \frac{1}{(2p-1)2p}\right]$$

[*Integrate by parts.*]

ANSWERS

1. (i) $\frac{1}{2}\pi$. (ii) does not exist.
2. (i) $\frac{1}{2}\log 3$. (ii) $\frac{1}{2}$.
3. (i) principal value is 0. (ii) principal value is 0.
4. (i) does not exist. (ii) does not exist.
5. (i) $\log 2$. (ii) does not exist.
6. (i) π. (ii) $\frac{5}{32}\pi$.
7. (i) $\frac{1}{4}$. (ii) 0.
8. (i) does not exist. (ii) $\log 2$.

23. (i) $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots\dots\dots,\; x^2 < 1$

$\log 2 + \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^2}\dots\dots\dots,\; x^2 < 1$

(ii) $-[x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots]$

(iii) $x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \dots$

27. (i) $\log 2$.

CHAPTER VIII

IRRATIONAL FUNCTIONS

8.1. In the previous chapters we have discussed simple cases of integrals of irrational functions. We shall now consider here some harder types of such integrals.

8.2. If the integrand contains only fractional powers of x, *i.e.*, if the integrand be of the form

$$F(x^{\frac{1}{n}}),$$

where $F(z)$ is a rational function of z,

the *substitution* is $x = z^n$,

where n is the least common multiple of the denominators of the fractional exponents of x. [See *Ex. 1 of Examples VIII.*]

8.3. If the integrand contains only fractional powers of $(a + bx)$, *i.e.*, if the integrand be of the form

$$F[(a + bx)^{\frac{1}{n}}],$$

where $F(z)$ is a rational function of z,

the *substitution* is $a + bx = z^n$,

where n is the least common multiple of the denominators of the fractional exponents of $(a + bx)$. [See *Ex. 2 & 3 of Examples VIII.*]

8.4. Let the integral be of the form

$$\int x^m (a + bx^n)^p \, dx,$$

where m, n, p are rational numbers.

(A) If *p be a positive integer*, expand $(a + bx^n)^p$ by the Binomial Theorem and integrate term by term.

[See *Ex. 4(i) of Examples VIII* .]

(B) If *p be a fraction*, say, equal to r/s, where r and s are integers and s is positive, *Case I.* If $\dfrac{m+1}{n}$ = an integer or zero ,

the *substitution* is $a + bx^n = z^s$.

If $\dfrac{m+1}{n} \neq$ an integer or zero, we apply the following

Case II.

Case II. If $\dfrac{m+1}{n} + \dfrac{r}{s}$ = an integer or zero ,

the *general substitution* is $a + bx^n = z^s x^n$ . ... (1)

If, however, the *integer is positive or zero* ,
alternative substitution is $a + bx^n = z^s$.

If the *integer is negative* ,
the *alternative substitution is* $ax^{-n} + b = z^s$,
which is practically the same as (1) of Case II, sometimes facilitates the calculation. [See *Ex. 1 of Art. 8 .8.*]

8.5. The integral of the form $\displaystyle\int \frac{dx}{(ax^2 + b)\sqrt{(cx^2 + d)}}$

Here the *substitution* is $cx^2 + d = x^2 z^2$.

Sometimes *trigonometrical substitutions* like
$x = k \tan\theta$, $x = k \sin\theta$, $x = k \sec\theta$, etc. facilitate integration.
[See *Ex. 28 (ii) of Examples II (A) and Ex. 8(i) and Ex. 8(ii) of Examples VIII* .]

8.6. The integral of the form

$$\int \frac{dx}{(px^2 + qx + r)\sqrt{(ax^2 + bx + c)}}$$

Here we shall consider two cases only.

Case I. If $px^2 + qx + r$ breaks up into two linear factors of the forms $(mx + n)$ and $(m'x + n')$, then we resolve $\{1/(mx + n)(m'x + n')\}$ into two partial fractions and the integral then transforms into the sum (or difference) of two integrals of the type (B) of Art. 2.8. [See *Ex. 13 of Examples VIII*.]

Case II. If $px^2 + qx + r$ is a perfect square, say, $(lx + m)^2$, then the *substitution* is $lx + m = 1/z$.

In some cases *trigonometrical substitutions*, as in Art. 8.5, are effective.

If $q = 0$, $b = 0$ the integral reduces to the form given in the Art. 8.5.

In all these cases, the *general substitution* is

$$\sqrt{\left(\frac{ax^2 + bx + c}{px^2 + qx + r}\right)} = z.$$

Briefly, we have considered integrals of the type

$$\int \frac{dx}{P\sqrt{Q}},$$

where P and Q are both *linear* functions of x and *P linear, Q quadratic* [See *Art. 2.8(A) and 2.8(B).*] and *P quadratic, Q quadratic*. [*See Art. 8.5 and 8.6.*]

If P be *quadratic* and Q *linear*, put $Q = z^2$.

Also we have considered integrals of the type

$$\int \frac{f(x)}{P\sqrt{Q}}\,dx,$$

where *f (x) is a polynomial, P , Q being linear or quadratic*. [See *Ex. 11 to 15 of Examples VIII.*]

8.7. The integral of the form

$$\int \frac{f(x)}{\sqrt{(ax^4 + 2bx^3 + cx^2 + 2bx + a)}}\,dx,$$

where $f(x)$ is a rational function of x.

The denominator can be written as

$$x\sqrt{\left\{a\left(x^2 + \frac{1}{x^2}\right) + 2b\left(x + \frac{1}{x}\right) + c\right\}}$$

and hence the *substitution* is

$$x + \frac{1}{x} = z \quad \text{or} \quad x - \frac{1}{x} = z,$$

according as $f(x)$ is expressible in the form

$$\left(x - \frac{1}{x}\right)\phi\left(x + \frac{1}{x}\right) \quad \text{or} \quad \left(x + \frac{1}{x}\right)\phi\left(x - \frac{1}{x}\right).$$

If $b = 0$, *the substitution*

$$x^2 + \frac{1}{x^2} = z \quad \text{or} \quad x^2 - \frac{1}{x^2} = z$$

is sometimes useful. [See *Ex. 19 of Examples VIII.*]

8.8. Illustrative Examples.

Ex. 1. *Integrate* $\displaystyle\int \frac{dx}{x^3 \sqrt[3]{(1 + x^3)}}$.

Comparing it with the form of Art. 8.4, we find here

$$m = -3,\ n = 3,\ r = -1,\ s = 3.$$

Now, $\dfrac{m+1}{n} \neq$ an integer, but

$$\frac{m+1}{n} + \frac{r}{s} = -1, \text{ (an integer)}. \qquad \ldots \quad (1)$$

$\therefore$ by Art. 8.4, Case II, we put $1 + x^3 = z^3 x^3$.

$$\therefore \quad x^3(z^3 - 1) = 1. \quad \therefore x = \frac{1}{(z^3 - 1)^{1/3}}. \qquad \ldots \quad (2)$$

$$\therefore \quad dx = -\frac{z^2}{(z^3 - 1)^{4/3}}\, dz. \qquad \ldots \quad (3)$$

$$\therefore \text{ denominator} = x^4 z = \frac{z}{(z^3 - 1)^{4/3}}.$$

$$\therefore I = -\int z\, dz = -\tfrac{1}{2} z^2 = -\tfrac{1}{2}\frac{(1 + x^2)^{2/3}}{x^2}.$$

Alternatively, since (1) is a negative integer, we can put

$$x^{-3} + 1 = z^3.$$

Thus, $\displaystyle I = \int \frac{dx}{x^3\left\{x^3\left(1 + \frac{1}{x^3}\right)\right\}^{1/3}} = \int x^{-4}(x^{-3} + 1)^{-1/3}\, dx.$

Since $x^{-3} + 1 = z^3$, $\therefore - x^{-4}\, dx = z^2\, dz$.

$\therefore \quad I = -\int z^{-1} z^2\, dz =$ etc.

Ex. 2. *Integrate* $\displaystyle\int \frac{dx}{(x^2 - 2x + 1)\sqrt{(x^2 - 2x + 3)}}$.

It is of the form Case II of Art. 8.6.

$$I = \int \frac{dx}{(x-1)^2 \sqrt{\{(x-1)^2 + 2\}}}$$

$$= \int \frac{dz}{z^2 \sqrt{(z^2 + 2)}}, \text{ putting } z = x - 1.$$

It is of the form of Art. 8.5.

$$\therefore \quad I = \int \frac{\sqrt{2} \sec^2 \theta\, d\theta}{2 \tan^2 \theta . \sqrt{2} \sec \theta}, \text{ putting } z = \sqrt{2} \tan \theta,$$

$$= \frac{1}{2} \int \operatorname{cosec} \theta \cot \theta\, d\theta = -\tfrac{1}{2} \operatorname{cosec} \theta.$$

Since $\tan \theta = \dfrac{1}{\sqrt{2}} z$, $\operatorname{cosec} \theta = \dfrac{\sqrt{(z^2 + 2)}}{z}$,

$$\therefore \quad I = -\frac{1}{2} \frac{\sqrt{(z^2 + 2)}}{z} = -\frac{1}{2} \frac{\sqrt{(x^2 - 2x + 3)}}{x - 1}.$$

Ex. 3. *Integrate the following :*

(i) $\displaystyle\int \frac{x^2 + 1}{x^4 + 1}\, dx$. (ii) $\displaystyle\int \frac{x^2 - 1}{x^4 + 1}\, dx$. (iii) $\displaystyle\int \frac{1}{x^4 + 1}\, dx$.

(i) $\displaystyle I = \int \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2}}\, dx$ (dividing the numerator and denominator by x^2)

$$= \int \frac{\left(1 + \frac{1}{x^2}\right) dx}{\left(x - \frac{1}{x}\right)^2 + 2} = \int \frac{dz}{z^2 + 2} \text{ [on putting } x - \frac{1}{x} = z \text{]}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{z}{\sqrt{2}} = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x^2 - 1}{x\sqrt{2}}\right).$$

(ii) It is similar to (i) .

$$I = \int \frac{\left(1 - \frac{1}{x^2}\right) dx}{x^2 + \frac{1}{x^2}} = \int \frac{1 - \frac{1}{x^2}}{\left(x + \frac{1}{x}\right)^2 - 2} dx$$

$$= \int \frac{dz}{z^2 - 2} \text{ [on putting } x + \frac{1}{x} = z \text{]}$$

$$= \frac{1}{2\sqrt{2}} \log \frac{z - \sqrt{2}}{z + \sqrt{2}} = \frac{1}{2\sqrt{2}} \log \frac{x^2 + 1 - x\sqrt{2}}{x^2 + 1 + x\sqrt{2}}.$$

(iii) $I = \frac{1}{2}\int \frac{(x^2 + 1) - (x^2 - 1)}{x^4 + 1} dx$

$$= \frac{1}{2}\int \frac{x^2 + 1}{x^4 + 1} dx - \frac{1}{2}\int \frac{x^2 - 1}{x^4 + 1} dx$$

$$= \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{x^2 - 1}{x\sqrt{2}}\right) - \frac{1}{4\sqrt{2}} \log \frac{x^2 + 1 - x\sqrt{2}}{x^2 + 1 + x\sqrt{2}}.$$

[*by (i) and (ii)*]

Ex. 4. *Integrate* $\int \frac{1 - x^2}{1 + x^2} \frac{dx}{\sqrt{(1 + x^2 + x^4)}}$.

$$I = \int \frac{-x^2\left(1 - \frac{1}{x^2}\right) dx}{x\left(x + \frac{1}{x}\right)\sqrt{\left\{x^2\left(x^2 + \frac{1}{x^2} + 1\right)\right\}}}$$

$$= -\int \frac{\left(1 - \frac{1}{x^2}\right) dx}{\left(x + \frac{1}{x}\right)\sqrt{\left\{\left(x + \frac{1}{x}\right)^2 - 1\right\}}}.$$

$$= -\int \frac{dz}{z\sqrt{(z^2 - 1)}} \qquad \text{[putting } x + \frac{1}{x} = z \text{]}$$

$$= \int \frac{\operatorname{cosec}\theta \cot\theta}{\operatorname{cosec}\theta \cot\theta} d\theta \text{ [putting } z = \operatorname{cosec}\theta \text{]}$$

$$= \int d\theta = \theta = \operatorname{cosec}^{-1} z = \operatorname{cosec}^{-1}\left(\frac{x^2 + 1}{x}\right) = \sin^{-1}\left(\frac{x}{1 + x^2}\right)$$

EXAMPLES VIII

Integrate the following :-

1. $\int \frac{1+\sqrt[4]{x}}{\sqrt[4]{x^3}\,(1+\sqrt{x})}\,dx.$ [*Put* $x = z^4$.]

2. $\int \frac{dx}{\sqrt{(x+2)}+\sqrt[4]{(x+2)}}.$ [*Put* $x+2 = z^4$.]

3. $\int \frac{dx}{\sqrt{(2+x)}+\{\sqrt{(2+x)}\}^3}.$

4. (i) $\int \sqrt{x}\,(1+\sqrt[5]{x})^2\,dx.$ (ii) $\int \sqrt{(2+\sqrt{x})}\,dx.$

5. (i) $\int \frac{x^3}{(1+x^2)^{3/2}}\,dx.$ (ii) $\int \frac{dx}{x^4\,(2+x^2)^{1/2}}$

6. (i) $\int \frac{\sqrt[3]{(1+x^3)}}{x^5}\,dx.$ (ii) $\int \frac{dx}{x^n\,(1+x^n)^{1/n}}.$

   (iii) $\int \frac{\sqrt{(1+x^4)}}{x^3}\,dx.$

7. (i) $\int \frac{\sqrt{(x-x^2)}}{x^3}\,dx.$ (ii) $\int \frac{\sqrt{x}\,\sqrt{(1-2x)}}{x^4}\,dx.$

8. (i) $\int \frac{dx}{(x^2+1)\sqrt{(x^2+4)}}.$ (ii) $\int \frac{dx}{(x^2-1)\sqrt{(x^2-9)}}$

   [*Put* (*i*) $x = 2\tan\theta$; (ii) $x = 3\sec\theta$]

9. (i) $\int \frac{x^2\,dx}{(x-1)\sqrt{(x+2)}}.$ (ii) $\int \frac{dx}{(x-2)^{3/2}\,(x-5)^{1/2}}$

10. (i) $\int \frac{\sqrt{(1+x+x^2)}}{1+x}\,dx.$ (ii) $\int \frac{\{x+\sqrt{(a^2+x^2)}\}^n}{\sqrt{(a^2+x^2)}}\,dx$

11. $\int \frac{x^2+2x+4}{(x+1)\sqrt{(x^2+1)}}\,dx.$

12. $\int \frac{dx}{(4x^2+4x+1)\sqrt{(4x^2+4x+5)}}$

13. $\int \frac{dx}{(2x^2+9x+9)\sqrt{(x^2+3x+2)}}$

14. $\int \frac{x+3}{(x^2+5x+7)\sqrt{(x+2)}}\,dx$.

15. (i) $\int \frac{dx}{(x^2+5x+7)\sqrt{(x+2)}}$. (ii) $\int \frac{(x^2+4x+4)\,dx}{(x^2+5x+7)\sqrt{(x+2)}}$

16. (i) $\int \frac{x^2+1}{x^4+x^2+1}\,dx$. (ii) $\int \frac{x^2-1}{x^4+x^2+1}\,dx$.

17. $\int \frac{x^2}{x^4+x^2+1}\,dx$.

18. $\int \frac{1}{x^4+x^2+1}\,dx$.

19. $\int \frac{(1-x^2)\,dx}{(1+x^2)\sqrt{(x^4+1)}}$.

20. $\int \frac{(1+x^2)\,dx}{(1-x^2)\sqrt{(1-3x^2+x^4}}$

21. $\int \frac{(x^2-x^{-2})}{x\sqrt{(x^2+x^{-2}+1)}}\,dx$.

22. $\int \frac{x^2-x^{-2}}{x(x^{-2}-x^2)^{3/2}}\,dx$.

23. $\int \frac{1+x^{-2}}{\sqrt{(x^2+x^{-2}-1)}}\,dx$.

24. Integrate $\int \frac{dx}{x\sqrt{(x^2-x+2)}}$,

by the substitution $z = x + \sqrt{(x^2-x+2)}$

and show that the value is $\frac{1}{\sqrt{2}} \log \frac{\sqrt{(x^2-x+2)}+x-\sqrt{2}}{\sqrt{(x^2-x+2)}+x+\sqrt{2}}$

25. Integrate $\int \frac{dx}{x\sqrt{(x^2+2x-1)}}$, by the substitution $z = x + \sqrt{(x^2+2x-1)}$ and show that the value is $2\tan^{-1}\{x+\sqrt{(x^2+2x-1)}\}$.

ANSWERS

1. $4\left[\tan^{-1}(\sqrt[4]{x}) + \frac{1}{2}\log(1+\sqrt{x})\right]$.

2. $2(x+2)^{1/2} - 4(x+2)^{1/4} + 4\log\{1+(x+2)^{1/4}\}$.

3. $2\tan^{-1}(2+x)^{1/2}$. 4. (i) $\frac{2}{3}x^{2/3} + \frac{12}{11}x^{11/6} + \frac{6}{13}x^{13/6}$

(ii) $\frac{4}{5}(2+x)^{5/2} - \frac{8}{3}(2+\sqrt{x})^{3/2}$.

5. (i) $\frac{x^2+2}{\sqrt{(x^2+1)}}$. (ii) $-\frac{1}{12}\frac{(2+x^2)^{3/2}}{x^3} + \frac{1}{4}\frac{(2+x^2)^{1/2}}{x}$

6. (i) $-\frac{1}{4}\frac{(1+x^3)^{4/3}}{x^4}$. (ii) $\frac{1}{1-n}\frac{(1+x^n)^{(n-1)/n}}{x^{n-1}}$.

(iii) $\frac{1}{2}\left[\log(x^2+\sqrt{(1+x^4)}) - \frac{\sqrt{(1+x^4)}}{x^2}\right]$.

7. (i) $-\frac{2}{3}\frac{(x-x^2)^{3/2}}{x^3}$. (ii) $-\frac{2}{5}\frac{(1-2x)^{5/2}}{x^{5/2}} - \frac{4}{3}\frac{(1-2x)^{3/2}}{x^{3/2}}$.

8. (i) $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x\sqrt{3}}{\sqrt{(x^2+4)}}\right)$. (ii) $\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{(x^2-9)}}{2\sqrt{2}x}\right)$.

9. (i) $\frac{2}{3}(x+2)^{3/2} - 2(x+2)^{1/2} + \frac{1}{\sqrt{3}}\log\frac{\sqrt{(x+2)}-\sqrt{3}}{\sqrt{(x+2)}+\sqrt{3}}$.

(ii) $\frac{2}{3}\sqrt{\left(\frac{x-5}{x-2}\right)}$.

10. (i) $\sqrt{(1+x+x^2)} - \frac{1}{2}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) - \sinh^{-1}\left(\frac{1-x}{\sqrt{3}(1+x)}\right)$.

(ii) $\frac{1}{n}[x+\sqrt{(a^2+x^2)}]^n$.

11. $\sqrt{(x^2+1)} + \sinh^{-1}x - \frac{3}{\sqrt{2}}\sinh^{-1}\left(\frac{1-x}{1+x}\right)$.

12. $-\frac{1}{8}\frac{\sqrt{(4x^2+4x+5)}}{2x+1}$.

13. $\frac{2}{3}\sec^{-1}(2x+3) + \frac{1}{3\sqrt{2}}\cosh^{-1}\left(\frac{5+3x}{x+3}\right)$.

14. $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+1}{\sqrt{3}\sqrt{(x+2)}}\right)$.

15. (i) $\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x+1}{\sqrt{3}\sqrt{(x+2)}}\right) - \frac{1}{2}\log \frac{x+3-\sqrt{(x+2)}}{x+3+\sqrt{(x+2)}}$

(ii) $2\sqrt{(x+2)} - \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{x+1}{\sqrt{3}\sqrt{(x+2)}}\right)$.

16. (i) $\frac{1}{\sqrt{3}} \tan^{-1} \frac{x^2-1}{x\sqrt{3}}$. (ii) $\frac{1}{2} \log \frac{x^2-x+1}{x^2+x+1}$

17. $\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x\sqrt{3}}{1-x^2}\right) + \frac{1}{4} \log \frac{x^2-x+1}{x^2+x+1}$.

18. $\frac{1}{2\sqrt{3}} \tan^{-1}\left(\frac{x\sqrt{3}}{1-x^2}\right) - \frac{1}{4}\log \frac{x^2-x+1}{x^2+x+1}$

19. $\frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{x\sqrt{2}}{1+x^2}\right)$. 20. $\sin^{-1}\left(\frac{x}{x^2-1}\right)$. 21. $\frac{\sqrt{(x^4+x^2+1)}}{x}$

22. $\frac{x}{\sqrt{(1-x^4)}}$. 23. $\sinh^{-1}\left(\frac{x^2-1}{x}\right)$.

—

MISCELLANEOUS EXAMPLES I

1. Integrate the following functions with respect to x :-

(i) $\dfrac{x^2 + \cos^2 x}{x^2 + 1} . \operatorname{cosec}^2 x$. (ii) $\dfrac{\sin x}{\sin (x - \alpha)}$. [*H.S. '84, '87*]

(iii) $\dfrac{\cos 8x - \cos 7x}{1 + 2\cos 5x}$. (iv) $\dfrac{\tan \alpha - \tan x}{\tan \alpha + \tan x}$.

(v) $\sec^{\frac{3}{5}} x \operatorname{cosec}^{\frac{7}{5}} x$. (vi) $x^3 (\log x)^2$

(vii) $\sec x \log (\sec x + \tan x)$. (viii) $x^3 \cos x$

(ix) $\sec x \tan x \sqrt{(2 + \tan^2 x)}$. (x) $x \cos^{-1} x$.

(xi) $(\log x)^3$. (xii) $\tan^{-1} (\sqrt{x})$. (xiii) $\log (1 + x^2)$

(xiv) $x^2 \sin^{-1} x$. (xv) $2^x \cos x$. (xvi) $e^x x^4$

Integrate the following :-

2. (i) $\displaystyle\int \left(\frac{x - 1}{x^2 + 1}\right)^2 dx$. (ii) $\displaystyle\int \frac{x^3}{(1 + x^2)^2} dx$

3. (i) $\displaystyle\int \frac{\log (1 + x)}{x^2} dx$. (ii) $\displaystyle\int \frac{\sin (\log x)}{x^3} dx$.

4. (i) $\displaystyle\int \frac{dx}{(e^x + e^{-x})^2}$. (ii) $\displaystyle\int \frac{dx}{(1 + e^x)(1 + e^{-x})}$

5. (i) $\displaystyle\int (a + x) \sqrt{a^2 + x^2}\, dx$. (ii) $\displaystyle\int (a^2 + x^2) \sqrt{a + x}\, dx$.

6. (i) $\displaystyle\int \frac{dx}{(1 + x^2) \sqrt{(1 - x^2)}}$. (ii) $\displaystyle\int \frac{dx}{(1 - x^2) \sqrt{(1 + x^2)}}$

7. (i) $\displaystyle\int \frac{dx}{(x^2 - 4) \sqrt{(x^2 - 1)}}$. (ii) $\displaystyle\int \frac{dx}{(x^2 + 1) \sqrt{(x^2 + 4)}}$

8. (i) $\displaystyle\int \frac{dx}{x^2 (1 + x^2)^2}$. (ii) $\displaystyle\int \frac{dx}{x^2 \sqrt{(x^2 + 1)}}$

9. (i) $\displaystyle\int \frac{dx}{x (x^2 + 1)}$. (ii) $\displaystyle\int \frac{dx}{x^4 \sqrt{(x^2 + 1)}}$

10. (i) $\int \frac{dx}{x^3\sqrt{(x^2-1)}}$. (ii) $\int \frac{dx}{x(x+1)^2}$.

11. (i) $\int \frac{x+1}{x^4(x-1)}\,dx$. (ii) $\int \frac{\sqrt{(x^2+1)}}{x^2}\,dx$.

12. (i) $\int \frac{x}{1+\sin x}\,dx$. (ii) $\int e^x \frac{2+\sin 2x}{1+\cos 2x}\,dx$.

13. (i) $\int \frac{\cdot\, x\tan^{-1}x}{(1+x^2)^{3/2}}\,dx$. (ii) $\int \frac{e^{a\tan^{-1}x}}{(1+x^2)^{3/2}}\,dx$.

14. (i) $\int \frac{\sqrt{1+\sin 2x}}{1+\cos 2x}\,dx$. (ii) $\int \frac{dx}{(a\sin x+b\cos x)^2}$

15. (i) $\int \frac{dx}{\cos x\cos 2x}$. (ii) $\int \sin\left(2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right)dx$

16. (i) $\int \frac{\sqrt{x}\,dx}{(x+1)(x+2)}$. (ii) $\int \frac{dx}{x(x-1)^2(x^2+1)}$

17. (i) $\int \frac{x^2\,dx}{(x-1)^2(x^2+1)}$. (ii) $\int \frac{dx}{x\sqrt{(x^2+x-6)}}$.

18. (i) $\int \frac{ax}{\sin x+\tan x}$ (ii) $\int \frac{\sin x\,dx}{3\cos x+2\sin x}$.

19. (i) $\int \frac{e^x-n}{e^{-x}+1}\,dx$. (ii) $\int \frac{dx}{x\sqrt{(5x^2-4x+1)}}$.

20. (i) $\int \frac{(x-1)(x-4)}{(x-2)(x-3)}\,dx$. (ii) $\int \frac{3x^2-2x-3}{(x-1)(x-2)(x-3)}\,d$.

21. (i) $\int \frac{dx}{x^4+18x^2+81}$ (ii) $\int \frac{dx}{(x^2+2x+5)^2}$.

22. (i) $\int \frac{dx}{(1+x)^{3/2}+(1+x)^{1/2}}$ (ii) $\int \sqrt{x+\sqrt{(x^2+2)}}\,dx$.

Evaluate the following :

23. (i) $\int_0^1 x^{\frac{3}{2}}(1-x)^{\frac{1}{2}}d$. (ii) $\int_0^{\frac{1}{2}\pi} x^3\sin 3x\,dx$.

24. (i) $\int_{0} x \log\left(1 + \frac{1}{2}x\right) dx$. (ii) $\int_{0}^{\pi} \log(1 + \cos x)\, dx$.

25. (i) $\int_{0}^{\infty} \frac{dx}{(1 + x^2)^2}$. (ii) $\int_{0}^{\infty} \frac{\sqrt{x}}{1 + x^2}\, dx$.

26. (i) $\int_{1}^{\infty} \frac{dx}{x(1 + x^2)}$. (ii) $\int_{0}^{\infty} \frac{x\, dx}{(1 + x)(1 + x^2)}$.

27. (i) $\int_{0} \frac{dx}{1 - x + x^2}$. (ii) $\int_{0}^{1} \frac{dx}{(1 + x^2)\sqrt{(1 - x^2)}}$.

28. (i) $\int_{-1}^{+1} \frac{x^2 - 1}{(x^2 + 1)^2}\, dx$. (ii) $\int_{1}^{\sqrt{2}} \frac{x^2 + 1}{x^4 + 1}\, dx$.

Show that :-

29. $$\int_{0}^{1} \frac{dx}{(1 + x)(2 + x)} = 0.288 \text{ (nearly)}.$$

30. $$\int_{0}^{a} \frac{dx}{x + \sqrt{(a^2 - x^2)}} = \frac{\pi}{4}.$$

31. $$\int_{\frac{3}{2}}^{\frac{5}{2}} \frac{dx}{\sqrt{[(x - 1)(3 - x)]}} = \frac{\pi}{3}.$$

32. $$\int_{0}^{a} \sqrt{\frac{a^2 - x^2}{a^2 + x^2}}\, dx = \left(\frac{1}{4}\pi - \frac{1}{2}\right) a^2. \text{ [Put } x^2 = a^2 \cos 2\theta. \text{]}$$

33. $$\int_{0}^{\pi} \frac{dx}{3 + 2\sin x + \cos x} = \frac{\pi}{4}.$$

34. $$\int_{0}^{\infty} \frac{dx}{a^2 e^x + b^2 e^{-x}} = \frac{1}{ab} \tan^{-1} \frac{b}{a}.$$

35. $$\int_0^{\infty} \log\left(x + \frac{1}{x}\right) \frac{dx}{1 + x^2} = \pi \log 2 .$$ [*Put* $x = \tan\theta$.]

36. If $C_0, C_1, C_2, \ldots\ldots, C_n$ denote the coefficients in the expansion of $(1 + x)^n$, where n is a positive integer, show that

$$\frac{C_0}{1} + \frac{C_1}{2} + \frac{C_2}{3} + \ldots\ldots \frac{C_n}{n+1} = \frac{2^{n+1} - 1}{n + 1} .$$

ANSWERS

1. (i) $(-\cot x + \tan^{-1} x)$. (ii) $x \cos\alpha + \sin\alpha \log \sin(x - \alpha)$.
   (iii) $\frac{1}{3}\sin 3x - \frac{1}{2}\sin 2x$. (iv) $\sin 2\alpha \log \sin(x + \alpha) - x \cos 2\alpha$.
   (v) $-\frac{5}{2}\cot^{2/5} x$. (vi) $\frac{1}{4}x^4 \{(\log x)^2 - \frac{1}{2}\log x + \frac{1}{8}\}$.
   (vii) $\frac{1}{2}\{\log(\sec x + \tan x)\}^2$. (viii) $(x^3 - 6x)\sin x + 3(x^2 - 2)\cos$.
   (ix) $\frac{1}{2}\sec x \sqrt{(1 + \sec^2 x)} + \frac{1}{2}\log(\sec x + \sqrt{\sec^2 x + 1})$.
   (x) $\frac{1}{12}x \sin 3x + \frac{1}{36}\cos 3x + \frac{3}{4}x \sin x + \frac{3}{4}\cos x$.
   (xi) $x(t^3 - 3t^2 + 6t - 6)$, when $t = \log x$.
   (xii) $(x + 1)\tan^{-1}(\sqrt{x}) - \sqrt{x}$.
   (xiii) $x \log(1 + x^2) - 2x + 2\tan^{-1} x$.
   (xiv) $\frac{1}{3}x^3 \sin^{-1} x + \frac{1}{3}\sqrt{(1 - x^2)} - \frac{1}{9}(1 - x^2)^{\frac{3}{2}}$.
   (xv) $\dfrac{2^x}{\sqrt{\{1 + (\log 2)^2\}}}\cos\{x - \cot^{-1}(\log 2)\}$.
   (xvi) $e^x(x^4 - 4x^3 + 12x^2 - 24x + 24)$.

2. (i) $\tan^{-1} x + \dfrac{1}{1 + x^2}$. (ii) $-\dfrac{1 + 2x^2}{4(1 + x^2)^2}$

3. (i) $\log x - \left(1 + \dfrac{1}{x}\right)\log(1 + x)$. (ii) $-\frac{1}{5}x^{-2}(\cos\log x + 2\sin\log x)$

4. (i) $-\frac{1}{2}(1 + e^{2x})^{-1}$. (ii) $-(1 + e^x)^{-1}$.

5. (i) $\frac{1}{6}(2x^2 + 3ax + 2a^2)\sqrt{(a^2 + x^2)} + \frac{1}{2}a^3 \log\{x + \sqrt{(a^2 + x^2)}\}$.
   (ii) $\dfrac{2}{105}(x + a)^{\frac{3}{2}}(15x^2 - 12ax + 43a^2)$.

6. (i) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x\sqrt{2}}{\sqrt{(1-x^2)}}\right)$. (ii) $\frac{1}{2\sqrt{2}}\log\frac{\sqrt{(1+x^2)}+x\sqrt{2}}{\sqrt{(1+x^2)}-x\sqrt{2}}$.

7. (i) $\frac{1}{4\sqrt{3}}\log\frac{2\sqrt{(x^2-1)}-x\sqrt{3}}{2\sqrt{(x^2-1)}+x\sqrt{3}}$. (ii) $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x\sqrt{3}}{\sqrt{(x^2+4)}}\right)$.

8. (i) $-\frac{2+3x^2}{2x(1+x^2)}-\frac{3}{2}\tan^{-1}x$. (ii) $-\frac{\sqrt{(1+x^2)}}{x}$. 9. (i) $\log\frac{x}{\sqrt{(x^2+1)}}$

(ii) $\frac{2x^2-1}{3x^3}\sqrt{(x^2+1)}$. 10. (i) $\frac{1}{2}\sec^{-1}x+\frac{\sqrt{(x^2-1)}}{2x^2}$.

(ii) $\log\frac{x}{1+x}-\frac{x}{1+x}$.

11. (i) $\frac{1+3x+6x^2}{3x^3}+2\log\frac{x-1}{x}$. (ii) $\log\{(x+\sqrt{(x^2+1)})\}-\frac{\sqrt{(1+x^2)}}{x}$

12. (i) $x(\tan x-\sec x)+\log(1+\sin x)$. (ii) $e^x\tan x$.

13. (i) $\frac{x-\tan^{-1}x}{\sqrt{(1+x^2)}}$. (ii) $\frac{(a+x)e^{a\tan^{-1}x}}{(1+a^2)\sqrt{(1+x^2)}}$.

14. (i) $\frac{1}{2}\{\sec x+\log(\sec x+\tan x)\}$. (ii) $\frac{-\cos x}{a(a\sin x+b\cos x)}$.

15. (i) $\frac{1}{\sqrt{2}}\log\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}-\frac{1}{2}\log\frac{1+\sin x}{1-\sin x}$

(ii) $\frac{1}{2}\{x\sqrt{(1-x^2)}-\cos^{-1}x\}$.

16. (i) $2\sqrt{2}\tan^{-1}\sqrt{\frac{x}{2}}-2\tan^{-1}\sqrt{x}$.

(ii) $\log\frac{x}{x-1}+\frac{1}{2}\frac{1}{1-x}+\frac{1}{2}\tan^{-1}x$.

17. (i) $\frac{1}{2}\log(x-1)-\frac{1}{2(x-1)}-\frac{1}{4}\log(x^2+1)$.

(ii) $\sqrt{\frac{\text{[illegible]}}{3}}\cos^{-1}\sqrt{\frac{2(x+3)}{5x}}$

18. (i) $\frac{1}{2}\log\tan\frac{1}{2}x-\frac{1}{4}\tan^2\frac{1}{2}x$.

(ii) $\frac{1}{13}\{2x-3\log(3\cos x+2\sin x)\}$.

19. (i) $e^x-(n+1)\log(e^x+1)$. (ii) $\sinh^{-1}\left(\frac{2x-1}{x}\right)$.

20. (i) $x + 2\{\log(x-2) - \log(x-3)\}$.

(ii) $9\log(x-3) - 5\log(x-2) - \log(x-1)$.

21. (i) $\dfrac{1}{54}\left\{\tan^{-1}\tfrac{1}{3}x + \dfrac{3x}{x^2+9}\right\}$.

(ii) $\dfrac{1}{16}\tan^{-1}\dfrac{x+1}{2} + \dfrac{x+1}{8(x^2+2x+5)}$.

22. (i) $2\tan^{-1}(1+x)^{1/2}$. (ii) $\dfrac{2}{3}\dfrac{x^2 + x\sqrt{(2+x^2)} - 2}{\sqrt{\{x+\sqrt{(2+x^2)}\}}}$.

23. (i) $\frac{1}{16}\pi$. (ii) $\frac{2}{27} - \frac{1}{12}\pi^2$. **24.** (i) $\frac{3}{4}(1 - 2\log\frac{3}{2})$. (ii) $\pi\log\frac{1}{2}$.

25. (i) $\frac{1}{4}\pi$. (ii) $\pi/\sqrt{2}$. **26.** (i) $\frac{1}{2}\log 2$. (ii) $\frac{1}{4}\pi$.

27. (i) $\dfrac{2\pi}{9}\sqrt{3}$. (ii) $\dfrac{\pi}{2\sqrt{2}}$. **28.** (i) -1. (ii) $\dfrac{1}{\sqrt{2}}\cot^{-1}2$.

CHAPTER IX

INTEGRATION BY SUCCESSIVE REDUCTION AND BETA AND GAMMA FUNCTIONS

9.1. Reduction Formulæ.

It has been mentioned in Art. 1.6 that, in some cases of integration, we take recourse to the method of successive reduction of the integrand which mostly depends on the repeated application of integration by parts. This is specially the case when the integrands are complicated in nature and depend on certain parameter or parameters. These parameters may be positive, negative, negative or fractional indices, as for example, $x^n e^{ax}$, $\tan^n x$, $(x^2+a^2)^{n/2}$, $\sin^m x \cos^n x$, etc. To obtain a complete integral of these trigonometric or algebraic functions, we first of all define these integrals by the letters I, J, U, etc., introducing the parameter or parameters as suffixes, and connect them with certain similar other integral or integrals whose suffixes are lower than that of the original integral. Then by repeatedly changing the value of the suffixes, the original integral can be made to rest on much simpler integrals. This last integral can be easily evaluated and knowing the value of this last integral, by the process of repeated substitution, the value of the original integral can be found out. The formula in which a certain integral involving some parameters is connected with some integrals of lower order is called a *Reduction Formula*. In most of the cases the reduction formula is obtained by the process of *integration by parts*. Of course, in some cases the method of differentiation (See § 9.19 below) or other special devices are adopted (See § 9.20). In the next few pages methods of finding the reduction formula of certain integrals are discussed.

Case I. Integrals involving one parameter.

9.2. *Obtain a reduction formula for* $\int x^n e^{ax}\,dx$.

Let $I_n = \int x^n e^{ax}\,dx$. ... (1)

Integrating by parts,

$$\int x^n e^{ax}\,dx = x^n \frac{e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax}\,dx, \qquad \ldots (2)$$

$$\text{or,}\quad I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}. \qquad \ldots (3)$$

Note 1. It may be observed that the integral on the right-hand side of (2) is of the same form as the integral in (1) except for the power of x, which is $n-1$, and which can be obtained from (1) replacing n by $n-1$ on both sides. If n be a positive integer, proceeding successively as above, I_n will finally depend upon $I_0 = \int e^{ax}\,dx = e^{ax}/a$, and is thus known.

Note 2. In evaluating (3) from (1) we could integrate x^n first, but in that case I_n would have been connected with I_{n+1}, *i.e.*, with an integral whose suffix is greater than that of the original one, which is not usually desirable. A little practice will enable the students to choose the right function.

9.3. *Obtain reduction formula for*

$$\text{(i)}\quad \int \sin^n x\,dx\,;\ \int_0^{\pi/2} \sin^n x\,dx.$$

$$\text{(ii)}\quad \int \cos^n x\,dx\,;\ \int_0^{\pi/2} \cos^n x\,dx.$$

(i) As in Article 6.23A(1) of the book,

$$I_n = \int \sin^n x\,dx$$

$$= -\frac{\sin^{n-1} x \,.\, \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$$

$$\therefore\ I_n = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2} \qquad \ldots (1)$$

is the required reduction formula.

Also by (1), taking limits of integration from 0 to $\frac{1}{2}\pi$,

$$J_n = \int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n} J_{n-2} \; (n > 1) \quad \ldots (2)$$

Similarly,

(ii) $$I_n = \int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2} \ldots (3)$$
[C. P. '86]

and $$J_n = \int_0^{\pi/2} \cos^n x \, dx = \frac{n-1}{n} J_{n-2} \quad (n > 1). \quad \ldots (4)$$

Note. If the integrand be $\sinh^n x$ or $\cosh^n x$ a similar process may be adopted.

9.4. *Obtain reduction formula for*

(i) $\int \tan^n x \, dx$; (ii) $\int_0^{\pi/4} \tan^n x \, dx$. [C. P. '89]

(*n , a positive integer*) .

Here, $$I_n = \int \tan^n x \, dx = \int \tan^{n-2} x . \tan^2 x \, dx$$
$$= \int \tan^{n-2} x . (\sec^2 x - 1) \, dx$$
$$= \int \tan^{n-2} x . \sec^2 x \, dx - \int \tan^{n-2} x \, dx$$
$$= \frac{\tan^{n-1} x}{n-1} - I_{n-2} .$$

Thus ,

$$I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2} . \quad \ldots (1)$$

Also, taking limits from 0 to $\frac{1}{4}\pi$,

$$J_n = \int_0^{\pi/4} \tan^n x \, dx = \left[\frac{\tan^{n-1} x}{n-1} \right]_0^{\pi/4} - \int_0^{\pi/4} \tan^{n-2} x \, dx \text{ by (1)}$$
$$= \frac{1}{n-1} - J_{n-2} . \quad \ldots (2)$$

Note 1. If n be a positive integer,

$$\int \tan^n x\, dx = \frac{\tan^{n-1} x}{n-1} - \frac{\tan^{n-3} x}{n-3} + \frac{\tan^{n-5} x}{n-5} - \ldots$$

If n be odd, the last term is $(-1)^{(n-1)/2} \int \tan x\, dx$

$$= (-1)^{(n-1)/2} \log \sec x .$$

If n be even, the last term is $(-1)^{(n+2)/2} \int \tan^2 x\, dx$

$$= (-1)^{(n+2)/2} (\tan x - x) .$$

Note 2. If the integrands be $\cot^n x$, $\tanh^n x$, $\coth^n x$ the same process may be adopted.

9.5. *Obtain a reduction formula for* $\int \sec^n x\, dx$.

$$I_n = \int \sec^n x\, dx = \int \sec^{n-2} x . \sec^2 x\, dx .$$

Integrating by parts,

$$I_n = \sec^{n-2} x . \tan x - \int (n-2) \sec^{n-3} x \sec x . \tan x . \tan x\, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1)\, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \left[\int \sec^n x\, dx - \int \sec^{n-2} x\, dx \right]$$

Transposing and simplifying,

$$I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2} . \qquad \ldots (1)$$

Note. If the integrands are $\operatorname{cosec}^n x$, $\operatorname{sech}^n x$, $\operatorname{cosech}^n x$ then proceeding as above we can get the reduction formula for each of them.

9.6. *Obtain a reduction formula for* $\int e^{ax} \cos^n x\, dx$.

Let $I_n = \int e^{ax} \cos^n x\, dx$.

Integrating by parts,

$$I_n = \frac{e^{ax} \cos^n x}{a} + \frac{n}{a} \int e^{ax} \cos^{n-1} x . \sin x\, dx$$

$$= \frac{e^{ax}\cos^n x}{a} + \frac{n}{a}\left[\frac{e^{ax}}{a}\cos^{n-1}x\,.\sin x - \frac{1}{a}\int e^{ax}\right.$$

$$\left.\times\left\{(n-1)\cos^{n-2}x\,(-\sin x)\,.\sin x + \cos^{n-1}x.\cos x\right\}dx\right]$$

$$= \frac{e^{ax}}{a}\cos^n x + \frac{ne^{ax}}{a^2}\cos^{n-1}x\,.\,sin\,x$$

$$- \frac{n}{a^2}\int e^{ax}\left\{(n-1)\cos^{n-2}x\,(\cos^2 x - 1) + \cos^n x\right\}dx$$

$$= \frac{e^{ax}\cos^{n-1}x\,(a\cos x + n\sin x)}{a^2}$$

$$- \frac{n}{a^2}\left[n\int e^{ax}\cos^n x\,dx - (n-1)\int e^{ax}\cos^{n-2}x\,dx\right].$$

Transposing,

$$\left(1 + \frac{n^2}{a^2}\right)I_n = \frac{e^{ax}\cos^{n-1}x\,(a\cos x + n\sin x)}{a^2} + \frac{n(n-1)}{a^2}I_{n-2},$$

$$\text{or, } I_n = \frac{e^{ax}\cos^{n-1}x\,(a\cos x + n\sin x)}{n^2 + a^2} + \frac{n(n-1)}{n^2 + a^2}I_{n-2}.$$

9.7. *Obtain a reduction formula for* $\int (x^2 + a^2)^n\,dx$.

Let $I_n = \int (x^2 + a^2)^n\,dx$.

Integrating by parts (taking 1 as the second factor),

$$I_n = x(x^2 + a^2)^n - \int n(x^2 + a^2)^{n-1}\,.\,2x\,.\,x\,dx$$

$$= x(x^2 + a^2)^n - 2n\int (x^2 + a^2)^{n-1}(x^2 + a^2 - a^2)\,dx$$

$$= x(x^2 + a^2)^n - 2n\int (x^2 + a^2)^n\, dx$$

$$+ 2na^2\int (x^2 + a^2)^{n-1}\, dx.$$

Transposing,

$$(1 + 2n)\, I_n = x(x^2 + a^2)^n + 2na^2 I_{n-1}.$$

$$\therefore\quad I_n = \frac{x(x^2 + a^2)^n}{2n + 1} + \frac{2na^2}{2n + 1} I_{n-1}.$$

Note. It may be noted that here n need not be an integer. Put $n = \frac{1}{2}$ and compare with § 3.5 (C).

9.8. *Obtain a reduction formula for* $\int (ax^2 + bx + c)^n\, dx$

Let $I_n = \int (ax^2 + bx + c)^n\, dx$.

If a be *positive,*

$$I_n = a^n \int (z^2 \pm k^2)^n\, dx, \text{ where } z = x + \frac{b}{2a},$$

$$\text{and } k^2 = \frac{4ac - b^2}{4a^2}; \qquad \ldots (1)$$

and if a be *negative,* say, $= -a'$,

$$I_n = (a')^n \int (k^2 - z^2)^n\, dx,$$

$$\text{where}\quad z = x - \frac{b}{2a'}, \text{ and } k^2 = \frac{4a'c + b^2}{4a'^2}. \qquad \ldots (2)$$

But (1) and (2) are similar to that of § 9.7 above, and can be evaluated by the same process.

9.9. *Obtain a reduction formula for* $\int \frac{dx}{(x^2 + a^2)^n}$. $[n \neq 1]$

Let $I_n = \int \frac{dx}{(x^2 + a^2)^n}$, then $I_{n-1} = \int \frac{dx}{(x^2 + a^2)^{n-1}}$.

Integrating by parts,

$$I_{n-1} = \frac{x}{(x^2 + a^2)^{n-1}} - \int \frac{-(n-1).2x.x}{(x^2+a^2)^n}\,dx$$

$$= \frac{x}{(x^2+a^2)^{n-1}} + 2(n-1)\int \frac{x^2+a^2-a^2}{(x^2+a^2)^n}\,dx$$

$$= \frac{x}{(x^2+a^2)^{n-1}} + 2(n-1)I_{n-1} - 2(n-1)a^2 I_n .$$

Transposing,

$$2(n-1)a^2 I_n = \frac{x}{(x^2+a^2)^{n-1}} + (2n-3)I_{n-1},$$

i.e., $$I_n = \frac{1}{2(n-1)a^2}\frac{x}{(x^2+a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2} I_{n-1} .$$

9.10. *Obtain a reduction formula for* $\int \frac{dx}{(ax^2+bx+c)^n}$.

Let $$I_n = \int \frac{dx}{(ax^2+bx+c)^n} . \qquad \ldots (1)$$

If a be *positive,*

$$I_n = \frac{1}{a^n}\int \frac{dz}{(z^2 \pm k^2)^n},$$

where $$z = x + \frac{b}{2a}, \; k^2 = \frac{4ac - b^2}{4a^2}, \qquad \ldots (2)$$

and if a be *negative,* say, $= -a'$,

$$I_n = \frac{1}{(a')^n}\int \frac{dz}{(k^2 - z^2)^n},$$

where $$z = x - \frac{b}{2a'}, \text{ and } k^2 = \frac{4a'c + b^2}{4a'^2} . \qquad \ldots (3)$$

Both (2) and (3) can be integrated by the same process as in § 9.9 above.

Note. In Article 5.1, *Case IV* of the book, we have remarked that when the integrand is a rational fraction in which the denominator contains factors real, quadratic but some repeated, in general a reduction formula is required. Thus, to integrate such functions, separate repeated and non-repeated quadratic factors and for repeated quadratic factors, use the result of the above Article.

9.11. *Obtain a reduction formula for* $\int \frac{x^n\,dx}{\sqrt{(ax^2+bx+c)}}$ *where* n *is any positive integer.*

Let $I_n = \int \frac{x^n\,dx}{\sqrt{(ax^2+bx+c)}}$.

Noting that $x^n = \frac{2ax+b-b}{2a} x^{n-1}$,

$$I_n = \frac{1}{2a}\int \frac{(2ax+b)}{\sqrt{(ax^2+bx+c)}} x^{n-1}\,dx - \frac{b}{2a}\int \frac{x^{n-1}}{\sqrt{(ax^2+bx+c)}}\,dx.$$

Now, $\int \frac{(2ax+b)}{\sqrt{(ax^2+bx+c)}} x^{n-1}\,dx$

$$= 2\sqrt{ax^2+bx+c}\,.\,x^{n-1} - \int 2(n-1)x^{n-2}\sqrt{ax^2+bx+c}\,dx$$

$$= 2x^{n-1}\sqrt{ax^2+bx+c} - 2(n-1)\int \frac{x^{n-2}(ax^2+bx+c)}{\sqrt{(ax^2+bx+c)}}\,dx$$

$$= 2x^{n-1}\sqrt{ax^2+bx+c} - 2(n-1)[aI_n + bI_{n-1} + cI_{n-2}].$$

$$\therefore\ I_n = \frac{x^{n-1}}{a}\sqrt{ax^2+bx+c} - \frac{n-1}{a}\left[aI_n + bI_{n-1} + cI_{n-2}\right] - \frac{b}{2a} I_{n-1}$$

$$= \frac{x^{n-1}}{a}\sqrt{ax^2+bx+c} - (n-1)I_n - \frac{(2n-1)b}{2a} I_{n-1} - \frac{(n-1)c}{a} I_{n-2}.$$

Transposing and simplifying,

$$I_n = \frac{x^{n-1}}{an}\sqrt{ax^2+bx+c} - \frac{(2n-1)b}{2an} I_{n-1} - \frac{(n-1)c}{an} I_{n-2}$$

Case II. Reduction formulæ involving two parameters.

9.12. *Obtain a reduction formula for* $\int x^m (\log x)^n \, dx$

(n , *a positive integer*).

Here, since two parameters m, n are involved, we shall define the integral by the symbol $I_{m,n}$.

$$I_{m,n} = \int x^m (\log x)^n \, dx.$$

Integrating by parts,

$$I_{m,n} = \frac{x^{m+1}}{m+1}(\log x)^n - \frac{1}{m+1}\int n (\log x)^{n-1}\frac{1}{x} \cdot x^{m+1}\, dx$$

$$= \frac{x^{m+1}}{m+1}(\log x)^n - \frac{n}{m+1}\int x^m (\log x)^{n-1}\, dx$$

$$= \frac{x^{m+1}}{m+1}(\log x)^n - \frac{n}{m+1} I_{m,n-1}.$$

i.e., $$I_{m,n} = \frac{x^{m+1}}{m+1}(\log x)^n - \frac{n}{m+1} I_{m,n-1}.$$

Note 1. Here we have connected $I_{m,n}$ with $I_{m,n-1}$ and by successive change the power of log x can be reduced to zero, *i.e.*, after n operations we shall get a term $I_{m,0}$, *i.e.*, $\int x^m \, dx$, which is easily integrable. Thus, by step by step substitution, $I_{m,n}$ can be evaluated. It may be noted that when two parameters are involved this is the usual practice.

Note 2. Students must be cautious in defining these integrals. Here, as for illustration, $I_{m,n} \neq I_{n,m}$.

9.13. *Obtain reduction formulæ for*

(i) $\int \frac{(a+bx)^m}{x^n}\, dx$; (ii) $\int \frac{dx}{x^m (a+bx)^n}$

(i) Let $I_{m,n} = \int \frac{(a + bx)^m}{x^n} dx$. $[n \neq 1]$

Integrating by parts,

$$I_{m,n} = \frac{(a + bx)^m}{-(n-1)x^{n-1}} + \frac{mb}{n-1} \int \frac{(a + bx)^{m-1}}{x^{n-1}} dx.$$

$$\therefore I_{m,n} = -\frac{(a + bx)^m}{(n-1)x^{n-1}} + \frac{mb}{n-1} I_{m-1,n-1}. \quad \ldots (1)$$

(ii) Let $I_{m,n} = \int \frac{dx}{x^m (a + bx)^n}$.

Integrating by parts,

$$I_{m,n} = \frac{1}{-(m-1)x^{m-1}(a+bx)^n} - \frac{nb}{m-1} \int \frac{dx}{x^{m-1}(a+bx)^{n+1}}$$

$$= -\frac{1}{(m-1)x^{m-1}(a+bx)^n}$$

$$- \frac{n}{m-1} \int \frac{(a+bx)-a}{x^m (a+bx)^{n+1}} dx \quad \ldots (2)$$

$$= -\frac{1}{(m-1)x^{m-1}(a+bx)^n}$$

$$- \frac{n}{m-1} I_{m,n} + \frac{an}{m-1} I_{m,n+1}.$$

$$\therefore \frac{an}{m-1} I_{m,n+1} = \frac{1}{(m-1)x^{m-1}(a+bx)^n} + \frac{m+n-1}{m-1} I_{m,n}.$$

Changing n to $n-1$ on both sides,

$$I_{m,n} = \frac{1}{a(n-1)x^{m-1}(a+bx)^{n-1}} + \frac{m+n-2}{(n-1)a} I_{m,n-1}. \quad (3)$$

Note. Formula (2) or (3) can be taken as the reduction formula for (ii). (3) is more rapidly converging. The other ways in which these integrals can be expressed are left to the students. [See also § 2.2, *Ex.* 9 .]

9.14. *Obtain reduction formulæ for*

$$\text{(i)} \int x^m (1-x)^n \, dx; \qquad \text{(ii)} \int_0^1 x^m (1-x)^n \, dx.$$

(i) Let $I_{m,n} = \int x^m (1-x)^n \, dx$

$$= \frac{x^{m+1}}{m+1} . (1-x)^n + \frac{n}{m+1} \int x^{m+1} . (1-x)^{n-1} \, dx$$

$$= \frac{x^{m+1}(1-x)^n}{m+1} + \frac{n}{m+1} \int x^m (1-x)^{n-1} \{1-(1-x)\} \, dx$$

$$= \frac{x^{m+1}(1-x)^n}{m+1} + \frac{n}{m+1} \left[I_{m,n-1} - I_{m,n} \right].$$

Transposing and simplifying,

$$I_{m,n} = \frac{x^{m+1}(1-x)^n}{m+n+1} + \frac{n}{m+n+1} I_{m,n-1} .$$

(ii) If $J_{m,n} = \int_0^1 x^m (1-x)^n \, dx$, by above, this

$$= \left[\frac{x^{m+1}(1-x)^n}{m+n+1} \right]_0^1 + \frac{n}{m+n+1} J_{m,n-1}$$

$$\therefore \quad J_{m,n} = \frac{n}{m+n+1} J_{m,n-1} .$$

Note. In Integral Calculus $J_{m,n}$ is usually denoted as $\beta_{m,n}$, the *first Eulerian integral*. It is also referred to as the Beta-function.

[See § 9.21 *below.*]

It is interesting to note that $J_{m,n} = J_{n,m}$,
i.e., $\beta_{m,n} = \beta_{n,m}$ although $I_{m,n} \neq I_{n,m}$.

9.15. *Obtain reduction formulæ for*

$$\text{(i)} \; I_{m,n} = \int \sin^m x \cos^n x \, dx;$$

$$\text{(ii)} \; J_{m,n} = \int_0^{\frac{1}{2}\pi} \sin^m x \cos^n x \, dx$$

(m , n being positive integers .

in Article 6.23, Case [illegible], and obtained

$$I_{m,n} = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}$$

$$\text{or} \quad = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}$$ in a similar way, and when m and n are positive integers.

$$J_{m,n} = \frac{n-1}{m+n} J_{m,n-2} = \frac{m-1}{m+n} J_{m-2,n}.$$

Using § 6.19 (iv), we also see that $J_{m,n} = J_{n,m}$.

9.16. *Obtain a reduction formula for* $\int \frac{\sin^m x}{\cos^n x} dx$. $[n \neq 1]$

Let $I_{m,n} = \int \sin^m x \cos^{-n} x \, dx.$ ... (1)

Consider $I'_{p,q} = \int \sin^p x \cos^q x \, dx$

$$= \frac{\sin^{p+1} x \cos^{q-1} x}{p+q} + \frac{q-1}{p+q} I'_{p,q-2}.$$

[by § 9.15 *above*]

Changing q to $q+2$,

$$I'_{p,q+2} = \frac{\sin^{p+1} x \cos^{q+1} x}{p+q+2} + \frac{q+1}{p+q+2} I'_{p,q}.$$

Transposing,

$$I'_{p,q} = -\frac{\sin^{p+1} x \cos^{q+1} x}{q+1} + \frac{p+q+2}{q+1} I'_{p,q+2}. \quad \dots (2)$$

$[q+1 \neq 0]$

Now replace p by m and q by $-n$ in (2) and use the definition (1).

Then, (2) becomes

$$I_{m,n} = \frac{1}{n-1} \frac{\sin^{m+1} x}{\cos^{n-1} x} - \frac{m-n+2}{n-1} I_{m,n-2}.$$

9.17. *Obtain a reduction formula for* $\int \frac{dx}{\sin^m x \cos^n x}$. $[n \neq 1]$

Let $I_{m,n} = \int \frac{dx}{\sin^m x \cos^n x}$

Consider, as before,

$$I'_{p,q} = \int \sin^p x \cos^q x \, dx$$

$$= -\frac{\sin^{p+1} x \cos^{q+1} x}{q+1} + \frac{p+q+2}{q+1} I'_{p,q+2}.$$

[as in § 9.16 *(2) above*]

Replacing p by $-m$ and q by $-n$ and using the def. of $I_{m,n}$,

$$I_{m,n} = \frac{1}{n-1} \frac{1}{\sin^{m-1} x \cos^{n-1} x} + \frac{m+n-2}{n-1} I_{m,n-2}.$$

9.18. *Obtain a reduction formula for*

$$I_{m,n} = \int \cos^m x \cos nx \, dx,$$

connecting with (i) $I_{m-1,n-1}$, (ii) $I_{m-2,n}$. $(m \neq \pm n)$

(i) Let

$$I_{m,n} = \int \cos^m x \cos nx \, dx$$

$$= \frac{\cos^m x \cdot \sin nx}{n} - \frac{m}{n} \int \cos^{m-1} x \cdot (-\sin x) \sin nx \, dx \qquad \ldots (1)$$

Since $\sin nx \sin x = \cos(n-1)x - \cos nx \cos x$,

$$\therefore I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n} \int \cos^{m-1} x \cdot \{\cos(n-1)x$$
$$- \cos nx \cos x\} \, dx$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n} \left[I_{m-1,n-1} - I_{m,n} \right].$$

Simplifying,

$$I_{m,n} = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1} .$$

(ii) From (1),

$$I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int (\cos^{m-1} x \sin x) . \sin nx \, dx .$$

Again integrating by parts,

$$I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\Bigg[-\frac{\cos^{m-1} x \sin x \cos nx}{n}$$

$$+ \frac{1}{n}\int \{(m-1) \cos^{m-2} x (-\sin x) \sin x$$

$$+ \cos^{m-1} x . \cos x \} \cos nx \, dx \Bigg]$$

$$= \frac{\cos^m x \sin nx}{n} - \frac{m (\cos^{m-1} x \cos nx \sin x)}{n^2}$$

$$+ \frac{m}{n^2}\int \{(m-1) \cos^{m-2} x (\cos^2 x - 1)$$

$$+ \cos^m x \} \cos nx \, dx$$

$$= \frac{\cos^{m-1} x (n \sin nx \cos x - m \cos nx \sin x)}{n^2}$$

$$+ \frac{m}{n^2}\int \{(m-1+1) \cos^m x \cos nx$$

$$- (m-1) \cos^{m-2} x \cos nx \} \, dx$$

$$= \frac{\cos^{m-1} x (n \sin nx \cos x - m \cos nx \sin x)}{n^2}$$

$$+ \frac{m}{n^2}\Big[m I_{m,n} - (m-1) I_{m-2,n} \Big]$$

Transposing and dividing,

$$I_{m,n} = \frac{n \sin nx \cos x - m \cos nx \sin x}{n^2 - m^2} \cos^{m-1} x$$

$$- \frac{m (m-1)}{n^2 - m^2} I_{m-2,n} .$$

There are three other integrals of a similar type.

$$\text{(i)} \int \cos^m x \sin nx\, dx, \quad \text{(ii)} \int \sin^m x \cos nx\, dx$$

and $\quad \text{(iii)} \int \sin^m x \sin nx\, dx,$

which can be treated in a similar manner, and connected by a reduction formula either with $I_{m-1,n-1}$ or with $I_{m-2,n}$ in each case.

For instance,

$$(m+n)\int \cos^m x \sin nx\, dx = -\cos^m x \cos nx + m\, I_{m-1,n-1}.$$

$$(n^2 - m^2)\int \sin^m x \cos nx\, dx$$

$$= (n \sin nx \sin x + m \cos nx \cos x)\sin^{m-1} x$$

$$- m(m-1)\, I_{m-2,n};\ \text{etc.}$$

Case III. Special devices.

9.19. *Obtain a reduction formula for* $\int \frac{dx}{(a + b\cos x)^n}$.

Let $\quad I_n = \int \frac{dx}{(a + b\cos x)^n}$

Consider $P = \frac{\sin x}{(a + b\cos x)^{n-1}} \qquad \ldots (1)$

$\therefore \quad \frac{dP}{dx}$

$$= \frac{\cos x(a + b\cos x)^{n-1} - (n-1)(a + b\cos x)^{n-2}(-b\sin x).\sin x}{[(a + b\cos x)^{n-1}]^2}$$

$$= \frac{\cos x(a + b\cos x) + (n-1)\, b(1 - \cos^2 x)}{(a + b\cos x)^n}$$

$$= \frac{(n-1)b + a\cos x - (n-2)\, b\cos^2 x}{(a + b\cos x)^n}$$

$$= \frac{A + B(a + b\cos x) + C(a + b\cos x)^2}{(a + b\cos x)^n}\ (\text{say}). \qquad \ldots (2)$$

Then comparing the coefficients,

$A + B \cdot a + Ca^2 = (n-1)b$, $B \cdot b + 2Cab = a$, $Cb^2 = -(n-2)b$

Solving

$$A = -(n-1)\frac{a^2-b^2}{b},\ B = (2n-3)\frac{a}{b},\ C = -\frac{n-2}{b} \qquad \ldots(3)$$

$\therefore$ substituting these values of A, B, C in (2), we get

$$\frac{dP}{dx} = -\frac{(n-1)(a^2-b^2)}{b}\frac{1}{(a+b\cos x)^n}$$

$$+\frac{(2n-3)a}{b}\frac{1}{(a+b\cos x)^{n-1}} - \frac{n-2}{b}\frac{1}{(a+b\cos x)^{n-2}}$$

Integrating both sides with respect to x, and using the definition of I_n,

$$P = -\frac{(n-1)(a^2-b^2)}{b}I_n + \frac{(2n-3)a}{b}I_{n-1} - \frac{n-2}{b}I_{n-2}$$

$$\therefore\ I_n = -\frac{b}{(n-1)(a^2-b^2)}\frac{\sin x}{(a+b\cos x)^{n-1}}$$

$$+\frac{(2n-3)a}{(n-1)(a^2-b^2)}I_{n-1} - \frac{(n-2)}{(n-1)(a^2-b^2)}I_{n-2}.$$

Alternative method.

Let $P = \dfrac{\sin x}{(a+b\cos x)^{n-1}}$ and $V = a + b\cos x$.

$$\therefore\ \cos x = \frac{V-a}{b},$$

$$\therefore\ \frac{dP}{dx} = \frac{d}{dx}\left(\frac{\sin x}{V^{n-1}}\right) = \frac{\cos x}{V^{n-1}} - (n-1)\frac{\sin x(-b\sin x)}{V^n}$$

$$= \frac{V-a}{bV^{n-1}} + \frac{(n-1)b}{V^n}\left[1-\left(\frac{V-a}{b}\right)^2\right]$$

$$= -\frac{(n-1)(a^2-b^2)}{bV^n} + \frac{a(2n-3)}{bV^{n-1}} - \frac{(n-2)}{bV^{n-2}}.$$

Integrating both sides *w. r. t.* x and using the definition

$$I_n = \int \frac{dx}{V^n}, \text{ the result follows.}$$

Note. When n is a positive integer, by a repeated application of the above reduction formula, I_n will ultimately depend on I_1, which is easily integrable (See § 4.2).

9.20. *Obtain reduction formulæ for* $\int \mathbf{x^m(a+bx^n)^p\,dx}$.

In this integral, usually denoted as *binomial differentials*, three parameters are involved and this integral, written as

$I_{m,n,p} = \int x^m (a + bx^n)^p\,dx$, can be connected with any one of the integrals below :

(i) $I_{m+n,n,p-1} = \int x^{m+n} (a + bx^n)^{p-1}\,dx$,

(ii) $I_{m,n,p-1} = \int x^m (a + bx^n)^{p-1}\,dx$,

(iii) $I_{m,n,p+1} = \int x^m (a + bx^n)^{p+1}\,dx$,

(iv) $I_{m-n,n,p+1} = \int x^{m-n} (a + bx^n)^{p+1}\,dx$,

(v) $I_{m-n,n,p} = \int x^{m-n} (a + bx^n)^p\,dx$,

(vi) $I_{m+n,n,p} = \int x^{m+n} (a + bx^n)^p\,dx$

(i) $I_{m,n,p} = \int x^m (a + bx^n)^p\,dx$. Integrating by parts,

$$I_{m,n,p} = \frac{x^{m+1}}{m+1}(a + bx^n)^p - \frac{1}{m+1}\int p(a + bx^n)^{p-1}\,nbx^{n-1}\,x^{m+1}\,dx$$

$$= \frac{x^{m+1}}{m+1}(a + bx^n)^p - \frac{nbp}{m+1} I_{m+n,n,p-1} \quad \ldots (1)$$

Again, as above,

$$I_{m,n,p} = \frac{x^{m+1}}{m+1}(a+bx^n)^p$$

$$- \frac{nbp}{m+1}\int \frac{x^m}{b}(a+bx^n-a)(a+bx^n)^{p-1}\,dx$$

$$\left[\text{writing } x^{m+n} = \frac{1}{b}x^m(a+bx^n-a)\right].$$

Transposing and simplifying,

$$I_{m,n,p} = \frac{x^{m+1}(a+bx^n)^p}{np+m+1} + \frac{anp}{np+m+1}I_{m,n,p-1} \quad .(2)$$

Changing p to $p+1$ in (2) and transposing, we get a connection with the integral (iii), *viz.*,

$$I_{m,n,p} = -\frac{x^{m+1}(a+bx^n)^{p+1}}{an(p+1)}$$

$$+ \frac{n(p+1)+m+1}{an(p+1)}I_{m,n,p+1}. \quad \ldots (3)$$

Also changing m to $m-n$ and p to $p+1$ in (1) and transposing, we get

$$I_{m,n,p} = \frac{x^{m-n+1}(a+bx^n)^{p+1}}{nb(p+1)}$$

$$- \frac{m-n+1}{nb(p+1)}I_{m-n,n,p+1}. \quad \ldots (4)$$

To get a connection with $I_{m-n,n,p}$ and $I_{m+n,n,p}$ write

$$x^m = \frac{1}{nb}\left(x^{m-n+1}.\,nbx^{n-1}\right).$$

$$\therefore\ I_{m,n,p} = \frac{1}{nb}\int x^{m-n+1}.(a+bx^n)^p.\,nb\,.\,x^{n-1}\,dx.$$

Integrating by parts and simplifying,

$$I_{m,n,p} = \frac{x^{m-n+1}(a+bx^n)^{p+1}}{b(np+m+1)} - \frac{a(m-n+1)}{b(np+m+1)} I_{m-n,n,p}. \quad \ldots (5)$$

Changing m to $m+n$ in (5) and transposing,

$$I_{m,n,p} = \frac{x^{m+1}(a+bx^n)^{p+1}}{a(m+1)} - \frac{b(np+m+n+1)}{a(m+1)} I_{m+n,n,p}. \quad \ldots (6)$$

These six formulæ of $I_{m,n,p}$ can be obtained by another method

Write $P = x^{\lambda+1}(a+bx^n)^{\mu+1}$,
where λ and μ are the smaller indices of x and $(a+bx^n)$ respectively in the two expressions whose integrals are to be connected.

Find $\frac{dP}{dx}$ and express it as a linear combination of the two integrands. On integration the result can be obtained.

To illustrate the above statements we shall find a connection of $I_{m,n,p}$ with $I_{m+n,n,p}$.

Here evidently $\lambda = m$, $\mu = p$. $\therefore$ $P = x^{m+1}(a+bx^n)^{p+1}$.

$$\therefore \frac{dP}{dx} = (m+1)x^m(a+bx^n)^{p+1} + (p+1)x^{m+1}.nbx^{n-1}(a+bx^n)^p$$

$$= (m+1)x^m(a+bx^n)^p.(a+bx^n) + nb(p+1)x^{m+n}(a+bx^n)^p$$

$$= (m+1)ax^m(a+bx^n)^p + b(np+n+m+1)x^{m+n}(a+bx^n)^p.$$

Integrating with respect to x,

$$P = (m+1)aI_{m,n,p} + b(np+n+m+1)I_{m+n,n,p}.$$

$$\therefore I_{m,n,p} = \frac{x^{m+1}(a+bx^n)^{p+1}}{a(m+1)} - \frac{b(np+n+m+1)}{a(m+1)} I_{m+n,n,p}$$

which is the same as (6).

Similarly, the other five results can be obtained.

For another illustration see sum no. 7, § 9.22.

9.21. Beta and Gamma functions.

In many problems in the applications of Integral Calculus, the use of the Beta and Gamma functions often facilitates calculations. So we give below an account of those functions —their definitions and important properties, some of which are, however, mentioned without any proof.*

Definitions :

(A) $\int_0^1 x^{m-1} (1 - x)^{n-1} dx$ *denoted by* B (m , n) [$m > 0$, $n > 0$]

is called the *First Eulerian integral* or Beta function.

(B) $\int_0^\infty e^{-x} x^{n-1} dx$ *denoted by* Γ (n) [$n > 0$]

[C. P. '84, '88]

is called the *Second Eulerian integral* or Gamma function.

Here m and n are positive but they need not be integers.

Properties :

(i) By property (iv) of Art. 6.19, we get

$$\int_0^1 x^{m-1} (1 - x)^{n-1} dx = \int_0^1 x^{n-1} (1 - x)^{m-1} dx .$$

$\therefore$ B (m , n) = B (n , m) .

(ii) $\Gamma(1) = \int_0^\infty e^{-x} dx = 1$. [See Ex.1 of Art. 7.2.]

$\therefore$ **Γ (1) = 1.**

* Results (v), (vi) and (vii) are given without any proof here. The proofs are based on "double integration" which is treated in chapter 20 of the present book. Nevertheless, the results are extremely important in applications and are to be carefully remembered.

(iii) As in Ex. 9, Illustrative Examples Art. 7.2, it can be shown that even *when n is not a positive integer,*

$$\int_0^\infty e^{-x} x^n \, dx = n \int_0^\infty e^{-x} x^{n-1} dx$$ [C. P. '80]

$\therefore \quad \Gamma(n+1) = n\Gamma(n).$

When n is a positive integer,

$\Gamma(n+1) = n!$ [C. P. '85, '88]

(iv) Writing kx for x in (B), we easily get

$$\int_0^\infty e^{-kx} x^{n-1} dx = \frac{\Gamma(n)}{k^n}, \; [k > 0, n > 0]$$ [C. P. '83]

(v) $B(m, n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ [C. H. '86]

(vi) $\Gamma(m)\Gamma(1-m) = \dfrac{\pi}{\sin m\pi} \; (0 < m < 1).$

(vii) Putting $m = \frac{1}{2}$ in (vi), we get [C. H. '86]

$\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}) = \dfrac{\pi}{\sin \frac{1}{2}\pi} = \pi.$

$\therefore \quad \Gamma(\frac{1}{2}) = \sqrt{\pi}.$ [C. P. '82]

Alternatively, we can deduce the value of $\Gamma(\frac{1}{2})$ in the following way.

Putting $m = n = \frac{1}{2}$ in (v),

$$\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)} = B(\tfrac{1}{2}, \tfrac{1}{2}) = \int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}} dx$$

$$= 2\int_0^{\frac{1}{2}\pi} d\theta \; [\text{ on putting } x = \sin^2\theta]$$

$= \pi.$ [C. P. '83]

Hence the result.

(viii) $B(m, n) = \int_0^\infty \frac{x^{m-1}\,dx}{(1+x)^{m+n}} = \int_0^\infty \frac{x^{n-1}\,dx}{(1+x)^{m+n}}$

9.22. Standard Integrals.

(1) $\int_0^{\frac{1}{2}\pi} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)} \quad \begin{bmatrix} p>-1 \\ q>-1 \end{bmatrix}.$

Left side $= \int_0^{\frac{1}{2}\pi} (\sin^2\theta)^{p/2}(1-\sin^2\theta)^{q/2}\,d\theta$

$$= \frac{1}{2}\int_0^1 x^{\frac{p+1}{2}-1}(1-x)^{\frac{q+1}{2}-1}\,dx$$

[on putting $x = \sin^2\theta$]

$$= \frac{1}{2}B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \text{Right side by (v).}$$

[Compare § 6.23 *B.*]

(2) $\int_0^{\frac{1}{2}\pi} \sin^p\theta \, \delta\theta = \int_0^{\frac{1}{2}\pi} \cos^p\theta d\theta = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)}$

The proof is similar to (1). [Compare § 6.23 *A.*]

(3) $\int_0^\infty e^{-x^2}dx = \frac{1}{2}\sqrt{\pi}.$ [*C. P.* '83]

Left side $= \frac{1}{2}\int_0^\infty e^{-z} z^{\frac{1}{2}-1}dz$ [on putting $x^2 = z$]

$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$ by (*B*) $= \frac{1}{2}\sqrt{\pi}$ by (vii). [Compare *Art.* 7.3]

9.23. Illustrative Examples.

Ex. 1. *Obtain a reduction formula for* $\int \tan^n x\, dx$ *and hence or otherwise find the values of (i)* $\int \tan^5 x\, dx$; *(ii)* $\int \tan^6 x\, dx$.

From § 9.4 formula (1), $I_n = \int \tan^n x\, dx = \dfrac{\tan^{n-1} x}{n-1} - I_{n-2}$.

(i) $\therefore\ I_5 = \int \tan^5 x\, dx = \frac{1}{4}\tan^4 x - I_3$,

$$I_3 = \tfrac{1}{2}\tan^2 x - I_1, \text{ where } I_1 = \int \tan x\, dx = \log \sec x.$$

$$\therefore\ I_5 = \tfrac{1}{4}\tan^4 x - \tfrac{1}{2}\tan^2 x + \log \sec x.$$

(ii) $I_6 = \frac{1}{5}\tan^5 x - I_4$; $I_4 = \frac{1}{3}\tan^3 x - I_2$;

$$\therefore\ I_2 = \frac{\tan^5 x}{1} - I_0, \text{ where } I_0 = \int dx = x.$$

$$\therefore\ I_6 = \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \frac{\tan x}{1} - x.$$

[Compare § 9.4., *Note 1 in these two cases* .]

Ex. 2. *Obtain a reduction formula for* $\int \sec^n x\, dx$.

Hence find the values of (i) $\int \sec^6 x\, dx$. *(ii)* $\int \sec^7 x\, dx$.

From § 9.5, $I_n = \int \sec^n x\, dx = \dfrac{\sec^{n-2} x \tan x}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$.

(i) $\therefore\quad I_6 = \int \sec^6 x\, dx = \dfrac{\sec^4 x \tan x}{5} + \dfrac{4}{5} I_4$;

$$I_4 = \frac{\sec^2 x \tan x}{3} + \frac{2}{3} I_2 ; I_2 = \int \sec^2 x\, dx = \tan x.$$

$$\therefore\quad I_6 = \frac{\sec^4 x \tan x}{5} + \frac{4}{5}\frac{\sec^2 x \tan x}{3} + \frac{2.4}{3.5}\tan x.$$

(ii) Also, $I_7 = \int \sec^7 x\, dx = \dfrac{\sec^5 x \tan x}{6} + \dfrac{5}{6} I_5$;

$$I_5 = \frac{\sec^3 x \tan x}{4} + \frac{3}{4} I_3 \; ; \; I_3 = \frac{\sec x \tan x}{2} + \frac{1}{2} I_1 \; ;$$

$$I_1 = \int \sec x\, dx = \log(\sec x + \tan x).$$

$$I_7 = \frac{\sec^5 x \tan x}{6} + \frac{5}{6}\frac{\sec^3 x \tan x}{4} + \frac{3.5}{4.6}\frac{\sec x \tan x}{2} + \frac{1.3.5}{2.4.6} \log(\sec x + \tan x).$$

Ex. 3. *Obtain a reduction formula for* $\int_0^\infty e^{-ax} \cos^n x\, dx, \quad (a > 0)$

and hence find the value of $\int_0^\infty e^{-4x} \cos^5 x\, dx$.

From § 9.6, replacing a by $-a$,

$$I_n = \int_0^\infty e^{-ax} \cos^n x\, dx$$

$$= \left[\frac{e^{-ax} \cos^{n-1} x(-a \cos x + n \sin x)}{n^2 + a^2} \right]_0^\infty + \frac{n(n-1)}{n^2 + a^2} I_{n-2}$$

$$= \frac{a}{n^2 + a^2} + \frac{n(n-1)}{n^2 + a^2} I_{n-2} \quad [\text{ since } \underset{x \to \infty}{Lt}\, e^{-ax} \to 0 \text{ for } a > 0]$$

is the required reduction formula.

$$\therefore \quad I_5 = \frac{4}{5^2 + 4^2} + \frac{5.4}{5^2 + 4^2} I_3 = \frac{4}{41} + \frac{20}{41} I_3 \; ;$$

$$I_3 = \frac{4}{3^2 + 4^2} + \frac{3.2}{3^2 + 4^2} I_1 = \frac{4}{25} + \frac{6}{25} I_1 \; ;$$

$$I_1 = \frac{4}{1^2 + 4^2} = \frac{4}{17}. \quad \therefore \; I_5 = \frac{708}{3485}$$

Ex. 4. *Obtain a reduction for* $\int \dfrac{dx}{(x^2 + a^2)^{n/2}}$.

Hence find the value of $\int \dfrac{dx}{(x^2 + a^2)^{7/2}}$.

Let $I_n = \int \frac{dx}{(x^2 + a^2)^{n/2}}$. Integrating by parts,

$$I_n = \frac{x}{(x^2 + a^2)^{n/2}} + \frac{n}{2} \int \frac{x}{(x^2 + a^2)^{(n/2)+1}} \, 2x \, dx$$

$$= \frac{x}{(x^2 + a^2)^{n/2}} + n \int \frac{x^2 + a^2 - a^2}{(x^2 + a^2)^{(n/2)+1}} \, dx$$

$$= \frac{x}{(x^2 + a^2)^{n/2}} + nI_n - na^2 I_{n+2}.$$

Changing n to $n - 2$ on both sides,

$$I_{n-2} = \frac{x}{(x^2 + a^2)^{(n-2)/2}} + (n-2) I_{n-2} - (n-2) a^2 I_n.$$

$$\therefore \; I_n = \frac{1}{(n-2)a^2} \frac{x}{(x^2 + a^2)^{(n-2)/2)}} + \frac{n-3}{(n-2)a^2} I_{n-2}.$$

The result can be obtained from § 9.7 by substituting $(n/2)$ in place of n and changing the definition of I_n.

$$\therefore \; I_7 = \int \frac{dx}{(x^2 + a^2)^{7/2}} = \frac{1}{5a^2} \frac{x}{(x^2 + a^2)^{5/2}} + \frac{4}{5a^2} I_5 \; ;$$

$$I_5 = \frac{1}{3a^2} \frac{x}{(x^2 + a^2)^{3/2}} + \frac{2}{3a^2} I_3 \; ; \; I_3 = \frac{1}{a^2} \frac{x}{(x^2 + a^2)^{1/2}}.$$

$$\therefore \; I_7 = \frac{1}{5a^2} \frac{x}{(x^2 + a^2)^{5/2}} + \frac{4}{3.5a^4} \frac{x}{(x^2 + a^2)^{3/2}} + \frac{2.4}{3.5a^6} \frac{x}{(x^2 + a^2)^{1/2}}.$$

Ex. 5. *With the help of a reduction formula, find the value of*

$$\int \frac{\sin^5 x}{\cos^6 x} \, dx.$$

From § 9.16, we get the general form of the reduction formula as

$$I_{m,n} = \int \frac{\sin^m x}{\cos^n x} dx = \frac{1}{n-1} \frac{\sin^{m+1} x}{\cos^{n-1} x} - \frac{m - n + 2}{n - 1} I_{m,n-2} \; [n \neq 1].$$

$$\therefore I_{5,6} = \frac{1}{5} \frac{\sin^6 x}{\cos^5 x} - \frac{1}{5} I_{5,4} \; ; \; I_{5,4} = \frac{1}{3} \frac{\sin^6 x}{\cos^3 x} - \frac{3}{3} I_{5,2} \; ;$$

$$I_{5,2} = \frac{\sin^6 x}{\cos x} - \frac{5}{1} I_{5,0}.$$

Also $I_{5,0} = \int \sin^5 x\, dx$

$$= -\frac{\sin^4 x \cos x}{5} - \frac{4}{5}\frac{\sin^2 x \cos x}{3} - \frac{4}{5}\frac{2}{3} \cdot \cos x$$

[from § 9.3 (i)].

$$\therefore\ I_{5,6} = \frac{1}{5}\frac{\sin^6 x}{\cos^5 x} - \frac{1}{15}\frac{\sin^6 x}{\cos^3 x} + \frac{1}{5}\frac{\sin^6 x}{\cos x} + \frac{1}{5} \cdot \sin^4 x \cos x + \frac{4}{5}\frac{\sin^2 x \cos x}{3} + \frac{4}{5}\frac{2}{3}\cos x.$$

Ex. 6. *From the reduction formula for $\int \cos^m x \cos nx\, dx$ obtain the value of*

$$\int \cos^3 x \cos 5x\, dx.$$

From § 9.18 (i), $I_{m,n} = \int \cos^m x \cos nx\, dx$

$$= \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}\,.$$

Here $m = 3, n = 5$;

$$\therefore I_{3,5} = \int \cos^3 x \cos 5x\, dx = \frac{\cos^3 x \sin 5x}{8} + \frac{3}{8} I_{2,4}\,;$$

$$I_{2,4} = \frac{\cos^2 x \sin 4x}{6} + \frac{2}{6} I_{1,3}\,; I_{1,3} = \frac{\cos x \sin 3x}{4} + \frac{1}{4} I_{0,2}\,;$$

$$I_{0,2} = \int \cos 2x\, dx = \frac{\sin 2x}{2}\,.$$

$$\therefore I_{3,5} = \frac{\cos^3 x \sin 5x}{8} + \frac{\cos^2 x \sin 4x}{16} + \frac{\cos x \sin 3x}{32} + \frac{\sin 2x}{64}$$

Ex. 7. *With the help of the different reduction formulæ for*

$$\int x^m (a + bx^n)^p\, dx, \textit{ find the values of}$$

(i) $\int x^3 (a + bx^2)^4\, dx$. *(ii)* $\int \frac{x^3}{(a+bx^2)^4}\, dx$.

(i) Here $m = 3$, $n = 2$, $p = 4$, and since $p = 4$ is positive,

$\therefore$ (i) can be connected with § 9.20 (1) or (2).

Using (1),

$$I_{3,2,4} = \frac{x^4(a+bx^2)^4}{4} - \frac{2.b.4}{4} I_{5,2,3}.$$

$$I_{5,2,3} = \frac{x^6(a+bx^2)^3}{6} - \frac{2.b.3}{6} I_{7,2,2}.$$

$$I_{7,2,2} = \frac{x^8(a+bx^2)^2}{8} - \frac{2.b.2}{8} I_{9,2,1}.$$

$$I_{9,2,1} = \frac{x^{10}(a+bx^2)}{10} - \frac{2.b}{10} I_{11,2,0}.$$

$$I_{11,2,0} = \int x^{11}\,dx = \frac{x^{12}}{12}.$$

$$I_{3,2,4} = \frac{x^4(a+bx^2)^4}{4} - \frac{bx^6(a+bx^2)^3}{3} + \frac{b^2x^8(a+bx^2)^2}{4} - \frac{b^3x^{10}(a+bx^2)}{10} + \frac{b^4}{5}\frac{x^{12}}{12}.$$

Using § 9.20 (2) the result can be obtained in a different form.

(ii) For this, the suitable formulæ are § 9.20 (3) or (4).

Using (3), replacing p by -4,

$$I_{3,2,4} = -\frac{1}{2a(-3)}\frac{x^4}{(a+bx^2)^3} + \frac{2(-3)+3+1}{2a(-3)} I_{3,2,3}$$

$$= \frac{1}{6a}\frac{x^4}{(a+bx^2)^3} + \frac{1}{3a} I_{3,2,3};$$

$$I_{3,2,3} = -\frac{x^4}{2a(-2)}\frac{1}{(a+bx^2)^2} + \frac{2(-2)+3+1}{2a(-2)} I_{3,2,2}$$

$$= \frac{x^4}{4a(a+bx^2)^2}.$$

$$\therefore I_{3,2,4} = \frac{1}{6a}\frac{x^4}{(a+bx^2)^3} + \frac{1}{12a^2}\frac{x^4}{(a+bx^2)^2}.$$

Ex. 8. *Find the reduction formula for* $\int \frac{x^m\,dx}{(a+2bx+cx^2)^n}$ $(n \neq -1)$, *and hence obtain the value of* $\int_0^2 \frac{x^3\,dx}{(x^2-4x+5)^4}$.

Let $I_{m,n} = \int \frac{x^m\,dx}{(a + 2bx + cx^2)^n}$.

Consider $I_{m-2,n} = \frac{x^{m-2}\,dx}{(a + 2bx + cx^2)^n}$. Integrating by parts,

$$I_{m-2,n} = \frac{x^{m-1}}{(m-1)(a + 2bx + cx^2)^n} + \frac{n}{m-1}\int \frac{x^{m-1}(2cx + 2b)}{(a + 2bx + cx^2)^{n+1}}\,dx$$

$$= \frac{x^{m-1}}{(m-1)(a + 2bx + cx^2)^n} + \frac{n}{m-1}\left\{2c\int \frac{x^m\,dx}{(a + 2bx + cx^2)^{n+1}} + 2b\int \frac{x^{m-1}\,dx}{(a + 2bx + cx^2)^{n+1}}\right\}$$

Changing n to $(n - 1)$ on both sides,

$$I_{m-2,n-1} = \frac{x^{m-1}}{(m-1)(a + 2bx + cx^2)^{n-1}} + \frac{n-1}{m-1}\cdot[2c\,I_{m,n} + 2b\,I_{m-1,n}].$$

Dividing and transposing,

$$I_{m,n} = -\frac{x^{m-1}}{2c(n-1)(a + 2bx + cx^2)^{n-1}} + \frac{m-1}{2c(n-1)}I_{m-2,n-1} - \frac{b}{c}I_{m-1,n}. \qquad \ldots (1)$$

Also, $I_{m-2,n-1} = \int \frac{x^{m-2}\,dx}{(a + 2bx + cx^2)^{n-1}} = \int \frac{x^{m-2}(a + 2bx + cx^2)}{(a + 2bx + cx^2)^n}\,dx$

$$= a\,I_{m-2,n} + 2b\,I_{m-1,n} + c\,I_{m,n}.$$

Substituting and simplifying,

$$I_{m,n} = -\frac{x^{m-1}}{c(2n-m-1)(a + 2bx + cx^2)^{n-1}} + \frac{2b(m-n)}{c(2n-m-1)}I_{m-1,n} + \frac{a(m-1)}{c(2n-m-1)}I_{m-2,n}. \qquad \ldots (2)$$

Either of (1) or (2) may be regarded as a reduction formula.

Hence, using (2), ($a = 5$, $b = -2$, $c = 1$, here).

$$I_{3,4} = \left[-\frac{x^2}{4(x^2-4x+5)^3} \right]_0^2 + \frac{-4(-1)}{4} I_{2,4} + \frac{5.2}{4} I_{1,4},$$

$$I_{2,4} = \left[-\frac{x}{5(x^2-4x+5)^3} \right]_0^2 + \frac{-4(-2)}{5} I_{1,4} + \frac{5}{5} I_{0,4},$$

$$I_{1,4} = \left[-\frac{1}{6(x^2-4x+5)^3} \right]_0^2 + \frac{-4(-3)}{6} I_{0,4},$$

$$I_{0,4} = \int_0^2 \frac{dx}{(x^2-4x+5)^4} = \int_0^2 \frac{dx}{[(x-2)^2+1]^4}$$

$$= \int_0^2 \frac{dz}{(z^2+1)^4} \quad [\text{putting } z = 2 - x]$$

$$= \frac{433}{3000} + \frac{5}{16}\tan^{-1} 2 \; [\text{using § 9.9 successively}] = \lambda \text{ (say)}.$$

Then,
$$I_{1,4} = -\frac{124}{6.5^3} + 2\lambda,$$

$$I_{2,4} = -\frac{2}{5} - \frac{124 \times 4}{3.5^4} + \frac{21}{5}\lambda,$$

$$I_{3,4} = -\frac{3896}{3.5^4} + \frac{46}{5}\lambda$$

$$= -\frac{3896}{3.5^4} + \frac{46 \times 433}{3.5^4.8} + \frac{46}{16}\tan^{-1} 2$$

$$= -\frac{3}{4} + \frac{23}{8}\tan^{-1} 2.$$

Ex. 9. *If* $u_n = \int_0^{\pi/2} x^n \sin x \, dx \; (n > 0)$, *then prove that*

$$u_n + n(n-1)u_{n-2} = n\left(\tfrac{1}{2}\pi\right)^{n-1}$$

Integrating by parts,

$$u_n = \left[-x^n \cos x \right]_0^{\pi/2} + n\int_0^{\pi/2} x^{n-1} \cos x \, dx$$

$$= n\left\{\left[x^{n-1}\sin x\right]_0^{\pi/2} - (n-1)\int_0^{\pi/2} x^{n-2}\sin x\,dx\right\}$$

$$= n\left(\tfrac{1}{2}\pi\right)^{n-1} - n(n-1)\,u_{n-2}\,.$$

$$\therefore\quad u_n + n(n-1)\,u_{n-2} = n\left(\tfrac{1}{2}\pi\right)^{n-1}.$$

Ex. 10. *If* $S_n = \int_0^{\frac{1}{2}\pi} \frac{\sin(2n-1)x}{\sin x}\,dx$, $V_n = \int_0^{\frac{1}{2}\pi} \left(\frac{\sin nx}{\sin x}\right)^2 dx$,

being an integer, then show that

$$S_{n+1} = S_n = \tfrac{1}{2}\pi,\quad V_{n+1} - V_n = S_{n+1}\,.$$

Obtain the value of V_n .

Here $S_{n+1} - S_n = \int_0^{\pi/2} \frac{\sin(2n+1)x - \sin(2n-1)x}{\sin x}\,dx$

$$= \int_0^{\pi/2} \frac{2.\cos 2nx.\sin x}{\sin x}\,dx = 2\int_0^{\pi/2} \cos 2nx\,dx$$

$$= 2\left[\frac{\sin 2nx}{2n}\right]_0^{\pi/2} = 0 \text{ for all integral values of } n.$$

$$\therefore\quad S_{n+1} = S_n = S_{n-1} \ldots\ldots\ldots = S_1\,.$$

Now, $S_1 = \int_0^{\pi/2} \frac{\sin x}{\sin x}\,dx = \int_0^{\pi/2} dx = \frac{\pi}{2}$.

$$\therefore\quad S_{n+1} = S_n = \tfrac{1}{2}\pi\,.$$

Also, $V_{n+1} - V_n = \int_0^{\pi/2} \frac{\sin^2(n+1)x - \sin^2 nx}{\sin^2 x}\,dx$

$$= \int_0^{\pi/2} \frac{\sin(2n+1)x\,.\sin x}{\sin^2 x}\,dx$$

$$= \int_0^{\pi/2} \frac{\sin(2n+1)x}{\sin x}\,dx = S_{n+1}$$

$\therefore V_n - V_{n-1} = S_n = \frac{1}{2}\pi,\ V_{n-1} - V_{n-2} = \frac{1}{2}\pi, \ldots, V_2 - V_1 = \frac{1}{2}\pi.$

$\therefore$ adding, $V_n - V_1 = (n-1)\pi/2$.

Since $V_1 = \int_0^{\pi/2} dx = \frac{1}{2}\pi, \qquad \therefore V_n = \frac{1}{2}n\pi$.

Ex. 11. *Show that*

(i) $\Gamma(\frac{7}{2}) = \frac{15}{8}\sqrt{\pi}$; (ii) $\Gamma(\frac{1}{3})\Gamma(\frac{2}{3}) = \frac{2}{\sqrt{3}}\pi$;

(iii) $\int_0^{\frac{1}{2}\pi} \sin^4\theta \cos^6\theta \, d\theta = \int_0^{\frac{1}{2}\pi} \sin^6\theta \cos^4\theta \, d\theta = \frac{3}{512}\pi$.

(i) $\Gamma(\frac{7}{2}) = \Gamma(\frac{5}{2}+1) = \frac{5}{2}\Gamma(\frac{5}{2})$

[since $\Gamma(n+1) = n\Gamma(n)$, *Art. 9.21 (iii)*]

$= \frac{5}{2}\Gamma(\frac{3}{2}+1) = \frac{5}{2}\frac{3}{2}\Gamma(\frac{3}{2}) = \frac{5}{2}\frac{3}{2}\Gamma(\frac{1}{2}+1) = \frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})$

$= \frac{15}{8}\sqrt{\pi}$. [*by Art. 9.21 (vii)*].

(ii) Left side $= \Gamma(\frac{1}{3})\Gamma(1-\frac{1}{3}) = \frac{\pi}{\sin\frac{1}{3}\pi}$ [*by Art. 9.21 (vi)*]

(iii) By Art. 9.21 (A) (1), $= \frac{2}{\sqrt{3}}\pi$.

First integral $= \frac{1}{2}\cdot\frac{\Gamma(\frac{5}{2})\Gamma(\frac{7}{2})}{\Gamma(6)} = \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{5!} = \frac{3}{512}\pi$.

By Art.6.19 (iv), Second Integral = First Integral.

Ex. 12. *Show that* $\Gamma(n+\frac{1}{2}) = \frac{\Gamma(2n+1)\sqrt{\pi}}{2^{2n}\Gamma(n+1)}$. [*C. H. '85*]

$\Gamma(n+\frac{1}{2}) = \Gamma\left(\frac{2n+1}{2}\right) = \Gamma\left(\frac{2n-1}{2}+1\right)$

$= \frac{2n-1}{2}\Gamma\left(\frac{2n-1}{2}\right)$ [*by Art. 9.21 (iii)*]

$= \frac{2n-1}{2}\Gamma\left(\frac{2n-3}{2}+1\right)$

$= \frac{2n-1}{2}\frac{2n-3}{2}\Gamma\left(\frac{2n-3}{2}\right)$

$$= \frac{2n-1}{2}\,\frac{2n-3}{2}\,\frac{2n-5}{2}\,\ldots\ldots\frac{5}{2}\,\frac{3}{2}\,\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)$$

[*By repeated application of the result of the above Article.*]

$$= \frac{(2n-1)(2n-3)(2n-5)\ldots\ldots 5.3.1}{2^n}\sqrt{\pi} \quad \ldots \quad (1)$$

Now multiply the numerator and denominator of (1) by

$2n(2n-2)(2n-4)\ldots 4.2.$

$$\therefore \Gamma(n+\tfrac{1}{2}) = \frac{2n(2n-1)(2n-2)(2n-3)\ldots\ldots 5.4.3.2.1}{2^n.2.n.2(n-1).2(n-2).\ \ldots 2.2.2.1}\sqrt{\pi}$$

$$= \frac{\Gamma(2n+1)}{2^n.2^n\,n(n-1)(n-2)\ldots\ldots 2.1}\sqrt{\pi}$$

$$= \frac{\Gamma(2n+1)\sqrt{\pi}}{2^{2n}.\Gamma(n+1)}.$$

Note. The above result can be written in the form

$$\Gamma(\tfrac{1}{2})\Gamma(2n) = 2^{2n-1}\Gamma(n)\Gamma(n+\tfrac{1}{2})$$

It is an important result often used in Higher Mathematics.

Note. 2. The right side of (1) can be written as $(\frac{1}{2})_n\ \Gamma(\frac{1}{2})$ where the notation $(a)_n$ denotes $a(a+1)(a+2)\ldots(a+n-1)$

$$\therefore \Gamma(n+\tfrac{1}{2}) = (\tfrac{1}{2})_n\,\Gamma(\tfrac{1}{2}).$$

Ex. 13. *Show that* $B(m,n)B(m+n,l) = B(n,l)B(n+l,m)$

Left side $= \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\,\dfrac{\Gamma(m+n)\Gamma(l)}{\Gamma(l+m+n)} = \dfrac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n)}$

Similarly, right side $= \dfrac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n)}$

Hence the result.

Ex. 14. *Evaluate*

$$\int_0^t x^{\alpha+k-1}\,(t-x)^{\beta+k-1}\,dx$$

and find its value when $\alpha = \beta = \frac{1}{2}$

Put $x = ty$ $\therefore dx = t\,dy$, when $x = 0$, $y = 0$, $x = t$ $y = 1$.

$$\therefore I = \int_0^1 t^{\alpha+\beta+2k-1}\ y^{\alpha+k-1}(1-y)^{\beta+k-1}\,dy$$

$$= t^{\alpha+\beta+2k-1}\ \frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{\Gamma(\alpha+\beta+2k)}$$

When $\alpha = \beta = \frac{1}{2}$,

$$I = t^{2k}\ \frac{\Gamma(k+\frac{1}{2})\Gamma(k+\frac{1}{2})}{\Gamma(2k+1)}$$

$$= t^{2k}\ \frac{\Gamma(2k+1)\sqrt{\pi}\,(\frac{1}{2})_k\,\Gamma(\frac{1}{2})}{2^{2k}\,\Gamma(2k+1)\Gamma(k+1)}$$

[*by Ex. 12 and Note (2) of Art 9.23*]

$$= t^{2k}\ \frac{(\frac{1}{2})_k\,\pi}{2^{2k}\,k!}$$

EXAMPLES IX

1. Obtain a reduction formula for $\int x^n e^{ax}\,dx$, $(n \neq -1)$ and hence find the value of $\int x^4 e^{ax}\,dx$.

2. Show that $\int x^3 e^{ax}\,dx = \dfrac{e^{ax}}{a^4}(a^3x^3 - 3a^2x^2 + 6ax - 6)$.

3. Find the reduction formulæ for

(i) $\int \cot^n x\,dx$. (ii) $\int \operatorname{cosec}^n x\,dx$.

4. If $I_n = \int \sinh^n\theta\,d\theta$, then show that

$$nI_n = \sinh^{n-1}\theta\cosh\theta - (n-1)\,I_{n-2}.$$

5. Obtain the reduction formulæ for

(i) $\int \tanh^n\theta\,d\theta$. (ii) $\int \operatorname{sech}^n\theta\,d\theta$.

6. Show that if $I_n = \int e^{ax}\sin^n bx\,dx$, then

$$I_n = \frac{a\sin bx - nb\cos bx}{a^2+n^2b^2}\,e^{ax}\sin^{n-1}bx + \frac{n(n-1)b^2}{a^2+n^2b^2}\,I_{n-2}.$$

7. If $I_n = \int x^n \cos bx\, dx$ and $J_n = \int x^n \sin bx\, dx$, then show that

(i) $bI_n = x^n \sin bx - nJ_{n-1}$.

(ii) $bJ_n = -x^n \cos bx + nI_{n-1}$;

(iii) $b^2 I_n = x^{n-1}(bx \sin bx + n \cos bx) - n(n-1) I_{n-2}$.

(iv) $b^2 J_n = x^{n-1}(n \sin bx - bx \cos bx) - n(n-1) J_{n-2}$.

8. Find the values of the integrals :

(i) $\int (x^2 - 6x + 7)^5\, dx$. (ii) $\int \frac{dx}{(x^2+1)^4}$.

(iii) $\int \frac{dx}{(x^2+x+1)^3}$. (iv) $\int \frac{x^3\, dx}{\sqrt{(x^2-2x+2)}}$.

9. Show that

$$I_n = \int (a^2 + x^2)^{n/2}\, dx = \frac{x(a^2+x^2)^{n/2}}{n+1} + \frac{na^2}{n+1} I_{n-2} ;$$

find also I_3.

10. If $I_n = \int (1+x^2)^n e^{ax}\, dx \; (n > 1)$, deduce that

$$I_n = \frac{1}{a} e^{ax}(1+x^2)^n - \frac{2nx}{a^2} e^{ax}(1+x^2)^{n-1} + \frac{2n(2n-1)}{a^2} I_{n-1} - \frac{4n(n-1)}{a^2} I_{n-2}.$$

11. Show that if $u_n = \int x^n \sqrt{a^2 - x^2}\, dx$, then

$$u_n = -\frac{x^{n-1}(a^2-x^2)^{3/2}}{n+2} + \frac{n-1}{n+2} a^2 u_{n-2}.$$

12. Find the reduction formulæ for

(i) $\int \frac{x^n\, dx}{\sqrt{(2ax - x^2)}}$; (ii) $\int \frac{dx}{x^n \sqrt{(x^2-1)}}$.

13. If $I_n = \int x^n \sqrt{a-x}\, dx$, then prove that

$$(2n+3) I_n = 2an\, I_{n-1} - 2x^n (a-x)^{3/2}.$$

Hence, evaluate $\int_0^a x^3 \sqrt{ax - x^2}\, dx$.

14. If $u_n = \int \frac{x^n\, dx}{\sqrt{(ax^2 + bx + c)}}$, then show that

$$(n+1)\,au_{n+1} + \tfrac{1}{2}(2n+1)\,bu_n + ncu_{n-1} = x^n \sqrt{ax^2 + bx + c}$$

15. If $I_n = \int (\sin x + \cos x)^n dx$, then show that

$$nI_n = -(\sin x + \cos x)^{n-2}.\cos 2x + 2(n - 1)\,I_{n-2}$$

16. Show that

(i) $I_n = \int_0^\infty \frac{dx}{(1 + x^2)^n} = \frac{2n - 3}{2n - 2}\, I_{n-1}$.

(ii) $\int_0^\infty \frac{dx}{(1 + x^2)^5} = \frac{1.3.5.7}{2.4.6.8}\,\frac{\pi}{2}$

17. Show that, if $I_n = \int_0^{\pi/2} \cos^n x\, dx$ and $J_n = \int_0^{\pi/2} \sin^n x\, dx$

(i) $I_n = J_n$. (ii) $I_n = \frac{n - 1}{n}\, I_{n-2}\ (n > 2)$.

18. With a suitable substitution, using the previous example find the values of

(i) $\int_0^1 \frac{x^n}{\sqrt{(1 - x^2)}}\, dx$; (ii) $\int_0^\infty \frac{dx}{(1 + x^2)^n}$.

(n being a positive integer)

19. Prove that if $u_n = \int_0^1 x^n \tan^{-1} x\, dx$, then

$$(n + 1)\,u_n + (n - 1)\,u_{n-2} = \frac{\pi}{2} - \frac{1}{n}.$$

20. If $n \geq 2$ and $I_n = \int_{-1}^{+1} (1 - x^2)^n \cos mx \, dx$, then show that $m^2 I_n = 2n(2n - 1) I_{n-1} - 4n(n-1) I_{n-2}$.

21. If $U_n = \int_0^{\pi/2} \theta \sin^n \theta \, d\theta$ and $n > 1$, then prove that

$$U_n = \frac{n-1}{n} U_{n-2} + \frac{1}{n^2}.$$

22. (i) Obtain a reduction formula for $\int \frac{dx}{(1+x^2)^n \sqrt{(1+x^2)}}$

and (ii) evaluate $\int_0^{\infty} \frac{dx}{(1+x^2)^n \sqrt{(1+x^2)}}$. [Put $x = \tan \theta$.]

23. If $\phi(n) = \int_0^{\infty} e^{-x} x^{n-1} \log x \, dx$, then show that

$$\phi(n+2) - (2n+1)\phi(n+1) + n^2 \phi(n) = 0.$$

24. If $I_n = \int_0^{\pi/4} \tan^n \theta \, d\theta$, then prove that

$$n(I_{n+1} + I_{n-1}) = 1.$$

25. Show that $\int_0^1 x^{a-1} (\log x)^n \, dx = \frac{(-1)^n n!}{a^{n+1}}$

26. If $\beta_{m,n} = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx$, then show that

$$\beta_{m,n} = \beta_{n,m} = \frac{(m-1)!(n-1)!}{(m+n-1)!}$$

(m and n being integers, each > 1).

27. If m, n are positive integers, then show that

$$I_{m,n} = \int_a^b (x-a)^m (b-x)^n \, dx = \frac{n(b-a)}{m+n+1} I_{m,n-1}.$$

Hence prove that $I_{m,n} = \dfrac{m!\,n!\,(b-a)^{m+n+1}}{(m+n+1)!}$.

28. Find the values of

(i) $\displaystyle\int_0^{\pi/2} \sin^6 x \cos^4 x \, dx$ (ii) $\displaystyle\int_0^{\pi/2} \sin^{3/2} x \cos^3 x \, dx$.

(iii) $\displaystyle\int \frac{\cos^5 x}{\sin^4 x} \, dx$. (iv) $\displaystyle\int \frac{dx}{\sin^{5/2} x \cos^{3/2} x}$

29. If $I_{m,n} = \int \cos^m x \sin^n x \, dx$, then show that

$$(m+n)(m+n-2) I_{m,n}$$
$$= \{(n-1)\sin^2 x - (m-1)\cos^2 x\}\cos^{m-1} x \sin^{n-1} x$$
$$+ (m-1)(n-1) I_{m-2,n-2}.$$

30. Obtain a reduction formula for

$I_{m,n} = \int \cos^m x \sin nx \, dx$, and deduce the value of

$$\int_0^{\frac{1}{2}\pi} \cos^5 x \sin 3x \, dx.$$

31. If $I_{m,n} = \int \sin^m x \cos nx \, dx$, then show that

$$I_{m,n} = \frac{m \cos x \cos nx + n \sin x \sin nx}{n^2 - m^2} \sin^{m-1} x$$
$$- \frac{m(m-1)}{n^2 - m^2} I_{m-2,n}$$

32. If $I_{m,n} = \displaystyle\int_0^{\pi/2} \sin^m x \cos nx \, dx$ and

$$J_{m,n} = \int_0^{\pi/2} \sin^m x \sin nx \, dx \text{, then show that}$$

$$(m+n) I_{m,n} = \sin \tfrac{1}{2} n\pi - m J_{m-1,n-1} \; (m > 1).$$

33. If $f(m,n) = \int_0^{\pi/2} \cos^m x \cos nx \, dx$, then show that

$$f(m,n) = \frac{m}{m+n} f(m-1, n-1) = \frac{m(m-1)}{m^2-n^2} f(m-2, n)$$

$$= \frac{m}{m-n} f(m-1, n+1),$$

and hence show that $f(m,m) = \dfrac{\pi}{2^{m+1}}$.

34. Obtain a reduction formula for $\int \dfrac{dx}{(a+b\sin x)^n}$.

35. Find the values of

(i) $\int_0^{\pi} \dfrac{dx}{(1+\cos\alpha\cos x)^3}$. (ii) $\int \dfrac{dx}{(1+k\sin x)^2}$ $(k<1)$.

36. Using the integral $\int x^m (a+bx^n)^p \, dx$, find the values of

(i) $\int x^5 (1+x^2)^{7/2} \, dx$. (ii) $\int \dfrac{x^5}{(1+2x^3)^{1/2}} \, dx$.

[*Use* § 9.20 (5).] [*Use* § 9.20 (4).]

(iii) $\int_{\frac{1}{2}}^{1} \dfrac{dx}{x^4 \sqrt{(1-x^2)}}$. [*Use* § 9.20 (6).]

37. Find the reduction formula for $\int x^m \sqrt{2ax - x^2} \, dx$.

Hence show that $\int_0^{2a} x^m \sqrt{2ax-x^2} \, dx = \pi \dfrac{a^{m+2}}{2^m} \dfrac{(2m+1)!}{(m+2)! \, m!}$.

38. If $I_m = \int_0^\infty x^m e^{-x} \cos x \, dx$ and

$$J_m = \int_0^\infty x^m e^{-x} \sin x \, dx,$$

then prove that (m being an integer > 1)

(i) $I_m = \frac{1}{2} m (I_{m-1} - J_{m-1})$. (ii) $J_m = \frac{1}{2} m (I_{m-1} + J_{m-1})$.

(iii) $I_m - mI_{m-1} + \frac{1}{2} m (m - 1) I_{m-2} = 0$.

39. Show that $\int_0^{\frac{1}{2}\pi} \sin 2nx \cot x \, dx = \frac{1}{2}\pi$.

40. (i) If $u_n = \int \cos n\theta \operatorname{cosec} \theta \, d\theta$, then show that

$$u_n - u_{n-2} = \frac{2 \cos (n - 1) \theta}{n - 1}.$$

(ii) If $P_n = \int \frac{\sin (2n - 1) x}{\sin x} dx$, $Q_n = \int \frac{\sin^2 nx}{\sin^2 x} dx$,

then show that $n (P_{n+1} - P_n) = \sin 2nx$

and $Q_{n+1} - Q_n = P_{n+1}$.

41. Prove that, if

$$J_n = \int_0^\pi \frac{1 - \cos nx}{1 - \cos x} dx, \text{ where } n \text{ is a positive integer}$$

or zero, then $J_{n+2} + J_n = 2J_{n+1}$.

Hence prove that $\int_0^{\pi/2} \frac{\sin^2 n\theta}{\sin^2 \theta} d\theta = \frac{n\pi}{2}$.

42. (i) Prove that $\int_0^\pi \frac{\sin n\theta}{\sin \theta} d\theta = 0$ or π according as n is an even or odd positive integer.

(ii) By means of a reduction formula or otherwise, prove that

$$\int_0^{\pi} \frac{\sin^2 n\theta}{\sin^2 \theta}\, dx = n\pi \text{, } n \text{ being a positive integer.}$$

43. Show that if n is a positive integer, then

$$\int_0^{2\pi} \frac{\cos (n-1) x - \cos nx}{1 - \cos x}\, dx = 2\pi$$

and deduce that $\displaystyle\int_0^{2\pi} \left(\frac{\sin \frac{1}{2} nx}{\sin \frac{1}{2} x} \right)^2 dx = 2n\pi$.

44. If $I_{m,n} = \displaystyle\int_0^{\pi/2} \cos^m x \sin nx\, dx$, then show that

$$I_{m,m} = \frac{1}{2^{m+1}} \left[2 + \frac{2^2}{2} + \frac{2^3}{3} + \ldots + \frac{2^m}{m} \right].$$

45. Show that $\displaystyle\int_0^{\infty} e^{-ax} \sin^n x\, dx$

$$= \frac{n(n-1)(n-2) \ldots 3.2}{(a^2+n^2)\{a^2+(n-2)^2\} \ldots (a^2+3^2)} \frac{1}{a^2+1}$$

if n is odd ;

$$= \frac{n(n-1)(n-2) \ldots 2.1}{(a^2+n^2)\{a^2+(n-2)^2\} \ldots (a^2+2^2)} \frac{1}{a}$$

if n is even .

46. If $I_n = \displaystyle\int_0^{\pi/2} (a \cos \theta + b \sin \theta)^n\, d\theta$, then prove that

$$nI_n = ab(a^{n-2} + b^{n-2}) + (n-1)(a^2+b^2) I_{n-2} .$$

47. If $I_n = \int (a\cos^2 x + 2h\sin x\cos x + b\sin^2 x)^{-n}\,dx$, then prove that

$$4(n+1)(ab-h^2)I_{n+2} - 2(2n+1)(a+b)I_{n+1} + 4nI_n$$

$$= 2\,\frac{h(\cos^2 x - \sin^2 x) + (b-a)\sin x\cos x}{(a\cos^2 x + 2h\sin x\cos x + b\sin^2 x)^{n+1}}.$$

[*Apply the alternative method of* § 9.19.]

48. Show that

$$\text{(i)} \int_{-1}^{1} (1+x)^p(1-x)^q\,dx = 2^{p+q+1}\,\frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}$$

[$p > -1,\ q > -1$]

[*Put* $1 + x = 2y$.]

$$\text{(ii)} \int_a^b (x-a)^m(b-x)^n\,dx = (b-a)^{m+n+1}\,\frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}.$$

[$m > -1,\ n > -1$]

[*Put* $x - a = (b-a)y$.]

49. Show that

$$\int_0^\infty e^{-x^2}x^\alpha\,dx = \tfrac{1}{2}\Gamma\left(\frac{\alpha+1}{2}\right).\ [\alpha > -1]$$

[*Put* $x^2 = y$.]

50. Show that

$$\int_0^\infty e^{-x^4}x^2\,dx \times \int_0^\infty e^{-x^4}\,dx = \frac{\pi}{8\sqrt{2}}.$$

[*Put* $x^4 = z$.]

51. Show that

$$B(m,n)\,B(m+n,l) = B(n,l)\,B(n+l,m)$$
$$= B(l,m)\,B(l+m,n).$$

52. Show that

$$\Gamma\left(\tfrac{1}{9}\right)\Gamma\left(\tfrac{2}{9}\right)\ldots\ldots\Gamma\left(\tfrac{8}{9}\right)=\tfrac{3}{16}\pi^4$$

[*Combine 1st and last factor, 2nd and last but one etc. and apply formula (vi), § 9.21.*]

53. Show that

$$\int_0^1 \frac{dx}{(1-x^6)^{1/6}}=\frac{\pi}{3}. \quad [\textit{ Put } x^6 = z.]$$

54. Show that the sum of the series

$$\frac{1}{n+1}+m\frac{1}{n+2}+\frac{m(m+1)}{2!}\frac{1}{n+3}$$

$$+\frac{m(m+1)(m+2)}{3!}\frac{1}{n+4}+\ldots\text{ to }\infty$$

$$=\frac{\Gamma(n+1)\Gamma(1-m)}{\Gamma(n-m+2)}, \text{ where } n>-1 \text{ and } m<1.$$

$$\left[R.S.=B(n+1,1-m)=\int_0^1 x^n(1-x)^{-m}\,dx, \text{ etc.}\right]$$

55. Show that

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^{2m-1}\theta\cos^{2n-1}\theta}{(a\sin^2\theta+b\cos^2\theta)^{m+n}}\,d\theta=\frac{1}{2}\frac{\Gamma(m)\Gamma(n)}{a^m b^n\Gamma(m+n)}.$$

[*Apply Art. 9.21 (viii).*]

ANSWERS

1. $I_n=-\dfrac{x^n e^{-ax}}{a}+\dfrac{n}{a}I_{n-1}$.

$$I_4=-\frac{e^{-ax}}{a^5}[x^4a^4+4x^3a^3+12x^2a^2+24xa+24].$$

3. (i) $I_n=-\dfrac{\cot^{n-1}x}{n-1}-I_{n-2}$. (ii) $I_n=-\dfrac{\cot x\operatorname{cosec}^{n-2}x}{n-1}+\dfrac{n-2}{n-1}I_{n-2}$.

5. (i) $I_n = -\frac{\tanh^{n-1}\theta}{n-1} + I_{n-2}$. (ii) $I_n = \frac{\text{sech}^{n-2}\theta \tanh\theta}{n-1} + \frac{n-2}{n-1} I_{n-2}$.

8. (i) $(x-3)\left[\frac{(x^2-6x+7)^5}{11} - \frac{20}{11.9}(x^2-6x+7)^4 + \frac{20.16}{11.9.7}(x^2-6x+7)^3\right.$

$\left. - \frac{20.16.12}{11.9.7.5}(x^2-6x+7)^2 + \frac{20.16.12.8}{20.9.7.5.3}(x^2-6x+7) - \frac{20.16.12.8.4}{11.9.7.5.3}\right]$.

(ii) $\frac{x}{6(x^2+1)^3} + \frac{5x}{24(x^2+1)^2} + \frac{5x}{16(x^2+1)} + \frac{5}{16}\tan^{-1}x$.

(iii) $\frac{2x+1}{6(x^2+x+1)^2} + \frac{2x+1}{3(x^2+x+1)} + \frac{4}{3\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$.

(iv) $\frac{2x^2+5x+7}{6}\sqrt{x^2-2x+2} - \frac{1}{2}\sinh^{-1}(x-1)$.

9. $\frac{1}{4}x(a^2+x^2)^{3/2} + \frac{3}{8}a^2x(a^2+x^2)^{1/2}$

$+ \frac{3}{8}a^4\log(x+\sqrt{a^2+x^2})$.

12. (i) $nI_n = -x^{n-1}\sqrt{2ax-x^2} + (2n-1)aI_{n-1}$.

(ii) $I_n = \frac{\sqrt{x^2-1}}{(n-1)x^{n-1}} + \frac{n-2}{n-1}I_{n-2}$. 13. $\frac{7\pi a^5}{256}$.

18. (i) $\frac{n-1}{n}\frac{n-3}{n-2}\cdots\cdots\frac{2}{3}$, if n is odd

and $\frac{n-1}{n}\frac{n-3}{n-2}\cdots\cdots\frac{1}{2}\frac{\pi}{2}$, if n is even.

(ii) $\frac{2n-3}{2n-2}\frac{2n-5}{2n-4}\cdots\cdots\frac{1}{2}\frac{\pi}{2}$, if $n > 1$

and $\frac{\pi}{2}$, if $n = 1$.

22. (i) $I_n = \frac{1}{2n-1}\frac{x}{(1+x^2)^{n-1}\sqrt{(1+x^2)}} + \frac{2n-2}{2n-1}I_{n-1}$.

(ii) $\frac{2n-2}{2n-1}\frac{2n-4}{2n-3}\cdots\cdots\frac{2}{3}$.

28. (i) $\frac{5\pi}{4096}$. (ii) $\frac{8}{45}$. (iii) $-\frac{\cos^4 x}{3\sin^3 x} + \frac{4\cos^2 x}{3\sin x} + \frac{8\sin x}{3}$

(iv) $2\left[\frac{1}{5}\tan^{5/2}x + 2\tan^{1/2}x - \frac{1}{3}\cot^{3/2}x\right]$.

30. $I_{m,n} = \dfrac{-\cos^m x \cos nx}{m+n} + \dfrac{m}{m+n} I_{m-1,n-1}$; $\dfrac{1}{3}$

34. $(n-1)(a^2-b^2) I_n = \dfrac{b \cos x}{(a+b\sin x)^{n-1}} + (2n-3)\, a\, I_{n-1} - (n-2)\, I_{n-2}$

35. (i) $\dfrac{\pi}{2}(2+\cos^2\alpha)\operatorname{cosec}^5\alpha$.

(ii) $\dfrac{1}{1-k^2}\dfrac{k\cos x}{1+k\sin x} + \dfrac{2}{(1-k^2)^{3/2}}\tan^{-1}\left\{\dfrac{\tan\frac{1}{2}x+k}{\sqrt{(1-k^2)}}\right\}$.

36. (i) $\dfrac{(1+x^2)^{9/2}}{9\cdot 11\cdot 13}\left[99x^4-36x^2+8\right]$.

(ii) $\frac{1}{9}(1+2x^3)^{1/2}(x^3-1)$. (iii) $2\sqrt{3}$.

37. $I_m = -\dfrac{x^{m-1}(2ax-x^2)^{3/2}}{m+2} + \dfrac{(2m+1)a}{m+2} I_{m-1}$.

CHAPTER X

AREAS OF PLANE CURVES

[*Quadrature* *]

10.1. Areas in Cartesian Co-ordinates.

Suppose we want to determine the area A_1 bounded by the curve $y = f(x)$, the x-axis and two fixed ordinates $x = a$ and $x = b$ The function $f(x)$ is supposed to be single-valued, finite and continuous in the interval (a, b).

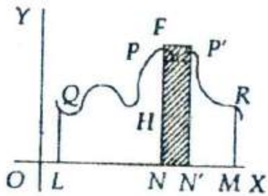

Fig.1

Consider the variable area $QLNP = A$, say, bounded by the curve $y = f(x)$, the x-axis, the fixed ordinate $\overline{QL}$ where $OL = a$ and a variable ordinate $\overline{PN}$ where $ON = x$. Clearly, A has a definite value for each value of x and is thus a function of x. When x is increased by an amount Δx ($= NN'$), A assumes an increment ΔA = the area $PNN'P'$. Now, if $f(x_1)$ and $f(x_2)$ be the greatest and the least ordinates in the interval Δx,

such that $x \le x_1 \le x + \Delta x$, $x \le x_2 \le x + \Delta x$,

clearly the area ΔA lies between the inscribed and circumscribed rectangles HN' and FN'

$$i.e., \quad f(x_2)\,\Delta x < \Delta A < f(x_1)\,\Delta x.$$

$$\therefore \quad f(x_2) < \frac{\Delta A}{\Delta x} < f(x_1). \qquad \ldots (1)$$

* The process of finding the area bounded by any defined contour line is called *Quadrature*, the term meaning 'the investigation of the size of a square which shall have the same area as that of the region under consideration'

Now, as Δx approaches zero, by the continuity of the function $f(x)$ at x $f(x_1)$ and $f(x_2)$ both approach $f(x)$, and also $\frac{\Delta A}{\Delta x}$ tends to $\frac{dA}{dx}$. Hence, as the relation (1) is always true, we get in the limit

$$\frac{dA}{dx} = f(x).$$

$\therefore$ by definition, $A = \int f(x)\,dx + C \equiv F(x) + C$, where C is an arbitrary constant, and $F(x)$ an indefinite integral of $f(x)$. Now, when $x = a$, $\overline{PN}$ coincides with $\overline{QL}$, and the area becomes zero. Also, when $x = b$, the area A becomes the required area A_1.

$$\therefore\ 0 = F(a) + C \text{ and } A_1 = F(b) + C.$$

$$\therefore\ A_1 = F(b) - F(a) = \int_a^b f(x)\,dx.$$

The definite integral

$$\int_a^b f(x)\,dx\text{, i.e., } \int_a^b y\,dx$$

therefore, represents the area bounded by the curve $y = f(x)$, the x-axis and the two fixed ordinates $x = a$ and $x = b$.

Note. An alternative method of proof of the above result, depending on the definition of a definite integral as a summation, has been given in Art. 6.11.

Cor. 1. In the same way, it can be shown that the *area bounded by any curve, two given abscissæ ($y = c$, $y = d$) and the y-axis is*

$$\int_c^d x\,dy.$$

Cor. 2. *If the axes be oblique,* ω being the angle between them, the corresponding formula for the areas would be

$$\sin\omega \int_a^b y\,dx \text{ and } \sin\omega \int_c^d x\,dy \text{ respectively.}$$

10.2. Illustrative Examples.

Ex. 1. *Find the area of the quadrant of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *between the major and minor axes.*

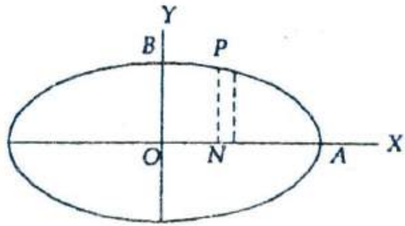

Fig.2

Clearly, the area being bounded by the curve, the x-axis and the ordinates $x = 0$ and $x = a$, the required area

$$=\int_0^a y\,dx$$

$$=\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx \qquad \left[\text{since } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ for the curve}\right]$$

$$=\frac{b}{a}\int_0^{\pi/2} a\cos\theta \,.\, a\cos\theta\, d\theta \text{ (putting } x = a\sin\theta \text{)}$$

$$=\frac{ab}{2}\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta=\frac{ab}{2}\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$=\frac{ab}{2}\cdot\frac{\pi}{2}=\frac{1}{4}\pi ab.$$

Cor. 1. The area of the whole ellipse is clearly four times the above, *i.e.*, $=\pi ab$.

Cor. 2. Putting $b = a$ and proceeding exactly as before, the area of a quadrant of the circle $x^2+y^2=a^2$ is $\frac{1}{4}\pi a^2$, and the area of the whole circle $=\pi a^2$.

Ex. 2. *Determine the area bounded by the parabola $y^2 = 4ax$ and any double ordinate of it, say, $x = x_1$.*

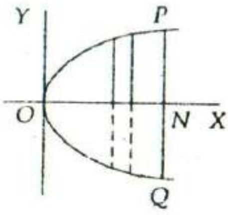

Fig.3

The area OPN is bounded by the curve $y^2 = 4ax$, the x-axis and the two ordinates $x = 0$ and $x = x_1$.

$$\therefore \text{ area } OPN = \int_0^{x_1} y\, dx = \int_0^{x_1} \sqrt{4ax}\, dx$$

[The *positive value of* y is taken since we are considering the positive side of the y-axis.]

$$= \sqrt{4a} \left[\tfrac{2}{3} x^{3/2} \right]_0^{x_1}$$

$$= \sqrt{4a} \cdot \tfrac{2}{3} x_1^{3/2} = \tfrac{2}{3} x_1 y_1 \qquad (\text{ where } y_1 = PN = \sqrt{4ax_1}\ ).$$

The parabola being symmetrical about the x-axis, the required area POQ

$= 2 \cdot \frac{2}{3} x_1 y_1 = \frac{4}{3} x_1 y_1$

$= \frac{2}{3}$ the area of the rectangle contained by PQ and ON,

i.e., $= \frac{2}{3}$ the area of the circumscribed rectangle.

Cor. The area bounded by the parabola and its latus rectum $= \frac{8}{3} a^2$

Ex. 3. *Find the whole area of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, bounded by its base.*

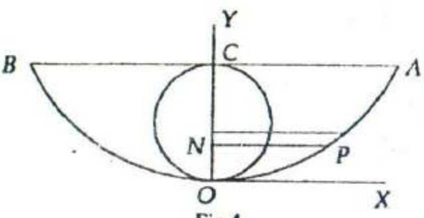

Fig.4

The area of half the cycloid, *viz.*, area AOC, is evidently bounded by the curve, the y-axis and the abscissæ $y = 0$ and $y = 2a$. Hence, this area is given by

$$\int_0^{2a} x\, dy$$

$$= \int_0^{\pi} a(\theta + \sin\theta) \cdot a \sin\theta\, d\theta \qquad \left[\begin{array}{r} \text{since } y = a(1 - \cos\theta) \\ x = a(\theta + \sin\theta) \end{array} \right]$$

$$= a^2 \left[-\theta \cos\theta + \sin\theta + \tfrac{1}{2}(\theta - \tfrac{1}{2}\sin 2\theta) \right]_0^{\pi} = \tfrac{3}{2}\pi a^2 .$$

Hence, the whole area of the cycloid is $3\pi a^2$.

Note. It should be noted here that if $\overline{AM}$ be drawn perpendicular from A on $\overrightarrow{OX}$, the expression $\int_0^{OM} y\,dx$ represents the area OAM, and not the area OAC.

Ex. 4. *Find the area of the loop of the curve*

$$xy^2 + (x+a)^2(x+2a) = 0.$$

Here let us first of all trace the curve. The equation can be put in the form $y^2 = -\{(x+a)^2(x+2a)\}/x$. We notice that $y = 0$ at the points B and A where $x = -a$ and $x = -2a$, and $y \to \pm\infty$ when $x \to 0$. For positive values of x, as also for negative values of x less than $-2a$, y^2 is negative and so y is imaginary. There is thus no part of the curve beyond O to the right, or beyond A ($x = -2a$) to the left. From A to B, for each value of x, y has two equal and opposite finite values and a loop is thus formed within this range, symmetrical about the x-axis. From B to O, each value of x gives two equal and opposite values of y which gradually increase in magnitude to ∞ as x aproaches 0. The curve, therefore, is as shown in the figure.

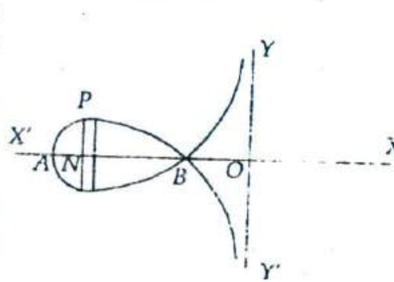

Fig.5

The required area of the loop = 2 . area APB

$$= 2 \cdot \int_{-2a}^{-a} y\,dx = 2\int_{-2a}^{-a} \sqrt{-\frac{(x+a)^2(x+2a)}{x}}\,dx$$

and substituting z for $x + 2a$, this reduces to

$$2\int_0^a (a - z)\sqrt{\frac{z}{2a - z}}\,dz$$

$$= 2\int_0^{\pi/2} a\cos\theta \frac{\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta} \, . \, 2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\, d\theta$$

$$\left[\text{putting } x = 2a\sin^2\frac{\theta}{2}\right]$$

$$= 2a^2\int_0^{\pi/2} \cos\theta\,(1 - \cos\theta)\, d\theta = 2a^2\left(1 - \frac{\pi}{4}\right)$$

$$= \tfrac{1}{2}a^2\,(4 - \pi).$$

10.3. Area between two given curves and two given ordinates.

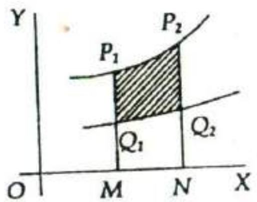

Fig.6

Let the area required be bounded by two given curves $y = f_1(x)$ and $y = f_2(x)$ and two given ordinates $x = a$ and $x = b$, indicated by $Q_1 Q_2 P_2 P_1 Q_1$ in the (figure - 6), where $OM = a$ and $ON = b$.

Clearly, area $Q_1 Q_2 P_2 P_1 Q_1$ = area $P_1 M P_2$ – area $Q_1 MNQ_2$

$$= \int_a^b f_1(x)\,dx - \int_a^b f_2(x)\,dx$$

$$= \int_a^b \{f_1(x) - f_2(x)\}\,dx$$

$$= \int_a^b (y_1 - y_2)\,dx,$$

where y_1 and y_2 denote the ordinates of the two curves $P_1 P_2$ and $Q_1 Q_2$ corresponding to the same abscissa x.

10.4. Illustrative Examples.

Ex. 1. *Find the area above the x-axis, included between the parabola* $y^2 = ax$ *and the circle* $x^2 + y^2 = 2ax$. [J. E. E. '89]

The abscissæ of the common points of the curves $y^2 = ax$ and $x^2 + y^2 = 2ax$ are given by $x^2 + ax = 2ax$, *i.e.*, $x = 0$ and $x = a$.

We are thus to find out the area between the curves and the ordinates $x = 0$ and $x = a$ above the x-axis (*i.e.*, for positive values only of the ordinates).

The required area is therefore

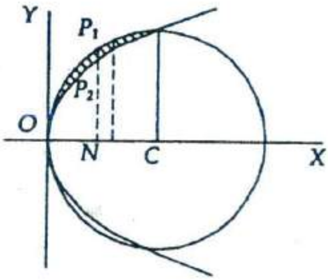

Fig.7

$$\int_0^a (y_1 - y_2)\, dx \quad [\text{ where } y_1^2 = 2ax - x^2 \text{ and } y_2^2 = ax]$$

$$= \int_0^a (\sqrt{2ax - x^2} - \sqrt{ax})\, dx .$$

Now, putting $x = 2a \sin^2\theta$,

$$\int_0^a \sqrt{2ax - x^2}\, dx = \int_0^{\pi/4} 2a \sin\theta\cos\theta \,.\, 4a\sin\theta\cos\theta\, d\theta$$

$$= a^2 \int_0^{\pi/4} (1 - \cos 4\theta)\, d\theta = a^2 \left[\theta - \frac{\sin 4\theta}{4} \right]_0^{\pi/4} = \frac{\pi}{4} a^2$$

Also, $\displaystyle\int_0^a \sqrt{ax}\, dx = \sqrt{a} \left[\frac{2}{3} x^{3/2} \right]_0^a = \frac{2}{3} a^2$.

Hence, the required area is $\displaystyle \frac{\pi}{4} a^2 - \frac{2}{3} a^2 = a^2 \left(\frac{\pi}{4} - \frac{2}{3} \right)$.

Ex. 2. *Find, by integration, the area of the ellipse*

$$ax^2 + 2hxy + by^2 = 1 .$$

The equation can be put in the form

$$by^2 + 2hxy + (ax^2 - 1) = 0.$$

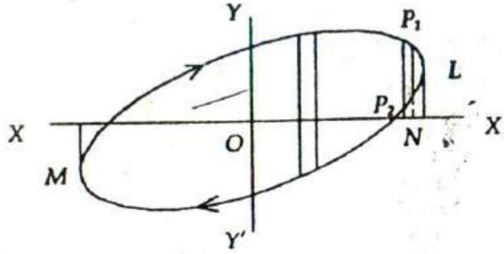

Fig. 8

If y_1, y_2 be the values of y corresponding to any value of x, we have

$$y_1 - y_2 = \frac{2}{b}\sqrt{h^2x^2 - b(ax^2 - 1)} = \frac{2}{b}\sqrt{b - (ab - h^2)x^2},$$

$ab - h^2$ being positive here, since the conic is an ellipse.

The extreme values of x, where the ordinates touch the ellipse, are given by

$$y_1 - y_2 = 0, \text{ i.e., } x = \pm\sqrt{\frac{b}{ab - h^2}}.$$

The required area can be treated as bounded by two curves, MP_1L, LP_2M respectively, both satisfying the given equation, but one having a single value y_1 for y coresponding to any value of x, and the other also having a single value y_2 for the same value of x.

Hence, the area required

$$= \int_{-\sqrt{\frac{b}{ab-h^2}}}^{+\sqrt{\frac{b}{ab-h^2}}} (y_1 - y_2)\,dx = \frac{2}{b}\int_{-\sqrt{\frac{b}{ab-h^2}}}^{+\sqrt{\frac{b}{ab-h^2}}} \sqrt{b - (ab - h^2)x^2}\,dx$$

and putting $\sqrt{(ab - h^2)}\,x = \sqrt{b}\sin\theta$, this becomes

$$= \frac{2}{\sqrt{(ab - h^2)}}\int_{-\pi/2}^{\pi/2} \cos^2\theta\,d\theta = \frac{\pi}{\sqrt{(ab - h^2)}}.$$

Note. The area of the ellipse can also be obtained as follows :

Assuming the equation of the ellipse referred to its major and minor axes as axes of co-ordinates to be $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$, by the theory of invariants as given in Conic Sections, we know that $\frac{1}{\alpha^2} \cdot \frac{1}{\beta^2} = ab - h^2$.

Now [from Ex. 1, Cor. 1, Art. 10.2] the area of the ellipse is

$$\pi\alpha\beta = \frac{\pi}{\sqrt{(ab - h^2)}}.$$

Ex. 3. *Find the area between the curve* $y^2 = \{(a - x)^3/(a + x)\}$ *and the asymptote.*

To trace the curve, we notice that y is imaginary for values of x greater than a or less than $-a$. At $x = a$, $y = 0$, and from a to $-a$, for each value of x, y has two equal and opposite values, tending to $\pm\infty$ as x aproaches $-a$. At $x = a$, the x-axis touches both the branches. The figure is, therefore, as shown, symmetrical about the x-axis.

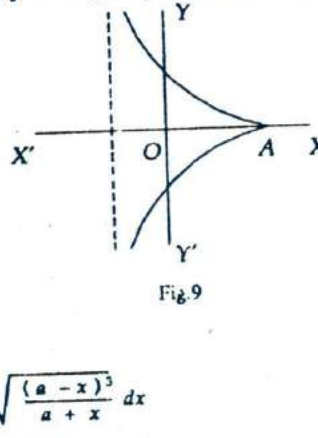

Fig.9

The required area between the curve and its asymptote is therefore

$$2\int_{-a}^{a} y\,dx = 2\int_{-a}^{a} \sqrt{\frac{(a - x)^3}{a + x}}\,dx$$

and substituting z for $a + x$ this reduces to

$$2\int_{0}^{2a} (2a - z)\sqrt{\frac{2a - z}{z}}\,dz$$

$$= 2\int_{0}^{\frac{1}{2}\pi} 2a\cos^2\theta\,\frac{\cos\theta}{\sin\theta} \cdot 4a\sin\theta\cos\theta\,d\theta \quad [\text{ where } z = 2a\sin^2\theta\ ]$$

$$= 16a^2\int_{0}^{\frac{1}{2}\pi} \cos^4\theta\,d\theta = 16a^2\,\frac{1.3}{2.4}\,\frac{\pi}{2} = 3\pi a^2.$$

10.5. Areas in Polar co-ordinates.

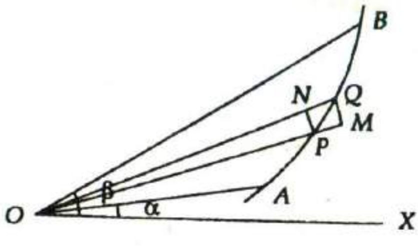

Fig. 10

Let $r = f(\theta)$ be a curve APB, where $f(\theta)$ is supposed to be a finite, continuous and single-valued function in the interval $\alpha < \theta < \beta$. The area bounded by the curve, and the radii vectors $\theta = \alpha$ and $\theta = \beta$ is given by the definite integral

$$\frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta, \quad i.e., \frac{1}{2}\int_{\alpha}^{\beta} \{f(\theta)\}^2\, d\theta .$$

Let A denote the area POA, bounded by the curve, the given radius vector $\overline{OA}$, *i.e.*, $\theta = \alpha$, and the variable radius vector $\overline{OP}$ at vectorial angle θ. $\alpha < \theta < \beta$. Then for each value of θ, A has a definite value and so A is a function of θ. If Q be the neighbouring point $r + \Delta r, \theta + \Delta\theta$ on the curve, we have

ΔA = the infinitesimal change in A due to a change $\Delta\theta$ in θ

= the elementary area POQ

and this clearly lies between the circular sectorial areas OPN and OQM, where PN and QM are arcs of circles with centre O.

Thus, $\frac{1}{2}r^2\,\Delta\theta < \Delta A < \frac{1}{2}(r + \Delta r)^2\,\Delta\theta$,

i.e., $$\frac{1}{2}\{f(\theta)\}^2 < \frac{\Delta A}{\Delta\theta} < \frac{1}{2}\{f(\theta + \Delta\theta)\}^2 .$$

Now, proceeding to the limit, and remembering that $f(\theta)$ being continuous, $f(\theta + \Delta\theta) \to f(\theta)$ as $\Delta\theta \to 0$, we get

$$\frac{dA}{d\theta} = \frac{1}{2}\{f(\theta)\}^2, \quad i.e., \frac{1}{2}r^2 .$$

Thus, $A = \frac{1}{2}\int r^2\, d\theta + C = F(\theta) + C$, say.

Now, taking P coincident with A and B respectively and denoting the required area AOB by A_1, we get

$$0 = F(\alpha) + C \text{ and } A_1 = F(\beta) + C.$$

where $$A_1 = F(\beta) - F(\alpha) = \tfrac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta.$$

Note. 1. The curve APB is here assumed as concave towards O. A similar proof with corresponding modifications holds even if the curve be convex, or partly concave and partly convex or wavy, in fact of any form.

Note. 2. As in the case of area in Cartesian co-ordinates, the above result can also be deduced directly from the definition of a definite integral as a summation.

Cor. *The area bounded by the two curves* $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$ *and two given radii vectors* $\theta = \alpha$ *and* $\theta = \beta$ *is*

$$\frac{1}{2}\int_{\alpha}^{\beta} (r_2^{\,2} - r_1^{\,2})\, d\theta.$$

Alternative proof

Let AB be the curve, $\overline{OA}$ and $\overline{OB}$ be the radii vectors corresponding to $\theta = \alpha$ and $\theta = \beta$.

Divide $\beta - \alpha$ into n parts, each equal to h, and draw the corresponding radii vector s. Let P and Q be the points on the curve corresponding to $\theta = \alpha + rh$ and $\theta = \alpha + (r+1)h$ and let us suppose θ goes on increasing from α to β. With centre O and radii OP, OQ respectively draw arcs PN, QM as in the figure. Then the area OPQ lies in magnitude between

$$\tfrac{1}{2}OP^2.h \text{ and } \tfrac{1}{2}OQ^2.h,$$

i.e., between $\frac{1}{2}[f(\alpha + rh)]^2 h$ and $\frac{1}{2}[f(\alpha + (r+1)h)]^2 h$.

Hence, adding up all the areas like OPQ, it is clear that the area AOB lies between

$$\tfrac{1}{2}\sum_{r=0}^{n-1} [f(\alpha + rh)]^2 h \text{ and } \tfrac{1}{2}\sum_{r=0}^{n-1} [f(\alpha + (r+1)h)]^2 h.$$

Now, let $n \to \infty$, so that $h \to 0$; then as the limit of each of the above two sums is

$$\frac{1}{2}\int_{\alpha}^{\beta}\{f(\theta)\}^2\,d\theta,$$

it follows that the area AOB is also equal to the definite integral.

10.6. Illustrative Examples.

Ex. 1. *Find the area bounded by the cardioide* $r = a(1 - \cos\theta)$.

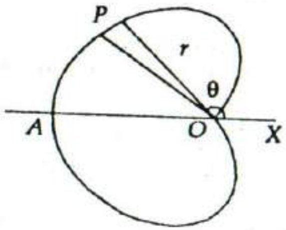

Fig. 11

The curve is symmetrical about the initial line, since replacing θ by $-\theta$, r does not alter. Beginning from $\theta = 0$ and gradually increasing θ to π, the corresponding values of r are noticed, and the curve is easily traced in the figure - 11.

Now, the required area is evidently, from the above article,

$$2\cdot\frac{1}{2}\int_0^{\pi} r^2\,d\theta = a^2\int_0^{\pi}(1-\cos\theta)^2\,d\theta = a^2\cdot\frac{3}{2}\pi = \frac{3}{2}\pi a^2.$$

Note. It should be noted that the area bounded by the cardioide whose equation is $r = a(1 + \cos\theta)$ is also $\frac{3}{2}\pi a^2$.

Ex. 2. *Find the area of a loop of the curve* $r = a\cos 2\theta$.

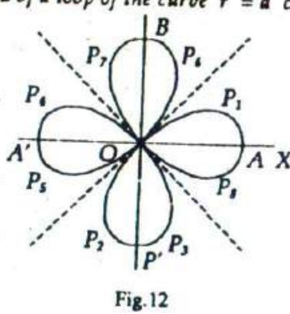

Fig. 12

In tracing the curve, we notice that, as θ increases from 0 to $\frac{1}{4}\pi$, r diminishes from a to 0, the portion AP_1O being thus traced. As θ increases from $\frac{1}{4}\pi$ to $\frac{3}{4}\pi$, r is negative throughout, and the corresponding portion of the curve which is traced is OP_2BP_3O. Then as θ increases from $\frac{3}{4}\pi$ to $\frac{5}{4}\pi$, r remains positive and the portion $OP_4A'_5O$ of the curve is traced. As θ increases from $\frac{5}{4}\pi$ to $\frac{7}{4}\pi$, r is again negative and we get the portion $OP_6B'P_7O$ of the curve. Finally, when θ increases from $\frac{7}{4}\pi$ to 2π, r is positive and the portion OP_8AO of the curve is described. The curve thus consists of four equal loops as shown in the figure.

It is now clear from the figure that, area of one loop

$$= 2 \,.\, \text{area } AP_1O$$

$$= 2 \,.\, \frac{1}{2}\int_0^{\frac{1}{4}\pi} r^2\, d\theta = a^2 \int_0^{\frac{1}{4}\pi} \cos^2 2\theta\, d\theta = \tfrac{1}{8}\pi a^2 .$$

Cor. Hence, the entire area of the curve, *i.e.*, the sum of the areas of the 4 loops $= \frac{1}{2}\pi a^2$.

Note. All curves of the type $\mathbf{r = a \sin n\theta}$, or $\mathbf{r = a \cos n\theta}$ may be similarly traced, by dividing each quadrant into n equal parts, and increasing θ successively through each division. If r be found positive, the traced portion of the curve will be in the same division ; if r be negative, the traced part will be in the diametrically opposite division. Anyway, when the curve is completely traced, it will be found to consist of *n equal loops if n be odd, and $2n$ equal loops if n be even.*

Ex. 3. *(i) Find the area of the loop of the folium of Descartes,*

$$x^3 + y^3 = 3axy .$$

(ii) Find also the area included between the folium and its asymptote and show that it is equal to the area of the loop.

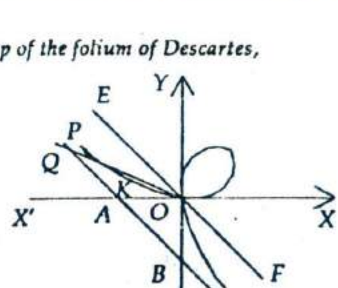

Fig. 13

(i) Transforming to corresponding polar co-ordinates by putting $x = r\cos\theta$, $y = r\sin\theta$, the polar equation to the curve becomes

$$r = \frac{3a\cos\theta\sin\theta}{\cos^3\theta + \sin^3\theta} \qquad \ldots \quad (1)$$

As θ increases from 0 to $\frac{1}{2}\pi$, r at first increases from 0 to $(3a/\sqrt{2})$, reaching the maximum at $\theta = \frac{1}{4}\pi$, and then diminishes to 0 again, thus forming a loop in the first quadrant.

The required area of this loop is

$$= \frac{1}{2}\int_0^{\frac{1}{2}\pi} r^2\, d\theta = \frac{9a^2}{2}\int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta\cos^2\theta}{(\sin^3\theta + \cos^3\theta)^2}\, d\theta$$

$$= \frac{9a^2}{2}\int_0^{\infty} \frac{t^2\, dt}{(1+t^3)^2} \quad [\text{ putting } t = \tan\theta\ ]$$

$$= \frac{9a^2}{2} \underset{\varepsilon \to \infty}{Lt} \int_0^{\varepsilon} \frac{t^2\, dt}{(1+t^3)^2} = \frac{3a^2}{2} \underset{\varepsilon \to \infty}{Lt} \left[-\frac{1}{1+\varepsilon^3} + 1\right]$$

$$= \tfrac{3}{2}a^2 .$$

(ii) The equation of the asymptote of the folium is

$$x + y + a = 0. \qquad \ldots (2)$$

Its polar equation is $r = \dfrac{-a}{\sin\theta + \cos\theta}$. $\qquad \ldots (3)$

Now, $r \to \infty$ if $(\sin\theta + \cos\theta) \to 0$, *i.e.*, if $\tan\theta \to -1$,

i.e., if $\theta \to \frac{3}{4}\pi$.

$\therefore$ the direction of the asymptote is $\theta \to \frac{3}{4}\pi$.

The asymptote intersects the two axes at A and B, where

$OA = a$ and $OB = a$, *i.e.*, $OA = OB$.

Hence, the area of $\Delta OAB = \frac{1}{2}a^2$. $\qquad \ldots (4)$

Area between the folium and its asymptote = the triangular area OAB + the limiting value of twice the area between the curve and the asymptote in the second quadrant (from symmetry)

$= \frac{1}{2}a^2$ + the limiting value of twice the curvilinear area $OKPQAO$

$= \frac{1}{2}a^2 + 2\sigma$ (say). $\qquad \ldots (5)$

Draw a radius vector $\overline{OPQ}$ making an angle θ with the x-axis, such that $\frac{3}{4}\pi < \theta < \pi$. Suppose it cuts the curve and the asymptote at P and Q respectively.

Let us denote the curvilinear area $OKPQAO$ by S, the triangular area $OQAO$ by S_1, and the curvilinear area $OKPO$ by S_2.

$$\therefore \quad S = S_1 - S_2 .$$

$$\therefore \quad \sigma = \underset{\theta \to \frac{3}{4}\pi}{Lt} S = \underset{\theta \to \frac{3}{4}\pi}{Lt} (S_1 - S_2).$$

Now, applying the formula for area in polar co-ordinates, *i.e.*, $\frac{1}{2}\int r^2 \, d\theta$ and using equations (1) and (3), we get

$$S = \frac{1}{2}\left[\int_\theta^\pi \frac{a^2 \, d\theta}{(\sin\theta + \cos\theta)^2} - \int_\theta^\pi \frac{9a^2 \sin^2\theta \cos^2\theta}{(\sin^3\theta + \cos^3\theta)^2} \, d\theta\right]$$

$$= \tfrac{1}{2}a^2 \; (I_1 - I_2), \text{ say}.$$

$$\text{Now, } \int \frac{d\theta}{(\sin\theta + \cos\theta)^2} = \int \frac{\sec^2\theta \, d\theta}{(1 + \tan\theta)^2}$$

(on multiplying the numerator and denominator by $\sec^2\theta$)

$$= \int \frac{dt}{t^2} \qquad [\text{putting } t = 1 + \tan\theta]$$

$$= -\frac{1}{t} = -\frac{1}{1 + \tan\theta},$$

$$\therefore I_1 = -\left[\frac{1}{1 + \tan\theta}\right]_\theta^\pi = \frac{1}{1 + \tan\theta} - 1 .$$

$$\text{Again, } \int \frac{\sin^2\theta \cos^2\theta \, d\theta}{(\sin^3\theta + \cos^3\theta)^2} = \int \frac{\tan^2\theta \sec^2\theta}{(1 + \tan^3\theta)^2} \, d\theta$$

on multiplying the numerator and denominator by $\sec^6\theta$)

$$= \int \frac{dt}{3t^2} \qquad [\text{putting } 1 + \tan^3\theta = t]$$

$$= -\frac{1}{3}\frac{1}{t} = -\frac{1}{3}\frac{1}{(1 + \tan^3\theta)} .$$

$$\therefore \quad I_2 = 9\left(-\frac{1}{3}\right)\left[\frac{1}{1 + \tan^3\theta}\right]_\theta^\pi = \frac{3}{1 + \tan^3\theta} - 3 .$$

$$\therefore \quad S = \frac{1}{2}a^2\left[2 + \frac{1}{1 + \tan\theta} - \frac{3}{1 + \tan^3\theta}\right]$$

$$= \frac{1}{2}a^2\left[2 + \frac{\tan^2\theta - \tan\theta - 2}{1 + \tan^3\theta}\right]$$

$$= \frac{1}{2}a^2\left[2 + \frac{(\tan\theta + 1)(\tan\theta - 2)}{(1 + \tan\theta)(1 - \tan\theta + \tan^2\theta)}\right]$$

$$= \frac{1}{2}a^2\left[2 + \frac{\tan\theta - 2}{1 - \tan\theta + \tan^2\theta}\right].$$

Now, $\sigma = \underset{\theta \to \frac{3\pi}{4}}{Lt} S = \frac{1}{2}a^2$.

$\therefore$ the required area $= \frac{1}{2}a^2 + 2\sigma = \frac{3}{2}a^2 =$ the area of the loop.

Ex. 4. *Show that the area between the folium of Descartes and its asymptote is equal to the area of its loop, each being equal to* $\frac{3}{2}a^2$.

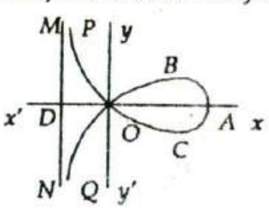

Fig.14

The equation of the folium is $x^3 + y^3 = 3axy$.

Turn the axes through $\frac{1}{4}\pi$; that is substitute $(x-y)/\sqrt{2}$ and $(x+y)/\sqrt{2}$ for x and y respectively. Then the given equation transforms into

$$y^2 = \frac{x^2}{3}\,\frac{3c - x}{c + x}, \text{ where } c = \frac{1}{\sqrt{2}}a.$$

$$\therefore \qquad y = \pm\frac{1}{\sqrt{3}}\,x\sqrt{\left(\frac{3c - x}{c + x}\right)}.$$

Here $c + x = 0$, *i.e.*, $x = -c$ is the equation of the asymptote $\overline{MN}$

$$OA = 3c,\ OD = -c.$$

$\therefore$ the required area σ between the Folium and the asymptote

$$= 2\underset{t \to c}{Lt}\int_{-t}^{0} y\,dx = \frac{2}{\sqrt{3}}\underset{t \to c}{Lt}\int_{-t}^{0} x\sqrt{\left(\frac{3c - x}{c + x}\right)}\,dx$$

$$= \frac{2}{\sqrt{3}}\underset{t \to c}{Lt}\int_{-t}^{0}\frac{x(3c - x)}{\sqrt{\{(x + c)(3c - x)\}}}\,dx.$$

Let $I = \int\frac{x(3c - x)}{\sqrt{\{(x + c)(3c - x)\}}}\,dx = 2c^2\int(1 - 2\cos\theta)(1 + \cos\theta)\,d\theta$

$$\left[\text{on putting } x = c - 2c\cos\theta, \text{ so that } \cos\theta = \frac{c - x}{2c}\right]$$

$$= -2c^2 \int (\cos\theta + \cos 2\theta)\, d\theta$$

$$= -2c^2 (\sin\theta + \tfrac{1}{2}\sin 2\theta)$$

$$= -2c^2 \left\{ \sin\left(\cos^{-1}\frac{c-x}{2c}\right) + \frac{1}{2}\sin\left(2\cos^{-1}\frac{c-x}{2c}\right) \right\}.$$

$$\therefore\ \sigma = \frac{2}{\sqrt{3}}\left(-2c^2\right) \underset{t \to c}{Lt} \left[\sin\left(\cos^{-1}\frac{c-x}{2c}\right) + \tfrac{1}{2}\sin\left(2\cos^{-1}\frac{c-x}{2c}\right)\right]_{-t}^{0}$$

$$= -2a^2 . \tfrac{3}{4}\left[\text{on putting } c = \frac{1}{\sqrt{2}}a\right] = -\tfrac{3}{2}a^2 .$$

Again L, the area of the loop $OBAC$,
= 2 area of the portion OBA

$$= 2\int_0^{3c} y\, dx = \frac{2}{\sqrt{3}}\int_0^{3c} \frac{x(3c-x)}{\sqrt{\{(x+c)(3c-x)\}}}\, dx .$$

Putting, as before, $x = c - 2c\cos\theta$,

$$L = \frac{2}{\sqrt{3}}\left(-2c^2\right)\left[\sin\left(\cos^{-1}\frac{c-x}{2c}\right) + \tfrac{1}{2}\sin\left(2\cos^{-1}\frac{c-x}{2c}\right)\right]_0^{3c}$$

$$= \frac{4c^2}{\sqrt{3}}\ \frac{3}{2}\ \frac{\sqrt{3}}{2} = \tfrac{3}{2}a^2\left[\text{on putting } c = \frac{1}{\sqrt{2}}a\right].$$

Ex. 5. *Find the area between the cissoid* $r = \dfrac{a\sin^2\theta}{\cos\theta}$ *and its asymptote.*

The curve may be traced either from its polar equation or by converting it to Cartesian form, and the figure will be as shown. The asymptote is easily found to be the line $x = a$ or in polar co-ordinates $r\cos\theta = a$. Now, let $\overline{OPQ}$ be any radius vector at an angle θ to the x-axis, intersecting the curve and its asymptote at P and Q respectively.

Fig.15

$$\text{Area } OAQPO = \frac{1}{2}\int_0^{\theta} (r_1{}^2 - r_2{}^2)\, d\theta \quad [\text{ where } r_1 = OQ,\ r_2 = OP\ ]$$

$$= \frac{1}{2}\int_0^{\theta} \left(\frac{a^2}{\cos^2\theta} - a^2\frac{\sin^4\theta}{\cos^2\theta}\right) d\theta$$

$$= \frac{a^2}{2}\int_0^{\theta} (1 + \sin^2\theta)\, d\theta$$

$$= \frac{a^2}{2}\left\{\frac{3}{2}\theta - \frac{\sin 2\theta}{4}\right\}.$$

Now, the required area between the curve and the asymptote is clearly (there being symmetry about the x-axis, and since the direction of the asymptote is given by $\theta = \frac{1}{2}\pi$)

$$\underset{\theta \to \frac{1}{2}\pi}{Lt}\left[2\,\frac{a^2}{2}\left(\frac{3}{2}\theta - \frac{\sin 2\theta}{4}\right)\right] = a^2\left(\tfrac{3}{2}\cdot\tfrac{1}{2}\pi\right) = \frac{3}{4}\pi a^2.$$

Ex. 6. *Find the area common to the Cardioide $r = a(1 + \cos\theta)$ and the circle $r = \frac{3}{2}a$, and also the area of the remainder of the Cardioide.*

Fig.16

At the common point P of the two curves, we have

$$\tfrac{3}{2} = 1 + \cos\theta.$$

$$\therefore \cos\theta = \tfrac{1}{2},\ \text{or},\ \theta = \tfrac{1}{3}\pi.$$

The required area is easily seen to be

2 [area OCP + area PQO]

$$= 2\left\{\frac{1}{2}\int_0^{\frac{1}{3}\pi} \left(\tfrac{3}{2}a\right)^2 d\theta + \frac{1}{2}\int_{\frac{1}{3}\pi}^{\pi} a^2(1 + \cos\theta)^2\, d\theta\right\}$$

$$= \tfrac{9}{4}a^2\cdot\tfrac{1}{3}\pi + a^2\left\{\tfrac{3}{2}\left(\pi - \tfrac{1}{3}\pi\right) + 2\left(\sin\pi - \sin\tfrac{1}{3}\pi\right) + \tfrac{1}{4}\left(\sin 2\pi - \sin\tfrac{2}{3}\pi\right)\right\}$$

$$= \left(\frac{7}{4}\pi - \frac{9\sqrt{3}}{8}\right)a^2.$$

Again, the area of the remainder of the Cardioide, *i.e.*, *APCR*

$$= 2 \,.\, \text{area } APC = 2 \,.\, \frac{1}{2} \int_0^{\frac{1}{3}\pi} (r_1^{\,2} - r_2^{\,2}) \, d\theta$$

$$= \int_0^{\frac{1}{3}\pi} \{ a^2 (1 + \cos\theta)^2 - \tfrac{9}{4} a^2 \} \, d\theta$$

$$= a^2 \int_0^{\frac{1}{3}\pi} (2\cos\theta + \tfrac{1}{2}\cos 2\theta - \tfrac{3}{4}) \, d\theta$$

$$= a^2 \left\{ 2 \, \frac{\sqrt{3}}{2} + \frac{1}{4} \, \frac{\sqrt{3}}{2} - \frac{3}{4} \, \frac{1}{3} \pi \right\} = a^2 \left\{ \frac{9\sqrt{3}}{8} - \frac{1}{4} \pi \right\}.$$

Note. The whole area of the Cardioide is evidently the sum of these two, *i.e.*, $= \frac{3}{2} \pi a^2$. [See *Ex. 1 above.*]

10.7. The sign of an area.

In the expression $\int_a^b y \, dx$ for an area we tacitly assume that the ordinate y is positive throughout the range (a, b) and that x increases from a to b, *i.e.*, $b > a$. In this case the area calculated by the above formula will be positive. If, however, y be negative or if $b < a$ while y is positive, *i.e.*, in moving along the curve from $x = a$ to $x = b$ we are moving parallel to the negative direction of the x-axis, the calculated area will be negative.

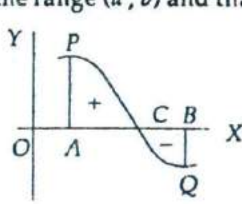

Fig. 17

If, therefore, we proceed to calculate the total area where, in the range (a, b), y is positive for some portion and negative for the rest, as in the above figure, by using the formula

$\int_a^b y \, dx$, the calculated result will give us the difference of the magnitude of the two areas ACP and CQB, which may be positive or negative or even zero if the magnitudes of the two areas are equal.

Hence, if our object be to get the sum total of the magnitudes of the two areas, we should calculate each part separately by the formula of the type $\int_a^c y\,dx$, $\int_c^b y\,dx$, the results being found to be associated with their proper signs. We shall now discard the signs and consider the sum of the magnitudes.

In each individual case, therefore, we should first of all have a clear idea of the figure and the area to be calculated, and then we should proceed. For instance, notice that in the figure -18, area $PACR$ is +, area $CRSD$ is − and area $SDBQ$ is +, and that for the range DC of the x-axis, y is three-valued and in calculating the area $PACR$ we are to use one value of y for the portion in the formula $\int_a^c y\,dx$, for calculating the area $CRSD$ we are to use a second value of y in the formula $\int_c^d y\,dx$, the upper limit d being less than c for this part, and lastly for the area $SDBQ$ we are to use the third value of y for this part in the formula $\int_d^b y\,dx$. If we take the algebraic sum of the three areas, with their proper signs, we get the area bounded by the curve, the x-axis and the ordinates $\overline{AP}$ and $\overline{BQ}$.

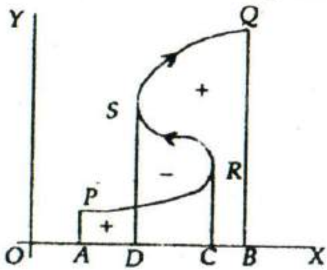

Fig. 18

Similarly, in the formula $\frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta$ in polar co-ordinates if $\beta < \alpha$, *i.e.*, if θ diminishes in moving along the curve from $\theta = \alpha$ to $\theta = \beta$, the calculated area will be negative. Then area OPR is +, area ORS is −, area OSQ is +, the area bounded by $PRSQ$ and the radii vectors $\overline{OP}$, $\overline{OQ}$ being their algebraic sum. Also for the range SOR, for each value of θ, r has three values, and we must use the right value in each case for that part when moving along PR or along RS or along SQ in the expression $r^2 d\theta$.

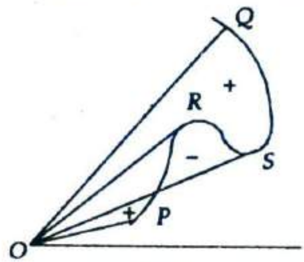

Fig.19

10.8. Areas of closed curves.

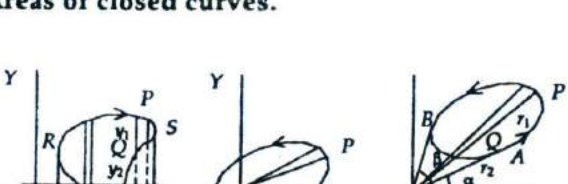

Fig.20

In a closed curve given by Cartesian equation, clearly for each value of x there will be two values of y, say, y_1 and y_2 [See *figure* (i)]. The extreme values of y, say, a and b, are obtained by puting $y_1 = y_2$. Now $\int_a^b (y_1 - y_2)\,dx$ will give the positive value of the required area, provided $b > a$ and $y_1 > y_2$. This amounts, as it were, to the determination of the area between two curves having the same equation as the given one, but y being single-valued in each, the proper value being chosen for each part. The method has been illustrated in Ex. 2, Art. 10.4.

In polar curves, if the origin be within the curve See [*figure* (ii)],

$\frac{1}{2}\int_0^{2\pi} r^2\, d\theta$ gives the desired area.

If the origin be outside, corresponding to each value of θ there are two values of r, say, r_1 and r_2 See [*figure* (iii)]. The extreme values of θ, namely α and β, are obtained by putting $r_1 = r_2$. Now, if $r_1 > r_2$ and $\beta > \alpha$, the positive value of the area will be

given by the expression $\frac{1}{2}\int_\alpha^\beta (r_1^2 - r_2^2)\, d\theta$.

In fact, the area $OAPB$ is given by $\int_\alpha^\beta r_1^2\, d\theta$ and is positive,

while $\frac{1}{2}\int_\beta^\alpha r_2^2\, d\theta$ gives the area $OBQA$ with negative sign, the algebraic sum of the two giving the desired area.

In the case of closed curves there is another method of calculating the area. Let (x, y) be the Cartesian co-ordinates of a point on the curve whose polar co-ordinates are (r, θ), then $x = r\cos\theta$, $y = r\sin\theta$.

Now, if t be a single variable parameter in terms of which x, y and, therefore, r, θ of any point on the curve can be expressed, we have

$$\frac{dx}{dt} = \frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt},$$

$$\frac{dy}{dt} = \frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt},$$

$$\therefore \quad x\frac{dy}{dt} - y\frac{dx}{dt} = r^2\frac{d\theta}{dt}.$$

Hence, *the area which is expressed by the integral* $\frac{1}{2}\int r^2\, d\theta$ *can as well be expressed by the line integral*

$$\frac{1}{2}\int\left(x\,\frac{dy}{dt}-y\,\frac{dx}{dt}\right)dt$$

along the curve, the limits of t for the closed curve being such that the point (x, y) returns to its initial position. The rule of signs for the area is that the above expression is positive when the area lies to the left of a point describing the curve in the direction in which t increases.

10.9. Approximate evaluation of a definite integral : Simpson's rule.

In many cases, a definite integral cannot be obtained either because the quantity to be integrated cannot be expressed as a mathematical function or because the indefinite integral of the function itself cannot be determined directly. In such cases formula of approximation are used. One such important formula is Simpson's rule. By this rule the definite integral of any function (or the area bounded by a curve, the x-axis and two extreme ordinates) is expressed in terms of the individual values of any number of ordinates within the interval, by assuming that the function within each of the small ranges into which the whole interval may be divided can be represented, to a sufficient degree of approximation, by a parabolic function.

Simpson's Rule : An *approximate value* of the definite integral

$$\int_a^b y\,dx\text{, where } y = f(x)$$

$$= \tfrac{1}{3}h\,[(y_1 + y_{2n+1}) + 2(y_3 + y_5 + \ldots + y_{2n-1})$$

$$+ 4(y_2 + y_4 + \ldots + y_{2n})]$$

where $h = \dfrac{b-a}{2n}$ and $y_1, y_2, y_3, \ldots$ are the values of y when $x = a, a+h, a+2h, \ldots$.

In words, the above rule can be written as

$\frac{1}{3}h$ [sum of the extreme ordinates + 2 . sum of the remaining odd ordinates + 4 . sum of the remaining even ordinates].

Let PQ be the curve $y = f(x)$ and $\overline{PL}$, $\overline{QM}$ be the ordinates $x = a$, $x = b$. Divide the interval LM into $2n$ equal intervals each

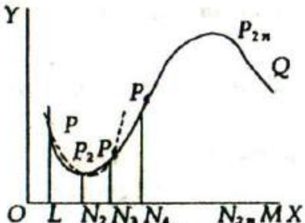

Fig. 21

of length h by the points N_2, N_3, ... so that $h = (b - a)/2n$ and let P_2N_2, P_3N_3, ... be the ordinates at N_2, N_3, ... Then $PL = y_1$, $P_2N_2 = y_2$, $P_3N_3 = y_3$, ... Through PP_2P_3 draw a parabola having its axis parallel to the y-axis, and let its equation referred to parallel axes through $N_2(a + h, 0)$ be

$$y = a + bx + cx^2 . \qquad \ldots (1)$$

Then, the area bounded by the parabolic arc PP_2P_3, the ordinates of P, P_3 and the x-axis (such to be called hereafter shortly as the area under the parabola)

$$= \int_{-h}^{h} (a + bx + cx^2)\, dx = 2h\left(a + \tfrac{1}{3}ch^2\right). \qquad \ldots (2)$$

Since $P(-h, y_1)$, $P_2(0, y_2)$, $P_3(h, y_3)$ are points on the parabola (1),

$\therefore\ y_1 = a - bh + ch^2$, $y_2 = a$, $y_3 = a + bh + ch^2$

from which we get $a = y_2$, $c = \dfrac{y_1 - 2y_2 + y_3}{2h^2}$.

$\therefore$ from (2), area under the parabola $= \frac{1}{3}h(y_1 + 4y_2 + y_3)$.

Now, area of the 1st strip (ordinates y_1, y_2, y_3) under the curve $y = f(x)$ is approximately = area under the parabola

$$= \frac{1}{3}h(y_1 + 4y_2 + y_3).$$

Similarly, area of the 2nd strip (ordinates y_3, y_4, y_5) under the curve is approximately $= \frac{1}{3}h(y_3 + 4y_4 + y_5)$; area of the 3rd strip (ordinates y_5, y_6, y_7) under the curve is approximately $= \frac{1}{3}h(y_5 + 4y_6 + y_7)$; and area of the nth strip under the curve is approximately

$$= \frac{1}{3}h(y_{2n-1} + 4y_{2n} + y_{2n+1}).$$

$\therefore$ summing all these, *area under the curve*, i.e., $\int_a^b y\,dx$ is approximately

$$= \frac{1}{3}h[(y_1 + y_{2n+1}) + 2(y_3 + y_5 + \dots + y_{2n-1}) + 4(y_2 + y_4 + \dots + y_{2n})].$$

Note. It should be noted that the closer the ordinates, the more approximate is the value.

Simpon's rule is sometimes called *Parabolic rule*.

Ex. *Given* $e^0 = 1$, $e^1 = 2.72$, $e^2 = 7.39$, $e^3 = 20.09$, $e^4 = 54.60$; *verify Simpson's rule by finding an approximate value of* $\int_0^4 e^x\,dx$, *taking 4 equal intervals, and compare it with its exact value.*

Hence, $a = 0, b = 4, n = 2, h = 1, y = f(x) = e^x$.

$\therefore$ by Simpson's rule we get the approximate value

$$\frac{1}{3}h[(y_1 + y_5) + 2y_3 + 4(y_2 + y_4)]$$
$$= \frac{1}{3}h[(e^0 + e^4) + 2e^2 + 4(e^1 + e^3)]$$
$$= \frac{1}{3}h[1 + 54.60 + 2 \times 7.39 + 4(2.72 + 20.09)]$$
$$= 53.87$$

Exact value $= \left[e^x \right]_0^4 = e^4 - 1 = 54.60 - 1 = 53.60$.

$\therefore$ *error* $= 53.87 - 53.60 = 0.27$ (approximately).

EXAMPLES X

1. Find the area of a hyperbola $xy = c^2$ bounded by the x-axis, and the ordinates $x = a$, $x = b$.

2. Find the area of the segment of the parabola $y = (x-1)(4-x)$ cut off by the x-axis.

3. Find the area bounded by the x-axis and one arc of the sine curve $y = \sin x$.

4. In the logarithmic curve $y = ae^x$, show that the area between the x-axis and any two ordinates is proportional to the difference between the ordinates.

5. Find, by integration, the area of the triangle bounded by the line $y = 3x$, the x-axis and the ordinate $x = 2$. Verify your result by finding the area as half the product of the base and the altitude.

6. Show that the area bounded by the parabola $\sqrt{x} + \sqrt{y} = \sqrt{a}$ and the co-ordinate axes is $\frac{1}{6}a^2$.

7. Show that the area bounded by the semi-cubical parabola $y^2 = ax^3$ and a double ordinate is $\frac{2}{5}$ of the area of the rectangle formed by this ordinate and the abscissa.

8. Show that the area of

(i) the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ is $\frac{3}{8}\pi a^2$;

(ii) the hypo-cycloid $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$ is $\frac{3}{8}\pi ab$;

(iii) the evolute $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$ is $\frac{3}{8}\pi\frac{(a^2-b^2)^2}{ab}$.

9. Find the area enclosed by the curves : $(a > 0)$

(i) $x(1+t^2) = 1 - t^2$; $y(1+t^2) = 2t$.

(ii) $x = 3\cos t$; $y = 2\sin t$.

(iii) $x = a\cos t(1-\cos t)$; $y = a\sin t(1-\cos t)$.

(iv) $x = a(2\cos t + \cos 2t)$; $y = a(\sin t + \sin 2t)$.

10. Find the area of the segment cut off from $y^2 = 4x$ by the line $y = x$.

11. Find the area bounded by the curve $y^2 = x^3$ and the line $y = 2x$.

12. Find the area of the portion of the circle $x^2 + y^2 = 1$, which lies inside the parabola $y^2 = 1 - x$.

13. (i) Show that the area bounded by the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$.

(ii) Find the area bounded by the curves

$$y^2 - 4x - 4 = 0 \text{ and } y^2 + 4x - 4 = 0.$$

14. Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the square bounded by $x = 0, x = 4, y = 0, y = 4$ into three equal areas.

15. The curves $y = 4x^2$ and $y^2 = 2x$ meet at the origin O and at the point P, forming a loop. Show that the straight line $\overline{OP}$ divides the loop into two parts of equal area.

16. (i) Find the area included between the ellipses $x^2 + 2y^2 = a^2$ and $2x^2 + y^2 = a^2$.

(ii) Show that the area common to the two ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ and } \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \ (a > b)$$

is $2ab \tan^{-1} \dfrac{2ab}{a^2 - b^2}$.

17. Find the area of the following curves : $(a > 0)$

(i) $a^2y^2 = a^2x^2 - x^4$.
(ii) $(y - x)^2 = a^2 - x^2$. [See *Ex. 2, Art. 10.4.*]
(iii) $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.
(iv) $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$.
[Transform *(iii)* and *(iv)* to Polar.]
(v) $x = a\cos\theta + b\sin\theta,\ y = a'\cos\theta + b'\sin\theta$.
(vi) $x = a\sin 2t,\ y = a\sin t$.

18. Find the area of the loop of each of the following curves : $(a > 0)$

(i) $y^2 = x(x-1)^2$.

(ii) $ay^2 = x^2(a-x)$.

(iii) $y^2 = x^2(x+a)$.

(iv) $x = \frac{1-t^2}{1+t^2}, y = t\frac{1-t^2}{1+t^2}, (-1 \leq t \leq 1)$.

(v) $x = a(1-t^2), y = at(1-t^2), (-1 \leq t \leq 1)$.

19. Find the area of the loop or one of two loops (where such exist) of the folowing curves : $(a > 0)$

(i) $x(x^2+y^2) = a(x^2-y^2)$.

(ii) $y^2(a^2+x^2) = x^2(a^2-x^2)$.

(iii) $y^2(a-x) = x^2(a+x)$.

(iv) $y^2 = x^2(4-x^2)$.

(v) $x^2 = y^2(2-y)$.

20. Find the whole area included between each of the following curves and its asymptote : $(a > 0)$

(i) $x^2y^2 = a^2(y^2-x^2)$.

(ii) $y^2(a-x) = x^3$.

(iii) $y^2(a-x) = x^2(a+x)$.

(iv) $x^2y^2 + a^2b^2 = a^2y^2$.

(v) $xy^2 = 4a^2(2a-x)$.

21. Find the area of the following curves : $(a > 0)$

(i) $r = a\sin\theta$.

(ii) $r^2 = a^2\sin 2\theta$; $r^2 = a^2\cos 2\theta$.

(iii) $r^2(a^2\sin^2\theta + b^2\cos^2\theta) = a^2b^2$.

(iv) $r = a \sin 3\theta$.

(v) $r = a(\sin 2\theta + \cos 2\theta)$.

(vi) $r^2 = a^2 \cos^2\theta + b^2 \sin^2\theta$.

(vii) $r = 3 + 2\cos\theta$.

22. Show that

(i) the area included between the hyperbolic spiral $r\theta = a$ and any two radii vectors is proportional to the difference between the lengths of those radii vectors ;

(ii) the area included between the logarithmic spiral $r = e^{a\theta}$ and any two radii vectors is proportional to the difference between the squares of those radii vectors.

23. Find the area of a loop of the following curves : $(a > 0)$

(i) $x^4 + y^4 = 2a^2xy$. [*Transform to Polar*.]

(ii) $r^2 = a^2 \cos 2\theta$.

(iii) $r^2 = a^2 \cos 4\theta$.

24. Find the area of the ellipse $9x^2 + 4y^2 - 18x - 16y - 1 = 0$.

25. If for the curve $x(x^2 + y^2) = a(x^2 - y^2)$ $(a > 0)$, A be the area between the curve and its asymptote and L be the area of its loop, show that $A + L = 4a^2$.

26. Show that for the curve

$$y^2(a + x) = x^2(3a - x) \quad (a > 0),$$

the area of its loop and the area between the curve and its asymptote are both equal to $(3\sqrt{3})a^2$.

27. Show that the area included between one of the branches of the curve $x^2y^2 = a^2(x^2 + y^2)$ $(a > 0)$ and the asymptote is equal to the total area of the curve $(x^2 + y^2)^2 = a^2(x^2 - y^2)$ $(a > 0)$.

28. If $p = f(r)$ be the equation of a curve, show that its

$$\text{area} = \frac{1}{2}\int \frac{pr}{\sqrt{(r^2 - p^2)}}\,dr$$

taken between the proper limits.

29. If $p = f(\psi)$ be the equation of a curve, show that its

$$\text{area} = \frac{1}{2}\int p\left(p + \frac{d^2 p}{d\psi^2}\right) d\psi$$

taken between the proper limits.

30. (i) Show that the sectorial area of the equi-angular spiral $p = r\sin\alpha$ included between the two radii vectors r_1 and r_2 is $\frac{1}{4}(r_2^2 - r_1^2)\tan\alpha$.

(ii) Show that the area of the lemniscate $a^2 p = r^3$ is a^2.

[*For half a loop* r *varies from* 0 *to* a.]

31. Find an approximate value of

$$\int_0^{0.2} (1 - 2x^2)^{1/3}\,dx, \text{ taking 2 equal intervals.}$$

Given $f(0.1) = 0.99334$, $f(0.2) = 0.9725$

where $f(x) = (1 - 2x^2)^{1/3}$.

32. Find the approximate value of

$$\int_1^2 \frac{dx}{x}, \text{ taking 10 equal intervals, and calculate the error.}$$

Given
$f(1.1) = 0.90909$ $f(1.6) = 0.62500$
$f(1.2) = 0.83333$ $f(1.7) = 0.58824$
$f(1.3) = 0.76923$ $f(1.8) = 0.55556$
$f(1.4) = 0.71429$ $f(1.9) = 0.52632$
$f(1.5) = 0.66667$

where $f(x) = \dfrac{1}{x}$.

33. Evaluate

$$\int_0^{\frac{1}{2}\pi} \sqrt{(2 + \sin x)}\, dx \text{, using 4 equal intervals,}$$

given when $x = 0°0', 22°30', 45°0', 67°30', 90°0'$,

$$\sqrt{(2 + \sin x)} = 1.414, 1.544, 1.645, 1.710, 1.732.$$

34. Obtain an approximate value of

$\int_0^1 \frac{dx}{1 + x^2}$ taking 4 equal intervals, and hence obtain an approximate value of π correct to four places of decimals.

35. A river is 80 metre wide. The depth d in metre at a distance x metre from one bank is given by the following table :

$x =$	0	10	20	30	40	50	60	70	80
$d =$	0	4	7	9	12	15	14	8	3

Find approximately the area of the cross-section.

36. Use Simpson's rule, taking five ordinates, to find approximately to two places of decimal the value of

$$\int_1^2 \sqrt{(x - 1/x)}\, dx .$$

ANSWERS

1. $c^2 \log \frac{b}{a}$ 2. $4\frac{1}{2}$. 3. 2. 9. (i) π. (ii) 6π.
(iii) $\frac{3}{2}\pi a^2$. (iv) $6\pi a^2$. 10. $\frac{8}{3}$. 11. 3.2. 12. $\frac{1}{2}\pi + \frac{4}{3}$.
13. (ii) $\frac{16}{3}$. 16. (i) $2\sqrt{2}a^2 \sin^{-1}\frac{1}{\sqrt{3}}$. 17. (i) $\frac{4}{3}a^2$. (ii) πa^2.
(iii) a^2. (iv) $\frac{1}{2}\pi(a^2 + b^2)$. (v) $\pi(ab' - a'b)$. (vi) $\frac{8}{3}a^2$.
18. (i) $\frac{8}{15}$. (ii) $\frac{8}{15}a^2$. (iii) $\frac{8}{15}a^{5/2}$. (iv) $2 - \frac{1}{2}\pi$. (v) $\frac{8}{15}a^2$.

19. (i) $2a^2(1-\frac{1}{4}\pi)$. (ii) $a^2(\frac{1}{2}\pi-1)$. (iii) $2a^2(1-\frac{1}{4}\pi)$. (iv) $\frac{16}{3}$.
(v) $\frac{32}{15}\sqrt{2}$. **20.** (i) $4a^2$. (ii) $\frac{3}{4}\pi a^2$. (iii) $2a^2(1+\frac{1}{4}\pi)$. (iv) $2\pi ab$
(v) $4\pi a^2$. **21.** (i) $\frac{1}{4}\pi a^2$. (ii) a^2 ; a^2. (iii) πab. (iv) $\frac{1}{4}\pi a^2$.
(v) πa^2. (vi) $\frac{1}{2}\pi(a^2+b^2)$. (vii) 11π.

23. (i) $\frac{1}{4}\pi a^2$. (ii) $\frac{1}{2}a^2$. (iii) $\frac{1}{4}a^2$. **24.** 6π. **31.** 0.1982.

32. 0.69315 ; error = 0.00001. **33.** 2.546. **34.** 3.1416.

35. 710 m^2. **36.** 0.84.