chapter I

Network Concepts

Most of this textbook will be devoted to the analysis of networks which are energized with sine-wave voltages or currents. Before considering these time-varying sources, however, it will be advantageous to review some basic network concepts: concepts which are equally applicable to time-varying sources or non-time-varying sources.

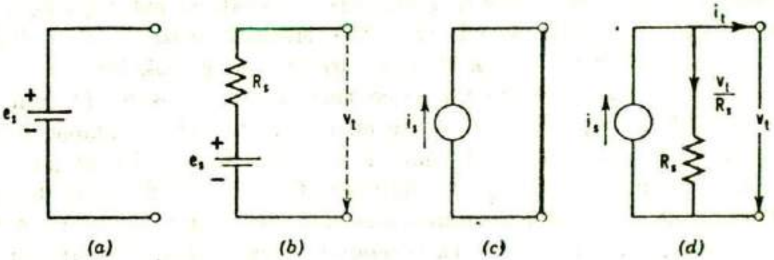


Fig. 1. Voltage and current sources.

Sources. The common non-time-varying voltage sources are batteries and direct-current generators. Since the reader undoubtedly has an intuitive understanding of how these sources are employed to energize electric circuits, we start with them. In later chapters, sources which develop time-varying voltages and currents will be employed almost exclusively. For the present, only the sources indicated in Fig. 1 will be used, and the battery symbol will indicate a non-time-varying voltage source regardless of the exact nature of the voltage source. Unless specifically noted, this voltage source is assumed to possess zero internal resistance. Where it is desirable to simulate the internal loss of a voltage source, a resistance will be placed in series with the ideal voltage source as indicated in Fig. 1*b*.

Where a voltage source, e_s, is specified, it will be understood that a potential difference of e_s volts is maintained between (or across) the terminals of the ideal source regardless of the current that may pass through this ideal source. The actual terminal voltage of a voltage source having internal resistance R_s which is *delivering* current to a network is

$$v_t = e_s - R_s i \tag{1}$$

where i is the current flowing in the − to + direction. See Fig. 2a where $i = i_t$.

A voltage source is idle when it is operating open-circuited; $i \doteq 0$. Otherwise it is delivering or absorbing power to the extent of

$$P_s = e_s i \quad \text{watts} \tag{2}$$

depending upon the direction of i relative to the polarity of e_s. The power delivered by a voltage source possessing internal resistance is

$$P_t = v_t i = (e_s - R_s i)i = e_s i - R_s i^2 \tag{3}$$

$R_s i^2$ is the heat power developed internally and as such is not available for distribution to the rest of the network.

Where a current source, i_s, is specified in circuit theory, it will be understood that the source delivers this specified current regardless of the resistance which is placed across the terminals of the source. It is, of course, unrealistic to ask that a current source look into an open circuit (or infinite resistance) since this situation results in infinite power (Ri^2) being delivered to the open circuit. This example, however, illustrates an important point: if a current source of i_s amperes is specified, then by definition this number of amperes is delivered to the network regardless of the resistance placed across the generator terminals. (A contradiction of definitions, of course, occurs when current generators of different specified currents are connected in series.)

A current source is idle when it is short-circuited as indicated in Fig. 1c. In this case the power delivered is zero owing to the fact that the specified current circulates through zero external resistance. When a finite resistance, R, appears across the terminals, the ideal current source delivers

$$P_s = Ri_s^2 \quad \text{watts} \tag{4}$$

A practical current source can be simulated by incorporating an internal resistance, R_s, across the terminals as illustrated in Fig. 1d. Under these conditions the terminal current is

$$i_t = i_s - \frac{v_t}{R_s} \tag{5}$$

where v_t is the terminal voltage developed when the current source is connected to a load resistance. This generator develops a terminal voltage of $R_s i_s$ when $i_t = 0$, that is, when the current source is operating open-circuited. If R_L is placed across the current source, the terminal voltage is

$$v_t = R_L i_t = R_L i_s - \frac{R_L}{R_s} v_t \tag{6}$$

and the power delivered to R_L is

$$R_L i_t^2 = (R_L i_t)i_s - \frac{R_L}{R_s} v_t i_t \tag{7}$$

The total power generated is $(R_L i_t)i_s = v_t i_s$ of which $\frac{R_L}{R_s} v_t i_t = \frac{v_t^2}{R_s}$ watts are dissipated in the internal resistance, R_s.

Although sources have been designated as voltage sources and current sources, it is evident that either one energizes the network with both

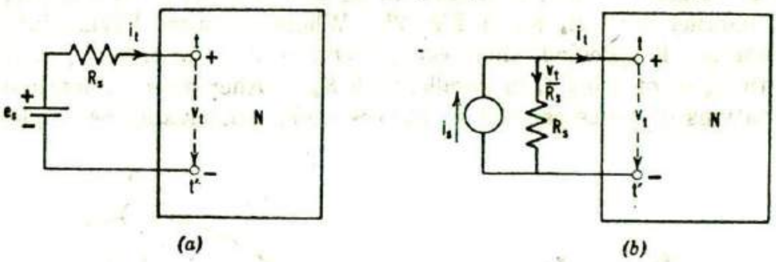


Fig. 2. Equivalent sources.

current and voltage. Indeed the two sources are entirely interchangeable when a finite internal resistance is present. The voltage source depicted in Fig. 2*a*, for example, supplies the network N with i_t amperes and v_t volts. The current, i_t, may be expressed as

$$\frac{e_s - v_t}{R_s} = i_t \tag{8}$$

which may be rearranged as

$$\frac{e_s}{R_s} - \frac{v_t}{R_s} = i_t \tag{8a}$$

Thus a specified e_s in series with a specified resistance R_s results in a specific current source

$$i_s = \frac{e_s}{R_s}$$

The conditions imposed by equation (8*a*) are satisfied by the circuit configuration shown in Fig. 2*b* where a current source i_s delivers i_t amperes to the network at v_t volts. Substitution of i_s permits equation (8*a*) to be written as

$$i_s = \frac{v_t}{R_s} + i_t \tag{8b}$$

where v_t/R_s is the current which is *lost* to the network as a result of the internal resistance R_s and the terminal voltage v_t.

Reference to Fig. 2 and to equations (8), (8*a*), and (8*b*) shows that a voltage source having an internal resistance of R_s ohms may be replaced with an ideal current source of e_s/R_s amperes in parallel with a resistive path of R_s ohms or that a current source i_s in parallel with R_s may be replaced with a voltage source ($e_s = R_s i_s$) in series with a resistance of R_s ohms. The rest of the network, that is, the portion of the network to the right of terminals tt' in Fig. 2, cannot tell whether it is energized with e_s in series with R_s as in Fig. 2*a* or with $i_s = e_s/R_s$ in parallel with R_s as in Fig. 2*b*. Where a source having internal resistance is specified, there exists a choice of using either e_s in series with R_s or of using i_s in parallel with R_s. Other more elaborate combinations of series-parallel resistances could conceivably be employed.

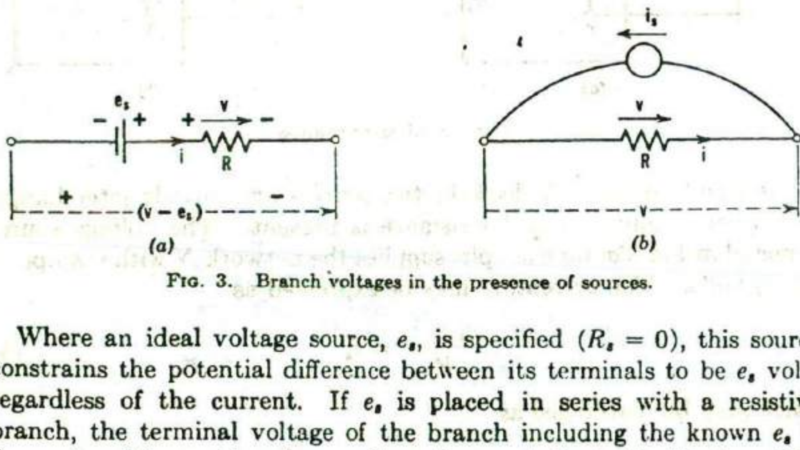


FIG. 3. Branch voltages in the presence of sources.

Where an ideal voltage source, e_s, is specified ($R_s = 0$), this source constrains the potential difference between its terminals to be e_s volts regardless of the current. If e_s is placed in series with a resistive branch, the terminal voltage of the branch including the known e_s is $(v - e_s)$ and is considered as a voltage drop as indicated in Fig. 3*a*. The inclusion of e_s does not increase the number of unknowns since e_s is specified. Where an ideal current source is placed across a resistive branch as illustrated in Fig. 3*b*, it can either be associated with R to form a series branch equivalent to that shown in Fig. 3*a* or be left as a fixed or specified current between the two terminals.

An ideal voltage source has zero internal series resistance. An ideal current source possesses infinite internal series resistance. This conclusion may be deduced from the definition of an ideal current source, namely, a source which delivers i_s regardless of the finite load resistance, R_L, which is placed across the terminals of the source. To satisfy this definition, it is evident that the internal voltage, say e_s, as well as the internal series resistance R_{int} must approach infinity. The specified current may be considered to be

$$i_s = \frac{k_1 e_s}{k_2 R_{\text{int}} + R_L} = \frac{k_1}{k_2} \tag{9}$$

as both e_s and R_{int} approach infinity, R_L remaining finite.

Superposition. A linear circuit element is one in which the current through the element is directly proportional to the voltage across the terminals of the element. Linear networks consist of linear elements and fixed (or specified) voltage and current sources.

One reason for the rapid strides which have been made in the analysis of linear networks is that the principle of superposition can be applied to these networks. With the aid of this principle, the voltage or current response in any part of a linear network resulting from two or more sources may be determined by:

(1) Finding the component response developed by each individual source.

(2) Adding (algebraically) the component responses to obtain the actual response.

The truth of the principle of superposition is almost self-evident since effects are proportional to causes in linear systems where the principle applies. In any event, a general proof will be left for the reader *after* the subjects of determinants and general network solutions have been considered.

A simple application of superposition is illustrated in Fig. 4 where the current in the resistance $R = 2$ ohms is found as the sum of the current in R due to e_s, namely, I_{R1}, and the current due to i_s, namely, I_{R2}. In determining I_{R1} (Fig. 4*b*), i_s is de-energized either by opening the i_s branch[1] (for purposes of analysis) or by letting $i_s = 0$ and recognizing that a current source possesses infinite internal resistance. The value of the current in resistance R due to $e_s = 23$ volts is

$$I_{R1} = \frac{e_s}{R} = \frac{23}{1+2} = \frac{23}{3} \quad \text{amperes}$$

In determining I_{R2}, the current in R due to i_s (Fig. 4*c*), e_s, is replaced by a short circuit since an ideal voltage source has zero internal resistance. Application of Kirchhoff's voltage law to the two parallel branches in Fig. 4*c* shows that

$$2I_{R2} = 1(4 - I_{R2})$$

[1] A branch is a conducting path terminated at either end by one of the network junctions or nodes.

or

$$I_{R2} = \frac{4}{3} \text{ amperes}$$

The actual current in resistance R is

$$I_R = I_{R1} + I_{R2} = \frac{23}{3} + \frac{4}{3} = 9 \text{ amperes}$$

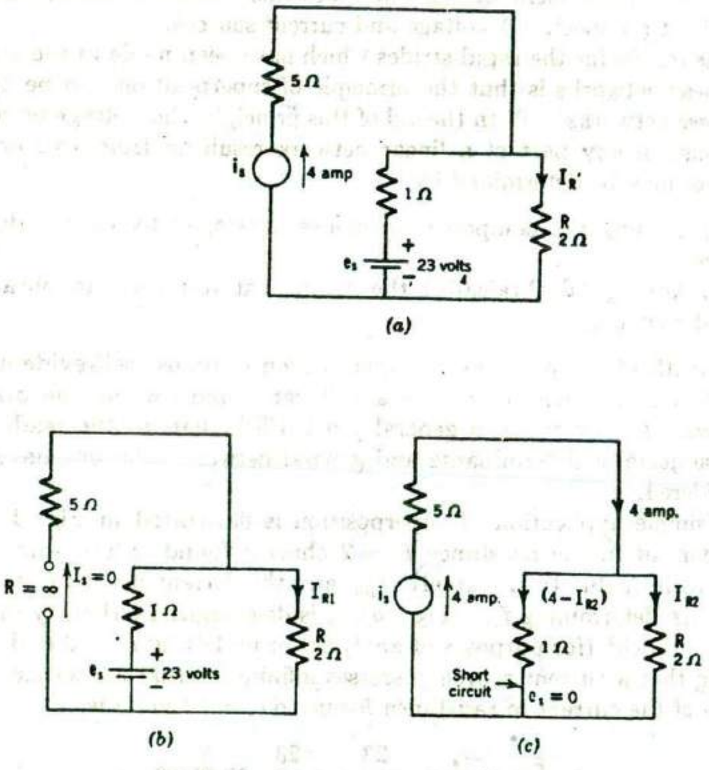


FIG. 4. An example of superposition: $I_R = I_{R1} + I_{R2}$.

The principle of superposition will be employed later in developing certain general methods of analysis where component responses due to the independent variables as well as those due to the sources are combined to establish general equilibrium equations for the network.

Network Variables. In a network consisting of b branches, there are in general $2b$ unknowns: b unknown branch currents, i_b, and b unknown branch voltages, v_b. A direct relationship exists, however, between each branch current and the associated branch voltage. Where the branches are resistive in character for example,

$$v_b = R_b i_b \quad \text{or} \quad i_b = G_b v_b \tag{10}$$

where R_b is the branch resistance and G_b is the branch conductance.

After the application of either of the volt-ampere relationships [given in equation (10)] to each of the branches there remain only b unknowns. Evaluation of these unknowns requires that b independent relationships be established. If the network has a total of n_t nodes or junctions, Kirchhoff's current law may be applied independently

$$(n_t - 1) = n \tag{11}$$

times. These n relationships together with $(b - n)$ relationships established by the application of the voltage law are sufficient in number to effect solutions for the b unknowns.

Systematized methods of network analysis ordinarily employ either linear combinations of branch currents or of branch voltages rather than the branch quantities themselves because we can write the reduced number of equations directly from the network map. Network variables which are linearly related to branch currents and branch voltages are respectively *loop currents* and *node-pair voltages*, the subjects of the following two articles.

Loop Currents. A loop current as the name implies traverses a closed path. Ordinarily the closed path is so selected that the associated loop current is a *measurable* current of the network, that is, a current which could be measured physically with the aid of an ammeter. It is not, however, essential to analysis that loop currents be measurable currents, nor is the closed path necessarily restricted to a single passage through any branch. Simple closed paths are usually easier to handle and are therefore to be preferred. The direction of the fictitious loop currents is arbitrary provided that the sense is taken care of algebraically in the summation.

In Fig. 5 are illustrated three loop currents, i_1, i_2, and i_3, together with the six branch currents i_{b1}, i_{b2}, i_{b3}, i_{b4}, i_{b5}, and i_{b6}. The linear relationship can most easily be visualized from

$$i_b = \sum i_{\text{loop}} \tag{12}$$

Any particular branch current, i_b, is the algebraic sum of the loop currents traversing this branch. Thus in Fig. 5

$$i_{b1} = i_1 \qquad i_{b5} = i_2 \qquad i_{b6} = i_3$$

$$i_{b2} = i_1 - i_3 \quad i_{b3} = i_1 - i_2 \quad i_{b4} = i_2 - i_3$$

In effect, the six branch currents have been replaced with three loop currents for the purpose of analysis. This reduction in the number of variables is accomplished at the expense of the current-law relationships. The manner in which loop currents automatically satisfy $\sum i = 0$ at the junctions is illustrated below. As applied to Fig. 5, we note that

At node ①: $i_{b1} - i_{b2} - i_{b6} = i_1 - (i_1 - i_3) - i_3 = 0$ (13)

At node ②: $i_{b2} - i_{b3} - i_{b4} = (i_1 - i_3) - (i_1 - i_2) - (i_2 - i_3) = 0$ (14)

At node ③: $i_{b4} - i_{b5} + i_{b6} = (i_2 - i_3) - i_2 + i_3 = 0$ (15)

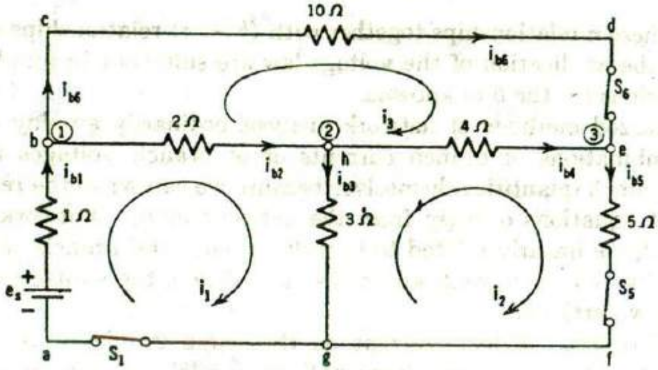


FIG. 5. Loop currents employed to replace branch currents.

In a four-junction network, $n_t = 4$, the current law can be applied independently only three times, and it will be observed from equations (13) through (15) that the loop currents automatically establish three independent relationships between the branch currents.

That loop currents can always be selected as "measurable" currents will be evident after network topology has been considered. In Fig. 5, for example, ammeters placed at the S_1, S_5, and S_6 positions would measure respectively loop currents i_1, i_2, and i_3.

Since the current-law relationships are satisfied with loop currents, the voltage-law relationships ($\sum v = 0$) must be applied $(b - n_t + 1)$ times. Obviously these $(b - n_t + 1) = (b - n)$ voltage equations must be independent relationships. One method of establishing independent voltage equations is to think of opening all loops except one and then establish the voltage law for this particular loop invoking the principle of superposition with the loop currents considered as independent variables. In other words, the sum of the voltage drops around any loop will be obtained employing one loop current at a time and then all of these voltage drops will be summed to equal zero in accordance with Kirchhoff's voltage law. As applied to the loop traversed by i_1 of Fig. 5, we think of opening switches S_5 and S_6 and sum the voltage drops occasioned by i_1 (and e_s if a source is specified). Thus

$$(1 + 2 + 3)i_1 - e_s = 0 \quad \text{or} \quad 6i_1 = e_s \tag{16}$$

The resistance of loop 1 through which i_1 flows is 6 ohms. This resistance is called the self-resistance of loop 1 to distinguish it from the mutual resistances or the resistances of loop 1 which are common to loops 2 and 3.

The voltage equation given in equation (16) does not include the voltages developed in loop 1 by loop currents i_2 and i_3. To account for the effect of i_2, we think of closing switch S_5 and observe that loop current i_2 circulates through a portion of loop 1, that is, through the 3-ohm resistance. The direction of i_2 through the 3-ohm resistance is such as to establish a voltage *rise* in loop 1 as seen from the tracing direction employed for loop 1. Taking into account the voltage rise established in loop 1 by loop current i_2, we expand equation (16) to read

$$6i_1 - 3i_2 = e_s \tag{17}$$

Next, switch S_6 is closed and the effect of loop current i_3 on the voltage equilibrium of loop 1 is observed. The current i_3 circulates through the 2-ohm resistor of loop 1 in such a direction as to produce a voltage rise in the tracing direction of loop 1. The final voltage equation for loop 1 in terms of loop currents i_1, i_2, and i_3 takes the form

$$6i_1 - 3i_2 - 2i_3 = e_s \tag{18}$$

An important aspect of equation (18) is that it can be brought into being with the aid of superposition employing elementary physical concepts. Exactly the same method may be employed to show that the voltage equation for loop 2 is

$$-3i_1 + 12i_2 - 4i_3 = 0 \tag{19}$$

and for loop 3

$$-2i_1 - 4i_2 + 16i_3 = 0 \tag{20}$$

In using superposition to establish the voltage equations, we have taken i_1, i_2, and i_3 as *independent* variables and considered their effects one at a time. Although the establishment of voltage equations with loop currents soon becomes a routine procedure, we should realize that this procedure is in effect based upon the superposition principle.

Equations (18), (19), and (20) may be solved simultaneously for i_1, i_2, and i_3. Then any particular branch current can be found from the algebraic sum of the loop currents flowing through the particular branch. That is

$$i_{\text{branch}} = \sum i_{\text{loop}}$$

Or equations (13), (14), and (15) may be employed to find i_{branch} in terms of i_{loop}, but this procedure is unnecessarily laborious.

Ordinarily, the closed paths employed in establishing the voltage-law equations coincide in contour and direction with the paths selected for the loop currents. This is a matter of convenience but not of necessity since any three independent closed paths may be employed to obtain three independent voltage equations. Independent closed paths can always be obtained by including successively a branch *not previously traversed*. Assume, for example, that paths 1 and 2 of Fig. 5 follow i_1 and i_2 respectively. A third voltage equation may be obtained by summing the voltage drops around the path *abhefga*. Thus

$$3i_1 + 9i_2 - 6i_3 = e_s \tag{21}$$

which is the sum of equations (18) and (19) and hence not independent of these equations. A third independent voltage equation may be obtained [in place of equation (20)] by summing around a closed path which includes the *cd* path or branch. If the *abcdefga* path is selected, there is obtained

$$1i_1 + 5i_2 + 10i_3 = e_s \tag{22}$$

This equation [which is the sum of equations (18), (19), and (20)] may be used in conjunction with equations (18) and (19) to find the values of i_1, i_2, and i_3.

The coefficients of the independent variables of equations (18), (19), and (20) may be arranged in an orderly fashion as shown below:

$$\begin{bmatrix} 6 & -3 & -2 \\ -3 & 12 & -4 \\ -2 & -4 & 16 \end{bmatrix} \tag{23}$$

Except for the sources, this ordered array of numbers completely characterizes the network to which it is applicable. In this type of characterization, the first column represents the coefficients of i_1; the second column represents the coefficients of i_2, and so on. An ordered array (or arrangement) of numbers or symbols is called a *matrix*. In general a matrix consists of m rows and n columns as, for example,

$$A = A_{(m,n)} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

A rather complete algebra involving matrices has been developed, but here we are concerned only with the orderly arrangement of $n \times m$ symbols or numbers which characterize a network, also with the evaluation of the determinant of the matrix. Brackets will be employed to designate matrices, whereas straight bars will be used to designate the determinant. (It is expected that the reader understands the elementary algebra of determinants including the application of Cramer's rule which is widely used in solving simultaneous equations.)

The matrix representing the coefficients of the i's in equations (18), (19), and (20) is written as indicated above in matrix (23). The determinant of this matrix is written as

$$\begin{vmatrix} 6 & -3 & -2 \\ -3 & 12 & -4 \\ -2 & -4 & 16 \end{vmatrix} = 816 \text{ ohms}^3 \tag{23a}$$

In this case the matrix is called the resistance system matrix and the determinant of this matrix has a numerical value of 816 ohms^3. If equations (18), (19), and (22) were employed, the resistance matrix would take the form

$$\begin{bmatrix} 6 & -3 & -2 \\ -3 & 12 & -4 \\ 1 & 5 & 10 \end{bmatrix} \quad \text{and} \quad \begin{vmatrix} 6 & -3 & -2 \\ -3 & 12 & -4 \\ 1 & 5 & 10 \end{vmatrix} = 816 \text{ ohms}^3 \tag{24}$$

If measurable currents are selected as the loop currents and the paths traversed in writing the voltage equations coincide with the current paths, the determinant of the resistance matrix of a network has the same numerical value regardless of the paths traversed by the loop currents. Only by selecting involved multiple loops will the network determinant differ from its base value. (For an example of what is meant here, see Problem 10 and Fig. 26b at the end of the chapter.)

If the numerical value of i_2 per unit e_s in Fig. 5 were required, it could be obtained with the aid of Cramer's rule and equations (18), (19), and (20) as

$$i_2 = \frac{\begin{vmatrix} 6 & 1 & -2 \\ -3 & 0 & -4 \\ -2 & 0 & 16 \end{vmatrix}}{816} = \frac{56}{816} = \frac{7}{102} \text{ ampere}$$

or, if equations (18), (19), and (22) are employed, as

$$i_2 = \frac{\begin{vmatrix} 6 & 1 & -2 \\ -3 & 0 & -4 \\ 1 & 1 & 10 \end{vmatrix}}{816} = \frac{56}{816} = \frac{7}{102} \quad \text{ampere}$$

If a network has l independent loop currents, the resistance matrix will have l columns; one column for each loop current. Since l equations are required to obtain a unique solution, the matrix must also have l rows. Thus $l \times l$ matrices are involved in network solutions where (as previously considered) $l = b - n$. The matrix may be written down from an inspection of the network if proper physical interpretation is given to each of the elements in the general matrix

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} & \cdots & R_{1l} \\ R_{21} & R_{22} & R_{23} & \cdots & R_{2l} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ R_{l1} & R_{l2} & R_{l3} & \cdots & R_{ll} \end{bmatrix} \tag{25}$$

The use of brackets in matrix (25) implies that *only the ordered arrangement of the coefficients* of the general voltage equations is being portrayed. The *determinant of the matrix* is indicated with straight side bars and implies that the actual value of this array is being considered.

If reasonably simple paths are selected for the loop currents, the elements of the matrix may be given such physical meanings that the numerical values of these elements can be read directly from the diagram of the network. R_{11} is the self-resistance of loop 1 through which loop current 1 flows, and in general R_{jj} is the self-resistance of loop j through which loop current j flows, and in general R_{jk} is that part of the resistance of loop j through which loop current k flows. If the closed paths selected for the establishment of the voltage relationships coincide with the paths traversed by the loop currents and if j and k are integers from 1 to l inclusive,

$$R_{jk} = R_{kj} \qquad (\text{for } j \neq k) \tag{26}$$

A situation where $R_{jk} = R_{kj}$ is given in equation (23a), and a situation where $R_{jk} \neq R_{kj}$ is given in equation (24). Ordinarily the closed paths employed to establish the voltage equations are the same as the paths traversed by measurable loop currents. Under these conditions the system matrix is symmetrical about the main diagonal. The main diagonal consists of $R_{11}, R_{22}, R_{33}, \cdots, R_{ll}$.

Example. Let it be required to find the form of the resistance matrix of the network given in Fig. 6*a* when the voltage relationships are established by following the closed paths mapped out by the designated loop currents.

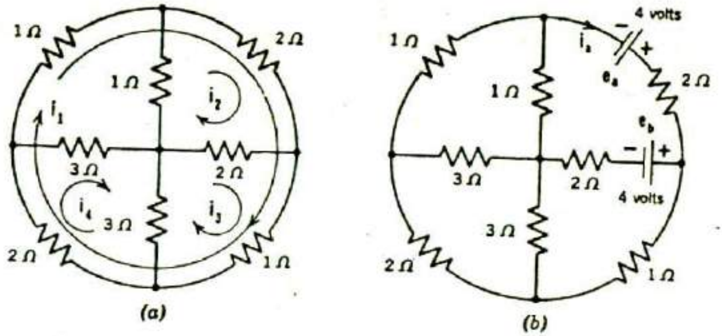


FIG. 6. Example of network analysis employing loop currents.

From an inspection of the network resistance and remembering that the sign of a resistance must be considered negative when the loop currents in the resistance are in opposite directions, we find directly that

$$
\begin{aligned}
R_{11} &= 6 \text{ ohms} \qquad & R_{12} &= R_{21} = 2 \text{ ohms}\\
R_{22} &= 5 & R_{13} &= R_{31} = 1\\
R_{33} &= 6 & R_{14} &= R_{41} = 2\\
R_{44} &= 8 & R_{23} &= R_{32} = -2\\
& & R_{24} &= R_{42} = 0\\
& & R_{34} &= R_{43} = -3
\end{aligned}
$$

The system determinant is

$$
\Delta = \begin{vmatrix} 6 & 2 & 1 & 2\\ 2 & 5 & -2 & 0\\ 1 & -2 & 6 & -3\\ 2 & 0 & -3 & 8 \end{vmatrix} = 506 \text{ ohms}^4
$$

The system determinant expressed in matrix form is simply a shorthand way of expressing the voltage relationships

$$
\begin{aligned}
6i_1 + 2i_2 + 1i_3 + 2i_4 &= 0 \qquad &\text{(loop 1)}\\
2i_1 + 5i_2 - 2i_3 + 0i_4 &= 0 &\text{(loop 2)}\\
1i_1 - 2i_2 + 6i_3 - 3i_4 &= 0 &\text{(loop 3)}\\
2i_1 + 0i_2 - 3i_3 + 8i_4 &= 0 &\text{(loop 4)}
\end{aligned}
$$

The right-hand members of these equations are zero because no voltage sources were specified in Fig. 6a. If two 4-volt sources are employed to energize the network as indicated in Fig. 6b,

$$e_{s1} = 4, \quad e_{s2} = 4 - 4 = 0, \quad e_{s3} = 4, \quad e_{s4} = 0 \text{ volts}$$

where the subscript s indicates a source voltage and the numerical subscript refers to the number of the loop to which the driving voltage is applicable. After we incorporate these driving voltages into the voltage equations given above, any or all of the loop currents may be found. Loop current i_1, for example, may be found with the aid of Cramer's rule as indicated below:

$$i_1 = \frac{\begin{vmatrix} 4 & 2 & 1 & 2 \\ 0 & 5 & -2 & 0 \\ 4 & -2 & 6 & -3 \\ 0 & 0 & -3 & 8 \end{vmatrix}}{506} = \frac{244}{506} = 0.482 \text{ ampere}$$

If the analysis requires the power delivered to the network by the source e_a, the actual branch current, i_a, flowing through e_a will have to be evaluated as

$$i_a = \sum i_{\text{loop}} = i_1 + i_2$$

$$i_2 = \frac{\begin{vmatrix} 6 & 4 & 1 & 2 \\ 2 & 0 & -2 & 0 \\ 1 & 4 & 6 & -3 \\ 2 & 0 & -3 & 8 \end{vmatrix}}{506} = \frac{40}{506} \text{ ampere}$$

$$i_a = \frac{244}{506} + \frac{40}{506} = \frac{284}{506} = 0.561 \text{ ampere}$$

The power delivered to the network by the e_a source is

$$P_a = e_a i_a = 4 \times 0.561 = 2.244 \text{ watts}$$

The voltage equations used to effect a network solution are not restricted to the equations obtained by traversing the paths mapped out by the loop currents. If in Fig. 6a, for example, we should choose to write voltage equations around the four *inside* meshes, the voltage equations (in terms of i_1, i_2, i_3, and i_4 of Fig. 6a) take the form

$$\begin{aligned} 1i_1 - 1i_2 + 0i_3 - 3i_4 &= 0 \\ 2i_1 + 5i_2 - 2i_3 + 0i_4 &= 0 \\ 1i_1 - 2i_2 + 6i_3 - 3i_4 &= 0 \\ 2i_1 + 0i_2 - 3i_3 + 8i_4 &= 0 \end{aligned}$$

The resistance matrix under these conditions takes the unsymmetrical form

$$R = \begin{bmatrix} 1 & -1 & 0 & -3 \\ 2 & 5 & -2 & 0 \\ 1 & -2 & 6 & -3 \\ 2 & 0 & -3 & 8 \end{bmatrix}$$

The determinant of this matrix, however, has the same numerical value (506 ohms^4) as the system determinant previously employed.

Node-Pair Voltages. The potential difference between any two nodes or junctions of a network is called a node-pair voltage. If properly selected, node-pair voltages may be used as the independent variables in network analysis in place of loop currents. This procedure is sometimes referred to as nodal analysis. In certain network configurations, the use of node-pair voltages has distinct advantages over the use of loop currents. The concept of node-pair voltages as network variables will be first illustrated in a particular case before any attempt is made at generalizations. To this end, we propose to determine branch voltage v_a in Fig. 7*a*, employing node-pair voltages as the independent network variables.

In nodal analysis, it is convenient to relate branch currents and branch voltages by way of branch conductance; that is

$$i_b = G_b v_b \tag{27}$$

where $G_b = 1/R_b$. Before proceeding with any analysis it is desirable to combine the simple series and parallel combinations of resistances to form a single branch conductance between nodes. The two 1-ohm resistances which are in series between nodes ① and ③ of Fig. 7*a*, for example, are combined to form a 2-ohm resistance and converted to a 0.5-mho conductance in Fig. 7*b*. It is also desirable to transform voltage sources associated with series resistance to equivalent current sources since the network solution is to be based upon current-law equations. Thus $e_{s2} = 2$ volts in series with 0.5 ohm in Fig. 7*a* is replaced with 4-ampere current generator in parallel with a 2-mho conductance as indicated in Fig. 7*b*. (See page 3.)

The correct number of node-pair voltages to employ in nodal analysis is equal to the total number of network nodes less one or

$$(n_t - 1) = n$$

The justification for this statement will become evident when it is recognized that a nodal analysis involves the establishment of current-law equations only. It will be remembered that, in a network having

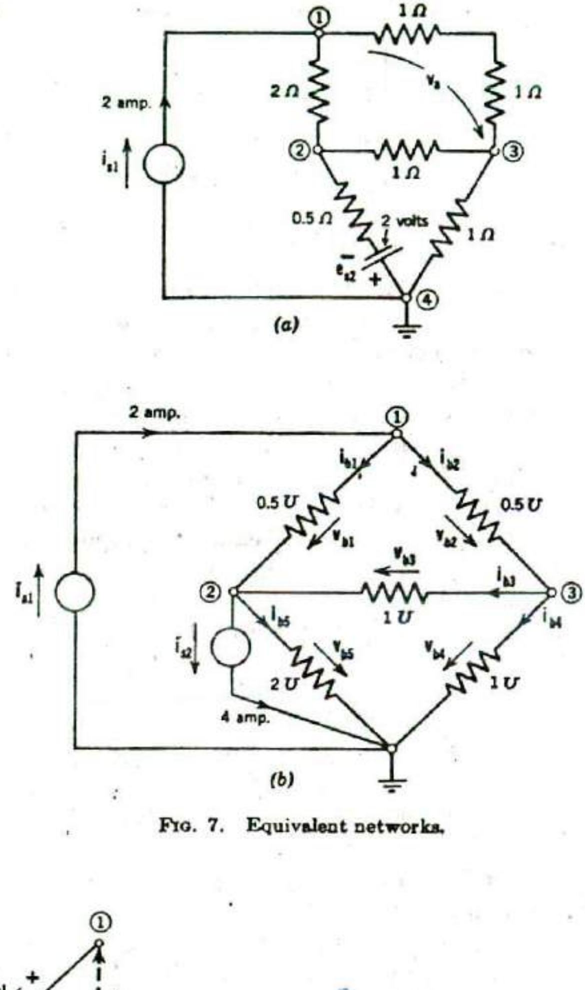


FIG. 7. Equivalent networks.

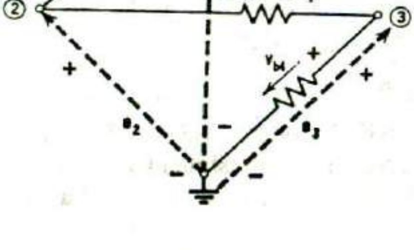


FIG. 8. Node-pair voltages e_1, e_2, and e_3 employed in the analysis of the network given in Fig. 7.

$n_t = (n + 1)$ junctions or nodes, only n independent current equations can be established. Therefore n independent node-pair voltages must be employed in the analysis.

The network of Fig. 7 has four nodes. Hence three node-pair voltages (e_1, e_2, and e_3 of Fig. 8) are selected as the independent variables upon which to base the analysis. The node-pair voltages selected must not of themselves form a closed path because in this case $(e_1 + e_2 + e_3)$ would equal zero, thus exhibiting a dependency. In Fig. 8, e_1, e_2, and e_3 are so selected that they have only one node in common. This particular selection yields independent node-pair voltages and results in certain simplifications as will become evident presently.

It will be observed from Figs. 7 and 8 that all the branch voltages can be expressed as linear combinations of ϵ_1, e_2, and e_3. The equations will be established by setting the voltage drops, represented by the v's, equal to the sum of the voltage rises, designated by the e's, when tracing from a node in the direction of the voltage drop, thence through the voltage rises back to the starting point. Thus

$$\begin{aligned} v_{b1} &= e_1 - e_2 \qquad & v_{b4} &= e_3 \\ v_{b2} &= e_1 - e_3 \qquad & v_{b5} &= e_2 \\ v_{b3} &= e_3 - e_2 \end{aligned} \tag{28}$$

Following a closed path, for example, $v_{b1} - v_{b3} - v_{b2} = (e_1 - e_2) - (e_3 - e_2) - (e_1 - e_3) = 0$. The result of using the e's as independent network variables is that the voltage-law relationships of the network have in effect been used and there remains only three current-law relationships to be established. These latter relationships may be obtained by applying Kirchhoff's current law at nodes ①, ②, and ③ of Fig. 7b. We observe first, however, that the branch currents are related to the e's as follows:

$$\begin{aligned} i_{b1} &= 0.5v_{b1} = 0.5e_1 - 0.5e_2 \\ i_{b2} &= 0.5v_{b2} = 0.5e_1 - 0.5e_3 \\ i_{b3} &= 1v_{b3} = 1e_3 - 1e_2 \\ i_{b4} &= 1v_{b4} = 1e_3 \\ i_{b5} &= 2v_{b5} = 2e_2 \end{aligned} \tag{29}$$

The current-law relationships are

At node ①: $\quad i_{b1} + i_{b2} = 1e_1 - 0.5e_2 - 0.5e_3 = i_{s1}$ (30)

At node ②: $-i_{b1} - i_{b3} + i_{b5} = -0.5e_1 + 3.5e_2 - 1e_3 = -i_{s2}$ (31)

At node ③: $-i_{b2} + i_{b3} + i_{b4} = -0.5e_1 - 1e_2 + 2.5e_3 = 0$ (32)

Since i_{s1} and i_{s2} are known quantities, the numerical values of the e's may be obtained straightforwardly, and, from the e's, the branch voltages follow directly. In the present example, we set out to determine the numerical value of $v_a = v_{b2}$ in Fig. 7.

$$v_{b2} = e_1 - e_3 = \frac{\begin{vmatrix} 2 & -0.5 & -0.5 \\ -4 & 3.5 & -1 \\ 0 & -1 & 2.5 \end{vmatrix} - \begin{vmatrix} 1 & -0.5 & 2 \\ -0.5 & 3.5 & -4 \\ -0.5 & -1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 3.5 & -1 \\ -0.5 & -1 & 2.5 \end{vmatrix}}$$

$$v_a = v_{b2} = e_1 - e_3 = \frac{8.5 - (-0.5)}{5.75} = 1.566 \text{ volts}$$

The method outlined above is elegant in its simplicity, but, with more general choices of the e's, the physical phenomena involved may become obscured. It will prove instructive to solve the problem outlined above making use of the principle of superposition. This principle has already been employed in the establishment of the voltage equations of the loop-current method of analysis. There, all loops but the pertinent one were open-circuited, and the component voltage drops around each loop were evaluated using one loop current at a time. A similar method of attack will be employed here in establishing the required number of current equations, but in this case we shall let all of the e's but the pertinent one equal zero in finding the current directed away from nodes ①, ②, and ③. In this way we shall be able to interpret the elements of the conductance matrix of a network in light of measurable conductances.

When we apply Kirchhoff's current law at each of the three marked nodes of Fig. 7b, it will be convenient to think of placing an ammeter at the pertinent node as indicated in Fig. 9. In Fig. 9a, the current directed away from node ① for a 1-volt rise of e_1 (and for $e_2 = e_3 = 0$) may be determined

$$I_{11} = 0.5e_1 + 0.5e_1 = 1e_1$$

Insofar as e_1 and i_{s1} are concerned, the current equation at node ① reads

$$\underset{\text{(away)}}{I_{11}} = 1e_1 = \underset{\text{(toward node 1)}}{i_{s1}} \tag{33}$$

This relationship, of course, does not account for the effect upon the current at node ① caused by e_2, e_3, or i_{s2}. To find the effect of e_2 upon the current at node ①, we short-circuit e_1 and e_3 as indicated in Fig. 9b and note that

$$I_{12} = -0.5e_2 \tag{34}$$

The minus sign is required since an increase in e_2 produces $0.5e_2$ ampere directed *toward* node ① and current *away from* that node has been taken as positive. (See Fig. 9a and equation 33.) Next, the effect of e_3 upon the current at node ① is observed. In making this observation, we short-circuit e_1 and e_2 as indicated in Fig. 9c and find

$$I_{13} = -0.5e_3 \tag{35}$$

The component currents of equations (33), (34), and (35) may be combined in accordance with the principle of superposition to obtain the current-law relationship which exists at node ①.

$$1e_1 - 0.5e_2 - 0.5e_3 = i_{s1} \tag{36}$$

i_{s1} is the source current directed *toward* node ① with $e_1 = e_2 = e_3 = i_{s2} = 0$. With a more general choice of the e's, the effect of i_{s2} might make a contribution to the current at node ①. Since i_{s2} is connected directly across the terminal points of e_2 in this instance and since e_2 is replaced by a short circuit for this particular evaluation, i_{s2} produces a zero component current at node ① or at the I_1 position. (It will, of course, be recognized that the ammeter connected to node ① is merely an artifice for helping us keep track of the various component currents established at this node by e_1, e_2, e_3, i_{s1}, and i_{s2}.)

In establishing the current-law relationship which exists at node ② of Fig. 7b, we make use of Figs. 9d, 9e, and 9f to obtain

$$-0.5e_1 + 3.5e_2 - 1e_3 = -i_{s2} \tag{37}$$

In a similar manner, we find that the current law applied to node ③ yields

$$-0.5e_1 - 1e_2 + 2.5e_3 = 0 \tag{38}$$

The coefficients of equations (36), (37), and (38) indicate that $G_{11} =$ 1 mho, $G_{22} = 3.5$ mhos, and $G_{33} = 2.5$ mhos. Reference to Fig. 7b will show that these conductances are precisely the conductances connected to nodes ①, ②, and ③ respectively. Further examination will show that the mutual conductances like G_{12}, G_{13}, G_{21}, G_{23}, etc., are the negatives of the connecting conductances. This method of finding the G's is widely used in cases where the e's have a common terminal as in Figs. 7 and 8.

Equations (36), (37), and (38) have resulted from the application of the principle of superposition and in this particular case are identical in

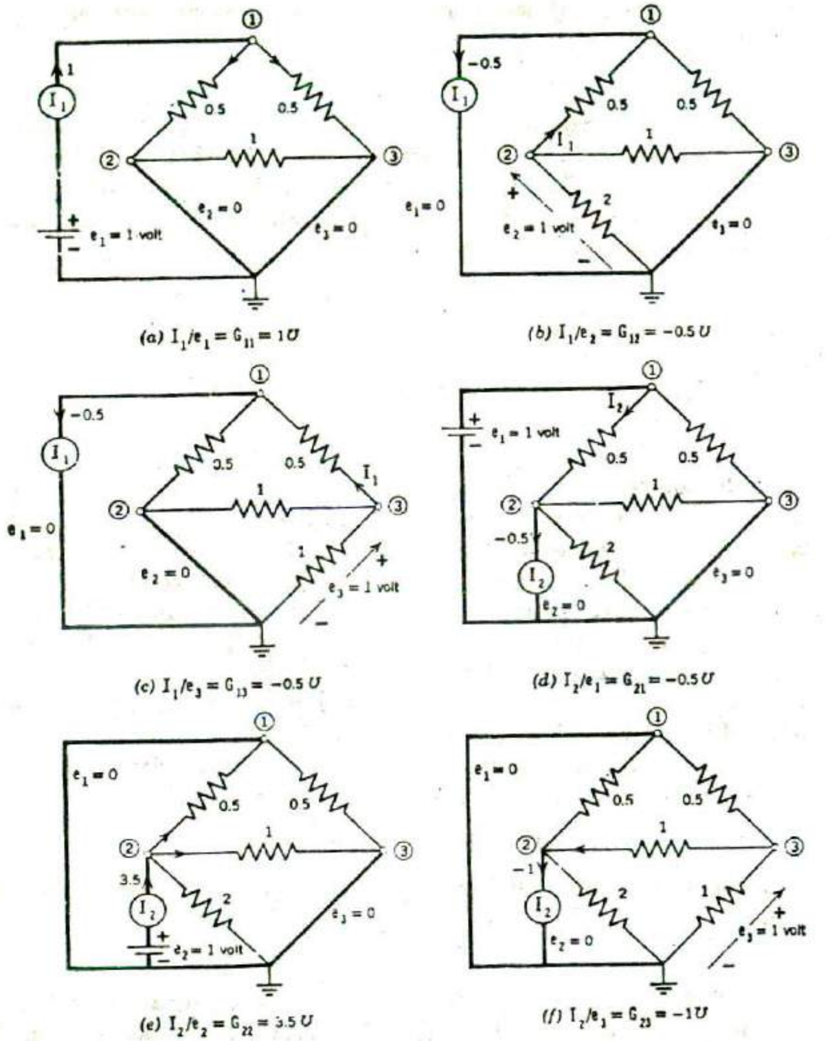


FIG. 9. Component currents at nodes ①, ②, and ③ produced by unit steps of the node pair voltages e_1, e_2, and e_3. Numbers on resistances are mhos.

form to equations (30), (31), and (32). (For a more general choice of the e's, the two sets of equations might differ in numerical form.) One advantage which accrues from the use of superposition is that the elements of the conductance matrix have values which can be measured

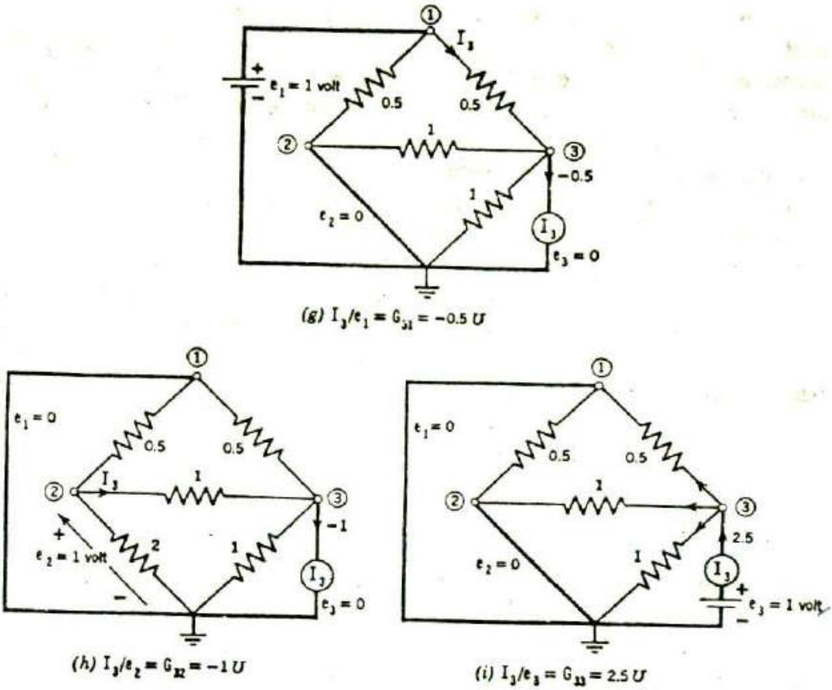


FIG. 9 (*Continued*)

with the aid of an ideal ammeter and a 1-volt source. In general, this matrix takes the form

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} & \cdots & G_{1n} \\ G_{21} & G_{22} & G_{23} & \cdots & G_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ G_{n1} & G_{n2} & G_{n3} & \cdots & G_{nn} \end{bmatrix} \tag{39}$$

If the scheme outlined in Fig. 9 is followed, the meanings of the G's are clear. For j equal to any number from 1 to n inclusive, G_{jj} is the current flowing from node j into the network per unit voltage increase in e_j. (e_j is the node-pair voltage, the arrow end of which terminates at node j.) In Fig. 9*a* ammeter I_1 is employed to measure G_{11}, in Fig. 9*e* ammeter I_2 is employed to measure G_{22}, and so on. All node-pair voltages except e_j are set equal to zero during this measurement since the scheme employed here makes use of the principle of superposition. G_{jk} is the current flowing into the network from node j per unit voltage increase in e_k; $j \neq k$. (e_k is the node-pair voltage, the arrow end of which terminates at node k.) In Fig. 9*b*, for example, ammeter I_1 is employed to measure G_{12}, the current flowing into the network per unit voltage increase in e_2 with all other independent node-pair voltages (e_1 and e_3) set equal to zero.

After the conductance matrix has been established and the source currents properly accounted for, the nodal solution is complete except for routine manipulations.

A deeper insight into the nodal method will be obtained, however, if the independent node-pair voltages selected do not have a common node. This subject will be pursued after the meaning of a *topological tree* has been established.

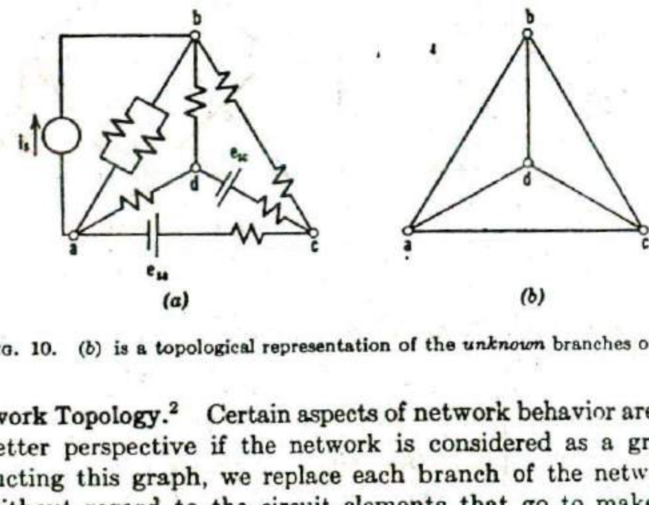


FIG. 10. (*b*) is a topological representation of the *unknown* branches of (*a*).

Network Topology.[2] Certain aspects of network behavior are brought into better perspective if the network is considered as a graph. In constructing this graph, we replace each branch of the network by a line, without regard to the circuit elements that go to make up this branch. Simple parallel elements may also be combined. The graph of the network given in Fig. 10*a*, for example, is illustrated in Fig. 10*b*. Where a branch consists solely of a current source, this branch may be omitted from the graph because it represents neither an unknown voltage nor an unknown current. For purposes of analysis, the *b* unknown branch voltages or the *b* unknown branch currents are of im-

[2] Topology, generally, is concerned with the form or structure of a geometrical entity, not with the precise size or shape of this entity. Network topology is concerned with the line graph formed by the interconnected network branches and not with the size, shape, or operating characteristics of the network elements that go to form the branches. In this sense, network topology is network geometry.

mediate importance. The known sources may be incorporated into the equilibrium equations at any appropriate stage of the analysis.

The network graph illustrated in Fig. 10*b* has four nodes, six branches, and three *inside loops* or *meshes* and is mappable on a plane. The graph may be considered as separating the entire area of the plane into four bounded areas, the three *inside* meshes and the *outside* area or *outside* mesh. In this connection, any undivided area having a boundary composed of branch lines is called a mesh. Since the outside area has such a boundary, it can be classed as a mesh. When the graph is mapped on a sphere, any one of the inside meshes of a plane graph like Fig. 10*b* can become the *outside* mesh. The process whereby this is accomplished is called topological warping.

A network solution based on loop currents requires that the correct number of *independent voltage equations* be employed. If based on node-pair voltages, the solution requires that the correct number of *independent current equations* be employed. In simple networks, *independent* equations can be obtained readily by inspection or by methods previously considered. Certain general aspects of this problem can be brought to light by the use of a topological *tree*.

A *tree* is a set of branches such that each node (or terminal) has connected to it at least one branch, the set contains no closed loops, and a single (unique) path can be found which joins any two nodes of the graph to which the tree is applicable.

Four open-ended graphs based on the circuit configuration of Fig. 10 are presented in Fig. 11. Since each of these open-ended graphs satisfies all the requirements of a tree, each graph is a tree corresponding to the network of Fig. 10.

In forming a tree (corresponding to a particular network) certain branches are of necessity opened. The branches thus opened are called links or link branches. The *links* of Fig. 11*a*, for example, are branches *ab*, *bc*, and *ca* and of Fig. 11*b* are *ab*, *dc*, and *da*. Obviously, the link branches and the tree branches combine to form the graph of the entire network.

The identification of the link currents with the loop currents leads directly to *measurable* loop currents. In cases where interest centers around particular currents, as, for example, around the input and output currents of the network, the input and output branches may be selected as links, the reason being that only one loop current traverses a link branch. Only one loop current is then required to obtain the current in this branch. Thus the tree may actually be selected with ulterior motives of this kind in mind.

Once a topological tree has been formed for a particular network, the determination of *independent* loop currents is a straightforward procedure. Simply close one link as, for example, link ab of Fig. 11a, and employ the loop thus formed as the path for loop current number 1. In this case

$$I_{\text{loop}} = I_{\text{link}} = I_{abda} = I_1$$

Then *open this link* and close another link to obtain the path of a second loop current and repeat this process until each link-branch current has been identified with a loop current. Thus we obtain loops for which

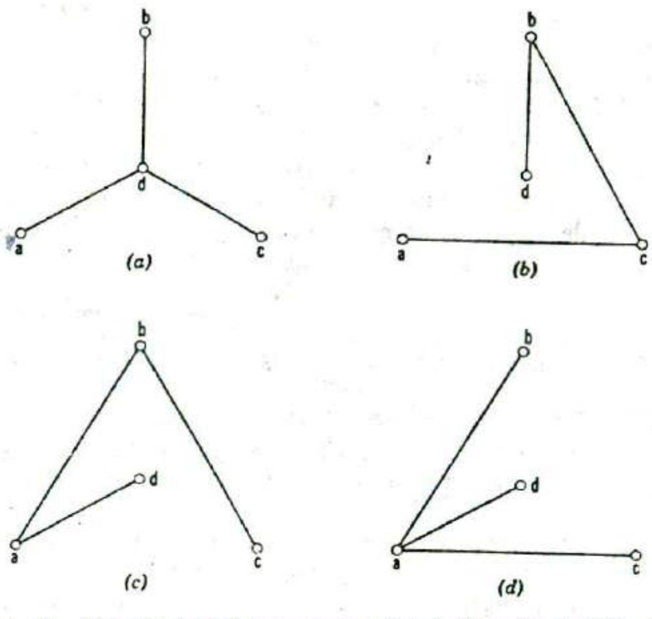


FIG. 11. Four topological trees corresponding to the network of Fig. 10.

independent voltage equations can be written. The loop currents are independent inasmuch as each can be measured with an ammeter in a different link branch. The correct number of independent loop currents is obtained since all the loop currents thus selected are required to obtain a network solution, and more than this number of loop currents will lead to voltage equations which are not independent of those already established.

The independent node-pair voltages required to effect a nodal solution can also be found readily from a topological tree. In elementary nodal analysis we ordinarily select one node of the network to be common to each of the node-pair voltages employed. In Fig. 11*a*, for example, we might select node *d* as common and use

$$e_1 = v_{ad}, \quad e_2 = v_{bd}, \quad \text{and} \quad e_3 = v_{cd}$$

as the three independent node-pair voltages required to effect a solution. Or we might select node *c* as common and use

$$e_1 = v_{ac}, \quad e_2 = v_{bc}, \quad \text{and} \quad e_3 = v_{dc}$$

as the required node-pair voltages. It should be noted that this method of selecting node-pair voltages automatically leads to $(n_t - 1)$ or n voltages, the correct number required to obtain a network solution. The *independence* of the node-pair voltages thus selected follows from:

1. One path only exists between the common node and any other node by way of tree branches.
2. The nodes are separated in potential one from the other by at least the potential difference of one tree branch.

One advantage of the topological approach to circuit analysis is that it opens up avenues of attack that might otherwise be overlooked. For example, the tree-branch voltages themselves form an independent set of node-pair voltages that can be used in a nodal analysis to effect a network solution. There are n_t nodes and, except for the first tree

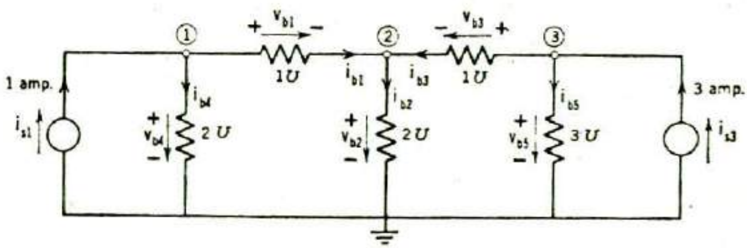


FIG. 12. For illustrative example.

branch (which will be considered to have two nodes incident upon it), every other tree branch utilizes one additional node in its specification. Thus n tree branches exist in a given tree, and hence n independent node-pair voltages can be obtained directly from the tree-branch voltages. The one requirement in selecting a set of node-pair voltages with which to carry out a network analysis is that these node-pair voltages correspond to the node-pair voltages of a topological tree.

In order to further illustrate the nodal method, the network given in Fig. 12 will be analyzed in three different ways employing node-pair voltages. First the e's of the tree shown in Fig. 13a will be taken as the independent node-pair voltages. Where a common node is employed, the self-conductances and mutual conductances may be obtained directly from an inspection of the network. Thus in mhos

$$\begin{array}{lll} G_{11} = 3 & G_{12} = -1 & G_{13} = 0 \\ G_{21} = -1 & G_{22} = 4 & G_{23} = -1 \\ G_{31} = 0 & G_{23} = -1 & G_{33} = 4 \quad \text{mhos} \end{array}$$

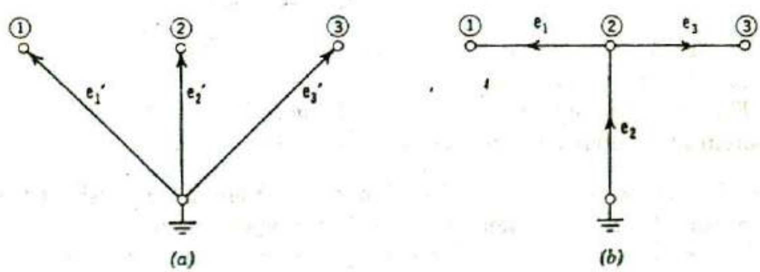


FIG. 13. Two trees corresponding to the network of Fig. 12.

Let it be required to find the voltages of nodes ① and ② relative to ground.

$$\text{Potential of node ①} = e_1' = \frac{\begin{vmatrix} 1 & -1 & 0 \\ 0 & 4 & -1 \\ 3 & -1 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{vmatrix}} = \frac{18}{41} \quad \text{volt}$$

$$\text{Potential of node ②} = e_2' = \frac{\begin{vmatrix} 3 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 3 & 4 \end{vmatrix}}{41} = \frac{13}{41} \quad \text{volt}$$

If the tree given in Fig. 13*b* is used in the analysis, it is found that

$$\left.\begin{aligned} v_{b1} &= e_1 & i_{b1} &= 1e_1 \\ v_{b2} &= e_2 & i_{b2} &= 2e_2 \\ v_{b3} &= e_3 & i_{b3} &= 1e_3 \\ v_{b4} &= e_1 + e_2 & i_{b4} &= 2e_1 + 2e_2 \\ v_{b5} &= e_2 + e_3 & i_{b5} &= 3e_2 + 3e_3 \end{aligned}\right\} \tag{40}$$

At node ①: $$i_{b1} + i_{b4} = 3e_1 + 2e_2 + 0e_3 = 1 \tag{41}$$

At node ②: $$-i_{b1} + i_{b2} - i_{b3} = -1e_1 + 2e_2 - 1e_3 = 0 \tag{42}$$

At node ③: $$i_{b3} + i_{b5} = 0e_1 + 3e_2 + 4e_3 = 3 \tag{43}$$

Solving for e_1 and e_2

$$e_1 = \frac{\begin{vmatrix} 1 & 2 & 0 \\ 0 & 2 & -1 \\ 3 & 3 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & 2 & 0 \\ -1 & 2 & -1 \\ 0 & 3 & 4 \end{vmatrix}} = \frac{5}{41} \quad \text{volt}$$

$$e_2 = \frac{\begin{vmatrix} 3 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 3 & 4 \end{vmatrix}}{41} = \frac{13}{41} \quad \text{volt}$$

The potential of node ① of Fig. 12 relative to ground is

$$v_{b4} = e_1 + e_2 = \frac{18}{41} \quad \text{volt}$$

If the node-pair voltages e_1, e_2, and e_3 of Fig. 13*b* are employed in conjuuction with the principle of superposition, the self-conductances and mutual conductances are determined from the physical considerations outlined in Fig. 14. If it is recognized that $G_{21} = G_{12}$, $G_{31} = G_{13}$, $G_{32} = G_{23}$, and $G_{33} = 4$ mhos, the conductance matrix becomes

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 7 & 3 \\ 0 & 3 & 4 \end{bmatrix} \tag{44}$$

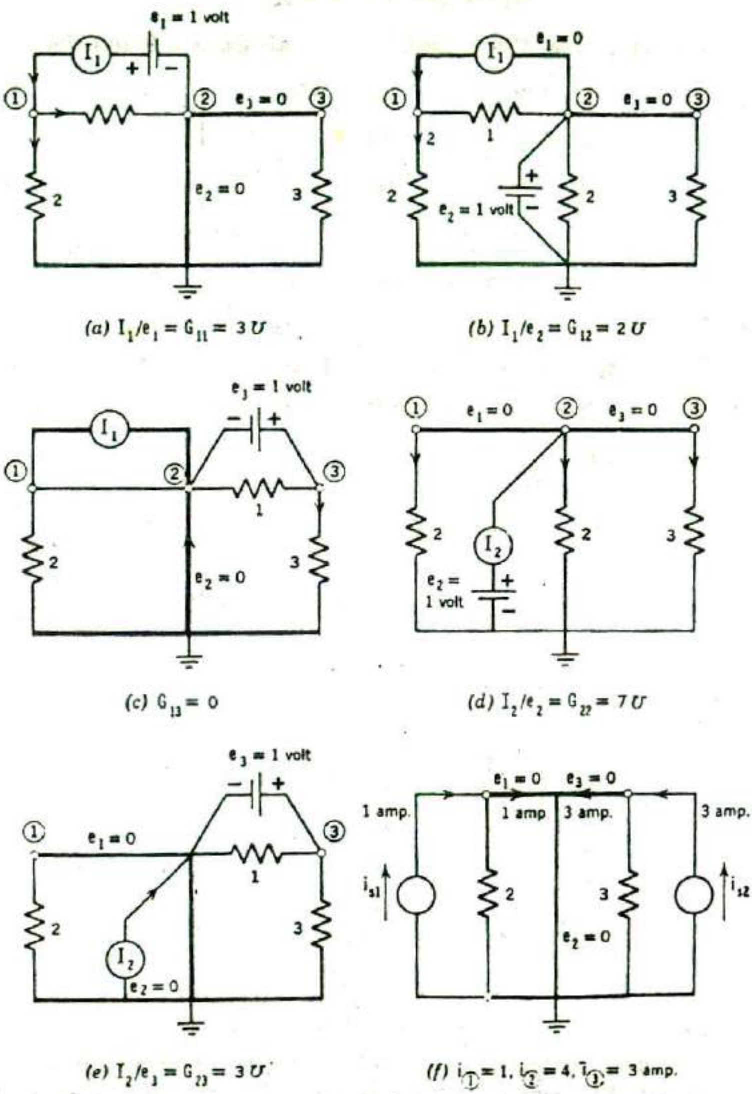


FIG. 14. Evaluation of self-conductance and mutual conductance of the network of Fig. 12 employing the node-pair voltages e_1, e_2, and e_3 of Fig. 13b.

Since the law of superposition is being employed in the establishment of the current equations at nodes ①, ②, and ③, it is necessary to include the currents directed toward these nodes from all the current sources with $e_1 = e_2 = e_3 = 0$. From Fig. 14f, we find that 1 ampere is directed toward node ① from the sources, 4 amperes are directed toward node ② from the sources, and 3 amperes toward node ③. Since the component currents resulting from e_1, e_2, and e_3 have been taken as positive away from the nodes, the three current equations may be written as follows:

$$3e_1 + 2e_2 + 0e_3 \doteq 1 \tag{45}$$

$$2e_1 + 7e_2 + 3e_3 = 4 \tag{46}$$

$$0e_1 + 3e_2 + 4e_3 = 3 \tag{47}$$

Solving for e_1 and e_2

$$e_1 = \frac{\begin{vmatrix} 1 & 2 & 0 \\ 4 & 7 & 3 \\ 3 & 3 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & 2 & 0 \\ 2 & 7 & 3 \\ 0 & 3 & 4 \end{vmatrix}} = \frac{5}{41} \quad \text{volt}$$

$$e_2 = \frac{\begin{vmatrix} 3 & 1 & 0 \\ 2 & 4 & 3 \\ 0 & 3 & 4 \end{vmatrix}}{41} = \frac{13}{41} \quad \text{volt}$$

From the three examples outlined above (and from others that can be developed) it is evident that node-pair voltages may be used in a variety of ways to effect network solutions. The same may be said for the use of loop currents. Ingenious combinations of node-pair voltages and loop currents as well as ingenious network theorems are often employed to obtain desired solutions. One of the fascinating aspects of network analysis is the variety of attack available to the analyst.

Duality. Where circuit elements are in series as in Fig. 15*a*, the natural choice for independent variable is current since it is common to each element. For the case considered

$$R_1 i_b + R_2 i_b + R_3 i_b = v_b \tag{48}$$

or

$$R_b = (R_1 + R_2 + R_3) = \frac{v_b}{i_b} \tag{48a}$$

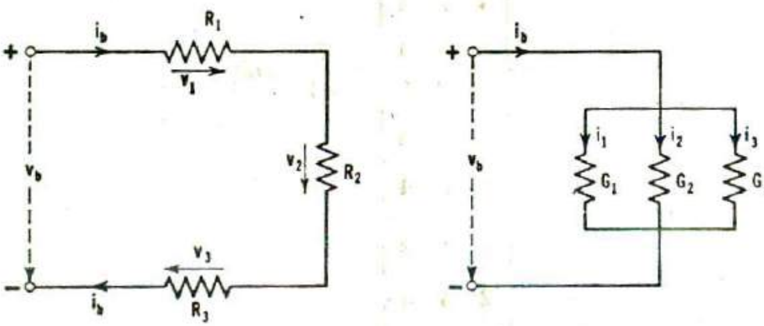
Where elements are in parallel as in Fig. 15*b*, the natural choice for independent variable is the voltage which is common to (or across) each of the elements. In Fig. 15*b*

$$G_1v_b + G_2v_b + G_3v_b = i_b \tag{49}$$

or

$$G_b = (G_1 + G_2 + G_3) = \frac{i_b}{v_b} \tag{49a}$$

The similarity in form of equations (48) and (49) is evident. In one, the voltage law is used to establish the basic relationship between i_b and v_b; in the other, the current law is employed. In one, resistances are used; in the other, conductances.



(*a*) (*b*)

FIG. 15. $v_b = v_1 + v_2 + v_3$ and $i_b = i_1 + i_2 + i_3$.

This dualism extends throughout the two fundamental methods of network analysis. One method utilizes loop currents, resistances, voltage equations, and voltage sources. The other utilizes node-pair voltages, conductances, current equations, and current sources. Measurable independent loop currents may be identified with the current flowing in the link branches of the network whereas measurable independent node-pair voltages may be identified with the tree-branch voltages. The equilibrium equations in one method of analysis are based upon

$$\sum v_{\text{around a closed loop}} = 0$$

The equilibrium equations of the other are based upon

$$\sum i_{\text{directed toward a node}} = 0$$

Wherever all the elements of one system can be put into a one-to-one correspondence with the elements of another system, the correspondence is referred to as *duality*. Duality can therefore exist between the loop-current and node-pair-voltage methods of analysis, one method being the dual of the other. From an algebraic point of view, two networks are *duals* if the nodal equations of the one are of the same form as the loop equations of the other. The equations of equilibrium for the network of Fig. 16*a* which has two independent loop currents are, for example,

$$\begin{aligned} R_{11}i_1 + R_{12}i_2 &= e_{s1} \\ R_{21}i_1 + R_{22}i_2 &= e_{s2} \end{aligned} \tag{50}$$

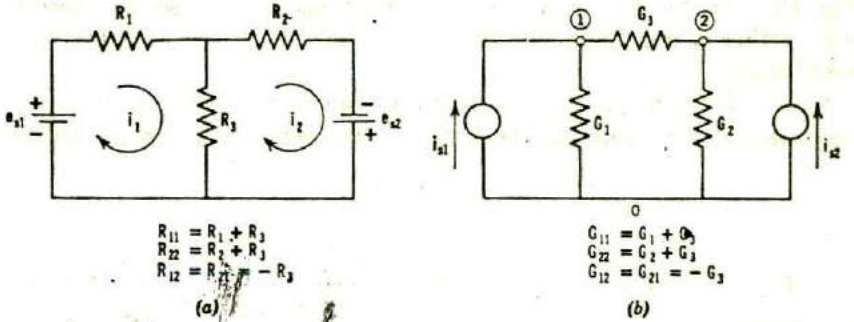


FIG. 16. Dual networks.

The equations of equilibrium for the network of Fig. 16*b* which has two independent node-pair voltages are of the form

$$\begin{aligned} G_{11}e_1 + G_{12}e_2 &= i_{s1} \\ G_{21}e_1 + G_{22}e_2 &= i_{s2} \end{aligned} \tag{51}$$

Except for the interpretations given to the symbols in equations (50) and (51), these equations are identical. The fact that the forms of the equations are identical makes them duals. Obviously, duality is a mutual relationship. Equations (50) are as much the dual of equations (51) as equations (51) are the dual of equations (50).

From a graphical point of view, two networks are duals when meshes (around which $\sum v = 0$) in one network are in a one-to-one correspondence with the nodes (at which $\sum i = 0$) in the other network. In this connection a mesh is regarded as a region or area bounded by network branches or voltage drops. With this interpretation of the term mesh, *a network branch divides exactly two meshes* (or regior provided that the network graph can be mapped on a plane or (without cross-overs). Correspondingly, *a network branch* *two nodes*. It will be remembered that a network posse nodes at which independent current relationships can be established. The dual of this network will possess $(m_t - 1) = l$ meshes or loops around which independent voltage relationships can be established. (m_t symbolizes the total number of meshes or regions of a particular graph.) The graph of Fig. 16*a*, for example, is composed of three meshes, *two inside meshes* around which i_1 and i_2 circulate and *one outside mesh* (or region) bounded by the $e_{s1} - R_1$ and $R_2 - e_{s2}$ branches. The outside region is, of course, as much a mesh as either of the inside regions since it is bounded by network branches. Furthermore, if Fig. 16*a* were mapped on a sphere and topologically warped (by stretching), either of the present inside meshes could be made to be the "outside" mesh.

Some of the major correspondences which exist between the loop-current and node-pair-voltage solutions are listed in Table I. Others

TABLE I

Element Involved	Solution: Loop Current	Solution: Node-Pair Voltage
Equilibrium equations	voltage $\sum v = 0$	current $\sum i = 0$
Number of independent equations	$b - n = l$	$(b - l) = n$
Basic constituent	branch voltage	branch current
Energizing element	voltage source	current source
Network variable	loop current	node-pair voltage
Independent-network variables	link-branch currents	tree-branch voltages
Circuit parameter	resistance	conductance
Parameters add	in series	in parallel
Infinite parameter	$R = \infty$ (open circuit)	$G = \infty$ (short circuit)
Zero parameter	$R = 0$ (short circuit)	$G = 0$ (open circuit)
Topographical entity	mesh	node
Any topological branch	*divides* exactly two regions (or meshes) providing the graph will map on a plane (or sphere)	*joins* exactly two nodes

will become evident as we proceed. It should be noted that, whereas $l = n$ in dual networks, l is not necessarily equal to n in a particular network.

Graphical Construction of Dual Networks. In constructing a network which is to be the dual of a specified network, all of the voltage drops encountered on the contour of a mesh of one network are transformed current paths emanating from the corresponding node of the other, or vice versa. A simple graphical scheme for developing the correspondence between $\sum v = 0$ in one network with $\sum i = 0$ in the other is depicted in Fig. 17c. The original network in this case is Fig. 17a

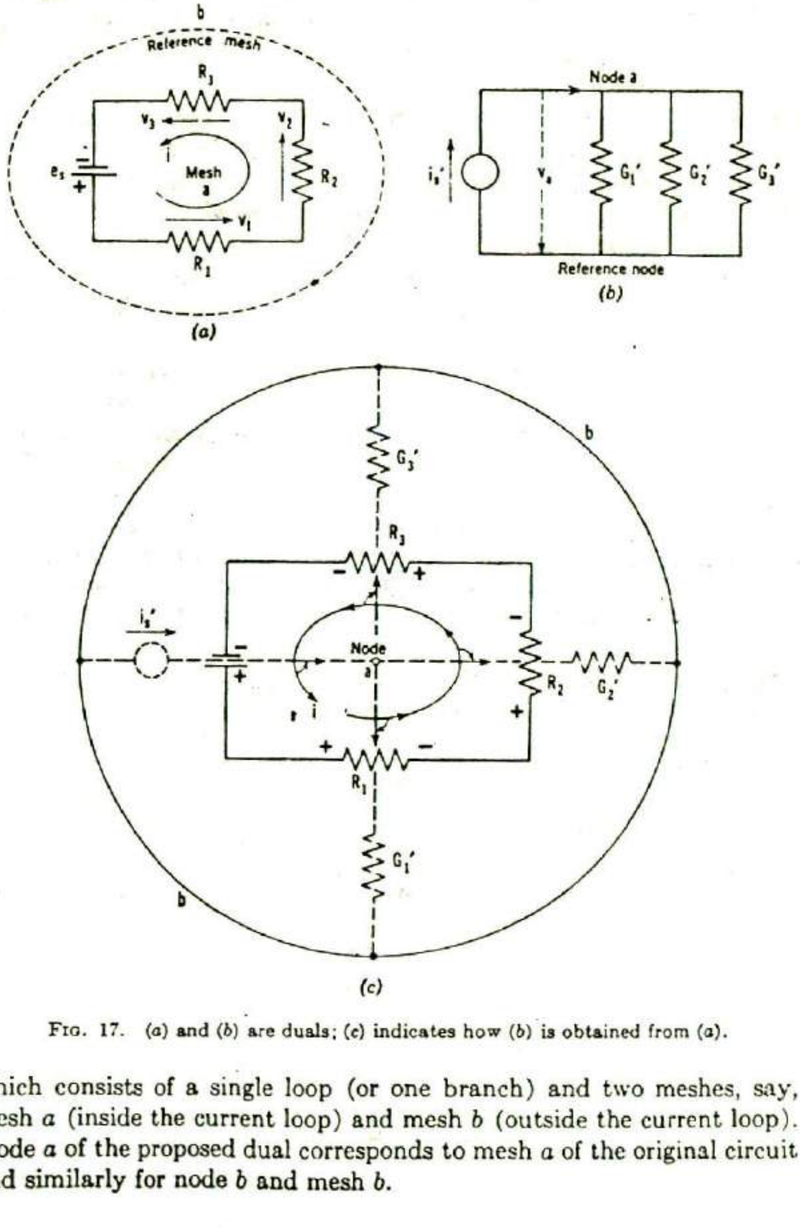


FIG. 17. (a) and (b) are duals; (c) indicates how (b) is obtained from (a).

which consists of a single loop (or one branch) and two meshes, say, mesh a (inside the current loop) and mesh b (outside the current loop). Node a of the proposed dual corresponds to mesh a of the original circuit and similarly for node b and mesh b.

The details involved in the graphical construction of a dual are illustrated in Fig. 17*c*. From node *a* in Fig. 17*c*, for example, a line is so drawn that it connects node *a* and the reference node as it passes through one element (or voltage drop) of the original loop. This process is repeated for each voltage drop involved in $\sum v = 0$ with due regard for positive direction. Some orderly convention must be adopted for correlating positive directions in the dual with those which have been selected for analysis of the original. The simple scheme illustrated in Fig. 17*c* consists of turning the arrow direction of the loop current (as it crosses the line connecting node *a* and the reference node) in the direction we select for positive current flow in the branch of the dual which is being generated. For the case considered, the positive direction of current flow is selected as being *from* node *a to* the reference node. Thus the loop-current direction in being turned clockwise for each of the three voltage drops (v_1, v_2, and v_3 of Fig. 17*a*) determines the positive direction of the current flow in the three corresponding paths of the dual as being *from node a to the reference node*. In applying this scheme to the voltage source e_s, we note that the loop-current direction coincides with a voltage rise as it passes through e_s. The positive direction of the current source, i_s, in the dual which replaces e_s of the original network is therefore obtained by turning the loop-current direction in the counter*clockwise* direction. The positive direction of i_s' is thus determined to be *from* the reference node *to* node *a* as indicated in Fig. 17*b* or Fig. 17*c*. (Any other scheme for determining positive circuit directions in the dual is as good provided that it is used consistently.)

The numerical values of the mhos in the dual network are related to the ohmic values in the original network by the normalizing factor g_n^2. Thus

$$G_j' = g_n^2 R_j \tag{52}$$

where g_n is arbitrarily selected.

A current source, i_s', of the dual network is made to correspond to a voltage source of the original network by a normalizing factor g_n if the power delivered by i_s' is to be equal to the power delivered by e_s. Thus for P_{es} to equal P_{is}

$$P_{es} = \frac{e_s^2}{R} = \frac{i_s'^2}{G'} = P_{is}' \tag{53}$$

from which

$$\frac{i_s'}{e_s} = \sqrt{\frac{G'}{R}} = g_n \tag{54}$$

If, for example, in Fig. 17, $R_1 = 2$, $R_2 = 1$, and $R_3 = 3$ ohms and $e_s = 12$ volts, the equation for equilibrium is

$$2i + 1i + 3i = 12 \text{ volts} \qquad (i = 2 \text{ amperes})$$

If a normalizing factor, g_n^2, of 4 is arbitrarily selected,

$$G_1' = 8, \quad G_2' = 4, \quad \text{and} \quad G_3' = 12 \text{ mhos}$$

Also $i_s' = (2 \times 12)$ amperes and the equation for equilibrium of the dual network is

$$8v_a + 4v_a + 12v_a = 24 \text{ amperes} \qquad (v_a = 1 \text{ volt})$$

In Fig. 17*a*

$$P_{es} = 12 \times 2 = 24 \text{ watts}$$

In Fig. 17*b*

$$P_{is}' = 1 \times 24 = 24 \text{ watts}$$

The graphical process illustrated in Fig. 17 is extended to a four-mesh network in Fig. 18. It will be observed that all of the elements common to loop 1 of Fig. 18*a* appear as elements which are common to node ① of the dual network; similarly for the other loops and corresponding nodes. The dual network contains the same number of branches as the original network if the three parallel paths which connect to node ① (and which are derived from a single series branch of the original network) are counted as a single branch. It is, of course, evident that for algebraic duality l (the number of independent loop currents) of one network must equal n (the number of independent nodes) of the other. For $l = n$

$$m_t = l + 1 = n + 1 = n_t$$

where m_t is the total number of meshes and n_t is the total number of nodes.

The manner in which the graphical process described above may be reversed is illustrated in Fig. 19. Here the dual of a dual is constructed to obtain the original network. (See Fig. 18.) Since duality is a mutual relationship, the construction of a dual goes *from* meshes *to* nodes (if the original network is viewed as consisting of meshes) or *from* nodes *to* meshes (if the original network is viewed in light of nodes as the topological entities). An example of the latter situation is given in Fig. 19*a*. Each current directed away from node ① corresponds to a voltage drop in mesh 1 of Fig. 19*b*; similarly for the other corresponding nodes and meshes.

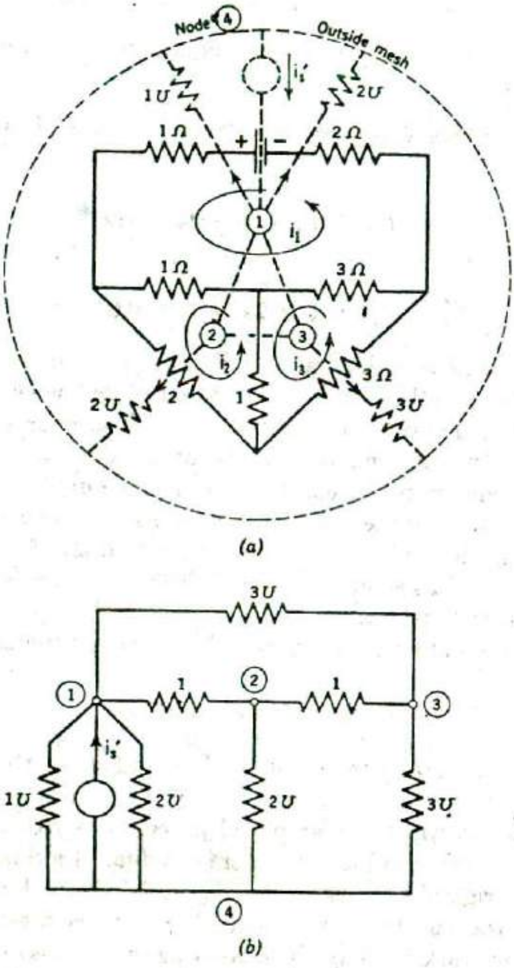


FIG. 18. (a) Original network. (b) Dual network, $g_n = 1$.

A qualification has previously been made that, if a geometrical dual of a network is to be constructed, the graph of the original network must be mappable on a plane or sphere. The reason for this qualification is that the construction requires the network branches to be so oriented

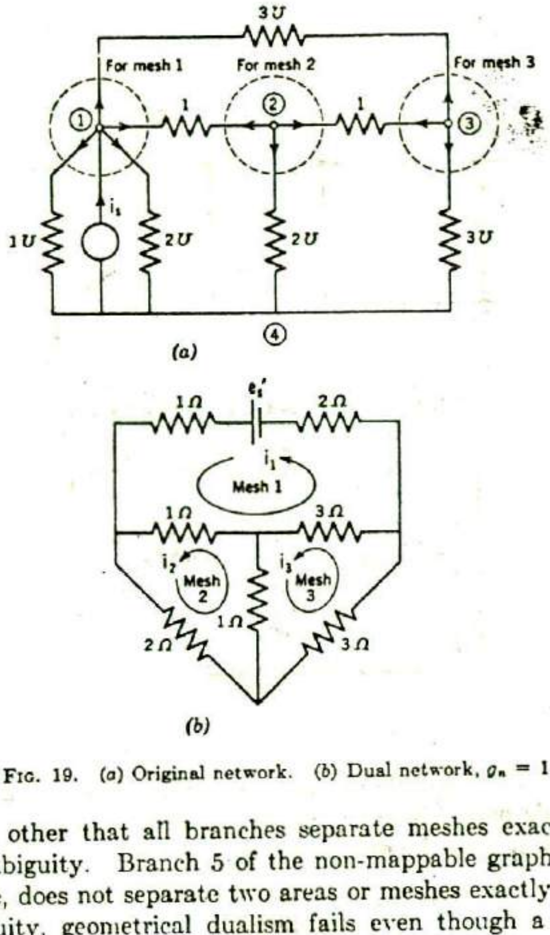


FIG. 19. (a) Original network. (b) Dual network, $g_n = 1$.

one to the other that all branches separate meshes exactly, that is, without ambiguity. Branch 5 of the non-mappable graph of Fig. 20b, for example, does not separate two areas or meshes exactly. Owing to this ambiguity, geometrical dualism fails even though a dual set of equilibrium equations may be established. If, for example, the numbers on the graphs of Fig. 20 refer to ohms resistance, the three equilibrium equations for either network are

$$\begin{aligned} 7i_1 - 2i_2 - 3i_3 &= 0 \\ -2i_1 + 11i_2 - 5i_3 &= 0 \\ -3i_1 - 5i_2 + 14i_3 &= 0 \end{aligned} \tag{55}$$

A dual set of equations may be written as

$$\begin{aligned} 7e_1 - 2e_2 - 3e_3 &= 0 \\ -2e_1 + 11e_2 - 5e_3 &= 0 \\ -3e_1 - 5e_2 + 14e_3 &= 0 \end{aligned} \tag{56}$$

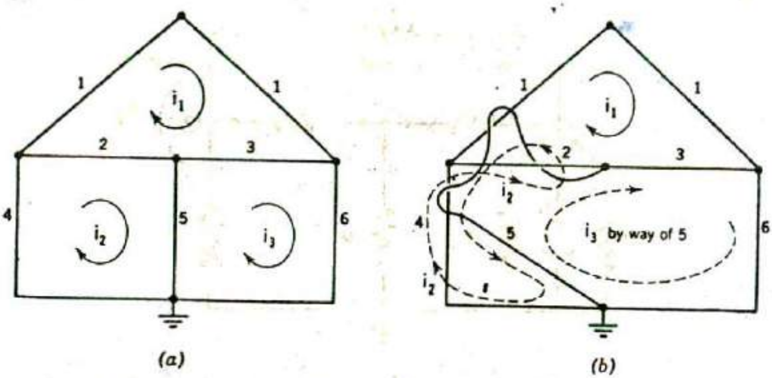


FIG. 20. (a) A mappable graph. (b) A non-mappable graph.

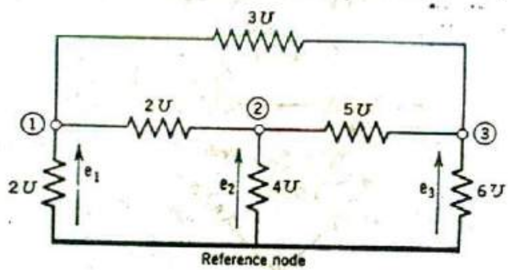


FIG. 21. Dual of Fig. 20a. (Prob. 19.)

A network to which equations (56) are applicable is given in Fig. 21, where the node-pair voltages e_1, e_2, and e_3 are the voltages of nodes ①, ②, and ③ relative to the reference node.

PROBLEMS

1. A three-branch network is given in Fig. 22 where the branch voltages are

$$v_{b1} = (-2 + 3i_{b1}) \qquad v_{b2} = (-4 + 2i_{b2}) \qquad v_{b3} = 2i_{b3}$$

(a) Write the required number of current and voltage equations (to effect a network solution) employing i_{b1}, i_{b2}, and i_{b3} as independent variables, and evaluate i_{b2} therefrom.

(b) Write two voltage equations employing loop currents i_1 and i_2 as independent

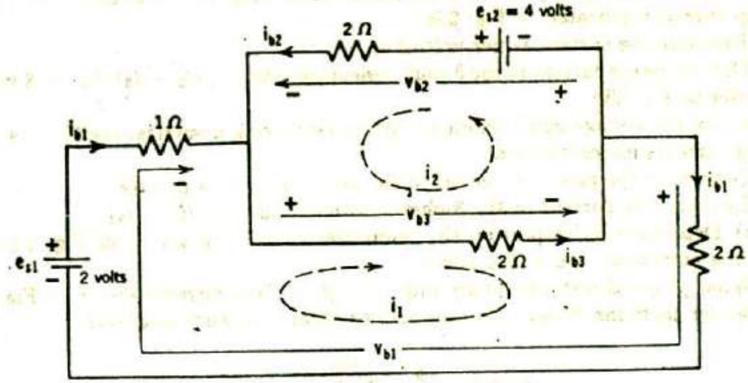


FIG. 22. Prob. 1.

variables starting with

$$3i_{b1} + 2i_{b3} = 2 \qquad 2i_{b2} + 2i_{b3} = 4$$

Evaluate i_{b3} as $(i_1 + i_2)$.

2. In Fig. 20*a*, page 38, is given a six-branch network where the numbers alongside the branches indicate ohms of resistance as well as the designations of the branches. Thus

$$R_{b1} = 2, \quad R_{b2} = 2, \quad R_{b3} = 3, \quad R_{b4} = 4, \quad R_{b5} = 5, \quad R_{b6} = 6 \text{ ohms}$$

The energizing sources are not shown in Fig. 20, the assumption being that any one or all of the branches may have voltage sources in series with the branch resistances.

Write three voltage equations employing the loop currents i_1, i_2, and i_3 as the independent variables. Let the source voltages in loop 1 be $E_1 = e_{s1} + e_{s2} + e_{s3}$; the source voltages in loop 2 be $E_2 = e_{s4} + e_{s5} - e_{s2}$; and the source voltages in loop 3 be $E_3 = e_{s6} - e_{s5} - e_{s3}$.

Note: With only a very little practice, voltage equations of this kind can be written down directly from an inspection of the network by mental applications of the principle of superposition.

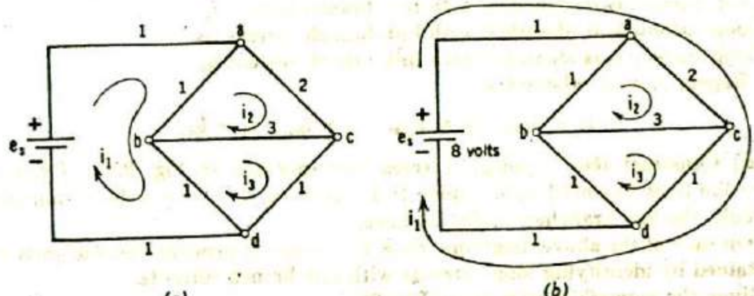


FIG. 23. The numerical values placed alongside the branches (or parts of branches) refer to ohms resistance.

3. Refer to Fig. 23*a*.

(*a*) Determine by inspection the numerical values of b and n_t, and specify n and l numerically.

(b) Write the voltage equilibrium equations employing numerical coefficients and the loop currents indicated in Fig. 23a.

(c) Evaluate the current i_1 per volt of e_s.

(d) Determine the current in the 3-ohm resistance, namely, $(i_2 - i_3)$ if $e_s = 8$ volts.

4. Refer to Fig. 23b.

(a) Write the voltage equilibrium equations employing numerical coefficients and the loop currents indicated there.

(b) Determine the power delivered to the network by $e_s = 8$ volts.

(c) Evaluate the current in the 3-ohm resistance, namely, $(i_2 - i_3)$.

5. (a) Determine by inspection the numerical values of b and n_t of Fig. 24 and specify the numerical values of n and l.

(b) What physical restrictions are imposed by the loop currents shown in Fig. 24 which render them insufficient (in number) to effect a network solution?

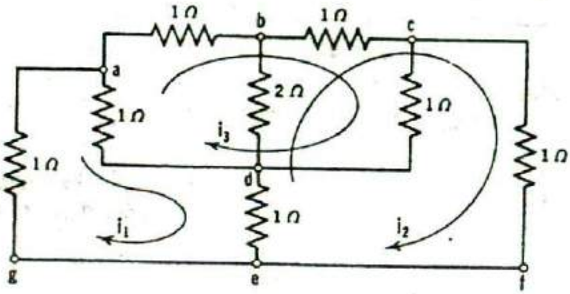


FIG. 24. Problems 5, 6, and 18.

(c) What is the correct numerical value of the resistance determinant of the network employing measurable currents as loop currents? By *resistance determinant* is meant the determinant of the resistance matrix which characterizes the network.

6. (a) Construct a topological tree corresponding to the network shown in Fig. 24 such that

1. Loop current i_1 is identified with link-branch current i_{ega}.
2. Loop current i_2 is identified with link-branch current i_{cfe}.
3. Loop current i_3 is identified with link-branch current i_{ab}.
4. Loop current i_4 is identified with link-branch current i_{bc}.

(b) Repeat part (a) above for

$$i_1 = i_{ega}, \quad i_2 = i_{cfe}, \quad i_3 = i_{bd}, \quad i_4 = i_{bc}$$

7. (a) Construct four topological trees corresponding to Fig. 23a. Draw the tree in solid lines (oriented with respect to the nodes a, b, c, d) and the remainder of the circuit, the link branches, in dotted lines.

(b) On each of the above diagrams, show the three independent loop currents that are obtained by identifying loop currents with link-branch currents.

8. Given the network illustrated in Fig. 25.

(a) Calculate the current through the branch ab which contains the 1-volt battery using the loop currents shown in Fig. 25a.

(b) Again calculate the current through branch ab employing the loop currents shown in Fig. 25b. All resistance values remain at 1 ohm as indicated in Fig. 25a.

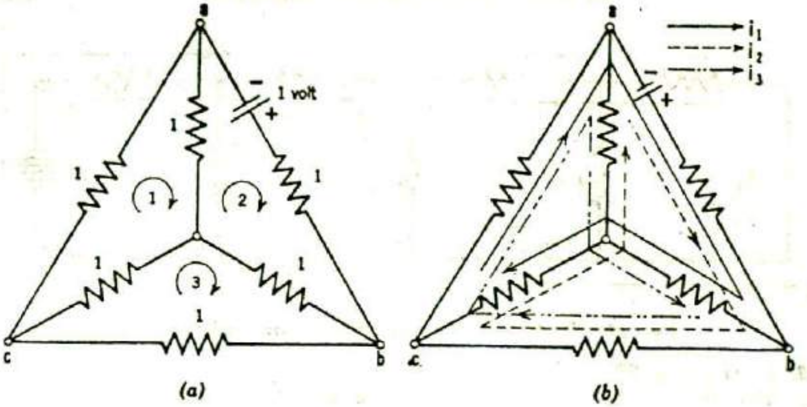


FIG. 25. Prob. 8. (Resistance values refer to ohms.)

9. (*a*) Write the voltage equilibrium equations for the network illustrated in Fig. 26*a* for the loop currents indicated.

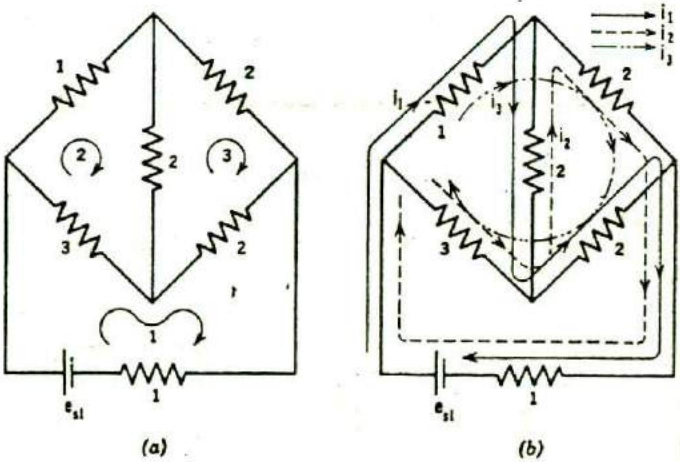


FIG. 26. Problems 9 and 10. (Resistance values refer to ohms.)

(*b*) What is the numerical value of the resistance determinant of the network, that is, the determinant of the resistance matrix which characterizes the network?

10. (*a*) Write the voltage equilibrium equations for the network given in Fig. 26*b* for the loop currents indicated.

(*b*) What is the numerical value of the determinant of the resistance matrix which characterizes the network?

11. (*a*) In Fig. 27, a resistance matrix is formed which corresponds to the loop currents shown there. What is the numerical value of the determinant of this matrix?

(*b*) What is the correct value of the resistance determinant of the network?

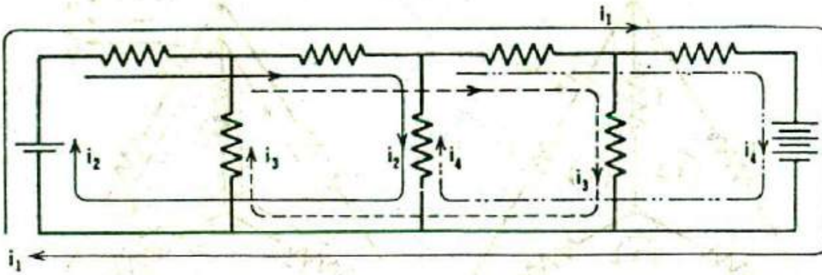


FIG. 27. Prob. 11. (All resistance values are 1 ohm.)

12. Refer to Fig. 28.

(a) Find the potential of node x relative to ground.

(b) Find the potential of point y relative to ground.

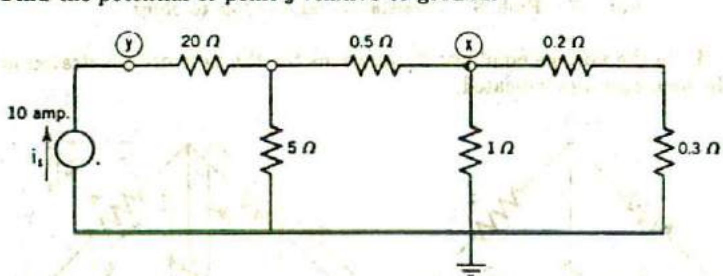


FIG. 28. Prob. 12.

13. (a) Determine by inspection the numerical values of b and n_t of the network illustrated in Fig. 29 and specify the numerical values of n (the number of independent nodes) and l (the number of independent loops).

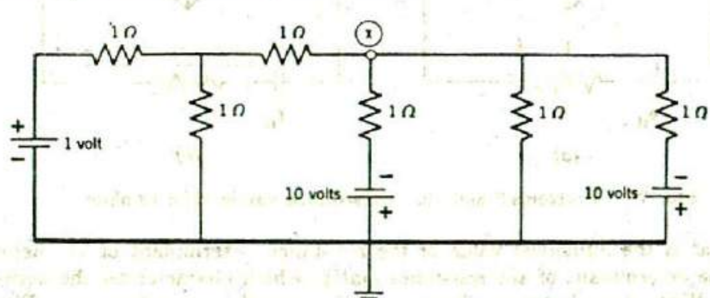


FIG. 29. Problems 13 and 14.

(b) Transform the three voltage sources and associated series resistances to equivalent current sources with due regard for positive directions, and draw the equivalent network incorporating the three current sources.

14. Find the voltage of node x relative to ground in the network given in Fig. 29.

15. Find the potentials of nodes ① and ② of Fig. 30 relative to ground, employing v_1 and v_2 as independent node-pair voltages.

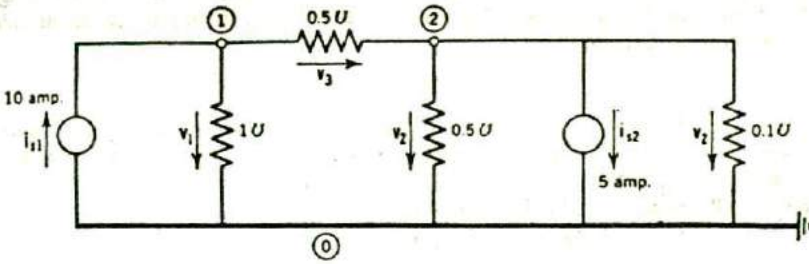


Fig. 30. Problems 15 and 16.

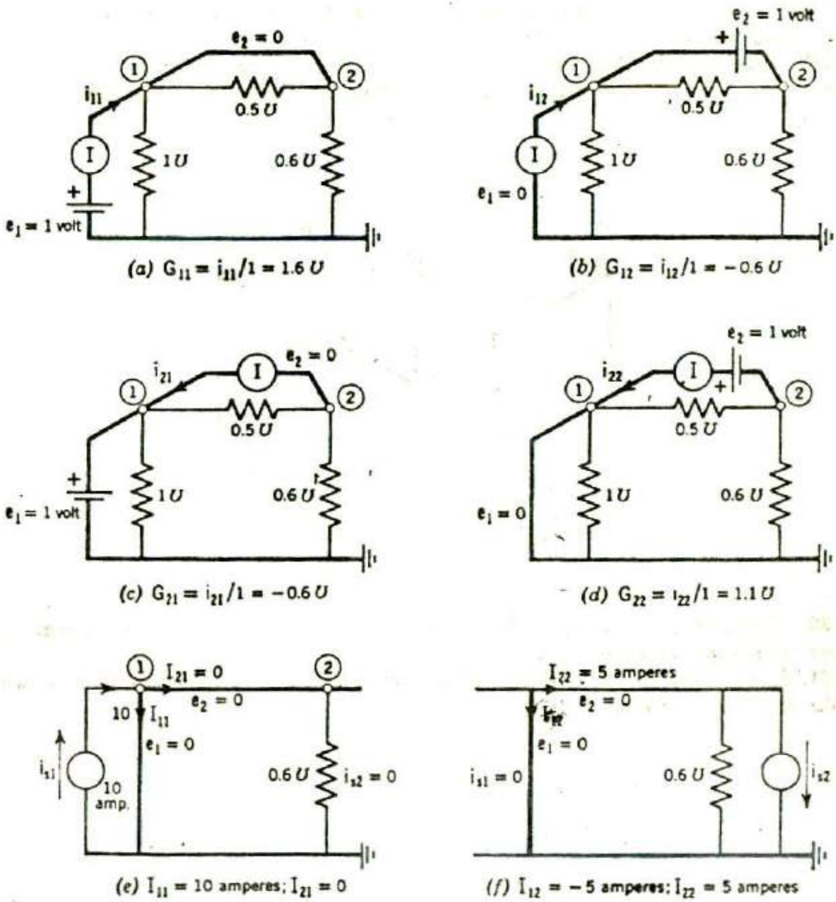


Fig. 31. Prob. 17. For use in solving a problem by superposition.

16. Repeat Problem 15 employing v_1 and v_3 as independent node-pair voltages.

17. Determine the potentials of nodes ① and ② in Fig. 30 employing the principle of superposition as it applies to

$$e_1 = v_1 \quad e_2 = v_3 \quad i_{s1} \quad i_{s2}$$

Sketches showing G_{11}, G_{12}, G_{21}, G_{22} and the component currents at nodes ① and ② are given in Fig. 31. This exercise in superposition is designed to show how the effects of e_1, e_2, i_{s1}, and i_{s2} may be considered separately in the analysis of the circuit. When all effects are combined it will be found that

$$1.6e_1 - 0.6e_2 = I_{11} + I_{12} = 5 \text{ amperes}$$

$$-0.6e_1 + 1.1e_2 = I_{22} + I_{21} = 5 \text{ amperes}$$

18. Construct the dual of the network given in Fig. 24, page 40, without regard to sources with the construction going *from* meshes *to* nodes. Employ a normalizing factor (g_n^2) of 4.

19. Construct the dual of the network illustrated in Fig. 21, page 38, with the construction going *from* nodes *to* meshes. Let g_n^2, the normalizing factor, equal unity.

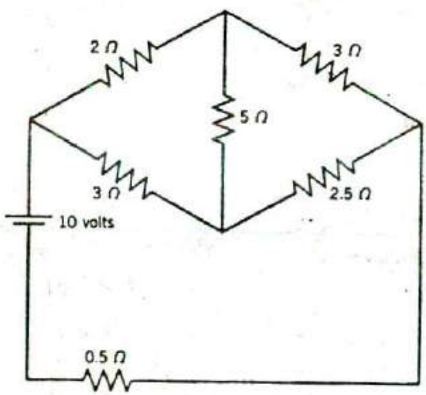


FIG. 32. Prob. 20.

20. Construct the dual of the network shown in Fig. 32 with the construction going *from* meshes *to* nodes. $g_n^2 = 2$.

21. Evaluate v_a in Fig. 7, page 16, employing one known loop current and two unknown loop currents.

chapter II Instantaneous Current, Voltage, and Power

Large segments of circuit analysis are devoted to the steady-state responses of circuits which are energized with alternating currents or voltages having approximate sinusoidal time variations. Several definitions or conventions involving alternating quantities of this kind must be learned and several concepts must be mastered before alternating currents and voltages can be handled with facility.

Early History. The first successful electrical power system in the United States was probably Edison's direct-current installation in New York City. This station was performing creditably in 1885. Alternating-current power systems began commercially with the Great Barrington (Massachusetts) installation in 1886.

During the decade 1907–1917, which followed the invention of the three-electrode vacuum tube, sustained oscillatory currents at high frequencies became a reality. These high-frequency oscillatory or alternating currents are essential to all modern radio, television, and radar forms of communication.

The outstanding advantage of a-c systems (as contrasted with d-c systems) is the relative ease with which alternating potential differences can be generated, amplified, and otherwise transformed in magnitude. The result is that, at the present time, approximately 95 per cent of the electrical energy consumed in the United States is generated, transmitted, and actually utilized in the form of alternating current. In the power field the annual energy consumption amounts to about 600 billion kilowatthours. In the communication field several thousand broadcast stations (of the AM, FM, and television variety) employ alternating potential differences to generate their carrier waves.

Generation of Alternating Potential Differences. When magnets are moved relative to electrical conductors as shown in Fig. 1, there is induced in the conductors a potential difference or emf. In accordance with Faraday's law, $e = -N \frac{d\phi}{dt}$ or its equivalent $e = N'Blv$ and the emf varies with time. For the instant depicted in Fig. 1, the application of one of the rules for finding the magnitude and direction of an induced emf will show that the emf induced in the armature conductors is zero, since at that instant no flux is being cut by these conductors. One-eighth revolution later, however, the induced emf is of maximum magnitude and of such a direction as to establish a voltage rise *from* terminal *a to* terminal *d*. One-quarter of a revolution after the position shown in Fig. 1 the induced emf will again be zero. Three-eighths of

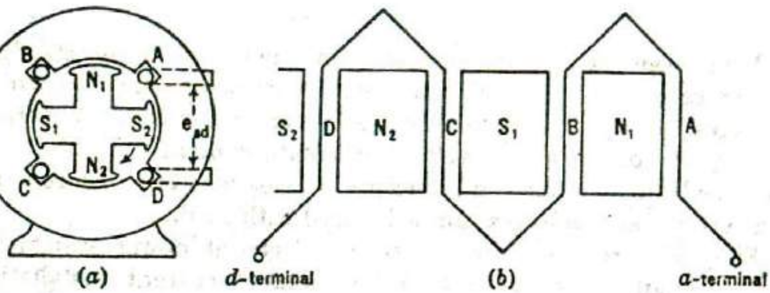


FIG. 1. (*a*) A four-pole, four-conductor a-c generator of the revolving field type. (*b*) Developed diagram showing method of connecting conductors *A*, *B*, *C*, and *D*. Pole faces are toward the reader.

a revolution from the reference position the emf will again be of maximum magnitude but so directed as to establish a voltage rise *from* terminal *d to* terminal *a*.

Thus the terminals *a* and *d* of the generator become alternately positive and negative relative to each other, and a time-varying potential difference of the general nature shown in Oscillogram 1 (page 51) is developed.

In communication systems, vacuum tubes or transistors (working in conjunction with suitable electrical circuits) produce alternating currents of higher frequencies than those obtainable with rotating equipment. A common triode oscillator circuit is shown schematically in Fig. 2. The a-c energy developed across the output terminals is actually derived from the d-c supply voltage labeled E_{bb}, but it is not expected that the reader will understand the conversion from direct current to alternating current which takes place in Fig. 2 until after he has studied the subject of electrical resonance. The only purpose in mentioning the triode oscillator at this stage is to acquaint the reader with the fact that high-frequency alternating currents can be produced with very simple circuit configurations. Many simple circuit configurations other than that shown in Fig. 2 may be used for this purpose.

Definition of Alternating Current. An alternating current, as the name implies, goes through a series of different values both positive

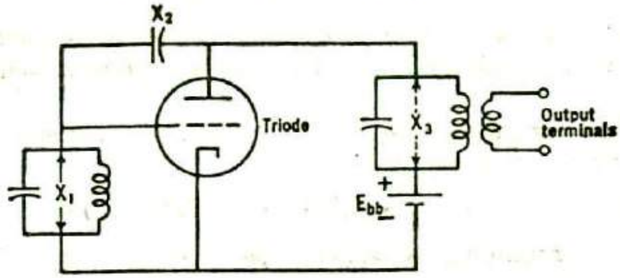


FIG. 2. Circuit arrangement of a simple triode oscillator.

and negative in a period of time T, after which it continuously repeats this same series of values in a cyclic manner as indicated in Fig. 3*c*.

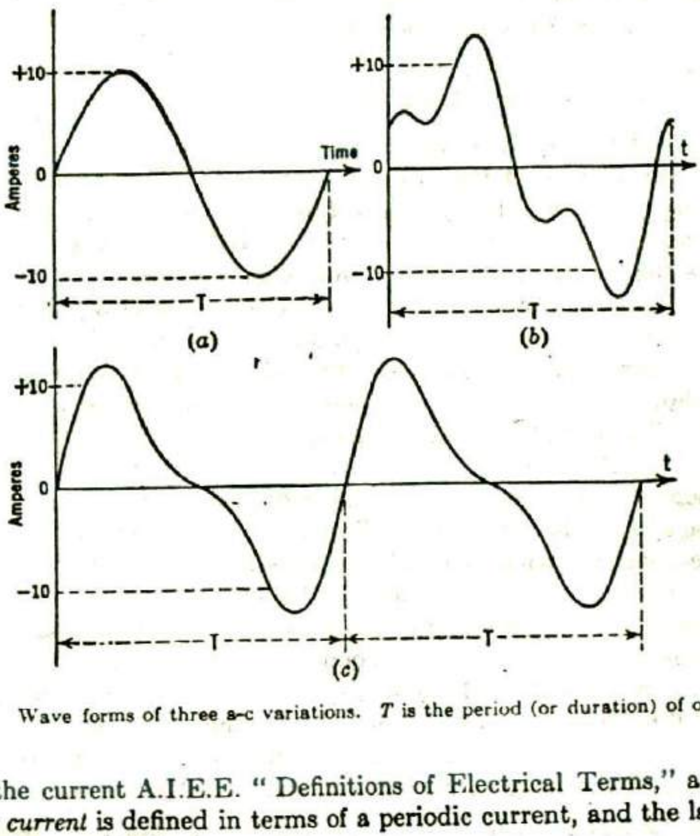


FIG. 3. Wave forms of three a-c variations. T is the period (or duration) of one cycle.

In the current A.I.E.E. "Definitions of Electrical Terms," an *alternating current* is defined in terms of a periodic current, and the latter in terms of an oscillating current.

"An *oscillating current* is a current which alternately increases and decreases in magnitude with respect to time according to some definite law.

"A *periodic current* is an oscillating current the values of which recur at equal intervals of time. Thus

$$i = I_0 + I_1 \sin(\omega t + \alpha_1) + I_2 \sin(2\omega t + \alpha_2) + \cdots \tag{1}$$

where i = the instantaneous value of a periodic current at time t

$I_0, I_1, I_2, \alpha_1, \alpha_2$ = constants (positive, negative, or zero)

$\omega = \dfrac{2\pi}{T}$ (T being the period)

"An *alternating current* is a periodic current, the average value of which over a period is zero. The equation for an alternating current is the same as that for a periodic current except that $I_0 = 0$."

Examples. In Fig. 3*a*, $i = 10 \sin \omega t$ amperes; in Fig. 3*b*, $i = 10 \sin \omega t + 4 \sin(3\omega t + 90°)$ amperes; and, in Fig. 3*c*, $i = 10 \sin \omega t + 4 \sin 2\omega t$ amperes.

Period and Cycle. The *period* of an alternating current or voltage is the smallest value of time which separates recurring values[1] of the alternating quantity. The period of time which separates these recurring values is shown in Fig. 3 as T, the symbol normally employed to designate the *period* of one cycle of an alternating quantity.

One complete set of positive and negative values of an alternating quantity is called a *cycle*. Thus Figs. 3*a* and 3*b* each depict one cycle. A cycle is sometimes specified in terms of angular measure since, as will be shown presently, ω in equation (1) actually represents angular velocity. One complete cycle is then said to extend over 360° or 2π radians of angular measure.

Frequency. Frequency is the number of cycles per second. Unless otherwise stated, the term "cycles" implies "cycles per second."

In the rotating machine of Fig. 1, it is apparent that a complete cycle is produced in the armature conductors when these conductors are cut by the flux from a pair of poles or, in this case, one-half revolution of the rotating field. Each conductor will be cut by two pairs of poles in one revolution of the field structure, and two complete cycles of emf will be developed in the armature winding per revolution.

In general, for a p-pole machine the number of cycles per revolution is $p/2$, and, if the speed of rotation in revolutions per second is repre-

[1] The mathematical meaning of "recurring values" is implied in this definition, namely, that at least one complete *set* of values intervenes between two recurring values.

sented by rps, the equation for frequency is

$$f = \frac{p\,(\text{rps})}{2} \quad \text{cycles per second} \tag{2}$$

Since T is the time (or duration) of one cycle, it is plain that

$$f = \frac{1}{T} \quad \text{cycles per second} \tag{3}$$

if T is expressed in seconds.

Example. Let it be required to find the *frequency* and the *period* of the emf generated in the armature winding of Fig. 1 if the speed of rotation is 1800 rpm.

$$f = \frac{4}{2} \times \frac{1800}{60} = 60 \text{ cycles per second}$$

$$T = \frac{1}{f} = \frac{1}{60} \quad \text{second}$$

The common power plant frequencies in use today are 60, 50, and 25 cycles, the first mentioned being by far the most prevalent in this country. Abroad 50 cycles is very common, and some foreign railways use frequencies considerably less than 25 cycles. A 25-cycle variation causes a noticeable flicker in incandescent lamps; hence it is undesirable for lighting. Formerly 25 cycles was used for power work but, with the advent of a better understanding of the laws governing a-c power transmission and the design of machinery, this frequency is rapidly being superseded. In general, 60-cycle apparatus is lighter and costs less than 25-cycle equipment. The difference is similar to that between high- and low-speed d-c machines.

Audio frequencies range from approximately 16 cycles to approximately 20,000 cycles, voice frequencies occupying the range from about 200 to 2500 cycles. Carefully engineered audio systems, like some theater installations, are designed to accommodate frequencies from 30 to 12,000 cycles.

Radio frequencies range from about 50,000 cycles to 10^{10} cycles, the AM program broadcast band being from 540 to 1600 kilocycles, and the FM and television broadcast bands being from about 50 to 200 megacycles. Radar systems often operate with a carrier frequency of 3000 or 10,000 megacycles.

At the 1947 Atlantic City Conference it was agreed to express frequencies as employed by radio engineers in kilocycles per second at and below 30,000 kilocycles per second and in megacycles per second above this frequency. The present FCC standard band designations follow.

VLF	(very low frequency)	less than 30 kilocycles per second
LF	(low frequency)	30–300 kilocycles per second
MF	(medium frequency)	300–3000 kilocycles per second
HF	(high frequency)	3000–30,000 kilocycles per second
VHF	(very high frequency)	30,000 kilocycles per second–300 megacycles per second
UHF	(ultra high frequency)	300–3000 megacycles per second
SHF	(super high frequency)	3000–30,000 megacycles per second
EHF	(extremely high frequency)	30,000–300,000 megacycles per second

Wave Form. The shape of the curve resulting from a plot of instantaneous values of voltage or current as ordinate against time as abscissa is its *wave form* or *wave shape*. It has been shown that the passage of a pair of poles past a given reference point on the stator of Fig. 1 produced a complete cycle of generated or induced emf. This corresponded to 2π electrical radians, or 360 electrical degrees. In other words, one cycle occurs in or occupies 2π radians, or 360°. The abscissa, instead of being expressed in terms of time in seconds, can be and is quite frequently expressed in terms of radians or degrees. Thus one cycle occurs in 2π radians, or 360°.

Angular Velocity or Angular Frequency. In the preceding article a complete cycle was seen to correspond to 2π radians. The time for a complete cycle was defined as the period T. Hence the angular velocity ω in radians per second is $2\pi/T$. Therefore

$$\omega = \frac{2\pi}{T} = 2\pi f \tag{4}$$

Equation (4) specifies angular velocity in terms of frequency, and this velocity is called electrical[2] angular velocity or *angular frequency*.

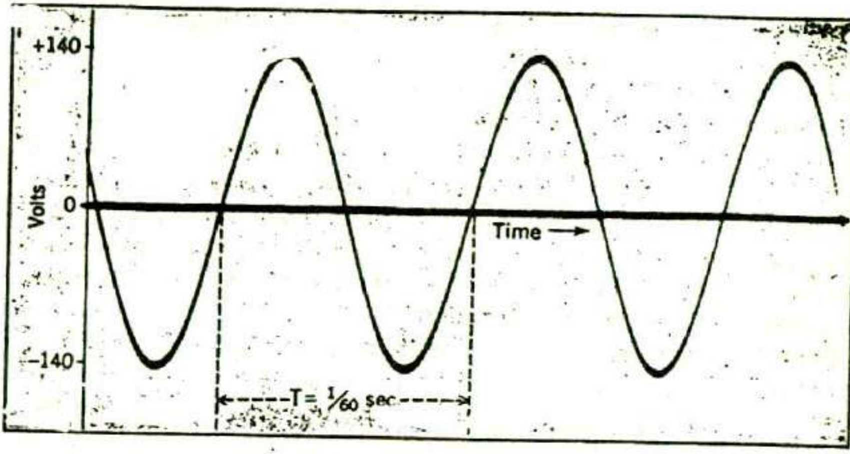
If equations (2) and (4) are combined,

$$\omega = 2\pi f = 2\pi \frac{p}{2} \text{ (rps)} = \frac{p}{2}\,[2\pi \text{ (rps)}] \tag{5}$$

Equation (5) shows that electrical angular velocity equals (pairs of poles) times (mechanical angular velocity) in generators of the type shown in Fig. 1.

Alternating Voltages and Currents Represented by Sine Waves. Whereas the foregoing has referred to waves of any shape, the usual

[2] Mechanical angular velocity, 2π(rps) radians per second, is not to be confused with electrical angular velocity. In Fig. 1 the two are related by the factor $p/2$, but in vacuum tube oscillators of the type shown in Fig. 2 the electrical angular velocity or angular frequency is defined almost solely by the inductance and capacitance employed at the X_1 and X_2 positions in the circuit.



OSCILLOGRAM 1. Emf of a sine-wave generator.

attempt is to secure a sine wave. Oscillogram 1 is a photographic record of the potential difference produced by a so-called sine-wave generator.

Many of the alternating waves met with in practice approximate a sine wave very closely. Alternating-voltage and -current calculations are therefore based on sine waves. (The method whereby non-sinusoidal waves are expressed so as to be calculated according to the laws of sine waves is explained in Chapter VI.) A true sine wave is shown in Fig. 4. The equation for it is

$$i = I_m \sin \omega t \qquad (6)$$

where ωt is expressed in radians and is called the time angle, i is the instantaneous value of current, and I_m is the maximum value of the sinusoidal variation. Since ωt represents an angle, equation (6) may be expressed in terms of radians or degrees. Thus

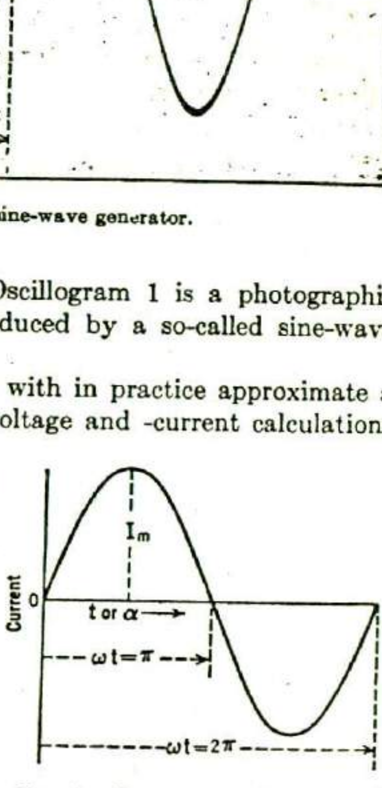


FIG. 4. Sine wave may be expressed as $I_m \sin \alpha$ or as $I_m \sin \omega t$.

$$i = I_m \sin \alpha \qquad (7)$$

where α is in degrees or radians. Equation (6) expresses the current as a sinusoidal variation with respect to time, whereas equation (7) expresses it as a function of angular measure.

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Alternating Potential Difference. Alternating voltage or potential difference may take the form of a generated (or induced) emf or the form of a potential drop, sometimes abbreviated p.d. In the interest of clear thinking these two forms of voltage should be distinguished from one another. Instantaneous values of generated or induced emf's will be designated by e, and instantaneous values of potential drops by the symbol v. Similarly E_m and V_m will be used to distinguish a maximum value of induced voltage from a maximum value of potential drop. Corresponding distinctions will be made between other particular values of induced voltages and voltage drops.

Phase. Phase (as the term is defined by the A.I.E.E.) is the fractional part of a period through which time or the associated time angle ωt has advanced from an arbitrary reference. In the case of a simple sinusoidal variation, the origin is usually taken as the last previous passage through zero from the negative to the positive direction. Thus one phase of a sine wave is $\frac{1}{12}$ of a period (or 30° from the origin) where the ordinate is one-half the maximum ordinate; another phase is $\frac{1}{4}$ of a period (or 90° from the origin) where the ordinate has its maximum positive value; and so on for any other fractional part of T (or of $\omega T = 2\pi$).

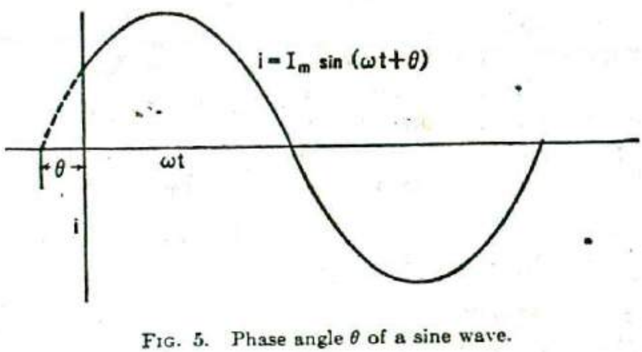


FIG. 5. Phase angle θ of a sine wave.

In accordance with the above definition, the phase angle of a single wave is the angle from the zero point on the wave to the value at the point from which time is reckoned. Thus $i = I_m \sin(\omega t + \theta)$ represents a sine wave of current with a phase angle θ. The phase of the wave from which time is reckoned (i.e., when $t = 0$) is $i = I_m \sin \theta$. The angle θ is the phase angle of the current with respect to the point where $i = 0$ as a reference. These principles are illustrated in Fig. 5.

The phase angle when used in connection with a single alternating quantity merely provides a simple analytical method of starting the variation in question at any point along the wave. As such it is of little importance in steady-state analysis in contrast with its great usefulness in the analysis of transient conditions.

Phase Difference. The phase angle is a very important device for properly locating different alternating quantities with respect to one another. For example, if the applied voltage is

$$v = V_m \sin \omega t \tag{8}$$

and it is known from the nature and magnitude of the circuit parameters that the current comes to a corresponding point on its wave before the voltage wave by θ degrees, the current can be expressed as

$$i = I_m \sin (\omega t + \theta) \tag{9}$$

Figure 6 illustrates the phase positions of v and i for $\theta = 45°$. The current in this case is said to lead the voltage by 45°, or the voltage is said to lag the current by 45°. A given alternating quantity lags

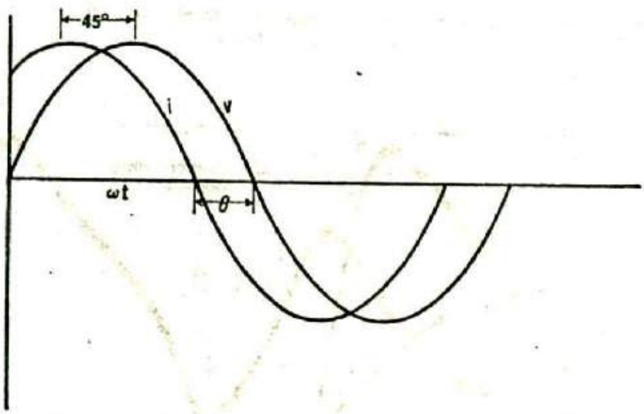
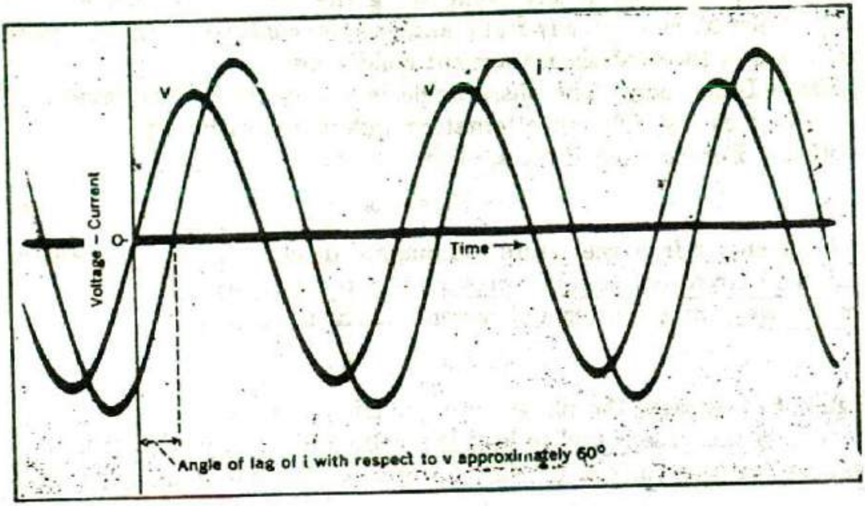


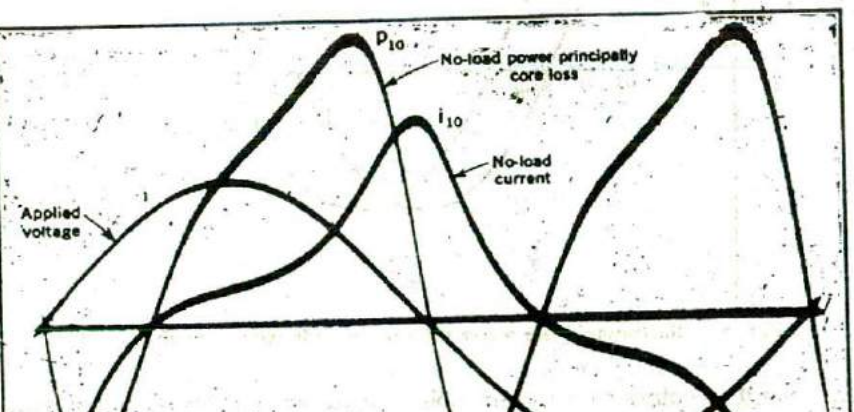
FIG. 6. Illustrating a case where the i wave leads the v wave by $\theta = 45°$.

another if it comes to a certain point on its wave later than the other one comes to the corresponding point on its wave. Another way of saying the same thing is that the positive maximum of the leading quantity occurs before the positive maximum of the lagging quantity. Thus it is said that there is a phase difference of 45° between the two waves. The angle of phase difference is the difference of the phase angles of the two waves. Thus, if $e = 100 \sin (\omega t + 45°)$ and $i = 10 \sin (\omega t - 15°)$, the angle of phase difference is $45° - (-15°) = 60°$.

Oscillogram 2 illustrates the actual phase relation between an applied sinusoidal voltage and the resulting current that flows in a particular



Oscillogram 2. Photographic record of voltage and current for a circuit containing resistance and inductance.



Oscillogram 3. Oscillographic records of the no-load current and no-load power taken by the primary of an iron-core transformer. The applied voltage variation, v_1, is also shown.

circuit. Inspection of the oscillogram will show that the current lags the voltage in this particular case by approximately 60°. Oscillogram 3 illustrates a case where the current and power waves are distinctly non-sinusoidal.

Examples. If a voltage is described as having sinusoidal wave form, a maximum value of 200 volts, and an angular frequency of 377 radians per second (60 cycles per second), and it is desired to reckon time from the point of zero voltage where dv/dt

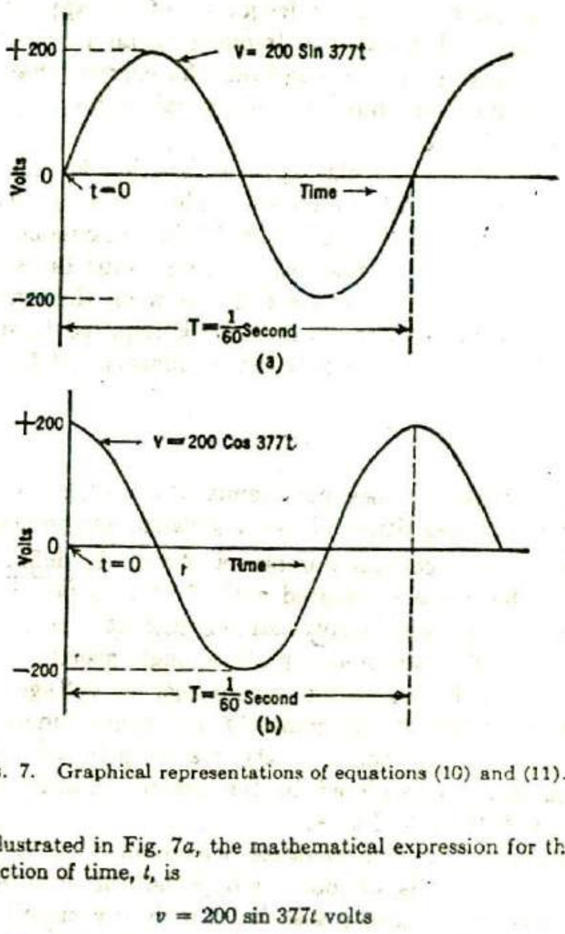


FIG. 7. Graphical representations of equations (10) and (11).

is positive, as illustrated in Fig. 7a, the mathematical expression for the alternating voltage as a function of time, t, is

$$v = 200 \sin 377t \text{ volts} \tag{10}$$

If it is desired to reckon time from some other point along the voltage wave, it is simply necessary to add to the angle $377t$ in the above equation an angle equal to the angular displacement between $v = 0$ (dv/dt positive) and the point on the voltage wave from which it is desired to reckon time. If it is assumed that time is to be reckoned from the point of positive maximum voltage, the angular displacement referred to above is $+90°$, and the expression for voltage becomes

$$v = 200 \sin (377t + 90°) = 200 \cos 377t \text{ volts} \tag{11}$$

This type of variation is shown in Fig. 7b.

Equations (10) and (11) describe exactly the same type of voltage variation except for the $t = 0$ reference.

The current that flows in a circuit as a result of applying a sinusoidal voltage is governed in magnitude and phase by the circuit parameters (resistance R, self-inductance L, capacitance C, and mutual inductance M) and the angular velocity or frequency of the applied voltage. In one sense of the word the angular frequency is an a-c circuit parameter. If the circuit parameters are constant, the current that flows will be of sinusoidal wave form but will, in general, differ in phase from the sinusoidal applied voltage.

Mathematically a particular type of function is required to relate voltage and current in an a-c circuit. The one generally employed is called the impedance function or simply the impedance of the circuit. The impedance function must tell two important facts: (1) the ratio of V_m to I_m,[3] and (2) the phase angle between the waves of voltage and current. A special type of notation is required to signify the two properties of the impedance function in abbreviated form. One such type of notation is

$$Z\underline{/\text{angle}}$$

The above expression does not signify the multiplication of Z and $\underline{/\text{angle}}$. Z is the magnitude of the impedance and in a particular case is represented by a certain number of ohms. It defines the ratio of V_m to I_m. The angle associated with Z, if it is positive, defines the *lead* of the voltage with respect to the current. In accordance with the convention thus adopted a positive angle specifies the number of degrees or radians by which the current lags the voltage.

The determination of the complete impedance function for various combinations of R, L, and C is the first step in a-c circuit analysis. The combinations considered in the present chapter are shown in diagrammatic fashion in Fig. 8.

The *R* Branch. The consideration of a circuit element which possesses only ohmic resistance is, of course, a hypothetical venture because some self-inductance is inevitably associated with any circuit configuration. However, the case may be approached in practice to a degree comparable to the accuracy of ordinary measurements. It is well known that

[3] It will be shown in Chapter III that the magnitude of the impedance Z defines the ratio of $V_{\text{effective}}$ to $I_{\text{effective}}$ as well as the ratio V_m to I_m.

resistance impedes the motion of electricity and causes an irreversible transformation of electrical energy into heat energy in accordance with Joule's law.

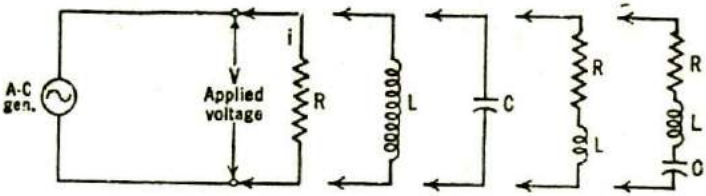


FIG. 8. Elementary circuit arrangements of R, L, and C.

Impedance. The impedance of a simple R branch may be expressed as

$$R\angle 0^\circ \text{ ohms}$$

The reason follows directly from Kirchhoff's emf law. If a voltage, $v = V_m \sin \omega t$, is applied to a branch of R resistance, Fig. 9, the equation for dynamic equilibrium is

$$v = Ri = V_m \sin \omega t \quad (12)$$

from which

$$i = \frac{V_m}{R} \sin \omega t = I_m \sin \omega t \quad (13)$$

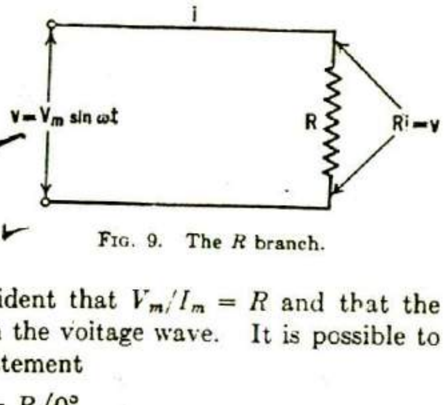


FIG. 9. The R branch.

From the above equation it is evident that $V_m/I_m = R$ and that the current wave is in time phase with the voltage wave. It is possible to express these facts in the single statement

$$\mathbf{Z}_R = R\angle 0^\circ$$

In general, R is expressed directly in ohms, in which case $\mathbf{Z}_R$ is in ohms.

Power. The determination of the rate at which electrical energy is generated or absorbed is, in general, an important problem. Instantaneous power is symbolized by the lower-case letter p.

$$p = ei \quad \text{(generated power)}$$

$$p = vi \quad \text{(absorbed power)}$$

The present discussion confines itself to the determination of instantaneous absorbed power wherein positive values of p indicate that the circuit under consideration is receiving energy from the supplying source.

Negative values of p indicate that the reactive elements of the circuit, if such are present, are actually releasing energy at a rate which is greater than the rate at which energy is being received.

In the present case, that of the simple R branch, all the energy produced by the instantaneous power absorbed is converted into heat. Presumably no reactive elements, inductance coils or condensers, are present. The instantaneous power is given by the product of equations (12) and (13).

$$p = vi = V_m I_m \sin^2 \omega t \tag{14}$$

Since $\sin^2 \omega t = \frac{1}{2} - \frac{1}{2} \cos 2\omega t$, it follows that

$$p = \frac{V_m I_m}{2} - \frac{V_m I_m}{2} \cos 2\omega t \tag{15}$$

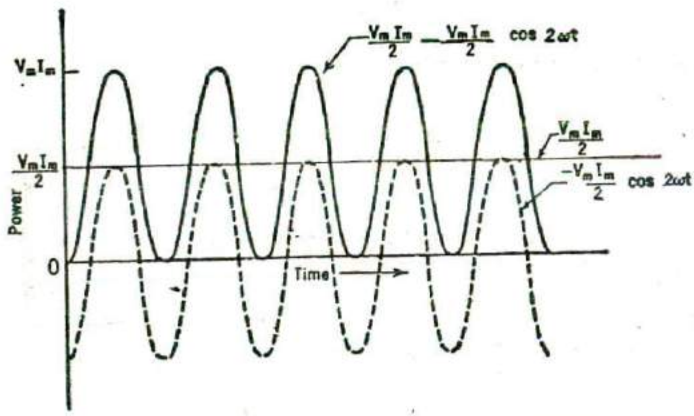
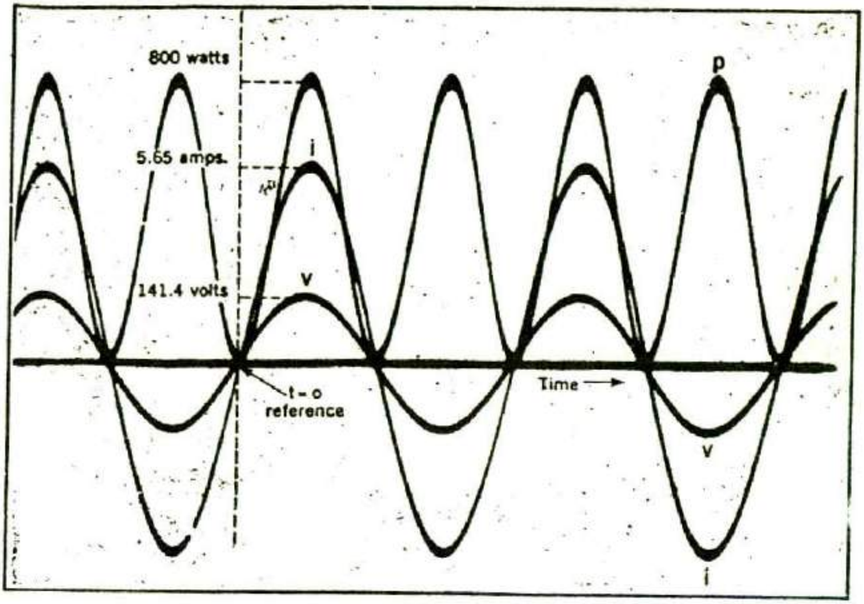


FIG. 10. Graphical representation of equation (15).

Figure 10 illustrates the component parts of equation (15). It will be observed from the above equation that the instantaneous power wave is a double-frequency variation, with respect to the frequency of the current or the voltage, which has an average positive value of $\frac{V_m I_m}{2}$. The cos $2\omega t$ term causes the instantaneous power to acquire periodically zero and $V_m I_m$ values. At no time does the power reach instantaneous negative values.

Photographic records of v, i, and p in a branch which approximates the purely resistive case are shown in Oscillogram 4. The oscillogram illustrates in a graphical manner the relations which have been derived for the R branch and substantiates the physical fact that voltage and current are in time phase in a resistive circuit.



OSCILLOGRAM 4. Voltage, current, and power variations in a resistive circuit element. $R = 25$ ohms. If time is reckoned from the point of zero voltage (dv/dt positive): $V = 141.4 \sin 377t$ volts, $i = 5.65 \sin 377t$ amperes, $p = 400 - 400 \cos 754t$ watts, average power $= 400$ watts.

The L Branch. If a circuit element of pure inductance, Fig. 11, is considered, the equation for dynamic equilibrium is

$$v = L\frac{di}{dt} = V_m \sin \omega t \tag{16}$$

$$di = \frac{V_m}{L} \sin \omega t \, dt \tag{17}$$

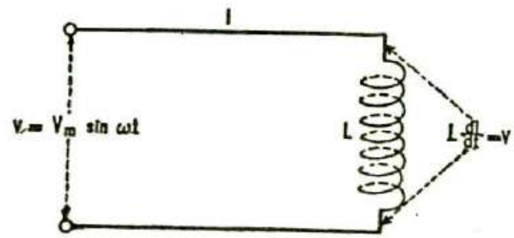


FIG. 11. The L branch.

After both sides of the above equation are integrated it follows that

$$i = -\frac{V_m}{\omega L} \cos \omega t + c_1 \tag{18}$$

The constant of integration c_1 will be considered to be equal to zero since only the steady-state current symmetrical about the zero axis is o be considered.[4]

Under the above conditions equation (18) reduces to

$$i = \frac{V_m}{\omega L} \sin (\omega t - 90°) = I_m \sin (\omega t - 90°) \tag{19}$$

Impedance. Inductance opposes the *rate of change* of current, and for this reason it is sometimes called *electrical inertia.* Since the inductance, L, limits the rate at which the current can change, it follows logically that L actually governs the maximum value of the current in an a-c circuit which is energized by a voltage of specified angular velocity.

It will be observed from equation (19) that $V_m/I_m = \omega L$ and that i lags v by one-quarter of a cycle or 90°. The impedance of a pure L branch is according to the convention previously adopted

$$\mathbf{Z}_L = \omega L \underline{/90°}$$

The reason for using the positive angle in connection with impedances that cause lagging currents will become more evident when the rules of vector algebra and the conventions pertaining to vector diagrams are considered.

The magnitude of the above impedance, ωL, is called *inductive reactance.* Inasmuch as the inductive reactance is directly proportional to the angular velocity of the driving voltage, $2\pi f$, it is obvious that the magnitude of the impedance offered to the flow of alternating current by a coil of fixed self-inductance, L, is directly proportional to frequency. When ω is expressed in radians per second and L is expressed in henrys, the inductive reactance, X_L, is in ohms.

$$X_L = \omega L = 2\pi f L \tag{20}$$

Example. The inductive reactance of a 10-millihenry inductance coil in a 60-cycle circuit is

$$X_L = 2\pi \times 60 \times 0.010 = 3.77 \text{ ohms}$$

and

$$\mathbf{Z} = 3.77 \underline{/90°} \text{ ohms}$$

The inductive reactance of the same coil in a 60,000-cycle circuit is

$$X_L = 2\pi \times 60{,}000 \times 0.010 = 3770 \text{ ohms}$$

If a 60-cycle sinusoidal voltage of maximum value equal to 100 volts is applied to

[4] In a general analysis, c_1 would be evaluated in terms of the boundary conditions under which the circuit is initially closed. Determined in this manner, c_1 would define the transient component of the current. c_1 is neglected here because transient components of the current are not to be considered at this time. In a physically realizable circuit the transient component is of short duration.

the 10-millihenry inductance coil,

$$v = 100 \sin 377t \text{ volts}$$

and

$$i = \frac{100}{3.77} \sin (377t - 90^\circ) \text{ amperes}$$

Power and Energy. The instantaneous power delivered to the pure inductance branch as obtained by multiplying equation (16) by equation (19) is

$$p = vi = [V_m \sin \omega t][I_m \sin (\omega t - 90^\circ)] \tag{21}$$

from which

$$p = V_m I_m (-\sin \omega t \cos \omega t) \tag{22}$$

or

$$p = -\frac{V_m I_m}{2} \sin 2\omega t \tag{23}$$

Figure 12 illustrates the v, i, and p variations in a purely inductive branch. It will be observed that the power variation is again a double-

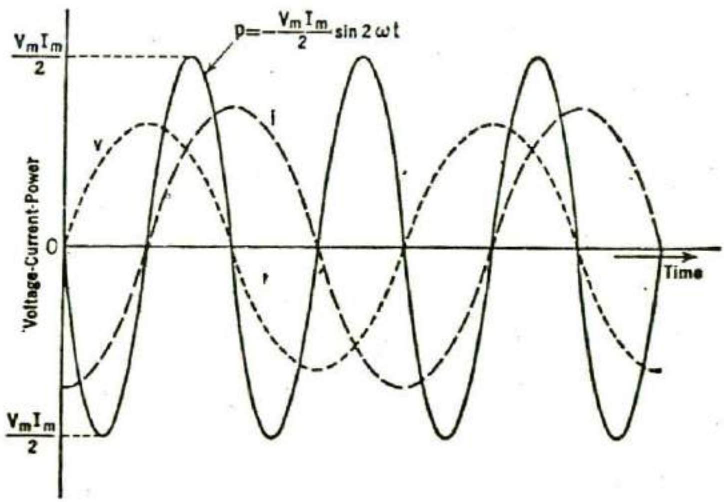


FIG. 12. Voltage, current, and power variations in a purely inductive branch.

frequency variation with respect to the frequency of the driving voltage. The fact that equation (23) indicates negative power during the first one-quarter of a cycle of the driving voltage, that is, from $t = 0$ to $t = T/4$, is the direct result of the choice of the time reference.[5]

[5] The assumption of sinusoidal driving voltage, $v = V_m \sin \omega t$, automatically imposes the condition of $t = 0$ at the point of $v = 0$ (dv/dt positive). The beginner should not confuse the $t = 0$ reference of a steady-state variation with the time at which the circuit is initially energized.

Since steady-state conditions have been assumed, the circuit has presumably adjusted itself to the relative phase relations indicated by equations (16), (19), and (23).

Under the conditions which have been assumed, namely, a steady-state sinusoidal driving voltage and a purely inductive circuit, the power variation is symmetrical about the zero power axis. The average power absorbed is equal to zero. The implication is that the inductive element receives energy from the source during one-quarter of a cycle of the applied voltage and returns exactly the same amount of energy to the driving source during the next one-quarter of a cycle. The exact amount of energy delivered to the circuit during a quarter of a cycle may be obtained by integrating any positive loop of the power wave, for example, integrating p between the limits of $t = T/4$ and $t = T/2$.

$$W_L = \int_{T/4}^{T/2} -\frac{V_m I_m}{2} \sin 2\omega t \, dt$$

$$= \frac{V_m I_m}{2\left(\frac{4\pi}{T}\right)} \left[\cos \frac{4\pi}{T} t\right]_{T/4}^{T/2}$$

$$= \frac{V_m I_m}{2\omega}$$

Since $V_m = \omega L I_m$,

$$W_L = \frac{(\omega L I_m) I_m}{2\omega} = \frac{L I_m^2}{2} \tag{24}$$

If L is expressed in abhenrys and I_m in abamperes, the above energy is in ergs. If L and I_m are expressed in henrys and amperes respectively, W_L is given in joules.

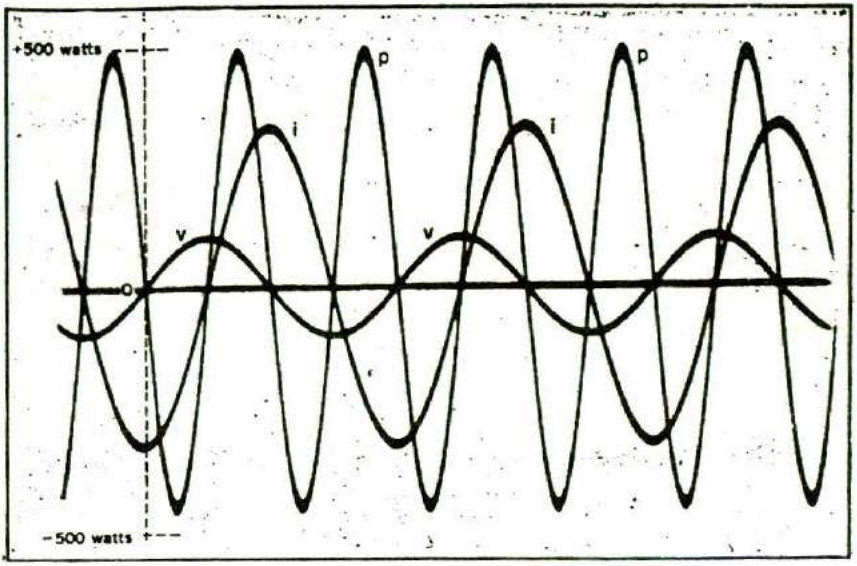
Oscillogram 5 illustrates the relative phase relations in a circuit which approaches, to a fair degree of accuracy, the purely inductive arrangement that has been described mathematically.

The *C* Branch. If it assumed that a sinusoidal voltage, $V_m \sin \omega t$, is applied to an ideal capacitor as indicated in Fig. 13, the expression for steady-state equilibrium is

$$v = \frac{q}{C} = V_m \sin \omega t \tag{25}$$

When the above equation is differentiated with respect to time, it follows that

$$\frac{dq}{dt} = V_m \omega C \cos \omega t \tag{26}$$



OSCILLOGRAM 5. Voltage, current, and power variations in a highly inductive circuit element. $L = 0.056$ henry, $f = 60$ cycles, $X_L = 21.2$ ohms, $R = 1.0$ ohm, $V_{max} = 141.4$ volts, $I_{max} = 6.66$ amperes, $P_{av} = 25$ watts approximately. Note the lag of the i wave with respect to the v wave; also the large negative power loops. Positive power peaks of approximately 500 watts are present even though the average power dissipated in the circuit element is only about 25 watts.

or

$$i = \frac{V_m}{\frac{1}{\omega C}} \sin (\omega t + 90°) = I_m \sin (\omega t + 90°) \qquad (27)$$

Impedance. The ratio of V_m to I_m in the pure C branch is $1/\omega C$, and the current leads the applied voltage by one-quarter of a cycle or 90°. In accordance with the convention which has been adopted, the impedance of the C branch is

$$\mathbf{Z}_C = \frac{1}{\omega C} \angle -90°$$

The magnitude of the impedance, $1/\omega C$, is called *capacitive reactance,* and it is evident from the nature of the expression that capacitive reactance is inversely proportional to the frequency of the driving voltage and also inversely proportional to the capacitance of the capacitor, C. A series circuit in which no capac-

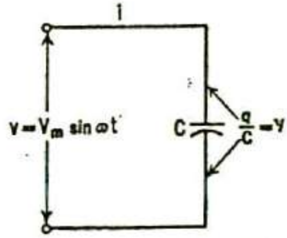


FIG. 13. The C branch.

itor is present has infinite capacitance and, hence, zero capacitive reactance.

The impedance of a capacitor causes the current to lead the voltage by 90°, whereas the impedance of an inductance causes the current to lag the voltage 90°. The effects of the two types of reactive elements as regards the phase of the resulting current are exactly opposite.

If, in the expression for capacitive reactance, ω is expressed in radians per second and C is expressed in farads, the resulting capacitive reactance is in ohms. If the capacitance of the capacitor is expressed in microfarads (abbreviated μf), the expression for capacitive reactance takes the form

$$X_C = \frac{10^6}{\omega C_{\mu f}} \quad \text{ohms}$$

Example. The capacitive reactance of a 15-μf capacitor in a 25-cycle circuit is

$$X_C = \frac{10^6}{2\pi \times 25 \times 15} = 425 \text{ ohms}$$

and

$$Z_C = 425 \underline{/-90^\circ} \text{ ohms}$$

The capacitive reactance of the same capacitor to a 250-cycle driving voltage is

$$X_C = \frac{10^6}{2\pi \times 250 \times 15} = 42.5 \text{ ohms}$$

If a 25-cycle sinusoidal voltage of maximum value equal to 200 volts is applied to the 15-μf capacitor

$$v = 200 \sin (157t) \text{ volts}$$

and

$$i = \frac{200}{425} \sin (157t + 90^\circ) \text{ amperes}$$

Power and Energy. The instantaneous power delivered to the C branch is

$$p = vi = [V_m \sin \omega t][I_m \sin (\omega t + 90^\circ)] \tag{28}$$

from which

$$p = V_m I_m (\sin \omega t \cos \omega t) \tag{29}$$

or

$$p = \frac{V_m I_m}{2} \sin 2\omega t \tag{30}$$

The phase relations of v, i, and p in a purely capacitive branch are shown in Fig. 14. The double-frequency power variation is, as in the pure L branch, symmetrical about the zero power axis. In the present case the capacitor receives energy from the source during the first quarter of a cycle of the voltage variation and returns the same amount

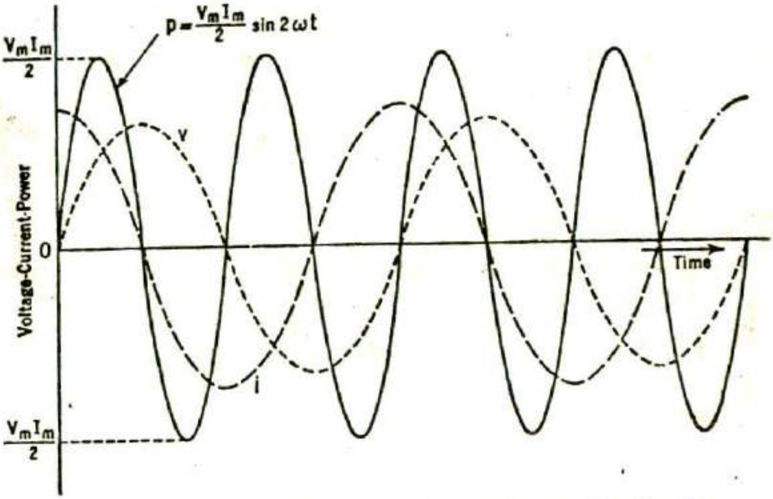


FIG. 14. Voltage, current, and power in a purely capacitive branch.

during the second quarter cycle, etc. The average power absorbed over an integral number of half cycles is, obviously, equal to zero.

The amount of energy received by the capacitor during a quarter cycle may be determined by integrating the power wave over any positive loop; for example, integrating equation (30) between the limits of $t = 0$ and $t = T/4$.

$$W_C = \int_0^{T/4} \frac{V_m I_m}{2} \sin 2\omega t \, dt$$

$$= \frac{V_m I_m}{2\left(\frac{4\pi}{T}\right)} \left[-\cos \frac{4\pi}{T} t \right]_0^{T/4}$$

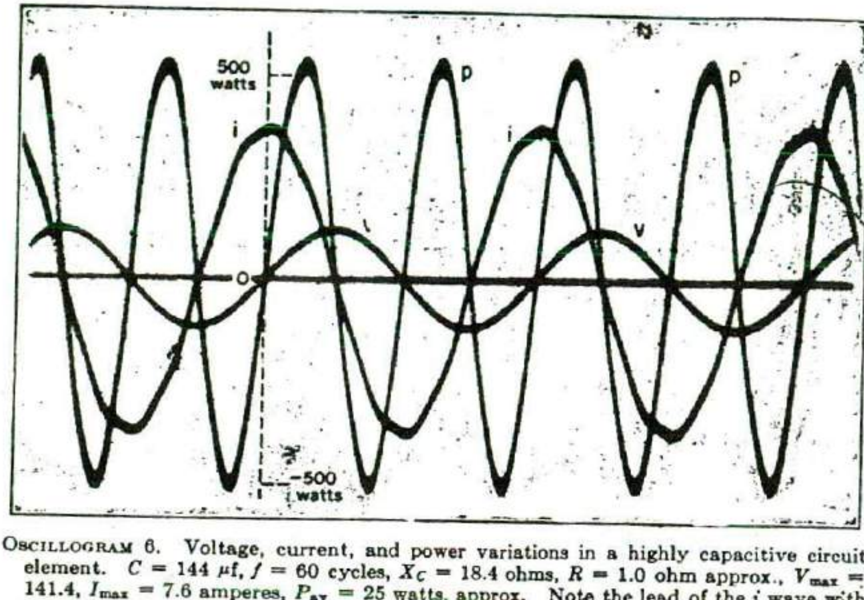
$$= \frac{V_m I_m}{\left(\frac{4\pi}{T}\right)} = \frac{V_m I_m}{2\omega}$$

Since $I_m = \omega C V_m$,

$$W_C = \frac{V_m(\omega C V_m)}{2\omega} = \frac{V_m^2 C}{2} \tag{31}$$

If V_m and C are expressed in volts and farads respectively, the above expression for energy is in joules. W_C is the maximum amount of energy stored in the electric field of the capacitor at any one time.

Comparison of equations (30) and (23) will show that the capacitive



OSCILLOGRAM 6. Voltage, current, and power variations in a highly capacitive circuit element. $C = 144\ \mu f$, $f = 60$ cycles, $X_C = 18.4$ ohms, $R = 1.0$ ohm approx., $V_{max} = 141.4$, $I_{max} = 7.6$ amperes, $P_{av} = 25$ watts, approx. Note the lead of the i wave with respect to the v wave.

element receives energy from the supplying source during the periods in which the inductive element returns energy to the source, and vice versa. When capacitive elements and inductive elements are both present in a given circuit, there is, in general, a natural tendency for the elements to exchange energy. In certain circuit arrangements relatively large amounts of energy oscillate between the electromagnetic fields of the inductances and the electric fields of the capacitors.

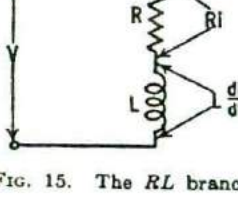


FIG. 15. The *RL* branch.

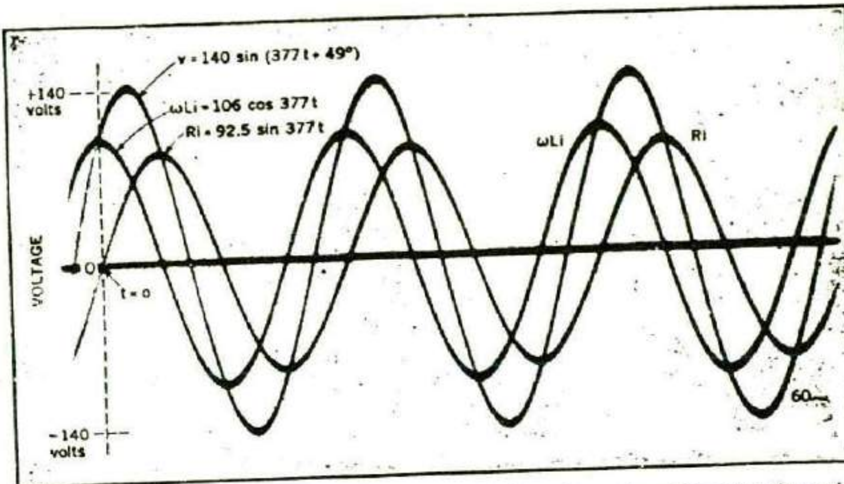
Oscillogram 6 illustrates the v, i, and p variations in a branch which approaches, to a close degree of accuracy, a purely capacitive circuit element.

The *RL* Branch. If it is assumed that a sinusoidal driving voltage, $V_m \sin \omega t$, is applied to a series combination of a resistive element and an inductive element, Fig. 15, the equation for voltage balance is

$$v = Ri + L\frac{di}{dt} = V_m \sin \omega t \qquad (32)$$

This is one form of Kirchhoff's emf law applied to instantaneous voltages. It states that the instantaneous voltage drop across the resistive element plus the instantaneous voltage drop across the inductive element equals the instantaneous voltage drop across the RL branch.

A straightforward solution of equation (32) for i in terms of the applied voltage and circuit parameters requires a certain knowledge of differential equations on the part of the reader which is not essential to the problem at hand. The problem in which we are particularly



OSCILLOGRAM 7. Illustrating the manner in which the voltage drop Ri across the resistance and the voltage drop ωLi across an inductance coil combine to equal the applied voltage v. $R = 18.5$ ohms connected in series with $X_L = 21.1$ ohms. $RI_{\max} = 92.5$ volts, $\omega LI_{\max} = 106$ volts, $V_{\max} = 140$ volts.

interested at this point is the evaluation of the ratio V_m/I_m together with the time-phase difference between the voltage and current in an RL branch. Provided that R and L are constant, a current of sinusoidal wave form will flow in the branch if a sinusoidal voltage is applied. A critical inspection of equation (32) will help to establish the mathematical reasons for this physical fact.

If it is assumed that a sinusoidal current, $i = I_m \sin \omega t$, flows through a series branch consisting of a resistive element, R, and an inductive element, L, then

$$Ri + L\frac{di}{dt} = \text{voltage applied, } v \tag{33}$$

or

$$RI_m \sin \omega t + \omega LI_m \cos \omega t = v \tag{34}$$

Equations (33) and (34) state that the instantaneous component voltage drops, Ri and $L\,di/dt$, add together to form the combined voltage drop across the RL branch. Oscillogram 7 illustrates the manner in which the Ri component ($RI_m \sin \omega t$) and the $L\,di/dt$ component ($\omega LI_m \cos \omega t$) combine to equal the applied voltage (v) in a particular RL branch.

Since sine and cosine waves are 90° out of time phase with respect to one another, the RI_m and the ωLI_m components may be related as shown in Fig. 16a, that is, as the two right-angle sides of a right triangle.

If both sides of the equation are divided by $\sqrt{R^2 + (\omega L)^2}$, equation (34) takes the following form:[6]

$$I_m\left[\sin \omega t \frac{R}{\sqrt{R^2 + (\omega L)^2}} + \cos \omega t \frac{\omega L}{\sqrt{R^2 + (\omega L)^2}}\right] = \frac{v}{\sqrt{R^2 + (\omega L)^2}} \quad (35)$$

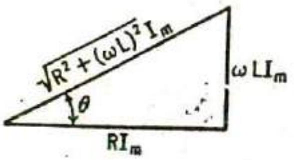


FIG. 16a. The addition of RI_m and ωLI_m.

From Fig. 16a,

$$\cos \theta = \frac{R}{\sqrt{R^2 + (\omega L)^2}} \quad (36)$$

and

$$\sin \theta = \frac{\omega L}{\sqrt{R^2 + (\omega L)^2}} \quad (37)$$

Then

$$I_m[\sin \omega t \cos \theta + \cos \omega t \sin \theta] = \frac{v}{\sqrt{R^2 + (\omega L)^2}} \quad (38)$$

from which

$$v = I_m\sqrt{R^2 + (\omega L)^2} \sin (\omega t + \theta)$$

or

$$v = I_m Z \sin (\omega t + \theta) = V_m \sin (\omega t + \theta) \quad (39)$$

[6] The method of combination here employed requires only a knowledge of trigonometry. Since the combination of

$$(A \sin x + B \cos x)$$

occurs frequently in a-c circuit analysis, a simpler method of combination is often used. This scheme consists of representing the sines and cosines by revolving vectors

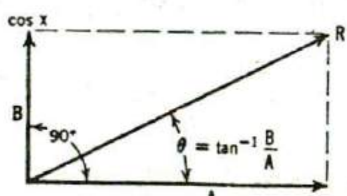


FIG. 16b. Vector representation of sine and cosine functions.

or phasors as explained on pages 90 and 91 which most students will remember was done in physics when sine waves were used to represent simple harmonic motion.

It is thus shown that (1) $Z = \sqrt{R^2 + (\omega L)^2} = V_m/I_m$, (2) $\theta = \tan^{-1} \omega L/R$, and (3) v leads i in the RL branch by $\theta°$.

Impedance.

$$\mathbf{Z}_{RL} = \sqrt{R^2 + (\omega L)^2} \underline{/ \tan^{-1} \frac{\omega L}{R}} \tag{40}$$

The above expression for $\mathbf{Z}_{RL}$ implies that the numerical ratio of V_m to I_m in the RL branch is $\sqrt{R^2 + (\omega L)^2}$ and that the current lags the applied voltage by the angle whose tangent is $\omega L/R$. In general, R is expressed in ohms, ω in radians per second, and L in henrys, in which case $\sqrt{R^2 + (\omega L)^2}$ is given in ohms. In determining the phase angle it is, of course, only necessary that ωL and R be expressed in similar units.

The expression for the impedance of a pure R branch is at once obtainable from $\mathbf{Z}_{RL}$ by assuming that $L = 0$, in which case $\mathbf{Z}_{RL}$ reduces to $R\underline{/0°}$. If the assumption is made that $R = 0$, $\mathbf{Z}_{RL}$ reduces immediately to the expression which has previously been derived for the impedance of a pure L branch, namely, $\omega L\underline{/90°}$.

An examination of the two factors which combine to form $\mathbf{Z}_{RL}$ will show that R is the factor which directly impedes or opposes the flow of current, whereas ωL is the factor which impedes or opposes any change in current. For a resulting sinusoidal current these two factors act in time quadrature with respect to one another. For example, when the current is zero the R factor has zero effect and the L factor has its greatest effect because it is when $i = 0$ that $[di/dt]$ for a sine wave is at its maximum value. When the current is at its maximum value, I_m, the R factor has its greatest effect and the L factor has zero effect because $[di/dt]$ for a sine wave is zero at the point of maximum current. It is the time quadrature nature (90° time-phase displacement) of the individual impedance effects that makes possible a simple vector algebra method of analyzing a-c circuits.[7]

Example. If $R = 20$ ohms and $L = 0.056$ henry, the 60-cycle impedance of the RL branch which is formed by placing R in series with L is

$$\mathbf{Z} = \sqrt{20^2 + (377 \times 0.056)^2} \underline{/ \tan^{-1} \frac{21.1}{20}}$$

$$= 29.1 \underline{/46.5°} \text{ ohms}$$

Through employing such methods, the $\sin x$ component may be represented by a horizontal vector of magnitude A. Since counterclockwise is the standard direction for positive or forward rotation, the $\cos x$ component (which leads the $\sin x$ component by 90°) will then be drawn to a magnitude of B vertically upward. Thus Fig. 16*b* is obtained and the resultant R is readily seen to be $\sqrt{A^2 + B^2} \sin (x + \theta)$.

[7] The vector or phasor method of analysis is considered in Chapter IV.

If

$$v = 200 \sin (377t) \text{ volts}$$

$$i = \frac{200}{29.1} \sin (377t - 46.5°)$$

$$= 6.87 \sin (377t - 46.5°) \text{ amperes}$$

It will be observed that the instantaneous current is obtained from the instantaneous voltage ($200 \sin 377t$) and the impedance function ($29.1 \underline{/46.5°}$) by *two distinct operations* which are performed in a single step. These are:

(*a*) The maximum magnitude of the voltage (200) is divided by the magnitude of the impedance (29.1) to obtain the maximum magnitude of the current, 6.87 amperes.

(*b*) The correct angular displacement of the current wave with respect to the voltage wave is obtained by subtracting the impedance angle (46.5°) from the time angle of the voltage wave, namely, $377t$.

Note: In evaluating the correct angular displacement between the instantaneous current and voltage waves in terms of the impedance angle, it is better to combine the angles in such a way as to yield the relation between current and voltage waves which are known to exist from a knowledge of the physical characteristics of the circuit. This process should not be obscured by any elaborate mathematical conventions.

Power. The instantaneous power or, as it is sometimes called, the instantaneous volt-amperes, delivered to the RL branch may be obtained from

$$p = vi = [V_m \sin (\omega t + \theta)][I_m \sin \omega t] \tag{41}$$

After the $\sin (\omega t + \theta)$ term is expanded, the above equation can be written in the following forms:

$$p = V_m I_m \sin \omega t [\sin \omega t \cos \theta + \cos \omega t \sin \theta]$$

$$= V_m I_m \sin^2 \omega t \cos \theta + V_m I_m (\sin \omega t \cos \omega t) \sin \theta$$

$$= \frac{V_m I_m}{2} \cos \theta - \frac{V_m I_m}{2} [\cos 2\omega t] \cos \theta + \frac{V_m I_m}{2} [\sin 2\omega t] \sin \theta \tag{42}$$

Figure 17 is a graphical representation of the component parts of equation (42) together with the resultant graph of instantaneous power. It should be plain that the average value with respect to time of either the [$\cos 2\omega t$] or the [$\sin 2\omega t$] term is equal to zero when considered over a time interval equal to an integral number of cycles. The average value with respect to time of the power when considered over an integral number of cycles is, therefore, equal to

$$P_{av} = \frac{V_m I_m}{2} \cos \theta$$

The above expression for average power may also be obtained by finding

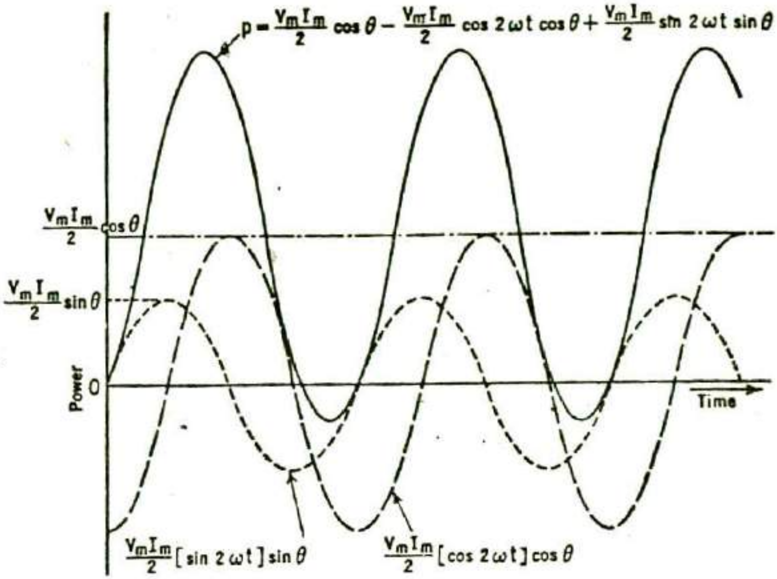


FIG. 17. Graphical representation of equation (42) for the particular case of $\theta = 30°$.

the average value of the right-hand member of equation (41) as follows:

$$P_{av} = \frac{1}{T} \int_0^T V_m \sin(\omega t + \theta) I_m \sin \omega t \, dt$$

$$= \frac{V_m I_m}{2} \cos\theta \tag{43}$$

Real Power and Reactive Power or Reactive Volt-Amperes. A detailed analysis of the component parts of equation (42) will aid in understanding why electrical power is treated in terms of real and reactive components and why these two components are sometimes represented as the legs of a right triangle.

Real Power. *Instantaneous real power* refers to $\left[\frac{V_m I_m}{2} \cos\theta - \frac{V_m I_m}{2} (\cos 2\omega t) \cos\theta\right]$, the first two terms on the right-hand side of equation (42). Reference to Fig. 17 will show that these two terms combine to form an instantaneous power variation which contains no negative values; hence this portion of equation (42) is called the *instantaneous real power.*

Unless qualified to mean *instantaneous* real power, the expression *real power* refers only to $\frac{V_mI_m}{2}\cos\theta$, the average value of the total instantaneous power with respect to time. [See equations (42) and (43).]

Reactive Power or Reactive Volt-Amperes. The third term on the right-hand side of equation (42), $\left[\frac{V_mI_m}{2}(\sin 2\omega t)\sin\theta\right]$, is variously called *instantaneous reactive power, instantaneous quadrature power, instantaneous reactive volt-amperes*, etc., for the reason that the area under the $\left[\frac{V_mI_m}{2}(\sin 2\omega t)\sin\theta\right]$ curve represents the energy which oscillates between the driving source and the reactive (either inductive or capacitive) elements of the receiving circuit. It will be observed from Fig. 17 that the *instantaneous reactive power* is that portion of the total instantaneous power variation which has equal positive and negative loops, and which contains the sine of the phase angle between v and i as a factor.

Unless qualified to mean *instantaneous* reactive power or *instantaneous* reactive volt-amperes, the expressions *reactive power* and *reactive volt-amperes* refer simply to $\frac{V_mI_m}{2}\sin\theta$, the maximum instantaneous value of the third term on the right-hand side of equation (42).

Units of reactive volt-amperes in the practical system of units are called *vars*. (See pages 98 and 99.)

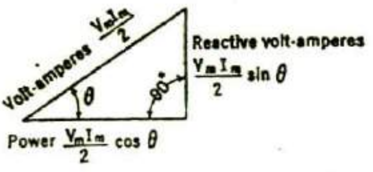
Volt-Amperes. Both the real power, $\frac{V_mI_m}{2}\cos\theta$, and the reactive volt-amperes, $\frac{V_mI_m}{2}\sin\theta$, are important quantities, and they are often measured independently, a wattmeter being used to measure $\frac{V_mI_m}{2}\cos\theta$ and a reactive volt-ampere meter, called a varmeter, being used to measure $\frac{V_mI_m}{2}\sin\theta$.

The real power and the reactive power may be combined to yield the volt-amperes of the circuit, namely, $\frac{V_mI_m}{2}$

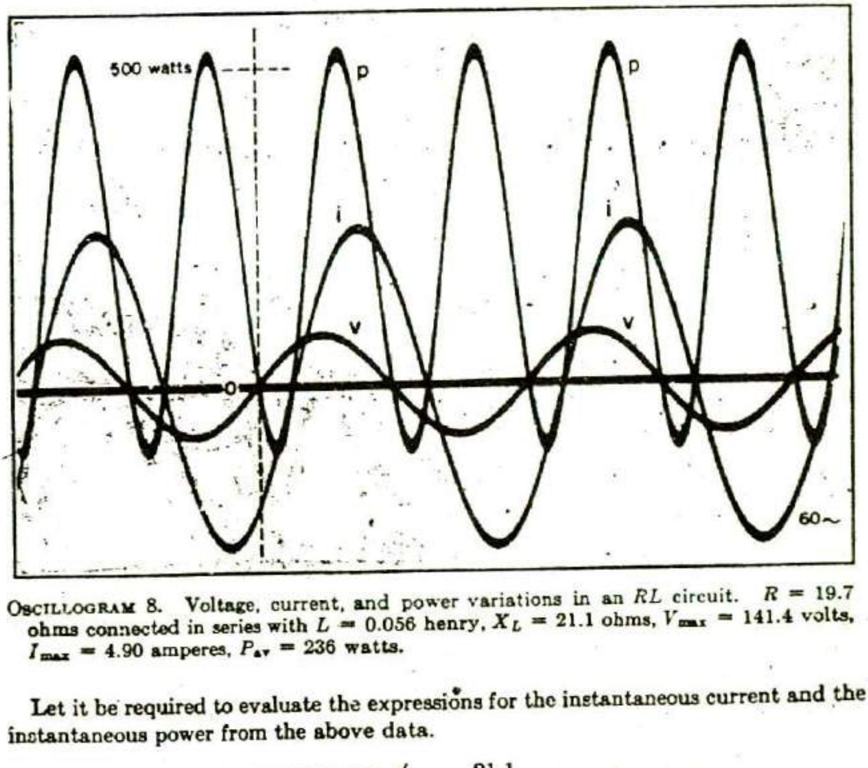
$$\sqrt{\left[\frac{V_mI_m}{2}\cos\theta\right]^2+\left[\frac{V_mI_m}{2}\sin\theta\right]^2}=\frac{V_mI_m}{2}$$

The above relationship is illustrated graphically in Fig. 18 and will be encountered in later chapters in a more universally used form.

FIG. 18. Relation of power, reactive volt-amperes, and volt-amperes.



Example. Consider the RL circuit whose voltage, current, and power variations are depicted in Oscillogram 8. $R = 19.7$ ohms, $\omega L = 21.1$ ohms, and $v = 141.4 \sin 377t$ volts.



OSCILLOGRAM 8. Voltage, current, and power variations in an RL circuit. $R = 19.7$ ohms connected in series with $L = 0.056$ henry, $X_L = 21.1$ ohms, $V_{max} = 141.4$ volts, $I_{max} = 4.90$ amperes, $P_{av} = 236$ watts.

Let it be required to evaluate the expressions for the instantaneous current and the instantaneous power from the above data.

$$Z = \sqrt{19.7^2 + 21.1^2} \Big/ \tan^{-1}\frac{21.1}{19.7} = 28.85 \underline{/47^\circ} \text{ ohms}$$

The instantaneous current is

$$i = \frac{141.4}{28.85} \sin(377t - 47^\circ) = 4.9 \sin(377t - 47^\circ) \text{ amperes}$$

The expression for the instantaneous power is, by equation (42),

$$p = 236 - 236 \cos 754t + 253 \sin 754t \text{ watts}$$

In this expression,

> [236 − 236 cos 754t] is called the *instantaneous* real power
> 253 sin 754t is the *instantaneous* reactive volt-amperes
> 236 watts is the real power
> 253 vars is the reactive power or reactive volt-amperes.

The *RLC* Branch. If a current of sinusoidal wave form, $i = I_m \sin \omega t$, is assumed to flow through the RLC branch shown in Fig. 19, it is plain that

$$v_R = Ri = RI_m \sin \omega t \tag{44}$$

$$v_L = L\frac{di}{dt} = \omega L I_m \cos \omega t \tag{45}$$

and

$$v_C = \frac{q}{C} = \frac{\int i\,dt}{C} = \frac{\int I_m \sin \omega t\,dt}{C} = \frac{-I_m}{\omega C}\cos \omega t \qquad (46)^8$$

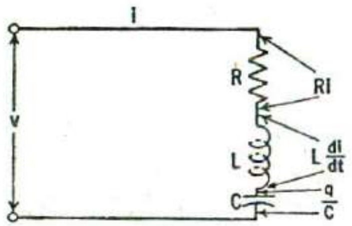


FIG. 19. The RLC branch.

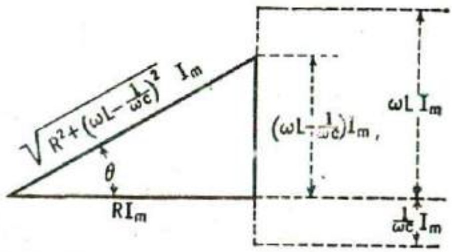


FIG. 20. Illustrating the manner in which the three voltage drops RI_m, $\omega L I_m$, and $\frac{1}{\omega C}I_m$ combine to form the voltage drop $\sqrt{R^2 + (\omega L - 1/\omega C)^2}\,I_m$.

The voltage applied to the branch is, physically, the sum of the three component voltages. In the form of an equation

$$RI_m \sin \omega t + \omega L I_m \cos \omega t - \frac{1}{\omega C} I_m \cos \omega t = v \tag{47}$$

or

$$RI_m \sin \omega t + \left(\omega L - \frac{1}{\omega C}\right) I_m \cos \omega t = v \tag{48}$$

The combination of the sine and cosine terms of the above equation may be effected in the same manner as previously outlined for the sine and cosine components. In the present case RI_m and $\left(\omega L - \frac{1}{\omega C}\right) I_m$ are considered as the two legs of the right triangle shown in Fig. 20.

[8] The reason for neglecting the constant of integration is similar to that given in the footnote on page 60.

It will be remembered from the discussion of the purely inductive and the purely capacitive branches that these two reactive elements cause exactly opposite phase displacements of the current with respect to the voltage. Since ωL has arbitrarily been considered to be a positive quantity, it becomes necessary to consider $1/\omega C$ a negative quantity. It should be recognized that, of and by itself, there is nothing inherently negative about the quantity $1/\omega C$. The fact that it acts oppositely to the quantity ωL in governing current flow requires that $1/\omega C$ be treated negatively if ωL is treated positively.

Impedance. If equation (48) is manipulated as indicated on page 68, the impedance of the RLC branch is found to be

$$\mathbf{Z}_{(RLC)} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \Bigg/ \tan^{-1} \frac{\left(\omega L - \frac{1}{\omega C}\right)}{R} \qquad (49)$$

If R is in ohms, L in henrys, C in farads, and ω in radians per second, $\mathbf{Z}$ is in ohms. Given R, L, C, and ω, the complete impedance function can be evaluated. The general expression for $\mathbf{Z}_{(RLC)}$ is of considerable importance in a-c circuit theory because all the impedance functions which have thus far been treated are directly deducible from this expression.

In a branch where ωL is negligibly small as compared with R and $1/\omega C$, the ωL term may be considered to be equal to zero, in which case the branch reduces to a resistance and capacitance branch.

$$\mathbf{Z}_{(RC)} = \sqrt{R^2 + \left(-\frac{1}{\omega C}\right)^2} \Bigg/ \tan^{-1} \frac{\left(-\frac{1}{\omega C}\right)}{R} \qquad (50)$$

The negative angle implies that the current wave lags the voltage wave by a *negative* angle. The correct physical interpretation is that the current wave leads the voltage wave by the angle whose tangent is $\left(\frac{1}{\omega C}\right) \Big/ R$.

With respect to its terminals the RLC branch will, in general, simulate the behavior of either the RL or the RC branch. If $\omega L > 1/\omega C$, the RLC branch responds to an impressed voltage at its terminals exactly as would an equivalent RL branch, the inductive reactance of which is $(\omega L - 1/\omega C)$. Similarly, if $1/\omega C > \omega L$, the RLC branch will respond to an impressed voltage at its terminals exactly as would an equivalent RC branch, the capacitive reactance of which is $(1/\omega C - \omega L)$. In either of the cases referred to, there will be interchanges of energy taking place between the two reactive elements.

The singular case, wherein $\omega L = 1/\omega C$, is of particular interest because the impedance here reduces to $R/\underline{0^\circ}$. With respect to its terminals the RLC branch, under the condition of $\omega L = 1/\omega C$, responds s would a purely resistive branch. If R is assumed to be a fixed uantity, the above condition may be obtained by the proper adjustment of L, C, or ω, and when $\omega L = 1/\omega C$ the impedance of the branch will be a minimum.

Example. If $R = 10$ ohms, $L = 0.056$ henry, and $C = 50$ μf, the impedance of the RLC branch at 60 cycles is

$$Z = \sqrt{10^2 + \left(377 \times 0.056 - \frac{10^6}{377 \times 50}\right)^2} \Big/ \underline{\tan^{-1} \frac{(21.1 - 53.0)}{10}}$$

$$= 33.4 \Big/ \underline{\tan^{-1} (-3.19)}$$

$$= 33.4 \Big/ \underline{-72.6^\circ} \text{ ohms}$$

If

$$v = 200 \sin 377t \text{ volts}$$

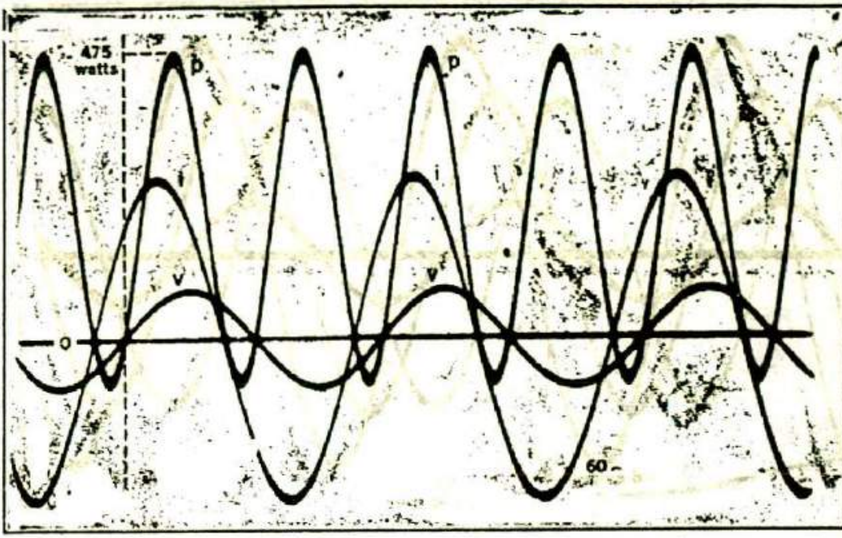
$$i = \frac{200}{33.4} \sin (377t + 72.6^\circ) \text{ amperes}$$

Power. Since $i = I_m \sin \omega t$ and $v = V_m \sin (\omega t + \theta)$, the expression for the instantaneous power delivered to the RLC branch takes the same form as equation (42), namely,

$$p = \frac{V_m I_m}{2} \cos \theta - \frac{V_m I_m}{2} [\cos 2\omega t] \cos \theta + \frac{V_m I_m}{2} [\sin 2\omega t] \sin \theta \quad (51)$$

In the present case θ may presumably take any value between $+90^\circ$ and -90°. The average power delivered to the RLC branch is in any case $\frac{V_m I_m}{2} \cos \theta$. [See equation (43).] The maximum value of the instantaneous reactive volt-amperes, $\left[\frac{V_m I_m}{2} \sin 2\omega t \sin \theta\right]$, is directly proportional to $\sin \theta$. Since the $\sin 2\omega t$ factor causes the instantaneous reactive volt-amperes to be alternately positive and negative, the absolute meaning of the sign of the reactive power term is not highly significant.

According to the convention of signs which has been employed in the present discussion, positive reactive volt-amperes — that is, a positive coefficient of $[\sin 2\omega t]$ in equation (51) — indicate inductive reactive volt-amperes, whereas negative reactive power indicates capacitive reactive volt-amperes. These signs are merely the result of choosing ωL positive and $1/\omega C$ negative. Further consideration of signs of reactive power will be given in the next chapter.

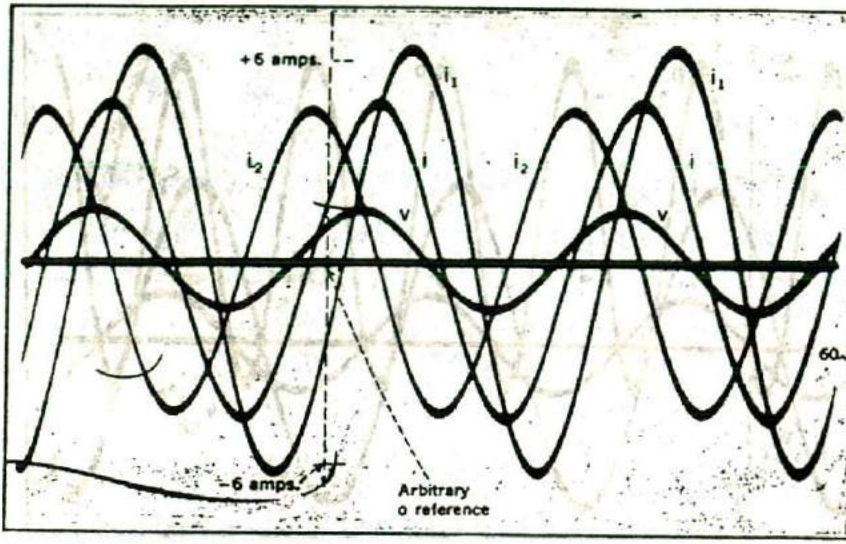


OSCILLOGRAM 9. Voltage, current, and power variations in an RLC circuit. $R = 20$ ohms, $L = 0.042$ henry, $C = 78\ \mu$f, $X_L = 15.8$ ohms, $X_C = 34$ ohms, $V_{max} = 141.4$ volts, $I_{max} = 5.23$ amperes, $P_{av} = 275$ watts.

The term $\left[\frac{V_m I_m}{2} \sin 2\omega t \sin \theta\right]$ is equal to zero at all times when $\theta = 0$, that is, when $\omega L = 1/\omega C$. In this case the reactive volt-amperes required by the inductive element are furnished by the capacitive element, and vice versa. Relatively large amounts of energy may oscillate between the reactive elements even though the RLC branch simulates a purely resistive branch at its terminals.

Oscillogram 9 illustrates the variations of v, i, and p in a particular RLC circuit. In the case shown $\omega L < 1/\omega C$ and the lead of the current with respect to the voltage is clearly indicated.

Impedance Functions. It should be understood from the foregoing analyses that impedance functions for any combinations of R, L, and C are independent of the point on the wave from which time is reckoned. In addition, the functions are entirely independent of whether the voltage or current wave is made the dependent wave. Thus in the RL branch a current wave $i = I_m \sin \omega t$ was assumed and the voltage wave $v = V_m \sin (\omega t + \theta)$ was found to lead the current by $\theta = \tan^{-1}(\omega L/R)$. If a voltage $v = V_m \sin \omega t$ is assumed impressed upon the circuit, the impedance function is the same, and it states that the voltage wave must lead the current by $\tan^{-1}(\omega L/R)$. Hence the current wave may be written as $i = I_m \sin (\omega t - \theta)$. Similar interpretations apply to any combination of R, L, and C. When the impedance function is found,



OSCILLOGRAM 10. Photographic records of the applied voltage and the three branch currents of the circuit arrangement shown in Fig. 22.

the *relation* between the voltage drop and the current is thereby determined. If one is assumed, the other may be determined from the impedance function as illustrated by the examples in the preceding articles.

Instantaneous Currents Combine Algebraically. The concept of adding instantaneous voltage drops across series elements to obtain the total voltage applied to a series circuit has been considered. Kirchhoff's emf law applies to a-c circuits if instantaneous values of voltage or their equivalents are considered. Likewise Kirchhoff's current law applies to a-c circuits provided instantaneous values of current or their equivalents[9] are employed. Figure 21 illustrates the principle in a simple case. Kirchhoff's current law states that the current flowing toward a junction, which in the present case is i, is equal to the current flowing away from the junction, namely, $i_1 + i_2$.

In general

$$\sum i_{\text{toward a junction}} = \sum i_{\text{away from the junction}} \tag{52}$$

or, if current *away* from the junction is considered as *negative* current *toward* the junction,

$$\sum i_{\text{toward a junction}} = 0 \tag{53}$$

[9] The equivalents referred to are the vector forms that are employed to replace instantaneous values. See Chapters III and IV.

If the currents are measured by devices which do not respond to instantaneous values, the combined measurements, in general, will not satisfy the above current law, for the simple reason that the devices employed fail to account for the relative phase positions of the currents involved.

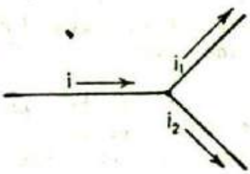


FIG. 21. Instantaneous current toward a junction is equal to the instantaneous current away from the junction.

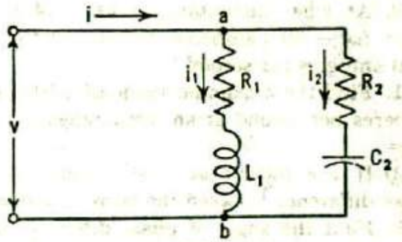


FIG. 22. R_1L_1 branch in parallel with R_2C_2 branch.

Oscillogram 10 shows how the instantaneous currents i_1 and i_2 of Fig. 22 add algebraically to yield the resultant current i. The analytical method of finding the expression for i from i_1 and i_2 will be explained in Chapter VI.

PROBLEMS

1. (a) What is the frequency of a 10-pole alternator when running at 360 rpm? (b) At what speeds should a 6-pole alternator run to yield 25, 30, 50, and 60 cycles per second?

2. How many poles are required on an alternator which runs at 300 rpm to develop 50 cycles per second?

3. What is the mechanical angular velocity of the machine in Problem 2? What is the electrical angular velocity or the angular frequency?

4. Write the expression for a sine-wave current having a maximum value of 1.732 amperes and a frequency of 1591 kilocycles. The $t = 0$ reference is to be selected at a point where di/dt is positive and $i = +1.5$ amperes.

5. Express as a sine function of time a 50-cycle alternating current which has a maximum value of 10 amperes. What is the angular velocity of this current wave?

6. Express an alternating current of 10 amperes maximum value which has an angular velocity of 377 radians per second as a cosine function of time. What is the frequency of this wave?

7. Express the equation of the current wave of Problem 5 if time is reckoned from the positive maximum value of the wave. Also express it for each possibility when time is reckoned from the negative 5-ampere value of the wave.

8. The time variation of a voltage wave is given by $e = 100 \sin 157t$ volts, where t is expressed in seconds.

(*a*) What is the maximum value of the voltage?

(*b*) What is the frequency of the voltage variation?

(*c*) If $e = 100 \sin (157t + 30°)$, what is the maximum value of the voltage? the frequency?

9. What are the maximum and minimum rates of change of the voltage depicted in Oscillogram 1, page 51, if the maximum voltage is 140 volts? Express results in volts per second.

10. At what instantaneous value of current is the 60-cycle current wave $i = 10 \sin (\omega t - 30°)$ amperes changing at the rate of 3265 amperes per second? (*b*) at 2260 amperes per second?

11. Find the maximum value of a 50-cycle current wave that is changing at 2000 amperes per second at an instantaneous value 30° from the maximum value of the wave.

12. If $v = 100 \sin (\omega t - 30°)$ and $i = 10 \sin (\omega t - 60°)$, what is the angle of phase difference between the current and voltage waves? Which wave leads?

13. Find the angle of phase difference between $v = 100 \cos (\omega t - 30°)$ and $i = -10 \sin (\omega t - 60°)$. Which wave lags?

14. A voltage has for its equation $v = 100 \cos \omega t$. Write the equation of a current wave of 10 amperes maximum which leads the specified voltage wave by $\frac{1}{8}$ of a cycle. Let angular measure be expressed in radians in this particular case.

15. (*a*) Given a sine-wave signal, the analytical expression of which is $[k \sin (2\pi f t)]$. If this wave is sampled (or tested) at $t = 0$ and at equal time intervals thereafter of $\Delta t = 1/f_s = 3/(2f)$, what is the nature of the sampled signal? (f_s represents the sampling frequency.)

(*b*) Make a rough plot of 6 or 8 cycles of a sine-wave voltage. Let this signal be sampled first at 135° after $v = 0$ (dv/dt positive) and thereafter at equal intervals of 315° of the signal voltage. If the sampled data is interpreted as representing points of a sine wave, what is the sampled frequency relative to the frequency of the actual signal?

16. (*a*) Find the instantaneous value of a sinusoidal alternating current having a maximum value of 90 amperes, 60° after the current passes through its zero value going positive; 225° after the current passes through its zero value going positive.

(*b*) Find the difference in time between the 60° value of current and the 225° value of current if the frequency is 50 cycles.

17. The current through a particular filter choke may be represented approximately by the equation

$$i = 1.0 + 0.50 \sin 1885t - 0.10 \cos 3770t \text{ amperes}$$

or

$$i = 1.0 + 0.50 \sin \alpha - 0.10 \cos 2\alpha \text{ amperes}$$

where $\alpha = 1885t$ radians if t is expressed in seconds.

(*a*) What is the frequency of the sine term? of the cosine term?

(*b*) What are the maximum and minimum values of current?

(*c*) Graph the current i with respect to time t or with respect to angular measure α.

18. A voltage $v = 150 \cos 314t$ volts is applied to a purely resistive branch of $R = 30$ ohms.

(*a*) Write the expression for i as a function of time, employing numerical coefficients.

Ans.: $i = 5 \cos 314t$ amperes.

(*b*) What is the frequency of the voltage and current variations?

Ans.: 50 cycles.

(c) Write the expression for p as a function of time, employing numerical coefficients.

Ans.: $p = 750 \cos^2 314t = 375 + 375 \cos 628t$ watts.

(d) What is the frequency of the power variation?

Ans.: 100 cycles.

19. A current $i = 5 \sin (110t + 30°)$ amperes flows in a purely resistive branch of 20 ohms.

(a) Write the expression for v as a function of time employing numerical coefficients.

(b) What is the frequency of the voltage variation?

(c) Write the expression for p as a function of time, employing numerical coefficients.

(d) What is the frequency of the power variation?

20. A voltage $v = 100 \cos (\omega t + 60°)$ volts is impressed upon a pure resistance circuit of 10 ohms.

(a) Write the equation with respect to time of the current wave and employ numerical coefficients.

(b) Find the equation with respect to time of the power wave.

(c) What is the maximum instantaneous power?

(d) What is the minimum instantaneous power?

(e) What is the average value of the power wave?

21. (a) What is the maximum time rate of change of a 60-cycle alternating current of sine form, the maximum value of which is 10 amperes?

(b) If this current flows through a pure inductance of 100 millihenrys, find the maximum value of the voltage across the terminals of the inductance.

22. A voltage $v = -150 \sin 377t$ volts is applied to a particular ciruit element, and it is found, by oscillographic analysis, that $i = 10 \cos 377t$ amperes. Make a sketch of the v and i waves. Find the nature and magnitude of the circuit parameter.

Ans.: $L = 0.0398$ henry.

23. A voltage drop $v = 100 \sin (377t + 30°)$ volts is across a pure inductance of 0.02654 henry.

(a) Use numerical coefficients and express the current through the coil as a function of time.

(b) Find the equation with respect to time of the power wave. Express the result as a single sine function.

(c) What is the average power?

(d) What is the first value of time at which maximum energy is stored in the inductance?

(e) What is the maximum amount of energy stored in the inductance during a cycle? State units.

24. A current of $5 \sin 300t$ amperes flows through a pure inductive branch of 0.2 henry.

(a) Find the impedance function and express numerically.

(b) How many joules are stored in the magnetic field about the inductance when $t = 0.05$ second?

(c) Write the expression for v as a function of time employing numerical coefficients.

25. A voltage $v = 200 \cos (157t + 30°)$ volts is applied to a particular circuit element, and it is found, by oscillographic analysis, that $i = 5 \sin (157t - 150°)$ amperes. Sketch the v and i waves. Find the nature and magnitude of the circuit parameter.

26. A voltage $v = 100 \sin 377t$ volts is impressed on a pure capacitance of 530.5 μf.

(*a*) Write the expression for i as a function of time employing numerical coefficients.

(*b*) Find the expression for the power wave as a function of time, employing numerical coefficients.

(*c*) How many joules are stored in the condenser when the current is zero? when the current is a maximum?

27. A voltage $v = 200 \sin 377t$ volts is applied to an inductive branch, and the maximum current is found, by oscillographic analysis, to be 10 amperes.

(*a*) Find the value of L in millihenrys.

Ans.: 53.1 millihenrys.

(*b*) If it is known that this inductance coil actually possesses 1.0 ohm resistance, what is the true value of L, assuming that $V_m = 200$ volts and $I_m = 10$ amperes?

Ans.: $L_{\text{true}} = \sqrt{20^2 - 1^2}/377 = 53.04$ millihenrys.

28. $R = 10$ ohms and $L = 0.05$ henry are connected in series and energized by a 25-cycle sinusoidal voltage, the maximum value of which is 150 volts.

(*a*) Find the complete impedance expression for the RL branch.

(*b*) Write the expression for the supply voltage as a function of time, making $v = 0$ (dv/dt positive) at $t = 0$.

(*c*) Write the expression for current as a function of time, assuming that the voltage in (*b*) is applied to the branch. Employ numerical coefficients.

(*d*) Write the expression for the instantaneous power delivered to the branch as a function of time. Express the result in three terms — a constant term, a single cosine term, and a single sine term. What is the average power?

(*e*) What are the reactive volt-amperes or vars?

(*f*) Sketch the v, i, and p variations in rectangular coordinates.

29. $R = 10$ ohms and $L = 0.05$ henry are connected in series and energized by a 25-cycle sinusoidal voltage, the maximum value of which is 150 volts.

(*a*) Find the complete impedance expression for the RL branch.

Ans.: $12.7\underline{/38.2°}$ ohms.

(*b*) Write the expression for the supply voltage as a function of time, making $v = 75$ (dv/dt positive) at $t = 0$.

Ans.: $v = 150 \sin (157t + 30°)$ volts.

(*c*) Write the expression for current as a function of time, assuming that the voltage in (*b*) is applied to the branch. Employ numerical coefficients.

Ans.: $i = 11.8 \sin (157t - 8.2°)$ amperes.

(*d*) Write the expression for the instantaneous power delivered to the branch as a function of time. Express the result in three terms — a number, one cosine, and one sine term. What is the average power delivered?

Ans.: $p = 695 - 820 \cos 314t + 328 \sin 314t$ watts.
$P_{av} = 695$ watts.

30. A resistive element of 30 ohms is connected in series with an inductance coil, the self-inductance of which is 50 millihenrys and the ohmic resistance of which is 4.5 ohms. A voltage $v = 100 \cos 377t$ volts is connected to the series branch.

(*a*) Evaluate the expression for i.

(*b*) Evaluate the expression for p.

(*c*) Write the expression for the real power as a function of time, employing numerical coefficients. What is the average value of the instantaneous real power?

(*d*) Write the expression for the reactive volt-amperes as a function of time, employing numerical coefficients. What is the average value of the instantaneous reactive power?

(e) What is the inductive reactance of the branch in ohms?

31. A current $i = 10 \cos 157t$ amperes flows in an RL circuit containing $R = 15$ ohms and $L = 0.0637$ henry.

(a) Write the equation of v as a function of time, employing numerical coefficients.

(b) Write the expression for the power wave as a function of time.

32. (a) What is the capacitive reactance of an 8-μf capacitor at 60 cycles?

(b) What is the capacitive reactance of an 800-$\mu\mu$f capacitor at 6 megacycles?

33. A resistive element of 151 ohms is connected in series with a capacitor of 4 μf capacitance. A 500-cycle sinusoidal voltage, the maximum value of which is 15 volts, energizes the RC branch.

(a) Write the expression for the supply voltage, choosing the $t = 0$ reference at the point of maximum positive voltage.

(b) Evaluate $\mathbf{Z}_{RC}$ completely.

(c) Evaluate the expression for i.

(d) Evaluate the expression for p which corresponds to the product of voltage and current employed here, and express all trigonometric terms with exponents no higher than unity.

34. Assume that the current $i = I_m \sin \omega t$ flows through a given RC branch. Show that the voltage across the branch is

$$v = I_m Z \sin(\omega t + \theta) = V_m \sin(\omega t + \theta)$$

where

$$Z = \frac{V_m}{I_m} = \sqrt{R^2 + \left(\frac{-1}{\omega C}\right)^2}$$

and

$$\theta = \tan^{-1} \frac{\left(\frac{-1}{\omega C}\right)}{R} \qquad \textit{Hint: } \frac{q}{C} = \frac{\int i\,dt}{C}$$

35. A resistance of 10 ohms is in series with a 303-μf capacitor. If the voltage drop across the capacitor is $150 \sin (220t - 60°)$ volts, find the equation with respect to time of the voltage drop across the entire series circuit. Find also the expression for the current at any time t.

36. A 2000-cycle alternating voltage of sine form when impressed across the terminals of a condenser establishes a current of 0.01 ampere (maximum value). If the maximum value of the voltage is 20 volts find the capacitance of the condenser in microfarads.

37. Consider a series RLC branch wherein $R = 10$ ohms, $L = 0.10$ henry, and C is 200 μf. Assume that the current $i = 10 \sin (157t)$ amperes flows through the RLC branch.

(a) Write the expression for the voltage drop across R, namely, Ri, employing numerical coefficients.

(b) Write the expression for the voltage drop across L, namely, $L\,di/dt$, employing numerical coefficients.

(c) Write the expression for the voltage drop across C, namely, q/C, employing numerical coefficients.

(d) Add (a), (b), and (c) to find the voltage drop across the RLC branch. Express the result as a single sine function of time.

(e) What is the numerical value of the impedance of the series RLC branch?

38. Assume that the current $i = I_m \cos \omega t$ flows through a given RLC branch.

Show that the voltge across the branch is

$$v = I_m Z \cos(\omega t + \theta) = V_m \cos(\omega t + \theta)$$

where

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

and

$$\theta = \tan^{-1} \frac{\left(\omega L - \frac{1}{\omega C}\right)}{R}$$

39. In the following exercise, it is assumed that a coil having L henrys of inductance and R ohms of series resistance is placed in series with a condenser of C farads of capacitance. A current of $i = I_m \sin(t\sqrt{LC})$ amperes flows in the circuit. Show that the energy $w_L + w_C$ = constant, and evaluate this constant.

40. A resistive element of 20 ohms, an inductance coil of L = 300 millihenrys and R_L = 10 ohms, and a condenser of 50 μf capacitance are connected in series to form an RLC branch. A voltage $v = 100 \sin 157t$ volts is applied to the RLC branch.

(*a*) What is the numerical value of Z_{RLC}?

(*b*) Write the expression for i, employing numerical coefficients.

(*c*) Write the expression for p, employing numerical coefficients, and express all trigonometric functions with exponents no higher than unity.

(*d*) What is the average value of the power delivered to the branch?

(*e*) What is the maximum value of the reactive volt-amperes?

(*f*) Write the expression for the voltage drop across the 20-ohm resistive element as a function of time, employing numerical coefficients.

(*g*) Write the expression for the instantaneous power delivered to the 20-ohm resistor as a function of time, employing numerical coefficients.

41. A voltage $v = 282.8 \sin 500t$ volts is applied to a series circuit, and the resulting current is found to be $i = 5.656 \sin(500t - 36.87°)$ amperes. One element of this series combination is known to be a capacitor which has a capacitance of 100 μf. Determine the magnitudes of the other series elements present.

chapter III Effective Current and Voltage—Average Power

Except for the maximum values of sinusoidal wave variations attention has been given only to general instantaneous values of current and voltage. The only practicable method of measuring instantaneous values of current, voltage, and power is by means of an oscillograph, a very useful instrument in many respects but one which is relatively inaccurate, cumbersome, and costly. It was shown in the previous chapter that instantaneous values are inconvenient to manipulate analytically, and in general they fail to specify concisely the magnitudes of the quantities involved. In this chapter the values of currents and voltages usually dealt with will be considered.

Ampere Value of Alternating Current. Alternating currents are defined so as to make applicable to them essentially the same laws that govern heating and transfer of power by direct current. An alternating current which produces heat in a given resistance at the same average rate as I amperes of direct current is said to have a value of I amperes. The average rate of heating produced by an alternating current during one cycle is $\frac{1}{T}\int_0^T Ri^2\,dt$. The average rate of producing heat by I amperes of direct current in the same resistance is RI^2. Hence by definition

$$RI^2 = \frac{1}{T}\int_0^T Ri^2\,dt$$

and

$$I = \sqrt{\frac{1}{T}\int_0^T i^2\,dt} = \sqrt{\text{average } i^2} \tag{1}$$

The current given in equation (1) which defines the alternating current in terms of its average rate of producing heat in a resistance is called the root mean square (abbreviated rms) value. It is also called the effective or virtual value. The graphical evaluation of the rms value of an alternating current is illustrated in Fig. 1. When the equation of the wave is not known or when it is inconvenient to determine it, the graphical means, suggested by Fig. 1, of evaluating equation (1) becomes a useful method to employ.

Problem 1. Find the effective value of a current that starts at zero, rises instantaneously, then remains at a value of 20 amperes for 10 seconds, then decreases instantaneously, remaining at a value of 10 amperes for 20 seconds, and then repeats this cycle. *Ans.:* 14.14 amperes.

The rms value of an alternating current may be measured with the ordinary dynamometer type of meter. This meter consists of two coils in series, one of them being movable. The force tending to turn

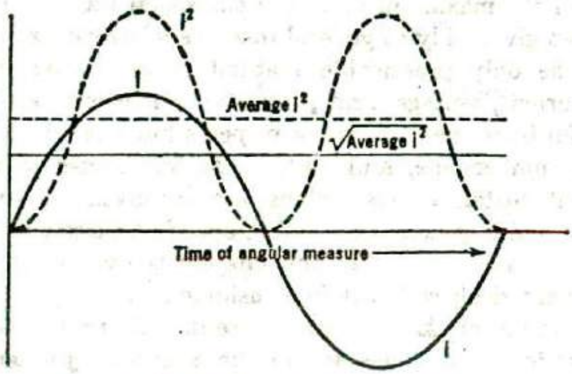


Fig. 1. Graphical evaluation of rms value.

the movable coil from any fixed position is proportional to the product of the currents in the two coils. Since the coils are in series and the same current flows in each, the force for any given position of the coils is proportional to i^2. Since the coil has a relatively high inertia, it cannot follow the variation in the force produced, and therefore takes a position corresponding to the average force or average i^2. If a suitable square root scale is placed under the pointer, the pointer will indicate the square root of the average square, or the rms value. Other types of a-c ammeters are also used to indicate effective values of current. (See Chapter X.)

Alternating Volt. An alternating volt is the value of a wave of alternating potential which maintains an alternating current of 1 rms ampere through a non-inductive resistance of 1 ohm. It therefore follows that the volt value of a wave is measured by the square root of the average square of the instantaneous values of the voltage wave.

Average Values. The average value of any a-c wave which is symmetrical about the zero axis is zero. However, when average value is applied to alternating quantities, it usually means the average of either the positive or negative loop of the wave. This value represents the d-c equivalent for electrolytic action of the alternating wave *abcde*, Fig. 2, if the wave were commutated (or rectified) and made the same as the wave *abcfe*. Since the average ordinate multiplied by the base is equal to the area under the curve, it follows directly that

$$\text{Average value} = \frac{2}{T}\int_0^{T/2} i\,dt \tag{2}$$

Equation (2) is applicable only when the wave passes through zero at the time $t = 0$. For any other condition the time t_1 at which the instantaneous value of the wave is zero must be determined and the average value found from

$$\text{Average value} = \frac{2}{T}\int_{t_1}^{(t_1+T/2)} i\,dt \tag{3}$$

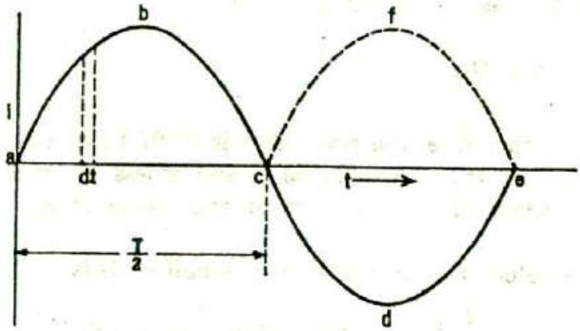


FIG. 2. Rectified a-c wave shown dotted.

If the average values of the positive and negative loops are different, the actual average value taken over a complete cycle represents the value of a d-c component in the wave. For example, the average

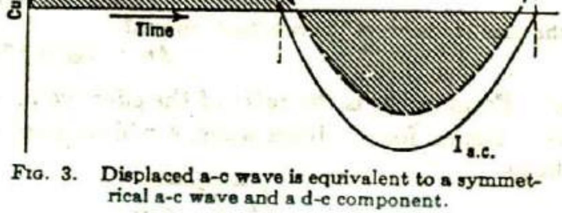


FIG. 3. Displaced a-c wave is equivalent to a symmetrical a-c wave and a d-c component.

value of the cross-hatched wave in Fig. 3 is I_{dc}. Inspection will show that the dotted wave is the sum of the alternating wave I_{ac} and the direct current I_{dc}.

Effective and Average Values of a Sinusoid. Through the use of equation (1) the effective value of any wave may be found. If the equation of the wave is not known, the integration must be performed graphically. When the equation is known, the analytical solution is generally to be preferred. Consider the sinusoid,

$$i = I_m \sin \omega t$$

$$I^2_{(\text{rms})} = \frac{1}{T}\int_0^T i^2\, dt = \frac{1}{T}\int_0^T I_m^{\,2} \sin^2 \omega t\, dt = \frac{I_m^{\,2}}{T}\int_0^T (\tfrac{1}{2} - \tfrac{1}{2}\cos 2\omega t)\, dt$$

$$= \frac{I_m^{\,2}}{T}\left[\frac{t}{2} - \frac{1}{4\omega}\sin 2\left(\frac{2\pi}{T}\right)t\right]_0^T = \frac{I_m^{\,2}}{2}$$

$$I_{(\text{rms})} = \frac{I_m}{\sqrt{2}} = 0.707 I_m \tag{4}$$

For a *sine wave*, therefore, the rms value is 0.707 times the maximum. In general, $I_{(\text{rms})}$ is written simply as I, and unless otherwise specified the symbol I refers to the effective or rms value of an alternating current.

The average value of a sinusoid over one-half cycle is

$$I_{\text{av}} = \frac{2}{T}\int_0^{T/2} I_m \sin \omega t\, dt = \frac{2}{\pi} I_m = 0.636 I_m \tag{5}$$

Problem 2. A resultant current wave is made up of two components, a 5-ampere, d-c component and a 60-cycle, a-c component which is of sinusoidal wave form and which has a maximum value of 4 amperes.

(*a*) Draw a sketch of the resultant current wave.

(*b*) Write the analytical expression for the resultant current wave, choosing the $t = 0$ reference at a point where the a-c component is at zero value and where di/dt is positive.

Ans.: $i = [5 + 4 \sin (377t)]$ amperes.

(*c*) What is the average value with respect to time of the resultant current over a complete cycle?

Ans.: $I_{\text{av}} = 5$ amperes.

(*d*) What is the effective value of the resultant current?

Ans.: $I_{\text{eff}} = 5.75$ amperes.

Form Factor. Form factor is the ratio of the effective to the average value of a wave. Hence, for a voltage wave, e, which has equal positive and negative loops:

$$\text{Form factor} = \frac{\sqrt{\dfrac{1}{T}\displaystyle\int_0^T e^2\, dt}}{\dfrac{2}{T}\displaystyle\int_0^{T/2} e\, dt} \tag{6}$$

Equation (6) is subject to the same limitations as those explained for equation (2). Form factor has very little physical significance. It gives no certain indication of wave shape or wave form. Although a peaked wave will usually have a higher form factor than a flat-topped wave, it cannot be conclusively stated that one wave is more peaked than another because it has a higher form factor. That form factor tells nothing of the shape of a wave is evident from the fact that a sine wave and the wave $e = E_m \sin \omega t + (5/12)E_m \sin 5\omega t$, shown in Fig. 4, have the

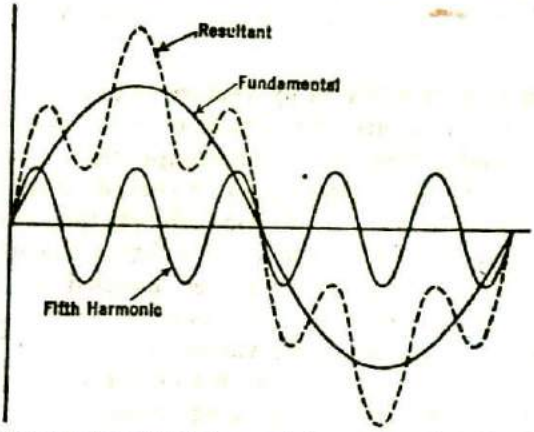


FIG. 4. Form factor of dotted wave is the same as that of a sine wave.

same form factor, namely, 1.11. However, form factor does give some indication of the relative hysteresis loss that will exist when a voltage is impressed on a coil wound on an iron core. Also some use is made of form factor in determining effective voltages induced in such coils when a known non-sinusoidal flux wave is present in the iron core.

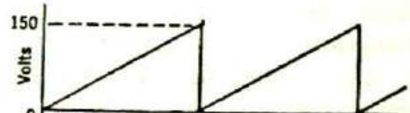


FIG. 5. Sawtooth wave form of voltage for Problems 3 and 4.

Problem 3. Find the form factor of the sawtooth wave form shown in Fig. 5. *Hint:* Between the limits of $t = 0$ and $t = T = 3$ seconds, the analytical expression for the voltage is $e = 50t$ volts. In a case of this kind,

$$E_{av} = \frac{1}{T}\int_0^T e\,dt$$

Ans.: 1.155.

Crest or Peak Factor. The crest, peak, or amplitude factor is the ratio of the maximum value of a voltage wave to the effective value. For the dotted wave shown in Fig. 4 the crest factor is 1.85. A knowledge of this factor is necessary when using an ordinary voltmeter to measure a voltage employed in insulation testing. The dielectric stress to which insulation is subjected depends upon the maximum value of the voltage attained. Since waves of the same effective value may have different maximum values, it is obvious that a knowledge of crest factor is required when making dielectric tests. The crest factor of a *sine wave* is

$$\frac{E_m}{0.707E_m} = \sqrt{2}$$

Problem 4. Find the crest factor of the sawtooth wave form shown in Fig. 5.
Ans.: 1.732

Representation of Sine Waves by Vectors or Phasors. It has previously been stated that an attempt is made to secure sine waves of alternating currents and potentials. Alternating-current computations are often based upon the assumption of sine waves of voltage and current. When non-sinusoidal quantities are encountered, they are expressed in terms of a number of sine components of different magnitudes and frequencies, and these components are then handled according to the methods applicable to sine waves. In general, it would be cumbersome continually to handle instantaneous values in the form of equations of the waves. A more convenient means is to employ a vector method of representing these sine waves. The directed lines or vectors that are employed to represent sinusoidally time-varying quantities in a coplanar system are called phasors. Actually for the purposes in this book there is no difference between considering these representations as vectors or phasors. This distinction is made to avoid confusion in some of the more advanced work involving vector analysis as defined in mathematics. Since in elementary circuit analysis a vector diagram and phasor diagram mean the same thing, the terms will be used interchangeably. The phasor or vector representations of sine functions may be manipulated instead of the sine functions themselves to secure the desired result.

The sine wave of current $i = I_m \sin \omega t$ is shown in Fig. 6*a*. All the ordinates of this wave at the various times t may be represented by the projection of the revolving vector OA on the vertical axis of Fig. 6*b*. This projection is $I_m \sin \omega t$ if OA has a magnitude of I_m. This is the equation of the wave shown in Fig. 6*a*.

If two sine waves are related as shown in Fig. 7, each may be represented by the projections of counterclockwise[1] revolving vectors on the

[1] Counterclockwise is assumed the positive direction of rotation of vectors. The counterclockwise direction of rotation has been arbitrarily used by engineers in the United States and many foreign countries. Some foreign countries have used clock-

vertical. A little study will show that the angle of phase difference for the two waves must also be the angular displacement between the two vectors OA and OB representing them. If OA and OB are added vec-

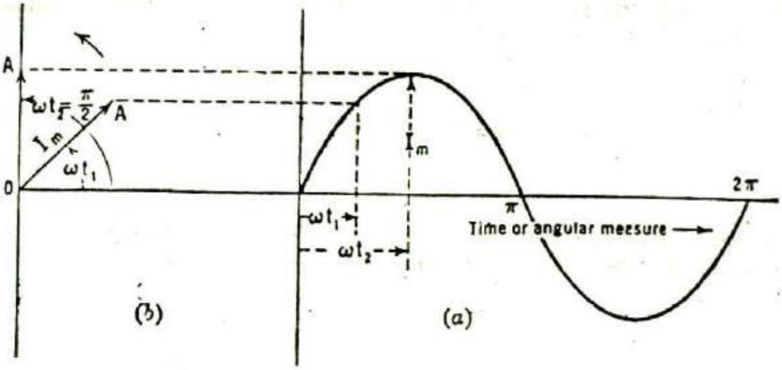


FIG. 6. Projection of a revolving vector represents a sine wave.

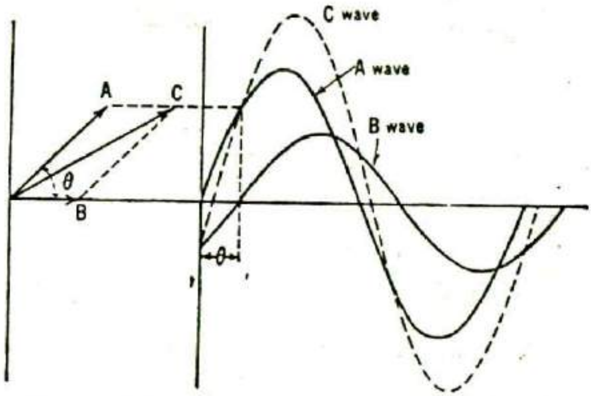


FIG. 7. Addition of sine waves by the use of vectors.

torially, a resultant OC is obtained whose projection on the vertical will represent the instantaneous values of the algebraic sum of the sine waves A and B.

Example 1. Add the following currents as waves and as vectors:

$$i_1 = 5 \sin \omega t$$

$$i_2 = 10 \sin (\omega t + 60°)$$

---

wise as positive. To avoid errors the student must always consider counterclockwise as the positive direction of rotation of all vectors in this book. One vector is said to be ahead or leading another when it is farther advanced counterclockwise than the other.

-7

As waves:

$$\begin{aligned}\text{Sum} &= i_0 = i_1 + i_2 = 5 \sin \omega t + 10 \sin (\omega t + 60^\circ) \\ &= 5 \sin \omega t + 10 \sin \omega t \cos 60^\circ + 10 \cos \omega t \sin 60^\circ \\ &= 10 \sin \omega t + 8.66 \cos \omega t\end{aligned}$$

Refer to the right triangle shown in Fig. 8*a*. If the previous equation is multiplied and divided by 13.23, there results

$$\begin{aligned}i_0 &= 13.23 \left[\frac{10}{13.23} \sin \omega t + \frac{8.66}{13.23} \cos \omega t\right] \\ &= 13.23 [\cos \alpha \sin \omega t + \sin \alpha \cos \omega t] \\ &= 13.23 \sin (\omega t + \alpha) \\ &= 13.23 \sin (\omega t + 40.9^\circ)\end{aligned}$$

As vectors: A wave of relative phase represented by $\sin \omega t$ will be represented by a vector along the reference axis. Positive angles will be assumed to be measured

FIG. 8.

counterclockwise. The two waves are then represented by vectors as shown in Fig. 8*b*. The sum will be found by adding x and y components.

$$\begin{aligned}\textstyle\sum x &= 5 + 10 \cos 60^\circ = 10 \\ \textstyle\sum y &= 10 \sin 60^\circ = 8.66 \\ \text{Sum} &= \sqrt{\overline{\textstyle\sum x}^2 + \overline{\textstyle\sum y}^2} = \sqrt{10^2 + 8.66^2} = 13.23 \\ \alpha &= \tan^{-1} \frac{\sum y}{\sum x} = \tan^{-1} \frac{8.66}{10} = 40.9^\circ\end{aligned}$$

Since the resultant is counterclockwise (positive) from the reference, the equation may be written as

$$i_0 = 13.23 \sin (\omega t + 40.9^\circ)$$

Problem 5. Subtract i_2 from i_1 in example 1 by both methods shown above.
Ans.: $8.66 \sin (\omega t - 90^\circ)$.

It is apparent that these coplanar vectors are merely convenient representations of sine waves, the independent variable of which is time. As such, they are time vectors and do not have any meaning so far as space relations are concerned. When the lengths of the two vectors represent maximum values of the waves respectively, the resultant vector will represent the maximum value of the resultant of the two waves.

Effective or rms values of voltages and currents are ordinarily used.

For sine waves these have been shown to be equal to the maximum value divided by $\sqrt{2}$. Thus maximum values of the vectors could be handled vectorially and the resultant divided by $\sqrt{2}$ to obtain the effective value. Instead, all the initial vectors could have their maximum values multiplied by 0.707 and the resultant of these would then be the resultant maximum divided by $\sqrt{2}$. If the latter procedure is followed, the vectors can be considered to represent effective values. Vectorial representation of effective values is customary, in which case the results are given directly in terms of effective values, the ones usually desired.

In drawing vector diagrams certain conventions must be observed. First, a convenient reference axis should be established. The vectors have their relations to one another fixed but they may be represented with respect to any axis. In Fig. 7, the vectors OA and OB were considered to revolve in order to represent the waves. The resultant OC was obtained by adding the two vectors when OB was along the axis of reference. Obviously, the same result would have been obtained had OA and OB been added when stopped in any other position with respect to the reference axis, provided that their magnitudes and the angle θ between them were not changed. Second, it must be observed that *counterclockwise is considered the positive direction of rotation of vectors* and that a vector rotated through an angle of lead or ahead of another vector must be rotated counterclockwise. It then follows that an angle of lag from a given axis must be in the clockwise direction. A vector thus rotated is said to be behind the axis in question.

To illustrate the use of these conventions, the vector diagrams of voltage and current for a pure resistance, a pure inductance, and a pure capacitance circuit will be drawn. The waves shown on Oscillogram 4, page 59, for a pure resistance circuit, indicate that the applied voltage is in phase with the current. With current taken as, or along, the reference axis the vector diagram is shown in Fig. 9.

FIG. 9. Resistance branch and vector diagram.

It was shown in Chapter II and experimentally illustrated in Oscillogram 5, page 63, that the wave of voltage drop across a circuit containing only inductance leads the current by 90°. This relation is illustrated vectorially with the current as the reference in Fig. 10 of the present chapter.

Reference to Oscillogram 6, page 66, will show that Fig. 11 of the present chapter represents vectorially the relations previously explained for the purely capacitive circuit.

Current was taken as the reference in the three previous diagrams. This was not necessary. The current could just as well have been drawn at any angle with respect to the reference axis, but for any partic-

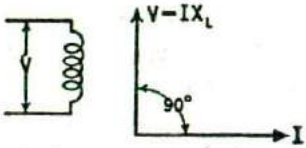


FIG. 10. Inductance branch and vector diagram.

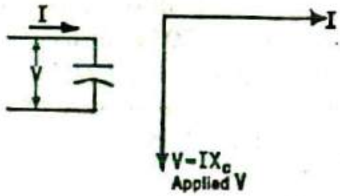


FIG. 11. Capacitance branch and vector diagram.

ular case the relation between current and voltage must remain the same, that is, the resistance drop must always be in phase with the current, the drop across the inductance must always lead the current by exactly 90°, and the drop across the capacitance must always lag the current by exactly 90°. The reference axis that appears to be the most convenient for the particular problem at hand should be chosen.

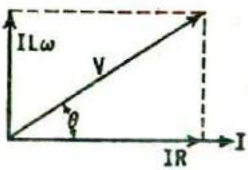


FIG. 12. Addition of voltage drops across L and R.

Vector Diagrams as Determined by Resistance and Reactance Drops and Impedance Functions. If a current $i = I_m \sin \omega t$ is assumed to flow in a circuit containing R and L, Kirchhoff's emf law states that $v = Ri + L\, di/dt$. Therefore $v = RI_m \sin \omega t + I_m L\omega \cos \omega t$. Since $RI_m \sin \omega t$ is of the same phase as $I_m \sin \omega t$, the resistance drop is shown in phase with the current in the vector diagram of Fig. 12. It will also be noted that $I_m L\omega \cos \omega t$ is 90° ahead of $I_m \sin \omega t$. Hence it is so drawn on the vector diagram. The vector sum of these two components is the resultant applied voltage **V**. The angle between **V** and **I** is $\theta = \tan^{-1} \omega L/R$. The same relation between **V** and **I** is obtained from the impedance function $Z\underline{/\theta}$. As explained in Chapter II, a positive angle θ means that the applied voltage leads the current or that the current lags the applied voltage by the phase angle θ. Thus the relation of **V** and **I** shown in the vector diagram could have been shown directly from the impedance function where the angle tells the phase and V/Z gives the magnitude of **I**. It should be noted that effective values were used exclusively in Fig. 12. Through the same procedure the student can show that Fig. 13 represents the vector diagram for an R and C circuit. The vector diagram of the R, L, and C circuit combines the vector diagrams in Figs. 12 and 13 the results of which appear in Fig. 14.

Problem 6. A 60-cycle current of 15 amperes flows in a circuit of 5 ohms resistance, 10/377 henry inductance, and $1/(377 \times 15)$ farad capacitance. Draw the vector diagram, and calculate the applied voltage and the phase angle between it and the current. *Ans.:* 106 volts; angle 45°.

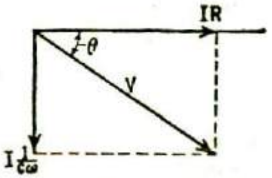


FIG. 13. Addition of voltage drops across C and R.

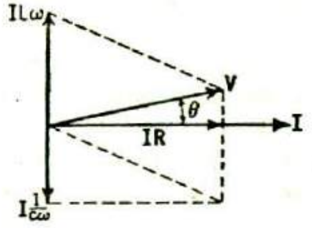


FIG. 14. Addition of voltage drops across L, C, and R.

Significance of Currents Flowing in the Direction of Voltage Rises and Drops. If the potential becomes greater in the direction of tracing a circuit, a voltage rise is being encountered. For example, assume the polarities of a circuit at some instant to be as indicated in Fig. 15. When tracing from a to b through the generator, the tracing is in the direction of increasing potential (from minus to plus) or in the direction of a voltage rise. In a similar way, when tracing through the load from c to d, the tracing is in the direction of a fall of potential or a voltage drop. Since the generator is the "pump," the current will flow from minus to plus through the generator, whereas in the external circuit it flows from plus to minus. It is evident, then, that a current flowing in the general direction of a potential rise represents electrical power generated or delivered. Also, when the current flows in the direction of a potential fall or drop, as it does through a load, power is being consumed or taken. If, then, a voltage rise is assumed positive, the generated power would be positive. A voltage drop is then negative and, since the same current flows in the direction of the voltage drop through the load, the power determined would be negative. These are the usual conventions employed when power generated and power consumed are simultaneously considered. If a voltage drop is assumed positive, then positive current in conjunction with the positive drop would yield positive power and

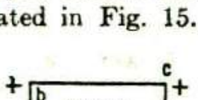


FIG. 15. Polarities of an a-c generator and load at some instant.

under these conditions power absorbed is positive. It is immaterial which conventions are used; that which is the most expedient is the one to choose. Physically, the same results are obtained. Although the above conventions are the most common, it is possible to establish other systems.

If a voltage rise is assumed positive, the question sometimes arises: Will generated power still be positive if the tracing direction is reversed? The answer is yes, as may be shown by the following considerations. Assume the tracing direction in Fig. 15 is *badc*. Then a voltage drop is encountered in the tracing direction through the generator. Since a voltage rise was considered positive, this drop through the generator will be negative. Since current flows through a generator in the general direction of increasing potential, the current will be in a direction opposite to that of the tracing direction. Hence it must be called a negative current. The product of the voltage drop through the generator, which was negative, by the negative current (opposite to the tracing direction) is positive. The sign of power generated is therefore unchanged. Similarly, it may be shown that the sign of the power dissipated by the load is unchanged. Hence the choice of the tracing direction does not affect the signs of generated and dissipated power. These are fixed by the signs assumed for voltage rises and drops in conjunction with the current.

Power, Real and Reactive. In Chapter II it was shown that the general expression for average power, when waves of voltage and current are sinusoidal, is $\frac{V_m I_m}{2} \cos\theta$. Since the maximum value of a sine wave divided by the square root of 2 is the effective value, the equation for average power may be written

$$P = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \cos\theta = VI \cos\theta \tag{7}$$

When V is in volts and I is in amperes, the power is expressed in watts. As previously shown, the power in a single-phase circuit is not constant. The instantaneous power from equation (42), Chapter II, is

$$p = \left[\frac{V_m I_m}{2} \cos\theta - \frac{V_m I_m}{2} \cos\theta \cos 2\omega t\right] + \frac{V_m I_m}{2} \sin\theta \sin 2\omega t \tag{8}$$

The first two terms of the right side of equation (8) represent instantaneous real power. When $2\omega t$ is an odd multiple of π, the value of the real power is

$$\frac{2V_m I_m}{2} \cos\theta = 2VI \cos\theta$$

When $2\omega t$ is a multiple of 2π, real power is 0. Hence real power in a single-phase circuit fluctuates between 0 and $2VI\cos\theta$ and has an average value of $VI\cos\theta$ (shown in Chapter II). The third term of the right-hand member of equation (8) represents what has been called instantaneous reactive power, or, preferably, instantaneous reactive volt-amperes. Its equation is

$$p_X = \left(\frac{V_m I_m}{2}\sin\theta\right)\sin 2\omega t \tag{9}$$

Thus instantaneous reactive volt-amperes fluctuate between $+\frac{V_m I_m}{2}\sin\theta$ and $-\frac{V_m I_m}{2}\sin\theta$. Whereas the average value of the instantaneous reactive volt-amperes is zero, the maximum value is $\frac{V_m I_m}{2}\sin\theta$. This is the value referred to when reactive volt-amperes are considered.[2] Hence

$$P_X = \frac{V_m}{\sqrt{2}}\frac{I_m}{\sqrt{2}}\sin\theta = VI\sin\theta \tag{10}$$

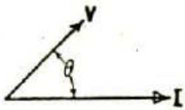


FIG. 16. Angle θ is positive when voltage leads current if current is along the reference axis.

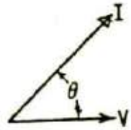


FIG. 17. Angle θ is positive when current leads voltage if voltage is along the reference axis.

It is plain that reactive volt-amperes as determined from equation (10) will be positive when θ is positive. As interpreted from vector diagrams when current is taken as a reference, Fig. 16, θ is positive when the voltage leads the current or for inductive loads. If voltage is taken along the reference, Fig. 17, θ is positive when the current leads the generated voltage. In the former case reactive volt-amperes are positive for inductive loads or lagging currents, whereas in the second case positive reactive volt-amperes are obtained when the load is capacitive or where the current leads the voltage. Another basis for determining the sign of reactive power was given in Chapter II. It is apparent that inductive reactive power or volt-amperes can be defined as positive or

[2] It should be recognized that this discussion refers to components of the resultant power wave. These components do not exist as separate entities but they are convenient components to consider for purposes of analysis. Actually a single wave, as shown in Chapter II, is the only power wave which has a physical existence.

negative depending upon the basis employed. Which of these signs to adopt has been a subject of discussion for many years, and each sign has been employed at various times. At the 1934 Paris meeting of the Committee on Electrical Units, reactive power caused by a current lagging the voltage was defined as negative reactive power. However this definition encountered several inconsistencies and after a great deal of discussion a committee of the American Institute of Electrical Engineers as reported in the January 1948 issue of *Electrical Engineering* recommended that inductive reactive volt-amperes be defined as positive reactive power. This convention is consistent with the sign obtained if the sign of the angle in the impedance function is employed in the formula for reactive volt-amperes. This convention is also consistent with the sign obtained by calculating I^2X where the sign of X for an inductive reactance is positive. This is the standard sign employed for inductive reactance. It will be seen in the next chapter that, when counterclockwise is considered as the direction for measuring positive angles of rotation, the use of complex numbers requires the adoption of the positive sign for inductive reactance. As a result of these and other considerations the present standard in the United States is to call inductive reactive volt-amperes positive. This is the sign which has been recommended to the International Committee on Electrical and Magnetic Units for adoption although no official action by that body has yet been taken. In any event it is desirable to label the reactive volt-amperes as inductive or capacitive. In combining the two in analytical work the important requirement is to consider one positive and the other negative regardless of the convention employed.

Reactive volt-amperes are expressed in vars, a term coined from the first letters of the words "volt amperes reactive." Reactive volt-amperes considered over a period of time represent oscillations of energy between the source and the load. Their function is to supply the energy for magnetic fields and charging capacitors, and to transfer this energy back to the source when the magnetic field collapses or when the capacitor discharges. Although reactive volt-amperes, as such, require no average energy input to the generators, they do necessitate a certain amount of generator volt-ampere capacity and thereby limit the available power output of the generators. Reactive volt-amperes cannot be transferred without incurring a copper loss. Although this i^2r loss is caused by the transfer of the reactive volt-amperes, it is not a part of the reactive volt-amperes. Reactive volt-amperes are due to quadrature components of voltage and current and as such represent zero average power. These additional losses must be supplied by an average energy input to the alternators.

From equation (42), Chapter II, instantaneous real power was found to be $V_m I_m \cos\theta \sin^2 \omega t$. This may be considered to consist of a voltage $V_m \sin \omega t$ and a current $(I_m \cos\theta) \sin \omega t$, which is in phase with the voltage. The current $I_m \cos\theta$ is called the in-phase component, power component, active component, or energy component of current with respect to voltage. In terms of root mean square values the power is due to a voltage V and a component of current $I \cos\theta$ in phase with V, as shown in Fig. 18. Since the product of the voltage V and energy component of current $I \cos\theta$ is $VI \cos\theta$, the same expression as equation (7) for power, it is evident that power may be determined in this

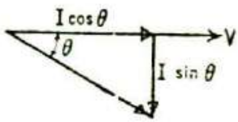


FIG. 18. In-phase and quadrature components of current with respect to voltage.

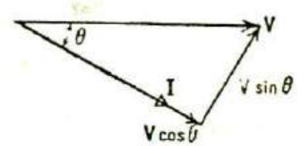


FIG. 19. In-phase and quadrature components of voltage with respect to current.

manner. If $\cos\theta$ is grouped with V, then $V \cos\theta$ may be viewed as the in-phase component, active component, energy component, or power component of voltage with respect to current, as shown in Fig. 19. Obviously, power may also be obtained by multiplying the in-phase component of voltage with respect to current by the current. Similarly, $I \sin\theta$ in Fig. 18 is the "out-of-phase component," quadrature component, or reactive component of current with respect to voltage. This component multiplied by the voltage gives reactive volt-amperes, as may be seen by comparison with equation (10). Also, $V \sin\theta$ is the quadrature, reactive, or wattless component of voltage with respect to current. This component of voltage multiplied by current also yields the reactive volt-amperes, or vars.

Volt-Amperes. The product of effective voltage by effective current in an a-c circuit is called volt-amperes. A larger unit is kilovolt-amperes, abbreviated kva. Obviously, a given number of volt-amperes may represent any number of different values of power, depending upon the value of $\cos\theta$ in equation (7). Cosine θ is therefore a factor by which volt-amperes are multiplied to give power. Hence cosine θ is called power factor. As an equation

$$\text{Power factor} = \cos\theta = \frac{\text{power}}{\text{volt-amperes}} \tag{11}$$

Reference to equation (10) will show that $\sin\theta$ is the factor by which volt-amperes are multiplied to yield reactive volt-amperes or vars.

Hence $\sin \theta$ is called the reactive factor:

$$\text{Reactive factor} = \sin \theta = \frac{\text{reactive volt-amperes}}{\text{volt-amperes}} \tag{12}$$

Since $\sin^2 \theta + \cos^2 \theta = 1$, reactive factor $= \sqrt{1 - (\text{p.f.})^2}$ and power factor $= \sqrt{1 - (\text{r.f.})^2}$.

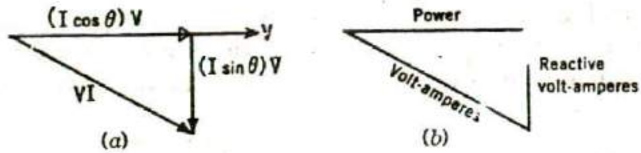


FIG. 20. Relation of power, reactive volt-amperes, and resultant volt-amperes.

If the current and each of its two components in Fig. 18 are multiplied by V, a relationship between power, reactive volt-amperes, and volt-amperes is obtained, as shown in Figs. 20*a* and *b*. Hence

$$\text{Volt-amperes} = \sqrt{(\text{power})^2 + (\text{reactive va})^2}$$

This relation is very useful in problems involving correction of power factor.

Example 2. One hundred and ten volts are applied to a series circuit consisting of 8 ohms resistance, 0.0531 henry inductance, and 189.7 μf capacitance. When

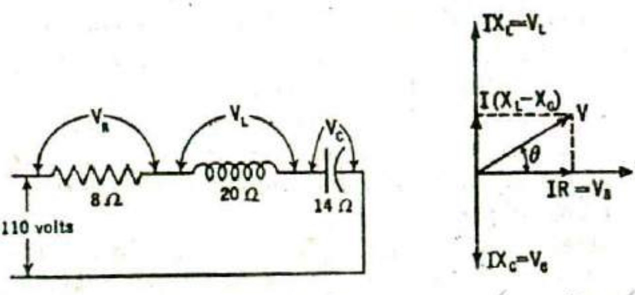


FIG. 21. R, L, and C in series and the corresponding vector diagram.

the frequency is 60 cycles, calculate current, power, power factor, vars, reactive factor, and volt-amperes. Also calculate the voltage drop across each circuit element. The circuit and vector diagrams are shown in Fig. 21.

$$X_L = 2\pi f L = 2\pi\, 60 \times 0.0531 = 20 \text{ ohms}$$

$$X_C = \frac{1}{2\pi f C} = \frac{1}{2\pi\, 60 \times 189.7 \times 10^{-6}} = 14 \text{ ohms}$$

$$X = X_L - X_C = 20 - 14 = 6 \text{ ohms}$$

$$R = 8 \text{ ohms}$$

$$Z = \sqrt{R^2 + X^2} = \sqrt{8^2 + 6^2} = 10 \text{ ohms}$$

$\tan^{-1}\frac{6}{10}$

$$I = \frac{110}{10} = 11 \text{ amperes}$$

$\angle 0'$ $\angle 36.96$

$$\text{P.f.} = \cos\theta = \frac{IR}{V} = \frac{IR}{IZ} = \frac{R}{Z} = \frac{8}{10} = 0.8$$

$$P = VI\cos\theta = 110 \times 11 \times 0.8 = 968 \text{ watts}$$

Also

$$P = I^2R = 11^2 \times 8 = 968 \text{ watts}$$

$$\text{Reactive va} = VI\sin\theta = 110 \times 11 \times \frac{XI}{ZI} = 110 \times 11 \times \frac{6}{10} = 726 \text{ vars}$$

$$\text{va} = VI = 110 \times 11 = 1210 = \sqrt{968^2 + 726^2}$$

$$V_R = IR = 11 \times 8 = 88 \text{ volts}$$

$$V_L = IX_L = 11 \times 20 = 220 \text{ volts}$$

$$V_C = IX_C = 11 \times 14 = 154 \text{ volts}$$

It will be noted that the arithmetic sum of these three voltages is much greater than the applied voltage. Alternating voltages of the same frequency can be added but they *must* be added *vectorially* with due regard for phase relation. Thus

$$220 - 154 = 66 \text{ volts in quadrature with } \mathbf{I}$$

$$V_R = 88 \text{ volts in phase with } \mathbf{I}$$

Therefore

$$V = \sqrt{88^2 + 66^2} = 110 \text{ volts, which checks the applied voltage.}$$

Example 3. Given the parallel circuit shown in Fig. 22, find I, I_1, I_2, and total power consumed.

Solution. The impedance functions of branches 1 and 2 are

$$\mathbf{Z}_1 = \sqrt{6^2 + 8^2}\ \angle\tan^{-1}\frac{8}{6} = 10\ \angle 53.17^\circ \text{ ohms}$$

$$\mathbf{Z}_2 = \sqrt{5^2 + 5^2}\ \angle\tan^{-1}\left(\frac{-5}{5}\right) = 7.07\ \angle -45^\circ \text{ ohms}$$

$$I_2 = \frac{100}{7.07} = 14.14 \text{ amperes}$$

$$I_1 = \frac{100}{10} = 10 \text{ amperes}$$

$$\mathbf{I} = \mathbf{I}_2 + \mathbf{I}_1$$

The vector diagram is drawn as shown in Fig. 23.

The currents may be added by using $\sum x$ and $\sum y$ components or by the cosine law. The former will be used and a tabulation of results made. The x-axis will be taken along V. This is arbitrary. Any other position may be used.

Current	x components	y components
I_2	$I_2 \cos 45 = 10$	$I_2 \sin 45 = 10$
I_1	$I_1 \cos(-53.17^\circ) = 6$	$I_1 \sin(-53.17^\circ) = -8$

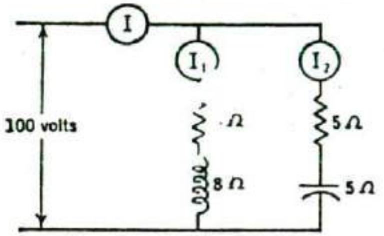


Fig. 22. Parallel branches.

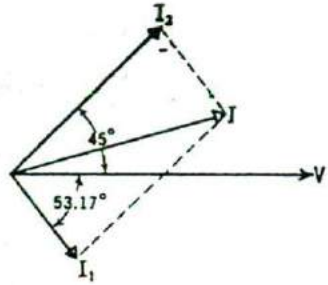


Fig. 23. Vector diagram of Fig. 22.

$$\sum x = 16 \qquad\qquad \sum y = 2$$
$$I = \sqrt{16^2 + 2^2} = 16.13 \text{ amperes}$$
$$P = VI_{\text{in-phase}} = 100 \times \sum x = 100 \times 16 = 1600 \text{ watts}$$

An alternative is

$$P = I_1{}^2R_1 + I_2{}^2R_2 = 10^2 \times 6 + 14.14^2 \times 5$$
$$= 600 + 1000 = 1600 \text{ watts}$$

Vector Combination of Voltages. Thus far, only currents have been added and subtracted vectorially. Since vector combinations are based upon the assumption of sine waves, it is apparent that sinusoidal voltage waves can be added and subtracted vectorially. For example,

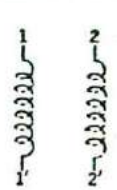


Fig. 24. Coils in which a-c voltages are induced.

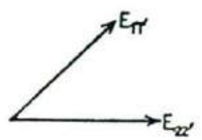


Fig. 25. Voltages induced in coils of Fig. 24.

the coils shown in Fig. 24 are assumed to have induced voltages which are phase-displaced by 45°, as shown in Fig. 25. The voltage $E_{12'}$ is desired when 1′ and 2 of Fig. 24 are connected. In general, the difference of potential between two points of a winding or circuit is found by adding all the potential drops (rises are negative drops) encountered in *tracing through* the winding from one point in question to the other. This statement follows from the definition of potential difference. The voltages are denoted by subscripts, and the order in which the subscripts are written must be the same as the order in which they are encountered as the circuit is being traced. Thus for Fig. 24, when 1′ and 2 are connected, $\mathbf{E}_{12'} = \mathbf{E}_{11'} + \mathbf{E}_{22'}$. This vector addition is shown in Fig. 26

If 1′ were connected to 2′ in Fig. 24, the emf $\mathbf{E}_{12}$ would be $\mathbf{E}_{12} = \mathbf{E}_{11'} + \mathbf{E}_{2'2}$. This result is obtained by adding the voltage vectors, $\mathbf{E}_{11'}$ and $\mathbf{E}_{2'2}$, as shown in Fig. 27.

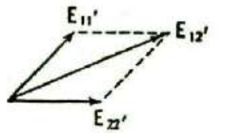


FIG. 26. E_{12}' for Fig. 24 when 1 and 2 are connected.

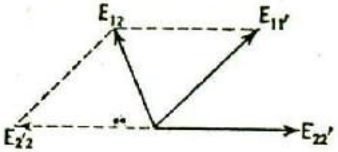


FIG. 27. E_{12} for Fig. 24 when 1′ and 2′ are connected.

Problem 7. Two coils on the armature of an alternator are displaced 60 electrical degrees. The emf of each coil is 100 volts. What is the resultant emf of the two coils when connected series adding and also when series subtracting?

Ans.: 173.2 volts, 100 volts.

PROBLEMS

8. An elevator motor takes 20 amperes for 15 seconds. Power is then cut off for 45 seconds, after which the cycle is repeated. If rated full-load current of the motor is 12 amperes, will it overheat on a continuation of this cycle? What is the equivalent continuous current which will yield the same average rate of heating?

9. A motor takes 50 amperes for 10 seconds, after which power is off for 20 seconds. It then takes 60 amperes for 5 seconds, after which power is cut off for 1 minute. What will the continuous rated current have to be so that the motor will not overheat?

10. (a) What is the average value of the pulsating current shown in Fig. 28?
(b) What is the effective value?

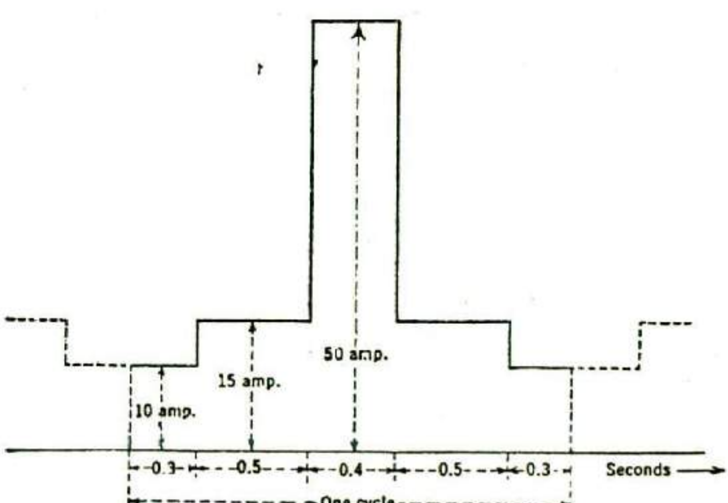


FIG. 28. See Problems 10, 11, 12, and 25.

11. (a) If the current shown in Fig. 28 flows through a d-c ammeter in series with an effective reading a-c ammeter, what will be the reading of each instrument, assuming perfect calibration of the instruments?

(b) If the resistance of the circuit is constant (the pulsating current being produced by a pulsating voltage), which of the above readings should be employed in finding the power by the I^2R formula?

12. (a) If the current shown in Fig. 28 flows through a 5-ohm resistance, what number of joules of heat energy is produced each cycle? what number of gram calories?

(b) What power is dissipated in the above resistance over any integral number of cycles?

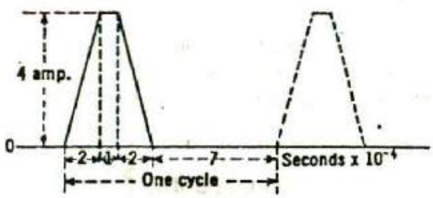


FIG. 29. See Problem 13.

13. The plate current of a triode operating as an oscillator takes the general form shown in Fig. 29.

(a) What is the frequency of oscillation depicted in Fig. 29?

(b) What is the average value of the pulsating current?

(c) What is the effective value of the pulsating current?

Note: The current during the first 2×10^{-4} second shown in Fig. 29 may be represented by the equation $i = 2 \times 10^4 t$ amperes. Utilize symmetry.

14. A current in a circuit starts at zero and increases linearly until a value of 12 amperes is attained. It then drops to zero in negligible time and repeats the cycle. What will an effective reading a-c ammeter in this circuit read?

15. A current starts abruptly at 10 amperes and decreases linearly to zero and then repeats this cycle. Find the rms value without changing the orientation of the wave from that given.

16. Find the rms value of a current in terms of radius ρ whose instantaneous values make semicircles of radius ρ above and below the x-axis.

17. A current has a positive loop which follows a semicircle of radius 1 ampere and the diameter of this semicircle lies on the x-axis. If through the addition of a constant current of 1 ampere the resultant current is represented by the semicircle with its diameter raised 1 ampere above the x-axis, find the rms value of the resultant current.

18. Calculate the form factor of the current wave in Problem 14.

19. Find the rms value of $e = 100 \sin \omega t + 60 \sin (5\omega t + 30°)$ volts by integration.

20. Calculate the form factor of $e = 100 \sin \omega t + 60 \cos 3\omega t$.

21. Find the rms value of $e = 100 \sin \omega t - 40 \sin 3\omega t$ volts.

22. Calculate the form factor of the voltage wave in Problem 21.

23. Find a wave other than that given in the text which is not a sine wave but which has the same form factor as a sine wave.

24. Calculate the peak factor of (a) a sine wave, (b) a rectangular wave, (c) a symmetrical triangular wave whose positive and negative halves are symmetrical about their respective midordinates if the angle at the peak is 60°, and (*d*) a triangular wave whose angle at the peak is 90°.

25. Calculate the crest factor for the wave shown in Fig. 28.

26. Calculate the crest factor and form factor of a wave whose positive and negative loops are semicircles.

27. The respective branch currents flowing toward a junction of two parallel branches of a circuit are $i_1 = 30 \sin (\omega t + 60°)$ amperes and $i_2 = 20 \sin (\omega t - 20°)$ amperes. Find the resultant current leaving the junction in terms of a single sine wave. Find also the effective value of the current.

28. One branch current of $i_1 = 40 \sin (\omega t - 40°)$ amperes combines with a second branch current to yield a resultant of $50 \sin (\omega t + 80°)$ amperes. Find the equation of the second branch current. Find also the effective value.

29. A motor requires 25 amperes and 220 volts at a lagging power factor of 0.88. Find the power, vars, reactive factor, and the volt-amperes taken.

30. A motor requires 10 amperes and 220 volts at a power factor of 0.8 lag. Find the power, reactive volt-amperes, reactive factor, and the volt-amperes required.

31. The voltage of a circuit is $v = 200 \sin (\omega t + 30°)$, and the current is $i = 50 \sin (\omega t + 60°)$. What are the average power, volt-amperes, and power factor?

32. A motor takes 15 amperes and 220 volts at a lagging power-factor angle of 72° when running at no load. Find the number of watts, vars, and volt-amperes it is taking.

33. How many resultant volt-amperes will be taken from the line when the two motors in Problems 29 and 32 are operating simultaneously as stated from the same line? What is the resultant line current and power supplied?

34. One motor takes 250 amperes at 0.8 power factor lag while another motor takes 50 kw at 0.5 leading power factor from a line of 220 volts. What is the resultant line current for these two motors? What is the power factor of the combined loads? Is it leading or lagging?

35. The voltage of a circuit is $v = 200 \sin \omega t$ volts, and the current is $i = 50 \cos (\omega t - 30°)$ amperes. What are the average power, vars, and power factor?

36. A varmeter in a circuit indicates 600 vars, and a wattmeter in the same circuit shows 800 watts. Find the volt-amperes, power factor, and reactive factor of the circuit.

37. A series circuit has 8 ohms resistance and 20 millihenrys inductance. If 110 volts at 60 cycles are impressed, calculate the current and power.

38. One branch of a parallel circuit consists of 6 ohms resistance, 48 ohms inductive reactance, and 40 ohms capacitive reactance, while the other branch consists of a resistance of 7 ohms and a capacitive reactance of 2 ohms. Find the current delivered to the combination when 100 volts are impressed across the entire circuit. Calculate the total power and that consumed by each branch.

39. (*a*) Find the readings of ammeters I_1, I_2, and I, and of wattmeter W of Fig. 30. Compare the reading of W with $I_1^2R_1 + I_2^2R_2$.

(*b*) Draw the vector diagram of $\mathbf{V}$, $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}$, $\mathbf{I}_1\mathbf{R}_1$, $\mathbf{I}_1\mathbf{X}_{L1}$, $\mathbf{I}_2\mathbf{R}_2$, and $\mathbf{I}_2\mathbf{X}_{C2}$.

(*c*) Assuming that $\mathbf{V}$ represents a potential drop from a to b through the circuit branches, find the potential drop from d to c, or $\mathbf{V}_{dc}$.

(*d*) Assuming that $\mathbf{V}$ represents a potential drop from b to a through the circuit branches, find the potential drop from d to c, or $\mathbf{V}_{dc}$.

40. Work Problem 39 if the parameters are changed to $R_1 = 8$ ohms, $L_1 = 0.025$ henry, $R_2 = 10$ ohms, and $C_2 = 120\ \mu f$.

41. Find the readings of the ammeter I and of the wattmeter W in Fig. 30 if an

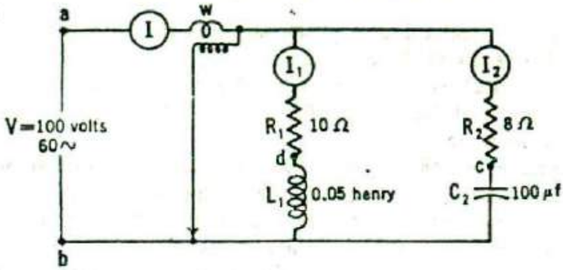


FIG. 30. See Problems 39, 40, and 41.

additional branch R_3L_3 is placed in parallel with the R_1L_1 and R_2C_2 branches. $R_3 = 15$ ohms and $L_3 = 0.12$ henry.

42. Find the readings of the ammeter I and of the wattmeter W in Fig. 31 for the parameters specified.

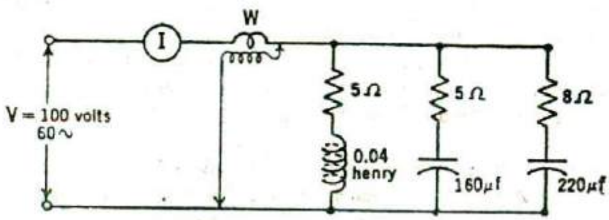


FIG. 31. See Problem 42.

43. A type of alternator much used in laboratories has six coils spaced about the armature at intervals of 30 electrical degrees. The two leads of each coil are brought out to a terminal board, making available six voltages. Because of the 30 electrical

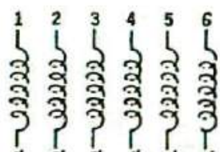


FIG. 32. Six coils of an a-c generator. Adjacent coils are displaced 30 electrical degrees.

degrees of space displacement of the coils on the armature, the individual coil voltages have phase differences of 30°. Let Fig. 32 represent the six coils, and assume that adjacent coils in the figure are electrically adjacent coils on the alternator armature. Assume also that the coil voltages are sinusoidal and that leads 1, 2, 3, 4, 5, and 6 are corresponding ends of the coils, and that $\mathbf{E}_{1'1}$ is 30° behind $\mathbf{E}_{2'2}$, $\mathbf{E}_{2'2}$ is 30° behind $\mathbf{E}_{3'3}$, and so on.

(a) Draw the vector diagram of $\mathbf{E}_{1'1}$, $\mathbf{E}_{2'2}$, $\mathbf{E}_{3'3}$, $\mathbf{E}_{4'4}$, $\mathbf{E}_{5'5}$, and $\mathbf{E}_{6'6}$ when $\mathbf{E}_{1'1}$ is laid off along the $+x$-axis. Each coil has an effective emf of 50 volts.

(b) Find $\mathbf{E}_{13'}$ when 1′ is connected to 3.

(c) Find $\mathbf{E}_{13}$ when 1′ is connected to 3′.

(d) Find the greatest voltage that can be obtained by connecting all coils in series.

(e) Draw the vector diagram that represents the three voltages, $\mathbf{E}_{12'}$, $\mathbf{E}_{34'}$, and $\mathbf{E}_{56'}$, assuming that 1′ is connected to 2, 3′ to 4, and 5′ to 6.

chapter **IV**

Phasor Algebra (as Applied to A-C Circuit Analysis)

The Operator j. Since the complex quantities normally employed in a-c circuit analysis to simplify calculations are added and subtracted like coplanar vectors, they are often referred to as vectors. However such coplanar vectors which represent sinusoidally time-varying quantities are now more properly called phasors.

It is well known that a plane vector can be specified in magnitude and direction in terms of its x-axis projection and its y-axis projection. For example, if the x-axis projection of the phasor or vector **A**[1] in Fig. 1 is known as x_A and the y-axis projection is known as y_A then the magnitude of the phasor **A** is

$$A = \sqrt{x_A{}^2 + y_A{}^2} \tag{1}$$

From the geometry of Fig. 1 it is plain that the angle, θ_A, between the direction of phasor **A** and the direction of the positive x-axis is

$$\theta_A = \tan^{-1}\frac{y_A}{x_A} \tag{2}$$

In order to specify a phasor in terms of its x and y components, some means must be employed to distinguish between the x-axis projection and the y-axis projection. Inasmuch as the $+y$-axis projection is $+90°$ from the $+x$-axis, a convenient operator for the purpose at hand is one which will, when applied to a phasor, rotate it 90° counterclockwise without changing the magnitude of the phasor.

Let j be an operator which produces 90° counterclockwise rotation of any phasor to which it is applied as a multiplying factor. The physical significance of the operator j can best be appreciated by first considering that it operates on a given phasor **A**, the direction of which is along the $+x$-axis. Then, by definition, when the phasor **A** of Fig. 2 is multiplied by j a new phasor, j**A**, 90° counterclockwise from **A**, will be obtained. If the operator j is applied to the phasor j**A** it will, by definition, rotate j**A** 90° in the counterclockwise direction. The result

[1] Bold-face type is used to represent a phasor in both magnitude and phase, whereas light-face italics represent the magnitude only.

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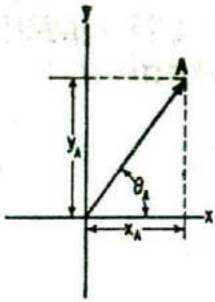

Fig. 1. Resolution of phasor **A** into its x-axis and y-axis components.

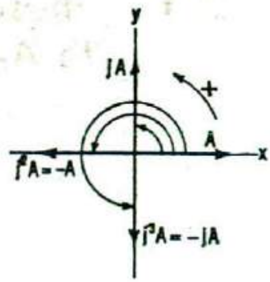

Fig. 2. Effects produced by successive applications of the operator j upon a phasor **A**, the original position of which is along the $+x$-axis.

is $jj\mathbf{A} = j^2\mathbf{A}$ as shown in Fig. 2. Also from Fig. 2

$$j^2\mathbf{A} = -\mathbf{A}'$$

Hence:

$$j^2 = -1$$

and

$$j = \sqrt{-1} \tag{3}$$

If the operator j is applied to the phasor $j^2\mathbf{A}$ the result is $j^3\mathbf{A} = -jA$. The phasor $j^3\mathbf{A}$ is 270° counterclockwise from the reference axis, directly opposite the phasor $j\mathbf{A}$ in Fig. 2. If the phasor $j^3\mathbf{A}$, in turn, is operated on by j, the result is $j^4\mathbf{A} = j^2j^2\mathbf{A} = \mathbf{A}$. It will be observed that successive applications of the operator j to the phasor **A** produce successive 90° steps of rotation of the phasor in the counterclockwise direction without affecting the magnitude of the phasor.

From Fig. 2 it is apparent that multiplying **A** by $-j$ yields $-j\mathbf{A}$, a phasor of identical magnitude rotated clockwise 90° from **A**. Hence $-j$ is an operator which produces clockwise rotation of 90°.

The Cartesian Form of Notation. A phasor in any quadrant can be completely specified in a cartesian or rectangular form of notation, as shown below.

$$\mathbf{A} = \pm a \pm ja' \tag{4}$$

where a is the x-axis projection and a' is the y-axis projection of the phasor. In any case the magnitude of the phasor **A** is

$$A = \sqrt{a^2 + a'^2} \tag{5}$$

The phase position of a first-quadrant vector is conveniently described in terms of the positive acute angle measured in a ccw direction from the $+x$-axis to the position of the phasor. In equation form

$$\theta_{1st} = \tan^{-1} \frac{(+a')}{(+a)} \tag{6}$$

The phase position of a fourth-quadrant phasor is conveniently described in terms of the negative acute angle measured in a cw direction from the $+x$-axis to the position of the phasor.

$$\theta_{4th} = \tan^{-1} \frac{(-a')}{(+a)} \tag{7}$$

A fourth-quadrant phasor can, of course, be specified in terms of the positive angle $(360° - \theta_{4th})$, where θ_{4th} is the magnitude of the angle measured in a negative or clockwise direction from the $+x$-axis to the position of the phasor.

Phase positions of second- and third-quadrant phasors are easily located in terms of the a and a' components by first finding the acute angle, the tangent of which is a'/a, without regard to sign, and then subtracting this angle from or adding it to 180°, depending upon whether the a' component is positive or negative.

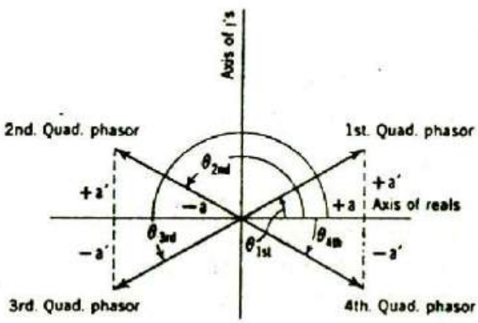

FIG. 3. Phasors in any quadrant can be specified in terms of their real (x-axis) and j (y-axis) components.

Figure 3 illustrates how phasors in any quadrant can be specified in magnitude and phase position in terms of real and j components. In determining the phase angle it is necessary to know the individual signs of the a and a' components in order to locate the angle θ correctly.

The Operator ($\cos \theta \pm j \sin \theta$). Reference to Fig. 3 will show that the x-axis projection of a phasor in any quadrant is $A \cos \theta$. The angle θ may be measured either positively or negatively from the $+x$-axis in determining the x-axis projection, since $\cos \theta = \cos(-\theta)$.

The y-axis projection of the phasor in any quadrant is $A \sin \theta$ if θ

is measured in the ccw direction from the $+x$-axis. The y-axis projection is $-A \sin \theta$ if θ is measured in the cw direction from the $+x$-axis to the position of the phasor. Therefore,

$$\mathbf{A} = A\ (\cos \theta \pm j \sin \theta) \tag{8}$$

is equivalent to the form shown in equation (4). The plus sign is used if θ is measured counterclockwise from the reference axis, the minus sign if θ is measured clockwise.

Equation (8) shows that $(\cos \theta + j \sin \theta)$ operating on a real magnitude A, that is a phasor of A units magnitude along the $+x$-axis, rotates this phasor through a $+\theta$-angle from its initial position. Similarly the operator $(\cos \theta - j \sin \theta)$ rotates the original phasor through a $-\theta$-angle.

It may be shown that the operator $(\cos \theta \pm j \sin \theta)$ rotates any phasor to which it is attached as a multiplying factor through $+\theta$ or $-\theta$ degrees, depending whether the plus or minus sign is employed. Consider a phasor in an initial position such that $a = A \cos \alpha$ and $a' = A \sin \alpha$.

$$\mathbf{A}\ \text{(initially)} = a + ja' = A\ (\cos \alpha + j \sin \alpha) \tag{9}$$

Let $\mathbf{A}' = \mathbf{A}$ [operated on by $(\cos \theta + j \sin \theta)$].

$$\mathbf{A}' = A\ (\cos \alpha + j \sin \alpha)\ (\cos \theta + j \sin \theta) \tag{10}$$

$$\begin{aligned} \mathbf{A}' &= A\ (\cos \alpha \cos \theta + j \cos \alpha \sin \theta + j \sin \alpha \cos \theta + j^2 \sin \alpha \sin \theta) \\ &= A\ [(\cos \alpha \cos \theta - \sin \alpha \sin \theta) + j\ (\sin \alpha \cos \theta + \cos \alpha \sin \theta)] \\ &= A\ [\cos\ (\alpha + \theta) + j \sin\ (\alpha + \theta)] \end{aligned} \tag{11}$$

Equation (11) shows that $\mathbf{A}'$ is a phasor equal in magnitude to the phasor $\mathbf{A}$ but advanced θ degrees from the $\mathbf{A}$ position since it now makes an angle of $(\alpha + \theta)$ with the reference axis.

In similar manner it may be shown that the operator $(\cos \theta - j \sin \theta)$ rotates any phasor to which it is attached through $-\theta$ degrees.

Exponential Form of the Operator $(\cos \theta \pm j \sin \theta)$. An important relationship is contained in the following equation:

$$(\cos \theta \pm j \sin \theta) = \epsilon^{\pm j\theta} \tag{12}$$

Equation (12), known as Euler's equation, follows directly from an inspection of the Maclaurin series expansions[2] of $\cos \theta$, $\sin \theta$, and $\epsilon^{j\theta}$.

[2] Certain functions, among which are $\cos(\theta)$, $\sin(\theta)$, and $\epsilon^{\pm j\theta}$, can be expanded into series form by means of Maclaurin's theorem. The theorem states that

$$f(\theta) = f(0) + \frac{f'(0)\theta}{1} + \frac{f''(0)\theta^2}{\underline{/2}} + \frac{f'''(0)\theta^3}{\underline{/3}} + \cdots \text{etc.}$$

where $f(\theta)$ is the particular function of θ that is to be expanded, $f(0)$ is the value of

Expanded into series form

$$\cos\theta = 1 - \frac{\theta^2}{\underline{/2}} + \frac{\theta^4}{\underline{/4}} - \frac{\theta^6}{\underline{/6}} + \cdots \tag{13}$$

$$\sin\theta = \theta - \frac{\theta^3}{\underline{/3}} + \frac{\theta^5}{\underline{/5}} - \frac{\theta^7}{\underline{/7}} + \cdots \tag{14}$$

$$\epsilon^{j\theta} = 1 + j\theta + \frac{(j\theta)^2}{\underline{/2}} + \frac{(j\theta)^3}{\underline{/3}} + \frac{(j\theta)^4}{\underline{/4}} + \frac{(j\theta)^5}{\underline{/5}} + \frac{(j\theta)^6}{\underline{/6}} + \cdots \tag{15}$$

All quantities involving even powers of j reduce to real numbers since $j^2 = -1$, $j^4 = 1$, $j^6 = -1$, etc. All quantities involving odd powers of j reduce to first-degree j terms because $j^3 = -j$, $j^5 = j$, etc. If the j terms are properly evaluated, equation (15) may be arranged as follows:

$$\epsilon^{+j\theta} = \left(1 - \frac{\theta^2}{\underline{/2}} + \frac{\theta^4}{\underline{/4}} - \frac{\theta^6}{\underline{/6}} + \cdots\right) + j\left(\theta - \frac{\theta^3}{\underline{/3}} + \frac{\theta^5}{\underline{/5}} - \frac{\theta^7}{\underline{/7}} + \cdots\right) \tag{16}$$

Therefore

$$\epsilon^{j\theta} = \cos\theta + j\sin\theta \tag{17}$$

and

$$A\epsilon^{j\theta} = A\ (\cos\theta + j\sin\theta) \tag{18}$$

In a similar manner it may be shown that

$$\epsilon^{-j\theta} = \cos\theta - j\sin\theta \tag{19}$$

Polar Form of the Operator ($\cos\theta \pm j\sin\theta$). The exponential form of the operator ($\cos\theta \pm j\sin\theta$) is very often written in a simplified form. It has been shown that

$$\epsilon^{\pm j\theta} = (\cos\theta \pm j\sin\theta) \tag{20}$$

Therefore

$$A\epsilon^{\pm j\theta} = A\ (\cos\theta \pm j\sin\theta) \tag{21}$$

By definition

$$\epsilon^{\pm j\theta} = 1\,\underline{/\pm\theta} \tag{22}$$

and

$$A\epsilon^{\pm j\theta} = A\,\underline{/\pm\theta} \tag{23}$$

Therefore

$$\epsilon^{j\theta} = \underline{/\theta} = (\cos\theta + j\sin\theta) \tag{24}$$

$$\epsilon^{-j\theta} = \underline{/-\theta} = (\cos\theta - j\sin\theta) \tag{25}$$

this particular function when θ is set equal to zero, $f'(0)$ is the value of the first derivative of the function when θ is set equal to zero, f'' is the value of the second derivative of the function when θ is set equal to zero, etc.

and

$$A\epsilon^{j\theta} = A \underline{/\theta} = A\ (\cos\theta + j\sin\theta) \tag{26}$$

$$A\epsilon^{-j\theta} = A \underline{/-\theta} = A\ (\cos\theta - j\sin\theta) \tag{27}$$

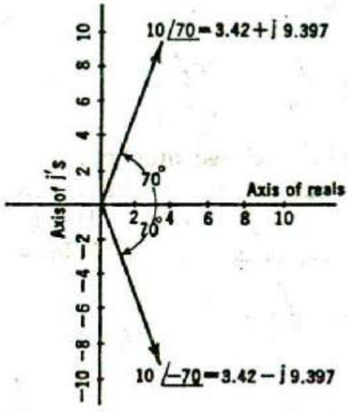

FIG. 4. Phasor representation of equations (26) and (27) for the case of $A = 10$ units and $\theta = 70°$.

Equations (26) and (27) state the equivalence of the three forms of notation that are commonly employed to define a given phasor in magnitude and phase position. Graphical representations of equations (26) and (27) for particular values of A and θ are shown in Fig. 4. The exponential and polar forms are identical by definition and find their greatest use in the processes of multiplication, division, extraction of roots, and raising phasors to given powers. Both these forms express a phasor in terms of polar coordinates. $A \underline{/\theta}$ is simply a shorthand or symbolic style of writing $A\epsilon^{j\theta}$. Common usage distinguishes between the two forms by calling $A\epsilon^{j\theta}$ the exponential form and $A \underline{/\theta}$ the polar form.

The rectangular or cartesian form, $A\ (\cos\theta \pm j\sin\theta)$, is indispensable in the processes of addition or subtraction of phasors if the j form of phasor algebra is employed.

Problem 1. Write the equivalent polar form of the phasor $3 + j4$ where the numbers refer to unit lengths. Illustrate the phasor by means of a diagram.

Ans.: $5\epsilon^{j53.1°} = 5 \underline{/53.1°}$.

Problem 2. A phasor is given in the form of $10\epsilon^{-j120°}$. Write the symbolic polar and cartesian forms of the phasor, and illustrate, by means of a phasor diagram, the magnitude and phase position of the phasor.

Ans.: $10 \underline{/-120°} = -5 - j8.66$.

Addition of Phasors. The phasor sum of two phasors **A** and **B** is a third phasor which is defined in magnitude and phase position by the diagonal of the parallelogram which has for two of its sides the phasors **A** and **B**. The particular diagonal of the parallelogram thus formed, which represents the phasor sum, **A** + **B**, is indicated in Fig. 5.

Each phasor may be considered as having a tail and a head. If the arrow heads in Fig. 5 indicate the heads of the phasors, then the phasor sum of two phasors is the line which joins the tail of the first phasor and the head of the second phasor after the second phasor has been placed so that its tail coincides with the head of the first phasor.

The fact that

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{28}$$

is obvious from the definition that has been given for the phasor sum of two phasors.

The process of adding two phasors may be extended to include any number of phasors simply by first adding any two of the phasors involved and then adding to this phasor sum, which is in itself a phasor, the third phasor, etc. The order in which the addition is carried out is immaterial. For example

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{B} + \mathbf{C} + \mathbf{A} = \mathbf{C} + \mathbf{A} + \mathbf{B} \tag{29}$$

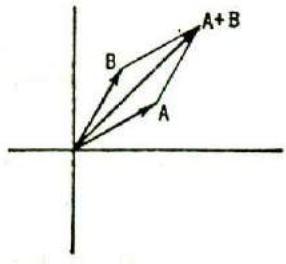

FIG. 5. Addition of the phasors **A** and **B**.

Phasors are written in the rectangular $(a + ja')$ form when addition is to be performed, since the exponential or polar forms do not lend themselves to the addition process. If $\mathbf{A} = a + ja'$, $\mathbf{B} = b + jb'$, and $\mathbf{C} = c + jc'$,

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = (a + b + c) + j\,(a' + b' + c') \tag{30}$$

The magnitude of the resultant phasor is

$$D = \sqrt{(a + b + c)^2 + (a' + b' + c')^2} \tag{31}$$

The phase position of the resultant phasor is

$$\theta_D = \tan^{-1} \frac{(a' + b' + c')}{(a + b + c)} \tag{32}$$

Any or all of the component parts of the phasors **A**, **B**, and **C** in the above example may be negative. The process that has been given for the addition of three phasors can, of course, be extended.

Example. Let it be required to add

$$\mathbf{A} = 10\underline{/36.9^\circ} = 8 + j6 \quad \text{and} \quad \mathbf{B} = 6\underline{/120^\circ} = -3 + j5.20$$

$$\mathbf{A} + \mathbf{B} = \mathbf{C} = (8 - 3) + j\,(6 + 5.2)$$

$$\mathbf{C} = 5 + j11.2$$

The magnitude of the **C** phasor is

$$C = \sqrt{5^2 + 11.2^2} = 12.27 \text{ units}$$

The position of the phasor **C** with respect to the $+x$-axis is

$$\theta_C = \tan^{-1} \frac{11.2}{5} = \tan^{-1} 2.24 = 65.95^\circ$$

Figure 6 illustrates the phasor addition of **A** and **B** for the particular values that have been employed in this example.

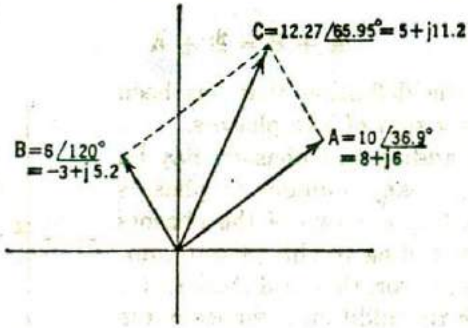

FIG. 6. Phasor addition in a particular numerical case.

Problem 3. Add the phasors $14\,\underline{/60^\circ}$ and $20\,\underline{/15^\circ}$. State the result in both rectangular and polar forms, and illustrate, by means of a phasor diagram, the operation that has been performed.

Ans.: $26.3 + j17.3 = 31.5\,\underline{/33.35^\circ}$.

Problem 4. Given the following three phasors:

$$\mathbf{A} = 40\epsilon^{j120^\circ},\ \mathbf{B} = 20\,\underline{/-40^\circ},\ \mathbf{C} = 26.46 + j0$$

find $\mathbf{A} + \mathbf{B} + \mathbf{C}$ and illustrate the three phasors, together with their phasor sum, by means of a phasor diagram.

Ans.: $21.78 + j21.78 = 30.8\,\underline{/45^\circ}$.

Subtraction of Phasors. In ordinary algebra the operation or process of subtraction is accomplished by changing the sign of the quantity to be subtracted and proceeding as in addition. In phasor algebra the phasor which is to be subtracted is rotated through 180° and then added. To rotate a phasor through 180° the operator $j^2 = -1$ may be applied or 180° may be added or subtracted from the original phase angle of the phasor. Thus a phasor $\mathbf{A} = A\,\underline{/\theta}$ rotated through 180° becomes

$$\mathbf{A}' = j^2A\,\underline{/\theta} = -A\,\underline{/\theta} = A\,\underline{/\theta \pm 180^\circ}$$

and a phasor $\mathbf{B} = b + jb'$ rotated through 180° becomes

$$\mathbf{B}' = j^2(b + jb') = -b - jb'$$

Figure 7*a* illustrates the subtraction of phasor **D** from phasor **C**. Symbolically, the operation may be indicated as: $\mathbf{C} - \mathbf{D} = \mathbf{E}$. After the phasor which is to be subtracted has been rotated through 180°, the phasor thus resulting is added to the phasor from which the subtraction is being made.

Figure 7*b* illustrates the subtraction of phasor **C** from phasor **D**. It will be observed that $(\mathbf{D} - \mathbf{C})$ is of equal magnitude and 180° removed from $(\mathbf{C} - \mathbf{D})$. In general

$$(\mathbf{C} - \mathbf{D}) = -(\mathbf{D} - \mathbf{C}) \tag{33}$$

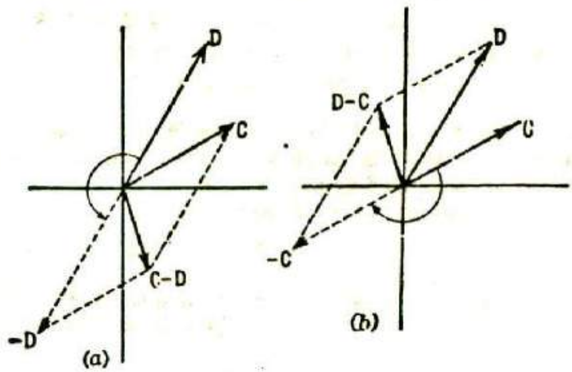

FIG. 7. Illustrating phasor subtraction.

The difference of two phasors might have been defined in terms of one of the diagonals of the parallelogram formed by the two phasors.

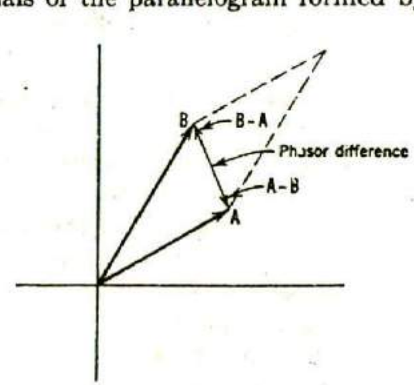

FIG. 8. The diagonal which defines the difference between two phasors. (The sense or direction of the diagonal is dependent upon the particular phasor difference in question.)

Figure 8 illustrates the particular diagonal which represents the difference between phasors **B** and **A**. The diagonal concept is useful in certain types of phasor diagrams, but for general calculations the method which has previously been described is to be preferred.

Examples. Given the phasors

$$\mathbf{A} = 30\underline{/60^\circ}$$

and

$$\mathbf{B} = 21\ (\cos 160° - j \sin 160°)$$

let it be required to subtract phasor **B** from phasor **A**. The first step is to write the phasors in cartesian form.

$$\mathbf{A} = 30\underline{/60°} = 30\ (\cos 60° + j \sin 60°) = 15 + j26$$
$$\mathbf{B} = 21\ (\cos 160° - j \sin 160°) = -19.75 - j7.18$$
$$\mathbf{A} - \mathbf{B} = (15 + j26) - (-19.75 - j7.18)$$
$$= 34.75 + j33.18 = 48\underline{/43.6°}$$

For the particular case considered, the difference $(\mathbf{A} - \mathbf{B})$ is somewhat greater in magnitude than either of the original phasors. This condition is in general true if the original phasors are separated by more than 90°.

Let it be required to subtract phasor **A** from phasor **B**.

$$(\mathbf{B} - \mathbf{A}) = (-19.75 - j7.18) - (15 + j26) = -34.75 - j33.18 = 48\underline{/223.6°}$$

Problem 5. Draw a phasor diagram showing the phasors **A** and **B** of the above illustrative example, together with the phasors $(\mathbf{A} - \mathbf{B})$ and $(\mathbf{B} - \mathbf{A})$.

Problem 6. Given the following three phasors:

$$\mathbf{A} = 42\epsilon^{j200°}$$
$$\mathbf{B} = 20\underline{/-40°}$$
$$\mathbf{C} = 24.25 + j14$$

find $(\mathbf{A} + \mathbf{C}) - \mathbf{B}$ analytically and draw the phasor diagram.

Ans.: $32.95\underline{/157.7°}$.

Multiplication of Phasors and Complex Quantities. In a-c circuit analysis it is often desirable to operate on a phasor current with an impedance function so that the resulting voltage may be obtained. Similarly, it is sometimes desirable to operate on a phasor voltage with an admittance function, i.e., the reciprocal of the impedance function, to obtain the resulting current. The process of operating on a current (or voltage) phasor with a complex impedance (or admittance) function is called complex or phasor multiplication.

The complex product of two phasors, **A** and **B**, in so far as a-c circuit analysis is concerned, is a third phasor which has a magnitude equal to AB and a phase position with respect to the reference axis which is equal to the sum of the individual phase angles of **A** and **B**, namely, $(\alpha_A + \alpha_B)$. It will be shown presently why this particular definition of a complex product is especially suited to the phasor manipulations that are universally employed in a-c circuit theory. A graphical interpretation of the definition is given in Fig. 9 for the particular case of $\mathbf{A} = 2\underline{/40°}$ and $\mathbf{B} = 3\underline{/100°}$.

Analytically, the product of two phasors can be formed most conveniently when the phasors are expressed in exponential or polar form.

For example, the product of the **A** and **B** phasors shown in Fig. 9 is simply

$$\mathbf{AB} = 2\epsilon^{j40^\circ} \cdot 3\epsilon^{j100^\circ} = 6\epsilon^{j(40^\circ+100^\circ)} = 6\epsilon^{j140^\circ}$$

or

$$\mathbf{AB} = 2 \times 3\underline{/40^\circ + 100^\circ} = 6\underline{/140^\circ}$$

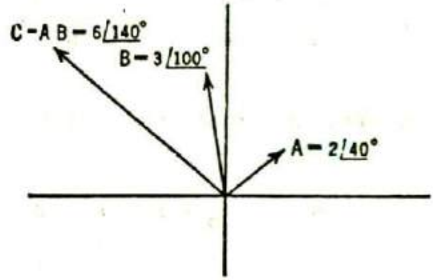

FIG. 9. Illustrating phasor multiplication.

From the definition which has been given for the complex product it is evident that the order in which the multiplication is carried out is immaterial. That is

$$\mathbf{AB} = \mathbf{BA} \tag{34}$$

Furthermore, the definition which has been given is capable of extension to any number of phasors or complex quantities. For example,

$$\mathbf{ABC} = ABC\underline{/\alpha_A + \alpha_B + \alpha_C} \tag{35}$$

and

$$\mathbf{ABC} = \mathbf{BCA} = \mathbf{CAB}, \text{ etc.} \tag{36}$$

The product of two phasors expressed in rectangular style can be formed by taking the cross-products of the component parts as in ordinary algebra. The proper interpretation must, of course, be given to the terms which involve j. If the phasors are given as $\mathbf{A} = a + ja'$ and $\mathbf{B} = b + jb'$, the product is formed exactly in accordance with the rules of ordinary algebra.

$$\mathbf{F} = \mathbf{AB} = (a + ja')(b + jb') = (ab - a'b') + j(a'b + ab') \tag{37}$$

The magnitude of the resulting phasor is

$$\begin{aligned} F &= \sqrt{(ab - a'b')^2 + (a'b + ab')^2} \\ &= \sqrt{a^2b^2 - 2aba'b' + a'^2b'^2 + a'^2b^2 + 2a'bab' + a^2b'^2} \\ &= \sqrt{(a^2 + a'^2)(b^2 + b'^2)} = \sqrt{A^2B^2} = AB \end{aligned} \tag{38}$$

The magnitude of **F** is thus shown to be equal to the product of the magnitudes of the phasors whose product is being formed. It remains to be shown that the phase angle of **F** as defined by the real and j components of equation (37) agrees with the definition that has been given for the product of two phasors. From equation (37) the phase angle of **F** takes the following form:

$$\alpha_F = \tan^{-1}\frac{(a'b + ab')}{(ab - a'b')} = \tan^{-1}\frac{\dfrac{a'b + ab'}{AB}}{\dfrac{ab - a'b'}{AB}}$$

It is evident from the definitions that have been given to a, a', b, and b' that

$$\frac{a'}{A} = \sin\alpha_A, \quad \frac{a}{A} = \cos\alpha_A, \quad \frac{b'}{B} = \sin\alpha_B, \quad \text{and} \quad \frac{b}{B} = \cos\alpha_B$$

Therefore

$$\alpha_F = \tan^{-1}\frac{\sin\alpha_A\cos\alpha_B + \cos\alpha_A\sin\alpha_B}{\cos\alpha_A\cos\alpha_B - \sin\alpha_A\sin\alpha_B}$$

$$\alpha_F = \tan^{-1}\frac{\sin(\alpha_A + \alpha_B)}{\cos(\alpha_A + \alpha_B)} = \tan^{-1}\tan(\alpha_A + \alpha_B)$$

$$\alpha_F = \alpha_A + \alpha_B \tag{39}$$

Equations (38) and (39) show that the product of two phasors may be formed by ordinary algebraic multiplication when the factors are expressed in cartesian form.

Example. Given the phasors:

$$\mathbf{A} = 2\,(\cos 40^\circ + j\sin 40^\circ) = 1.532 + j1.286$$
$$\mathbf{B} = 3\,(\cos 100^\circ + j\sin 100^\circ) = -0.521 + j2.954$$

let it be required to find the product of **A** and **B** by the algebraic multiplication of the cartesian forms.

$$\begin{aligned}\mathbf{F} = \mathbf{AB} &= (1.532 + j1.286)\,(-0.521 + j2.954)\\ &= -0.799 + j4.525 - j0.670 + j^2 3.798\\ &= (-0.799 - 3.798) + j(-0.670 + 4.525)\\ &= -4.597 + j3.855\end{aligned}$$

$$\mathbf{F} = \sqrt{-4.597^2 + 3.855^2}\,\Big/\tan^{-1}\frac{3.855}{-4.597}$$
$$= 6.0\,\underline{/180^\circ - 40^\circ} = 6\,\underline{/140^\circ}$$

The graphical representations of the phasors **A**, **B**, and **F** are given in Fig. 9.

Problem 7. Find the complex product of

$$\mathbf{A} = 5 - j4 \quad \text{and} \quad \mathbf{B} = 2 + j3$$

by algebraic multiplication of the cartesian forms and draw the phasor diagram. Change **A** and **B** to polar form and perform the multiplication process, **BA**.

Ans.: $22 + j7 = 23.09\underline{/17.65^\circ}$.

Problem 8. Given the following three phasors:

$$\mathbf{A} = 20 + j20,\ \mathbf{B} = 30\underline{/-120^\circ},\ \mathbf{C} = 6 + j0$$

perform the following indicated operations:

$$(a)\ \ \mathbf{A} + \mathbf{B} + \mathbf{C}, \qquad (b)\ \ (\mathbf{A} + \mathbf{B})\mathbf{C}, \qquad (c)\ \ \mathbf{ABC}.$$

Draw a phasor diagram of **A**, **B**, and **C**, together with the phasors which represent the results of the above indicated operations.

Ans.: (a) $11.67\underline{/-31^\circ}$, (b) $39.05\underline{/-50.2^\circ}$, (c) $4242\underline{/-75^\circ}$.

Division of Complex Quantities (or Phasors). For the purposes of a-c circuit theory the division of one complex quantity by another is carried out algebraically, as shown below, when the quantities are expressed in exponential form.

$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{A\epsilon^{j\alpha_A}}{B\epsilon^{j\alpha_B}} = \frac{A}{B}\,\epsilon^{j\alpha_A}\,\epsilon^{-j\alpha_B} = \frac{A}{B}\,\epsilon^{j(\alpha_A-\alpha_B)} \tag{40}$$

That is, the process of dividing one phasor, **A**, by a second phasor, **B**, results in a third phasor, the magnitude of which is the quotient of the magnitudes of the phasors **A** and **B**, namely A/B. The phase position of the resulting phasor with respect to the reference axis is the algebraic difference between the individual phase angles of the phasors **A** and **B** with respect to the reference axis, namely, $\alpha_A - \alpha_B$. It should be noted that the angle of the phasor in the denominator is always subtracted from the angle of the phasor in the numerator. Due regard is taken for the inherent signs of the individual phase angles, α_A and α_B, during the process of forming the algebraic difference. In symbolic polar form division is carried out as shown below:

$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{A\underline{/\alpha_A}}{B\underline{/\alpha_B}} = \frac{A}{B}\underline{/\alpha_A - \alpha_B} \tag{40a}$$

Examples. The processes of division in two particular cases are shown below:

$$\mathbf{F} = \frac{\mathbf{A}}{\mathbf{B}} = \frac{20\underline{/60^\circ}}{5\underline{/30^\circ}} = 4\underline{/30^\circ}$$

$$\mathbf{G} = \frac{\mathbf{C}}{\mathbf{D}} = \frac{12\epsilon^{j90^\circ}}{4\epsilon^{-j30^\circ}} = 3\epsilon^{j120^\circ}$$

The graphical interpretations of the above operations are contained in Fig. 10*a* and Fig. 10*b*.

The process of division can be carried out very conveniently when the phasors are expressed in exponential or polar form. However, it is entirely possible and in some cases desirable to perform the operation with the phasors expressed in rectangular form. If $\mathbf{A} = a + ja'$ and $\mathbf{B} = b + jb'$, then

$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{a + ja'}{b + jb'} = \frac{(a + ja')\,(b - jb')}{(b + jb')\,(b - jb')} \tag{41}$$

Both numerator and denominator of the above expression are multiplied by $(b - jb')$, the conjugate of $(b + jb')$. The conjugate of a given phasor is a second phasor, the real component of which is identical with the real component of the given phasor and the j part of which is equal in magnitude but reversed in sign from the j component of the given phasor.

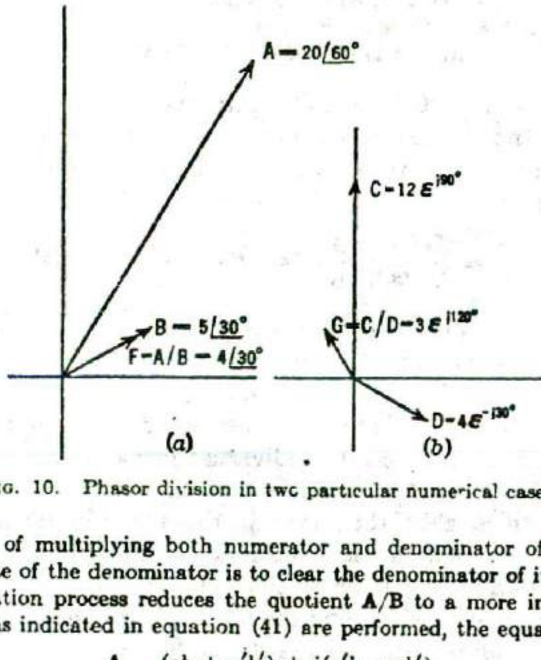

Fig. 10. Phasor division in two particular numerical cases.

The purpose of multiplying both numerator and denominator of equation (41) by the conjugate of the denominator is to clear the denominator of its j component. This rationalization process reduces the quotient $\mathbf{A}/\mathbf{B}$ to a more intelligible form. If the operations indicated in equation (41) are performed, the equation reduces to

$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{(ab + a'b') + j(a'b - ab')}{(b^2 + b'^2)} \tag{42}$$

By a process which is somewhat similar to that employed on pages 117–118 it may be shown that

$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{A}{B}\Big/\tan^{-1}\left[\frac{\sin(\alpha_A - \alpha_B)}{\cos(\alpha_A - \alpha_B)}\right] = \frac{A}{B}\underline{/\alpha_A - \alpha_B} \tag{43}$$

Example. If $\mathbf{A} = 10 + j17.3$ and $\mathbf{B} = 4.33 + j2.5$, let it be required to find $\mathbf{A}/\mathbf{B}$ by the method given in equations (41) and (42).

$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{10 + j17.3}{4.33 + j2.5} = \frac{(10 + j17.3)\,(4.33 - j2.5)}{(4.33 + j2.5)\,(4.33 - j2.5)}$$

$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{(43.3 + 43.3) + j(75 - 25)}{4.33^2 + 2.5^2}$$

Reduced to polar form

$$\frac{\mathbf{A}}{\mathbf{B}} = \sqrt{3.465^2 + 2.0^2} \Big/ \tan^{-1}\frac{2.0}{3.465} = 4.0 \underline{/30°}$$

Problem 9. Given $\mathbf{A} = 40 \underline{/105°}$ and $\mathbf{B} = 5 + j8.66$, find $\mathbf{A}/\mathbf{B}$, and draw a phasor diagram illustrating **A**, **B**, and **A**/**B**.

Ans.: $4 \underline{/45°}$.

Problem 10. Given the following three phasors:

$$\mathbf{A} = 20 + j20, \quad \mathbf{B} = 30 \underline{/-120°}, \quad \mathbf{C} = 5 + j0$$

perform the following indicated operations:

$$(a)\ \frac{\mathbf{A} + \mathbf{B}}{\mathbf{C}} \qquad\qquad (b)\ \frac{\mathbf{BC}}{\mathbf{A}}$$

Draw a phasor diagram of **A**, **B**, and **C**, together with the phasors which represent the results of the above indicated operations.

Ans.: (a) $1.56 \underline{/-50.2°}$. (b) $5.3 \underline{/-165°}$.

Raising a Phasor to a Given Power. A phasor or preferably a complex quantity may be raised to a given power n; where n is an integer, by multiplying the phasor by itself n times. For example, if $\mathbf{A} = A \underline{/\alpha_A}$,

$$\mathbf{A}^n = A^n \underline{/n\alpha_A} \tag{44}$$

The nth power of **A** is a phasor whose magnitude is A^n and whose phase position with respect to the reference is $n\alpha_A$. The concept of successive applications of a given operator follows directly from the successive multiplication of the operator by itself. Obviously the process involved is accomplished most easily with the phasor or operator in exponential or polar form.

From the rules which have been given for multiplication it is evident that

$$\mathbf{A}^n\mathbf{B}^n = A^nB^n \underline{/n\alpha_A + n\alpha_B} \tag{45}$$

Example. An operator which is commonly used successively is the one which rotates a given phasor through $+120°$. This operator is

$$\mathbf{a} = 1\ (\cos 120° + j \sin 120°) = -0.50 + j0.866$$

In polar form

$$\mathbf{a} = 1 \underline{/120°}$$

$$\mathbf{a}^2 = 1 \underline{/240°}$$

$$\mathbf{a}^3 = 1 \underline{/360°} = 1 \underline{/0°}$$

$$\mathbf{a}^4 = 1 \underline{/480°} = 1 \underline{/120°}$$

The above operators are widely used in three-phase circuit problems because, under balanced conditions, the individual phase voltages (and currents) are displaced from one another by 120°. Figure 11 illustrates **a**, **a**^{2}, and **a**^{3} diagrammatically. Incidentally, the three values indicated in Fig. 11, $\left(-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)$, $\left(-\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)$, and $(1 + j0)$, are the three roots of $\sqrt[3]{1}$ because each of these roots cubed equals unity.

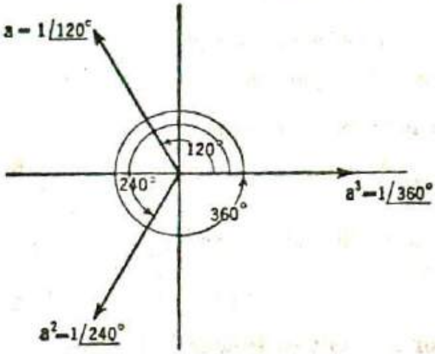

FIG. 11. Illustrating the operator $\mathbf{a} = (-0.50 + j0.866)$, together with $\mathbf{a}^2$ and $\mathbf{a}^3$.

Problem 11. Raise the phasor $(8.66 + j5.0)$ to the second power; to the fifth power. *Ans.*: $100\,\underline{/60°}$; $100{,}000\,\underline{/150°}$.

Extracting the Roots of a Phasor. The inverse of the process of raising a phasor to a given power is employed in the extraction of the roots of a particular phasor. If $\mathbf{A} = A\,\underline{/\alpha_A}$ it follows that one of the n roots of $\sqrt[n]{\mathbf{A}}$ is $\sqrt[n]{A}\,\underline{\Big/\frac{\alpha_A}{n}}$ because the latter value muitiplied by itself n times will equal **A**. The remaining $(n-1)$ roots are found by adding $2\pi q$ radians or $360q$ degrees to α_A before the division by n is performed. q is any integer and is used as $1, 2, 3, \cdots$, and $(n-1)$ to obtain the remaining roots. It should be noted that the addition of any multiple of 360° to the angle of the phasor does not change the phasor although it does provide a systematic method of evaluating the $(n-1)$ remaining roots. In this method only positive magnitudes are employed, as

$$\sqrt[n]{\mathbf{A}} = \sqrt[n]{A}\,\underline{\Big/\frac{\alpha_A + 2\pi q}{n}} \qquad [q = 0, 1, 2, \cdots (n-1)] \tag{46}$$

The cartesian form of the above equation is

$$\sqrt[n]{\mathbf{A}} = \sqrt[n]{A}\left[\cos\left(\frac{\alpha_A + 2\pi q}{n}\right) + j\sin\left(\frac{\alpha_A + 2\pi q}{n}\right)\right] \tag{47}$$

Example. Let it be required to find the square roots of **A** where $\mathbf{A} = 3.08 + j8.455$.

For convenience the phasor is first transformed into polar form.

$$\mathbf{A} = \sqrt{3.08^2 + 8.455^2} \;\Big/ \tan^{-1}\frac{8.455}{3.08} = 9.0\underline{/70^\circ}$$

The first root is $\sqrt{9.0}\;\Big/\frac{70^\circ}{2} = 3\underline{/35^\circ}$.

The second root is $\sqrt{9.0}\;\Big/\frac{70^\circ + 360^\circ}{2} = 3\underline{/215^\circ}$.

Figure 12 illustrates the phasor **A** together with its two roots. It will be noted that either root multiplied by itself yields the phasor **A**.

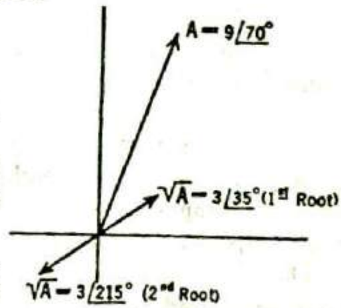

FIG. 12. Phasor $9\underline{/70^\circ}$ and its two roots.

Problem 12. Find the cube roots of the phasor $(8 + j0)$, and draw a complete phasor diagram of the phasor and its three roots.

Ans.: $2\underline{/0^\circ}$, $2\underline{/120^\circ}$, $2\underline{/240^\circ}$.

The Logarithm of a Phasor. Certain definitions in long-line and recurrent network theory utilize logarithms of phasor quantities. The general concept of the logarithm of a phasor is similar to that of the logarithm of an ordinary number. The logarithm of a phasor **A** is the inverse of the exponential of **A**. In other words, the logarithm of the phasor $\mathbf{A} = A\epsilon^{j\theta}$ to the base ϵ is defined as the power to which ϵ must be raised to equal $A\epsilon^{j\theta}$. By definition

$$\log_\epsilon A\epsilon^{j\theta} = \log_\epsilon A + \log_\epsilon \epsilon^{j\theta} = \log_\epsilon A + j\theta \log_\epsilon \epsilon = \log_\epsilon A + j\theta \quad (48)$$

It will be noted that the logarithm of the phasor $\mathbf{A} = A\underline{/\theta}$ is itself a phasor. In rectangular form, when the logarithm is taken to the base ϵ, the real component is $\log_\epsilon A$; that is, the logarithm to the base ϵ of the magnitude of the phasor **A** and the j component is θ (radians) in magnitude. In this connection, θ, the phase angle of the phasor **A**, *must be considered in radians.*

Example. If $\mathbf{A} = 52\underline{/70^\circ}$, let it be required to find $\log_\epsilon \mathbf{A}$.

$$\log_\epsilon 52\underline{/70^\circ} = \log_\epsilon 52 + j\frac{70^\circ}{57.3^\circ} = 3.95 + j1.22$$

Problem 13. Perform the following indicated operations:

$$\frac{15\underline{/70^\circ}}{(3 - j4)} + \log_\epsilon (8 + j5)$$

Draw a phasor diagram including each of the three original phasors together with the $\log_\epsilon (8 + j5)$ and the phasor which represents the result of the indicated operations.

Ans.: $0.60 + j3.07$.

Impedance Expressed in Polar Form. It was shown in Chapter III that the currents and voltages in an a-c circuit can be conveniently represented by phasors. With the aid of phasor algebra it is a simple matter to represent these currents and voltages analytically. However, the great benefit to be derived from the use of phasor or complex algebra is the simple algebraic relations that can be established between the voltages and currents by using the impedance function as a complex quantity. Although the impedance function may take the form of a phasor or vector, it is not a phasor in the same sense that alternating voltages or currents are phasors. From an algebraic point of view the impedance function is merely a complex quantity which properly relates phasor voltages and phasor currents one to the other. As such it is a most important operator in circuit analysis.

The physical considerations concerning the impedance function have been explained in Chapters II and III. If the polar form of the impedance function which was used throughout Chapter II is manipulated in accordance with the rules of phasor algebra, the results obtained will agree with physical facts. For example, it has been shown that the impedance function of a series RLC branch is

$$\mathbf{Z} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \Bigg/ \underline{\tan^{-1} \frac{\left(\omega L - \frac{1}{\omega C}\right)}{R}} \tag{49}$$

The abbreviated form is

$$\mathbf{Z} = Z\underline{/\theta} \tag{50}$$

where $+\theta$ represents a lead of the voltage with respect to the current or a lag of the current with respect to the voltage. If a phasor voltage $\mathbf{V} = V\underline{/\alpha}$ is applied to the above branch the resulting current is

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{V\underline{/\alpha}}{Z\underline{/\theta}} = \left[\frac{V}{Z}\right]\underline{/(\alpha - \theta)} \tag{51}$$

The phasor quotient $\mathbf{V}/\mathbf{Z}$ results in a phasor current which is V/Z in magnitude and θ degrees behind $\mathbf{V}$ regardless of the position that $\mathbf{V}$ has with respect to the reference axis. Thus $\mathbf{I}$ is correctly defined in magnitude and phase position.

In a similar manner it may be shown that $\mathbf{IZ} = \mathbf{V}$. If it be assumed that a current $\mathbf{I} = I\underline{/\beta}$ flows through an RLC branch, the impedance of which is $\mathbf{Z} = Z\underline{/\theta}$,

$$\mathbf{IZ} = (I\underline{/\beta})\,(Z\underline{/\theta}) = [IZ]\underline{/(\beta + \theta)} = \mathbf{V} \tag{52}$$

The product of the phasors $\mathbf{IZ}$ yields a phasor voltage $\mathbf{V}$, which is (IZ) in magnitude and θ degrees in advance of the current $\mathbf{I}$. It will be remembered that θ has been defined as

$$\tan^{-1}\frac{\left(\omega L - \dfrac{1}{\omega C}\right)}{R}$$

If $\omega L < 1/\omega C$, θ is a negative angle, in which case **V** actually lags **I**.

Example. A given RL branch has $R = 3.5$ ohms and $L = 0.092$ henry. Find the complex expression for the current which flows through the branch if a 60-cycle voltage, $\mathbf{V} = 110\underline{/30°}$ volts, is applied to the RL branch. (The phase angle which is associated with V is wholly arbitrary in a simple series circuit. For simplicity it might have been taken as zero degrees.)

$$\mathbf{Z} = \sqrt{R^2 + (\omega L)^2}\,\underline{\Big/\tan^{-1}\frac{(\omega L)}{R}}$$

$$\mathbf{Z} = \sqrt{3.5^2 + (377 \times 0.092)^2}\,\underline{\Big/\tan^{-1}\frac{(377 \times 0.092)}{3.5}}$$

$$\mathbf{Z} = 34.8\underline{/84.25°} \text{ ohms}$$

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{110\underline{/80°}}{34.8\underline{/84.25°}} = 3.16\underline{/-54.25°} \text{ amperes}$$

Figure 13 is a phasor diagram of **V** and **I** for the particular RL branch that has been considered.

Problem 14. An RLC series branch consists of $R = 12.9$ ohms, $L = 0.056$ henry, and $C = 78\ \mu$f. (*a*) What is the complex impedance of the RLC branch at 60 cycles? (*b*) If a 60-cycle current, $\mathbf{I} = 10\underline{/30°}$ amperes, flows through the branch, find the voltage phasor **V** across the terminals of the series branch. Draw a phasor diagram illustrating the phasor positions of **I** and **V** and the magnitude of the phase angle of **V** with respect to **I**.

Ans.: (*a*) $12.9 + j(21.1 - 34) = 12.9 - j12.9 = 18.24\underline{/-45°}$ ohms.
(*b*) $182.4\underline{/-15°}$ volts.

Impedance Expressed in Cartesian Form. The cartesian form of the impedance function of a given branch or circuit is, in general,

$$\mathbf{Z} = R + j(X_L - X_C) \tag{53}$$

where R is the equivalent resistance of the branch or circuit with respect to the terminals considered and $(X_L - X_C)$ is the equivalent reactance of the branch or circuit with respect to the terminals considered.

In accordance with previous definitions, $X_L = 2\pi fL$ and $X_C = \dfrac{1}{2\pi fC}$.
A simple method of showing the validity of equation (53) is to employ a phasor diagram in which are represented the RI, X_LI, and X_CI voltage drops which combine vectorially to equal the applied voltage **V**.

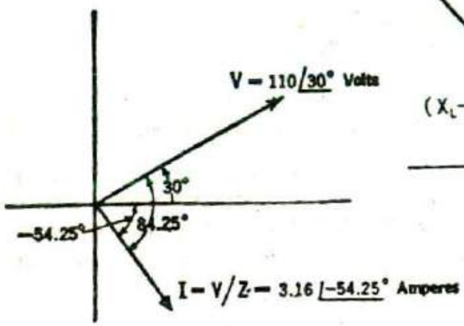

Fig. 13. Phasor diagram of V and I, in a particular RL series circuit.

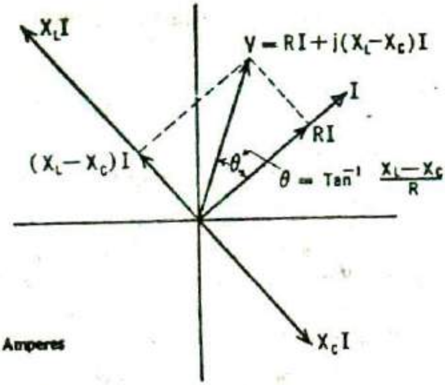

Fig. 14. Phasor addition of drops equals applied voltage.

In order to agree with physical facts: (1) the RI drop must be in phase with I; (2) the $X_L I$ drop must be 90° in advance of I; (3) the $X_C I$ drop must be 90° behind I.

Reference to Fig. 14 will show that the voltage

$$\mathbf{V} = RI + (X_L - X_C)I \quad \text{as phasors} \tag{54}$$

or

$$\mathbf{V} = R\mathbf{I} + j(X_L - X_C)\mathbf{I} \tag{55}$$

from which the complex impedance function is

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = R + j(X_L - X_C) \tag{56}$$

Obviously the relations stated in equations (54), (55), and (56) are independent of the phasor diagram position of I.

The cartesian or rectangular form of the complex expression for **Z** can be transformed to the polar form of **Z** by the method of complex algebra, and the transformation is, of course, reversible.

$$R + j(X_L - X_C) = \sqrt{R^2 + (X_L - X_C)^2} \,\Big/ \tan^{-1} \frac{(X_L - X_C)}{R} \tag{57}$$

The rectangular form of the impedance function is, in general, essential in combining impedances because impedances cannot be added or subtracted in polar form.

Example. The terminals of an a-c generator which has an internal resistance of 2 ohms and an equivalent internal inductive reactance of 6 ohms are connected to a particular RLC series branch, the R of which is 10 ohms, the ωL of which is 20 ohms, and the $1/\omega C$ of which is 40 ohms. If the magnitude of the internally generated emf is 500 volts, find the current that flows in the series circuit and the terminal voltage of the generator.

The internal impedance of the generator is

$$\mathbf{Z}_g = 2 + j6 = 6.32\underline{/71.6^\circ} \text{ ohms}$$

The total impedance of the series circuit is

$$\begin{aligned}\mathbf{Z}_t &= \mathbf{Z}_g + \mathbf{Z}_{RLC}\\ &= (2 + j6) + [10 + j(20 - 40)]\\ &= 12 - j14 = 18.44\underline{/-49.4^\circ} \text{ ohms}\end{aligned}$$

The generated emf, $\mathbf{E}_g$, is arbitrarily chosen to coincide with the reference axis. Therefore

$$\mathbf{E}_g = 500 + j0 = 500\underline{/0^\circ} \text{ volts}$$

The current that flows in the series circuit is

$$\mathbf{I} = \frac{\mathbf{E}_g}{\mathbf{Z}_t} = \frac{500\underline{/0^\circ}}{18.44\underline{/-49.4^\circ}} = 27.1\underline{/49.4^\circ} \text{ amperes}$$

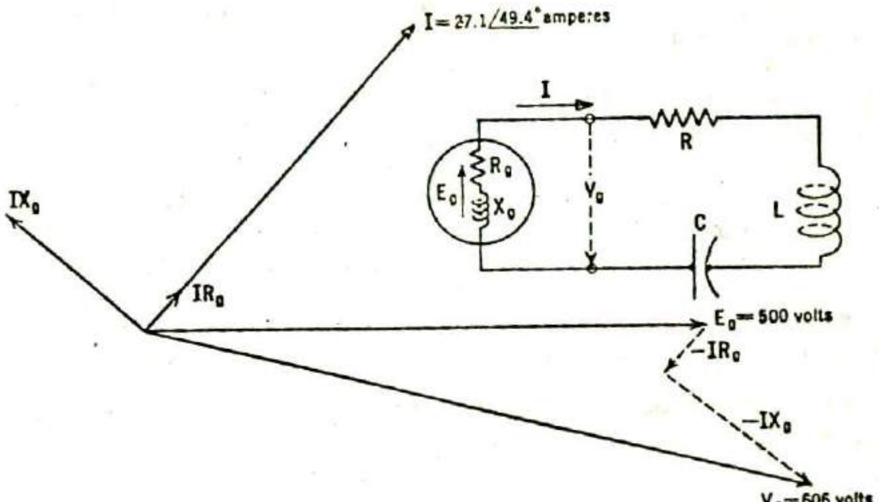

FIG. 15. Voltage relations for a generator supplying a leading power-factor load.

The terminal voltage of the generator considered as a voltage drop across the external circuit is

$$(1)\quad \mathbf{V}_g = \mathbf{E}_g - \mathbf{I}\mathbf{Z}_g \quad \text{or} \quad (2)\quad \mathbf{V}_g = \mathbf{I}\mathbf{Z}_{RLC}$$

$$\begin{aligned}\mathbf{V}_g &= (500\underline{/0^\circ}) - (27.1\underline{/49.4^\circ})\,(6.32\underline{/71.6^\circ})\\ &= 500\underline{/0^\circ} - 171.3\underline{/121^\circ}\\ &= (500 + j0) - (-88.3 + j147)\\ &= 588.3 - j147 = 606\underline{/-14^\circ} \text{ volts}\end{aligned}$$

$$\mathbf{V}_g = (27.1\underline{/49.4^\circ})\,(22.36\underline{/-63.4^\circ}) = 606\underline{/-14^\circ} \text{ volts}$$

A phasor diagram illustrating $\mathbf{E}_g$, $\mathbf{I}$, IR_g, IX_g, and $\mathbf{V}_g$ is given in Fig. 15. It will be observed that the terminal voltage of the generator ($\mathbf{V}_g$) is greater in magnitude than the internally generated emf (E_g) owing to the manner in which the voltage phasor $\mathbf{I}X_g$ subtracts from $(\mathbf{E}_g - \mathbf{I}R_g)$ to form phasor $\mathbf{V}_g$.

Problem 15. (a) Draw a phasor diagram illustrating $\mathbf{E}_g$, $\mathbf{I}$, $\mathbf{IR}$, $\mathbf{IX}_L$, $\mathbf{IX}_C$, and $\mathbf{V}_g$ of the above numerical example and show how $\mathbf{IR}$, $\mathbf{IX}_L$, and $\mathbf{IX}_C$ combine vectorially to form $\mathbf{V}_g$.

(b) Calculate the total power generated and the total power absorbed by the external RLC branch. Compare $V_gI\cos\theta\Big]_{\mathbf{I}}^{\mathbf{V}_g}$ plus I^2R_g with $E_gI\cos\theta\Big]_{\mathbf{I}}^{\mathbf{E}_g}$

Ans.: Total power = 8810 watts; branch power = 7345 watts.

Addition and Subtraction of Voltages and Currents. Correctly written, the complex expressions for voltages and currents specify both the magnitudes and relative phase positions of these quantities. Therefore, in complex form:

1. Voltage drops in series may be added to obtain the combined voltage drop of the series elements considered. If the combined voltage drop and one component are known, the remaining voltage drop may be determined by subtracting the component in question from the combined voltage drop.

2. Generated emf's connected in additive or subtractive series may be added or subtracted, depending upon the relative polarities of the terminals which are joined together to form the series connection. Series connections of generated emf's will be considered in more detail when polyphase systems are studied.

3. Two or more currents flowing away from a junction may be added to find the current flowing toward the junction, or vice versa.

Circuit Directions of Voltages and Currents. It has been shown that the average power absorbed by a branch or circuit is

$$P = VI\cos\theta\Big]_{\mathbf{I}}^{\mathbf{V}} \tag{58}$$

where V is the magnitude of the voltage drop across the branch or circuit,

I is the magnitude of the current flowing through the branch or circuit in the same circuit direction as that which has been taken for the $+V$ direction.

$\theta\Big]_{\mathbf{I}}^{\mathbf{V}}$ is the angle of lag (or lead) of $\mathbf{I}$ with respect to $\mathbf{V}$. In a normal dissipative type of branch or circuit, θ will not be as great as $\pm 90°$.

Similarly, the average power generated by a generating device is

$$P = EI\cos\theta\Big]_{\mathbf{I}}^{\mathbf{E}} \tag{59}$$

where E is the magnitude of the generated voltage,

I is the magnitude of the current flowing in the same circuit direction as that which has been taken for the $+E$ direction.

$\theta\Big]_{\mathbf{I}}^{\mathbf{E}}$ is the angle of lag (or lead) of $\mathbf{I}$ with respect to $\mathbf{E}$. In case the generating device is actually delivering power, $\theta\Big]_{\mathbf{I}}^{\mathbf{E}}$ will be less than 90° in magnitude. This, in general, is the condition that exists when only one generator is present. Average negative generated power indicates that the generating device in question is actually absorbing power from some other generator.

A single generator connected to a dissipative branch is shown in Fig. 16. If the $+E$ circuit direction is assumed to be from b to a *through* the generator, the positive circuit direction of the current is from b to a *through* the generator, and from a to b *through* the dissipative branch. The positive circuit direction of a voltage drop through a dissipative branch defines the positive circuit direction of the current through the branch, or vice versa. In Fig. 16, therefore, the $+V$ direction is from a to b through the external branch. With the aid of these elementary concepts, the correct phase relations of all quantities involved may be conveniently determined. If E_g is taken as reference,

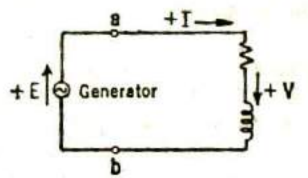

FIG. 16. Illustrating an arbitrarily assigned positive circuit direction of the generated voltage, E, together with the resulting positive circuit directions of I and V.

$$\mathbf{I} = \frac{E_g\underline{/0^\circ}}{\mathbf{Z}_{\text{gen}} + \mathbf{Z}_{\text{load}}} = I\underline{/\alpha} \tag{60}$$

$$\mathbf{V} = E_g\underline{/0^\circ} - \mathbf{I}\mathbf{Z}_{\text{gen}} = V\underline{/\beta} \tag{61}$$

$$\text{Average generated power} = E_g I \cos \alpha \tag{62}$$

$$\text{Average power absorbed by the external branch} = VI \cos (\beta - \alpha) \tag{63}$$

Thus it will be seen that the current in a series loop may be associated with the generated voltage to obtain the generated power and with a particular voltage drop across a given part of the circuit to obtain the power absorbed by this particular part of the circuit. Unless otherwise specified, the current in a series loop having only one generator is assumed to flow in the positive direction of voltage rise through the generator and in the positive direction of voltage drop through the load portion of the circuit.

Example of Two Generators. Figure 17 illustrates two a-c generators which are connected in parallel with respect to the load terminals but are connected in subtractive series with respect to the series loop joining the two generators. If no load

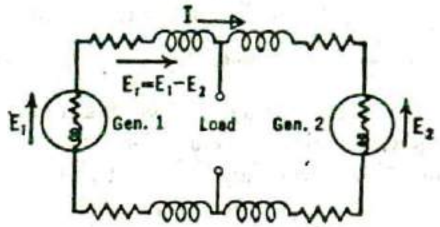

FIG. 17. Two generated emf's connected in parallel with respect to the load terminals. $\mathbf{E}_1$ and $\mathbf{E}_2$ are in subtractive series with respect to the series loop which joins the two generators.

is placed across the load terminals, the series loop is the only path in which current flows. If it is assumed that the generators are driven by separate prime movers and controlled by separate voltage regulators, it is entirely possible for the voltages to differ in magnitude and phase position.

Let $E_1 = 1350\underline{/0^\circ}$ volts and $E_2 = 1300\underline{/-10^\circ}$ with respect to the load terminals. The impedance of each generator is $(1 + j3)$ ohms and each of the series loop connecting lines has $(2 + j1)$ ohms impedance. Find the magnitude and phase position of the current which circulates in the series loop under the above conditions.

The resultant generated emf which acts to send current through the series loop in the $+E_1$ direction is

$$\mathbf{E}_r = \mathbf{E}_1 - \mathbf{E}_2 = (1350 + j0) - (1280 - j226) = 70 + j226 \text{ volts}$$

The positive circuit direction of E_r is the same as that which has been arbitrarily assigned to $\mathbf{E}_1$, since the phasor difference $\mathbf{E}_1 - \mathbf{E}_2$ has been employed in defining $\mathbf{E}_r$. The current that flows in the direction of $\mathbf{E}_r$ is

$$\begin{aligned}\mathbf{I} &= \frac{\mathbf{E}_r}{\mathbf{Z}_{\text{loop}}} = \frac{70 + j226}{6 + j8} \\ &= \frac{(70 + j226)\,(6 - j8)}{(6 + j8)\,(6 - j8)} \\ &= (22.28 + j7.96) = 23.65\underline{/19.65^\circ} \text{ amperes}\end{aligned}$$

The power generated by the $\mathbf{E}_1$ generator is

$$P_{g1} = 1350 \times 23.65 \cos 19.65^\circ = 30{,}110 \text{ watts}$$

The power generated by the $\mathbf{E}_2$ is

$$\begin{aligned}P_{g2} &= 1300 \times 23.65 \cos [180^\circ - (10^\circ + 19.65^\circ)] \\ &= -26{,}750 \text{ watts}\end{aligned}$$

In calculating the power generated by the $\mathbf{E}_2$ machine, either the voltage or the current is reversed in phase position so that the $\mathbf{E}_2$ and $\mathbf{I}$ circuit directions coincide. The physical interpretation of the negative generated power found for machine 2 is that machine 2 is actually receiving power from machine 1. A phasor diagram of $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_r$, and $\mathbf{I}$ is shown in Fig. 18.

In general, a circulating current between the two generators may exist as a result of difference in the *magnitude* of the two generated voltages, or a difference in *phase*, or *both*.

A further insight into the power relations of the circuit arrangement shown in Fig. 17 may be obtained by adding to the power absorbed by machine 2 the total I^2R loss of the series loop and comparing the result with the total power generated by machine 1.

$$(23.65^2 \times 6) + 26{,}750 = 30{,}110 \text{ watts}$$

The physical interpretation of the above equation is that machine 1 generates 30,110 watts, of which 3360 are dissipated in the form of heat in the resistance of the series loop and 26,750 watts are absorbed by machine 2 in the form of electromagnetic motor power.

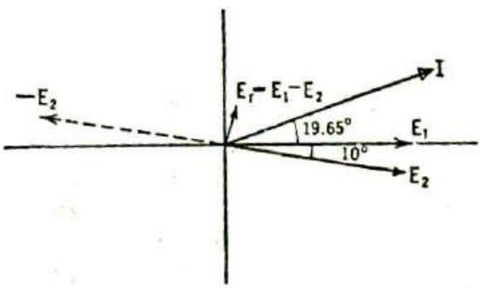

FIG. 18. Phasor diagram of two-generator problem.

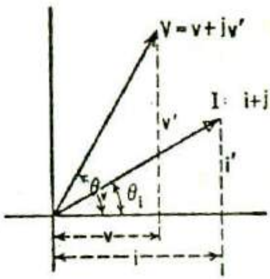

FIG. 19.

Power Calculations Employing Complex Forms. If voltage and current are expressed in rectangular complex form, the average absorbed or generated power may be calculated in terms of the components of the voltage and current which are involved. Reference to Fig. 19 will show that

$$P = VI \cos \theta \Big]_{\mathbf{I}}^{\mathbf{V}} \tag{64}$$

or

$$\begin{aligned} P &= VI \cos (\theta_v - \theta_i) = VI \cos (\theta_i - \theta_v) \\ &= VI\,[\cos \theta_v \cos \theta_i + \sin \theta_v \sin \theta_i] \\ &= (V \cos \theta_v)\,(I \cos \theta_i) + (V \sin \theta_v)\,(I \sin \theta_i) \end{aligned} \tag{65}$$

In rectangular form

$$\mathbf{V} = V \cos \theta_v + jV \sin \theta_v = v + jv' \tag{66}$$

$$\mathbf{I} = I \cos \theta_i + jI \sin \theta_i = i + ji' \tag{67}$$

If the above components of V and I in equation (65) are employed, it follows that

$$P = vi + v'i' \text{ (absorbed power)} \tag{68}$$

If the voltage in question is a generated voltage,

$$P = ei + e'i' \text{ (generated power)} \tag{69}$$

Due regard must be taken for the sign of each component in equations (68) and (69) when these power equations are employed.

Example. If, at a certain stage in the solution of a problem, it is found that $\mathbf{E} = (200 + j40)$ volts and that the current flowing in the positive circuit direction of $\mathbf{E}$ is $\mathbf{I} = (30 - j10)$ amperes, the power generated is

$$\begin{aligned} P = ei + e'i' &= (200)\,(30) + (40)\,(-10) \\ &= 6000 - 400 = 5600 \text{ watts} \end{aligned}$$

The same result could, of course, be obtained by first evaluating the magnitudes of E, I, and $\theta\Big]_{\mathbf{I}}^{\mathbf{E}}$ and then making use of the more familiar relation

$$P = EI\cos\theta\Big]_{\mathbf{I}}^{\mathbf{E}}$$

Reactive Volt-Ampere Calculations Employing Complex Forms. Reactive volt-amperes or reactive power, P_X, may also be calculated in terms of the rectangular components of the voltage and current involved. If the voltage phasor and the current phasor shown in Fig. 19 are considered,

$$\mathbf{V} = v + jv'$$

$$\mathbf{I} = i + ji'$$

As defined in Chapters II and III,

$$P_X = VI\sin\theta\Big]_{\mathbf{I}}^{\mathbf{V}} \tag{70}$$

In accordance with a convention which is in common use, θ is the angle of *lead* of the voltage with respect to the current. If this convention of signs is employed, reactive power is a positive quantity for lagging currents and a negative quantity for leading currents. (See Chapter III, page 98.) If the angle θ in equation (70) is considered as the angle of lead of the voltage with respect to the current, then

$$\begin{aligned} P_X &= VI\sin(\theta_v - \theta_i) \\ &= VI(\sin\theta_v\cos\theta_i - \cos\theta_v\sin\theta_i) \\ &= (V\sin\theta_v)(I\cos\theta_i) - (V\cos\theta_v)(I\sin\theta_i) \end{aligned} \tag{71}$$

From the definitions which have been attached to v, v', i, and i', equation (71) reduces directly to

$$P_X = v'i - vi' \tag{72}$$

Example. If $\mathbf{V} = 200\underline{/30°} = (173.2 + j100)$ volts and $\mathbf{I} = 10\underline{/60°} = (5 + j8.66)$ amperes, find the real power, the reactive volt-amperes, and the total volt-amperes involved.

$$P = vi + v'i' = 866 + 866 = 1732 \text{ watts}$$
$$P_X = v'i - vi' = 500 - 1500 = -1000 \text{ vars}$$

The minus sign in connection with P_X merely indicates that the reactive power in question is the result of a leading current.

The volt-amperes associated with $\mathbf{V}$ and $\mathbf{I}$ can be obtained directly from the product of V and I, or as follows:

$$Va = \sqrt{P^2 + P_X{}^2} = \sqrt{1732^2 + (-1000)^2}$$
$$= 2000 \text{ volt-amperes}$$

The Conjugate Method of Calculating Real and Reactive Power. The question naturally arises as to the significance of the product of phasor voltage and phasor current. The answer is to be found in the definition that has been given to the product of two complex numbers. The magnitude of the product of voltage and current, even in complex form, represents the volt-amperes which are associated with **V** and **I**. The component parts of the cartesian expression for **VI** are, however, meaningless. For this reason, phasor voltage times phasor current cannot be used directly to calculate real power or reactive volt-amperes.

A method of conjugates is sometimes employed in the determination of real power and reactive volt-amperes. It affords a convenient means of calculating these quantities when both the voltage and current are expressed in cartesian form.

If the conjugate of the current, that is, the cartesian expression of the current with the sign of the j component reversed, is multiplied by the voltage in cartesian form, the result is a complex quantity the real part of which is the real power and the j part of which is the reactive volt-amperes.

Let $\quad \mathbf{V} = v + jv' \quad \text{and} \quad \mathbf{I} = i + ji'$

The conjugate of **I** is $(i - ji')$ and

$$(v + jv')(i - ji') = (vi + v'i') + j(v'i - vi') \tag{73}$$

or

$$(v + jv')\,(i - ji') = P + jP_X \tag{74}$$

If the conjugate of **V** is multiplied by **I** in complex form, the result is

$$(v - jv')\,(i + ji') = (vi + v'i') - j(v'i - vi') \tag{75}$$

or

$$(v - jv')\,(i + ji') = P - jP_X \tag{76}$$

The real power, as obtained by the method of conjugates, is the same regardless of whether **V** or **I** is conjugated. The sign of the reactive volt-amperes, however, is dependent upon the choice of the **V** or **I** conjugate as shown by equations (74) and (76). To be consistent with convention of signs employed in equation (70), the conjugate of the *current* must be employed. Also to be in accord with the discussion in the previous chapter (page 98) and the recommendation to the International Committee on Electrical and Magnetic Units the conjugate of current must be employed. In any event $P_X = v'i - vi'$ or $vi' - v'i$ with the sign being a matter of definition. The present situation indicates that the current should be conjugated or that $P_X = v'i - vi'$ should be used and that inductive vars should be called positive.

Example. Given $\mathbf{V} = 173.2 + j100$ volts and $\mathbf{I} = 5.0 + j8.66$ amperes, find the real power and the reactive volt-amperes by the method of conjugates. Employing the conjugate of the current,

$$\begin{aligned} P_{va} &= (173.2 + j100)\ (5.0 - j8.66) \\ &= 866 - j1500 + j500 + 866 \\ &= 1732 - j1000 \end{aligned}$$

This above result may be interpreted, in light of equations (73) and (74), to mean that $P = 1732$ watts and that $P_X = -1000$ vars. The negative sign indicates capacitive vars when the conjugate of the current is used.

Transmission Expressed as a Complex Number. The term "transmission" will be used here as a general designation of the effect (say the current or power) in a receiver station produced by a generator at the sending station. (See Fig. 20.[3]) In low-power communication networks, particular attention centers on the *change* in magnitude (and the phase shift) of the receiver current *relative* to the receiver current which could be obtained under optimum conditions of operation. These relative changes are due to two causes, namely:

(1) R_2 in Fig. 20 not being equal to the generator resistance R_1, the latter being fixed by the generator characteristics.

(2) The network intervening between the sending-station generator E_1 and the receiver-station resistance R_2.

The intervening network will usually take the form of a transmission line, transformer, selective filter, attenuator, or amplifier. Various combinations of these basic four-terminal networks may be employed between the generator terminals ($11'$ in Fig. 20) and the receiver terminals $22'$, but until the detailed operation of these devices has been studied we shall represent them simply as a box having four terminals as shown in the figure.

[3] If any series or shunt reactance is associated with either the generator or load, it may be placed within the four-terminal network for the purposes of analysis, thus making Fig. 20 a more general case than is apparent from the diagram.

It will be accepted here, subject to later proof, that the most efficient possible transmission between E_1 and R_2 will occur when the impedance looking to right of terminals $11'$ is equal to R_1, that is, when $V_1/I_1 = R_1$. (Amplifiers are excluded from the foregoing statement because these

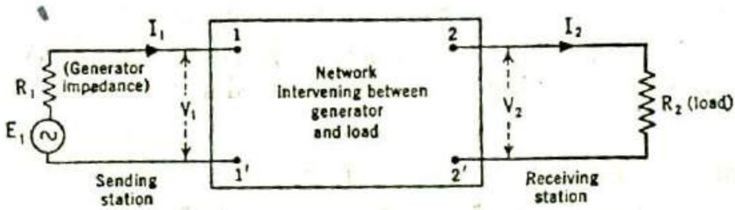

FIG. 20. Four-terminal network intervening between a generator and a resistive load.

devices draw power from sources other than the E_1 generator.) Under this condition of operation the generator resistance is said to match the impedance looking to the right of terminals $11'$ and

$$I_{1(\text{matched})} = \frac{E_1}{2R_1}$$

The ratio of the powers entering and leaving the network under the condition that $\mathbf{V}_1/\mathbf{I}_1 = R_1$ is

$$\frac{\text{Power entering terminals } 11'}{\text{Power leaving terminals } 22'} = \frac{V_1 I_1}{V_2 I_2} = \frac{(E_1/2)(E_1/2R_1)}{(R_2 I_2)(I_2)} \tag{77}$$

If now we define the *transfer impedance* from E_1 to R_2 under any condition of operation as

$$\mathbf{Z}_T = \frac{\mathbf{E}_1}{\mathbf{I}_2} \tag{78}$$

we note that the value of $\mathbf{Z}_T$ which will make the power ratio of equation (77) unity is

$$\mathbf{Z}_{T(\text{opt})} = 2\sqrt{R_1 R_2} \tag{79}$$

In other words, for fixed values of R_1 and R_2, all the power entering terminals $11'$ in Fig. 20 will leave terminals $22'$ if the intervening network is such that $\mathbf{E}_1/\mathbf{I}_2 = 2\sqrt{R_1 R_2}$.

In describing the transmission characteristics of an arbitrary four-terminal network of the kind shown in Fig. 20, it is desirable that the receiver current, $\mathbf{I}_2$, be measured relative to its optimum value, $\mathbf{E}_1/2\sqrt{R_1 R_2}$. Both the magnitude and phase of $\mathbf{I}_2$ relative to this base can be measured in terms of the real and j components of the transmission constant, $\boldsymbol{\gamma}$, if the latter is defined as

$$\boldsymbol{\gamma} = \alpha + j\beta = \log_e \frac{\mathbf{Z}_T}{2\sqrt{R_1 R_2}} = \log_e \frac{\mathbf{E}_1/2\sqrt{R_1 R_2}}{\mathbf{I}_{2(\text{general})}} = \log_e \frac{\mathbf{I}_{2(\text{opt})}}{\mathbf{I}_{2(\text{general})}} \quad (80)$$

where $\mathbf{Z}_T = \mathbf{E}_1/\mathbf{I}_2$ for any arbitrary intervening network.

α is the attenuation (to be described in more detail later).
β is the phase shift (also to be described in detail later).

The transmission constant is thus defined as a logarithmic measure of $\mathbf{Z}_T$ relative to $\mathbf{Z}_{T(\text{opt})}$. Since $E_1/2\sqrt{R_1 R_2}$ in Fig. 20 is considered to be a constant, it is plain that α is a logarithmic measure of $I_{2(\text{opt})}/I_{2(\text{general})}$ and that β is the phase angle difference between $\mathbf{I}_{2(\text{general})}$ and $\mathbf{I}_{2(\text{opt})}$. The phase angle of $\mathbf{I}_{2(\text{opt})}$ would normally be zero, since the reference would normally be $\mathbf{E}_1 = E_1\underline{/0°}$ and $\mathbf{I}_{2(\text{opt})}$ is in phase with $\mathbf{E}_1$, being equal to $\mathbf{E}_1/2\sqrt{R_1 R_2}$.

Attenuation, α. It will be noted from equation (80) that the attenuation can be written as

$$\alpha = \log_e \frac{\sqrt{I^2_{(\text{opt})}}}{\sqrt{I^2_{(\text{general})}}} = \frac{1}{2}\log_e \frac{{I_2}^2_{(\text{opt})}R_2}{{I_2}^2_{(\text{general})}R_2} \quad \text{nepers} \quad (81)$$

Attenuation in this case is an inverse logarithmic measure of the power received by R_2 under general conditions of operation to that which is received by R_2 under optimum conditions of operation. The fact that logarithmic measure is employed in the definition of $\boldsymbol{\gamma}$ makes $\alpha = 0$ if ${I_2}^2_{(\text{general})}R_2$ is equal to ${I_2}^2_{(\text{opt})}R_2$, and as the former decreases in value owing to losses in the intervening network α grows larger logarithmically. If $\log_e$ is employed as in equation (81), the units of α are called nepers.

Another common definition of attenuation as it applies to general transmission characteristics is

$$\alpha_{\text{db}} = 10 \log_{10} \frac{{I_2}^2_{(\text{opt})}R_2}{{I_2}^2_{(\text{general})}R_2} \quad \text{decibels} \quad (82)$$

Plainly

$$\frac{(\text{No. of})\alpha_{\text{db}}}{\alpha_{\text{nepers}}} = \frac{10 \log_{10} K}{\frac{1}{2}\log_e K} = \frac{20 \log_{10} K}{2.303 \log_{10} K} = 8.686$$

where K is any power ratio. The above relationship indicates that the number of decibels per neper is 8.686. It is a matter of indifference which unit of attenuation is used, since engineers generally understand that the decibel is by definition a unit of attenuation which is 8.686 times smaller in magnitude than the neper, there being 8.686 decibels of attenuation for each neper of attenuation in any particular specification of attenuation.

Phase Shift, β. In taking the logarithm indicated in equation (80) it will be noted that

$$\boldsymbol{\gamma} = \alpha + j\beta = \log_\epsilon \frac{\mathbf{I}_{2(\mathrm{opt})}}{\mathbf{I}_{2(\mathrm{general})}} = \log_\epsilon \frac{I_{2(\mathrm{opt})}}{I_{2(\mathrm{general})}} + j[\theta_{(\mathrm{opt})} - \theta_{(\mathrm{general})}] \quad (83)$$

Thus, if $\mathbf{E}_1$ is selected as a reference, $\theta_{(\mathrm{opt})} = 0$ and $\beta = -\theta_{(\mathrm{general})}$. Regardless of the reference selected, β specifies the phase difference between $\mathbf{I}_2$ under optimum conditions and $\mathbf{I}_2$ under general operating conditions.

If the evaluation of $\alpha + j\beta$ is to be carried no further than that shown in equation (83), it is a matter of choice whether β is stated in radians or degrees. If $\boldsymbol{\gamma}$ is to be expressed in polar form, however, β must be expressed in radians.

Examples. In Fig. 20, let $R_1 = 100$ ohms, $R_2 = 25$ ohms, and assume that terminal 1 is connected directly to terminal 2 and terminal 1′ directly to 2′. Let it be required to find the attenuation and phase shift relative to the optimum operating conditions.

If equation (80) is to be employed, we note that

$$\mathbf{I}_{2(\mathrm{opt})} = \frac{\mathbf{E}_1}{2\sqrt{100 \times 25}} = \frac{\mathbf{E}_1}{100}$$

$$\mathbf{I}_{2(\mathrm{actual})} = \frac{\mathbf{E}_1}{125}$$

$$\alpha + j\beta = \log_\epsilon \frac{\mathbf{E}_1/100}{\mathbf{E}_1/125} = 0.223 + j0$$

Thus $\alpha = 0.223$ neper or 1.938 decibels. This attenuation results from R_2 not being equal to R_1.

$\beta = 0$ since no phase difference exists between the two conditions of operation.

As a check on the arithmetic we might employ equation (82) as

$$\alpha_{\mathrm{db}} = 20 \log_{10} \frac{I_{2(\mathrm{opt})}}{I_{2(\mathrm{actual})}} = 20 \log_{10} \frac{E_1/100}{E_1/125}$$

$$\alpha_{\mathrm{db}} = 20 \log_{10} \frac{125}{100} = 20 \times 0.0969 = 1.938 \text{ decibels}$$

As a second example of the use of equation (80) let it be assumed that $R_1 = 25$ ohms, $R_2 = 100$ ohms, and that, for $\mathbf{E}_1 = 10\underline{/0^\circ}$ volts, $\mathbf{V}_2 = 3.53\underline{/-45^\circ}$ volts.

It is required that the transmission constant, $\boldsymbol{\gamma}$, be found from the above data.

$$\mathbf{I}_{2(\mathrm{opt})} = \frac{\mathbf{E}_1}{2\sqrt{R_1R_2}} = \frac{10\underline{/0^\circ}}{2\sqrt{2500}} = 0.1\underline{/0^\circ} \text{ ampere}$$

$$\mathbf{I}_{2(\mathrm{actual})} = \frac{\mathbf{V}_2}{R_2} = \frac{3.53\underline{/-45^\circ}}{100} = 0.0353\underline{/-45^\circ} \text{ ampere}$$

$$\mathbf{Z}_T = \frac{\mathbf{E}_1}{\mathbf{I}_2} = \frac{10\,\underline{/0^\circ}}{0.0353\,\underline{/-45^\circ}} = 283\,\underline{/45^\circ}\text{ ohms}$$

$$\mathbf{\Upsilon} = \alpha + j\beta = \log_e \frac{\mathbf{Z}_T}{2\sqrt{R_1R_2}} = \log_e \frac{283\,\underline{/45^\circ}}{100} = 1.04 + j45^\circ$$

or

$$\mathbf{\Upsilon} = \alpha + j\beta = \log_e \frac{\mathbf{I}_{2(\text{opt})}}{\mathbf{I}_{2(\text{actual})}} = \log_e \frac{0.1\,\underline{/0^\circ}}{0.0353\,\underline{/-45^\circ}} = 1.04 + j0.785\text{ radians}$$

Thus

$$\mathbf{\Upsilon} = 1.304\,\underline{/0.647\text{ (radians)}} = 1.304\,\underline{/37.05^\circ}$$

If $\mathbf{\Upsilon}$ is specified in polar form as above, we obtain α and β as the real and j terms directly by changing the polar form of $\mathbf{\Upsilon}$ to rectangular form.

A significant point which should not be overlooked in the foregoing discussion of *attenuation* and *phase shift* is that, as applied to the four-terminal network shown in Fig. 20, these quantities were obtained from

$$\alpha + j\beta = \log_e \frac{\mathbf{Z}_{T(\text{general})}}{\mathbf{Z}_{T(\text{opt})}} = \frac{1}{2}\log_e \frac{{I_2}^2_{(\text{opt})}R_2}{{I_2}^2_{(\text{general})}R_2} + j[\theta_{Z_{T(\text{general})}}]$$

where $\mathbf{Z}_{T(\text{opt})}$ was an *arbitrarily selected* base which yielded maximum power delivered to the load resistance, R_2. (It was assumed that R_1 was fixed by the characteristics of the E_1 generator and that R_2 was fixed by the characteristics of the receiving device.) The base selected here is that which is normally employed when we wish to take account of the possible mismatch between R_1 and R_2 as well as the loss and phase shift introduced by the intervening four-terminal network. It also permits the possible mismatch between R_1 and R_2 to be rectified by the intervening network if the latter is designed for this purpose.

In general circuit analysis, *attenuation* and *phase shift* are used in a wide variety of different ways to describe loss (or gain) and phase difference relative to other *arbitrarily selected* bases. Attenuation and phase shift are meaningful quantities only when the base is clearly understood, since attenuation and phase shift are measures of power loss (or gain) and phase relative to the base which is selected as being most appropriate for the problem at hand.

PROBLEMS

16. Perform the following indicated operations: (*a*) $(5 + j8) + (-2 - j4)$; (*b*) $(-12 + j6) - (30 - j20)$; (*c*) $(16 - j12)(-5 + j8)$; (*d*) $(-5 + j8.66) + (5 - j8.66)$; (*e*) $(2 - j3) + (-1$

17. Two impedances, $Z_1 = 2 + j3$ ohms and $Z_2 = 3 - j7$ ohms, are connected in a circuit so that they are additive. Find the equivalent impedance of the two in polar form.

18. Write the cartesian and polar expressions for a phasor, the magnitude of which is 100 units and the phase position of which is:

(*a*) 30° behind the reference axis.
(*b*) 45° behind the reference axis.
(*c*) 120° behind the reference axis.
(*d*) 180° behind the reference axis.
(*e*) 60° ahead of the reference axis.
(*f*) 120° ahead of the reference axis.
(*g*) 210° ahead of the reference axis.

19. Find the magnitude and angular position with respect to the reference axis of the phasors which are represented by:

(*a*) $8.0 + j6.0$.
(*b*) $-10 + j10.0$.
(*c*) $38.3 - j31.14$.
(*d*) $-57.36 + j81.92$.
(*e*) $-76.6 - j64.3$.
(*f*) $-50.0 - j86.6$.

20. (*a*) Rotate the phasor $(8.66 + j5.0)$ through $+40°$ by multiplying it by the correct operator.

(*b*) Rotate the phasor $(-5.0 - j8.66)$ through $-30°$.

(*c*) Express the results of (*a*) and (*b*) in both cartesian and polar forms.

21. Perform the following indicated operations:

(*a*) $(8 + j6)(10\,\underline{/-120°})(\cos 36.87° - j \sin 36.87°)(0.1\epsilon^{+j60°})$.

(*b*) $\dfrac{[34.2 + j94][10\epsilon^{-j30°}][30(\cos 60° + j \sin 60°)]}{[20\,\underline{/40°}][50\,(\cos 30° + j \sin 30°)]}$.

22. Express each of the following as a single complex number in cartesian and polar forms:

(*a*) $[\sqrt{4.5 - j7.79} + \log_e 10\,\underline{/172°}]$.

(*b*) $\sqrt{\dfrac{(940 + j342)}{10\epsilon^{j10°}}}$.

(*c*) $\dfrac{(-8.66 + j5.0)(50\,\underline{/-100°})(2\epsilon^{j70°})}{j5}$.

(*d*) $50\epsilon^{-j\omega t}$ at $t = \dfrac{1}{480}$ second if $\omega = 377$ radians per second.

(*e*) $\dfrac{30 + j10}{6 - j3} \cdot \sqrt{4.5 - j7.794}$.

23. Find all possible roots of

$$\sqrt[3]{\frac{10\,\underline{/45°}5\epsilon^{j60°}(-4.047 - j2.94)}{1 - j1.732}}$$

24. The series impedance of a transmission line is $Z_a = 10\,\underline{/68°}$ ohms, and the shunt impedance of the line is $Z_b = 25{,}000\,\underline{/-90°}$ ohms.

(*a*) Find the characteristic impedance of the line which is defined as $Z_0 = \sqrt{Z_a Z_b}$.

(*b*) Find the propagation constant of the line which is defined as $\gamma = \sqrt{Z_a / Z_b}$.

25. A voltage of $125\,\underline{/40°}$ volts is impressed across a series combination of 2.0 ohms resistance and 8.0 ohms inductive reactance. Find the magnitude and phase position of the current with respect to the reference axis employed in stating the voltage phasor.

26. Two impedances, $\mathbf{Z}_1 = (1 - j3)$ ohms and $\mathbf{Z}_2 = (3 + j6)$ ohms, are connected in parallel. The magnitude of the current through $\mathbf{Z}_1$ is known to be 10 amperes.

(a) Find the complex polar expression for the current through $\mathbf{Z}_2$ with respect to $\mathbf{I}_1 = 10\,\underline{/0^\circ}$ as a reference.

(b) Find $\mathbf{I}_0 = \mathbf{I}_1 + \mathbf{I}_2$ in cartesian form.

(c) Draw a phasor diagram of $\mathbf{V}$, $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_0$, employing $\mathbf{I}_1$ as reference.

27. The characteristic impedance of a T-section filter is $\mathbf{Z}_{0T} = \sqrt{\mathbf{Z}_1\mathbf{Z}_2 + \dfrac{\mathbf{Z}_1^2}{4}}$, where $\mathbf{Z}_1$ is the full series arm impedance and $\mathbf{Z}_2$ is the shunt impedance of the filter section. If $\mathbf{Z}_1 = 30\,\underline{/86.0^\circ}$ ohms and $\mathbf{Z}_2 = 10.0\,\underline{/-90^\circ}$ ohms, find $\mathbf{Z}_{0T}$ from the above definition of $\mathbf{Z}_{0T}$.

28. Express $\log_\epsilon \sqrt{\dfrac{125\,\underline{/-90^\circ}}{5\,\underline{/90^\circ}}}$ in rectangular form.

Ans.: $1.61 \mp j\pi/2$.

29. An equation which is useful in filter circuit analysis is

$$\alpha + j\beta = 2\log_\epsilon\left(\sqrt{1 + \frac{\mathbf{Z}_1}{4\mathbf{Z}_2}} + \sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}}\right)$$

If $\mathbf{Z}_1 = 25.14\,\underline{/-90^\circ}$ ohms and $4\mathbf{Z}_2 = 795\,\underline{/+90^\circ}$, evaluate α and β.

30. Find α and β in Problem 29 if

$$\mathbf{Z}_1 = 4 \times 10^3\,\underline{/-90^\circ} \quad \text{ohms}$$
$$4\mathbf{Z}_2 = 1000\,\underline{/90^\circ} \quad \text{ohms}$$

31. Given the equation

$$\mathbf{V}_m = \mathbf{V} - \mathbf{ZI}$$

where $\mathbf{V} = 100\,\underline{/0^\circ}$ volts, $\mathbf{Z} = 15\,\underline{/80^\circ}$ ohms, $\mathbf{I} = 10\,\underline{/-3^\circ}$ amperes. Express $\mathbf{V}_m$ in polar form.

32. (a) Solve the following equation for a and for b:

$$(12 + a) + jb = 20 + j10$$

(b) Solve the following equation for a and for β:

$$(a + 10) + j50 = 100(\cos\beta + j\sin\beta)$$

(c) Given: $(100 + j0) + 5R\,\underline{/-45^\circ} = 200\,\underline{/-\theta^\circ}$; find R and θ.

33. (a) Plot $A\epsilon^{+j\omega t}$ and $A\epsilon^{-j\omega t}$ in polar coordinates for $\omega = 157$ radians per second at $t = 0.005$, $t = 0.010$, $t = 0.015$, $t = 0.020$, and $t = 0.04$ second.

(b) Plot $\dfrac{A\epsilon^{j\omega t} + A\epsilon^{-j\omega t}}{2}$ in polar coordinates and also in rectangular coordinates versus ωt for one complete cycle.

(c) Show that a simple harmonic oscillating variation, such as $A\cos\omega t$, can be represented by two oppositely rotating phasors, each of which has the same angular velocity as the oscillating phasor and each of which has a magnitude equal to one-half the magnitude of the oscillating phasor.

34. (a) A voltage $\mathbf{V} = 100 - j50$ volts across a circuit causes a current $\mathbf{I} = -2 - j8$ amperes to flow. Calculate the power absorbed by the circuit, employing equation (68).

(b) Calculate power if $\mathbf{V} = -50 + j100$ volts and $\mathbf{I} = -6 - j2$ amperes.

(c) Calculate power if $\mathbf{V} = -50 + j100$ volts and $\mathbf{I} = -8 + j3$ amperes.

35. Calculate the vars for each of the parts of Problem 34, employing equation (72).

36. Calculate the power and vars by the method of conjugates for each part of Problem 34.

37. The voltage applied to two parallel branches is $40\underline{/80^\circ}$ volts. The current through branch 1 is $5\underline{/30^\circ}$ amperes, and the current through branch 2 is $(-6 + j8)$ amperes. Find the real power, P, and the reactive volt-amperes, P_x, supplied to the parallel combination by the method of conjugates. *Note:* Check results against $VI = 40 \times 10.62 = \sqrt{P^2 + P_x^2}$.

38. In Fig. 20, page 135, $R_1 = 200$ ohms, $R_2 = 20{,}000$ ohms, and $\mathbf{V}_2 = (0.1\underline{/114.6^\circ})\mathbf{E}_1$. Find the attenuation and phase shift which are produced by the combination of the mismatch of R_1 and R_2 and the intervening network.

39. In Fig. 20, $R_1 = 200$ ohms, $R_2 = 20{,}000$ ohms, and $\mathbf{I}_2 = \mathbf{E}_1/4000$ amperes. Find the attenuation and phase shift which are produced by the combination of the mismatch of R_1 and R_2 and the intervening network.

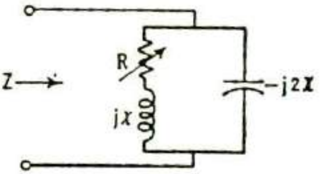

Fig. 21. See Problem 40.

40. For the circuit shown in Fig. 21,

$$\mathbf{Z} = \frac{(R + jX)(-j2X)}{R + jX - j2X}$$

Plot Z and θ of $\mathbf{Z} = Z\underline{/\theta}$ versus R, employing $R = 0$, $R = X/2$, $R = X$, $R = 2X$, $R = 5X$, $R = 10X$, and $R = \infty$.

chapter V

Sinusoidal Single-Phase Circuit Analysis

Impedances in Series. A series circuit of three impedances is shown in Fig. 1. In a circuit of this kind it is evident that only a single current

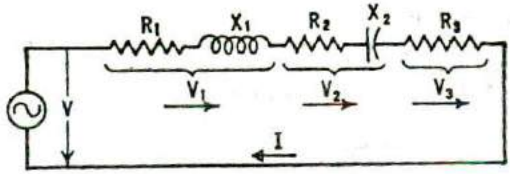

FIG. 1. Impedances in series.

can exist at any instant and that the current throughout all impedances is the same.[1] Kirchhoff's emf law states that

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3 \tag{1}$$

or

$$\mathbf{V} = \mathbf{IZ}_1 + \mathbf{IZ}_2 + \mathbf{IZ}_3 \tag{2}$$

and

$$\mathbf{V} = \mathbf{I}(\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3) = \mathbf{IZ} \tag{3}$$

Equation (3) shows that series impedances are added in complex form to obtain the equivalent impedance. Thus

$$\mathbf{Z} = \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3 = (R_1 + jX_1) + (R_2 + jX_2) + (R_3 + j0)$$

or $$\mathbf{Z} = (R_1 + R_2 + R_3) + j(X_1 + X_2) = R + jX \tag{4}$$

Equation (4) shows that the resultant resistance R of a simple series circuit is obtained by arithmetically adding the separate resistances. When it is remembered that inductive reactances are considered positive and capacitive reactances are negative, equation (4) also shows that the resultant reactance X of a series circuit is the algebraic sum of the separate reactances.

If current is taken as the reference, the vector diagram of the circuit of Fig. 1 appears as shown in Fig. 2. Such a vector diagram is called a funicular or string diagram. Another type of vector diagram which

[1] The assumption is made that the current is confined to the series circuit. Maxwellian space displacement currents are neglected.

represents the same circuit is shown in Fig. 3. This is called a polar diagram. The distinguishing characteristic of a string vector diagram is that certain component vectors are combined head-to-tail to form a resultant vector as, for example, the component voltages $\mathbf{I}R_1$, $\mathbf{I}X_1$, $\mathbf{I}R_2$,

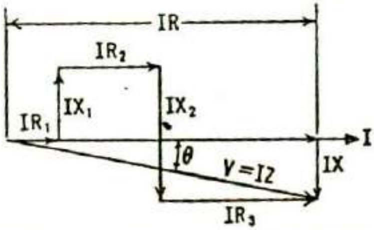

FIG. 2. Funicular or string vector diagram of circuit in Fig. 1.

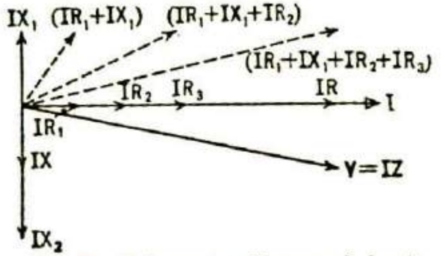

FIG. 3. Polar vector diagram of circuit in Fig. 1.

$\mathbf{I}X_2$, and $\mathbf{I}R_3$ are combined head-to-tail to form the resultant voltage vector **V**. In a polar vector diagram, all vectors are started from a common origin as shown in Fig. 3.

Either type of diagram may be used since they represent the same thing. The one which appears to be the simpler in any particular case should be used. In certain cases the funicular diagram shows the quantities to better advantage, whereas for others the polar diagram is more suggestive of the relationships and more convenient to use.

In general, for a series circuit of n impedances

$$\mathbf{V} = \mathbf{I}(\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3 + \cdots + \mathbf{Z}_n) \tag{5}$$

and $$\mathbf{Z} = (R_1 + R_2 + R_3 + \cdots + R_n) + j(X_1 + X_2 + X_3 + \cdots + X_n) \tag{6}$$

$$\mathbf{Z} = \sqrt{(R_1 + R_2 + R_3 + \cdots + R_n)^2 + (X_1 + X_2 + X_3 + \cdots + X_n)^2} \Big/ \tan^{-1} \frac{X_1 + X_2 + X_3 + \cdots + X_n}{R_1 + R_2 + R_3 + \cdots + R_n} \tag{7}$$

In Chapter II the impedance angle was shown to be the phase angle between the current and the voltage. In Chapter III power factor was shown to be the cosine of this angle. Hence, for a series circuit, Fig. 2 shows

$$\text{Power factor} = \cos\theta = \frac{IR}{IZ} = \frac{R}{Z}$$

$$= \frac{R_1 + R_2 + R_3 + \cdots + R_n}{\sqrt{(R_1 + R_2 + R_3 + \cdots + R_n)^2 + (X_1 + X_2 + X_3 + \cdots + X_n)^2}} \tag{8}$$

Example 1. Calculate the current, voltage drops V_1, V_2, and V_3, power consumed by each impedance, and the total power taken by the circuit with the constants shown in Fig. 4. The impressed voltage will be taken along the reference axis.

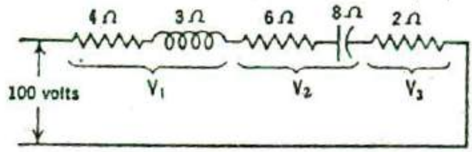

FIG. 4. Circuit for example 1.

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{100 + j0}{4 + j3 + 6 - j8 + 2} = \frac{100\ (12 + j5)}{(12 - j5)\ (12 + j5)} = 7.1 + j2.96 \text{ amperes}$$

$$\mathbf{V}_1 = \mathbf{IZ}_1 = (7.1 + j2.96)\ (4 + j3) = 19.53 + j33.14 \text{ volts}$$

$$\mathbf{V}_2 = \mathbf{IZ}_2 = (7.1 + j2.96)\ (6 - j8) = 66.27 - j39.06 \text{ volts}$$

$$\mathbf{V}_3 = \mathbf{IZ}_3 = (7.1 + j2.96)\ (2 + j0) = \underline{14.2 + j5.92 \text{ volts}}$$

Check: $\mathbf{V} = 100 + j0$ volts

Note that the drops are added *vectorially* to check the impressed voltage.

$$P_1 = RI^2 = 4(\sqrt{7.1^2 + 2.96^2})^2 = 4 \times 7.69^2 = 237 \text{ watts}$$

$$P_2 = 6 \times 7.69^2 = 355 \text{ watts}$$

$$P_3 = 2 \times 7.69^2 = \underline{118 \text{ watts}}$$

$$\text{Total power} = 710 \text{ watts}$$

The total power is also $(vi + v'i') = 100 \times 7.1 = 710$ watts.

Problem 1. (*a*) Find the current through the circuit in Fig. 5 and the voltage drops $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{cd}$.

Ans.: $\mathbf{I} = 10 \angle 0°$ amperes, $\mathbf{V}_{ab} = 20 - j40 = 44.7 \angle -63.45°$ volts.
$\mathbf{V}_{bc} = 30 + j110 = 114 \angle 74.75°$ volts.
$\mathbf{V}_{cd} = 20 + j0 = 20 \angle 0°$ volts.

(*b*) Draw a string vector diagram of $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{cd}$, including both $\mathbf{V}$ and $\mathbf{I}$ on the diagram.

(*c*) Draw a polar vector diagram of $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, $\mathbf{V}_{cd}$, $\mathbf{V}$, and $\mathbf{I}$.

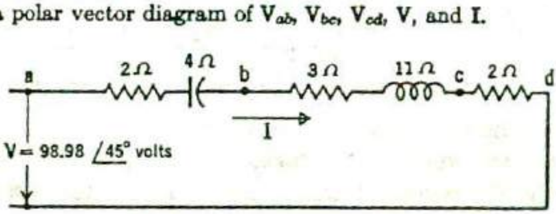

FIG 5. See Problems 1 and 2.

Problem 2. Calculate the total power dissipated in Fig. 5 from (I^2R), from $(VI \cos \theta)$, and from $(vi + v'i')$. *Ans.:* $P = 700$ watts.

Series Resonance. A series circuit containing R, L, and C is in resonance when the resultant reactance is zero. Since the drop across the inductance leads the current by 90° whereas that across the condenser lags the current by 90°, the two drops are opposite. If they are made equal as in Fig. 6, the reactive voltage drops neutralize and the impressed voltage is equal only to the resistance drop. This condition is called series resonance. Inspection of the vector diagram of Fig. 6 shows that the applied voltage is in phase with the current. The power factor is unity, and the circuit is in resonance. Thus for series resonance

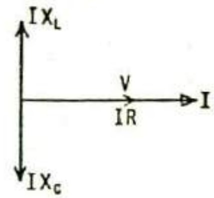

FIG. 6. Vector diagram of series circuit in resonance.

$$IX_L = IX_C \quad \text{or} \quad X_L = X_C \tag{9}$$

Since $2\pi fL = 1/2\pi fC$ at the point of series resonance, the series resonant frequency is

$$f_m = \frac{1}{2\pi\sqrt{LC}} \tag{10}$$

where f_m is in cycles per second when L is expressed in henrys and C in farads. It is apparent that series resonance can be produced in a series circuit by varying either L, C, or f. The current is always given by

$$I = \frac{V}{Z} = \frac{V}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{V}{\sqrt{R^2 + \left(2\pi fL - \dfrac{1}{2\pi fC}\right)^2}} \tag{11}$$

For any value of current the drop across the resistance is

$$V_R = IR = \frac{VR}{\sqrt{R^2 + \left(2\pi fL - \dfrac{1}{2\pi fC}\right)^2}} \tag{12}$$

Similarly, the drops across the inductance and capacitance are respectively

$$V_L = IX_L = \frac{VX_L}{\sqrt{R^2 + \left(2\pi fL - \dfrac{1}{2\pi fC}\right)^2}} \tag{13}$$

and

$$V_C = IX_C = \frac{VX_C}{\sqrt{R^2 + \left(2\pi fL - \dfrac{1}{2\pi fC}\right)^2}} \tag{14}$$

The general characteristics of a circuit in resonance are the same regardless of which parameter is varied to produce resonance. For instance, in all cases the power factor at resonance is 1. The power is simply the impressed voltage times the current. The current is V/R, the maximum possible value for the resistance which is in the circuit. The general shape of the current curve before, at, and after resonance is shown in Fig. 7. Resonance occurs at the point C. Limited as it is only by the resistance of the circuit, the current at the resonant point C will be large if the resistance is small. When the resultant reactance is large as it is at point A there will be only a small current flowing. Hence

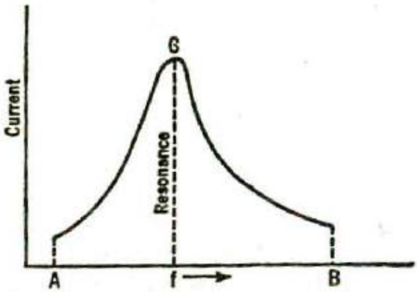

FIG. 7. Variation of current with frequency in the range series resonance.

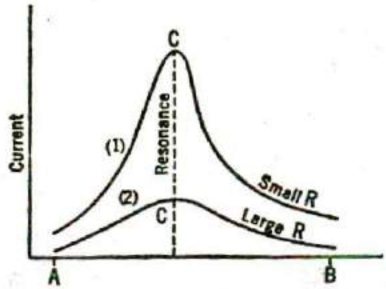

FIG. 8. Effect of resistance on current variation in the range of series resonance.

there is a rapid rise in current from point A to point C. Conversely, when the resistance is large, the amount of the change in current from point A to C will be small. In the former case the current peak will be sharper than in the latter, as illustrated in Fig. 8. Hence the small resistance is said to give sharp tuning and the large resistance broad tuning. More accurately, the ratio of L to R governs the sharpness of tuning. This is shown later. The preceding statements are true for all methods of securing resonance. The various ways of securing resonance will now be considered in somewhat more detail.

Varying Inductance. When L is varied to produce resonance, a series of curves shown in Fig. 9 is obtained. Equations (11), (12), (13), and (14) are the equations of the current and potential drop curves shown. It will be noted that V_C becomes a maximum at resonance whereas *the maximum value of* V_L *occurs after resonance.* This result is expected. Since $V_C = IX_C$ and X_C is constant, the maximum drop across the condenser will occur when the current is a maximum. In the case of $V_L = IX_L$, both I and X_L are increasing before resonance and the product must be increasing. At resonance, I is not changing but X_L is increasing, and hence the drop is increasing. The drop continues to increase until the reduction in the current offsets the increase in X_L. This point can be determined from $dV_L/dX_L = 0$. Differentiating equation (13) and setting the result equal to zero yield

$$\frac{dV_L}{dX_L} = \frac{[R^2+(X_L-X_C)^2]^{\frac{1}{2}}V - VX_L\frac{1}{2}[R^2+(X_L-X_C)^2]^{-\frac{1}{2}}2(X_L-X_C)}{R^2+(X_L-X_C)^2} = 0$$

and

$$X_L = \frac{R^2+X_C{}^2}{X_C}$$

or

$$L = \frac{1}{2\pi f}\left(\frac{R^2+X_C{}^2}{X_C}\right) = C(R^2+X_C{}^2) \tag{15}$$

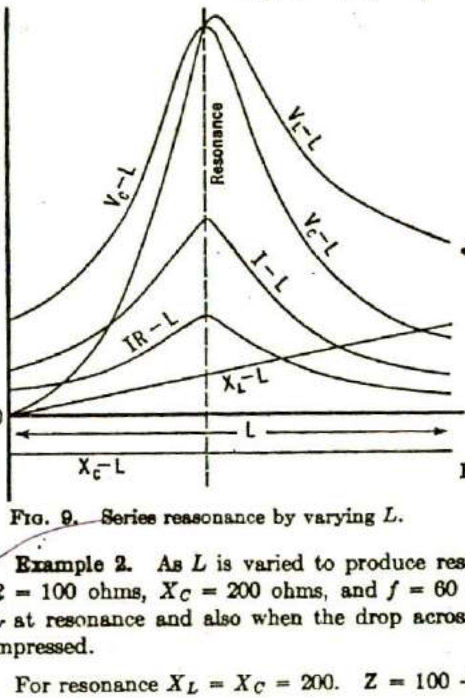

FIG. 9. Series resonance by varying L.

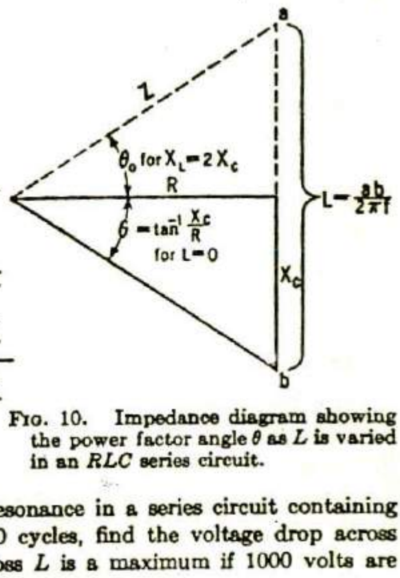

FIG. 10. Impedance diagram showing the power factor angle θ as L is varied in an RLC series circuit.

Example 2. As L is varied to produce resonance in a series circuit containing $R = 100$ ohms, $X_C = 200$ ohms, and $f = 60$ cycles, find the voltage drop across L at resonance and also when the drop across L is a maximum if 1000 volts are impressed.

For resonance $X_L = X_C = 200$. $\mathbf{Z} = 100 + j200 - j200 = 100 + j0$ ohms.

$I = \dfrac{1000}{100} = 10$ amperes.

V_L (at resonance) $= IX_L = 10 \times 200 = 2000$ volts.

For maximum V_L $\quad 2\pi fL = \dfrac{R^2 + X_C{}^2}{X_C} = \dfrac{100^2 + 200^2}{200} = 250$ ohms.

I (for maximum V_L) $= \dfrac{1000}{\sqrt{100^2 + (250 - 200)^2}} = 8.94$ amperes.

Maximum $V_L = 8.94 \times 250 = 2235$ volts.

The variation in phase angle between $\mathbf{V}$ and $\mathbf{I}$ as L is varied is easily obtained from the impedance diagram in Fig. 10. The angle can be seen to vary from $\tan^{-1}\dfrac{X_C}{R}$

(a negative angle) when L is zero to $+90°$ when L becomes ∞. Hence the power factor varies from $\dfrac{R}{\sqrt{R^2 + X_C{}^2}}$ (when L is 0) to 0 (when L becomes infinite).

Problem 3. (*a*) Find the value of inductive reactance and the value of inductance which will make the power factor of the above series circuit equal to 0.866, current leading.

Hint: Problems of this type are most easily solved when it is recognized that $\dfrac{\sum X}{\sum R} = \pm \tan \theta$. *Ans.:* $X_L = 142.3$ ohms, $L = 0.377$ henry.

(*b*) Find the value of inductive reactance which will make the p.f. equal to 0.866, current lagging.

Ans.: $X_L = 257.7$ ohms.

Varying Capacitance. When C is varied to produce resonance, curves as shown in Fig. 11 are obtained. As before, the equations of these curves are equations (11), (12), (13), and (14). Here the drop across the inductance is a maximum when the current is a maximum, since X_L is constant. The maximum drop across the condenser occurs before resonance. At resonance, X_C is decreasing whereas the current is not changing (slope being zero). The drop IX_C must, therefore, be decreasing. Consequently, the drop must have been a maximum before resonance. At resonance the drops across the inductance and the capacitance are equal and opposite. The conditions for maximum V_C may be determined analytically by setting the first derivative of equation (14) with respect to C or X_C equal to zero, similarly to the procedure illustrated when L was varied. This derivation is left to the student.

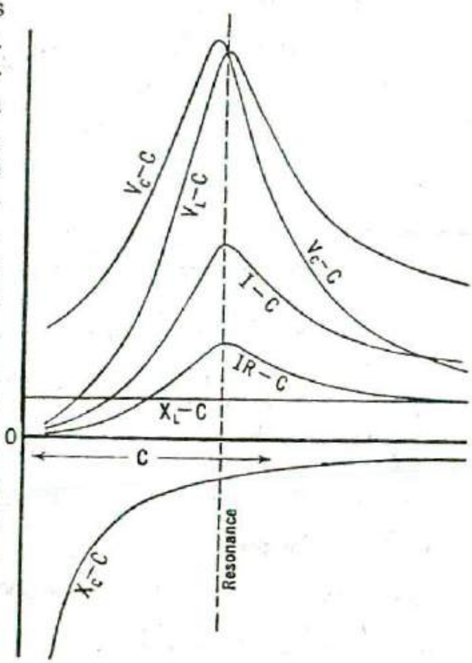

FIG. 11. Series resonance by varying capacitance.

The impressed voltage equals the IR drop, the power factor is unity, and the current is a maximum at resonance. For zero capacitance the capacity reactance is infinite and the current is therefore zero. For infinite capacitance the capacity reactance is zero and the current is $\frac{V}{\sqrt{R^2 + X_L{}^2}}$. The phase angle between the current and the applied voltage varies between the limits indicated in Fig. 12. The power factor varies from $\frac{R}{\sqrt{R^2 + X_L{}^2}}$, when C is infinite, to zero when C is zero.

Resonance is usually obtained by varying capacitance since it is only necessary to make alternate plates of a condenser movable to secure variable capacitance. This is easily and simply accomplished, and the variation of capacitance can be made extremely smooth and gradual.

Problem 4. When varying C to produce resonance in a circuit containing 100 ohms resistance and 200 ohms inductive reactance at 60 cycles, find the maximum drop across the capacitance if the impressed voltage on the circuit is 100 volts.

Ans.: 223.5 volts.

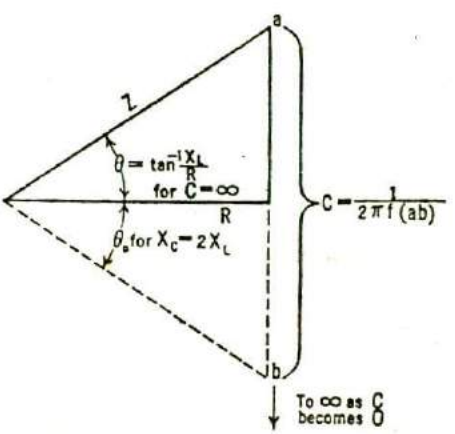

FIG. 12. Impedance diagram indicating range of power factor angle θ as C is varied in an *RLC* series circuit.

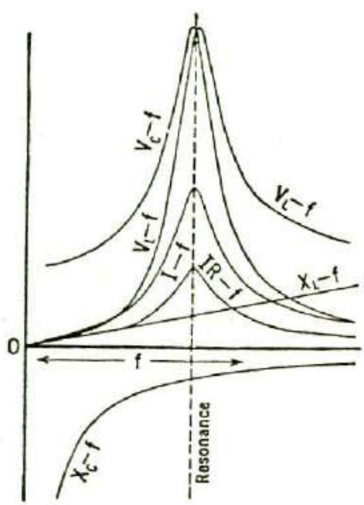

FIG. 13. Series resonance by varying frequency.

Varying Frequency. When frequency is varied to produce resonance, the curves shown in Fig. 13 are obtained. Here neither the inductance nor the capacitance has the maximum drop across it at resonance. Inspection of Figs. 9, 11, and 13 will show that this method of securing resonance partakes of both the methods previously discussed. The student can explain these curves by considering the principles previously presented. The current is zero for both zero frequency and infinite frequency. The phase angle between current and voltage varies between $-90°$ to $+90°$, as may be seen by studying the impedance triangles portrayed in Fig. 14. It will be observed that, for all methods of producing resonance, the current is a maximum and dependent only upon the impressed voltage and the resistance of the circuit, that the power factor is 1, and that the power is a maximum and equal to the volt-amperes at the point of resonance.

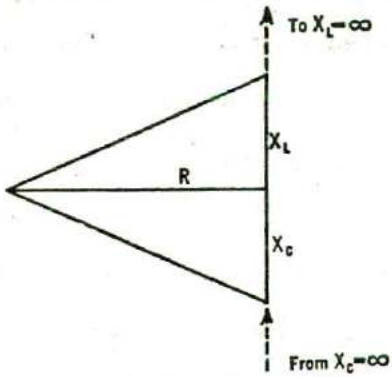

FIG. 14. Impedance triangle indicating variation of phase angle from $-90°$ to $+90°$ as frequency is varied in an *RLC* series circuit.

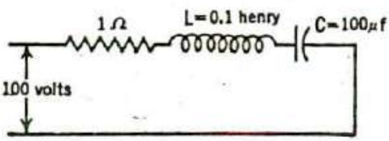

FIG. 15. Circuit for example 3.

Example 3. For the circuit arrangement and constants shown in Fig. 15 calculate the frequency, power, power factor, and voltage drop across each part of the circuit at resonance.

$$f_m = \frac{1}{2\pi}\sqrt{\frac{1}{LC}} = \frac{1}{2\pi}\sqrt{\frac{1}{0.1 \times 0.000100}} = 50.4 \text{ cycles}$$

Check:
$$\begin{cases} X_L = 2\pi\, 50.4 \times 0.1 = 31.6 \text{ ohms} \\ X_C = \dfrac{1}{2\pi\, 50.4 \times 0.0001} = 31.6 \text{ ohms} \end{cases}$$

$$I = \frac{100}{\sqrt{1^2 + (31.6 - 31.6)^2}} = 100 \text{ amperes}$$

$$P = 100 \times 100 = 10{,}000 \text{ watts}$$

$$\text{P.f.} = \frac{\text{watts}}{\text{va}} = \frac{10{,}000}{100 \times 100} = 1$$

$$V_R = 100 \times 1 = 100 \text{ volts}$$

$$V_L = 100 \times 31.6 = 3160 \text{ volts}$$

$$V_C = 100 \times 31.6 = 3160 \text{ volts}$$

Problem 5. (*a*) What is the resonant frequency of a series circuit consisting of 2 ohms resistance, 150 microhenrys, and 200 μμf capacitance? (*b*) What is the resonant frequency if $R = 3$ ohms, $L = 300$ microhenrys, and $C = 100$ μμf? (*c*) What is the impedance of each of the combinations at 1000 kilocycles?

Ans.: (*a*) 920 kilocycles, (*b*) 920 kilocycles, (*c*) 147 ohms and 294 ohms.

The Series *RLC* Circuit as a Selector. Even though the *RLC* circuit passes all waves of finite frequency to some extent, it has been shown to have the lowest impedance for the resonant frequency. As Fig. 7 shows, the *RLC* circuit passes frequencies near the resonant frequency more readily than other frequencies. The circuit thus has selective properties. The band of frequencies which is passed quite readily is called

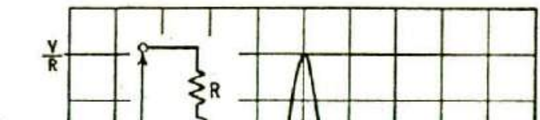

FIG. 16. The *RLC* series branch, as a band selector, graphed for $R = 10$ ohms, $L = 0.01$ henry, and $C = 4.0\ \mu$f.

the pass band. The pass band is sometimes arbitrarily considered to be the range of frequency over which the current is equal to or greater than $V/\sqrt{2}R$, as indicated in Fig. 16. Within this range, the power (I^2R) is equal to or greater than $V^2/2R$. This range will now be determined. From equation (11)

$$I = \frac{V}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} \tag{16}$$

The maximum current (V/R) and the maximum power V^2/R occur at the resonant frequency or when

$$\omega_m = \frac{1}{\sqrt{LC}} \tag{17}$$

where ω_m is 2π times the resonant frequency f_m. Let ω_x be the angular velocities at which

$$I = \frac{V}{\sqrt{2}R}$$

Since at these points the power is exactly one-half the maximum power which occurs at resonance, they are called the half-power points. Substituting the above current in equation (16) gives

$$\frac{V}{\sqrt{2}R} = \frac{V}{\sqrt{R^2 + (\omega_x L - 1/\omega_x C)^2}} \tag{18}$$

From which $R = \pm(\omega_x L - 1/\omega_x C)$.

Note that at these points the resistance of the circuit equals the resultant reactance, the phase angle between the applied voltage and current is 45°, and the power factor 0.707.

When solved for ω_x the above equation yields

$$\omega_x = \pm\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}} \tag{19}$$

In a selective RLC branch, $(R/2L)^2$ is usually much smaller than $1/LC$. Hence, neglecting this term, equation (19) becomes

$$\omega_x \approx \pm R/2L \pm \sqrt{1/LC} \tag{20}$$

But $\sqrt{1/LC}$ is the angular velocity ω_m corresponding to the resonant frequency. Therefore

$$\omega_x \approx \pm\frac{R}{2L} \pm \omega_m \tag{21}$$

and, if only positive values of ω_m are considered,

$$\omega_x = \omega_m \pm \frac{R}{2L} \tag{22}$$

Let

$$\omega_1 = \omega_m - \frac{R}{2L} \tag{23}$$

and

$$\omega_2 = \omega_m + \frac{R}{2L} \tag{24}$$

The width of the pass band as shown on Fig. 16 is

$$\Delta\omega = \omega_2 - \omega_1 = \frac{R}{L} \text{ radians per second} \tag{25}$$

The frequency range for the pass band, as here defined, is $\Delta f = f_2 - f_1 = R/2\pi L$. The per unit band width is defined as $\Delta f/f_m$. If we arbitrarily select a band width other than that shown in Fig. 16, as we shall have occasion to do later, we make appropriate changes in our definition of Δf.

Example 4. Let it be required to find the decibel (db) current response at the half-power points of Fig. 16 (relative to the response at ω_m) if by definition we take

$$\text{db} = 20 \log \frac{I}{\dfrac{V}{R}}$$

where I is the current response at any point on the graph shown in the figure.

Since $I = V/\sqrt{2}R$ at the points in question,

$$\text{db} = 20 \log \frac{\dfrac{V}{\sqrt{2}R}}{\dfrac{V}{R}} = -20 \log 1.414 = -3$$

The above arithmetic shows why the half-power points are sometimes referred to in the literature as the -3 db points.

The Q of a Series Circuit. The degree of selectivity of a circuit, that is, the narrowness of the band width shown in Fig. 16, is usually expressed in terms of the symbol Q. Although several different forms of the definition of Q appear in the literature, they are all intended to convey the same meaning. We shall employ the following definition since it ties in closely with experimental procedures:

$$Q = \frac{\omega_m}{\omega_2 - \omega_1} = \frac{\omega_m}{\Delta\omega} = \frac{f_m}{\Delta f} \tag{26}$$

See Fig. 16 for the meanings of ω_1, ω_2, and ω_m.

In the case of the series RLC circuit

$$Q_s = \frac{\omega_m}{\Delta\omega} = \frac{\omega_m}{\dfrac{R_s}{L}} = \frac{\omega_m L}{R_s} = \frac{1}{\omega_m C R_s} = \frac{1}{\dfrac{1}{\sqrt{LC}} C R_s} = \frac{1}{R_s}\sqrt{\frac{L}{C}} \tag{27}$$

where R_s is the total equivalent series resistance of the circuit. Since the equivalent series circuit resistance of the capacitor is usually negligibly small in comparison with the series circuit resistance of the coil, it is customary to speak of the Q of the coil alone, the assumption being that the coil will be resonated at some specified frequency with a capacitor of suitable size.

From equation (27) $Q_s = \dfrac{\omega_m L}{R_s}$. If the numerator and denominator of the right member of this equation are each multiplied by the current at resonance, I_{res}

$$Q_s = \frac{\omega_m L I_{res}}{R_s I_{res}} = \frac{\text{voltage drop across } L}{\text{applied voltage}} \tag{28}$$

Thus Q_s is a multiple of the applied circuit voltage that will exist across each of the reactive elements at resonance.

Example 5. The per unit band width between the half-power (or -3 db) points in Fig. 16 is to be 0.02. Find the Q of the coil required.

$$\text{Per unit band width} = \frac{\Delta\omega}{\omega_m} = \frac{1}{Q}$$

$$Q = \frac{1}{0.02} = 50$$

If the coil to be employed has an inductance of 10 millihenrys and the resonant frequency is 20 kc, find the values of R_s and C.

$$R_s = \frac{\omega_m L}{Q_s} = \frac{2\pi \times 20{,}000 \times 0.01}{50} = 8\pi = 25.1 \text{ ohms}$$

$$C = \frac{1}{\omega_m{}^2 L} = \frac{1}{4\pi^2(20{,}000)^2 \times 0.01} = 0.00633 \times 10^{-6} \text{ farad}$$

The use of Q (or the reciprocal of Q) in circuit analysis will take on more importance and significance in radio-frequency circuits where Q_s is essentially constant than in low-frequency circuits where R_s is essentially constant. [It should be noted that R_s has been tacitly assumed constant in equation (27) as well as in Fig. 16.] In analyzing tuned radio-frequency circuits near resonant frequency, $\omega_m = 1/\sqrt{LC}$, we obtain greater accuracy by writing

$$Z = R_s + j\left(\omega L - \frac{1}{\omega C}\right)$$

as

$$Z = \omega_m L\left[\frac{R_s}{\omega_m L} + j\left(\frac{\omega}{\omega_m} - \frac{\omega_m}{\omega}\right)\right]$$

or

$$Z = \omega_m L\left(\frac{1}{Q} + jF\right) = \sqrt{\frac{L}{C}}\left(\frac{1}{Q} + jF\right) \tag{28}$$

since Q is considerably more constant over a reasonable frequency range centered on ω_m than is R_s. It is plain that $F = (\omega/\omega_m - \omega_m/\omega)$.

If L, C, and Q in equation (28) are essentially constant, then $F = (\omega/\omega_m - \omega_m/\omega)$ is the only variable involved, and it should be plain that the current response versus ω will take the same shape as that shown in Fig. 16 since in one case the response is based upon

$$I = \frac{V}{\sqrt{R_s^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

and the other upon

$$I = \frac{V}{\sqrt{\frac{L}{CQ^2} + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

which is obtained by substitution for R_s its value obtained from equation (27). In the low-frequency case we assume that R_s is constant, which is essentially true, and in the high-frequency case we assume that L/CQ^2 is essentially constant. Cases arise where neither assumption is justified, but cases of this kind are reserved for more advanced courses.

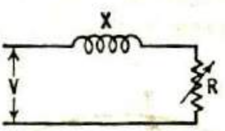

FIG. 17. Series circuit with variable R.

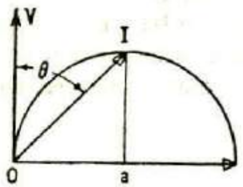

FIG. 18. Circle diagram of Fig. 17 for constant V and X but with variable R.

Circle Diagram of Series Circuit. Circle diagrams are often employed as an aid in analyzing the operating characteristics of circuits which under some conditions are used in representing transmission lines and some types of a-c machinery. The basis of representing a series circuit by means of a circle diagram will be derived with reference to Fig. 17.

The resistance R of the circuit in Fig. 17 will be considered a variable, whereas the applied voltage and reactance will be assumed constant. The power-factor angle is designated by θ. If R is zero, I is obviously equal to V/X, and this value of I will lag $\mathbf{V}$ by 90° if X is inductive. (See Fig. 18.) As R is increased from its zero value, the magnitude of $\mathbf{I}$ becomes less than V/X and θ becomes less than 90° and finally, when R equals ∞, I equals zero and θ equals zero. The fact that the locus of the vector $\mathbf{I}$ traces out a semicircle, as indicated in Fig. 18, may be seen from the following derivation.

In general,

$$I = \frac{V}{Z} \tag{29}$$

and

$$\sin\theta = \frac{X}{Z} \tag{30}$$

or

$$Z = \frac{X}{\sin\theta} \tag{31}$$

Substituting (31) in (29),

$$I = \frac{V}{X}\sin\theta \tag{32}$$

For constant V and X, equation (32) is the polar equation of a circle of diameter V/X. Figure 18 shows a plot of equation (32) with respect to V as a reference and for positive angles θ, representing inductive loads, measured clockwise. These conventions are employed because they are the ones most commonly used for such circle diagrams in a-c machinery. Since Ia in Fig. 18 is $OI \cos\theta$, it is apparent that Ia is proportional to the power consumed by the circuit. If the diagram is drawn to a certain current scale as, for example, I amperes per inch, the watt scale will be VI watts per inch.

A simple transmission line circuit in which the capacitance and leakance are assumed negligible may be represented by Fig. 19, where

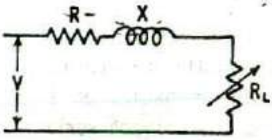

FIG. 19. Series circuit, R and X assumed constant, R_L variable.

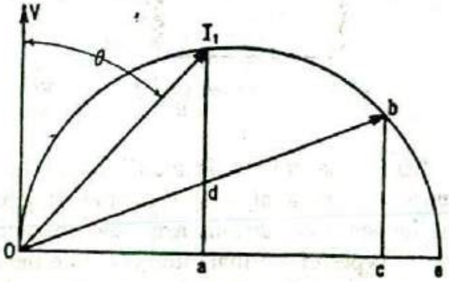

FIG. 20. Circle diagram of Fig. 19 for constant V, R, and X and variable R_L.

R and X are, respectively, the series resistance and reactance of the line and R_L is the load resistance. If R is constant and R_L is varied, the current follows the equation $I = (V/X)\sin\theta$ as in the previous case. The distance Ia in Fig. 18 again represents the total power consumed by the circuit, but the total power dissipated is consumed in both R and R_L. The power dissipated by each resistance can easily be represented on the diagram.

If the resistance R_L is assumed to be zero, all power must be dissipated in the resistance R. For this condition the power is represented by bc in Fig. 20 and Ob represents the corresponding current. For some finite value of R_L other than zero, the current is OI_1 and the total power consumed is proportional to I_1a. Of this total, da is the amount consumed in R and I_1d is dissipated by R_L. To prove that da represents the power dissipated in R, it is only required to show that da and bc are proportional to the respective squares of the currents OI_1 and Ob for the two conditions.

From similar triangles $$\frac{da}{bc} = \frac{Oa}{Oc}$$

Since $$Oa = OI_1 \cos aOI_1$$

$$= OI_1 \frac{OI_1}{Oe} = \frac{(OI_1)^2}{Oe}$$

and $$Oc = Ob \cos cOb = Ob \frac{Ob}{Oe} = \frac{(Ob)^2}{Oe}$$

$$\frac{da}{bc} = \frac{\dfrac{(OI_1)^2}{Oe}}{\dfrac{(Ob)^2}{Oe}} = \frac{(OI_1)^2}{(Ob)^2}.$$

Therefore, for any current such as OI_1, I_1d represents the power consumed in R_L, da shows the watts lost in R, and the total power input to the circuit is given by I_1a. If I^2R_L is considered as the output of the circuit (the power transmitted by the line), the efficiency must be

$$\text{Efficiency} = \frac{\text{output}}{\text{input}} = \frac{I_1d}{I_1a}$$

The power factor at the input end is $\cos\theta$. It is also I_1a/OI_1.

The maximum power that can be transmitted by a circuit like Fig. 19 under conditions of constant R and X occurs when the extremity of OI_1 in Fig. 20 coincides with the point of tangency to the circle of a line drawn parallel to Ob. It is a matter of simple geometry to show that V times I_1d under these conditions yields the result for maximum power as given by equation (59) if $X_r = 0$ [which requires that k in equation (59) be zero]. Since I_1d may be employed as a quantitative measure of the power delivered to the load resistance R_L, it is plain from Fig. 20 that this load power varies from zero (when $R_L = 0$) to a maximum and back to zero again (when $R_L = \infty$).

The details of circle diagram constructions which apply to circuits of the kind shown in Fig. 19 may be readily comprehended from a numerical problem like the following.

Problem 6. Refer to Fig. 19. R and X are constant at the values $R = 2$ ohms and $X = 3.464$ ohms. V is constant at 346.4 volts.

(a) Lay off $OV = V$ graphically in a vertical position to any convenient voltage scale as, for example, 100 volts per inch.

(b) Lay off Oe (of Fig. 20) equal to V/X in a horizontal position to a scale of not more than 20 amperes per inch. (A scale of 10 amperes per inch will give more accurate results.)

(c) Lay off Ob (of Fig. 20) equal to I when $R_L = 0$.

Ans.: $I = 346.4/4 = 86.6$ amperes, 60° behind V.

(d) Draw a tangent to the semicircle which is parallel to Ob and construct OI_1 from O to this point of tangency. What is the magnitude of the current and what is the p.f. at this point of operation? *Ans.:* $I = 50$ amperes, p.f. $= 0.86$.

(e) What is the maximum power that can be delivered to R_L?

Ans.: $P_{\max} = V \times I_1 d_{\max} = 10{,}000$ watts.

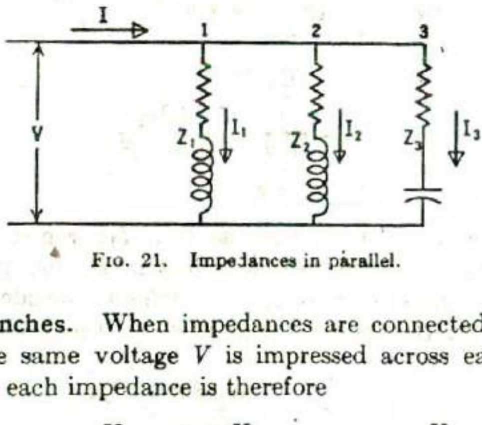

Fig. 21. Impedances in parallel.

Parallel Branches. When impedances are connected in parallel, as in Fig. 21, the same voltage V is impressed across each impedance. The current in each impedance is therefore

$$\mathbf{I}_1 = \frac{\mathbf{V}}{\mathbf{Z}_1}, \quad \mathbf{I}_2 = \frac{\mathbf{V}}{\mathbf{Z}_2}, \quad \text{and} \quad \mathbf{I}_3 = \frac{\mathbf{V}}{\mathbf{Z}_3}$$

From Kirchhoff's current law,

$$\begin{aligned} \mathbf{I} &= \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 \\ &= \frac{\mathbf{V}}{\mathbf{Z}_1} + \frac{\mathbf{V}}{\mathbf{Z}_2} + \frac{\mathbf{V}}{\mathbf{Z}_3} = \mathbf{V}\left(\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3}\right) \\ &= \mathbf{V}(\mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3) = \mathbf{V}\mathbf{Y}_0 \end{aligned} \tag{33}$$

where the symbol **Y** represents the reciprocal of impedance and is called admittance. Equation (33) shows that the resultant current flowing through several impedances in parallel is the product of the voltage and the sum of the reciprocals of the several branch impedances. In other words, the voltage is multiplied by the sum of the admittances of the several branches. Equation (33) shows that *admittances* are *added* for *parallel* branches. For branches in series it will be remembered that *impedances* are added. Since both admittance and impedance are complex quantities, all additions involving either of them *must* be made in *complex* form. Arithmetic addition should not be attempted. In only one case is arithmetic addition correct, and in this case the addition in complex form will give the same result. If equation (33) is solved for impedance $\mathbf{Z}_0$ by obtaining the ratio of $\mathbf{V}$ to $\mathbf{I}$, we obtain

$$\mathbf{Z}_0 = \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3} = \frac{1}{\mathbf{Y}_0} \tag{34}$$

Equation (34) shows that the resultant impedance of several parallel branches is the reciprocal of the resultant admittance. Since the unit of impedance is the ohm and admittance is the reciprocal of impedance, the unit of admittance is the reciprocal ohm or mho (ohm spelled backwards).

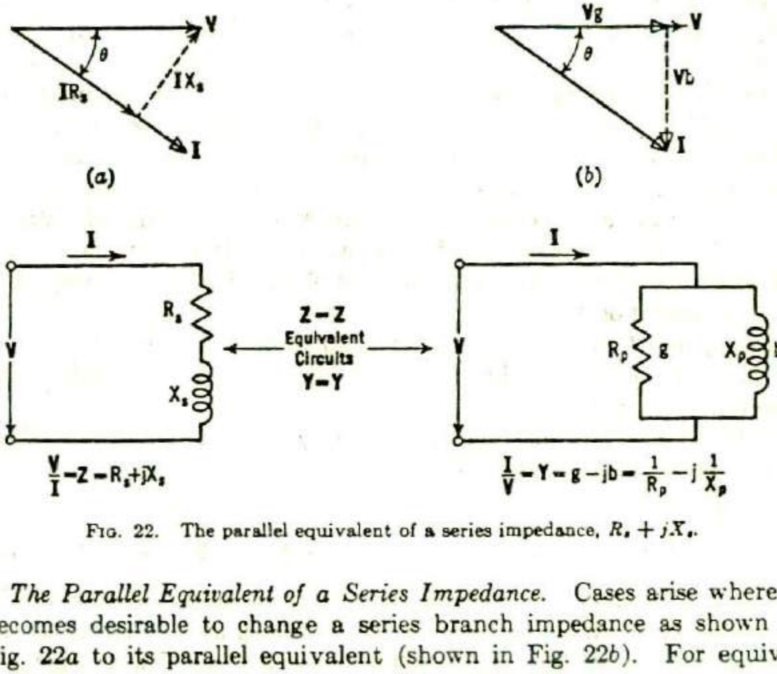

FIG. 22. The parallel equivalent of a series impedance, $R_s + jX_s$.

The Parallel Equivalent of a Series Impedance. Cases arise where it becomes desirable to change a series branch impedance as shown in Fig. 22*a* to its parallel equivalent (shown in Fig. 22*b*). For equivalence, $\mathbf{Y}$ of Fig. 22*a* must equal $\mathbf{Y}$ of Fig. 22*b*. Therefore

$$\mathbf{Y} = \frac{1}{R_s + jX_s} = \frac{1}{R_p} + \frac{1}{jX_p}$$

or, upon rationalizing,

$$\frac{R_s}{R_s^2 + X_s^2} - j\frac{X_s}{R_s^2 + X_s^2} = \frac{1}{R_p} - j\frac{1}{X_p} \tag{35}$$

$R_s/(R_s^2 + X_s^2)$ is called the conductance of the series impedance Z_s and is denoted by the symbol g. $X_s/(R_s^2 + X_s^2)$ is called the susceptance of the series impedance Z_s and is denoted by the symbol b. Employing the symbols g and b, we have

$$\mathbf{Y} = g - jb = \frac{1}{R_p} - j\frac{1}{X_p} \tag{36}$$

The physical significance of g and b may be interpreted as follows. If equation (36) is multiplied by $\mathbf{V}$ to obtain the current $\mathbf{I}$, we have:

$$\mathbf{I} = \mathbf{V}g - j\mathbf{V}b = \frac{\mathbf{V}}{R_p} - j\frac{\mathbf{V}}{X_p}$$

It will be seen that $\mathbf{V}g$ shown on the vector diagram, Fig. 22*b*, is the component of current in phase with the voltage and is the current $\mathbf{V}/R_p$ in the resistive branch of the parallel equivalent of $\mathbf{Z}_s$. Also $\mathbf{V}b$ shown on the vector diagram is the component of current in quadrature with the voltage and is the component $\mathbf{V}/X_p$ in the inductive branch of the parallel equivalent of $\mathbf{Z}_s$. Hence the conductance $1/R_p$ of the resistive branch of the equivalent parallel circuit is the conductance g of the admittance $\mathbf{Y} = g - jb = 1/\mathbf{Z}_s$, and the susceptance $1/X_p$ of the inductive branch is the susceptance b of the admittance $\mathbf{Y} = 1/\mathbf{Z}_s$. It is important to observe that conductance g in the circuits of Fig. 22 is the reciprocal of R_p but *not* of R_s. Similarly susceptance is the reciprocal of X_p but *not* of X_s.

Since g and b are components of admittance and either g, b, or Y multiplied by voltage yields a current, they are all expressed in the same units, namely, mhos.

If the admittances in equation (33) are expressed in terms of their conductances and susceptances, we have

$$\begin{aligned}\mathbf{I} &= \mathbf{V}(g_1 - jb_1 + g_2 - jb_2 + g_3 - jb_3)\\ &= \mathbf{V}[(g_1 + g_2 + g_3) - j(b_1 + b_2 + b_3)] = \mathbf{V}(g_0 - jb_0)\end{aligned} \tag{37}$$

Equation (37) shows that conductances may be added arithmetically to obtain the resultant conductance while susceptance must be added algebraically to obtain the resultant susceptance. That algebraic addition of susceptances is required is evident from the expression $X/(R^2 + X^2)$ for susceptance when it is remembered that X may be positive or negative depending upon whether it is inductive or capacitive, respectively.

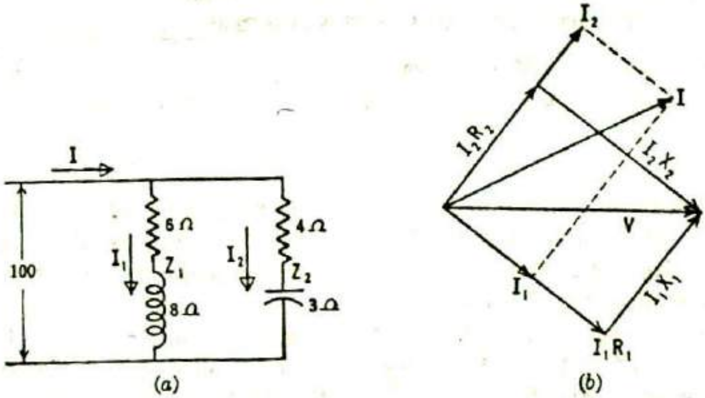

FIG. 23. (a) Circuit for example 6. (b) Phasor diagram of (a).

Example 6. For the circuit of Fig. 23a with the parameters shown, the following are desired: (a) conductance and susceptance of each branch; (b) the resultant conductance and susceptance; (c) the vector or phasor diagram.

$$\mathbf{I}_1 = \frac{100 + j0}{6 + j8} = 6 - j8 = 10\underline{/-53.2^\circ} \text{ amperes}$$

$$\mathbf{I}_2 = \frac{100 + j0}{4 - j3} = 16 + j12 = 20\underline{/36.9^\circ} \text{ amperes}$$

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = 22 + j4 = 22.35\underline{/10.3^\circ} \text{ amperes}$$

$$\mathbf{Y}_1 = \frac{1}{\mathbf{Z}_1} = \frac{1}{(6 + j8)}\,\frac{(6 - j8)}{(6 - j8)} = 0.06 - j0.08 \text{ mho}$$

from which

$$g_1 = 0.06 \text{ mho}, \quad b_1 = 0.08 \text{ mho}$$

or, as an alternative method,

$$g_1 = \frac{R_1}{Z_1^2} = \frac{6}{100}, \quad b_1 = \frac{X_1}{Z_1^2} = \frac{8}{100}$$

$$\mathbf{Y}_2 = \frac{1}{Z_2} = \frac{1}{(4 - j3)}\,\frac{4 + j3}{(4 + j3)} = 0.16 + j0.12 \text{ mho}$$

from which

$$g_2 = 0.16 \text{ mho}, \quad b_2 = -0.12 \text{ mho}$$

or, as an alternative method,

$$g_2 = \frac{R_2}{Z_2^2} = \frac{4}{25}, \quad b_2 = \frac{X_2}{Z_2^2} = \frac{-3}{25}$$

The vector or phasor diagram is shown in Fig. 23b.

Another way to obtain the resultant current is shown below:

$$g = g_1 + g_2 = 0.06 + 0.16 = 0.22 \text{ mho}$$
$$b = b_1 + b_2 = 0.08 - 0.12 = -0.04 \text{ mho}$$

$$\mathbf{Y} = g - jb = 0.22 - j(-0.04) = 0.22 + j0.04 \text{ mho}$$
$$\mathbf{I} = \mathbf{VY} = 100\ (0.22 + j0.04) = 22 + j4 = 22.35\underline{/10.3°} \text{ amperes}$$

Or admittances may be added as follows:

$$\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2 = 0.06 - j0.08 + 0.16 + j0.12 = 0.22 + j0.04$$

and

$$I = \mathbf{VY} = 22 + j4 \text{ amperes}$$

The calculation of admittances from the reciprocals of impedances and their addition in complex form is generally the most direct procedure. Experience has shown that students make fewer errors in signs when following this procedure.

Instead of representing admittance in general as $g - jb$ and then using $g = R/Z^2$ and $b = X/Z^2$, many prefer to call it $g + jb$ and then to use $g = R/Z^2$ and b as $-X/Z^2$. Both give the same result for admittance. In either case, X is substituted as a positive value for inductance and negative for capacitance. In a dissipative circuit conductance is always positive. To *avoid confusion* in signs it is *best* to obtain admittance from $1/(R + jX)$ rather than from calculations of conductance and susceptance. Knowing how to calculate and use conductances and susceptances expedites the solution of some types of problems, although they may be solved by other means. The special case of two parallel impedances $\mathbf{Z}_1$ and $\mathbf{Z}_2$ occurs often in electrical engineering. For this case, $\mathbf{Y}_1 = 1/\mathbf{Z}_1$ and $\mathbf{Y}_2 = 1/\mathbf{Z}_2$. Hence

$$\mathbf{Y} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} \quad \text{and} \quad \mathbf{Z} = \frac{1}{\mathbf{Y}} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}$$

This expression, which is analogous to the much used expression for the resultant of two parallel resistances in direct currents, is very useful in alternating currents. When all reactances are zero, the expression reduces to the d-c case of $R_1R_2/(R_1 + R_2)$.

Problem 7. Three impedances $\mathbf{Z}_1$, $\mathbf{Z}_2$, and $\mathbf{Z}_3$ are connected in parallel across a 60-cycle voltage the magnitude of which is 40 volts.

$$\mathbf{Z}_1 = 10 + j0, \quad \mathbf{Z}_2 = 20 + j20, \quad \mathbf{Z}_3 = 30 - j40 \text{ ohms}$$

(*a*) Find g_1, b_1, g_2, b_2, g_3, and b_3.

(*b*) Find the resultant g and the resultant b of the three parallel branches.

Ans.: $g = 0.137$, $b = 0.009$ mho.

(*c*) What is the in-phase component of the resultant current; the quadrature component of the resultant current?

Ans.: $Vg = 5.48$ amperes, $Vb = 0.36$ amperes.

Resonance in Parallel Branches. Parallel branches containing inductance and capacitance are in resonance when the reactive current in the inductive branch is equal to the reactive current in the capacitive branch. The resultant reactive current for the circuit as a whole is therefore zero. For resonance

$$Vb_L = Vb_C$$

$$b_L = b_C \tag{38}$$

Hence the resultant current flowing is in phase with the applied voltage, and the power factor of the whole circuit is 1. This is sometimes called unity-power-factor resonance. Figure 24 shows a circuit and the corresponding vector diagram for this condition. From an inspection of the vector diagram it will be noted that the reactive components of current

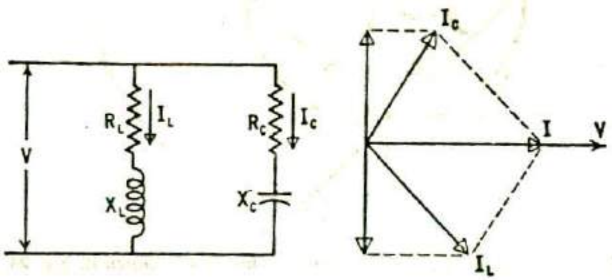

FIG. 24. Circuit and corresponding vector diagram for parallel resonance.

contribute nothing to the total current. Only the components of current in phase with the voltage exist in the resultant current. It might be inferred from this that the resultant current is a minimum at resonance. This is true if the *conductances* are *constant*. It is approximately true if the conductances are negligibly small, as they usually are in selective circuits as used in radio. An example will be considered later wherein minimum current does not occur at resonance.

The parameters possible of variation to make equation (38) true may be seen when the susceptances are replaced by their equivalent values, as shown in equation (39).

$$\frac{2\pi fL}{R_L^2 + (2\pi fL)^2} = \frac{\dfrac{1}{2\pi fC}}{R_C^2 + \left(\dfrac{1}{2\pi fC}\right)^2} \tag{39}$$

The quantities that may be varied are L, C, f, R_L, or R_C.

Resonance by varying L. In the following discussion L will be varied by a means which will not change the resistance of the inductive circuit. Let OV, Fig. 25, be the voltage impressed on a circuit like the one shown in Fig. 24. A current, I_C, will then flow in the condenser branch whose parameters are held constant. When L is zero, the current through the inductive branch is V/R_L and it is in phase with the applied voltage. The applied voltage is equal to $I_L R_L$ under these conditions. When L is increased from zero, the current through the

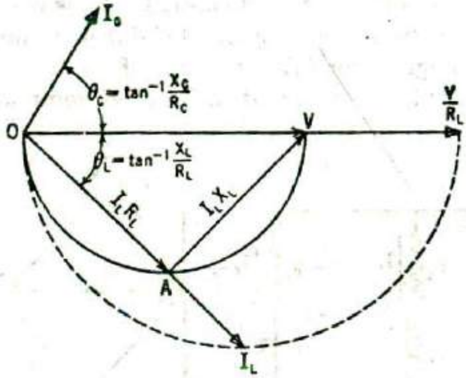

FIG. 25. Locus of I as L is varied in the circuit shown in Fig. 24.

inductive branch lags V by the $\tan^{-1}(X_L/R_L)$, as illustrated in Fig. 25 by OI_L. For any value of I_L, the $I_L R_L$ drop and the $I_L X_L$ drop must add at right angles to give the applied voltage. These component drops are OA and AV, respectively. Since they are always at right angles and their sum must be OV, the locus of the $I_L R_L$ drop must be a semicircle OAV. Since I_L is proportional to the $I_L R_L$ drop and in phase with it, the locus of I_L must also be a semicircle.

When the $I_L R_L$ drop coincides with the diameter of its circle, the current I_L must also coincide with the diameter of its own circle. The diameter of the latter must, therefore, be V/R_L. Hence the dotted circle drawn with V/R_L as a diameter must be the locus of I_L. Since the resultant current is $I_C + I_L$, this addition is performed by drawing the semicircle OI_LB with the left extremity of its diameter starting at I_C as shown in Fig. 26. For example, a particular sum of I_C and I_L is represented by CC. As L is varied, the locus of the resultant current is, therefore, the circle I_CCb. Hence, as L is increased from 0 to ∞, the resultant current varies from Ob to Oe, which is one point of resonance; thence to Od, which is a second resonant point; and then to OI_C. Neither of the resonant points gives either a maximum or minimum current, but they do yield unity power factor. The minimum current is OI_m, the value where the resultant current is normal to the circle

I_CCb. For any particular problem the values of I_C, θ_C, and I_Cb, which is equal to V/R_L, can be calculated directly from the parameters. Any other values of current can then be calculated trigonometrically from the geometry of the figure. A few facts should be observed. First, if $V/2R_L$ (the radius of the circle I_CCb) is less than $I_C \sin \theta_C$, parallel resonance cannot be obtained regardless of the value of L. This is in contrast to the series circuit, where some value of L will yield resonance for any value of R or C. Second, if $V/2R_L = I_C \sin \theta_C$,

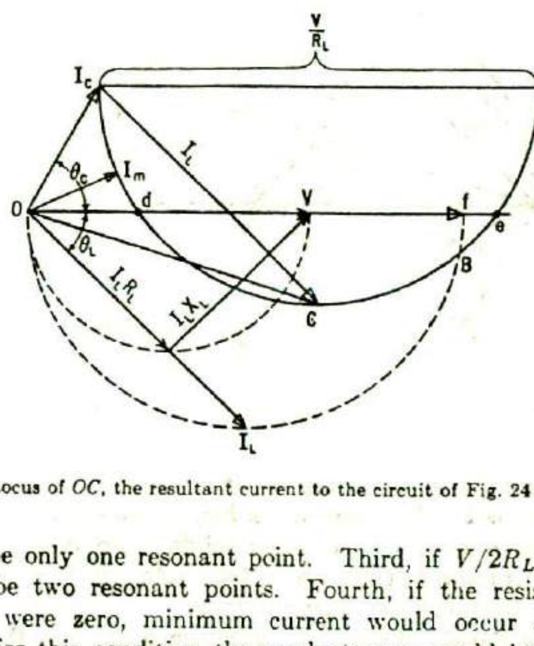

FIG. 26. Locus of OC, the resultant current to the circuit of Fig. 24 as L is varied.

there will be only one resonant point. Third, if $V/2R_L > I_C \sin \theta_C$, there will be two resonant points. Fourth, if the resistance of the inductance were zero, minimum current would occur at resonance. Note that for this condition the conductances would be constant for the two branches.

Resonance by Varying C. Through a similar procedure to that outlined above, the student can develop the graphical representation for the case where resonance is produced by varying C while R_L, L, R_C, and f are held constant. The graphical representation is shown in Fig. 27. The locus of the resultant current is the circle *adce*. Again it will be noted that resonance which occurs at d and e is not the condition for minimum current. Minimum current occurs at I_m, where the resultant current is normal to the circle *adce*. If R_C is zero, the radius of the circle *adce* becomes infinite, or, what is the same thing, the current I_C is in quadrature with the voltage V. Under this condition there is but one point of resonance and it corresponds to minimum current. The conductance of the capacitor circuit is zero, whereas that of the inductive branch is constant. This constant conductance makes the current at resonance a minimum, and hence the impedance a maximum. Since most selective circuits employ constant inductance and variable capacitance and the resistances of the capacitive branches are very small, maximum impedance or minimum current at resonance is practically realized in these circuits. Since at resonance the current is

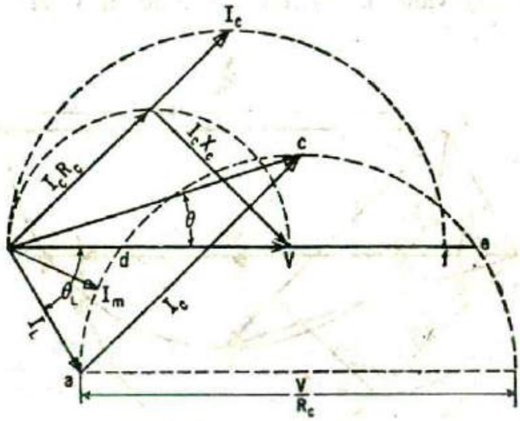

FIG. 27. Locus of resultant current to the circuit of Fig. 24 is the circle *adce* as C is varied.

simply the conductance times the voltage impressed, it is evident that the power factor is 1. An inspection of Fig. 27 will reveal the manner in which the phase angle θ between the resultant current and the applied voltage varies as the resultant current follows the circle *adce*. Between points d and e, leading power factor obtains.

Resonance by Varying Frequency. From equation (39) the frequency for parallel resonance is found to be

$$f_m = \frac{1}{2\pi\sqrt{LC}}\left[\frac{R_L{}^2C - L}{R_C{}^2C - L}\right]^{1/2} \qquad (40)$$

When $R_L{}^2C > L$ and $R_C{}^2C < L$, the quantity $\left[\frac{R_L{}^2C - L}{R_C{}^2C - L}\right]^{1/2}$ is imaginary and therefore no real frequency will yield resonance. The same situation results if both inequality signs are reversed. If R_L and R_C are equal, equation (40) for resonance becomes

$$f_m = \frac{1}{2\pi\sqrt{LC}}$$

which is the same as that for series resonance. This equation is also correct when $R_L = R_C = 0$ and may therefore be used as a close approximation when R_L and R_C are very small. It should be apparent that there are values of R_L, C, R_C, and L in a parallel circuit for which parallel resonance is impossible, regardless of frequency. This is in contrast to the series circuit containing R, L, and C where there is

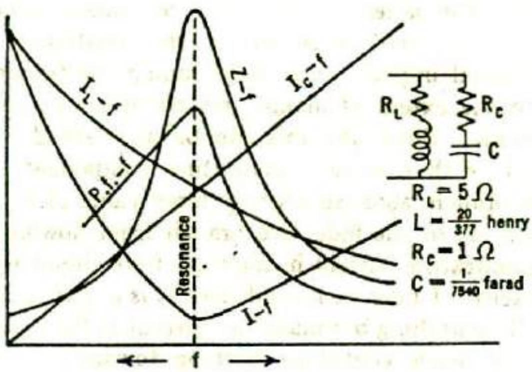

FIG. 28. Parallel resonance by varying frequency.

always some real resonant frequency for any values of the three parameters. The trends of various quantities as frequency is varied from a value too small to produce resonance to a value higher than that required for resonance are shown in Fig. 28 for a condition where resonance is obtainable.

Resonance by Varying R_L or R_C. When equation (40) is solved for R_L, the following equations result:

$$R_L = \sqrt{\frac{LC\omega^2 (R_C{}^2 C - L) + L}{C}} \tag{41}$$

$$R_L = \sqrt{LC\omega^2 R_C{}^2 - L^2\omega^2 + \frac{L}{C}} \tag{42}$$

$$R_L = \sqrt{\frac{X_L}{X_C} R_C{}^2 - X_L{}^2 + \frac{L}{C}} \tag{43}$$

When the parameters are such as to make the expressions under the above radical positive, R_L takes on definite positive values. It is thus shown that within limits there are definite values of R_L which will bring the circuit to resonance at some particular values of frequency, L, C, and R_C. Also, for resonance,

$$R_C = \sqrt{\frac{R_L^2}{\omega^2 LC} - \frac{1}{\omega^2 C^2} + \frac{L}{C}} \tag{44}$$

Equation (44) shows that, for those values of parameters which make the quantity under the radical positive, resonance may be produced by choosing the proper value of R_C.

In contrast to the series circuit, where resistances have no part in determining the frequency of resonance, the resistances of a parallel circuit are of signal importance in determining the frequency of resonance, even to the extent of making resonance either possible or impossible to attain. Physically this can be understood when it is remembered that, with a certain quadrature component of current in the capacitive branch, some sufficiently large value of R_L will prevent a resultant current in the inductive branch from flowing, which is as much as the quadrature current in the capacitive circuit even when the inductance is zero. Under such conditions it is apparent that inserting inductance will do nothing but make the current in the inductive branch still smaller and hence contribute nothing toward making resonance possible. Such a case was discussed with reference to Fig. 26 when $I_C \sin \theta_C$ was greater than $V/2R_L$. Figure 26, which is simply a phasor diagram, shows that $I_L \sin \theta_L$ can never be made as large as $I_C \sin \theta_C$ if $V/2R_L$ is less than $I_C \sin \theta_C$. A similar situation obtains for the capacitive branch.

Problem 8. Draw the phasor diagram and show the locus of I_L as X_L is varied, when $R_C = 1$ ohm, $X_C = 10$ ohms, $R_L = 6$ ohms, and the impressed voltage 100 volts for a circuit as shown in Fig. 24. Repeat the problem when R_L is changed to 4 ohms. What is the largest possible quadrature component of current in the inductive branch as X_L is varied in each case? In which case can resonance be produced? Why?

Ans.: 8.33 amperes, 12.5 amperes, resonance for 4-ohm case only.

Duality. The principle of duality (pages 29–38) may be extended to series and parallel resonance as shown below:

Series Resonance	Parallel Resonance
a. Reactive components of voltage combine to equal zero.	*a.* Reactive components of current combine to equal zero.
b. Voltage source constant in maximum magnitude.	*b.* Current source constant in maximum magnitude.
c. Current maximum for constant resistance.	*c.* Voltage maximum for constant conductance.
d. Impedance at minimum value.	*d.* Admittance at minimum value.
e. Inductive and capacitive reactances equal in magnitude.	*e.* Inductive and capacitive susceptances equal in magnitude.

From the above tabulation it will be noted that the dual elements are

Series	Parallel
a. Reactive voltage	*a*. Reactive current
b. Voltage	*b*. Current
c. Current	*c*. Voltage
d. Impedance	*d*. Admittance
e. Resistance	*e*. Conductance
f. Reactance	*f*. Susceptance

Recognition of duality will often yield a deeper understanding of circuit behavior than would otherwise be the case. It may also help to reduce the time required for an understanding of the physical operation of circuits. If, for example, series resonance is thoroughly understood, it is a simple matter to extend this knowledge to parallel resonance by way of the duality principle.

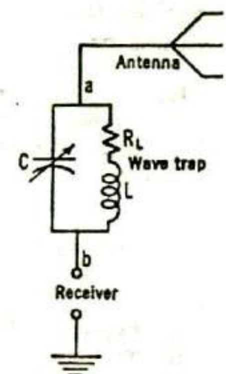

Fig. 29. Simple form of wave trap.

A Simple Form of Wave Trap. Resonance phenomena as presented in the foregoing articles form the basis upon which many circuits used in both wire and wireless communication operate. They are especially adapted to selective circuits such as those for filters and oscillators. A parallel combination of capacitance and inductance, along with its incidental resistance, can be made into an effective band eliminator, suppressor, or wave trap. The impedance of such a circuit (from a to b in Fig. 29), where the resistance of the capacitance is negligibly small and R_L is very small compared to ωL, is most easily found by taking the reciprocal of the resultant admittance. Since the branches are tuned for parallel resonance, the resultant admittance is conductance only. Thus

$$Y_m = \frac{R_L}{Z_L^2} \tag{45}$$

and

$$Z_m = \frac{1}{Y_m} = \frac{Z_L^2}{R_L} \tag{46}$$

Since $R_L^2 \ll \omega^2 L^2$,

$$Z_m = \frac{\omega^2 L^2}{R_L} \tag{47}$$

In a previous article it was shown that when $R_L = R_C = 0$ the resonant frequency is practically

$$f_m = \frac{1}{2\pi\sqrt{LC}} \quad \text{or} \quad \omega = \omega_m = 2\pi f_m = \frac{1}{\sqrt{LC}} \tag{48}$$

Substituting (48) in (47) gives the impedance at resonance

$$Z_m = \frac{L}{CR_L} \tag{49}$$

When used as a wave trap, the parallel combination of inductance and capacitance is placed in series with the antenna lead as shown in Fig. 29. At the resonant frequency the dynamic resistance of the wave trap is very nearly equal to L/CR_L [equation (49)]. Experience has shown that within the standard broadcast band the dynamic resistance at the frequency f_m can be made about 10 times the impedance at frequencies ± 20 kc from f_m. Thus the wave trap acts as a band suppressor or eliminator.

Problem 9. A typical coil used in the broadcast band for a wave trap like that in Fig. 29 has $L = 250 \times 10^{-6}$ henry and a ratio of reactance to resistance at 10^6 cycles of 170. Assuming the resistance of the condenser to be zero, calculate the following:

(a) C to produce resonance at 1000 kc from equation (39).

(b) C to produce resonance at 1000 kc from equation (48).

(c) Impedance of the wave trap from a to b when adjusted for parallel resonance at 1000 kc.

(d) Impedance of the wave trap to 990 kc when in resonance for 1000 kc.

(e) The ratio of the impedances for (c) to (d).

Ans.: 101.3 $\mu\mu$f, 101.3 $\mu\mu$f, 267,000 ohms, 75,100 ohms, 3.56.

A Singular Case of Parallel Resonance. For some values of the parameters R_L, R_C, L, and C connected as in Fig. 24, the circuit is in resonance for all frequencies. This may be shown as follows. From equation (39) the condition for parallel resonance is

$$\frac{\omega L}{R_L{}^2 + \omega^2 L^2} = \frac{\dfrac{1}{\omega C}}{R_C{}^2 + \dfrac{1}{\omega^2 C^2}}$$

$$= \frac{1}{\omega C}\,\frac{\omega^2 C^2}{R_C{}^2\omega^2 C^2 + 1} = \frac{\omega C}{1 + \omega^2 C^2 R_C{}^2}$$

or

$$\frac{1}{\dfrac{R_L{}^2}{L} + \omega^2 L} = \frac{1}{\dfrac{1}{C} + \omega^2 C R_C{}^2} \tag{50}$$

To be independent of frequency, an inspection of equation (50) will show that the following two conditions must be imposed simultaneously

Condition 1 $$\frac{R_L^2}{L} = \frac{1}{C} \quad \text{or} \quad R_L = \sqrt{\frac{L}{C}}$$

Condition 2 $$CR_C^2 = L \quad \text{or} \quad R_C = \sqrt{\frac{L}{C}}$$

Hence for resonance at all frequencies

$$R_L = R_C = \sqrt{\frac{L}{C}} \tag{51}$$

Since the circuit is in resonance (resultant susceptance = 0), its admittance must be the resultant conductance. Therefore

$$Y_m = g_m = \frac{R_L}{Z_L^2} + \frac{R_C}{Z_C^2} = \frac{\sqrt{\frac{L}{C}}}{\frac{L}{C} + \omega^2 L^2} + \frac{\sqrt{\frac{L}{C}}}{\frac{L}{C} + \frac{1}{\omega^2 C^2}} = \sqrt{\frac{C}{L}}$$

and

$$Z_m = \sqrt{\frac{L}{C}} \tag{52}$$

Equation (52) shows that the impedance of the circuit is also independent of frequency. The preceding demonstration has shown that, when $R_L = R_C = \sqrt{L/C}$, a circuit arrangement like that in Fig. 24 is in resonance for all frequencies and offers the same impedance $\sqrt{L/C}$ to all frequencies.

It has been shown that under certain conditions the network of Fig. 24 is equivalent to a single series resistance of value $\sqrt{L/C}$ at all frequencies. For general information it may be stated that it is possible to find networks that are equivalent to a given network at all frequencies although in contrast with the one discussed the impedances of the different networks, while being the same for any given frequency, will not remain constant at the various frequencies. A detailed study of such circuits is left for courses covering the theory of networks.

The Q of Parallel Circuits. In vacuum tube circuit analysis one frequently encounters the circuit arrangement which reduces essentially to that shown in Fig. 30*a*, namely, a coil and capacitor connected in parallel and energized with a current source. In the practical cases which will be encountered, the resistance of the coil, R_s, is very small compared to ωL; therefore

$$R_s^2 \ll \omega^2 L^2$$

Under these conditions the transformation of the series R_s and L to a parallel combination of g and b_L as suggested in Fig. 22 transforms Fig. 30a to that shown in Fig. 30b, where

$$g = \frac{1}{R_p} \approx \frac{R_s}{\omega^2 L^2}$$

$$b_L \approx \frac{1}{\omega L} \quad \text{and} \quad b_C = \omega C$$

It should be noted that b_L and b_C are magnitudes of the inductive and capacitive susceptances, respectively. Where purely reactive

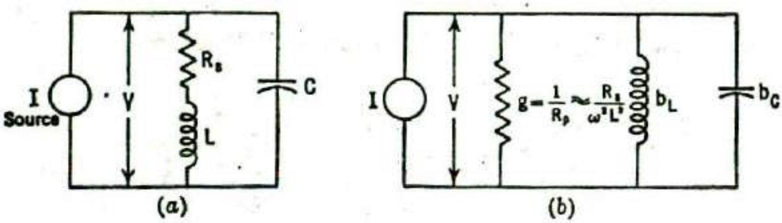

FIG. 30. Circuit shown in (b) is the equivalent of that shown in (a).

branches are placed in parallel, as in Fig. 30b, it is convenient to write $\mathbf{Y} = g + j(b_C - b_L)$ and thereby obtain an expression which is directly analogous to $\mathbf{Z} = R + j(X_L - X_C)$. In Fig. 30b we find

$$V = \frac{I}{\sqrt{g^2 + (b_C - b_L)^2}} = \frac{I}{\sqrt{g^2 + \left(\omega C - \dfrac{1}{\omega L}\right)^2}} \tag{53}$$

Comparing the above equation with equation (16), we observe a correspondence which allows us to interpret Fig. 16 as the voltage response versus ω. This response has a maximum value of I/g, and the analysis following equation (16) can with a few obvious changes in notation be employed to determine the band width of the selective circuit shown in Fig. 30.

Since g in equation (53) corresponds to R in equation (16), and C to L, and L to C, we may write for the parallel circuit

$$\Delta\omega = \omega_2 - \omega_1 = \frac{g}{C} \tag{54}$$

either by analogy with equation (25) or by direct computation.

Employing the same definition of Q as given on page 153 (namely, $Q = \omega_m/\Delta\omega$) and remembering that $\omega_m \approx 1/\sqrt{LC}$ when the resistances of the parallel branches are small relative to the reactances, we find that for the parallel circuit

$$Q_p = \frac{\omega_m}{\Delta\omega} = \frac{\omega_m C}{g} = \frac{1}{g\omega_m L} = \frac{1}{g}\sqrt{\frac{C}{L}} \tag{55}$$

In elementary analytical calculations, it is quite customary to treat both R_s of equation (27) and g of equation (55) as constants, that is, independent of frequency. Neither of these approximations, however, agrees with the physical facts as accurately as treating Q as constant over a reasonable frequency range centered on the resonant frequency, f_m, since R_s increases with increases in ω. Over certain ranges of the radio-frequency band, R_s varies almost linearly with respect to ω, and under these conditions we may set $R_s = k\omega$ with the following results:

$$Q_s = \frac{\omega L}{R_s} = \frac{\omega L}{k\omega} = \text{constant}$$

$$Q_p = \frac{1}{g\omega L} = \frac{\omega^2 L^2}{R_s \omega L} = \text{constant}$$

Example 7. In Fig. 30*a* it will be assumed that the coil has a series resistance, R_s, of 25.1 ohms and a self-inductance of 10 millihenrys. This coil is to be resonated at 20 kc with the capacitor C.

Let it be required to find the equivalent parallel circuit resistance, $1/g$, the tuning capacitance, the Q of the parallel circuit, and maximum voltage response per milliampere of current I.

$$g = \frac{R_s}{\omega_m^2 L^2} = \frac{25.1}{(2\pi \times 20{,}000)^2 \times (0.01)^2} = 1.59 \times 10^{-5} \text{ mho}$$

$$R_p = \frac{1}{g} = 62{,}900 \text{ ohms}$$

$$C = \frac{1}{L\omega_m^2} = \frac{1}{0.01\,(2\pi \times 20{,}000)^2} = 0.00633 \times 10^{-6} \text{ farad}$$

$$Q = \frac{\omega_m C}{g} = \frac{\omega_m \omega_m^2 L^2}{R_s \omega_m^2 L} = \frac{\omega_m L}{R_s} = 50$$

$$\text{Maximum voltage response} = \frac{0.001}{g} = 62.9 \text{ volts per milliampere}$$

A certain class of vacuum tube, namely, the pentode, can under certain operating conditions be made to function as current source supplying up to several milliamperes of alternating current simply by energizing one of its electrodes (the control grid) with a small a-c voltage. Since this small a-c voltage is often considerably less than 1 volt in magnitude, it is plain that large voltage amplifications may be obtained from the circuit configuration shown in Fig. 30*b* if the current source takes the form of a pentode. Moreover this circuit has a reasonable degree of selectivity since the band width between the $0.707V_{max}$ points on the response curve is

$$\Delta\omega = \frac{g}{C} = \frac{1.59 \times 10^{-5}}{0.00633 \times 10^{-6}} = 2510 \text{ radians per second}$$

On this basis of reckoning, the per unit band width is

$$\frac{\Delta\omega}{\omega_m} = \frac{2510}{2\pi \times 20{,}000} = 0.02$$

Series-Parallel Circuits. The series-parallel circuit illustrated in Fig. 31 is a combination of the series and parallel circuits which have been discussed previously. The principles previously considered apply to the analysis of series-parallel circuits. These are (1) impedances in series are added in complex form and (2) admittances of those branches which are in parallel must be added in complex form. To illustrate, consider Fig. 31. The admittances of impedances Z_4 and Z_5 are added in complex form, and the reciprocal of the resultant admittance is then the equivalent impedance of section B. An alternative method of finding the impedance of section B, as was previously shown, is to use $\mathbf{Z}_B = \mathbf{Z}_4\mathbf{Z}_5/(\mathbf{Z}_4 + \mathbf{Z}_5)$. Through a similar procedure the impedance of section A is determined. The impedances of section A, section B, and Z_1 are in series and are, therefore, added in complex form. This procedure yields the equivalent or resultant impedance $\mathbf{Z}_e$ of the series-parallel circuit. The current $\mathbf{I}$ may then be found from $\mathbf{V}/\mathbf{Z}_e$.

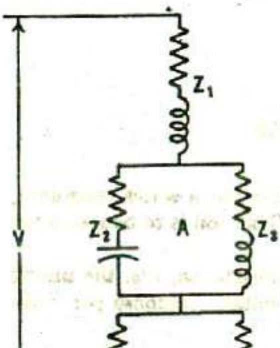

FIG. 31. Impedances in series-parallel.

Determination of Branch Currents and Voltages. After the resultant current is determined, the process is reversed to determine branch voltages and currents. The general procedure is to subtract the voltage drop calculated for the known current and the impedance through which it flows from the applied voltage to obtain the voltage drop across the remainder of the circuit, or to calculate the drops across various sections from the resultant current and the equivalent impedance of the branch through which the current flows. For example, in Fig. 31, the drop across section A is the product of equivalent impedance $\mathbf{Z}_A$ of that section and the current $\mathbf{I}$. The current through each of the parallel impedances is then determined by dividing this drop by the impedance of the particular branch or, if the admittances have been determined, by multiplying the voltage drop across the branch by the particular branch admittance. A similar procedure can be followed for section B, and so on.

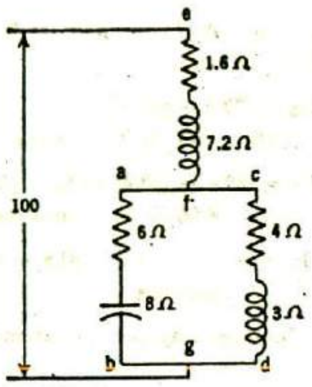

FIG. 32. Circuit for example 8.

Example 8. Calculate current, power, and power factor for each impedance shown in Fig. 32, and the total current and power and the power factor of the whole combination.

$$\mathbf{Y}_{ab} = \frac{1}{6 - j8} = 0.06 + j0.08 \text{ mho}$$

$$\mathbf{Y}_{cd} = \frac{1}{4 + j3} = 0.16 - j0.12 \text{ mho}$$

$$\mathbf{Y}_{fg} = \mathbf{Y}_{ab} + \mathbf{Y}_{cd} = 0.22 - j0.04 \text{ mho}$$

$$\mathbf{Z}_{fg} = \frac{1}{\mathbf{Y}_{fg}} = \frac{1}{(0.22 - j0.04)} \frac{(0.22 + j0.04)}{(0.22 + j0.04)} = 4.4 + j0.8 \text{ ohms}$$

An alternative method is

$$\mathbf{Z}_{fg} = \frac{\mathbf{Z}_{ab}\mathbf{Z}_{cd}}{\mathbf{Z}_{ab} + \mathbf{Z}_{cd}} = \frac{(6 - j8)(4 + j3)}{6 - j8 + 4 + j3} = 4.4 + j0.8 \text{ ohms}$$

$$\mathbf{Z}_{eg} = \mathbf{Z}_{ef} + \mathbf{Z}_{fg} = 1.6 + j7.2 + 4.4 + j0.8 = 6 + j8 \text{ ohms}$$

$$\mathbf{I} = \frac{100\underline{/0^\circ}}{6 + j8} = 6 - j8 = 10\underline{/-53.2^\circ} \text{ amperes}$$

$$P = vi + v'i' = 6 \times 100 + 0 \times 8 = 600 \text{ watts}$$

$$\text{P.f.} = \frac{600}{100 \times 10} = 0.6 \quad \text{or} \quad \frac{R}{Z} = \frac{6}{10} = 0.6 \text{ lag}$$

$$\mathbf{V}_{ef} = \mathbf{I}_{ef}\mathbf{Z}_{ef} = (6 - j8)\,(1.6 + j7.2) = 67.2 + j30.4$$
$$= 73.8\underline{/24.4^\circ} \text{ volts}$$

$$\mathbf{V}_{fg} = \mathbf{V} - \mathbf{I}_{ef}\mathbf{Z}_{ef} = 100 - 67.2 - j30.4 = 32.8 - j30.4$$
$$= 44.7\underline{/-42.8^\circ} \text{ volts}$$

Or, more directly,

$$\mathbf{V}_{fg} = \mathbf{I}\mathbf{Z}_{fg} = (6 - j8)\,(4.4 + j0.8) = 32.8 - j30.4$$
$$= 44.7\underline{/-42.8^\circ} \text{ volts}$$

$$\mathbf{I}_{ab} = \mathbf{V}_{fg}\mathbf{Y}_{ab} = (32.8 - j30.4)\,(0.06 + j0.08)$$
$$= 4.4 + j0.8 = 4.48\underline{/10.3^\circ} \text{ amperes}$$

$$\mathbf{I}_{cd} = \mathbf{V}_{fg}\mathbf{Y}_{cd} = (32.8 - j30.4)\,(0.16 - j0.12)$$
$$= 1.6 - j8.8 = 8.95\underline{/-79.7^\circ} \text{ amperes}$$

or

$$\mathbf{I}_{cd} = \mathbf{I} - \mathbf{I}_{ab} = 6 - j8 - 4.4 - j0.8 = 1.6 - j8.8$$
$$= 8.95\underline{/-79.7^\circ} \text{ amperes}$$

The powers in the various branches may now be determined in terms of principles previously considered.

$$P_{ab} = vi + v'i' = (32.8)\,(4.4) + (-30.4)\,(0.8)$$
$$= 144.32 - 24.32 = 120 \text{ watts}$$

$$P_{cd} = (32.8)\,(1.6) + (-30.4)\,(-8.8)$$
$$= 52.48 + 267.52 = 320 \text{ watts}$$

$$P_{ef} = (67.2)\,(6) + (30.4)\,(-8) = 403.2 - 243.2 = 160 \text{ watts}$$

or

$$P_{ef} = I^2r = (6^2 + 8^2)\,(1.6) = 160 \text{ watts}$$

$$P_{eg} = 100 \times 6 = 600 \text{ watts}$$

Check:

$$P = P_{ab} + P_{cd} + P_{ef} = 120 + 320 + 160 = 600 \text{ watts}$$

$$Pf_{ab} = \frac{R_{ab}}{Z_{ab}} = \frac{6}{\sqrt{6^2 + 8^2}} = 0.6 \text{ lead}$$

$$Pf_{cd} = \frac{R_{cd}}{Z_{cd}} = \frac{4}{\sqrt{4^2 + 3^2}} = 0.8 \text{ lag}$$

Problem 10. Study through the details of the above example and draw a vector diagram of $\mathbf{V}$, $\mathbf{I}$, $\mathbf{V}_{ef}$, $\mathbf{I}_{ab}$, $\mathbf{I}_{cd}$, and $\mathbf{V}_{fg}$. Employ a voltage scale of 25 volts per inch and a current scale of 2 amperes per inch.

Series-Parallel Tuning. It has been shown that for certain conditions parallel resonance yields maximum impedance and that series resonance gives minimum impedance. These facts suggest that a combination of these two phenomena may be used to exaggerate the effect of some certain frequency and minimize the effect of another. An arrangement that does this is shown in Fig. 33. This procedure is known as series-parallel tuning. To illustrate, assume that two waves, one of 10,000 cycles and the other of 20,000 cycles, are impressed at *ab* and that it is desired to detect the 10,000-cycle wave at *D*. Obviously as much 10,000-cycle current through *D* as can be obtained is desired, and as little as possible of the 20,000-cycle wave is to be tolerated. Hence the parallel branches of capacitance and inductance are adjusted for parallel resonance at 20,000 cycles. Then the 20,000-cycle wave encounters a high impedance, and little current due to it will flow through *D*. For the 10,000-cycle wave a little thought will show that the parallel circuit acts as an inductance. If a capacitance is placed in series with the parallel circuit *de* and its reactance for the 10,000-cycle frequency is made equal to the equivalent inductive reactance of the parallel circuit *de* for this same frequency, the circuit from *a* to *b* will be in series resonance for the 10,000-cycle wave. The current through *D* for the 10,000-cycle wave, therefore, will be large, whereas parallel resonance from *d* to *e* for the 20,000-cycle frequency will allow only a small 20,000-cycle current to flow through *D*.

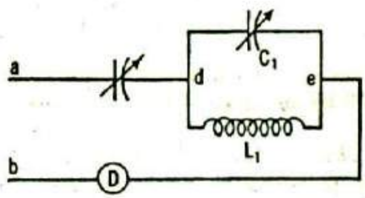

FIG. 33. Series-parallel tuning circuit.

Example 9. Assume L_1 to have 0.005 henry inductance and 50 ohms resistance. Neglect resistance of the condensers. Parallel resonance for 20,000 cycles obtains when

$$b_L = b_C$$

$$\frac{\omega 0.005}{50^2 + \omega^2(0.005)^2} = \omega C_1$$

where

$$\omega = 2\pi\, 20{,}000 = 12.57 \times 10^4 \text{ radians per second}$$

$$C_1 = \frac{0.005}{50^2 + 0.005^2\omega^2} = 1.257 \times 10^{-8} \text{ farad}$$

$$\mathbf{Y}_{de} = g = \frac{50}{50^2 + 0.005^2\omega^2} = \frac{50}{397{,}300} \text{ mho}$$

$$Z_{de} = \frac{397{,}300}{50} = 7946 \text{ ohms}$$

For 10,000 cycles,

$$\mathbf{Y}_{C1} = j2\pi\, 10{,}000 \times 1.257\, 10^{-8} = j79 \times 10^{-5} \text{ mho}$$

$$\mathbf{Y}_{L1} = \frac{1}{50 + j0.005 \times 2\pi\, 10{,}000} = 49.3 \times 10^{-5} - j310 \times 10^{-5} \text{ mho}$$

$$\mathbf{Y}_{de} = \mathbf{Y}_{C1} + \mathbf{Y}_{L1} = 49.3 \times 10^{-5} - j231 \times 10^{-5} \text{ mho}$$

$$\mathbf{Z}_{de} = \frac{10^5}{49.2 - j231} = 88.1 + j413 \text{ ohms}$$

Since 413 ohms is the equivalent reactance of the divided circuit, a capacitive reactance of 413 ohms is required to produce series resonance. Then $\mathbf{Z}_{ab} = 88.1$ ohms for 10,000 cycles.

For 20,000 cycles,

$$\mathbf{Z}_{ad} = -\frac{j413}{2} = -j206.5 \text{ ohms}$$

$$\mathbf{Z}_{ab} = 7946 - j206.5 \quad \text{or} \quad 7946 \text{ ohms approximately}$$

$$\frac{\mathbf{Z}_{ab20,000}}{\mathbf{Z}_{ab10,000}} = \frac{7946}{88.1} = 90.2$$

Hence for equal impressed voltages across ab, the value of the 20,000-cycle current will be about $\frac{1}{90}$ of the value of the 10,000-cycle current.

The student should devise the explanation to show that if the 10,000-cycle wave is to be suppressed and the 20,000-cycle wave detected, an inductance would have to be substituted for the capacitance between a and d.

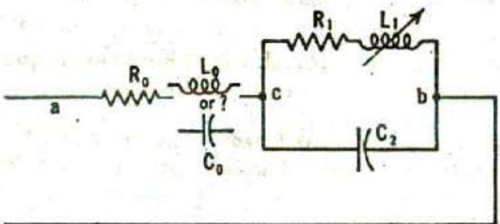

FIG. 34. See Problem 11.

Problem 11. The circuit ab of Fig. 34 is to pass a 45,000-cycle current with minimum impedance and is to block a 15,000-cycle current as effectively as possible. $R_0 = 20$ ohms, $R_1 = 40$ ohms, and $C_2 = 0.05$ μf are fixed. R_2, the resistance of the C_2 branch, is assumed to be negligibly small. L_1 is capable of being varied over the required range, it being assumed that the resistance of branch 1 is 40 ohms when L_1 is set at the desired value. Either a fixed C_0 *or* a fixed L_0 (of negligibly small resistance presumably) is to be placed in series with R_0 to accomplish the above-stated tuning effect.

(a) Solve for L_1, which will put the parallel circuit bc into parallel resonance at 15,000 cycles.

(b) Calculate the equivalent impedance from b to c at 45,000 cycles with L_1 set at its 15,000-cycle resonant value. Is bc predominantly capacitive or predominantly inductive at 45,000 cycles?

(c) What type of reactance (inductive or capacitive) must be placed in series with R_0 to put the branch ab into series resonance? Calculate the value of L_0 or C_0 which is required to put the branch ab into series resonance at 45,000 cycles.

(d) Assuming that the branch ab has been put into series resonance at 45,000 cycles, what is the actual impedance from a to b at 45,000 cycles? at 15,000 cycles?

Outline the above procedure for the reverse tuning effect, that is, for circuit *ab* to pass 15,000 cycles and block 45,000 cycles.

Ans.: (a) $L_1 = 2.17$ or 0.0835 millihenry. Use 2.17 for lower conductance.
(b) $Z_{bc} = 0.69 - j79.9$ ohms, predominantly capacitive.
(c) $L = 0.283$ millihenry.
(d) $Z_{ab45,000} = 20.69$ ohms. $Z_{ab15,000} = 1103$ ohms.

Complex Frequency. As applied to sinusoidal wave forms of current or voltage, $i = I_m \sin(\omega t + \theta)$ or $v = V_m \sin(\omega t + \theta)$, we might define complex angular frequency as

$$j\omega = \frac{di/dt}{i} = \frac{dv/dt}{v} = \frac{\omega \cos(\omega t + \theta)}{\sin(\omega t + \theta)} \tag{56}$$

all of which has the requisite dimension, namely a number per second. In this connection, we recognize j as an operator which advances the real quantity $[\sin(\omega t + \theta)]$ through 90° to yield $\cos(\omega t + \theta)$. That is

$$\omega[j \sin(\omega t + \theta)] = \omega \cos(\omega t + \theta)$$

An extension of the above definition to complex exponential currents and voltages provides us with the general concept of *complex frequency*. A complex exponential may be represented in any one of several different ways; for example:

$$\mathrm{i} = I\epsilon^{(st+j\theta)} = \mathbf{I}\epsilon^{st} = \mathbf{I}\epsilon^{\alpha t}(\cos \omega t + j \sin \omega t) \tag{57}$$

where $\mathbf{I} = I\epsilon^{j\theta}$ and $\mathbf{s} = \alpha + j\omega$. Depending upon the manner in which it is used, I may be expressed either as the maximum or rms value of the sinusoidal component of the complex exponential.

It will be observed that the complex exponential is capable of representing any of the four wave forms shown in Fig. 35 with either the real part, $\mathcal{R}$, or imaginary part, $\mathcal{I}$, of i. In this connection

$$\mathcal{R}(\mathrm{i}) = I\epsilon^{\alpha t} \cos(\omega t + \theta) \tag{58}$$

and

$$\mathcal{I}(\mathrm{i}) = I\epsilon^{\alpha t} \sin(\omega t + \theta) \tag{59}$$

In later courses, analyses will often be carried through in terms of complex exponentials. Then either the real or imaginary portion of the final result will be used. The interesting aspect of this approach is that the analysis, in terms of complex exponentials, is usually simpler to write than is the analysis of either the real or imaginary component alone. Consider, for example, the *LRC* series branch shown in Fig. 19, page 74. If the steady-state branch current, i, is to be represented as a complex exponential it will be expressed as: $\mathrm{i} = \mathbf{I}\epsilon^{st}$. Since in linear circuits, current is directly proportional to voltage the voltage drop

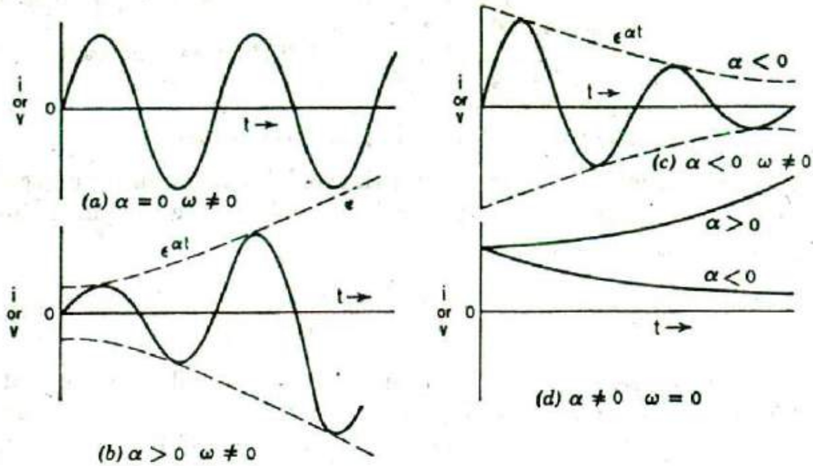

FIG. 35. Wave forms which can be represented by complex exponentials.

across the branch will be $\mathbf{V}\epsilon^{st}$ as is evident from a detailed study of the voltage equation

$$L\frac{d\mathrm{i}}{dt} + R\mathrm{i} + \frac{\int \mathrm{i}\,dt}{C} = \mathbf{V}\epsilon^{st} \tag{60}$$

Substitution of $\mathrm{i} = \mathbf{I}\epsilon^{st}$ into the left-hand side of this equation will show that

$$\left(L\mathrm{s} + R + \frac{1}{C\mathrm{s}}\right)\mathbf{I} = \mathbf{V} \tag{61}$$

The impedance of the LRC series circuit $(\mathbf{V}/\mathbf{I})$ in terms of the complex frequency s is usually written as $\mathbf{Z}(\mathrm{s})$, meaning $\mathbf{Z}$ expressed as a function of s. Thus

$$\mathbf{Z}(\mathrm{s}) = \frac{\mathbf{V}}{\mathbf{I}} = \left(L\mathrm{s} + R + \frac{1}{C\mathrm{s}}\right) \tag{62}$$

Where the circuit parameters L, R, and C are constants, it is evident that complex exponentials satisfy Kirchhoff's laws in rather elegant fashion. The associated complex frequency is

$$\mathrm{s} = \frac{d\mathrm{i}/dt}{\mathrm{i}} = \frac{d\mathrm{v}/dt}{\mathrm{v}} = \alpha + j\omega \tag{63}$$

which may be verified as follows. From $\mathrm{i} = \mathbf{I}\epsilon^{st}$, $d\mathrm{i}/dt = \mathrm{s}\mathbf{I}\epsilon^{st} = \mathrm{si}$. Therefore $\mathrm{s} = \dfrac{d\mathrm{i}/dt}{\mathrm{i}}$. A similar procedure using $\mathrm{v} = \mathbf{V}\epsilon^{st}$ will also yield s. The real part of s, namely α, accounts for an exponential increase or decrease of the current or voltage whereas the imaginary part, ω, defines or specifies the angular frequency of the current or voltage.

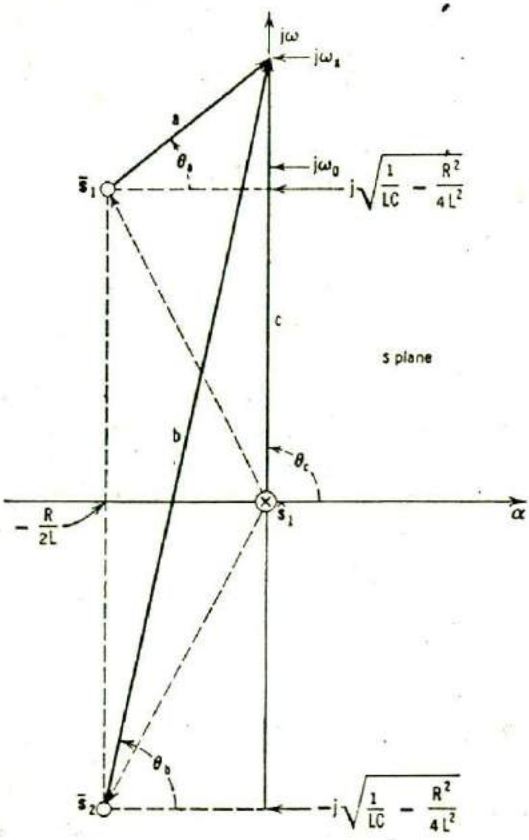

FIG. 36. Illustrating the pole and zeros of $\mathbf{Z}(\mathbf{s}) = L\,\dfrac{(\mathbf{s} - \bar{\mathbf{s}}_1)(\mathbf{s} - \bar{\mathbf{s}}_2)}{(\mathbf{s} - \mathbf{s}_1)}$

Since $\mathbf{s}$ is a complex number, it is natural to employ an $\mathbf{s}$ plane in circuit analysis with α measured along the axis of reals and ω measured along the axis of imaginaries. In terms of this convention, real angular frequency ω is plotted along the j axis of the complex $\mathbf{s}$ plane as indicated in Fig. 36.

Poles and Zeros. Network behavior is sometimes characterized by the poles and zeros of the impedance function $\mathbf{Z}(\mathbf{s})$ of the network.[2]

[2] More generally, the transfer characteristics of the network, $\mathbf{V}_{out}/\mathbf{V}_{in}$, $\mathbf{V}_{out}/\mathbf{I}_{in}$, $\mathbf{I}_{out}/\mathbf{I}_{in}$, and $\mathbf{I}_{out}/\mathbf{V}_{in}$, are characterized by the poles and zeros of these transfer functions, all of which are ratios of polynomials in $\mathbf{s}$.

A *pole* of $\mathbf{Z}(\mathbf{s})$ is defined as the value of complex frequency at which $\mathbf{Z}(\mathbf{s})$ becomes infinite and a zero of $\mathbf{Z}(\mathbf{s})$ is defined as the value of $\mathbf{s}$ at which $\mathbf{Z}(\mathbf{s})$ equals zero. For example, the impedance of the *LRC* series circuit derived in the previous article may be expressed as:

$$\mathbf{Z}(\mathbf{s}) = L\mathbf{s} + R + \frac{1}{C\mathbf{s}} = \frac{L\left(\mathbf{s}^2 + \frac{R}{L}\mathbf{s} + \frac{1}{LC}\right)}{\mathbf{s}}$$

or

$$\mathbf{Z}(\mathbf{s}) = \frac{L(\mathbf{s} - \bar{\mathbf{s}}_1)(\mathbf{s} - \bar{\mathbf{s}}_2)}{(\mathbf{s} - \hat{\mathbf{s}}_1)} \tag{64}$$

$\hat{\mathbf{s}}_1 = 0$ is the single pole of $\mathbf{Z}(\mathbf{s})$

$\bar{\mathbf{s}}_{1,2} = -\frac{R}{2L} \pm j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$ are the two zeros of $\mathbf{Z}(\mathbf{s})$.

If the pole and zeros of $\mathbf{Z}(\mathbf{s})$ are plotted on the $\mathbf{s}$ plane as in Fig. 36, it is evident just how the magnitude and phase angle of $\mathbf{Z}(\mathbf{s})$ could be evaluated for any value of $\mathbf{s}$ with the aid of a scale and a protractor. Ordinarily, interest lies only in values of $\mathbf{s}$ which are on the real frequency axis, that is, the $j\omega$ axis of the s plane. At any value of $\mathbf{s} = j\omega_x$, for example,

$$\mathbf{Z}(\mathbf{s}) = \mathbf{Z}(j\omega_x) = L\frac{(j\omega_x - \bar{\mathbf{s}}_1)(j\omega_x - \bar{\mathbf{s}}_2)}{j\omega_x}$$

or

$$\mathbf{Z}(j\omega_x) = L\frac{ab}{c}\underline{/\theta_a + \theta_b - \theta_c} \tag{65}$$

where $a = |j\omega_x - \bar{\mathbf{s}}_1|$, $b = |j\omega_x - \bar{\mathbf{s}}_2|$, and $c = \omega_x$, all of which may be measured with the aid of a suitable scale. θ_a, θ_b, and θ_c are the angles of the three phasors $(j\omega_x - \bar{\mathbf{s}}_1)$, $(j\omega_x - \bar{\mathbf{s}}_2)$, and $j\omega_x$ respectively measured from the $+\alpha$-axis direction.

In order to illustrate the pole-zero method of analysis (as well as to point out some of its shortcomings), we shall evaluate $\mathbf{Z}(\mathbf{s})$ at $\mathbf{s} = j\omega_0 = j\frac{1}{\sqrt{LC}}$ from the location of the zeros and pole in Fig. 36.

At $\mathbf{s} = j\omega_0 = j\frac{1}{\sqrt{LC}}$

$$a = \sqrt{\left(\frac{R}{2L}\right)^2 + \left[\frac{1}{\sqrt{LC}} - \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}\right]^2}$$

$$\mathrm{a} = \sqrt{\frac{2}{LC} - 2\sqrt{\frac{1}{L^2C^2} - \frac{R^2}{4L^3C}}}$$

$$\mathrm{b} = \sqrt{\frac{2}{LC} + 2\sqrt{\frac{1}{L^2C^2} - \frac{R^2}{4L^3C}}}$$

$$\mathrm{c} = \frac{1}{\sqrt{LC}}, \quad \theta_c = \pi/2 \text{ radians}$$

$$\theta_a = \tan^{-1}\frac{\dfrac{1}{\sqrt{LC}} - \sqrt{\dfrac{1}{LC} - \dfrac{R^2}{4L^2}}}{R/2L}, \quad \theta_b = \tan^{-1}\frac{\dfrac{1}{\sqrt{LC}} + \sqrt{\dfrac{1}{LC} - \dfrac{R^2}{4L^2}}}{R/2L}$$

$$\begin{aligned}\mathbf{Z}(\mathrm{s}) &= L\frac{ab}{c}\underline{/\sum\theta} = L\frac{\sqrt{\dfrac{4}{L^2C^2} - 4\left(\dfrac{1}{L^2C^2} - \dfrac{R^2}{4L^3C}\right)}}{1/\sqrt{LC}}\underline{/0^\circ} \\ &= R\underline{/0^\circ} \text{ ohms} \end{aligned} \tag{66}$$

In arriving at $\sum\theta = 0$, we make use of the well-known and easily derived relationship: $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\dfrac{x+y}{1-xy}$ Obviously, no advantage accrues from the use of poles and zeros in this simple case.

In complicated filter circuits, the phase characteristics ($\sum\theta$ versus ω) are often evaluated by the graphical method (or with the aid of an electronic computer) since the analytical expressions for the phase characteristics can become extremely cumbersome.

Example. Let it be required to find the frequency response of the so-called stagger-tuned amplifier circuit shown in Fig. 37. By frequency response in this case we mean the manner in which $\mathbf{E}_{out}/\mathbf{E}_{in} = \mathbf{E}_3/\mathbf{E}_1$ varies with angular frequency, ω. If we let

$$\frac{\mathbf{E}_{out}}{\mathbf{E}_{in}} = \frac{\mathbf{E}_3}{\mathbf{E}_1} = A\underline{/\theta}$$

we might plot A versus ω to show how the magnitude of $\mathbf{E}_3/\mathbf{E}_1$ varies with ω and plot θ versus ω to show how the phase of $\mathbf{E}_3$ (relative to $\mathbf{E}_1$) varies with ω. The latter plot is sometimes referred to as .e phase characteristic.

In the circuit of Fig. 37 $g_{m1}\mathbf{E}_1$ and $g_{m2}\mathbf{E}_2$ are the current-generator representations of the two tubes. The details of how these current generators represent the amplifying properties of the tubes are incidental to the present problem.

If the impedance of L is represented by Ls and the impedance of C is represented by $1/C$s as derived on page 180, the application of Kirchhoff's voltage law to the $\mathbf{I}_1$ loop shows that

$$\left(L_1\mathrm{s} + R_1 + \frac{1}{C_1\mathrm{s}}\right)\mathbf{I}_1 = \frac{-g_{m1}\mathbf{E}_1}{C_1\mathrm{s}}$$

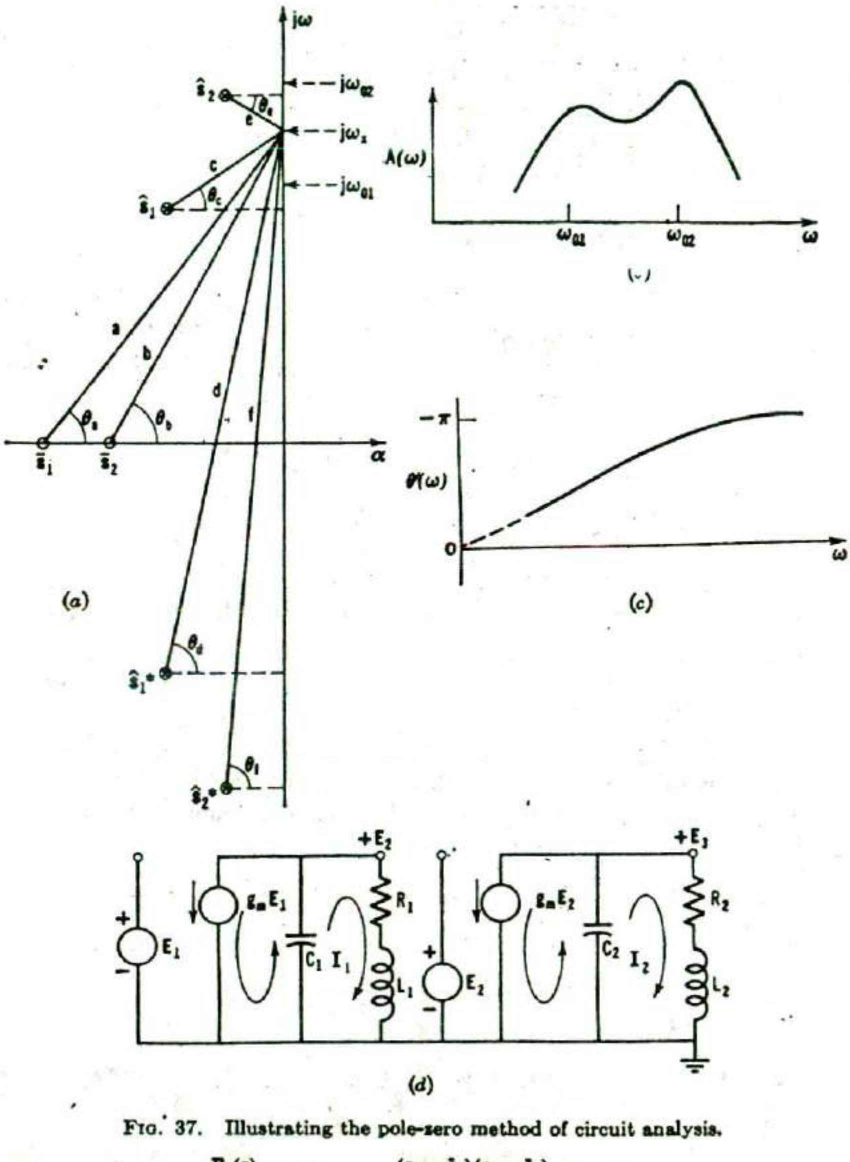

FIG. 37. Illustrating the pole-zero method of circuit analysis.

$$\frac{\mathbf{E}_3(s)}{\mathbf{E}_1(s)} = k\,\frac{(s-\bar{s}_1)(s-\bar{s}_2)}{(s-\hat{s}_1)(s-\hat{s}_1^*)(s-\hat{s}_2)(s-\hat{s}_2^*)}$$

From which

$$\mathbf{E}_2 = (L_1 s + R_1)\mathbf{I}_1 = \frac{-(L_1 s + R_1)g_{m1}\mathbf{E}_1}{C_1 s\left(L_1 s + R_1 + \dfrac{1}{C_1 s}\right)}$$

and

$$\frac{\mathbf{E}_2}{\mathbf{E}_1} = \frac{-g_{m1}}{C_1} \times \frac{L_1\left(\mathbf{s} + \frac{R_1}{L_1}\right)}{L_1\left(\mathbf{s}^2 + \frac{R_1}{L_1}\mathbf{s} + \frac{1}{L_1C_1}\right)} = \frac{-g_{m1}}{C_1} \times \frac{(\mathbf{s} - \bar{\mathbf{s}}_1)}{(\mathbf{s} - \hat{\mathbf{s}}_1)\,(\mathbf{s} - \mathbf{s}_1^*)}$$

where $\bar{\mathbf{s}}_1 = -R_1/L_1$ is a zero on the negative α axis of the $\mathbf{s}$ plane.

$$\hat{\mathbf{s}}_1 = -\frac{R_1}{2L_1} + j\sqrt{\frac{1}{L_1C_1} - \frac{R_1^2}{4L_1^2}}, \text{ a second-quadrant pole.}$$

$$\hat{\mathbf{s}}_1^* = -\frac{R_1}{2L_1} - j\sqrt{\frac{1}{L_1C_1} - \frac{R_1^2}{4L_1^2}}, \text{ the conjugate of } \hat{\mathbf{s}}_1.$$

In a similar manner

$$\frac{\mathbf{E}_3}{\mathbf{E}_2} = \frac{-g_{m2}}{C_2} \times \frac{(\mathbf{s} - \bar{\mathbf{s}}_2)}{(\mathbf{s} - \hat{\mathbf{s}}_2)\,(\mathbf{s} - \hat{\mathbf{s}}_2^*)}$$

where $\bar{\mathbf{s}}_2 = -R_2/L_2$ is a zero on the negative α axis of the $\mathbf{s}$ plane.

$$\hat{\mathbf{s}}_2 = -\frac{R_2}{2L_2} + j\sqrt{\frac{1}{L_2C_2} - \frac{R_2^2}{4L_2^2}}, \text{ a second-quadrant pole.}$$

$$\hat{\mathbf{s}}_2^* = -\frac{R_2}{2L_2} - j\sqrt{\frac{1}{L_2C_2} - \frac{R_2^2}{4L_2^2}}, \text{ the conjugate of } \hat{\mathbf{s}}_2.$$

It follows directly that

$$\frac{\mathbf{E}_3}{\mathbf{E}_1} = \frac{g_{m1}g_{m2}}{C_1C_2} \times \frac{(\mathbf{s} - \bar{\mathbf{s}}_1)\,(\mathbf{s} - \bar{\mathbf{s}}_2)}{(\mathbf{s} - \hat{\mathbf{s}}_1)\,(\mathbf{s} - \hat{\mathbf{s}}_1^*)\,(\mathbf{s} - \hat{\mathbf{s}}_2)\,(\mathbf{s} - \hat{\mathbf{s}}_2^*)} = A\,\underline{/\theta}$$

Let ω_x be *any* value of ω. Then $\mathbf{s} = \mathbf{s}_x = j\omega_x$ and the following quantities may be measured from the pole and zero plot:

$$|j\omega_x - \bar{\mathbf{s}}_1| = a \qquad |j\omega_x - \bar{\mathbf{s}}_2| = b$$
$$|j\omega_x - \hat{\mathbf{s}}_1| = c \qquad |j\omega_x - \hat{\mathbf{s}}_1^*| = d$$
$$|j\omega_x - \hat{\mathbf{s}}_2| = e \qquad |j\omega_x - \hat{\mathbf{s}}_2^*| = f$$

θ_a = angle associated with $(j\omega_x - \bar{\mathbf{s}}_1) = \mathbf{a}$

θ_b = angle associated with $(j\omega_x - \bar{\mathbf{s}}_2) = \mathbf{b}$

θ_c = angle associated with $(j\omega_x - \hat{\mathbf{s}}_1) = \mathbf{c}$

θ_d = angle associated with $(j\omega_x - \hat{\mathbf{s}}_1^*) = \mathbf{d}$

θ_e = angle associated with $(j\omega_x - \hat{\mathbf{s}}_2) = \mathbf{e}$

θ_f = angle associated with $(j\omega_x - \hat{\mathbf{s}}_2^*) = \mathbf{f}$

Then

$$A(\omega) = \frac{g_{m1}g_{m2}}{C_1C_2} \times \frac{ab}{cdef}$$

where a, b, c, d, e, and f depend upon ω for their values:

$$\theta(\omega) = \underline{/\theta_a + \theta_b - \theta_c - \theta_d - \theta_e - \theta_f}$$

If the coils are of the high-Q variety

$$\omega_{01} \gg \frac{R_1}{L_1} \quad \text{and} \quad \omega_{02} \gg \frac{R_2}{L_2}$$

Under these conditions a/d and b/f are each approximately equal to 1/2, and ab/df of the $A(\omega)$ expression reduces to 1/4. Then

$$A(\omega) \doteq \frac{g_{m1}g_{m2}}{4C_1C_2} \times \frac{1}{ce}$$

It is also evident from Fig. 37*a* that $(\theta_a + \theta_b - \theta_d - \theta_f) \doteq 0$ so that

$$\theta(\omega) \doteq -(\theta_c + \theta_e)$$

The results are indicated schematically in Fig. 37.

The great advantage of the pole-zero method of circuit analysis is that the general behavior of the circuit is displayed without detailed and laborious calculations. The method is generally more suitable for advanced courses than it is for a first course in circuit analysis. On pages 572–575, the use of complex frequency, poles, and zeros in finding both the steady-state and transient responses of a circuit is given.

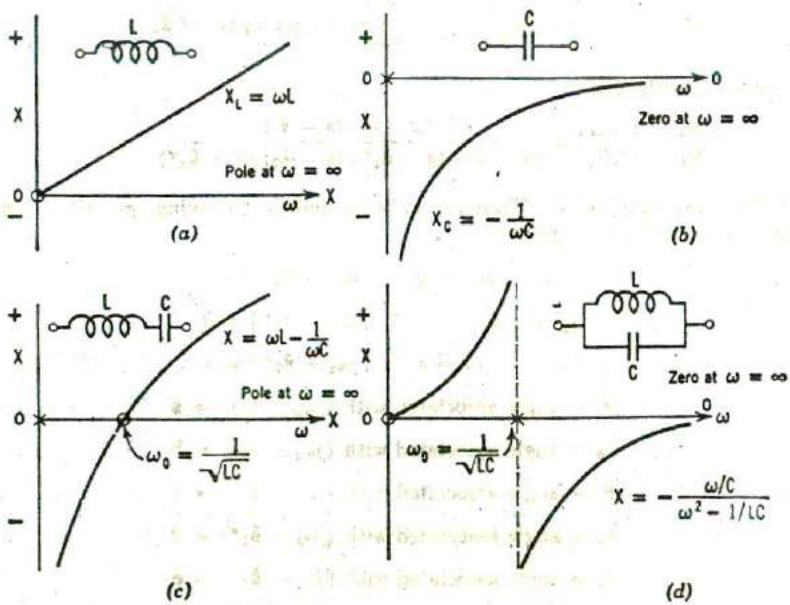

FIG. 38. Reactance of four elementary circuits plotted against ω as the independent variable.

Pure Reactance Circuits. Four characteristics of pure reactance circuits are illustrated in Fig. 38:

(1) Either a pole or zero exists at $\omega = 0$.

(2) Either a pole or zero exists at $\omega = \infty$.

(3) Poles and zeros alternate along the real frequency axis.

(4) The slope of the reactance curves is always positive, that is, $dX/d\omega > 0$ for all finite ω.

It will prove instructive to investigate these properties of reactance circuits employing the s-plane method of attack.

The poles and zeros of a reactance network are confined to the $j\omega$-axis of the s plane since there are no dissipative elements like R and G present to give the critical frequencies a real component. In Fig. 38*c*, series resonance occurs at $\omega L = 1/\omega C$ or at a zero of $\mathbf{Z}(\mathrm{s}) = L\mathrm{s} + 1/C\mathrm{s}$. Parallel resonance occurs in Fig. 38*d* at a pole of

$$\mathbf{Z}(\mathrm{s}) = \frac{1}{\mathbf{Y}(\mathrm{s})} = \frac{1}{1/L\mathrm{s} + C\mathrm{s}} = \frac{\mathrm{s}/C}{\mathrm{s}^2 + 1/LC} \tag{67}$$

The latter expression has a zero at $\mathrm{s} = 0$ (or at $j\omega = 0$) and poles at $\hat{\mathrm{s}} = \pm j\dfrac{1}{\sqrt{LC}}$. The poles and zeros of functions which are plotted against real ω are often indicated by crosses and circles respectively as in Fig. 38.

An illustration of multiple resonance is given in Fig. 39 where

$$\mathbf{Z}(\omega) = \frac{1}{\mathbf{Y}_1} + \mathbf{Z}_2 = \frac{1}{j\left(\omega C_1 - \dfrac{1}{\omega L_1}\right)} + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)$$

or

$$\mathbf{Z}(\omega) = \frac{j\omega L_2\left[\omega^4 - \left(\dfrac{1}{L_1C_1} + \dfrac{1}{L_2C_2} + \dfrac{1}{L_2C_1}\right)\omega^2 + \dfrac{1}{L_1L_2C_1C_2}\right]}{\omega^2\left(\omega^2 - \dfrac{1}{L_1C_1}\right)} \tag{68}$$

It will be observed that $\mathbf{Z}(\omega)$ is a pure reactance that has a pole at $\omega = 0\left(\text{equal to } -j\dfrac{1}{\omega C_2}\right)$ and a pole at $\omega = \infty$ (equal to $j\omega L_2$) as well as internal poles at $\omega_2 = \pm\sqrt{\dfrac{1}{L_1C_1}}$. Poles and zeros at $\omega = 0$ and at $\omega = \infty$ are referred to as *external* poles and zeros, whereas those between these limits are called *internal* poles and zeros. The bracket term in the numerator of equation (68) contains four zeros: two between $\omega = 0$ and $\omega = \infty$ (designated in Fig. 39) and their negatives (not shown) which, of course, lie between $\omega = 0$ and $\omega = -\infty$. It will be observed that the poles and zeros alternate along the ω-axis which is equivalent to their alternating along the $j\omega$-axis of the s plane. It will prove instructive to investigate this property in more general terms.

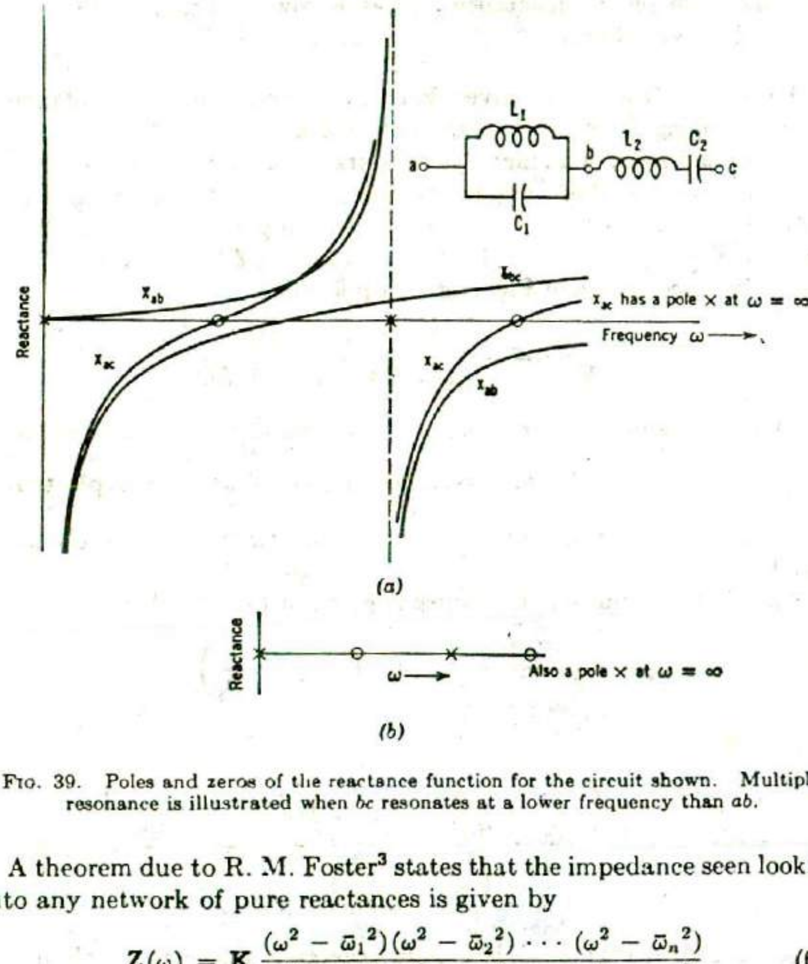

FIG. 39. Poles and zeros of the reactance function for the circuit shown. Multiple resonance is illustrated when bc resonates at a lower frequency than ab.

A theorem due to R. M. Foster[3] states that the impedance seen looking into any network of pure reactances is given by

$$\mathbf{Z}(\omega) = \mathbf{K}\,\frac{(\omega^2 - \bar{\omega}_1{}^2)(\omega^2 - \bar{\omega}_2{}^2)\cdots(\omega^2 - \bar{\omega}_n{}^2)}{(\omega^2 - \hat{\omega}_1{}^2)(\omega^2 - \hat{\omega}_2{}^2)\cdots(\omega^2 - \hat{\omega}_m{}^2)} \tag{69}$$

where $\mathbf{K}$ is equal to $j\omega H$ or to $\dfrac{H}{j\omega}$. H is a real number which depends upon the values of certain L's and C's in the network. In effect, this theorem states that the shape of the curve for a given impedance function is entirely determined by the poles and zeros. In other words, through

[3] "A Reactance Theorem" by R. M. Foster, B.S.T.J., April 1924, pp. 259–267.

the same poles and zeros only curves of the same shape can be drawn. These curves differ only by the scale factor H.

$\bar{\omega}_1, \bar{\omega}_2, \cdots, \bar{\omega}_n$ are the internal zeros or the values of ω at which $\mathbf{Z}(\omega) = 0$, not counting the possibility of a zero at $\omega = 0$ or at $\omega = \infty$.

$\hat{\omega}_1, \hat{\omega}_2, \cdots, \hat{\omega}_m$ are the internal poles or the values of ω at which $\mathbf{Z}(\omega) = \infty$, not counting the possibility of a pole at $\omega = 0$ or $\omega = \infty$. It will be observed that m and n represent the numbers of internal poles and zeros respectively.

In manipulating equation (69), four cases each of which is defined and illustrated by the sketches shown in Fig. 40 must be allowed for, namely:

LL circuits which have an external zero at $\omega = 0$ and an external pole at $\omega = \infty$,

CC circuits which have an external pole at $\omega = 0$ and an external zero at $\omega = \infty$,

LC circuits which have external zeros at $\omega = 0$ and at $\omega = \infty$,

CL circuits which have external poles at $\omega = 0$ and at $\omega = \infty$.

Case I: $\mathbf{K} = j\omega H$ and $n = m$. The *LL* circuit where

$$\mathbf{Z}(\omega) = j\omega H \frac{(\omega^2 - \bar{\omega}_1^{\,2}) \cdots (\omega^2 - \bar{\omega}_n^{\,2})}{(\omega^2 - \hat{\omega}_1^{\,2}) \cdots (\omega^2 - \hat{\omega}_m^{\,2})} \tag{70}$$

The reactance versus ω graph is given in Fig. 40*a*.

Case II: $\mathbf{K} = \dfrac{H}{j\omega}$ and $n = m$. The *CC* circuit where

$$\mathbf{Z}(\omega) = \frac{H}{j\omega} \frac{(\omega^2 - \bar{\omega}_1^{\,2}) \cdots (\omega^2 - \bar{\omega}_n^{\,2})}{(\omega^2 - \hat{\omega}_1^{\,2}) \cdots (\omega^2 - \hat{\omega}_m^{\,2})} \tag{71}$$

See Fig. 40*b*.

Case III: $\mathbf{K} = j\omega H$ and $m = n + 1$. The *LC* circuit where

$$\mathbf{Z}(\omega) = j\omega H \frac{(\omega^2 - \bar{\omega}_1^{\,2}) \cdots (\omega^2 - \bar{\omega}_n^{\,2})}{(\omega^2 - \hat{\omega}_1^{\,2}) \cdots (\omega^2 - \hat{\omega}_{n+1}^{\,2})} \tag{72}$$

See Fig. 40*c*.

Case IV: $\mathbf{K} = \dfrac{H}{j\omega}$ and $m = n - 1$. The *CL* circuit where

$$\mathbf{Z}(\omega) = \frac{H}{j\omega} \frac{(\omega^2 - \bar{\omega}_1^{\,2}) \cdots (\omega^2 - \bar{\omega}_n^{\,2})}{(\omega^2 - \hat{\omega}_1^{\,2}) \cdots (\omega^2 - \hat{\omega}_{n-1}^{\,2})} \tag{73}$$

See Fig. 40*d*.

In order to show that, in general, the poles and zeros alternate along

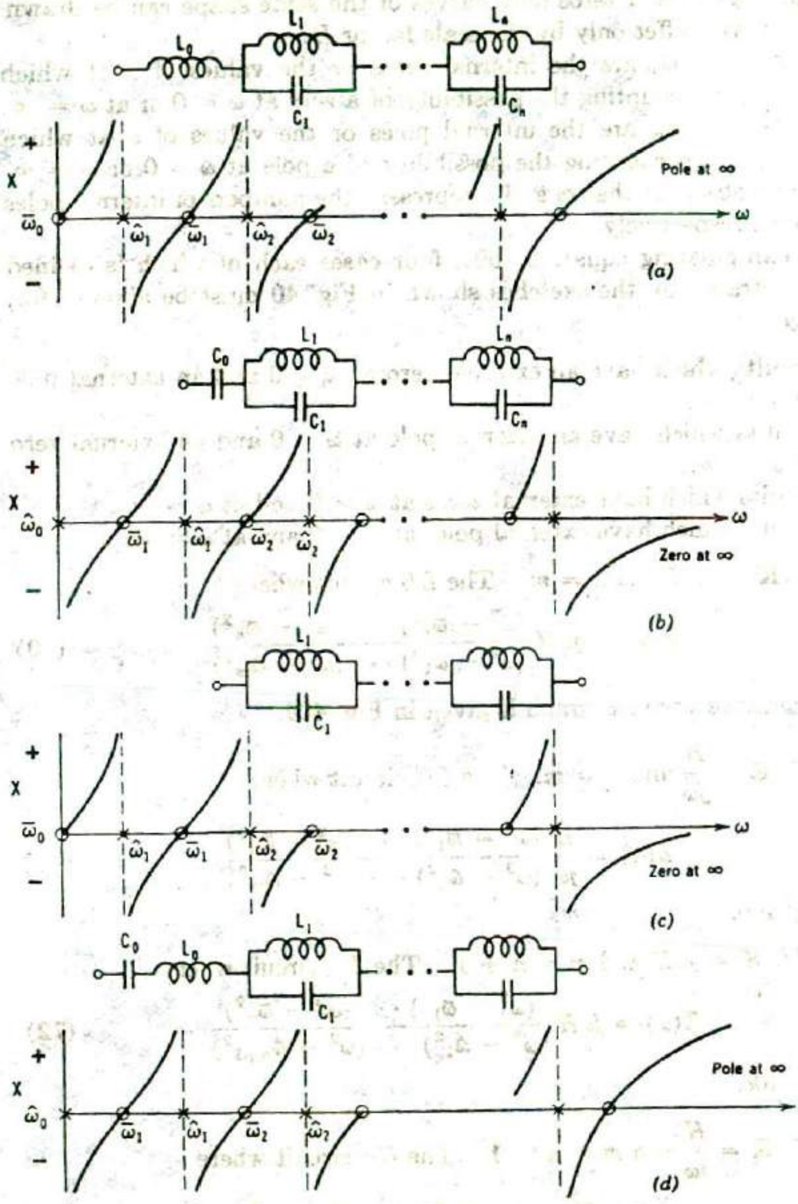

FIG. 40. Reactance graphs of *LL*, *CC*, *LC*, and *CL* networks. (Of the many possible circuit configurations which might be employed to obtain the reactance graphs, one simple configuration is indicated for each of the four cases.)

the $j\omega$-axis of the s plane, we should first investigate the behavior of the reactance in the *immediate* vicinity of a zero and in the *immediate* vicinity of a pole. Reference to Fig. 41*a* will show that just below the zero $\bar{s}_1$ where $\omega_- < \bar{\omega}_1$ the reactance is governed essentially by $(s - \bar{s}_1)$ regardless of the other zeros and poles. For $s = j\omega_-$ (just short of the zero $\bar{s}_1$) the reactance of the network is negative.

$$k(s - \bar{s}_1) = j(\omega_- - \bar{\omega}_1)k = -jX$$

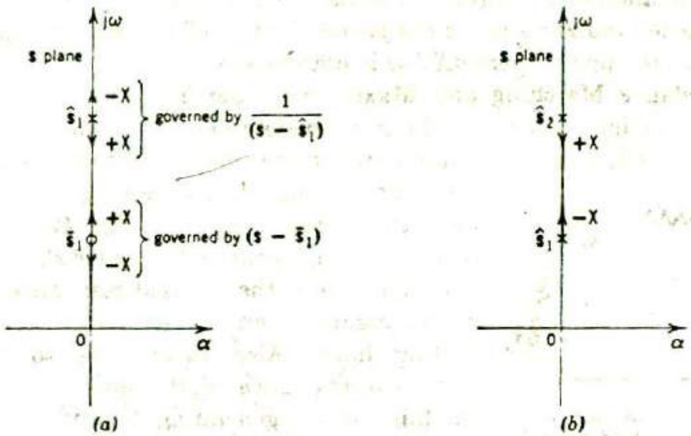

FIG. 41. For use in proving that the poles and zeros alternate along the $j\omega$ axis.

where k is a positive real number for zeros on the positive $j\omega$-axis $(\bar{\omega}_1 > 0)$.

Just above $\bar{s}_1$ where $\omega_+ > \bar{\omega}_1$ the reactance of the network is positive since the governing factor is $(s - \bar{s}_1)$. That is

$$k(s - \bar{s}_1) = j(\omega - \bar{\omega}_1)k = +jX$$

The effects of the other poles and zeros do not affect these results because the zero *very* near $s = j\omega$ dominates completely the behavior of $Z(j\omega)$ in this neighborhood.

Reference to Fig. 41*a* will also show that just below a pole, say $\hat{s}_1$,

$$Z(j\omega) \rightarrow \frac{k}{(s - \hat{s}_1)} = \frac{k}{j(\omega_- - \hat{\omega}_1)} = +jX \tag{74}$$

At $j\omega = j\hat{\omega}_1$, $Z(j\omega) = \pm j\infty$
Just above $\hat{s}_1$ in Fig. 41*a*

$$Z(j\omega) \rightarrow \frac{k}{(s - \hat{s}_1)} = \frac{k}{j(\omega_+ - \hat{\omega}_1)} = -jX \tag{75}$$

In passing through a pole in the direction of increasing $(j\omega)$, **Z** changes from an infinitely large positive reactance to an infinitely large negative reactance. Suppose now that two poles, $\hat{\mathbf{s}}_1$ and $\hat{\mathbf{s}}_2$, appeared consecutively along the $j\omega$-axis as illustrated in Fig. 41*b*. Since $\mathbf{s} = j\omega$ is allowed to vary *from* $\hat{\mathbf{s}}_1$ *to* $\hat{\mathbf{s}}_2$, the reactance would have to change continuously from $-\infty$ to $+\infty$. As applied to a physical circuit arrangement, this change in reactance (from $-\infty$ to $+\infty$) requires that the reactance, X, be zero somewhere between $\hat{\mathbf{s}}_1$ and $\hat{\mathbf{s}}_2$ or that the poles be separated from one another by zeros. A natural consequence of the alternation of the poles and zeros along the $j\omega$-axis is that $dX/d\omega$ is always positive except *at* the poles where $dX/d\omega$ is not defined.

Impedance Matching and Maximum Power Transfer A common problem in impedance matching is to determine the load impedance which will allow the maximum power to be transferred to the load from some generating device having a constant generated voltage, E_g. Let Fig. 42 represent such an arrangement and consider R_1 to represent the sum of the internal resistance of the generating device and the resistance of the connecting lines. Also assume X_1 to be the combined reactance of the line and internal reactance of the generating device. The solution is obtained by expressing the power at the receiver algebraically and then finding the maximum value of the expression. Let the receiver impedance be represented by R_r and X_r. If the receiver is a two-terminal network, R_r and X_r are its equivalent series parameters. Thus

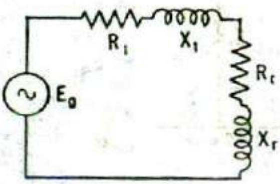

FIG. 42. Generator connected to a load through line impedance.

$$I = \frac{E_g}{\sqrt{(R_1 + R_r)^2 + (X_1 + X_r)^2}}$$

$$P_r = I^2R_r = \frac{E_g{}^2R_r}{(R_1 + R_r)^2 + (X_1 + X_r)^2} \tag{76}$$

In order to make the derivation easily applicable to all conditions, the ratio of X_r/R_r will be represented by k. Then

$$X_r = kR_r$$

and

$$P_r = \frac{E_g{}^2R_r}{(R_1 + R_r)^2 + (X_1 + kR_r)^2} \tag{77}$$

Setting $dP_r/dR_r = 0$ and solving for R_r give

$$R_r = \frac{Z_1}{\sqrt{1 + k^2}} \tag{78}$$

where $Z_1 = \sqrt{R_1^2 + X_1^2}$. Substituting equation (78) in equation (77), expanding the terms in the denominator, and simplifying give

$$P_{\max} = \frac{E_g^2}{2Z_1\sqrt{1 + k^2} + 2(R_1 + kX_1)} \tag{79}$$

Equation (79) gives the maximum power for any value of k, the ratio of X_r/R_r. To find the value of k that yields the greatest maximum power, it is necessary simply to set $dP_{\max}/dk = 0$ and solve for k. Then

$$k = \pm \frac{X_1}{R_1} \tag{80}$$

Substituting equation (80) in equation (79) yields

$$P_{\max\max} = \frac{E_g^2}{4R_1 + \dfrac{2}{R_1}(X_1^2 \pm X_1^2)} \tag{81}$$

It is obvious from equation (81) that the greatest maximum power will occur when the minus sign is used or when $k = -X_1/R_1$. For this case

$$P_{\max\max} = \frac{E_g^2}{4R_1} \tag{82}$$

Since R_r cannot be negative in a dissipative network, X_r must be minus to make k negative. Hence X_r is capacitive if X_1 is inductive, and vice versa. Also for this condition, from equation (78),

$$R_r = \frac{\sqrt{R_1^2 + X_1^2}}{\sqrt{1 + X_1^2/R_1^2}} = \frac{R_1\sqrt{R_1^2 + X_1^2}}{\sqrt{R_1^2 + X_1^2}} = R_1$$

Also for the greatest maximum power $X_r = kR_r = -(X_1/R_1)\,R_r = -(X_1/R_1)\,R_1 = -X_1$. Hence the receiver impedance must equal the generator plus line impedance, and the reactances must be of opposite signs. In short, the receiver impedance must be the conjugate of the combined generator and line impedance. As would be expected, the circuit is tuned for series resonance. Since R_1 and R_r are equal and the current is the same in both, one-half the power input is dissipated in the generator and line, and one-half is given to the receiver. The efficiency of transmission for the greatest maximum power is, therefore, 50 per cent.

Constant potential power systems are not designed to operate on the basis of maximum power transfer, but most low-current circuits are so designed. Impedance matching is, therefore, of considerable importance in all communication networks, and much attention has been given to this phase of circuit analysis by communication engineers.

Problem 12. A generating device has an impedance of $0.5 + j1$ ohms and is connected to a load by a line of $1.5 + j4$ ohms. At what load will maximum power transfer be realized? If the generated voltage is 20 volts, what is the power received by the load when adjusted for maximum power transfer? Find the line loss and loss in the generating device.

Ans.: $Z_{load} = 2 - j5$ ohms. $P_{max\ max} = 50$ watts at receiver. $P_{line\ loss} = 37.5$ watts. $P_{gen\ loss} = 12.5$ watts.

Problem 13. If a load impedance having a ratio of $X/R = 5$ is used at the end of the line in Problem 12, find the load impedance for maximum power transfer. What is the maximum power the load can receive?

Ans.: For positive k, $P = 3.675$ watts. $Z_L = 1.056 + j5.28$ ohms.
For negative k, $P = 45.2$ watts. $Z_L = 1.056 - j5.28$ ohms.

Networks. Resistors, inductors, capacitors, vacuum tubes, and sources of emf may be linked together in all conceivable forms. Most of the combinations, and almost all of those which contain emf's in more than one branch, cannot be solved by simple series-parallel circuit theory alone, as previously outlined in this chapter Such combinations may be classed as networks. Networks that contain sources of emf or power are sometimes called active, whereas those that do not contain any internal emf's or sources of power are called passive networks. Networks are said to be linear when the current in all branches is directly proportional to the driving voltage or emf impressed. Thus a network containing iron-core inductance coils and resistances that vary with current strength are non-linear. Networks may be composed of bilateral or unilateral elements. *Bilateral* elements are those circuit elements like inductance, resistance, and capacitance which transmit current equally well in either direction. *Unilateral* elements are those circuit elements like rectifiers and vacuum tubes which transmit effectively in only one direction.

Through the application of a few simple network theorems, certain combinations of circuit elements which are not solvable by ordinary series-parallel circuit theory directly may be solved quite readily.

The Superposition Theorem. The current which flows at any point or the voltage between any two points in a linear network, as a result of the simultaneous action of a number of emf's distributed throughout the network, is the sum of the currents or voltages at these points which would exist if each source of emf were considered separately, each of the other sources being replaced at that time by their internal impedances. This theorem states that each emf in a network may be treated as acting independently and the current in any branch of a network due to the simultaneous action of all emf's is the vector sum of the currents in the particular branch produced by each emf acting separately. It is important to keep all circuit elements closed or connected as they are in the network. All the emf's except the one for which currents are being calculated are assumed to be zero. Any impedances associated with the source of emf must be left connected in the network whether the emf is assumed to be zero or whether it is the one considered as an independent driving voltage.

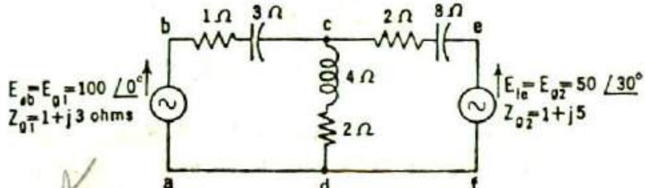

FIG. 43. See example 10.

Example 10. Calculate the current in branch *bc* for the network of Fig. 43
Solution: Assume

$$E_{g2} = 0$$

$$Z_{cf} = \frac{Z_{cef}Z_{cd}}{Z_{cef} + Z_{cd}} = \frac{(3 - j3)\,(2 + j4)}{(3 - j3 + 2 + j4)} = 3.69 + j0.462 \text{ ohms}$$

$$Z_{bf} = Z_{abc} + Z_{cf} = 1 + j3 + 1 - j3 + 3.69 + j0.462$$
$$= 5.69 + j0.462 \text{ ohms}$$

$$I_{bc1} = \frac{100 + j0}{5.69 + j0.462} = 17.43 - j1.417 \text{ amperes}$$

Now assume

$$E_{g1} = 0$$

$$Z_{ca} = \frac{Z_{cba}Z_{cd}}{Z_{cba} + Z_{cd}} = \frac{(1 - j3 + 1 + j3)\,(2 + j4)}{2 + 2 + j4} = 1.5 + j0.5 \text{ ohms}$$

$$Z_{ea} = Z_{fec} + Z_{ca} = 1 + j5 + 2 - j3 + 1.5 + j0.5 = 4.5 - j2.5 \text{ ohms}$$

$$I_{ec} = \frac{50\,\underline{/30^\circ}}{4.5 - j2.5} = 5 + j8.34 \text{ amperes}$$

$$I_{cb2} = \frac{I_{ec}Z_{ca}}{Z_{cba}} = \frac{(5 + j8.34)\,(1.5 + j0.5)}{2} = 1.66 + j7.50 \text{ amperes}$$

$$I_{bc} = I_{bc1} + I_{bc2} = I_{bc1} - I_{cb2} = 17.43 - j1.417 - 1.66 - j7.50$$
$$= 15.77 - j8.917 \text{ amperes}$$

Problem 14. Calculate the current in branch *ac* of Fig. 44.

Ans.: $\mathbf{I}_{ac} = 1.76 - j3.14$ amperes.

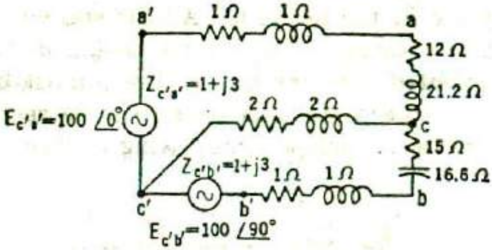

FIG. 44. See Problem 14.

Reciprocity Theorem. If any source of emf, **E**, located at one point in a network composed of linear bilateral circuit elements, produces a current **I** at a second point in the network, the same source of emf, **E**, acting at the second point will produce the same current **I** at the first point.

Example 11. The application of the above theorem may be illustrated as follows. Given the network shown in Fig. 45. The reciprocity theorem states that, if 100 volts are inserted in *bc* and branch *ef* is left closed, the current flowing in *ef* will

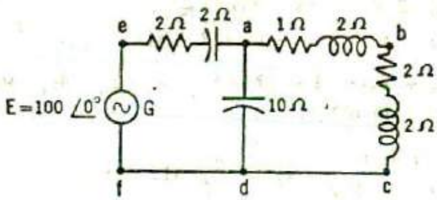

FIG. 45. See example 11.

then be exactly the same as the current that flowed in *bc* when this same voltage was applied at *ef*. To verify this theorem the current in *bc* will be calculated for the 100 volts at *ef*.

$$\mathbf{Z}_{ac} = \frac{\mathbf{Z}_{abc}\mathbf{Z}_{ad}}{\mathbf{Z}_{abc} + \mathbf{Z}_{ad}} = \frac{(3 + j4)\,(-j10)}{3 + j4 - j10} = 6.67 + j3.33 \text{ ohms}$$

$$\mathbf{Z}_{ec} = \mathbf{Z}_{ea} + \mathbf{Z}_{ac} = 2 - j2 + 6.67 + j3.33 = 8.67 + j1.33 \text{ ohms}$$

$$\mathbf{I}_{ec} = \frac{\mathbf{V}_{fe}}{\mathbf{Z}_{ec}} = \frac{100 + j0}{8.67 + j1.33} = 11.27 - j1.732 \text{ amperes}$$

$$\mathbf{V}_{ac} = \mathbf{I}_{ec}\mathbf{Z}_{ac} = (11.27 - j1.732)\,(6.67 + j3.33) = 81 + j26 \text{ volts}$$

$$\mathbf{I}_{bc} = \frac{\mathbf{V}_{ac}}{\mathbf{Z}_{abc}} = \frac{81 + j26}{3 + j4} = 13.88 - j9.84 \text{ amperes}$$

Now assume that 100 volts are inserted in branch *cb* and that *ef* remains closed. The current in *ef* will be calculated by a procedure similar to that shown above.

$$\mathbf{Z}_{af} = \frac{(2 - j2)(-j10)}{(2 - j2 - j10)} = 1.352 - j1.892 \text{ ohms}$$

$$\mathbf{Z}_{caf} = 3 + j4 + 1.352 - j1.892 = 4.352 + j2.108 \text{ ohms}$$

$$\mathbf{I}_{caf} = \frac{100 + j0}{4.352 + j2.108} = 18.6 - j9.02 \text{ amperes}$$

$$\mathbf{V}_{af} = (18.6 - j9.02)(1.352 - j1.892) = 8.07 - j47.4 \text{ volts}$$

$$\mathbf{I}_{ef} = \frac{8.07 - j47.4}{2 - j2} = 13.88 - j9.84 \text{ amperes}$$

which is the same as the current $\mathbf{I}_{bc}$ above.

From the reciprocity theorem it follows that the ratio of the emf in branch 1 of a linear bilateral network to the current it causes in branch 2 is the same as the ratio of a voltage placed in branch 2 to the current it would cause in branch 1. This ratio of voltage in one branch to the current in another branch is called the *transfer impedance*.

Problem 15. Make use of the first set of calculations for Fig. 45 when the emf is inserted in *fe* and with the aid of the reciprocity theorem find the current in *fe* if 100 volts are inserted in branch *ad*. Verify your result by actually calculating the current in *fe* when 100 volts are inserted in branch *ad*.

Ans.: $-2.6 + j8.1$ amperes.

Thévenin's Theorem. If an impedance **Z** is connected between any two points of an energized network, the resulting current **I** through this impedance is the potential difference **V** between these points, prior to connection, divided by the sum of the connected impedance **Z** and the impedance $\mathbf{Z}_0$, where $\mathbf{Z}_0$ is the impedance of the rest of the network looking back into the network from the points across which impedance **Z** is connected. In evaluating $\mathbf{Z}_0$ all sources of emf must be assumed to be zero and replaced by their internal impedances.

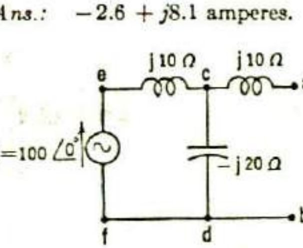

Fig. 46. See example 12.

Example 12. For the network shown in Fig. 46 the voltage drop at *ab* is found as follows:

$$\mathbf{I}_{fecd} = \frac{100\angle 0°}{10\angle -90°} = 10\angle 90° \text{ amperes}$$

$$\mathbf{V}_{cd} = \mathbf{V}_{ab} = (10\angle 90°)(20\angle -90°) = 200\angle 0° \text{ volts}$$

Now suppose that the current through a load impedance $\mathbf{Z}_L = 30\underline{/0°}$ ohms connected across ab is desired. According to Thévenin's theorem, the current is $\mathbf{V}_{ab}$ divided by the sum of the load impedance and the impedance looking into the network at ab. Thus the impedance looking into the network at ab (designated by $\mathbf{Z}_0$) when the emf in the branch ef is assumed zero is:

$$\mathbf{Z}_0 = j10 + \frac{(j10)(-j20)}{j10 - j20} = j30 \text{ ohms}$$

According to Thévenin's theorem

$$\mathbf{I}_{\text{load}} = \frac{\mathbf{V}_{ab}}{\mathbf{Z}_0 + \mathbf{Z}_L} = \frac{200\underline{/0°}}{j30 + 30\underline{/0°}} = 4.72\underline{/-45°} \text{ amperes}$$

This result may be checked by the usual series-parallel circuit theory as follows:

$$\mathbf{Z}_{cb} = \frac{(30 + j10)(-j20)}{30 + j10 - j20} = 12 - j16 \text{ ohms}$$

$$\mathbf{Z}_{eb} = j10 + 12 - j16 = 12 - j6 \text{ ohms}$$

$$\mathbf{I}_{ec} = \frac{100 + j0}{12 - j6} = 6.667 + j3.333 \text{ amperes}$$

$$\mathbf{V}_{cb} = (6.667 + j3.333)(12 - j16) = 133.3 - j66.67 \text{ volts}$$

$$\mathbf{I}_{ab} = \frac{133.3 - j66.67}{30 + j10} = 3.333 - j3.333 = 4.72\underline{/-45°} \text{ amperes}$$

which is the same as that obtained by Thévenin's theorem.

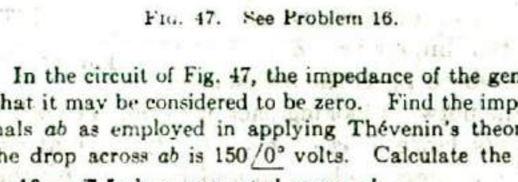

FIG. 47. See Problem 16.

Problem 16. In the circuit of Fig. 47, the impedance of the generator is assumed low enough so that it may be considered to be zero. Find the impedance $\mathbf{Z}_0$ looking into the terminals ab as employed in applying Thévenin's theorem. As may be easily shown, the drop across ab is $150\underline{/0°}$ volts. Calculate the current in a load impedance $\mathbf{Z}_L = 10 - j7.5$ ohms connected across ab.

Ans.: $\mathbf{Z}_0 = j7.5$ ohms, $\mathbf{I}_L = 15\underline{/0°}$ amperes.

The Nodal Method. The method ordinarily employed in analyzing circuits consists in establishing the necessary number of *voltage equilibrium equations* and solving for the currents. In many cases, particularly in vacuum tube circuits, it is desirable to employ *current equilibrium equations* and solve for the voltages. The latter method, known as the nodal method, consists essentially in writing Kirchhoff's current law at the nodes or junctions of the network the required number of times to effect a solution for various voltages in which we might be interested. See Chapter I.

In its simplest sense, a *node* of a network is any accessible terminal which is at a significant potential difference with respect to the other terminals. In this sense, the network shown in Fig. 48 *might be* considered a four-node network having nodes a, b, c, and d. Only the junction points (c and d) of the network, however, *need be* considered nodes, since the number of independent nodes is the number of junctions minus one. This will become more evident as we proceed.

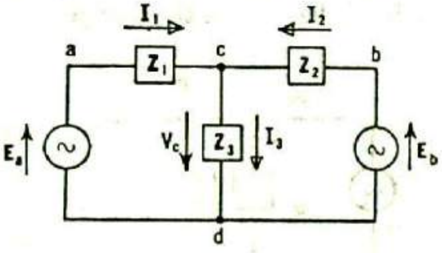

FIG. 48. Voltage sources may be transformed to equivalent current sources shown in Figs. 49 and 50.

Before the nodal method of analysis can be applied to voltage sources having internal impedance, these voltage sources must be transformed to equivalent current sources in accordance with the following principles. (If a specified voltage source is assumed to have zero impedance, it follows that the potential difference between the terminals of the generator is specified and hence does not enter the analysis as an unknown potential difference.)

In order to illustrate the transformation of a voltage source having internal impedance to an equivalent current source, let us suppose that $\mathbf{Z}_1$ of Fig. 48 is actually the internal impedance of the E_a voltage generator, thus eliminating point a as a node. Let V_c be the potential of node c relative to node d. Applying Kirchhoff's voltage law we have

$$\mathbf{I}_1\mathbf{Z}_1 + \mathbf{V}_c = \mathbf{E}_a \tag{83}$$

or

$$\mathbf{I}_1 = \frac{\mathbf{E}_a}{\mathbf{Z}_1} - \frac{\mathbf{V}_c}{\mathbf{Z}_1} \tag{84}$$

It will be observed in equation (83) that the inclusion of the potential of node d (V_d) is unnecessary and in general any node may be selected as a reference node from which to reckon all other nodal potentials.

If $\mathbf{E}_a$ and $\mathbf{Z}_1$ are specified quantities, equation (84) states that the current flowing into node c (I_1) is equal to a specified current ($\mathbf{E}_a/\mathbf{Z}_1$) minus a current ($\mathbf{V}_c/\mathbf{Z}_1$). The specified current ($\mathbf{E}_a/\mathbf{Z}_1$) may be considered as a current source across nodes c and d, provided that a $\mathbf{Z}_1$ path is placed in parallel with this source to account for the $(\mathbf{V}_c/\mathbf{Z}_1)$ current in equation (84). Thus the voltage source $\mathbf{E}_a$ in series with $\mathbf{Z}_1$ shown in Fig. 48 may be replaced with the circuit configuration shown in Fig. 49. In a similar manner the $\mathbf{E}_b$ source and the impedance $\mathbf{Z}_2$ may be replaced with the configuration shown in Fig. 50.

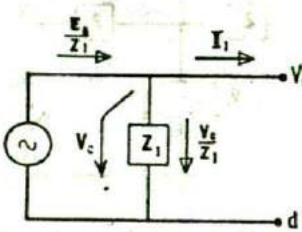

FIG. 49. Equivalent current source of E_a voltage source of Fig. 48.

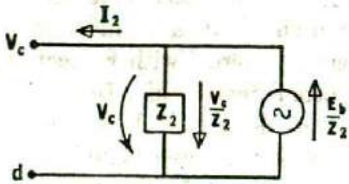

FIG. 50. Equivalent current source of E_b voltage source of Fig. 48.

If now these equivalent current sources are used in Fig. 48 instead of the voltage sources, Fig. 48 takes the form shown in Fig. 51. Employing Fig. 51, the current equation for the node c can be written in terms of voltage drops and admittances as follows:

$$\underset{\text{(current leaving node } c)}{\mathbf{Y}_1\mathbf{V}_c + \mathbf{Y}_3\mathbf{V}_c + \mathbf{Y}_2\mathbf{V}_c} = \underset{\text{(current entering node } c)}{\mathbf{Y}_1\mathbf{E}_a + \mathbf{Y}_2\mathbf{E}_b} \tag{85}$$

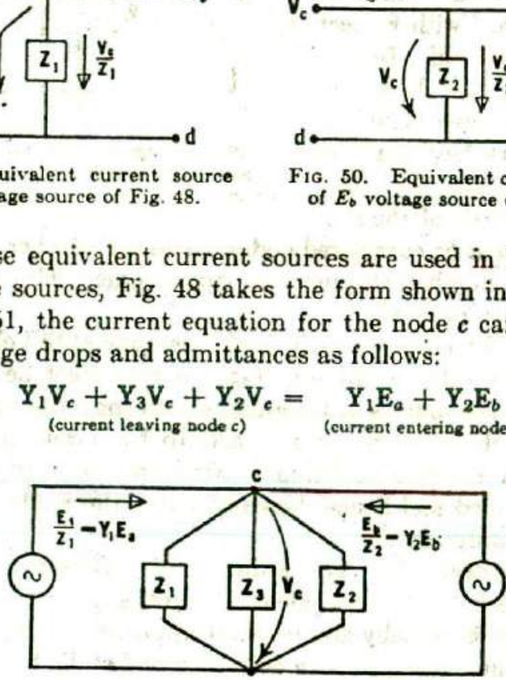

FIG. 51. Transformation of the circuit shown in Fig. 48.

$\mathbf{V}_c$ can be obtained from equation (85) directly in terms of known quantities and all currents thereby calculated.

Example 13. Assume the data for Fig. 48 to be as follows: $\mathbf{E}_a = 100\underline{/0^\circ}$ volts, $\mathbf{E}_b = 50\underline{/90^\circ}$ volts, $\mathbf{Z}_1 = 5\underline{/0^\circ}$ ohms, $\mathbf{Z}_2 = 10\underline{/36.9}$ ohms, and $\mathbf{Z}_3 = 20\underline{/53.1^\circ}$ ohms. Find the voltage $\mathbf{V}_c$ and currents $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$.

From equation (85),

$$\mathbf{V}_c(\mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3) = \mathbf{Y}_1\mathbf{E}_a + \mathbf{Y}_2\mathbf{E}_b$$

$$\mathbf{V}_c\left(\frac{1}{5\underline{/0^\circ}} + \frac{1}{10\underline{/36.9^\circ}} + \frac{1}{20\underline{/53.1^\circ}}\right) = \frac{100\underline{/0^\circ}}{5\underline{/0^\circ}} + \frac{50\underline{/90^\circ}}{10\underline{/36.9^\circ}}$$

$$\mathbf{V}_c(0.2 + 0.08 - j0.06 + 0.03 - j0.04) = 20 + 5\underline{/53.1^\circ}$$

$$\mathbf{V}_c = 71.6\underline{/27.76^\circ} \text{ volts}$$

$$\mathbf{I}_3 = \mathbf{V}_c\mathbf{Y}_3 = (71.6\underline{/27.76^\circ})\ (0.05\underline{/-53.1^\circ}) = 3.58\underline{/-25.34^\circ} \text{ amperes}$$

As seen from Fig. 49,

$$\mathbf{I}_1 = \mathbf{E}_a\mathbf{Y}_1 - \mathbf{V}_c\mathbf{Y}_1 = (0.2\underline{/0^\circ})\ (100\underline{/0^\circ} - 71.6\underline{/27.76^\circ}) = 7.35 - j6.66 \text{ amperes}$$

and, from Fig. 50,

$$\mathbf{I}_2 = \mathbf{E}_b\mathbf{Y}_2 - \mathbf{V}_c\mathbf{Y}_2 = (0.1\underline{/-36.9^\circ})\ (50\underline{/90^\circ} - 71.6\underline{/27.76^\circ})$$
$$= -4.05 + j5.134 \text{ amperes}$$

The nodal method of analysis is usually superior to the mesh-current method if the number of nodes (after transformation to current sources) does not exceed the number of meshes or loops. If N represents the number of nodes in a network, only $N - 1$ independent node equations are required, and these are obtained by applying Kirchhoff's current law to $N - 1$ nodes.

To arrive at the method of formulating a general system of nodal equations, assume that Fig. 52*a* is the network to be solved. First, replace the voltage sources by constant-current sources as shown in Fig. 52*b*. Assume one node as the reference node, node 4 in this case. The output of the constant-current generator a is $\mathbf{E}_a/\mathbf{Z}_a = \mathbf{I}_1$. Similarly the output of constant-current generator b is $\mathbf{E}_b/\mathbf{Z}_b = \mathbf{I}_3$. To obtain the current in any impedance, the voltage drop across the impedance is multiplied by the admittance. The voltage drop can always be obtained in terms of the nodal voltages. Remembering that the voltage drop from node 1 to node 2 is the sum of the drops encountered in going from node 1 to 2 by any path, we may write $\mathbf{V}_{12} = \mathbf{V}_{14} + \mathbf{V}_{42} = \mathbf{V}_1 - \mathbf{V}_2$. Hence $\mathbf{I}_{12} = (\mathbf{V}_1 - \mathbf{V}_2)\mathbf{Y}_{12}$. Application of Kirchhoff's current law to node 1 yields

$$\mathbf{Y}_a\mathbf{V}_1 + \mathbf{Y}_1\mathbf{V}_1 + \mathbf{Y}_{12}(\mathbf{V}_1 - \mathbf{V}_2) + \mathbf{Y}_{13}(\mathbf{V}_1 - \mathbf{V}_3) = \mathbf{I}_1 \qquad (86)$$

or

$$(\mathbf{Y}_a + \mathbf{Y}_1 + \mathbf{Y}_{12} + \mathbf{Y}_{13})\mathbf{V}_1 - \mathbf{Y}_{12}\mathbf{V}_2 - \mathbf{Y}_{13}\mathbf{V}_3 = \mathbf{I}_1 \qquad (87)$$

The sum of all the admittances from node 1 to all other nodes is called the self-admittance and is designated by $\mathbf{Y}_{11}$. The admittance of the impedance connecting node 1 to any other node, say n, is called the mutual admittance, $\mathbf{Y}_{1n}$. Thus $\mathbf{Y}_{12}$, $\mathbf{Y}_{13}$, etc., are mutual admittances. When these notations are used, equation (87) becomes

$$\mathbf{Y}_{11}\mathbf{V}_1 - \mathbf{Y}_{12}\mathbf{V}_2 - \mathbf{Y}_{13}\mathbf{V}_3 = \mathbf{I}_1 \qquad (88)$$

Similarly, for node 3,

$$\mathbf{Y}_{33}\mathbf{V}_3 - \mathbf{Y}_{32}\mathbf{V}_2 - \mathbf{Y}_{31}\mathbf{V}_1 = \mathbf{I}_3 \tag{89}$$

where $\mathbf{Y}_{33} = \mathbf{Y}_{13} + \mathbf{Y}_3 + \mathbf{Y}_{23} + \mathbf{Y}_b$. And, for node 2,

$$\mathbf{Y}_{22}\mathbf{V}_2 - \mathbf{Y}_{21}\mathbf{V}_1 - \mathbf{Y}_{23}\mathbf{V}_3 = 0 \tag{90}$$

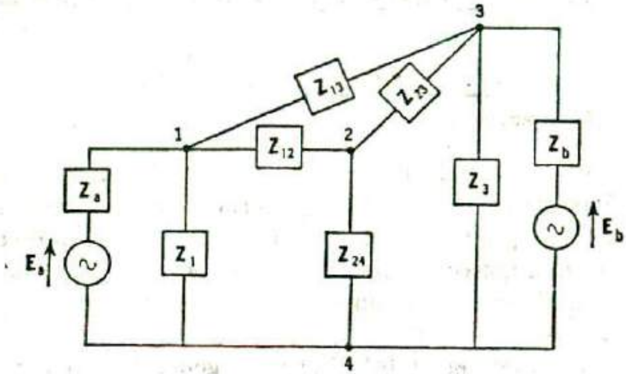

FIG. 52a. A network having two voltage sources.

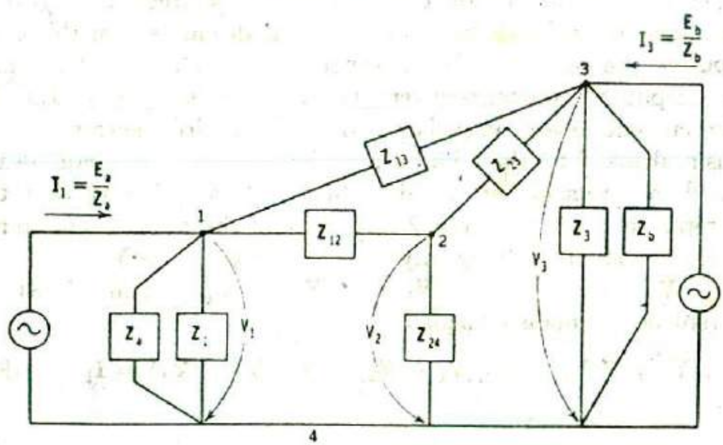

FIG. 52b. Transformation of circuit shown in Fig. 52a.

An extension of equations (88), (89), and (90) will yield the genera system of nodal equations for an n-node system as follows.

$$\left.\begin{aligned} \mathbf{Y}_{11}\mathbf{V}_1 - \mathbf{Y}_{12}\mathbf{V}_2 - \mathbf{Y}_{13}\mathbf{V}_3 - \cdots - \mathbf{Y}_{1n}\mathbf{V}_n &= \mathbf{I}_1 \\ -\mathbf{Y}_{21}\mathbf{V}_1 + \mathbf{Y}_{22}\mathbf{V}_2 - \mathbf{Y}_{23}\mathbf{V}_3 - \cdots - \mathbf{Y}_{2n}\mathbf{V}_n &= \mathbf{I}_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ -\mathbf{Y}_{n1}\mathbf{V}_1 - \mathbf{Y}_{n2}\mathbf{V}_2 - \mathbf{Y}_{n3}\mathbf{V}_3 - \cdots + \mathbf{Y}_{nn}\mathbf{V}_n &= \mathbf{I}_n \end{aligned}\right\} \begin{array}{l}\text{General system}\\ \text{of nodal}\\ \text{equations where a}\\ \text{common node}\\ \text{is employed}\end{array}$$

As previously defined, $\mathbf{I}_1, \mathbf{I}_2, \cdots$, and $\mathbf{I}_n$ are the output currents of the constant-current generators directed toward the various nodes. The nodal voltages in the general system of equations above may be solved for by determinants. After some practice with these systematic forms of solution, the determinant forms can be established from an inspection of the network after all specified voltage generators have been transformed to equivalent current generators. The writing of the current equations as shown above can therefore be dispensed with and the analysis reduced to a simple routine procedure.

In order to appreciate fully the usefulness of the nodal method, one should apply it to vacuum tube circuits where the plate-to-cathode path of the tube functions as a current sink (or negative current source). This application, however, presupposes an elementary knowledge of the functioning of a vacuum tube, and for this reason the following example may be omitted without loss of continuity by those readers who have no knowledge of the performance of a vacuum tube.

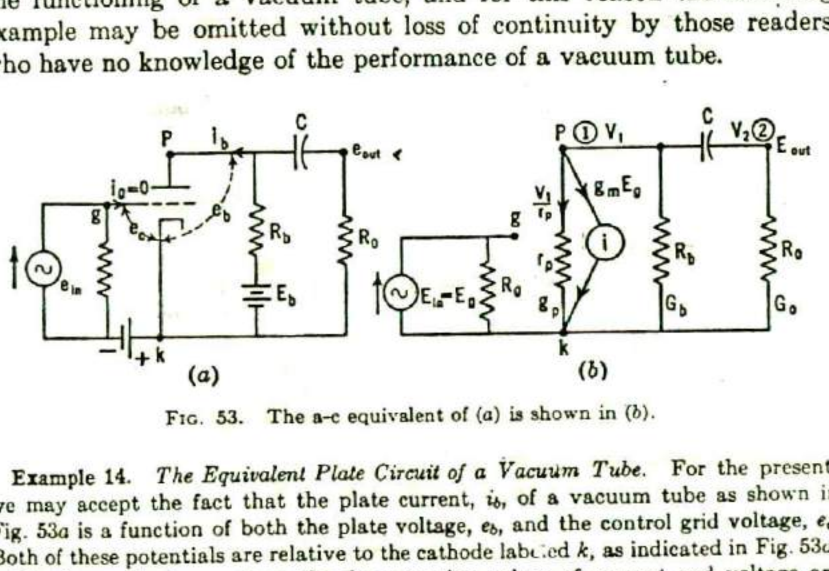

FIG. 53. The a-c equivalent of (a) is shown in (b).

Example 14. *The Equivalent Plate Circuit of a Vacuum Tube.* For the present, we may accept the fact that the plate current, i_b, of a vacuum tube as shown in Fig. 53a is a function of both the plate voltage, e_b, and the control grid voltage, e_c. Both of these potentials are relative to the cathode labeled k, as indicated in Fig. 53a.

If only small *changes* from the d-c operating values of current and voltage are involved, we may write

$$\Delta i_b = \frac{\partial i_b}{\partial e_b}\,\Delta e_b + \frac{\partial i_b}{\partial e_c}\,\Delta e_c \tag{91}$$

and, if the change in plate current Δi_b is called i_p, if the change in plate voltage Δe_b is called e_p, and if the change in grid voltage Δe_c is called e_g, we have

$$i_p = \frac{e_p}{r_p} + g_m e_g \tag{92}$$

where $r_p = \partial e_b/\partial i_b$ is called the variational or plate resistance of the vacuum tube, and $g_m = \partial i_b/\partial e_c$ is called the mutual conductance or transconductance. For a particular condition of d-c operation both r_p and g_m are usually known. The plate current of the vacuum tube so biased that the control grid current is zero is given by equation (92), and it is this equation which permits the use of the equivalent uit shown in Fig. 53*b* for the plate-to-cathode portion of the vacuum tube shown ig. 53*a*.

In Fig. 53 we may replace the instantaneous values of the e's and the i's with fective values if a sinusoidal time variation of e_{in} is assumed and if e_{in} is at no time so large as to permit the contro grid to draw current. It will be observed that the vacuum tube functions as a current sink $(g_m e_g)$ in parallel with a resistance path, namely, the r_p path in Fig. 53.

In order to illustrate further the application of the nodal method in a numerical case let it be required to find E_{out} in Fig. 53 if:

$$
\begin{aligned}
e_{in} &= 0.707 \sin 3770t \text{ volt} \quad \text{or} & E_{in} &= E_g = 0.5\underline{/0^\circ} \text{ volt} \\
g_m &= 2000 \text{ micromhos} & g_m &= 200 \times 10^{-5} \text{ mho} \\
r_p &= 20{,}000 \text{ ohms} & g_p &= 5 \times 10^{-5} \text{ mho} \\
R_b &= 50{,}000 \text{ ohms} & G_b &= 2 \times 10^{-5} \text{ mho} \\
R_0 &= 200{,}000 \text{ ohms} & G_0 &= 0.5 \times 10^{-5} \text{ mho} \\
C &= 0.00265\ \mu f & \mathbf{Y}_{12} &= \mathbf{Y}_{21} = j\omega C = j10^{-5} \text{ mho}
\end{aligned}
$$

Applying Kirchhoff's current law to node 1 in Fig. 53*b*, we obtain

$$g_p\mathbf{V}_1 + G_b\mathbf{V}_1 + j\omega C(\mathbf{V}_1 - \mathbf{V}_2) = -g_m\mathbf{E}_g = \mathbf{I}_1$$

or

$$\mathbf{Y}_{11}\mathbf{V}_1 - \mathbf{Y}_{12}\mathbf{V}_2 = -g_m\mathbf{E}_g = -100 \times 10^{-5} \text{ ampere}$$

where, in this particular case,

$$
\begin{aligned}
\mathbf{Y}_{11} \text{ (the self-admittance of node 1)} &= g_p + G_b + j\omega C \\
&= (7 + j1)10^{-5} \text{ mho}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{Y}_{12} = \mathbf{Y}_{21} \text{ (the mutual admittance between nodes 1 and 2)} & \\
&= j\omega C \\
&= j10^{-5} \text{ mho}
\end{aligned}
$$

Applying the current law to node 2,

$$G_0\mathbf{V}_2 + j\omega C(\mathbf{V}_2 - \mathbf{V}_1) = 0$$

or

$$-\mathbf{Y}_{21}\mathbf{V}_1 + \mathbf{Y}_{22}\mathbf{V}_2 = 0$$

where

$$
\begin{aligned}
\mathbf{Y}_{22} \text{ (the self-admittance of node 2)} &= G_0 + j\omega C \\
&= (0.5 + j1) \times 10^{-5} \text{ mho}
\end{aligned}
$$

The detailed applications of the current law can be dispensed with as soon as the systematized procedure implied by the subscripts attached to the Y's is understood. The determinant form of the solution for $\mathbf{V}_2$ is

$$
\mathbf{V}_2 = \frac{\begin{vmatrix} \mathbf{Y}_{11} & \mathbf{I}_1 \\ -\mathbf{Y}_{21} & 0 \end{vmatrix}}{\begin{vmatrix} \mathbf{Y}_{11} & -\mathbf{Y}_{12} \\ -\mathbf{Y}_{21} & \mathbf{Y}_{22} \end{vmatrix}} = \frac{\begin{vmatrix} (7 + j1) & -100 \\ -j1 & 0 \end{vmatrix} \times 10^{-10}}{\begin{vmatrix} (7 + j1) & -j1 \\ -j1 & (0.5 + j1) \end{vmatrix} \times 10^{-10}}
$$

$$\mathbf{V}_2 = \frac{-j100}{3.5 + j7.5} = \frac{100\underline{/-90^\circ}}{8.27\underline{/65^\circ}} = 12.08\underline{/-155^\circ} \text{ volts}$$

The amplification of the circuit arrangement shown in Fig. 53 is

$$\frac{\mathbf{E}_{\text{out}}}{\mathbf{E}_{\text{in}}} = \frac{12.08\underline{/-155^\circ}}{0.5\underline{/0^\circ}} = 24.16\underline{/-155^\circ}$$

which indicates that the magnitude of the output voltage is 24.16 times that of the input voltage and that the output voltage lags the input voltage by 155° or 155/360 part of a cycle.

Norton's Theorem. This theorem states that with respect to any pair of terminals of any active network, the active network may be replaced with a single current source in parallel with an impedance equal to the impedance which is seen looking back into the network

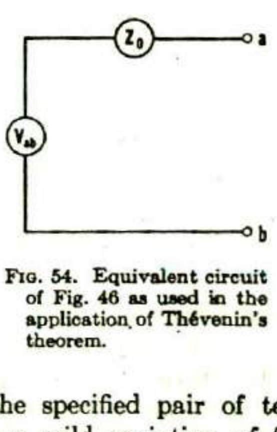

FIG. 54. Equivalent circuit of Fig. 46 as used in the application of Thévenin's theorem.

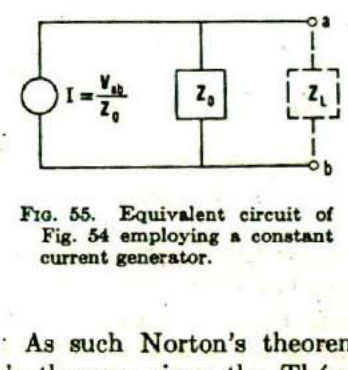

FIG. 55. Equivalent circuit of Fig. 54 employing a constant current generator.

from the specified pair of terminals. As such Norton's theorem is merely a mild variation of Thévenin's theorem since the Thévenin equivalent of an active network (Fig. 54) is readily transformed to the configuration shown in Fig. 55. In this latter figure

$$\mathbf{I} = \mathbf{I}_{\text{source}} = \mathbf{V}_{ab}/\mathbf{Z}_0$$

where $\mathbf{V}_{ab}$ is the open-circuit voltage which appears across the selected terminals and $\mathbf{Z}_0$ is equal to the series impedance of the Thévenin equivalent circuit, Fig. 54. The transformation from Fig. 54 to Fig. 55 is contained essentially in equation (84), page 199.

Example 15. Norton's theorem will be applied to example 12. From example 12 the voltage $\mathbf{V}_{ab} = 200\underline{/0^\circ}$ and the impedance looking back at ab was $\mathbf{Z}_0 = j30$ ohms. This yields a circuit shown by Fig. 54 which was employed in Thévenin's theorem. Converted to a constant-current generator in accordance with the principles shown in the previous article, the circuit of Fig. 55 is obtained. If an impedance load

$Z_L = 30\underline{/0^\circ}$ is connected across terminals ab the following solution results:

$$\mathbf{I} = \frac{\mathbf{V}_{ab}}{\mathbf{Z}_0} = \frac{200\underline{/0^\circ}}{j30} = -j6.66 \text{ amperes (current source)}$$

$$\mathbf{I}_{ab} = \frac{\dfrac{\mathbf{Z}_L\mathbf{Z}_0}{\mathbf{Z}_L + \mathbf{Z}_0}(-j6.66)}{\mathbf{Z}_L} = \frac{\mathbf{Z}_0}{\mathbf{Z}_L + \mathbf{Z}_0}(-j6.66)$$

$$= \frac{j30}{30 + j30}(-j6.66) = \frac{200}{30 + j30} = 4.72\underline{/-45^\circ} \text{ amperes}$$

which is the same result as obtained in example 12. Thus in accordance with Norton's theorem the circuit of Fig. 55 may be used between terminals ab to replace Fig. 46.

Either Thévenin's or Norton's theorem is often applied where complicated networks relative to a pair of terminals are being analyzed.

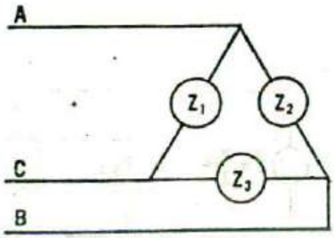

Fig. 56. Delta.

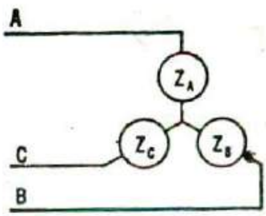

Fig. 57. Wye.

Equivalence of Special Circuits (Wyes and Deltas). Figures 56 and 57 show two types of circuits which are very commonly encountered in the reduction of electrical networks. The first is called a delta system; Fig. 57 is called a wye. It is possible to substitute a wye-connected system of impedances for a delta system, and vice versa, if proper values are given to the substituted impedances. Suppose that it is desired to substitute a wye for a given delta. The two systems will be exactly equivalent if the impedance between any pair of lines A, B, and C, Fig. 58, for the delta is the same as that between the corresponding pair for the wye when the third line is broken. If this condition is imposed, the following equations are obtained:

Line A open: $$\mathbf{Z}_C + \mathbf{Z}_B = \frac{\mathbf{Z}_3(\mathbf{Z}_1 + \mathbf{Z}_2)}{\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3} \tag{93}$$

Line B open: $$\mathbf{Z}_A + \mathbf{Z}_C = \frac{\mathbf{Z}_1(\mathbf{Z}_2 + \mathbf{Z}_3)}{\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3} \tag{94}$$

Line C open: $$\mathbf{Z}_A + \mathbf{Z}_B = \frac{\mathbf{Z}_2(\mathbf{Z}_1 + \mathbf{Z}_3)}{\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3} \tag{95}$$

Solution of these three equations simultaneously for $\mathbf{Z}_A$, $\mathbf{Z}_B$, and $\mathbf{Z}_C$ in terms of the impedances $\mathbf{Z}_1$, $\mathbf{Z}_2$, and $\mathbf{Z}_3$ gives the following:

$$\mathbf{Z}_A = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3} \tag{96}$$

$\Delta \to Y$

$$\mathbf{Z}_B = \frac{\mathbf{Z}_2\mathbf{Z}_3}{\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3} \tag{97}$$

$$\mathbf{Z}_C = \frac{\mathbf{Z}_1\mathbf{Z}_3}{\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3} \tag{98}$$

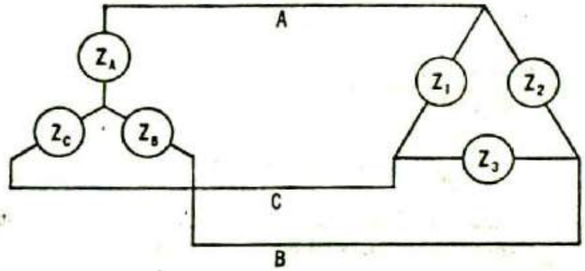

FIG. 58. Circuit for establishment of equivalence between wye and delta systems of impedances.

From equations (96), (97), and (98), the values of the wye impedances $\mathbf{Z}_A$, $\mathbf{Z}_B$, and $\mathbf{Z}_C$ that will replace a system of delta impedances $\mathbf{Z}_1$, $\mathbf{Z}_2$, and $\mathbf{Z}_3$ may be found. These results are easily remembered when it is observed that the denominators are all the same and equal to the sum of the three delta impedances. The numerator for $\mathbf{Z}_A$ is the product of the two delta impedances which connect to $\mathbf{Z}_A$. Similarly the numerator for $\mathbf{Z}_B$ is the product of $\mathbf{Z}_2$ and $\mathbf{Z}_3$.

It should be noticed that the special case of balanced delta impedances yields wye impedances, which are also balanced and equal to

$$\mathbf{Z}_Y = \frac{\mathbf{Z}_\Delta^{\,2}}{3\mathbf{Z}_\Delta} = \frac{\mathbf{Z}_\Delta}{3}$$

and

$$\mathbf{Z}_\Delta = 3\mathbf{Z}_Y$$

Example 16. Find I for the circuit and constants shown in Fig. 59. First a wye is substituted for the delta abc. The wye and its corresponding impedances are shown dotted.

$$\mathbf{Z}_A = \frac{(1 + j12)\,(4 - j6)}{(4 - j6) + (3 + j0) + (1 + j12)} = 8.6 - j1.2 \text{ ohms}$$

now

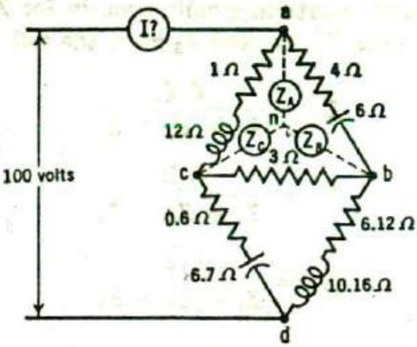

FIG. 59. See example 16.

$$Z_B = \frac{(4 - j6)\ (3)}{8 + j6} = -0.12 - j2.16 \text{ ohms}$$

$$Z_C = \frac{(1 + j12)\ (3)}{8 + j6} = 2.4 + j2.7 \text{ ohms}$$

After the above impedances are substituted, the circuit appears as shown in Fig. 60. It is apparent that a series-parallel circuit results, the method of solution of

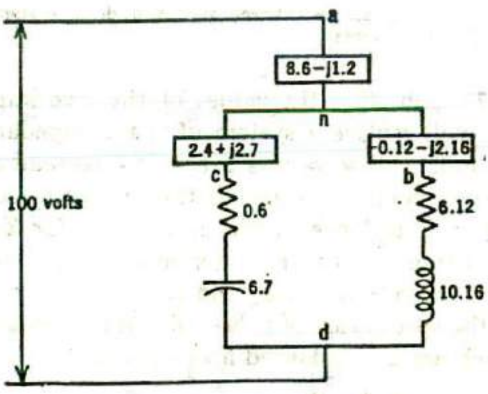

FIG. 60. Reduction from Fig. 59.

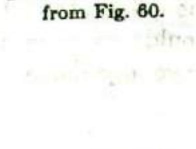

FIG. 61. Reduction from Fig. 60.

which has been given in a previous article. Combining the parallel branches results in the circuit shown in Fig. 61. Thus

$$Z_{ncd} = 3 - j4 \text{ ohms}$$

$$Z_{nbd} = 6 + j8 \text{ ohms}$$

$$Z_{nd} = \frac{(6 + j8)\ (3 - j4)}{(3 - j4) + (6 + j8)} = \frac{50(9 - j4)}{(9 + j4)\ (9 - j4)}$$

$$= \frac{450 - j200}{81 + 16} = 4.645 - j2.065 \text{ ohms}$$

$$\mathbf{I} = \frac{100 + j0}{13.245 - j3.265} = 7.14 + j1.76 = 7.355\underline{/14^\circ} \text{ amperes}$$

To find the currents in the various branches, the steps are retraced as follows:

$$\begin{aligned}
\mathbf{V}_{nd} &= \mathbf{I}\mathbf{Z}_{nd} = (7.14 + j1.76)\,(4.645 - j2.065) \\
&= 36.73 - j6.57 \text{ volts}
\end{aligned}$$

$$\mathbf{I}_{ncd} = \frac{(36.73 - j6.57)\,(3 + j4)}{(3 - j4)\,(3 + j4)} = 5.45 + j5.09 \text{ amperes}$$

$$\mathbf{I}_{nbd} = \frac{(36.73 - j6.57)\,(6 - j8)}{(6 + j8)\,(6 - j8)} = 1.69 - j3.33 \text{ amperes}$$

$$\mathbf{V}_{an} = \mathbf{I}\mathbf{Z}_{an} = (7.14 + j1.76)\,(8.6 - j1.2) = 63.51 + j6.57 \text{ volts}$$

$$\begin{aligned}
\mathbf{V}_{nc} &= \mathbf{I}_{ncd}\mathbf{Z}_{nc} = (5.45 + j5.09)\,(2.4 + j2.7) \\
&= -0.64 + j26.9 \text{ volts}
\end{aligned}$$

$$\begin{aligned}
\mathbf{V}_{nb} &= \mathbf{I}_{nbd}\mathbf{Z}_{nb} = (1.69 - j3.33)\,(-0.12 - j2.16) \\
&= -7.403 - j3.25 \text{ volts}
\end{aligned}$$

$$\begin{aligned}
\mathbf{V}_{ac} &= \mathbf{V}_{an} + \mathbf{V}_{nc} = 63.51 + j6.57 - 0.64 + j26.9 \\
&= 62.87 + j33.47 \text{ volts}
\end{aligned}$$

$$\begin{aligned}
\mathbf{V}_{ab} &= \mathbf{V}_{an} + \mathbf{V}_{nb} = 63.51 + j6.57 - 7.403 - j3.25 \\
&= 56.11 + j3.32 \text{ volts}
\end{aligned}$$

$$\mathbf{I}_{ac} = \frac{(62.87 + j33.47)\,(1 - j12)}{(1 + j12)\,(1 - j12)} = 3.19 - j4.96 \text{ amperes}$$

$$\mathbf{I}_{ab} = \frac{(56.11 + j3.32)\,(4 + j6)}{(4 - j6)\,(4 + j6)} = 3.93 + j6.73 \text{ amperes}$$

Check: $3.19 - j4.96 + 3.93 + j6.73 = 7.12 + j1.77$, which is within slide-rule accuracy of $7.14 + j1.76$ amperes.

$$\begin{aligned}
\mathbf{V}_{cb} &= \mathbf{V}_{cn} + \mathbf{V}_{nb} = 0.64 - j26.9 - 7.403 - j3.25 \\
&= -6.763 - j30.15 \text{ volts}
\end{aligned}$$

$$\mathbf{I}_{cb} = \frac{-6.763 - j30.15}{3} = -2.254 - j10.05 \text{ amperes}$$

$$\begin{aligned}
\mathbf{I}_{cd} &= \mathbf{I}_{ac} - \mathbf{I}_{cb} = 3.19 - j4.96 + 2.254 + j10.05 \\
&= 5.444 + j5.09 \text{ amperes}
\end{aligned}$$

which checks $\mathbf{I}_{ncd}$.

$$\begin{aligned}
\mathbf{I}_{bd} &= \mathbf{I}_{cb} + \mathbf{I}_{ab} = -2.254 - j10.05 + 3.93 + j6.73 \\
&= 1.68 - j3.32 \text{ amperes}
\end{aligned}$$

There are a few occasions when it is convenient and desirable to substitute an equivalent delta for a wye. This is simply the problem of finding the values of $\mathbf{Z}_1$, and $\mathbf{Z}_2$, and $\mathbf{Z}_3$ that will replace the values of $\mathbf{Z}_A$, and $\mathbf{Z}_B$, and $\mathbf{Z}_C$ in Fig. 58. The solution is obtained when equations (93), (94), and (95) are solved algebraically for the impedances $\mathbf{Z}_1$, $\mathbf{Z}_2$, and $\mathbf{Z}_3$ in terms of the impedances $\mathbf{Z}_A$, $\mathbf{Z}_B$, and $\mathbf{Z}_C$. It will usually be found simpler to solve for these quantities from equations (96), (97), and (98), which were derived from equations (93), (94), and (95). The solution gives

$$\mathbf{Z}_1 = \frac{\mathbf{Z}_A\mathbf{Z}_B + \mathbf{Z}_B\mathbf{Z}_C + \mathbf{Z}_C\mathbf{Z}_A}{\mathbf{Z}_B} \tag{99}$$

$$\mathbf{Z}_2 = \frac{\mathbf{Z}_A\mathbf{Z}_B + \mathbf{Z}_B\mathbf{Z}_C + \mathbf{Z}_C\mathbf{Z}_A}{\mathbf{Z}_C} \tag{100}$$

$$\mathbf{Z}_3 = \frac{\mathbf{Z}_A\mathbf{Z}_B + \mathbf{Z}_B\mathbf{Z}_C + \mathbf{Z}_C\mathbf{Z}_A}{\mathbf{Z}_A} \tag{101}$$

Equations (99), (100), and (101) are easy to write when it is observed that the numerator of each is the same and equal to the sum of all

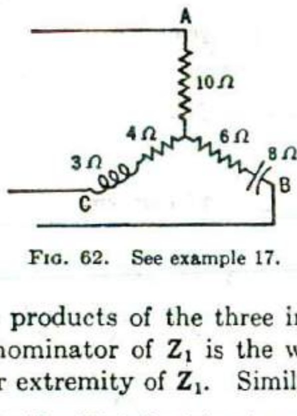

FIG. 62. See example 17.

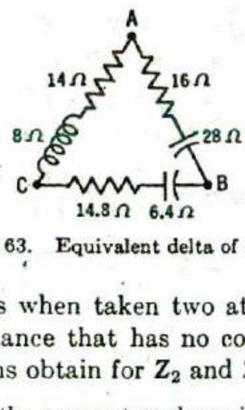

FIG. 63. Equivalent delta of Fig. 62.

possible products of the three impedances when taken two at a time. The denominator of $\mathbf{Z}_1$ is the wye impedance that has no connection to either extremity of $\mathbf{Z}_1$. Similar relations obtain for $\mathbf{Z}_2$ and $\mathbf{Z}_3$.

Example 17. Find the delta that will replace the wye system shown in Fig. 62.

$$\mathbf{Z}_{AB\Delta} = \frac{(10)\,(6 - j8) + (6 - j8)\,(4 + j3) + (10)\,(4 + j3)}{4 + j3}$$

$$= \frac{148 - j64}{4 + j3} = 16 - j28 \text{ ohms}$$

$$\mathbf{Z}_{BC\Delta} = \frac{148 - j64}{10} = 14.8 - j6.4 \text{ ohms}$$

$$\mathbf{Z}_{CA\Delta} = \frac{148 - j64}{6 - j8} = 14 + j8 \text{ ohms}$$

From these three impedances the equivalent delta is found as shown in Fig. 63.

Two commonly used types of networks are the T and π configurations shown, respectively, in Fig. 64*a* and Fig. 64*b*. Viewed as three-terminal networks, these configurations will be recognized as the wye and delta, respectively. The same formulas derived for changing a wye to an equivalent delta are therefore applicable for changing a T to an equivalent π. Likewise formulas for changing a delta to an equivalent wye may be used to change a π to an equivalent T.

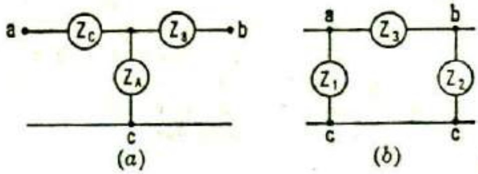

FIG. 64. (a) T network. (b) π network.

T- and π-sections are used extensively in transmission line and filter-section calculations. In cases of this kind, the T- and π-sections shown in Fig. 64 are usually considered as four-terminal networks because these sections are inserted into a two-wire circuit and are considered to have a pair of "input" terminals and a pair of "output" terminals. The manipulation of T- and π-sections as four-terminal networks will be considered in detail in Chapters X and XI.

PROBLEMS

17. Calculate the current through the impedances of Fig. 65. Find voltage drops across *ab*, *bc*, and *cd*. Draw the vector diagram showing the current and the voltage drop across each resistance or reactance. Calculate the power factor of the complete circuit.

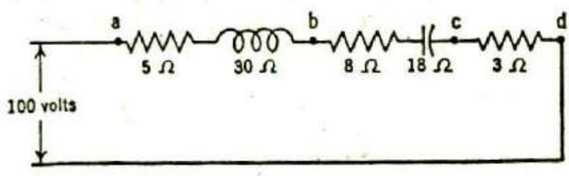

FIG. 65. See Problems 17, 18, and 24.

18. Find all possible values of pure reactance which, when placed in series with the circuit of Fig. 65, will make the overall power factor 0.6. Find the power dissipated in the circuit for this condition.

19. A particular 110-volt, 60-cycle, $\frac{1}{4}$-hp, single-phase induction motor has an efficiency of 60 per cent and a power factor of 0.6 lagging at full load. This motor is to be used temporarily on a 220-volt, 60-cycle line. A resistor (non-inductive) of suitable current capacity and of proper resistance is to be placed in series with the motor.

(*a*) What value of resistance is required if the motor is to have 110 volts across its terminals at rated full load?

(*b*) Draw the complete phasor diagram ($\mathbf{V}_{motor}$, $\mathbf{IR}_{external}$, $\mathbf{I}$, and $\mathbf{V}_{line}$) with $\mathbf{V}_{motor}$ as reference.

20. A single-phase lagging-power-factor load takes 300 watts and 5 amperes at 120 volts. Find the reactance of a pure capacitor that may be placed in series with this load so that it will operate normally from a 240-volt source.

21. Two single-phase motors are connected in parallel across a 110-volt, 60-cycle source of supply. Motor 1 is a split-phase induction type which takes a lagging current, and motor 2 is a capacitor type which takes a leading current. Find the total power, the combined line current, and the resultant power factor of the two motors operating in parallel from the following data:

Motor	Horsepower Output	Per Unit Efficiency	Per Unit Power Factor
1	$\frac{1}{3}$	0.60	0.70 (lagging)
2	$\frac{1}{2}$	0.75	0.95 (leading)

22. A series circuit on which 100 volts is impressed consists of a 10-ohm resistance, a 5-ohm condenser, a resistance R in which is lost 50 watts, and a reactance X taking 100 inductive vars. Calculate all values of R and X to satisfy the conditions stated and the corresponding currents for each of the combinations.

23. A toaster operates at 115 volts, 60 cycles, and 10 amperes and absorbs 1150 watts at its terminals. A choke coil is to be wound with a ratio of X_L to R of 5, so that, if placed in series with the toaster on a 230-volt, 60-cycle line, the toaster will have 115 volts across its terminals.

(*a*) What is the impedance of the choke coil required? State $\mathbf{Z}$ in polar and in rectangular complex form.

(*b*) Draw the complete vector diagram with $V_{toaster}$ as reference.

(*c*) What is the power factor of the combined toaster and choke coil in series?

24. Find the inductance or capacitance which may be inserted in the circuit of Fig. 65 to put the entire circuit in resonance. Frequency 60 cycles.

25. (*a*) If the impressed voltage on a series circuit containing 5 ohms resistance, 100 ohms inductive reactance at 60 cycles, and a variable capacitance is 100 volts, find the maximum drop across the capacitance and the value of the capacitance for this condition.

(*b*) Repeat the calculation if, instead of the 5-ohm resistance, a 100-ohm resistance is used. Compare the results in the two cases.

26. A series circuit dissipates 800 watts and also requires 1000 volt-amperes when the impressed voltage is 100 volts. Find the equivalent series resistance and possible reactances of this circuit.

27. The frequency range of the pass band as previously defined in this chapter for an RLC circuit is 100 cycles when a coil having a Q of 50 is used. All resistance of the circuit is assumed in the coil.

(*a*) Find the upper and lower frequency limits of the pass band.

(*b*) If a coil with a Q of 200 is used at the same resonant frequency as in (*a*), what will be the frequency range of the pass band?

28. Given the RLC series circuit shown in Fig. 66.

(*a*) Find the resonant frequency of the series circuit.

(*b*) Find the Q of the series circuit at the resonant frequency.

(c) At what angular velocities do the half-power points occur?

(d) Assuming that L is varied to obtain resonance, at what value of L would V_L be maximum? Assume the frequency in this case to be constant at 159 kc.

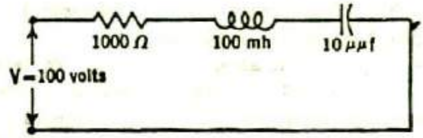

FIG. 66. See Problem 28.

29. Given the circuit shown in Fig. 67.

(a) What are the values of X_L that will produce resonance?

(b) Find the magnitude of the maximum impedance obtainable with this circuit. Assume that the frequency is held fixed.

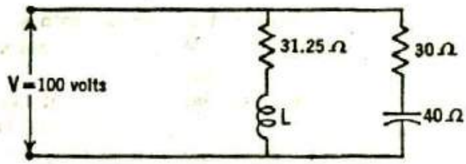

FIG. 67. See Problem 29.

(c) If R_L is changed to 30 ohms (R_C remaining the same) and L and C are made 9 millihenrys and 10 μf, respectively, what is the impedance looking into the circuit at 100 cycles per second and 10,000 cycles per second?

(d) At what frequency will the circuit as designated in part (c) be in resonance?

30. In the following exercises, it is assumed that a coil having L henrys of inductance and R_s ohms of series resistance is placed in resonance with a series capacitor C, so that $\omega_m = 1/\sqrt{LC}$.

(a) Show that $Q_s = \omega_m L/R_s$ is

$$Q_s = \frac{\text{reactive factor (of the coil)}}{\text{power factor (of the coil)}}$$

(b) Show that

$$\text{Power factor (of the coil)} = \frac{1}{\sqrt{Q_s^2 + 1}}$$

(c) Show that

$$Q_s = \frac{\omega_m w}{R_s I^2}$$

where w is the reactive energy stored in L and C at any time and $R_s I^2$ is the average dissipated power of the circuit. *Note:* $w = (Li^2/2) + (Cv_c^2/2) = \text{constant}$.

31. An impedance $Z_1 = 8 - j5$ is in parallel with an impedance $Z_2 = 3 + j7$ ohms. Find the resultant impedance of the combination. What is the overall power factor?

32. If 100 volts are impressed on the parallel impedances of Problem 31, find I_1, I_2, and the resultant current. Draw the vector diagram of the circuit, showing each current, and the voltage drop across each parameter.

33. An impedance load consisting of 12 ohms resistance and 16 ohms inductive reactance is connected across a 60-cycle, 100-volt source. Find the capacitance of a capacitor which may be paralleled with this load to bring the power factor to 1. Assume negligible resistance for the capacitor.

34. Work Problem 33 if a final power factor of 0.8 instead of 1 is desired. Obtain solutions for leading and lagging power factors.

35. Find the value of pure resistance which would be required in parallel with the impedance load of Problem 33 to bring the resultant power factor to 0.8.

36. A capacitor branch having a ratio of X to R of 5 is paralleled with an impedance consisting of 4 ohms resistance and 3 ohms inductive reactance. The power factor of the resulting circuit is 0.8 lead. Find the size of the capacitor in microfarads if the frequency is 60 cycles.

37. A single-phase load on 200 volts takes 5 kw at 0.6 lagging power factor. Find the kva size of capacitor which may be connected in parallel with this motor to bring the resultant power factor to 1.

38. Work Problem 37 if it is desired to bring the power factor to 0.9 lag instead of to 1.

39. The load of Problem 37 is operated in parallel with a synchronous motor that takes 8 kw at 0.5 leading power factor. What are the resultant current supplied by the line and the power factor of the combination?

40. Over the period of a year, an industrial establishment takes an average load of 2000 kw continuously at a (current) lagging power factor of 0.80.

(*a*) What is the annual fixed charge on the kva capacity required to serve this establishment if 1 kva of installed capacity (boiler, generator, transmission line, and transformers) costs $200? The fixed charge (consisting of interest, taxes, and depreciation) may be taken as 8 per cent of the investment.

(*b*) Repeat part (*a*) assuming that the power factor of the establishment is unity.

41. What value of resistance should be placed in parallel with a 50-μf capacitor to give a combined power factor of 0.6 on a 60-cycle system? (Neglect the resistance of the capacitor.)

42. Find the series-circuit resonant frequency of a 100-microhenry inductance and a 400-$\mu\mu f$ capacitance.

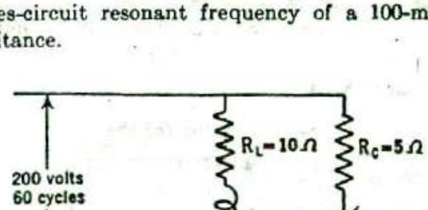

FIG. 68. See Problems 43, 44, and 45.

43. Find C to produce resonance in Fig. 68. How much power is dissipated in R_C at resonance?

44. Find the value of C in Fig. 68 which will yield maximum impedance for the whole circuit.

45. What minimum value of R_C in Fig. 68 would prevent the possibility of attaining resonance by varying C?

46. A fixed condenser is placed in parallel with a fixed resistance and variable inductance of negligible resistance as shown in Fig. 69. Show that the general expression for X_L which will produce unity-power-factor resonance is:

$$X_L = \frac{X_C}{2} \pm \sqrt{\frac{X_C^2}{4} - R^2}$$

Hint: For unity p.f., $b_L = b_C$.

47. Refer to Fig. 69.

(*a*) Draw a to-scale vector diagram of **V**, $\mathbf{I}_C$ and $\mathbf{I}_{RL}$ for $X_L = 0$.

(*b*) On the above diagram draw the loci of $\mathbf{I}_{RL}$ and **I** for X_L variable from 0 to ∞.

(*c*) Determine the values of X_L which will produce unity-power-factor resonance either graphically or analytically.

(*d*) Determine the minimum value of **I** either graphically or analytically, and find the value of X_L which produces this minimum value of **I**.

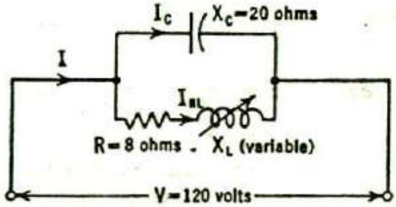

FIG. 69. See Problems 46 and 47.

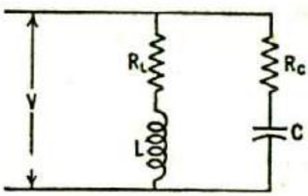

FIG. 70. See Problem 49.

48. A 2-μf capacitance is connected in parallel with a 20-ohm resistance. Plot the magnitudes of the admittance and impedance of the parallel combination against frequency for frequencies of 0, 10,000, 100,000, and 1,000,000 cycles.

49. (*a*) If $L = 0.050$ henry, $C = 200\ \mu$f, and $R_L = R_C = 1.0$ ohm, find the resonant frequency of the parallel branches shown in Fig. 70.

(*b*) If $R_L = 20$ ohms, $L = 0.050$ henry, $C = 100\ \mu$f, find the value of R_C which will yield parallel resonance of the two branches at a frequency of 45 cycles.

(*c*) If $C = 100\ \mu$f, $R_L = 20$ ohms, and $R_C = 20$ ohms, find the value of L that will place the branches in parallel resonance irrespective of frequency.

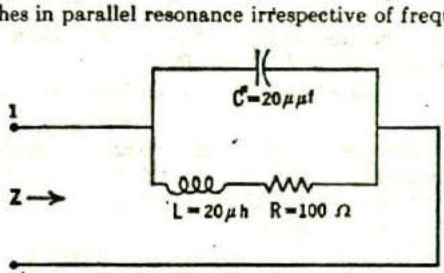

FIG. 71. See Problems 50, 53, 54, and 56.

50. (*a*) Transform the circuit shown in Fig. 71 to that shown in Fig. 72, employing numerical values of g, b_L, and b_C and assuming that the operating angular frequency is 5×10^7 radians per second. (Results which are accurate to within 1 per cent will be considered satisfactory.)

(*b*) If terminals 11′ of Fig. 71 are energized with a current of 2 milliamperes

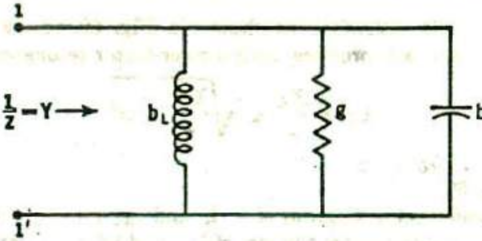

FIG. 72. See Problem 50.

(at $\omega = 5 \times 10^7$ radians per second), what voltage will be developed across these terminals?

(c) What is the Q_p of the circuit?

(d) Assuming that R is constant, find the resistance component of $\mathbf{Z}$ in Fig. 71 in terms of L, R, C, and ω.

$$Ans.: \quad R_Z = \frac{R}{(LC\omega^2 - 1)^2 + R^2\omega^2C^2}.$$

51. Given: $R = 2$ ohms, $L = 1$ henry, and $C = 0.1$ farad.

(a) If R, L, and C are connected in series, find the pole and zeros of the series impedance, $\mathbf{Z}(\mathbf{s})$, numerically. Evaluate $\mathbf{Z}(\omega)$ at $\omega = 2$ radians per second (or at $\mathbf{s} = j2$ radians per second) graphically from a plot of $\bar{\mathbf{s}}_1$, $\bar{\mathbf{s}}_2$, and $\hat{\mathbf{s}}_1$ and compare the result thus obtained with $\mathbf{Z}(2) = 2 + j(2 - 5) = 3.61\underline{/-56.3^\circ}$ ohms.

(b) If R, L, and C are connected in parallel as in Fig. 72, find the pole and zeros of $\mathbf{Y}(\mathbf{s})$ numerically. Evaluate $\mathbf{Y}(\omega)$ at $\omega = 1$ radian per second (or at $\mathbf{s} = j1$ radian per second) graphically from a plot of $\bar{\mathbf{s}}_1$, $\bar{\mathbf{s}}_2$, and $\hat{\mathbf{s}}_1$ and compare the result with

$$\mathbf{Y}(1) = 0.5 + j(0.1 - 1) = 1.03\underline{/-60.9^\circ} \text{ mhos.}$$

(c) Repeat part (b) for $\omega = 4$ radians per second, and compare with

$$\mathbf{Y}(4) = 0.5 + j(0.4 - 0.25) = 0.522\underline{/16.7^\circ} \text{ mho.}$$

52. (a) Find the angular frequency at which R_Z of Problem 50 has its maximum value, employing literal values of L, C, and R.

(b) What is the numerical value of the angular frequency for (a)?

Ans.: 4.987×10^7 radians per second.

(c) Compare the above result with the approximate value of $1/\sqrt{LC}$.

53. What is the maximum numerical value of the resistance component of Z in Fig. 71 as ω is varied from zero to infinity? (A result which is accurate to within 1 per cent will be considered satisfactory.)

54. The series resistance of the 20-microhenry coil shown in Fig. 71 is $R = 100$ ohms. What is the Q of the coil at $\omega = 0.1/\sqrt{LC}$ and at $\omega = 1/\sqrt{LC}$?

55. A coil having L henrys of inductance and R_s ohms of series resistance is placed in resonance with a parallel capacitor, C, having no appreciable series resistance at an angular frequency of ω_m which is essentially equal to $1/\sqrt{LC}$. $R_s^2 \ll \omega_m^2L^2$.

Show that $Q_p = \omega_m C/g$ is essentially equal to

$$Q_p = \frac{\omega_m w}{V^2 g}$$

where V is the effective voltage across the parallel branches, w is the reactive energy stored in L and C at any time, and V^2g is the average dissipated power of the circuit. *Note:* In terms of instantaneous values and letting $v_c = v$, the instantaneous applied voltage,

$$w = \frac{Li_L^2}{2} + \frac{Cv^2}{2} = \text{constant}$$

56. It will be assumed here that the capacitor shown in Fig. 71 has a series resistance of 10 ohms.

(a) What is the equivalent parallel resistance of the capacitor at $\omega_m \approx 1/\sqrt{LC}$?

(b) What is the equivalent parallel resistance of the two branches at $\omega_m = 1/\sqrt{LC}$?

57. Given the circuit arrangement shown in Fig. 73a, where the voltage generator has an internal resistance of 20,000 ohms as indicated:

(a) Transform the circuit to that shown in Fig. 73b.

(b) What is the Q_p of the parallel branches facing the current generator in Fig. 73b at $\omega = 5 \times 10^7$ radians per second?

(c) Compare the result obtained in (b) with the Q of the coil itself at $\omega = 5 \times 10^7$ radians per second. The coil has a resistance of 50 ohms as indicated.

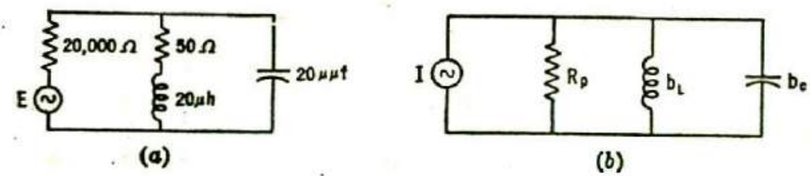

FIG. 73. See Problems 57 and 58.

58. (a) If the generator voltage in Fig. 73a is 200 volts at $\omega = 5 \times 10^7$ radians per second, what is the magnitude of the current of the equivalent current generator employed in Fig. 73b?

(b) What voltage is developed across the parallel branches by the current generator at $\omega = 5 \times 10^7$ radians per second?

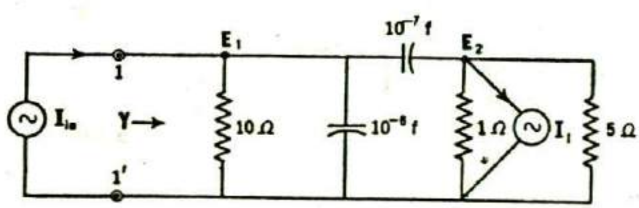

FIG. 74. See Problem 59.

59. Find the admittance Y (looking to the right of terminals 11′) in Fig. 74, and express the result in terms of a resistance R_p in parallel with a condenser C, where R_p and C are expressed numerically in ohms and microfarads, respectively. $\mathbf{I}_1 = 0.1\mathbf{E}_1$.

The I_{in} and I_1 current generators have the polarities indicated, and the operating angular frequency is 10^6 radians per second. *Note:* Current generators are always considered to have infinite internal impedance or zero internal admittance.

60. The parameters in Fig. 75 are:

$$Z_1 = R_1 + jX_{L1} = 10 + j30 \text{ ohms}$$

$$Z_2 = R_2 + jX_{L2} = 5 + j10 \text{ ohms}$$

$$Z_3 = R_3 - jX_{C2} = 4 - j16 \text{ ohms}$$

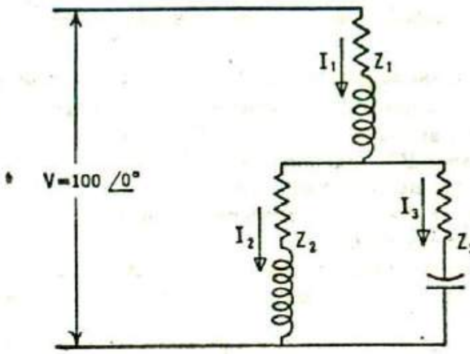

FIG. 75. See Problems 60, 61, and 70.

(*a*) Find I_1, I_2, I_3, V_1, and V_{23} in complex polar form with respect to applied voltage (100° /0° volts) as a reference.

(*b*) Draw a complete phasor diagram of the above voltages and currents.

(*c*) Find the watts and vars input to the entire circuit.

61. Find the power dissipated in each branch of Fig. 75 for the parameters given in Problem 60.

62. Find the pure reactance or reactances X in Fig. 76 which will make the overall power factor 0.707.

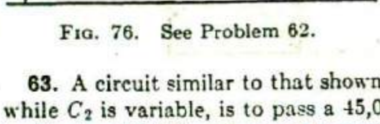

FIG. 76. See Problem 62.

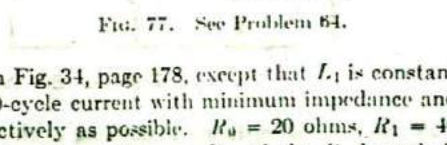

FIG. 77. See Problem 64.

63. A circuit similar to that shown in Fig. 34, page 178, except that L_1 is constant while C_2 is variable, is to pass a 45,000-cycle current with minimum impedance and to block a 15,000-cycle current as effectively as possible. $R_0 = 20$ ohms, $R_1 = 40$ ohms, and $L_1 = 0.002$ henry are fixed. The resistance, R_2, of the C_2 branch is assumed to be negligibly small. Either a fixed C_0 or a fixed L_0 (of negligibly small resistance) is to be placed in series with R_0 to accomplish the desired tuning effect.

(*a*) Solve for C_2 which will put the parallel circuit *bc* into parallel resonance at 15,000 cycles.

(*b*) Calculate the equivalent impedance from *b* to *c* at 45,000 cycles with C_2 set at its 15,000-cycle resonant value. Is *bc* predominantly capacitive or inductive at 45,000 cycles?

(c) Must an inductance L_0 or a capacitance C_0 be used to put the branch *ab* into series resonance for 45,000 cycles? Calculate its value.

(d) Assuming that branch *ab* has been put into series resonance at 45,000 cycles, what is the actual impedance from *a* to *b* at 45,000 cycles? at 15,000 cycles?

64. Given the circuit shown in Fig. 77, determine the impedance looking into terminals *ab* at 1592 cycles per second.

65. A generating device has an impedance of $0.5 + j1$ ohms and is connected to a load by a line of $0.25 + j2$ ohms. At what load will maximum power transfer be realized? If the generated voltage is 20 volts, what is the power received by the load when adjusted for maximum power transfer? Find the line loss and the loss in the generating device.

66. (a) If the resistance of the load in Problem 65 is fixed at 0.75 ohm and only inductive reactance is permitted in the load, for what value of load reactance will maximum load power to the load be realized?

(b) What is the maximum load power under these conditions?

67. Work Problem 65 if the receiver impedance is restricted to pure resistance.

68. If a load impedance having a ratio of $X/R = 5$ is used at the end of the line in Problem 65, find the load impedance for maximum power transfer. What is the maximum power the load can receive?

69. Calculate $\mathbf{I}_2$ in Fig. 78 by the superposition theorem if $E_1 = 100\underline{/0^\circ}$ and $E_2 = 50\underline{/60^\circ}$ volts.

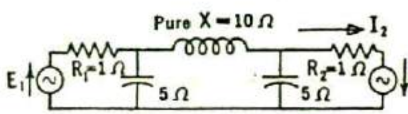

FIG. 78. See Problem 69.

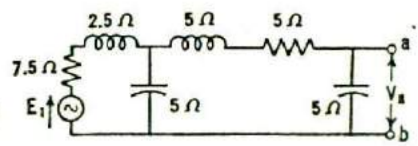

FIG. 79. See Problem 71.

70. The voltage $\mathbf{V} = 100\underline{/0^\circ}$ volts is removed from branch 1 in Fig. 75 and inserted in branch 3. If the upper terminal of $\mathbf{Z}_1$ is connected to the lower common terminal of $\mathbf{Z}_2$ and $\mathbf{Z}_3$, calculate the current $\mathbf{I}_1$. How does this compare with $\mathbf{I}_3$ as calculated in Problem 60? By what theorem could this conclusion be reached?

71. Calculate $\mathbf{V}_R$ in Fig. 79 if $\mathbf{E}_1 = 200\underline{/0^\circ}$ volts. Then use Thévenin's theorem to calculate the current in an impedance $\mathbf{Z}_{ab} = 1.46 + j6.78$ ohms if it is connected to the terminals *ab*.

72. Given the circuit shown in Fig. 80.

(a) Using the superposition theorem, determine the current through the resistor marked *A*.

(b) Using Thévenin's theorem, determine the current through an impedance $Z_{ab} (= 3 + j3$ ohms) that is presumed to be placed across terminals *ab*.

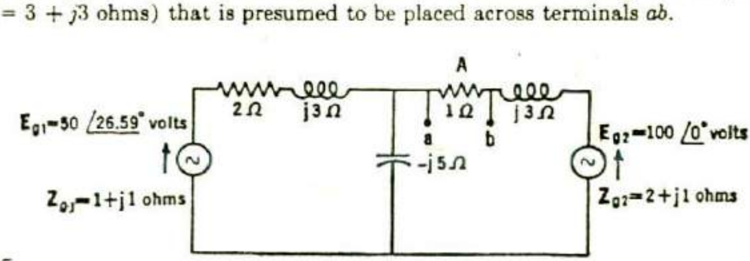

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FIG. 80. See Problem 72.

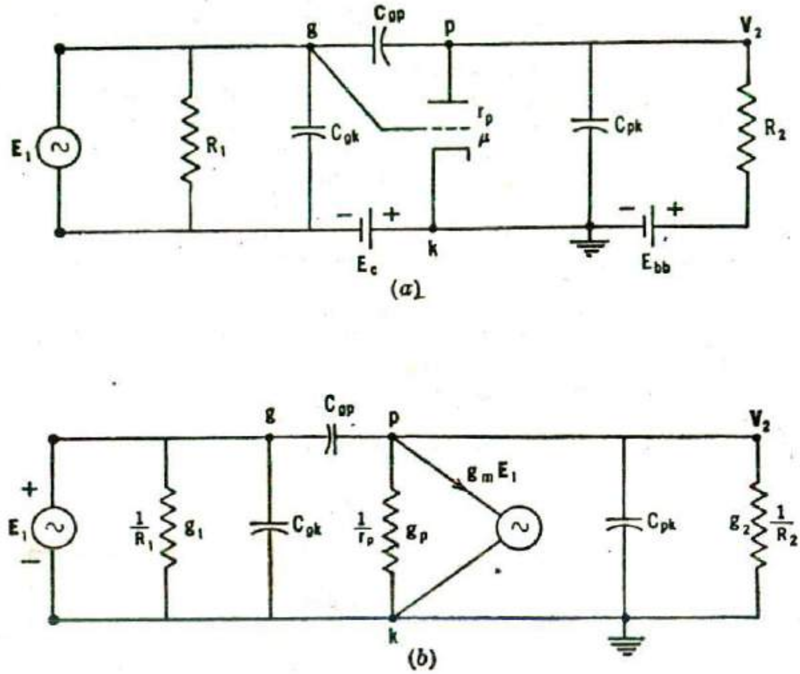

Fig. 81. See Problems 73 and 74.

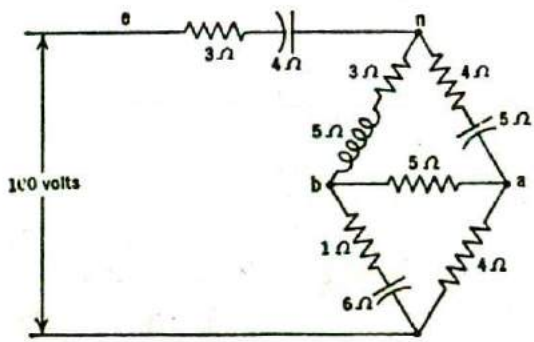

Fig. 82. See Problems 75 and 77.

73. In Fig. 81b: $R_1 = 10^5$ ohms, $R_2 = 5 \times 10^4$ ohms, r_p(of the tube) $= 10^4$ ohms; $C_{gk} = C_{pk} = 40\ \mu\mu\text{f}$, $C_{gp} = 5\ \mu\mu\text{f}$; μ(of the tube) $= 20$; g_m(of the tube) $= \mu/r_p = 2 \times 10^{-3}$ mho.

Find the voltage, V_2, relative to ground if $E_1 = 1\underline{/0°}$ volt. The operating angular frequency is 10^6 radians per second.

Note: In Fig. 81b: $g_1 = 1/R_1$, $g_p = 1/r_p$, $g_2 = 1/R_2$; $j\omega C_{gk} = j\omega C_{pk} = j4 \times 10^{-5}$ mho, and $j\omega C_{gp} = j0.5 \times 10^{-5}$ mho, which is a hint that the problem should probably be solved on the nodal basis, employing E_1 as a known voltage.

74. Find the admittance **Y** (looking to the right of the $\mathbf{E}_1$ generator terminals in Fig. 81b), and express the result in terms of a resistance R_p in parallel with a capacitor C where R_p and C are expressed numerically in ohms and microfarads, respectively.

The parameters and the operating angular frequency are given in Problem 73, and if this problem has been worked $\mathbf{V}_2$ will be a known voltage of $15.6\underline{/159.32°}$ volts.

75. Reduce the impedances shown in Fig. 82 to a single equivalent series impedance. Find the current in branch *ab*.

76. Derive the expressions shown in equations (99), (100), and (101), page 210.

77. Find the equivalent delta system of impedances which will replace the wye *an*, *bn*, *cn*, in Fig. 82.

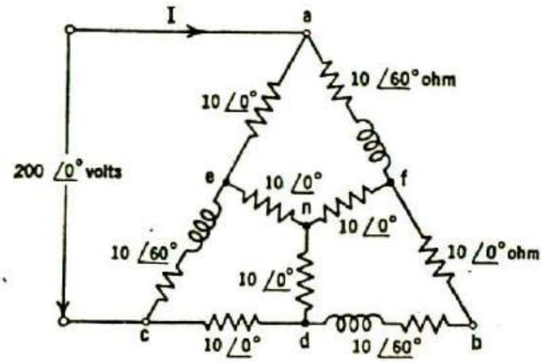

Fig. 83. See Problem 78.

78. Find the voltages $\mathbf{V}_{de}$, $\mathbf{V}_{ef}$, and $\mathbf{V}_{fd}$ in Fig. 83. What is the phase displacement between these voltages?

79. What relationship between the Z's of Fig. 84 will make $I_3 = 0$ regardless of the magnitude of E_{in}? *Hint:* A simple method of solution is to transform the

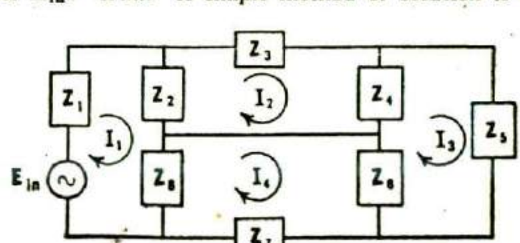

Fig. 84. See Problem 79.

Z_2-Z_3-Z_4 and Z_6-Z_7-Z_8 deltas to equivalent wyes and make the Z_{24} and Z_{68} legs of the latter the negatives of each other to produce a short circuit across the load.

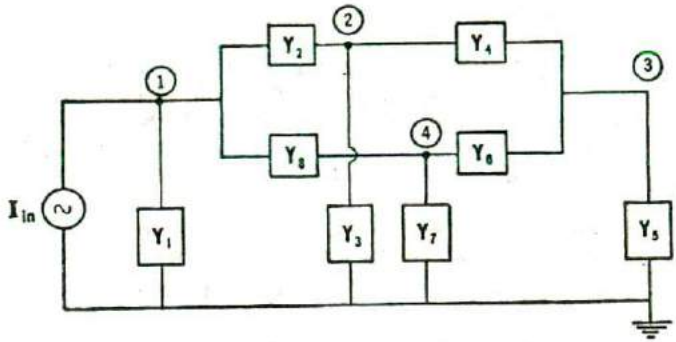

FIG. 85. See Problem 80.

80. What relationship between the Y's of Fig. 85 will make the voltage V_3 (relative to ground) equal to zero regardless of the magnitude of I_{in}?