CHAPTER XI

# LENGTH OF PLANE CURVES

[ *Rectification* ]*

## 11.1. Lengths determined from Cartesian Equations.

We know from Differential Calculus that if $s$ be the length of the arc of a curve measured from a fixed point $A$ on it to any point $P$, whose Cartesian co-ordinates are $(a, b)$ and $(x, y)$ respectively, then

$$\frac{ds}{dx} = \sec\psi = \sqrt{1 + \tan^2\psi} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

$\psi$ denoting the angle made by the tangent at $P$ to the $x$-axis.

Thus, we can write

$$s = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx + C,$$

where $\frac{dy}{dx}$ is expressed in terms of $x$ from the equation to the curve and $C$ is the integration constant. If the indefinite integral

$$\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

be denoted by $F(x)$, then since $s = 0$ when $P$ coincides with $A$, *i.e.*, when $x = a$, we get

$$0 = F(a) + C, \text{ whence } C = -F(a).$$

$$\text{Thus,} \quad s = F(x) - F(a) = \int_a^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

* The process of finding the length of an arc of a curve, *i.e.*, 'of finding a straight line whose length is the same as that of a specified arc' is called *Rectification*. For the definition of the *length* of an arc of a curve, see Authors' *Differential Calculus*, Appendix.

Hence, between two points having $x_1$ and $x_2$ as abscissæ the length of the curve is given by

$$s_2 - s_1 = \int_a^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx - \int_a^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx. \qquad \ldots (1)$$

If it be convenient to get $\frac{dy}{dy}$, and accordingly $\frac{dx}{dy}$, in terms of $y$, instead of $x$, from the equation to the curve, we can use the result

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2},$$

whence the length $AP$ is given by

$$s = \int_b^y \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy,$$

where $\frac{dx}{dy}$ is expressed in terms of $y$.

Also the length of the curve between the two points whose ordinates are $y_1$ and $y_2$ respectively will be

$$s_2 - s_1 = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy. \qquad \ldots (2)$$

If both $x$ and $y$ are expressed in terms of a common variable parameter $t$ and so $s$ is also a function of $t$, we can write

$$\frac{ds}{dt} = \frac{ds}{dx}\frac{dx}{dt} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \frac{dx}{dt}$$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \quad \left(\text{since } \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}\right)$$

Thus, as before, the length of the curve between two points on it for which $t = t_1$ and $t = t_2$ respectively will be given by

$$s_2 - s_1 = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt . \qquad \ldots (3)$$

All the above cases can be included in a single result in the differential form

$$ds = \sqrt{dx^2 + dy^2} , \qquad \ldots (4)$$

where the right-hand side is expressed in the differential form in terms of a single variable from the given equation to the curve . This , when integrated between proper limits , gives the desired length of the curve .

Note . In the above formula (1) , (2) and (3) , it is assumed that $\frac{dy}{dx}, \frac{dx}{dy}, \frac{dx}{dt}, \frac{dy}{dt}$ are all continuous in the range of integration .

## 11 . 2. Illustrative Examples .

Ex. 1. *Find the length of the arc of the parabola $y^2 = 4ax$ measured from the vertex to one extremity of the latus rectum .*

Here, $2y \frac{dy}{dx} = 4a$, or, $\frac{dy}{dx} = \frac{2a}{y} = \frac{2a}{\sqrt{(4ax)}} = \sqrt{\frac{a}{x}}$

The abscissæ of the vertex and one extremity of the latus rectum are 0 and $a$ respectively. Hence , the required length

$$s = \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^a \sqrt{\frac{x + a}{x}}\, dx$$

$$= \int_0^a \frac{x + a}{\sqrt{\{x(x + a)\}}}\, dx$$

$$= \left[ \sqrt{x(x + a)} + a \log(\sqrt{x} + \sqrt{(x + a)}) \right]_0^a$$

$$= a\{\sqrt{2} + \log(1 + \sqrt{2})\} .$$

Ex. 2. *Determine the length of an arc of the cycloid* $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, *measured from the vertex (i.e., the origin).*

Here,
$$\frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$$
$$= a\sqrt{(1 + \cos\theta)^2 + \sin^2\theta} = 2a\cos\tfrac{1}{2}\theta.$$

Also at the origin $\theta = 0$. Hence the required length, from $\theta = 0$ to any point $\theta$, is

$$s = \int_0^\theta 2a\cos\tfrac{1}{2}\theta\, d\theta = 4a\sin\tfrac{1}{2}\theta.$$

Cor. 1. Since at the extremity of the cycloid (i.e., at the cusp) $y = 2a$ we have $\theta = \pi$ there. Thus, *the length of a complete cycloid* being double the length from the vertex to the extremity is $2.4a\sin\frac{1}{2}\pi = 8a$.

Cor. 2. $s^2 = 16a^2\sin^2\frac{1}{2}\theta = 8a.a(1 - \cos\theta) = 8ay$.

Ex. 3. *Find the whole length of the loop of the curve*

$$3ay^2 = x(x - a)^2.$$

We notice here that, for negative values of $x$, $y$ is imaginary and so there is no part of the curve on the negative side of the $x$-axis. Again at the points where $x = 0$ and $x = a$, we have $y = 0$. Between these two points, for every value of $x$ there are equal and opposite values of $y$, a loop being thereby formed. For each value of $x$ greater than $a$, $y$ has two equal and opposite values, and with $x$ increasing, $y$ continually increases in magnitude. The curve is thus traced as in the adjoining figure. The extremities of the loop are given by $x = 0$ and $x = a$.

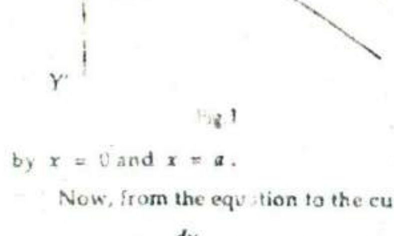

Fig. 1

Now, from the equation to the curve,

$$6ay\frac{dy}{dx} = (x - a)^2 + 2x(x - a) = (x - a)(3x - a);$$

$$\therefore \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{(x-a)^2(3x-a)^2}{36a^2y^2}}$$

$$= \sqrt{1 + \frac{(3x-a)^2}{12ax}} = \frac{3x+a}{2\sqrt{(3ax)}} ;$$

$\therefore$ the half length of the loop is

$$\int_0^a \frac{3x+a}{2\sqrt{(3ax)}}\,dx = \frac{1}{2\sqrt{(3a)}}\left[3 \cdot \frac{2}{3}\,x^{3/2} + a \cdot 2\sqrt{x}\right]_0^a$$

$$= \frac{1}{\sqrt{(3a)}}\left[2a^{3/2}\right] = \frac{2a}{\sqrt{3}} = \frac{2}{3}\sqrt{3}a .$$

The whole length of the loop, therefore, from the symmetry of the curve, $= \frac{4}{3}\sqrt{3}a$ .

## 11.3. Lengths determined from polar equations.

From the formulæ

$$\tan\phi = r\frac{d\theta}{dr} , \ \cos\phi = r\frac{dr}{ds} , \ \sin\phi = r\frac{d\theta}{ds}$$

in Differential Calculus, where $s$ represents the length of the arc of a curve from any fixed point $A$ of it to a variable point $P$ whose polar co-ordinates are $(r, \theta)$ and $\phi$ denotes the angle between the radius vector to the point and the tangent at the point, we can write

$$\frac{1}{r}\frac{ds}{d\theta} = \operatorname{cosec}\phi = \sqrt{1 + \cot^2\phi} = \sqrt{1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2} ,$$

whence $$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}. \qquad \ldots (1)$$

Again,

$$\frac{ds}{dr} = \sec\phi = \sqrt{1 + \tan^2\phi} = \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}. \qquad \ldots (2)$$

From (1) and (2), the length of an arc of the curves can be expressed in either of the forms

$$s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta ,$$

or, $$s = \int_{r_1}^{r_2} \sqrt{1 + r^2 \left(\frac{d\theta}{dr}\right)^2}\, dr,$$

where $r_1, \theta_1$ and $r_2, \theta_2$ are the polar co-ordinates of the extremities of the required arc. In the first form, $r$ as also $\frac{dr}{d\theta}$ are expressed in terms of $\theta$ from the given polar equation to the curve. In the second from, $\frac{d\theta}{dr}$ is expressed in terms of $r$.

Both (1) and (2) can be combined in a single differential form

$$ds = \sqrt{dr^2 + r^2 d\theta^2}.$$

**Note.** It is assumed in the above formula that $\frac{dr}{d\theta}, \frac{d\theta}{dr}$ are continuous in the range of integration.

**Ex.** *Find the perimeter of the Cardioide $r = a(1 - \cos\theta)$, and show that the arc of the upper half of the curve is bisected by $\theta = \frac{2}{3}\pi$.* [C. P. 1949]

Here, since $r = a(1 - \cos\theta)$, $\frac{dr}{d\theta} = a\sin\theta$.

Hence, the length of any arc of the curve, measured from the origin, where $\theta = 0$, to any point, is given by

$$s = \int_0^{\theta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\theta} \sqrt{a^2(1 - \cos\theta)^2 + a^2\sin^2\theta}\, d\theta$$

$$= \int_0^{\theta} 2a\sin\frac{\theta}{2}\, d\theta = 4a\left[-\cos\frac{\theta}{2}\right]_0^{\theta} = 4a\left(1 - \cos\frac{\theta}{2}\right).$$

Thus, *the length of the upper half of the curve*, which clearly extends from $\theta = 0$ to $\theta = \pi$, is $4a(1 - \cos\frac{1}{2}\pi) = 4a$. [See Figure Ex 1. Art 10.6]

*The whole perimeter* is clearly double of this, and thus $= 8a$

Again, the length of the curve from $\theta = 0$ to $\theta = \frac{2}{3}\pi$ is $4a(1 - \cos\frac{1}{3}\pi) = 2a$, and so the line $\theta = \frac{2}{3}\pi$ bisect the arc of the upper nalf of the curve.

## 11.4. Lengths determined from pedal equations.

From the formulæ $\frac{dr}{ds} = \cos\phi$ and $p = r\sin\phi$ in Differential Calculus, we can write

$$\frac{ds}{dr} = \frac{1}{\cos\phi} = \frac{1}{\sqrt{(1-\sin^2\phi)}} = \frac{1}{\sqrt{\left(1-\frac{p^2}{r^2}\right)}} = \frac{r}{\sqrt{(r^2-p^2)}},$$

whence the length of an arc of the curve extending from $r = r_1$ to $r = r_2$ will be given by

$$s = \int_{r_1}^{r_2} \frac{r\,dr}{\sqrt{(r^2-p^2)}},$$

where $p$ is to be replaced in terms of $r$ from the given pedal equation to the curve.

**Ex.** *Find the length of the arc of the parabola $p^2 = ar$ from $r = a$ to $r = 2a$.*

The required length is given by

$$= \int_a^{2a} \frac{r\,dr}{\sqrt{(r^2-p^2)}} = \int_a^{2a} \frac{r\,dr}{\sqrt{(r^2-ar)}}$$

$$= \left[\sqrt{r^2-ar} + a\log(\sqrt{r}+\sqrt{r-a})\right]_a^{2a}$$

$$= a\sqrt{2} + a\log(\sqrt{2}+1) = a\left[\sqrt{2}+\log(1+\sqrt{2})\right].$$

## 11.5. Length of an arc of an Evolute.

We know from Differential Calculus that the difference between the radii of curvature at two points of a given curve is equal to the length of the corresponding arc of its evolute.

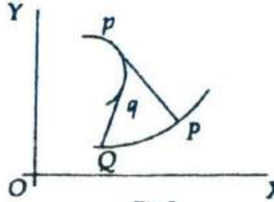

Fig.2

Thus, if $\rho_1$ and $\rho_2$ be the radii of curvature at $P$ and $Q$ of a given curve $PQ$, $p$ and $q$ being the corresponding points n the evolute, the length of the arc $pq$ of the evolute $= \rho_1 - \rho_2$.

In fact $p, q$ are the centres of curvature and so $\overline{Pp}$ and $\overline{Qq}$ are the radii of curvature at $P$ and $Q$ of the curve $PQ$, and if the evolute be regarded as a rigid curve, and a string be unwound from it, being kept tight, then the points of the unwinding string describe a system of parallel curves, one of which is the given curve $PQ$, of which $pq$ is the evolute. $PQ$ is called the involute of $pq$.

Ex. *Calculate the entire length of the evolute of the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad [\text{C. P. 1918}]$$

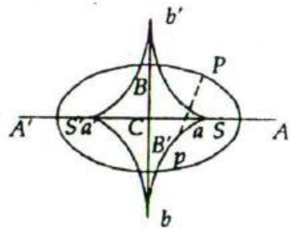


Fig.3

$a, b, a', b'$ being the centres of curvature of the ellipse at $A, B, A', B'$ respectively, the evolute, as shown in the figure, consists of four similar portions, the portion $apb$ corresponding to the part $APB$ of the given ellipse.

Now, from Differential Calculus, it is known that at any point on the ellipse, the radius of curvature

$$\rho = \frac{a^2b^2}{p^3},$$

where $p$ is the perpendicular from the centre on the tangent at the point.

Thus, the length of the arc $apb$ of the evolute

$$= \rho_B - \rho_A = \frac{a^2b^2}{b^3} - \frac{a^2b^2}{a^3} = \frac{a^2}{b} - \frac{b^2}{a}.$$

Hence, the entire length of the evolute of the ellipse

$$= 4\left(\frac{a^2}{b} - \frac{b^2}{a}\right).$$

## 11.6. Intrinsic Equation to a Curve.

If $s$ denotes the length of an arc of a plane curve measured from some fixed point $A$ on it up to an arbitrary point $P$, and if $\psi$ be the inclination of the tangent to the curve at $P$ to any fixed line on the plane ( *e.g.*, the $x$-axis ), the relation between $s$ and $\psi$ is called the *Intrinsic Equation* of the curve.

It should be noted that the intrinsic equation of a curve determines only the form of the curve, and not its position on the plane.

**(A) Intrinsic Equation derived from Cartesian Equation.**

Let the Cartesian equation to the curve be $y = f(x)$. Then $\psi$ denoting the angle between the tangent at any point $P$ and the *x-axis*,

$$\tan \psi = \frac{dy}{dx} = f'(x). \quad \ldots(1)$$

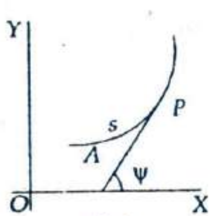

Fig4

$$\text{Also, } s = \text{arc } AP = \int_a^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_a^x \sqrt{1 + [f'(x)]^2}\, dx = F(x), \text{ say}, \quad \ldots (2)$$

'$a$' denoting the abscissa of $A$, and '$x$' that of $P$.

Now, the $x$-eliminant between (1) and (2), ( which will be a relation between $s$ and $\psi$ ), will be the required intrinsic equation of the curve.

If the equation to the curve be given in the *parametric form* $x = f(t)$, $y = \phi(t)$, we can write

$$\tan \psi = \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{\phi'(t)}{f'(t)}. \quad \ldots (1)$$

Also $$s = \int_{t_1}^{t} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_{t_1}^{t} \sqrt{[f'(t)]^2 + [\phi'(t)]^2}\, dt$$

$$= F(t), \text{ say}, \qquad \ldots (2)$$

where $t_1$ is the value of the parameter $t$ at $A$.

The $t$-eliminant between (1) and (2) will be the required intrinsic equation to the curve.

**(B) Intrinsic Equation derived from Polar Equation.**

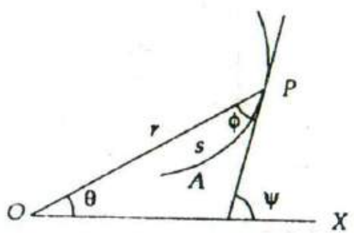


Fig5

Let $r = f(\theta)$ be the polar equation to a curve.

Let $\phi$ denote the angle between the tangent and the radius vector at any point $P(r, \theta)$, $\psi$ the angle made by the tangent with the initial line, and $s$ the length of the arc $AP$ where $A(a, \alpha)$ is a fixed point on the curve.

Then, $$\tan\phi = r\frac{d\theta}{dr} = \frac{f(\theta)}{f'(\theta)}, \qquad \ldots (1)$$

$$\psi = \theta + \phi, \qquad \ldots (2)$$

and $$s = \int_{\alpha}^{\theta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_{\alpha}^{\theta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta = F(\theta), \text{ say}. \qquad \ldots (3)$$

Now, eliminating $\phi$ and $\theta$ between (1), (2) and (3), we get a relation between $s$ and $\psi$, which is the required intrinsic equation of the curve.

**(C) Intrinsic Equation derived from Pedal Equation.**

Let $p = f(r)$ be the pedal equation to the curve.

Then, as in *Art. 11.4*,

$$s = \int_a^r \frac{r\,dr}{\sqrt{(r^2 - p^2)}} = \int_a^r \frac{r\,dr}{\sqrt{(r^2 - \{f(r)\}^2)}} = F(r), \text{ say.} \quad \ldots (1)$$

Also, from Differential Calculus, $\rho$ denoting the radius of curvature,

$$\frac{ds}{d\psi} = \rho = r\frac{dr}{dp} = \frac{r}{f'(r)}. \quad \ldots (2)$$

Eliminating $r$ between (1) and (2), we get a relation of the form

$$\frac{ds}{d\psi} = \phi(s), \text{ or, } \frac{d\psi}{ds} = \frac{1}{\phi(s)} \quad \therefore \psi = \int \frac{ds}{\phi(s)},$$

which, when the right side is integrated, will give the required intrinsic equation.

## 11.7. Illustrative Examples.

Ex. 1. *Obtain the intrinsic equation of the Catenary $y = c\cosh\frac{x}{c}$ in the form $s = c\tan\psi$.*

Here, $\tan\psi = \frac{dy}{dx} = \sinh\frac{x}{c}$. $\quad \ldots (1)$

Also, measuring $s$ from the vertex, where $x = 0$,

$$s = \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^x \sqrt{1 + \sinh^2\frac{x}{c}}\,dx$$

$$= \int_0^x \cosh\frac{x}{c}\,dx = \left[c\sinh\frac{x}{c}\right]_0^x = c\sinh\frac{x}{c}.$$

Hence, from (1), $s = c\tan\psi$.

**Ex. 2.** *Obtain the intrinsic equation of the cycloid*

$$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$$

*taking the vertex as the fixed point and the tangent at that point as the fixed line.*

As shown in Ex. 2, Art. 11.2, the length of the arc of the above cycloid measured from the vertex is given by

$$s = 4a \sin\frac{\theta}{2} \qquad \ldots (1)$$

Also, $\tan\psi = \dfrac{dy}{dx} = \dfrac{dy}{d\theta} \Big/ \dfrac{dx}{d\theta} = \dfrac{a\sin\theta}{a(1+\cos\theta)} = \tan\dfrac{\theta}{2}$.

$\therefore\quad \psi = \dfrac{\theta}{2}$. Hence, from (1), $s = 4a\sin\psi$,

which is the required intrinsic equation

**Ex. 3.** *Find the intrinsic equation of the Cardioide*

$$r = a(1 - \cos\theta),$$

*the arc being measured from the cusp ( i.e., where $\theta = 0$ ).*

Here, $\psi = \theta + \phi \qquad \ldots (1)$

and $\tan\phi$, *i.e.*, $r\dfrac{d\theta}{dr} = \dfrac{1-\cos\theta}{\sin\theta} = \tan\dfrac{\theta}{2}$.

$\therefore \phi = \frac{1}{2}\theta. \qquad \ldots (2)$

Also by the Ex., Art. 11.3, we have

$$s = \int_0^\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = 4a\left(1 - \cos\frac{\theta}{2}\right). \qquad \ldots (3)$$

Since, from (1) and (2), $\psi = \theta + \frac{1}{2}\theta = \frac{3}{2}\theta$, i.e., $\theta = \frac{2}{3}\psi$,

$\therefore$ from (3), $s = 4a(1 - \cos\frac{1}{3}\psi)$, the required intrinsic equation.

**Ex. 4.** *Find the Cartesian equation of the curve for which the intrinsic equation is $s = a\psi$.*

Here, $\dfrac{dx}{d\psi} = \dfrac{dx}{ds}\dfrac{ds}{d\psi} = \cos\psi . a$.

$\therefore\quad dx = a\cos\psi\, d\psi. \qquad \therefore x = a\sin\psi + c. \qquad \ldots (1)$

Again, $\dfrac{dy}{d\psi} = \dfrac{dy}{ds}\dfrac{ds}{d\psi} = \sin\psi . a$.

$\therefore\quad dy = a\sin\psi\, d\psi. \qquad \therefore y = -a\cos\psi + d. \qquad \ldots (2)$

From (1) and (2), eliminating $\psi$, we get

$(x - c)^2 + (y - d)^2 = a^2$, the required Cartesian equation.

## EXAMPLES XI

1. Find the lengths of the following :

(i) the perimeter of the circle $x^2 + y^2 = a^2$ ;

(ii) the arc of the catenary $y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$ from the vertex to the point $(x_1, y_1)$ ;

(iii) the perimeter of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ ;

(iv) the perimeter of the hypo-cycloid $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$ ;

(v) the perimeter of the evolute

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3} ;$$

(vi) the arc of the semi-cubical parabola $ay^2 = x^3$ from the cusp to any point $(x, y)$.

2. If $s$ be the length of an arc of $3ay^2 = x(x - a)^2$ measured from the origin to the point $(x, y)$, show that $3s^2 = 4x^2 + 3y^2$.

3. Show that the length of the arc of the parabola $y^2 = 4ax$ which is intercepted between the points of intersection of the parabola and the straight line $3y = 8x$ is $a\left(\log 2 + \frac{15}{16}\right)$.

4. Show that the complete perimeter of the curve

$$x = \frac{1 - t^2}{1 + t^2},\ y = \frac{2t}{1 + t^2} \text{ is } 2\pi$$

5. If for a curve

$$x \sin\theta + y \cos\theta = f'(\theta)$$

and $$x \cos\theta - y \sin\theta = f''(\theta),$$

show that $$s = f(\theta) + f''(\theta) + c, \text{ where } c \text{ is a constant.}$$

**6.** Find the length of the arcs of the following curves :

(i) $\left.\begin{array}{l} x = e^{\theta} \sin \theta \\ y = e^{\theta} \cos \theta \end{array}\right\}$ from $\theta = 0$ to $\theta = \frac{1}{2}\pi$.

(ii) $\left.\begin{array}{l} x = a(\cos \theta + \theta \sin \theta) \\ y = a(\sin \theta - \theta \cos \theta) \end{array}\right\}$ from $\theta = 0$ to $\theta = \theta_1$.

(iii) $\left.\begin{array}{l} x = c \sin 2\theta (1 + \cos 2\theta) \\ y = c \cos 2\theta (1 - \cos 2\theta) \end{array}\right\}$

from the origin to any point.

**7.** Show that the perimeter of the ellipse $x = a \cos \theta$, $y = b \sin \theta$, is given by

$$2a\pi \left\{ 1 - \left(\frac{1}{2}\right)^2 \frac{e^2}{1} - \left(\frac{1.3}{2.4}\right)^2 \frac{e^4}{3} - \left(\frac{1.3.5}{2.4.6}\right)^2 \frac{e^6}{5} - \dots \right\}.$$

**8.** Compare the perimeters of the two conics

$$\frac{x^2}{9} + \frac{y^2}{7} = 1 \text{ and } \frac{x^2}{36} + \frac{y^2}{28} = 1.$$

**9.** Find the lengths of the loop of each of the following curves:

(i) $9y^2 = (x + 7)(x + 4)^2$ ;

(ii) $x = t^2, y = t - \frac{1}{3}t^3$.

**10.** Find the lengths of the following :

(i) a quadrant of the circle $r = 2a \sin \theta$ ;

(ii) the arc of the parabola $r(1 + \cos \theta) = 2$ from $\theta = 0$ to $\theta = \frac{1}{2}\pi$.

(iii) the arc of the equi-angular spiral $r = ae^{\theta \cot \alpha}$ between the radii vectors $r_1$ and $r_2$.

**11.** If $s$ be the length of the curve $r = a \tanh \frac{1}{2}\theta$ between the origin and $\theta = 2\pi$, and $\Delta$ the area between the same points, show that $\Delta = a(s - a\pi)$.

12. Show that the area between the curve

$$y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right),$$

the $x$-axis and the ordinates at two points on the curve is equal to $a$ times the length of the arc terminated by those points.

13. Show that in the astroid $x^{2/3} + y^{2/3} = a^{2/3}$,

(i) $s \propto x^{2/3}$,

(ii) $\rho^2 + 4s^2 = 6as$,

$s$ being measured from the point for which $x = 0$.

14. Show that

(i) in the cycloid $x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$,

$$\rho^2 + s^2 = 16a^2,$$

the arc being measured from the vertex ( where $\theta = 0$ );

(ii) in the catenary $y = c \cosh(x/c)$,

$$y^2 = c\rho = c^2 + s^2,$$

the arc being measured from the vertex ;

(iii) in the cardioide $r = a(1 + \cos\theta)$, $s^2 + 9\rho^2 = 16a^2$, the arc being measured from the vertex ( *i.e.*, $\theta = 0$ ).

15. Show that the length of the arc of the hyperbola $xy = a^2$ between the points $x = b$ and $x = c$ is equal to the arc of the curve $p^2(a^4 + r^4) = a^4r^2$ between the limits $r = b$ and $r = c$.

16. Show that the length of the arc of the evolute $27ay^2 = 4(x - 2a)^3$ of the parabola $y^2 = 4ax$, from the cusp to one of the points where the evolute meets the parabola, is $2a(3\sqrt{3} - 1)$.

17. Find the intrinsic equation of each of the following curves, the fixed point from which the arc is measured being indicated in each case :

(i) the parabola $y^2 = 4ax$ ... (vertex),

(ii) the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ ... (one of the cusps),

(iii) the semi-cubical parabola $ay^2 = x^3$ ... (cusp),

(iv) the curve $y = a \log \sec (x/a)$ ... (origin),

(v) the equi-angular spiral $r = ae^{\theta \cot \alpha}$ ... (point $a$, 0),

(vi) the involute of the circle, *viz.*,

$$\theta = \frac{\sqrt{r^2 - a^2}}{a} - \cot^{-1} \frac{a}{r} \ldots \text{(point } a, 0).$$

**18.** Find the intrinsic equation of each of the following curves:

(i) $p = r \sin \alpha$,

(ii) $p^2 = r^2 - a^2$.

**19.** Find the intrinsic equation of the curve for which the length of the arc measured from the origin varies as the square root of the ordinate. Also obtain the Cartesian co-ordinates of any point on the curve in terms of any parameter.

**20.** If $s = c \tan \psi$ is the intrinsic equation of a curve, show that the Cartesian equation is $y = c \cosh (x/c)$, given that when $\psi = 0$, $x = 0$ and $y = c$.

## ANSWERS

**1.** (i) $2\pi a$; (ii) $\frac{1}{2}a\left(e^{x_1/a} - e^{-x_1/a}\right)$; (iii) $6a$;

(iv) $4\,\frac{a^2 + ab + b^2}{a + b}$; (v) $4\left(\frac{a^2}{b} - \frac{b^2}{a}\right)$;

(vi) $\frac{8a}{27}\left\{\left(1 + \frac{9x}{4a}\right)^{3/2} - 1\right\}$.

**6.** (i) $\sqrt{2}\,(e^{\pi/2} - 1)$ (ii) $\frac{1}{2}\,a\theta_1^2$. (iii) $\frac{4}{3}c \sin 3\theta$.

**8.** (i) 1 : 2. **9.** (i) $4\sqrt{3}$; (ii) $4\sqrt{3}$.

**10.** (i) $\frac{1}{2}\pi a$, (ii) $\sqrt{2} + \log(\sqrt{2} + 1)$; (iii) $(r_2 - r_1)\sec \alpha$.

**17.** (i) $s = a \operatorname{cosec} \psi \cot \psi + a \log(\operatorname{cosec} \psi + \cot \psi)$;

(ii) $s = \frac{3}{2}a \sin^2 \psi$;

(iii) $27s = 8a(\sec^3\psi - 1)$; (iv) $s = a \log \tan(\frac{1}{2}\psi + \frac{1}{4}\pi)$;

(v) $s = a \sec\alpha \{e^{(\psi-\alpha)\cot\alpha} - 1\}$; (vi) $s = \frac{1}{2}a\psi^2$.

18. (i) $s = Ce^{\psi\cot\alpha}$; (ii) $s = \frac{1}{2}a\psi^2$.

19. $s = 4a \sin\psi$; $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$.

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CHAPTER XII

# VOLUMES AND SURFACE-AREAS OF SOLIDS OF REVOLUTION

## 12.1. Solids of revolution, the axis of revolution being the x-axis.

Let a curve $LM$, whose Cartesian equation is given by $y = f(x)$ say, be rotated about the x-axis so as to form a solid of

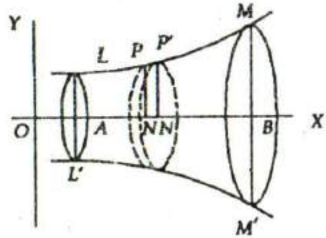



Fig. 1

revolution, and let us consider the portion $LL'M'M$ of this solid bounded by $x = x_1$ and $x = x_2$. We can imagine this solid to be divided into an infinite number of infinitely thin circular slices by planes perpendicular to the axis of revolution $\overrightarrow{OX}$. If $\overline{PN}$ and $P'N'$ be two adjacent ordinates of the curve, where the co-ordinates of $P$ and $P'$ are $(x, y)$ and $(x + \Delta x, y + \Delta y)$ respectively, the volume of the corresponding slice, which has its thickness $\Delta x$, is ultimately equal to $\pi y^2 \Delta x$.*

Hence, the total volume of the solid considered ( bounded by $x = x_1$ and $x = x_2$ ) is given by

$$V = \underset{\Delta x \to 0}{Lt} \Sigma\, \pi y^2 \Delta x = \pi \int_{x_1}^{x_2} y^2\, dx.$$

* Strictly, the volume of the slice between $\pi y_1{}^2 \Delta x$ and $\pi y_2{}^2 \Delta x$ where $y_1$ and $y_2$ are the greatest and the least values of $y$ within the range $PP'$ and thus equals $\pi y^2 \Delta x$, where $y$ lies between $y_1$ and $y_2$ and is thus the ordinate for some point within the range $PP'$ ( not necessarily of $P$ ). Thus, $Lt\ \Sigma \pi y^2 \Delta x = \int y^2\, dx$. [ See *Art. 6.2. Note 2.* ]

length measured upto $P$ from any fixed point on the curve $LM$, the surface-area of the ring-shaped element generated by rotating $PP'$ is ultimately $2\pi y \,.\, \Delta s$.

Hence, the required surface-area is given by

$$S = \underset{\Delta s \to 0}{Lt} \Sigma ( 2\pi y \Delta s ) = 2\pi \int_{s_1}^{s_2} y \, ds$$

[ $s_1$ , $s_2$ being the values of $s$ for the points $L$, $M$ ]

$$= 2\pi \int_{x_1}^{x_2} y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx .$$

**Cor. 1.** When the *axis of revolution is the y-axis*, and we consider the portion of the solid bounded by $y = y_1$ and $y = y_2$ respectively,

$$V = \pi \int_{y_1}^{y_2} x^2 \, dy ,$$

and $$S = 2\pi \int_{s_1}^{s_2} x \, ds = 2\pi \int_{y_1}^{y_2} x \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy.$$

**Cor. 2.** Even if the curve revolved be given by *its polar equation ( the axis of revolution being the initial line )*, and the portion of the volume considered be bounded by two parallel planes perpendicular to the initial line, we may change to corresponding Cartesian co-ordinates, with the initial line as the $x$-axis, by writing $x = r \cos \theta$, $y = r \sin \theta$.

Thus,

$$V = \pi \int_{x_1}^{x_2} y^2 \, dx = \pi \int_{\theta_1}^{\theta_2} r^2 \sin^2 \theta \,.\, d ( r \cos \theta ),$$

$$S = 2\pi \int_{s_1}^{s_2} y \, ds = 2\pi \int_{\theta_1}^{\theta_2} r \sin \theta \,.\, \sqrt{dr^2 + r^2 d\theta^2},$$

where $r$ is expressed in terms of $\theta$ from the given equation of the curve, or, if convenient, we may use $r$ as the independent variable and express $\theta$ in terms of $r$ from the equation, the limits being the corresponding values of $r$.

*Alternative proof of*

( *i* ) *Volume of a solid of revolution.*

Let a curve $CD$, whose equation is $y = f(x)$, be rotated about the $x$-axis so as to form a solid of revolution. To find the volume of the solid generated by the revolution about the $x$-axis, of the area $ABDC$ bounded by the curve $y = f(x)$, the ordinates at $A$ and $B$ and the $x$-axis, let $a$ and $b$ be the abscissæ of $C$ and $D$.

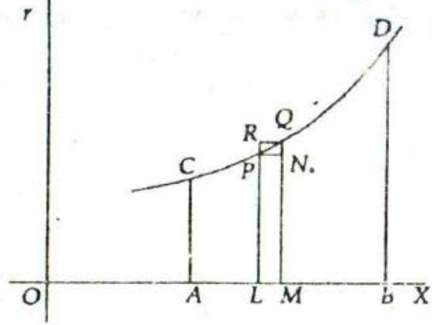



Fig.2

Divide $\overline{AB}$ into $n$ equal parts, each equal to $h$, and draw ordinates at the points of division. Let the ordinates at $x = a + rh$ and $x = a + (r+1)h$ be $PL$ and $QM$, and let us suppose $y$ goes on increasing as $x$ increases from $a$ to $b$.

Draw $\overline{PN}$ perpendicular on $\overline{QM}$, and $\overline{QR}$ perpendicular on $\overline{LP}$ produced. Then the volume of the solid generated by the revolution of the area $LMQP$ lies in magnitude between the volumes generated by the rectangles $LMNP$ and $LMQR$,

*i.e.*, between $\pi[f(a+rh)]^2 h$ and $\pi[f(a+(r+1)h)]^2 h$.

Hence, adding up the volumes generated by all areas like $LMQP$, it is clear that the required volume lies in magnitude between

$$\pi \sum_{r=0}^{n-1} [f(a+rh)]^2 h \text{ and } \pi \sum_{r=0}^{n-1} [f(a+(r+1)h)]^2 h.$$

Now let $n \to \infty$, so that $h \to 0$; then as the limit of each of the above two sums is

$$\pi \int_a^b [f(x)]^2 \, dx \text{, i.e., } \pi \int_a^b y^2 \, dx,$$

it follows that the required volume is also equal to this definite integral.

( ii ) *Surface-area of a solid of revolution.*

Let the length of the arc from $C$ upto any point $P(x, y)$ be $s$ and suppose that the surface-area of the solid generated by the revolution of the arc $CD$ about the $x$-axis is required. As in the case of the volume, divide $\overline{AB}$ into $n$ equal parts, each equal to $h$, and erect ordinates at the points of division. Let the ordinates at $x = a + rh$ and $x = a + (r + 1)h$ be $PL$ and $QM$, and let the arc $PQ$ be equal to $l$. The surface-area of the solid generated by the revolution of $LMQP$ about the $x$-axis lies in magnitude between the curved surface of two right circular cylinders, *each of thickness* $l$ one of radius $PL$ and the other of radius $QM$, *i.e.*, between

$$2\pi f(a + rh) l \text{ and } 2\pi f\{a + (r + 1)h\} l.$$

Hence, adding up all surface-areas generated by elementary areas like $PQ$, it is clear that the required surface-area lies in magnitude between

$$2\pi \sum_{r=0}^{n-1} \frac{l}{h} f(a + rh) h \text{ and } 2\pi \sum_{r=0}^{n-1} \frac{l}{h} f\{a + (r + 1)h\} h.$$

Now let $n \to \infty$, so that $h \to 0$; then $l / h$ tending to $\frac{ds}{dx}$, the limit of each of the above two sums is

$$2\pi \int_a^b f(x) \frac{ds}{dx} \, dx \text{, i.e., } 2\pi \int_a^b y \, ds.$$

Hence, the required surface-area is also equal to this definite gral.

## 12.2. Illustrative Examples.

**Ex. 1.** *Find the volume and area of the curved surface of a paraboloid of revolution formed by revolving the parabola $y^2 = 4ax$ about the x-axis, and bounded by the section $x = x_1$.*

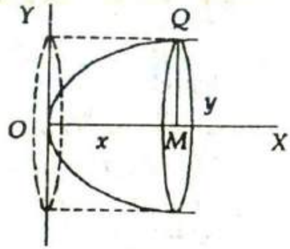


Fig.3

Here, $y = 2\sqrt{ax}$.

$\therefore \frac{dy}{dx} = \sqrt{\frac{a}{x}}$.

Now the required volume

$$V = \pi \int_0^{x_1} y^2\, dx = \pi \int_0^{x_1} 4ax\, dx = 2\pi a x_1^2 = \tfrac{1}{2}\pi x_1 y_1^2$$

( where $y_1$ is the extreme ordinate, so that $y_1^2 = 4ax_1^2$ ) $= \frac{1}{2} . \pi y_1^2 x_1 = \frac{1}{2}$ . ( *the volume of the corresponding cylinder*, with the extreme circular section as the base and height equal to be abscissa ).

Also, the required surface-area

$$S = 2\pi \int_0^{x_1} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_0^{x_1} \sqrt{4ax} \sqrt{1 + \frac{a}{x}}\, dx$$

$$= 4\pi \sqrt{a} \int_0^{x_1} \sqrt{a + x}\, dx = \frac{8}{3}\pi \sqrt{a}\, \{(a + x_1)^{3/2} - a^{3/2}\}.$$

**Ex. 2.** *The part of the parabola $y^2 = 4ax$ bounded by the latus rectum revolves about the tangent at the vertex. Find the volume and the area of the curved surface of the reel thus generated.*

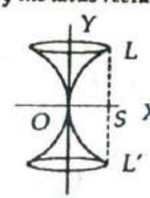


Fig.3

Here the axis of revolution being the y-axis, and the extreme values of $y$ being evidently $\pm 2a$,

the required volume

$$V = \pi \int_{-2a}^{+2a} x^2\, dy = \pi \int_{-2a}^{+2a} \frac{y^4}{16a^2}\, dy \qquad [\text{ since } y^2 = 4ax ]$$

$$= \frac{\pi}{16a^2}\, 2\, \frac{(2a)^5}{5} = \frac{4}{5}\pi a^3$$

Also the required surface-area

$$S = 2\pi \int x\, ds = 2\pi \int_{-2a}^{+2a} x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= 2\pi \int_{-2a}^{+2a} \frac{y^2}{4a} \sqrt{1 + \frac{y^2}{4a^2}}\, dy \quad \left[\text{since } \frac{dx}{dy} = \frac{y}{2a}\right]$$

$$= 4\pi a^2 \int_{-\frac{1}{4}\pi}^{+\frac{1}{4}\pi} \tan^2\theta \sec^3\theta\, d\theta \quad [\text{putting } y = 2a\tan\theta]$$

$$= 4\pi a^2 \int_{-\frac{1}{4}\pi}^{+\frac{1}{4}\pi} (\sec^3\theta - \sec^2\theta)\, d\theta$$

$$= 4\pi a^2 \left[\tfrac{1}{4}\tan\theta\sec^2\theta - \tfrac{1}{8}\tan\theta\sec\theta - \tfrac{1}{8}\log\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\theta)\right]_{-\frac{1}{4}\pi}^{+\frac{1}{4}\pi}$$

$$= 4\pi a^2 [\tfrac{3}{4}\sqrt{2} - \tfrac{1}{4}\log\cot\tfrac{1}{8}\pi] = \pi a^2 [3\sqrt{2} - \log(\sqrt{2}+1)].$$

Ex. 3. *Find the volume and the surface-area of the solid generated by revolving the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ about its base.*

The equations show that the cycloid has the $x$-axis as its base ; the extreme values of $x$ are given by $\theta = \pm\pi$, *i.e.,* $x = \pm a\pi$.

The required volume

$$V = \pi \int_{-a\pi}^{a\pi} y^2\, dx = \pi a^3 \int_{-\pi}^{\pi} (1 + \cos\theta)^3\, d\theta$$

$$= 8\pi a^3 \int_{-\pi}^{\pi} \cos^6 \tfrac{1}{2}\theta\, d\theta = 8\pi a^3 \cdot \frac{5}{8}\pi = 5\pi^2 a^3.$$

The required surface-area

$$S = 2\pi \int y\, ds = 2\pi \int y\sqrt{dx^2 + dy^2}$$

$$= 2\pi \int_{-\pi}^{\pi} a(1+\cos\theta) . \sqrt{\{a(1+\cos\theta)d\theta\}^2 + (-a\sin\theta\, d\theta)^2}$$

$$= 2\pi a^2 \int_{-\pi}^{\pi} (1+\cos\theta)\sqrt{2(1+\cos\theta)}\, d\theta$$

$$= 8\pi a^2 \int_{-\pi}^{\pi} \cos^3 \tfrac{1}{2}\theta\, d\theta = 8\pi a^2 . \frac{8}{3} = \frac{64}{3}\pi a^2 .$$

**Ex. 4.** *Find the volume and the surface-area of the solid generated by evolving the cardioide $r = a(1-\cos\theta)$ about the initial line.*

Here, since the curve is symmetrical about the initial line, the solid of revolution might as well be considered to be formed by revolving the upper half of the curve about the initial line. The extreme points of the curve are given by $\theta = 0$ and $\theta = \pi$.

The required volume

$$V = \pi \int y^2\, dx = \pi \int r^2 \sin^2\theta . d(r\cos\theta)$$

$$= \pi a^3 \int (1-\cos\theta)^2 \sin^2\theta . d\{(1-\cos\theta)\cos\theta\}$$

$$= \pi a^3 \int_{\pi}^{0} (1-\cos\theta)^2 \sin^2\theta(-\sin\theta + 2\sin\theta\cos\theta)\, d\theta$$

[ *x increases as θ diminishes from π to 0* ]

$$= \pi a^3 \int_{-1}^{+1} (1-z)^2(1-z^2)(1-2z)\, dz \quad [\text{putting } z = \cos\theta]$$

$$= \tfrac{8}{3}\pi a^3 .$$

The required surface-area

$$S = 2\pi \int y\, ds = 2\pi \int r\sin\theta . \sqrt{dr^2 + r^2 d\theta^2}$$

$$= 2\pi \int_0^{\pi} a(1-\cos\theta)\sin\theta . \sqrt{(a\sin\theta\, d\theta)^2 + a^2(1-\cos\theta)^2\, d\theta^2}$$

$$= 2\pi a^2 \int_0^{\pi} (1-\cos\theta)\sin\theta\sqrt{2(1-\cos\theta)}\, d\theta$$

$$= 2\sqrt{2}\pi a^2 \int_0^2 z^{3/2}\, dz \qquad [\text{ putting } z = 1 - \cos\theta\,]$$

$$= 2\sqrt{2}\pi a^2 \cdot \tfrac{2}{5}(2)^{5/2} = \tfrac{32}{5}\pi a^2 .$$

## 12.3. Solids of revolution, axis of revolution being any line in the plane.

If the given curve $LM$ be revolved about any line $\overline{AB}$ its plane,

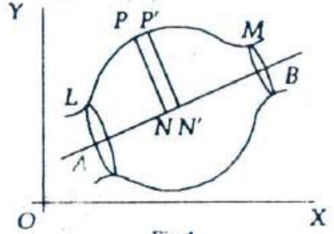



Fig.4

and the portion considered of the solid of revolution formed be bounded by the planes perpendicular to $\overline{AB}$ through the points $A$ and $B$ respectively, then $\overline{PN}$ being the perpendicular on $\overline{AB}$ from any point $P$ on the curve, $\overline{P'N'}$ the contiguous perpendicular, the volume of the portion considered is given by

$$V = Lt\ \Sigma\pi \,.\, PN^2 \,.\, NN' = \pi \int_0^{AB} PN^2 \,.\, d\,(AN)\,.$$

Also, the surface-area of the portion considered is given by

$$S = Lt\, \Sigma 2\pi \,.\, PN \,.\, (\text{ elementary arc } PP'\,) = 2\pi \int PN \,.\, ds\,.$$

From the given equation of the curve and of the line, $\overline{AB}$, $\overline{PN}$, as also $\overline{AN}$ and $ds$ are expressed in terms of a single variable, and the corresponding values of the variable for the points $A$ and $B$ are taken as the limits of integration

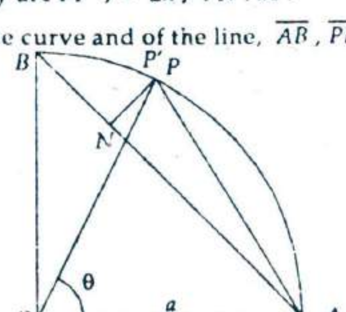



Fig.5

**Ex.** *A quadrant of a circle, of ·s a, revolves round its chord. · volume and the surface-area lid spindle thus generated.*

$P$ being any point on the quadrant $APB$, where $m\,\angle AOP = \theta$, clearly $AP = 2a \sin \frac{1}{2}\theta$ and $m\,\angle PAN = \frac{1}{2}\, m\,\angle POB = (\frac{1}{2}\pi - \theta)$.

$\therefore\ PN = 2a \sin \frac{1}{2}\theta \sin(\frac{1}{4}\pi - \frac{1}{2}\theta) = a\,\{\cos(\theta - \frac{1}{4}\pi) - \cos\frac{1}{4}\pi\}$;

$AN = 2a \sin \frac{1}{2}\theta \cos(\frac{1}{4}\pi - \frac{1}{2}\theta) = a\,\{\sin\frac{1}{4}\pi + \sin(\theta - \frac{1}{4}\pi)\}$.

Elementary arc $PP' = a\,d\theta$.

Also, for the solid formed, limits of $\theta$ are 0 and $\frac{1}{2}\pi$ respectively.

Hence, $V = \pi \int PN^2 \,.\, d(AN)$

$$= \pi a^3 \int_0^{\frac{1}{2}\pi} \{\cos(\theta - \tfrac{1}{4}\pi) - \cos\tfrac{1}{4}\pi\}^2 \cos(\theta - \tfrac{1}{4}\pi)\, d\theta$$

$$= \pi a^3 \int_0^{\frac{1}{2}\pi} [\cos^2(\theta - \tfrac{1}{4}\pi) - \sqrt{2}\cos^2(\theta - \tfrac{1}{4}\pi) + \tfrac{1}{2}\cos(\theta - \tfrac{1}{4}\pi)]\, d\theta$$

$$= \pi a^3 \int_0^{\frac{1}{2}\pi} \Big[ \tfrac{1}{4}\cos(3\theta - \tfrac{3}{4}\pi) + \tfrac{3}{4}\cos(\theta - \tfrac{1}{4}\pi) - \frac{1}{\sqrt{2}}\{\cos(2\theta - \tfrac{1}{2}\pi) + 1\} \Big]\, d\theta$$

$$= \pi a^3 \Big[ \frac{1}{12}\sin(3\theta - \tfrac{3}{4}\pi) - \frac{1}{2\sqrt{2}}\sin(2\theta - \tfrac{1}{2}\pi) + \frac{5}{4}(\theta - \tfrac{1}{4}\pi) - \frac{1}{\sqrt{2}}\theta \Big]_0^{\frac{1}{2}\pi}$$

$$= \pi a^3 \left( \frac{10 - 3\pi}{6\sqrt{2}} \right).$$

Also, $S = 2\pi \int_0^{\frac{1}{2}\pi} PN \,.\, a\, d\theta$

$$= 2\pi a^2 \int_0^{\frac{1}{2}\pi} \{\cos(\theta - \tfrac{1}{4}\pi) - \cos\tfrac{1}{4}\pi\}\, d\theta$$

$$= 2\pi a^2 \Big[ \sin(\theta - \tfrac{1}{4}\pi) - \frac{1}{\sqrt{2}}\theta \Big]_0^{\frac{1}{2}\pi}$$

$$= 2\pi a^2 \left( \frac{2}{\sqrt{2}} - \frac{\pi}{2\sqrt{2}} \right) = \pi a^2 \left( \frac{4 - \pi}{\sqrt{2}} \right).$$

## 12.4. Theorem of Pappus or Guldin.

*If a plane area bounded by a closed curve revolves through any angle about a straight line in its own plane, which does not intersect the curve, then*

*(I) The volume of the solid generated is equal to the product of the revolving area and the length of the arc described by the centroid of the area.*

*(II) The surface-area of the solid generated is equal to the product of the perimeter of the revolving area into the length of the arc described by the centroid of that perimeter.*

*Proof.*

(I) Let $\delta A$ be any element of the area whose distance from the axis of rotation is $z$. Then, $\theta$ being the angle through which the

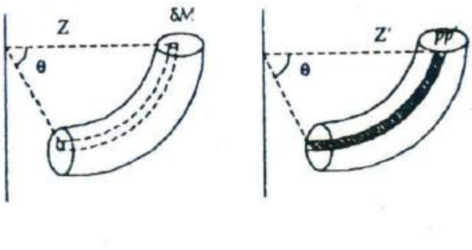



Fig.6

area is rotated, the length of the arc described by $\delta A$ is $z\theta$, and hence the elementary volume described by the element $\delta A$ is $z\theta \cdot \delta A$.

The whole volume described by the given area therefore

$= \Sigma z\theta\ \delta A = \theta \Sigma z \cdot \delta A = \theta \bar{z} A$ ( from *Elementary Statics* )
[ *where* $A$ is the total area of the curve and $\bar{z}$ is the distance of its centroid from the axis of revolution ]

$= A\bar{z}\theta$ = area of the closed curve × length of the arc described by its centroid.

(II) Let $\delta s$ be the length of any element $PP'$ of the perimeter of the given curve, and $z'$ its distance from the axis of revolution. The elementary surface traced out by the element $\delta s$ is ultimately $z'\theta \cdot \delta s$.

The total surface-area of the solid generated is therefore

$$= \Sigma z'\theta.\delta s = \theta \Sigma z' . \delta s = \theta \bar{z}' s \quad (\text{ from } \textit{Elementary Statics} )$$

[ where $s$ is the whole perimeter of the curve, and $\bar{z}'$ the distance of the centroid of this perimeter from the axis ]

$= s\bar{z}'\theta$ = perimeter × length of the arc described by its centroid.

Note. The above results hold even if the axis of rotation touches the closed curve.

**Ex. 1.** *Find the volume and surface-area of a solid tyre, a being the radius of its section, and b that of the core.*

The tyre is clearly generated by revolving a circle of radius $a$ about an axis whose distance from the centre of the circle is $b$

The centre of the circle is the centroid of both the area of the circle as also of the perimeter of the circle, and the length of the path described by it is evidently $2\pi b$.

Hence, the required volume $= \pi a^2 \times 2\pi b = 2\pi^2 a^2 b$
and the required surface-area $= 2\pi a \times 2\pi b = 4\pi^2 ab$.

**Ex. 2.** *Show that the volume of the solid formed by the rotation about the line $\theta = 0$ of the area bounded by the curve $r = f(\theta)$ and the lines $\theta = \theta_1$, $\theta = \theta_2$ is*

$$\tfrac{2}{3}\pi \int_{\theta_1}^{\theta_2} r^3 \sin\theta \, d\theta .$$

*Hence, find the volume of the solid generated by revolving the cardioide $r = a(1 - \cos\theta)$ about the initial line.*

Dividing the area in question into an infinite number of elementary areas ( as in the *figure, § 10.5* ) by radial lines through the origin, let us consider one such elementary area bounded by the radii vectors inclined at angles $\theta$ and $\theta + d\theta$ to the initial line, their lengths being $r$ and $r + dr$, say. This elementary area is ultimately in the form of a triangle, whose area is $\frac{1}{2} r ( r + dr ) \sin d\theta$, *i.e.*, $\frac{1}{2} r^2 \, d\theta$ upto the first order. Its C.G. is, neglecting infinitesimals, at a distance $\frac{2}{3} r$ from the origin and its perpendicular distance from the initial line is ultimately $\frac{2}{3} r \sin \theta$. The elementary volume obtained by revolving the elementary area about the initial line is therefore, by Pappus' theorem, ultimately equal to

$$2\pi \cdot \tfrac{2}{3} r \sin\theta \cdot \tfrac{1}{2} r^2 \, d\theta = \tfrac{2}{3} \pi r^3 \sin\theta \, d\theta .$$

Hence, integrating between the extreme limits $\theta = \theta_1$ and $\theta = \theta_2$, the total volume of the solid of revolution in question is

$$\tfrac{2}{3}\pi \int_{\theta_1}^{\theta_2} r^3 \sin\theta \, d\theta.$$

In case of the cardioide $r = a(1 - \cos\theta)$, the extreme limits for $\theta$ are easily seen to be 0 and $\pi$, and so the volume of the solid of revolution generated by it is

$$\tfrac{2}{3}\pi \int_0^{\pi} a^3 (1 - \cos\theta)^3 \sin\theta \, d\theta \text{, which on putting } 1 - \cos\theta = z$$

easily reduces to

$$\tfrac{2}{3}\pi a^3 \int_0^2 z^3 \, dz = \frac{2}{3}\pi a^3 \cdot \frac{2^4}{4} = \frac{8}{3}\pi a^3 .$$

## EXAMPLES XII

**1.** Find the volumes of the solids generated by revolving, about the $x$-axis, the areas bounded by the following curves and lines :

(i) $y = \sin x ; x = 0 ; x = \pi$.

(ii) $y = 5x - x^2 ; x = 0 ; x = 5$.

(iii) $y^2 = 9x ; y = 3x$.

(iv) $\sqrt{x} + \sqrt{y} = \sqrt{a} ; x = 0 ; y = 0$.

**2.** Show that the volume of a right circular cone of height $h$ and base of radius $a$ is $\frac{1}{3}\pi a^2 h$.

3. The circle $x^2 + y^2 = a^2$ revolves round the $x$-axis ; show that the surface-area and the volume of the whole sphere generated are respectively $4\pi a^2$ and $\frac{4}{3}\pi a^3$.

4. Prove that the surface area and the volume of the ellipsoid formed by the revolution of the ellipse $x^2/a^2 + y^2/b^2 = 1$

(i) round its major axis are respectively

$$2\pi ab \{ \sqrt{1 - e^2} + e^{-1} \sin^{-1} e \} \text{ and } \tfrac{4}{3}\pi ab^2,$$

and (ii) round its minor axis are respectively

$$2\pi \left\{ a^2 + \frac{b^2}{e} \log \sqrt{\frac{1 + e}{1 - e}} \right\} \text{ and } \frac{4}{3}\pi a^2 b.$$

5. Show that the curved surface and volume of the catenoid formed by the revolution, about the $x$-axis, of the area bounded by the catenary $y = \frac{a}{2}\left( e^{x/a} + e^{-x/a} \right)$, the $y$-axis, the $x$-axis, and an ordinate are respectively

$$\pi (sy + ax) \text{ and } \tfrac{1}{2}\pi a (sy + ax),$$

$s$ being the length of the arc between $(0, a)$ and $(x, y)$.

6. The arc of the astroid $x = a \cos^3\theta$, $y = a \sin^3\theta$, from $\theta = 0$ to $\theta = \frac{1}{2}\pi$, revolves about the $x$-axis ; show that the volume and the surface-area of the solid generated are respectively $\frac{16}{105}\pi a^3$ and $\frac{6}{5}\pi a^2$.

7. A cycloid revolves round the tangent at the vertex ; show that the volume and the surface-area of the solid generated are $\pi^2 a^3$ and $\frac{32}{3}\pi a^2$ respectively, $a$ being the radius of the generating circle.

8. The portion between the two consecutive cusps of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ is revolved about the $x$-axis ; show that the area of the surface so formed is to the area of the cycloid as $64 : 9$. [ *Nagpur, 1934* ]

9. Show that the surface-area of the spherical zone contained between two parallel planes $= 2\pi a \times$ the distance between the two planes, where $a$ is the radius of the sphere.

10. Show that the volume of the solid generated by the revolution of the upper-half of the loop of the curve $y^2 = x^2 (2 - x)$ about $\overline{OX}$ is $\frac{4}{3}\pi$.

11. Show that the volume of the solid produced by the revolution of the loop of the curve $y^2 (a + x) = x^2 (a - x)$ about the $x$-axis is $2\pi a^3 (\log 2 - \frac{2}{3})$. [ *P. P. 1935* ]

12. Show that the surface-area and the volume of the solid generated by the revolution about the $x$-axis of the loop of the curve $x = t^2, y = t - \frac{1}{3}t^3$ are respectively $3\pi$ and $\frac{3}{4}\pi$.

13. The smaller of the two arcs into which the parabola $y^2 = 8ax$ divides the circle $x^2 + y^2 = 9a^2$ is rotated about the $x$-axis. Show that the volume of the solid generated is $\frac{28}{3}\pi a^3$.

14. If the curve $r = a + b \cos\theta$ ($a > b$) revolves about the initial line, show that the volume generated is $\frac{4}{3}\pi a (a^2 + b^2)$.

15. The following curves revolve round their asymptotes ; find the volume generated in each case :

(i) $y^2 (2a - x) = x^3$ ;

(ii) $y (a^2 + x^2) = a^3$ ; [ *P. P. 1933* ]

(iii) $(a - x) y^2 = a^2 x$.

16. An arc of a parabola is bounded at both ends by the latus rectum of length $4a$. Find the volume generated when the arc is rotated about the latus rectum. [ *Nagpur, 1935* ]

17. Show that the volume of the solid formed by revolving the ellipse $x = a\cos\theta, y = b\sin\theta$ about the line $x = 2a$ is $4\pi^2 a^2 b$.

18. Show that, if the area lying within the cardioide $r = 2a (1 + \cos\theta)$ and outside the parabola $r (1 + \cos\theta) = 2a$ revolves about the initial line, the volume generated is $18\pi a^3$

19. Show that the volume of the solid generated by revolution about $\overline{OY}$ of the area bounded by $\overline{OY}$, the curve $y^2 = x^3$ and the line $y = 8$ is $\frac{384}{7}\pi$.

20. The arc of a parabola from the vertex to one extremity of the latus rectum is revolved about the corresponding chord. Prove that the volume of the spindle so formed is $(2\sqrt{5} / 75)\pi a^3$.

## ANSWERS

1. (i) $\frac{1}{2}\pi^2$. (ii) $\frac{625}{6}\pi$. (iii) $\frac{3}{2}\pi$. (iv) $\frac{1}{15}\pi a^3$.

15. (i) $2\pi^2 a^3$; (ii) $\frac{1}{2}\pi^2 a^3$; (iii) $\frac{1}{2}\pi^2 a^3$. 16. $\frac{32}{15}\pi a^3$.

CHAPTER XIII

# CENTROIDS AND MOMENTS OF INERTIA

## 13.1. Centroid.

It has been proved in elementary statics that if a system of particles having masses $m_1, m_2, m_3, \ldots\ldots$ have their distances parallel to any co-ordinate axis given by $x_1, x_2, x_3 \ldots\ldots$, then the corresponding co-ordinate of their centre of mass will be given by

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \ldots}{m_1 + m_2 + \ldots} = \frac{\Sigma mx}{\Sigma m}.$$

Similarly, $\bar{y} = \dfrac{\Sigma my}{\Sigma m}$, etc.

Now, if, instead of a system of stray particles, we get a continuous body, we may consider it to be formed of an infinite number of infinitely small elements of masses, and in this case it may be shown, as in the other cases, *viz.*, determination of lengths, areas, etc., the summation, $\Sigma$, will be replaced by the integral sign.

Thus, if $\delta m$ be an element of mass of the body at a point whose co-ordinates are $(x, y)$ (or, in three dimensions, $x, y, z$) the position of the centre of mass of the body will be given by

$$\bar{x} = \frac{\int x\,dm}{\int dm}, \quad \bar{y} = \frac{\int y\,dm}{\int dm},$$

the limits of integration being such as to include the whole body.

In practice, the elementary mass $\delta m$ is proportional to the element of length $\delta s$, or element of area, or element of volume of the corresponding element, according as we proceed to find the centroid of an arc, or area, or volume, and the limits of integration then will be the limits of the corresponding element.

## 13.2. Centroid of a thin rod.

(i) When the rod is *uniform*.

Let $\overline{OA}$ be a rod of length $a$ and let us take $\overleftrightarrow{OAX}$ as $x$-axis.

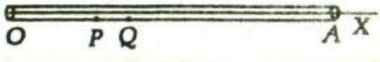



Fig. 1

Let $P$, $Q$ be two neighbouring points on the rod at distances $x$ and $x + \delta x$ from $O$, so that $PQ = \delta x$. Let $\rho$ be the density and $\alpha$ be the uniform cross-section of the rod. Then the element of mass $\delta m$ at $P = \alpha . \delta x \rho$, where $\alpha$ and $\rho$ are constants.

Let $\bar{x}$ be the distance of its C. G. from $O$. Then taking moment about $O$, we have

$$\bar{x} . \Sigma \alpha \delta x \rho = \Sigma \alpha \delta x \rho . x,$$

*i.e.*, $\bar{x} \Sigma \delta x = \Sigma x \delta x$ ( on dividing both sides by the constants $\alpha$, $\rho$ ).

$$\therefore \quad \bar{x} = \frac{\int_0^a x\,dx}{\int_0^a dx} = \frac{\left[\frac{1}{2} x^2\right]_0^a}{\left[x\right]_0^a} = \frac{1}{2} a. \qquad \ldots (1)$$

The limits of integration are taken as such, since for the whole rod $x$ varies from 0 to $a$.

*Thus, the* C. G. *of a uniform thin rod is at its mid-point.*

(ii) When the rod is of *variable density.*

Suppose the density $\rho$ at the point $P$ be a known function of its distance from one end, say, $O$. Then $\rho = f(x)$.

Here, proceeding as above, the element of mass $\delta m$ at $P$ $= \alpha . \delta x . \rho = \alpha . \delta x f(x)$.

$$\therefore \quad \bar{x} \Sigma \alpha \delta x f(x) = \Sigma \alpha \delta x f(x) . x,$$

*i.e.*, $\bar{x} \Sigma f(x) \delta x = \Sigma x f(x) \delta x$, dividing by the constant $\alpha$.

$$\therefore \quad \bar{x} = \frac{\int_0^a x f(x)\,dx}{\int_0^a f(x)\,dx}. \qquad \ldots (2)$$

Substituting the known value of $f(x)$, in any case, and integrating, the final value of $\bar{x}$ is obtained.

For example, if the density at any point of the rod varies as the distance from the extremity $O$, then $f(x) = kx$, where $k$ is a constant, and therefore

$$\bar{x} = \int_0^a x^2\,dx \Big/ \int_0^a x\,dx = \frac{2}{3}a. \qquad \ldots (3)$$

Note. If $\alpha$ be the cross-section of a rod at a point $P$ on it and $\rho$ be the density there, then $\alpha\rho$ ( *i.e.*, mass per unit length ) is called the *line-density* of the rod at $P$. By the single word 'density' is usually meant volume-density, *i.e.*, mass per unit volume.

If in the case (ii) it is given that *the line-density* $\lambda$ *at any point* $P$ *varies as its distance from* $O$, then $\delta m$ ( the element of mass ) at $P$ would be $\lambda\,\delta x$. Now we can proceed as in (3).

## 13.3. Centroid of an arc.

Let $(x, y)$ be the co-ordinates of any point $P$ on the arc $AB$, and $\rho$ be the density at $P$. Let $s$ be the length of the arc $CP$ measured from a fixed point $C$ on the arc. Then $\delta s$ = elementary arc $PQ$ at $P$, and hence

$\rho\,.\,\delta s$ = element of mass at $P\,(=\delta m)$.

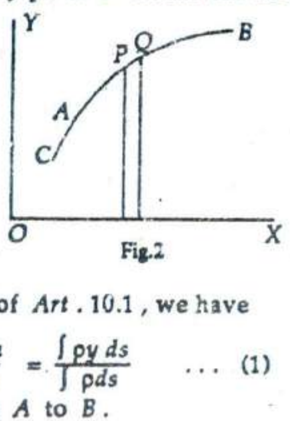


Fig.2

Let $(\bar{x}, \bar{y})$ be the co-ordinates of the C.G. of the arc $AB$. Then, as in (4) of *Art*. 10.1, we have

$$\bar{x} = \frac{\int x\,dm}{\int dm} = \frac{\int \rho x\,ds}{\int \rho\,ds}, \quad \bar{y} = \frac{\int y\,dm}{\int dm} = \frac{\int \rho y\,ds}{\int \rho\,ds} \qquad \ldots (1)$$

the limits of integration extending from $A$ to $B$.

When $\rho$ *is constant*, the formula (1) becomes

$$\bar{x} = \frac{\int x\,ds}{\int ds}, \quad \bar{y} = \frac{\int y\,ds}{\int ds}. \qquad \ldots (2)$$

The formulæ (1) and (2) are fundamental formulæ for the determination of the C.G. of an arc and this can be easily transformed when the equation of the curve is given in Cartesian co-ordinates (general or parametric), or in polar co-ordinates.

**Note 1.** In the application of the above integrals the following results should be noted. When the equation of the curve is

(i) $y = f(x)$, $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,.\,dx$.

(ii) $x = f(y)$, $ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,.\,dy$.

(iii) $x = \phi(t)$, $y = \psi(t)$, $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,.\,dt$.

(iv) $f(r, \theta) = 0$, $\left.\begin{aligned} ds &= \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,.\,d\theta, \\ ds &= \sqrt{1 + \left(r\frac{d\theta}{dr}\right)^2}\,.\,dr, \end{aligned}\right\}$

and $x = r\cos\theta$, $y = r\sin\theta$.

**Note 2.** The C. G. in such cases is generally not on the arc $AB$.

## 13.4. Centroid of a plane area.

*Case I. Cartesian.*

Suppose the area is bounded by the curve $y = f(x)$, the axis of $x$ and the ordinates $x = x_1$, $x = x_2$.

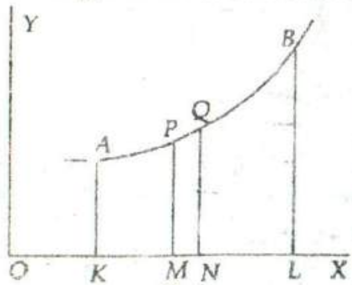


Fig.3

Let $(x, y)$, $(x + \delta x, y + \delta y)$ be the co-ordinates of $P$ and a neighbouring point $Q$ on the curve. Divide the whole area into elementary strips like $PMNQ$, by drawing lines parallel to the $y$-axis. The area of the strip $= y\,dx$ ultimately, since $\delta x$ is very small.

Let the area be homogeneous and let $\rho$ be the *surface-density* of the strip $PMNQ$. Then $\delta m$, the element of mass of the strip $PMNQ = y\delta x\,\rho$ and the C.G. of the strip $PMNQ$ is ultimately at the point $(\bar{x}, \frac{1}{2}y)$ ( with sufficient accuracy for our purpose ). Let $(\bar{x}, \bar{y})$ be the C. G. of the area $AKLB$. Then, taking moments about $\overleftrightarrow{OY}$ and $\overleftrightarrow{OX}$ respectively, we have

$$\bar{x}\,.\,\Sigma \rho y\,.\,\delta x = \Sigma \rho y\,\delta x\,.\,x;\ \bar{y}\,.\,\Sigma \rho y\,.\,\delta x = \Sigma \rho y\,\delta x\,.\,\tfrac{1}{2}y.$$

Cancelling out the constant $\rho$ from both sides, we get in the limit

$$\bar{x} = \frac{\int_{x_1}^{x_2} xy\,dx}{\int_{x_1}^{x_2} y\,dx},\quad \bar{y} = \frac{1}{2}\frac{\int_{x_1}^{x_2} y^2\,dx}{\int_{x_1}^{x_2} y\,dx},$$

where $y$ has to be expressed in terms of $x$ from the equation of the curve.

**Note.** The *surface-density* $\rho$ at any point of an area is $\sigma\lambda$, where $\sigma$ is the volume-density and $\lambda$ is the thickness at the point.

*Case II. Polar.*

Let the area $AOB$ be bounded by the curve $r = f(\theta)$ and the radii vectors $\overline{OA}$, $\overline{OB}$ ( $\theta = \alpha$ and $\theta = \beta$ ) so that $m\angle XOA = \alpha$, $m\angle XOB = \beta$.

Let $O$ be the origin, $\overleftrightarrow{OX}$ the initial line and $\overleftrightarrow{OY}$ the $y$-axis.

Let the whole area be divided into elementary triangular strips like $OPQ$ by radii vectors drawn from $O$. Let the co-ordinates of $P$, $Q$ be $(r, \theta)$, $(r + \delta r, \theta + \delta\theta)$. Then $m\angle POQ = \delta\theta$.

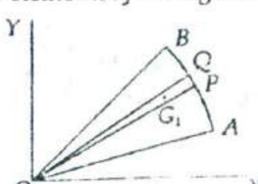


Fig 4

Now, area of the strip $OPQ = \frac{1}{2}r^2\,\delta\theta$ ultimately, since $\delta\theta$ is very small. Then the C.G. of the strip $OPQ$ is a point $G_1$ in $OPQ$, whose co-ordinates are ultimately $(\frac{2}{3} r \cos\theta, \frac{2}{3} r \sin\theta)$ ( with sufficient degree of accuracy for our purpose ). Let $\rho$ be the surface-density of the strip. Then elementary mass $\delta m$ of the strip $OPQ$ is $\frac{1}{2} r^2 \delta\theta \cdot \rho$, situated at $G_1$. Let $(\bar{x}, \bar{y})$ be the co-ordinates of the C.G. of the area $AOB$.

Therefore, taking moments about the $y$-axis and the $x$-axis, we have

$$\bar{x} \cdot \Sigma \tfrac{1}{2} r^2 \rho\, \delta\theta = \Sigma \tfrac{1}{2} r^2 \rho \cdot \tfrac{2}{3} r \cos\theta . \delta\theta ;$$

$$\bar{y} \cdot \Sigma \tfrac{1}{2} r^2 \rho\, \delta\theta = \Sigma \tfrac{1}{2} r^2 \rho \cdot \tfrac{2}{3} r \sin\theta . \delta\theta .$$

Cancelling out from both sides $\frac{1}{2}\rho$, since $\rho$ is constant, we have finally in the limit

$$\bar{x} = \frac{2}{3} \frac{\int_\alpha^\beta r^3 \cos\theta\, d\theta}{\int_\alpha^\beta r^2\, d\theta}, \quad \bar{y} = \frac{2}{3} \frac{\int_\alpha^\beta r^3 \sin\theta\, d\theta}{\int_\alpha^\beta r^2\, d\theta}.$$

where $r = f(\theta)$ from the equation to the bounding curve.

## 13.5. Centroid of the volume and surface of revolution of a uniform solid.

Suppose a solid is formed by the revolution of the curve $y = f(x)$ about the $x$-axis $\overleftrightarrow{OX}$ and suppose it is bounded by two ordinates $\overline{AL}$, $\overline{BM}$ corresponding to $x = x_1$ and $x = x_2$.

(i) The volume generated by the element of area $PNN'P'$, where $(x, y)$ are the co-ordinates of $P$, is ( the area of the circle described by $\overline{PN}$ ) $\times$ ( the thickness between the two circles described by $\overline{PN}$ and $\overline{P'N'}$ ) and $= \pi y^2 \delta x$ ultimately [ since $PN = y$, and $\delta x$ is very small ]. If $\rho$ be the density of the slice bounded

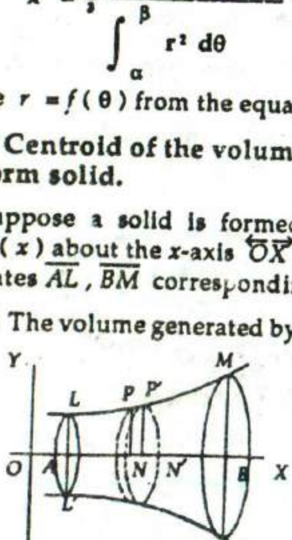



Fig.5

by the two circles, then $\delta m$ the element, of mass of the strip $= \rho.\pi y^2\ \delta x$. The C. G. of the element from symmetry, lies on $\overleftrightarrow{OX}$, and is ultimately at a distance $x$ from $O$. Hence, if $(\bar{x}, \bar{y})$ be the co-ordinates of the C. G. of the volume generated by the area $ALMB$, then, taking moment about the y-axis, we have

$$\bar{x}\,.\,\Sigma\,\rho\pi\,y^2\,\delta x = \Sigma\,\rho\pi\,y^2\,\delta x\,.\,x\,.$$

As the solid is of uniform density, cancelling out $\rho\pi$ from both sides, we get

$$\bar{x} = \frac{\Sigma\,y^2 x\,\delta x}{\Sigma\,y^2\,\delta x} = \frac{\int_{x_1}^{x_2} y^2 x\,dx}{\int_{x_1}^{x_2} y^2\,dx}\,,$$

and from symmetry, $\bar{y} = 0$.

(ii) The area of the surface generated by the revolution of the arc $PP'$ ( $= \delta s$ ) about $\overleftrightarrow{OX}$ is ( the circumference of the circle described by $\overline{PN}$ ) $\times$ ( length of the arc $PP'$ ) and $= 2\pi y\,.\,\delta s$ ultimately, since $PN = y$ and $\delta s$ is small. If $\rho$ be the surface-density, then $\delta m$ the element of mass of the belt $= \rho\,.\,2\pi y\,.\,\delta s$.

The C. G. of the belt from symmetry lies on $\overleftrightarrow{OX}$ and is ultimately at a distance $x$ from $O$. Hence, if $(\bar{x}, \bar{y})$ be the co-ordinates of the C. G. of the surface generated by $LM$, then, taking moment about the y-axis, we have

$$\bar{x}.\Sigma\,\rho.2\pi y\,\delta s = \Sigma\,\rho.2\pi y\,\delta s.x\,.$$

As the surface is of uniform density, cancelling out $2\pi\rho$ from both sides, we get

$$\bar{x} = \frac{\Sigma\,y\,\delta s\,.x}{\Sigma\,y\,\delta s} = \frac{\int yx\,ds}{\int y\,ds}\,.$$

In the integration, the limits for $s$ correspond to $x = x_1$ and $x_2$.

Cor. When the equation of the curve is given in polar co-ordinates, say, $r = f(\theta)$, the above formulæ can easily be transformed into the following forms by the relations between Cartesian and polar co-ordinates, *viz.*, $x = r\cos\theta$, $y = r\sin\theta$.

$$\textit{Solid}: \left\{ \bar{x} = \frac{\int r^3 \sin^2\theta \cos\theta \frac{d}{d\theta}(r\cos\theta)\,.\,d\theta}{\int r^2 \sin^2\theta \cos\theta \frac{d}{d\theta}(r\cos\theta)\,.\,d\theta}, \bar{y} = 0 \right.$$

taken between proper limits .

$$\textit{Surface}: \left\{ \bar{x} = \frac{\int r^2 \sin\theta \cos\theta \frac{ds}{d\theta}\,.\,d\theta}{\int r \sin\theta \frac{ds}{d\theta}\,.\,d\theta}, \bar{y} = 0 \right.$$

taken between proper limits.

## 13.6. Illustrative Examples.

**Ex. 1.** *Find the centroid of an wire in the form of a circular arc.*

Let $AB$ be a wire in the form of a circular arc of radius '$a$', which subtends an angle $2\alpha$ at its centre $O$.

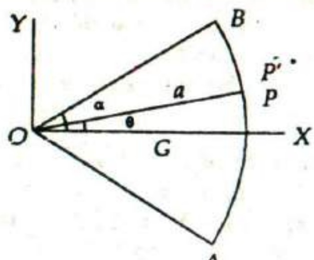



Fig.6

Take $O$ as origin, and $\overleftrightarrow{OX}$, which bisects the arc $AB$, as $x$-axis.

Then, by symmetry, the centroid $G$ lies somewhere on $\overleftrightarrow{OX}$.

Now, $\theta$ denoting the vectorial angle of the point $P$ on the arc, the element $PP'$ there has a length $a\,d\theta$, and the abscissa of $P$ is $a\cos\theta$. Also, to cover the whole arc, $\theta$ extends between the limits $-\alpha$ to $\alpha$. Hence, the abscissa $OG$ of the centroid $C$ is given by

$$\bar{x} = \frac{\int x\,dm}{\int dm} = \frac{\int_{-\alpha}^{\alpha} a\cos\theta\,.\,\rho a\,d\theta}{\int_{-\alpha}^{\alpha} \rho\, a\,d\theta}$$

( $\rho$ denoting the linear density of the wire )

$$= a \frac{\int_{-\alpha}^{\alpha} \cos\theta \, d\theta}{\int_{-\alpha}^{\alpha} d\theta} = a \frac{2 \sin\alpha}{2\alpha} = a \frac{\sin\alpha}{\alpha} .$$

**Cor.** The distance of the *centroid of a semi-circular arc* from the centre is $2a / \pi$.

**Ex. 2.** *Find the centre of gravity of a uniform lamina bounded by parabola and a double ordinate of it.*

Let the lamina be bounded by a parabola $y^2 = 4ax$ and a double ordinate $\overline{RMR'}$ given by $x = x_1$.

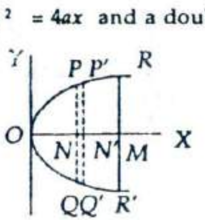



Fig.7

By symmetry, the centroid lies on the x-axis and hence $\bar{y} = 0$.

Divide the lamina into elementary strips by lines paralel to the y-axis. Consider the strip $PQQ'P'$, where the co-ordinates of $P$ are $(x, y)$. The length $PQ$ is $2y$ and the breadth $NN'$ is $\delta x$. Hence the area of the strip is ultimately $2y\, \delta x$. The limits of $x$, to cover the area considered, are clearly $0$ to $x_1$.

Hence, for the required centre of gravity,

$$\bar{x} = \frac{\int x \, dm}{\int dm} = \frac{\int_0^{x_1} x \cdot 2y \, dx \cdot \sigma}{\int_0^{x_1} 2y \, dx \cdot \sigma}$$

( where $\sigma$ is the surface-density of the lamina )

$$= \frac{\int_0^{x_1} x . 2\sqrt{4ax} \, dx \cdot \sigma}{\int_0^{x_1} 2\sqrt{4ax} \, dx \cdot \sigma} = \frac{\int_0^{x_1} x^{3/2} \, dx}{\int_0^{x_1} x^{1/2} \, dx} = \frac{\frac{2}{5} x_1^{5/2}}{\frac{2}{3} x_1^{3/2}} = \frac{3}{5} x_1 .$$

Thus, the centre of gravity divides the length $OM$ in the ratio of $3 : 5$

. Ex. 3. *Find the centre of gravity of a uniform lamina in the form of a quadrant of the ellipse* $x^2/a^2 + y^2/b^2 = 1$. [ *P. P. 1935* ]

Let $AOB$ be the quadrant considered. Divide it into elementary strips by lines parallel to the $y$-axis. The area of the elementary strip corresponding to the point $P$, whose co-ordinates are $(x, y)$, is ultimately $y\,\delta x$, and the centroid of this element is at the middle point of the strip (which is supposed infinitely thin ) and thus has its co-ordinates $(x, y/2)$. The limits of $x$ for the quadrant considered are evidently 0 and $a$.

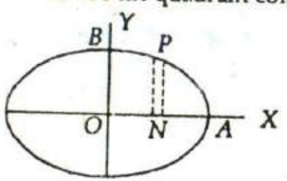



Fig.8

Hence, the C. G. of the area considered will be given by, $(x', y')$ denoting the co-ordinates of the centroid of the element $dm$ which is taken here as the strip,

$$\bar{x} = \frac{\int x'\,dm}{\int dm} = \frac{\int_0^a x\,.\,y dx\,.\,\sigma}{\int_0^a y\,dx\,.\,\sigma}$$ [ $\sigma$ being the surface-density of the lamina ]

$$= \frac{\int_0^a x\,\frac{b}{a}\sqrt{a^2 - x^2}\,dx\,.\,\sigma}{\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx\,.\,\sigma} \quad \left[\text{ since } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right]$$

$$= \frac{\int_0^a x\sqrt{a^2 - x^2}\,dx}{\int_0^a \sqrt{a^2 - x^2}\,dx} = a\,\frac{\int_0^{\frac{1}{2}\pi} \sin\theta\cos^2\theta\,d\theta}{\int_0^{\frac{1}{2}\pi} \cos^2\theta\,d\theta}$$

[ putting $x = a\sin\theta$ ]

$$= a\,\frac{\frac{1}{3}}{\frac{1}{2}\,\frac{\pi}{2}} = \frac{4a}{3\pi}.$$

$$\bar{y} = \frac{\int y'\,dm}{\int dm} = \frac{\int_0^a \frac{y}{2}\cdot y\,dx\,.\,\sigma}{\int_0^a y\,dx\,.\,\sigma} = \frac{1}{2}\,\frac{\int_0^a \frac{b^2}{a^2}\,(a^2 - x^2)\,dx}{\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx}$$

$$= \frac{1}{2}\,b\,\frac{\int_0^{\frac{1}{2}\pi} \cos^3\theta\,d\theta}{\int_0^{\frac{1}{2}\pi} \cos^2\theta\,d\theta} = \frac{1}{2}\,b\,\frac{\frac{2}{3}}{\frac{1}{2}\,\frac{\pi}{2}} = \frac{4b}{3\pi}\,.$$

**Cor.** *The centroid of half the ellipse bounded by the minor axis is on the major axis* at a distance $4a/3\pi$ from the centre.

Also, the centroid of a *semi-circular area of radius 'a'* is on the radius bisecting it, at a distance $4a/3\pi$ from the centre.

**Ex. 4.** *Find the centre of gravity of a solid hemisphere.*

Clearly, the hemisphere may be supposed to be generated by revolving a circular quadrant $APB$ about one bounding radius $\overline{OA}$, which we may choose as $x$-axis. By symmetry, the centre of gravity of the hemisphere will be on $\overleftrightarrow{OX}$. Now divide the hemisphere into infinitely thin circular slices by planes perpendicular to the axis of revolution $\overleftrightarrow{OX}$. An element of such slice, corresponding to the point $P$, has its volume ultimately equal to $\pi y^2\,\delta x$ ($x, y$ being the Cartesian co-ordinates of $P$), and the $x$ co-ordinate of its centre is $\bar{x}$.

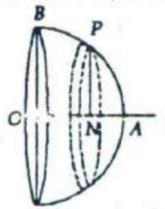



Fig.9

Hence, if $\rho$ be the density of the solid hemisphere and $a$ its radius, the position of the C. G. is given by

$$\bar{x} = \frac{\int_0^a x\,.\,\pi y^2\,dx\,.\,\rho}{\int_0^a \pi y^2\,dx\,.\,\rho} = \frac{\int_0^a x\,(a^2 - x^2)\,dx}{\int_0^a (a^2 - x^2)\,dx} \quad [\text{since } x^2 + y^2 = a^2]$$

$$= \frac{a^2 \dfrac{a^2}{2} - \dfrac{a^4}{4}}{a^2 . a - \dfrac{a^3}{3}} = \frac{3}{8} a .$$

## 13.7. Moment of Inertia.

If a system of particles have masses $m_1$, $m_2$, $m_3$ . . and if $r_1$, $r_2$ $r_3$ . . be their distances from a given line, then $\Sigma\, mr^2$ is defined as the *moment of inertia* of the system of particles about the given line

If $M$ be the total mass of the system $m_1$, $m_2$, etc., it is usual to express the moment of inertia of the system about any line in the form $Mk^2$, where $k$ represents a length and is called the *radius of gyration* of the system about the given line.

If, instead of a system of particles, it is a body in the form of a thin wire or a lamina or a solid of which we want to find the moment of inertia about a given line, we may consider the body to be made up of an infinite number of infinitely small elements of masses, and then the summation $\Sigma mr^2$ reduces to the integral $\int r^2\, dm$, where the limits are such as to cover the whole body.

## 13.8. Two important theorems of moments of inertia.

*( a ) If a thin lamina ( thickness negligible ) has its moments of inertia about two perpendicular axes in its plane respectively equal to $I_1$ and $I_2$, then the moment of inertia about a normal to the plane through their point of intersection is $I_1 + I_2$ .*

Take the given perpendicular axes as $x$-axis and $y$-axis, $I_1$ and $I_2$ being the respective moments of inertia. Consider an element of mass $\delta M = \sigma dxdy$ at the point $P$ ( $\sigma$ = surface-density). Its moment of inertia about $\overleftrightarrow{OX} = dI_1 = y^2 \sigma dxdy$ . Similarly, $dI_2$ ( that about $\overleftrightarrow{OY}$ ) $= x^2 \sigma dxdy$ . Also, $dI$ = moment of inertia of the element about a normal to the plane at the origin ( point of intersection of the given axes ) $= ( x^2 + y^2 ) \sigma dxdy$ ; for the distance of $P ( x , y )$ from the origin ( or the normal to the plane ) $= \sqrt{(x^2 + y^2)}$ . Thus, $dI = dI_1 + dI_2$ is true for every point $P$ . Therefore, integrating between proper limits over the whole area,

$$I = \iint (x^2 + y^2)\,\sigma dxdy = \iint x^2\,\sigma dxdy + \iint y^2\,\sigma dxdy = I_2 + I_1 .$$

N. B. This theorem is true even if $\sigma$ is a function of $(x, y)$.

( *b* ) *The theorem of parallel axes in the case of a lamina is : The moment of inertia of a thin lamina ( thickness negligible ) about any given line in its plane is equal to that about a parallel line through its C. G., together with the moment of inertia of the whole mass concentrated at the C. G. about the given line.*

$G$ is the centre of gravity. Take $\overleftrightarrow{Gx}$ ( parallel to the line $\overline{BA}$ ) as $x$-axis. Let $\overleftrightarrow{Gy}$ ( $y$-axis ) cut $\overline{BA}$ at $B$. Let $GB = h$ = distance of the given line from $G$. Consider an element of mass $\sigma dxdy$ at $P(x, y)$, $\sigma$ = surface-density. $PN$ = perpendicular on $\overline{BA} = h - y$. $\therefore\ PN^2 = (h - y)^2$. Then,

$$\begin{aligned} I \text{ ( moment of inertia about } \overline{BA} \text{ )} &= \iint \sigma (h - y)^2\, dxdy \\ &= h^2 \iint \sigma dxdy - 2h \iint y\sigma dxdy + \iint \sigma y^2\, dxdy \\ &= Mh^2 - 0 + I_0 \ \ldots \ (i), \text{ where } M = \iint \sigma dxdy . \\ &= \text{mass of the lamina ,} \end{aligned}$$

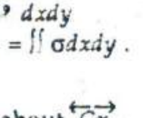
$I_0 = \iint \sigma y^2\, dxdy$ = moment of intertia about $\overleftrightarrow{Gx}$,
also, $\bar{y}$ ( distance of C. G. from $\overleftrightarrow{Gx}$ = 0 , here )
$= \iint \sigma y dxdy \,/\, M$.

$$\therefore \iint \sigma y dxdy = 0 .$$

Thus, from (i) we get the theorem.

## 13.9. Illustrative Examples.

Ex. 1. *Find the moment of inertia of a thin uniform straight rod of mass M and length 2a about its perpendicular bisector.*

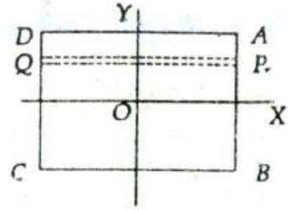



An infinitesimal element of length $\delta x$ at $P$, whose distance from the middle point of the rod is $x$, has its mass $M\,\delta x / (2a)$. *Hence, the moment of inertia of the rod about the perpendicular bisector* $\overleftrightarrow{OY}$ is given by

$$I = \int_{-a}^{+a} x^2 \frac{M}{2a}\, dx = \frac{M}{2a}\,\frac{2a^3}{3} = M\,\frac{a^2}{3} .$$

**Ex. 2.** *Find the moment of inertia of a thin uniform lamina in the form of a rectangle about an axis of symmetry through its centre.*

Let $2a$ and $2b$ be the lengths of the adjacent sides $\overline{AD}$ and $\overline{AB}$ of the rectangular lamina $ABCD$, and $\overleftrightarrow{OX}$, $\overleftrightarrow{OY}$ the axes of symmetry through its centre $O$, which are parallel to them.




$M$ being the mass of the lamina, the surface-density is clearly $M/(4ab)$. Now, divide the lamina into thin strips parallel to $\overleftrightarrow{OX}$, and consider any strip $\overline{PQ}$ at a distance $y$ from $\overleftrightarrow{OX}$, whose breadth is $\delta y$. The mass of the strip is then evidently $\{M/(4ab)\}\, 2a\,\delta y$. Every portion of it being ultimately at the same distance $y$ from $\overleftrightarrow{OX}$, the moment of inertia of the whole lamina about the $x$-axis is given by

$$I_x = \int_{-b}^{+b} y^2 \cdot \frac{M}{4ab}\, 2a\, dy$$

$$= M\frac{b^2}{3}.$$

Similarly, the moment of inertia of the lamina about $\overleftrightarrow{OY}$ is given by

$$I_y = M\frac{a^2}{3}.$$

**Ex. 3.** *Find the moment of inertia of a thin uniform elliptic lamina about its axes.*

Let $x^2/a^2 + y^2/b^2 = 1$ be the equation to the ellipse. Its area is known to be $\pi ab$, if $M$ be its mass, the surface-density is $M/(\pi ab)$. Dividing the lamina into thin strips by lines parallel to the $x$-axis, an elementary strip at a distance $y$ from the $x$-axis has its length $2x = (2a/b)\sqrt{(b^2 - y^2)}$ from the equation of the elliptic boundary. Thus, $\delta y$ being the breadth of the strip, its mass is

$$\frac{M}{\pi ab} \cdot 2\frac{a}{b}\sqrt{b^2 - y^2}\,\delta y.$$

Hence, the moment of inertia of the lamina about the $x$-axis is given by

$$I_x = \int_{-b}^{+b} y^2 \cdot \frac{M}{\pi ab}\, 2\, \frac{a}{b} \sqrt{b^2 - y^2}\, dy$$

$$= \frac{2Mb^2}{\pi} \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \sin^2\theta \cos^2\theta\, d\theta \quad [\text{ putting } y = b \sin\theta\,]$$

$$= \frac{2Mb^2}{\pi}\, \frac{\pi}{8} = M\, \frac{b^2}{4}.$$

Similarly, the moment of inertia of the lamina about the $y$-axis is given by

$$I_y = M\, \frac{a^2}{4}\ .$$

**Cor.** The moment of inertia of a thin uniform *circular disc* of mass $M$ and radius $a$ about any diameter is $M\,(a^2 / 4)$.

**Ex. 4.** *Find the moment of inertia of a thin uniform circular plate about an axis through its centre perpendicular to its plane.*

Let $M$ be the mass and $a$ the radius of the circular lamina, so that its surface-density is $M/(\pi a^2)$.

Divide the lamina into infinitely thin concentric rings by circles concentric with the boundary. Any elementary ring between circles of radii $r$ and $r + \delta r$ has its area ultimately equal to $2\pi r\ \delta r$ and so its mass is $[\,M/(\pi a^2)]\ 2\pi r\,\delta r$. As every part of the ring is ultimately at the same distance $r$ from the axis in question which is perpendicular to its plane through the centre, the moment of inertia of the ring about the axis is ultimately $[\,M/(\pi a^2)]\ 2\pi r\ \delta r\ .\ r^2$.

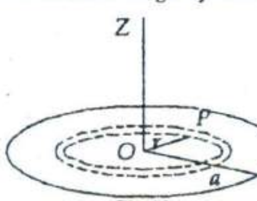


Fig.12

Hence, the required moment of inertia of the disc about the axis is given by

$$I = \int^{a} \frac{M}{\pi a^2}\ .\ 2\pi r\ .\ dr\ .\ r^2$$

$$= \frac{2M}{a^2} \int_0^{a} r^3\, dr = \frac{2M}{a^2}\, \frac{a^4}{4} = M\, \frac{a^2}{2}\ .$$

Ex. 5. *Find the moment of inertia of a sphere about a diameter.*

If $M$ be the mass and $a$ the radius of the sphere, the volume of the sphere is known to be $\frac{4}{3}\pi a^3$, and hence its density is $M/(\frac{4}{3}\pi a^3)$.

Take the diameter about which the moment of inertia is required to be the $x$-axis. Divide the sphere into infinitely thin circular slices by planes perpendicular to this axis. An elementary slice between the planes $x$ and $x + \delta x$ has its volume ultimately equal to $\pi(a^2 - x^2)\delta x$, since its radius is $\sqrt{(a^2 - x^2)}$. [See *Fig. Ex.* **4**, *Art.* 13.6.] Hence the moment of inertia of this slice about the $x$-axis, which is perpendicular to its plane through its centre, is ultimately

$$\frac{M}{\frac{4}{3}\pi a^3} \cdot \pi(a^2 - x^2)\delta x \cdot \frac{a^2 - x^2}{2}. \quad \text{[ see } Ex.\ 4 \text{ above. ]}$$

Hence, the required moment of inertia of the whole sphere about the diameter is given by

$$I = \int_{-a}^{+a} \frac{M}{\frac{4}{3}\pi a^3} \cdot \pi(a^2 - x^2)\,dx\,\frac{a^2 - x^2}{2}$$

$$= \frac{3}{8}\frac{M}{a^3}\int_{-a}^{+a}(a^4 - 2a^2x^2 + x^4)\,dx$$

$$= \frac{3}{8}\frac{M}{a^3}\left(a^4 \cdot 2a - 2a^2 \cdot \frac{2a^3}{3} + \frac{2a^5}{5}\right) = \frac{2}{5}Ma^2.$$

## EXAMPLES XIII

1. Show that the C.G. of a thin hemispherical shell is at the middle point of the radius perpendicular to its bounding plane.

2. Show that the C.G. of (i) a solid right circular cone is on the axis at a distance from the base equal to $\frac{1}{4}$ of the height of the cone ; (ii) a thin hollow cone without base is on the axis at a distance from the base equal to $\frac{1}{3}$ of the height of the cone.

3. Find the centroid of the whole arc of the cardioide $r = a(1 + \cos\theta)$.

4. Find the centroid of the area bounded by the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ and its base.

5. Find the centroid of the sector of a circle.

6. Find the centroid of the arc of the parabola $y^2 = 4ax$ included between the vertex and one extremity of the latus rectum.

7. Find the position of the centroids of the following areas :

(i) A loop of the curve $y^2(a + x) = x^2(a - x)$.

(ii) Area bounded by the curve $y^2(2a - x) = x^3$, and its asymptote.

(iii) Area bounded by $y^2 = 4ax$ and $y = 2x$.

(iv) One loop of $r = a \cos 2\theta$.

8. Find the C. G. of the arc which is in the first quadrant of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$.

9. Find the centroid of the area of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ lying in the first quadrant.

10. Find the centroid of the area between the sine curve $y = \sin x$ and $y = 0$, where $0 \le x \le \pi$.

11. Find the C. G. of the area of the parabola

$$\left(\frac{x}{a}\right)^{1/2} + \left(\frac{y}{b}\right)^{1/2} = 1$$ between the curve and the axes.

12. Find the centroid of the area of half the cardioide $r = a(1 + \cos\theta)$ bounded by $\theta = 0$.

13. Find the centroid of the area of the right loop of the lemniscate $r^2 = a^2 \cos 2\theta$.

14. Find the C. G. of the solid formed by the revolution of the quadrant of the ellipse $(x^2/a^2 + y^2/b^2) = 1$ about its (i) major axis, (ii) minor axis.

15. Find the centroid of the (i) surface and (ii) solid generated by revolving half of the cardioide $r = a(1 + \cos\theta)$ bounded by $\theta = 0$ about the initial line.

**16.** Find the C. G. of the surface formed by the revolution of the parabola $y^2 = 2x$ cut off by the line $x = 4$ about the axis of the parabola.

**17.** An equilateral triangle of side '$a$' revolves round its base which is fixed. Find the volume of the solid generated.

**18.** Find the moment of inertia of a solid right circular cylinder of radius $a$ about its axis.

**19.** Obtain the moment of inertia of a solid right circular cone of height $h$ and semi-vertical angle $\alpha$ about its axis.

**20.** Prove that the moment of inertia about an axis through the centre perpendicular to the plane of a thin circular ring whose outer and inner radii are $a$ and $b$ is $\frac{1}{2}M(a^2 + b^2)$, where $M$ denotes the mass of the ring.

**21.** Find the moment of inertia of a rectangular parallelopiped, the lengths of whose edges are respectively $2a$, $2b$, $2c$ about an axis through its centre parallel to the edge $2a$.

**22.** Show that the moment of inertia of a thin hollow spherical shell of radius $a$ and mass $M$ about a diameter is $M(2a^2/3)$.

**23.** Show that the moment of inertia of a parabolic area of latus rectum $4a$, cut off by an ordinate at a distance $h$ from the vertex, is $\frac{3}{7}Mh^2$ about the tangent at the vertex, and $\frac{4}{5}Mah$ about the axis, $M$ being the mass of the area.

**24.** Show that if a thin lamina has its moments of inertia about two perpendicular axes in its plane respectively equal to $I_1$ and $I_2$, then the moment of inertia about a normal to the plane through their point of intersection is $I_1 + I_2$.

**25.** Prove the *Theorem of parallel axes* in the case of a lamina, namely, that the moment of inertia of a thin lamina about any given line in its plane is equal to that about a parallel line through its C. G. together with the moment of inertia of the whole mass concentrated at the C.G. about the given line.

26. Find the moment of inertia of the circumference of the circle of radius '$a$' about a diameter.

27. Find the radius of gyration of a circle of radius '$a$' about a diameter.

28. Find the moment of inertia of the surface of a sphere of radius '$a$' about a diameter.

29. Find the moment of inertia of a truncated cone about its axis, the radius of its ends being '$a$' and '$b$'

30. Find the moment of inertia of an isosceles triangle, each of whose equal sides is '$a$' about the perpendicular from the vertex upon the opposite side.

31. Find the moment of inertia of the area bounded by $r^2 = a^2 \cos 2\theta$ about its axis.

32. Find the moment of inertia of a circular area of radius '$a$' about the line whose perpendicular distance from its centre is $d$.

33. Find the moment of inertia of a rectangular parallelopiped whose sides are $2a$, $2b$, $2c$ about its edge $2a$.

34. Show that the moment of inertia of a thin uniform rod of length $2a$ and mass $M$ about a line through one end perpendicular to the rod is $M \frac{4}{3} a^2$.

35. Show that the moment of inertia of a thin uniform lamina in the form of a rectangle whose sides are $2a$ and $2b$ about an axis perpendicular to the plane of the lamina at the point of intersection of the diagonals of the lamina is $\frac{1}{3}(a^2 + b^2) M$.

36. Show that the moment of inertia of a thin uniform elliptic lamina whose semi-axes are $a$ and $b$ about the line through the centre of the ellipse and perpendicular to its plane is $\frac{1}{4}(a^2 + b^2) M$.
[ See *Ex*. 39. ]

37. Show that the moment of intertia of the area of a lemniscate of Bernoulli $r^2 = c^2 \cos 2\theta$ about the line in its plane through the origin and perpendicular to its axis is $Mc^2 (3\pi + 8)/48$.

**38.** *ABC* is a uniform triangular lamina and $l$ is the length of the perpendicular drawn from $A$ on $\overline{BC}$. If $M$ be the total mass of the lamina, then show that the moment of inertia of the lamina about $\overline{BC}$ is $\frac{1}{6}Ml^2$.

**39.** (i) Show that the moment of inertia of a uniform elliptic lamina of mass $M$, the equation of the ellipse being $x^2/a^2 + y^2/b^2 = 1$, about a diameter making an angle $\theta$ with the major axis is $M(a^2 \sin^2\theta + b^2 \cos^2\theta)/4$.

(ii) If $r$ be the length of the semi-diameter of the ellipse in the above case, then show that the moment of inertia is $M\ a^2 b^2/(4r^2)$.

(iii) If, in the above case, $p$ be the length of the perpendicular from the centre of the ellipse on the tangent parallel to the semi-diameter, then show that the moment of inertia about the tangent is $\frac{5}{4}Mp^2$.

## ANSWERS

3. $\bar{x} = \frac{4}{3}a,\ \bar{y} = 0.$  4. $\bar{x} = 0,\ \bar{y} = \frac{7}{6}a.$

5. On the radius, bisecting the sector, at a distance $\frac{2}{3}a\ \frac{\sin\alpha}{\alpha}$ from the centre, $2\alpha$ being the angle of the sector at the centre, and $a$ the radius.

6. $\bar{x} = \frac{a}{4}\ \frac{3\sqrt{2} - \log(\sqrt{2}+1)}{\sqrt{2} + \log(\sqrt{2}+1)},\ \bar{y} = \frac{4a}{3}\ \frac{2\sqrt{2} - 1}{\sqrt{2} + \log(\sqrt{2}+1)}.$

7. (i) $\bar{x} = \frac{a}{3}\ \frac{3\pi - 8}{4 - \pi},\ \bar{y} = 0.$  (ii) $\bar{x} = \frac{5a}{3},\ \bar{y} = 0.$

(iii) $\bar{x} = \frac{2}{5}a,\ \bar{y} = a.$  (iv) $\bar{x} = \frac{128\sqrt{2}}{105}\ \frac{a}{\pi},\ \bar{y} = 0.$

8. $\bar{x} = (\pi - \frac{4}{3})a,\ \bar{y} = \frac{2}{3}a.$  9. $\bar{x} = \bar{y} = \frac{256a}{315\pi}.$

10. $\bar{x} = \frac{1}{2}\pi,\ \bar{y} = \frac{1}{8}\pi.$  11. $\frac{\bar{x}}{a} = \frac{\bar{y}}{b} = \frac{1}{5}.$

12. $\bar{x} = \frac{5a}{6},\ \bar{y} = \frac{16a}{9\pi}.$  13. $\bar{x} = \frac{\pi a\sqrt{2}}{8},\ \bar{y} = 0.$

14. (i) $\bar{x} = \frac{3}{8}a$, $\bar{y} = 0$; (ii) $\bar{x} = 0$, $\bar{y} = \frac{3}{8}b$.

15. (i) For surface $\bar{x} = \frac{50}{63}a$, $\bar{y} = 0$; (ii) For volume $\bar{x} = \frac{4a}{5}$, $\bar{y} = 0$.

16. $\bar{x} = \frac{149}{65}$, $\bar{y} = 0$. 17. $\frac{\pi a^3}{4}$.

18. $M\frac{a^2}{2}$. 19. $\frac{3}{10} M h^2 \tan^2 \alpha$. 21. $M\frac{b^2 + c^2}{3}$.

26. $\frac{Ma^2}{2}$. 27. $\frac{a}{2}$ 28. $\frac{2Ma^2}{3}$. 29. $\frac{3M}{10}\left(\frac{a^5 - b^5}{a^3 - b^3}\right)$.

30. $\frac{Ma^2}{6}$. 31. $\frac{Ma^2}{16}\left(\pi - \frac{8}{3}\right)$.

32. $M\left(d^2 + \frac{1}{4}a^2\right)$. 33. $\frac{4M}{3}(b^2 + c^2)$.

N.B. $M$ is the mass in each case.

## CHAPTER XIV

## ON SOME WELL-KNOWN CURVES

**14.1. We give below diagrams, equations, and a few characteristics of some well-known curves which have been used in the preceding pages in obtaining their properties. The student is supposed to be familiar with conic sections and graphs of circular functions, so they are not given here.**

### 14.2. Cycloid.

**The *cycloid* is the curve traced out by a point on the circumference of a circle which rolls ( without sliding ) on a straight line.**

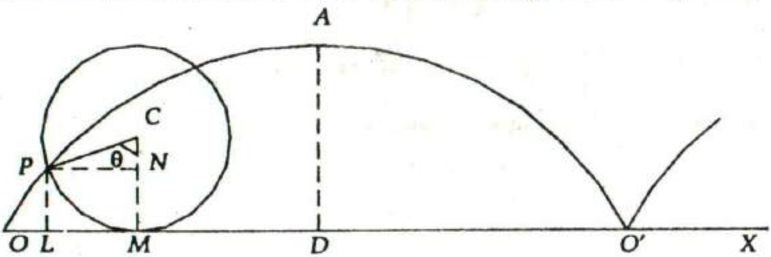



Fig.1

$x = a(\theta - \sin\theta)$ $y = a(1 - \cos\theta)$

Let $P$ be the point on the circle $MP$, called the *generating circle*, which traces out the cycloid. Let the line $\overrightarrow{OMX}$ on which the circle rolls be taken as $x$-axis and the point $O$ on $\overrightarrow{OX}$, with which $P$ was in contact when the circle began rolling, be taken as origin.

Let $a$ be the radius of the generating circle and $C$ its centre, $P$ the point $(x, y)$ on it, and let $m\angle PCM = \theta$. Then $\theta$ is the angle through which the circle turns as the point $P$ traces out the locus.

$\therefore\ OM = arc\ PM = a\theta.$

Let $\overline{PL}$ be drawn perpendicular to $\overrightarrow{OX}$.

$$\therefore\ x = OL = OM - LM = a\theta - PN = a\theta - a\sin\theta$$
$$= a(\theta - \sin\theta).$$

$$y = PL = NM = CM - CN = a - a\cos\theta$$
$$= a(1 - \cos\theta).$$

Thus, *the parametric equations of the cycloid with the starting point as origin and the line on which the circle rolls,* called *base, as x-axis*, are

$$x = a(\theta - \sin\theta), y = a(1 - \cos\theta). \quad \ldots (1)$$

The point $A$ at the greatest distance from the base $\overrightarrow{OX}$ is called *vertex*. Thus, for the vertex, $y$, *i.e.*, $a(1 - \cos\theta)$ is maximum. Hence, $\cos\theta = -1$, *i.e.*, $\theta = \pi$.

$\therefore AD = a(1 - \cos\pi) = 2a$. $\therefore$ vertex is $(a\pi, 2a)$.

For $O$ and $O'$, $y = 0$. $\therefore \cos\theta = 1$. $\therefore \theta = 0$ and $2\pi$.

As the circle rolls on, arches like $OAO'$ are generated over and over again, and any single arch is called a cycloid.

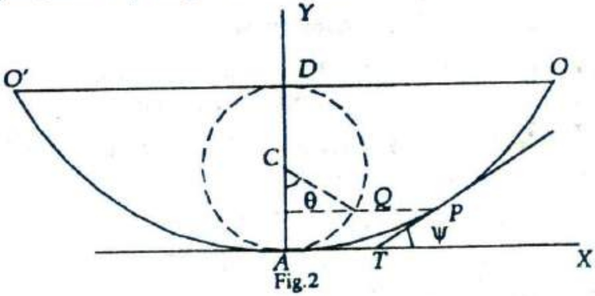



Fig.2

$x = a(\theta + \sin\theta)$ $\quad$ $y = a(1 - \cos\theta)$

Since the vertex is the point $(a\pi, 2a)$, the equation of the cycloid with the vertex as the origin and the tangent at the vertex as the $x$-axis can be obtained from the previous equations by transferring the origin to $(a\pi, 2a)$ and turning the axes through $\pi$, *i.e.*, by writing

$a\pi + x'\cos\pi - y'\sin\pi$ and $2a + x'\sin\pi + y'\cos\pi$ for $x$ and $y$ respectively.

Hence, $a(\theta - \sin\theta) = a\pi - x'$,

or, $\quad x' = a(\pi - \theta) + a\sin\theta = a(\theta' + \sin\theta'),$
where $\theta' = \pi - \theta,$

and $\quad a(1 - \cos\theta) = 2a - y',$

or, $\quad y' = 2a - a + a\cos\theta = a + a\cos\theta$
$= a - a\cos(\pi - \theta) = a(1 - \cos\theta').$

Hence, (dropping dashes) *the equation of the cycloid with the vertex as origin and the tangent at the vertex as x-axis is*

$$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta). \qquad \ldots (2)$$

In this equation, $\theta = 0$ *for vertex*, $\theta = \pi$ *for 0* and $\theta = -\pi$ *for O'*.

The *characteristic properties are :*

(i) For the cycloid $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$, radius of curvature = twice the length of the normal.

(ii) The evolute of the cycloid is an equal cycloid.

(iii) For the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, $\psi = \frac{1}{2}\theta$ and $s^2 = 8ay$, $s$ being measured from the vertex.

(iv) The length of the above cycloid included between the two cusps is $8a$.

(v) Intrinsic equation is $s = 4a\sin\psi$.

**Note.** The above equation (2) can also be obtained from the Fig. (1) geometrically as follows :

If $(x', y')$ be the co-ordinates of $P$ referred to the vertex as origin and the tangent at the vertex as x-axis,

$x' = LD = OD - OL = a\pi - x = a(\pi - \theta) + a\sin\theta,$
$y = AD - PL = 2a - y = 2a - a(1 - \cos\theta) = a(1 + \cos\theta).$

Hence, writing $\theta'$ (or $\theta$) for $\pi - \theta$, etc.

## 14.3. Catenary.

The *catenary* is the curve in which a uniform heavy flexible string will hang under the action of gravity when suspended from two points. It is also called the *chainette*.

Its equation, as shown in books on Statics, is

$$y = c \cosh \frac{x}{c} = \frac{c}{2}\left(e^{x/c} + e^{-x/c}\right).$$

C is called the *vertex* ; $OC = c . \overrightarrow{OX}$ is called the directrix .

The *characteristic properties* are :

(i) The perpendicular from the foot of the ordinate upon the tangent at any point is of constant length.

(ii) Radius of curvature at any point = length of the normal at the point ( the centre of curvature and the $x$-axis being on the opposite sides of the curve ).

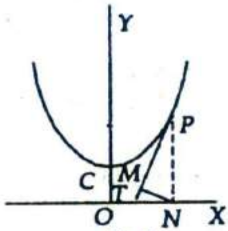


Fig.3

(iii) $y^2 = c^2 + s^2$, $s$ being measured from the vertex C .

(iv) $s = c \tan \psi$, $y = c \sec \psi$ . (v) $x = c \log(\sec \psi + \tan \psi)$.

## 14.4. Tractrix.

Its equation is

$$x = \sqrt{a^2 - y^2} + \frac{a}{2} \log \frac{a - \sqrt{a^2 - y^2}}{a + \sqrt{(a^2 - y^2)}},$$

or, $\quad x = a(\cos t + \log \tan \frac{1}{2} t), y = a \sin t$ .

Here, $OA = a$ . The *characteristic properties* are :

(i) The portion of the tangent intercepted between the curve and the $x$-axis is constant.

(ii) The radius of curvature varies inversely as the

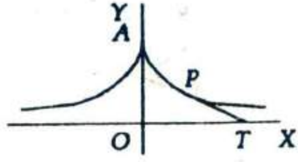


Fig.4

normal (the centre of curvature and the $x$-axis being on the opposite sides of the curve ).

(iii) The evolute of the tractrix is the catenary

$$y = a \cosh ( x / a ) .$$

## 14.5. Four-cusped Hypocycloid.

Its equation is $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$,

or, $x = a \cos^3\phi$,
$y = b \sin^3\phi$.

Here, $OA = OA' = a$;
$OB = OB' = b$.

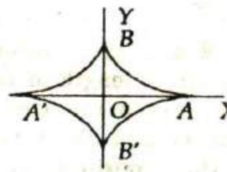


Fig.5

The perimeter of the hypo-cycloid $ABA'B'$ is $4 \dfrac{a^2 + ab + b^2}{a + b}$
The astroid is a special case of this, when $a = b$.

## 14.6. Astroid.

Its equation is $x^{2/3} + y^{2/3} = a^{2/3}$,

or, $x = a \cos^3\theta$, $y = a \sin^3\theta$.

Here, $OA = OB = OA' = OB = a$.

The whole figure lies completely within a circle of radius $a$ and centre $O$. The points $A$, $A'$, $B$, $B'$ are called cusps. It is a special type of a four-cusped hypocycloid. [ See § 14.5 ]

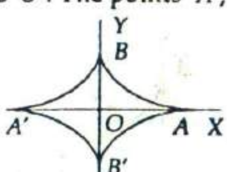


Fig.6

The *characteristic property* of this curve is that the tangent at any point to the curve intercepted between the axes is of constant length.

The perimeter of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ is $6a$.

## 14.7. Evolutes of Parabola and Ellipse.

(i) The equation of the evolute of the parabola $y^2 = 4ax$ is

$$27ay^2 = 4(x - 2a)^3.$$

The curve is called a *semi-cubical parabola*.

Transferring the origin to $(2a, 0)$, its equation assumes the form $y^2 = kx^3$ where $k = 4/(27a)$, which is the standard equation of the semi-cubical parabola with its vertex at the origin.

Hence, the vertex $C$ of the evolute is $(2a, 0)$.

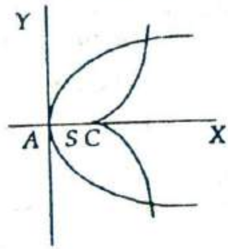


Fig.7

(ii) The equation of the evolute of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3},$$

which can be written in the form

$$\left(\frac{x}{\alpha}\right)^{2/3} + \left(\frac{y}{\beta}\right)^{2/3} = 1,$$

where $\alpha = (a^2 - b^2)/a$, $\beta = (a^2 - b^2)/b$.

The area of the evolute is $\frac{3}{8}\pi \dfrac{(a^2 - b^2)^2}{ab}$.

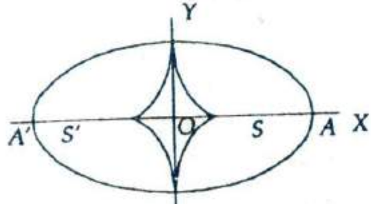



Fig.8

The length of the evolute is $4\left(\frac{a^2}{b}-\frac{b^2}{a}\right)$.

Hence, it is a four-cusped hypo-cycloid.

## 14.8. Folium of Descartes.

Its equation is $x^3 + y^3 = 3axy$.

It is symmetrical about the line $y = x$.

The axes of co-ordinates are tangents at the origin, and there is a loop in the first quadrant.

It has an asymptote $x + y + a = 0$ and its radii of curvatures at the origin are each $= \frac{3}{2}a$.

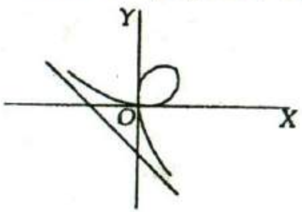


Fig.9

The area included between the curve and its asymptote

= the area of the loop of the curve

$= \frac{3}{2}a^2$.

## 14.9. Logarithmic and Exponential Curves.

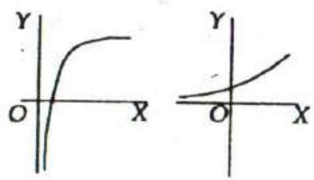


Fig.10

(i) $y = \log x$ (ii) $y = e^x$.

(i) $x$ is always positive; $y = 0$ when $x = 1$, and as $x$ becomes smaller and smaller, $y$, being negative, becomes numerically larger and larger. *For $x > 0$, the curve is continuous.*

(ii) $x$ may be positive or negative, but $y$ is always positive and $y$ becomes smaller and smaller, as $x$, being negative, becomes numerically larger and larger. *The curve is continuous for all values of* $x$.

## 14.10. Probability Curve.

The equation of the probability curve is $y = e^{-x^2}$.

The $x$-axis is an asymptote.

The *area between the curve and the asymptote is*

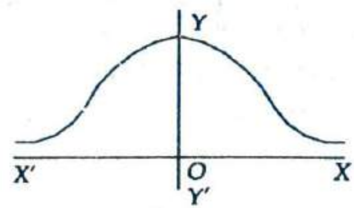


Fig. 11

$$= 2\int_0^\infty e^{-x^2}\,dx = 2 \cdot \tfrac{1}{2}\sqrt{\pi} = \sqrt{\pi}.$$

## 14.11. Cissoid of Diocles.

Its Cartesian equation is

$$y^2(2a - x) = x^3.$$

$OA = 2a$ ; $x = 2a$ is an asymptote.

Its polar equation is

$$r = \frac{2a \sin^2\theta}{\cos\theta}.$$

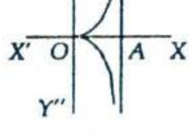


Fig. 12

## 14.12. Strophoid.

The equation of the curve is

$$y^2 = x^2\,\frac{a + x}{a - x}.$$

$$OA = OB = a.$$

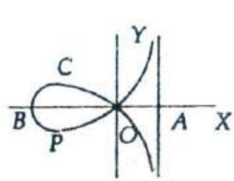


Fig. 13

*OCBPO* is a loop .

$x = a$ is an asymptote.

The curve $y^2 = x^2 \dfrac{a - x}{a + x}$ is similar, just the reverse of strophoid, the loop being on the right side of the origin and the asymptote on the left side.

## 14.13. Witch of Agnesi.

The equation of the curve is

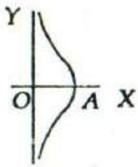



Fig.14

$$xy^2 = 4a^2 (2a - x) .$$

Here, $OA = 2a$ .

This curve was first discussed by the Italian lady mathematician Maria Gactaua Agnesi, Professor of Mathematics at Bologna.

## 14.14. Logarithmic (or Equiangular) spiral.

Its equation is $r = ae^{\theta \cot \alpha}$ ( or, $r = ae^{m\theta}$ ) .
where cot $\alpha$ or $m$ is constant.

(i) The tangent at any point makes a constant angle with the radius vector ( $\phi = \alpha$ ) .

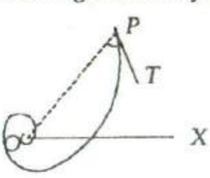



Fig.15

(ii) Its pedal, inverse, polar reciprocal and evolute are all equiangular spirals.

(iii) The radius of curvature subtends a right angle at the pole.

**Note.** Because of the property (i) , the spiral is called *equiangular*.

## 14.15. Spiral of Archimedes.

Its equation is $r = a\theta$ .

Its *characteristic property* is that its polar subnormal is constant.

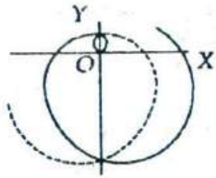



Fig.16

## 14.16. Cardioide.

Its equation is (i) $r = a(1 + \cos\theta)$, or (ii) $r = a(1 - \cos\theta)$.

In (i), $\theta = 0$ for $A$, and $\theta = \pi$ for $O$.

In (ii), $\theta = \pi$ for $A$, and $\theta = 0$ for $O$.

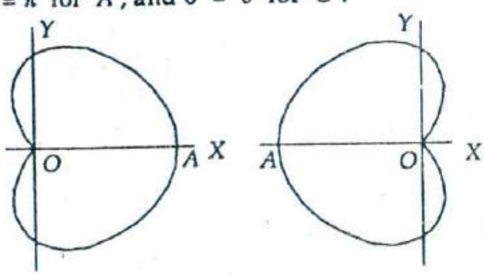



Fig.17

(i) $r = a(1 + \cos\theta)$. (ii) $r = a(1 - \cos\theta)$.

In both cases, the curve is symmetrical about the initial line which divides the whole curve into two equal halves and for the upper half $\theta$ varies from 0 to $\pi$, and $OA = 2a$.

The curve (ii) is really the same as (i) turned through 180°.

The curve passes through the origin, its tangent there being the initial line, and the tangent at $A$ is perpendicular to the initial line.

The evolute of the cardioide is a cardioide.

The perimeter of the cardioide is $8a$ .

**Note.** Because of its shape like a human heart, it is called a cardioide. The cardioide $r = a(1 + \cos\theta)$ is the pedal of the circle $r = 2a\cos\theta$ with respect to a point on the circumference of the circle and inverse of the parabola $r = a/(1 + \cos\theta)$ .

## 14.17. Limacon.

The equation of the curve is

$$\mathbf{r} = \mathbf{a} + \mathbf{b}\cos\theta .$$

When $a > b$ , we have the outer curve, and when $a < b$ , we have the inner curve with the loop.

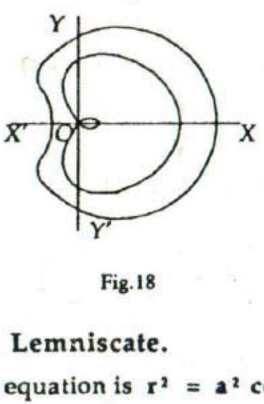



Fig.18

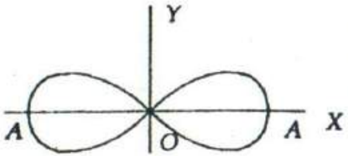
When $a = b$ , the curve reduces to a cardioide. [ See *fig. in § 14.16.*.]

Limacon is the pedal of a circle with respect to a point outside the circumference of the circle.

## 14.18. Lemniscate.

Its equation is $\mathbf{r}^2 = \mathbf{a}^2 \cos 2\theta$ ,

or, $(x^2 + y^2)^2 = a^2 (x^2 - y^2)$ .




It consists of two equal loops, each symmetrical about the initial line which divides each loop into two equal halves.

$$OA = OA' = a .$$

Fig.19

$r^2 = a^2 \cos 2\theta$ .

The tangents at the origin are $y = \pm x$.

For the upper half of the right-hand loop $\theta$ varies from 0 to $\frac{1}{4}\pi$.

A *characteristic property* of it is that the product of the distances of any point on it from $(\pm a/\sqrt{2}, 0)$ is constant.

The area of the lemniscate is $a^2$.

The lemniscate is the pedal of the rectangular hyperbola $r^2 \cos 2\theta = a^2$. The curve represented by $r^2 = a^2 \sin 2\theta$ is also sometimes called lemniscate or *rose lemniscate*, to distinguish it from the first lemniscate, which is sometimes called *Lemniscate of Bernoulli* after the name of the mathematician J. Bernoulli who first studied its properties.

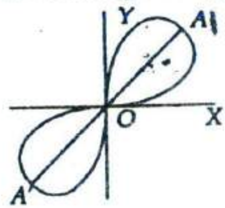



Fig.20

$r^2 = a^2 \sin 2\theta$.

The curve consists of two equal loops, situated in the first and third quadrants, and symmetrical about the line $y = x$. It is the first curve turned through 45°.

The tangents at the origin are the axes of $x$ and $y$.

The area of the curve is $a^2$.

## 14.19. Rose-Petals ( $r = a \sin n\theta$, $r = a \cos n\theta$ ).

The curve represented by $r = a \sin 3\theta$, or $r = a \cos 3\theta$ is called a *three-leaved rose*, each consisting of three equal loops. The order in which the loops are described is indicated in the figures by numbers. In each case, $OA = OB = OC = a$, and $m\angle AOB = m\angle BOC = m\angle COA = 120°$.

The curve represented by $r = a \sin 2\theta$, or $r = a \cos 2\theta$ is called a *four-leaved rose*, each consisting of four equal loops. In each case, $OA = OB = OC = OD = a$ and $m\angle AOB = m\angle BOC = m\angle COD = m\angle DOA = 90°$.

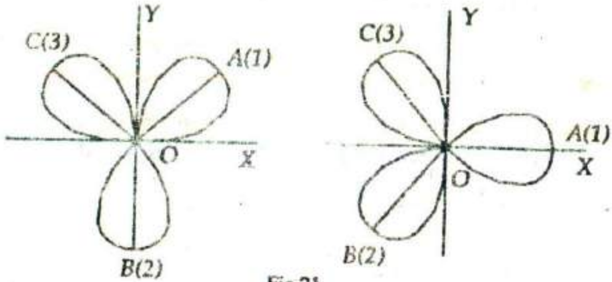



Fig.21

$r = a \sin 3\theta$. $r = a \cos 3\theta$.

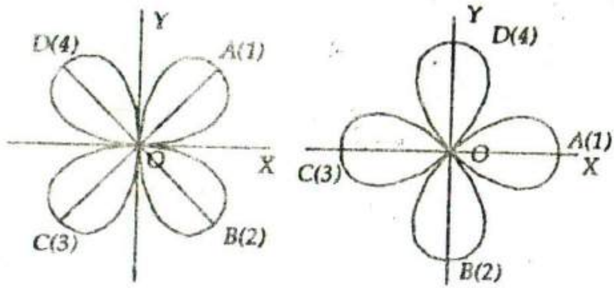



Fig.22

$r = a \sin 2\theta$. $r = a \cos 2\theta$.

The class of curves represented by $r = a \sin n\theta$, or $r = a \cos n\theta$, where $n$ is a positive integer, is called *rose-petal*, there being $n$ or $2n$ *equal loops* according as $n$ is *odd* or *even*, all being arranged symmetrically about the origin and lying entirely within a circle whose centre is the pole and radius $a$.

## 14.20. Sine Sipral ( $r^n = a^n \sin n\theta$, or, $r^n = a^n \cos n\theta$ ).

The class of curves represented by (i) $r^n = a^n \sin n\theta$, or (ii) $r^n = a^n \cos n\theta$ is called *sine spiral* and embraces several important and well-known curves as particular cases.

Thus, for the values $n = -1, 1, -2, +2, -\frac{1}{2}$ and $\frac{1}{2}$ the sine spiral is respectively a straight line, a circle, a rectangular hyperbola, a lemniscate, a parabola and a cardioide.

For (i) $\phi = n\theta$; for (ii) $\phi = \frac{1}{2}\pi + n\theta$.

The pedal equation in both the cases is

$$p = r^{n+1} / a^n .$$

—

# DIFFERENTIAL EQUATIONS

## CHAPTER XV

## INTRODUCTION AND DEFINITIONS

### 15.1. Definition and classification.

A *differential equation* is an equation involving differentials (or differential coefficients) with or without the variables from which these differentials (or differential coefficients) are derived.

The following are examples of differential equations :

$$\frac{dy}{dx} = e^{x} \qquad \ldots (1)$$

$$\left(\frac{dy}{dx}\right)^2 = ax^2 + bx + c \qquad \ldots (2)$$

$$\frac{d^2y}{dx^2} = 0 \qquad \ldots (3)$$

$$\left(\frac{d^3y}{dx^3}\right)^2 = x^2 \frac{dy}{dx} \qquad \ldots (4)$$

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^2 + 2y = 0 \qquad \ldots (5)$$

$$x\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0 \qquad \ldots (6)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \qquad \ldots (7)$$

Differential equations are divided into two classes, *viz., Ordinary* and *Partial.*

An *ordinary differential equation* is one in which all the differentials (or derivatives) involved have reference to a single independent variable.

A *partial differential equation* is one which contains partial differentials (or derivatives) and as such involves two or more independent variables.

Thus, in the above set, equations (1), (2), (3), (4) and (5) are ordinary diferential equations and equations (6) and (7) are partial differential equations.

In order to facilitate discussions, differential equations are classified according to order and degree.

The *order* of a differential equation is the order of the highest derivative (or differential) in the equation. Thus, equations (1) and (2) are of the first order, (3) and (5) are of the second order, and (4) is of the third order.

The *degree* of an algebraic differential equation is the degree of the derivative (or differential) of the highest order in the equation, after the equation is freed from radicals and fractions in its derivatives. Thus, equations (2) and (4) are of the second degree.

**Note.** Strictly speaking, the term 'degree' is used with reference to those differential equations only which can be written as polynomials in the derivatives.

We shall consider in this treatise only ordinary differential equations of different orders and degrees.

## 15.2. Formation of ordinary Differential Equations.

Let $$f(x, y, c_1) = 0 \qquad \ldots (1)$$
be an equation containing $x$, $y$ and one arbitrary constant $c_1$.

Differentiating (1), we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0. \qquad \ldots (2)$$

Equation (2) will, in general, contain $c_1$. If $c_1$ be eliminated between (1) and (2), we shall get a relation involving $x$, $y$ and $\frac{dy}{dx}$, which will evidently be a differential equation of the first order.

Similarly, if we have an equation

$$f(x, y, c_1, c_2) = 0 \qquad \ldots (3)$$
containing two arbitrary constants $c_1$ and $c_2$, then by differentiat-

ing this twice we shall get two equations. Now, between these two equations and the given equation, in all three equations, if the two arbitrary constants $c_1$ and $c_2$ be eliminated, we shall evidently get a differential equation of the second order.

In general, if we have an equation

$$f(x, y, c_1, c_2, \ldots, c_n) = 0 \qquad \ldots (4)$$

containing $n$ arbitrary constants $c_1, c_2, \ldots, c_n$, then by differentiating this $n$ times we shall get $n$ equations. Now, between these $n$ equations and the given equation, in all $(n + 1)$ equations, if the $n$ arbitrary constants $c_1, c_2, \ldots, c_n$ be eliminated, we shall evidently get a differential equation of the $n$th order*, for there being $n$ differentiations the resulting equation must contain a derivative of the $n$th order.

**Note.** From the process of forming a differential equation from a given primitive, it is clear that since the equation obtained by varying the arbitrary constants in the primitive represents a certain system or family of curves, the differential equation (in which the constants do not appear) expresses some properties common to all those curves. We may thus say that a *differential equation represents a family of curves all satisfying some common properties*. This can be considered as the geometrical interpretation of the differential equation.

## 15.3. Solution of a Differential Equation.

Any relation connecting the variables of an equation and not involving their derivatives, which satisfies the given differential equation, *i.e.*, from which the given differential equation can be derived, is called a *solution* of the differential equation. Thus,

$y = e^x + C$, where $C$ is any arbitrary constant,

and $y = Ax + B$, where $A$ and $B$ are arbitrary constants, are respectively the solutions of the differential equations (1) and (3) of Art. 15.1.

From the above, it is clear that a differential equation may have an unlimited number of solutions, for each of the different relations

* A relation containing $n$ arbitrary constants may, in certain cases, give rise to a differential equation of order less than $n$.

obtained by giving particular values to the arbitrary constant or constants in the solution of the equation satisfies the equation and hence, is a solution of the equation ; thus $y = x - \sqrt{11}$ , $y = 2x - 3$ , $y = -\frac{5}{2}x$ , etc. are all solutions of the differential equation (3) of Art. 15.1.

The arbitrary constants $A$ , $B$, $C$ appearing in the solution are called *arbitrary constants of integration.*

The solution of a differential equation in which the number of *independent arbitrary constants* is equal to the order of the equation is called the *general* or *complete solution* ( or *complete primitive* ) of the equation.

The solution obtained by giving particular values to the arbitrary constants of the general solution is called a *particular solution* of the equation.

Thus, $y = Ax + B$ is the general solution, and $y = x - \sqrt{11}$ , $y = 2x - 3$, $y = -\frac{5}{2}x$ are all particular solutions of the equation (3) of Art. 15.1.

There is another kind of solution called the *singular solution,* which will be discussed in a subsequent chapter. [ See *Art. 17.5* ]

By a proper manipulation of the arbitrary constants in the general solution of a differential equation, the general solution is very often written in different forms ; it should be noted, however, that each of these forms determines the same relation between the variables. This will be subsequently illustrated in the worked out examples.

When an equation is to be solved, it is generally implied that the complete solution is required.

It sometimes happens that the process of solving a differential equation leads to integrals which cannot be evaluated in terms of known elementary functions. In such a case, the equation is considered as having been solved when it has been reduced to an expression involving integrals and it is then said that the *solution of the equation has been reduced to quadrature.*

**Note. 1.** The arbitrary constants in the solution of a differential equation are said to be *independent,* when it is impossible to deduce from the solution an equivalent relation containing fewer arbitrary constants. Thus, the two arbitrary constants $A, B$ in the equation $y = Ae^{x+B}$ are not independent since the equation can be written as $y = Ae^{B} \cdot e^{x} = Ce^{x}$.

**Note. 2.** In this elementary treatise, we shall not concern ourselves with the question whether a differential equation has a solution or what are the conditions under which it will have a solution of a particular character ; in fact, we shall assume without proof the following fundamental theorem of differential equation, *viz.,*

*An ordinary differential equation of order $n$ has a solution involving $n$ independent arbitrary constants, and this solution is unique.*

## 15.4. Illustrative Examples.

**Ex. 1.** *Find the differential equation of all straight lines passing through the origin.*

Let $y = mx$ ... (1)
be the equation of any straight line passing through the origin.

Differentiating (1), $\frac{dy}{dx} = m$. ... (2)

Eliminating $m$ between (1) and (2), we get

$$y = x\frac{dy}{dx}$$, the required differential equation.

**Ex. 2.** *Find the differential equation from the relation*

$$x = a\cos t + b\sin t,$$

*$a$ and $b$ being arbitrary constants.*

Differentiating the given relation twice with respect to $t$, we get

$x_1 = -a\sin t + b\cos t$ and
$x_2 = -a\cos t - b\sin t = -(a\cos t + b\sin t) = -x$.

$\therefore x_2 + x = 0$, *i.e.,* $\frac{d^2x}{dt^2} + x = 0$ is the required differential equation.

**Ex. 3.** *Eliminate $a$ and $b$ from $y = a\tan^{-1}x + b$.*

Differentiating the given relation with respect to $x$,

$$y_1 = \frac{a}{1+x^2}. \qquad \therefore (1+x^2)y_1 = a.$$

Differentiating, $(1 + x^2) y_2 + 2xy_1 = 0$.

This is the required eliminant.

## EXAMPLES XV

1. Show that the differential equation of a system of concentric circles having the centres at the origin is $x\,dx + y\,dy = 0$. Interpret the result geometrically.

2. Prove that the differential equation of all circles touching the $x$-axis at the origin is $(x^2 - y^2)\,dy - 2xy\,dx = 0$.

3. (i) Show that the differential equation of all parabolas

(a) having their axes parallel to the $y$-axis is $y_3 = 0$;

(b) with foci at the origin and axes along the $x$-axis is

$$yy_1^2 + 2xy_1 - y = 0;$$

(ii) Show that the differential equation of the family of circles $x^2 + y^2 + 2gx + 2fy + c = 0$ is $(1 + y_1^2) y_3 - 3y_1 y_2^2 = 0$;

(iii) Show that the differntial equation of the family of cardioides $r = a(1 + \cos\theta)$ is $(1 + \cos\theta)\,dr + r\sin\theta\,d\theta = 0$.

4. Show that the differential equation of the system of rectangular hyperbolas $xy = c^2$ is $x\,dy + y\,dx = 0$, and interpret the result geometrically; deduce that the tangent intercepted between the axes is bisected at the point of contact.

5. Vertify that $y + x + 1 = 0$ is a solution of the differential equation $(y - x)\,dy - (y^2 - x^2)\,dx = 0$.

6. Show that $V = (A/r) + B$ is a solution of the differential equation

$$\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = 0.$$

7. Find the differential equation from the relation

(i) $y = A\sin x + B\cos x + x\sin x$; [*J. E. E.* '89]

(ii) $y = Ae^{x} + Be^{-x}$; [*J. E. E.* '84]

(iii) $y = A\cos x + B\sin x + C\cosh x + D\sinh x$, where $A, B, C, D$ are arbitrary constants.

8. Eliminate $a$ and $b$ from each of the relations

(i) $y = a\log x + b$; (ii) $xy = ae^{x} + be^{-x}$;

(iii) $ax^2 + by^2 = 1$; [ *C. P. 1945* ]

(iv) $r = a + b\cos\theta$.

9. (i) Show that the differential equation, whose genera solution is $y = c_1 x + c_2 x^2$, is $y = xy_1 - \frac{1}{2}x^2 y_2$.

(ii) Show that

$y = \cos x$, $y = \sin x$, $y = c_1 \cos x$, $y = c_2 \sin x$ are all solutions of the differential equation $y_2 + y = 0$.

[ *In (i) and (ii),* $c_1$ , $c_2$ *are arbitrary constants.* ]

10. Show that the differential equations, whose general solutions are

(i) $y = A\sin x + B\cos x$, [ *C. P. '88* ]

(ii) $y = A\sinh x + B\cosh x$,

where $A$ and $B$ are arbitrary constants, are respectively

$$\frac{d^2y}{dx^2} + y = 0 \text{ and } \frac{d^2y}{dx^2} - y = 0.$$

## ANSWERS

1. The radius vector and the tangent at any point are mutually perpendicular.

4. The radius vector and the tangent at any point are equally inclined to the $x$-axis.

7. (i) $y + y_2 = 2\cos x$. (ii) $y_2 - y = 0$. (iii) $y_4 - y = 0$.

8. (i) $xy_2 + y_1 = 0$. (ii) $xy_2 + 2y_1 = y$.

(iii) $x(yy_2 + y_1^2) = yy_1$. (iv) $r_2 = r_1 \cot\theta$.

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