CHAPTER XVI

EQUATIONS OF THE FIRST ORDER AND THE FIRST DEGREE

16.1. A differential equation of the first order and the first degree can be put in the form

$$M\ dx + N\ dy = 0,$$

where both M and N are functions of x and y, or constants not involving the derivatives. The general solution of an equation of this type contains only one arbitrary constant. In this chapter we shall consider only certain special types of equations of the first order and the first degree.

16.2. Separation of Variables.

If the equation $M\,dx + N\,dy = 0$ can be put in the form

$$f_1(x)\,dx + f_2(y)\,dy = 0,$$

then it can be immediately solved by integrating each term separately. Thus, the solution of the above equation is

$$\int f_1(x)\,dx + \int f_2(y)\,dy = C.$$

The process of reducing the equation $M\,dx + N\,dy = 0$ to the form $f_1(x)\,dx + f_2(y)\,dy = 0$ is called the *Separation of the Variables.*

Note. Sometimes *transformation to the polar co-ordinates* facilitates separ-ation of variables. In this connection it is convenient to remember the following differentials.

If $x = r\cos\theta$, $y = r\sin\theta$,

(i) $x\,dx + y\,dy = r\,dr$, (ii) $dx^2 + dy^2 = dr^2 + r^2\,d\theta^2$,

(iii) $x\,dy - y\,dx = r^2\,d\theta$.

[*For illustrations, see Ex. 8(ii) and (iii) of Examples XVI(A).*]

16.3. Illustrative Examples.

Ex. 1. *Solve* $(1 + y^2)\,dx + (1 + x^2)\,dy = 0$.

Dividing by $(1 + x^2)(1 + y^2)$, we get

$$\frac{dx}{1+x^2} + \frac{dy}{1+y^2} = 0.$$

$\therefore$ integrating, $\tan^{-1}x + \tan^{-1}y = C$. ... (1)

Note. Writing the arbitrary constant C in the form $\tan^{-1}a$, the above solution can be written as $\tan^{-1}x + \tan^{-1}y = \tan^{-1}a$,

or, $\tan^{-1}\dfrac{x+y}{1-xy} = \tan^{-1}a$, or, $x + y = a(1-xy)$. ... (2)

Both forms of solutions (1) and (2) are perfectly general ; and any one of these can be considered as the complete solution of the given equation. [See *Art.* 15.3.]

Ex. 2. *Solve* $x(y^2+1)\,dx + y(x^2+1)\,dy = 0$.

Dividing both sides by $(x^2+1)(y^2+1)$, we have

$$\frac{x}{x^2+1}\,dx + \frac{y}{y^2+1}\,dy = 0.$$

$\therefore$ integrating, we have

$\frac{1}{2}\log(x^2+1) + \frac{1}{2}\log(y^2+1) = C$.

Writing $\frac{1}{2}\log A$ in the place or C, the above solution can be written in the form

$$(x^2+1)(y^2+1) = A.$$

Note. In order to express the solution in a neat form, we have taken $\frac{1}{2}\log A$ (A being a constant) in the place of the arbitrary constant C.

Ex. 3. *Solve* $(x+y)^2\dfrac{dy}{dx} = a^2$.

Put $x + y = v$, *i.e.,* $y = v - x$. $\therefore\ \dfrac{dy}{dx} = \dfrac{dv}{dx} - 1$.

$\therefore$ the equation reduces to

$$v^2\left(\frac{dv}{dx} - 1\right) = a^2, \text{ or, } \frac{dv}{dx} = 1 + \frac{a^2}{v^2} = \frac{a^2+v^2}{v^2}$$

$$\therefore\ dx = \frac{v^2}{a^2+v^2}\,dv = \left(1 - \frac{a^2}{a^2+v^2}\right)dv.$$

integrating, $\displaystyle\int dx = \int dv - a^2\int\frac{dv}{a^2+v^2}$, or, $x + C = v - a^2 \cdot \frac{1}{a} \tan^{-1} \frac{v}{a} = x + y - a \tan^{-1} \frac{x + y}{a}$

$\therefore$ $y = a \tan^{-1} \frac{x + y}{a} + C$ is the required solution.

Ex. 4. *Find the foci of the curve which satisfies the differential equation* $(1 + y^2)\,dx - xy\,dy = 0$ *and passes through the point* $(1, 0)$.

Separating the variables of the equation, we have

$$\frac{dx}{x} - \frac{y\,dy}{1 + y^2} = 0.$$

$\therefore$ integrating, $\log x - \frac{1}{2}\log(1 + y^2) = \log C$,

or, $\log \frac{x}{\sqrt{(1 + y^2)}} = \log C$. $\therefore$ $x = C\sqrt{1 + y^2}$.

This is the equation of any curve satisfying the given differential equation. If the curve passes through $(1, 0)$, we have $1 = C$.

$\therefore$ the equation of the required curve is $x^2 - y^2 = 1$.

It is a rectangular hyperbola, and its foci are evidently $(\pm\sqrt{2}, 0)$.

Ex. 5. *Show that all curves for which the length of the normal is equal to the radius vector are either circles or rectangular hyperbolas.*

Since the length of the normal $= y\sqrt{(1 + y_1^2)}$ and the radius vector $= \sqrt{(x^2 + y^2)}$,

$\therefore$ $y^2(1 + y_1^2) = x^2 + y^2$, or, $y^2 y_1^2 = x^2$, or, $yy_1 = \pm x$.

$$\therefore \quad \frac{dy}{dx} = \pm\frac{x}{y}. \quad \therefore \ x\,dx \pm y\,dy = 0.$$

$\therefore$ integrating, $x^2 \pm y^2 = a^2$, a^2 being the arbitrary constant of integration.

Thus, the curves are either circles or rectangular hyperbolas.

Ex. 6. *Show that by substituting* $ax + by + c = z$ *in the equation* $\frac{dy}{dx} = f(ax + by + c)$ *the variables can be separated.*

Since $ax + by + c = z$, $\therefore$ $a + b\frac{dy}{dx} = \frac{dz}{dx}$.

$$\therefore \frac{dy}{dx} = \frac{1}{b}\left(\frac{dz}{dx} - a\right).$$

Hence, the given equation transforms into

$$\frac{1}{b}\left(\frac{dz}{dx}-a\right)=f(z),$$

$$i.e.,\quad \frac{dz}{a+bf(z)}=dx.$$

Thus, the variables are separated.

EXAMPLES XVI (A)

Solve the following differential equations (**Ex. 1 - 10**) :

1. (i) $\dfrac{dy}{dx}=\dfrac{x^2+x+1}{y^2+y+1}$. (ii) $x^2\dfrac{dy}{dx}+y=1$.

(iii) $\dfrac{dy}{dx}+\dfrac{y(y-1)}{x(x-1)}=0$.

2. (i) $y\,dx+(1+x^2)\tan^{-1}x\,dy=0$.

(ii) $e^{x-y}\,dx+e^{y-x}\,dy=0$.

3. (i) $x\sqrt{1-y^2}\,dx+y\sqrt{1-x^2}\,dy=0$.

(ii) $x^2(y-1)\,dx+y^2(x-1)\,dy=0$.

4. $\dfrac{dy}{dx}+\dfrac{y^2+y+1}{x^2+x+1}=0$.

5. (i) $\dfrac{dy}{dx}+\sqrt{\dfrac{1-y^2}{1-x^2}}=0$. (ii) $\dfrac{dy}{dx}=\dfrac{x(1+y^2)}{y(1+x^2)}$.

(iii) $\dfrac{dy}{dx}+\dfrac{\sqrt{(x^2-1)(y^2-1)}}{xy}=0$.

6. (i) $\sec^2 x\tan y\,dx+\sec^2 y\tan x\,dy=0$.

(ii) $x\cos^2 y\,dx-y\cos^2 x\,dy=0$. [*H. S. '85*]

(iii) $\dfrac{\log(\sec x+\tan x)}{\cos x}dx=\dfrac{\log(\sec y+\tan y)}{\cos y}dy$.

7. $(x^2-yx^2)\,dy+(y^2+xy^2)\,dx=0$. [*C. P. '88*]

8. (i) $y\,dx - x\,dy = xy\,dx$.

(ii) $x^2(x\,dx + y\,dy) + 2y(x\,dy - y\,dx) = 0$.

(iii) $\dfrac{x + yy_1}{xy_1 - y} = \sqrt{\left(\dfrac{1 - x^2 - y^2}{x^2 + y^2}\right)}$.

9. (i) $\dfrac{dy}{dx} + 1 = e^{x-y}$. (ii) $\dfrac{dy}{dx} = \sqrt{y - x}$.

10. (i) $\sin^{-1}\left(\dfrac{dy}{dx}\right) = x + y$. (ii) $\log\left(\dfrac{dy}{dx}\right) = ax + by$.

11. Find the particular solution of

$$\cos y\,dx + (1 + 2e^{-x})\sin y\,dy = 0.$$

when $x = 0$, $y = \frac{1}{4}\pi$.

12. Find the equation of the curve for which.

(i) the cartesian subtangent is constant.

(ii) the cartesian subnormal is constant.

(iii) the polar subtangent is constant.

(iv) the polar subnormal is constant.

13. Show that the curve for which the normal at every point passes through a fixed point is a circle.

14. Show that the curve for which the radius of curvature at every point is constant is a circle.

15. Show that the curve for which the tangent at every point makes a constant angle with the radius vector is an equiangular spiral.

16. Show that the curve in which the angle between the tangent and the radius vector at every point is one-half of the vectorial angle is a cardioide.

17. Show that the curve in which the angle between the tangent and the radius vector at every point is one-third of the inclination of the tangent to the initial line is a cardioide.

18. Show that the curve in which the portion of the tangent included between the co-ordinate axes is bisected by the point of contact is a rectangular hyperbola.

ANSWERS

1. (i) $\frac{1}{3}(x^3 - y^3) + \frac{1}{2}(x^2 - y^2) + x - y = C$. (ii) $y = 1 + Ce^{1/x}$.
(iii) $xy = c(x-1)(y-1)$. **2.** (i) $y \tan^{-1} x = C$.
(ii) $e^{2x} + e^{2y} = C$. **3.** (i) $\sqrt{1-x^2} + \sqrt{1-y^2} = C$.
(ii) $(x+1)^2 + (y+1)^2 + 2\log(x-1)(y-1) = C$.
4. $2xy + x + y + C(x+y+1) = 1$. **5.** (i) $\sin^{-1} x + \sin^{-1} y = C$.
(ii) $1 + y^2 = C(1+x^2)$. (iii) $\sqrt{(x^2-1)} - \sec^{-1} x + \sqrt{(y^2-1)} = C$.
6. (i) $\tan x \tan y = C$.
(ii) $x \tan x - \log \sec x = y \tan y - \log \sec y + C$.
(iii) $[\log(\sec x + \tan x)]^2 - [\log(\sec y + \tan y)]^2 = C$.
7. $\log \frac{x}{y} - \frac{x+y}{xy} = C$. **8.** (i) $ye^x = Cx$. (ii) $(x^2+y^2)(x+2)^2 = Cx^2$
(iii) $x^2 + y^2 = \sin^2 \alpha$, where $\alpha = \tan^{-1}(y/x) + C$.
9. (i) $e^y + \frac{1}{2}e^x + Ce^{-x}$. (ii) $\sqrt{y-x} + \log(\sqrt{y-x} - 1) = \frac{1}{2}x + C$.
10. (i) $\tan(x+y) - \sec(x+y) = C + x$. (ii) $ae^{-by} + be^{ax} = C$.
11. $(e^x + 2)\sec y = 3\sqrt{2}$. **12.** (i) $y = Ce^{x/a}$;
(ii) $y^2 = 2ax + C$. (iii) $r(C - \theta) = a$. (iv) $r = a\theta + C$.

16.4. Homogeneous Equations.

If M and N of the equation $M\,dx + N\,dy = 0$ are both of the same degree in x and y and are homogeneous, the equation is said to be *homogeneous*. Such an equation can be put in the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right).$$

Every homogeneous equation of the above type can be easily solved by putting $y = vx$ where v is a function of x, and consequently $\frac{dy}{dx} = v + x\left(\frac{dv}{dx}\right)$, whereby it reduces to the and consequently $\frac{dy}{dx} = v + x\left(\frac{dv}{dx}\right)$, whereby it reduces to the form $v + x\left(\frac{dv}{dx}\right) = f(v)$, *i.e.*, $\frac{dy}{x} = \frac{dv}{f(v) - v}$ in which the variables are separable as shown below.

Ex. *Solve* $(x^2 + y^2)\,dx - 2xy\,dy = 0$.

The equation can be written as

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}.$$

Putting $y = vx$, so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$, we have

$$v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{2vx^2} = \frac{1 + v^2}{2v}.$$

$$\therefore \quad x\frac{dv}{dx} = \frac{1 + v^2}{2v} - v = \frac{1 - v^2}{2v}.$$

$$\therefore \quad \frac{dx}{x} - \frac{2v}{1 - v^2}\,dv = 0.$$

$\therefore$ integrating, $\log x + \log(1 - v^2) = \log C$.

$\therefore$ $x(1 - v^2) = C$.

Re-substituting y / x for v and simplifying, we get the solution

$$x^2 - y^2 = Cx.$$

16.5. A Special Form.

The equation of the form

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\left(\frac{a_1}{a_2} \neq \frac{b_1}{b_2}\right) \qquad \ldots (1)$$

can be easily solved by putting $x = x' + h$ and $y = y' + k$, where h and k are constants, so that $dx = dx'$ and $dy = dy'$ and choosing h, k in such a way that

$$\left.\begin{aligned} a_1h + b_1k + c_1 &= 0 \\ \text{and} \quad a_2h + b_2k + c_2 &= 0 \end{aligned}\right\}. \qquad \ldots (2)$$

For, now the equation reduces to the form

$$\frac{dy'}{dx'} = \frac{a_1x' + b_1y'}{a_2x' + b_2y'}$$

which is homogeneous in x' and y' and hence solvable by the method of the previous article.

Note 1. The above method obviously fails, if $a_1 / a_2 = b_1 / b_2$; for in this case h and k cannot be determined from equation (2).

Let the equation be

$$\frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2} \left(\frac{a_1}{a_2} = \frac{b_1}{b_2} \right) \quad \ldots \quad (3)$$

Let $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{1}{m}$. $\therefore a_2 = a_1 m , \; b_2 = b_1 m$,

where m is a non-zero constant.

Assuming this to be the case, let the common value of these ratios be denoted by $1/m$, so that $a_2 = a_1 m$ and $b_2 = b_1 m$.

The equation (3) becomes

$$\frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{m (a_1 x + b_1 y) + c_2} .$$

Now, putting $a_1 x + b_1 y = v$, the variables can be easily separated and hence the equation can be solved. [See Ex. 2, *below*.]

Note 2. If in the equation (1), $a_2 = - b_1$, then the equation can be solved more easily by grouping the terms suitably. [See *Ex. 1(iv) of Examples XVI(C)* .]

16.6. Illustrative Examples.

Ex. 1. *Solve* $\frac{dy}{dx} = \frac{6x - 2y - 7}{2x + 3y - 6}$.

Putting $x = x' + h$, $y = y' + k$, so that $dx = dx'$, $dy = dy'$, we have

$$\frac{dy'}{dx'} = \frac{6x' - 2y' + 6h - 2k - 7}{2x' + 3y' + 2h + 3k - 6} .$$

Putting $6h - 2k - 7 = 0$ and $2h + 3k - 6 = 0$, and solving these two equations, we have $h = \frac{3}{2}$, $k = 1$.

$\therefore$ the equation becomes $\frac{dy'}{dx'} = \frac{6x' - 2y'}{2x' + 3y'}$.

Since the equation is now homogeneous, putting $y' = vx'$ and hence $\frac{dy'}{dx'} = v + x' \frac{dv'}{dx'}$, and simplifying, the equation reduces to

$$\frac{dx'}{x'} = - \frac{1}{2} \frac{6v + 4}{3v^2 + 4v - 6} \, dv \text{, which on integration gives}$$

$$- \log Ax' = \tfrac{1}{2} \log (3v^2 + 4v - 6).$$

$$\therefore \quad (Ax')^{-1} = (3v^2 + 4v - 6)^{1/2}.$$

Now, restoring the values of x' and v', where $x' = x - \frac{3}{2}$ and $v = y'/x' = 2(y - 1)/(2x - 3)$, we get the solution in the form

$$3y^2 + 4xy - 6x^2 - 12y + 14x = C.$$

Ex. 2. *Solve* $\frac{dy}{dx} = \frac{6x - 2y - 7}{3x - y + 4}$.

Since here $a_1/a_2 = b_1/b_2$, $\therefore$ putting $3x - y = v$, we get

$$3 - \frac{dy}{dx} = \frac{dv}{dx}$$, and hence the given equation gives

$$\frac{dv}{dx} = 3 - \frac{2v - 7}{v + 4} = \frac{v + 19}{v + 4}.$$

$$\therefore \quad dx = \frac{v + 4}{v + 19} dv = \left(1 - \frac{15}{v + 19}\right) dv.$$

$$\therefore \quad x + C = v - 15 \log (v + 19).$$

On restoring the value of v, we get the solution in the form

$$2x - y - 15 \log (3x - y + 19) = C.$$

Ex. 3. *Show that in an equation of the form*

$$y f_1 (xy) \, dx + x f_2 (xy) \, dy = 0,$$

the variables can be separated by the substitution $xy = v$.

Since $xy = v$, $y = \frac{v}{x}$ and $d(xy) = dv$, i.e., $y\,dx + x\,dy = dv$

and $dy = \frac{x\,dv - v\,dx}{x^2}$, i.e., $x\,dy = dv - \frac{v}{x}\,dx$.

$$\therefore \quad \frac{v}{x} f_1 (v)\,dx + f_2 (v) \{dv - \frac{v}{x}\,dx\} = 0.$$

$$\therefore \quad \frac{f_2 (v)\,dv}{v \{f_1 (v) - f_2 (v)\}} + \frac{dx}{x} = 0.$$

Thus, the variables are separated.

[See *Ex. 14, 15, 16 of Examples XVI(B)*.]

We can as well form an equation in v and y, by taking $xy = v$, $x = v/y$

and $dx = \frac{y\,dv - v\,dy}{y^2}$

[For illustration see *Alternative proof of Ex. 4 of Art. 16.7.*]

EXAMPLES XVI(B)

Solve (Ex. 1 - 15) :-

1\. (i) $x + y\dfrac{dy}{dx} = 2y$. (ii) $\dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{y^2}{x^2}$.

2\. (i) $\dfrac{dy}{dx} = \dfrac{y(x - 2y)}{x(x - 3y)}$. (ii) $\dfrac{dy}{dx} = \dfrac{x^2 y}{x^3 + y^3}$.

3\. $(x^2 + y^2)\,dy = xy\,dx$.

4\. (i) $\dfrac{dy}{dx} = \dfrac{x - y}{x + y}$. (ii) $\dfrac{dy}{dx} = \dfrac{y(y + x)}{x(y - x)}$.

5\. (i) $\{3x \sinh(y/x) + 5y \cosh(y/x)\}\,dx - 5x \cosh(y/x)\,dy = 0$

(ii) $(1 + 3e^{x/y})\,dx + 3e^{x/y}(1 - x/y)\,dy = 0$.

6\. $(x^2 - 2xy)\,dy + (x^2 - 3xy + 2y^2)\,dx = 0$.

7\. $y^2\,dx + (x^2 + xy)\,dy = 0$.

8\. (i) $\dfrac{dy}{dx} = \dfrac{y}{x} + \tan\dfrac{y}{x}$. [*H.S.'81, '83, '89*] (ii) $\dfrac{dy}{dx} = \dfrac{3x + 2y}{2x - 3y}$.

9\. $(6x - 5y + 4)\,dy + (y - 2x - 1)\,dx = 0$.

10\. $(x - 3y + 4)\,dy + (7y - 5x)\,dx = 0$.

11\. $(2x - 2y + 5)\,dy - (x - y + 3)\,dx = 0$. [*H. S. '83*]

12\. $(x + y + 1)\,dx - (2x + 2y + 1)\,dy = 0$. [*J. E. E. '89*]

13\. $y(2xy + 1)\,dx + x(1 + 2xy + x^2y^2)\,dy = 0$.

14\. $x^2y^3\,dx + 3x^2y\,dy + 2y\,dx = 0$.

15\. $(1 + xy\cos xy)\,dx + x^2\cos xy\,dy = 0$.

16\. Show that $(4x + 3y + 1)\,dx + (3x + 2y + 1)\,dy = 0$ represents hyperbolas having as asymptotes

$$x + y = 0,\ 2x + y + 1 = 0 .$$

ANSWERS

1. (i) $y = x + Ce^{x/(y-x)}$. (ii) $2x - y = Cx^2y$.

2. (i) $y^3e^{x/y} = Cx^2$. (ii) $y^3 = Ce^{x^3/y^3}$.

3. $y = Ce^{x^2/2y^2}$. 4. (i) $y^2 + 2xy - x^2 = C$. (ii) $xy = Ce^{y/x}$.

5. (i) $x^3 = C\sinh^5(y/x)$. (ii) $x + 3ye^{x/y} = C$.

6. $y = x\log(Cx^{-1})$. 7. $xy^2 = C(x + 2y)$.

8. (i) $x = C\sin\frac{y}{x}$. (ii) $3\log(x^2 + y^2) = 4\tan^{-1}\frac{y}{x} + C$.

9. $(5y - 2x - 3)^4 = C(4y - 4x - 3)$. 10. $(3y - 5x + 10)^2 = C(y - x + 1)$

11. $2y - x + C = \log(x - y + 2)$. 12. $6y - 3x = \log(3x + 3y + 2) + C$.

13. $2x^2y^2\log y - 4xy - 1 = Cx^2y^2$. 14. $x(xy - 2)^3 = C(xy - 1)^3$.

15. $xe^{\sin xy} = C$.

16.7. Exact Equation.

The differential equation $M\,dx + N\,dy = 0$, where both M and N are functions of x and y, is said to be *exact* when there is a function u of x, y, such that $M\,dx + N\,dy = du$, *i.e.*, when $M\alpha x + N\,dy$ becomes *a perfect differential*.

Now, we know from Differential Calculus that $M\,dx + N\,dy$ should be a perfect differential if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Hence, the *condition that $M\,dx + N\,dy = 0$ should be an exact differential equation is* $\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x}$.

The method of solving an exact equation of the type $M\,dx + N\,dy = 0$ is as follows :

"First integrate the terms in $M\,dx$ as if y were constant, then integrate the terms in $N\,dy$ considering x as constant, and, rejecting the terms already obtained, equate the sum of these integrals to a constant". This will be the solution of the required equation.

Ex. 1 *Solve* $(2x^3 + 4y)\,dx + (4x + y - 1)\,dy$.

Here, $M = 2x^3 + 4y$, $N = 4x + y - 1$.

$$\frac{\partial M}{\partial y} = 4 = \frac{\partial N}{\partial x}\ ; \text{ hence it is an exact equation.}$$

$$\int M\,dx = \int (2x^3 + 4y)\,dx = 2\frac{x^4}{4} + 4yx = \tfrac{1}{2}x^4 + 4xy, \quad \dots \quad (1)$$

$$\int N\,dy = \int (4x + y - 1)\,dy = 4xy + \tfrac{1}{2}y^2 - y. \quad \dots \quad (2)$$

Rejecting the term 4xy in (2) which already occurs in (1) and then adding (1) and (2) and equating the sum to a constant, we get the general solution to be

$$\tfrac{1}{2}x^4 + \tfrac{1}{2}y^2 + 4xy - y = C.$$

An exact differential equation can often be solved by *inspection*, by picking out the terms of $M\,dx + N\,dy$ that obviously form a perfect differential and then integrating this. This is illustrated in the following worked out examples. While grouping the terms suitably to form a perfect differential it will be convenient to remember the following differentials.

$$y\,dx + x\,dy = d\,(xy),\ \frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right),\ \frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right).$$

16.8. Integrating Factors.

If a differential equation when multiplied by a factor (a function of x, y) becomes exact, this factor is called an *Integrating Factor* of the equation. An integrating factor is sometimes shortly writen as I. F.

Integrating factors can often be obtained by *inspection*. This is illustrated in Ex. 2 below. An equation can have more that one integrating factor, this is also illustrated in Ex. 3 below.

16.9. Rules for determining Integrating Factors.

Let the differential equation be

$$M\,dx + N\,dy = 0. \quad \dots \quad (1)$$

The condition that it should be *exact*, is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \dots \quad (2)$$

Rule (I). If $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}$ is a function of x only, say $f(x)$, then $e^{\int f(x)dx}$ will be an *integrating factor of* (1).

If $M\,dx + N\,dy = 0$, be an *exact* equation, when it is multiplied by $e^{\int f(x)dx}$ we must have

$$\frac{\partial}{\partial y}\left(Me^{\int f(x)dx}\right) = \frac{\partial}{\partial x}\left(Ne^{\int f(x)dx}\right)$$

i.e., $\dfrac{\partial M}{\partial y}e^{\int f(x)dx} = \dfrac{\partial N}{\partial x}e^{\int f(x)dx} + Ne^{\int f(x)dx}f(x)$

i.e., $$\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N} = f(x).$$

Rule (II). If $\dfrac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M} = f(y)$, (a function of y alone) $e^{\int f(y)dy}$ is an *integrating factor.*

Proof is similar to that given above.

Rule (III). If M and N are both homogeneous functions in x, y of degree n (say), then

$$\frac{1}{Mx + Ny}, \ (Mx + Ny \neq 0)$$

is an *integrating factor* of the equation (1).

We can easily show that

$$\frac{\partial}{\partial y}\left(\frac{M}{Mx + Ny}\right) = \frac{\partial}{\partial x}\left(\frac{N}{Mx + Ny}\right)$$

if we remember that M and N are homogeneous functions of degree n and hence $x\dfrac{\partial M}{\partial x} + y\dfrac{\delta M}{\partial y} = nM$.

and $x\dfrac{\partial N}{\partial x} + y\dfrac{\partial N}{\partial y} = nN$.

If $Mx + Ny = 0$, then $\frac{M}{N} = -\frac{y}{x}$ and the equation reduces to $y\,dx - x\,dy = 0$ which can be easily solved.

Rule (IV). If the equation (1) is of the form

$$y\,f(xy)\,dx + x\,g(xy)\,dy = 0,$$

then

$$\frac{1}{Mx - Ny}, \ (Mx - Ny \neq 0)$$

is an *integrating factor* of (1).

We can easily show that

$$\frac{\partial}{\partial y}\left(\frac{M}{Mx - Ny}\right) = \frac{\partial}{\partial x}\left(\frac{N}{Mx - Ny}\right),$$

$$i.e., \frac{\partial}{\partial y}\left[\frac{y\,f(xy)}{xy[f(xy) - g(xy)]}\right] = \frac{\partial}{\partial x}\left[\frac{x\,g(xy)}{xy[f(xy) - g(xy)]}\right],$$

provided we remember $y\,\frac{\partial}{\partial y}F(xy) = x\,\frac{\partial}{\partial x}F(xy)$.

If, however, $Mx - Ny = 0$, then $\frac{M}{N} = \frac{y}{x}$ and the equation reduces to $x\,dy + y\,dx = 0$ which can be easily solved.

16.10. Illustrative Examples.

Ex. 1. *Solve*: $(2x^2 + y^2 + x)\,dx + xy\,dy = 0$.

Here $\frac{\partial M}{\partial y} = 2y$; $\frac{\partial N}{\partial x} = y$. $\quad\therefore$ the equation is not exact.

Now,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y - y}{xy} = \frac{1}{x}.$$

$\therefore$ *by Rule (I)*, I. F. $= e^{\int(1/x)\,dx} = e^{\log x} = x$.

Multiplying both sides of the given equation by x, we have

$$(2x^3 + xy^2 + x^2)\,dx + x^2y\,dy = 0,$$

or, $2x^3\,dx + x^2\,dx + xy\,(y\,dx + x\,dy) = 0,$

or, $2x^3\,dx + x^2\,dx + xy\,d\,(xy) = 0,$

or, $2x^3\,dx + x^2\,dx + z\,dz = 0$, where $z = xy$.

$\therefore$ integrating, $\frac{x^4}{2} + \frac{x^3}{3} + \frac{z^2}{2} = c_1$.

$\therefore$ the required solution is $3x^4 + 2x^3 + 3x^2y^2 = c$.

Ex. 2. *Solve* : $(x^3 + y^3)\,dx - xy^2\,dy = 0$.

Here, $\frac{\partial M}{\partial y} = 3y^2 ; \frac{\partial N}{\partial x} = -y^2$. $\therefore$ the equation is not exact.

Now, $\frac{1}{Mx + Ny} = \frac{1}{x^4 + xy^3 - xy^3} = \frac{1}{x^4}$.

The equation is homogeneous.

$\therefore$ *by Rule (III)*, $1/x^4$ is an integrating factor.

Multiplying both sides of the given equation by $1/x^4$, we have

$$\left(\frac{1}{x} + \frac{y^3}{x^4}\right)dx - \frac{y^2}{x^3}\,dy = 0.$$

This is exact.

Now, $\int M\,dx = \int \left(\frac{1}{x} + \frac{y^3}{x^4}\right)dx = \log x - \frac{1}{3}\frac{y^3}{x^3}$

$$\int N\,dy = -\frac{1}{3}\frac{y^3}{x^3}.$$

$\therefore$ *by Art. 16.7*, the solution is

$$\log x - \frac{1}{3}\frac{y^3}{x^3} = c, \text{ i.e., } y^3 = 3x^3 \log x + cx^3.$$

Ex. 3. *Solve* $(y + x)\,dx + x\,dy = 0$.

The equation can be written as

$(y\,dx + x\,dy) + x\,dx = 0$, or, $d\,(xy) + x\,dx = 0$.

$\therefore$ integrating, $xy + \frac{1}{2}x^2 = C$ is the required solution.

Ex. 4. *Solve* $y(1 + xy)\,dx - x\,dy = 0$.

The equation can be written as

$$y\,dx - x\,dy + y^2 x\,dx = 0.$$

Dividing by y^2, $\dfrac{y\,dx - x\,dy}{y^2} + x\,dx = 0$ or, $d\left(\dfrac{x}{y}\right) + x\,dx = 0$.

Hence integrating, the solution is $\dfrac{x}{y} + \dfrac{1}{2}x^2 = C$

Note. Here $1/y^2$ is an *integrating factor*.

Ex. 5. *Solve* $y\,dx - x\,dy = 0$.

Multiplying the given equation by $\dfrac{1}{y^2}$, this can be written as

$$\frac{y\,dx - x\,dy}{y^2} = 0, \text{ i.e., } d\left(\frac{x}{y}\right) = 0, \text{ i.e., } \frac{x}{y} = C, \text{ i.e., } x = Cy \ldots \quad (1)$$

Again, multiplying the equation by $1/x^2$, this can be written as

$$\frac{x\,dy - y\,dx}{x^2} = 0, \text{ i.e., } d\left(\frac{y}{x}\right) = 0, \text{ i.e., } \frac{y}{x} = C, \text{ i.e., } y = Cx \ldots \quad (2)$$

Again, multiplying the equation by $1/xy$, this can be written as

$$\frac{y\,dx - x\,dy}{xy} = 0, \text{ i.e., } \frac{dx}{x} - \frac{dy}{y} = 0. \quad \therefore \log x - \log y = \log C,$$

$$\text{i.e., } \log\frac{x}{y} = \log C.$$

$$\therefore \frac{x}{y} = C, \text{ i.e., } x = Cy. \quad \ldots \quad (3)$$

Further the equation can be written as

$$\frac{dy}{dx} - \frac{1}{x}y = 0. \quad \ldots \quad (4)$$

This belongs to the linear form (See Art. 16.11).

Note. Thus, we see from (1), (2) and (3) that the number of integrating factors of an equation is *more than one* and from (4) we find that a differential equation can be solved by *different methods*.

Ex. 6. *Solve*

$$(x^3y^3 + x^2y^2 + xy + 1)\,y\,dx + (x^3y^3 - x^2y^2 - xy + 1)\,x\,dy = 0.$$

Now, $(x^3y^3 + x^2y^2 + xy + 1) = x^2y^2(xy + 1) + (xy + 1)$
$= (xy + 1)(x^2y^2 + 1)$

and $x^3y^3 - x^2y^2 - xy + 1 = x^2y^2(xy - 1) - (xy - 1)$
$= (xy - 1)(x^2y^2 - 1) = (xy - 1)^2(xy + 1)$

$\therefore$ the given equation becomes

$(xy + 1)(x^2y^2 + 1)\,y\,dx + (xy - 1)^2(xy + 1)\,x\,dy = 0,$

or, $(x^2y^2 + 1)\,y\,dx + (xy - 1)^2\,x\,dy = 0,$ ... (1)

[cancelling the common factor $(xy + 1)$]

or, $(x^2y^2 + 1)\,y\,dx + (x^2y^2 - 2xy + 1)\,x\,dy = 0,$

or, $x^2y^2(y\,dx + x\,dy) + (y\,dx + x\,dy) - 2x^2y\,dy = 0,$

or, $x^2y^2\,d(xy) + d(xy) - 2x^2y\,dy = 0,$

or, $d(xy) + \dfrac{d(xy)}{x^2y^2} - \dfrac{2}{y}\,dy = 0$ [on dividing by x^2y^2],

or, $dv + \dfrac{dv}{v^2} - \dfrac{2}{y}\,dy = 0$ [putting $xy = v$]

or, $v - \dfrac{1}{v} - 2\log y = C,$

or, $xy - \dfrac{1}{xy} - 2\log y = C.$

Alternative Method :

Putting $xy = v$, so that $x = \dfrac{v}{y}$, $dx = \dfrac{y\,dv - v\,dy}{y^2}$ in (1),
we get, on simplification,

$$\left(1 + \frac{1}{v^2}\right)dv - \frac{2dy}{y} = 0.$$

$\therefore$ integrating, $v - \dfrac{1}{v} - 2\log y = C,$

i.e., $xy - \dfrac{1}{xy} - 2\log y = C.$

EXAMPLES XVI(C)

Solve :-

1. (i) $(2x - y + 1)\,dx + (2y - x - 1)\,dy = 0$.

(ii) $\dfrac{dy}{dx} + \dfrac{ax + hy + g}{hx + by + f} = 0$.

(iii) $(1 - x^2)\dfrac{dy}{dx} - 2xy = x - x^3$.

(iv) $\dfrac{dy}{dx} = \dfrac{4x - 5y + 3}{5x - 6y + 2}$.

2. (i) $x\dfrac{dy}{dx} + y = y^2 \log x$.

(ii) $x\dfrac{dy}{dx} = y + \cos\dfrac{1}{x}$.

3. (i) $x\,dx + y\,dy + (x^2 + y^2)\,dy = 0$.

(ii) $x^2y_1 + xy + 2\sqrt{(1 - x^2y^2)} = 0$.

4. (i) $x\,dy - y\,dx + a(x^2 + y^2)\,dx = 0$.

(ii) $x\,dy - y\,dx - 2\sqrt{(x^2 - y^2)}\,dx = 0$.

5. (i) $x\,dx + y\,dy + \dfrac{x\,dy - y\,dx}{x^2 + y^2} = 0$.

(ii) $(x + x^5 + 2x^3y^2 + xy^4)\,dx + y\,dy = 0$.

6. (i) $(x + y)\,dy + (y - x)\,dx = 0$.

(ii) $(x + y)(dx - dy) = dx + dy$.

(iii) $(2 - xy)\,y\,dx - (2 + xy)\,x\,dy = 0$.

(iv) $2xy\,dx - (x^2 - y^2)\,dy = 0$.

(v) $(xy^2 + 3e^{x^{-3}})\,dx - x^2y\,dy = 0$.

7. (i) $(x^3 + 3xy^2)\,dx + (y^3 + 3x^2y)\,dy = 0$.

(ii) $(x^3y^2 - y)\,dx - (x^2y^3 + x)\,dy = 0$.

(iii) $(x^2 + y^2 + 2x)\,dx + xy\,dy = 0$.

8. (i) $\frac{dy}{dx} \sin x - y \cos x + y^2 = 0$.

(ii) $(xy \cos xy + \sin xy)\, dx + x^2 \cos xy\, dy = 0$.

(iii) $x \cos(y/x)(y\, dx + x\, dy) = y \sin(y/x)(x\, dy - y\, dx)$.

(iv) $(\cos y + y \cos x)\, dx + (\sin x - x \sin y)\, dy = 0$.

9. (i) $y(1 + xy)\, dx + x(1 - xy)\, dy = 0$.

(ii) $(x + 2y^3) \frac{dy}{dx} = y$.

(iii) $(x^2y^2 + xy)\, y\, dx + (x^2y^2 - 1)\, x\, dy = 0$.

10. $(x^2 + y^2 + 4)\, x\, dx + (x^2 - y^2 + 9)\, y\, dy = 0$.

11. $(1 + 3x^2 + 6xy^2)\, dx + (1 + 3y^2 + 6x^2y)\, dy = 0$.

12. Solve $(1 + x^2)\, y_1 + 2xy = 4x^2$, and obtain the cubic curve satisfying the equation and passing through the origin.

ANSWERS

1. (i) $x^2 + y^2 - xy + x - y = C$. (ii) $ax^2 + by^2 + 2hxy + 2gx + 2fy = C$.
(iii) $y(1 - x^2) = \frac{1}{2}x^2 - \frac{1}{4}x^4 + C$. (iv) $5xy - 3y^2 + 2y - 2x^2 - 3x = C$.

2. (i) $1 + Cxy = y(1 + \log x)$. (ii) $\frac{y}{x} + \sin \frac{1}{x} = C$.

3. (i) $x^2 + y^2 = Ce^{-2y}$. (ii) $\sin^{-1}(xy) + 2 \log x = C$.

4. (i) $\tan^{-1} \frac{y}{x} + ax = C$. (ii) $y = x \sin \log(Cx^2)$.

5. (i) $\frac{1}{2}(x^2 + y^2) + \tan^{-1} \frac{y}{x} = C$. (ii) $(C + x^2)(x^2 + y^2) = 1$.

6. (i) $x^2 - y^2 - 2xy = C$. (ii) $x - y + C = \log(x + y)$.
(iii) $2 \log(x/y) - xy = C$. (iv) $x^2 + y^2 = Cy$.
(v) $y^2 / x^2 = 2e^{1/x^3} + C$.

7. (i) $x^4 + y^4 + 6x^2y^2 = C$. (ii) $x^2 - y^2 + 2/xy = C$.
(iii) $3x^4 + 8x^3 + 6x^2y^2$.

8. (i) $\sin x = y(x + C)$. (ii) $x \sin xy = C$.

(iii) $xy \cos(y/x) = C$. (iv) $x \cos y + y \sin x = C$.

9. (i) $\log(x/y) - 1/xy = C$. (ii) $x = y^3 + Cy$. (iii) $xy - \log y = C$.

10. $x^4 - y^4 + 8x^2 + 18y^2 + 2x^2y^2 = C$.

11. $x + y + x^3 + y^3 + 3x^2y^2 = C$.

12. $y(1 + x^2) = \frac{4}{3}x^3 + C$; $3y(1 + x^2) = 4x^3$.

16.11. Linear Equations.

An equation of the form

$$\frac{dy}{dx} + Py = Q$$

in which *P and Q are functions of x alone or constants* is called *a linear equation of the first order.*

The general solution of the above equation can be found as follows. Multiply both sides of the equation by $e^{\int P dx}$.

$$\therefore \frac{dy}{dx} e^{\int P dx} + Pye^{\int P dx} = Qe^{\int P dx},$$

$$i.e., \frac{d}{dx}\left(ye^{\int P dx}\right) = Qe^{\int P dx}.$$

$\therefore$ integrating, $ye^{\int P dx} = \int Qe^{\int P dx}\,dx + C$,

or, $y = e^{-\int P dx}\left[\int Qe^{\int P dx}\,dx + C\right]$ is the required solution.

Cor. 1. If in the above equation Q is zero, the general solution is $y = Ce^{-\int P dx}$.

Cor. 2. If P be a constant and equal to $-m$, then the solution is

$$y = e^{mx}\left[\int e^{-mx} Q\,dx + C\right].$$

Note. Here the factor $e^{-\int P dx}$, which renders the lefthand member of the equation a perfect differential, is called an *Integrating Factor*. It is sometimes shortly written as $I.F$.

Ex. *Solve* $\cos x \frac{dy}{dx} + y \sin x = 1$.

Dividing by $\cos x$, $\frac{dy}{dx} + y \tan x = \sec x$. ... (1)

Here $\int P\,dx = \int \tan x\,dx = \log \sec x$.

$\therefore$ the integrating factor is $e^{\log \sec x} = \sec x$.

Multiplying both sides of (1) by $\sec x$, we have

$$\sec x \frac{dy}{dx} + y \sec x \tan x = \sec^2 x, \text{ or, } \frac{d}{dx}(y \sec x) = \sec^2 x.$$

$\therefore$ integrating, $y \sec x = \tan x + C$ is the required solution

16.12. Bernoulli's Equation.

An equation of the form

$$\frac{dy}{dx} + Py = Qy^n,$$

where P *and* Q *are functions of* x alone, is known as *Bernoulli's equation*. It is easily reduced to the linear form of Art. 16.11 as is shown below.

Dividing both sides by y^n, we get

$$y^{-n} \frac{dy}{dx} + Py^{-n+1} = Q.$$

Putting $y^{-n+1} = v$, and hence $(-n+1)\,y^{-n} \dfrac{dy}{dx} = \dfrac{dv}{dx}$, the equation reduces to

$$\frac{dv}{dx} + (1-n)Pv = (1-n)Q.$$

This being linear in v can be solved by the method of the previous article.

Note. Here n is a rational number

16.13. Illustrative Examples.

Ex. 1. Solve $\dfrac{dy}{dx} + \dfrac{x}{1-x^2}\,y = x\sqrt{y}$

Dividing both sides of the equation by $\sqrt{y}$ we have

$$y^{-1/2} \frac{dy}{dx} + \frac{x}{1-x^2}\,y^{1/2} = x \qquad \ldots \quad (1)$$

Put $\quad y^{1/2} = v. \quad \therefore \frac{1}{2} y^{-1/2} \dfrac{dy}{dx} = \dfrac{dv}{dx}.$

∴ (1) reduces to $\frac{dv}{dx} + \frac{x}{2(1-x^2)} v = \frac{x}{2}$ ... (2)

which is of the linear form.

Here $\int P\,dx = \int \frac{x\,dx}{2(1-x^2)} = -\frac{1}{4}\log(1-x^2) = \log(1-x^2)^{-1/4}$

∴ I. F. $= e^{\log(1-x^2)^{-1/4}} = (1-x^2)^{-1/4}$.

∴ multiplying (2) by $(1-x^2)^{-1/4}$ and integrating, we have

$$\frac{v}{(1-x^2)^{1/4}} = \int \frac{x}{2(1-x^2)^{1/4}}\,dx = -\frac{1}{3}(1-x^2)^{3/4} + C.$$

∴ substituting the value of v, the required solution is

$$\sqrt{y} = -\frac{1}{3}(1-x^2) + C(1-x^2)^{1/4}.$$

Ex. 2. *Solve* $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$.

Dividing by $\cos^2 y$, we have

$$\sec^2 y\,\frac{dy}{dx} + 2x\tan y = x^3. \qquad \ldots (1)$$

Putting $\tan y = z$, so that $\sec^2 y\,\frac{dy}{dx} = \frac{dz}{dx}$, we have

$$\frac{dz}{dx} + 2xz = x^3. \qquad \ldots (2)$$

This is of the linear form. Here I. F. $= e^{\int 2x\,dx} = e^{x^2}$.

∴ multiplying (2) by e^{x^2} and integrating,

$ze^{x^2} = \int x^3 e^{x^2}\,dx + C = \frac{1}{2}e^{x^2}(x^2-1) + C$ (integrating by parts)

∴ $e^{x^2}\tan y = \frac{1}{2}e^{x^2}(x^2-1) + C$ is the required solution.

EXAMPLES XVI(D)

Solve (Ex. 1 – 14) :-

1. (i) $\frac{dy}{dx} + y = x$. (ii) $\frac{dy}{dx} + \frac{1-2x}{x^2}y = 1$.

2. $\frac{dy}{dx} + y\cot x = 2\cos x$.

3. $\cos^2 x \dfrac{dy}{dx} + y = \tan x$.

4. (i) $(1 - x^2)\dfrac{dy}{dx} - xy = 1$. (ii) $\dfrac{dy}{dx} + xy = x$.

(iii) $\dfrac{dy}{dx} + \dfrac{y}{x} = \sin x^2$.

5. $(x^3 - x)y_1 - (3x^2 - 1)y = x^5 - 2x^3 + x$.

6. $(x\cos x)y_1 + y(x\sin x + \cos x) = 1$.

7. $(1 + y^2)dx - (\tan^{-1}y - x)dy = 0$.

8. (i) $y^2 + \left(x - \dfrac{1}{y}\right)\dfrac{dy}{dx} = 0$. (ii) $(x + y + 1)\dfrac{dy}{dx} = 1$.

(iii) $(x^2y^3 + 2xy)dy = dx$.

[*Write as a linear equation in* x]

9. $\dfrac{dy}{dx} + \dfrac{1}{x}y = y^2$. 10. $y_1 - 2y\tan x + y^2\tan^4 x = 0$.

11. (i) $\dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{y^2}{x^2}$. (ii) $\dfrac{dy}{dx} + \dfrac{1}{x} = \dfrac{e^y}{x^2}$.

12. $\dfrac{dy}{dx} + \dfrac{1}{x}\sin 2y = x^3\cos^2 y$. [*Put* $\tan y = z$.]

13. $\dfrac{dy}{dx} + \dfrac{y}{x}\log y = \dfrac{y}{x^2}(\log y)^2$.

14. $\dfrac{dy}{dx} - \dfrac{\tan y}{1 + x} = (1 + x)e^x \sec y$.

15. Solve $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2$, given $y = 1$ when $x = 1$.

16. Show that the equation of the curve whose slope at any point is equal to $y + 2x$ and which passes through the origin is $y = 2(e^x - x - 1)$.

17. Find the curve for which the sum of the reciprocals of the radius vector and the polar subtangent is constant.

18. Show that the curves for which the radius of curvature varies as the square of the perpendicular upon the normal belong

to the class whose pedal equation is

$$r^2 - p^2 = \frac{p}{k} + \frac{1}{2k^2} + Ae^{2kp}$$

k being a given constant and A an arbitrary constant.

ANSWERS

1. (i) $y = x - 1 + Ce^{-x}$. (ii) $y = x^2(1 + Ce^{1/x})$.

2. $y = \sin x + C(\sin x)^{-1}$. 3. $y = (\tan x - 1) + Ce^{-\tan x}$.

4. (i) $\sqrt{1 - x^2} = C + \sin^{-1} x$. (ii) $y = 1 + Ce^{x^2/2}$.

(iii) $xy + \frac{1}{2}\cos x^2 = C$.

5. $y = (x^3 - x)\log Cx$. 6. $yx \sec x = \tan x + C$.

7. $x + 1 = Ce^{-\tan^{-1} y} + \tan^{-1} y$. 8. (i) $x - 1 = y^{1} + Ce^{1/y}$.

(ii) $x + y + 2 = Ce^{y}$. (iii) $2x^{1} = 1 - y^2 + Ce^{-y^2}$.

9. $y^{-1} = Cx - x\log x$. 10. $5\sec^2 x = y(\tan^5 x + C)$.

11. (i) $x = \frac{1}{2}y + Cx^2 y$. (ii) $2x = e^{y}(1 + Cx^2)$.

12. $6x^2 \tan y = x^6 + C$. 13. $x = \log y(Cx^2 + \frac{1}{2})$.

14. $\sin y = (1 + x)(e^{x} + C)$ 15. $4xy = x^4 + 3$.

17. $r^{1} = a^{-1} + Ce^{\theta}$.

CHAPTER XVII

EQUATIONS OF THE FIRST ORDER BUT NOT OF THE FIRST DEGREE

17.1. The typical equation of the first order and the nth degree can be written as

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \ldots + P_{n-1} p + P_n = 0, \quad \ldots \quad (1)$$

where p stands for $\frac{dy}{dx}$ and $P_1, P_2, \ldots\ldots, P_n$ are functions of x and y.

The complete solution of such an equation would involve only one arbitrary constant.

Two cases may arise :-

Case I. When the left side of the equation can be resolved into rational factors linear in the derivative.

Case II. When the left side cannot be thus factorised.

17.2. Left side resolvable into factors.

In this case, the equation (1) takes the form

$$\{p - f_1(x, y)\}\{p - f_2(x, y)\} \ldots \{p - f_n(x, y)\} = 0. \quad \ldots \quad (2)$$

It is evident from above that a solution of any one of the equations

$$p - f_1(x, y) = 0, p - f_2(x, y) = 0, \text{etc.} \quad \ldots \quad (3)$$

is also a solution of (1).

Let the solutions of the equations (3) be

$$\phi_1(x, y, C_1) = 0, \phi_2(x, y, C_2) = 0, \ldots, \phi_n(x, y, C_n) = 0,$$

where $C_1, C_2, \ldots, C_n$ are arbitrary constants of integration.

These solutions are evidently just as general, if $C_1 = C_2 = \ldots = C_n$, since the individual solutions are all independent of one another and all the C's can have any one of an infinite number of values. All the solutions can thus, without loss of generality, be combined into

$$\phi_1 (x, y, C) . \phi_2 (x, y, C) \ldots\ldots \phi_n (x, y, C) = 0 .$$

17.3. Left side not resolvable into factors.

Here, we shall consider only the following cases :

(A) Equations solvable for y.*

Suppose the equation is put in the form

$$y = f (x, p) .$$

Differentiating this with respect to x, we shall get an equation in two variables x and p ; suppose the solution of the latter equation is $\phi (x , p \; C) = 0$.

The p-eliminant between this relation and the original equation gives a relation between x, y and C, which is the required solution.

(B) Equations solvable for x.

$$x = f(y, p) .$$

Differentiating this with respect to y, and noting that $\frac{dx}{dy} = 1/p$, we shall get an equation in two variables y and p. If p be eliminated between the solution of the latter equation (which contains an arbitrary constant) and the original equation, we shall get the required solution.

(C) Equations in which either x or y is absent.

In such cases, the variables are easily separable, or they may be solvable for y or for x.

Note. In case the elimination of p cannot be effected, or it leads to complicated expressions, it is customary to express x and y separately in terms of p, and these values of x and y, as if parametric equations, constitute the solution of the equation.

---

* Clairaut's Equation $y = px + f(p)$ [*Art.* 17.5] as also its extended form $y = x\phi(p) + f(p)$ [*Ex.* 2, *Art.* 17.5] belong to this type.

17.4 . Illustrative Examples.

Ex. 1. *Solve* $p^2 + 2px + py + 2xy = 0$. [*C. P. '88*]

The equation can be written as

$$(p + 2x)(p + y) = 0.$$

$\therefore$ either $p + 2x = 0$, *i.e.*, $\frac{dy}{dx} + 2x = 0$, *i.e.*, $dy + 2x\,dx = 0$, whence, integrating, $y + x^2 = C$;

otherwise, $p + y = 0$, or, $\frac{dy}{dx} + y = 0$, *i.e.*, $\frac{dy}{y} + dx = 0$, whence, integrating, $\log y + x = C$.

$\therefore$ the required solution is $(y + x^2 - C)(x + \log y - C) = 0$.

Ex. 2. *Solve* $4xp^2 - 8yp - x = 0$.

From the equation, $y = \frac{1}{8}x\left(4p - \frac{1}{p}\right)$.

$\therefore$ differentiating with respect to x, we have

$$p = \frac{1}{8}\left\{\left(4p - \frac{1}{p}\right) + x\frac{dp}{dx}\left(4 + \frac{1}{p^2}\right)\right\},$$

which on simplification reduces to $p\,dx = x\,dp$ the integral of which is

$$p = Cx. \qquad (1)$$

Now, eliminating p between (1) and the given equation, the required solution is $4C^2x^2 - 8Cy - 1 = 0$.

Ex. 3. *Solve* $x = y - p^2$.

Differentiating with respect to y, we have

$$\frac{1}{p} = 1 - 2p\frac{dp}{dy}.$$

$$\therefore dy = 2\frac{p^2}{p-1}dp = 2\left\{p + 1 + \frac{1}{p-1}\right\}dp.$$

$$\therefore \text{integrating}, \; y = p^2 + 2p + 2\log(p-1) + C \quad \ldots (1)$$

$$\text{and hence, } x = 2p + 2\log(p-1) + C. \quad \ldots (2)$$

(1) and (2) taken together, or the p-eliminant of them constitutes the general solution of the given equation.

Ex. 4. *Solve* $4yp^2 - 2px + y = 0$.

We can write the equation as

$$x = 2yp + \frac{y}{2p} \qquad \ldots \quad (1)$$

Differentiating this with respect to y, **we have**

$$\frac{1}{p} = 2p + 2y\,\frac{dp}{dy} + \frac{1}{2p} - \frac{y}{2p^2}\,\frac{dp}{dy},$$

$$\text{or, } \frac{4p^2 - 1}{2p}\left[1 + \frac{y\,dp}{p\,dy}\right] = 0.$$

$$\therefore\ p\,dy + y\,dp = 0, \qquad \textit{i.e.},\ py = C. \qquad \ldots \quad (2)$$

Substituting the value of p obtained from (2) in (1), we get the solution

$$y^2 = 2Cx - 4C^2.$$

Note. It will be noted in this connection that, in solving examples of this type, the factors containing derivatives which are omitted often give rise to other solutions of the differential equations which are not included in the general solution. Such solutions are termed singular solutions. [See *Art. 17.5*]

17.5. Clairaut's Equation.

An equation of the form

$$y = px + f(p), \text{ where } p = \frac{dy}{dx},$$

is called *Clairaut's equation*.

Differentiating both sides of the equation with respect to x, we have

$$p = p + x\,\frac{dp}{dx} + f'(p)\,\frac{dp}{dx}, \text{ or, } \frac{dp}{dx}\,\{x + f'(p)\} = 0.$$

$$\therefore \text{ either } \frac{dp}{dx} = 0, \qquad \ldots \quad (1)$$

$$\text{or, } \quad x + f'(p) = 0. \qquad \ldots \quad (2)$$

$$\text{From (1), } \quad p = C. \qquad \ldots \quad (3)$$

Now, if p be eliminated between (3) and the original equation, we get $y = Cx + f(C)$ as the *general* or *complete solution* of the equation.

Again, if p be eliminated between (2) and the original equation, we shall obtain a relation between x and y which also satisfies the differential equation and as such, can be called a solution of the given equation. Since this solution does not contain any arbitrary constant nor can it be derived from the complete solution by giving any particular value to the arbitrary constant, it is called the **singular solution** of the differential equation.

Thus, we see that the equation of Clairaut's form has two kinds of solutions,

(i) the *complete solution* (linear in x and y) containing one arbitrary constant ;

and (ii) the *singular solution* containing no arbitrary constant.

Now, to eliminate p between

$$y = px + f(p) \text{ and } 0 = x + f'(p)$$

is the same as to eliminate C between

$$y = Cx + f(C) \text{ and } 0 = x + f'(C).$$

i.e., the same as the process of finding the envelope of the line $y = Cx + f(C)$ for different values of C.

Thus, *the singular solution represents the envelope of the family of straight lines represented by the complete solution.*

Note. It is beyond the scope of the present treatise to enter into the details of the theory of singular solutions.

Ex. 1. *Solve* $y = px + p - p^2$.

Differentiating both sides with respect to x,

$$p = p + x\frac{dp}{dx} + \frac{dp}{dx} - 2p\frac{dp}{dx}.$$

$$\therefore \quad \frac{dp}{dx}(x + 1 - 2p) = 0.$$

$\therefore$ either $\dfrac{dp}{dx} = 0$, i.e., $p = C$, ... (1)

or, $x + 1 - 2p = 0$, i.e., $p = \frac{1}{2}(x + 1)$. ... (2)

Eliminating p between (1) and the given equation, we get

$$y = Cx + C - C^2 \text{ as the complete solution}$$

and eliminating p between (2) and the given equation, we get

$$y = \tfrac{1}{2}(x + 1)x + \tfrac{1}{2}(x + 1) - \tfrac{1}{4}(x + 1)^2 = \tfrac{1}{4}(x + 1)^2,$$

i.e., $4y = (x + 1)^2$ as the singular solution.

Note. It can easily be verified that the family of straight lines represented by the complete solution touches the parabola represented by the singular solution.

Ex. 2. *Solve* $y = (1 + p)x + ap^2$.

Differentiating with respect to x, we have

$$p = (1 + p) + (x + 2ap)\frac{dp}{dx}.$$

$$\frac{dx}{dp} + x = -2ap.$$

This is a linear equation in x and p. [See *Art. 16.11.*]

$\therefore$ multiplying both sides by $e^{\int dp}$, i.e., e^p, we get

$$e^p\frac{dx}{dp} + e^p \cdot x = -2ap \cdot e^p,$$

or, $\dfrac{d}{dp}(xe^p) = -2ap \cdot e^p$.

$\therefore$ integrating, $xe^p = -2a\int pe^p dp + C = -2ae^p(p - 1) + C$,

or, $x = 2a(1 - p) + Ce^{-p}$.

$\therefore$ $y = 2a - ap^2 + (1 + p)Ce^{-p}$ from the given equation.

The p-eliminant of these two constitutes the solution.

EXAMPLES XVI

Solve the following and find the singular solutions of Ex. 5 to only :-

1. (i) $p^2 + p - 6 = 0$. (ii) $p^2 + 2xp - 3x^2 = 0$.

2. (i) $p^2 - p(e^x + e^{-x}) + 1 = 0$.

(ii) $p^2 y - p(xy + 1) + x = 0$.

(iii) $p(p^2 + xy) = p^2(x + y)$.

3. (i) $p^2 - (a + b)p + ab = 0$. (ii) $p(p + x) = y(x + y)$.

4. (i) $xyp^2 - (x^2 - y^2)p - xy = 0$.

(ii) $p^3 - p(x^2 + xy + y^2) + x^2y + xy^2 = 0$.

(iii) $p^3 - (x^2 + xy + y^2)p^2 + (x^3y + x^2y^2 + xy^3)p - x^3y^3 = 0$.

5. (i) $y = px + a/p$. (ii) $y = px + \sqrt{(a^2p^2 + b^2)}$.

(iii) $y = px + p^n$.

6. (i) $y = px + ap(1 - p)$. (ii) $py = p^2(x - b) + a$.

7. $(x - a)p^2 + (x - y)p - y = 0$.

8. $(y + 1)p - xp^2 + 2 = 0$.

9. (i) $p^3x - p^2y - 1 = 0$. (ii) $y = yp^2 + 2px$.

10. $\sin y \cos px - \cos y \sin px - p = 0$.

11. (i) $x = 4p + 4p^3$. (ii) $p^2 - 2xp + 1 = 0$.

12. (i) $e^y - p^3 - p = 0$. (ii) $y = p \cos p - \sin p$.

13. (i) $y = p^2x + p$. (ii) $y = (p + p^2)x + p^{-1}$.

14. (i) $x + yp = ap^2$. (ii) $y = 2px + p^2$.

15. $p^2 - p(y + 3) + x = 0$. **16.** $y = Ap^3 + Bp^2$.

ANSWERS

1. (i) $(y + 3x - c)(y - 2x - c) = 0$.

(ii) $(2y + 3x^2 - c)(2y - x^2 - c) = 0$.

2. (i) $(y - e^x - c)(y + e^{-x} - c) = 0$.

(ii) $(2y - x^2 - c)(2x - y^2 - c) = 0$.

(iii) $(y - c)(2y - x^2 - c)(y - ce^x) = 0$.

3. (i) $(y - ax - c)(y - bx - c) = 0.$
(ii) $(y - ce^{x})(y + x - ce^{-x} - 1) = 0.$

4. (i) $(xy - c)(x^2 - y^2 - c) = 0.$

(ii) $(2y - x^2 - c)(y - ce^{x})(y + x - 1 - ce^{-x}) = 0.$

(iii) $(x^3 - 3y + c)(e^{x^2/2} + cy)(xy + cy + 1) = 0.$

5. (i) $y = cx + \dfrac{a}{c}$; $y^2 = 4ax.$

(ii) $y = cx + \sqrt{(a^2c^2 + b^2)}$; $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

(iii) $y = cx + c^n$; $n^n y^{n-1} + x^n (n - 1)^{n-1} = 0.$

6. (i) $y = cx + ac(1 - c)$; $(x + a)^2 = 4ay.$

(ii) $cy = c^2(x - b) + a$; $y^2 = 4a(x - b).$

7. $(x - a)c^2 + (x - y)c - y = 0$; $(x + y)^2 = 4ay.$

8. $(y + 1)c - c^2x + 2 = 0$; $(y + 1)^2 + 8x = 0.$

9. (i) $c^3x - c^2y - 1 = 0.$ (ii) $y^2 = 2cx + c^2.$

10. $y = cx + \sin^{-1}c.$

11. (i) $x = 4p + 4p^3$
$y = 2p^2 + 3p^4 + c.$

(ii) $x = \frac{1}{2}(p + p^{-1})$
$y = \frac{1}{4}p^2 - \frac{1}{2}\log p + c$

12. (i) $x = 2\tan^{-1} p - p^{-1} + c$
$y = \log(p^3 + p).$

(ii) $x = c + \cos p$
$y = p\cos p - \sin p.$

13. (i) $y = p^2x + p$
$x = \dfrac{\log p - p + c}{(p - 1)^2}.$

(ii) $y = (p + p^2)x + p^{-1}$
$x = \dfrac{1 + ce^{1/p}}{p^2}.$

14. (i) $x + yp = ap^2$

$$x(1 + p^2)^{1/2} = p[c + a\log\{p + (1 + p^2)^{1/2}\}].$$

(ii) $(3xy + 2x^3 + c)^2 - 4(x^2 + y)^3 = 0.$

15. $y(1 - p^2)^{1/2} + (1 - p^2)^{3/2} = c$, with the given relation

16. $y = Ap^3 + Bp^2$
$x = \frac{3}{2}Ap^2 + 2Bp + c.$

CHAPTER XVIII

LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

18.1 Equations of the Second Order.

We shall first consider linear differential equations with constant coefficients of the second order, since they occur very frequently in many branches of applied mathematics. The typical form of such equations is

$$\frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = X, \qquad \ldots (1)$$

or, symbolically $(D^2 + P_1 D + P_2)\, y = X$,
where P_1, P_2 are constants and X is a function of x only or a constant. Two forms of this equation usually present themselves, namely, when the right-hand member is zero, and when the right-hand member is a function of x. We shall first consider the first form and then the second.

18.2 Equations with right-hand member zero.

Let the equation be

$$\frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0. \qquad \ldots (2)$$

As a *trial solution** of (2), let us take $y = e^{mx}$. Then, if we put $y = e^{mx}$ in the left side of (2), it must satisfy the equation, *i.e.*, we must have

$$(m^2 + P_1 m + P_2)\, e^{mx} = 0,$$

or, since $e^{mx} \neq 0$, $m^2 + P_1 m + P_2 = 0$. $\qquad \ldots$ (3)
The equation (3) is called the *Auxiliary equation of* (2).

Let m_1, m_2 be the two roots of the equation (3).

Then, $y = e^{m_1 x}$ and $y = e^{m_2 x}$ are obviously solutions of (2). Also, it can be easily verified by direct substitution that

---

* This trial solution is suggested by the solution of the first order linear equation $y_1 + Py = 0$, which is of the same form.

$y = C_1 e^{m_1 x}$, $y = C_2 e^{m_2 x}$ and $y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$ satisfy the equation (2), and, as such are solutions of (2).

We [illegible] now consider the nature of the general solution of the equation (2) according as the roots of the auxiliary equation (3) are (i) *real and distinct*, (ii) *real and equal* and (iii) *imaginary*.

(i) Auxiliary equation having real and distinct roots.

If m_1 and m_2 are real and distinct, then $y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$ is the general solution, since it satisfies the equation, and contains two independent arbitrary constants equal in number to the order of the equation.

(ii) Auxiliary equation having two equal roots.

If the auxiliary equation has two equal roots, the method of the preceeding paragraph does not lead to the general solution. For, if $m_1 = m_2 = \alpha$ say, then the solution of the preceding paragraph assumes the form

$$y = (C_1 + C_2) e^{\alpha x} = C e^{\alpha x} \text{, when } C_1 + C_2 = C,$$

which is not the general solution, since it involves only one independent constant and the equation is of the second order.

A method will now be devised for finding the general solution in the case under discussion. Since the auxiliary solution (3) has two equal roots each being equal to α, it follows that the differential equation (2) assumes the form

$$\frac{d^2y}{dx^2} - 2\alpha \frac{dy}{dx} + \alpha^2 y = 0.$$

Let $y = e^{\alpha x} v$, where v is a function of x, be a trial solution of this equation. Substituting this value of y in the left side of the above equation, we have

$$e^{\alpha x} \frac{d^2v}{dx^2} = 0, \quad i.e., \quad \frac{d^2v}{dx^2} = 0 \text{, since } e^{\alpha x} \neq 0.$$

Now, integrating this twice, we get $v = C_1 + C_2 x$.

Hence, the solution of (2) in this case is

$$y = (C_1 + C_2 x) e^{\alpha x}.$$

This is the general solution of (2), since it satisfies (2), and contains two independent arbitrary constants.

(iii) Auxiliary equation having a pair of complex roots.

If $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, then the general solution of (2) is

$$y = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}.$$

The above solution, by adjusting the arbitrary constants, can be put in a more convenient form not involving imaginary expressions; thus we have

$$\begin{aligned} y &= e^{\alpha x} [C_1 e^{i\beta x} + C_2 e^{-i\beta x}] \\ &= e^{\alpha x} [C_1 (\cos \beta x + i \sin \beta x) + C_2 (\cos \beta x - i \sin \beta x)] \\ &= e^{\alpha x} [(C_1 + C_2) \cos \beta x + i (C_1 - C_2) \sin \beta x] \\ &= e^{\alpha x} [A \cos \beta x + B \sin \beta x], \end{aligned}$$

where $A = C_1 + C_2$ and $B = i (C_1 - C_2)$ are the arbitrary constants which may be given any real values we like.

Again, by adjusting the arbitrary constants A and B suitably, *i.e.*, by putting $C \cos \varepsilon$ for A and $- C \sin \varepsilon$ for B, the general solution can also be written in the form

$$y = Ce^{\alpha x} \cos (\beta x + \varepsilon),$$

where C and ε are the two arbitrary constants.

Ex. 1. *Solve* $\dfrac{d^2y}{dx^2} + 3 \dfrac{dy}{dx} + 2y = 0$.

Let $y = e^{mx}$ be a solution of the above equation ;

then $e^{mx} (m^2 + 3m + 2) = 0$. $\therefore$ $m^2 + 3m + 2 = 0$, since $e^{mx} \neq 0$.

$\therefore$ $(m + 1)(m + 2) = 0$, $\therefore$ $m = - 1$, or, $- 2$.

$\therefore$ the general solution is $y = C_1 e^{-x} + C_2 e^{-2x}$.

Ex. 2. *Solve* $\dfrac{d^2y}{dx^2} - 2a \dfrac{dy}{dx} + a^2 y = 0$.

Let $y = e^{mx}$ be a solution of the above equation ;

then $e^{mx}(m^2-2am+a^2)=0$, or, $m^2-2am+a^2=0$, since $e^{mx}\neq 0$.

$\therefore\ (m-a)^2=0$.

Since the auxiliary equation has repeated roots here,

$\therefore$ the general solution is $y=(C_1+C_2x)e^{ax}$.

Ex. 3. *Solve* $(D^2+2D+5)y=0$. [C. P. '61]

The equation is $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+5y=0$.

Let $y=e^{mx}$ be a solution of the equation;

then $e^{mx}(m^2+2m+5)=0$. $\therefore m^2+2m+5=0$, since $e^{mx}\neq 0$,

$\therefore\quad m=-1\pm 2i$;

$\therefore$ the general solution is $y=C_1e^{(-1+2i)x}+C_2e^{(-1-2i)x}$ which, as shown in Art. 18.2(iii), can be put in the form

$$y=e^{-x}(A\cos 2x+B\sin 2x).$$

EXAMPLES XVIII(A)

Solve :-

1. $\dfrac{d^2y}{dx^2}+5\dfrac{dy}{dx}+4y=0$.

2. $\dfrac{d^2y}{dx^2}-7\dfrac{dy}{dx}+12=0$.

3. $\dfrac{d^2y}{dx^2}-3\dfrac{dy}{dx}+2y=0$.

4. $\dfrac{d^2y}{dx^2}+(a+b)\dfrac{dy}{dx}+aby=0$.

5. (i) $2\dfrac{d^2y}{dx^2}-3\dfrac{dy}{dx}+y=0$. (ii) $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}+y=0$.

6. $y_2-4y_1+4y=0$.

7. (i) $(D^2+D)y=0$. (ii) $(D^2+6D+25)y=0$.

8. $(D^2-2mD+m^2+n^2)y=0$.

9. (i) $(D^2 - 4D + 13)y = 0$. (ii) $(D^2 - n^2)y = 0$.

10. (i) $\frac{d^2s}{dt^2} + 4\frac{ds}{dt} + 13s = 0$. (ii) $(D + 3)^2\, y = 0$.

11. Solve in the particular cases :-

(i) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$; when $x = 0$, $y = 3$ and $\frac{dy}{dx} = 0$.

(ii) $\frac{d^2x}{dx^2} + y = 0$; when $x = 0$, $y = 4$; when $x = \frac{1}{2}\pi$, $y = 0$.

(iii) $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0$; when $t = 0$, $x = 0$ and $\frac{dx}{dt} = 0$.

(iv) $\frac{d^2x}{dt^2} + n^2x = 0$; when $t = 0$, $\frac{dx}{dt} = 0$ and $x = a$.

12. Find the curve for which the curvature is zero at every point.

13. Show that if $l\frac{d^2\theta}{dt^2} + g\theta = 0$, and if $\theta = \alpha$ and $\frac{d\theta}{dt} = 0$, when $t = 0$, then $\theta = \alpha\cos\{t\sqrt{g/l})\}$.

14. Show that the solution of

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + ux = 0$$

is $x = e^{-kt/2}(A\cos nt + B\sin nt)$, if $k^2 < 4\mu$ and $n^2 = \mu - \frac{1}{4}k^2$.

ANSWERS

1. $y = c_1e^{x} + c_2e^{4x}$.   2. $y = c_1e^{3x} + c_2e^{4x}$.

3. $y = c_1e^{x} + c_2e^{2x}$.   4. $y = c_1e^{ax} + c_2e^{-bx}$.

5. (i) $y = c_1e^{x} + c_2e^{x/2}$ (ii) $y = (A + Bx)e^{-x}$.

6. $y = e^{2x}(A + Bx)$. 7. (i) $y = A + Be^{-x}$.

(ii) $y = e^{3x}(A\cos 4x + B\sin 4x)$.

8. $y = e^{mx}(A\cos nx + B\sin nx)$.

9. (i) $y = e^{2x} (A \cos 3x + B \sin 3x)$. (ii) $y = Ae^{nx} + Be^{-nx}$

10. (i) $s = e^{2t} (A \cos 3t + B \sin 3t)$. (ii) $y = e^{-3x} (A + Bx)$

11. (i) $y = 2e^{x} + e^{2x}$. (i) $y = 4 \cos x$. (iii) $x = 0$.

(iv) $x = a \cos nt$. 12. A straight line.

18.3. Right-hand member a function of x.

We shall now consider the solution of the general form

$$\frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = X . \qquad \ldots (1)$$

If $y = \phi (x)$ be the general solution of

$$\frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0 \qquad \ldots (2)$$

and $y = \psi (x)$ be any particular solution of (1), then

$y = \phi (x) + \psi (x)$ is the general solution of (1).

This result can be established by direct substitution.

Thus, substituting $y = \phi (x) + \psi (x)$ in the left side of (1), we have

$$\left\{ \frac{d^2\phi}{dx^2} + P_1 \frac{d\phi}{dx} + P_2 \phi \right\} + \left\{ \frac{d^2\psi}{dx^2} + P_1 \frac{d\psi}{dx} + P_2 \psi \right\} .$$

The first group of terms is zero, since $y = \phi (x)$ is a solution of (2), and the second group of terms is equal to X, since $y = \psi (x)$ is a solution of (1).

Hence, $y = \phi (x) + \psi (x)$ is a solution of (1), and it is the general solution, since the number of independent arbitrary constants in it is two, $\phi (x)$ being the general solution of (2).

Thus, we see that the process of solving equation (1) is naturally divided into two parts : the first is to find the general solution of (2), say $\phi (C_1 , C_2 , x)$, and the next is to find any particular solution of (1), say $\psi (x)$ not containing any arbitrary constant. Then

$$y = \phi (C_1 , C_2 , x) + \psi (x)$$

will be the general solution of (1).

The expression $\phi(C_1, C_2, x)$ is called the *Complementary function* and $\psi(x)$, *i.e.*, any particular solution of (1) is called the *Particular Integral* of the equation (1).

18.4. Symbolical Oprators.

We have already shown in Art. 18.2 how to obtain the Complementary function : now we shall consider how to obtain the Particular Integral. In order to discuss methods of finding a particular integral, it would be convenient to introduce certain symbolical operators and their properties.

With the usual notations of Differential Calculus $\frac{d^2x}{dx^2}$, etc. will be denoted by the symbols D, D^2, etc. Also $1/D$ (or, D^{-1}), $1/D^2$ (or, D^{-2}), etc. will be used to denote the inverse operators, *i.e.*, the operators which integrate a function, with respect to x, once, twice, etc. Let us write the equation

$$\frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = X \qquad \ldots (1)$$

in its symbolic form

$$(D^2 + P_1 D + P_2) y = X. \qquad \ldots (2)$$

or, more briefly as $f(D) y = X$. $\qquad \ldots$ (3)

The expression $\frac{1}{f(D)} X$ will be used to denote a function of x not involving arbitrary constants, such that the result of operating upon it with $f(D)$ is X, and as such $\frac{1}{f(D)}$ and $f(D)$ denote two inverse operators.

Thus, *the function* $\frac{1}{f(D)} X$ *is clearly a Particular Integral of the equation* $f(D) y = X$.

As a particular case, when $f(D) = D$, $\frac{1}{D} X$ will denote a function of x, obtained by integrating X once with respect to x, which does not contain any arbitrary constant of integration ; similarly $\frac{1}{D^2} X$ will denote a function of x, obtained by integrating X twice with respect to x, and not containing any arbitrary constant of integration. For example,

$$\frac{1}{D}\,x^4 = \frac{1}{5}\,x^5;\ \frac{1}{D^2}\,x^3 = \frac{1}{20}\,x^5;\ \frac{1}{D}\,1 = x;\ \frac{1}{D^2}\,1 = \frac{1}{2}\,x^2.$$

Important Results on Symbolical Operators.

If $F(D)$ be any rational integral function of D,

i.e., if $F(D) = D^n + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n$, then

(i) $F(D)e^{ax} = F(a)e^{ax}$,

(ii) $F(D)e^{ax}V = e^{ax}F(D+a)V$, V being a function of x

(iii) $F(D^2)\begin{cases}\sin(ax+b)\\ \cos(ax+b)\end{cases} = F(-a^2)\begin{cases}\sin(ax+b)\\ \cos(ax+b).\end{cases}$

By actual differentiation, we can easily verify the above results.

18.5. Methods of finding Particular Integrals.

We shall discuss here the methods of obtaining particular integrals, *i.e.*, the methods of evaluating $\frac{1}{f(D)}X$, when X has special forms.

(*a*) $X = x^m$, *m being a positive integer.*

Expand $1/f(D)$, *i.e.*, $\{f(D)\}^{-1}$ in ascending powers of D and operate on x^m with the result. It is clear that in the expansion no terms beyond the one containing D^m need be retained, since $D^{m+1}x^m = 0$.

Note. The justification of the above method lies in the fact that the function of x which we shall get by operating on x^m by the series of powers of D obtained by expanding $\{f(D)^{-1}\}$, when operated upon by $f(D)$, will give x^m. For example,

$$\frac{1}{D^2+1}\,x^4 = (1+D^2)^{-1}x^4 = (1 - D^2 + D^4 - \ldots)x^4 = x^4 - 12x^2 + 24.$$

Now, $(D^2+1)(x^4 - 12x^2 + 24) = 12x^2 - 24 + x^4 - 12x^2 + 24 = x^4$.

(*b*) $X = e^{ax}V$, *where* V *is a function of* x, *or a constant*.

If V_1 is a function of x, we have, from Art. 18.4(ii),

$$f(D)e^{ax}V_1 = e^{ax}f(D+a)V_1 = e^{ax}V\text{, say,}$$

---

* For proof see Authors' *Differential Calculus.*

so that $f(D+a)V_1 = V$, i.e., $V_1 = \frac{1}{f(D+a)} V$.

Thus, $\frac{1}{f(D)} e^{ax} V = e^{ax} V_1 = e^{ax} \frac{1}{f(D+a)} V$.

Again, noticing that $f(D+a)k'$ where k' is a constant is evidently a constant $= k$ say, and proceeding exactly as above we can show that

$$\frac{1}{F(D)} e^{ax} k = e^{ax} \frac{1}{f(D+a)} k = k e^{ax} \frac{1}{f(D+a)} 1.$$

(*c*) $X = e^{ax}$, *where a is any constant.*

If $f(a) \neq 0$, $f(D)\left\{\frac{e^{ax}}{f(a)}\right\} = \frac{1}{f(a)} f(a) e^{ax} = e^{ax}$.

[From *Art. 18.4 (i)*]

$\therefore \quad \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)}$, *provided* $f(a) \neq 0$.

If $f(a) = 0$, then $(D-a)$ is a factor of $f(D)$.

$\therefore$ either $f(D) = (D-a)\phi(D)$, where $\phi(a) \neq 0$, ... (1)

or else, $= (D-a)^2$. ... (2)

(i) $\frac{1}{f(D)} e^{ax} = \frac{1}{D-a} \frac{1}{\phi(D)} e^{ax} = \frac{1}{(D-a)} \frac{e^{ax}}{\phi(a)} = \frac{1}{\phi(a)} \frac{e^{ax}}{D-a}$

$$= \frac{e^{ax}}{\phi(a)} \frac{1}{D} 1 \ [\text{by } (b)] = \frac{x e^{ax}}{\phi(a)}.$$

(ii) $\frac{1}{f(D)} e^{ax} = \frac{1}{(D-a)^2} e^{ax} = e^{ax} \frac{1}{D^2} 1 \ [\text{by } (b)] = e^{ax} \frac{x^2}{2}$.

(*d*) $X = \sin(ax+b)$ or $\cos(ax+b)$.

If $f(D)$ contains only even powers of D, let us denote it by $\phi(D)^2$. Then, if $\phi(-a^2) \neq 0$, we get, by Art. *18.4.(iii)*,

$$\phi(D^2) \frac{\sin(ax+b)}{\phi(-a^2)} = \frac{\phi(-a^2)\sin(ax+b)}{\phi(-a^2)} = \sin(ax+b)$$

$\therefore \frac{1}{\phi(D^2)} \sin(ax+b) = \frac{1}{\phi(-a^2)} \sin(ax+b)$, *if* $\phi(-a^2) \neq 0$

Similarly, $\frac{1}{\phi(D^2)}\cos(ax+b) = \frac{1}{\phi(-a^2)}\cos(ax+b)$, *if* $\phi(-a^2) \neq 0$.

If $\phi(-a^2) = 0$, or if $f(D)$ contains both the first and the second powers of D, the method of procedure that is to be adopted in such cases is illustrated in Ex. 5 and Ex. 6 of § 18.7 below.

(*e*) $X = x^m \sin(ax + b)$ or $x^m \cos(ax + b)$.

In evaluating particular integrals of this type, it is convenient o replace sin ($ax + b$) and cos ($ax + b$) by their exponential values and then proceed as in case (b).

(*f*) X = xV, *where V is any function of x*.

$$\frac{1}{f(D)}xV = \left\{ x - \frac{1}{f(D)}f'(D) \right\} \frac{1}{f(D)}V.$$

Proof :

We have $D(xV) = xDV + V$,

$D^2(xV) = D(xDV) + DV = xD^2V + 2DV$,

and similarly, $D^n(xV) = xD^nV + nD^{n-1}V$

$$= xD^nV + \left(\frac{d}{dD}D^n\right)V. \quad \ldots (1)$$

Hence, $f(D)xV = xf(D)V + f'(D)V$. $\quad \ldots (2)$

Now, put $f(D)V = V_1$; hence $V = \frac{1}{f(D)}V_1$.

$\therefore$ (2) becomes

$$f(D)x\frac{1}{f(D)}V_1 = xV_1 + f'(D)\frac{1}{f(D)}V_1,$$

i.e., $x\frac{1}{f(D)}V_1 = \frac{1}{f(D)}xV_1 + \frac{1}{f(D)}f'(D)\frac{1}{f(D)}V_1$.

Transposing, we get

$$\frac{1}{f(D)}xV_1 = \left\{ x - \frac{1}{f(D)}f'(D) \right\} \frac{1}{f(D)}V_1.$$

Dropping the suffix, we get

$$\frac{1}{f(D)}xV = \left\{ x - \frac{1}{f(D)}f'(D) \right\} \frac{1}{f(D)}V.$$

Note. It should be noted that, when X is the sum or difference of two or more functions of x, say $X = X_1 \pm X_2 \pm X_3$, then the particular integral

$$= \frac{1}{f(D)}(X_1 \pm X_2 \pm X_3) = \frac{1}{f(D)}X_1 \pm \frac{1}{f(D)}X_2 \pm \frac{1}{f(D)}X_3$$

18.6. Alternative method of finding $\frac{1}{f(D)}X$.

When the auxiliary equation has real and distinct roots, corresponding to each such root m, there will be a partial fraction of the form $A/(D-m)$, where A is a known constant and hence

$$\frac{1}{f(D)}X \text{ can be written in the form}$$

$$\frac{A_1}{D - m_1}X + \frac{A_2}{D - m_2}X + \ldots\ldots$$

each term of which can be evaluated by the method shown below.

$$\text{Now, } \frac{1}{D - m}X = \frac{1}{D - m}e^{mx}e^{-mx}X = e^{mx}\frac{1}{D}e^{-mx}X.$$

$$\therefore \quad \frac{1}{D - m}X = e^{mx}\int e^{-mx}X\,dx. \qquad \ldots \quad (1)$$

This method is illustrated in *Ex. 8 of Art. 18.7.*

18.7. Illustrative Examples.

Ex. 1. *Solve* $(D^2 + 4)y = x^2$.

Here, the auxiliary equation $m^2 + 4 = 0$ has roots $m = \pm 2i$.

$\therefore$ the complementary function $= A\cos 2x + B\sin 2x$.

$$\text{Particular Integral} = \frac{1}{D^2 + 4}x^2 = \frac{1}{4(1 + \frac{1}{4}D^2)}x^2$$

$$= \tfrac{1}{4}(1 + \tfrac{1}{4}D^2)^{-1}x^2$$

$$= \tfrac{1}{4}(1 + \tfrac{1}{4}D^2 + \tfrac{1}{16}D^4 - \ldots)x^2 = \tfrac{1}{4}(x^2 - \tfrac{1}{2}).$$

$\therefore$ the required general solution is

$$y = A\cos 2x + B\sin 2x + \tfrac{1}{4}(x^2 - \tfrac{1}{2}).$$

Ex. 2. *Solve* $(D-3)^2 y = 2e^{4x}$.

Here, the auxiliary equation $(m-3)^2 = 0$ has roots 3, 3

$\therefore$ C. F. $= (A + Bx)e^{3x}$.

$$\text{P. I.} = \frac{1}{(D-3)^2}2e^{4x} = \frac{2e^{4x}}{(4-3)^2} = 2e^{4x}.$$

$\therefore$ the general solution is $y = (A + Bx)e^{3x} + 2e^{4x}$.

Ex. 3. *Solve* $(D-2)^2 y = 6e^{2x}$.

Here, the auxiliary equation $(m-2)^2 = 0$ has roots 2, 2.

$\therefore$ C. F. $= (A + Bx)e^{2x}$.

$$\text{P. I.} = \frac{1}{(D-2)^2}6e^{2x} = 6e^{2x}\frac{1}{D^2}1 = 6e^{2x}\tfrac{1}{2}x^2 = 3x^2e^{2x}.$$

$\therefore$ the general solution is $y = (A + Bx)e^{2x} + 3x^2e^{2x}$.

Ex. 4. *Solve* $\dfrac{d^2y}{dx^2} + y = \cos 2x$.

The equation can be written as $(D^2 + 1)y = \cos 2x$.

The auxiliary equation $m^2 + 1 = 0$ has roots $\pm i$.

$\therefore$ C. F. $= A\cos x + B\sin x$.

$$\text{P. I.} = \frac{1}{D^2+1}\cos 2x = \frac{\cos 2x}{-2^2+1} = -\frac{1}{3}\cos 2x.$$

$\therefore$ the general solution is $y = A\cos x + B\sin x - \frac{1}{3}\cos 2x$.

Ex. 5. *Solve* $(D^2 + 1)y = \cos x$.

As in Ex. 4, C. F. $= A\cos x + B\sin x$.

But the method of obtaining the particular integral employed in Ex. 4 fails here. We may, however, substitute the exponential value of $\cos x$ and proceed. Alternatively, we may proceed as follows :

$$\text{Let} \quad Y = \frac{1}{D^2+1}\cos x \text{ and } Z = \frac{1}{D^2+1}\sin x.$$

$$\therefore Y + iZ = \frac{1}{D^2+1}(\cos x + i\sin x) = \frac{1}{D^2+1}e^{ix}$$

$$= e^{ix}\frac{1}{(D+i)^2+1}1 = e^{ix}\frac{1}{2iD + D^2}1$$

$$= e^{ix}\frac{1}{2iD}\left(1 + \frac{D}{2i}\right)^{-1}.1 = e^{ix}\frac{1}{2i}\frac{1}{D}.1$$

$$= e^{ix}\frac{x}{2i} = \frac{x}{2i}(\cos x + i\sin x).$$

$\therefore$ equating the real part, $Y = \frac{1}{2}x\sin x$.

$\therefore$ the general solution is $y = A\cos x + B\sin x + \frac{1}{2}x\sin x$.

Ex. 6. *Solve* $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 5y = 10\sin x$.

The equation can be written as $(D^2 - 2D + 5)y = 10\sin x$.

The auxiliary equation $m^{\cdot} - 2m + 5 = 0$ has roots $1 \pm 2i$.

$\therefore$ C. F. $= e^x(A\cos 2x + B\sin 2x)$;

$$\text{P. I.} = \frac{1}{D^2 - 2D + 5}10\sin x = \frac{(D^2 + 5) + 2D}{(D^2 + 5)^2 - 4D^2}10\sin x$$

$$= \frac{D^2 + 2D + 5}{(-1^2 + 5)^2 + 4}10\sin x = \frac{1}{2}(D^2 + 2D + 5)\sin x$$

$$= \tfrac{1}{2}(-\sin x + 2\cos x + 5\sin x) = 2\sin x + \cos x.$$

$\therefore$ the general solution is $y = e^x(A\cos 2x + B\sin 2x) + 2\sin x + \cos x$.

Ex. 7. *Solve* $(D^2 - 4D + 4)y = x^3e^{2x}$.

The auxiliary equation $m^2 - 4m + 4 = 0$ has roots 2, 2.

$\therefore$ C. F. $= (Ax + B)e^{2x}$;

$$\text{P. I.} = \frac{1}{D^2 - 4D + 4}x^3e^{2x} = \frac{1}{(D-2)^2}x^3e^{2x}$$

$$= e^{2x}\frac{1}{D^2}x^3 = e^{2x}\frac{x^5}{20}:$$

$\therefore$ the general solution is $y = (Ax + B)e^{2x} + \frac{1}{20}e^{2x}x^5$.

Ex. 8. *Evaluate* $\dfrac{1}{D^2 + 3D + 2}e^{e^x}$.

$$\text{Given expression} = \frac{1}{(D+1)(D+2)}e^{e^x} \qquad \ldots (1)$$

$$= \left[\frac{1}{D+1} - \frac{1}{D+2}\right]e^{e^x}$$

$$= \frac{1}{D+1}e^{e^x} - \frac{1}{D+2}e^{e^x}$$

$$= e^{-x}\int e^x e^{e^x}\,dx - e^{-2x}\int e^{2x}e^{e^x}\,dx. \quad \ldots (2)$$

Let $I_1 = \int e^x e^{e^x} dx$ and $I_2 = \int e^{2x} e^{e^x} dx$.

Put $e^x = z$. $\therefore$ $e^x dx = dz$.

$$\therefore \quad I_1 = \int e^z dz = e^z = e^{e^x}$$

$$I_2 = \int z e^z dz = z e^z - \int e^z dz = z e - e^z = e^z (z - 1) = e^{e^x}(e^x - 1)$$

$\therefore$ from (2), the given expression

$$= e^{-x} e^{e^x} - e^{-2x} . e^{e^x} (e^x - 1)$$

$$= e^{-2x} e^{e^x} .$$

18.8. Two special types of the Second Order equations.

(A) $\dfrac{d^2y}{dx^2} = f(x).$

Integrating both sides with respect to x, we have

$$\frac{dy}{dx} = f(x)\,dx + A = \phi(x) + A, \text{ say}.$$

Integrating again,

$$y = \int \phi(x)\,dx + Ax + B = \psi(x) + Ax + B, \text{ say}.$$

Note. As a generalization of the above method, we can solve the equation $\dfrac{d^n y}{dx^n} = f(x)$ and in particular $\dfrac{d^n y}{dx^n} = 0$, by successive integraion.

(B) $\dfrac{d^2y}{dx^2} = f(y).$

Multiplying both sides by $2\frac{dy}{dx}$, we get

$$2\,\frac{dy}{dx}\,\frac{d^2y}{dx^2} = 2f(y)\,\frac{dy}{dx},$$

or, $$\frac{d}{dx}\left\{\frac{dy}{dx}\right\}^2 = 2f(y)\,\frac{dy}{dx}.$$

Now, integrating both sides with respect to x, we have

$$\left(\frac{dy}{dx}\right)^2 = 2\int f(y)\,\frac{dy}{dx}\,dx + C_1 = 2\int f(y)\,dy + C_1 .$$

Let $2\int f(y)\,dy = \phi(y)$.

$$\therefore \quad \frac{dy}{dx} = \pm\sqrt{\phi(y) + C_1}\,.$$

$\therefore\ dx = \pm\int \dfrac{dy}{\sqrt{[\phi(y) + C_1]}}$, whence, integrating,

$$x = \pm\,\psi(y, C_1) + C_2 \text{ (say)}.$$

18.9. Illustrative Examples.

Ex. 1. *Solve* $\dfrac{d^2y}{dx^2} = \cos nx$.

Integrating both sides with respect to x, we have

$$\frac{dy}{dx} = \frac{1}{n}\sin nx + A.$$

Integrating again, $y = -\dfrac{1}{n^2}\cos nx + Ax + B$,

which is the general solution.

Ex. 2. *Solve* $\dfrac{d^2y}{dx^2} = \dfrac{a}{y^3}$.

Multiplying both sides by $2\dfrac{dy}{dx}$, we get

$$2\frac{dy}{dx}\frac{d^2y}{dx^2} = 2\frac{a}{y^3}\frac{dy}{dx}, \quad \text{or, } \frac{d}{dx}\left(\frac{dy}{dx}\right)^2 = 2a\frac{1}{y^3}\frac{dy}{dx}.$$

Now, integrating both sides with respect to x, we have

$$\left(\frac{dy}{dx}\right)^2 = 2a\int\frac{1}{y^3}\,dy + C_1$$

$$= -\frac{2a}{y^2}\frac{1}{2} + C_1 = C_1 - \frac{a}{y^2} = \frac{C_1y^2 - a}{y^2}.$$

$$\therefore\ \frac{dy}{dx} = \pm\frac{\sqrt{C_1y^2 - a}}{y}, \quad \text{or, } \int dx = \pm\int\frac{y\,dy}{\sqrt{(C_1y^2 - a)}}.$$

$$\therefore\ x = \pm\frac{1}{C_1}\sqrt{C_1y^2 - a} + C_2.$$

$$\therefore\ x - C_2 = \pm\frac{1}{C_1}\sqrt{C_1y^2 - a}.$$

$$\therefore\ C_1{}^2(x - C_2)^2 = C_1y^2 - a.$$

This is the general solution.

Note. An alternative method of procedure for solution of the equation of the above type, *i.e.*, of the type $\frac{d^2y}{dx^2} = f(y)$ is indicated below.

Put $\frac{dy}{dx} = p$. $\therefore \frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$.

$\therefore p\frac{dp}{dy} = \frac{a}{y^3}$, or, $p\,dp = ay^{-3}\,dy$.

$\therefore$ integrating. $\frac{1}{2}p^2 = -\frac{a}{2}y^{-2} + \frac{C_1}{2}$.

$\therefore$ p^2, *i.e.*, $\left(\frac{dy}{dx}\right)^2 = C_1 - \frac{a}{y^2}$.

Now the rest is the same as before.

Ex. 3. *Solve* $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + n^2y = 0$.

Put $x = e^z$, so that $z = \log x$;

then $\frac{dx}{dz} = e^z = x$.

$\therefore \frac{dy}{dz} = \frac{dy}{dx}\frac{dx}{dz} = x\frac{dy}{dx}$ and $\frac{d^2y}{dz^2} = \frac{d}{dx}\left(x\frac{dy}{dx}\right)\frac{dx}{dz} = \left(x\frac{d^2y}{dx^2} + \frac{dy}{dx}\right)x$,

i.e., $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = \frac{d^2y}{dz^2}$.

$\therefore$ the given equation reduces to

$$\frac{d^2y}{dz^2} + n^2y = 0.$$

Multiplying by $2\frac{dy}{dz}$ and integrating with respect to z,

$$\left(\frac{dy}{dz}\right)^2 + n^2y^2 = \text{constant} = n^2a^2 \text{ (say)}.$$

$$\therefore \quad \frac{dy}{dx} = \pm n\sqrt{a^2 - y^2},$$

or, $\quad \pm\frac{dy}{\sqrt{(a^2 - y^2)}} = n\,dz$.

$\therefore$ integrating, $\mp\cos^{-1}\frac{y}{a} = nz + \varepsilon$,

whence $y = a\cos(nz + \varepsilon)$, or, $y = a\cos(n\log x + \varepsilon)$ is the required solution, a and ε being arbitrary constants of integration.

18.10. Equations of the types

$$(A)\ F\left(\frac{d^n y}{dx^n}, \ldots\ldots \frac{dy}{dx}, x\right) = 0,$$

$$(B)\ F\left(\frac{d^n y}{dx^n}, \ldots\ldots \frac{dy}{dx}, y\right) = 0.$$

(A) These equations do not contain y directly. The *substitution* is $\frac{d^r y}{dx^r}$ (derivative of the lowest order) $= q$.

(B) These equations do not contain x directly. The *substituion* is $\frac{dy}{dx} = p$.

Then $\frac{d^2 y}{dx^2} = p\frac{dp}{dy}$; $\frac{d^3 y}{dx^3} = p^2 \frac{d^2 p}{dy^2} + p\left(\frac{dp}{dy}\right)^2$, etc.

18.11. Illustrative Examples.

Ex. 1. *Solve* $2x\frac{d^3 y}{dx^3}\frac{d^2 y}{dx^2} - \left(\frac{d^2 y}{dx^2}\right)^2 + 1 = 0.$

Put $\frac{d^2 y}{dx^2} = q$. $\therefore \frac{d^3 y}{dx^3} = \frac{dq}{dx}$.

$\therefore$ the given equation becomes

$$2x\frac{dq}{dx}q - q^2 + 1 = 0.$$

$$\therefore \quad \frac{2q}{q^2 - 1}dq = \frac{dx}{x}, \text{ or, } \log(q^2 - 1) = \log(c_1 x).$$

$$\therefore \quad q^2 - 1 = c_1 x.$$

$$\therefore q, \text{ i.e., } \frac{d^2 y}{dx^2} = \sqrt{(1 + c_1 x)}.$$

$$\therefore \frac{dy}{dx} = \frac{2}{3c_1}(1 + c_1 x)^{3/2} + c_2.$$

$$\therefore \quad y = \frac{2}{3c_1}\frac{2}{5c_1}(1 + c_1 x)^{5/2} + c_2 x + c_3,$$

$$= \frac{4}{15c_1^2}(1 + c_1 x)^{5/2} + c_2 x + c_3.$$

$$\therefore \quad 15c_1^2 y = 4(1 + c_1 x)^{5/2} + c_2 x + c_3.$$

Ex. 2. *Solve* $y\dfrac{d^2y}{dx^2}-\left(\dfrac{dy}{dx}\right)^2+\left\{\left(\dfrac{dy}{dx}\right)^2-\left(\dfrac{d^2y}{dx^2}\right)^2\right\}^{1/2}=0$.

Put $\dfrac{dy}{dx}=p$. $\therefore\ \dfrac{d^2y}{dx^2}=p\,\dfrac{dp}{dy}$.

$\therefore$ the equation transforms into

$$yp\,\frac{dp}{dy}-p^2+\left\{p^2-p^2\left(\frac{dp}{dy}\right)^2\right\}^{1/2}=0.$$

$\therefore\ p=qy+(1-q^2)^{1/2}$, where $q=\dfrac{dp}{dy}$.

This is *Clairaut's form.*

$\therefore\ p=Ay+(1-A^2)^{1/2}=Ay+k$ (say), where $k=(1-A^2)^{1/2}$

$$\therefore\quad dx=\frac{dy}{Ay+k}.$$

$$\therefore\ x+B=\frac{1}{A}\log(Ay+k)=\frac{1}{A}\log\{Ay+(1-A^2)^{1/2}\}.$$

EXAMPLES XVIII(B)

Solve the following equations :-

1. (i) $\dfrac{d^2y}{dx^2}+4y=2x+3$. (ii) $\dfrac{d^2y}{dx^2}+y=x^3$.

2. (i) $\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}=x^2$. (ii) $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}-6y=x$.

3. (i) $(D+3)^2y=25e^{2x}$. (ii) $(D^2+9)y=9e^{3x}$.

4. (i) $\dfrac{d^2y}{dx^2}-a^2y=e^{ax}$. (ii) $\dfrac{d^2y}{dx^2}-y=e^{2x}$.

   (iii) $\dfrac{d^2y}{dx^2}-4\dfrac{dy}{dx}+3y=2e^{3x}$.

5. (i) $(D^2-4)y=\sin 2x$. (ii) $(D^2+4)y=\sin 2x$.

6. (i) $\dfrac{d^2y}{dx^2}+y=\sin x$. (ii) $\dfrac{d^2y}{dx^2}+4y=x\cos x$.

   (iii) $\dfrac{d^2y}{dx^2}+y=\cos^2 x$ (iv) $\dfrac{d^2y}{dx^2}+4y=x\sin^2 x$.

7. (i) $(D^2 - 1)y = xe^{3x}$. (ii) $(D^2 - 9)y = e^{3x}\cos x$.

8. (i) $(D^2 + 2D + 2)y = xe^{-x}$.

   (ii) $(D^2 - 1)y = e^x \sin \frac{1}{2}x$.

   (iii) $(D^2 + 1)y = \sin x \sin 2x$.

   (iv) $(D^2 - D - 2)y = \sin 2x$.

   (v) $(D - 2)^2 y = x^2 e^{3x}$. [C. P. '86]

9. (i) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = e^x + e^{-x}$.

   (ii) $\dfrac{d^2y}{dx^2} - 2k\dfrac{dy}{dx} + k^2 y = e^x$. (iii) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = e^x$.

   (iv) $\dfrac{d^2y}{dx^2} - y = \cosh x$. (v) $\dfrac{d^2y}{dx^2} - y = xe^x \sin x$.

10. $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = \log x$. [*Put* $x = e^z$] [C. P. '85]

11. $(x^2D^2 + xD + 1)y = \sin(\log x^2)$. [*Put* $x = e^z$]

12. (i) Show that the general solution of the equation for S. H. M., *viz.*,

$$\frac{d^2x}{dt^2} = -n^2x, \text{ is } x = A\cos(nt + \varepsilon).$$

(ii) Evaluate $\dfrac{1}{D}e^{ax}\cos bx$ and hence show that

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx).$$

13. Solve in the particular cases :-

   (i) $\dfrac{d^2y}{dx^2} + y = \sin 2x$; when $x = 0$, $y = 0$ and $\dfrac{dy}{dx} = 0$.

   (ii) $y_2 - 5y_1 + 6y = 2e^x$; when $x = 0$, $y = 1$ and $y_1 = 1$.

   (iii) $(D^2 - 4D + 4)y = x^2$; when $x = 0$, $y = \frac{3}{8}$ and $Dy = 1$.

   (iv) $(D^2 - 1)y = 2$; given $Dy = 3$, when $y = 1$; and $x = 2$, when $y = -1$.

Solve : -

14. (i) $x \frac{d^2y}{dx^2} = 1$. (ii) $\frac{d^2y}{dx^2} = xe^x$.

15. (i) $y_2 \cos^2 x = 1$. (ii) $y^3 y_2 = a$

16. $y'' = \tan y \sec^2 y$, given $y_1 = 0$, when $y = 0$.

17. (i) $\frac{d^2y}{dx^2} = \frac{1}{\sqrt{y}}$. (ii) $\frac{d^2y}{dx^2} + \frac{a^2}{y^2} = 0$.

18. (i) $\frac{d^2y}{dx^2} = x^2 \sin x$. (ii) $\frac{d^2x}{dt^2} = e^{2t}$.

19. (i) $x \frac{d^2y}{dx^2} = 2 \frac{dy}{dx}$. (ii) $a \frac{d^2y}{dx^2} = \frac{dy}{dx}$.

20. $\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^x$.

21. $(1 + x^2) y_2 + 2xy_1 = 2$.

22. $y \frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 + y^2 \log y = 0$.

23. $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} = 0$.

24. $y_2 - (y_1)^2 = 0$.

25. $yy_2 + (y_1)^2 = 2$.

26. $\frac{d^4y}{dx^4} - \frac{d^2y}{dx^2} = 0$.

ANSWERS

1. (i) $y = A\cos 2x + B\sin 2x + \frac{1}{4}(2x + 3)$.
   (ii) $y = A\cos x + B\sin x + (x^3 - 6x)$.
2. (i) $y = Ae^{-2x} + B + \frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{4}x$.
   (ii) $y = Ae^{2x} + Be^{-3x} - \frac{1}{6}(x + \frac{1}{6})$.

3. (i) $y = (C_1 + C_2x)e^{-3x} + e^{2x}$.

(ii) $y = A\cos 3x + B\sin 3x + \frac{1}{3}e^{3x}$.

4. (i) $y = C_1e^{ax} + C_2e^{-ax} + \frac{x}{2a}e^{ax}$.

(ii) $y = Ae^{x} + Be^{-x} + \frac{1}{3}e^{2x}$.

(iii) $y = C_1e^{x} + C_2e^{3x} + xe^{3x}$.

5. (i) $y = Ae^{2x} + Be^{-2x} - \frac{1}{8}\sin 2x$.

(ii) $y = A\cos 2x + B\sin 2x - \frac{1}{4}x\cos 2x$.

6. (i) $y = A\cos x + B\sin x - \frac{1}{2}x\cos x$.

(ii) $y = C_1\cos 2x + C_2\sin 2x + \frac{1}{3}x\cos x + \frac{2}{9}\sin x$.

(iii) $y = A\cos x + B\sin x + \frac{1}{2} - \frac{1}{6}\cos 2x$.

(iv) $y = A\cos 2x + B\sin 2x + \frac{1}{8}x - \frac{1}{32}x\cos 2x - \frac{1}{16}x^2\sin 2x$.

7. (i) $y = C_1e^{x} + C_2e^{-x} + \frac{1}{4}e^{2x}(3x - 4)$.

(ii) $y = C_1e^{3x} + C_2e^{-3x} + \frac{1}{37}e^{3x}(6\sin x - \cos x)$.

8. (i) $y + e^{-x}(A\cos x + B\sin x + x)$.

(ii) $y = Ae^{x} + Be^{-x} - \frac{4}{17}e^{x}(\sin\frac{1}{2}x + 4\cos\frac{1}{2}x)$.

(iii) $y = A\cos x + B\sin x + \frac{1}{4}x\sin x + \frac{1}{16}\cos 3x$.

(iv) $y = Ae^{-x} + Be^{2x} + \frac{1}{20}(\cos 2x - 3\sin 2x)$.

(v) $y = e^{2x}(A + Bx + \frac{1}{12}x^4)$.

9. (i) $y = e^{-x}(C_1 + C_2x + \frac{1}{2}x^2) + \frac{1}{4}e^{x}$.

(ii) $y = (A + Bx)e^{kx} + e^{x}(1 - k)^{-2}$.

(iii) $y = (A + Bx + \frac{1}{2}x^2)e^{x}$.

(iv) $y = Ae^{x} + Be^{-x} + \frac{1}{2}x\sinh x$.

(v) $y = Ae^{x} + Be^{-x} - \frac{e^{x}}{25}\{(10x + 2)\cos x + (5x - 14)\sin x\}$.

10. $y = (A + B\log x)x + \log x + 2$.

11. $y = A\cos\log x + B\sin\log x - \frac{1}{3}\sin\log x^2$.

13. (i) $y = \frac{2}{3}\sin x - \frac{1}{3}\sin 2x$. (ii) $y = e^{x}$.

(iii) $y = \frac{1}{2}xe^{2x} + \frac{1}{4}x^2 + \frac{1}{2}x + \frac{3}{8}$. (iv) $y + 2 = e^{x-2}$.

14. (i) $y = x\log x + Ax + B$. (ii) $y = (x - 2)e^{x} + Ax + B$.

15. (i) $y = \log\sec x + Ax + B$. (ii) $C_1{}^2y^2 = a + (C_2 \pm C_1{}^2x)^2$.

16. $(\sin y + Ce^{x})(\sin y + Ce^{-x}) = 0$.

17. (i) $3x = 2(\sqrt{y} - 2C_1)(\sqrt{y} + C_1)^{1/2} + C_2$.

(ii) $\sqrt{C_1y^2 + y} - \frac{1}{\sqrt{C_1}}\log(\sqrt{C_1y} + \sqrt{1 + C_1y}) = aC_1\sqrt{2}\,.\,x + C_2$

18. (i) $y = C_1 + C_2x + (6 - x^2)\sin x - 4x\cos x$.

(ii) $x = \frac{1}{4}e^{2t} + C_1t + C_2$. 19. (i) $y = \frac{1}{3}Ax^3 + B$.

(ii) $a\log(y + B) = x + C$. 20. $y = C_1e^{-x} + C_2 + \frac{1}{2}e^{x}$.

21. $y = \log(1 + x^2) + A\tan^{-1}x + B$. 22. $y = e^{A\sin x + B\cos x}$.

23. $e^{x}(C_1 - e^{y}) = C_2$. 24. $e^{y}(C_1x + C_2) = 1$.

25. $y^2 = 2x^2 + C_1x + C_2$. 26. $y = C_1e^{x} + C_2e^{-x} + C_3x + C_4$.

18.12. Equation of the nth order.

The linear differential equation of the *n*th order with constant coefficients is

$$\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + \ldots + P_{n-1}\frac{dy}{dx} + P_ny = X, \quad \ldots \quad (1)$$

or, symbolically $(D^n + P_1D^{n-1} + P_2D^{n\ 2} + \ldots + P_n)\,y = X$, (2)

or, more briefly $f(D)\,y = X$, ... (3)

where $P_1, P_2, \ldots, P_n$ are constants, and X is a function of x only, or a constant.

The method adopted in the case of the solution of the second order equation admits of easy extension to the above case. Thus, the general solution of (1) consists of two parts (i) *the Complementary Function* and (ii) *the Particular Integral*, the complementary function being the general solution of

$$f(D)\,y = 0 \quad \ldots \quad (4)$$

and the particular integral being the value of $\frac{1}{f(D)}X$.

Assuming, as before, $y = e^{mx}$ as a trial solution of (4), we shall find that $y = e^{mx}$ will be a solution of (4),

$$\text{if } f(m) = 0, \text{ i.e., if } m^n + P_1 m^{n-1} + \ldots + P_n = 0. \quad \ldots \quad (5)$$

Equation (5) is then the auxiliary equation of (4).

If the auxiliary equation (5) has n *real and distinct roots*, *viz.*, $m_1, m_2, \ldots, m_n$, then the complete solution of (4) is

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \ldots + C_n e^{m_n x}.$$

If the auxiliary equation has a *multiple real root of order* r, and if this root be α, then $f(D)$ contains $(D - \alpha)^r$ as a factor, and the corresponding part of the complementary function will be the solution of $(D - \alpha)^r y = 0$.

Assuming, as before, $y = e^{\alpha x} v$,

$$(D - \alpha)^r y = (D - \alpha)^r e^{\alpha x} v = e^{\alpha x} D^r v$$

and the solution of $D^r v = 0$ is, by successive integration,

$$v = (C_0 + C_1 x + C_2 x^2 + \ldots + C_{r-1} x^{r-1}),$$

whence $y = (C_0 + C_1 x + C_2 x^2 + \ldots + C_{r-1} x^{r-1}) e^{\alpha x}$ is the corresponding part of the complementary function.

If the auxiliary equation has *complex roots* $\alpha \pm i\beta$, the corresponding part of the solution is, as before,

$$y = e^{\alpha x} (A \cos \beta x + B \sin \beta x),$$

and *if* $\alpha \pm i\beta$ *are double roots of the auxiliary equation*, the corresponding part of the solution will be

$$e^{\alpha x} [(A_1 + A_2 x) \cos \beta x + (B_1 + B_2 x) \sin \beta x].$$

The method of obtaining the particular integral of (1), when X has those special forms [See *Art. 18.5*], is essentially the same as shown in the case of the second order equations.

18.13. Illustrative Examples.

Ex. 1. *Solve* $(D^3 + 3D^2 + 3D + 1) y = e^{-x}$.

Here, the auxiliary equation is $m^3 + 3m^2 + 3m + 1 = 0$ of which the roots are $-1, -1, -1$. $\therefore$ C. F. $= e^{-x} (C_0 + C_1 x + C_2 x^2)$.

$$\text{P. I.} = \frac{1}{(D^3 + 3D^2 + 3D + 1)} e^{-x} = \frac{1}{(D + 1)^3} e^{-x}$$
$$= e^{-x} \frac{1}{(D - 1 + 1)^3} 1 = e^{-x} \frac{1}{D^3} 1 = e^{-x} \frac{1}{6} x^3 .$$

$\therefore$ **the general solution is** $y = e^{-x}(C_0 + C_1 x + C_2 x^2 + \frac{1}{6} x^3)$.

Ex. 2. *Solve* $(D^4 + 2D^3 + 3D^2 + 2D + 1) y = xe^x$.

The equation can be written as $(D^2 + D + 1)^2 y = xe^x$.

Here, the auxiliary equation is $(m^2 + m + 1)^2 = 0$; it has double **complex roots** $-\frac{1}{2} \pm i\frac{1}{2}\sqrt{3}$, $-\frac{1}{2} \pm i\frac{1}{2}\sqrt{3}$.

$\therefore$ **C. F. is** $e^{-x/2} [(A_1 + A_2 x)\cos(\frac{1}{2}\sqrt{3}x) + (B_1 + B_2 x)\sin(\frac{1}{2}\sqrt{3}x)]$.

$$\text{P. I.} = \frac{1}{(D^2 + D + 1)^2} xe^x = e^x \frac{1}{[(D + 1)^2 + (D + 1) + 1]^2} x$$
$$= e^x \frac{1}{(D^2 + 3D + 3)^2} x = e^x \frac{1}{9} \left[\frac{1}{\{1 + D(1 + \frac{1}{3}D)\}^2}\right] x$$
$$= e^x . \tfrac{1}{9}\{1 + D(1 + \tfrac{1}{3}D)\}^{-2} x = e^x . \tfrac{1}{9}\{1 - 2D + \ldots + \ldots\} x$$
$$= \tfrac{1}{9} e^x (x - 2).$$

$\therefore$ **the general solution is**

$$y = e^{-x/2} [(A_1 + A_2 x)\cos(\tfrac{1}{2}\sqrt{3}x) + (B_1 + B_2 x)\sin(\tfrac{1}{2}\sqrt{3}x)] + \tfrac{1}{9} e^x (x - 2).$$

Ex. 3. *Solve* $\frac{d^8 y}{dx^8} - 2\frac{d^4 y}{dx^4} + y = \sin(2x + 3)$.

The equation can be written as

$$(D^4 - 1)^2 y = \sin(2x + 3).$$

The auxiliary equation is $(m^4 - 1)^2 = 0$; its roots are $1, 1, -1, -1$ $i, i, -i, -i$. Hence,

$$\text{C. F. is } e^x (A_1 + A_2 x) + e^{-x} (B_1 + B_2 x) + (C_1 + C_2 x)\cos x + (D_1 + D_2 x)\sin x . \quad \ldots \quad (1)$$

$$\text{P. I.} = \frac{1}{(D^4 - 1)^2} \sin(2x + 3) = \frac{1}{\{(-2^2)^2 - 1\}^2} \sin(2x + 3)$$
$$= \frac{1}{225} \sin(2x + 3). \quad \ldots \quad (2)$$

Adding (1) and (2), we get the general solution.

EXAMPLES XVIII(C)

Solve :-

1. (i) $\frac{d^3y}{dx^3} - y = 0$. [C. P. 1946] (ii) $\frac{d^4y}{dx^4} - y = 0$.

2. (i) $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0$.

   (ii) $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$.

   (iii) $\frac{d^4y}{dx^4} + 4\frac{d^3y}{dx^3} + 8\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 4y = 0$.

   (iv) $(D+1)^3(D^2+1)y = 0$.

3. (i) $\frac{d^3y}{dx^3} - y = x^3 - x^2$. (ii) $\frac{d^3y}{dx^3} + \frac{d^2y}{ax^2} = x^2$.

4. (i) $(D^3 - D)y = e^x + e^{-x}$.

   (ii) $(D^3 - 1)y = \sin(3x+1)$.

5. $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2y = 0$.

6. $(D^3 + D^2 - D - 1)y = \sin^2 x$.

7. $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4y = e^{3x}$.

8. $\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + 4y = e^x \sin\frac{x}{2}$.

9. $(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x$.

10. $(D^4 - 4D^3 + 3D^2 + 4D - 4)y = e^{2x}$.

11. $(D^4 + 1)y = 2\cos^2\frac{1}{2}x - 1 + e^{-x}$.

12. $(D^4 + 2D^2 + 1)y = \cos x$.

13. $(D - 1\,^2(D^2 + 1)^2 y = e^x + \sin^2\frac{1}{2}x$.

14. $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{2x} + x$.

15. $\frac{d^4y}{dx^4} + 5\frac{d^2y}{dx^2} + 4y = 360\sin\frac{7}{2}x\cos\frac{x}{2}$

ANSWERS

1. (i) $y = Ae^{x} + e^{-x/2}\left(B\sin\frac{1}{2}\sqrt{3}x + C\cos\frac{1}{2}\sqrt{3}x\right)$.

(ii) $y = Ae^{x} + Be^{-x} + C\cos x + D\sin x$.

2. (i) $y = e^{x}(A + Bx) + Ce^{-2x}$. (ii) $y = A + Be^{2x} + Ce^{-x}$

(iii) $y = e^{-x}[(A + Bx)\cos x + (C + Dx)\sin x]$.

(iv) $y = e^{-x}(A + Bx + Cx^{2}) + D\cos x + E\sin x$.

3. (i) $y = Ae^{x} + e^{-x/2}\left(B\sin\frac{1}{2}\sqrt{3}x + C\cos\frac{1}{2}\sqrt{3}x\right) - x^{3} + x^{2} - 6$.

(ii) $y = A + Bx + Ce^{-x} + \frac{1}{12}x^{4} - \frac{1}{3}x^{3} + x^{2}$.

4. (i) $y = A + Be^{x} + Ce^{-x} + \frac{1}{2}x(e^{x} + e^{-x})$.

(ii) $$y = e^{-x/2}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right) + Ce^{x} + \frac{27}{730}\cos(3x + 1) - \frac{1}{730}\sin(3x + 1).$$

5. $y = (A_1 + A_2x)e^{x} + A_3e^{2x}$.

6. $y = C_1e^{x} + (C_2 + C_3x)e^{-x} + \frac{1}{25}\sin 2x + \frac{1}{50}\cos 2x - \frac{1}{2}$.

7. $y = e^{2x}(C_1 + C_2x) + C_3e^{-x} + \frac{1}{4}e^{3x}$.

8. $$y = C_1e^{-2x} + e^{x}(C_2\cos x + C_3\sin x) - \frac{16}{111}e^{x}\left(\frac{1}{2}\cos\frac{1}{2}x + 3\sin\frac{1}{2}x\right).$$

9. $y = e^{x}(C_1 + C_2\cos x + C_3\sin x) + xe^{x} + \frac{1}{10}(\cos x + 3\sin x)$

10. $y = (C_1 + C_2x)e^{2x} + C_3e^{x} + C_4e^{-x} + \frac{1}{6}x^{2}e^{2x}$.

11. $$y = e^{-ax}[C_1\cos ax + C_2\sin ax] + e^{ax}[C_3\cos ax + C_4\sin ax] + \frac{1}{2}(\cos x + e^{-x}),$$ where $a = 1/\sqrt{2}$.

12. $y = (C_1 + C_2x)\sin x + (C_3 + C_4x)\cos x - \frac{1}{8}x^{2}\cos x$.

13. $$y = (C_1 + C_2x)e^{x} + (C_3 + C_4x)\cos x + (C_5 + C_6x)\sin x + \frac{1}{8}e^{x}x^{2} + \frac{1}{2} - \frac{1}{32}x^{2}\sin x.$$

14. $y = (C_1 + C_2 x) e^x + C_3 + \frac{1}{2} e^{2x} + \frac{1}{2} x^2 + 2x.$

15. $y = C_1 \cos x + C_2 \sin x + C_3 \cos 2x.$
$+ C_4 \sin 2x + \sin 4x + \frac{9}{2} \sin 3x$

18.14. Homogeneous Linear Equation.

An equation of the form

$$x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1}}{dx^{n-1}} + \dots$$
$$+ P_{n-1} x \frac{dy}{dx} + P_n y = X, \qquad \dots (1)$$

or symbolically, $(x^n D^n + P_1 x^{n-1} D^{n-1} + \dots$

$$+ P_{n-1} x D + P_n) y = X, \qquad \dots (2)$$

where $P_1, P_2, \dots, P_n$ are constants and X is a function of x alone, is called a *homogeneous linear equation*.

The *substitution*

$$x = e^z, \text{ i.e., } z = \log x$$

will transform the above equation into an equation with *constant coefficients*, which has already been discussed in Art. 18.12. Here the independent variable will be z.

Now, $$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}; \qquad \dots (3)$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2 y}{dz^2}\frac{1}{x}$$

$$= \frac{1}{x^2}\left(\frac{d^2 y}{dz^2} - \frac{dy}{dz}\right). \qquad \dots (4)$$

Similarly, $$\frac{d^3 y}{dx^3} = \frac{1}{x^3}\left(\frac{d^3 y}{dz^3} - 3\frac{d^2 y}{dz^2} + 2\frac{dy}{dz}\right). \qquad \dots (5)$$

Let us write δ for $\frac{d}{dx}$; with this notation (3), (4), (5) can be written as

$$x \frac{dy}{dx} = \delta y, \qquad \dots (6)$$

$$x^2 \frac{d^2 y}{dx^2} = \delta(\delta - 1) y, \qquad \dots (7)$$

$$x^3 \frac{d^3 y}{dx^3} = \delta(\delta - 1)(\delta - 2)y \qquad \ldots (8)$$

$$x^n \frac{d^n y}{dx^n} = \delta(\delta - 1)(\delta - 2)\ldots(\delta - n + 1)y \qquad \ldots (9)$$

Note. This is sometimes called *Cauchy equation.*

18.15. Equation reducible to the Homogeneous Linear form.

An equation of the form

$$(ax + b)^n \frac{d^n y}{dx^n} + P_1 (ax + b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots$$

$$\ldots + P_{n-1}(ax + b)\frac{dy}{dx} + P_n y = X, \qquad \ldots (10)$$

where $P_1, P_2, \ldots, P_n$ are constants and X is a function of x alone can be reduced to a linear equation with constant coefficients *by the substitution* $ax + b = z$.

Note. This is sometimes called *Legendre equation*.

18.16. Illustrative Examples.

Ex. 1. *Solve* $x^3 \frac{d^3 y}{dx^3} + 3x^2 \frac{d^2 y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^2$.

Put $x = e^z$, *i.e.*, $z = \log x$.

Then by Art. 18.14, the equation transforms into

$$[\delta(\delta - 1)(\delta - 2) + 3\delta(\delta - 1) - 2\delta + 2]y = e^{2z}, \qquad \ldots (1)$$

where $\delta = \frac{d}{dz}$, or, $(\delta - 1)^2(\delta + 2)y = e^{2z}$.

$\therefore$ the roots of the auxiliary equation are 1, 1, − 2.

The C. F. is $y = (C_1 + C_2 z)e^z + C_3 e^{-2z}$.

And P. I. is $\frac{1}{(\delta - 1)^2(\delta + 2)} e^{2z} = \frac{1}{4}e^{2z}$.

$\therefore$ the general solution of (1) is

$$y = (C_1 + C_2 z)e^z + C_3 e^{-2z} + \tfrac{1}{4}e^{2z}.$$

Hence, the general solution of the given equation is

$$y = (C_1 + C_2 \log x)x + C_3 x^{-2} + \tfrac{1}{4}x^2.$$

Ex. 2. *Solve* $(x^2 D^2 + 2xD)y = xe^x$.

Put $x = e^z$, *i.e.*, $z = \log x$.

$\therefore$ *by Art. 18.14*, the equation transforms into

$$[\delta(\delta - 1) + 2\delta]y = e^z e^{e^z}, \qquad (1)$$

where $\delta = \frac{d}{dz}$, or, $(\delta^2 + \delta)y = e^z e^{e^z}$.

$\therefore$ the roots of the auxiliary equation are $0, -1$.

$\therefore$ the C. F. is $y = C_1 + C_2 e^{-z}$.

$$\text{P. I.} = \frac{1}{\delta(\delta + 1)} e^z e^{e^z}$$

$$= \left(\frac{1}{\delta} - \frac{1}{(\delta + 1)}\right) e^z e^{e^z}$$

$$= \frac{1}{\delta} e^z e^{e^z} - \frac{1}{\delta + 1} e^z e^{e^z}$$

$$= \int e^z e^{e^z} dz - e^{-z} \int e^{2z} e^{e^z} dz. \qquad [\text{By Art. } 18.6]$$

$$= e^{e^z} - e^{-z} \{(e^z - 1)e^{e^z}\} = e^{-z} e^{e^z}$$

[See Ex. 8 *of Art. 18.7.*]

$\therefore$ the general solution of (1) is

$$y = C_1 + C_2 e^{-z} + e^{-z} e^{e^z}.$$

Hence, the general solution of the given equation is

$$y = C_1 + C_2 x^{-1} + x^{-1} e^x.$$

EXAMPLES XVIII(D)

Solve the following equations :-

1. $x^2 - \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x$.

2. $(x^2 D^2 + xD - 1)y = \sin(\log x) + x\cos(\log x)$.

3. $x^2 \frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 4y = x^4$.

4. $(x^2D^2 - 2)\,y = x^2 + \dfrac{1}{x}$.

5. $x^3 \dfrac{d^3y}{dx^3} + 2x^2 \dfrac{d^2y}{dx^2} - 2x \dfrac{dy}{dx} = 0$.

6. $(x^3 D^3 + xD - 1)\,y = x^2$.

7. $x^3 \dfrac{d^3y}{dx^3} + x\dfrac{dy}{dx} - y = x$.

8. $(x + 2)^2 \dfrac{d^2y}{dx^2} - 4(x + 2)\dfrac{dy}{dx} + 6y = x$.

9. $(x^4D^4 + 6x^3D^3 + 9x^2D^2 + 3xD + 1)\,y = 0$.

10. $x^4 \dfrac{d^3y}{dx^3} + 3x^3 \dfrac{d^2y}{dx^2} - 2x^2 \dfrac{dy}{dx} + 2xy = \log x$.

ANSWERS

1. $y = C_1x^2 + C_2x^3 + \frac{1}{2}x$.

2. $y = C_1x + C_2x^{-1} - \frac{1}{2}\sin(\log x)$
$$+ \frac{x}{5}\{2\sin(\log x) - \cos(\log x)\}$$

3. $y = (C_1 + C_2\log x)\,x^{-2} + \frac{1}{36}x^4$.

4. $y = C_1x^{-1} + C_2x^2 + \frac{1}{3}x^2\log x - \frac{1}{3}x^{-1}\log x$.

5. $y = C_1x^2 + C_2x^{-1} + C_3$.

6. $y = \{C_1 + C_2\log x + C_3(\log x)^2\}\,x + x^2$.

7. $y = \{C_1 + C_2\log x + C_3(\log x)^2\}\,x + \frac{1}{6}x(\log x)^3$.

8. $y = C_1(x + 2)^2 + C_2(x + 2)^3 + \frac{1}{6}(3x + 4)$.

9. $y = (C_1 + C_2\log x)\cos(\log x)$
$$+ (C_3 + C_4\log x)\sin(\log x)$$

10. $y = (C_1 + C_2\log x)\,x + C_3x^{-2} + \frac{1}{4}x^{-1}\log x$.

CHAPTER XIX

APPLICATIONS

19.1. We have already considered in the preceding chapters some applications of differential equations to geometrical problems. Here we shall have some other applications of differential equations.

19.2. Orthogonal Trajectories.

If every member of a family of curves cuts the members of a given family at right angles, each family is said to be a set of *orthogonal trajectories* of the other.

(A) Rectangular Co-ordinates.

Supose we have one-parameter family of curves

$$f(x, y, c) = 0, \qquad \dots \quad (1)$$

c being the variable parameter.

Let us first form the differential equation of the family by differentiation of (1) with respect to x and by elimination of c [See *Art. 15.2*], and let the differential equation be

$$\phi\left(x, y, \frac{dy}{dx}\right) = 0. \qquad \dots \quad (2)$$

If the two curves cut at right angles, and if ψ, ψ' be the angles which the tangents to the given curve and the trajectory at the common point of intersection, (say x, y), make with the x-axis, we have $\psi - \psi' = \frac{1}{2}\pi$, and, therefore, $\tan\psi = -\cot\psi'$. Since $\tan\psi = \frac{dy}{dx}$, it follows that the differential equation of the system of trajectories is obtained by substituting

$$-1 \bigg/ \frac{dy}{dx}, \quad \text{i.e.,} \ -\frac{dx}{dy} \text{ for } \frac{dy}{dx} \text{ in (2)}.$$

Thus, *the differential equation of the system of orthogonal trajec-ries* is

$$\phi\left(x, y, -\frac{dx}{dy}\right) = 0. \qquad \dots \quad (3)$$

Integrating (3) we shall get the equation in the ordinary form.

(B) Polar Co-ordinates.

Suppose the equation of a given one-parameter family of curves be

$$f(r, \theta, c) = 0 \qquad \dots (1)$$

and the corresponding differential equation, obtained by eliminating the arbitrary parameter c, be

$$F\left(r, \theta, \frac{dr}{d\theta}\right) = 0. \qquad \dots (2)$$

If ϕ, ϕ' denote the angles which the tangents to the given curve and the trajectory at the common point of intersection, (say r, θ), make with the radius vector to the common point, we have, as before, $\tan \phi = -\cot \phi'$.

Since $\tan \phi = r\,(d\theta/dr)$, it follows that the differential equation of the system of orthogonal trajectories is obtained by substituting

$$-\frac{1}{r}\frac{dr}{d\theta} \text{ for } r\frac{d\theta}{dr}, \quad i.e., -r^2\frac{d\theta}{dr} \text{ for } \frac{dr}{d\theta} \text{ in (2)}.$$

Hence, *the differential equation of the required system of orthogonal trajectories* is

$$F\left(r, \theta, -r^2\frac{d\theta}{dr}\right) = 0. \qquad \dots (3)$$

Integrating (3) we shall get the equation in the ordinary form.

19.3. Illustrative Examples.

Ex. 1. *Find the orthogonal trajectories of the rectangular hyperbolas* $xy = a^2$.

Differentiating $xy = a^2$ with respect to x, we have the differential equation of the family of curves

$$x\frac{dy}{dx} + y = 0 \qquad \dots (1)$$

and hence, for the orthogonal trajectories, the differential equation is

$$-x\frac{dx}{dy} + y = 0, \quad \text{or, } x\,dx - y\,dy = 0.$$

Integrating this, we have $x^2 - y^2 = c^2$, the required equation of the orthogonal trajectories. It represents a system of rectangular hyperbolas.

Ex. 2. *Find the orthogonal trajectories of the cardioides*

$$r = a(1 - \cos\theta).$$

Since $r = a(1 - \cos\theta)$, $\therefore \log r = \log a + \log(1 - \cos\theta)$.

Differentiating with respect to θ, we get the differential equation of the family of curves

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1 - \cos\theta}.$$

$\therefore$ the differential equation of the system of orthogonal trajectories is

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \frac{\sin\theta}{1 - \cos\theta},$$

$$\text{or,}\quad \frac{dr}{r} + \frac{1 - \cos\theta}{\sin\theta}\,d\theta = 0,$$

$$\text{or,}\quad \frac{dr}{r} + \frac{\sin\theta}{1 + \cos\theta}\,d\theta = 0.$$

$\therefore$ integrating, $\log\dfrac{r}{1 + \cos\theta} = \log c$;

i.e., $r = c(1 + \cos\theta)$

represents the required orthogonal trajectories.

Ex. 3. *Find the orthogonal trajectories of the system of curves*

$$r^n = a^n\cos n\theta.$$

Since $r^n = a^n\cos n\theta$, $\therefore n\log r = n\log a + \log\cos n\theta$

Differentiating with respect to θ, (and thereby eliminating a), we get the differential equation of the family of curves

$$n\frac{1}{r}\frac{dr}{d\theta} = -n\frac{\sin n\theta}{\cos n\theta}.$$

$\therefore$ the differential equation of the system of orthogonal trajectories is

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = -\frac{\sin n\theta}{\cos n\theta}.$$

$$\therefore\quad \frac{dr}{r} - \frac{\cos n\theta}{\sin n\theta}\,d\theta = 0.$$

$\therefore$ integrating, $\log r - \dfrac{1}{n}\log\sin n\theta = \log c$,

$$i.e., \log \frac{r}{(\sin n\theta)^{1/n}} = \log c .$$

$$\therefore \quad r^n = c^n \sin n\theta .$$

19.4. Velocity and Acceleration of a moving particle.

If a particle be moving along a straight line, and if at any instant t the position P of the particle be given by the distance s measured along the path from a suitable fixed point A on it, then, v denoting the velocity and f the acceleration of the particle at that instant, we have

$$\begin{aligned} \mathbf{v} &= \text{rate of displacement} \\ &= \text{rate of change of } s \text{ with respect to time} \\ &= \frac{ds}{dt} ; \end{aligned}$$

$$\text{and } \mathbf{f} = \text{rate of change of velocity with respect to time}$$

$$= \frac{dv}{dt} = \frac{d^2s}{dt^2} .$$

If, instead of moving in a straight line, the particle be moving in any manner in a plane, the position of the particle at any instant t being given by the cartesian co-ordinates x, y, referred to a fixed set of axes, the components of velocity and acceleration parallel to those axes will similarly be given by

$$v_x = \text{rate of displacement parallel to } x\text{-axis} = \frac{dx}{dt} ,$$

$$v_y = \text{rate of displacement parallel to } y\text{-axis} = \frac{dy}{dt} ,$$

$$f_x = \text{rate of change of } v_x = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} ,$$

$$f_y = \text{rate of change of } v_y = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2} .$$

The applications of these results are illustrated in the following illustrative examples.

19.5. Illustrative Examples.

Ex. 1. *A particle starting with velocity u moves in a straight line with a uniform acceleration f. Find the velocity and distance travelled in any time.*

s denoting the distance travelled by the particle in time t, the acceleration of the particle is given by the expression d^2s / dt^2, and so, in this case, $d^2s / dt^2 = f$;

$\therefore$ integrating, $\frac{ds}{dt} = ft + A$, where A is the integration constant. Now, ds / dt is the expression for the velocity v of the particle at time t, and when $t = 0$, *i.e.*, at start $v = u$. $\therefore u = 0 + A$.

Hence, $$v = \frac{ds}{dt} = ft + u. \qquad \ldots (1)$$

Integrating (1), $s = \frac{1}{2}ft^2 + ut + B$,
where the integration constant B is found in this particular case from the fact that $s = 0$ when $t = 0$, $\therefore B = 0$.

Hence, $$s = \frac{1}{2}ft^2 + ut = ut + \frac{1}{2}ft^2.$$

Ex. 2. *A particle is projected with a velocity u at an angle α to the horizon. Find the path.*

Taking the starting point as origin, and taking the axes of co-ordinates horizontal and vertical respectively, if x, y denote the co-ordinates of the particle at any time t, since there is no force and therefore no acceleration in the horizontal direction, and since the vertical acceleration in the horizontal direction is always the same $= g$ downwards, we have in this case

$$\frac{d^2x}{dt^2} = 0, \frac{d^2y}{dt^2} = -g.$$

Hence, integrating, $$\frac{dx}{dt} = A, \frac{dy}{dt} = -gt + B. \qquad \ldots (1)$$

But, dx / dt, dy / dt represent the horizontal and the vertical components of velocity respectively, and these, at start when $t = 0$, are given by $u \cos \alpha$ and $u \sin \alpha$.

$\therefore u \cos \alpha = A, u \sin \alpha = 0 + B$,
whereby the integration constants are obtained.

Thus, (1) gives

$$\frac{dx}{dt} = u \cos \alpha, \frac{dy}{dt} = u \sin \alpha - gt.$$

Integrating again, $x = ut \cos \alpha + C$
and $y = ut \sin \alpha - \frac{1}{2}gt^2 + D$.

Now, since $x = y = 0$ when $t = 0$, we get from above $C = D = 0$.

Hence, $x = ut \cos \alpha$
and $y = ut \sin \alpha - \frac{1}{2} gt^2$.

Eliminating t, the path of the particle is given by

$$y = x \tan \alpha - \frac{1}{2}g \frac{x^2}{u^2 \cos^2 \alpha}$$

which is evidently a parabola.

19.6. Miscellaneous Applications.

The examples below will illustrate some other applications of differential equations.

Ex. 1. *The population of a country increases at the rate proportional to the number of inhabitants. If the population doubles in 30 year, in how many year will it treble ?*

Let x be the population in t year.

$\therefore \frac{dx}{dt} = kx$, $\therefore$ solving, $x = Ce^{kt}$.

Let $x = x_0$, when $t = 0$. $\therefore C = x_0$; $\therefore x = x_0 e^{kt}$.

When $x = 2x_0$, $t = 30$; $\therefore 2x_0 = x_0 e^{30k}$; $\therefore 2 = e^{30k}$.

When $x = 3x_0$, let $t = T$; $\therefore 3x_0 = x_0 e^{kT}$; $\therefore 3 = e^{kt}$.

$$\left.\begin{aligned} \therefore\ 30k &= \log_e 2 \\ \text{and } kT &= \log_e 3 \end{aligned}\right\} \qquad \therefore \frac{T}{30} = \frac{\log_e 3}{\log_e 2} = \frac{48}{30} \text{ approximately.}$$

$\therefore \quad T = 30 \times \frac{48}{30} = 48$ year approximately.

Ex. 2. *After how many years will Rs. 100, placed at the rate of 5% continuously compounded, amount to Rs.1000 ?*

Let x be the amount in t years.

$\therefore \frac{dx}{dt} = \frac{5}{100} x = kx$ say, where $k = \frac{1}{20}$.

$\therefore$ solving, $x = Ce^{kt}$.

When $t = 0, x = 100$; $\therefore C = 100$. $\therefore x = 100e^{kt}$.

When $x = 1000$, let $t = T$. $\therefore 1000 = 100e^{kt}$. $\therefore e^{kT} = 10$.

$\therefore kT = \log_e 10 = 2.30$ approximately.

$\therefore T = (1/k) \times 2.30 = 20 \times 2.30 = 46$ nearly.

$\therefore$ the required time is 46 years nearly.

EXAMPLES XIX

Find the orthogonal trajectories of the following families of curves :-

1. (i) $y = mx$. (ii) $y = ax^n$.
(iii) $x^2 + y^2 = 2ay$. (iv) $y^2 = 4ax$.
(v) $ay^2 = x^3$ (vi) $x^2 + 2y^2 = a^2$.
(vii) $x^{2/3} + y^{2/3} = a^{2/3}$. (viii) $x^2 + y^2 + a^2 = 1 + 2axy$.
(ix) $r = a \cos\theta$. (x) $r^2 = a^2 \cos 2\theta$.
(xi) $r(1 + \cos\theta) = 2a$. (xiii) $r^n \sin n\theta = a^n$.

2. (i) Show that the orthogonal trajectories of a system of concurrent straight lines form a system of concentric circles, and conversely.

[*Take the point of concurrence as origin.*]

(ii) Show that the orthogonal trajectories of the system of co-axial circles

$$x^2 + y^2 + 2\lambda x + c = 0$$

form another system of co-axial circles

$$x^2 + y^2 + 2\mu y - c = 0,$$

where λ and μ are parameters and c is a given constant.

(iii) Show that the orthogonal trajectories of the system of circles touching a given straight line at a given point form another system of circles which pass through the given point and whose centres lie on the given line.

3. (*a*) Show that every member of the first set of curves cuts orthogonally every member of the second

(i) $\dfrac{dy}{dx} = \dfrac{x^2 + x + 1}{y^2 + y + 1}$, (ii) $\dfrac{dy}{dx} + \dfrac{y^2 + y + 1}{x^2 + x + 1} = 0$.

(*b*) Show that

(i) the family of parabolas $y^2 = 4a(x + a)$ is self-orthogonal.

(ii) the family of confocal conics

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1 \quad (\lambda \text{ being the parameter})$$

is self-orthogonal.

4. (i) Find the curve in which the radius of curvature is proportional to the arc measured from a fixed point, and identify it.

(ii) Find the curve for which the tangent at any point cuts off from the co-ordinate axes intercepts whose sum is constant, and identify it.

5. Find the cartesian equation of a curve for which the tangent is of constant length.

6. A particle is said to execute a *Simple Harmonic Motion* when it moves on a straight line with its acceleration always directed towards a fixed point on the line and proportional to the distance from it in any position. If it starts from rest at a distance a from the fixed point, find its velocity in any position, and the time for that position. Deduce that the motion is oscillatory, and find the periodic time.

7. A particle falls towards the earth, starting from rest at a height h above the surface. If the attraction of the earth varies inversely as the square of the distance from its centre, find the velocity of the particle on reaching the earth's surface, given a the radius of the earth and g the value of the aceleration due to gravity at the surface of the earth.

8. A particle falls in a vertical line under gravity (supposed constant), and the force of air resistance to its motion is proportional to its velocity. Show that its velocity cannot exceed a particular limit.

9. A particle moves in an ellipse with an acceleration directed towards its centre. Show that the acceleration is proportional to its distance from the centre.

10. In a certain culture, the number of bacteria is increasing at a rate proportional to the number present. If the number doubles in 3 hour, how many may be expected at the end of 12 hour ?

11. After how many year will a sum of money, placed at the rate of 5% continuously compounded, double itself ?

12. Radium disappears at a rate proportional to the amount present. If 5% of the original amount disappears in 50 year, how much will remain at the end of 100 year ?

13. A tank consists of 50 litre of fresh water. Two litre of brine each containing 5 gram of dissolved salt are run into the tank per minute ; the mixture is kept uniform by stirring, and runs out at the rate of one litre per minute. If m gram of salt are present in the tank after t minute, express m in terms of t and find the amount of salt present after 10 minute.

14. The electric current I through a coil of resistance R and inductance L satisfies the equation $RI + L(dI/dt) = V$, where V is the potential difference between the two ends of the coil. A potential difference $V = \alpha \sin \omega t$ is applied to the coil from time $t = 0$ to the time $t = \pi/\omega$, where α, ω are positive constants. The current is zero at $t = 0$ and V is zero after $t = \pi/\omega$; find the current at any time both before and after $t = \pi/\omega$.

15. A horizontal beam of length $2l$ m, carrying a uniform load of w kg per m of length, is freely supported at both ends, satisfying the differential equation

$$EI \frac{d^2y}{dx^2} = \frac{1}{2}wx^2 - w/x,$$

y being the deflection at a distance x from one end. If $y = 0$ at $x = 0$, and $y_1 = 0$ at $x = l$, find the deflection at any point ; also find the maximum deflection.

16. A horizontal beam of length l simply supported at its end subject only to its own weight satisfies the equation

$$EI \frac{d^4y}{dx^4} = w,$$

where E, I, w are constants. Given [illegible] $= y = 0$ at $x = 0$ and at $x = l$, express the deflection y in terms of x.

17. A harmonic oscillator consists of an inductance L, a condenser of capacitance C and an *e.m.f.* E. Find the charge q and the current i when $E = E_0 \cos \omega t$ and initial conditions are $q = q_0$ and $i = i_0$ at $t = 0$; i, q satisfying the equations

$$\frac{d^2q}{dt^2} + \frac{q}{LC} = \frac{E_0}{L} \cos \omega t, \; i = \frac{dq}{dt}.$$

What hapens if $\omega = \dfrac{1}{\sqrt{(LC)}}$?

ANSWERS

1. (i) $x^2 + y^2 = a^2$. (ii) $x^2 + ny^2 = c^2$. (iii) $x^2 + y^2 = 2cx$.
   (iv) $2x^2 + y^2 = c^2$. (v) $2x^2 + 3y^2 = c^2$. (vi) $y = cx^2$.
   (vii) $x^{4/3} - y^{4/3} = c^{4/3}$.
   (viii) $y\sqrt{(1 - y^2)} \pm x\sqrt{(1 - x^2)} + \sin^{-1} y \pm \sin^{-1} x = c$.
   (ix) $r = c \sin \theta$. (x) $r^2 = c^2 \sin 2\theta$.
   (xi) $r(1 - \cos \theta) = c$. (xii) $r^n \cos n\theta = c$.

4. (i) Equiangular spiral. (ii) Parabola.

5. $x = \sqrt{a^2 - y^2} + \frac{1}{2}a\{\log(a - \sqrt{a^2 - y^2}) - \log(a + \sqrt{a^2 - y^2})\}$,
   if $y = a$, when $x = 0$.

6. $v = \sqrt{\mu(a^2 - x^2)}$, $t = \dfrac{1}{\sqrt{\mu}} \cos^{-1} \dfrac{x}{a}$, when μ is the acceleration at a unit distance. Period $\dfrac{2\pi}{\sqrt{\mu}}$.

7. $\sqrt{\dfrac{2agh}{a + h}}$

10. 16 times the original number.

11. 14 year nearly. 12. $\frac{361}{400}$ of the original amount.

13. $5t\left(1+\frac{50}{50+t}\right)$ gram ; $91\frac{2}{3}$ gram.

14. For $t<\frac{\pi}{\omega}$, $I=\frac{\alpha}{L^2\omega^2+R^2}\left[R\sin\omega t-\omega L\left(\cos\omega t-e^{-\frac{Rt}{T}}\right)\right]$

and for $t>\frac{\pi}{\omega}\ I,=\frac{\alpha\omega L}{L^2\omega^2+R^2}\left(1+e^{\frac{R\pi}{\omega L}}\right)e^{-\frac{Rt}{T}}$

15. $y=\frac{w}{24EI}(x^4-4ix^3+8l^3x)$; $y_{max}=\frac{5wl^4}{24EI}$

16. $y=\frac{w}{24EI}\left(x^4-2lx^3+l^3x\right)$.

17. $$q=\left(q_0-\frac{E_0C}{1-\omega^2LC}\right)\cos\frac{1}{\sqrt{(LC)}}t+\sqrt{LC}\,i_0\sin\frac{1}{\sqrt{(LC)}}t$$
$$+\frac{E_0C}{1-\omega^2LC}\cos\omega t.$$

$$i=i_0\cos\frac{1}{\sqrt{(LC)}}t-\frac{1}{\sqrt{(LC)}}\left(q_0-\frac{E_0C}{1-\omega^2LC}\right)\sin\frac{1}{\sqrt{(LC)}}t$$
$$-\frac{E_0C\omega}{1-\omega^2LC}\sin\omega t.$$

If $\omega=\frac{1}{\sqrt{(LC)}}$, *i.e.*, frequency of *e. m. f.* = natural frequency, oscillatior *i.e.*, resonance will take place and the circuit will be destroyed. Before destroying

$$q=q_0\cos\omega t+\frac{i_0}{\omega}\sin\omega t+\frac{E_0}{2L\omega}t\sin\omega t;$$

$$i=i_0\cos\omega t-q_0\,\omega\sin\omega t+\frac{E_0}{2L}\left(\frac{1}{\omega}\sin\omega t+t\cos\omega t\right)$$

—

CHAPTER XX

THE METHOD OF ISOCLINES

20.1. It is only in a limited number of cases that a differential equation may be solved analytically by the preceding methods. In many practical cases where the solution of a differential equation is needed under given initial conditions, and the above methods fail, a graphical method, the *method of isoclines*, is sometimes adopted. We proceed to explain below this method in the case of simple differential equations of the first order.

Let us consider an equation of the type

$$\frac{dy}{dx} = f(x, y). \qquad \ldots \quad (1)$$

As already explained before, the general solution of this equation involves one arbitrary constant of integration, and hence represents a family of curves, and, in general, one member of the family passes through a given point (x , y).

Now, if in (1) we replace $\frac{dy}{dx}$ by m, we get an equation $f(x, y) = m$, which for any particular numerical value of m represents a curve, at every point of which the value of $\frac{dy}{dx}$, *i.e.*, the slope of the tangent line to the family of curves represented by the general solution of (1) is the same as that numerical value of m. This curve $f(x, y) = m$ is called an *isoclinal* or isocline. For different numerical values of m we get different isoclinals, which may be graphically constructed on a graph paper. Through different points on any one isocline, short parallel lines are drawn having their common slope equal to the particular value of m for that isocline. Similar short parallel lines are drawn through points on other isoclinals. If the number of isoclines drawn be large, so that they are sufficiently close to one another, the short lines will ultimately join up and appear to form a series of curves which represent the family of curves giving the general solution of (1), and a particular number of the family passing through a given point represents the particular solution wanted. All necessary information regarding

the particular solution may now be obtained from the graph.

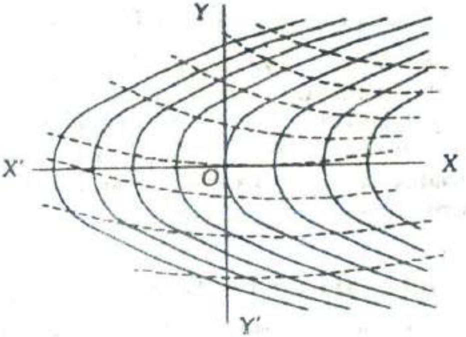

Fig. 1

As an example, let us consider the differential equation $\frac{dy}{dx} = x - y^2$. The isoclinals are given by $m = x - y^2$, or $y^2 = x - m$, a series of equal parabolas shifted left or right from $y^2 = x$, (which corresponds to $m = 0$), as shown in the figure. The dotted curves represent graphically the solutions of the differential equation.

CHAPTER XXI

DOUBLE AND TRIPLE INTEGRALS

21.1. In Chapter VI we have discussed what is meant by the definite integral of a function of a single variable with respect to that variable taken between two prescribed bounds. We shall now discuss briefly about the double intergation of a function of two independent variables taken over a two dimensional region and that of a function of three variables taken over a three-dimensional region.

21.2. Double integral over a rectangle.

First of all we confine our discussion of double integral of a function of two variables over a finite rectangular region and then we shall extend our idea to any finite region other than rectangles.

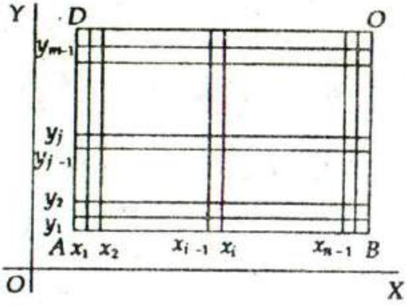

Fig.1

Let $f(x, y)$ be a bounded function of two independent variables x and y defined over the rectangle $ABCD$, bounded by the lines $x = a$, $x = b$, $y = c$, $y = d$. This rectangle will be denoted by $R[a, b; c, d]$, or, simply by R.

Let $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$

and $c = y_0 < y_1 < y_2 < \ldots < y_{m-1} < y_m = d$.

We draw the lines $x = x_1 , x = x_2 , \dots , x = x_i , \dots , x = x_{n-1}$ which are parallel to $\overrightarrow{OY}$ and the lines $y = y_1 , y = y_2 , y = y_j , \dots , y = y_{m-1}$ which are parallel to $\overrightarrow{OX}$ to divide the rectangle R into mn sub-rectangles.

Let us denote the sub-rectangle $R\ [x_{i-1} , x_i ; y_{j-1} , y_j]$ by R_{ij} and its area by A_{ij}. We have $A_{ij} = (x_i - x_{i-1})(y_j - y_{j-1})$.

Let m_{ij} and M_{ij} be the lower and upper bounds of $f (x , y)$ in R_{ij}. We next form the sums

$$s = \sum_{i=1}^{n} \sum_{j=1}^{m} m_{ij} A_{ij} ;$$

$$S = \sum_{i=1}^{n} \sum_{j=1}^{m} M_{ij} A_{ij} .$$

It is evident that for every mode of sub-division of R into sub-rectangles R_{ij}, $m (b-a)(d-c) \leq s \leq S \leq M (b-a)(d-c)$, where M, m are the upper and lower bounds of $f (x , y)$ in R. Thus, we can say that the two sets of upper and lower sums S and s are bounded for *all* modes of division of R into a finite number of sub-rectangles R_{ij} as defined above.

The lower bound of the set of upper sums is defined as the upper integral of $f (x , y)$ over R and is denoted by I and the upper bound of the set of lower sums is defined as the lower integral of $f (x , y)$ over R and is denoted by J. We write :

$$I = \underline{\iint}_R f (x , y) \, dx \, dy , \ J = \overline{\iint}_R f (x , y) \, dx \, dy .$$

If I and J be equal, then the function $f (x , y)$ is said to be integrable over the rectangle and the common value denoted by

$\iint_R f (x , y) \, dx \, dy$ or $\iint_R f (x , y) \, dA$ is defined as the *double integral* of $f (x , y)$ over the rectangle R.

Note. Norm of a division of a rectangle.

The norm of the sub-division D of a rectangle R is denoted by $\|D\|$ or Δ and may be defined to be the greatest diagonal of sub-rectangles, *i.e.*, $\|D\| = \max. \sqrt{\{(x_i - x_{i-1})^2 + (y_j - y_{j-1})^2\}}$, the maximum is to be taken of all the diagonals of the sub-rectangles of R.

21.3. Condition of integrability.

We state here, without proof, the necessary and sufficient condition for the integrability of a bounded function $f(x, y)$ over a rectangle R. The condition is that to every positive number ε, there corresponds a positive number δ, such that for every division of R whole norm is $< \delta$, the oscillation $S - s$ is less than ε.

21.4. Simple properties.

If $f(x, y)$ and $g(x, y)$ are integrable functions over a rectangle R, then the functions $f(x, y) \pm g(x, y)$ and $cf(x, y)$, where c is a constant, are also integrable over the same rectangle R and

(i) $$\iint_R \left\{ f(x, y) \pm g(x, y) \right\} dx\, dy$$
$$= \iint_R f(x, y)\, dx\, dy \pm \iint_R g(x, y)\, dx\, dy.$$

(ii) $$\iint_R cf(x, y)\, dx\, dy = c \iint_R f(x, y)\, dx\, dy.$$

(iii) If $f(x, y) \le g(x, y)$ in
then $$\iint_R f(x, y)\, dx\, dy \le \iint_R g(x, y)\, dx\, dy.$$

(iv) If $R = R_1 \cup R_2$ when $R_1 \cap R_2 = \phi$,

then $$\iint_R f(x, y)\, dx\, dy$$
$$\iint_{R_1} f(x, y)\, dx\, dy + \iint_{R_2} f(x, y)\, dx\, dy$$

Note 1. If R_1 and R_2 have a common region, as shown by the shaded area in the adjoining figure, when we calculate

$$\iint_{R_1} f(x,y)\,dx\,dy + \iint_{R_2} f(x,y)\,dx\,dy$$

we are really integrating $f(x,y)$ *twice* over the common region $EFDC$ so that the result (iv) is not valid. We, therefore, introduce the condition $R_1 \cap R_2 = \phi$, so that R_1 and R_2 have no common region.

Note 2. It is easily seen that the results (i), (ii), (iii) and (iv) in § 21.4 above remain valid for double integrals over any finite region E, provided the functions are integrable over the region E.

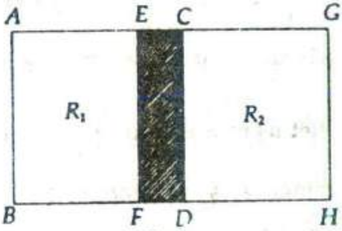

Fig.2

21.5. Calculation of a double integral. Equivalence of a double integral with repeated integrals.

Theorem. If the double $\iint_R f(x,y)\,dx\,dy$ exists, R being the rectangle $[a,b;c,d]$ and if the integral $\int_a^b f(x,y)\,dx$ also exists for all values of y in (c,d), then the repeated integral $\int_c^d dy \int_a^b f(x,y)\,dx$ exists and is equal to the double integral.

Proof. Let us divide the rectangle $R\,[a,b;c,d]$ into mn sub-rectangles by the lines $x = x_1, x = x_2, \ldots, x = x_{n-1}$ and $y = y_1$, $y = y_2, \ldots, y = y_{m-1}$, where $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ and $c = y_0 < y_1 < y_2 < \ldots\ldots < y_m = d$. If m_{ij} and M_{ij} be the lower and upper bounds of $f(x,y)$ in the sub-rectangle $R_{ij}\,[x_{i-1}, x_i; y_{j-1}, y_j]$, we have

$$m_{ij} \leq f(x,y) \leq M_{ij}. \qquad \ldots (1)$$

If y remains fixed, then $f(x, y)$ can be regarded as a function of one variable x only and then by using the mean-value theorem of Integral Calculus we have

$$m_{ij}(x_i - x_{i-1}) \leq \int_{x_{i-1}}^{x_i} f(x, y)\,dx \leq M_{ij}(x_i - x_{i-1}), \qquad \ldots (2)$$

which holds for all values of y in (y_{j-1}, y_j).

Let us now denote $\int_{x_{i-1}}^{x_i} f(x, y)\,dx$ by $g(y)$.

Since $g(y)$ is bounded in (y_{j-1}, y_j), we have

$$m_{ij}(x_i - x_{i-1})(y_j - y_{j-1}) \leq I_0$$

$$J_0 \leq M_{ij}\, x_i - x_{i-1})(y_j - y_{j-1}),$$

$$\text{i.e.,} \quad m_{ij} A_{ij} \leq I_0,\ J_0 \leq M_{ij} A_{ij} \qquad \ldots (3)$$

where I_0, J_0 are respectively the lower and upper integrals of $g(y)$ in (y_{j-1}, y_j), i.e., $I_0 = \underline{\int}_{y_{j-1}}^{y_j} g(y)\,dy$ and $J_0 = \overline{\int}_{y_{j-1}}^{y_j} g(y)\,dy$.

Now, taking summation with respect to i and j, we get

$$\sum_{i=1}^{n} \sum_{j=1}^{m} m_{ij} A_{ij} \leq \underline{\int}_{c}^{d} g(y)\,dy,$$

$$\overline{\int}_{c}^{d} g(y)\,dy \leq \sum_{i=1}^{n} \sum_{j=1}^{m} M_{ij} A_{ij},$$

$$\text{i.e.,} \quad s \leq \underline{\int}_{c}^{d} g(y)\,dy,\ \overline{\int}_{c}^{d} g(y)\,dy \leq S, \qquad \ldots (4)$$

where s and S denote respectively the lower and upper sums for the double integral of $f(x, y)$ over R.

(4) can be expressed as

$$s \leq \underline{\int}_c^d dy \int_a^b f(x, y)\, dx, \quad \overline{\int}_c^d dy \int_a^b f(x, y)\, dx \leq S.$$

$$\left.\begin{aligned} \therefore \quad I &\leq \underline{\int}_c^d dy \overline{\int}_a^b f(x, y)\, dx \leq J \\ \text{and} \quad I &\leq \underline{\int}_c^d dy \int_a^b f(x, y)\, dx \leq J \end{aligned}\right\} \qquad \ldots \quad (5)$$

Since the double integral exists, $I = J$ and therefore

$\int_c^d dy \int_a^b f(x, y)\, dx$ also exists and is equal to the double integral.

Hence, $\int_c^d dy \int_a^b f(x, y)\, dx = \iint_R f(x, y)\, dx\, dy$

Cor. 1. If $\iint_R f(x, y)\, dx\, dy$ exists and $\int_c^d f(x, y)\, dy$ exists, then also

$\int_a^b dx \int_c^d f(x, y)\, dy$ exists and $= \iint_R f(x, y)\, dy\, dx$.

Cor. 2. If the double integral exists, the two repeated integrals cannot exist without being equal.

Thus, if the double integral exists, then the repeated integrals

$\int_a^b dx \int_c^d f(x, y)\, dy, \int_c^d dy \int_a^b f(x, y)\, dx$ both exist and they are equal.

21.6. Double integration as a limit.

Let $f(x, y)$ be a continuous function in the rectangle $R[a, b; c, d]$. Let us divide R into sub-rectangles in the process as described in § 21. 2. Let (x_r, y_r) be any point of the sub-rectangle R_r, whose area is A_r. Now, form the sum $\Sigma f(x_r, y_r) . A_r$.

We can show that, as the norm of sub-division $\| D \| \to 0$, the limit of the above sum will be the double integral of $f(x, y)$ over R,

$$\text{i.e., } \underset{\| D \| \to 0}{Lt} \sum f(x_r, y_r) A_r = \iint_R f(x, y)\, dx\, dy .$$

21.7. Geometrical interpretation of double integral.

Let us consider the double integral $\iint_R f(x, y)\, dx\, dy$, where R is the rectangle $[a, b; c, d]$. Let $z = f(x, y)$ be the given function, which graphically represents a surface.

We divide R into mn sub-rectangles by the lines $x = x_1$, $x = x_2$, ..., $x = x_{n-1}$, and $y = y_1$, $y = y_2$, ..., $y = y_{m-1}$ where $a = x_0 < x_1 < x_2 < .. < x_n = b$, $c = y_0 < y_1 < y_2 < ... < y_m = d$. Let R_{ij} be the rectangle $[x_{i-1}, x_i; y_{j-1}, y_j]$ whose area is A_{ij}. Now, lines paraliel to the z-axis are drawn from points of R_{ij} upto the surface $z = f(x, y)$ to form a prism. Let V_{ij} be the volume of this prism and m_{ij}, M_{ij} the lower and upper bounds of $f(x, y)$ in R_{ij}. It is evident that

$$m_{ij} A_{ij} \le V_{ij} \le M_{ij} A_{ij} .$$

$$\therefore \sum_{i=1}^{n} \sum_{i=1}^{m} m_{ij} A_{ij} \le \sum_{i=1}^{n} \sum_{j=1}^{m} V_{ij} \le \sum_{i=1}^{n} \sum_{j=1}^{m} M_{ij} A_{ij} .$$

As the double integral $\iint_R f(x, y)\, dx\, dy$ exists, it is equal to

$$Lt \sum\sum m_{ij} A_{ij} = Lt \sum\sum M_{ij} A_{ij}$$

$$= Lt \sum\sum V_{ij} .$$

$$\therefore \iint_R f(x, y)\, dx\, dy = Lt \sum\sum V_{ij} = V , V$$ being the volume of the cylinder whose base is R, the generators of which are parallel to the z-axis drawn from points of the sides of R upto the surface $z = f(x, y)$.

21.8. Double integral over any finite region.

We have already defined the double integral of a function over a rectangle R. Now, we are going to define the double integral of a function over a given finite region E. As E is finite, we can construct a rectangle which can enclose the given region E. Let us define a function $g(x, y)$ over R as follows :

$$g(x, y) = f(x, y) \text{ for all points of } E,$$
$$= 0 \text{ outside } E.$$

The function $f(x, y)$ is said to be integrable over E if $g(x, y)$ be integrable over R. We have then

$$\iint_E f(x, y)\, dx\, dy = \iint_R g(x, y)\, dx\, dy .$$

Note. Let us take the z-axis perpendicular to the plane of E. If lines parallel to the z-axis be drawn from points on the boundary of E upto the surface $z = f(x, y)$, we get a cylinder. Geometrically $\iint_E f(x, y)\, dx\, dy$ represents the volume of the above cylinder.

21.9. Evaluation of double integral.

Let E be the region bounded by the curves $y = u(x)$, $y = v(x)$ and the ordinates $x = a$, $x = b$. If $u(x)$, $v(x)$ be continuous functions and $u(x) \le v(x)$ in (a, b) and $f(x, y)$ be a continuous function in E, then

$$\iint_E f(x, y)\, dy\, dx = \int_a^b dx \int_{u(x)}^{v(x)} f(x, y)\, dy .$$

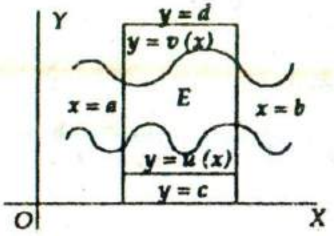

Fig.3

Let $R\ [a, b; c, d]$ be the rectangle which encloses the given region E and let us define the function $g(x, y)$ in R as follows :

$$g(x, y) = f(x, y) \text{ at all points of } E,$$
$$= 0 \text{ outside } E.$$

Now, $\iint_E f(x, y)\, dy\, dx = \iint_R g(x, y)\, dy\, dx$

$$= \int_a^b dx \int_c^d g(x, y)\, dy$$

$$= \int_a^b dx \left[\int_c^{u(x)} g(x, y)\, dy + \int_{u(x)}^{v(x)} g(x, y)\, dy + \int_{v(x)}^d g(x, y)\, dy \right]$$

$= \int_a^b dx \int_{u(x)}^{v(x)} g(x, y)\, dy$, (the other two integrals being zero)

$$= \int_a^b dx \int_{u(x)}^{v(x)} f(x, y)\, dy.$$

Note. If E be the region bounded by the continuous curves $x = U(y), x = V(y)$, the straight lines $y = c, y = d$ and $f(x, y)$ be a continuous function in E, then

$$\iint_E f(x, y)\, dx\, dy = \int_c^d dy \int_{U(y)}^{V(y)} f(x, y)\, dx.$$

21.10. Area of a region.

In the definition of double integral, if we put $f(x,y) = 1$, we have the area A of the region bounded by the curves $y = u(x)$, $y = v(x)$, the straight lines $x = a$, $x = b$ given by

$$A = \int_a^b \int_{u(x)}^{v(x)} dy\, dx .$$

21.11. Jacobian.

If $u_1, u_2, \ldots, u_n$ be n functions of n independent variables $x_1, x_2, \ldots, x_n$ and have partial derivatives of the first order at every point of the common domain in which the functions are defined, then the *Jacobian* of $u_1, u_2, \ldots, u_n$ with respect to $x_1, x_2, \ldots, x_n$ is denoted by

$$\frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)}, \text{ or, } J\left(\frac{u_1, u_2, \ldots, u_n}{x_1, x_2, \ldots, x_n}\right)$$

and defined to be the determinant

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

Cor. If $x = r \cos\theta$, $y = r\sin\theta$, then

$$J\left(\frac{x, y}{r, \theta}\right) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

21.12. Change of variable in a double integral.

Sometimes double integrals can easily be evaluated by changing the independent variables by suitable transformations.

Let $x = \phi(\xi, \eta), y = \psi(\xi, \eta)$ ... (1)

be two functions of ξ, η defined in a region E' of the $\xi - \eta$ plane bounded by a curve C'. Moreover we assume that

(i) the above two functions ϕ and ψ have continuous first order partial derivatives at all points of E' and C';

(ii) the equations (1) transform the region E' bounded by C' into a region E of the xy plane bounded by a curve C in such a way that there exists a one-one correspondence between E, E' and C, C';

(iii) the Jacobian $\dfrac{\partial(x, y)}{\partial(\xi, \eta)}$ does not change sign at any point of E, but it may vanish at some points of C', then

$$\iint_E f(x, y)\, dx\, dy = \iint_{E'} f\{\phi(\xi, \eta), \psi(\xi, \eta)\} \left| \frac{\partial(x, y)}{\partial(\xi, \eta)} \right| d\xi\, d\eta.$$

[Proof of the theorem is beyond the scope of this elementary treaties.]

21.13. Application of double integral.

(a) Mass of a plate.

Let a plate be bounded by the curve C and let the mass per unit area (*i.e.*, the density) at the point (x, y) be given by $\rho = f(x, y)$. Divide the plate into elementary areas by lines parallel to the axes of co-ordinates. Let δA be one of these elementary areas with (x, y) its centre of mass. The mass M of the plane area is given by

$$M = Lt\ \Sigma \rho \delta A = \iint_E \rho\, dx\, dy.$$

(b) Centre of mass of a thin plate.

Let us divide the plate E into elementary areas by lines parallel to the axes of co-ordinates. Let δA be one of these elementary areas with $(\bar{x}, \bar{y})$ its centre of mass, then the co-ordinates of the centre of mass of the plate are given by

$$\bar{x} = \underset{\delta A \to 0}{Lt} \quad \frac{\Sigma x\rho\delta A}{\Sigma\rho\delta A} = \frac{\iint_E \rho x dA}{\iint_E \rho dA} = \frac{\iint_E \rho x\, dx\, dy}{\iint_E \rho dx\, dy}$$

$$\bar{y} = \underset{\delta A \to 0}{Lt} \quad \frac{\Sigma y\, \rho\delta A}{\Sigma\, \rho\delta A} = \frac{\iint_E \rho y dA}{\iint_E \rho dA} = \frac{\iint_E \rho y\, dx\, dy}{\iint_E \rho\, dx\, dy}$$

where $\rho = f(x, y)$ is the density of the plate at (x, y).

(c) Centre of pressure of a plane lamina.

If a plane lamina be immersed in a liquid, the point at which the resultant pressure acts is called the *Centre of Pressure* of the lamina.

Let a lamina be immersed vertically in a liquid. Let us take the axes of co-ordinates in the plane of the lamina, the x-axis horizontal and the y-axis vertical. Let us divide the lamina E into elementary areas by lines drawn parallel to the axes. Let δA be one of these elementary areas and (x, y) be its centroid. Let $p = f(x, y)$ be the pressure at (x, y).

Total pressure on the lamina

$$= Lt\, \Sigma p \,.\, \delta A = \iint_E p dA = \iint_E p\, dx\, dy\,.$$

If $(\bar{x}, \bar{y})$ be the centre of pressure of the lamina, by taking moments about the axes of co-ordinates, we get

$$\bar{x} = \underset{\delta A \to 0}{Lt} \quad \frac{\Sigma px\, \delta A}{\Sigma p\, \delta A} = \frac{\iint_E px dA}{\iint_E p dA} = \frac{\iint_E px\, dx\, dy}{\iint_E p\, dx\, dy}$$

$$\bar{y} = \underset{\delta A \to 0}{Lt} \quad \frac{\Sigma py\, \delta A}{\Sigma\, p\, \delta A} = \frac{\iint_E py dA}{\iint_E p dA} = \frac{\iint_E py\, dx\, dy}{\iint_E p\, dx\, dy}$$

(d) Moments and product of inertia of a lamina.

If $r_1, r_2, \ldots, r_n$ be the distances, from a fixed line, of particles of masses $m_1, m_2, \ldots, m_n$ respectively, then Σmr^2 is defined as the *Moment of Inertia* of the system of particles about the line.

Let E be the plane lamina. Take the axes in the plane of the lamina. Divide E into elementary areas by lines drawn parallel to the axes. Let δA be such an area whose centroid is (x, y). Let $\rho = f(x, y)$ be the density at (x, y).

$\therefore I_x$ = Moment of inertia about the x-axis

$$= \underset{\delta A \to 0}{Lt} \; \Sigma \rho \delta A \,.\, y^2 = \iint_E \rho y^2 \, dA = \iint_E \rho y^2 \, dx\, dy\,;$$

I_y = Moment of inertia about the y-axis

$$= \underset{\delta A \to 0}{Lt} \; \Sigma \rho \delta A \,.\, x^2 = \iint_E \rho x^2 \, dA = \iint_E \rho x^2 \, dx\, dy.$$

F = Product of inertia with respect to x- and y-axes

$$= \underset{\delta A \to 0}{Lt} \; \Sigma \rho \delta A \,.\, x \,.\, y = \iint_E \rho xy \, dA = \iint_E \rho xy \, dx\, dy\,.$$

21.14. Illustrative Examples.

Ex. 1. *Evaluate* $\displaystyle\int_0^{\pi/2}\int_0^{\pi} \cos(x + y)\, dx\, dy$.

We have $\displaystyle I = \int_0^{\pi/2} dy \int_0^{\pi} \cos(x + y)\, dx$

$$= \int_0^{\pi/2} dy \left[\sin(x + y) \right]_0^{\pi}$$

$$= \int_0^{\pi/2} \left[\sin(\pi + y) - \sin(0 + y) \right] dy$$

$$= \int_0^{\pi/2} (-\sin y - \sin y)\,dy \qquad = \Big[2\cos y \Big]_0^{\pi/2}$$

$$= 2\cos\frac{\pi}{2} - 2\cos 0 \qquad = -2.$$

Ex. 2. *Evaluate* $\int_0^1 \int_0^{1-y^2} \Big[(x-1)^2 + y^2 \Big]\,dx\,dy$.

Here, $I = \int_0^1 dy \int_0^{1-y^2} \Big[(x-1)^2 + y^2 \Big]\,dx$

$$= \int_0^1 dy \left[\frac{(x-1)^3}{3} + y^2 x \right]_0^{1-y^2}$$

$$= \int_0^1 dy \left[\frac{(-y^2)^3}{3} + y^2(1-y^2) - \frac{(-1)^3}{3} \right]$$

$$= \int_0^1 \left(-\frac{y^6}{3} + y^2 - y^4 + \frac{1}{3} \right) dy$$

$$= \left[-\frac{y^7}{21} + \frac{y^3}{3} - \frac{y^5}{5} + \frac{y}{3} \right]_0^1 \quad = -\frac{1}{21} + \frac{1}{3} - \frac{1}{5} + \frac{1}{3} = \frac{44}{105}.$$

Ex. 3. *Show that* $\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3}\,dy \neq \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3}\,dx$.

We have $\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3}\,dy$

$$= \int_0^1 dx \int_0^1 \frac{2x-(x+y)}{(x+y)^3}\,dy$$

$$= \int_0^1 dx \int_0^1 \left[\frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2} \right] dy$$

$$= \int_0^1 dx \left[-\frac{2x}{2(x+y)^2} + \frac{1}{x+y} \right]_0^1$$

$$= \int_0^1 dx \left[-\frac{x}{(x+1)^2} + \frac{1}{x+1} + \frac{x}{x^2} - \frac{1}{x} \right]$$

$$= \int_0^1 \frac{dx}{(x+y)^2} = \left[-\frac{1}{x+1} \right]_0^1 = -\frac{1}{2} + 1 = \frac{1}{2}$$

Again, $\int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3}\, dx$

$$= \int_0^1 dy \int_0^1 \frac{x+y-2y}{(x+y)^3}\, dx$$

$$= \int_0^1 dy \int_0^1 \left[\frac{1}{(x+y)^2} - \frac{2y}{(x+y)^3} \right] dx$$

$$= \int_0^1 dy \left[-\frac{1}{x+y} + \frac{2y}{2(x+y)^2} \right]_0^1$$

$$= \int_0^1 \left[-\frac{1}{y+1} + \frac{y}{(y+1)^2} + \frac{1}{y} - \frac{y}{y^2} \right] dy$$

$$= \int_0^1 - \frac{1}{(y+1)^2}\, dy = \left[\frac{1}{y+1} \right]_0^1 = \frac{1}{2} - 1 = -\frac{1}{2} .$$

Thus, the two given integrals are unequal ;

i.e., the result is proved.

Note. This result is not unexpected as the function $\{(x-y)/(x+y)^3\}$ is discontinuous at $(0, 0)$. [*Prove*]

Ex. 4. *Evaluate* $\iint \sqrt{(4x^2 - y^2)}\, dx\, dy$ *over the triangle formed by the straight lines* $y = 0$, $x = 1$, $y = x$. [*C. H. 1967*]

The given double integral can be expressed as

$$\int_0^1 dx \int_0^x \sqrt{4x^2 - y^2}\, dy$$

$$= \int_0^1 dx \left[\frac{y\sqrt{4x^2 - y^2}}{2} + \frac{4x^2}{2} \sin^{-1} \frac{y}{2x} \right]_0^x$$

$$= \int_0^1 \left(\frac{x.x\sqrt{3}}{2} + 2x^2 \sin^{-1} \frac{1}{2} \right) dx$$

$$= \int_0^1 \left(\frac{\sqrt{3}}{2} x^2 + 2x^2 \cdot \frac{\pi}{6} \right) dx$$

$$= \left[\frac{\sqrt{3}}{2} \frac{x^3}{3} + \frac{\pi}{3} \frac{x^3}{3} \right]_0^1 = \frac{\sqrt{3}}{6} + \frac{\pi}{9} .$$

Ex. 5. *Evaluate* $\iint [2a^2 - 2a(x + y) - (x^2 + y^2)]\, dx\, dy$ *, the region of integration being the circle* $x^2 + y^2 + 2a(x + y) = 2a^2$. [C . H . 1962]

Here R , the region of integration, is $x^2 + y^2 + 2a(x + y) = 2a^2$,

i.e., $(x + a)^2 + (y + a)^2 = 4a^2$.

Now, use the transformation $x + a = X$, $y + a = Y$,

i.e., $x = X - a$, $y = Y - a$.

$$\therefore J\left(\frac{x,\ y}{X,\ Y}\right) = \begin{vmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 .$$

$$\therefore I = \iint_R [4a^2 - (x + a)^2 - (y + a)^2]\, dx\, dy$$

$$= \iint_{R'} (4a^2 - X^2 - Y^2) \,.\, 1\, dX\, dY ,$$ where the new region R' is the circle $X^2 + Y^2 = 4a^2$.

Finally, use the polar transformation

$$X = r\cos\theta,\ Y = r\sin\theta \qquad \therefore J\left(\frac{X,\ Y}{r,\ \theta}\right) = r$$

$$\therefore\ I = \int_{\theta=0}^{2\pi}\int_{r=0}^{2a} (4a^2 - r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{2a} (4a^2 - r^2)\, r\, dr$$

$$= \Big[\, \theta \,\Big]_0^{2\pi} \Big[\, 4a^2 \frac{r^2}{2} - \frac{r^4}{4} \,\Big]_0^{2a}$$

$$= 2\pi\,(2a^2 \,.\, 4a^2 - 4a^4) = 8\pi a^4 .$$

Note. If the region of integration R be the complete circle $x^2 + y^2 = a^2$ and we use the polar transformation $x = r\cos\theta$, $y = r\sin\theta$, the limits of r will be 0 to a and those of θ will be 0 to 2π.

If R be the positive quadrant of the above circle, r will vary from 0 to a and θ from 0 to $\pi/2$.

If R be the upper half of this circle, r will vary from 0 to a and θ from 0 to π.

Ex. 6. *Evaluate* $\iint_R \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) dx\, dy$, *where* R *consists of points in the positive quadrant of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. [*C . H.* '63, '72]

Use the transformation $x = aX$, $y = bY$.

$$\therefore\ J\left(\frac{x,\ y}{X,\ Y}\right) = \begin{vmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab .$$

R transforms to R' which is the positive quadrant of the circle

$$X^2 + Y^2 = 1 .$$

$$\therefore\ I = \iint_{R'} (1 - X^2 - Y^2)\, ab\, dX\, dY = ab \iint_{R'} (1 - X^2 - Y^2)\, dX\, dY .$$

Finally use the transformation $X = r\cos\theta$, $Y = r\sin\theta$.

$$\therefore\ J\left(\frac{X,\ Y}{r,\ \theta}\right) = r .$$

$$\therefore \quad I = ab \int_{\theta = 0}^{\pi / 2} \int_{r = 0}^{1} (1 - r^2)\, r\, dr\, d\theta$$

$$= ab \int_{0}^{\pi / 2} d\theta \int_{0}^{1} (1 - r^2)\, r\, dr$$

$$= ab \Big[\theta \Big]_{0}^{\pi / 2} \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_{0}^{1} = ab \frac{\pi}{2} \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{\pi ab}{8}$$

Ex. 7. *Evaluate* $\int_{0}^{\infty} \int_{0}^{\infty} e^{-(x^2 + 2xy \cos \alpha + y^2)}\, dx\, dy$. $(0 \le \alpha \le \pi)$

Here the region of integration is the positive quadrant. Use the transformation $x = r \cos \theta$, $y = r \sin \theta$.

$$\therefore \quad J\left(\frac{x, y}{r, \theta} \right) = r.$$

$$\therefore \ I = \int_{\theta = 0}^{\pi / 2} \int_{r = 0}^{\infty} e^{-(r^2 + 2 r^2 \cos \alpha \cos \theta \sin \theta)}\, r\, dr\, d\theta$$

$$= \int_{0}^{\pi / 2} d\theta \int_{0}^{\infty} e^{-(1 + \cos \alpha \sin 2\theta) r^2}\, r\, dr$$

$$= \int_{0}^{\pi / 2} d\theta \left[- \frac{1}{2(1 + \cos \alpha \sin 2\theta)} e^{-(1 + \cos \alpha \sin 2\theta) r^2} \right]_{0}^{\infty}$$

$$= \int_{0}^{\pi / 2} \frac{1}{2(1 + \cos \alpha \sin 2\theta)}\, d\theta$$

$$= \frac{1}{2} \int_{0}^{\pi / 2} \frac{d\theta}{\cos^2 \theta + \sin^2 \theta + 2 \cos \alpha \sin \theta \cos \theta}$$

$$= \frac{1}{2} \int_{0}^{\pi / 2} \frac{\sec^2 \theta\, d\theta}{\tan^2 \theta + 2 \cos \alpha \tan \theta + 1}$$

$$= \frac{1}{2}\int_0^{\infty} \frac{dz}{z^2 + 2z\cos\alpha + 1} \qquad [\text{ where } z = \tan\theta\ ]$$

$$= \frac{1}{2}\int_0^{\infty} \frac{dz}{(z + \cos\alpha)^2 + \sin^2\alpha}$$

$$= \frac{1}{2}\,\frac{1}{\sin\alpha}\left[\tan^{-1}\frac{z + \cos\alpha}{\sin\alpha}\right]_0^{\infty} = \frac{1}{2\sin\alpha}\left(\frac{\pi}{2} - \tan^{-1}\cot\alpha\right)$$

$$= \frac{1}{2\sin\alpha}\cot^{-1}\cot\alpha = \frac{\alpha}{2\sin\alpha}.$$

Ex. 8. *The density at the point* (x, y) *of a lamina bounded by the circle* $x^2 + y^2 - 2ax = 0$ *is* $\rho = x$. *Find its mass.*

$$M = \text{mass of the lamina} = \iint_R \rho\; dx\; dy,$$

R being the circle $x^2 + y^2 - 2ax = 0$

$$= \iint_R x\, dx\, dy.$$

Use the polar transformation $x = r\cos\theta$, $y = r\sin\theta$. Then $J\ \frac{x, y}{r, \theta} = r$, and the equation of the circle becomes $r = 2a\cos\theta$.

$$\therefore \qquad I = \int_{\theta = -\pi/2}^{\pi/2}\int_{r=0}^{2a\cos\theta} r\cos\theta . r\, dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \cos\theta\,.\,d\theta\,.\left[\frac{r^3}{3}\right]_0^{2a\cos\theta}$$

$$= \int_{-\pi/2}^{\pi/2} \frac{8a^3}{3}\cos^4\theta\,.\,d\theta$$

$$= \frac{16a^3}{3}\int_0^{\pi/2} \cos^4\theta\,.\,d\theta = \frac{16a^3}{3}\,\frac{3}{4}\,\frac{1}{2}\,\frac{\pi}{2} = \pi a^3.$$

Ex. 9. *A plane lamina of uniform surface density is bounded by the upper half of the cardioide $r = a(1 + \cos\theta)$ and the initial line. Find the co-ordinates of the centre of mass.*

Let $(\bar{x}, \bar{y})$ be the centre of mass.

$$\therefore \quad \bar{x} = \frac{\iint_R \rho x \, dx \, dy}{\iint_R \rho \, dx \, dy}, \quad R \text{ being the upper half of the cardioide } r = a(1 + \cos\theta)$$

$$= \frac{\iint_R r\cos\theta \,.\, r \, dr \, d\theta}{\iint_R r \, dr \, d\theta}$$

$$= \frac{\int_{\theta=0}^{\pi} \cos\theta \, d\theta \int_{r=0}^{a(1+\cos\theta)} r^2 \, dr}{\int_{\theta=0}^{\pi} d\theta \int_{r=0}^{a(1+\cos\theta)} r \, dr}$$

$$= \frac{\int_0^{\pi} \cos\theta \, d\theta \left[\frac{1}{3} r^3\right]_0^{a(1+\cos\theta)}}{\int_0^{\pi} d\theta \left[\frac{1}{2} r^2\right]_0^{a(1+\cos\theta)}}$$

$$= \frac{\int_0^{\pi} \frac{1}{3} a^3 \cos\theta (1 + \cos\theta)^3 \, d\theta}{\int_0^{\pi} \frac{1}{2} a^2 (1 + \cos\theta)^2 \, d\theta}$$

$$= \frac{2a}{3} \frac{\int_0^{\pi/2} \cos 2\phi (1 + \cos 2\phi)^3 \,.\, 2d\phi}{\int_0^{\pi/2} (1 + \cos 2\phi)^2 \, 2d\phi} \qquad | \; \theta = 2\phi$$

$$= \frac{2a}{3} \frac{\int_0^{\pi/2} (2\cos^2\phi - 1)\, 8\cos^6\phi\, d\phi}{\int_0^{\pi/2} 8\cos^6\phi\, d\phi}$$

$$= \frac{2a}{3} \frac{\left[2\frac{7}{8}\frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2} - \frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2}\right]}{\frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2}} = \frac{2a}{3}\frac{3}{4} = \frac{a}{2}.$$

$$\bar{y} = \frac{\iint_R \rho\ y\, dx\, dy}{\iint_R \rho\, dx\, dy}$$

$$= \frac{\iint_R r\sin\theta\, .\, r\, dr\, d\theta}{\iint_R r\, dr\, d\theta}$$

$$= \frac{\int_0^{\pi} \left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)} \sin\theta\, d\theta}{\int_0^{\pi} \left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta}$$

$$= \frac{2a}{3} \frac{\int_0^{\pi} (1 + \cos\theta)^3 \sin\theta\, d\theta}{\int_0^{\pi} (1 + \cos\theta)^2\, d\theta}$$

$$= \frac{2a}{3} \frac{\left[-\frac{1}{4}(1 + \cos\theta)^4\right]_0^{\pi}}{\int_0^{\pi} 4\cos^4\frac{\theta}{2}\, d\theta}$$

$$= \frac{2a}{3} \frac{4}{4\int_0^{\pi/2} \cos^4\phi \,.\, 2d\phi} \qquad \left[\frac{\theta}{2} = \phi\right]$$

$$= \frac{2a}{3} \frac{1}{2} \frac{1}{\frac{3}{4}\frac{1}{2}\frac{\pi}{2}} = \frac{16a}{9\pi}$$

$\therefore$ co-ordinates of the centre of mass are $\left(\frac{a}{2}, \frac{16a}{9\pi}\right)$.

Ex. 10. *A semi-circular lamina of radius a is immersed vertically in a liquid, the pressure of which varies as the depth with the bounding diameter in the surface. Find the centre of pressure of the lamina.*

Let us take the centre of the lamina as the origin, the bounding diameter as x-axis and the vertical radius as y-axis.

p = pressure at the point $(x, y) = ky$. Let $(\bar{x}, \bar{y})$ be the centre of pressure.

$$\therefore \bar{x} = \frac{\iint_R px\,dx\,dy}{\iint_R p\,dx\,dy}, R : x^2 + y^2 \le a^2, y \ge 0$$

$$= \frac{\iint_R xy\,dx\,dy}{\iint_R y\,dx\,dy} \quad [\text{put } x = r\cos\theta, y = r\sin\theta]$$

$$= \frac{\int_0^\pi \int_0^a r^3 \sin\theta\cos\theta\,dr\,d\theta}{\int_0^\pi \int_0^a r^2 \sin\theta\,dr\,d\theta} = \frac{\frac{a^4}{4}\left[\frac{\sin^2\theta}{2}\right]_0^\pi}{\frac{a^3}{3}\left[-\cos\theta\right]_0^\pi} = 0.$$

$$\bar{y} = \frac{\iint_R py\,dx\,dy}{\iint_R p\,dx\,dy} = \frac{\iint_R y^2\,dx\,dy}{\iint_R y\,dx\,dy}$$

$$\frac{\int_0^a r^3 \sin^2\theta \; dr\, d\theta}{{}_0\int_0^a r^2 \sin\theta \, dr\, d\theta}$$

$$= \frac{\frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}\left[\frac{r^4}{4}\right]_0^{a}}{\left[-\cos\theta\right]_0^{\pi}\left[\frac{r^3}{3}\right]_0^{a}}$$

$$= \frac{\frac{1}{2}\cdot\frac{a^4}{4}.\pi}{2\,.\,\frac{a^3}{3}} = \frac{3\pi a}{16}.$$

$\therefore$ the centre of pressure is $\left(0, \frac{3\pi a}{16}\right)$.

EXAMPLES XXI(A)

1. Evaluate :

(i) $\int_1^2 \int_0^1 (x+y)^2 \, dy\, dx$.

(ii) $\int_0^4 \int_0^1 xy\,(x - y)\, dy\, dx$.

(iii) $\int_0^{\pi/2} \int_0^{\pi} \sin(x + y)\, dx\, dy$.

(iv) $\int_0^{\log_e 2} \int_{-1}^{1} y e^{xy}\, dx\, dy$.

(v) $$\int_0^{\pi/2} \int_{\pi/2}^{\pi} e^x \cos(y - x)\, dy\, dx.$$

(vi) $$\int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx.$$ (vii) $$\int_1^2 \int_0^x \frac{dy\, dx}{x^2 + y^2}.$$

(viii) $$\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)\, dy\, dx.$$

(ix) $$\int_0^a \int_0^{\sqrt{a^2 - y^2}} \sqrt{a^2 - x^2 - y^2}\, dx\, dy.$$

(x) $$\int_0^2 \int_0^{\sqrt{2x - x^2}} x\, dy\, dx.$$ (xi) $$\int_1^2 dy \int_3^4 \frac{dx}{(x + y)^2}.$$

(xii) $$\int_0^{\pi} \int_0^{a\cos\theta} r \sin\theta\, dr\, d\theta.$$ (xiii) $$\int_0^{\pi} \int_0^{a(1 + \cos\theta)} r\, dr\, d\theta.$$

(xiv) $\iint xy\, dx\, dy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$.

(xv) $\iint (x^2 + y^2)\, dx\, dy$ over the region in the positive quadrant for which $x + y \leq 1$.

(xvi) $\iint x\, dx\, dy$ over the ellipse $b^2x^2 + a^2y^2 = 1$.

(xvii) $\iint xy(x + y)\, dx\, dy$ over the area bounded by $y = x^2$ and $y = x$.

2. Prove, by evaluating the repeated integrals, that

$$\int_0^1 dx \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2}\, dy \neq \int_0^1 dy \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2}\, dx.$$

3. Evaluate, by using suitable transformations :

(i) $\iint (x^2 + y^2)\, dx\, dy$ over the region enclosed by the triangle having its vertices at $(0, 0)$, $(1, 0)$, $(1, 1)$.

[*C. H. 1965*]

(ii) $\displaystyle\int_0^a \int_0^{\sqrt{a^2 - y^2}} \sqrt{x^2 + y^2}\, dy\, dx$.

(iii) $\iint x^2 y^2\, dx\, dy$ extended over the region $x \geq 0$, $y \geq 0$, $x^2 + y^2 \leq 1$. [*C. H. 1969*]

(iv) $\iint x^2 y^2 dx\, dy$ over the circle $x^2 + y^2 \leq 1$.

[*C. H. 1964*]

(v) $\iint \sqrt{(4a^2 - x^2 - y^2)}\, dx\, dy$ taken over the upper half of the circle $x^2 + y^2 - 2ax = 0$. [*C. H. 1966*]

(vi) $\iint [2 - 2(x + y) - (x^2 + y^2)]\, dx\, dy$, the region of integration being the circle $x^2 + y^2 + 2(x + y) = 2$.

(vii) $\iint xy(x^2 + y^2)^{n/2}\, dx\, dy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$, $(n + 3 > 0)$.

(viii) $\displaystyle\int_0^a \int_y^a \frac{x^2\, dx\, dy}{\sqrt{(x^2 + y^2)}}$.

(ix) $\iint \sin \dfrac{(x^2 + y^2)}{\pi} dx\, dy$ over the region to the first quadrant bounded by $y = 0$, $y = x$ and $x^2 + y^2 = \pi^2$.

(x) $\iint r^2 \sin\theta\, dr\, d\theta$ over the upper half of the circle $r = 2a\cos\theta$.

(xi) $\iint (x + y)^2 dx\, dy$ over the ellipse $\{(x^2/a^2) + (y^2/b^2)\} = 1$.

[*C. H. 1977*]

(xii) $\iint x^2 y\, dx\, dy$ over the positive quadrant of the ellipse $(x^2/a^2 + y^2/b^2) = 1$. [*C. H. 1971*]

(xiii) $\iint \left(\dfrac{a^2b^2 - b^2x^2 - a^2y^2}{a^2b^2 + b^2x^2 + a^2y^2} \right)^{1/2} dx\, dy$, the field of integration being the positive quadrant of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 .$$

(xiv) $\iint \dfrac{dx\, dy}{(1 + x^2 + y^2)^2}$ over the triangle with vertices $(0, 0), (2, 0), (1, \sqrt{3})$. [*C. H. 1970*]

(xv) $\iint \dfrac{dx\, dy}{(1 + x^2 + y^2)^2}$ taken over one loop of the lemniscate $(x^2 + y^2)^2 = x^2 - y^2$. [*C. H. 1974*]

(xvi) $\iint x\, dx\, dy$ over the region $r = 2a(1 + \cos\theta)$.

(xvii) $\displaystyle\int_0^\infty \int_0^\infty e^{-(x^2 + y^2)}\, dx\, dy$.

(xviii) $\displaystyle\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{x^2\, dx\, dy}{\{1 + \sqrt{(x^2 + y^2)}\}^5}$.

(xix) $\displaystyle\int_0^\infty \int_0^\infty e^{-(x^2 + 2xy + 2y^2)}\, dy\, dx$.

(xx) $\iint xy\, dx\, dy$ over the region bounded by the parabolas $y^2 = 4x$, $y^2 = 8x$, $x^2 = 4y$, $x^2 = 8y$.

[*Put* $\dfrac{y^2}{x} = u$, $\dfrac{x^2}{y} = v$.]

4. Find, by double integration, the area of the region bounded by the curves :

(i) $y^2 = 4x$, $y^2 = 16x$, $x = 1$, $x = 16$ in the positive quadrant ;

(ii) $y^2 = 4ax$, $x^2 = 4ay$;

(iii) $x^2 + y^2 = 100, x^2 + y^2 = 64, y = \sqrt{3}\,x, \sqrt{3}\,y = x$ in the positive quadrant;

(iv) the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and its auxiliary circle;

(v) $r = a(1 + \cos\theta)$, the initial line and the line $\theta = \pi/3$;

(vi) $y^2 = 8x, y^2 = 16x, xy = 25, xy = 16$ in the positive quadrant.

5. For the thin plates bounded by the following curves find the mass, the centre of mass and the moments of inertia about the axes, ρ being the density at (x, y):

(i) $x \geq 0, y \geq 0, x^2 + y^2 \leq 1, \rho = kxy$;

(ii) the parabola $y^2 = 4ax$ and its latus rectum, where ρ = constant;

(iii) $y \geq 0, x^2 + y^2 \leq a^2, \rho = x^2 + y^2$;

(iv) positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $\rho = kx$;

(v) triangle whose vertices are $(0, 0), (1, \sqrt{3}), (2, 0)$, where ρ = constant;

(vi) upper half of the circle $x^2 + y^2 = 2ax$, where $\rho = \sqrt{x^2 + y^2}$.

6. (i) A quadrant of an ellipse of semi-axes a and b $(a > b)$ is just immersed vertically with the semi-major axis in the surface of a liquid in which the pressure varies as the depth. Find the centre of pressure. [C. H. 1962]

(ii) Find the position of the centre of pressure of a quadrant of a circle of radius a which is just immersed vertically, with one edge in the suface of a liquid, the pressure of which varies as the square of the depth. [C. H. 1964]

ANSWERS

1. (i) $\frac{25}{6}$. (ii) 8. (iii) 2. (iv) $\frac{1}{2}$.

(v) 1. (vi) $\frac{1}{2}$. (vii) $\frac{\pi}{4}\log 2$. (viii) $\frac{3}{35}$.

(ix) $\frac{\pi a^3}{6}$. (x) $\frac{1}{2}\pi$. (xi) $\log\frac{25}{24}$. (xii) $\frac{a^2}{3}$.

(xiii) $\frac{3}{4}\pi a^2$. (xiv) $\frac{1}{8}a^4$. (xv) $\frac{1}{6}$. (xvi) 0.

(xvii) $\frac{3}{54}$. **3.** (i) $\frac{1}{3}$. (ii) $\frac{1}{20}\pi a^5$. (iii) $\frac{1}{96}\pi$.

(iv) $\frac{\pi}{24}$. (v) $\frac{4}{9}a^3(3\pi - 4)$. (vi) 8π. (vii) $\frac{a^{n+4}}{2(n+4)}$.

(viii) $\frac{a^3}{3}\log(\sqrt{2} + 1)$. (ix) $\frac{\pi^2}{4}$. (x) $\frac{2a^3}{3}$.

(xi) $\frac{\pi ab(a^2 + b^2)}{4}$. (xii) $\frac{1}{15}a^3b^2$. (xiii) $\frac{\pi ab}{8}(\pi - 2)$

(xiv) $\frac{\sqrt{3}}{2}\tan^{-1}\frac{1}{2}$. (xv) $\frac{1}{4}(\pi - 2)$. (xvi) $10\pi a^3$.

(xvii) $\frac{\pi}{4}$. (xviii) $\frac{\pi}{4}$. (xix) $\frac{\pi}{8}$. (xx) 192.

4. (i) 84. (ii) $\frac{16}{3}a^2$. (iii) 3π. (iv) $\pi a(a - b)$;

(v) $\frac{a^2}{16}(4\pi + 9\sqrt{3})$; (vi) $3\log 2$. **5.** (i) $\frac{k}{8}$; $\left(\frac{8}{15}, \frac{8}{15}\right)$, $\frac{k}{24}$, $\frac{k}{24}$;

(ii) $\frac{8}{3}\rho a^2$, $(\frac{3}{5}a, 0)$, $\frac{32}{15}\rho a^4$, $\frac{8}{7}\rho a^4$; (iii) $\frac{\pi a^4}{4}$, $\left(0, \frac{8a}{5\pi}\right)$, $\frac{\pi a^6}{12}$, $\frac{\pi a^6}{12}$

(iv) $\frac{ka^2b}{3}$, $\left(\frac{3\pi\alpha}{16}, \frac{3b}{8}\right)$, $\frac{ka^2b^3}{15}$, $\frac{2ka^4b}{15}$;

(v) $\sqrt{3}\rho$, $\left(1, \frac{1}{\sqrt{3}}\right)$, $\frac{\sqrt{3}\rho}{2}$, $\frac{7\sqrt{3}\rho}{6}$;

(vi) $\frac{16a^3}{9}$, $\left(\frac{6a}{5}, \frac{9a}{20}\right)$, $\frac{256a^5}{525}$, $\frac{512a^5}{175}$.

6. (i) On the minor axis and at a depth $\frac{3\pi b}{16}$;

(ii) On the vertical radius, at a depth $\frac{32a}{15\pi}$

21.15. Triple integral over a rectangular parallelopiped.

Let $f(x, y, z)$ be a bounded function of three independent variables x, y, z over the rectangular parallelopiped bounded by the planes $x = a, x = b, y = c, y = d, z = e, z = f$. This region will be denoted by $R[a, b; c, d; e, f]$ or simply by R.

$$\begin{aligned} \text{Let} \quad a &= x_0 < x_1 < x_2 < \ldots < x_{m-1} < x_m = b, \\ c &= y_0 < y_1 < y_2 < \ldots < y_{n-1} < y_n = d, \\ e &= z_0 < z_1 < z_2 < \ldots < z_{p-1} < z_p = f. \end{aligned}$$

We now draw the planes $x = x_1, x = x_2, \ldots, x = x_{m-1}$ which are parallel to YZ plane, planes $y = y_1, y = y_2, \ldots, y = y_{n-1}$ which are parallel to ZX plane and planes $z = z_1, z = z_2, \ldots, z = z_{p-1}$ which are parallel to XY plane to divide R into mnp sub-regions, each of which being a rectangular parallelopiped.

Let us denote the sub-region

$R[x_{i-1}, x_i; y_{j-1}, y_j; z_{k-1}, z_k]$ by R_{ijk} and its volume $(x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1})$ by V_{ijk}.

Let m_{ijk} and M_{ijk} be the lower and upper bounds of $f(x, y, z)$ in R_{ijk}. We now form the sums

$$s = \sum_{k=1}^{p} \sum_{j=1}^{n} \sum_{i=1}^{m} m_{ijk}\, v_{ijk},$$

$$S = \sum_{k=1}^{p} \sum_{j=1}^{n} \sum_{i=1}^{m} M_{ijk}\, V_{ijk}.$$

It is clear that for every mode of sub-division of R into a finite number of sub-regions

$m(b - a)(d - c)(f - e) \le s \le S \le M(b - a)(d - c)(f - e)$, where m, M are the lower and upper bounds of $f(x, y, z)$ in R. Thus, the two sets of upper and lower sums S, s are bounded.

The lower bound of the set of upper sums is defined as the upper integral of $f(x, y, z)$ over R and is denoted by I and the upper bound of the set of lower sums is defined as the lower integral of $f(x, y, z)$ over R and is denoted by J and we write

$$I = \overline{\iiint}_R f(x, y, z)\, dx\, dy\, dz, \quad J = \iiint_R f(x, y, z)\, dx\, dy\, dz.$$

If I and J be equal, then the function $\overline{f(x}, y, z)$ is said to be integrable over R and the common value denoted by

$\iiint_R f(x, y, z)\, dx\, dy\, dz$ or $\iiint_R f(x, y, z)\, dV$ is defined

as the *triple integral* of $f(x, y, z)$ over R.

21.16. Condition of integrability.

The necessary and sufficient condition of integrability of a bounded function $f(x, y, z)$ over R is that to every positive number ε, there corresponds a positive number δ, such that for every division of R whose norm is $< \delta$, the oscillation $S - s$ is less than ε.

[*Proof is omitted.*]

21.17. Calculation of triple integral. Equivalence of a double integral with repeated integrals.

Theorem. If the triple integral $\iiint_R f(x, y, z)\, dx\, dy\, dz$ exists over $R[a, b; c, d; e, f]$ and if the double integral $\iint_S f(x, y, z)\, dx\, dy$ also exists for all values of z in (e, f),

then the repeated integral $\int_e^f dz \left[\iint_S f(x, y, z)\, dx\, dy \right]$ exists and is equal to the triple integral.

Proof. Similar to that of § 21.5 and left as an exercise to the student.

Cor. If $f(x, y, z)$ be continuous over R, we have

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \int_e^f dz \int_c^d dy \int_a^b f(x,y,z)\,dx,$$ where we can change the order of integration to suit our convenience.

21.18. Triple integral over any finite region.

Let E be a finite region bounded by any surface. We can construct a rectangular parallelopiped R enclosing E completely. Let us define a function $g(x,y,z)$ over R as follows :

$$\begin{aligned} g(x,y,z) &= f(x,y,z) \text{ at all points of } E, \\ &= 0 \text{ outside } E. \end{aligned}$$

The function $f(x,y,z)$ is said to be integrable over E if $g(x,y,z)$ be integrable over R. Then we have

$$\iiint_E f(x,y,z)\,dx\,dy\,dz = \iiint_R g(x,y,z)\,dx\,dy\,dz.$$

20.19. Evaluation of triple integral.

Let E be the region bounded by the sufaces $z - u(x,y)$, $z = v(x,y)$; $y = \phi(x)$, $y = \psi(x)$; $x = a$, $x = b$. If $f(x,y,z)$ be a continuous function in E, then

$$\iiint_E f(x,y,z)\,dx\,dy\,dz = \int_a^b dx \int_{\phi(x)}^{\psi(x)} dy \int_{u(x,y)}^{v(x,y)} f(x,y,z)\,dz.$$

[*Proof*. Similar to that of § 21.9.]

21.20. Change of variable in a triple integral.

Let $x = u(\xi,\eta,\zeta)$, $y = v(\xi,\eta,\zeta)$, $z = w(\xi,\eta,\zeta)$ . . . (1) be three functions of ξ, η, ζ, defined in a region E' of the $\xi\eta\zeta$ space bounded by a surface S'. Moreover, we assume that

(i) u, v, w possess continuous partial derivatives of the first order at each point of E' and S',

(ii) the equations (1) transform the region E' bounded by S' into a region E of the xyz space bounded by the surface S in such a way that there exists a one-one correspondence between E, E' and S, S',

(iii) Jacobian $\dfrac{\partial(x, y, z)}{\partial(\xi, \eta, \zeta)}$ does not change sign atany point of E', but it may vanish at some points of S', then

$$\iiint_E f(x, y, z)\, dx\, dy\, dz$$

$$= \iiint_E f\,[u(\xi,\eta,\zeta), v(\xi,\eta,\zeta), w(\xi,\eta,\zeta)] \left| \frac{\partial(x,y,z)}{\partial(\xi,\eta,\zeta)} \right| d\xi\, d\eta\, d\zeta .$$

21.21. Applications of triple integral.

(a) Mass of a solid.

Let a body E be bounded by the surface S, and $\rho = f(x, y, z)$ be the mass per unit volume (*i.e.*, the density at the point (x, y, z). We can show as in § 21.13 (a) that the mass of the body is given by

$$M = \iiint_E \rho\, dx\, dy\, dz .$$

(b) Centre of mass of a body.

Let $\rho = f(x, y, z)$ be the density at (x, y, z) of a solid body E. If $(\bar{x}, \bar{y}, \bar{z})$ be the centre of mass of the body, then

$$\bar{x} = \frac{\iiint_E \rho\, x\, dx\, dy\, dz}{\iiint_E \rho\, dx\, dy\, dz}, \quad \bar{y} = \frac{\iiint_E \rho\, y\, dx\, dy\, dz}{\iiint_E \rho\, dx\, dy\, dz},$$

$$\bar{z} = \frac{\iiint_E \rho\, z\, dx\, dy\, dz}{\iiint_E \rho\, dx\, dy\, dz}$$

(c) Moment of inertia of a body.

Let $\rho = f(x, y, z)$ be the density at (x, y, z) of a solid body E. Then I_x, I_y, I_z its moments of inertia about ox, oy, oz are given by

$$I_x = \iiint_E \rho (y^2 + z^2)\, dx\, dy\, dz,$$

$$I_y = \iiint_E \rho (z^2 + x^2)\, dx\, dy\, dz,$$

$$I_z = \iiint_E \rho (x^2 + y^2)\, dx\, dy\, dz.$$

21.22. Illustrative Examples.

Ex. 1. *Evaluate* $\int_0^a \int_0^{\sqrt{a^2 - x^2}} \int_0^b (y^2 + z^2)\, dz\, dy\, dx$. [C. H. '76]

We have $$I = \int_0^a dx \int_0^{\sqrt{a^2 - x^2}} dy \int_0^b (y^2 + z^2)\, dz$$

$$= \int_0^a dx \int_0^{\sqrt{a^2 - x^2}} dy \left[y^2 z + \frac{z^3}{3} \right]_0^b$$

$$= \int_0^a dx \int_0^{\sqrt{a^2 - x^2}} \left(by^2 + \frac{b^3}{3} \right) dy$$

$$= \int_0^a dx \left[\frac{by^3}{3} + \frac{b^3 y}{3} \right]_0^{\sqrt{a^2 - x^2}}$$

$$= \int_0^a \left[\frac{b(a^2 - x^2)^{3/2}}{3} + \frac{b^3 (a^2 - x^2)^{1/2}}{3} \right] dx$$

$$= \int_0^{\pi/2} \left(\frac{ba^3 \cos^3\theta}{3} + \frac{b^3 a \cos\theta}{3} \right) a\cos\theta \, d\theta \quad [\, x = a\sin\theta \,]$$

$$= \int_0^{\pi/2} \left(\frac{a^4 b}{3}\cos^4\theta + \frac{a^2 b^3}{3}\cos^2\theta \right) d\theta$$

$$= \frac{a^4 b}{3}\,\frac{3}{4}\,\frac{1}{2}\,\frac{\pi}{2} + \frac{a^2 b^3}{3}\,\frac{1}{2}\,\frac{\pi}{2} \qquad = \frac{\pi a^2 b\,(3a^2 + 4b^2)}{48}$$

Ex. 2. *Evaluate* $\iiint (x + y + z + 1)^4 \, dx\, dy\, dz$ *over the region defined by* $x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1$.

The region of integration is the tetrahedron as shown in the figure.

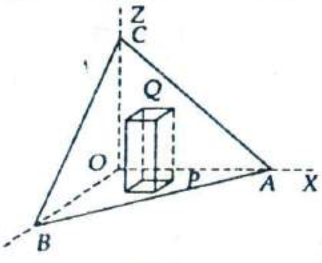

Fig.4

In this region z varies from 0 to $1 - x - y$, y varies from 0 to $1 - x$ and x varies from 0 to 1.

$$\therefore\ I = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} (x + y + z + 1)^4 \, dz$$

$$= \int_0^1 dx \int_0^{1-x} dy \, \frac{1}{5}\Big[(x + y + z + 1)^5 \Big]_0^{1-x-y}$$

$$= \frac{1}{5} \int_0^1 dx \int_0^{1-x} [\, 32 - (x + y + 1)^5 \,] \, dy$$

$$= \frac{1}{5} \int_0^1 dx \left[32y - \frac{1}{6}(x + y + 1)^6 \right]_0^{1-x}$$

$$= \frac{1}{5}\int_0^1 \left[32(1-x) - \frac{64}{6} + \frac{(1+x)^6}{6} \right] dx$$

$$= \frac{1}{5}\left[32\left(x - \frac{1}{2}x^2 \right) - \frac{32}{3}x + \frac{1}{42}(1+x)^7 \right]_0^1$$

$$= \frac{1}{5}\left[32 \cdot \frac{1}{2} - \frac{32}{3} + \frac{128}{42} - \frac{1}{42} \right] = \frac{351}{210} = \frac{117}{70}.$$

Ex. 3. *Find the mass of a solid in the form of the positive octant of the ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, *the density at* (x, y, z) *being* xyz.

M = Mass of the solid = $\iiint_E xyz\, dx\, dy\, dz$, where E is the positive octant of the ellipsoid.

Put $x = aX, y = bY, z = cZ$.

$$\therefore \frac{\partial(x, y, z)}{\partial(X, Y, Z)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc.$$

$\therefore M = \iiint_{E'} abc\ XYZ\, abc\, dX\, dY\, dZ$, where E' is the positive octant of the sphere $X^2 + Y^2 + Z^2 = 1$

$$= a^2b^2c^2 \iiint_{E'} XYZ\, dX\, dY\, dZ.$$

Put $X = r\sin\theta\cos\phi,\ Y = r\sin\theta\sin\phi,\ Z = r\cos\theta$.

$$\frac{\partial(X, Y, Z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta.$$

$$\therefore M = a^2b^2c^2 \int_0^1\int_0^{\pi/2}\int_0^{\pi/2} r^3\sin^2\theta\cos\theta\sin\phi\cos\phi\ r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= a^2b^2c^2\left[\frac{r^6}{6}\right]_0^1 \int_0^{\pi/2} \sin^3\theta\cos\theta\, d\theta \int_0^{\pi/2} \sin\phi\cos\phi\, d\phi$$

$$= \frac{1}{6} a^2 b^2 c^2 \left[\frac{\sin^4 \theta}{4} \right]_0^{\pi/2} \left[\frac{\sin^2 \phi}{2} \right]_0^{\pi/2}$$

$$= \frac{1}{6} a^2 b^2 c^2 \frac{1}{4} \frac{1}{2} = \frac{1}{48} a^2 b^2 c^2 .$$

Note. Generally in the case of a spherical region $x^2 + y^2 + z^2 = a^2$, we use the transformation $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$.

(i) limits for r, θ, ϕ are $0, a; 0, \pi; 0, 2\pi$ for the whole sphere;

(ii) these are $0, a; 0, \pi/2; 0, 2\pi$ for the upper hemisphere;

(iii) they are $0, a; 0, \pi/2;\ 0, \pi/2$ for the positive octant.

EXAMPLES XXI(B)

1. Evaluate :

(i) $\int_0^{3a} \int_0^{2a} \int_0^{a} (x + y + z)\, dx\, dy\, dz$.

(ii) $\int_0^{1} \int_{y^2}^{1} \int_0^{1-x} x\, dz\, dx\, dy$.

(iii) $\int_0^{a} \int_0^{x} \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$.

(iv) $\int_0^{2} \int_0^{z} \int_0^{x\sqrt{3}} \frac{x}{x^2 + y^2}\, dy\, dx\, dz$.

(v) $\iiint \frac{dx\, dy\, dz}{(1 + x + y + z)^2}$ extended over the tetrahedron bounded by the planes $x = 0, y = 0, z = 0, x + y + z = 1$.
[*C. H.* '70]

(vi) $\iiint x^2\, dx\, dy\, dz$ extended over the volume of the ball $x^2 + y^2 + z^2 \le a^2$. [*C. H.* '69]

2. Evaluate by using suitable transformations :

(i) $\iiint (x^2 + y^2 + z^2)\, xyz\, dx\, dy\, dz$ taken through the the sphere $x^2 + y^2 + z^2 \leq 1$. [*C. H.* '64]

(ii) $\displaystyle\iiint \frac{dx\, dy\, dz}{x^2 + y^2 + (z - 2)^2}$, over the sphere $x^2 + y^2 + z^2 \leq 1$. [*C. H.* '73]

(iii) $\displaystyle\iiint \sqrt{\frac{1 - x^2 - y^2 - z^2}{1 + x^2 + y^2 + z^2}}\, dx\, dy\, dz$ over the positive octant of the sphere $x^2 + y^2 + z^2 \leq 1$.

(iv) $\displaystyle\iiint \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}}\, dx\, dy\, dz$ taken over the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 .$$

(v) $\iiint (ax^2 + by^2 + cz^2)\, dx\, dy\, dz$ over the sphere $x^2 + y^2 + z^2 \leq R^2$.

3. For the solid bodies bounded by the following surfaces find the mass, centre of mass and the moments of intertia about the axes, ρ being the density at (x, y, z) :

(i) $x \geq 0,\ y \geq 0,\ z \geq 0,\ x^2 + y^2 + z^2 \leq a^2$; $\rho = kxyz$.

(ii) $x \geq 0,\ y \geq 0,\ z \geq 0,\ x + y + z \leq 1$; ρ = constant.

(iii) $z \geq 0,\ x^2 + y^2 + z^2 \leq a^2$; ρ = constant.

(iv) $x \geq 0,\ y \geq 0,\ z \geq 0,\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} \leq 1$; ρ = constant.

4. *A* mass *M* of gas is diffused through all space. If the density of the gas at (x, y, z) be $e^{-(x^2+y^2+z^2)}$, show that $M^2 = \pi^3$.

ANSWERS

1. (i) $18a^4$. (ii) $\frac{4}{35}$. (iii) $\frac{1}{8}(e^{4a} - 6e^{2a} + 8e^a - 3)$.

(iv) $\dfrac{2\pi}{3}$. (v) $\dfrac{1}{16}\log(256/e^5)$. (vi) $\dfrac{4\pi}{15}a^5$.

2. (i) 0. (ii) $\pi\left(2-\frac{3}{2}\log 3\right)$. (iii) $\frac{\pi}{8}\left[B\left(\frac{3}{4},\frac{1}{2}\right)-B\left(\frac{5}{4},\frac{1}{2}\right)\right]$.

(iv) πabc. (v) $\dfrac{4\pi(a+b+c)R^5}{15}$.

3. (i) $\dfrac{ka^6}{48}$; $\left(\dfrac{16a}{35},\dfrac{16a}{35},\dfrac{16a}{35}\right)$; $\dfrac{ka^8}{96},\dfrac{ka^8}{96},\dfrac{ka^3}{96}$.

(ii) $\frac{1}{6}\rho$; $\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$; $\frac{1}{30}\rho,\frac{1}{30}\rho,\frac{1}{30}$.

(iii) $\dfrac{2}{3}\pi\rho a^3$, $\left(0,0,\dfrac{3}{8}a\right)$; $\dfrac{4\pi a^5}{15}\rho,\dfrac{4\pi a^5}{15}\rho,\dfrac{4\pi a^5}{15}\rho$.

(iv) $\dfrac{1}{6}\pi abc\rho$; $\left(\dfrac{3a}{8},\dfrac{3b}{8},\dfrac{3c}{8}\right)$; $\dfrac{1}{30}\pi abc\rho(b^2+c^2)$,

$\dfrac{1}{30}\pi abc\rho(c^2+a^2),\dfrac{1}{30}\pi abc\rho(c^2+b^2)$.

Miscellaneous Examples II

Integrate the following (Ex. 1 to Ex. 21) :-

1. $\int \cos 2x \log \frac{\cos x + \sin x}{\cos x - \sin x}\, dx$.

2. $\int x \log [x + \sqrt{(x^2 + a^2)}]\, dx$.

3. $\int \frac{x\, dx}{\sqrt{(x + a)} + \sqrt{(x + b)}}$.

4. $\int \frac{\cot x}{(1 - \sin x)(\sec x + 1)}\, dx$.

5. (i) $\int \frac{\tan x}{1 + \sin x}\, dx$. (ii) $\int \frac{2 + \cos x}{\cos x (1 + \cos x)}\, dx$.

6. (i) $\int \sqrt{(\sec x - 1)}\, dx$. (ii) $\int \sqrt{(\cot x)}\, dx$.

7. $\int \frac{2x^2 + 3x + 7}{(x + 3)(x^2 + 2x + 5)}\, dx$.

8. $\int \frac{\tan x\, dx}{\sqrt{(a + b \tan^2 x)}}$, $a > b$.

9. $\int \frac{dx}{x (x^n + 1)}$.

10. $\int e^x \sin x \cos x \cos 2x \cos 4x\, dx$.

11. $\int \frac{\cosh x + \sinh x \sin x}{1 + \cos x}\, dx$.

12. $\int e^x . \frac{x^2 + 5x + 7}{(x + 3)^2}\, dx$.

13. $\int \frac{dx}{x \log x \log (\log x)}$.

14. $$\int \frac{x^3\,dx}{(x-a)(x-b)(x-c)}.$$

15. $$\int \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)(x^2+c^2)}.$$

16. $$\int \frac{x^7\,dx}{x^{12}-1}.$$

17. (i) $\displaystyle\int \frac{x^2+1}{x^4-x^2+1}\,dx$. (ii) $\displaystyle\int \frac{x^2-1}{x^4-x^2+1}\,dx$.

18. (i) $\displaystyle\int \frac{dx}{x^4-x^2+1}$. (ii) $\displaystyle\int \frac{x^2\,dx}{x^4-x^2+1}$

19. $$\int_0^1 \frac{2x^2+3x+3}{(x+1)(x^2+2x+2)}\,dx.$$

20. $$\int_0^{\frac{1}{2}\pi} \frac{\cos\theta\,d\theta}{1+2\sin\theta+2\sin^2\theta+2\sin^3\theta+\sin^4\theta}$$

21. $$\int_0^{\pi} \frac{x}{1+\cos^2 x}\,dx.$$

22. Show that

$$x < -\log(1-x) < \frac{x}{1-x} \quad (0 < x < 1).$$ [*C. H. 1963*]

23. If $n > 1$, then $0.5 < \displaystyle\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^{2n})}} < 0.524.$

[*C. H. 1963, '66*]

24. Show that $\log(m/n) = \log m - \log n$ from the definition of $\log x$ as a definite integral, *i.e.*, from

$$\log x = \int_1^x \frac{dt}{t}.$$

25. Find the area of the region included between two cardioides $r = a(1+\cos\theta)$, $r = a(1-\cos\theta)$, $a > 0$. [*C. H. 1967*]

26. Find the area of the loop of the curve

$$x = \frac{a\sin 3\theta}{\sin\theta}\ ,\ y = \frac{a\sin 3\theta}{\cos\theta}\ ,\ a > 0\ .$$

27. Find the area of the loop of the curve $r\cos\theta = a\cos 2\theta$.

28. (i) The area between the curves $y^2 = 4ax$ and $x^2 = 4ay$ $(a > 0)$ revolves about the axis of x. If V be the volume of the solid thus formed, then show that $5V = 96\pi a^3$.

(ii) If the curve $r = 2a\cos\theta$ revolves about the initial line, then show that the area of the surface of revolution is $4\pi a^2$.

29. If the area lying within the cardioide $r = 2a(1+\cos\theta)$ and without the parabola $r(1+\cos\theta) = 2a$ revolves about the initial line, then show that the volume generated is $18\pi a^3$.

30. Find the area common to the circle $r = a$ and the cardioide $r = a(1+\cos\theta)$.

31. Show that the area included between one of the branches of the curve $x^2y^2 = a^2(x^2+y^2)$ and its asymptotes is equal to one-fourth of the square formed by the asymptotes $(a > 0)$.

32. The distances of the vertices A, B, C of a scalene triangle of area S, from a fixed line $\overline{MN}$, are x_1, x_2, x_3. The line $\overline{MN}$ does not cut the triangle ABC. Find the volume generated by the revolution of the triangle ABC.

ANSWERS

1. $\frac{1}{2}\left[\sin 2x\log\frac{\cos x+\sin x}{\cos x-\sin x} + \log\cos 2x\right]$.

2. $\frac{x^2}{2}\log\left\{x+\sqrt{x^2+a^2}\right\} - \frac{x}{4}\sqrt{(x^2+a^2)} + \frac{a^2}{4}\sinh^{-1}\frac{x}{a}$.

3. $\frac{1}{a-b}\left[\frac{2}{5}(x+a)^{5/2} - \frac{2a}{3}(x+a)^{3/2}\right.$

$\left. - \frac{2}{5}(x+b)^{5/2} + \frac{2b}{3}(x+b)^{3/2}\right.$.

4. $\frac{1}{2}\log\tan\frac{1}{2}x + \frac{1}{4}\sec^2\frac{1}{2}x + \tan\frac{1}{2}x$.

5. (i) $\frac{1}{4}\log\frac{1+\sin x}{1-\sin x} + \frac{1}{2}\,\frac{1}{1+\sin x}$.

(ii) $2\log(\sec x + \tan x) - \tan\frac{x}{2}$.

6. (i) $-2\cosh^{-1}(\sqrt{2}\cos\frac{1}{2}x)$.

(ii) $\frac{1}{2\sqrt{2}}\log\frac{1+\sqrt{2\tan x}+\tan x}{1-\sqrt{(2\tan x)}+\tan x} + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2\tan x}}{1-\tan x}\right)$

7. $2\log(x+3) - \frac{1}{2}\tan^{-1}\{(x+1)/2\}$.

8. $-\frac{1}{\sqrt{(a-b)}}\sinh^{-1}\left(\frac{\sqrt{(a-b)}}{\sqrt{b}}\cos x\right)$.

9. $\log x - \frac{1}{n}\log(x^n + 1)$.

10. $\frac{1}{8}\frac{e^x}{\sqrt{(65)}}\sin(8x - \tan^{-1}8)$.

11. $\cosh x\tan\frac{1}{2}x$.

12. $e^x\,\frac{x+2}{x+3}$.

13. $\log\{\log(\log x)\}$.

14. $x + \sum\frac{a^3}{(a-b)(a-c)}\log(x-a)$.

15. $-\left[\sum\frac{a}{(a^2-b^2)(a^2-c^2)}\tan^{-1}\frac{x}{a}\right]$.

16. $\frac{1}{12}\log(x^4-1) - \frac{1}{24}\log(x^8+x^4+1)$

$$+\frac{1}{4\sqrt{3}}\tan^{-1}\left(\frac{2x^4+1}{\sqrt{3}}\right).$$

17. (i) $\tan^{-1}\frac{x^2-1}{x}$. (ii) $\frac{1}{2\sqrt{3}}\log\frac{x^2-\sqrt{3}.x+1}{x^2+\sqrt{3}.x+1}$.

18. (i) $\frac{1}{2}\tan^{-1}\frac{x^2-1}{x} - \frac{1}{4\sqrt{3}}\log\frac{x^2-\sqrt{3}.x+1}{x^2+\sqrt{3}.x+1}$.

(ii) $\frac{1}{2}\tan^{-1}\frac{x^2-1}{x} + \frac{1}{4\sqrt{3}}\log\frac{x^2-\sqrt{3}.x+1}{x^2+\sqrt{3}.x+1}$.

19. $\frac{1}{4}\pi + 2\log 2 - \tan^{-1}2$. 20. $\frac{1}{4}(\log 2 + 1)$. 21. $\frac{\pi^2}{2\sqrt{2}}$.

25. $2a^2(\frac{3}{4}\pi - 2)$. 26. $(3\sqrt{3})a^2$. 27. $2a^2(1 - \frac{1}{4}\pi)$.

30. $(\frac{5}{4}\pi - 2)a^2$. 32. $\frac{2}{3}\pi(x_1 + x_2 + x_3)S$.