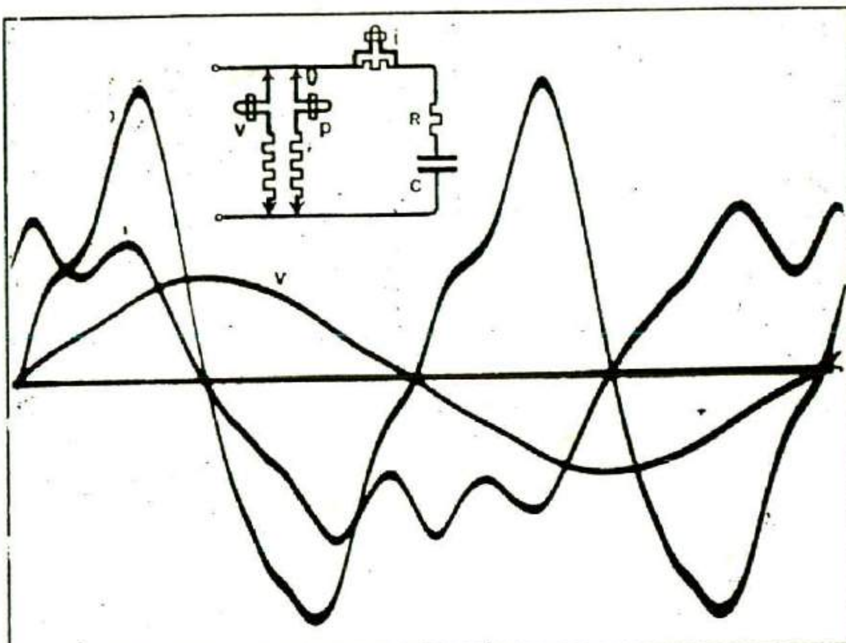
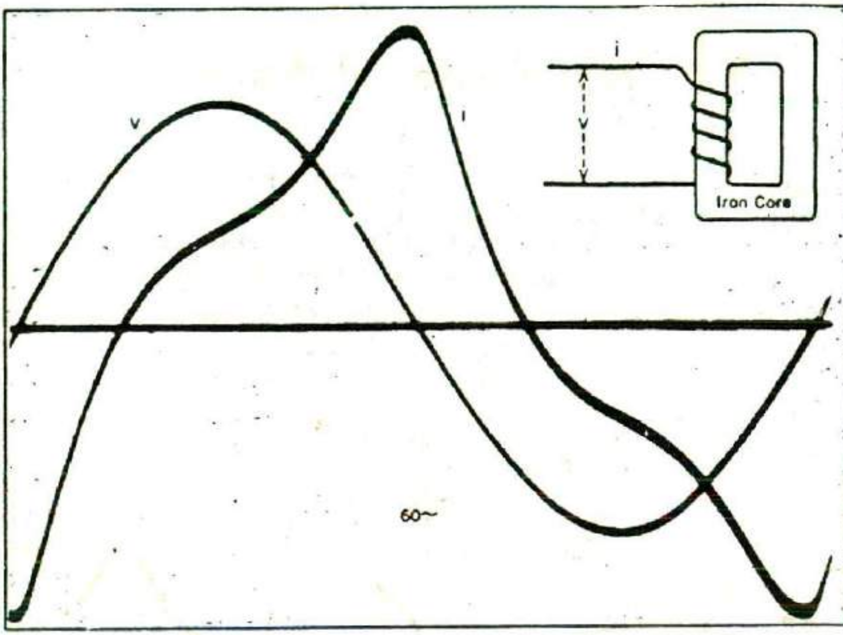
chapter **VI** # Non-Sinusoidal Waves

Complex Waves. The circuit theory that has been presented in the foregoing chapters has been based upon sine-wave variations of voltage and current, and only sine waves have been considered in the calculations. In many branches of electrical engineering non-sinusoidal waves are as common as sinusoidal waves, and in all branches non-sinusoidal

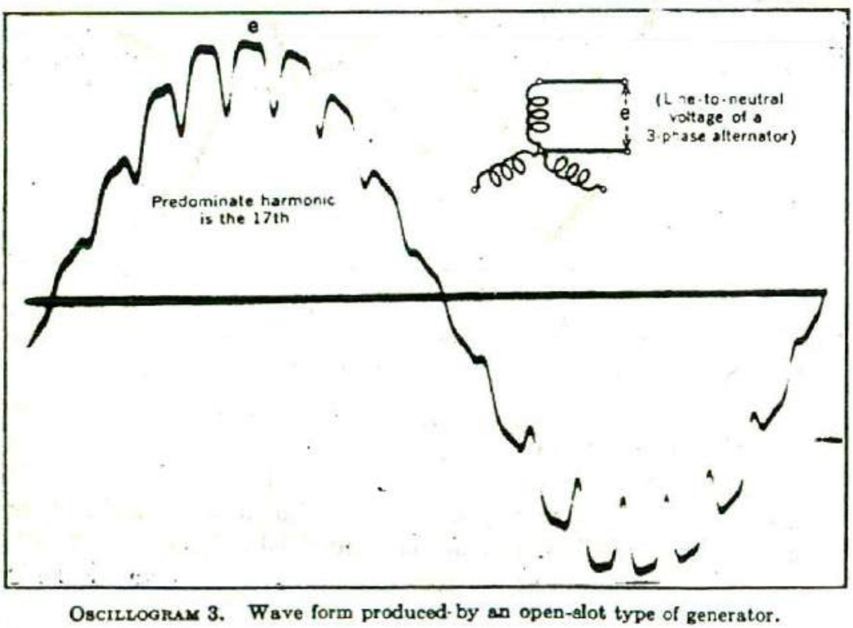


OSCILLOGRAM 1. v-wave form of voltage generated by a particular alternator. i-wave form of current which flows through a capacitive circuit element. Note the relatively larger harmonics in the current wave. p-wave form of instantaneous power. $E = 120$ volts (eff.), $I = 3.9$ amperes (eff.), $P_{av} = 20$ watts, $f = 60$ cycles.

waves must occasionally be given attention. Examples of non-sinusoidal waves are shown in Oscillograms 1, 2, and 3. Even though the voltage wave in Oscillogram 1 is nearly sinusoidal, the current through the capacitive circuit is greatly distorted. Also in Oscillogram 2 the current is non-sinusoidal even though the impressed voltage is practically



OSCILLOGRAM 2. Distorted current wave, i, results when a sine wave of voltage, v, is impressed on a particular coil with an iron core.



OSCILLOGRAM 3. Wave form produced by an open-slot type of generator.

a sinusoid. Oscillogram 3 shows the effect on the voltage wave form of an alternator due to open slots. The predominant harmonic in this case can easily be determined by the methods discussed in this chapter. The method of making circuit calculations when non-sinusoidal wave forms are encountered will also be given.

Most non-sinusoidal waves found in electrical engineering can be expressed in terms of sine-wave components of different frequencies. Under these conditions each sine component may be handled according to the laws governing the calculations of sine waves. The results of all component analyses are combined according to certain laws to form the composite or final analysis. There are, however, certain limitations to representing non-sinusoidal waves in terms of sine components.

Any periodic wave which is single-valued and continuous except for a finite number of finite discontinuities, and which does not have an infinite number of maxima or minima in the neighborhood of any point, may be represented by the sum of a number of sine waves of different frequencies. As an equation, the above theorem takes the following form and is known as a Fourier series:

$$y = f(x) = A_0 + A_1 \sin x + B_1 \cos x + A_2 \sin 2x + B_2 \cos 2x + A_3 \sin 3x + B_3 \cos 3x + \cdots + A_n \sin nx + B_n \cos nx \qquad (1)$$

Except in special cases an infinite number of components are theoretically required. Practically, however, only a few terms are necessary in most instances because of the relatively small effect of the terms of higher frequency. Since the wave which is represented by equation (1) is made up of a number of sine waves of different frequencies, it is called a complex wave. It is apparent that each component of this wave is sinusoidal and that each component in itself may be handled by the methods previously outlined for calculating sine waves. The facility with which sinusoidal components of a complex wave may be manipulated is sufficient justification for expressing a non-sinusoidal wave in such terms as equation (1) even though the equation of the wave may be known in terms of some other function of x.

Wave Analysis. Usually, a photographic record of the wave will be obtained through oscillographic analysis or other means. The determination of the Fourier equation which specifies a particular wave is called wave analysis. Wave analysis consists simply of determining the coefficients A_0, A_1, B_1, etc., of equation (1). These coefficients are determined by some operation on equation (1) that will eliminate all terms except the desired quantity. Then the desired coefficient may be evaluated. Thus, to determine A_0, it is necessary simply to multiply the equation by dx and to integrate between 0 and 2π, as shown below.

$$\int_0^{2\pi} y\,dx = \int_0^{2\pi} A_0\,dx + \int_0^{2\pi} A_1 \sin x\,dx + \int_0^{2\pi} B_1 \cos x\,dx + \int_0^{2\pi} A_2 \sin 2x\,dx + \int_0^{2\pi} B_2 \cos 2x\,dx + \int_0^{2\pi} A_3 \sin 3x\,dx + \int_0^{2\pi} B_3 \cos 3x\,dx + \cdots + \int_0^{2\pi} A_n \sin nx\,dx + \int_0^{2\pi} B_n \cos nx\,dx \quad (2)$$

or

$$\int_0^{2\pi} y\,dx = A_0 \int_0^{2\pi} dx = 2\pi A_0$$

and

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} y\,dx \quad (3)$$

To find A_1, equation (1) may be multiplied by $\sin x\,dx$ and integrated from 0 to 2π. Thus

$$\int_0^{2\pi} y \sin x\,dx = \int_0^{2\pi} A_0 \sin x\,dx + \int_0^{2\pi} A_1 \sin^2 x\,dx + \int_0^{2\pi} B_1 \cos x \sin x\,dx + \int_0^{2\pi} A_2 \sin 2x \sin x\,dx + \int_0^{2\pi} B_2 \cos 2x \sin x\,dx + \int_0^{2\pi} A_3 \sin 3x \sin x\,dx + \int_0^{2\pi} B_3 \cos 3x \sin x\,dx + \cdots + \int_0^{2\pi} A_n \sin nx \sin x\,dx + \int_0^{2\pi} B_n \cos nx \sin x\,dx \quad (4)$$

It is obvious that $\int_0^{2\pi} A_0 \sin x\,dx$ is zero since it represents the area under a sine wave for a complete cycle. There are four other types of terms. They are

(a) $\int_0^{2\pi} \sin^2 x\,dx = \int_0^{2\pi} (\frac{1}{2} - \frac{1}{2}\cos 2x)\,dx = \frac{2\pi}{2} = \pi,$

(b) $\int_0^{2\pi} \sin 2x \sin x\,dx$, which is of the general type:

$\int_0^{2\pi} \sin mx \sin nx\,dx = 0$, when m and n are different integers,[1]

(c) $\int_0^{2\pi} \cos mx \sin nx\,dx = 0$, when m and n are different integers,[2] and

(d) $\int_0^{2\pi} \cos x \sin x\,dx = 0.$

[1] This may be readily proved by substituting for $\sin mx \sin nx$ its equivalent $\frac{1}{2}[\cos(mx - nx) - \cos(mx + nx)]$.

[2] This may be readily proved by substituting for $\cos mx \sin nx$ its equivalent $\frac{1}{2}[\sin(mx + nx) - \sin(mx - nx)]$.

The student should prove statements *a*, *b*, *c*, and *d* by carrying out the operations indicated. If the above facts are used, equation (4) reduces to

$$\int_0^{2\pi} y \sin x \, dx = A_1 \pi$$

or

$$A_1 = \frac{1}{\pi} \int_0^{2\pi} y \sin x \, dx \tag{5}$$

To evaluate the coefficient of the cosine term B_1, equation (1) is multiplied by $\cos x \, dx$ and integrated from 0 to 2π. Thus

$$\int_0^{2\pi} y \cos x \, dx = \int_0^{2\pi} A_0 \cos x \, dx + \int_0^{2\pi} A_1 \sin x \cos x \, dx$$
$$+ \int_0^{2\pi} B_1 \cos^2 x \, dx + \int_0^{2\pi} A_2 \sin 2x \cos x \, dx + \int_0^{2\pi} B_2 \cos 2x \cos x \, dx$$
$$+ \int_0^{2\pi} A_3 \sin 3x \cos x \, dx + \int_0^{2\pi} B_3 \cos 3x \cos x \, dx + \cdots$$
$$+ \int_0^{2\pi} A_n \sin nx \cos x \, dx + \int_0^{2\pi} B_n \cos nx \cos x \, dx \tag{6}$$

If the relations stated above in *a*, *b*, *c*, and *d* are used, equation (6) becomes

$$\int_0^{2\pi} y \cos x \, dx = B_1 \int_0^{2\pi} \cos^2 x \, dx = B_1 \pi$$

or

$$B_1 = \frac{1}{\pi} \int_0^{2\pi} y \cos x \, dx \tag{7}$$

Similarly,

$$A_2 = \frac{1}{\pi} \int_0^{2\pi} y \sin 2x \, dx \tag{8}$$

$$B_2 = \frac{1}{\pi} \int_0^{2\pi} y \cos 2x \, dx \tag{9}$$

$$A_3 = \frac{1}{\pi} \int_0^{2\pi} y \sin 3x \, dx \tag{10}$$

$$B_3 = \frac{1}{\pi} \int_0^{2\pi} y \cos 3x \, dx \tag{11}$$

$$A_n = \frac{1}{\pi} \int_0^{2\pi} y \sin nx \, dx \tag{12}$$

$$B_n = \frac{1}{\pi} \int_0^{2\pi} y \cos nx \, dx \tag{13}$$

Various analytical and graphical methods may be employed to evaluate the coefficients of equations (3), (12), and (13). Two general methods are outlined below.

Analytical Method. If the equation of y in terms of x is known in some mathematical form, the wave may be analyzed analytically. This method is the least laborious but it cannot be employed if the function of x is not known analytically. The function of x employed need not throughout its entire range represent the particular wave to be analyzed. It is necessary to have the function of x only over the interval of periodicity, namely, 2π. Not even a single function of x is necessary. Several different ones may be used and the complete integral from 0 to 2π may be obtained from a sum of the integrals of the several functions, each taken over the interval in which it follows the curve to be analyzed.

The details connected with writing a Fourier series to represent a specified wave form are illustrated by the following examples.

Example 1. Let it be required to write the Fourier series which will represent the sawtooth wave form shown in Fig. 1. It will be observed that this wave form is

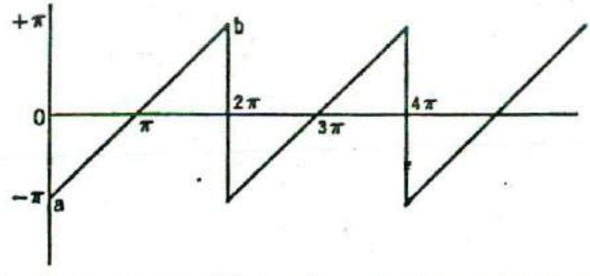


FIG. 1. A type of wave which is easily analyzed by analytical methods.

simply a straight-line variation, ranging from $y = -\pi$ to $y = +\pi$ over one complete cycle. This straight-line variation may be expressed analytically (between $x = 0$ and $x = 2\pi$) as:

$$y = f(x) = x - \pi$$

It should be noted that the above analytical expression for y in terms of x gives no indication of the various harmonics which are present in the wave, whereas a Fourier-series representation of the wave will yield this information.

From equation (3):

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} (x - \pi)\, dx = \frac{1}{2\pi}\left[\frac{x^2}{2} - \pi x\right]_0^{2\pi} = \frac{1}{2\pi}\left[\frac{4\pi^2}{2} - 2\pi^2\right] = 0$$

The fact that A_0 is zero could have been determined by inspection of Fig. 1 since it is obvious from the figure that the negative half of the wave is equal in area to the positive half.

From equation (12):

$$A_n = \frac{1}{\pi}\int_0^{2\pi}(x-\pi)\sin nx\,dx = \frac{1}{\pi}\left[\int_0^{2\pi} x\sin nx\,dx - \int_0^{2\pi}\pi\sin nx\,dx\right]$$

$$\int_0^{2\pi} x\sin nx\,dx = \left[-\frac{x\cos nx}{n} + \frac{1}{n^2}\sin nx\right]_0^{2\pi}$$

as may be proved by differentiation of the right member and $\int_0^{2\pi}\pi\sin nx\,dx = 0$ for all integral values of n. Therefore:

$$A_n = \frac{1}{\pi}\left[-\frac{x\cos nx}{n} + \frac{1}{n^2}\sin nx\right]_0^{2\pi} = -\frac{2}{n}$$

whence

$$A_1 = -\tfrac{2}{1};\quad A_2 = -\tfrac{2}{2};\quad A_3 = -\tfrac{2}{3};\quad A_4 = -\tfrac{2}{4};\quad \text{etc.}$$

From equation (13):

$$B_n = \frac{1}{\pi}\int_0^{2\pi}(x-\pi)\cos nx\,dx = \frac{1}{\pi}\left[\int_0^{2\pi} x\cos nx\,dx - \int_0^{2\pi}\pi\cos nx\,dx\right]$$

$$\int_0^{2\pi} x\cos nx\,dx = \left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{2\pi}$$

as may be proved by differentiation of the right member and $\int_0^{2\pi}\pi\cos nx\,dx = 0$ for all integral values of n. Therefore:

$$B_n = \frac{1}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{2\pi} = 0 \quad \text{(for all integral values of } n\text{)}$$

Hence all the coefficients B_1, B_2, B_3, etc., in equation (1) are 0 and the Fourier equation of the wave shown in Fig. 1 becomes:

$$y = -2\left(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x + \cdots + \frac{1}{n}\sin nx\right)$$

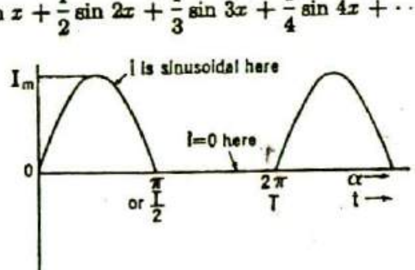


FIG. 2. Half-wave rectification of a sine wave. See example 2.

Example 2. Let it be required to write the first four terms of the Fourier series which will represent the wave form shown in Fig. 2. From Fig. 2, it is plain that i may be expressed analytically between the limits of 0 and 2π as two separate functions. That is:

$$i = I_m \sin\alpha \qquad [\text{between } \alpha \text{ (or } \omega t) = 0 \text{ and } \alpha \text{ (or } \omega t) = \pi]$$

and

$$i = 0 \qquad [\text{between } \alpha \text{ (or } \omega t) = \pi \text{ and } \alpha \text{ (or } \omega t) = 2\pi]$$

From equation (3):

$$A_0 = \frac{1}{2\pi}\left[\int_0^\pi I_m \sin\alpha\, d\alpha + \int_\pi^{2\pi} 0\, d\alpha\right] = \frac{I_m}{2\pi}\Big[-\cos\alpha\Big]_0^\pi$$

$$= \frac{I_m}{\pi} = 0.318 I_m$$

From equation (5):

$$A_1 = \frac{1}{\pi}\left[\int_0^\pi (I_m \sin\alpha)\sin\alpha\, d\alpha + \int_\pi^{2\pi} (0)\sin\alpha\, d\alpha\right]$$

$$= \frac{I_m}{\pi}\left[\int_0^\pi \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\alpha\right) d\alpha\right]$$

$$= \frac{I_m}{\pi}\left[\frac{\alpha}{2} - \frac{1}{4}\sin 2\alpha\right]_0^\pi = \frac{I_m}{\pi}\left[\frac{\pi}{2}\right] = 0.500 I_m$$

From equation (12) it follows that A_2, A_3, A_4, etc., are all zero because:

$$A_n = \frac{1}{\pi}\left[\int_0^\pi (I_m \sin\alpha)\sin n\alpha\, d\alpha\right] = 0 \text{ (for } n \neq 0 \text{ and } n \neq 1)$$

The above evaluation of A_n is evident if $(\sin\alpha \sin n\alpha)$ is replaced by its equivalent $\frac{1}{2}[\cos(n-1)\alpha - \cos(n+1)\alpha]$. Thus A_2, A_3, A_4, etc., are zero because:

$$A_n = \frac{1}{\pi}\int_0^\pi \tfrac{1}{2}[\cos(n-1)\alpha - \cos(n+1)\alpha]\, d\alpha$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n-1)\alpha}{(n-1)} - \frac{\sin(n+1)\alpha}{(n+1)}\right]_0^\pi = 0 \qquad \begin{cases}\text{for } n \neq 0\\ \text{and } n \neq 1\end{cases}$$

From equation (7):

$$B_1 = \frac{1}{\pi}\left[\int_0^\pi (I_m \sin\alpha)\cos\alpha\, d\alpha + \int_\pi^{2\pi} (0)\cos\alpha\, d\alpha\right]$$

$$= \frac{I_m}{\pi}\left[\int_0^\pi \frac{\sin 2\alpha}{2}\, d\alpha\right] = \frac{I_m}{\pi}\left[-\frac{\cos 2\alpha}{4}\right]_0^\pi = 0$$

From equation (13):

$$B_n = \frac{1}{\pi}\left[\int_0^\pi (I_m \sin\alpha)\cos n\alpha\, d\alpha\right]$$

$$= \frac{I_m}{\pi}\left[\int_0^\pi \left(\frac{\sin(\alpha + n\alpha)}{2} + \frac{\sin(\alpha - n\alpha)}{2}\right) d\alpha\right]$$

$$= \frac{I_m}{\pi}\left[-\frac{\cos(1+n)\alpha}{2(1+n)} - \frac{\cos(1-n)\alpha}{2(1-n)}\right]_0^\pi \qquad \begin{cases}\text{for } n \neq 0\\ \text{and } n \neq 1\end{cases}$$

For $n = 2$:

$$B_2 = \frac{I_m}{\pi}\left[-\frac{\cos 3\alpha}{6} - \frac{\cos(-\alpha)}{-2}\right]_0^\pi = \frac{I_m}{\pi}\left[+\frac{1}{6} + \frac{1}{6} - \frac{1}{2} - \frac{1}{2}\right]$$

$$= -\frac{2 I_m}{3\pi} = -0.212 I_m$$

Similarly for $n = 3$,

$$B_3 = 0$$

and for $n = 4$,

$$B_4 = -0.0424 I_m$$

The Fourier series which represents the wave form shown in Fig. 2 is therefore:

$$i = 0.318 I_m + 0.500 I_m \sin \alpha - 0.212 I_m \cos 2\alpha - 0.0424 I_m \cos 4\alpha - \cdots$$

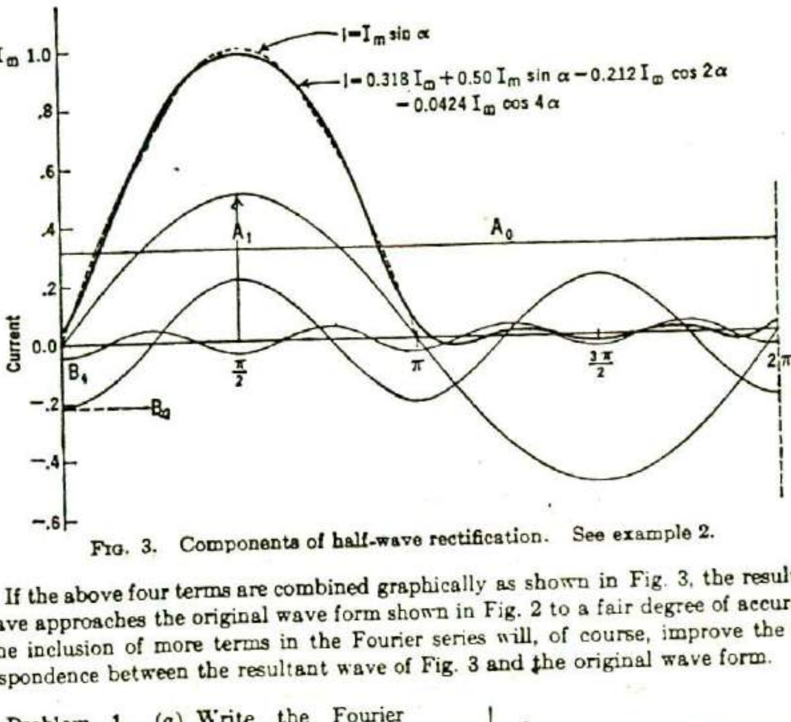


FIG. 3. Components of half-wave rectification. See example 2.

If the above four terms are combined graphically as shown in Fig. 3, the resultant wave approaches the original wave form shown in Fig. 2 to a fair degree of accuracy. The inclusion of more terms in the Fourier series will, of course, improve the correspondence between the resultant wave of Fig. 3 and the original wave form.

Problem 1. (a) Write the Fourier series which represents the wave form shown in Fig. 4 out to and including the A_3 term of the series. *Note:* $e = 100$ between $\alpha = 0$ and $\alpha = \pi$, and $e = 0$ between $\alpha = \pi$ and $\alpha = 2\pi$.

Ans.: $e = 50 + 63.7 \sin \alpha + 21.2 \sin 3\alpha$ volts.

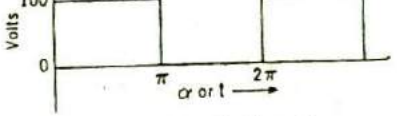


FIG. 4. See Problem 1.

(b) Show by means of a sketch the manner in which the above three components combine to approximate the flat-topped wave shown in Fig. 4.

Fourier Analysis of Symmetrical Triangular and Rectangular Waves. Symmetrical waves of triangular and rectangular shape are shown in Figs. 5 (solid lines) and 6 respectively. Since these wave forms are often used in the analyses of certain basic problems, it is convenient to have the Fourier equations of these waves readily available.

Triangular Wave. To facilitate analyzing, the triangular wave may be considered to be composed of several pieces, namely, the straight lines *oa*, *ac*, and *cd*. If the point slope form of equation for a straight line is applied, the equations of these lines will be found to be:

$$y_{oa} = \frac{2x}{\pi}; \quad y_{ac} = -\frac{2x}{\pi} + 2; \quad y_{cd} = \frac{2x}{\pi} - 4$$

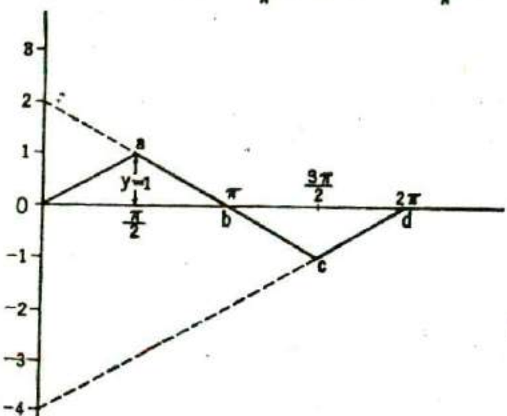


FIG. 5. Symmetrical triangular wave with a maximum value of 1.

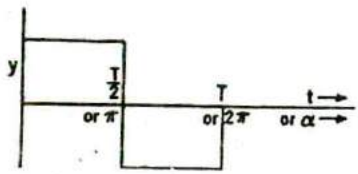


FIG. 6. Symmetrical rectangular wave.

Applying equations (3), (12), and (13) gives:

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} y\,dx = \frac{1}{2\pi}\left\{\int_0^{\pi/2}\frac{2x}{\pi}\,dx + \int_{\pi/2}^{3\pi/2}\left(\frac{-2x}{\pi}+2\right)dx + \int_{3\pi/2}^{2\pi}\left(\frac{2x}{\pi}-4\right)dx\right\}$$

$$A_n = \frac{1}{\pi}\int_0^{2\pi} y \sin nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi/2}\frac{2x}{\pi}\sin nx\,dx + \int_{\pi/2}^{3\pi/2}\left(\frac{-2x}{\pi}+2\right)\sin nx\,dx + \int_{3\pi/2}^{2\pi}\left(\frac{2x}{\pi}-4\right)\sin nx\,dx\right\}$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi} y \cos nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi/2}\frac{2x}{\pi}\cos nx\,dx + \int_{\pi/2}^{3\pi/2}\left(\frac{-2x}{\pi}+2\right)\cos nx\,dx + \int_{3\pi/2}^{2\pi}\left(\frac{2x}{\pi}-4\right)\cos nx\,dx\right.$$

Evaluation of the above for various values of n by ordinary calculus methods gives the equation of the wave in terms of a Fourier series as follows:

$$y = \frac{8}{\pi^2}\left(\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \frac{1}{7^2}\sin 7x + \cdots \text{etc.}\right) \quad (14)$$

It will be shown later how it is possible to determine from inspection, that, in certain classes of waves as typified by the above example, the terms represented by B_n must be zero.

The results of the above analysis may be generalized and the equation of a symmetrical triangular wave written as

$$y = A_1 \sin \omega t - \frac{A_1}{3^2} \sin 3\omega t + \frac{A_1}{5^2} \sin 5\omega t - \frac{A_1}{7^2} \sin 7\omega t + \cdots \text{etc.} \quad (14a)$$

where x of equation (14) has been replaced by ωt and A_1 equals $8/\pi^2$ times the maximum ordinate of the triangular wave. Theoretically, there is an infinite number of terms and the progression continues as the first four terms indicate.

Rectangular Wave. The rectangular wave is much used in the analysis of a-c machinery and has for its Fourier equation:

$$y = A_1 \sin \omega t + \frac{A_1}{3} \sin 3\omega t + \frac{A_1}{5} \sin 5\omega t + \frac{A_1}{7} \sin 7\omega t + \cdots \text{etc.} \quad (15)$$

where $A_1 = \frac{4}{\pi}$ times the height of the rectangle. Again there is an infinite number of terms which may be written as indicated by the first four terms shown. Figure 7 shows a graphical representation of the first three terms and illustrates that a fair approximation to the resultant wave is obtained by the addition of very few terms.

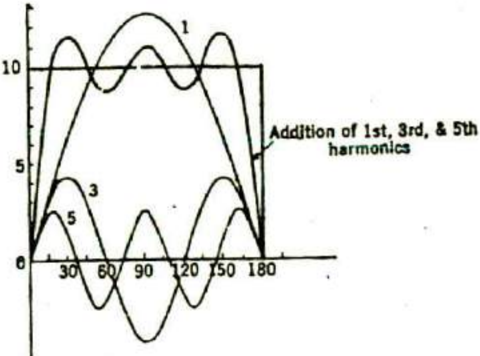


FIG. 7. The addition of only three harmonics gives a fair approximation of the rectangular wave.

Problem 2. Analyze the rectangular wave shown in Fig. 6 by the analytical method to prove the validity of equation (15).

Graphical Method. A second method of evaluating equations (3), (12), and (13) involves the evaluation of the integrals by a step-by-step method. The equation of y in terms of x is usually unknown, and for the majority of waves encountered it would be very cumbersome and laborious to establish equations which would yield pieces of the wave. It is under these conditions that the step-by-step method (sometimes called the graphical method) or its equivalent is employed. The details of this method follow.

Suppose the wave of Fig. 8 is to be analyzed. Equation (3) is simply the average height of the curve over 2π radians. It is found by dividing the area under the curve by the base. Any method of determining the area, such as counting squares or by use of a planimeter, may be employed. If the areas of the positive and negative loops are the same, A_0 is zero. Hence for waves having adjacent loops of the same shape and area with respect to some horizontal axis, the constant A_0 when present simply indicates how much the whole wave has been raised or lowered from symmetry about the axis of abscissas. For graphical analysis, equation (5) may be written

$$A_1 = \frac{1}{\pi}\sum_0^{2\pi} y \sin x \, \Delta x \tag{16}$$

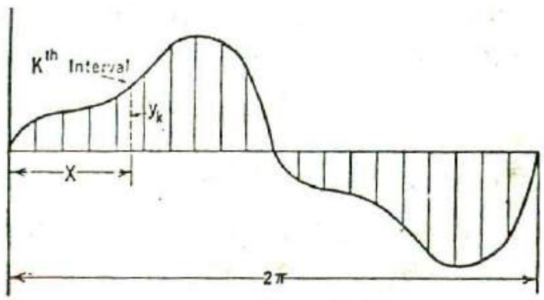


Fig. 8. Preparation of a wave for analysis by the graphical method.

Let 2π radians in Fig. 8 be divided into m equal parts. Then $\Delta x = \frac{2\pi}{m}$ and x, the distance to the midpoint of the kth interval, is $k\frac{2\pi}{m} - \frac{1}{2}\left(\frac{2\pi}{m}\right)$ or $(k - \frac{1}{2})\frac{2\pi}{m}$. Equation (16) now becomes

$$A_1 = \frac{1}{\pi}\sum_0^m \left[y_k \sin (k - \tfrac{1}{2})\frac{2\pi}{m}\right]\frac{2\pi}{m}$$

$$= \frac{2\pi}{m}\,\frac{1}{\pi}\sum_0^m y_k \sin\left[(k - \tfrac{1}{2})\frac{2\pi}{m}\right]$$

$$= \frac{2}{m}\sum_0^m y_k \sin\left[(k - \tfrac{1}{2})\frac{2\pi}{m}\right] \tag{17}$$

Similarly,

$$B_1 = \frac{2}{m}\sum_0^m y_k \cos (k - \tfrac{1}{2})\frac{2\pi}{m} \tag{18}$$

and

$$A_2 = \frac{2}{m}\sum_0^m y_k \sin 2\,(k - \tfrac{1}{2})\frac{2\pi}{m} \tag{19}$$

The first form of equation (17) shows that A_1 is $1/\pi$ times the area under a *new curve*, which would be obtained by plotting corresponding ordinates of the original curve multiplied by the sine of the angle to the ordinate in question. For A_n the ordinates of the new curve would be obtained by multiplying selected ordinates of the original curve by the sine of n times the fundamental angular distance to the respective ordinates. An analogous procedure is employed for cosine terms. Looked at in another way, equation (17) indicates that A_1 is twice the average ordinate of the *new* curve, which would be obtained by plotting corresponding ordinates of the original curve multiplied by the sine of the angle to the ordinate in question. Multiplying and dividing equation (16) or (17) by 2 makes this statement evident. Thus

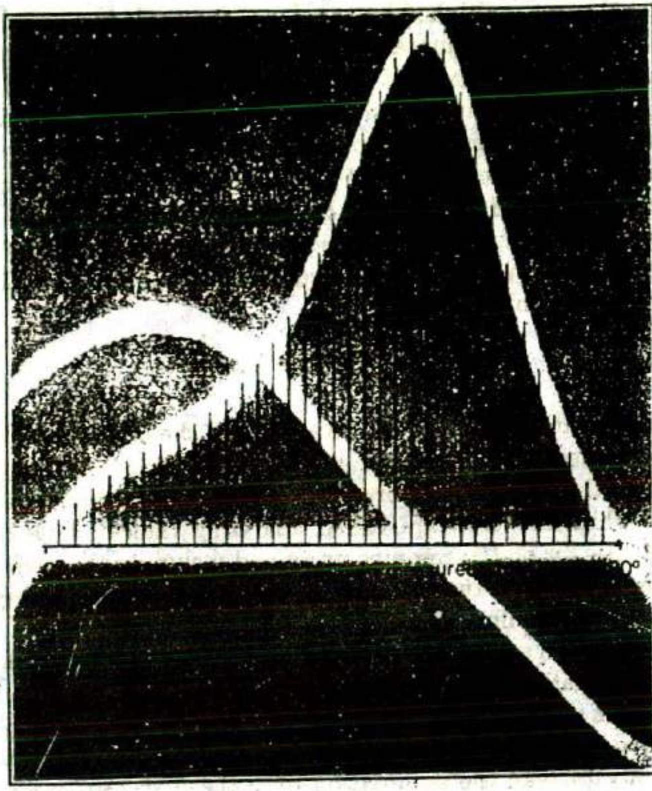
$$A_1 = 2\left[\frac{1}{2\pi}\sum_0^{2\pi} y \sin x \,\Delta x\right]$$

Similar interpretations may be drawn regarding the other coefficients of the Fourier series. The summations are best carried out in tabular form, and for this purpose a more or less standardized system is used. The tables which are used are called analyzing tables. One form of analyzing tables for odd harmonics up to and including the seventh are shown in heavy type on the following pages. (The light type refers to specific values for an illustrative example.)

It will be shown in a subsequent article that waves having symmetrical positive and negative loops cannot contain even harmonics. Under these conditions it is unnecessary to evaluate A_2, B_2, A_4, B_4, etc. Also, when the wave being analyzed consists of odd harmonics only, it is necessary only to take the summation over the first 180°. Since the summation over the second 180° would be the same as that over the first 180°, the total summation over 360° can be obtained by multiplying the summation over 180° by 2. If m is taken as the number of intervals in 360°, the summation over 180° may be multiplied by $4/m$ instead of multiplying the summation over 360° by $2/m$ as shown in equation (17). Whereas equations (17), (18), and (19) indicate that the midordinate of the interval selected should be used, it is customary to use the ordinate and the angle corresponding to those given in the tables. When the intervals are as small as 5°, the difference between the two schemes is negligible.

Example 3. Given the experimentally determined wave form shown in Oscillogram 4. Find the Fourier equation, employing analyzing tables similar to those given on pages 237–240.

Solution. Ordinates at every 5° are constructed as shown in Oscillogram 4. The magnitude of each is scaled and set in the column for ordinates opposite the corresponding angle in the column for angles. The product of the ordinates and the corresponding sines and cosines of n times the angles are obtained and tabulated as shown in the analyzing tables on pages 237–240.



OSCILLOGRAM 4. See example 3.

For the particular wave which is being analyzed:

$A_1 = 82.45$ units	$A_5 = -5.38$ units
$B_1 = -22.11$ units	$B_5 = -3.65$ units
$A_3 = -0.92$ unit	$A_7 = 2.01$ units
$B_3 = 26.2$ units	$B_7 = -1.29$ units

The Fourier equation of the wave is, therefore,

$$i = 82.45 \sin \omega t - 22.11 \cos \omega t - 0.92 \sin 3\omega t + 26.2 \cos 3\omega t - 5.38 \sin 5\omega t - 3.65 \cos 5\omega t + 2.01 \sin 7\omega t - 1.29 \cos 7\omega t$$

The fundamental frequency in this particular case is 60 cycles per second. Therefore ω is equal to 377 radians per second.

The actual number of terms in the Fourier equation in any particular case can usually be reduced because it is always possible to combine sine and cosine waves of the same frequencies. For example, consider the general wave

$$y = A_1 \sin \omega t + B_1 \cos \omega t + A_2 \sin 2\omega t + B_2 \cos 2\omega t + A_3 \sin 3\omega t + B_3 \cos 3\omega t$$

FUNDAMENTAL

1	2	3	4	5	6	7	8	9
sin x	Products (y sin x)		Ordinate No.	Angle x to ordinate	Meas. ordinate (y)	cos x	Products (y cos x)	
	+	−					+	−
.0872	0.5		1	5°	5.9	.9962	5.9	
.1736	1.7		2	10°	10.0	.9848	9.8	
.2588	3.5		3	15°	13.4	.9659	13.0	
.3420	5.6		4	20°	16.4	.9397	15.4	
.4226	8.2		5	25°	19.4	.9063	17.6	
.5000	10.9		6	30°	21.8	.8660	18.9	
.5736	13.5		7	35°	23.6	.8192	19.3	
.6428	16.6		8	40°	25.9	.7660	19.9	
.7071	19.9		9	45°	28.1	.7071	19.9	
.7660	23.5		10	50°	30.7	.6428	19.7	
.8192	27.8		11	55°	33.9	.5736	19.4	
.8660	32.4		12	60°	37.4	.5000	18.7	
.9063	38.1		13	65°	42.0	.4226	17.7	
.9397	43.9		14	70°	46.7	.3420	16.0	
.9659	51.0		15	75°	52.8	.2588	13.7	
.9848	59.1		16	80°	60.0	.1736	10.4	
.9962	67.5		17	85°	67.7	.0872	5.9	
1.0000	76.4		18	90°	76.4	.0000	0.0	
.9962	86.2		19	95°	86.5	−.0872		7.5
.9848	94.1		20	100°	95.5	−.1736		16.6
.9659	101.5		21	105°	105.1	−.2588		27.2
.9397	106.0		22	110°	112.8	−.3420		38.6
.9063	106.4		23	115°	117.4	−.4226		49.6
.8660	102.7		24	120°	118.5	−.5000		59.3
.8192	93.5		25	125°	114.2	−.5736		65.5
.7660	80.4		26	130°	104.9	−.6428		67.4
.7071	64.6		27	135°	91.4	−.7071		64.6
.6428	50.4		28	140°	78.3	−.7660		60.0
.5736	37.2		29	145°	65.0	−.8192		53.2
.5000	25.6		30	150°	51.1	−.8660		44.3
.4226	16.9		31	155°	40.0	−.9063		36.3
.3420	10.1		32	160°	29.4	−.9397		27.6
.2588	5.5		33	165°	21.3	−.9659		20.6
.1736	2.4		34	170°	14.0	−.9848		13.8
.0872	0.6		35	175°	7.1	−.9962		7.1
.0000	0.0		36	180°	0.0	−1.0000		0.0
Sum of products	1484.2	0					261.2	659.2
	1484.2						−398.0	

$$A_1 = \frac{1484}{36} \times 2 = 82.45$$

$$B_1 = \frac{-398.0}{36} \times 2 = -22.11$$

THIRD HARMONIC

1	2	3	4	5	6	7	8	9
sin 3x	Products (y sin $3x$)		Ordinate No.	Angle x to ordinate	Meas. ordinate (y)	cos 3x	Products (y cos $3x$)	
	+	−					+	−
.2588	1.5		1	5°	5.9	.9659	5.7	
.5000	5.0		2	10°	10.0	.8660	8.7	
.7071	9.5		3	15°	13.4	.7071	9.5	
.8660	14.2		4	20°	16.4	.5000	8.2	
.9659	18.8		5	25°	19.4	.2588	5.0	
1.0000	21.8		6	30°	21.8	.0000	0.0	
.9659	22.8		7	35°	23.6	−.2588		6.1
.8660	22.4		8	40°	25.9	−.5000		12.9
.7071	19.9		9	45°	28.1	−.7071		19.9
.5000	15.4		10	50°	30.7	−.8660		26.6
.2588	8.8		11	55°	33.9	−.9659		32.8
.0000	0.0		12	60°	37.4	−1.0000		37.4
−.2588		10.9	13	65°	42.0	−.9659		40.6
−.5000		23.4	14	70°	46.7	−.8660		40.5
−.7071		37.4	15	75°	52.8	−.7071		37.4
−.8660		52.0	16	80°	60.0	−.5000		30.0
−.9659		65.5	17	85°	67.7	−.2588		17.5
−1.0000		76.4	18	90°	76.4	−.0000		0.0
−.9659		83.7	19	95°	86.5	.2588	22.4	
−.8660		82.8	20	100°	95.5	.5000	47.8	
−.7071		74.4	21	105°	105.1	.7071	74.4	
−.5000		56.4	22	110°	112.8	.8660	97.7	
−.2588		30.4	23	115°	117.4	.9659	113.6	
.0000		0.0	24	120°	118.5	1.0000	118.5	
.2588	29.6		25	125°	114.2	.9659	110.4	
.5000	52.5		26	130°	104.9	.8660	90.9	
.7071	64.6		27	135°	91.4	.7071	64.6	
.8660	67.9		28	140°	78.3	.5000	39.2	
.9659	62.8		29	145°	65.0	.2588	16.8	
1.0000	51.1		30	150°	51.1	.0000	0.0	
.9659	38.7		31	155°	40.0	−.2588		10.4
.8660	25.5		32	160°	29.4	−.5000		14.7
.7071	15.1		33	165°	21.3	−.7071		15.1
.5000	7.0		34	170°	14.0	−.8660		12.1
.2588	1.8		35	175°	7.1	−.9659		6.9
.0000	0.0		36	180°	0.0	−1.0000		0.0
Sum of products	576.7	593.3					833.4	360.9
	−16.6						+472.5	

$$A_3 = \frac{2(-16.6)}{36} = -0.92$$

$$B_3 = \frac{2(472.5)}{36} = 26.2$$

FIFTH HARMONIC

1	2	3	4	5	6	7	8	9
sin $5x$	Products ($y \sin 5x$)		Ordinate No.	Angle x to ordinate	Meas. ordinate (y)	cos $5x$	Products ($y \cos 5x$)	
	+	−					+	−
.4226	2.5		1	5°	5.9	.9063	5.4	
.7660	7.7		2	10°	10.0	.6428	6.4	
.9659	13.0		3	15°	13.4	.2588	3.5	
.9848	16.2		4	20°	16.4	−.1736		2.8
.8192	15.9		5	25°	19.4	−.5736		11.1
.5000	10.9		6	30°	21.8	−.8660		18.9
.0872	2.1		7	35°	23.6	−.9962		23.5
−.3420		8.9	8	40°	25.9	−.9397		24.4
−.7071		19.9	9	45°	28.1	−.7071		19.9
−.9397		28.8	10	50°	30.7	−.3420		10.5
−.9962		33.8	11	55°	33.9	.0872	3.0	
−.8660		32.4	12	60°	37.4	.5000	18.7	
−.5736		24.1	13	65°	42.0	.8192	34.4	
−.1736		8.1	14	70°	46.7	.9848	46.0	
.2588	13.7		15	75°	52.8	.9659	51.0	
.6428	38.6		16	80°	60.0	.7660	46.0	
.9063	61.4		17	85°	67.7	.4226	28.6	
1.0000	76.4		18	90°	76.4	.0000	0.0	
.9063	78.5		19	95°	86.5	−.4226		36.6
.6428	61.4		20	100°	95.5	−.7660		73.2
.2588	27.2		21	105°	105.1	−.9659		101.6
−.1736		19.6	22	110°	112.8	−.9848		111.1
−.5736		67.4	23	115°	117.4	−.8192		96.2
−.8660		102.7	24	120°	118.5	−.5000		59.2
−.9962		114.0	25	125°	114.2	−.0872		10.0
−.9397		98.5	26	130°	104.9	.3420	35.8	
−.7071		64.6	27	135°	91.4	.7071	64.6	
−.3420		26.8	28	140°	78.3	.9397	73.6	
.0872	5.7		29	145°	65.0	.9962	64.8	
.5000	25.6		30	150°	51.1	.8660	44.3	
.8192	32.7		31	155°	40.0	.5736	23.0	
.9848	29.0		32	160°	29.4	.1736	5.1	
.9659	20.6		33	165°	21.3	−.2588		5.5
.7660	10.7		34	170°	14.0	−.6428		9.0
.4226	3.0		35	175°	7.1	−.9063		6.4
.0000	0.0		36	180°	0.0	−1.0000		0.0
Sum of products	552.8	649.6					554.2	619.9
	−96.8						−65.7	

$$A_5 = \frac{-96.8}{36} \times 2 = -5.38$$

$$B_5 = \frac{-65.7}{36} \times 2 = -3.65$$

SEVENTH HARMONIC

1	2	3	4	5	6	7	8	9
sin 7x	Products (y sin 7x)		Ordinate No.	Angle x to ordinate	Meas. ordinate (y)	cos 7x	Products (y cos 7x)	
	+	−					+	−
.5736	3.4		1	5°	5.9	.8192	4.8	
.9397	9.4		2	10°	10.0	.3420	3.4	
.9659	13.0		3	15°	13.4	−.2588		3.5
.6428	10.5		4	20°	16.4	−.7660		12.6
.0872	1.7		5	25°	19.4	−.9962		19.3
−.5000		10.9	6	30°	21.8	−.8660		18.9
−.9063		21.4	7	35°	23.6	−.4226		10.0
−.9848		24.4	8	40°	25.9	.1736	4.5	
−.7071		19.9	9	45°	28.1	.7071	19.9	
−.1736		5.3	10	50°	30.7	.9848	30.2	
.4226	14.3		11	55°	33.9	.9063	30.8	
.8660	32.4		12	60°	37.4	.5000	18.7	
.9962	41.9		13	65°	42.0	−.0872		3.7
.7660	35.8		14	70°	46.7	−.6428		30.0
.2588	13.7		15	75°	52.8	−.9659		51.0
−.3420		20.5	16	80°	60.0	−.9397		56.5
−.8192		55.5	17	85°	67.7	−.5736		38.8
−1.0000		76.4	18	90°	76.4	.0000		0.0
−.8192		70.9	19	95°	86.5	.5736	49.6	
−.3420		32.6	20	100°	95.5	.9397	89.8	
.2588	27.2		21	105°	105.1	.9659	101.6	
.7660	86.5		22	110°	112.8	.6428	72.5	
.9962	117.0		23	115°	117.4	.0872	10.2	
.8660	102.7		24	120°	118.5	−.5000		59.2
.4226	48.3		25	125°	114.2	−.9063		103.5
−.1736		18.2	26	130°	104.9	−.9848		103.2
−.7071		64.6	27	135°	91.4	−.7071		64.6
−.9848		77.1	28	140°	78.3	−.1736		13.6
−.9063		59.0	29	145°	65.0	.4226	27.5	
−.5000		25.6	30	150°	51.1	.8660	44.3	
.0872	3.5		31	155°	40.0	.9962	39.9	
.6428	18.9		32	160°	29.4	.7660	22.6	
.9659	20.6		33	165°	21.3	.2588	5.5	
.9397	13.6		34	170°	14.0	−.3420		4.8
.5736	4.1		35	175°	7.1	−.8192		5.8
.0000	0.0		36	180°	0.0	−1.0000		0.0
Sum of products	618.5	582.3					575.8	599.0
	36.2						−23.2	

$$A_7 = \frac{36.2}{36} \times 2 = 2.01$$

$$B_7 = \frac{-23.2}{36} \times 2 = -1.29$$

In Fig. 9 the vector OA of magnitude A_1 may be taken to represent the sin ωt. Remembering that the cosine wave leads the sine wave by 90°, the vector OB may be used to represent the cosine term. The vector sum OC of the two vectors OA and OB, therefore, represents the sum of $A_1 \sin \omega t$ and $B_1 \cos \omega t$ in both magnitude and phase. It leads the sin ωt position by $\tan^{-1} \frac{B_1}{A_1}$ and it also lags the cos ωt by $\tan^{-1} \frac{A_1}{B_1}$. The magnitude OC is $\sqrt{A_1^2 + B_1^2}$. The equation of the combination is $\sqrt{A_1^2 + B_1^2} \sin \left(\omega t + \tan^{-1} \frac{B_1}{A_1}\right)$, or $\sqrt{A_1^2 + B_1^2} \cos \left(\omega t - \tan^{-1} \frac{A_1}{B_1}\right)$.

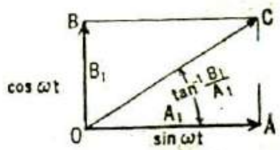


FIG. 9. Vector representation of sin ωt and cos ωt and their sum OC for particular magnitudes A_1 and B_1.

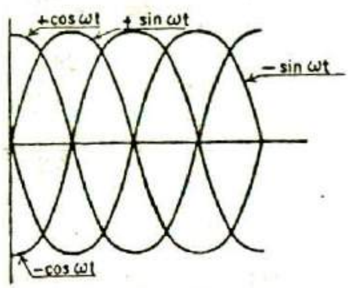


FIG. 10.

The vector representation of the positive and negative sines and cosines forms a convenient way to find trigonometric relations and to make combinations of these waves. For instance, the waves are shown in Fig. 10. The corresponding vector representation of the same waves is shown in Fig. 11. In Fig. 11 it can be seen that the

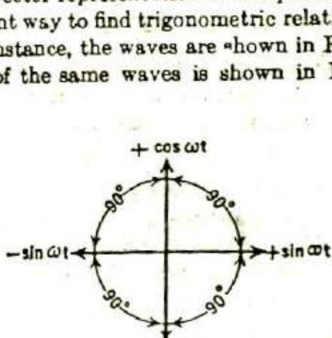


FIG. 11. Vector representation of waves shown in Fig. 10.

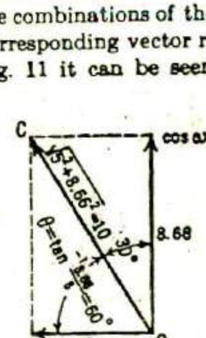


FIG. 12. Combination of $-5 \sin \omega t + 8.66 \cos \omega t$.

$\sin(\omega t + 90°)$ gives the $+\cos \omega t$, that $-\cos(\omega t - 90°)$ gives the $-\sin \omega t$, etc. By visualizing Fig. 11, all similar relations become apparent. In like manner, if $[-5 \sin \omega t + 8.66 \cos \omega t]$ is to be reduced to a single trigonometric term, the values would be laid off on Fig. 11 as shown in Fig. 12. The vector addition would then be performed to obtain the resultant OC. OC may be seen to lead the cos ωt by 30° or to lag the $-\sin \omega t$ by 60°. It also leads the $+\sin \omega t$ by 120°. Thus the equation of OC is any one of the following: $10 \cos(\omega t + 30°)$, $-10 \sin(\omega t - 60°)$, or $10 \sin(\omega t + 120°)$. There are also other equivalent expressions for the resultant wave.

Example 4. Express the equation obtained from the analysis of the wave of Oscillogram 4 in terms of positive sine components only. The results of the analysis show that:

$$A_1 = 82.45 \quad B_1 = -22.11 \quad C_1 = \sqrt{82.45^2 + (-22.11)^2} = 85.50 \text{ units}$$
$$A_3 = -0.92 \quad B_3 = 26.2 \quad C_3 = \sqrt{(-0.92)^2 + 26.2^2} = 26.2 \text{ units}$$
$$A_5 = -5.38 \quad B_5 = -3.65 \quad C_5 = \sqrt{(-5.38)^2 + (-3.65)^2} = 6.50 \text{ units}$$
$$A_7 = 2.01 \quad B_7 = -1.29 \quad C_7 = \sqrt{2.01^2 + (-1.29)^2} = 2.39 \text{ units}$$

With respect to the $+\sin \omega t$ position of Fig. 11 as a reference:

$$\alpha_1 = \tan^{-1} \frac{-22.11}{82.45} = \tan^{-1} -0.268 = -15°$$

$$\alpha_3 = \tan^{-1} \frac{26.2}{-0.92} = \tan^{-1} -28.5 = 92°$$

$$\alpha_5 = \tan^{-1} \frac{-3.65}{-5.38} = \tan^{-1} 0.678 = 214.2°$$

$$\alpha_7 = \tan^{-1} \frac{-1.29}{2.01} = \tan^{-1} -0.642 = -32.7°$$

It will be noted that the individual signs of the coefficients B and A must be considered in the evaluation of the phase angles.

The equation for the wave form shown in Oscillogram 4 is:

$$i = 85.50 \sin (\omega t - 15°) + 26.2 \sin (3\omega t + 92°) + 6.50 \sin (5\omega t + 214.2°) + 2.39 \sin (7\omega t - 32.7°)$$

It is desirable to draw figures, similar to that shown in Fig. 12, for each of the harmonics. This exercise is left to the student. The final test of the correctness of any wave analysis is whether the component parts found by the analysis can be combined to yield the original wave.

Problem 3. Evaluate i in the above equation at 30° intervals of ωt throughou one-half cycle, and plot the resultant curve. Compare the general wave shap thus found with that of the original wave form shown in Oscillogram 4.

Problem 4. Express the equation for the wave shape shown in Oscillogram in terms of positive cosine components.

Wave Analysis (Second Graphical Method). Although the funda mental basis of the previous method of analysis is simple, there are a number of methods which require less time for numerical computation One of these shorter methods follows.

Equation (1) may be written in the following form:

$$y = f(x) = A_0 + A_1 \sin x + A_2 \sin 2x + A_3 \sin 3x + \cdots + A_n \sin nx + B_1 \cos x + B_2 \cos 2x + B_3 \cos 3x + \cdots + B_n \cos nx \tag{20}$$

If q is a number equal to the order of the harmonic which is under investigation and $f(\pi/2q)$, $f(3\pi/2q)$, etc., are the values of $y = f(x)$ at $x = \pi/2q$, $x = 3\pi/2q$, etc., it can be shown that the following relations are true.[3]

$$2q(A_q - A_{3q} + A_{5q} - A_{7q} + \cdots) = f\left(\frac{\pi}{2q}\right) - f\left(\frac{3\pi}{2q}\right) + f\left(\frac{5\pi}{2q}\right) - \cdots - f\left[\frac{(4q-1)\pi}{2q}\right] \quad (21)$$

$$2q(B_q + B_{3q} + B_{5q} + \cdots) = f(0) - f\left(\frac{\pi}{q}\right) + f\left(\frac{2\pi}{q}\right) - f\left(\frac{3\pi}{q}\right) + \cdots - f\left[\frac{(2q-1)\pi}{q}\right] \quad (22)$$

When equations (21) and (22) are used, it must be remembered that the subscripts $3q$, $5q$, etc., represent the order of the harmonic obtained by multiplication of 3 times q, 5 times q, etc. Thus, if q is 3, B_{3q} would be B_9, B_{5q} would be B_{15}, etc.

Before proceeding to employ equations (21) and (22), it is necessary to estimate the maximum number of harmonics required in the analysis. The procedure is then to start with the highest harmonic and substitute the ordinates at the various angles indicated by the right members of equations (21) and (22). Since it is unlikely that all ordinates required will be given, it is usually necessary to plot the resultant wave in order that the required ordinates may be read from the curve. The necessity of having a graph of the curve will usually entail no extra work in practice because the method will usually be applied only when the resultant wave is obtained from an oscillogram similar to that illustrated in Oscillogram 4, page 236. After the harmonic coefficients are determined, A_0 is evaluated by substituting $x = 0$ in equation (20). Thus

$$f(0) = A_0 + B_1 + B_2 + B_3 + \cdots + B_n \quad (23)$$

$f(0)$ is read from the curve and, since everything except A_0 has been determined, A_0 can be calculated. As an example of the procedure, the wave employed in example 3 will be analyzed.

Example 5. Find the harmonic coefficients through the seventh harmonic for the wave given in Oscillogram 4, page 236, by employing equations (21), (22), and (23).

[3] See "Advanced Mathematics for Engineers," by Reddick and Miller, John Wiley & Sons, Inc., 2nd edition, 1947, p. 202.

For the seventh harmonic, $q = 7$ and equation (21) is used as follows:

$$(2 \times 7)A_7 = f\left(\frac{\pi}{14}\right) - f\left(\frac{3\pi}{14}\right) + f\left(\frac{5\pi}{14}\right) - f\left(\frac{7\pi}{14}\right) + f\left(\frac{9\pi}{14}\right) - f\left(\frac{11\pi}{14}\right)$$
$$+ f\left(\frac{13\pi}{14}\right) - f\left(\frac{15\pi}{14}\right) + f\left(\frac{17\pi}{14}\right) - f\left(\frac{19\pi}{14}\right) + f\left(\frac{21\pi}{14}\right)$$
$$- f\left(\frac{23\pi}{14}\right) + f\left(\frac{25\pi}{14}\right) - f\left(\frac{27\pi}{14}\right)$$

Note that, since the seventh harmonic is the highest required, $A_{3q} = A_{21}$, A_{5q}, etc., are all zero.

$$14A_7 = f(12.86°) - f(38.57°) + f(64.29°) - f(90°) + f(115.7°)$$
$$- f(141.4°) + f(167.2°) - f(193°) + f(218.7°) - f(244.3°)$$
$$+ f(270°) - f(296°) + f(321.5°) - f(347°)$$
$$= 12.4 - 24.5 + 40.5 - 76.4 + 117.6 - 74.6 + 17.5 - (-12.4)$$
$$+ (-24.5) - (-40.5) + (-76.4) - (-117.6) + (-74.6)$$
$$- (-17.5)$$
$$= 25$$

$$A_7 = \frac{25}{14} = 1.79$$

$$14B_7 = f(0) - f(25.7°) + f(51.4°) - f(77.1°) + f(103°) - f(128.7°)$$
$$+ f(154.3°) - f(180°) + f(205.5°) - f(231.3°) + f(257°)$$
$$- f(283°) + f(308.7°) - f(334.5°)$$
$$= 0 - 20 + 32 - 56 + 101 - 107 + 41 - 0 + -20 + 32 - 56$$
$$+ 101 - 107 + 41$$
$$= -18$$

$$B_7 = -\frac{18}{14} = -1.286$$

Because the wave is symmetrical about the 180° point, even harmonics cannot exist. If, however, equations (21) and (22) are used to find the sixth harmonic, zero will be obtained.

Equations (21) and (22) are now used to calculate A_5 and B_5 as follows.

$$(2 \times 5)A_5 = f\left(\frac{\pi}{10}\right) - f\left(\frac{3\pi}{10}\right) + f\left(\frac{5\pi}{10}\right) - f\left(\frac{7\pi}{10}\right) + f\left(\frac{9\pi}{10}\right) - f\left(\frac{11\pi}{10}\right)$$
$$+ f\left(\frac{13\pi}{10}\right) - f\left(\frac{15\pi}{10}\right) + f\left(\frac{17\pi}{10}\right) - f\left(\frac{19\pi}{10}\right)$$

$$10A_5 = f(18°) - f(54°) + f(90°) - f(126°) + f(162°) - f(198°)$$
$$+ f(234°) - f(270°) + f(306°) - f(342°)$$
$$= 15 - 33 + 76.4 - 113 + 26 - (-15) + (-33) - (-76.4)$$
$$+ (-113) - (-26)$$
$$= 2(15 - 33 + 76.4 - 113 + 26) = 2(-28.6) = -57.2$$

$$A_5 = -5.72$$

$$(2 \times 5)B_5 = f(0) - f\left(\frac{\pi}{5}\right) + f\left(\frac{2\pi}{5}\right) - f\left(\frac{3\pi}{5}\right) + f\left(\frac{4\pi}{5}\right) - f\left(\frac{5\pi}{5}\right) + f\left(\frac{6\pi}{5}\right) - f\left(\frac{7\pi}{5}\right) + f\left(\frac{8\pi}{5}\right) - f\left(\frac{9\pi}{5}\right)$$

$$\begin{aligned}
10B_5 &= f(0) - f(36^\circ) + f(72^\circ) - f(108^\circ) + f(144^\circ) - f(180^\circ) \\
&\quad + f(216^\circ) - f(252^\circ) + f(288^\circ) - f(324^\circ) \\
&= 0 - 24 + 49 - 110 + 68 - 0 + (-24) - (-49) + (-110) \\
&\quad - (-68) \\
&= -34 \\
B_5 &= -3.4
\end{aligned}$$

Determination of A_3 and B_3:

$$(2 \times 3)A_3 = f\left(\frac{\pi}{6}\right) - f\left(\frac{3\pi}{6}\right) + f\left(\frac{5\pi}{6}\right) - f\left(\frac{7\pi}{6}\right) + f\left(\frac{9\pi}{6}\right) - f\left(\frac{11\pi}{6}\right)$$

$$\begin{aligned}
6A_3 &= f(30^\circ) - f(90^\circ) + f(150^\circ) - f(210^\circ) + f(270^\circ) - f(330^\circ) \\
&= 21.8 - 76.4 + 51.1 - (-21.8) + (-76.4) - (-51.1) \\
&= -7 \\
A_3 &= -1.167
\end{aligned}$$

$$\begin{aligned}
6B_3 &= f(0) - f\left(\frac{\pi}{3}\right) + f\left(\frac{2\pi}{3}\right) - f\left(\frac{3\pi}{3}\right) + f\left(\frac{4\pi}{3}\right) - f\left(\frac{5\pi}{3}\right) \\
&= f(0) - f(60^\circ) + f(120^\circ) - f(180^\circ) + f(240^\circ) - f(300^\circ) \\
&= 0 - 37.4 + 118.5 - 0 + (-37.4) - (-118.5) = 162.2 \\
B_3 &= +27.03
\end{aligned}$$

For the fundamental, equations (21) and (22) become

$$(2 \times 1)(A_1 - A_3 + A_5 - A_7) = f\left(\frac{\pi}{2}\right) - f\left(\frac{3\pi}{2}\right)$$

$$2(A_1 - A_3 + A_5 - A_7) = f(90) - f(270) = 76.4 - (-76.4) = 152.8$$

Substituting the values of A_3, A_5, and A_7 found previously and solving for A_1 gives $A_1 = 82.74$.

In a similar way B_1 may be found as follows.

$$\begin{aligned}
(2 \times 1)(B_1 + B_3 + B_5 + B_7) &= f(0) - f(\pi) = 0 \\
2(B_1 + 27.03 - 3.4 - 1.286) &= 0 \\
B_1 &= -22.34
\end{aligned}$$

The foregoing method is easy to apply and entails less labor than the method employing analyzing tables. The accuracy, however, will vary with different wave shapes and will also be dependent upon the estimate of the number of harmonics required. It will be noted that the determination of the fundamental depends upon the values of the harmonics previously determined. It is therefore desirable to start with a high enough order of harmonic so that any higher-order components will be negligible so far as engineering accuracy is concerned. If only a single harmonic of some desired order is required, the method employing the analyzing tables may save time and be more accurate. The error in the method employing analyzing tables depends only upon the size of the intervals chosen and, obviously, approaches zero as the size of the interval is decreased and the number of them is increased. The determination of any one harmonic is independent of the determination of any other harmonics when analyzing tables are employed.

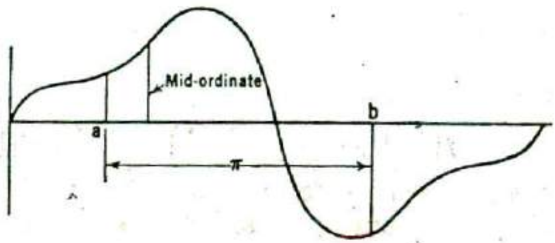


FIG. 13. Wave with unsymmetrical positive and negative loops.

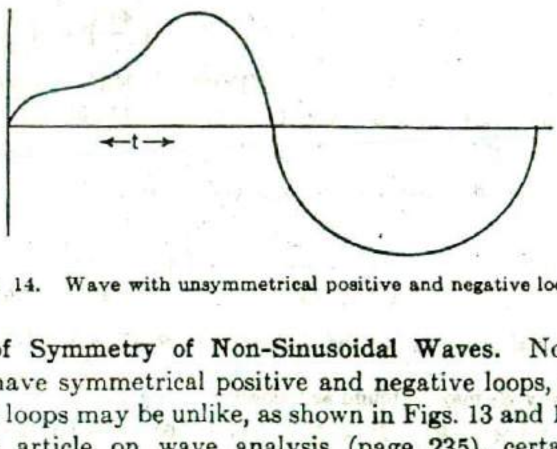


FIG. 14. Wave with unsymmetrical positive and negative loops.

Degrees of Symmetry of Non-Sinusoidal Waves. Non-sinusoidal waves may have symmetrical positive and negative loops, as shown in Fig. 8, or the loops may be unlike, as shown in Figs. 13 and 14. As indicated in the article on wave analysis (page 235), certain types of symmetry in a wave form will automatically eliminate the need for evaluating certain coefficients in the Fourier series which represents the wave.

When the variation from zero to 180° is repeated (except for sign) between 180° and 360°, the wave is said to possess half-wave symmetry. Mathematically a wave of this kind is described as having $[f(x + \pi) = -f(x)]$ symmetry. Expressed in another way, a wave has half-wave symmetry when any ordinate, such as b, Fig. 13, π radians distant from another ordinate, such as a, is equal in magnitude to that at point a but opposite in sign. Thus, the ordinate at any point a for a general wave is:

$$y_a = A_0 + C_1 \sin(\omega t + \alpha_1) + C_2 \sin(2\omega t + \alpha_2) + C_3 \sin(3\omega t + \alpha_3) + C_4 \sin(4\omega t + \alpha_4) + C_5 \sin(5\omega t + \alpha_5) + \cdots \quad (24)$$

The ordinate π radians distant from a is found by adding π radians to ωt. If this angle $(\omega t + \pi)$ is substituted and if it is remembered that $(\omega t + \pi)$ for the fundamental corresponds to $n(\omega t + \pi)$ for the nth harmonic, the following results:

$$y_b = A_0 + C_1 \sin(\omega t + \alpha_1 + \pi) + C_2 \sin(2\omega t + \alpha_2 + 2\pi) + C_3 \sin(3\omega t + \alpha_3 + 3\pi) + C_4 \sin(4\omega t + \alpha_4 + 4\pi) + C_5 \sin(5\omega t + \alpha_5 + 5\pi) + \cdots \quad (25)$$

Since the sine of any angle plus an even multiple of π radians is the same as the sine of the angle, and the sine of an angle plus any odd multiple of π radians is the same as the negative sine of the angle, equation (25) simplifies to:

$$y_b = A_0 - C_1 \sin(\omega t + \alpha_1) + C_2 \sin(2\omega t + \alpha_2) - C_3 \sin(3\omega t + \alpha_3) + C_4 \sin(4\omega t + \alpha_4) - C_5 \sin(5\omega t + \alpha_5) + \cdots \quad (26)$$

The ordinate y_b [equation (26)] would be exactly opposite to that of equation (24) if A_0 and all even harmonics in the wave were absent. Hence a wave is symmetrical with respect to the positive and negative loops if it contains no even harmonics and if A_0 is equal to zero. The converse of the foregoing statement is also true, that is, a wave which has $[f(\omega t) = -f(\omega t + \pi)]$ symmetry can contain neither even harmonics nor A_0. The effect of a second harmonic in destroying half-wave symmetry is shown graphically in Fig. 15. In analyzing waves possessing half-wave symmetry, the analysis need be carried through only ½ cycle or 180°.

A wave possessing half-wave symmetry as defined above may also be symmetrical about the midordinates of its positive and negative loops, namely, its 90° and 270° points. A wave of this kind is said to possess midordinate or quarter-wave symmetry, and the analysis need be carried through only ¼ cycle or 90°. The case where only the positive *or* negative loop is symmetrical about its midordinate is of relatively little importance. Thus the positive loop of the wave shown in Fig. 13 is not symmetrical about its midordinate, whereas that of Fig. 16 is symmetrical with respect to its midordinate. The wave will have the halves of its positive and negative loops symmetrical if its fundamental and all harmonics pass through zero values at the same time, and, further, if all even harmonics are absent. This fact is illustrated graphically in Fig. 17. The second harmonic, shown dotted, adds to the fundamental to the left of the midordinate of the positive loop and subtracts from it on the right-

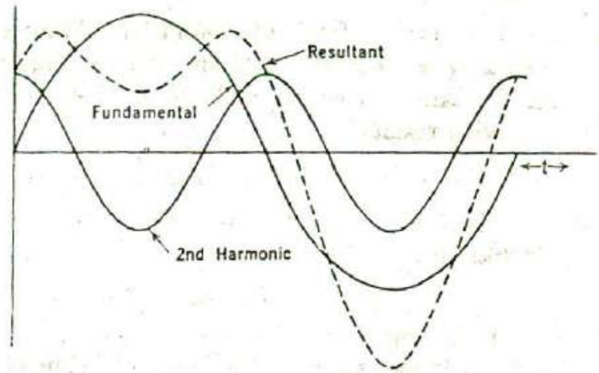


FIG. 15. Effect of second harmonic in destroying half-wave symmetry.

hand side. All the odd harmonics are symmetrical about the midordinate a when they pass through zero at the same time as the fundamental. If the zero-ordinate point of the complex wave is chosen as a

FIG. 16. Wave with positive and negative loops symmetrical about the midordinate (quarter-wave symmetry).

reference, it is plain that only odd sine terms can be present in the equation of a complex wave having quarter-wave symmetry.

Waves of Same Wave Shape. Waves are of the same wave shape if they contain the same harmonics, if the ratio of corresponding harmonics to their respective fundamentals is the same, and if the harmonics are spaced the same with respect to their fundamentals. Expressed in another way, for two waves of the same form the ratio of the magnitudes of corresponding harmonics must be constant, and, when the fundamentals are in phase, all the corresponding harmonics of the two waves must be in phase. The test is to note whether the ratio of corresponding harmonics is constant and then to shift one wave so that the fundamentals coincide. If the phase angles of corresponding harmonics in the two waves are then the same and if the first condition is also fulfilled, the waves are of the same wave shape or wave form.

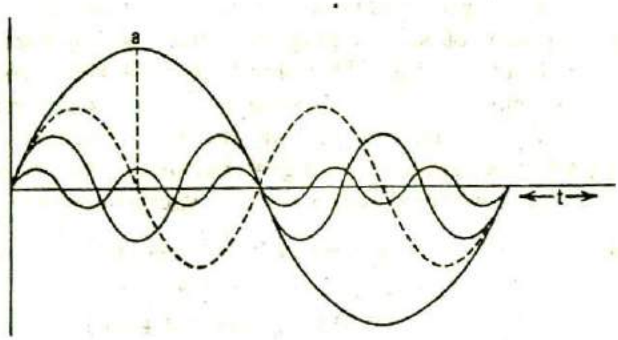


FIG. 17. Symmetry about the midordinate, a, is maintained if all odd harmonics are zero when the fundamental is zero. The second harmonic shown dotted will destroy this symmetry as will other even harmonics.

Example 6. Determine whether the following two waves are of the same shape:

$$e = 100 \sin (\omega t + 30°) - 50 \sin (3\omega t - 60°) + 25 \sin (5\omega t + 40°)$$

$$i = 10 \sin (\omega t - 60°) + 5 \sin (3\omega t - 150°) + 2.5 \cos (5\omega t - 140°)$$

Since all harmonics of the current wave are one-tenth of the corresponding harmonics in the voltage wave, the first requisite is fulfilled. Next, the fundamentals should be brought into phase by shifting the current wave forward 90° or the voltage wave backward 90°. The current wave will be shifted by adding 90° to the phase angle of its fundamental. Shifting the fundamental of a wave by $\alpha°$ corresponds to shifting the nth harmonic by $n\alpha°$. This may be verified by referring to Fig. 17. Suppose the reference axis is changed to the position marked a, thus shifting the wave ahead. This is a shift of 90°, or one quarter cycle for the fundamental. It is a shift of three quarter cycles for the third harmonic, or 270° and five quarter cycles for the fifth harmonic, or 450°. Hence, to maintain the same relation between the fundamental and all harmonics in the current waves, 3 × 90° or 270° will be added to the third, and 5 × 90° or 450° will be added to the fifth harmonic. Then:

$$\begin{aligned} i' &= 10 \sin (\omega t - 60° + 90°) + 5 \sin (3\omega t - 150° + 270°) \\ &\quad + 2.5 \cos (5\omega t - 140° + 450°) \\ &= 10 \sin (\omega t + 30°) + 5 \sin (3\omega t + 120°) + 2.5 \cos (5\omega t + 310°) \\ &= 10 \sin (\omega t + 30°) - 5 \sin (3\omega t - 60°) + 2.5 \sin (5\omega t + 40°) \end{aligned}$$

The corresponding harmonics of the current and voltage waves are hence in phase, and the two waves are of the same shape. Had either the third or fifth harmonic been out of phase with the corresponding harmonic in the voltage wave, the wave shapes would have been different.

The effect on wave shape of shifting a harmonic with respect to the fundamental can be understood through a study of Figs. 18, 19, and 20. In each figure the magnitudes of the fundamental and third harmonic are the same. As the third harmonic is shifted along the axis with respect to the fundamental, the wave form of the resultant is seen to change. This shifting of a harmonic with respect to the fundamental is sometimes spoken of as changing the phase of the harmonic with respect to the fundamental. This should not be construed to mean that there is a definite phase difference between a *vector* representing the fundamental and one representing the third harmonic. Vectors representing a fundamental and a higher harmonic cannot correctly be related on the same vector diagram without special interpretation.

Problem 5. Given the following equations for two wave forms of current:

$$i' = 10 \sin(\omega t + 30°) + 2 \sin 7\omega t$$
$$i'' = 35 \sin(\omega t - 10°) + 7 \sin(7\omega t + 80°)$$

Show that the wave form of the i' variation is like (or unlike) the wave form of the i'' variation. *Ans.:* Same form.

Effective Value of a Non-Sinusoidal Wave. In Chapter III the effective value of any wave was shown to be $\sqrt{\frac{1}{T}\int_0^T [f(t)]^2\, dt}$. Applying this expression to the general complex wave

$$i = I_0 + I_{m1} \sin \omega t + I_{m2} \sin(2\omega t + \alpha_2) + I_{m3} \sin(3\omega t + \alpha_3) + \cdots + I_{mn} \sin(n\omega t + \alpha_n)$$

gives

$$I = \left\{\frac{1}{T}\int_0^T [I_0 + I_{m1} \sin \omega t + I_{m2} \sin(2\omega t + \alpha_2) + I_{m3} \sin(3\omega t + \alpha_3) + \cdots + I_{mn} \sin(n\omega t + \alpha_n)]^2\, dt\right\}^{1/2}$$

$$= \sqrt{I_0^2 + \frac{I_{m1}^2 + I_{m2}^2 + I_{m3}^2 + I_{m4}^2 + \cdots + I_{mn}^2}{2}} \qquad (27)$$

Problem 6. Show by integration, including all steps, that the effective value of $= V_{m1} \sin(\omega t + \alpha_1) + V_{m3} \sin(3\omega t + 30°)$ is $\sqrt{\frac{V_{m1}^2 + V_{m3}^2}{2}}$.

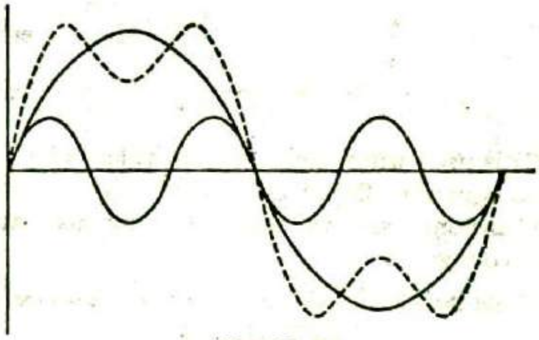


FIG. 18.

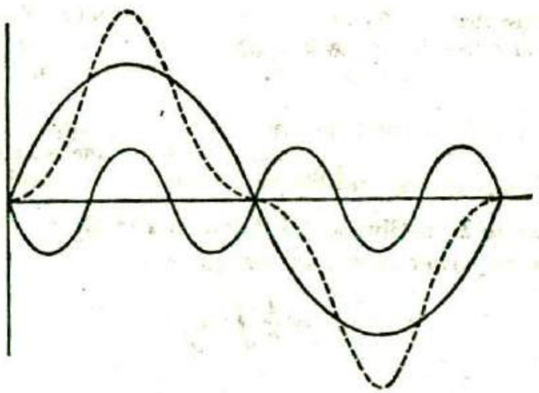


FIG. 19.

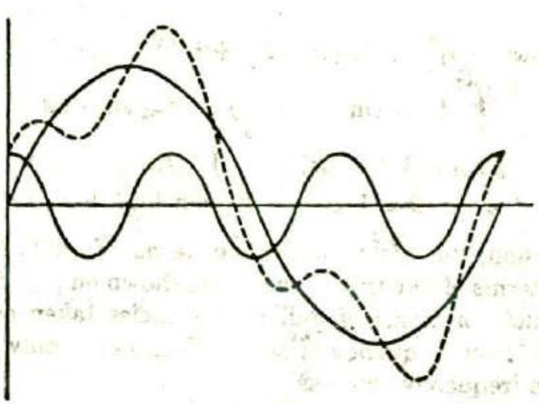


FIG. 20.

Figs. 18, 19, and 20 show the effect on wave shape of shifting a harmonic.

Since

$$\frac{I_{m1}}{\sqrt{2}} = I_1, \qquad \frac{I_{m2}}{\sqrt{2}} = I_2, \qquad \text{etc.}$$

$$I = \sqrt{I_0^2 + I_1^2 + I_2^2 + I_3^2 + I_4^2 + \cdots + I_n^2} \tag{28}$$

Equation (27) is used when the maximum values of the harmonics are given, whereas equation (28) gives the equivalent expression if effective values of the harmonics are available. It is obvious that similar expressions hold for voltages.

Example 7. Find the effective value of the voltage wave used in example 6.

$$E = \sqrt{\frac{100^2 + 50^2 + 25^2}{2}} = 81 \text{ volts}$$

It should be noted that the effective value is the square root of the *sum* of the squares of the maximum values divided by 2, irrespective of the phase angles or signs of the harmonics. A similar statement is true when effective values of the harmonics are used in equation (28).

For one method of analysis in a-c machinery, known as the Blondell two-reaction method, it is necessary to have the effective value of the rectangular wave given by equation (15), page 233. For this wave, effective value equals $A_1\pi/4$.

Power Due to Non-Sinusoidal Voltages and Currents. The expression for average power in general was given as

$$P = \frac{1}{T}\int_0^T ei\,dt.$$

When

$$e = E_{m1} \sin(\omega t + \alpha_1) + E_{m2} \sin(2\omega t + \alpha_2) + E_{m3} \sin(3\omega t + \alpha_3) + \cdots$$

and

$$i = I_{m1} \sin(\omega t + \alpha_1') + I_{m2} \sin(2\omega t + \alpha_2') + I_{m3} \sin(3\omega t + \alpha_3') + \cdots$$

$$P = \frac{1}{T}\int_0^T [E_{m1} \sin(\omega t + \alpha_1) + E_{m2} \sin(2\omega t + \alpha_2)$$
$$+ E_{m3} \sin(3\omega t + \alpha_3) + \cdots] \; [I_{m1} \sin(\omega t + \alpha_1')$$
$$+ I_{m2} \sin(2\omega t + \alpha_2') + I_{m3} \sin(3\omega t + \alpha_3') + \cdots]\,dt \tag{29}$$

Upon expansion, this yields products of terms of unlike frequencies and products of terms of like frequencies. As shown on page 226 the integral of the products of terms of unlike frequencies taken over a complete cycle of the lower frequency is zero. This leaves only the product of terms of like frequency, such as:

$$\frac{1}{T}\int_0^T A \sin(m\omega t + \alpha)\, B \sin(m\omega t + \alpha')\,dt$$

which gives

$$\frac{AB}{2}\cos(\alpha - \alpha') \tag{30}$$

Thus equation (29) becomes

$$P = \frac{E_{m1}I_{m1}}{2}\cos(\alpha_1 - \alpha_1') + \frac{E_{m2}I_{m2}}{2}\cos(\alpha_2 - \alpha_2') + \frac{E_{m3}I_{m3}}{2}\cos(\alpha_3 - \alpha_3') + \cdots \tag{31}$$

Or, since

$$\frac{E_{m1}I_{m1}}{2} = \frac{E_{m1}}{\sqrt{2}}\frac{I_{m1}}{\sqrt{2}} = E_1 I_1$$

$$P = E_1I_1\cos(\alpha_1 - \alpha_1') + E_2I_2\cos(\alpha_2 - \alpha_2') + E_3I_3\cos(\alpha_3 - \alpha_3') + \cdots \tag{32}$$

Average power when waves are non-sinusoidal is the algebraic sum of the powers represented by corresponding harmonics of voltage and current. No average power results from components of voltage and current of unlike frequency, provided that the time interval chosen is equal to an integral number of cycles of the lower-frequency variation. The foregoing statement can be proved either mathematically or graphically.

Example 8. Find the power represented by the following:

$$e = 100\sin(\omega t + 30°) - 50\sin(3\omega t + 60°) + 25\sin 5\omega t \text{ volts}$$

$$i = 20\sin(\omega t - 30°) + 15\sin(3\omega t + 30°) + 10\cos(5\omega t - 60°) \text{ amperes}$$

$$P = \frac{100 \times 20}{2}\cos[30° - (-30°)] + \frac{(-50)(15)}{2}\cos[60° - 30°] + \frac{25 \times 10}{2}\cos[-90° - (-60°)]$$

$$= 500 - 324.75 + 108.25$$

$$= 283.5 \text{ watts}$$

An alternative method of obtaining the power for the third-harmonic components follows.

$$e_3 = -50\sin(3\omega t + 60°) = +50\sin(3\omega t - 120°) \text{ volts}$$

$$i_3 = 15\sin(3\omega t + 30°) \text{ amperes}$$

$$P_3 = \frac{50 \times 15}{2}\cos(-120° - 30°) = 375\cos 150° = -324.75 \text{ watts}$$

Problem 7. Find the power delivered by the following:

$$e = 100 \sin \omega t + 50 \sin (5\omega t - 80°) - 40 \cos (7\omega t + 30°) \text{ volts}$$

$$i = 30 \sin (\omega t + 60°) + 20 \sin (5\omega t - 50°) + 10 \sin (7\omega t + 60°) \text{ amperes}$$

Ans.: 1083 watts.

Volt-Amperes. Volt-amperes are determined by the product of the effective voltage and effective current.

Example 9. Find the volt-amperes for the waves in example 8.

$$Va = EI = \sqrt{\frac{100^2 + 50^2 + 25^2}{2}} \sqrt{\frac{20^2 + 15^2 + 10^2}{2}} = 81 \times 19.03$$

$$= 1541 \text{ volt-amperes}$$

In general,

Volt-amperes =

$$\sqrt{\frac{E_{m1}^2 + E_{m2}^2 + E_{m3}^2 + \text{etc.}}{2}} \sqrt{\frac{I_{m1}^2 + I_{m2}^2 + I_{m3}^2 + \text{etc.}}{2}} \qquad (33)$$

Power Factor. Power factor for non-sinusoidal waves is defined as the ratio of the power to the volt-amperes. Hence

Power factor =

$$\frac{E_1 I_1 \cos(\alpha_1 - \alpha_1') + E_2 I_2 \cos(\alpha_2 - \alpha_2') + E_3 I_3 \cos(\alpha_3 - \alpha_3') + \text{etc.}}{\sqrt{E_1^2 + E_2^2 + E_3^2 + \text{etc.}} \ \sqrt{I_1^2 + I_2^2 + I_3^2 + \text{etc.}}} \qquad (34)$$

Example 10. Find the power factor for the waves given in example 8.

Power from example 8 = 283.5 watts

Volt-amperes from example 9 = 1541

$$\text{Power factor} = \frac{283.5}{1541} = 0.1837$$

The conditions under which the power factor is unity when waves are non-sinusoidal are found from equation (34). To make the power factor 1, the numerator (power) should be as large as possible. Hence

$$\cos(\alpha_1 - \alpha_1') = \cos(\alpha_2 - \alpha_2') = \cos(\alpha_3 - \alpha_3') + \text{etc.} = 1$$

Then

$$\text{p.f.} = \frac{E_1 I_1 + E_2 I_2 + E_3 I_3 + \cdots}{\sqrt{(E_1^2 + E_2^2 + E_3^2 + \text{etc.})(I_1^2 + I_2^2 + I_3^2 + \text{etc.})}}$$

This expression can equal unity only if $E_1/I_1 = E_2/I_2 = E_3/I_3$.

To simplify the algebra, consider only the fundamental and one harmonic.

$$\frac{E_1 I_1 + E_2 I_2}{\sqrt{(E_1^2 + E_2^2)(I_1^2 + I_2^2)}} = 1$$

$$E_1I_1 + E_2I_2 = \sqrt{E_1^2I_1^2 + E_1^2I_2^2 + E_2^2I_1^2 + E_2^2I_2^2}$$
$$E_1^2I_1^2 + 2E_1I_1E_2I_2 + E_2^2I_2^2 = E_1^2I_1^2 + E_1^2I_2^2 + E_2^2I_1^2 + E_2^2I_2^2$$
$$2E_1I_1E_2I_2 = E_1^2I_2^2 + E_2^2I_1^2$$

If $E_1/I_1 = E_2/I_2$, $E_1I_2 = E_2I_1$ and the above expression becomes $2E_2^2I_1^2 = 2E_2^2I_1^2$, under which conditions the premise is true. Hence, to have unity power factor, the voltage and current waves must be of the same wave shape and in phase. Even though the voltage and current waves pass through zero at the same instant, the power factor cannot be unity if any harmonic in one wave is absent in the other, or when its magnitude makes the wave shapes different.

Equivalent Sine Waves. Occasionally equivalent sine waves are used for certain calculations and comparisons. They must be used with discretion because calculations based upon them are usually in error by varying amounts. An equivalent sine wave of current or voltage is a sine wave the effective value of which is the same as the effective value of the non-sinusoidal wave which is being represented. When equivalent sine waves of corresponding non-sinusoidal voltages and currents are found, the phase angle between the equivalent sine waves is made such that the power and power factor are the same as those for the actual waves. Whether the equivalent angle of phase difference is one of lead or lag is determined by the angle between the fundamentals of the two waves. If the fundamental of current lags the fundamental of voltage, the equivalent sine wave of current must lag the equivalent sine wave of voltage. If the fundamentals are in phase and the power factor is not unity, the sign of the angle of equivalent phase difference is indeterminate.

Example 11. Find the equivalent sine waves for the current and voltage given in example 8.

$$\text{Effective voltage} = \sqrt{\frac{100^2 + 50^2 + 25^2}{2}} = 81 \text{ volts}$$

$$\text{Effective current} = \sqrt{\frac{20^2 + 15^2 + 10^2}{2}} = 19.03 \text{ amperes}$$

Power factor from example 10 = 0.1837

The angle of equivalent phase difference is $\cos^{-1} 0.1837 = 79.4°$. Since the fundamental of current lags the fundamental of voltage, the angle 79.4° is an angle of lag of current with respect to voltage for the equivalent sine waves. The equivalent sine waves of voltage and current, respectively, are:

$$e = \sqrt{2}\, 81 \sin \omega t \text{ volts}$$
$$i = \sqrt{2}\, 19.03 \sin (\omega t - 79.4°) \text{ amperes}$$

As indicated before, the use of equivalent sine waves in non-sinusoidal circuit analysis will generally lead to large errors, particularly in operations involving the addition or subtraction of the waves. Equivalent sine waves are sometimes used in specifying the deviation from a sine wave.

Problem 8. Find the equivalent sine waves for the waves given in Problem 7.
Ans.: 118.8 sin ωt volts; 37.4 sin $(\omega t + 60.8°)$ amperes.

Deviation Factor. Deviation factor is the ratio of the maximum difference between corresponding ordinates of an actual wave and an

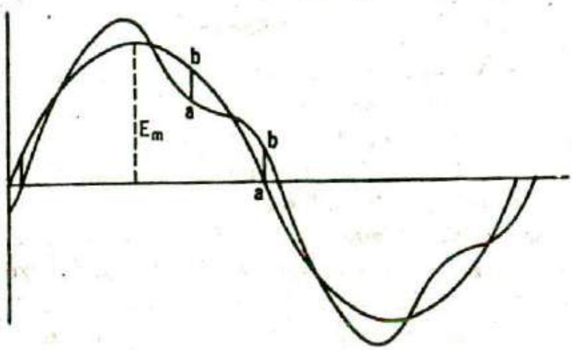


FIG. 21. Deviation of a distorted wave from an equivalent sine wave.

equivalent sine wave of the same length to the maximum ordinate of the equivalent sine wave when the two waves are superposed and shifted along the axis so as to make the maximum difference a minimum. For example, Fig. 21 shows a non-sinusoidal wave and an equivalent sine wave of the same period and length. These waves are shifted in such a way that the maximum difference between corresponding ordinates is as small as possible. In this particular case the maximum difference is ab. The ratio of ab to the maximum value E_m of the equivalent sine wave is the deviation factor. Deviation factor is sometimes used for specification purposes. A deviation factor of about 0.1 for commercial machines is usually allowable.

Series Circuit Analysis when Waves Are Non-Sinusoidal. The procedure is most readily understood from an example.

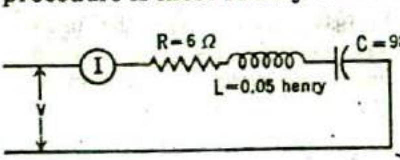


FIG. 22. See example 12.

Example 12. Given the circuit with the parameters shown in Fig. 22. When ω is 377 radians per second and the voltage $v = 141.4 \sin \omega t + 70.7 \sin (3\omega t + 30°) - 28.28 \sin (5\omega t - 20°)$ volts is impressed, find the current, I, that an ammeter would read. Also find the total power dissipated and the effective value of the voltage drop across the inductance. Also find the equation of the current wave.

Since the inductive and condensive reactances are different for different frequencies, each harmonic must be handled *separately*. Subscripts 1, 3, and 5 will designate the fundamental, third, and fifth harmonics, respectively. Either maximum or effective values may be used. If maximum values are used, maximum currents will result; when effective voltages are used, effective currents result. Whichever are used, the result can always be easily changed to give the other if desired. Since the effective values of the harmonic components of voltage in this particular case are more convenient numbers to handle, the solution will be negotiated through the use of effective values immediately.

Fundamental

$$V_1 = \frac{141.4}{\sqrt{2}} = 100 \text{ volts}$$

$$R_1 = 6 \text{ ohms}$$

$$X_{L1} = 377 \times 0.05 = 18.85 \text{ ohms}$$

$$X_{C1} = \frac{10^6}{377 \times 98.8} = 26.85 \text{ ohms}$$

$$\mathbf{Z}_1 = 6 + j18.85 - j26.85 = 6 - j8 \quad \text{or} \quad 10 \text{ ohms}$$

$$I_1 = \frac{V_1}{Z_1} = \frac{100}{10} = 10 \text{ amperes}$$

$$\mathbf{I}_1 \text{ leads } \mathbf{V}_1 \text{ by } \tan^{-1}\frac{8}{6} = 53.12°$$

$$P_1 = 10^2 \times 6 = 600 \text{ watts}$$

$$V_{L1} = I_1 X_{L1} = 10 \times 18.85 = 188.5 \text{ volts}$$

Third Harmonic

$$V_3 = \frac{70.7}{\sqrt{2}} = 50 \text{ volts}$$

$$R_3 = 6 \text{ ohms}$$

$$X_{L3} = 3X_{L1} = 3 \times 18.85 = 56.55 \text{ ohms}$$

$$X_{C3} = \frac{X_{C1}}{3} = \frac{26.85}{3} = 8.95 \text{ ohms}$$

$$\mathbf{Z}_3 = 6 + j56.55 - j8.95 = 6 + j47.6 \quad \text{or} \quad \sqrt{6^2 + 47.6^2} = 48.1 \text{ ohms}$$

$$I_3 = \frac{50}{48.1} = 1.04 \text{ amperes}$$

$$\mathbf{I}_3 \text{ lags } \mathbf{V}_3 \text{ by } \tan^{-1}\frac{47.6}{6} = 82.8°$$

$$P_3 = 1.04^2 \times 6 = 6.48 \text{ watts}$$

$$V_{L3} = 1.04 \times 56.55 = 58.9 \text{ volts}$$

Fifth Harmonic

$$V_5 = \frac{28.28}{\sqrt{2}} = 20 \text{ volts}$$

$$R_5 = 6 \text{ ohms}$$

$$X_{L5} = 5X_{L1} = 5 \times 18.85 = 94.25 \text{ ohms}$$

$$X_{C5} = \frac{X_{C1}}{5} = \frac{26.85}{5} = 5.37 \text{ ohms}$$

$$Z_5 = 6 + j94.25 - j5.37 = 6 + j88.88 \quad \text{or} \quad \sqrt{6^2 + 88.88^2} = 89 \text{ ohms}$$

$$I_5 = \frac{20}{89} = 0.225 \text{ ampere}$$

$$\mathbf{I}_5 \text{ lags } \mathbf{V}_5 \text{ by } \tan^{-1}\frac{88.88}{6} = 86.1°$$

$$P_5 = I_5^2R_5 = 0.225^2 \times 6 = 0.304 \text{ watt}$$

$$V_{L5} = 0.225 \times 94.25 = 21.2 \text{ volts}$$

$$I_{\text{total}} = \sqrt{I_1^2 + I_3^2 + I_5^2} = \sqrt{10^2 + 1.04^2 + 0.225^2} = 10.05 \text{ amperes}$$

$$P_{\text{total}} = P_1 + P_3 + P_5 = 600 + 6.48 + 0.304 = 606.8 \text{ watts}$$

$$V_L = \sqrt{188.5^2 + 58.9^2 + 21.2^2} = \sqrt{39510} = 198.8 \text{ volts}$$

Since the fundamental of current leads the fundamental of voltage by 53.12°, the equation of the fundamental of current must be $\sqrt{2}\,10 \sin(\omega t + 53.12°)$. Similarly, for the third harmonic,

$$i_3 = \sqrt{2}\,1.04 \sin(3\omega t + 30° - 82.8°)$$

or

$$i_3 = \sqrt{2}\,1.04 \sin(3\omega t - 52.8°) \text{ amperes}$$

Also

$$i_5 = -\sqrt{2}\,0.225 \sin(5\omega t - 20° - 86.1°)$$
$$= -\sqrt{2}\,0.225 \sin(5\omega t - 106.1°) \text{ amperes}$$

The complete equation is:

$$i = 14.14 \sin(\omega t + 53.12°) + 1.47 \sin(3\omega t - 52.8°) - 0.318 \sin(5\omega t - 106.1°)$$
$$= 14.14 \sin(\omega t + 53.12°) + 1.47 \sin(3\omega t - 52.8°) + 0.318 \sin(5\omega t + 73.9°) \text{ amperes}$$

Parallel Circuit Analysis when Waves Are Non-Sinusoidal. This is not appreciably different from the preceding series-circuit problem.

Example 13. Given the circuit shown in Fig. 23, with the 60-cycle constants as shown. When a voltage $v = 141.4 \sin \omega t + 70.7 \sin(3\omega t + 30°) - 28.28 \sin(5\omega t - 20°)$ volts is impressed, find the ammeter value of the total current, I, the current in each branch, power dissipated by each branch, total power dissipated, and the equation of the resultant current. $\omega = 377$ radians per second.

Fundamental

$$V_1 = \frac{141.4}{\sqrt{2}} = 100 \text{ volts magnitude}$$

$$\mathbf{V}_1 = 100 + j0 \text{ volts}$$

$$\mathbf{I}_{ab1} = \frac{100(5 + j15)}{(5 - j15)(5 + j15)} = 2 + j6 \quad \text{or} \quad 6.33 \text{ amperes}$$

$$\mathbf{I}_{cd1} = \frac{100}{10 + j2} = 9.62 - j1.925 \quad \text{or} \quad 9.82 \text{ amperes}$$

$$\mathbf{I}_{fe1} = \mathbf{I}_{ab1} + \mathbf{I}_{cd1} = 11.62 + j4.075 \quad \text{or} \quad 12.33 \text{ amperes}$$

$\mathbf{I}_{fe1}$ leads the fundamental of voltage by $\tan^{-1} \frac{4.075}{11.62} = 19.4°$

$$P_{ab1} = ei + e'i' = 100 \times 2 = 200 \text{ watts}$$

$$P_{cd1} = 100 \times 9.62 = 962 \text{ watts}$$

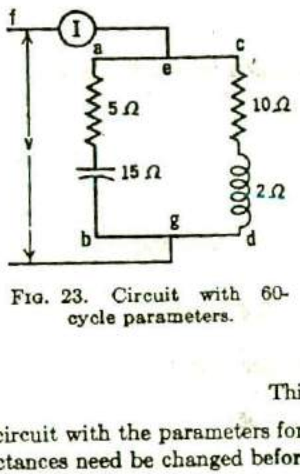


FIG. 23. Circuit with 60-cycle parameters.

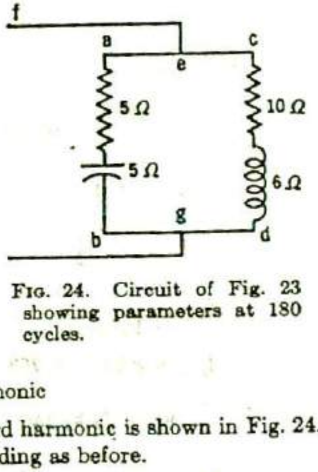


FIG. 24. Circuit of Fig. 23 showing parameters at 180 cycles.

Third Harmonic

The circuit with the parameters for the third harmonic is shown in Fig. 24. Only the reactances need be changed before proceeding as before.

$$V_3 = \frac{70.7}{\sqrt{2}} = 50 \text{ volts magnitude}$$

Take $\mathbf{V}_3$ along the reference axis for the third harmonic. (The most convenient reference axis should be chosen in any particular case in this type of analysis.)

$$\mathbf{V}_3 = 50 + j0 \text{ volts}$$

$$\mathbf{I}_{ab3} = \frac{50}{5 - j5} = 5 + j5 \quad \text{or} \quad 7.07 \text{ amperes}$$

$$\mathbf{I}_{cd3} = \frac{50}{10 + j6} = 3.68 - j2.21 \quad \text{or} \quad 4.3 \text{ amperes}$$

$$\mathbf{I}_{fe3} = 8.68 + j2.79 \quad \text{or} \quad 9.11 \text{ amperes}$$

$$\mathbf{I}_{fe3} \text{ leads } \mathbf{V}_3 \text{ by } \tan^{-1}\frac{2.79}{8.68} = 17.85°$$

$$P_{ab3} = 50 \times 5 = 250 \text{ watts}$$

$$P_{cd3} = 50 \times 3.68 = 184 \text{ watts}$$

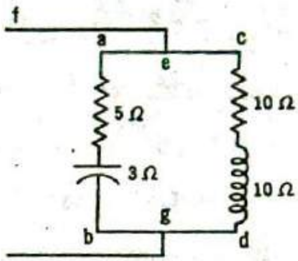


FIG. 25. Circuit of Fig. 23 showing parameters at 300 cycles.

Fifth Harmonic

The circuit with parameters for the fifth harmonic is shown in Fig. 25.

$$V_5 = \frac{28.28}{\sqrt{2}} = 20 \text{ volts}$$

Let $\mathbf{V}_5 = 20 + j0$ volts

$$\mathbf{I}_{ab5} = \frac{20}{5 - j3} = 2.94 + j1.763 \quad \text{or} \quad 3.43 \text{ amperes}$$

$$\mathbf{I}_{cd5} = \frac{20}{10 + j10} = 1 - j1 \quad \text{or} \quad 1.414 \text{ amperes}$$

$$\mathbf{I}_{fe5} = 3.94 + j0.763 \quad \text{or} \quad 4.01 \text{ amperes}$$

$$\mathbf{I}_{fe5} \text{ leads } \mathbf{V}_5 \text{ by } \tan^{-1}\frac{0.763}{3.94} = 10.95°$$

$$P_{ab5} = 20 \times 2.94 = 58.8 \text{ watts}$$

$$P_{cd5} = 20 \times 1 = 20.0 \text{ watts}$$

$$\text{Ammeter value of total current} = \sqrt{12.33^2 + 9.11^2 + 4.01^2} = 15.9 \text{ amperes}$$

$$\text{Ammeter value of current in } ab = \sqrt{6.33^2 + 7.07^2 + 3.43^2} = 10.1 \text{ amperes}$$

$$\text{Ammeter value of current in } cd = \sqrt{9.82^2 + 4.3^2 + 1.414^2} = 10.81 \text{ amperes}$$

$$P_{ab} = 200 + 250 + 58.8 = 508.8 \text{ watts}$$

$$P_{cd} = 962 + 184 + 20 = 1166 \text{ watts}$$

$$\text{Total power dissipated} = 1674.8 \text{ watts}$$

Since I_{fe1} leads V_1 by 19.4°, the equation for the fundamental of the current wave must lead the voltage wave 141.4 sin ωt by 19.4°. Hence

$$i_1 = \sqrt{2}\, 12.33 \sin (\omega t + 19.4°) \text{ amperes}$$

Similarly

$$i_3 = \sqrt{2}\, 9.11 \sin (3\omega t + 30° + 17.85°)$$
$$= \sqrt{2}\, 9.11 \sin (3\omega t + 47.85°) \text{ amperes}$$

and

$$i_5 = -\sqrt{2}\, 4.01 \sin (5\omega t - 20° + 10.95°)$$
$$= \sqrt{2}\, 4.01 \sin (5\omega t + 170.95°) \text{ amperes}$$

Therefore

$$i = i_1 + i_3 + i_5$$
$$= 17.45 \sin (\omega t + 19.4°) + 12.9 \sin (3\omega t + 47.85°)$$
$$+ 5.67 \sin (5\omega t + 171°) \text{ amperes}$$

Addition and Subtraction of Complex Waves. These operations are similar. Subtraction is performed by reversing the sign of the term to be subtracted and then adding. To illustrate, consider the bifurcated circuit shown in Fig. 26. Given

$$i_1 = 10 \sin (\omega t + 30°) - 5 \sin (3\omega t - 40°) \text{ amperes}$$
$$i_3 = 15 \sin (\omega t - 10°) + 10 \sin (3\omega t + 60°) \text{ amperes}$$

Find i_2.

From Kirchhoff's laws, $i_1 + i_2 = i_3$, or $i_2 = i_3 - i_1$.

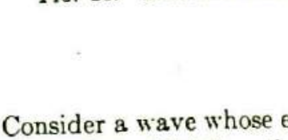


FIG. 26. Bifurcated line.

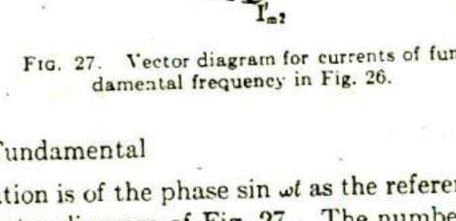


FIG. 27. Vector diagram for currents of fundamental frequency in Fig. 26.

Fundamental

Consider a wave whose equation is of the phase sin ωt as the reference. The solution will follow the vector diagram of Fig. 27. The number of primes on a symbol will indicate the order of the harmonic represented.

$$\mathbf{I}_{m1}' = 10 (\cos 30° + j \sin 30°) = 8.66 + j5$$
$$\mathbf{I}_{m3}' = 15 (\cos 10° - j \sin 10°) = 14.75 - j2.6$$
$$-\mathbf{I}_{m1}' = -8.66 - j5$$

$$\mathbf{I}_{m2}' = -\mathbf{I}_{m1}' + \mathbf{I}_{m3}' = 6.09 - j7.6 \quad \text{or} \quad 9.74 \text{ amperes}$$

$$\theta = \tan^{-1}\frac{-7.6}{6.09} = -51.3°$$

$$i_2' = 9.74 \sin(\omega t - 51.3°) \text{ amperes}$$

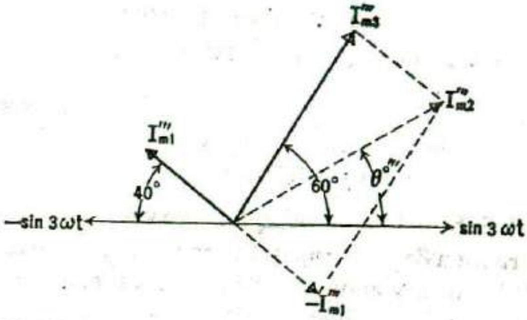


FIG. 28. Vector diagram for third harmonic currents in Fig. 26.

Third Harmonic

A wave of the phase of sin $3\omega t$ will be taken as the reference. Then the vector diagram representing the third-harmonic currents appears as shown in Fig. 28.

$$\mathbf{I}_{m1}''' = 5(\cos 140° + j \sin 140°) = -3.83 + j3.214$$

$$\mathbf{I}_{m3}''' = 10(\cos 60° + j \sin 60°) = 5 + j8.66$$

$$\mathbf{I}_{m2}''' = \mathbf{I}_{m3}''' - \mathbf{I}_{m1}''' = 5 + j8.66 + 3.83 - j3.214 = 8.83 + j5.446$$

or 10.37 amperes

$$\theta''' = \tan^{-1}\frac{5.446}{8.83} = 31.6°$$

$$i_2''' = 10.37 \sin(3\omega t + 31.6°) \text{ amperes}$$

The complete solution is

$$i_2 = i_2' + i_2'''$$

$$= 9.74 \sin(\omega t - 51.3°) + 10.37 \sin(3\omega t + 31.6°) \text{ amperes}$$

Introduction of Harmonics Due to Variation in Circuit Parameters. Harmonics in a current wave may exist even though the voltage causing it is a pure sinusoid. For example, consider a very thin filament of wire which has a high temperature coefficient of resistivity. If the wire is sufficiently thin so that it will heat and cool during a cycle as the current varies from zero to a maximum, the resistance will vary during the cycle. At the maximum point *a* on the voltage wave, Fig. 29, the resistance will be higher than at point *b*. The current at *a* will, therefore, fall below the value that would permit it to be proportional to the voltage. The wave i_1 shows the current wave for a constant resistance, whereas the dotted wave i_2 shows how it will vary when the resistance increases for the higher values of current during a cycle.

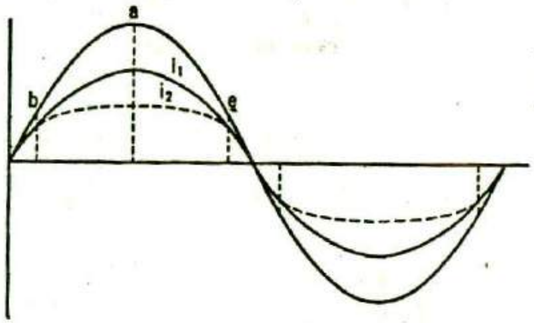


FIG. 29. Shape of i_2 wave is flatter than a sine wave owing to resistance increasing with current.

A very common example of harmonics in a current wave occurs when a sinusoidal voltage wave is impressed on an inductance coil with an iron core. As the current increases, the resulting operation on a higher part of the magnetization or saturation curve causes the inductance to become smaller. When the inductance becomes less, the inductive reactance is reduced and the current, therefore, rises more rapidly than it otherwise would. Thus the current wave becomes more peaked than a sinusoid. This is shown by Oscillogram 2, page 224 which was taken for an iron-core coil.

When the voltage on some device is to be reduced and it is desired to maintain the same wave form, a series resistance cannot be used if the current wave is not sinusoidal. The drop across the resistance will be non-sinusoidal, and this drop subtracted from an original sine wave of voltage will result in a non-sinusoidal wave across the device. In general, but not invariably, the subtraction of a non-sinusoidal voltage drop from a non-sinusoidal voltage will result in a non-sinusoidal wave of different shape from the original.

Modulated Waves. Modulated waves consist of a combination of waves of different frequencies and are, therefore, classified as complex or non-sinusoidal waves. The transmission of radio intelligence is usually accomplished by means of some combination of carrier and audio frequencies. Graphical representations of a carrier wave of relatively high frequency and of a modulating wave of relatively low frequency are shown in Fig. 30*a* and Fig. 30*b*, respectively. The carrier frequencies employed in the program broadcast band range from 540 to 1600 kc, and the modulating audio frequencies usefully employed at the transmitter range from about 30 to 10,000 cycles.

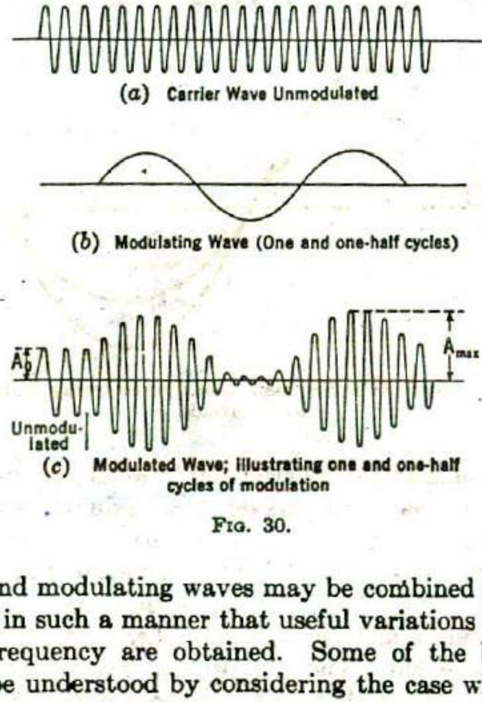


FIG. 30.

The carrier and modulating waves may be combined in a network at the transmitter in such a manner that useful variations in the resultant amplitude or frequency are obtained. Some of the basic principles involved may be understood by considering the case where the carrier frequency is generated by an ordinary type of alternator rather than by a vacuum tube oscillator. The carrier voltage will be represented by

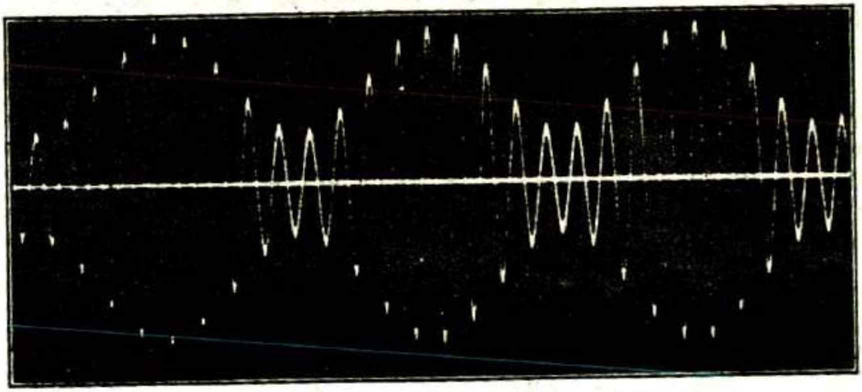
$$e_c = A_0' \sin \omega t \tag{35}$$

where A_0' is the maximum magnitude of the carrier voltage and ω is the carrier angular velocity. Either A_0' or ω may be varied in accordance with the intelligence to be transmitted, thus producing amplitude or frequency modulation. In the case of the ordinary alternator, A_0' could be made to vary by changing the field current sinusoidally and the resultant wave would correspond generally to that shown in Fig. 30*c* or in Oscillogram 5.

Amplitude modulation may be investigated conveniently by letting A_0' of equation (35) take the form $(A_0 + E_m' \sin \omega_1 t)$, where E_m' is the maximum amplitude of the modulating wave that is effectively superimposed on the carrier and ω_1 is the modulating angular velocity. E_m' is a measure of the degree of modulation (for a fixed value of A_0) and usually has values ranging from 50 to 100 per cent of A_0. Percentage modulation is defined as

$$\frac{E_m'}{A_0} \times 100 = \frac{A_{max} - A_0}{A_0} \times 100$$

where the A's refer to the amplitudes shown in Fig. 30c.



OSCILLOGRAM 5. Photograph of a sinusoidally modulated wave.

In general, the equation of a sinusoidally modulated wave is:

$$\begin{aligned} e &= (A_0 + E_m' \sin \omega_1 t) \sin \omega t \\ &= A_0 \sin \omega t + E_m' \sin \omega_1 t \sin \omega t \end{aligned} \tag{36}$$

The product of two sine waves of different frequencies may be expressed in terms of the following two well-known trigonometric relations.

$$\cos(\omega t - \omega_1 t) = \cos \omega t \cos \omega_1 t + \sin \omega t \sin \omega_1 t \tag{37}$$

$$\cos(\omega t + \omega_1 t) = \cos \omega t \cos \omega_1 t - \sin \omega t \sin \omega_1 t \tag{38}$$

Subtracting equation (38) from (37) gives

$$\cos(\omega t - \omega_1 t) - \cos(\omega t + \omega_1 t) = 2 \sin \omega t \sin \omega_1 t \tag{39}$$

Substituting the value of $\sin \omega t \sin \omega_1 t$ from equation (39) in equation (36) gives

$$\begin{aligned} e &= A_0 \sin \omega t + \frac{E_m'}{2} \cos(\omega t - \omega_1 t) - \frac{E_m'}{2} \cos(\omega t + \omega_1 t) \\ &= A_0 \sin \omega t + \frac{E_m'}{2} \cos 2\pi (f - f_1)t - \frac{E_m'}{2} \cos 2\pi (f + f_1)t \end{aligned} \tag{40}$$

Equation (40) consists of three terms. The first term, $A_0 \sin \omega t$, is of the same frequency as the original wave before modulation. This wave is called the carrier wave, and its frequency the carrier frequency. The second term, $(E_m'/2) \cos 2\pi (f - f_1)t$, has a frequency equal to $(f - f_1)$, the difference between the carrier frequency and the modulating frequency. This frequency $(f - f_1)$ is called the lower side-band frequency. The third term, $(E_m'/2) \cos 2\pi (f + f_1)t$, represents a frequency equal to $f + f_1$, the sum of the carrier and modulating frequencies. It is called the upper side-band frequency. Each of these three frequencies can be separated from the others in the resultant wave by the use of appropriate filters. If a carrier wave is modulated by a complex wave, each harmonic of the modulating wave gives rise to an upper and lower side-band frequency. Hence, in general, there are several different frequencies in each side band. The type of modulated wave presented above is primarily given as an example of non-sinusoidal waves. There are other types of modulated waves, but further discussion of them is beyond the scope of this text.

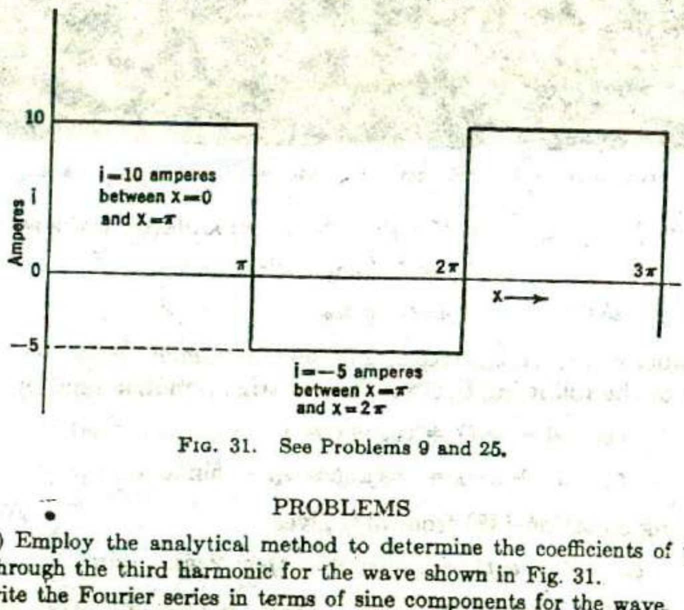


FIG. 31. See Problems 9 and 25.

PROBLEMS

9. (a) Employ the analytical method to determine the coefficients of the harmonics through the third harmonic for the wave shown in Fig. 31.

(b) Write the Fourier series in terms of sine components for the wave.

(c) Sketch the components, indicating the manner in which the components combine to approximate the original wave shape shown in Fig. 31.

10. (a) Employ the analytical method to determine the coefficients of the harmonics through the fifth harmonic for the wave shown in Fig. 32.

(b) Write the equation of the wave through the fifth harmonic.

(c) Sketch the components, indicating the manner in which the components combine to approximate the original wave shown in Fig. 32.

11. A certain current wave has a height of 1 from 0° to 30°, then increases linearly in a positive direction to a value of 3 at 60°, after which it remains at a height of 3 until 120° is reached. It then decreases linearly to a value of zero at 150° and then remains at zero value until 360°. The cycle is then repeated. Find A_0, A_1, and B_1 of the Fourier series terms which represent this wave.

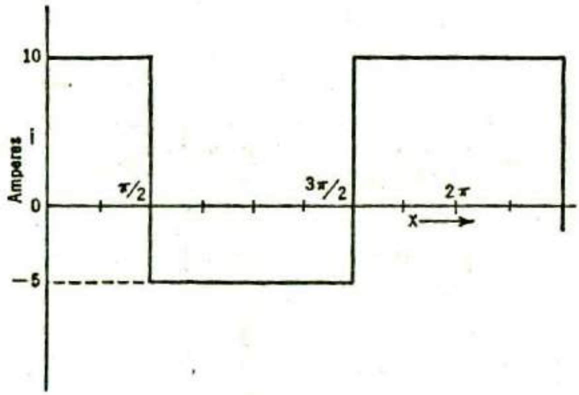


FIG. 32. See Problem 10.

12. A current wave is defined over one complete cycle by the following data:

x (in degrees)	i (in amperes)	x (in degrees)	i (in amperes)
0	−2.000	195	−3.613
15	+0.149	210	−5.000
30	+3.000	225	−6.364
45	+6.364	240	−7.660
60	+9.660	255	−8.634
75	+12.098	270	−9.000
90	+13.000	285	−8.634
105	+12.098	300	−7.660
120	+9.660	315	−6.364
135	+6.364	330	−5.000
150	+3.000	345	−3.613
165	+0.149	360	−2.000
180	−2.000	375	+0.149

(a) Employ the analyzing tables on pages 237 to 240, evaluate the Fourier series coefficients A_0, A_1, B_1, A_2, B_2, and A_3 of the above wave form, and write the Fourier series in equational form. (*Note*: Evaluations based on 15° intervals will be sufficiently accurate in this case since the actual Fourier series contains no terms beyond the A_3 term. Call any coefficient zero which is no greater in magnitude than the probable arithmetical error involved.)

(b) Graph each of the components and combine these components to form the resultant wave. Check various values on the resultant graph against the original data.

13. Employ the method of equations (21) and (22) and evaluate the Fourier series coefficients through the third harmonic for the wave given in Problem 12.

Write the following equation in terms of three sine components only:

$$v = 4.0 \sin \omega t - 3.0 \cos \omega t - 7.66 \sin 2\omega t + 6.43 \cos 2\omega t$$
$$- 2 \sin 3\omega t - 1.5 \cos 3\omega t$$

15. Given an a-c wave form as defined by the following table of measured ordinates:

Ordinate No.	Degrees	Measured Ordinate	Ordinate No.	Degrees	Measured Ordinate
0	0	0.0	19	95	7.1
1	5	0.8	20	100	7.4
2	10	1.7	21	105	8.0
3	15	2.7	22	110	9.0
4	20	3.6	23	115	10.5
5	25	4.5	24	120	12.0
6	30	5.6	25	125	13.2
7	35	6.9	26	130	14.0
8	40	8.2	27	135	14.0
9	45	9.7	28	140	13.0
10	50	10.7	29	145	11.6
11	55	11.0	30	150	10.0
12	60	11.0	31	155	8.0
13	65	10.4	32	160	5.8
14	70	9.8	33	165	4.0
15	75	9.2	34	170	2.5
16	80	8.5	35	175	1.0
17	85	7.8	36	180	0.0
18	90	7.0			

Negative loop similar to positive loop.

(a) Graph the wave and analyze it by the Fourier series method for fundamental the third, the fifth, and the seventh harmonics by the use of analyzing tables.

(b) Write the equation of the wave in terms of its sine and cosine components.

(c) Write the equation of the wave in terms of sine components only.

(d) Synthesize the components graphically, and compare the resultant with the original wave.

16. Employ equations (21) and (22) instead of analyzing tables, and find the sine and cosine coefficients of the Fourier series to include the seventh harmonic for the wave in Problem 15. Express the resultant wave in terms of four sine components only.

17. Given an a-c wave form as defined by the measured ordinates shown on page 269.

Analyze the wave by using equations (21) and (22) for the first seven harmonics, and write the Fourier series equation for the wave.

Degrees	Measured Ordinate	Degrees	Measured Ordinate
0	−0.6064	100	0.7848
10	0.1736	110	0.6767
20	0.9484	120	0.4966
30	1.4139	130	0.4200
40	1.4428	140	0.5669
50	1.149	150	0.8832
60	0.79	160	1.1420
70	0.5937	170	1.0880
80	0.6154	180	0.6064
90	0.737		

Negative loop similar to positive loop.

18. Show whether the following waves have symmetry with respect to the positive and negative loops:

$$e = 100 \sin (\omega t + 30°) - 50 \cos 2\omega t + 25 \sin (5\omega t + 150°) \text{ volts}$$

$$i = 20 \sin (\omega t + 40°) + 10 \sin (2\omega t + 30°) - 5 \sin (5\omega t - 50°) \text{ amperes}$$

19. Does either of the waves in Problem 18 possess symmetry about the mid-ordinate of the positive and negative loops? Why?

20. Are the following waves of the same wave form or shape? Give reason.

$$v = 100 \sin (\omega t + 70°) - 60 \sin (2\omega t - 30°) + 30 \sin (3\omega t - 60°)$$

$$i = 50 \cos (\omega t - 60°) + 30 \sin (2\omega t + 70°) - 15 \cos (3\omega t - 90°)$$

21. Are the following two waves of the same wave form? Give reason.

$$e = 100 \sin (\omega t - 20°) + 50 \sin (3\omega t + 60°) - 25 \cos (5\omega t - 30°) \text{ volts}$$

$$i = 20 \cos (\omega t - 60°) - 10 \sin (3\omega t + 15°) + 5 \sin (5\omega t - 70°) \text{ amperes}$$

22. Find the effective values of the voltage and current waves of Problem 18.

23. Find the effective value of:

$$v = 100 \sin (\omega t + 30°) - 40 \sin (2\omega t - 30°) + 40 \sin (2\omega t + 30°) + 20 \cos (5\omega t - 30°)$$

24. A complex wave has harmonics of the following effective values: fundamental 100 volts, third harmonic 70 volts, and fifth harmonic 50 volts. Find the voltmeter value of the complex wave.

25. The Fourier representation of the current variation shown in Fig. 31 is:

$$i = 2.5 + \frac{30}{\pi} \sin x + \frac{30}{3\pi} \sin 3x + \frac{30}{5\pi} \sin 5x + \frac{30}{7\pi} \sin 7x + \frac{30}{9\pi} \sin 9x + \cdots$$

Compare the effective value of the current as calculated by equation (27), page 250 (employing only the first six terms of the series given above), with the true effective value.

26. The current flowing through a particular filter choke is: $i = 5 + 2 \sin x$ amperes, where x ($= 754t$) represents angular measure. Sketch the wave shape of this current variation.

(a) What are the maximum, minimum, and average values of current?

(b) Does the maximum value of the a-c component satisfy the relation: $I_{m(ac)} = 0.5\ (I_{max} - I_{min})$?

(c) What is the effective value of the current: $i = 5 + 2 \sin x$ amperes?

27. Assuming that a pulsating direct current is composed of a d-c component (I_{dc}) and a single-frequency a-c component, the general expression for the current variation is: $i = I_{dc} + I_{m(ac)} \sin x$.

(a) If only the average and effective values of the pulsating current were known, would it be possible to find the maximum value of the a-c component, $I_{m(ac)}$?

(b) The average value of $i = I_{dc} + I_{m(ac)} \sin x$ is 4 amperes, and the effective value is 5 amperes. Find $I_{m(ac)}$.

28. Considering only second harmonic distortion, the plate current of one class of amplifiers (with sinusoidally varying grid-cathode excitation) is given by the equation:

$$i = I_0 + I_{m1} \sin x - I_{m2} \cos 2x$$

where $I_0 = I_b + I_{m2}$, I_b being the steady plate current with no a-c grid excitation.

(a) Sketch the wave form of the current variation for $I_0 = 0.2$, $I_{m1} = 0.1$, and $I_{m2} = 0.01$ ampere. Indicate the value of I_b on the sketch.

(b) What are the maximum (I_{max}), minimum (I_{min}), and average values of the wave form sketched in (a)? Does the average value of current (I_0) satisfy the relation: $0.5(I_{max} + I_{min})$?

29. Refer to the plate current variation given in Problem 28, namely,

$$i = I_0 + I_{m1} \sin x - I_{m2} \cos 2x$$

(a) If it is known that the average value of plate current changes from the steady value $I_b = I_0 - I_{m2}$ (with no a-c grid excitation) to the average value I_0 with a-c grid excitation, show either graphically or analytically that:

$$I_{max} \text{ (with a-c grid excitation)} = I_b + I_{m1} + 2I_{m2}$$

$$I_{min} \text{ (with a-c grid excitation)} = I_b - I_{m1} + 2I_{m2}$$

$$I_{m1} = 0.5\ (I_{max} - I_{min})$$

$$I_{m2} = \frac{(I_{max} + I_{min}) - 2I_b}{4}$$

(b) Show that the ratio of I_{m2} to I_{m1} expressed in per cent is:

$$\frac{I_{m2}}{I_{m1}} \times 100 = \frac{0.5(I_{max} + I_{min}) - I_b}{(I_{max} - I_{min})} \times 100$$

Note: The above ratio is called the per cent second harmonic distortion, and, since the values of I_{max}, I_{min}, and I_b may be readily measured under the conditions of steady grid bias, the above relation is sometimes used to determine the per cent second harmonic distortion where unsymmetrical positive and negative peaks of plate current are encountered.

(c) Determine the per cent harmonic distortion from $(I_{m2}/I_{m1}) \times 100$ and from the equation given in (b) if $I_0 = 0.2$, $I_{m1} = 0.1$, and $I_{m2} = 0.01$ ampere. ($I_b = 0.2 - 0.01$ ampere.)

30. Because of irregularities in the "straight" portion of the plate current-grid voltage characteristic of a vacuum tube, the equation for the plate current sometimes takes the general form

$$i = I_b + I_{m1} \sin x + I_{m3} \sin 3x$$

where I_b is the plate current corresponding to fixed values of grid-cathode and plate-cathode voltages. Find the maximum, the minimum, and the average values of i if $I_b = 0.2$, $I_{m1} = 0.07$, and $I_{m3} = 0.005$ ampere.

31. Calculate the power represented by the voltage and current in Problem 18.

32. Calculate the power represented by the current and voltage of Problem 21.

33. Calculate the power factor for the waves in Problem 18.

34. Determine the power factor for the waves in Problem 21.

35. Given: $v = 100 \sin (\omega t + 60°) - 50 \sin (3\omega t - 30°)$ volts

$i = 10 \sin (\omega t + 60°) + 5 \cos (3\omega t + 60°)$ amperes

(a) Calculate the power and power factor for the above waves.

(b) If only the magnitude of the third harmonic in the current wave is varied, what would be its value to bring the power factor for the composite waves to 0.8?

36. Determine the equivalent sine waves for the voltage and current in Problem 18.

37. Find the deviation factor for the voltage

$$e = 100 \sin (\omega t - 25.36°) + 50 \sin (3\omega t + 58.92°)$$

38. A voltage $v = 100 \sin (\omega t + 30°) - 50 \sin (3\omega t + 60°) + 30 \cos 5\omega t$ volts is impressed on a resistance of 6 ohms in series with a capacitance of 88.4 μf and an inductance of 0.01061 henry. Find the ammeter value of the current, the power dissipated by the circuit, the power factor of the whole circuit, and the voltage drop across the capacitance if $\omega = 377$ radians per second.

39. A current of $i = 10 \sin (\omega t - 60°) + 5 \sin (2\omega t + 20°)$ amperes flows in a series circuit consisting of 8 ohms resistance, 10 ohms 60-cycle capacitive reactance, and 4 ohms 60-cycle inductive reactance. Find the equation of the impressed voltage wave. $\omega = 377$ radians per second.

40. A branch containing 5 ohms resistance in series with an inductance of 0.00796 henry is in parallel with another branch consisting of a resistance of 6 ohms in series with a 60-cycle capacitive reactance of 15 ohms. For a voltage of $e = 100 \sin (\omega t + 30°) - 50 \cos (3\omega t - 30°)$ volts impressed on the combination, find the equation of the current wave required by the combination. $\omega = 377$ radians per second.

41. Find the ammeter readings in each branch and the supply line to the circuit of Problem 40.

42. Determine the power dissipated in each branch of the circuit of Problem 40 and the total power taken by the whole circuit.

43. Calculate the power factor of the whole circuit in Problem 40 and the power factor of each branch.

44. The following two currents flow toward a certain junction:

$$i_1 = 20 \sin (\omega t + 30°) - 10 \sin (2\omega t - 30°) + 5 \sin (3\omega t - 40°) \text{ amperes}$$

$$i_2 = 15 \cos \omega t + 10 \cos (2\omega t - 60°) + 10 \cos (3\omega t + 50°) \text{ amperes}$$

Find the equation of the current leaving the junction. What is the ammeter or effective value of each of the three currents?

45. Subtract i_2 from i_1 in Problem 44, and find the equation of the resultant.

46. At 60 cycles a certain impedance, Z_1, consists of 4 ohms resistance, 6 ohms capacitive reactance, and 3 ohms inductive reactance in series. Another identical

impedance, Z_2, is connected in parallel with Z_1. A third 60-cycle impedance (consisting of 1.5 ohms resistance and 2 ohms inductive reactance in series) is connected in series with the parallel combination of Z_1 and Z_2. If a voltage $v = 100 \sin 377t - 50 \sin 3(377t + 30°)$ volts is impressed on the entire series-parallel circuit, calculate: (*a*) the total rms current taken, (*b*) the rms current in each branch, (*c*) the equation of the current in branch Z_1, (*d*) the total power consumed, (*e*) the power factor of whole circuit.

47. The wave form given in Fig. 33 consists of a fundamental term $A_1 \sin x$ and one and only one other Fourier series term.

(*a*) What are the numerical values of the coefficients of the two terms?

(*b*) Write the equation of the wave. *Note:* It is suggested that the problem be solved by inspection and checked by the second graphical method of analysis, given on pages 242–246.

48. A capacitor having 20 $\mu\mu$f capacitance is connected in parallel with a coil having 20 microhenrys inductance and a series resistance as specified in (*a*) and (*b*) below. This parallel combination is energized with a pulse of current which is zero for $140° \lessgtr \omega t \lessgtr 40°$ during each cycle. The pulse reaches a maximum value of 100 milliamperes at $\omega t = 90°$ and

$$i(45°) = i(135°) = 18 \text{ milliamperes}$$

$$i(55°) = i(125°) = 49 \text{ milliamperes}$$

$$i(65°) = i(115°) = 73.5 \text{ milliamperes}$$

$$i(75°) = i(105°) = 90.5 \text{ milliamperes}$$

$$i(85°) = i(95°) = 99 \text{ milliamperes}$$

where $i(45°)$ means the value of i at $\omega t = 45°$.

Find the effective magnitude of the fundamental component of voltage developed across the parallel branches if $\omega = \frac{5}{3} \times 10^7$ radians per second. Compare this value of voltage with the third harmonic voltage developed across the parallel branches, recognizing the fact that the branches are tuned to the third harmonic. (*a*) Assume that $R = 10\,\Omega$ is the same for the fundamental and third harmonic. (*b*) Assume that $Q = \omega L/R$ is constant, R being $10\,\Omega$ for the fundamental.

chapter VII Coupled Circuits

Terminology. In electrical-engineering literature, the term "circuit" is used in a variety of ways. At times it is employed to designate a single branch of an electrical network; at other times it is used synonymously with the term "network" to mean a combination of two or more branches which are interrelated either electrically or magnetically, or both. In the present chapter the term "circuit" is employed to mean "any complete electrical loop around which Kirchhoff's emf law can be written."

Two circuits are said to be "coupled" when they are so related that energy interchanges can take place between them. More specifically this means that a potential difference appears in either of the two circuits which are coupled, if and when the other is energized. The circuits involved may be coupled conductively, electromagnetically, or electrostatically. Various combinations of these principal modes of coupling may exist between circuits. However, the great majority of the circuits in actual practice are coupled either conductively or electromagnetically.

Coupled circuits interact upon one another, and in general the movement of electricity in any particular circuit is governed, not only by the circuit parameters of that circuit, but to some extent by the parameters of all circuits to which the circuit in question is coupled.

Conductively Coupled Circuits. Two circuits which are conductively coupled are shown in Fig. 1. In a circuit arrangement of this kind, circuit 1 may be viewed as the driving or primary circuit and circuit 2 as the receiving or secondary circuit. $\mathbf{Z}_{12}$, the impedance of the branch which is common to both circuits, is called the mutual impedance between circuit 1 and circuit 2. The mutual impedance may consist, theoretically, of a pure resistance, a pure inductance, a pure capacitance, or some combination of these circuit elements.

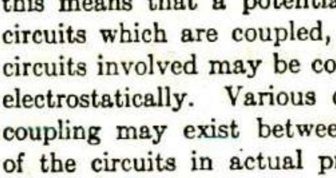


FIG. 1. Conductively coupled circuits.

If the exciting voltage and circuit parameters of Fig. 1 are given, the currents, component voltages, and component powers can be evaluated by simple circuit analysis.

In general the "loop current" method of solution[1] is particularly well suited to coupled circuit solutions. If this method of attack is employed, I_1 and I_2 are considered as the currents which flow around the complete loops of circuit 1 and circuit 2, respectively. The positive circuit directions assigned to I_1 and I_2 are, of course, arbitrary. If positive circuit directions are assigned to I_1 and I_2, as shown in Fig. 1, the actual current in the Z_{12} branch in the $+I_1$ direction is $I_1 - I_2$. The details of the "mesh current" method of solution as applied to Fig. 1 are given below. By definition:

$$\mathbf{Z}_{11} = \mathbf{Z}_1 + \mathbf{Z}_{12} \quad \text{(Impedance of circuit 1 to } I_1\text{)}$$

$$\mathbf{Z}_{22} = \mathbf{Z}_2 + \mathbf{Z}_{21} \quad \text{(Impedance of circuit 2 to } I_2\text{)}$$

If the circuit parameters are constant,

$$\mathbf{Z}_{12} = \mathbf{Z}_{21} \quad \text{(Mutual impedance between circuits 1 and 2)}$$

The application of Kirchhoff's emf law to circuits 1 and 2 of Fig. 1 results in:

$$\mathbf{Z}_{11}\mathbf{I}_1 - \mathbf{Z}_{12}\mathbf{I}_2 = \mathbf{E}_1 \tag{1}$$

$$-\mathbf{Z}_{21}\mathbf{I}_1 + \mathbf{Z}_{22}\mathbf{I}_2 = 0 \tag{2}$$

Employing elementary determinants, the expressions for $\mathbf{I}_1$ and $\mathbf{I}_2$ become:

$$\mathbf{I}_1 = \frac{\begin{vmatrix} \mathbf{E}_1 & -\mathbf{Z}_{12} \\ 0 & \mathbf{Z}_{22} \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_{11} & -\mathbf{Z}_{12} \\ -\mathbf{Z}_{21} & \mathbf{Z}_{22} \end{vmatrix}} = \frac{\mathbf{E}_1\mathbf{Z}_{22}}{\mathbf{Z}_{11}\mathbf{Z}_{22} - \mathbf{Z}_{12}^2} \tag{3}$$

$$\mathbf{I}_2 = \frac{\begin{vmatrix} \mathbf{Z}_{11} & \mathbf{E}_1 \\ -\mathbf{Z}_{21} & 0 \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_{11} & -\mathbf{Z}_{12} \\ -\mathbf{Z}_{21} & \mathbf{Z}_{22} \end{vmatrix}} = \frac{\mathbf{E}_1\mathbf{Z}_{21}}{\mathbf{Z}_{11}\mathbf{Z}_{22} - \mathbf{Z}_{12}^2} \tag{4}$$

The above method is generally applicable and may be extended to include any number of coupled circuits.

[1] In general circuit analysis many of the disagreeable details can be avoided by making use of this method. It is sometimes referred to as Maxwell's "cyclic current" method. See "A Treatise on Electricity and Magnetism," by Maxwell, Vol. 1, 3rd edition. See also Chapter I of this text.

Example 1. Let it be assumed that, in Fig. 1: $\mathbf{E}_1 = 100 \angle 0°$ volts, $\mathbf{Z}_1 = 3 + j4$ ohms, $\mathbf{Z}_{12} = 10 + j0$ ohms, and $\mathbf{Z}_2 = 4 - j8$ ohms. The impedance of the generator is considered to be negligibly small, or else its impedance is included in $\mathbf{Z}_1$.

$$\mathbf{Z}_{11} = (3 + j4) + (10 + j0) = 13 + j4 = 13.6 \angle 17.1° \text{ ohms}$$

$$\mathbf{Z}_{22} = (4 - j8) + (10 + j0) = 14 - j8 = 16.1 \angle -29.7° \text{ ohms}$$

$$\mathbf{Z}_{11}\mathbf{Z}_{22} = 219 \angle -12.6° = 214 - j47.8$$

$$\mathbf{Z}_{11}\mathbf{Z}_{22} - \mathbf{Z}_{12}^2 = 114 - j47.8 = 123.7 \angle -22.7°$$

$$\mathbf{I}_1 = \frac{(100 \angle 0°)(16.1 \angle -29.7°)}{123.7 \angle -22.7°} = 13.0 \angle -7° \text{ amperes}$$

$$\mathbf{I}_2 = \frac{(100 \angle 0°)(10 \angle 0°)}{123.7 \angle -22.7°} = 8.08 \angle 22.7° \text{ amperes}$$

The current in the $\mathbf{Z}_{12}$ branch in the direction of $\mathbf{I}_1$ is $\mathbf{I}_{12} = (\mathbf{I}_1 - \mathbf{I}_2)$.

$$\begin{aligned}\mathbf{I}_{12} &= 13.0\,(0.992 - j0.122) - 8.08\,(0.922 + j0.386)\\ &= (12.9 - j1.59) - (7.45 + j3.12)\\ &= 5.45 - j4.71 = 7.21 \angle -40.8° \text{ amperes}\end{aligned}$$

The total power generated by the generator E_1 is:

$$\begin{aligned}P_{gen} &= E_1 I_1 \cos\theta \Big]_{I_1}^{E_1} = 100 \times 13.0 \cos(-7°)\\ &= 1290 \text{ watts (approximately)}\end{aligned}$$

The total power absorbed by the network is:

$$\begin{aligned}I_1^2R_1 + I_2^2R_2 + I_{12}^2R_{12} &= 13.0^2 \times 3 + 8.08^2 \times 4 + 7.21^2 \times 10\\ &= 1288 \text{ watts (approximately)}\end{aligned}$$

Problem 1. Solve for $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_{12}$ in the above illustrative example by first reducing the coupled circuits to an equivalent series impedance. Draw the vector diagram of $\mathbf{E}_1$, $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_{12}$, $\mathbf{V}_{12}$, illustrating vectorially that $\mathbf{V}_{12} = \mathbf{E}_1 - \mathbf{I}_1\mathbf{Z}_1$.

Ans.: Given in the above illustrative example.

Mutual Impedance. Before proceeding with particular types of coupled circuits, we shall state some general definitions which will be useful later in this chapter and also in radio courses where the coefficient of coupling plays a far more prominent role than it does in a first course.

The mutual impedance between, say, circuits 1 and 2 of a general network is defined as the ratio of the voltage developed in circuit 2 per unit current in circuit 1 when all circuits except circuit 1 are open-circuited. This mutual impedance has already been employed in the foregoing section as $\mathbf{Z}_{21}$. If linear bilateral circuit elements are employed in the coupling of the two circuits, it should be plain that $\mathbf{Z}_{12}$, the ratio of the voltage developed in circuit 1 per unit current in circuit 2 with all circuits except circuit 2 open-circuited, is equal to $\mathbf{Z}_{21}$.

The definition given above for mutual impedance between two circuits can be generalized to apply to two pairs of terminals, 11′ and 22′, as shown in Fig. 2 where the network in the box may be any con-

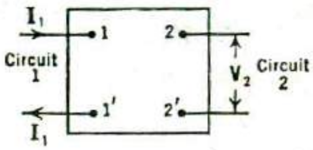


FIG. 2. Circuit 1 coupled to circuit 2 through an arbitrary network not shown.

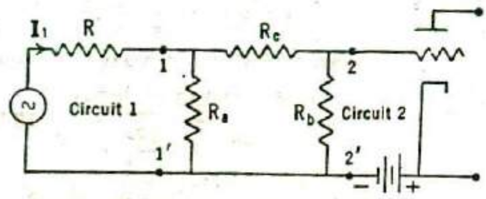


FIG. 3. Circuit 1 coupled to circuit 2 through a π set of resistances.

figuration of impedances. If, for example, the terminals 11′ and 22′ of Fig. 3 are selected, we would find upon measurement that

$$\mathbf{Z}_{21} = \frac{\mathbf{V}_2}{\mathbf{I}_1} = \frac{\mathbf{V}_b}{\mathbf{I}_1} = \frac{\dfrac{\mathbf{V}_a}{R_b + R_c} R_b}{\dfrac{\mathbf{V}_a(R_a + R_b + R_c)}{R_a(R_b + R_c)}} = \frac{R_a R_b}{R_a + R_b + R_c}$$

where $\mathbf{V}_b$ is the voltage developed across R_b (terminals 22′) and $\mathbf{V}_a$ is the voltage drop across R_a. The same result would have been obtained had the π set of resistors $(R_a - R_b - R_c)$ been transformed to an equivalent Y set of resistors.

In many networks, particularly in the field of radio, the direct currents must be confined to specified paths and a-c energy is transferred from

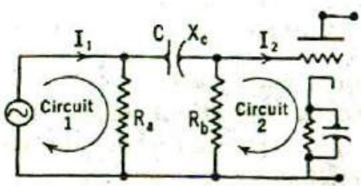


FIG. 4. Circuits coupled through R_a-C-R_b network.

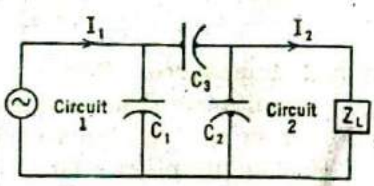


FIG. 5. Circuits coupled through C_1-C_2-C_3 network.

one circuit to another through the agency of an electric or magnetic field. In Fig. 4, for example, a-c energy may be transferred from circuit 1 to circuit 2 by way of the electric field existing between the plates of the coupling condenser, C.

A particular form of capacitive coupling is shown in Fig. 5. If the coupling reactance between circuit 1 and circuit 2 is defined as the voltage developed in circuit 2, namely, the voltage across C_2, per unit current in circuit 1, this coupling reactance is:

$$X_{\text{coupling}} = \frac{\dfrac{V_1}{X_2 + X_3} X_2}{\dfrac{V_1(X_1 + X_2 + X_3)}{X_1(X_2 + X_3)}} = \frac{X_1 X_2}{X_1 + X_2 + X_3}$$

where V_1 is the voltage across C_1 and the X's are the capacitive reactances of the respective condensers. The coupling capacitance between circuit 1 and circuit 2 (or vice versa) is:

$$C_{\text{coupling}} = \frac{1}{\omega X_{\text{coupling}}} = \frac{1}{\omega \dfrac{(1/\omega C_1)(1/\omega C_2)}{(1/\omega C_1) + (1/\omega C_2) + (1/\omega C_3)}}$$
$$= C_1 + C_2 + \frac{C_1 C_2}{C_3}$$

Problem 2. Show that the voltage developed across condenser C_1 per unit current flowing in circuit 2 of Fig. 5 is:

$$\frac{X_1 X_2}{X_1 + X_2 + X_3} = X_{\text{coupling}}$$

where $X_1 = 1/\omega C_1$, $X_2 = 1/\omega C_2$, and $X_3 = 1/\omega C_3$.

Problem 3. Consider R_a, R_b, and X_c of Fig. 4 to be a coupling device between circuit 1 and circuit 2. Show that the coupling impedance between the two circuits is:

$$\mathbf{Z}_{\text{coupling}} = \frac{(R_a^2 R_b + R_a R_b^2) + jR_a R_b X_c}{(R_a + R_b)^2 + X_c^2}$$

Note:

$$\mathbf{Z}_{\text{coupling}} = \frac{\mathbf{V}_b}{\mathbf{I}_1}$$

where $\mathbf{V}_b$ is the voltage developed across R_b by $\mathbf{I}_1$, or

$$\mathbf{Z}_{\text{coupling}} = \frac{\mathbf{V}_a}{\mathbf{I}_2}$$

where V_a is the voltage developed across R_a by $\mathbf{I}_2$.

Coefficient of Coupling. Given two pairs of terminals, $11'$ and $22'$, as shown in Fig. 2. The coefficient of coupling between circuit $11'$ and circuit $22'$ will be defined as:

$$\mathbf{k} = \frac{\mathbf{Z}_{12}}{\sqrt{\mathbf{Z}_{11'}\mathbf{Z}_{22'}}} = \frac{\mathbf{Z}_{21}}{\sqrt{\mathbf{Z}_{11'}\mathbf{Z}_{22'}}}$$

where $\mathbf{Z}_{12}$ is the mutual impedance between circuits 2 and 1. $\mathbf{Z}_{21} = \mathbf{Z}_{12}$.

$\mathbf{Z}_{11'}$ is the impedance seen looking into terminals 11′ with terminals 22′ open-circuited.

$\mathbf{Z}_{22'}$ is the impedance seen looking into terminals 22′ with terminals 11′ open-circuited.

Example 2. Consider terminals 11′ and 22′ of Fig. 3. Let it be required to find the coefficient of coupling between circuits 1 and 2.

It has been shown that

$$\mathbf{Z}_{21} = \mathbf{Z}_{12} = \frac{R_aR_b}{R_a + R_b + R_c}$$

$$\mathbf{Z}_{11'} = \frac{R_a(R_b + R_c)}{R_a + R_b + R_c}$$

$$\mathbf{Z}_{22'} = \frac{R_b(R_a + R_c)}{R_a + R_b + R_c}$$

$$\mathbf{k} = \frac{R_aR_b}{\sqrt{R_a(R_b + R_c)R_b(R_a + R_c)}}$$

If, for example, $R_c = 0$, the coefficient of coupling is unity. It should be noted that, with the general definition of coupling coefficient which has been given, $\mathbf{k}$ may be complex and greater than unity. In most cases, however, the coefficient of coupling is real and less than unity as in this example.

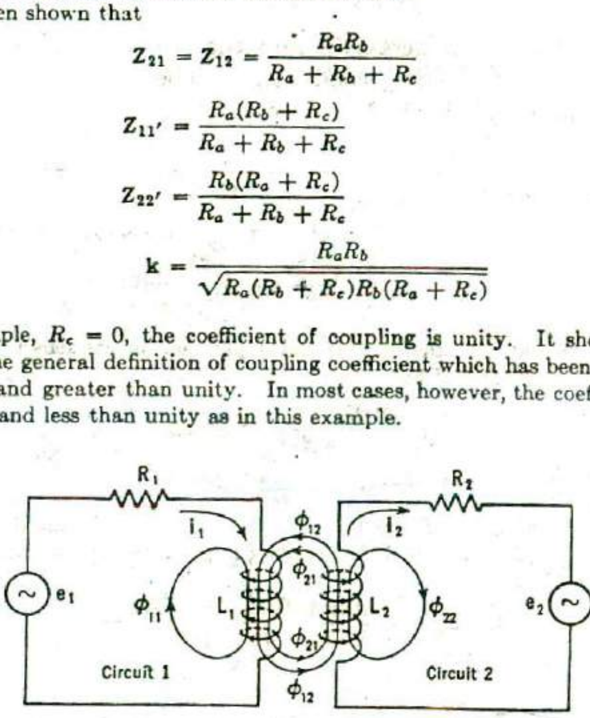


FIG. 6. Illustrating the four component fluxes ϕ_{11}, ϕ_{12}, ϕ_{22}, and ϕ_{21} into which the resultant magnetic field is separated for the purpose of analysis.

Magnetic Coupling. If a portion of the magnetic flux established by one circuit interlinks with a second circuit, the two circuits are coupled magnetically and energy may be transferred from one circuit to the other by way of the magnetic field which is common to the two circuits. The practical operation of many devices depends upon this type of coupling.

Separation of Magnetic Flux into Hypothetical Components. Magnetic coupling between two individual circuits is shown in Fig. 6. For the purpose of analysis, the total flux which is established by i_1, namely,

ϕ_1, is divided into two components. One component of ϕ_1 is that part which links with circuit 1 but not with circuit 2, namely, ϕ_{11}. The second component of ϕ_1 is ϕ_{12}, that part which links with both circuit 2 and circuit 1. In a similar manner, the flux established by i_2 is separated into two components for the sake of detailed analysis.

By definition:

$$\phi_1 = \phi_{11} + \phi_{12} \tag{5}$$

and

$$\phi_2 = \phi_{22} + \phi_{21} \tag{6}$$

The four component fluxes are shown in Fig. 6, and a recapitulation of their definitions is given below:

ϕ_{11} the fractional part of ϕ_1 which links *only* with the turns of circuit 1. This is the leakage flux of circuit 1 with respect to circuit 2.

ϕ_{12} the fractional part of ϕ_1 which links with the turns of circuit 2. This is the mutual flux produced by circuit 1.

ϕ_{22} the fractional part of ϕ_2 which links *only* with the turns of circuit 2. This is the leakage flux of circuit 2 with respect to circuit 1.

ϕ_{21} the fractional part of ϕ_2 which links with the turns of circuit 1. This is the mutual flux produced by circuit 2.

It should be recognized that the actual flux established by i_1 or i_2 does not conform to the simple configurations shown in Fig. 6. For example, part of ϕ_{11} links with only a fraction of the total turns of circuit 1, and likewise a part of ϕ_{12} links with only a fractional part of the turns of circuit 2. ϕ_{11} is a hypothetical flux which, when linking with all the turns, N_1, produces the same total flux linkages as the true flux linkages in question. Similar concepts are held for the other component fluxes, and, when used quantitatively in this manner, they represent accurately the true condition of affairs as regards induced voltages.

Mutual Inductance. In order to describe the magnetic interaction between circuits or between portions of the same circuit, the circuit parameter M is introduced. It is called the coefficient of mutual inductance, or simply mutual inductance, and is dimensionally equivalent to the coefficient of self-inductance, L. The similarity between the concept of mutual inductance of (or between) two circuits and the concept of self-inductance may be shown in the following manner. Refer to Fig. 6. For the purpose at hand we shall define the self-inductance of circuit 1 as:

$$L_1 = \frac{N_1\phi_1}{i_1} \left[\text{flux linkages of circuit 1 per unit current in circuit 1}\right] \quad (7)$$

On the same basis of reckoning, the mutual inductance of circuit 1 with respect to circuit 2 is:

$$M_{21} = \frac{N_1\phi_{21}}{i_2} \left[\text{flux linkages of circuit 1 per unit current in circuit 2}\right] \quad (8)$$

Also the mutual inductance of circuit 2 with respect to circuit 1 is:

$$M_{12} = \frac{N_2\phi_{12}}{i_1} \left[\text{flux linkages of circuit 2 per unit current in circuit 1}\right] \quad (9)$$

If the ϕ/i characteristics in equations (7), (8), and (9) are not straight lines, then L_1, M_{21}, and M_{12} are variable circuit parameters and for certain types of analyses can best be written in the forms:

$$L_1 = N_1 \frac{d\phi_1}{di_1} \quad (7a)$$

$$M_{21} = N_1 \frac{d\phi_{21}}{di_2} \quad (8a)$$

$$M_{12} = N_2 \frac{d\phi_{12}}{di_1} \quad (9a)$$

If, however, the flux is proportional to the current (i.e., permeability constant), both self-inductance and mutual inductance in equations (7), (8), and (9) are constant and as such are very useful circuit parameters in classical circuit theory.

Under the condition of constant permeability, the reluctance of the *mutual flux path* ($\mathcal{R}_{21}$ or $\mathcal{R}_{12}$) is a fixed quantity and $\mathcal{R}_{21} = \mathcal{R}_{12}$.

$$M_{21} = \frac{N_1\phi_{21}}{i_2} = \frac{KN_1N_2}{\mathcal{R}_{21}} \quad (10)$$

$$M_{12} = \frac{N_2\phi_{12}}{i_1} = \frac{KN_2N_1}{\mathcal{R}_{12}} \quad (11)$$

where K is a constant which depends for its value upon the units employed in evaluating $\phi = KNi/\mathcal{R}$. Therefore, if the permeability of the mutual flux path is constant, M_{21} and M_{12} are constant and $M_{21} = M_{12} = M$. This fact may also be proved in terms of the energies stored in the magnetic field when both circuits are energized.

If the permeability of the mutual flux path is not constant, neither M_{21} nor M_{12} will be constant and the following method of representing mutually induced voltages in terms of M loses much of its effectiveness. Unless otherwise stated, absence of ferromagnetic material will be assumed, in which case $M_{21} = M_{12} = M$.

The units in which mutual inductance is expressed are identical with the units in which self-inductance is expressed, usually the henry or millihenry. If the flux linkages in equations (8) or (9) are expressed in weber-turns (10^8 maxwell-turns) and the current is expressed in amperes, M is given in henrys.

Problem 4. Refer to Fig. 6, page 278, and assume that the L_1 coil consists of 50 turns and that the L_2 coil consists of 500 turns.

(a) What is the mutual inductance between the two circuits (in millihenrys) if 5 amperes in circuit 1 establishes a total equivalent flux (ϕ_1) of 30,000 maxwells 27,500 maxwells of which link with the turns of the L_2 coil?

(b) What is the self-inductance of the L_1 coil?

Ans.: (a) $M_{12} = 27.5$ millihenrys; (b) $L_1 = 3$ millihenrys.

Mutual Reactance, X_M. It is evident that any change in i_2 of Fig. 6 will cause a corresponding change in ϕ_{21}. In accordance with Lenz's law, any time rate of change of ϕ_{21} will manifest itself in circuit 1 in the form of a generated or induced voltage the value of which is:

$$e_{12} = -N_1 \frac{d\phi_{21}}{dt} \quad \text{or} \quad v_{12} = N_1 \frac{d\phi_{21}}{dt} \tag{12}$$

where e_{12} is considered as a voltage rise or generated voltage and v_{12} is considered as a voltage drop.

Similarly any change in i_1 will manifest itself in circuit 2 as:

$$e_{21} = -N_2 \frac{d\phi_{12}}{dt} \quad \text{or} \quad v_{21} = N_2 \frac{d\phi_{12}}{dt} \tag{13}$$

It is through the agency of these mutually induced voltages that the phenomenon known as mutual inductance can be taken into account in circuit analysis.

The basic equations of voltage for the two circuits shown in Fig. 6 are:

$$R_1 i_1 + N_1 \frac{d\phi_1}{dt} + N_1 \frac{d\phi_{21}}{dt} = e_1 \tag{14}$$

and

$$R_2 i_2 + N_2 \frac{d\phi_2}{dt} + N_2 \frac{d\phi_{12}}{dt} = e_2 \tag{15}$$

If the permeability of the flux paths is assumed constant, the above equations can be written in more convenient forms, since:

$$N_1\phi_1 = L_1 i_1 \quad \therefore N_1 \frac{d\phi_1}{dt} = L_1 \frac{di_1}{dt} \tag{16}$$

$$N_1\phi_{21} = M_{21} i_2 \therefore N_1 \frac{d\phi_{21}}{dt} = M_{21} \frac{di_2}{dt} \tag{17}$$

$$N_2\phi_2 = L_2 i_2 \quad \therefore N_2 \frac{d\phi_2}{dt} = L_2 \frac{di_2}{dt} \tag{18}$$

$$N_2\phi_{12} = M_{12} i_1 \therefore N_2 \frac{d\phi_{12}}{dt} = M_{12} \frac{di_1}{dt} \tag{19}$$

Equations (14) and (15) may, therefore, be written in the following manner:

$$R_1 i_1 + L_1 \frac{di_1}{dt} + M_{21} \frac{di_2}{dt} = e_1 \tag{14a}$$

$$R_2 i_2 + L_2 \frac{di_2}{dt} + M_{12} \frac{di_1}{dt} = e_2 \tag{15a}$$

It will be observed that the effects of mutual inductance are entered into the basic voltage equations as voltage drops ($+M\,di/dt$). If, for example, $i_1 = I_{m1} \sin \omega t$, the voltage drop in circuit 2 due to mutual inductance is:

$$M_{12} \frac{di_1}{dt} = \omega M_{12} I_{m_1} \cos \omega t = X_{M_{12}} I_{m_1} \cos \omega t \tag{20}$$

In general, $\omega M = X_M$. It is called the mutual reactance and is an impedance function which expresses the ratio of the voltage of mutual inductance to the exciting current. It will be noted that the voltage of mutual inductance leads the exciting current by 90°. Hence the vector expression for the mutual reactance is:

$$\mathbf{X}_M = j\omega M = \omega M \underline{/90^\circ} \tag{21}$$

Circuit configurations in which M may possess either a positive or negative sign will be considered presently.

Problem 5. An inductance coil has a resistance of 10 ohms, a self-inductance of 1/37.7 henry, and a mutual inductance of 0.02 henry with respect to a neighboring coil. ($M_{12} = M_{21}$.) A voltage of 50 sin 377t volts is impressed across the terminals of the primary coil. Find the ohmic value of the mutual reactance and the effective value of the voltage across the open-circuited terminals of the neighboring coil.

Ans.: $X_M = 7.54$ ohms, $V_2 = 18.85$ volts.

Problem 6. Let the effective values of the primary voltage and current of Problem 5 be known as V_1 and I_1, and draw a vector diagram illustrating $\mathbf{V}_1$, $\mathbf{I}_1$, $R_1\mathbf{I}_1$, $jX_{L1}\mathbf{I}_1$, $jX_M\mathbf{I}_1$, and $\mathbf{E}_{21}$. (*Note:* Considered as a generated voltage, $\mathbf{E}_{21}$ is 180° out of phase with $jX_M\mathbf{I}_1$, since the latter is a component voltage drop in circuit 2 in the same sense that $R\mathbf{I}_1$ and $jX_{L1}\mathbf{I}_1$ are component voltage drops in circuit 1.)

Ans.: $\mathbf{V}_1 = \frac{50}{\sqrt{2}} \underline{/0^\circ}$ volts, $\mathbf{I}_1 = 2.5\underline{/-45^\circ}$ amperes, $\mathbf{E}_{21} = 18.85\underline{/-135^\circ}$ volts.

Coefficient of Magnetic Coupling. The fractional part of ϕ_1 which links with N_2, ϕ_{12}/ϕ_1, and the fractional part of ϕ_2 which links with N_1, (ϕ_{21}/ϕ_2), are indices of the degree of coupling that exists between two windings. Where the windings are widely separated or are so situated in space that these fractions are small, the coupling is said to be loose. With closer proximity and proper orientation of the windings, ϕ_{12}/ϕ_1 and ϕ_{21}/ϕ_2 approach unity as a theoretical upper limit.

The coefficient of coupling between two windings which individually possess L_1 and L_2 units of self-inductance is defined as:

$$k_M = \sqrt{\left(\frac{\phi_{12}}{\phi_1}\right)\left(\frac{\phi_{21}}{\phi_2}\right)} = \sqrt{\frac{(M_{12}i_1/N_2)}{(L_1i_1/N_1)}\,\frac{(M_{21}i_2/N_1)}{(L_2i_2/N_2)}} = \sqrt{\left(\frac{M_{12}}{L_1}\right)\left(\frac{M_{21}}{L_2}\right)} \tag{22}$$

Under the condition of constant permeability, $M_{12} = M_{21} = M$. Therefore, if the permeability is constant,

$$k_M = \sqrt{\left(\frac{M}{L_1}\right)\left(\frac{M}{L_2}\right)} = \frac{M}{\sqrt{L_1L_2}} \tag{23}$$

Thus k_M is the geometric mean of the fractions (ϕ_{12}/ϕ_1) and (ϕ_{21}/ϕ_2) or between the fractions (M/L_1) and (M/L_2). Numerically the coefficient of coupling in practical installations may range from approximately 0.01 between certain types of radio circuits to as high as 0.98 or 0.99 between iron-core transformer windings.

Example 3. Let the number of turns of the two windings shown in Fig. 6 be $N_1 = 50$ and $N_2 = 500$. It will be assumed that 6000 maxwells link with the turns N_1, of circuit 1, per ampere of exciting current i_1, of which 5500 also link with N_2. Under the assumption of similar concentrated windings and of constant permeability of the flux paths, 60,000 maxwells will link with the turns N_2, of circuit 2, per ampere of exciting current i_2, and 55,000 of these flux lines will also link with N_1. The purpose of this numerical example is to specify the coefficient of coupling in terms of the fractions (ϕ_{12}/ϕ_1) and (ϕ_{21}/ϕ_2) and also in terms of the fractions (M_{12}/L_1) and (M_{21}/L_2). For 1 ampere of primary exciting current and for 1 ampere of secondary current:

$$\phi_1 = 6000 \text{ maxwells}$$
$$\phi_{12} = 5500 \text{ maxwells}$$
$$\phi_2 = 60{,}000 \text{ maxwells}$$
$$\phi_{21} = 55{,}000 \text{ maxwells}$$

$$k_M = \sqrt{\left(\frac{\phi_{12}}{\phi_1}\right)\left(\frac{\phi_{21}}{\phi_2}\right)} = \sqrt{\left(\frac{55}{60}\right)\left(\frac{55}{60}\right)} = 0.917$$

$$L_1 = \frac{N_1\phi_1}{i_1} = \frac{50 \times 6000}{1} \times 10^{-8} = 0.003 \text{ henry}$$

$$M_{12} = \frac{N_2\phi_{12}}{i_1} = \frac{500 \times 5500}{1} \times 10^{-8} = 0.0275 \text{ henry}$$

$$L_2 = \frac{N_2\phi_2}{i_2} = \frac{500 \times 60{,}000}{1} \times 10^{-8} = 0.30 \text{ henry}$$

$$M_{21} = \frac{N_1\phi_{21}}{i_2} = \frac{50 \times 55{,}000}{1} \times 10^{-8} = 0.0275 \text{ henry}$$

$$k_M = \sqrt{\left(\frac{M_{12}}{L_1}\right)\left(\frac{M_{21}}{L_2}\right)} = \frac{M}{\sqrt{L_1L_2}} = \frac{0.0275}{\sqrt{0.003 \times 0.30}} = 0.917$$

Problem 7. The individual self-inductances of two windings are 0.094 henry and 0.0108 henry. The coefficient of coupling between the windings is 0.805. Find the mutual inductance of the two windings. *Ans.:* 0.0256 henry.

Problem 8. A winding of 1000 turns has a (ϕ_1/i_1) characteristic of 9400 maxwells per ampere and is coupled magnetically to a second winding of 338 turns. Assuming constant permeability of the flux paths and similar concentrated windings, find L_1, L_2, and M in henrys if the coefficient of coupling is 0.805.

Ans. $L_1 = 0.094$ henry, $L_2 = 0.0108$ henry, $M = 0.0256$ henry.

Circuit Directions and the Sign of *M*. If only one circuit of an a-c network includes a generating device, the positive directions of the currents may be arbitrarily assigned if it is understood that the positive circuit direction given to the current through the generator arbitrarily defines the positive circuit direction of the generated voltage. When more than one generating device is present in an electrical network, the relative polarities and time phases of the generating devices with respect to the common branches must be taken into account in assigning the positive circuit directions of the currents in the coupled circuits.

In a given circuit or portion thereof the voltage of mutual inductance, $M\,di/dt$, may aid or oppose the voltage of self-inductance, $L\,di/dt$. If more than one circuit is involved, the currents are first given their positive circuit directions. When the positive circuit directions of the currents have been determined from the relative polarities of the several generating devices (if more than one generator exists), or when the positive circuit directions of the currents for a single generator have been arbitrarily assigned, the sign of M is considered positive if in a given winding the induced voltage of mutual inductance acts in the same direction as the induced voltage of self-inductance. If the induced voltage of mutual inductance opposes the induced voltage of self-inductance in a given winding, M is considered as a negative quantity.

In determining the sign of M, each particular case must be analyzed as to the relative positive circuit directions of the currents, the relative modes of winding of the coils involved, and the actual physical placement of one winding with respect to the other. It will be shown later that the sign of M between circuits which are not electrically connected and which are energized with a single generator in one circuit is wholly dependent upon the arbitrary positive circuit directions which are assigned to the currents in the separate circuits.

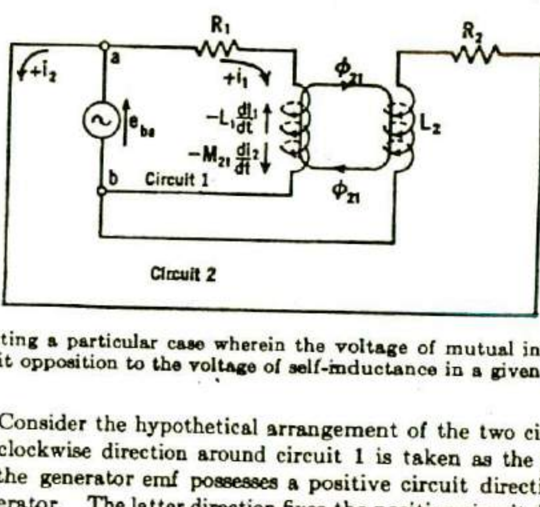


FIG. 7. Illustrating a particular case wherein the voltage of mutual inductance acts in circuit opposition to the voltage of self-inductance in a given coil.

Example 4. Consider the hypothetical arrangement of the two circuits shown in Fig. 7. If the clockwise direction around circuit 1 is taken as the positive circuit direction of i_1, the generator emf possesses a positive circuit direction from b to a through the generator. The latter direction fixes the positive circuit direction of i_2 as counter-clockwise around circuit 2.

By Lenz's law, the voltage of self-inductance in the L_1 coil *considered as an induced voltage* acts in a counter-clockwise direction around circuit 1 when di_1/dt is positive. If the positive circuit direction of i_2 and the modes of winding of the coils are considered, it is plain that voltage which is induced in the L_1 coil by the variation of ϕ_{21} is a clockwise direction around circuit 1 when di_2/dt or $d\phi_{21}/dt$ is positive.

Since $M\ di_2/dt$ acts oppositely to $L_1\ di_1/dt$ in circuit 1, M must be considered negative if L_1 is considered positive. The general equation for voltage equilibrium in circuit 1 is:

$$R_1 i_1 + L_1 \frac{di_1}{dt} + (-M) \frac{di_2}{dt} = e_{ba}$$

A simple way to determine the sign of M is to call M positive if the mmf's caused by the two currents combine to increase the total flux. If the mmf's oppose, the sign of M is negative.

Problem 9. Show, by means of detailed and independent analysis, that the general equation for voltage equilibrium in circuit 2 of Fig. 7 is:

$$R_2 i_2 + L_2 \frac{di_2}{dt} - M \frac{di_1}{dt} = e_{ba}$$

Instead of showing the actual modes of winding, a conventional method employing a dot-marked terminal, as shown in Fig. 8, is often used to yield the same information. This practice has been used for many years in the marking of iron-core instrument transformers, where the dots are known as polarity marks. The dots are placed so that a current entering the dot-marked terminal of any coil will produce a magnetomotive force and corresponding flux in the *same* direction around the magnetic circuit. Thus in Fig. 8 a current *entering* the dot-marked terminal of coil 1 causes a counter-clockwise flux in the magnetic circuit and a current *entering* the dot-marked terminal of coil 2 also causes a counter-clockwise flux in the same magnetic circuit. Hence the dots alone are sufficient to determine the relative modes

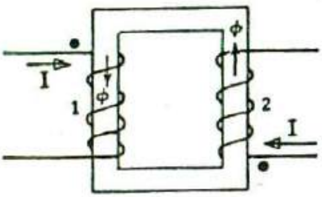


Fig. 8. Dot marks used to define relative polarities of two coils.

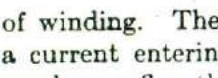


Fig. 9. Dot marks indicate $-M$.

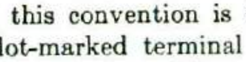


Fig. 10. Mode of winding and physical placement indicate $-M$.

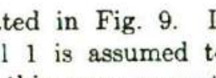


Fig. 11. Dot marks indicate $+M$.

of winding. The use of this convention is illustrated in Fig. 9. If a current entering the dot-marked terminal of coil 1 is assumed to produce a flux through the coils from left to right, this same current, since it is leaving the dot-marked terminal of coil 2, would cause a flux from right to left through the coils. Therefore, for the purpose of setting up an equation of voltage drops, M must be considered negative. Hence the relative modes of winding must be as shown in Fig. 10. If the coils of Fig. 9 were marked as shown in Fig. 11, a current entering the dot-marked terminal of coil 1 would also enter the dot-marked terminal of coil 2, the mmf's of the two coils would be additive, and the sign of M would be positive.

Mutual Inductance between Portions of the Same Circuit. Mutual inductance may be a significant factor in governing the flow of electricity in a single-series circuit where two or more portions of the circuit are coupled magnetically. A particular example is shown in Fig. 12. The arrangement consists of two magnetically coupled inductance coils connected in electrical series. Individually the coils possess L_a and L_b units of self-inductance together with R_a and R_b units of resistance, respectively.

If the coils are wound in the manner shown in Fig. 12, it is apparent that, in coil a, the voltage

$$-M_{ba}\frac{di}{dt} = -N_a\frac{d\phi_{ba}}{dt}$$

acts in the same circuit direction as the voltage $-L_a\,di/dt$. Likewise the voltage

$$-M_{ab}\frac{di}{dt} = -N_b\frac{d\phi_{ab}}{dt}$$

acts in the same circuit direction as $-L_b\,di/dt$. Hence M is positive.

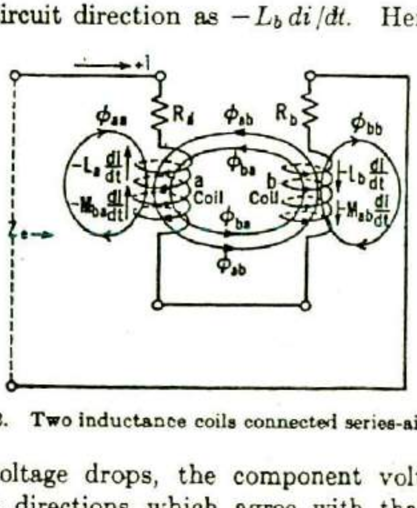


FIG. 12. Two inductance coils connected series-aiding.

Considered as voltage drops, the component voltages referred to above have circuit directions which agree with that of the applied voltage, v. Considered as voltage rises, the induced voltages are, of course, in circuit opposition to the applied voltage, v.

The facts involved can be stated in equation form as follows:

$$R_a i + L_a\frac{di}{dt} + M_{ba}\frac{di}{dt} + R_b i + L_b\frac{di}{dt} + M_{ab}\frac{di}{dt} = v \qquad (24)$$

If the mutual flux path is of constant permeability, the above equation reduces to:

$$(R_a + R_b)\,i + (L_a + L_b + 2M)\frac{di}{dt} = v \qquad (25)$$

If v varies sinusoidally with time and if all circuit parameters are constant, equation (25) may be written in terms of effective values as follows:

$$(R_a + R_b)\mathbf{I} + j\omega(L_a + L_b + 2M)\mathbf{I} = \mathbf{V} \tag{26}$$

It will be noted that M enters into the voltage equation in exactly the same manner as L. Hence ωM is a mutual reactance. The equivalent impedance of the series circuit shown in Fig. 12 follows directly from equation (26).

$$\mathbf{Z}_e = \frac{\mathbf{V}}{\mathbf{I}} = \sqrt{[R_a + R_b]^2 + [\omega(L_a + L_b + 2M)]^2} \Bigg/ \tan^{-1} \frac{\omega(L_a + L_b + 2M)}{(R_a + R_b)} \tag{27}$$

Equation (27) may also be written:

$$\mathbf{Z}_e = (R_a + R_b) + j\omega(L_a + L_b + 2M) = \mathbf{Z}_a + \mathbf{Z}_b + 2\mathbf{Z}_M \tag{27a}$$

where

$$\mathbf{Z}_a = R_a + j\omega L_a,\ \mathbf{Z}_b = R_b + j\omega L_b \quad \text{and} \quad \mathbf{Z}_M = 0 + j\omega M$$

If the two coils were connected together in the opposite sense, that is, with a polarity opposite to that shown in Fig. 12, the signs of the M terms in the above equations would be reversed.

Example 5. An inspection of equations (25), (26), and (27) will show that the equivalent inductance of the two coils connected in additive series is:

$$L_{e(\text{add})} = L_a + L_b + 2M$$

If the two coils are connected in subtractive series:

$$L_{e(\text{sub})} = L_a + L_b - 2M$$

The value of M may, therefore, be found experimentally by measuring $L_{e(\text{add})}$ and $L_{e(\text{sub})}$ since, from the above relations:

$$M = \frac{L_{e(\text{add})} - L_{e(\text{sub})}}{4}$$

Example 6. Let it be required to find the coefficient of coupling, the equivalent series-circuit impedance, and the magnitude of the current in a circuit arrangement similar to that shown in Fig. 12 if:

$R_a = 1.0$ ohm

$L_a = 4.0$ millihenrys

$R_b = 6.0$ ohms

$L_b = 9.0$ millihenrys

$M = +3$ millihenrys

$\omega = 1000$ radians per second

$V = 40.5$ volts, the applied voltage

(a) The coefficient of coupling is:

$$k = \frac{M}{\sqrt{L_a L_b}} = \frac{+3}{\sqrt{4 \times 9}} = +0.5$$

(b) The equivalent series-circuit impedance is:

$$\begin{aligned} Z_e &= (R_a + R_b) + j\omega(L_a + L_b + 2M) \\ &= (1 + 6) + j(1000)(0.004 + 0.009 + 0.006) \\ &= 7 + j19 = 20.25 \underline{/69.8^\circ} \text{ ohms} \end{aligned}$$

(c) The series current is:

$$I = \frac{40.5}{20.25} = 2.0 \text{ amperes}$$

A vector diagram of **V**, **I**, $\mathbf{V}_a$, and $\mathbf{V}_b$ is shown in Fig. 13 together with the component voltages of $\mathbf{V}_a$ and $\mathbf{V}_b$.

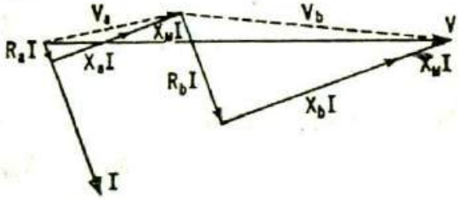


FIG. 13. Vector diagram of example 6.

Problem 10. Find the magnitude of the current in the above example if the two coils are connected in subtractive series, that is, $M = -3$ millihenrys. Draw a vector diagram illustrating the vector positions of **V**, **I**, $\mathbf{V}_a$, $\mathbf{V}_b$, and the various $R\mathbf{I}$ and $X\mathbf{I}$ component voltages. *Ans.*: $I = 4.09$ amperes.

Mutual Inductance between Parallel Branches. Reference to Fig. 14 will show that, in coil 1, $M_{21}\, di_2/dt$ acts in circuit opposition to $L_1\, di_1/dt$.

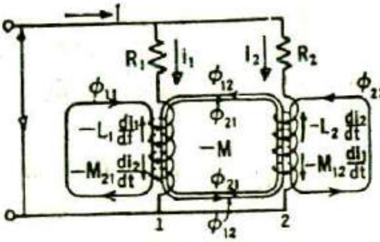


FIG. 14. Parallel arrangement of two inductance coils which are coupled magnetically. For the mode of winding shown and the assumed positive directions of currents as indicated, M is negative.

Similarly, in coil 2, $M_{12}\, di_1/dt$ acts in circuit opposition to $L_2\, di_2/dt$. In equation form:

$$R_1 i_1 + L_1 \frac{di_1}{dt} - M_{21} \frac{di_2}{dt} = v \tag{28}$$

$$R_2 i_2 + L_2 \frac{di_2}{dt} - M_{12} \frac{di_1}{dt} = v \tag{29}$$

It will be noted that the individual branch currents have been employed in the above equations.

If the circuit parameters are constant and a sinusoidal variation of v is assumed, the above equations may be written in terms of effective values as follows:

$$(R_1 + j\omega L_1)\mathbf{I}_1 - j\omega M\mathbf{I}_2 = \mathbf{V} \tag{30}$$

$$(R_2 + j\omega L_2)\mathbf{I}_2 - j\omega M\mathbf{I}_1 = \mathbf{V} \tag{31}$$

Let

$$(R_1 + j\omega L_1) = \mathbf{Z}_1 \tag{32}$$

$$(R_2 + j\omega L_2) = \mathbf{Z}_2 \tag{33}$$

$$0 + j\omega M = \mathbf{Z}_M \tag{34}$$

With the above abbreviations, equations (30) and (31) reduce to:

$$\mathbf{Z}_1\mathbf{I}_1 - \mathbf{Z}_M\mathbf{I}_2 = \mathbf{V} \tag{35}$$

$$-\mathbf{Z}_M\mathbf{I}_1 + \mathbf{Z}_2\mathbf{I}_2 = \mathbf{V} \tag{36}$$

The individual branch currents, $\mathbf{I}_1$ and $\mathbf{I}_2$, may be found from the simultaneous solutions of equations (35) and (36).

$$\mathbf{I}_1 = \frac{\begin{vmatrix} \mathbf{V} & -\mathbf{Z}_M \\ \mathbf{V} & \mathbf{Z}_2 \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_1 & -\mathbf{Z}_M \\ -\mathbf{Z}_M & \mathbf{Z}_2 \end{vmatrix}} = \frac{\mathbf{V}(\mathbf{Z}_2 + \mathbf{Z}_M)}{\mathbf{Z}_1\mathbf{Z}_2 - \mathbf{Z}_M^2} \tag{37}$$

$$\mathbf{I}_2 = \frac{\begin{vmatrix} \mathbf{Z}_1 & \mathbf{V} \\ -\mathbf{Z}_M & \mathbf{V} \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_1 & -\mathbf{Z}_M \\ -\mathbf{Z}_M & \mathbf{Z}_2 \end{vmatrix}} = \frac{\mathbf{V}(\mathbf{Z}_1 + \mathbf{Z}_M)}{\mathbf{Z}_1\mathbf{Z}_2 - \mathbf{Z}_M^2} \tag{38}$$

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = \frac{\mathbf{V}(\mathbf{Z}_1 + \mathbf{Z}_2 + 2\mathbf{Z}_M)}{\mathbf{Z}_1\mathbf{Z}_2 - \mathbf{Z}_M^2} \tag{39}$$

The equivalent impedance of the two parallel branches shown in Fig. 14 for the case of negative M is:

$$\mathbf{Z}_e = \frac{\mathbf{V}}{\mathbf{I}} = \frac{\mathbf{Z}_1\mathbf{Z}_2 - \mathbf{Z}_M^2}{\mathbf{Z}_1 + \mathbf{Z}_2 + 2\mathbf{Z}_M} \tag{40}$$

Example 7. In the circuit arrangement shown in Fig. 14 it will be assumed that:

$R_1 = 3.3$ ohms $\qquad L_2 = 0.0108$ henry

$L_1 = 0.094$ henry $\qquad M = -0.0256$ henry

$R_2 = 0.775$ ohm $\qquad \omega = 377$ radians per second

$\mathbf{V} = 50 /\underline{0°}$ volts

Let it be required to find $\mathbf{I}$, $\mathbf{I}_1$, $\mathbf{I}_2$, and the total power spent in the two parallel branches.

$\mathbf{Z}_1$ (individually) $= 3.3 + j35.4 = 35.5 /\underline{84.7°}$ ohms

$\mathbf{Z}_2$ (individually) $= 0.775 + j4.07 = 4.17 /\underline{79.25°}$ ohms

$\mathbf{Z}_M = 0 + j\omega M = 0 + j9.65 = 9.65 /\underline{90°}$ ohms

Note: $\mathbf{Z}_M$ is herein considered as inherently positive since the appropriate negative signs have been introduced into equations (30) and (31).

$$\mathbf{Z}_e = \frac{\mathbf{Z}_1\mathbf{Z}_2 - \mathbf{Z}_M{}^2}{\mathbf{Z}_1 + \mathbf{Z}_2 + 2\mathbf{Z}_M} = \frac{63.6\underline{/140^\circ}}{59.0\underline{/86^\circ}} = 1.078\underline{/54^\circ} \text{ ohms}$$

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_e} = \frac{50\underline{/0^\circ}}{1.078\underline{/54^\circ}} = 46.4\underline{/-54^\circ} \text{ amperes}$$

$$\mathbf{I}_1 = \frac{\mathbf{V}(\mathbf{Z}_2 + \mathbf{Z}_M)}{\mathbf{Z}_1\mathbf{Z}_2 - \mathbf{Z}_M{}^2} = \frac{(50\underline{/0^\circ})(13.73\underline{/86.8^\circ})}{63.6\underline{/140^\circ}}$$

$$\mathbf{I}_1 = 10.8\underline{/-53.2^\circ} \text{ amperes}$$

$$\mathbf{I}_2 = \frac{\mathbf{V}(\mathbf{Z}_1 + \mathbf{Z}_M)}{\mathbf{Z}_1\mathbf{Z}_2 - \mathbf{Z}_M{}^2} = \frac{(50\underline{/0^\circ})(45.1\underline{/85.8^\circ})}{63.6\underline{/140^\circ}}$$

$$\mathbf{I}_2 = 35.4\underline{/-54.2^\circ} \text{ amperes}$$

$$P = VI\cos\theta\Big]_{\mathbf{I}}^{\mathbf{V}} = 50 \times 46.4 \times \cos 54^\circ = 1365 \text{ watts}$$

Check:

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = 10.8\underline{/-53.2^\circ} + 35.4\underline{/-54.2^\circ}$$

$$\mathbf{I} = (6.46 - j8.65) + (20.8 - j28.8) = 27.26 - j37.45$$

$$\mathbf{I} = 46.4\underline{/-54^\circ} \text{ amperes}$$

$$P = I_1{}^2R_1 + I_2{}^2R_2 = 385 + 973 = 1358 \text{ watts}$$

Problem 11. Assume that the inductance coils in the above illustrative example are connected in parallel as shown in Fig. 14, except that the terminals of one coil are reversed from that shown in the figure. Show that, under these conditions:

$$\mathbf{Z}_e = 3.095\underline{/61.40^\circ} \text{ ohms}$$

$$\mathbf{I} = 16.16\underline{/-61.40^\circ} \text{ amperes } (\mathbf{V} \text{ as reference})$$

$$\mathbf{I}_1 = 4.43\underline{/-222.1^\circ} \text{ amperes}$$

$$\mathbf{I}_2 = 20.4\underline{/-57.30^\circ} \text{ amperes}$$

$$P = VI\cos\theta\Big]_{\mathbf{I}}^{\mathbf{V}} = 386 \text{ watts}$$

Draw the vector diagram of **V**, **I**, $\mathbf{I}_1$, and $\mathbf{I}_2$, and illustrate the manner in which the three component voltages in each branch combine vectorially to equal the applied voltage, **V**.

The Air-Core Transformer. In the conventional transformer arrangement shown schematically in Fig. 15, the individual circuits are not connected electrically. Circuit 1, energized by means of an alternating potential difference, is called the primary. Circuit 2 is called the secondary. As a result of the magnetic coupling between the circuits, circuit 2 has induced in it a voltage which is equal to:

$$-N_2 \frac{d\phi_{12}}{dt} = -M_{12} \frac{di_1}{dt} \tag{41}$$

The magnitude of the voltage induced in circuit 2 is proportional to the number of secondary turns, N_2, and is dependent upon the degree of coupling between the two windings.

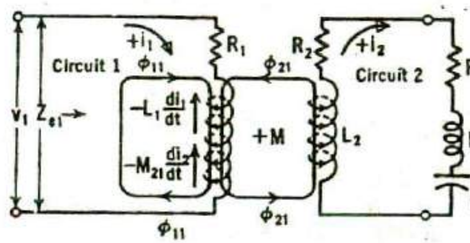


FIG. 15. Conventional air-core transformer arrangement.

The sign of M in the conventional transformer arrangement is dependent upon the arbitrary choice of the positive circuit direction of i_2. The majority of writers prefer to use the positive circuit direction of i_2 which allows them to employ the positive sign of M. For the relative modes of winding shown in Fig. 15, the positive clockwise direction of i_2 requires the use of $+M$, since under these conditions $M_{21}\, di_2/dt$ acts in the same circuit direction as $L_1\, di_1/dt$ in the primary winding. If the counter-clockwise direction around circuit 2 is taken as the positive circuit direction of i_2, then, of course, M must be considered negative. The resulting solutions will be identical in either case, except that all secondary currents and voltages will be reversed in sign. Experience with detailed solutions will convince the reader that the two different methods of attack yield identical physical results.

If the positive circuit directions are employed as indicated in Fig. 15, the mathematical analysis of the ordinary air-core transformer may be carried out as follows:

$$R_1 i_1 + L_1 \frac{di_1}{dt} + M_{21} \frac{di_2}{dt} = v_1 \tag{42}$$

$$(R_2 + R)i_2 + (L_2 + L)\frac{di_2}{dt} + \frac{\int i_2\, dt}{C_2} + M_{12} \frac{di_1}{dt} = 0 \tag{43}$$

If v_1 is assumed to have sinusoidal wave form and all circuit parameters are constant, the above equations may be written in terms of effective values as follows:

$$(R_1 + j\omega L_1)\mathbf{I}_1 + j\omega M\mathbf{I}_2 = \mathbf{V}_1 \tag{44}$$

$$(R_2 + j\omega L_2)\mathbf{I}_2 + \left[R + j\left(\omega L - \frac{1}{\omega C}\right)\right]\mathbf{I}_2 + j\omega M\mathbf{I}_1 = 0 \tag{45}$$

For the sake of simplicity in writing, the following abbreviations are adopted:

$\mathbf{Z}_1 = (R_1 + j\omega L_1)$ (Individual primary winding impedance) (46)

$\mathbf{Z}_2 = (R_2 + j\omega L_2)$ (Individual secondary winding impedance) (47)

$\mathbf{Z}_M = (0 + j\omega M)$ (Mutual impedance assuming no core loss) (48)

$\mathbf{Z} = \left[R + j\left(\omega L - \frac{1}{\omega C}\right)\right]$ (General expression for load impedance) (49)

Equations (44) and (45) become:

$$\mathbf{Z}_1\mathbf{I}_1 + \mathbf{Z}_M\mathbf{I}_2 = \mathbf{V}_1 \qquad (44)\text{–}(50)$$

$$\mathbf{Z}_M\mathbf{I}_1 + (\mathbf{Z}_2 + \mathbf{Z})\mathbf{I}_2 = 0 \qquad (45)\text{–}(51)$$

The simultaneous solutions of the above equations for $\mathbf{I}_1$ and $\mathbf{I}_2$ yield:

$$\mathbf{I}_1 = \frac{\begin{vmatrix} \mathbf{V}_1 & \mathbf{Z}_M \\ 0 & (\mathbf{Z}_2 + \mathbf{Z}) \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_1 & \mathbf{Z}_M \\ \mathbf{Z}_M & (\mathbf{Z}_2 + \mathbf{Z}) \end{vmatrix}} = \frac{\mathbf{V}_1(\mathbf{Z}_2 + \mathbf{Z})}{\mathbf{Z}_1(\mathbf{Z}_2 + \mathbf{Z}) - \mathbf{Z}_M^2} \tag{52}$$

$$\mathbf{I}_2 = \frac{\begin{vmatrix} \mathbf{Z}_1 & \mathbf{V}_1 \\ \mathbf{Z}_M & 0 \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_1 & \mathbf{Z}_M \\ \mathbf{Z}_M & (\mathbf{Z}_2 + \mathbf{Z}) \end{vmatrix}} = \frac{-\mathbf{V}_1\mathbf{Z}_M}{\mathbf{Z}_1(\mathbf{Z}_2 + \mathbf{Z}) - \mathbf{Z}_M^2} \tag{53}$$

If $\mathbf{I}_1$ has been evaluated, it may, in certain cases, be more convenient to solve for $\mathbf{I}_2$ directly from equation (51).

$$\mathbf{I}_2 = \frac{-\mathbf{Z}_M\mathbf{I}_1}{(\mathbf{Z}_2 + \mathbf{Z})} \tag{54}$$

The secondary terminal voltage, or the voltage which appears across the load impedance, is:

$$\mathbf{V}_2 = \mathbf{Z}\mathbf{I}_2 = -\mathbf{Z}_M\mathbf{I}_1 - \mathbf{Z}_2\mathbf{I}_2 \tag{55}$$

Also:

$$\mathbf{V}_2 = \frac{-\mathbf{V}_1\mathbf{Z}_M\mathbf{Z}}{\mathbf{Z}_1(\mathbf{Z}_2 + \mathbf{Z}) - \mathbf{Z}_M^2} \tag{56}$$

The above relations follow directly from equations (51) and (53). Equation (55) shows that the secondary circuit may be thought of as experiencing an induced voltage equal to $-\mathbf{Z}_M\mathbf{I}_1$, from which the internal secondary impedance drop, $\mathbf{Z}_2\mathbf{I}_2$, must be subtracted in order to obtain the secondary terminal voltage, $\mathbf{V}_2$.

Equivalent Impedance. The equivalent impedance of the transformer arrangement shown in Fig. 15 referred to the primary side is defined as the ratio of the applied voltage to the primary current. Thus:

$$\mathbf{Z}_{e1} = \frac{\mathbf{V}_1}{\mathbf{I}_1} = \frac{\mathbf{Z}_1(\mathbf{Z}_2 + \mathbf{Z}) - \mathbf{Z}_M^2}{(\mathbf{Z}_2 + \mathbf{Z})} \tag{57}$$

A more convenient form of the above equation is:

$$\mathbf{Z}_{e1} = \mathbf{Z}_1 - \frac{\mathbf{Z}_M^2}{(\mathbf{Z}_2 + \mathbf{Z})} = \mathbf{Z}_1 + \frac{\omega^2 M^2}{\mathbf{Z}_2 + \mathbf{Z}} \tag{58}$$

Equations (57) and (58) show that the air-core transformer, with respect to its primary terminals, is reducible to an equivalent series circuit.

Example 8 (for $\mathbf{Z} = 0$). It will be assumed that, in Fig. 16*a*:

$R_1 = 3.3$ ohms $\qquad M = 0.0256$ henry

$L_1 = 0.094$ henry $\qquad Z = 0$

$R_2 = 0.775$ ohm $\qquad \omega = 377$ radians per second

$L_2 = 0.0108$ henry $\qquad \mathbf{V}_1 = 50\underline{/0^\circ}$ volts

$$\mathbf{Z}_1 = 3.3 + j35.4 = 35.5\underline{/84.7^\circ} \text{ ohms}$$

$$\mathbf{Z}_2 = 0.775 + j4.07 = 4.14\underline{/79.25^\circ} \text{ ohms}$$

$$\mathbf{Z}_M = 0 + j9.65 = 9.65\underline{/90^\circ} \text{ ohms}$$

$$\mathbf{Z}_{e1} = \mathbf{Z}_1 - \frac{\mathbf{Z}_M^2}{\mathbf{Z}_2} = (3.3 + j35.4) + \frac{93.1\underline{/0^\circ}}{4.14\underline{/79.25^\circ}}$$

$$\mathbf{Z}_{e1} = (3.3 + j35.4) + (4.20 - j22.1) = 7.50 + j13.3 = 15.27\underline{/60.55^\circ} \text{ ohms}$$

$$\mathbf{I}_1 = \frac{\mathbf{V}_1}{\mathbf{Z}_{e1}} = \frac{50\underline{/0^\circ}}{15.27\underline{/60.55^\circ}} = 3.28\underline{/-60.55^\circ} \text{ amperes}$$

$$\mathbf{I}_2 = \frac{-\mathbf{I}_1\mathbf{Z}_M}{\mathbf{Z}_2} = \frac{(3.28\underline{/119.45^\circ})(9.65\underline{/90^\circ})}{4.14\underline{/79.25^\circ}}$$

$$\mathbf{I}_2 = 7.66\underline{/130.2^\circ} \text{ amperes}$$

The total power dissipated in the two circuits is:

$$P = V_1 I_1 \cos\theta\Big]_{I_1}^{V_1} = 50 \times 3.28 \times \cos(-60.55^\circ) = 80.8 \text{ watts}$$

or

$$P = I_1^2 R_1 + I_2^2 R_2 = 3.28^2 \times 3.3 + 7.66^2 \times 0.775 = 81.0 \text{ watts}$$

The vector diagram of $\mathbf{V}_1$, $\mathbf{I}_1$, $\mathbf{I}_2$, and $-\mathbf{Z}_M\mathbf{I}_1$ is shown in Fig. 16*b*. In the particular case shown in Fig. 16*b*, the voltage induced in circuit 2, $-\mathbf{Z}_M\mathbf{I}_1$, is balanced entirely by the internal secondary impedance drop, namely, $\mathbf{Z}_2\mathbf{I}_2$. If the counterclockwise direction around circuit 2 had been taken as the positive circuit direction, $\mathbf{I}_2$ and $\mathbf{Z}_M\mathbf{I}_1$ would appear on the vector diagram 180° from the positions shown in Fig. 16*b*.

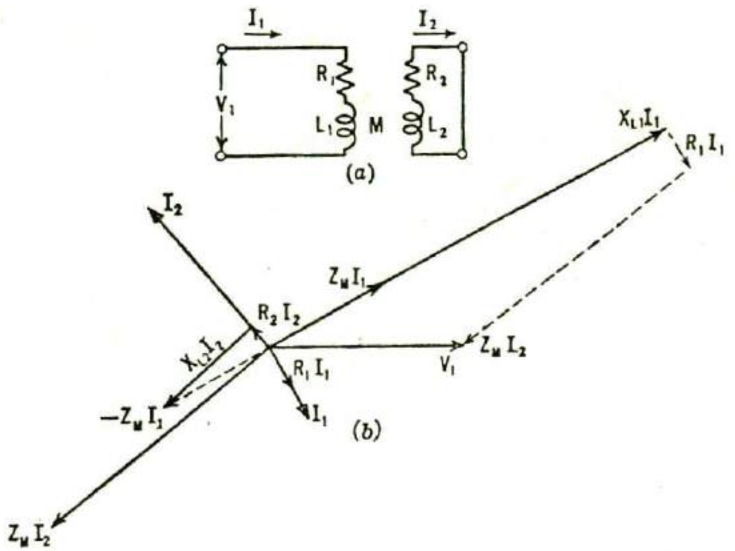


FIG. 16. Voltage and current relations in an air-core transformer the secondary of which is short-circuited. Note the manner in which $X_{L1}I_1$, R_1I_1, and Z_MI_2 combine vectorially to balance the applied voltage V_1.

Oscillogram 1 illustrates the instantaneous variations of v_1, i_1, and i_2 for the above numerical case. The salient features of the numerical solution are clearly shown. The primary current lags the applied voltage by approximately 60°, and the secondary current lags the primary current by approximately 170°. Within the limits of oscillographic accuracy, the maximum magnitudes of i_1 and i_2 agree with the results of the above numerical example.

Example 9 (for $\mathbf{Z} = 14.5 + j21.2$ ohms). It will be assumed that in Fig. 17*a*:

$R_1 = 3.3$ ohms $\qquad M = 0.0256$ henry

$L_1 = 0.094$ henry $\qquad \mathbf{Z} = 14.5 + j21.2$ ohms

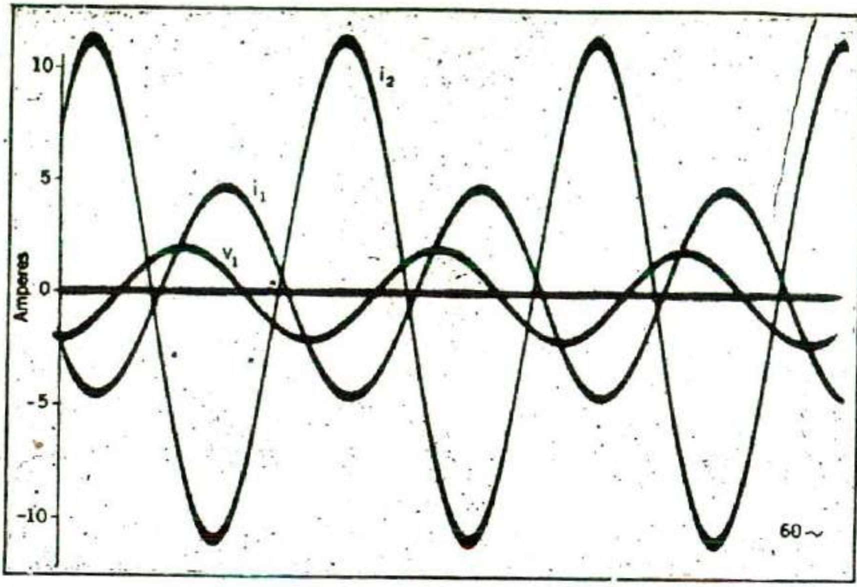
$R_2 = 0.775$ ohm $\qquad \omega = 377$ radians per second

$L_2 = 0.0108$ henry $\qquad \mathbf{V}_1 = 50\underline{/0°}$ volts

$$\mathbf{Z}_1 = 3.3 + j35.4 = 35.5\underline{/84.7°} \text{ ohms}$$

$$\mathbf{Z}_2 = 0.775 + j4.07 = 4.14\underline{/79.25°} \text{ ohms}$$

$$\mathbf{Z}_M = 0 + j9.65 = 9.65\underline{/90°} \text{ ohms}$$



OSCILLOGRAM 1. Illustrating the time phase relations of primary and secondary currents of an air-core transformer with respect to the applied voltage wave. (For a short-circuited secondary. See Fig. 16a.) $v_1 = 70.7 \sin 377t$ volts.

$$\mathbf{Z}_{e1} = \mathbf{Z}_1 - \frac{\mathbf{Z}_M{}^2}{\mathbf{Z}_2 + \mathbf{Z}} = 35.5\,\underline{/84.7^\circ} + \frac{93.1\,\underline{/0^\circ}}{15.28 + j25.3}$$

$$\mathbf{Z}_{e1} = (3.3 + j35.4) + (1.63 - j2.7) = 4.93 + j32.7$$

$$\mathbf{Z}_{e1} = 33.0\,\underline{/81.4^\circ} \text{ ohms}$$

$$\mathbf{I}_1 = \frac{\mathbf{V}_1}{\mathbf{Z}_{e1}} = \frac{50\,\underline{/0^\circ}}{33\,\underline{/81.4^\circ}} = 1.515\,\underline{/-81.4^\circ} \text{ amperes}$$

$$\mathbf{I}_2 = \frac{-\mathbf{I}_1\mathbf{Z}_M}{(\mathbf{Z}_2 + \mathbf{Z})} = \frac{(1.515\,\underline{/98.6^\circ})(9.65\,\underline{/90^\circ})}{29.6\,\underline{/58.9^\circ}}$$

$$\mathbf{I}_2 = 0.494\,\underline{/129.7^\circ} \text{ amperes}$$

$$\mathbf{V}_2 \text{ (terminal voltage)} = \mathbf{I}_2\mathbf{Z}$$

$$\mathbf{V}_2 = (0.494\,\underline{/129.7^\circ})(25.7\,\underline{/55.6^\circ}) = 12.7\,\underline{/185.3^\circ} \text{ volts}$$

The input power to the primary terminals is:

$$P_{\text{input}} = V_1 I_1 \cos\theta \Big]_{\mathbf{I}_1}^{\mathbf{V}_1} = 50 \times 1.515 \times \cos 81.4^\circ$$

$$= 50 \times 1.515 \times 0.1495 = 11.3 \text{ watts}$$

The power delivered to the load is:

$$P_{load} = V_2 I_2 \cos\theta\Big]_{I_2}^{V_2} = 12.7 \times 0.494 \cos 55.6°$$
$$= 12.7 \times 0.494 \times 0.565 = 3.55 \text{ watts}$$

The efficiency of this particular air-core transformer working under the conditions stated above is 3.55/11.3 or 31.4 per cent.

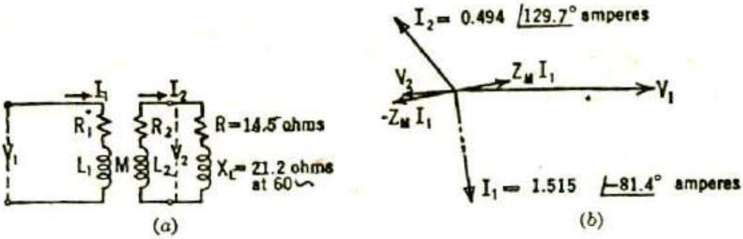
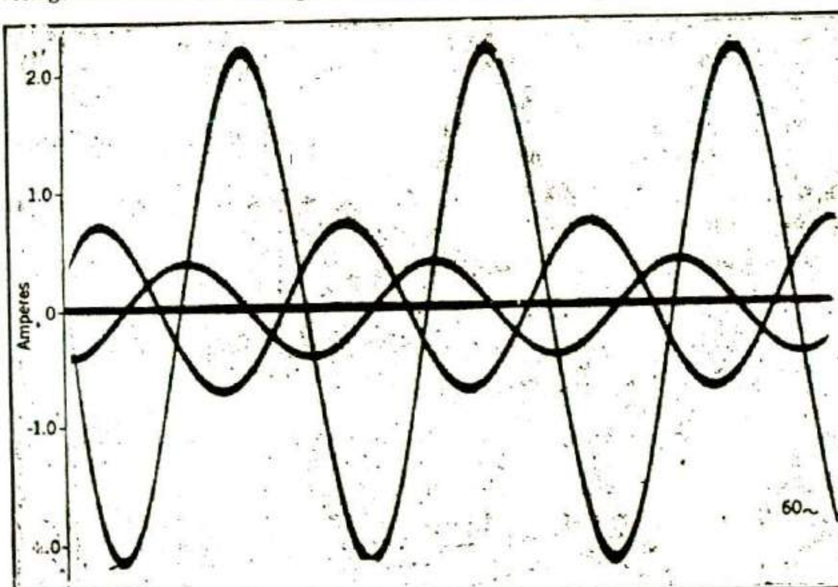


FIG. 17. Voltage and current relations in an air-core transformer the secondary of which is loaded as shown in (a).

Figure 17b is a vector diagram of $\mathbf{V}_1$, $\mathbf{I}_1$, $-\mathbf{Z}_M\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{V}_2$. Oscillogram 2 illustrates the variations of v_1, i_1, and i_2 for the particular case under discussion. The phase positions of the primary and secondary currents with respect to the applied voltage are shown in rectangular-coordinate form and agree with the calculated

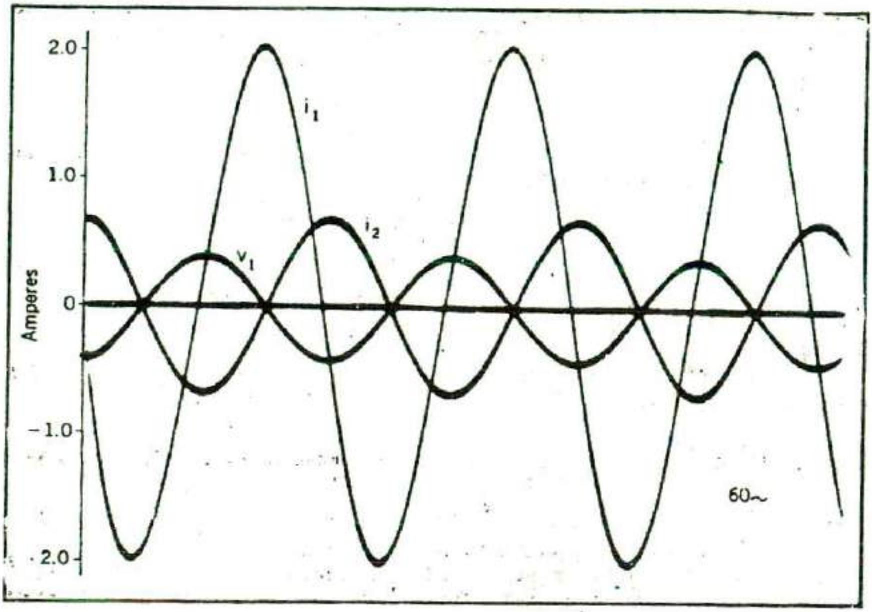


OSCILLOGRAM 2. Illustrating the time-phase relations of primary and secondary currents of an air-core transformer with respect to the applied voltage wave. (For an inductive-type load placed across the secondary terminals of the transformer. See Fig. 17a.)

v_1 represents the applied voltage wave (effective value = 50 volts)

i_1 represents the primary current wave (effective value = 1.5 amperes)

i_2 represents the secondary current wave (effective value = 0.5 ampere)



Oscillogram 3. Illustrating the time-phase relations of primary and secondary currents of an air-core transformer with respect to the applied voltage wave. (For a resistive-type load placed across the secondary terminals of the transformer. See Problem 12.)

values of these quantities. Likewise the wave shape and maximum values of the voltage and current waves are discernible.

Problem 12. Let the load impedance in the above numerical example be replaced with an impedance the value of which is $28.15\underline{/0^\circ}$ ohms.

(a) Show that, under this condition of operation,

$$\mathbf{Z}_{e1} = 35.5\underline{/79.5^\circ} \text{ ohms}$$

$$\mathbf{I}_1 = 1.409\underline{/-79.5^\circ} \text{ amperes} \quad (\mathbf{V}_1 \text{ as reference})$$

$$\mathbf{I}_2 = 0.465\underline{/182.4^\circ} \text{ amperes}$$

(b) Find the power input, the power output, and the efficiency of operation.

Ans.: $P_{\text{in}} = 12.8$ watts, $P_{\text{out}} = 6.08$ watts, efficiency $= 47.5\%$.

(c) Draw a vector diagram of $\mathbf{V}_1$, $\mathbf{I}_1$, $-\mathbf{I}_1Z_M$, $\mathbf{I}_2$, $\mathbf{I}_2R_2$, $\mathbf{I}_2(j\omega L_2)$, and $\mathbf{V}_2$.

(d) Compare the results obtained with those shown in Oscillogram 3. Oscillogram 3 is a photographic record of the variations of v_1, i_1, and i_2 in the air-core transformer arrangement considered in this particular problem.

Transferred Impedance. One of the primary considerations in communication circuits is that of transferring maximum power from a low-power generating device to a receiver. It has been shown in Chapter V that maximum power is transferred (for a fixed generator voltage) when the impedance of the receiver (in complex form) is the conjugate of the impedance of the generator and associated transmission lines. That is, if $\mathbf{Z}_{gen} = R + jX$, then $\mathbf{Z}_{rec}$ should equal $R - jX$ for maximum power transfer. For impedance matches which will prevent reflection losses, $\mathbf{Z}_{gen} = \mathbf{Z}_{rec}$. (See Chapters X and XI.)

At audio frequencies, iron-core transformers may be used successfully for transforming voltage magnitudes and for matching impedances, but at radio frequencies air-core transformers are generally used. In iron-core transformers where the coefficient of coupling is relatively high and where $(\omega L_2)^2 \gg R_2'^2$, a resistance, R, placed across an N_2-turn secondary, *may* appear at the terminals of an N_1-turn primary as $(N_1/N_2)^2R$, approximately. The term "*may* appear" is used because several conditions must be fulfilled simultaneously before the $(N_1/N_2)^2$ factor can be used successfully, as will be shown presently.

Classical methods will be employed to show how an impedance placed across the secondary terminals of an air-core transformer appears at the primary terminals in modified form.[2]

Reference to equation (58) will show that the equivalent impedance of an air-core transformer referred to the primary side is:

$$\mathbf{Z}_{e1} = \mathbf{Z}_1 - \frac{\mathbf{Z}_M^2}{\mathbf{Z}_2'} = (R_1 + jX_1) + \frac{X_M^2}{(R_2' + jX_2')} \tag{59}$$

where $\mathbf{Z}_2' = (\mathbf{Z}_2 + \mathbf{Z})$, the total secondary impedance.

Since $\mathbf{Z}_M^2 = -\omega^2 M^2$, and $\mathbf{Z}_2' = R_2' + j\omega L_2'$ (for a predominantly inductive secondary circuit), it follows that:

$$\mathbf{Z}_{e1} = (R_1 + j\omega L_1) + \left(\frac{\omega^2 M^2}{R_2' + j\omega L_2'}\right) \tag{60}$$

Rationalizing equation (60) yields:

$$\mathbf{Z}_{e1} = \left[R_1 + \frac{\omega^2 M^2 R_2'}{R_2'^2 + \omega^2 L_2'^2}\right] + j\omega\left[L_1 - \frac{\omega^2 M^2 L_2'}{R_2'^2 + \omega^2 L_2'^2}\right] \tag{61}$$

It will be observed that R_2' appears at the primary terminals in modified form, namely, as:

$$\left[\frac{\omega^2 M^2}{R_2'^2 + \omega^2 L_2'^2}\right] R_2' = \frac{X_M^2}{Z_2'^2} R_2'$$

[2] It should be recognized that classical methods are applicable only where $M_{21} = M_{12}$ = a constant. Where iron-core transformers are involved, the $(N_1/N_2)^2$ factor is often used as an approximation, but since detailed analyses of iron-core transformers are usually considered in a-c machinery courses they will not be given here.

If R_2' is very small compared with $\omega_2 L_2'$, if $L_2' = N_2\phi_2/i_2$, that is, if all of L_2' is concentrated in the secondary winding, and if $M = \sqrt{L_1 L_2'}$, then R_2' appears at the primary terminals as:

$$\left(\frac{N_1}{N_2}\right)^2 R_2' \quad \text{approximately}$$

Thus, if a high value of R_2' is to appear at the primary terminals at an apparently reduced value, N_1/N_2 must be made less than unity by the appropriate amount. The above transfer factor, $(N_1/N_2)^2$, can be theoretically approached only in the case of an ideal transformer the coefficient of coupling of which is unity. Even with unity coupling, R_2' is not actually transferred by the exact square of the turn ratio, N_1/N_2, as is sometimes supposed. In the iron-core transformer the conditions required to make $(N_1/N_2)^2$ the correct transfer factor are fulfilled to a degree which makes calculations fall well within engineering accuracy when this factor is applied. As a result, it is customary to use this factor in iron-core transformer practice.

Equation (61) reveals another interesting fact, namely, that the effective inductance at the primary terminals of a loaded transformer approaches zero only when $R_2'^2$ is negligibly small compared with $\omega^2 L_2'^2$ and when L_2' is entirely concentrated in the secondary winding. Under these conditions and if the coefficient of coupling is equal to unity,

$$\left[L_1 - \frac{\omega^2 M^2 L_2'}{\omega^2 L_2'^2}\right] = \left[L_1 - \frac{\omega^2 L_1 L_2'^2}{\omega^2 L_2'^2}\right] = 0$$

Example 10. Given an air-core (or constant-permeability) transformer, in which $N_1 = 500$ and $N_2 = 5000$. For the particular arrangement considered:

$$R_1 = 1.0 \text{ ohm} \qquad R_2 = 10 \text{ ohms}$$
$$L_1 = 0.03 \text{ henry} \qquad L_2 = 3.0 \text{ henrys}$$
$$M = 0.275 \text{ henry}$$
$$\mathbf{Z} = 90\underline{/0^\circ} \text{ ohms}$$

At 265.5 cycles per second, $\omega = 1667$ radians per second and

$$X_M = \omega M = 1667 \times 0.275 = 458.4 \text{ ohms}$$
$$X_M^2 = 458.4^2 = 210{,}000$$
$$\mathbf{Z}_2' = (10 + j5000) + (90 + j0) = 100 + j5000 \text{ ohms}$$
$$\mathbf{Z}_{e1} = (1 + j50) + \frac{210{,}000}{100 + j5000}$$
$$\mathbf{Z}_{e1} = (1 + j50) + (0.84 - j42) = 1.84 + j8.0 = 8.2\underline{/77^\circ} \text{ ohms}$$

It will be noted that $Z_2' = (100 + j5000)$ ohms appears at the primary terminals as $(0.84 - j42)$ ohms. This result emphasizes the wide discrepancy that may exist between ideal transformer operation and that actually obtained in an air-core transformer the coefficient of coupling of which is 0.917.

Under ideal conditions, the load impedance, $Z = 90\underline{/0^\circ}$ ohms, would appear at the primary terminals as

$$\left[\frac{N_1}{N_2}\right]^2 \times 90 = \frac{500^2}{5000^2} \times 90 = 0.90 \text{ ohm}$$

The ideal conditions referred to are: (1) perfect coupling, and (2) zero resistance in the transformer windings.

The reactive term in Z_{e1} may, of course, be neutralized with a series condenser in the primary circuit if a low resistive impedance at the primary circuit terminals is desired.

Problem 13. A generator which develops 10 volts (effective) at 265.5 cycles and which has an internal impedance of $2\underline{/0^\circ}$ ohms is to be used to energize the 90-ohm load resistance of the above example in the two following ways:

(a) Directly. That is, with the generator terminals directly across the terminals of the 90-ohm load.

(b) Through the transformer of the above example and a primary series condenser the capacitive reactance of which is 8 ohms.

Find the power delivered to the 90-ohm load in (a) and in (b).

Ans.: (a) 1.063 watts; (b) 5.13 watts.

Primary Unity-Power-Factor Resonance. The inductive reactance of Z_{e1} caused by the introduction of a transformer may be neutralized in any one of several different ways. If, upon evaluation in a particular case, Z_{e1} possesses an inductive reactive component, suitable neutralizing capacitors may be placed in either the primary or the secondary circuits, and these capacitors may be arranged either in series or in parallel with the transformer windings. For the sake of analysis, let Z_{e1} be written in the form given in equation (61).

$$Z_{e1} = \left[R_1 + \frac{\omega^2 M^2 R_2'}{R_2'^2 + \omega^2 L_2'^2}\right] + j\omega\left[L_1 - \frac{\omega^2 M^2 L_2'}{R_2'^2 + \omega^2 L_2'^2}\right] \quad (61)$$

R_2' is the total secondary circuit resistance. L_2' is the total secondary circuit self-inductance.

$$Z_{e1} = R_{e1} + jX_{e1} \quad (62)$$

where

$$X_{e1} = \left[\omega L_1 - \frac{\omega^3 M^2 L_2'}{R_2'^2 + \omega^2 L_2'^2}\right] = \left[X_1 - \frac{X_M{}^2 X_2'}{R_2'^2 + X_2'^2}\right] \quad (63)$$

Series Primary Capacitor. Primary unity power factor can be obtained by introducing a capacitor in series with the primary, which has a capacitive reactance equal in magnitude to tl inductive reactance represented in equation (63).

$$X_{C1(\text{series})} = \left[X_1 - \frac{X_M{}^2 X_2{}'}{R_2{}'^2 + X_2{}'^2}\right] \tag{64}$$

Parallel Primary Capacitor. A capacitor, placed in parallel with the primary terminals, can be used to produce primary unity power factor. It is simply necessary to make the susceptance (b_C) of the parallel capacitor equal in magnitude to the susceptance (b_L) of $\mathbf{Y}_{e1}$, where:

$$\mathbf{Y}_{e1} = \frac{1}{R_{e1} + jX_{e1}} = \frac{R_{e1}}{R_{e1}{}^2 + X_{e1}{}^2} - j\frac{X_{e1}}{R_{e1}{}^2 + X_{e1}{}^2} \tag{65}$$

The inductive susceptance of the uncompensated transformer looking into the primary terminals is given by the j component of the above equation. The capacitive susceptance of the parallel primary capacitor must, therefore, be equal to:

$$b_{C1(\text{parallel})} = \frac{X_{e1}}{R_{e1}{}^2 + X_{e1}{}^2} \tag{66}$$

Secondary Capacitors. Under the assumptions that have been made concerning equations (61), (62), and (63), $X_2{}'$ is an inductive reactance. The introduction of a capacitor in series with the secondary circuit or the introduction of a capacitor in parallel with the secondary load terminals will tend to neutralize the original inductive reactance and cause the net inductive $X_2{}'$ to be smaller in magnitude. If $R_2{}'^2$ is not too great, the lower value of $X_2{}'$ increases the magnitude of the subtractive term of equation (63), namely,

$$\left[\frac{X_M{}^2 X_2{}'}{R_2{}'^2 + X_2{}'^2}\right]$$

Provided $R_2{}'^2$ is sufficiently small in comparison with $X_2{}'^2$ to permit the required increase in the above expression, X_{e1} may be made equal to zero with the proper adjustment of the secondary capacitance. The correct value of secondary capacitance to employ in a particular case is not difficult to determine. However, the general algebraic expressions for the proper sizes of capacitors are of rather awkward algebraic form. In the circuits where this type of tuning is employed the desired effect is very often accomplished by means of a variable condenser which can be adjusted experimentally to the proper capacitance.

Adjustment of M. Assume that X_1 or $X_2{}'$ of equation (63) possesses a capacitive reactive component which is at least large enough to make

$$X_{e1} = \left[X_1 - \frac{X_M^2 X_2'}{R_2'^2 + X_2'^2}\right] = 0 \tag{67}$$

when the two windings are in their position of closest coupling. If now X_M is made smaller by decreasing the coefficient of coupling, X_{e1} will take on positive values, thus indicating a resulting inductive reactance. In general, the capacitive element employed would be adjusted to máke X_{e1} slightly capacitive for the condition of maximum X_M. The primary current could thus be made to lead or lag the primary voltage by adjusting the degree of coupling between the two transformer windings.

Example 11. Let it be required to find the condenser of proper size to place in parallel with the primary terminals of Fig. 17*a* to produce primary unity power factor. The circuit parameters, and so forth, are given on page 295. For the case considered: $\mathbf{Z}_1 = 3.3 + j35.4$, $\mathbf{Z}_M = 0 + j9.65$, and $\mathbf{Z}_2' = (\mathbf{Z}_2 + \mathbf{Z}) = 15.28 + j25.27$ ohms at 60 cycles. Without the condenser:

$$\mathbf{Z}_{e1} = 4.93 + j32.7 \text{ ohms}$$

$$\mathbf{Y}_{e1} = \frac{4.93}{1094} - \frac{j32.7}{1094} = (0.0045 - j0.0299) \text{ mho}$$

Neglecting the resistance of the capacitor which is to be used:

$$b_{e1\,(\text{parallel})} = \frac{1}{X_{C1}} = 2\pi fC$$

$$C = \frac{0.0299}{377} = 79.3 \times 10^{-6} \text{ farad} = 79.3\ \mu\text{f}$$

Problem 14. Find the primary series capacitance to employ in the above example to produce primary unity power factor. *Ans.:* 81.1 μf.

Problem 15. Solve equation (63) for the value of X_2' which will make $X_{e1} = 0$.

$$\textit{Ans.:}\ X_2' = \frac{X_M^2}{2X_1} \pm \sqrt{\frac{X_M^4}{4X_1^2} - R_2'^2}.$$

Problem 16. Can a secondary series capacitance be employed in example 11 to produce primary unity power factor?

Ans.: No; R_2' is too large for the specified values of X_1 and X_M.

Partial Resonance. In the coupled circuits of the type shown in Fig. 18, the two chief concerns are usually: (*a*) maximum value of I_2 (and of V_{C2}) for a given value of V_1; (*b*) sharply defined peak of I_2 for variable X_2, X_M, or ω.

In considering the salient features of these tuned coupled circuits, a slight modification in notation is desirable. Thus far we have distinguished between the impedance of the primary winding ($\mathbf{Z}_1$), the impedance of the secondary winding ($\mathbf{Z}_2$), and the impedance of the load ($\mathbf{Z}$). It is plain from the development preceding equations (52) and (53), page 293, that no restrictions have been imposed on the nature of $\mathbf{Z}_1$. $\mathbf{Z}_1$ is simply the equivalent series-circuit impedance of the primary circuit. Similarly $\mathbf{Z}_2 + \mathbf{Z}$ is the equivalent series-circuit impedance of the secondary circuit. The equations in the remainder of this chapter will be simpler to write and easier to grasp if $\mathbf{Z}_1$ is under-

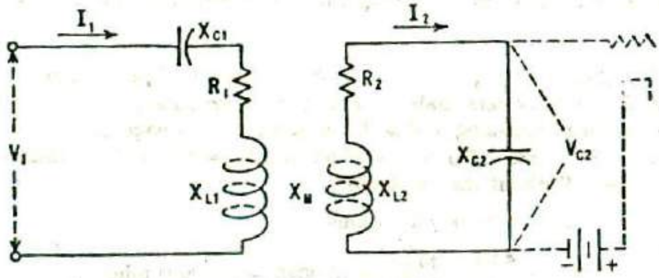


FIG. 18. A double-tuned circuit arrangement.

stood to be the total series impedance of the primary circuit and if $\mathbf{Z}_2$ is understood to be the total series impedance of the secondary circuit. Thus:

$$\mathbf{Z}_1 = R_1 + j(X_{L1} - X_{C1}) = R_1 + jX_1 \tag{68}$$

$$\mathbf{Z}_2 = R_2 + j(X_{L2} - X_{C2}) = R_2 + jX_2 \tag{69}$$

$$\mathbf{Z}_M = jX_M = j\omega M \quad \text{(as before)} \tag{70}$$

The equation for the secondary current $\mathbf{I}_2$ [as given in equation (53), page 293] becomes:

$$\mathbf{I}_2 = \frac{-\mathbf{V}_1\mathbf{Z}_M}{\mathbf{Z}_1\mathbf{Z}_2 - \mathbf{Z}_M{}^2} = \frac{-\mathbf{V}_1(jX_M)}{(R_1 + jX_1)(R_2 + jX_2) + X_M{}^2} \tag{71}$$

or

$$\mathbf{I}_2 = \frac{-\mathbf{V}_1 X_M[(X_1R_2 + X_2R_1) + j(R_1R_2 - X_1X_2 + X_M{}^2)]}{(X_1R_2 + X_2R_1)^2 + (R_1R_2 - X_1X_2 + X_M{}^2)^2} \tag{72}$$

For simplicity in writing, let

$$a = X_1R_2 + X_2R_1 \quad \text{and} \quad b = R_1R_2 - X_1X_2 + X_M{}^2$$

Then:

$$\mathbf{I}_2 = \frac{-\mathbf{V}_1 X_M(a + jb)}{a^2 + b^2} \tag{73}$$

The magnitude of $\mathbf{I}_2$ is:

$$I_2 = V_1 X_M \sqrt{\frac{a^2 + b^2}{(a^2 + b^2)^2}} = \frac{V_1 X_M}{\sqrt{a^2 + b^2}} \tag{74}$$

or

$$I_2 = \frac{V_1 X_M}{\sqrt{X_1^2 R_2^2 + X_2^2 R_1^2 + R_1^2 R_2^2 + 2R_1 R_2 X_M^2 + X_1^2 X_2^2 - 2X_1 X_2 X_M^2 + X_M^4}} \tag{75}$$

In solving for I_2, where numerical values are involved, it is often more convenient to use equation (71) than equation (75). This is particularly true where X_1 or X_2 is equal to zero. Equation (75), however, is useful in determining maximum values of I_2 that can be obtained by varying any one of the parameters.

Partial resonance in coupled circuits is obtained when any one parameter is so varied as to cause maximum effective secondary current, I_2, under the condition of constant applied voltage, V_1.

From equation (75) it is evident that partial resonance can be obtained by adjusting any one of the five parameters: R_1, R_2, X_1, X_2, or X_M. (For fixed values of R_1, L_1, C_1, M, R_2, L_2, and C_2, partial resonance may be obtained by adjustment of the frequency.) Partial resonance will obviously be produced by adjusting any parameter which appears only in the positive terms of the denominator of equation (75) to zero. Hence partial resonance obtains, theoretically, when either R_1 or R_2 is equal to zero. Practically, neither R_1 nor R_2 can be zero and, as will be shown presently, the value of $R_1 R_2$ determines the optimum value of I_2 that can be obtained.

The values of X_1, X_2, or X_M which will produce partial resonance may, in general, be found by differentiating the expression for I_2 [as given in equation (75)] with respect to the proper X and equating dI_2/dX equal to zero. For example, the value of X_1 which will produce partial resonance may be determined by equating dI_2/dX_1 equal to zero and solving for X_1 in terms of the other parameters. Thus:

$$\frac{dI_2}{dX_1} = 0 = -V_1 X_M \tfrac{1}{2}[2X_1(R_2^2 + X_2^2) - 2X_2 X_M^2] \tag{76}$$

The only useful relationship which can be derived from the above is:

$$X_1(R_2^2 + X_2^2) = X_2 X_M^2 \tag{77}$$

The value of X_1 which will produce partial resonance is, therefore:

$$X_{1(\text{res})} = \frac{X_2 X_M^2}{R_2^2 + X_2^2} = \frac{X_2 X_M^2}{Z_2^2} \tag{78}$$

Reference to equation (63), page 301, will show that the above value of X_1 is also the unity-power-factor-resonance value of X_1. In making this comparison it should be recognized that R_2 and X_2 of equation (78) mean the same as R_2' and X_2' of equation (63) because of the shift in notation which was made at the beginning of this section. In a similar manner, it may be shown that the value of X_2 for partial resonance is:

$$X_{2(\text{res})} = \frac{X_1 X_M^2}{R_1^2 + X_1^2} = \frac{X_1 X_M^2}{Z_1^2} \tag{79}$$

The interpretation of the above equation is that X_2 must have the value stated to produce maximum I_2. If $X_1 = 0$, then X_2 should be tuned to zero to produce maximum I_2 for a fixed value of X_M. If the primary circuit is not tuned to $X_{L1} - X_{C1} = 0$, then the secondary must be detuned to the value $X_1 X_M^2 / Z_1^2$. Where sharpness of secondary tuning is of more importance than an optimum value of I_2, the primary is often purposely detuned to effect a pronounced peak in the I_2 versus X_{C2} graph. (See Problem 17, page 309.)

If X_1 and X_2 are both equal to zero (by virtue of $X_{L1} - X_{C1} = 0$ and $X_{L2} - X_{C2} = 0$), equation (75) reduces to

$$I_{2(\max)} = \frac{V_1 X_M}{R_1 R_2 + X_M^2} \tag{80}$$

If, now, X_M is varied by changing the coefficient of coupling between the coils, the optimum value of I_2 is obtained when

$$\frac{dI_{2(\max)}}{dX_M} = \frac{V_1(R_1R_2 + X_M^2) - 2V_1X_M^2}{(R_1R_2 + X_M^2)^2} = 0 \tag{81}$$

or when

$$X_M = \omega M = \pm\sqrt{R_1 R_2} \quad \text{(called critical coupling)} \tag{82}$$

Under these conditions:

$$I_{2(\text{opt})} = \frac{V_1\sqrt{R_1R_2}}{R_1R_2 + R_1R_2} = \frac{V_1}{2\sqrt{R_1R_2}} \tag{83}$$

The relationships stated in equations (78), (79), (82), and (83) are of considerable importance in voltage amplification in radio circuits. Some of the essential features involved are illustrated numerically in the following examples and in graphical form in Figs. 19 and 20. For fixed values of the other parameters, there is a value of X_M or a coefficient of coupling which will produce maximum I_2 as shown in the graphs of Fig. 19. Frequency responses of coupled circuits for fixed values of R_1, L_1, C_1, M, R_2, L_2, and C_2 are shown in Fig. 20. Graphs of I_2 and V_{C2} versus X_{C2} are reserved for student exercises.

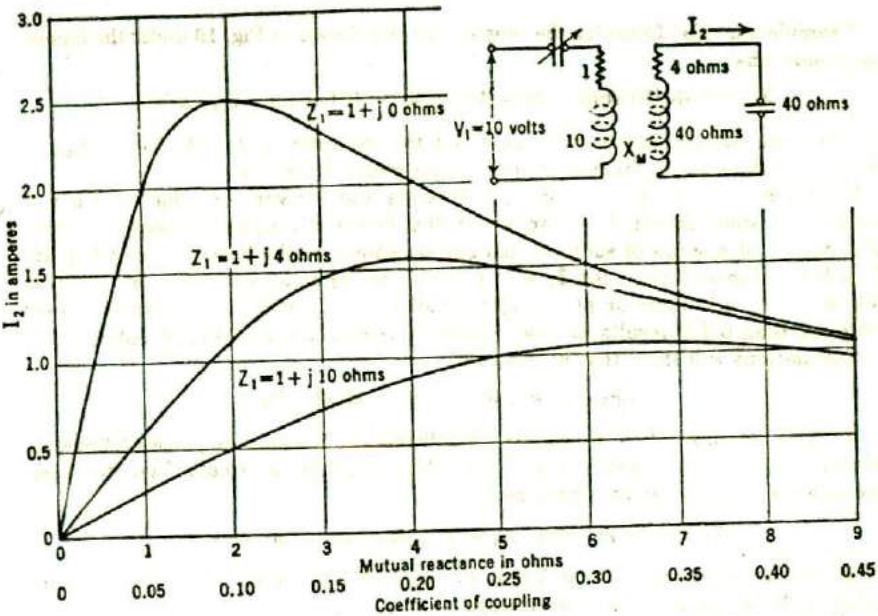


FIG. 19. Variation of secondary current with coefficient of coupling for different values of primary impedance. See example 12.

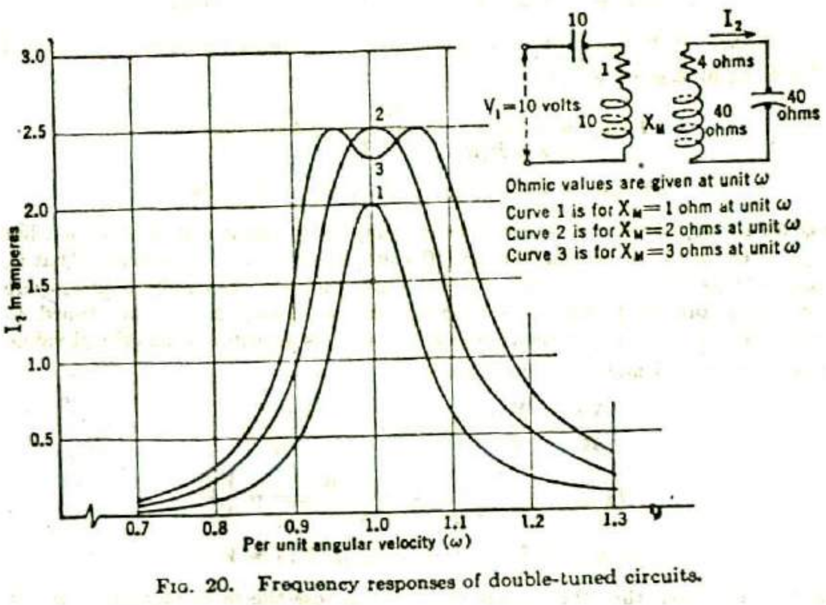


FIG. 20. Frequency responses of double-tuned circuits.

Example 12. (*a*) Consider the coupled circuits shown in Fig. 19 under the following conditions:

$$\mathbf{Z}_1 = 1 + j10 \text{ ohms} \quad \mathbf{Z}_2 = 4 + j(40 - 40) \text{ ohms} \quad X_M \text{ variable}$$

In this case the primary is not tuned and the secondary is tuned, that is, $X_{C2} = X_{L2} = 40$ ohms at the frequency of the impressed voltage, V_1.

Solutions of equation (71) for $V_1 = 10$ volts and for various values of X_M will show the manner in which I_2 varies with the degree of coupling between the coils. The results of a series of such calculations are shown in the lower curve of Fig. 19. It will be observed that, for $\mathbf{Z}_1 = 1 + j10$ ohms, I_2 attains a maximum value at X_M equal to 6.5 ohms or at a coefficient of coupling of 0.325. Closer or looser coupling than 0.325 results in lesser values of I_2 and hence of $V_{C2} = I_2 X_{C2}$.

Calculations will show that in this case

$$V_{C2(\max)} = 1.063 \times 40 = 42.52 \text{ volts}$$

(*b*) The response of I_2 to variable X_M when the primary is partially tuned is shown in the middle graph of Fig. 19. In this case, 6 ohms of capacitive reactance is employed in the primary circuit and

$$\mathbf{Z}_1 = 1 + j4 \text{ ohms} \quad \mathbf{Z}_2 = 4 + j0 \text{ ohms} \quad X_M \text{ variable}$$

I_2 attains a maximum value at $X_M = 4.3$ ohms of 1.565 amperes. The maximum value of the secondary condenser voltage is:

$$V_{C2(\max)} = 1.565 \times 40 = 62.6 \text{ volts}$$

(*c*) The upper graph of Fig. 19 shows the response of I_2 to a variable X_M when both primary and secondary are tuned.

$$\mathbf{Z}_1 = 1 + j0 \text{ ohms} \quad \mathbf{Z}_2 = 4 + j0 \text{ ohms}, \quad X_M \text{ variable}$$

In accordance with equations (82) and (83), I_2 attains its optimum value of $V_1/2\sqrt{R_1R_2}$ at $X_M = \sqrt{R_1R_2}$.

$$I_{2(\text{opt})} = \frac{V_1}{2\sqrt{R_1R_2}} = \frac{10}{2 \times 2} = 2.5 \text{ amperes}$$

$$V_{C2(\text{opt})} = I_{2(\text{opt})} X_{C2} = 2.5 \times 40 = 100 \text{ volts}$$

The Q (or $\omega L/R$) of the coils in this case is equal to 10, and it will be observed that $V_{C2(\text{opt})}$ is equal to the driving voltage (10 volts) times the Q of the coils. That is, $V_{C2(\text{opt})} = V_1 Q = 10 \times 10 = 100$ volts. This fact is generally true where $X_{L2} = 4X_{L1}$, provided that both primary and secondary circuits are tuned to resonance and provided that the coupling reactance is adjusted to its critical value, namely, $\sqrt{R_1R_2}$. Under these conditions,

$$Q = \frac{X_{L1}}{R_1} = \frac{X_{L2}}{R_2} \quad \text{and} \quad R_1R_2 = \frac{X_{L1}X_{L2}}{Q^2}$$

$$I_{2(\text{opt})} = \frac{V_1}{2\sqrt{R_1R_2}} = \frac{V_1Q}{\sqrt{4X_{L1}X_{L2}}} = \frac{V_1Q}{X_{L2}}$$

$$V_{C2(\text{opt})} = I_{2(\text{opt})} X_{C2} = I_{2(\text{opt})} X_{L2} = V_1Q$$

Thus it will be seen that the voltage developed across the secondary condenser of the coupled circuits shown in Fig. 18 may be equal to Q times the applied voltage. If, for example, the Q of the coils is 50, a voltage amplification of 50 can be obtained simply with the aid of the tuned coupled circuits. As indicated in Fig. 18, the voltage developed across the secondary condenser may be applied between the control grid and cathode of a vacuum tube in order to obtain further voltage amplification.

Example 13. The response of a coupled circuit to a constant driving voltage of variable frequency is shown in Fig. 20 for three different values of X_M. Since the critical coupling at unit angular velocity is 2 ohms, the graphs shown in Fig. 20 represent couplings which are less than, equal to, and greater than critical coupling.

In these graphs, unit angular velocity is called the angular velocity at which $X_{L1} - X_{C1} = 0$ and at which $X_{L2} - X_{C2} = 0$. At unit angular velocity,

$$\mathbf{Z}_1 = 1 + j(10 - 10), \quad \mathbf{Z}_2 = 4 + j(40 - 40) \quad X_M = 1, 2, \text{ or } 3 \text{ ohms}$$

At other values of ω, the X_L's and X_M vary directly as ω, and the X_C's vary inversely as ω.

For coupling less than critical coupling the maximum value of the secondary current is less than for critical coupling, and for couplings greater than critical coupling the current response is generally similar to the double-peaked curve shown in Fig. 20.

If a single pronounced peak of I_2 versus ω is desired, the coupling should not be greater than critical coupling, and the Q of the coils should be as high as practicable. If the Q of the coils is made higher than that used in Fig. 20, the peaks of the curves will be sharper and more clearly defined. Sharpness of tuning is particularly important in radio receiver circuits.

Problem 17. In the coupled circuits shown in Fig. 18, page 304:

$R_1 = 1.0$ ohm	$R_2 = 4.0$ ohms
$X_{L1} = 10$ ohms	$X_{L2} = 40$ ohms
$X_{C1} = 10$ ohms	X_{C2} is variable
$X_M = 2$ ohms	$V_1 = 10$ volts

Graph I_2 and V_{C2} versus X_{C2} between the limits of $X_{C2} = 20$ ohms and $X_{C2} =$ 60 ohms.

Ans.: $I_{2(\max)} = 2.5$ amperes at $X_{C2} = 40$ ohms.
$V_{C2(\max)} = 102$ volts at $X_{C2} = 41.7$ ohms, approximately.

Note: The fact that circuits of this kind tune more sharply but to lesser peak values when one member is partially detuned may be shown by repeating the above problem using $\mathbf{Z}_1 = 1 + j4$ ohms rather than $\mathbf{Z}_1 = 1 + j0$.

Double-Tuned Circuit Analysis and Design in Terms of $f/f_0 - f_0/f$. The double-tuned circuit shown in Fig. 21a is widely used in radio engineering practice, and it is the purpose of this section to derive design equations which will specify the Q's of the circuits and the coefficient of coupling in terms of the band width and the degree of irregularity which can be tolerated in the response characteristic. The current enerator (g_mE_g) in parallel with R_p is the plate circuit representation f a vacuum tube. $\left(di_b = \frac{\partial i_b}{\partial e_g} de_g + \frac{\partial i_b}{\partial e_b} de_b \text{ or } i_p = g_m e_g + \frac{e_p}{R_p}\cdot\right)$ See page 203.

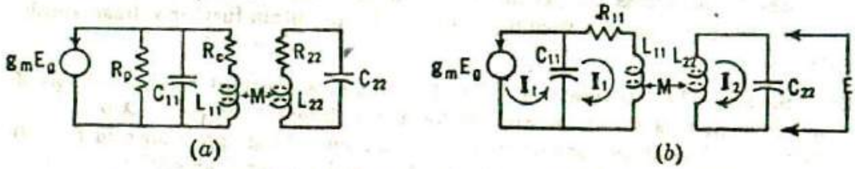


Fig. 21. The actual double-tuned circuit shown in (a) transforms readily to that shown in (b).

Wherever inductive and capacitive reactances are combined as shown in Fig. 22 the analysis is simplified considerably by letting

$$\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} = \frac{f}{f_0} - \frac{f_0}{f} = F \tag{84}$$

where $\omega_0 = 1/\sqrt{L_{11}C_{11}} = 1/\sqrt{L_{22}C_{22}}$ under the assumption that the primary and secondary circuits will be tuned to the same frequency.

It will be noted that F as defined above is the difference between two dimensionless quantities (f/f_0 and f_0/f) which individually characterize the variations of inductive and capacitive reactances relative to variations in frequency.

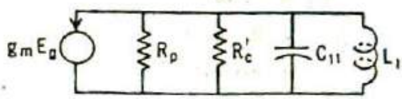


Fig. 22.

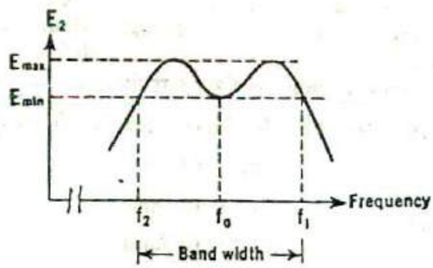


Fig. 23. Response curve of double-tuned circuit. $f_0 = \sqrt{f_1 f_2}$ is the center frequency.

As shown in Fig. 23, $f_1 - f_2$ will be called the band width and it will be assumed that $f_1 - f_2$ is small compared with f_0. For narrow-band responses of this kind, E_2 has a value of $E_{\min}$ within the pass band at

$$f_0 = \sqrt{f_1 f_2}$$

where f_1 and f_2 are the frequencies (other than f_0) at which the response, E_2, has values of $E_{\min}$. See Fig. 23. In this connection it will be noted that, if F_{min} symbolizes the value of F where $E_2 = E_{min}$, say at $f = f_1$, then

$$F_{min} = \frac{f_1}{f_0} - \frac{f_0}{f_1} = \frac{f_1}{f_0} - \frac{f_2}{f_0} \tag{85}$$

since $f_0 = \sqrt{f_1 f_2}$. If the band width is specified, F_{min} is known.

If we let $a = 1/Q_1$, $b = 1/Q_2$, and $k = M/\sqrt{L_{11}L_{22}}$:

$$\mathbf{Z}_{11}\text{(in Fig. 21}b\text{)} = R_{11} + j\left(\omega L_{11} - \frac{1}{\omega C_{11}}\right) = \omega_{01}L_{11}(a + jF_{11}) \tag{86}$$

$$\mathbf{Z}_{22} = \omega_{02}L_{22}(b + jF_{22}) \tag{87}$$

$$\mathbf{Z}_{12} = \mathbf{Z}_{21} = j\omega M = j\omega k\sqrt{L_{11}L_{22}} \tag{88}$$

We assume that C_{11} and C_{22} will be so adjusted that

$$\omega_{01} = \frac{1}{\sqrt{L_{11}C_{11}}} = \omega_{02} = \frac{1}{\sqrt{L_{22}C_{22}}} = \omega_0$$

where

$$\omega_0^2 = \frac{1}{\sqrt{L_{11}L_{22}C_{11}C_{22}}}$$

The problem is essentially that of expressing a, b, and k in terms of F_{min} and $(E_{max} - E_{min})$.

Employing the loop current method of analysis in Fig. 21b and treating $g_m E_g$ as a known value of current, say I_t, circulating in the left-hand loop, we have

$$\left.\begin{aligned} \mathbf{Z}_{11}\mathbf{I}_1 + \mathbf{Z}_{12}\mathbf{I}_2 &= j\frac{g_m E_g}{\omega C_{11}} = j\frac{I_t}{\omega C_{11}} \\ \mathbf{Z}_{21}\mathbf{I}_1 + \mathbf{Z}_{22}\mathbf{I}_2 &= 0 \end{aligned}\right\} \tag{89}$$

The output voltage is

$$\mathbf{E}_2 = -j\frac{1}{\omega C_{22}}\mathbf{I}_2 = \frac{\left(-j\frac{1}{\omega C_{22}}\right)\left(-j\frac{I_t}{\omega C_{11}}\right)(j\omega k\sqrt{L_{11}L_{22}})}{\omega_0^2 L_{11}L_{22}[(ab - F^2) + j(a + b)F] + \omega^2 k^2 L_{11}L_{22}} \tag{90}$$

$$\mathbf{E}_2 = \frac{\dfrac{-jI_t k\sqrt{L_{11}L_{22}}}{\omega C_{11}C_{22}}}{\omega_0^2 L_{11}L_{22}\left[\left(ab + \dfrac{\omega^2}{\omega_0^2}k^2 - F^2\right) + j(a + b)F\right]} \tag{91}$$

Since we are interested particularly in the region shown in Fig. 23 where any ω is close to ω_0 if the per unit band width is small, we may set $\omega^2/\omega_0^2 \approx 1$ in equation (91) and obtain

$$\mathbf{E}_2 = \frac{-jI_tk}{\omega\sqrt{C_{11}C_{22}}[(k^2 + ab - F^2) + j(a + b)F]} \quad (92)$$

At $\omega = \omega_0$, the center angular frequency $F = 0$ and

$$\mathbf{E}_{20} = \mathbf{E}_0 = \frac{-jI_tk}{\omega_0\sqrt{C_{11}C_{22}}\,(k^2 + ab)} \quad (93)$$

Consider now the ratio of the magnitudes of E_2 and E_0 and let the ratio ω/ω_0 again be reckoned as unity. Under these conditions

$$\left(\frac{E_2}{E_0}\right)^2 = \frac{1}{1 + \dfrac{F^4 + (a^2 + b^2 - 2k^2)F^2}{(k^2 + ab)^2}} \quad (94)$$

or

$$\frac{E_2}{E_0} = \frac{1}{\sqrt{1 + \dfrac{F^4 + (a^2 + b^2 - 2k^2)F^2}{(k^2 + ab)^2}}} \quad (95)$$

From equation (95) it is plain that the shape of the E_2 curve (reckoned in per unit values relative to E_0) will be determined by the relative magnitudes of $a^2 + b^2$ and $2k^2$. If $a^2 + b^2 \gtreqless 2k^2$, then a single-peaked curve is obtained since, as F takes on values greater than 0 (f different from f_0), the E_2/E_0 curve will decrease continuously from its maximum value of unity, the value of E_2/E_0 when $F = 0$ or when $f = f_0$.

If, however, $a^2 + b^2 < 2k^2$, the denominator of equation (95) takes on a minimum value or E_2/E_0 takes on a maximum value where $f(F^2) = F^4 + (a^2 + b^2 - 2k^2)F^2$ is a minimum. This minimum may be found from

$$\frac{df(F^2)}{d(F^2)} = 2(F^2) - (2k^2 - a^2 - b^2) = 0$$

or where

$$F^2 = F_{\max}{}^2 = \frac{2k^2 - a^2 - b^2}{2} \quad (96)$$

When plotted versus actual frequency, the response takes the form shown in Fig. 23 or, when plotted versus F, the form shown in Fig. 24.

We may write an expression for $(E_2/E_0)_{max} = E_{max}$ from equations (95) and (96), and, since E_{min} is taken as unity, we may write

$$\frac{E_{max}^2}{E_{min}^2} = \frac{1}{1 - \dfrac{(2k^2 - a^2 - b^2)^2}{4(k^2 + ab)^2}} \tag{97}$$

Let

$$\alpha^2 = 1 - \frac{E_{min}^2}{E_{max}^2} = \frac{(2k^2 - a^2 - b^2)^2}{4(k^2 + ab)^2} = \frac{F_{min}^4}{4(k^2 + ab)^2} \tag{98}$$

where $F_{min}^2 = (2k^2 - a^2 - b^2)$. [See Fig. 24 and equation (95.)] It follows that

$$\alpha = \frac{F_{min}^2}{2(k^2 + ab)} \tag{99}$$

and

$$\frac{E_2}{E_0} = \frac{1}{\sqrt{1 + \dfrac{4\alpha^2 F^2 (F^2 - F_{min}^2)}{F_{min}^4}}} \tag{100}$$ [3]

$F_{min} = \sqrt{2k^2 - a^2 - b^2} = (f_1 - f_2)/f_0$ is the value of F at the edge of the pass band where $E_2/E_0 = 1 = E_{min}$.

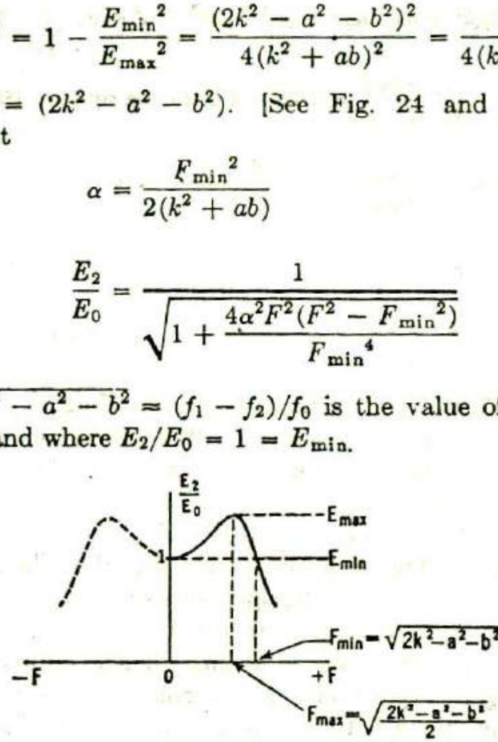


FIG. 24. A response curve, E_2/E_0 versus the variable F for $a^2 + b^2 < 2k^2$. $(F = f/f_0 - f_0/f.)$

Equation (100) is a convenient working equation since it includes α, a measure of the response irregularity which can be tolerated within the pass band $(f_1 - f_2)$, and F_{min}, a measure of the pass band width $(f_1 - f_2)$. From a design point of view, α and F_{min} would normally be specified (at least indirectly), and k, a, and b would then be so chosen

[3] These results are due to Dr. T. C. G. Wagner of the University of Maryland who has developed design formulas for double-, triple-, and quadruple-tuned circuits.

that the specified values of α and $f_1 - f_2$ would be obtained in the final design. See Problem 45, page 324, and example 14 for applications.

Example 14. Let it be required to design a double-tuned circuit which will have per unit band width $[(f_1 - f_2)/f_0]$ of 0.05 and a ratio of E_{max} to E_{min} equal to 1.25. If we make $a = b$ $(Q_1 = Q_2)$, we may readily show that:

$$a^2 = b^2 = \frac{F_{min}^2(1 - \alpha)}{4\alpha} \quad \text{and} \quad k^2 = \frac{F_{min}^2(1 + \alpha)}{4\alpha}$$

ince $\alpha = F_{min}^2/2(k^2 + ab)$ and $F_{min}^2 = 2k^2 - a^2 - b^2$. In the particular case ınder discussion

$$F_{min}^2 = \left(\frac{f_1 - f_2}{f_0}\right)^2 = 0.05^2 = 0.0025 \text{ [see equation (85)]}$$

ınd

$$\alpha^2 = 1 - \frac{E_{min}^2}{E_{max}^2} = 1 - \frac{16}{25} \quad \text{or} \quad \alpha = 0.6$$

Thus

$$a^2 = b^2 = \frac{0.0025(0.4)}{2.4} = 0.000417 \quad \text{and} \quad Q_1 = Q_2 = 49$$

$$k^2 = \frac{0.0025(1.6)}{2.4} = 0.00167 \quad \text{and} \quad k = 0.041$$

Component Fluxes and Voltages in the Air-Core Transformer. Figure 25*a* shows diagrammatically the flux components in an air-core transformer. The current I_2 in the secondary produces an mmf which may be considered to cause two component fluxes: one the leakage flux ϕ_{22}, which links the turns of winding 2 only, and ϕ_{21}, which links both windings 2 and 1. The same conditions regarding the flux linkages as explained on page 279 for Fig. 6 apply to the present discussion, namely, that ϕ_{22} is a hypothetical component which, when linking all the turns of winding 2, produces the same total flux linkages as obtained from the true flux linkages in question. The current I_1 causes two component fluxes, ϕ_{12}, which links both windings, and ϕ_{11}, which links winding 1 only. Reference to example 9 on page 295 and application of Lenz's law will reveal in a general way the reason for the phase angle shown between $\mathbf{I}_1$ and $\mathbf{I}_2$ in the vector diagram (Fig. 25*b*). The component fluxes produced by $\mathbf{I}_1$ and $\mathbf{I}_2$ are also shown. It is plain from Fig. 25*a* that the resultant mutual flux is $\phi_M = \phi_{12} + \phi_{21}$. The total flux through winding 2 is $\phi_{2R} = \phi_M + \phi_{22} = \phi_2 + \phi_{12}$. Also the total flux through winding 1 is $\phi_{1R} = \phi_M + \phi_{11} = \phi_1 + \phi_{21}$. All these combinations are shown on the vector diagram. Equal numbers of turns on windings 1 and 2 are assumed.

Since $e = -N(d\phi/dt)$, the induced voltage due to a flux lags the flux by 90 degrees. Thus, on the vector diagram, $\mathbf{E}_{2R}$ is caused by ϕ_{2R}, $\mathbf{E}_{M2}$

by ϕ_M, and E_{22} by ϕ_{22}. The resultant induced emf in winding 2 is therefore E_{2R}. Because of the resistance R_2 of winding 2 the terminal voltage must be less than E_{2R} by the I_2R_2 drop as shown. Hence V_2 is the secondary terminal voltage. It is seen to be ahead of I_2 by the secondary load power-factor angle.

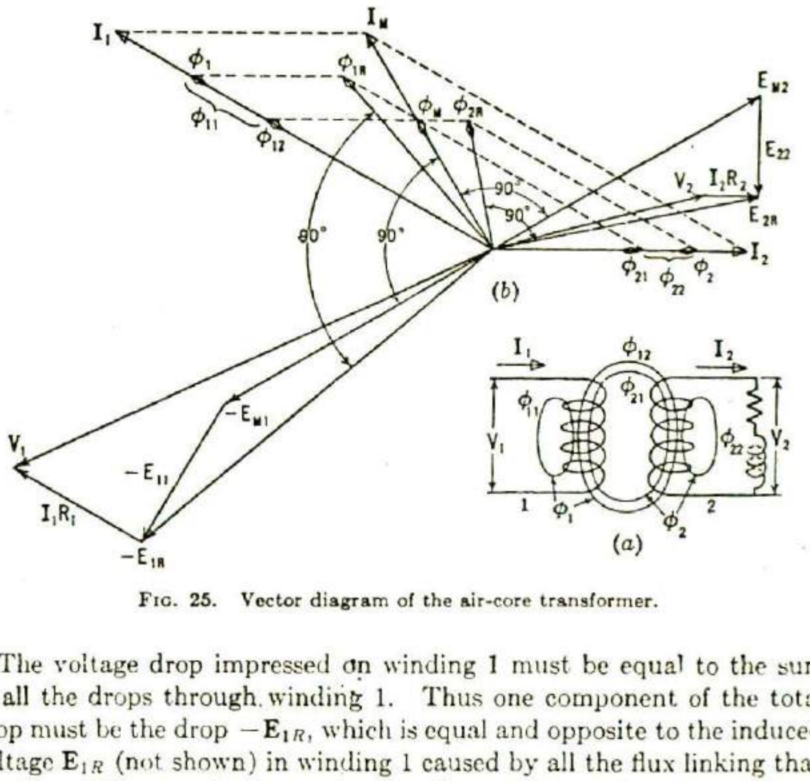


FIG. 25. Vector diagram of the air-core transformer.

The voltage drop impressed on winding 1 must be equal to the sum of all the drops through winding 1. Thus one component of the total drop must be the drop $-E_{1R}$, which is equal and opposite to the induced voltage E_{1R} (not shown) in winding 1 caused by all the flux linking that winding. The remaining component drop is the I_1R_1. Hence $V_1 = I_1R_1 + (-E_{1R})$. The components of $-E_{1R}$ are the voltage drops $-E_{11}$ and $-E_{M1}$, which overcome the induced voltages due to the primary leakage and mutual fluxes, respectively.

The leakage flux ϕ_{22} is (even for all practical purposes in iron-core transformers) proportional to the current I_2. E_{22} is an induced voltage rise and is directly proportional to I_2. The voltage $-E_{22}$ is opposite to E_{22} and therefore leads the current by 90 degrees. It is thus in the direction of a reactance drop, and, since it is proportional to the current, a constant reactance may be multiplied by the current I_2 to represent correctly the drop $-E_{22}$. Such a reactance which may be used to replace the effect of the leakage flux is called a leakage reactance, and the corresponding dron a leakage reactance drop. The vector diagram which is commonly used is shown in Fig. 26. Only the flux ϕ_M in Fig. 25 is shown, and the drops $-\mathbf{E}_{22}$ and $-\mathbf{E}_{11}$ are replaced by their corresponding leakage reactance drops $\mathbf{I}_2\mathbf{X}_2$ and $\mathbf{I}_1\mathbf{X}_1$, respectively.

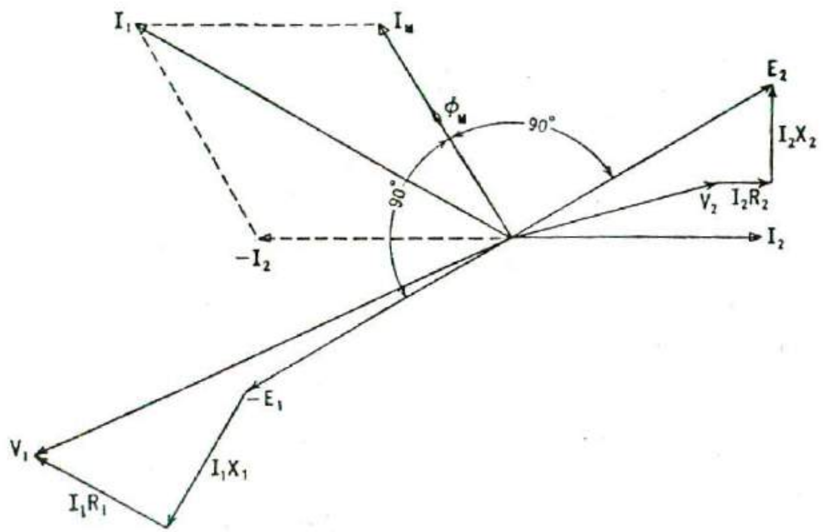


FIG. 26. Commonly used vector diagram for the air-core transformer shown in Fig. 25.

Leakage Reactance. Leakage reactance may be defined as $2\pi f$ times the leakage inductance. This may be shown as follows. By referring to Fig. 25a, leakage inductance

$$L_{S2} = \frac{N_2\phi_{22}}{I_2} \quad \text{or} \quad N_2\frac{d\phi_{22}}{di_2} \tag{101}$$

$$e_{22} = -N_2\frac{d\phi_{22}}{dt} \tag{102}$$

Dividing equation (102) by equation (101) gives

$$e_{22} = -L_{S2}\frac{di_2}{dt}$$

For sine waves

$$i_2 = I_{m2}\sin\omega t \tag{103}$$

and

$$e_{22} = -L_{S2}I_{m2}\omega\cos\omega t \tag{104}$$

Hence

$$E_{m22} = I_{m2}\omega L_{S2}$$

Also

$$E_{22} = \frac{I_{m2}}{\sqrt{2}} \omega L_{S2} = I_2 \omega L_{S2}$$

The magnitude of the leakage reactance drop has been defined equal to $E_{22} \doteq I_2 \omega L_{S2} \doteq I_2 X_2$. Therefore

$$X_2 = \omega L_{S2} \tag{105}$$

Since e_{22} in equation (104) is a voltage rise, the drop is $-e_{22} = L_{S2}\omega I_{m2} \cos \omega t$. Because this voltage drop is 90 degrees ahead of the current (equation 103), the complex expression for leakage reactance must be

$$X_2 = +j\omega L_{S2} \tag{106}$$

The Air-Core Autotransformer. Two inductance coils arranged as shown in Fig. 27 are called an autotransformer. If the driving voltage is applied to the terminals *ab* and the load connected across the terminals *ac*, the autotransformer functions as a step-up voltage device; whereas, if the driving voltage is applied to the terminals *ac* and the load connected to terminals *ab* or *bc*, it functions as a step-down voltage device. The mathematical analysis of the air-core autotransformer is reserved for student exercises. (See Problems 37, 38, and 39 at the end of this chapter.)

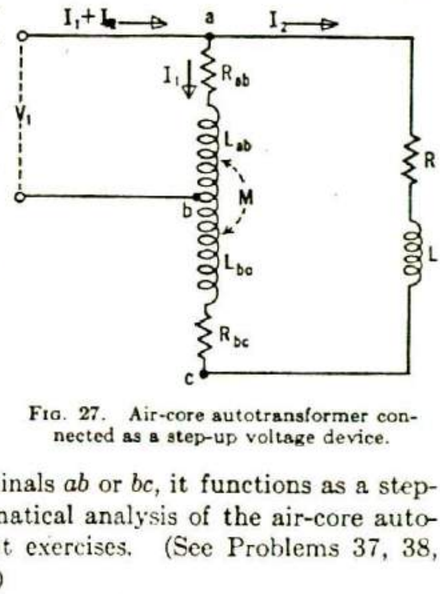


FIG. 27. Air-core autotransformer connected as a step-up voltage device.

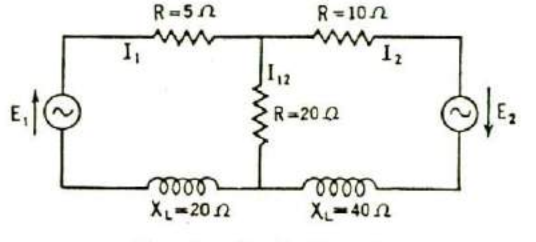


FIG. 28. See Problem 18.

PROBLEMS

18. In Fig. 28, $E_1 = 100/\underline{0°}$ volts and $E_2 = 100/\underline{+120°}$ volts. The physical meaning of the foregoing statement is that the E_2 generator develops a maximum generated emf ($\sqrt{2} \times 100$ volts) in its arrow direction $\frac{1}{3}$ of a cycle or 120° before

the E_1 generator develops its maximum generated emf in its arrow direction. Assuming that the resistances and reactances given in Fig. 28 include the generator impedances, find $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_{12}$.

19. In Fig. 2, page 276, it is found experimentally that $\mathbf{I}_1 = 1\underline{/90^\circ}$ ampere and $\mathbf{V}_{22'} = 4\underline{/0^\circ}$ volts (with terminals 22′ open-circuited) when $\mathbf{E}_1$ (the voltage applied to terminals 11′) is $6\underline{/0^\circ}$ volts. When a voltage of $6\underline{/0^\circ}$ volts is applied to terminals 22′ (with terminals 11′ open-circuited), $\mathbf{I}_2 = 1.5\underline{/90^\circ}$ amperes and $\mathbf{V}_{11}' = 6\underline{/0^\circ}$ volts.

(a) Find $\mathbf{Z}_{21}$ and $\mathbf{Z}_{12}$ from the above data.

(b) Find the coefficient of coupling between the two circuits.

(c) Draw a circuit configuration that might actually exist within the 11′2′2 box and that is consistent with the specified data.

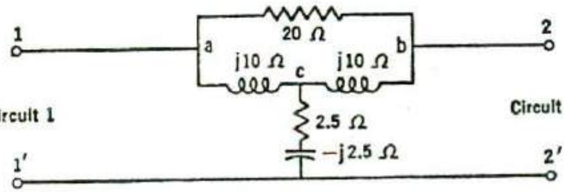


FIG. 29. See Problem 20.

20. Find the coefficient of coupling between circuits 1 and 2 in Fig. 29. *Hint:* Transform the *abc* delta to an equivalent wye, and then determine $\mathbf{Z}_{12}$ or $\mathbf{Z}_{21}$ of the equivalent circuit.

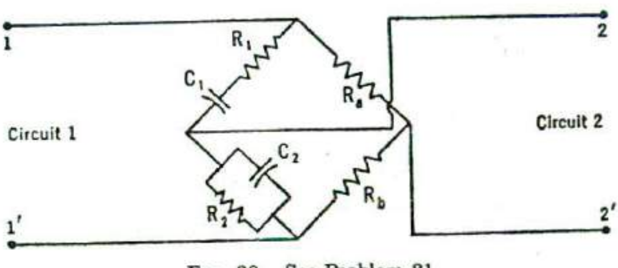


FIG. 30. See Problem 21.

21. Show that the coupling coefficient between circuits 1 and 2 in Fig. 30 is equal to zero if $\omega = 1/\sqrt{R_1R_2C_1C_2}$, $R_a = R_b$, $R_2 = 2R_1$, and $C_1 = 2C_2$.

22. Figures 31*a*, 31*b*, and 31*c* are the approximate equivalent circuits that are sometimes used in making voltage amplification calculations in resistance-coupled audio amplifiers. Show that the expressions given for E_2 in terms of μE_g are correct for each of the three configurations.

23. Two air-core inductance coils possess, individually, 60 and 30 millihenrys self-inductance, respectively. Measurements show that, if the two coils are connected in additive series as shown in Fig. 12, page 287, the equivalent self-inductance of the combination is 120 millihenrys.

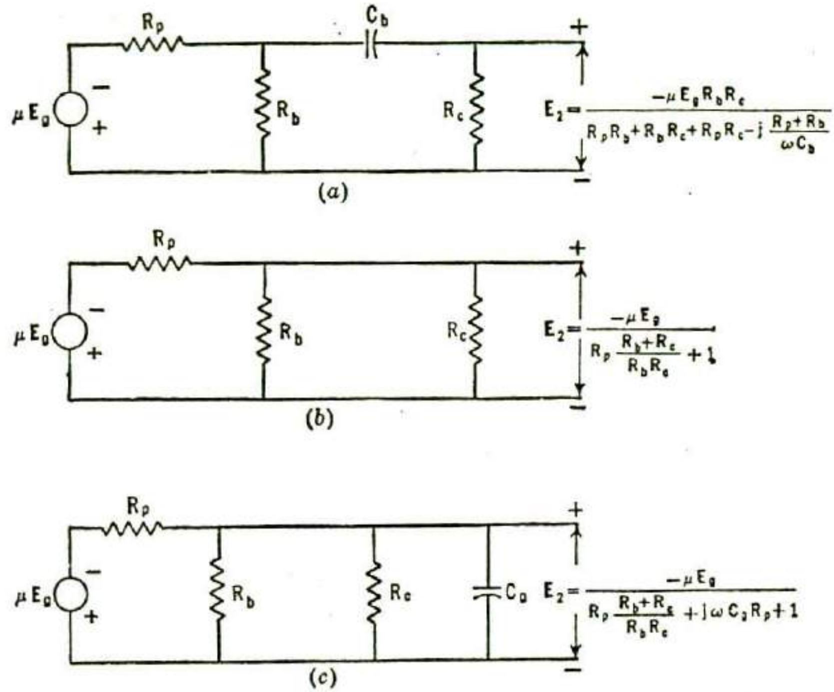


FIG. 31. Approximate equivalent circuits of resistance-capacitance coupled amplifiers. See Problem 22. (a) is for low-frequency range. (b) is for intermediate-frequency range, and (c) is for high-frequency range where the impedance of the blocking condenser C_b may be neglected.

(a) If the coils are connected in subtractive series, find the equivalent self-inductance of the combination.

(b) Find the coefficient of coupling between the coils.

24. Two inductance coils are connected in additive series. For 100 volts impressed on the combination, the current is 5 amperes and the power consumed is 200 watts. When the coils are reconnected in subtractive series and 100 volts are impressed, 8 amperes flow. Calculate the mutual inductance if the frequency for the above measurements is 69.5 cycles.

25. If the two coils in Problem 24 have equal resistances and the voltage drop across coil 1 is 36.05 volts for the additive series connection in Problem 24, (a) calculate L_1 and L_2 and the drop across coil 2 for this condition; (b) also calculate the coefficient of coupling.

26. The individual self-inductances of the two windings shown in Fig. 6, page 216, are 0.100 and 0.050 henry, respectively. The coefficient of coupling between the windings is 0.56. If the current in the 0.100-henry winding is a 60-cycle sinusoidal variation, the maximum magnitude of which is 10 amperes, find the effective value of voltage induced in the 0.050-henry winding as a result of the current variation in the 0.100-henry winding. Also find the magnitude of the rms induced voltage in the 0.1-henry winding.

27. In Fig. 32, $e_{ba} = 141.4 \sin 1131t$ volts and $e_{cd} = 70.7 \sin (1131t - 90°)$ volts.

(a) Find I_{ba} and I_{cd}, assuming that Fig. 32 correctly represents the modes of winding as well as the physical placement of the two inductance coils. The internal impedances of the generators may be assumed to be negligibly small.

(b) Find the power generated by each generator.

(c) Draw a vector diagram of $\mathbf{E}_{ba}$, $\mathbf{I}_{ba}$, $\mathbf{I}_{ba}\mathbf{R}_1$, $\mathbf{I}_{ba}\mathbf{X}_{L1}$, $\mathbf{E}_{cd}$, $\mathbf{I}_{cd}$, $\mathbf{I}_{cd}\mathbf{R}_2$, $\mathbf{I}_{cd}\mathbf{X}_{L2}$, $\mathbf{I}_{cd}\mathbf{X}_M$, and $\mathbf{I}_{ba}\mathbf{X}_M$.

28. Branch 1 of two parallel branches consists of a resistance of 2 ohms in series with an inductive reactance of 3 ohms. Branch 2 consists of a resistance of 5 ohms in series with an inductive reactance of 12 ohms. The coefficient of coupling between the two inductances is 0.8, and the inductances are wound so that the mmf's

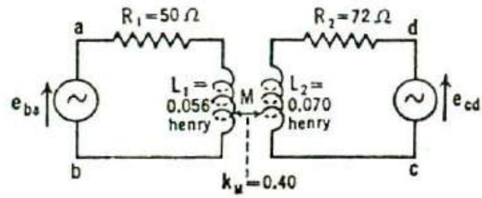


FIG. 32. See Problem 27.

due to I_1 and I_2 taken in the same direction from the junction are additive. If 100 volts are impressed on the two parallel branches, find I_1, I_2, the power supplied conductively to branch 2, the power supplied branch 2 electromagnetically, and the voltage drop across only the inductance of branch 2. What is the phase angle between the latter drop and the current in branch 2?

29. The coefficient of coupling for the coils in Fig. 33 is 0.5. Find the current in the resistance.

30. Calculate the phase and magnitude of the voltage drop V_{be} with respect to the total drop from a to c in Fig. 31. $X_{L1} = 5\ \Omega$; $X_{L2} = 5\ \Omega$; $X_M = 4\ \Omega$.

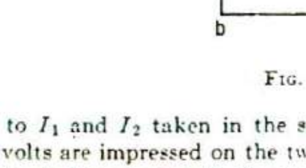


FIG. 33. See Problem 29.

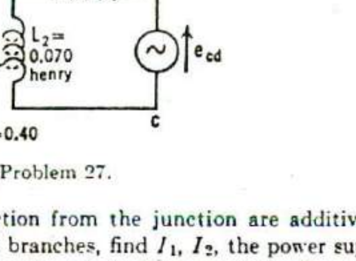


FIG. 34. See Problem 30.

31. In the coupled circuits shown in Fig. 18, page 304,

$R_1 = 4.0$ ohms	$R_2 = 10$ ohms
$X_{L1} = 40$ ohms	$X_{L2} = 100$ ohms
$X_{C1} = 40$ ohms	$X_{C2} = 120$ ohms
$X_M = 50$ ohms	$V_1 = 100$ volts

Find I_2 and V_{C2}.

32. In the coupled circuits shown in Fig. 18, page 304,

$R_1 = 4$ ohms	$R_2 = 10$ ohms
$X_{L1} = 40$ ohms	$X_{L2} = 100$ ohms
$X_{C1} = 40$ ohms	$X_{C2} = 120$ ohms
$X_M = 50$ ohms	$V_1 = 100$ volts

Find the equivalent primary impedance, Z_{e1}, of the coupled circuits and the ohmic value of the secondary-circuit impedance referred to the primary terminals. How many ohms reactance does the secondary reflect into the primary, and is it inductive or capacitive?

33. Assume that an 83-μf capacitance is placed in series with the primary of Fig. 17a. Except for the insertion of the 83-μf capacitance into the primary circuit, the parameters are as given on page 295. Find the value of M which will produce unity-power-factor resonance.

34. Show that the partial resonance which can be obtained by adjustment of the secondary reactance, X_2 (in coupled circuits of the kind shown in Fig. 18, page 304), occurs when $X_2 = X_1 X_M{}^2/Z_1{}^2$. (See equation 79, page 306.)

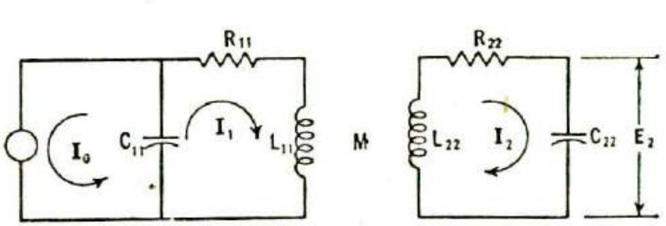


Fig. 35. See Problems 35, 41, 42, 43, 44, and 45.

35. In Fig. 35 $R_{11} = 10$, $L_{11} = 0.01$ henry, $L_{22} = 0.05$ henry, $M = 0.02$ henry, $R_{22} = 40\,\Omega$, $C_{22} = 20.0\ \mu$f, and $\omega = 1000$ radians per second. (a) Find the value of C_{11} that will make the whole circuit, looking into the lines connecting to the source, a pure resistance. (b) Find the value of the pure resistance.

36. Circuits 1 and 2 are inductively coupled. Circuit 1 consists of 2 ohms resistance in series with a coil of 16 ohms reactance and negligible resistance. Circuit 2 consists of 10 ohms resistance in series with an inductance coil of 100 ohms reactance and a capacitor of 100 ohms.

(a) If the coefficient of coupling is 0.05, what is the drop across the capacitor when 10 volts are applied to circuit 1?

(b) If a capacitor is placed in series with circuit 1 so as to tune circuit 1 to resonance ($\omega L_1 = 1/\omega C_1$), what will be the drop across the capacitor in circuit 2 for the same coefficient of coupling as before?

(c) If the coupling can be adjusted in part (b), what will be the greatest voltage drop across the secondary capacitor?

37. Write the general differential equations for voltage equilibrium in the two circuits shown in Fig. 27, page 317, in terms of R_{ab}, L_{ab}, R_{bc}, L_{bc}, M, R, and L, and the branch currents i_1 and i_2. Note that this is essentially two parallel branches which are coupled.

38. Assuming that v_1 varies sinusoidally, write the general voltage equations for Fig. 27, page 317, in terms of the effective values of the branch currents, I_1 and I_2. Solve the equations thus found for $\mathbf{I}_1$ and $\mathbf{I}_2$. What circuit considered earlier in this chapter has similar equations for $\mathbf{I}_1$ and $\mathbf{I}_2$?

39. Assume that, in Fig. 27, page 317,

$R_{ab} = 4.0$ ohms	$M = 0.02$ henry
$L_{ab} = 0.07$ henry	$R = 10$ ohms
$R_{bc} = 0.5$ ohm	$L = 0.00$ henry
$L_{bc} = 0.01$ henry	$\omega = 377$ radians per second

If $\mathbf{V}_1 = 100\underline{/0°}$ volts, find $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_1 + \mathbf{I}_2$. Also calculate the total power supplied and that dissipated in each of circuits 1 and 2. Draw the complete vector diagram of the voltages and currents.

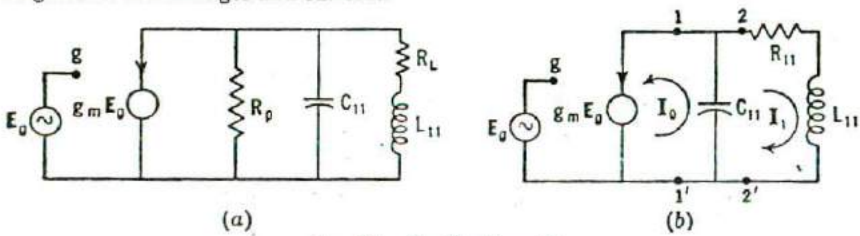


FIG. 36. See Problem 40.

40. Given the circuit arrangement shown in Fig. 36*a*, where the $g_m E_g$ current generator in parallel with R_p is the equivalent a-c circuit of a pentode which has a voltage of E_g volts applied to its control grid.

(*a*) If $R_p = 750{,}000$ ohms, $R_L = 12$ ohms, $L_{11} = 382$ microhenrys, and C_{11} is adjusted to resonate the $L_{11}C_{11}$ parallel branches at 500 kc, find R_{11} of the equivalent circuit shown in Fig. 36*b*.

(*b*) What is the Q of the coil itself, namely, $\omega_m L_{11}/R_L$, at 500 kc?

(*c*) What is the Q of the $C_{11} - R_{11}L_{11}$ parallel combination of Fig. 36*b* at 500 kc?

(*d*) Can I_1 in Fig. 36*b* be evaluated from the relation $\mathbf{Z}_{11}\mathbf{I}_1 = -(\mathbf{I}_0)\left(-j\dfrac{1}{\omega C_{11}}\right)$, where $\mathbf{Z}_{11} = R_{11} + j\left(\omega L_{11} - \dfrac{1}{\omega C_{11}}\right)$?

41. In Problem 40, it has been shown that the current generators of Fig. 36*b* and Fig. 35 can be replaced by equivalent voltage generators which have voltages of $-(\mathbf{I}_0)\left(-j\dfrac{1}{\omega C_{11}}\right)$.

Show that the equivalent primary impedance (including the reflected impedance from the secondary) which the equivalent voltage generator in Fig. 35 sees is:

$$\mathbf{Z}_{11\text{eq}} = \frac{j\dfrac{\mathbf{I}_0}{\omega C_{11}}}{\mathbf{I}_1} = \mathbf{Z}_{11} + \frac{\omega^2 M^2}{\mathbf{Z}_{22}} = \omega_{m1}L_{11}\left[(a + jF_{11}) + \frac{\omega^2 k^2}{\omega_{m1}\omega_{m2}(b + jF_{22})}\right]$$

where

$$\omega_{m1} = \frac{1}{\sqrt{L_{11}C_{11}}} \qquad \omega_{m2} = \frac{1}{\sqrt{L_{22}C_{22}}}$$

$$\mathbf{Z}_{11} = R_{11} + j\left(\omega L_{11} - \frac{1}{\omega C_{11}}\right) \qquad \mathbf{Z}_{22} = R_{22} + j\left(\omega L_{22} - \frac{1}{\omega C_{22}}\right)$$

$$a = \frac{1}{Q_1} = \frac{R_{11}}{\omega_{m1}L_{11}} \qquad b = \frac{1}{Q_2} = \frac{R_{22}}{\omega_{m2}L_{22}}$$

$$F_{11} = \left(\frac{\omega}{\omega_{m1}} - \frac{\omega_{m1}}{\omega}\right) \qquad F_{22} = \left(\frac{\omega}{\omega_{m2}} - \frac{\omega_{m2}}{\omega}\right)$$

$$k = \frac{M}{\sqrt{L_{11}L_{22}}}$$

42. The results of Problem 41 are to be employed in the following exercises.

(*a*) Show that a voltmeter across L_{11} of Fig. 35 will read a maximum value when C_{11} is adjusted to $1/L_{11}\omega^2$ if loop 2 is open-circuited and that this voltage will be

$$\mathbf{V}_{L_{11}\max} = \frac{\mathbf{K}}{\omega_{m1}L_{11}a}$$

where $\mathbf{K} = [-(\mathbf{I}_0/\omega C_{11})]\,(\omega L_{11})$.

(*b*) With C_{11} left at the value found above $(1/L_{11}\omega_{m1}^2)$, show that the voltmeter (which is across the L_{11} coil) will read a minimum value of

$$\mathbf{V}_{L_{11}\min} = \frac{\mathbf{K}}{\omega_{m1}L_{11}\left(a + \dfrac{k^2}{b}\right)}$$

when C_{22} is adjusted to $1/L_{22}\omega_{m1}^2$.

(*c*) Show that, if the experimental procedure outlined in (*a*) and (*b*) is followed, the coupling coefficient between the two coils is

$$k = \sqrt{ab\left(\frac{V_{L_{11}\max}}{V_{L_{11}\min}} - 1\right)}$$

43. In Fig. 35: $L_{11} = L_{22} = 500$ microhenrys; $C_{11} = C_{22} = 2000\ \mu\mu\text{f}$; $M = 8.66$ microhenrys; $a = R_{11}/\omega_{m1}L_{11} = b = R_{22}/\omega_{m2}L_{22} = 0.01$.

(*a*) Find the magnitude of the voltage across the C_{22} capacitor per milliampere of I_0 at $\omega = \omega_m = 1/\sqrt{L_{11}C_{11}}$ radians per second.

(*b*) Will the voltage found in part (*a*) be the maximum value of E_2 if the frequency is varied slightly about the value ω_m specified above?

44. (*a*) Make a sketch of $\dfrac{E_{C22}}{E_{C22(\omega=\omega_m)}}$ versus F for the circuit shown in Fig. 35 employing the circuit parameters specified in Problem 43. Calculate points for this sketch at

$$\omega = 1.010\omega_m \quad \text{or} \quad F = 2 \times 10^{-2}$$
$$\omega = 1.00707\omega_m \quad \text{or} \quad F = \sqrt{2} \times 10^{-2}$$
$$\omega = 1.005\omega_m \quad \text{or} \quad F = 10^{-2}$$
$$\omega = \omega_m \quad \text{or} \quad F = 0$$

using equation (95), namely:

$$\frac{E_{C22}}{E_{C22(\omega=\omega_m)}} = \frac{1}{\sqrt{1 + \dfrac{F^4 + (a^2 + b^2 - 2k^2)F^2}{(k^2 + ab)^2}}}$$

(b) Make a sketch of E_{C22} per milliampere of I_0 versus ω/ω_m employing the results of part (a). It may be assumed that the response curve is symmetrical about the center frequency ω_m.

45. Design a current-fed double-tuned circuit like that shown in Fig. 35 which has a per unit band width of 0.02 centered at $\omega_m = 10^6$ radians per second. Use $L_{11} = L_{22} = 500$ microhenrys. The permissible variation in the response curve over the pass band is 1.2516 decibels reckoned from E_{min} as reference. $(\alpha = 0.5)$

Note: Where $Q_1 = Q_2$, a design of this kind amounts simply to specifying some appropriate value for the Q's of the coils and then calculating the coefficient of coupling to employ between these coils to meet the conditions imposed. In this case, $F_{min}{}^2/\alpha = 0.0004/0.5 = 2(k^2 + ab) = 2(k^2 + a^2)$. In a more general case, one of the Q's may be chosen almost arbitrarily. Then $F_{min}{}^2/\alpha = 2(k^2 + ab)$ and $F_{min}{}^2 = (2k^2 - a^2 - b^2)$ may be solved simultaneously for k and the other Q to meet the specified values of F_{min} and α.

chapter VIII Balanced Polyphase Circuits

Generation of Polyphase Voltages. Polyphase voltages are generated in the same way as single-phase voltages. A polyphase system is simply several single-phase systems which are displaced in time phase from one another. The single-phase systems which form the polyphase systems are generally interconnected in some way.

In Fig. 1 is shown a single coil aa' on the armature of a two-pole machine. When the poles are in the position shown, the emf of conductor a of coil aa' is a maximum, and its direction is away from the reader. If a conductor is placed 120° from a at position b, it would

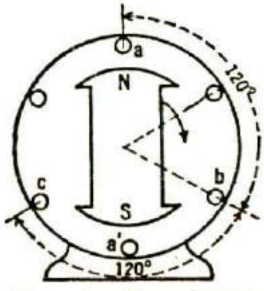


FIG. 1. Elementary three-phase generator.

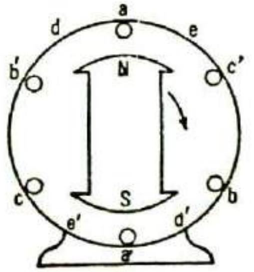


FIG. 2.

experience maximum emf in a direction away from the reader when the north pole axis was at b, or 120° later than when the pole axis was at a. In like manner, the maximum emf in the direction away from the reader for a conductor at c would occur 120° later than that at b, and 240° later than that at a. The placement of such conductors and the coils of which they are a part are shown in Fig. 2. Thus the coils aa', bb', and cc' would have emf's that are 120° out of time phase, as pictured in Fig. 3. This system is called three-phase because there are three waves of different time phase. In practice the space on the armature is completely covered with coils (except in single phase). For instance, the conductor of another coil could be placed in the slot to the right of conductor a in Fig. 2, and another to the left. The one to the right would have an emf which would lag that in a by the same angle that the one to the left would lead. The sum of the three emf's would give a resultant emf of the same phase as that in a. Conductors for phase a

would cover the periphery from d to e and from d' to e'. The distance from d to e is called a phase belt. The emf of all the coils in series for the whole phase would have the same phase relation as the emf of the center conductor of the phase belt. For this reason only the center conductors of the phase belts will be considered. It is apparent that any number of phases could be developed through properly spacing the coils on the stator.

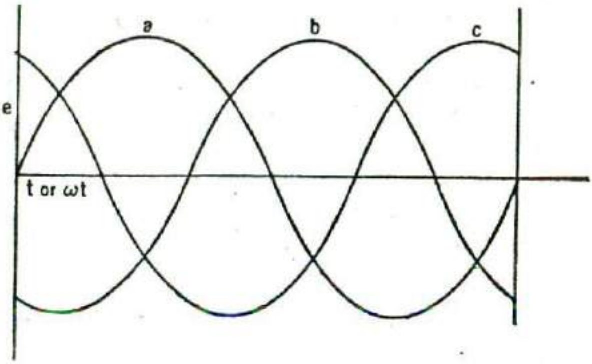


FIG. 3. Waves of emf generated by a three-phase generator.

In general, the electrical displacement between phases for a balanced n-phase system is $360/n$ electrical degrees. Three-phase systems are the most common, although for certain special applications a greater number of phases is used. For instance, practically all mercury-arc rectifiers for power purposes are either six- or twelve-phase. Most rotary converters are six-phase. Practically all modern generators are three-phase. Three-phase is also invariably used for transmitting large amounts of power. In general, three-phase apparatus is more efficient, uses less material for a given capacity, and costs less than single-phase apparatus. It will be shown later that, for a fixed amount of power to be transmitted a fixed distance at a fixed line loss with a fixed voltage between conductors, three-phase is more economical in the use of copper than any other number of phases.

In the development of the three-phase voltages in Fig. 3, clockwise rotation of the field structure of the alternator in Fig. 2 was assumed. This assumption made the emf of phase b lag that of a by 120°. Also, the emf of phase c lagged that of phase b by 120°. In other words, the order in which the emf's of phases a, b, and c came to their corresponding maximum values was abc. This is called the phase order or sequence abc. If the rotation of the field structure in Fig. 2 is reversed, the order in which the phases would attain their corresponding maximum voltages would be reversed. The phase sequence would be *acb*. This means that the emf of phase *c* would then lag that of phase *a* by 120° instead of by 240° as in the first case. In general, the phase sequence of the voltages applied to a load is fixed by the order in which the three-phase lines are connected. Interchanging any pair of lines reverses the phase sequence. For three-phase induction motors the effect of reversing the sequence is to reverse the direction of rotation. For three-phase unbalanced loads the effect is, in general, to cause a completely different set of values for line currents; hence when calculating such systems it is essential that phase sequence be specified or confusion may arise.

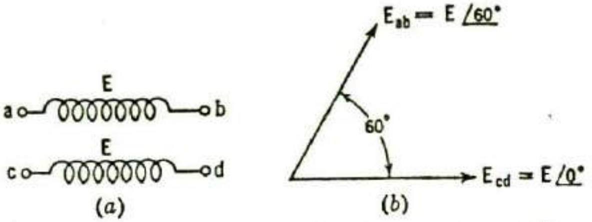


FIG. 4. Coils having induced emf's shown in part (*b*).

Vector Diagrams and Double-Subscript Notation. When drawing vector diagrams of polyphase circuits it is imperative that directions in which the circuit is being traced be noted and recorded. For example, let us assume that the two coils shown in Fig. 4*a* possess induced voltages or emf's that are 60° out of phase and that the coils are to be connected in additive series, that is, in such a manner that the emf's add at a 60°

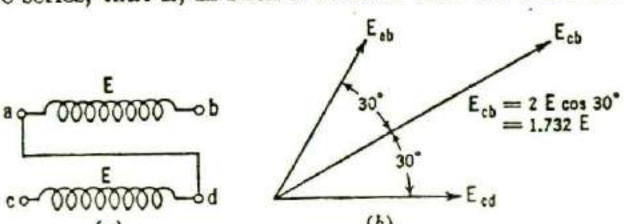


FIG. 5. Resultant emf shown in (*b*) for connection of coils shown in (*a*).

angle. From the information given it would be impossible to know whether terminal *a* should be connected to terminal *c* or terminal *d*. But if it were stated that the emf from *a* to *b* is 60° out of phase with that from *c* to *d* as shown in Fig. 4*b*, the way to connect the coils would be definitely fixed. Under such conditions, double-subscript notation is very convenient.

The order in which the subscripts are written denotes the direction in which the circuit is being traced. Thus the emf from *a* to *b* in Fig. 4*a* may be designated as $\mathbf{E}_{ab}$ and that from c to d as $\mathbf{E}_{cd}$. (See Fig. 4b.) If d is connected to a as shown in Fig. 5a, the emf from c to b is determined by adding all the emf's *in the directions encountered* as the circuit is traced from c to b. Hence $\mathbf{E}_{cb} = \mathbf{E}_{cd} + \mathbf{E}_{ab}$ as shown in Fig. 5b. This procedure will be further illustrated in succeeding articles.

Problem 1. In Fig. 4a, connect terminal b to terminal c and compare the resultant voltage $\mathbf{E}_{ad}$ with voltage $\mathbf{E}_{cb}$ of Fig. 5b.

Ans.: $\mathbf{E}_{ad} = \mathbf{E}_{cb}$.

Problem 2. (a) Connect terminal d to terminal b in Fig. 4a and find the voltage $\mathbf{E}_{ca}$ if $E = 120$ volts. $\mathbf{E}_{ab}$ and $\mathbf{E}_{cd}$ have the same vector relation as shown in Fig. 4b.

Ans.: $\mathbf{E}_{ca} = 120 \underline{/-60^\circ}$ volts.

(b) With terminal d connected to terminal b as above, find $\mathbf{E}_{ac}$.

Ans.: $\mathbf{E}_{ac} = 120 \underline{/120^\circ}$ volts.

A vector diagram is simply a means of representing certain electrical quantities that are related by a circuit. A vector diagram therefore must always be drawn in conjunction with a circuit. Sometimes circuits may be visualized instead of actually drawn, but without a definite picture of the circuit represented a vector diagram means nothing and cannot be intelligently drawn. It should be clearly recognized, however, that a circuit vector diagram of voltages and currents represents time-phase relations and not space relations of the circuit. This means that the space configuration of a circuit diagram is in no way indicative of the time-phase relations of the voltages or currents.

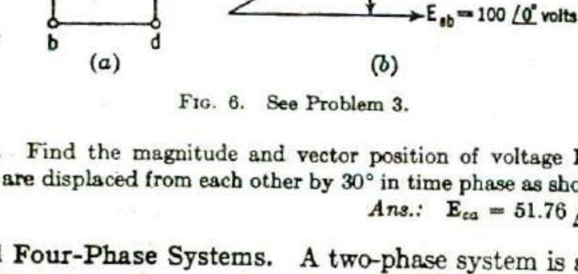


FIG. 6. See Problem 3.

Problem 3. Find the magnitude and vector position of voltage $\mathbf{E}_{ca}$ in Fig. 6a if $\mathbf{E}_{ab}$ and $\mathbf{E}_{cd}$ are displaced from each other by 30° in time phase as shown in Fig. 6b.

Ans.: $\mathbf{E}_{ca} = 51.76 \underline{/105^\circ}$ volts.

Two- and Four-Phase Systems. A two-phase system is an electrical system in which the voltages of the phases are 90° out of time phase. A two-phase system is pictured by the drum and Gramme ring windings in Figs. 7 and 8. From the position of the coils on the armature in Fig. 8 it can be seen that the emf's of the four coils are related in time phase as shown in Fig. 9. If the zero terminals of coils $a0$ and $c0$ are connected, the emf from a to c is $\mathbf{E}_{a0} + \mathbf{E}_{0c}$. This operation is shown in Fig. 10. Likewise, when the zeros of coils $b0$ and $d0$ are connected

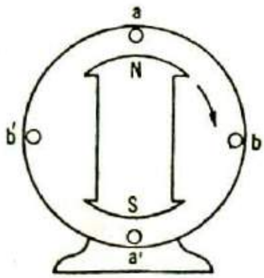


FIG. 7. Elementary drum-type two-phase generator.

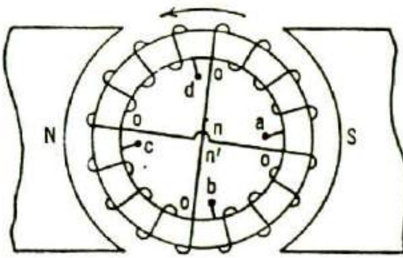


FIG. 8. Elementary Gramme ring-type two-phase generator.

$\mathbf{E}_{bd} = \mathbf{E}_{b0} + \mathbf{E}_{0d}$. This is also shown in Fig. 10. The emf's $\mathbf{E}_{ac}$ and $\mathbf{E}_{bd}$ are 90° apart in time phase, and the system shown in Fig. 8 constitutes a two-phase system. A two-phase system is the equivalent of two separate single-phase systems that are separated 90° in time phase.

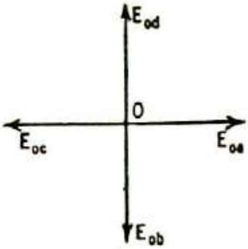


FIG. 9. Emf's of coils on generator in Fig. 8.

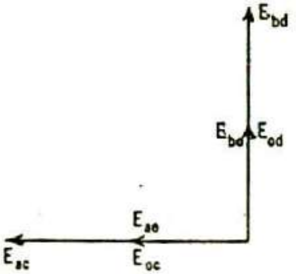


FIG. 10. Resultant emf's of two coils in series connected as shown in Fig. 8.

A four-phase and a two-phase system differ only in internal connections. Thus if connection is made between the two windings at n and n', the system would be called a four-phase system. The vector diagram of phase or coil voltages is shown in Fig. 9. Since there now is an electrical connection between the two groups of coils that constituted the two-phase system, there will be emf's between terminals d and a and also between b and c, as may be seen by studying the diagrammatic representation of the coils shown in Fig. 11. This connection is called a four-phase star. The voltages E_{da}, E_{ab}, E_{bc}, and E_{cd} are called the line voltages, while voltages E_{0a}, E_{0b}, E_{0c}, and E_{0d} are called the phase voltages, or voltages to neutral. From the circuit it is evident that

$\mathbf{E}_{da} = \mathbf{E}_{d0} + \mathbf{E}_{0a}$. This combination and similar ones for all the line voltages are shown in Fig. 12. Another method of showing the same thing is illustrated in Fig. 13. Thus, in the four-phase star, line voltage is the $\sqrt{2}$ times phase voltage and it is either 45° or 135° out of phase with the phase voltage, depending upon which voltages are considered.

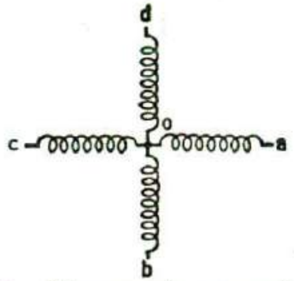


FIG. 11. Diagrammatic representation of Fig. 8 when n and n' are connected to form point o.

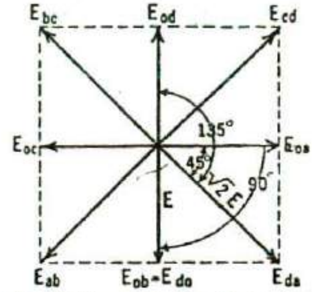


FIG. 12. Voltages of the four-phase star shown in Fig. 11.

Since $\mathbf{E}_{0a} + \mathbf{E}_{0b} + \mathbf{E}_{0c} + \mathbf{E}_{0d} = 0$, it would be possible to connect the four coils shown in Figs. 8 and 11 so that their voltages add in this way and no current would flow in the series circuit of the coils. This connection, shown in Fig. 14, is called a mesh connection, and in this case it would be known as a four-phase mesh. The line connections

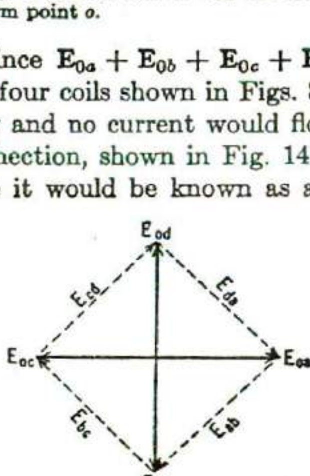


FIG. 13. Alternative representation of Fig. 12.

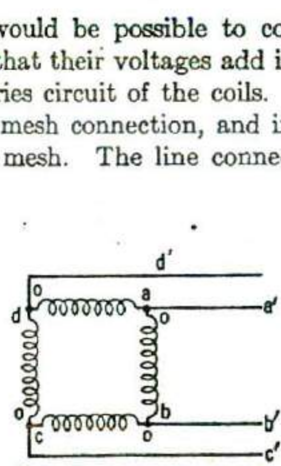


FIG. 14. Four-phase mesh.

are made at points a, b, c, and d. The vector diagram of the emf's for this system is shown in Fig. 15. For balanced loads the currents in adjacent phases are 90° out of phase as shown in Fig. 16. The aa' line current is $\mathbf{I}_{aa'} = \mathbf{I}_{da} + \mathbf{I}_{ba}$, as shown in Fig. 16. Thus line current of a balanced four-phase mesh is the $\sqrt{2}$ times phase current and is either 45° or 135° out of phase with the phase currents, according to which are being considered. Note that what was true about line and phase voltages in the star is true about line and phase currents in the mesh.

Inspection of the star system shows that line and phase currents must be identical, and the same thing is true regarding line and phase voltages in the mesh.

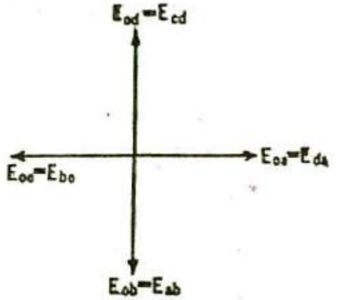


FIG. 15. Vector diagram of emf's of the four-phase mesh shown in Fig. 14.

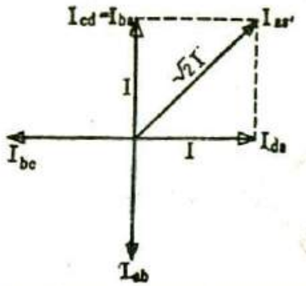


FIG. 16. Vector diagram of currents of the four-phase mesh shown in Fig. 14 under conditions of balanced load.

Sometimes a two-phase system is used with only three wires. When this is done, one wire is common to both phases. The circuit diagram of Fig. 8 when connected for such use is shown in Fig. 17, and the vector diagram is shown in Fig. 18. It will be noted that this is essentially half of the four-phase system shown in Fig. 11 when line wires are connected to points 0, d, and c.

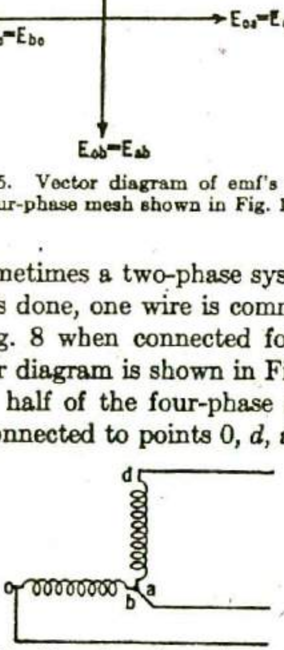


FIG. 17. Two-phase three-wire system.

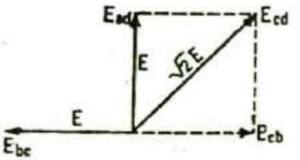


FIG. 18. Vector diagram of voltages for Fig. 17.

Three-Phase, Four-Wire Systems of Generated Emf's. The generation of three-phase was explained at the beginning of this chapter. If six wires were connected to terminals a, a', b, b', c, and c' of Fig. 2, the system might be called a six-wire, three-phase system. Such a generator could be loaded with three independent single-phase loads. Though such a system is not used, one that is widely used may be derived from it by making a common connection between terminals a', b', and c'. Four wires are all that would then be necessary, three for terminals a, b, and c, and one for the common connection $a'b'c'$. Such a system, called a four-wire, three-phase system, is shown diagrammatically in Fig. 19. This system is now extensively used for a-c networks and is rapidly displacing the formerly much used d-c networks in the downtown areas of large cities. The common wire connecting to n is called the neutral. Lighting loads are placed from line to neutral; motor and other three-phase power loads are connected between the three lines a, b, and c. The generated voltage waves of this system are shown in Fig. 3, and the vector diagram that portrays the same thing is shown in Fig. 20. The three voltages shown are called phase voltages or line-

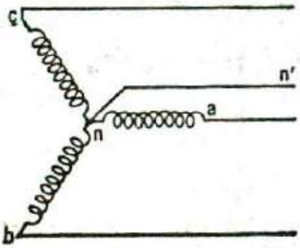


FIG. 19. Three-phase four-wire system.

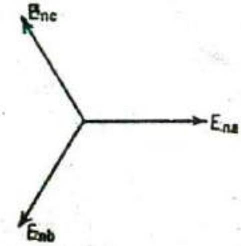


FIG. 20. Line-to-neutral voltages of Fig. 19.

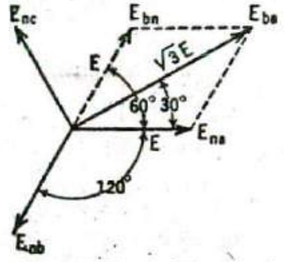


FIG. 21. Line voltage equals phase voltage times $\sqrt{3}$ in the wye connection.

to-neutral voltages. They are sometimes called the wye voltages of the system, and the connection of Fig. 19 is called a wye connection. The voltages between terminals a, b, and c are called the line or terminal voltages. Under balanced conditions they are definitely related to the phase voltages, as the following shows:

$$\mathbf{E}_{ba} = \mathbf{E}_{bn} + \mathbf{E}_{na}$$

This combination is shown in Fig. 21 where the magnitude of the phase

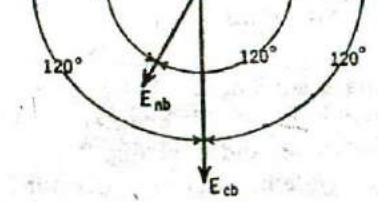


FIG. 22. Line and phase voltages of the wye connection (Fig. 19).

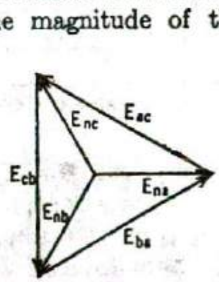
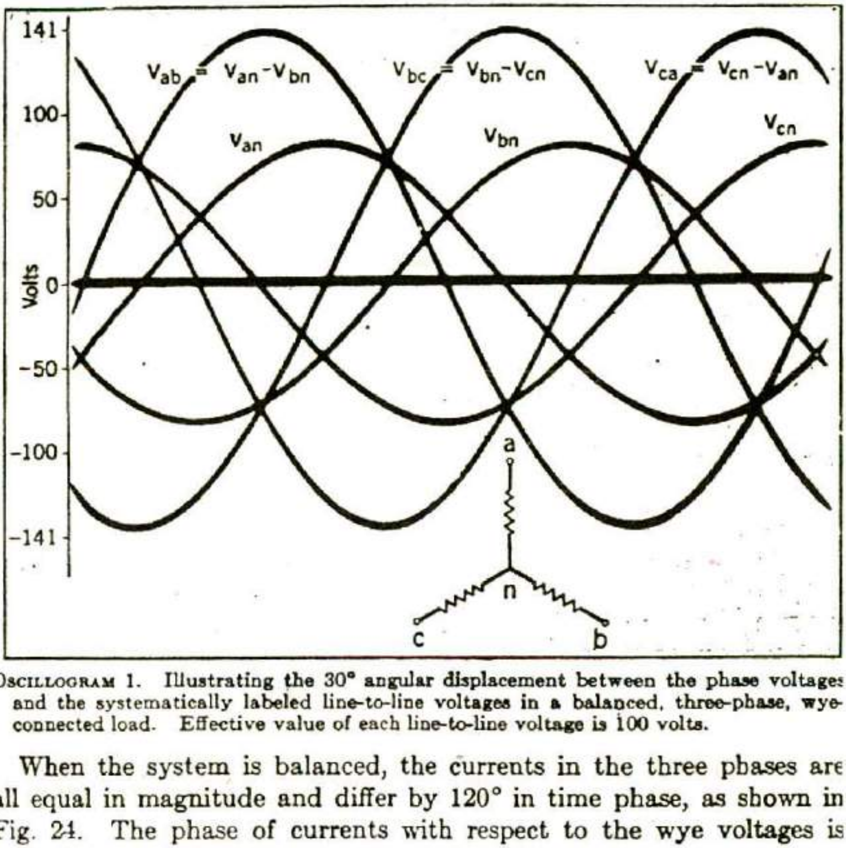


FIG. 23. Alternative representation of Fig. 22.

voltage is considered as E. Hence line voltage in the balanced three-phase star or wye connection is the $\sqrt{3}$ times the phase voltage and makes an angle with the component phase voltages of either 30° or 150°, depending upon which are considered. The complete vector diagram showing all line voltages is given in Fig. 22. Figure 23 shows the same system in terms of a polar vector diagram of phase voltages and a funicular diagram of line voltages. Oscillogram 1 shows these relationships as obtained from an actual load.



OSCILLOGRAM 1. Illustrating the 30° angular displacement between the phase voltages and the systematically labeled line-to-line voltages in a balanced, three-phase, wye-connected load. Effective value of each line-to-line voltage is 100 volts.

When the system is balanced, the currents in the three phases are all equal in magnitude and differ by 120° in time phase, as shown in Fig. 24. The phase of currents with respect to the wye voltages is defined by the circuit parameters in any particular case. An inspection of Fig. 19 shows that line and phase currents are identical. The current in the neutral wire is obtained through the application of Kirchhoff's current law. Thus

$$I_{n'n} = I_{na} + I_{nb} + I_{nc}$$

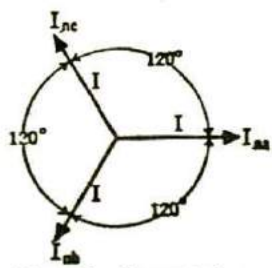


FIG. 24. Currents in a balanced-wye system.

If the system is balanced, I_{na}, I_{nb}, and I_{nc} are equal in magnitude and displaced from one another in time phase by 120° as shown in Fig. 24. Under these conditions it is apparent that the current in the neutral is zero since $I_{na} + I_{nb} + I_{nc} = 0$.

Problem 4. (*a*) Draw a polar (or single-origin) vector diagram which will represent the same phase voltages and the same line voltages as shown in Oscillogram 1 using $\mathbf{V}_{bn}$ as reference. Specify the effective magnitude of the phase voltages, the sequence of the phase voltages, and the sequence of the line voltages.

Ans.: V/phase = 57.7 volts.
Phase voltage sequence: *an-bn-cn*.
Line voltage sequence: *ab-bc-ca*.

(*b*) Draw a polar (or single-origin) vector diagram which will represent the same phase voltages as shown in Oscillogram 1, namely $\mathbf{V}_{an}$, $\mathbf{V}_{bn}$, and $\mathbf{V}_{cn}$, together with the line voltages $\mathbf{V}_{ba}$, $\mathbf{V}_{cb}$, and $\mathbf{V}_{ac}$, using $\mathbf{V}_{cn}$ as reference. Specify the sequence of these line voltages.

Ans.: Line voltage sequence: *ba-cb-ac*.

Three-Phase, Three-Wire Systems. The usual three-phase system consists of only three wires. In this event loads are not placed between the lines and neutral, and the neutral wire is therefore not brought out. The balanced relations discussed in the previous article are obviously unaffected by omitting the neutral wire and therefore apply to the three-phase, three-wire system.

The Delta Connection. If only three wires are used, the three-phase system may be connected in mesh similar to the four-phase system previously considered. Since

$$\mathbf{E}_{na} + \mathbf{E}_{nb} + \mathbf{E}_{nc} = 0$$

for the three-phase system, the three coils shown in Fig. 19 can be connected as shown in Fig. 25, and no current of fundamental frequency

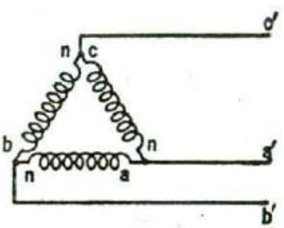


FIG. 25. Delta connection of the coils shown in Fig. 19.

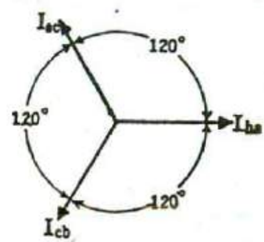


FIG. 26. Phase currents for the balanced delta of Fig. 25.

will flow around the series circuit of the three coils. This three-phase mesh connection is called a delta connection. It will be noted that star and mesh are general terms applicable to any number of phases, but wye and delta are special cases of the star and mesh when three-phase is considered. Inspection of Fig. 25 shows that phase voltages and line voltages are identical but that line and phase currents are different. The vector diagram of phase currents for a balanced load is shown in Fig. 26. Line currents are found through the application of Kirchhoff's current law. Thus

$$\mathbf{I}_{aa'} = \mathbf{I}_{ba} + \mathbf{I}_{ca}$$

This operation is carried out in Fig. 27. For a balanced system, line current is the $\sqrt{3}$ times phase current in magnitude and is out of phase with the component-phase currents by either 30° or 150°, depending

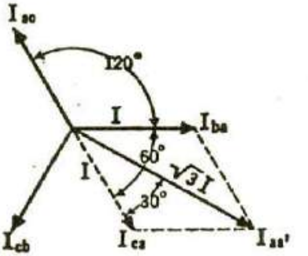


FIG. 27. Combination of phase currents gives line current for Fig. 25.

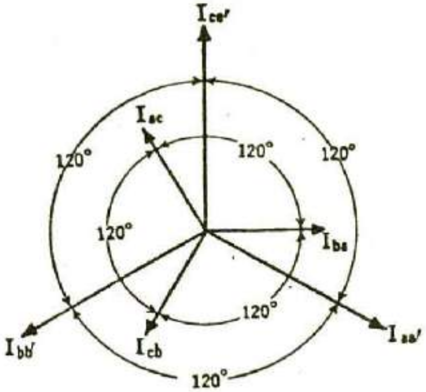
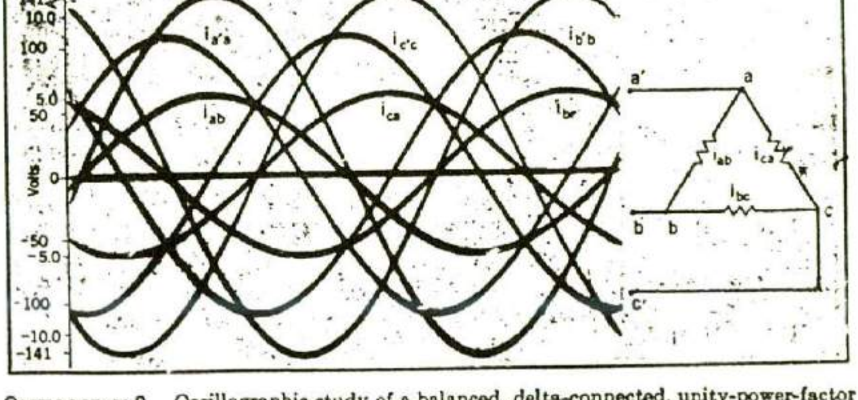


FIG. 28. Vector diagram of currents for a balanced delta is shown in Fig. 25.

upon which are considered. The complete vector diagram of currents for the three-phase balanced delta connection is shown in Fig. 28. Oscillogram 2 shows the relations discussed above as obtained from an actual load labeled as in the accompanying circuit diagram.



OSCILLOGRAM 2. Oscillographic study of a balanced, delta-connected, unity-power-factor load. The line-to-line voltages (or phase voltages) together with the phase currents and line currents are illustrated.

It should be understood that all the vectors on a vector diagram like that shown in Fig. 28 may be reversed, that is, changed individually through 180°, and, if a reversal in the order of subscripts accompanies this change, the resulting vector diagram will represent the same thing as does Fig. 28. As applied to the circuit shown on Oscillogram 2, for example, it is immaterial whether $\mathbf{I}_{ab}$ is considered to flow in the direction of $\mathbf{V}_{ab}$ or whether $\mathbf{I}_{ba}$ is considered to flow in the direction of $\mathbf{V}_{ba}$. Those who prefer to consider line voltages *ao*, *ca*, and *bc* rather than line voltages *ba*, *ac*, and *cb* will label a circuit diagram like that shown on Oscillogram 2, whereas those who prefer to consider line voltages *ba*, *ac*, and *cb* will employ $\mathbf{I}_{ba}$, $\mathbf{I}_{ac}$, and $\mathbf{I}_{cb}$ as the delta-phase currents.

Problem 5. Refer to Oscillogram 2. Draw a complete vector diagram of $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, $\mathbf{V}_{ca}$, $\mathbf{I}_{ab}$, $\mathbf{I}_{bc}$, $\mathbf{I}_{ca}$, $\mathbf{I}_{a'a}$, $\mathbf{I}_{b'b}$, and $\mathbf{I}_{c'c}$ employing $\mathbf{V}_{bc}$ as reference. From the scaled ordinates given on Oscillogram 2, determine the effective values of line (or phase) voltage, phase current, and line current.

Ans.: $V = 100$ volts; $I_p = 3.5$ amperes; $I_l = 6$ amperes.

The *n*-Phase Star and Mesh. The circuit and vector diagrams of two adjacent phases of an *n*-phase star system are shown in Figs. 29

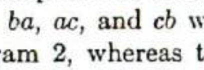


Fig. 29. Two adjacent phases of an *n*-phase star.

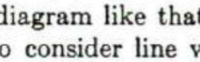


Fig. 30. Line-to-neutral voltages of adjacent phases of an *n*-phase star (Fig. 29).

and 30, respectively. The line voltage $\mathbf{E}_{ab}$ is $\mathbf{E}_{an} + \mathbf{E}_{nb}$. Remembering that the angle of phase difference between voltages of adjacent phases is $360°/n$, and calling the magnitude of phase voltage E_p, the general calculation of the line voltage can be understood from the vector relations shown in Fig. 31. Hence the line voltage is

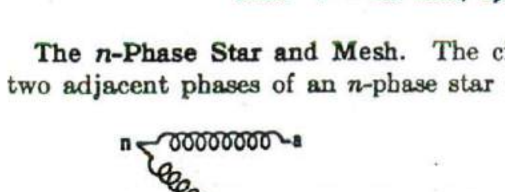


Fig. 31. Combination of line-to-neutral voltages to give line-to-line voltages in an *n*-phase star.

$$E_L = 2E_p \sin \frac{180°}{n} \quad (1)$$

From the circuit of Fig. 29 it is evident that line current and phase current are identical. Hence

$$I_L = I_p$$

From the circuit and vector diagrams shown for part of an n-phase mesh system in Fig. 32, the use of previously outlined principles will show that

$$E_L = E_p$$

and

$$I_L = 2I_p \sin \frac{180^\circ}{n} \tag{2}$$

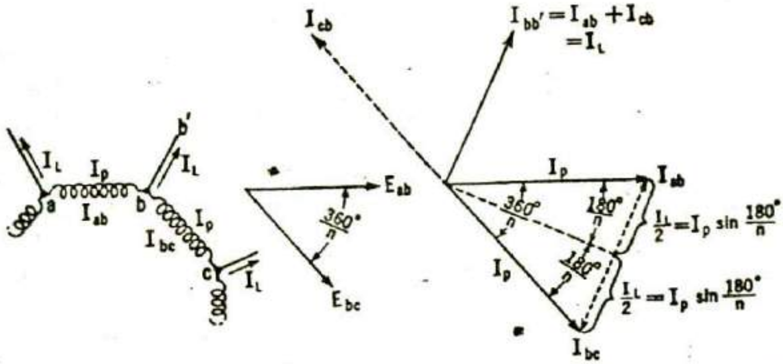


FIG. 32. Circuit diagram of adjacent phases and corresponding vector diagrams for an n-phase mesh.

Example 1. The line currents issuing from a balanced four-phase, mesh-connected generator (like that shown in Fig. 14, page 267) are known to be 70.7 amperes in magnitude. Let it be required to find the magnitude of the phase currents employing the general relationship stated in equation (2).

$$I_p = \frac{70.7}{2 \sin \frac{180^\circ}{4}} = \frac{70.7}{2 \sin 45^\circ} = \frac{70.7}{1.414} = 50 \text{ amperes}$$

Problem 6. Find the magnitude of the line currents issuing from a balanced six-phase, mesh-connected generator if the phase currents are known to be 100 amperes in magnitude. Illustrate solution by means of a vector diagram.

Ans.: $I_L = I_p = 100$ amperes.

Problem 7. Find the voltage between *adjacent* lines of a balanced twelve-phase, star-connected system if the phase voltages are 50 volts in magnitude. Illustrate solution by means of a vector diagram.

Ans.: 25.88 volts.

Problem 8. Find the voltage between *alternate* lines of a balanced six-phase, star-connected system if the phase voltages are 132.8 volts in magnitude.

Ans.: 230 volts.

Balanced Wye Loads. When three identical impedances are connected to a common point, n, Fig. 33, they constitute a balanced wye load. If balanced three-phase voltages are impressed on such a load, it would seem that all impedances should have equal voltage drops across them and that the ratio and phase of line and phase voltages should be the same as those discussed for the wye-connected generators. Application of Kirchhoff's laws as discussed in the next chapter shows that this is true. Hence the voltage drop V_p across each impedance in terms of the line voltage is

$$V_p = \frac{V_L}{\sqrt{3}}$$

The current, power, etc., may then be found in accordance with single-phase circuit analysis. As a general rule, all balanced three-phase circuits are calculated on a *per phase* basis in exactly the same manner as the corresponding calculations are made for any single-phase circuit. If this procedure is followed it is important that *per phase* values of V and I are not confused with line voltages and line currents even though line currents in a wye connection are the same as the phase currents, and the line voltages in a delta connection are the same as the phase voltages. As a general rule, all balanced three-phase circuits are calculated per phase just as the calculations were made for single-phase circuits.

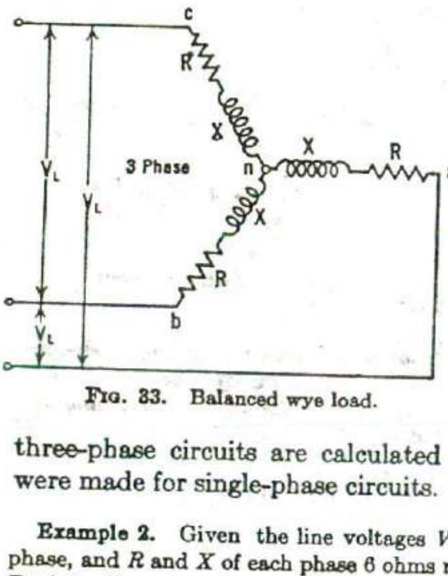


FIG. 33. Balanced wye load.

Example 2. Given the line voltages V_L in Fig. 33 as 220 volts balanced three-phase, and R and X of each phase 6 ohms resistance and 8 ohms inductive reactance. Find the line current, power per phase, and total power.

$$V_p = \frac{V_L}{\sqrt{3}} = \frac{220}{\sqrt{3}} = 127 \text{ volts}$$

$$I_L = I_p = \frac{127}{\sqrt{6^2 + 8^2}} = \frac{127}{10} = 12.7 \text{ amperes}$$

$$\text{Power per phase} = I_p{}^2R_p = 12.7^2 \times 6 = 968 \text{ watts}$$
$$\text{Total power} = 3 \times 968 = 2904 \text{ watts}$$

The example given could have been worked by means of complex numbers. Since there was no need for the vector expressions of voltages and currents, it was simpler to use magnitudes only. When it is necessary to combine the line current due to some particular load with that from another load, the vector expressions or their equivalents are required. To illustrate the vector method of handling the above example, assume the phase sequence V_{ba}, V_{cb}, V_{ac}. This means that $\mathbf{V}_{cb}$ lags $\mathbf{V}_{ba}$ by 120°. It would be possible to use any line voltage or any phase voltage as a reference. The vector diagram of a similar set of voltages to those required here is shown in Fig. 22 where E is used instead of V. The phase voltage of phase na will be taken as the reference (sometimes called the standard phase). Thus:

$$\mathbf{V}_{na} = 127 + j0 \text{ volts}$$
$$\mathbf{V}_{nb} = 127 \underline{/-120^\circ} = 127 (\cos 120^\circ - j \sin 120^\circ) = -63.5 - j110 \text{ volts}$$
$$\mathbf{V}_{nc} = 127 \underline{/120^\circ} = -63.5 + j110 \text{ volts}$$

If the vector expressions for line voltages are desired, they may be obtained by the following procedure.

$$\mathbf{V}_{ba} = \mathbf{V}_{bn} + \mathbf{V}_{na} = 63.5 + j110 + 127 + j0 = 190.5 + j110 \text{ volts, etc.}$$
$$\mathbf{I}_{na} = \frac{\mathbf{V}_{na}}{\mathbf{Z}_{na}} = \frac{127 + j0}{6 + j8} = 7.62 - j10.16 = 12.7 \underline{/-53.13^\circ} \text{ amperes}$$
$$\mathbf{I}_{nb} = \frac{\mathbf{V}_{nb}}{\mathbf{Z}_{nb}} = \frac{-63.5 - j110}{6 + j8} = \frac{127 \underline{/-120^\circ}}{10 \underline{/53.13^\circ}} = 12.7 \underline{/-173.13^\circ} \text{ amperes}$$
$$\mathbf{I}_{nc} = \frac{\mathbf{V}_{nc}}{\mathbf{Z}_{nc}} = \frac{127 \underline{/120^\circ}}{10 \underline{/53.13^\circ}} = 12.7 \underline{/66.87^\circ} \text{ amperes}$$
$$P_{na} = vi + v'i' = 127 \times 7.62 = 968 \text{ watts}$$

or

$$P_{nb} = 127 \times 12.7 \cos (120^\circ - 173.13^\circ) = 968 \text{ watts}$$

The vector diagram of the voltages and currents for this load as drawn from the vector solution is shown in Fig. 34.

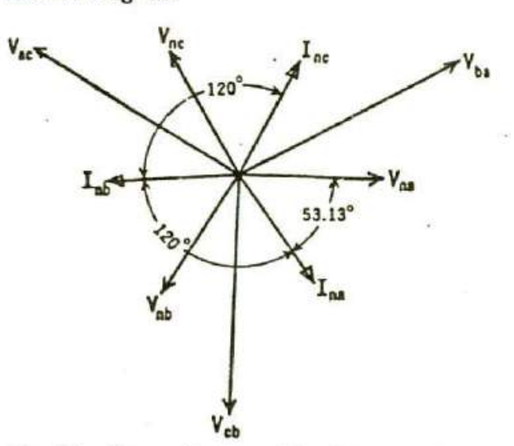


FIG. 34. Vector diagram of load in example 2.

Balanced Delta Loads. Three identical impedances connected as shown in Fig. 35 constitute a balanced delta load. The voltage drop across each impedance is known when the line voltage is given. Hence the phase currents may be determined directly as V_p/Z_p. The magnitudes of the line currents are simply phase currents multiplied by $\sqrt{3}$.

Example 3. Reconnect the impedances given in example 2 in delta, and calculate phase current, line current, phase power, and total power. ($R = 6$ ohms and $X = 8$ ohms per phase.)

$$V_L = V_p = 220 \text{ volts}$$
$$I_p = \frac{220}{\sqrt{6^2 + 8^2}} = 22 \text{ amperes}$$
$$I_L = \sqrt{3} \times 22 = 38.1 \text{ amperes}$$

Power per phase $= 22^2 \times 6 = 2904$ watts.

Total power $= 2904 \times 3 = 8712$ watts.

Alternative vector solution using sequence $\mathbf{V}_{ba}$, $\mathbf{V}_{cb}$, $\mathbf{V}_{ac}$. Use $\mathbf{V}_{ba}$ as the reference voltage.

$$\mathbf{V}_{ba} = 220 \underline{/0^\circ} \text{ volts}$$
$$\mathbf{V}_{cb} = 220 \underline{/-120^\circ} \text{ volts}$$
$$\mathbf{V}_{ac} = 220 \underline{/120^\circ} \text{ volts}$$
$$\mathbf{I}_{ba} = \frac{220 \underline{/0^\circ}}{10 \underline{/53.13^\circ}} = 22 \underline{/-53.13^\circ} = 13.2 - j17.6 \text{ amperes}$$
$$\mathbf{I}_{cb} = \frac{220 \underline{/-120^\circ}}{10 \underline{/53.13^\circ}} = 22 \underline{/-173.13^\circ} = -21.85 - j2.63 \text{ amperes}$$
$$\mathbf{I}_{ac} = \frac{220 \underline{/120^\circ}}{10 \underline{/53.13^\circ}} = 22 \underline{/66.87^\circ} = 8.65 + j20.2 \text{ amperes}$$
$$P_{ba} = 220 \times 22 \cos 53.13^\circ = 2904 \text{ watts}$$

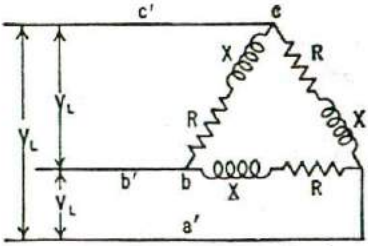


FIG. 35. Balanced delta load.

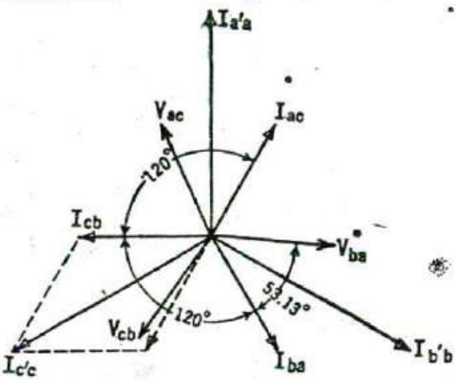


FIG. 36. Vector diagram for load of example 3.

Total power $= 3 \times 2904 = 8712$ watts.

$$\mathbf{I}_{c'c} = \mathbf{I}_{cb} + \mathbf{I}_{ca} = -30.5 - j22.8 = 38.1 \underline{/-143.13^\circ} \text{ amperes}$$
$$\mathbf{I}_{b'b} = \mathbf{I}_{bc} + \mathbf{I}_{ba} = +35.05 - j15 = 38.1 \underline{/-23.13^\circ} \text{ amperes}$$
$$\mathbf{I}_{a'a} = \mathbf{I}_{ab} + \mathbf{I}_{ac} = -4.55 + j37.8 = 38.1 \underline{/96.87^\circ} \text{ amperes}$$

The vector diagram of this delta load as drawn from the vector solution is shown in Fig. 36.

Three-Origin Vector Diagram of a Balanced Three-Phase System. Figure 37 shows a polar vector diagram of a three-phase balanced unity-power-factor wye load. Figure 38 shows a vector diagram of a

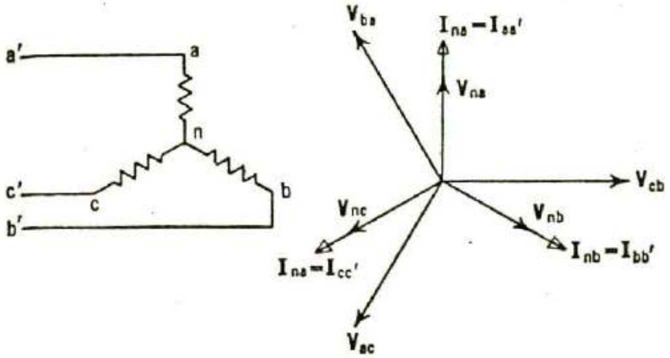


FIG. 37. Polar vector diagram of unity-power-factor, balanced wye-connected load.

balanced unity-power-factor delta load. A comparison of these two diagrams will show that the phase relation between *line* currents and *line* voltages is identical for both loads. Therefore a single vector diagram can be used to represent the relations between line currents and line

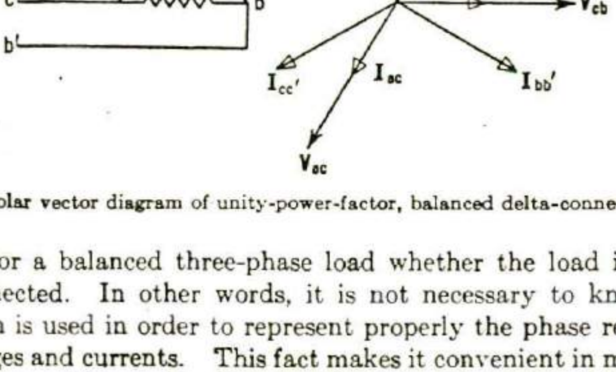


FIG. 38. Polar vector diagram of unity-power-factor, balanced delta-connected load.

voltages for a balanced three-phase load whether the load is wye- or delta-connected. In other words, it is not necessary to know which connection is used in order to represent properly the phase relations of line voltages and currents. This fact makes it convenient in many cases to use a three-origin vector diagram which is explained as follows.

If it is remembered that a vector can be translated without changing its value, the line voltages for the above loads may be arranged to form a closed triangle, as shown in Fig. 39. Also the line currents may be drawn from the corners of the triangle so formed as indicated. The three corners comprise the three origins; hence the name of the diagram.

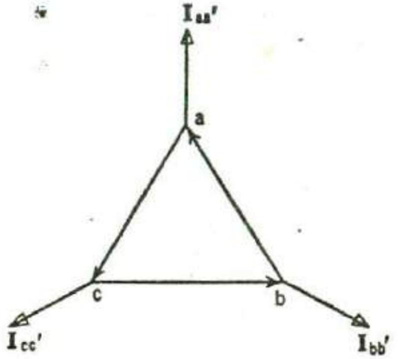


FIG. 39. Three-origin vector diagram of line voltages (*cb*-*ac*-*ba*) and line currents ($I_{bb'}$, $I_{cc'}$, $I_{aa'}$).

It will be observed that, at unity power factor, line current $I_{aa'}$ bisects the angle at origin *a* made by the line voltages at that point. A similar situation obtains for the other line currents. The bisectors of these angles may therefore be called the unity-power-factor positions of the line currents for a balanced three-phase load regardless of delta or wye connection. If a load having a power-factor angle of θ is to be represented, it is necessary only to let the three line currents swing from their unity-power-factor positions by the angle θ. That this is true is evident from a study of the changes in Figs. 37 and 38 when a load having a power-factor angle θ is represented.

It should be recognized that the three-origin diagram is essentially the equivalent wye diagram where the line voltages are drawn between extremities of the wye voltages to neutral, and these latter voltages, if shown, would be drawn from the corners of the triangle to the geometrical neutral. Inspection of the diagrams, Fig. 40*b* and *c*, shows the power-factor angle is actually the angle between the line current and the equivalent wye voltage or voltage to neutral. To show how the

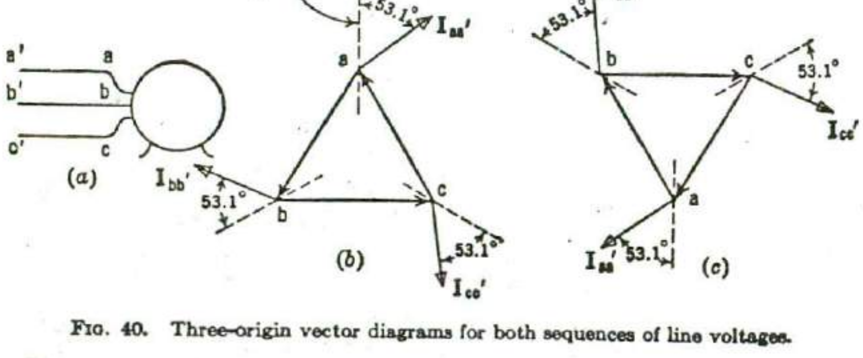


FIG. 40. Three-origin vector diagrams for both sequences of line voltages.

three-origin diagram might be used to represent a three-phase load, study the following example.

Example 4. A balanced three-phase, 0.6 p.f. lagging load takes 10 kva at 200 volts. Show the vector diagram of the line voltages and currents.

The load is represented by the circle, and the lines are labeled a, b, and c, as shown in Fig. 40. Assume $\mathbf{V}_{bc}$ as a reference, and complete the line voltage triangle as shown in (b) or (c) according to the sequence desired. The bisectors of the angles are shown dotted and are the unity-power-factor positions of the respective currents leaving points a, b, and c. The actual power-factor angle for the load is $\cos^{-1} 0.6 = 53.1°$, and the currents are therefore drawn lagging their unity-power-factor positions by this angle, as shown. Had the load operated at a leading power factor, the currents would have swung ahead of their unity-power-factor positions by 53.1°.

The above type of diagram lends itself to a simple visualization of line voltages and currents for a balanced three-phase load and contributes to an easy understanding of operating conditions in individual transformers for certain types of connections when supplying balanced loads. They may also be used to effect the proper combination of line currents from several balanced three-phase loads independent of whether the loads themselves are delta- or wye-connected. It should be recognized from this discussion that, as far as phase relations between line currents and line voltages are concerned, one is at liberty to assume a delta- or wye-connected load even though the actual type of connection is known or unknown. Also, if convenient, the directions of the currents shown in Fig. 40 may be reversed and so labeled.

Power Calculations in Balanced Systems. The determination of power in balanced polyphase systems is based upon calculations per phase. If the voltage per phase is V_p, the phase current I_p, and the angle between them θ_p, the power per phase is

$$P_p = V_p I_p \cos \theta_p \tag{3}$$

The power for all phases of an n-phase system is

$$P_t = nP_p = nV_p I_p \cos \theta_p \tag{4}$$

The universality of three-phase warrants the development of equation (4) to give power in terms of line current I_L and the line voltage V_L. Consider the wye connection. Then

$$\begin{aligned} P_t &= 3V_p I_p \cos \theta_p = 3 \frac{V_L}{\sqrt{3}} I_L \cos \theta_p \\ &= \sqrt{3} V_L I_L \cos \theta_p \end{aligned} \tag{5}$$

For the delta connection

$$\begin{aligned} P_t &= 3V_p I_p \cos \theta_p = 3V_L \frac{I_L}{\sqrt{3}} \cos \theta_p \\ &= \sqrt{3} V_L I_L \cos \theta_p \end{aligned} \tag{6}$$

The equations for power in terms of line voltages and line currents for *balanced three-phase* loads whether delta- or wye-connected are identical and equal to $\sqrt{3}V_L I_L \cos\theta_p$. In this expression, $\sqrt{3}V_L I_L \cos\theta_p$, for balanced three-phase power, it must be remembered that θ_p is the angle between *phase voltage and phase current* and *not* between line voltage and line current.

Problem 9. Three-phase line voltages of 2300 volts magnitude are impressed on a balanced wye-connected load which consists of 100 ohms resistance per phase in series with 173.2 ohms inductive reactance per phase. Find the line current and the total power taken by the three-phase load. Calculate P_t as $3I_p^2R_p$, as $3V_pI_p\cos\theta_p$, and as $\sqrt{3}V_LI_L\cos\theta_p$.

Ans.: $I_L = I_p = 6.64$ amperes, $P_t = 13.22$ kw.

Problem 10. Repeat Problem 9, assuming that the three impedances are connected in delta (rather than in wye) across the same line voltages.

Ans.: $I_L = 19.92$ amperes, $P_t = 39.66$ kw.

Volt-Amperes. The volt-amperes of a *balanced* three-phase system are defined as the sum of the volt-amperes of the separate phases or three times the number of volt-amperes per phase. Hence

$$\text{va}_t = 3\text{va}_p = 3V_pI_p$$

In terms of line voltage and line current, volt-amperes are

For delta:
$$3V_L\frac{I_L}{\sqrt{3}} = \sqrt{3}V_LI_L \tag{7}$$

For wye:
$$3\frac{V_L}{\sqrt{3}}I_L = \sqrt{3}V_LI_L \tag{8}$$

For an n-phase system under balanced conditions the total volt-amperes are n times the volt-amperes per phase.

Reactive Volt-Amperes. The reactive volt-amperes for a balanced three-phase system are defined as the sum of the reactive volt-amperes for each phase, or three times the reactive volt-amperes per phase. In terms of line voltage and line current the reactive volt-amperes or reactive power is

For wye:
$$P_X = 3V_pI_p\sin\theta_p = 3\frac{V_L}{\sqrt{3}}I_L\sin\theta_p$$
$$= \sqrt{3}V_LI_L\sin\theta_p \tag{9}$$

For delta:
$$P_X = 3V_pI_p\sin\theta_p = 3V_L\frac{I_L}{\sqrt{3}}\sin\theta_p$$
$$= \sqrt{3}V_LI_L\sin\theta_p \tag{10}$$

Summarizing for either *balanced* delta or wye, the totals for the systems are

$$P = \sqrt{3}V_L I_L \cos\theta_p \tag{11}$$

$$\text{va} = \sqrt{3}V_L I_L \tag{12}$$

$$P_X = \sqrt{3}V_L I_L \sin\theta_p \tag{13}$$

The sine of the angle between phase voltage and phase current ($\sin\theta_p$) is called the *reactive factor* of a *balanced* system.

Problem 11. Three-phase line voltages of 440 volts are impressed on a balanced delta-connected load which consists of 8 ohms resistance in series with 6 ohms inductive reactance per phase.

(*a*) Find the volt-amperes per phase, the reactive volt-amperes per phase, and the reactive factor of each phase.

Ans.: $\text{va}_p = 19{,}360$, $\text{rva}_p = 11{,}616$, r.f. = 0.6.

(*b*) Find the total volt-amperes of the system, the total reactive volt-amperes of the system, and the reactive factor of the system.

Ans.: $\text{va}_t = 58{,}080$, $\text{rva}_t = 34{,}848$, r.f. = 0.6.

Power Factor. The power factor of a balanced three-phase system, when the wave forms of voltage and current are sinusoidal, is defined as the cosine of the angle between *phase voltage* and *phase current* independent of whether the connection is delta or wye. It should be noted that the volt-amperes of equation (12) are equal to $\sqrt{P^2 + P_X{}^2}$. Thus

$$\begin{aligned}\text{va} &= \sqrt{(\sqrt{3}V_L I_L \cos\theta_p)^2 + (\sqrt{3}V_L I_L \sin\theta_p)^2} \\ &= \sqrt{3}V_L I_L \sqrt{\cos^2\theta_p + \sin^2\theta_p} = \sqrt{3}V_L I_L \end{aligned} \tag{14}$$

From equation (11),

$$\text{p.f.} = \cos\theta_p = \frac{P}{\sqrt{3}V_L I_L} \tag{15}$$

From equation (13),

$$\text{r.f.} = \sin\theta_p = \frac{P_X}{\sqrt{3}V_L I_L} \tag{16}$$

From equations (15) and (14),

$$\text{p.f.} = \frac{P}{\sqrt{P^2 + P_X{}^2}} \tag{17}$$

From equations (16) and (14),

$$\text{r.f.} = \frac{P_X}{\sqrt{P^2 + P_X{}^2}} \tag{18}$$

Example 5. A 5-horsepower, 220-volt, three-phase motor has an efficiency of 85 per cent and operates at 86 per cent power factor. Find the line current.

$$\text{Power input} = \sqrt{3}V_L I_L \,\text{p.f.} = \frac{5 \times 746}{0.85} = 4390 \text{ watts}$$

$$I_L = \frac{4390}{\sqrt{3}\,220 \times 0.86} = 13.4 \text{ amperes}$$

Balanced Three-Phase Loads in Parallel. The combination of a number of balanced loads which are in parallel may be effected through changing all loads to equivalent delta loads and then combining the impedances of corresponding phases according to the law governing parallel circuits. Also all loads may be changed to equivalent wye loads and the impedances of corresponding phases paralleled. In addition to these methods, the power of the several loads may be added arithmetically and the reactive volt-amperes may be added algebraically. The total volt-amperes will then be obtained as $\sqrt{P^2 + P_x^2}$.

Example 6. A 3-phase motor takes 10 kva at 0.6 power factor lagging from a source of 220 volts. It is in parallel with a balanced delta load having 16 ohms resistance and 12 ohms capacitive reactance in series in each phase. Find the total volt-amperes, power, line current, and power factor of the combination.

Solution a. Assume motor to be Y-connected.

$$\text{Motor line current} = \text{phase current} = \frac{10{,}000}{\sqrt{3}\,220} = 26.25 \text{ amperes}$$

$$\text{Equivalent impedance per phase of motor} = \frac{220}{\sqrt{3}\,26.25} = 4.84 \text{ ohms}$$

$$R = 4.84 \cos\theta = 4.84 \times 0.6 = 2.904 \text{ ohms}$$

$$X = 4.84 \sin\theta = 4.84 \times 0.8 = 3.872 \text{ ohms}$$

$$\text{Equivalent wye of delta load } \mathbf{Z}_p = \frac{16 - j12}{3} = 5.33 - j4 \text{ ohms}$$

$$\mathbf{Z}_0 = \frac{(5.33 - j4)\,(2.904 + j3.872)}{5.33 - j4 + 2.904 + j3.872} = 3.91\,\underline{/17.17^\circ} \text{ ohms}$$

$$I_0 = \frac{220}{\sqrt{3}\,3.91} = 32.5 \text{ amperes}$$

$$\text{va} = \sqrt{3}\,220 \times 32.5 = 12{,}370 \text{ volt-amperes}$$

$$\text{p.f.}_0 = \cos 17.17^\circ = 0.955$$

$$P = 12{,}370 \times 0.955 = 11{,}810 \text{ watts}$$

Solution b. The motor may be assumed delta-connected and the delta-phase impedances combined after which delta phase currents and line currents can be found. The remaining procedure is similar to that in solution *a*.

Solution c. Line currents for each load are determined and shown on a diagram of the type shown in Fig. 39 where the equivalent voltage to neutral V_{na} is drawn

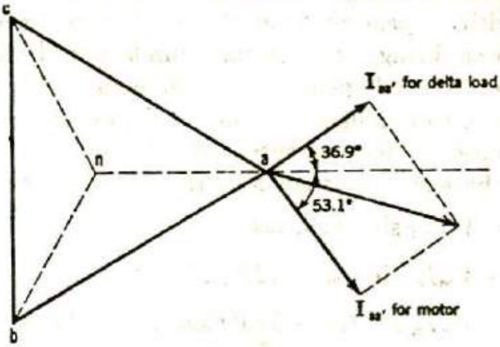


FIG. 41.

along the horizontal as shown in Fig. 41. Currents are then combined as indicated on Fig. 41.

$$\text{Motor line current} = \frac{10{,}000}{\sqrt{3}\,220} = 26.25 \text{ amperes}$$

$$\text{Delta-load line current} = \frac{220}{\sqrt{16^2 + 12^2}}\sqrt{3} = 19.05 \text{ amperes}$$

$$I_{aa'\text{motor}} = 26.25\underline{/-53.1^\circ} = 15.75 - j21$$

$$I_{aa'\text{delta load}} = 19.05\underline{/36.9^\circ} = 15.24 + j11.43$$

$$I_{aa'} = I_{aa'\text{motor}} + I_{aa'\text{delta load}} = 30.99 - j9.57 = 32.5\underline{/-17.17^\circ} \text{ amperes}$$

$$\text{va} = \sqrt{3}\,220 \times 32.5 = 12{,}370 \text{ volt-amperes}$$

$$\text{p.f.}_0 = \cos 17.17^\circ = 0.955$$

$$P = 12{,}370 \times 0.995 = 11{,}810 \text{ watts.}$$

Solution d. For the delta load, phase current is $220/\sqrt{16^2 + 12^2} = 11$ amperes.

$$P = 11^2 \times 16 \times 3 = 5810 \text{ watts for all phases}$$

$$P_X = 11^2 \times 12 \times 3 = 4350 \text{ vars for all phases (capacitive)}$$

For the motor

$$P = 10 \times 0.6 = 6 \text{ kw}$$

$$P_X = 10 \times 0.8 = 8 \text{ kilovars (inductive)}$$

$$\text{Summation of power} = 5.81 + 6 = 11.81 \text{ kw}$$

$$\text{Summation of kilovars} = 8 - 4.35 = 3.65 \text{ kilovars}$$

$$\text{kva}_0 = \sqrt{11.81^2 + 3.65^2} = 12.37$$

$$I_0 = \frac{12{,}370}{\sqrt{3}\,220} = 32.5 \text{ amperes}$$

$$\text{p.f.}_0 = \frac{11.81}{12.37} = 0.955$$

Of the four solutions, that which is most convenient for the quantities given should be employed.

Single-Phase and Balanced Three-Phase Power. A comparison of the variation with respect to time of instantaneous single-phase and three-phase power brings out certain fundamental differences. As shown in Chapter II, single-phase power follows a double-frequency sine law with respect to time plus a constant. The instantaneous power for each of three phases, when currents and voltages are sine waves, of a balanced three-phase system is given by the following general equations.

$$p_a = V_m I_m \sin \omega t \sin (\omega t - \theta)$$

$$p_b = V_m I_m \sin (\omega t - 120°) \sin (\omega t - 120° - \theta)$$

$$p_c = V_m I_m \sin (\omega t - 240°) \sin (\omega t - 240° - \theta)$$

The total three-phase power is

$$\begin{aligned} p_3 = P_a + P_b + P_c = V_m I_m [&\sin \omega t \sin (\omega t - \theta) \\ &+ \sin (\omega t - 120°) \sin (\omega t - 120° - \theta) \\ &+ \sin (\omega t - 240°) \sin (\omega t - 240° - \theta)] \end{aligned}$$

$$p_3 = 1.5 V_m I_m \cos \theta \tag{19}$$

For single-phase, say phase a,

$$\begin{aligned} p_1 &= V_m I_m \sin \omega t \sin (\omega t - \theta) \\ &= \frac{V_m I_m}{2} \cos \theta - \frac{V_m I_m}{2} \cos (2\omega t - \theta) \end{aligned} \tag{20}$$

Equation (19) shows the instantaneous value of three-phase power to be independent of time. In other words, balanced three-phase power under steady-state conditions is constant from instant to instant. In contrast, equation (20) for single-phase power shows it to follow a double-frequency variation with respect to time. This comparison is graphically illustrated in Fig. 42.

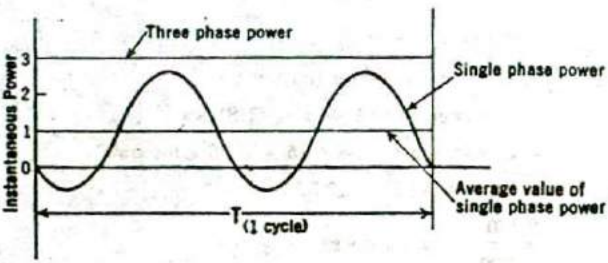


FIG. 42. Comparison of variations of single- and balanced three-phase power.

Power Measurement in Balanced Systems. A wattmeter gives a reading proportional to the product of the current through its current coil, the voltage across its potential coil, and the cosine of the angle between this voltage and current. Since the total power in a three-phase circuit is the sum of the powers of the separate phases, the total power could be measured by placing a wattmeter in each phase, as shown in Fig. 43. It is not generally feasible to break into the phases of a delta-connected load. Therefore the method shown in part (*a*) of Fig. 43

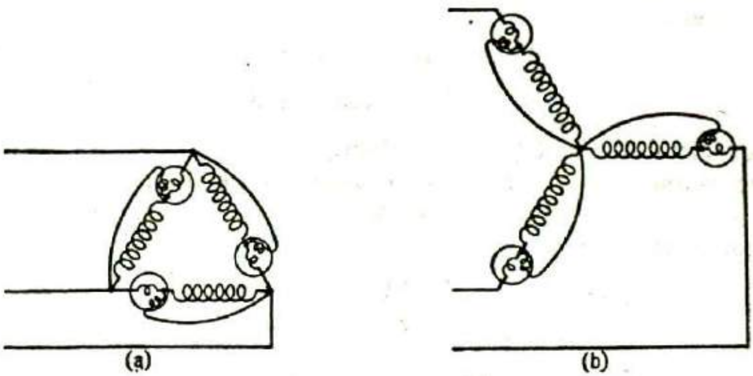


FIG. 43. A wattmeter in each phase may be used to measure three-phase power.

is not applicable. For the wye load shown in part (*b*), it is necessary to connect to the neutral point. This point is not always accessible. Hence another method making use of only two wattmeters is generally employed in making three-phase power measurements. This connection is shown in Fig. 44. To show that two such wattmeters may be used to measure power, the readings of each will be established and their sum compared with equation (11), which has been shown to be

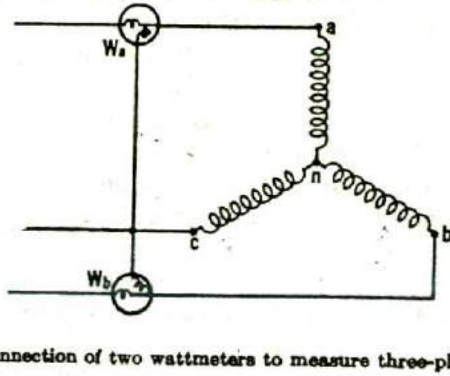


FIG. 44. Connection of two wattmeters to measure three-phase power.

correct for balanced three-phase power. It is important to take the direction of the voltage through the circuit the same as that taken for current when establishing wattmeter readings. Thus if the current coil of W_a, Fig. 44, is considered carrying current I_{an}, the potential across the voltage coil should be taken from a through the circuit, which in this particular case is V_{ac}. Figure 45 shows the vector diagram of the voltages and currents for a balanced system like that of Fig. 44. From this figure the power represented by the currents and voltages of each wattmeter is

$$W_a = V_{ac}I_{an} \cos (\theta - 30°) \tag{21}$$

$$W_b = V_{bc}I_{bn} \cos (\theta + 30°) \tag{22}$$

In equations (21) and (22) the subscripts serve only to assist in seeing which voltages and currents were used. Since the load is balanced, $V_{ac} = V_{bc}$, $I_{an} = I_{bn}$ and only magnitudes are involved. Dropping the subscripts gives

$$W_a = VI \cos (\theta - 30°) \tag{23}$$

$$W_b = VI \cos (\theta + 30°) \tag{24}$$

$$\begin{aligned} W_a + W_b &= VI \cos (\theta - 30°) + VI \cos (\theta + 30°) \\ &= VI [\cos \theta \cos 30° + \sin \theta \sin 30° + \cos \theta \cos 30° - \sin \theta \sin 30°] \\ &= \sqrt{3} VI \cos \theta \end{aligned} \tag{25}$$

Hence $W_a + W_b$ correctly measures the power in a balanced three-phase system of any power factor. As will be shown later, the algebraic sum of the readings of two wattmeters will give the correct value for power under any conditions of unbalance, wave form, or power factor.

For each value of θ (i.e., for each power factor) there is a definite ratio of W_a/W_b. If the ratio of the smaller to the larger reading is always taken and plotted against the corresponding $\cos \theta$ (i.e., power factor), a curve called the watt ratio curve results. This curve is shown in Fig. 46. Reference to the vector diagram of Fig. 45 and the curve of Fig. 46 shows that at 0.5 power factor one wattmeter reads zero. For the case under discussion 0.5 lagging power factor makes W_b read zero, while 0.5 leading power factor makes W_a read zero. When the power factor is zero, each wattmeter has the same deflection but the readings are of opposite signs. The foregoing facts are easily deducible from the vector diagram shown in Fig. 45 and also follow from equations (23) and (24). It is essential in the two-wattmeter method that the proper sign be given the wattmeter readings and that the sum be taken algebraically.

There are several ways to determine whether a wattmeter reading

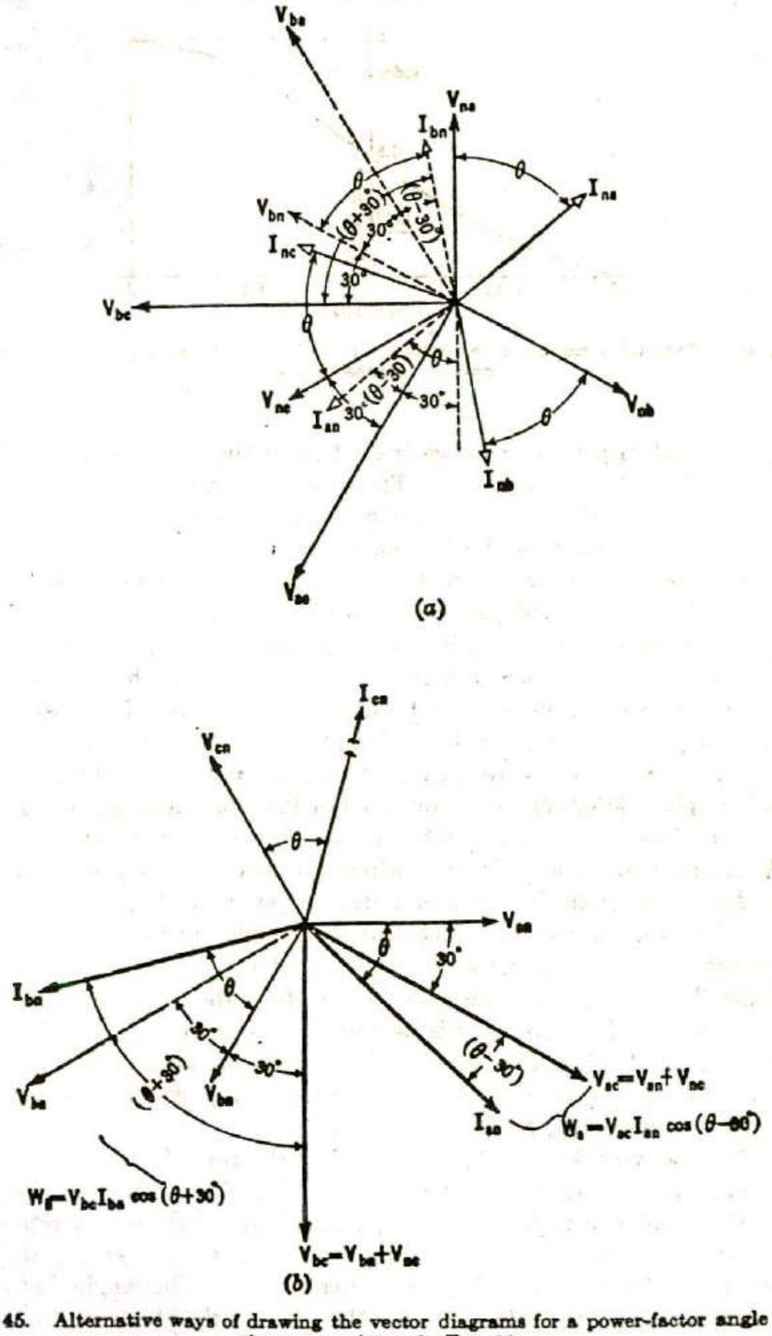


FIG. 45. Alternative ways of drawing the vector diagrams for a power-factor angle θ of the system shown in FIG. 44.

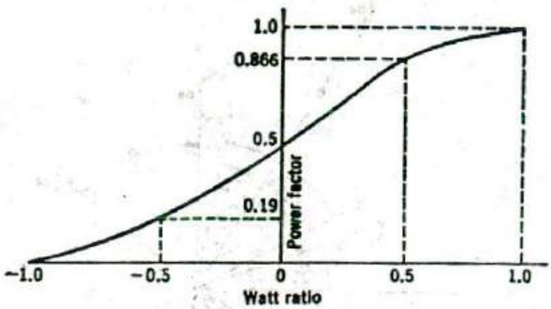


FIG. 46. Watt ratio curve for two-wattmeter method of measuring power (applicable only to balanced loads).

should be taken positive or negative. One of the best methods follows. Refer to Fig. 44. Open line a. Then all power must be transferred to the load over lines b and c. If wattmeter b is connected so that it reads "up scale," it will then be known to have this deflection when the power it reads is going to the load. Next reconnect line a and open line b. Then connect W_a so that it reads up scale. Now close line b. If at any time after this either wattmeter needle goes backward against the down-scale stop, power through this wattmeter channel is being transferred to the generator and this power must be of opposite sign to that registered by the other. Either the potential or current coil will have to be reversed to secure an up-scale reading. The foregoing test is applicable under any conditions of loading, although it may not always be feasible because of the necessity for opening the lines.

A second test applicable only when the load is practically *balanced* is to disconnect from the common potential point c of Fig. 44 the potential coil of the wattmeter which has the smaller reading and connect it to the line containing the current coil of the other wattmeter. If the needle goes against the down-scale stop, the wattmeter reading was negative. The foregoing is best explained through a consideration of the circuit diagram of Fig. 44 and the corresponding vector diagram of Fig. 45. As previously shown, W_a reads the power represented by V_{ac} and I_{an} while W_b reads that due to V_{bc} and I_{bn}. Since the angle $(\theta + 30°)$ between V_{bc} and I_{bn} is larger than the angle $(\theta - 30°)$ between V_{ac} and I_{an} for the load represented by Fig. 45, wattmeter W_b will have the smaller deflection. If the potential coil of W_b is now removed from line c in Fig. 44 and connected to line a, the meter will deflect because of the potential V_{ba} and current I_{bn}. The angle between V_{ba} and I_{bn} is seen to be $(\theta - 30°)$ or the same as that between the voltage and current for wattmeter W_a. W_a and W_b will then read alike.

Furthermore, since W_b was connected to read up scale when the angle between its voltage and current was less than 90°, it will continue to read up scale when it receives the potential V_{ba}. If, however, the power factor was below 0.5, the angle $(\theta + 30°)$ on Fig. 45 would be more than 90°. If the wattmeter W_b were made to read up scale under such conditions, it would reverse its deflection when given the potential V_{ba} as outlined above since it would then be subjected to a voltage and current of $(\theta - 30°)$, which is less than 90° out of phase. When the potential coil connection of W_b is moved from line c to a in Fig. 44, this wattmeter receives a potential of V_{ba}, while that for W_a (taken similarly from the line containing the current coil) is V_{ac}. These potentials are in the same order or direction around the diagram. Hence the potential coils are said to be connected in the same cyclic order about the circuit, and under these conditions both wattmeters would be expected to show the same deflection. This was found to be true in the above analysis.

Example 7. In a circuit like that shown in Fig. 44, W_a reads 800 and W_b reads 400 watts. When the potential coil of W_b is disconnected at c and connected at a, the needle goes against the down-scale stop.

Solution. The test indicates that W_b is reading -400 watts. Hence

$$P = W_a + W_b = 800 + (-400) = 400 \text{ watts}$$

$$\text{Watt ratio} = \frac{W_b}{W_a} = \frac{-400}{800} = -0.5$$

From a watt ratio curve like that shown on page 352, the power factor may be determined directly as 0.19.

The power factor, $\cos\theta$, could also have been calculated from a simultaneous solution of equations (23) and (24) since

$$\cos\theta = \cos\left(\tan^{-1}\frac{\sqrt{3}\,(W_a - W_b)}{W_a + W_b}\right)$$

This relation is made apparent in the next article.

Reactive Volt-Amperes. The reactive volt-amperes in a balanced three-phase circuit may be expressed by

$$P_X = \sqrt{3}\,(W_a - W_b) \tag{26}$$

This may be shown as follows:

$$\begin{aligned}\sqrt{3}\,(W_a - W_b) &= \sqrt{3}\,[VI\cos(\theta - 30°) - VI\cos(\theta + 30°)] \\ &= \sqrt{3}VI\,[\cos\theta\cos 30° + \sin\theta\sin 30° - \cos\theta\cos 30° \\ &\quad + \sin\theta\sin 30°] \\ &= \sqrt{3}VI\sin\theta\end{aligned}$$

This is the same as equation (13) for reactive power given on page 345. Since the ratio of the reactive volt-amperes, $\sqrt{3}V_L I_L \sin\theta$, to the power, $\sqrt{3}V_L I_L \cos\theta$, is the $\tan\theta$, it follows from equations (25) and (26) that

$$\tan\theta = \frac{\sqrt{3}\,(W_a - W_b)}{W_a + W_b} \tag{27}$$

where θ is the power-factor angle.

Example 8. The power factor in the preceding example could have been easily calculated by means of the relation stated in equation (26). Thus

$$P_X = \sqrt{3}\,(W_a - W_b) = \sqrt{3}\,[800 - (-400)] = 2078 \text{ vars}$$

$$(P = W_a + W_b = 800 - 400 = 400 \text{ watts})$$

$$\text{va} = \sqrt{P^2 + P_X{}^2} = \sqrt{400^2 + 2078^2} = 2114 \text{ volt-amperes}$$

$$\text{p.f.} = \frac{P}{\text{va}} = \frac{400}{2114} = 0.19$$

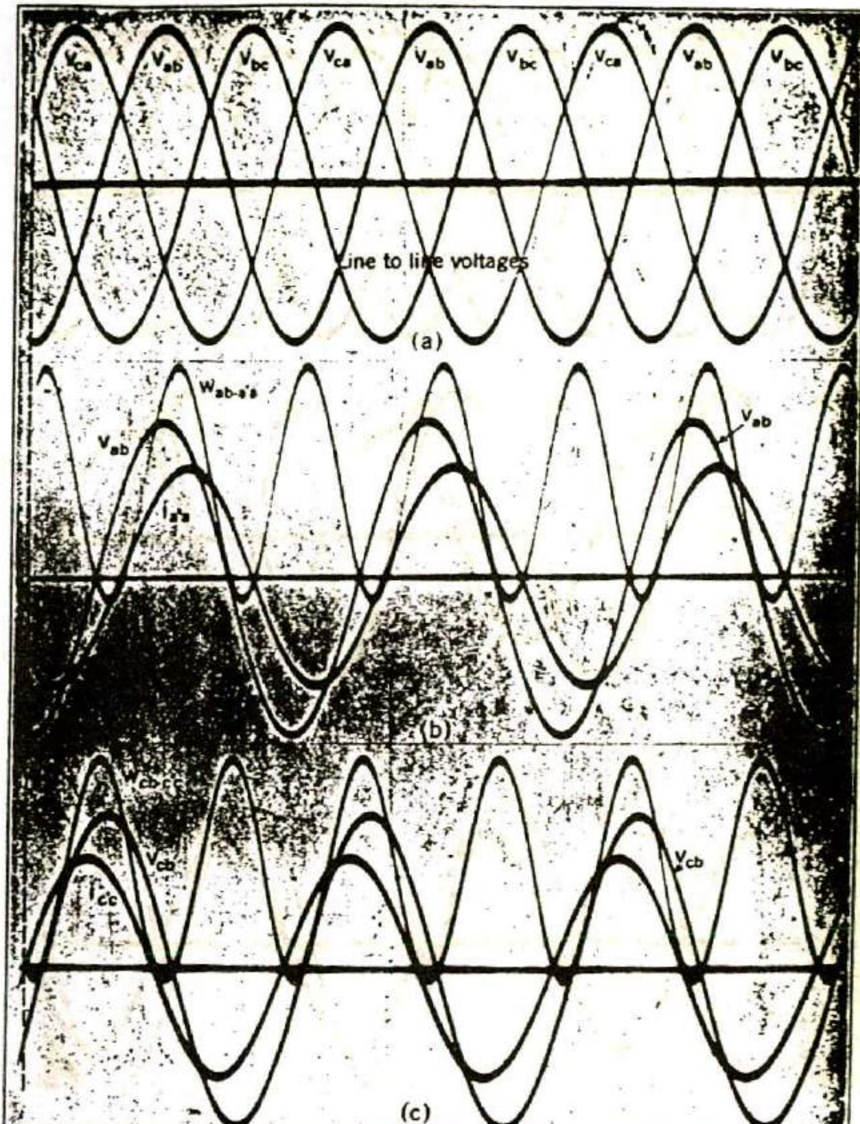
Three-Phase, Four-Wire Systems. If a three-phase, four-wire system is balanced, the fourth wire or neutral will carry no current. The system is the same as when the neutral is omitted, in which case it is the same as a balanced three-phase, three-wire system. It can therefore be metered as previously shown for the three-wire system. Another method is given later. Under any other conditions three meters or their equivalent are necessary. Unbalanced systems are considered in the next chapter.

Delta Systems. The measurement of power in a three-phase system was discussed with reference to a wye-circuit diagram and the corresponding vector diagram. When it is remembered that a delta system can always be replaced by an equivalent wye system, the preceding discussion will be seen to apply to the delta system. Furthermore only line voltages and line currents were involved in the discussion of the two-wattmeter method of measuring power, and there is no difference between these quantities for the delta and wye systems.

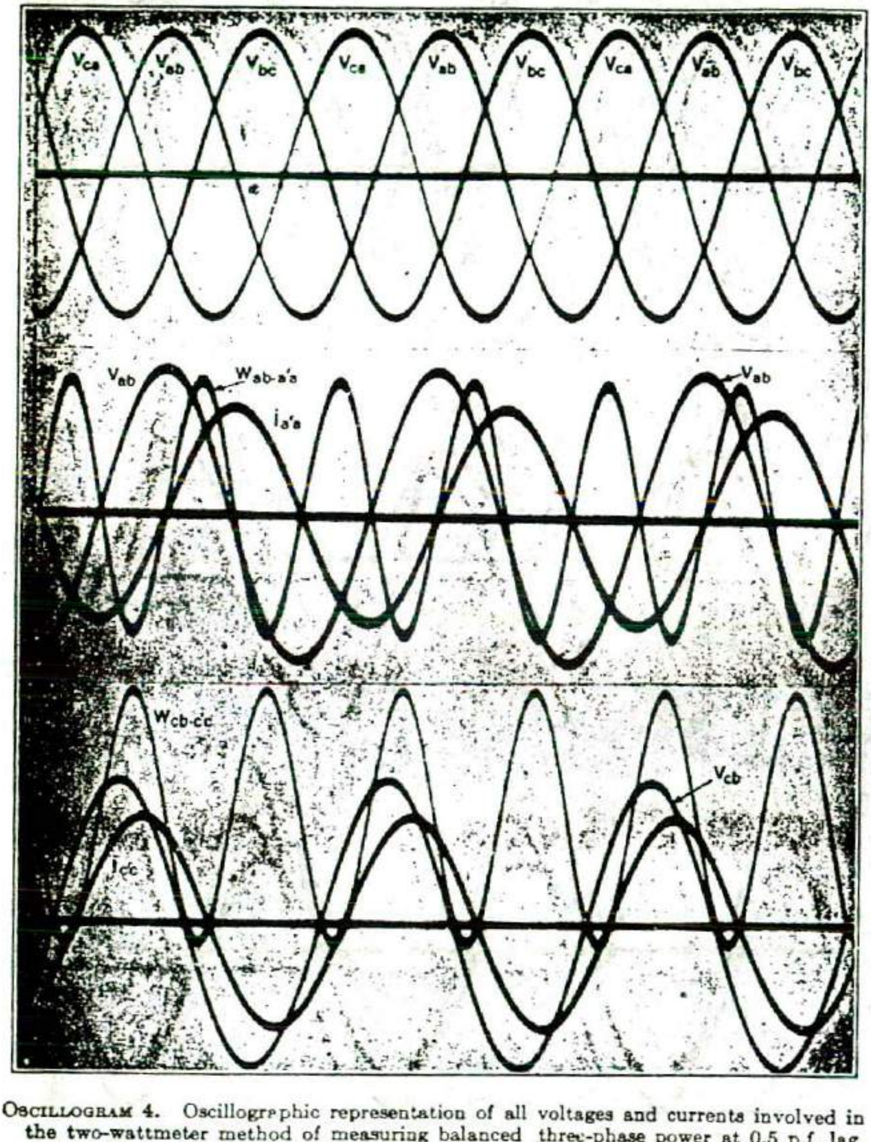
Oscillograms 3 and 4, which were obtained from a delta system as shown and labeled in Fig. 47, may be profitably studied.

Problem 12. Refer to Oscillogram 3. (*a*) If the line-to-line voltages have instantaneous maximum values of 155.5 volts and the delta-line currents have instantaneous maximum values of 14.14 amperes, find the average power readings of the wattmeters $W_{ab\text{-}a'a}$ and $W_{cb\text{-}c'c}$.

(*b*) Draw a vector diagram indicating all currents and voltages shown on Oscillogram 3. Use $\mathbf{V}_{ab}$ as reference, and include the delta-phase currents $\mathbf{I}_{ab}$, $\mathbf{I}_{bc}$, and



OSCILLOGRAM 3. Oscillographic representation of all voltages and currents involved in the two-wattmeter method of measuring balanced three-phase power at unity power factor. In (*a*) the sequence of line-to-line voltages is shown. v_{ca} is the voltage not used. In (*b*) $w_{ab-a'a}$ is a graph of the instantaneous driving torque of the wattmeter element which is operated by v_{ab} and $i_{a'a}$. In (*c*) $w_{cb-c'c}$ is a graph of the instantaneous driving torque of the wattmeter element which is operated by v_{cb} and $i_{c'c}$.



OSCILLOGRAM 4. Oscillographic representation of all voltages and currents involved in the two-wattmeter method of measuring balanced three-phase power at 0.5 p.f. lag, the condition under which one wattmeter reads zero. In the upper oscillogram, the sequence of line-to-line voltages is shown. The voltage v_{ca} is the voltage not used in the two-wattmeter method in this case. (In the center oscillogram, $w_{ab-a'a}$ is a graph of the instantaneous driving torque of the wattmeter element which is operated by v_{ab} and $i_{a'a}$. In the lower oscillogram, $w_{cb-c'c}$ is a graph of the instantaneous driving torque of the wattmeter element which is operated by v_{cb} and $i_{c'c}$.

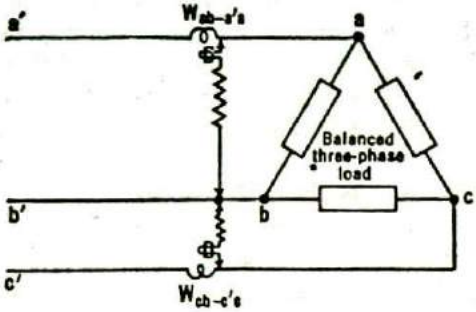


FIG. 47. Circuit arrangement for which Oscillograms 3 and 4 were taken.

I_{ca} which are not shown on the oscillogram but which combine to form the delta-line currents $I_{a'a}$ and $I_{c'c}$.

Ans.: (*a*) $W_{ab\text{-}a'a} = W_{cb\text{-}c'c} = 952.6$ watts.

(*b*) *ab-bc-ca* sequence of line-to-line voltages; I_{ab} in time phase with V_{ab}; $I_{a'a}$ lags V_{ab} by 30°; $I_{c'c}$ leads V_{cb} by 30°.

General n-Wire Balanced System. The total power taken by a balanced n-phase system is n times the power per phase. A single wattmeter connected to measure the product of the current, potential, and the cosine of the angle between the current and potential may be used to obtain the power of a balanced n-phase system. The wattmeter reading obtained is multiplied by n. If it is not possible to break into a phase of a mesh-connected load or to obtain the neutral of a star-connected one, power may still be measured with a single wattmeter. For the n-phase system, n equal resistances may be connected in star and then to the lines. A neutral is thus established, and power is measured as though the neutral wire of a star system were available. The method is shown in Fig. 48. If the number of phases is even, as, for example, in Fig. 48, only a single resistance is necessary provided that the potential coil of the wattmeter can be connected at the midpoint of this resistance. The resistance must be connected between two lines having the largest potential difference. The wattmeter reading must

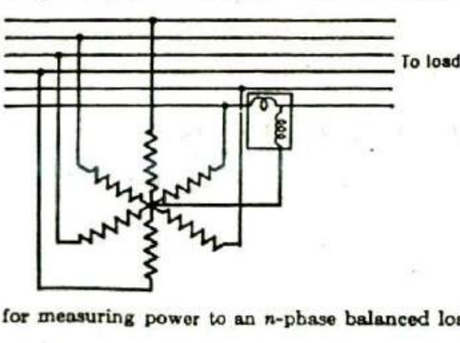


FIG. 48. A method for measuring power to an n-phase balanced load (load not shown).

be multiplied by n, the number of phases, to obtain the total power. If the number of phases is even, the potential coil may be connected from the line containing the current coil to the line which yields the highest potential difference. The total power is then the wattmeter indication multiplied by $n/2$. These connections may be used only for balanced systems.

Copper Required to Transmit Power under Fixed Conditions. All systems will be compared on the basis of a fixed amount of power transmitted a fixed distance with the same amount of loss and at the same maximum voltage between conductors. In all cases the total weight of copper will be directly proportional to the number of wires, since the distance is fixed, and inversely proportional to the resistance of each wire. First, three-phase will be compared with single-phase. Since the same voltage and power factor are to be assumed, the same respective symbols for these quantities for single- and three-phase will suffice.

$$P_1 = VI_1 \cos\theta$$

$$P_3 = \sqrt{3}VI_3 \cos\theta$$

Since

$$P_1 = P_3$$

$$VI_1 \cos\theta = \sqrt{3}VI_3 \cos\theta$$

$$I_1 = \sqrt{3}I_3$$

Also

$$I_1^2R_1 \times 2 = I_3^2R_3 \times 3$$

or

$$\frac{R_1}{R_3} = \frac{3I_3^2}{2I_1^2} = \frac{3I_3^2}{3I_3^2 \times 2} = \frac{1}{2}$$

$$\frac{\text{Copper three-phase}}{\text{Copper single-phase}} = \frac{\text{No. of wires three-phase}}{\text{No. of wires single-phase}} \times \frac{R_1}{R_3} = \frac{3}{2} \times \frac{1}{2} = \frac{3}{4}$$

The above shows that the same amount of power may be transmitted a fixed distance with a fixed line loss with only three-fourths of the amount of copper that would be required for single-phase, or one-third more copper is required for single-phase than would be necessary for three-phase.

Comparison of Three-Phase with Four-Phase.

$$P_3 = \sqrt{3}VI_3 \cos\theta$$

$$P_4 = 4\frac{V}{2}I_4 \cos\theta$$

(*Note:* V is highest voltage between any pair of wires.) Therefore

$$\sqrt{3}VI_3 \cos\theta = 4\frac{V}{2}I_4 \cos\theta$$

$$\sqrt{3}I_3 = \frac{4}{2}I_4$$

$$\frac{I_3}{I_4} = \frac{2}{\sqrt{3}}$$

$$3I_3^2R_3 = 4I_4^2R_4$$

$$\frac{R_4}{R_3} = \frac{3I_3^2}{4I_4^2} = \frac{3}{4} \times \frac{4}{3} = 1$$

$$\frac{\text{Copper three-phase}}{\text{Copper four-phase}} = \frac{3}{4} \times \frac{1}{1} = \frac{3}{4}$$

This is the same relation as shown for single-phase. If other systems are compared with three-phase in this manner, it will be found that three-phase is more economical in the use of copper than any other number of phases.

When a fixed amount of power is transmitted a fixed distance with a fixed loss for the same voltage to neutral, there is no difference between any of the systems. Consider three-phase and single-phase. The voltage to neutral single-phase is half the voltage between lines. This point is called the neutral, since the potential from either line to it is the same.

$$P_3 = P_1$$

$$3V_nI_3 \cos\theta = 2V_nI_1 \cos\theta$$

$$\frac{I_3}{I_1} = \frac{2}{3}$$

$$3I_3^2R_3 = 2I_1^2R_1$$

$$\frac{R_1}{R_3} = \frac{3I_3^2}{2I_1^2} = \frac{3}{2} \times \frac{4}{9} = \frac{2}{3}$$

$$\frac{\text{Copper three-phase}}{\text{Copper single-phase}} = \frac{3}{2} \times \frac{2}{3} = 1 \quad \text{(for same voltage to neutral)}$$

Comparison of Three-Phase with n-Phase for the Same Voltage to Neutral.

$$P_3 = P_n$$

$$3V_nI_3 \cos\theta = nV_nI_n \cos\theta$$

$$\frac{I_3}{I_n} = \frac{n}{3}$$

$$3I_3{}^2R_3 = nI_n{}^2R_n$$

$$\frac{R_n}{R_3} = \frac{3}{n}\frac{I_3{}^2}{I_n{}^2} = \frac{3}{n}\frac{n^2}{3^2} = \frac{n}{3}$$

$$\frac{\text{Copper three-phase}}{\text{Copper } n\text{-phase}} = \frac{3}{n}\frac{n}{3} = 1 \quad \text{(for same voltage to neutral)}$$

There is no difference in the amount of copper required between any of the systems if the voltage to neutral is fixed and if the same amount of power is transmitted a fixed distance at a fixed line loss.

Two-phase transmission was not considered in the above comparisons. When it is recognized that two-phase is the same as two independent single-phase systems, it is evident that two-phase, four-wire transmission requires the same amount of copper as single-phase. There are twice as many wires, but each is only one-half of the cross section of those necessary for single-phase.

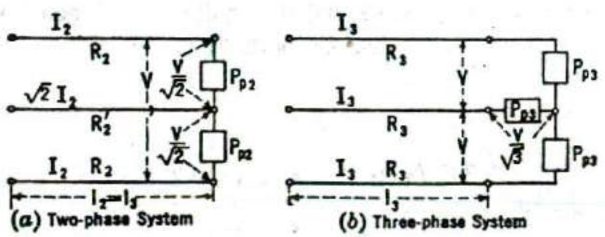


FIG. 49. See Problem 13.

Problem 13. Refer to Fig. 49. Find the ratio of the copper required for two-phase, three-wire transmission to that required for three-phase, three-wire transmission under the following conditions, all imposed simultaneously.

(*a*) A fixed amount of power transmitted.
(*b*) The same distance.
(*c*) With the same total line loss.
(*d*) With the same highest line voltage between any pair of lines in the two systems.
(*e*) With the same current density in the three two-phase conductors.

Hint:

From condition (*a*): $P_{2t} = 2V_{p2}I_2 \cos\theta = P_{3t} = 3V_{p3}I_3 \cos\theta$

From condition (*d*): $I_2 = \dfrac{\sqrt{3}}{\sqrt{2}} I_3$

From condition (*c*): $2I_2^2R_2 + (\sqrt{2}I_2)^2R_{2'} = 3I_3^2R_3$

From condition (*e*): Area of $R_{2'}$ wire $= \sqrt{2} \times$ area of R_2 wire

From condition (*b*): $R_{2'} = \dfrac{R_2}{\sqrt{2}}$ *Ans.:* 1.94.

Harmonics in the Wye System. An emf generated in a conductor will be sinusoidal only when the flux cutting the conductor varies according to a sine law. In a-c generators it is rather difficult, if not entirely impossible, to obtain an exact sine wave of distribution of the field flux. The slots and teeth change the reluctance of the path for the flux and cause ripples in the flux wave. Even if the distribution of the field flux were sinusoidal at no load, the distribution would be altered as the load came on, owing to the effect of the armature reaction of the current in the armature. The result is to induce in each phase an emf wave that is somewhat distorted from a true sine wave. In modern machines this distortion is relatively small. Through certain arrangements of the inductors on the armature and through certain ways of connecting them, some of the harmonics in the wave are reduced or are made to cancel entirely. When iron-core transformers are connected in wye, or any other way for that matter, the exciting current cannot be sinusoidal even though the impressed voltage is a perfect sine wave. This is due to the varying reluctance of the magnetic circuit with the consequent requirement of more ampere-turns to produce a given change in flux when the core operates at the higher flux densities. It therefore becomes of some importance to consider the effects of certain harmonics of currents and voltages in the phases of a three-phase system in affecting the line voltage of the system.

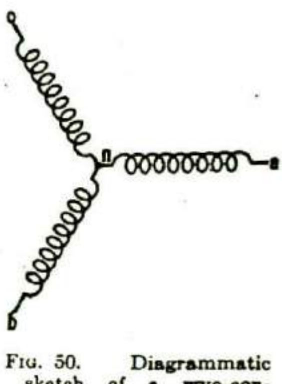


FIG. 50. Diagrammatic sketch of a wye-connected generator.

Assume that the emf induced in phase *a* of the wye-connected generator diagrammatically shown in Fig. 50 is

$$e_{na} = E_{m1}\sin\omega t + E_{m3}\sin(3\omega t + \alpha_3) + E_{m5}\sin(5\omega t + \alpha_5) + E_{m7}\sin(7\omega t + \alpha_7) \quad (28)$$

The sequence e_{na}, e_{nb}, e_{nc} will be used. Hence the fundamental of emf in phase nb will lag that in na by 120°, while that in phase nc will lag phase na by 240°. As usual, a shift of one degree for the fundamental will be a shift of n degrees for the nth harmonic. Then

$$\begin{aligned} e_{nb} &= E_{m1} \sin (\omega t - 120°) + E_{m3} \sin (3\omega t + \alpha_3 - 360°) \\ &\quad + E_{m5} \sin (5\omega t + \alpha_5 - 600°) + E_{m7} \sin (7\omega t + \alpha_7 - 840°) \\ &= E_{m1} \sin (\omega t - 120°) + E_{m3} \sin (3\omega t + \alpha_3) \\ &\quad + E_{m5} \sin (5\omega t + \alpha_5 - 240°) + E_{m7} \sin (7\omega t + \alpha_7 - 120°) \quad (29) \end{aligned}$$

$$\begin{aligned} e_{nc} &= E_{m1} \sin (\omega t - 240°) + E_{m3} \sin (3\omega t + \alpha_3) \\ &\quad + E_{m5} \sin (5\omega t + \alpha_5 - 120°) + E_{m7} \sin (7\omega t + \alpha_7 - 240°) \quad (30) \end{aligned}$$

The equations of the phase voltages show that all third harmonics are in phase. Also the phase sequence for the fifth harmonic is reversed from that of the fundamental. The sequence of the seventh is the same

TABLE 1

DISPLACEMENT BETWEEN VARIOUS HARMONICS IN THE PHASES OF FIG. 50

Displacement in electrical degrees

Harmonic	1	3	5	7	9	11	13
Phase A	0	0	0	0	0	0	0
Phase B	120	0	240	120	0	240	120
Phase C	240	0	120	240	0	120	240

as that for the fundamental. In general it will be found that the fundamental and all harmonics obtained by adding a multiple of 6 to the fundamental will have the same sequence. These are first, seventh, thirteenth, nineteenth, twenty-fifth, and so on. In like manner, the fifths, elevenths, seventeenths, twenty-thirds, etc., have like sequences but opposite to that of the fundamentals. Also the third, ninth, and all multiples of the third will be found to be in phase. These results are tabulated in Table 1. The relation between the fundamentals and third harmonics in each phase for $\alpha_3 = 0$ in equations (28), (29), and (30) is shown in Fig. 51.

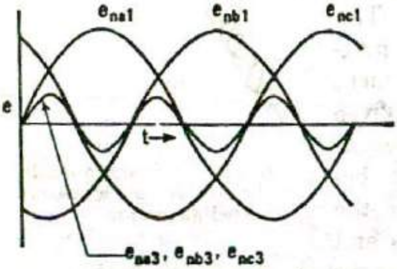


FIG. 51. Fundamental and third harmonic voltages.

The line voltage of the wye may be found by summing up the potentials encountered in passing through the circuit between the line terminals in question. With reference to Fig. 50,

$$e_{ba} = e_{bn} + e_{na}$$

Each harmonic must be handled separately. The combination of e_{bn} and e_{na} is shown by vector diagrams in Fig. 52. For the funda-

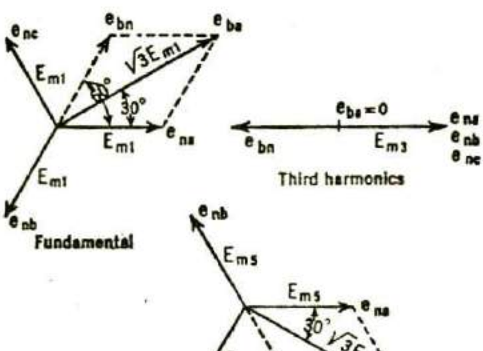


FIG. 52. Line voltages in Fig. 50 are found for each harmonic separately.

mental, e_{ba} is 30° ahead of e_{na}. Since $e_{na_1} = E_{m1} \sin \omega t$, $e_{ba_1} = \sqrt{3}E_{m1} \sin (\omega t + 30°)$. For the third harmonic, $e_{ba_3} = 0$. For the fifth, e_{ba_5} lags e_{na_5} by 30°. Hence $e_{ba_5} = \sqrt{3}E_{m5} \sin (5\omega t + \alpha_5 - 30°)$. The seventh-harmonic vector diagram is similar to that for the fundamental. The complete equation for the line voltage e_{ba} is

$$e_{ba} = \sqrt{3}E_{m1} \sin (\omega t + 30°) + \sqrt{3}E_{m5} \sin (5\omega t + \alpha_5 - 30°) + \sqrt{3}E_{m7} \sin (7\omega t + \alpha_7 + 30°) \quad (31)$$

Similarly,

$$e_{ac} = \sqrt{3}E_{m1} \sin (\omega t + 150°) + \sqrt{3}E_{m5} \sin (5\omega t + \alpha_5 - 150°) + \sqrt{3}E_{m7} \sin (7\omega t + \alpha_7 + 150°) \quad (32)$$

$$e_{cb} = \sqrt{3}E_{m1} \sin (\omega t - 90°) + \sqrt{3}E_{m5} \sin (5\omega t + \alpha_5 + 90°) + \sqrt{3}E_{m7} \sin (7\omega t + \alpha_7 - 90°) \quad (33)$$

The vector diagram of the third-harmonic voltages shows that the third harmonics in the two phases between any pair of terminals are in opposition and cancel. The third harmonics cannot contribute anything to line voltage, although they do contribute toward the total voltage between one terminal and neutral. The rms magnitude of the voltage to neutral in the example just considered is

$$E_{na} = \sqrt{\frac{E_{m1}{}^2 + E_{m3}{}^2 + E_{m5}{}^2 + E_{m7}{}^2}{2}}$$

The rms magnitude of the voltage between terminals is

$$E_{ba} = \sqrt{3}\sqrt{\frac{E_{m1}{}^2 + E_{m5}{}^2 + E_{m7}{}^2}{2}}$$

The ratio of line and phase voltage of a wye connection can be the $\sqrt{3}$ only when there is no third harmonic or its multiples in the wave of phase voltage.

Consider next the harmonics in the current waves for the wye. Kirchhoff's current law applied to the wye connection without a neutral wire connected states that

$$i_{na} + i_{nb} + i_{nc} = 0$$

Under balanced conditions this equation can be fulfilled only when the three currents are equal in magnitude and 120° apart in time phase, or when the magnitudes of each current are equal to zero. Since the third harmonics and their multiples are the only ones that are not 120° apart, each of them must be zero to fulfil the conditions imposed by Kirchhoff's current law. The vector diagrams for the harmonics of current appear exactly as those for phase voltages in Fig. 52. If, in each phase, e is replaced by i, the diagrams will represent currents. If the third harmonics of current do exist, there must be a neutral connection. This neutral or fourth wire furnishes the return path for the third harmonics of each phase. Since all third harmonics, in accordance with the diagram in Fig. 52, would have to be in phase, their arithmetic sum would flow in the neutral. A third-harmonic pressure or voltage may exist in each phase, but, unless a path through the neutral is provided, the three voltages do not have a closed circuit upon which they can act and, therefore, no third-harmonic current can flow. In a balanced wye-connected circuit without neutral connection, therefore, all harmonics except the third and its multiples can exist. In a four-wire, three-phase circuit (neutral wire connected) all harmonics in the current wave can exist.

Harmonics in the Delta System. If three coils having induced voltages as given by e_{na}, e_{nb}, and e_{nc} in the previous article are connected in delta, those voltages that do not add to zero around the loop will cause a circulating current to flow. Under any circumstances, in the delta of Fig. 53, the sum of the three terminal voltages taken in the same direction around the delta must be zero. Expressed algebraically,

$$v_{ca} + v_{ab} + v_{bc} = 0 \qquad (34)$$

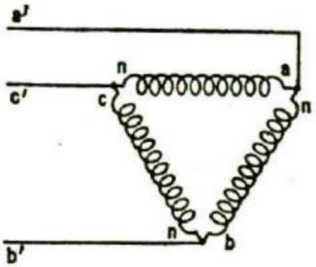


FIG. 53. Coils of Fig. 50 reconnected in delta.

Because the sum of the generated emf's, $e_{na} + e_{nb} + e_{nc}$, is equal to zero for all except triple-frequency voltages and its multiples, no circulatory current of other than triple frequency and its multiples can exist. Hence there will be no impedance drops at no load, and the generated voltages for all except the third harmonic and its multiples will appear across the terminals. For the third harmonic and its multiples the situation is different. Since the third-harmonic generated voltages of all phases of a three-phase system were shown to be equal and in phase,

$$e_{na_3} + e_{nb_3} + e_{nc_3} = 3E_{m3} \sin (3\omega t + \alpha_3)$$

will cause a current to circulate in the delta. This current multiplied by the impedance of the loop will be equal to the resultant third-harmonic voltage $3E_{m3} \sin (3\omega t + \alpha_3)$. Since the terminal voltage is equal to the generated voltage minus the internal drop, there will be no third-harmonic voltage between terminals in the delta if the phase emf's and impedances are balanced. In this way equation (34) is fulfilled for the third-harmonic voltages.

There is no third harmonic in the terminal voltage of the wye; neither is the wye connection subject to a third-harmonic circulating current. In the wye the third-harmonic voltages between terminals do not appear, as the result of their being in opposition between two terminals and neutralizing. In the delta, the third-harmonic voltage does not appear in the terminal voltage because it is short-circuited by the mesh connection and is consumed in the form of internal impedance drop. The equations of the terminal voltages of the delta generator or transformer at no load are the same as the generated voltages of each phase with the third-harmonic voltage and its multiples omitted. Thus

$$v_{ca} = E_{m1} \sin \omega t + E_{m5} \sin (5\omega t + \alpha_5) + E_{m7} \sin (7\omega t + \alpha_7) \quad (35)$$

$$v_{ab} = E_{m1} \sin (\omega t - 120°) + E_{m5} \sin (5\omega t + \alpha_5 - 240°) + E_{m7} \sin (7\omega t + \alpha_7 - 120°) \quad (36)$$

$$v_{bc} = E_{m1} \sin (\omega t - 240°) + E_{m5} \sin (5\omega t + \alpha_5 - 120°) + E_{m7} \sin (7\omega t + \alpha_7 - 240°) \quad (37)$$

Compare equations (35), (36), and (37) with equations (28), (29), and (30).

All harmonics of current are possible in the phases of the delta, since it is simply a closed series loop. Thus for phase *ca*, Fig. 53, we may have

$$i_{ca} = I_{m1} \sin \omega t + I_{m3} \sin (3\omega t + \alpha_3) + I_{m5} \sin (5\omega t + \alpha_5) + I_{m7} \sin (7\omega t + \alpha_7) \quad (38)$$

If the sequence is such that phase *ab* lags *ca* by 120°, the currents in the other phases are found by displacing the fundamentals by the usual 120° and the *n*th harmonic by *n* times this angle. Thus

$$\begin{aligned} i_{ab} &= I_{m1} \sin (\omega t - 120°) + I_{m3} \sin (3\omega t + \alpha_3 - 360°) \\ &\quad + I_{m5} \sin (5\omega t + \alpha_5 - 600°) + I_{m7} \sin (7\omega t + \alpha_7 - 840°) \\ &= I_{m1} \sin (\omega t - 120°) + I_{m3} \sin (3\omega t + \alpha_3) \\ &\quad + I_{m5} \sin (5\omega t + \alpha_5 - 240°) + I_{m7} \sin (7\omega t + \alpha_7 - 120°) \end{aligned} \quad (39)$$

$$i_{bc} = I_{m1} \sin (\omega t - 240°) + I_{m3} \sin (3\omega t + \alpha_3) + I_{m5} \sin (5\omega t + \alpha_5 - 120°) + I_{m7} \sin (7\omega t + \alpha_7 - 240°) \quad (40)$$

The line currents are obtained in terms of phase current as indicated below.

$$i_{a'a} = i_{ac} + i_{ab}$$

$$i_{b'b} = i_{ba} + i_{bc}$$

$$i_{c'c} = i_{ca} + i_{cb}$$

These operations are performed similarly to those illustrated in the vector diagrams of Fig. 52 for voltages. The results are

$$i_{a'a} = \sqrt{3} I_{m1} \sin (\omega t - 150°) + \sqrt{3} I_{m5} \sin (5\omega t + \alpha_5 + 150°) + \sqrt{3} I_{m7} \sin (7\omega t + \alpha_7 - 150°) \quad (41)$$

$$i_{b'b} = \sqrt{3} I_{m1} \sin (\omega t + 90°) + \sqrt{3} I_{m5} \sin (5\omega t + \alpha_5 - 90°) + \sqrt{3} I_{m7} \sin (7\omega t + \alpha_7 + 90°) \quad (42)$$

$$i_{c'c} = \sqrt{3} I_{m1} \sin (\omega t - 30°) + \sqrt{3} I_{m5} \sin (5\omega t + \alpha_5 + 30°) + \sqrt{3} I_{m7} \sin (7\omega t + \alpha_7 - 30°) \quad (43)$$

Equations (41), (42), and (43) show that no third-harmonic currents can exist in the lines of a delta. The third-harmonic current in one phase coming to a line connection exactly equals the third-harmonic current in the other phase leaving the junction. This leaves no third-harmonic current to flow in the line connection.

The magnitude of the phase current is

$$I_p = \sqrt{\frac{I_{m1}^2 + I_{m3}^2 + I_{m5}^2 + I_{m7}^2}{2}}$$

The magnitude of the line current is

$$I_L = \sqrt{\frac{(\sqrt{3}I_{m1})^2 + (\sqrt{3}I_{m5})^2 + (\sqrt{3}I_{m7})^2}{2}}$$

$$= \sqrt{3}\sqrt{\frac{I_{m1}^2 + I_{m5}^2 + I_{m7}^2}{2}}$$

The ratio of line to phase current can be $\sqrt{3}$ only when no third-harmonic currents exist.

Example 9. Only fundamentals and third harmonics are assumed to exist in the voltages of a wye connection like that shown in Fig. 50. Voltmeter readings as follows are obtained: $V_{na} = 150$, $V_{ba} = 220$. Calculate the magnitude of the third-harmonic voltage.

Solution. Since V_{ba} contains only fundamental voltage, the fundamental to neutral is $220/\sqrt{3} = 127$.

$$V_{na} = \sqrt{V_1^2 + V_3^2} \quad \text{or} \quad V_3 = \sqrt{150^2 - 127^2} = 79.9$$

The possibility of a third-harmonic circulating current in a delta makes this connection for a-c generators somewhat less desirable than the wye, although there are several other more important factors that make wye connection for generators predominate. Although the third-harmonic current is undesirable in the delta generator it *is* desirable in transformers, since there it acts as a component of the magnetizing current for the core which is essential if a sine wave of flux and induced voltage is to be obtained. Some high-voltage transformers which are connected wye on both primary and secondary have a third winding which is delta-connected to allow a third-harmonic circulating current to flow, thus supplying the transformers with the necessary triple-frequency component of magnetizing current. A delta-connected winding of this kind is called a tertiary winding.

PROBLEMS

14. What is the phase voltage and also the voltage between adjacent lines of a six-phase star connection if the greatest voltage between any pair of lines is 156 volts?

15. The voltage between adjacent lines of a twelve-phase star is 100 volts. Find the voltage to neutral, the voltage between alternate lines, and the greatest voltage between any pair of lines.

16. Find the phase current in a six-phase mesh if the line current is 10 amperes; also for a twelve-phase mesh for the same line current.

17. Given six coils each having an induced voltage of 63.5 volts. Adjacent coil voltages are 60° apart. In how many ways may you connect these coils to form a balanced three-phase wye system of voltages if all coils must be used for each system and if the magnitude of the line voltages of each system must be different? What are the line voltages for each wye system?

18. A generator has six coils, adjacent coils being displaced 30 electrical degrees. If each coil voltage is 114 volts, show how to connect them and calculate the line or terminal voltage for three-phase star. Repeat for three-phase mesh. Repeat for two-phase, where line voltage is taken as the phase voltage.

19. A generator has six coils, adjacent coils being displaced 30 electrical degrees. If all coils are used to form a three-phase mesh, what must be the emf of each coil to yield balanced three-phase voltages of 230 volts each? If all coils are connected for three-phase star, what must be the emf of each coil to give an emf between lines of 230 volts?

20. Draw vector diagrams which represent the currents and voltages shown in Oscillograms 3 and 4, pages 355 and 356, and label them in accordance with the labeling on the oscillogram.

21. Three-phase line voltages of 230 volts are impressed on a balanced wye load having 16 ohms resistance and 12 ohms reactance in series in each phase. Find the line current and total power. If the three impedances are reconnected in delta and placed across the same line voltages, what are the line and phase currents and the total power?

22. A current of 10 amperes flows in the lines to a twelve-phase mesh-connected load having 5 ohms resistance and 8 ohms capacitive reactance in series in each phase. What is the voltage between alternate lines on the load? Draw the vector diagram of the voltages and phase currents of two adjacent phases, and also show the line current from the junction of these two phases.

23. A balanced wye load consists of 3 ohms resistance and 4 ohms capacitive reactance in series per phase. Balanced three-phase voltages of 100 volts each are impressed across the lines at the load. If the load is connected to a generator through three lines of equal impedance, each line containing a resistance of 1 ohm and an inductive reactance of 4 ohms, find the voltage at the generator terminals.

24. A balanced wye load having 8 ohms resistance and 6 ohms inductive reactance in series in each phase is supplied through lines each having 1 ohm resistance and 2 ohms inductive reactance. If the sending-end voltage between lines is 250 volts, what will be the voltage between lines at the load?

25. A balanced delta load contains a resistance of 12 ohms and a capacitive reactance of 16 ohms in series in each phase. If the balanced impressed line voltages on the load are 115 volts each, calculate the line and phase currents.

26. A balanced delta load having 18 ohms resistance and 24 ohms capacitive reactance in series in each phase is supplied through lines each having 1 ohm resistance and 2 ohms inductive reactance. If the line-to-line voltage at the sending end is 250 volts, find the line-to-line voltage at the load terminals. Also find the total power consumed by the load.

27. A balanced wye inductive load takes 5.4 kw at 0.6 power factor at a line voltage of 200 volts. It is in parallel with a pure resistive balanced wye load taking 5 kw. Find the resultant line current supplied the combination.

28. The total power supplied two balanced three-phase loads in parallel is 12 kw at 0.8 power factor lagging. One of the loads takes 10 kva at 0.8 power-factor lead. The second load is a delta-connected balanced load. Find the res stance and reactance per phase of the delta load if the line voltage is 230 volts. If the unknown load were wye-connected, what would be the resistance and reactance per phase?

29. Each phase of a delta load has 6 ohms resistance and 9 ohms capacitive reactance in series. Each phase of a wye load has 8 ohms resistance and 6 ohms inductive reactance in series. The two loads are connected in parallel across three-phase line voltages of 100 volts. Calculate the resultant line current, the total power consumed, and the power factor of the combination.

30. A three-phase, 5-hp, 220-volt induction motor (balanced load) has an efficiency of 86 per cent and operates at 86.6 per cent lagging power factor. It is paralleled with a three-phase resistance furnace consisting of three 36-ohm resistances connected in delta. Find the kilovolt-amperes demanded by the combination, the power factor, and the line current.

31. A three-phase generator supplies balanced voltages of 230 volts each at it terminals when it carries a load which requires 10 amperes. If the power facto at the generator terminals is 0.8 leading, calculate the voltage at the load if the loa is connected through lines each having 1 ohm resistance and 5 ohms inductive re actance.

32. A balanced three-phase load requires 10 kva at 0.5 lagging power facto Find the kva size of a condenser bank which may be paralleled with the load t bring the power factor of the combination to 0.866 lag, and also to 0.866 lead.

33. If the line voltage for Problem 32 is 230 volts and the frequency 60 cycle find the capacitance in microfarads of capacitors required in each phase of th capacitor bank if they are delta-connected. What capacitance is required if the are wye-connected?

34. Three $15\underline{/60^\circ}$-ohm load impedances are connected in delta and supplied b lines, each line containing 1 ohm resistance and 1 ohm inductive reactance. If th line voltages on the supply side of the line impedances are balanced three-phase o 115 volts each, find the voltage across the load impedances. Also calculate the power loss in the supply lines and the power dissipated by the load itself.

35. If the current through each of the load impedances in Problem 34 is 20 amperes, find the required voltage on the supply side of the line impedances.

36. A three-phase line has three capacitors, each having a reactance of 300 ohms connected in delta across the lines at the source. Three similar capacitors are so connected between the lines at the load. Between these two sets of capacitors each line has a series inductive reactance of 10 ohms. If a balanced three-phase load of 100 kva at 0.6 power-factor lag requires 2300 volts between lines, what voltage between lines will be required at the source? What will be the power input to the lines and the power factor at the source?

37. The motor M in Fig. 54 has 2300 volts balanced three-phase voltages impressed at its terminals and takes 120 kva at 0.6 leading power factor. Calculate the line volts, power input, and the power factor at a, b, c.

38. If the motor in Fig. 54 is removed from the circuit and balanced three-phase line voltages of 2300 volts each are impressed at a, b, and c, how many volts will appear between lines at the motor end of the line?

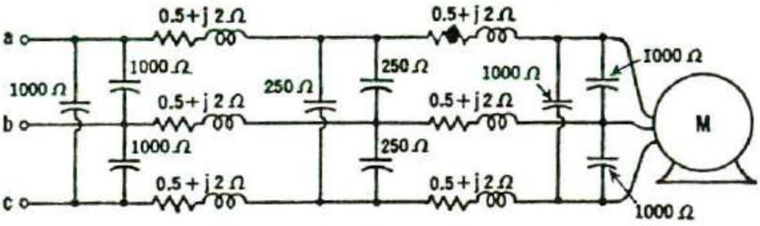


FIG. 54. See Problems 37 and 38.

39. A three-phase resonant shunt is connected to three-phase, 2300-volt lines to furnish a low impedance for a certain frequency so as to reduce the inductive interference with a telephone line. The shunt consists of three 10-kva, 60-cycle, 2300-volt capacitors connected in delta. In series with each line terminal from the delta is an inductance of 2.5 millihenrys. At what frequency does this three-phase combination resonate, that is, offer minimum impedance? Assume that resistances of capacitors and inductances are negligible.

40. (*a*) Three coils each having 36 ohms resistance and 100 millihenrys inductance are connected in delta. Find the microfarad capacitance of each capacitor which may be placed in each of the three lines from the delta to produce resonance (unity p.f.) of the system as a whole for a frequency of 800 cycles. This is a type of resonant shunt sometimes connected to power lines to reduce inductive interference with telephone circuits.

(*b*) Assume th · the capacitors calculated for each line in (*a*) are removed and connected in delta. Find how many henrys of inductance would be required in each line from this delta to bring the power factor of the combination to unity at 800 cycles.

41. Find the readings of W_a and W_b in Fig. 55 for the sequence V_{na}, V_{nc}, V_{nb}. Find the power dissipated in each phase.

42. A balanced three-phase load takes 5 kw and 20 reactive kva. Find the readings of two wattmeters properly connected to measure the total power.

43. In Fig. 55 find the reading of W_R. Also calculate the total reactive volt-amperes taken by the load. What is the ratio of the total reactive volt-amperes taken to the reading of W_R?

44. Prove that the ratio of the reading of W_R of Fig. 55 to the total reactive volt-amperes obtained in Problem 43 will obtain for all balanced loads when the impressed voltages are sinusoidal balanced three-phase.

45. (*a*) Calculate analytically the power-factor angle for a balanced three-phase circuit in which two wattmeters properly connected to measure three-phase power read +1000 and +800 watts, respectively.

(*b*) Also calculate the angle if the meters read +1000 and −800 watts, respectively.

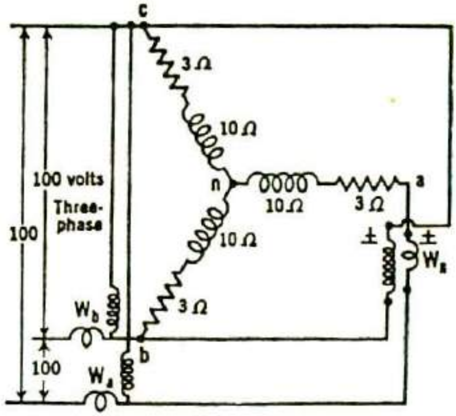


FIG. 55. See Problems 41, 43, and 44.

46. Two wattmeters measuring power to a balanced three-phase load read 1200 and −400 watts, respectively. How many volt-amperes does the load take? At what power factor?

47. The power to a balanced three-phase leading-power-factor load is measured by two wattmeters. The wattmeter having its current coil in line A and its potential coil from line A to line C indicates +1000 watts. The other wattmeter with its current coil in line B and its potential coil from line B to line C indicates +400 watts. What is the voltage sequence? What is the power factor of the load?

48. Each phase of a balanced twelve-phase star-connected load consists of 3 ohms resistance and 4 ohms inductive reactance in series. Balanced twelve-phase line voltages of 51.76 volts between adjacent lines are applied to the load. Calculate the line current, power factor, and total power consumed by the load.

49. The voltage induced in phase na of a three-phase wye-connected generator is

$$e_{na} = 127 \sin \omega t + 50 \sin (3\omega t - 30^\circ) + 30 \sin (5\omega t + 40^\circ)$$

If the sequence is e_{na}, e_{nb}, e_{nc}, find the equation with respect to time of the line voltage e_{ab}. *Note:* Phase voltages of polyphase generators differ only in phase angle.

50. If the phases of the generator in Problem 49 are reconnected in delta, what will be the equation with respect to time of the line voltage across phase na?

51. A wye-connected generator has a generated voltage per phase which contains only the fundamental, third, fifth, and seventh harmonics. The line voltage as measured by a voltmeter is 230 volts; the voltage to neutral is 160 volts. Calculate the magnitude of the third harmonic in the generated voltage.

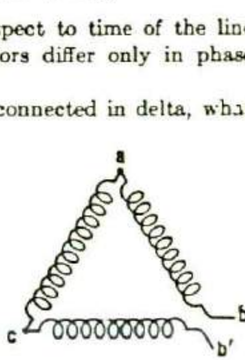


FIG. 56. See Problems 52 and 53.

52. The induced emf of a delta generator with one corner of the delta open as shown in Fig. 56 contains only odd harmonics up to the seventh. A voltmeter across ac reads 2500 volts, and, across bb' when negligible current flows, 1800 volts. Find the reading of a voltmeter connected from a to b'.

53. The induced phase voltage of a delta generator with one corner open as shown in Fig. 56 contains odd harmonics up to the seventh. A voltmeter connected from a to b' reads 2500 volts, and from a to c it reads 2200 volts when negligible current flows. What should it read from b to b'?

54. Figure 57 shows a generator connected to a balanced pure resistance load. An ammeter in the neutral reads 15 amperes, and the wattmeter shown reads 600

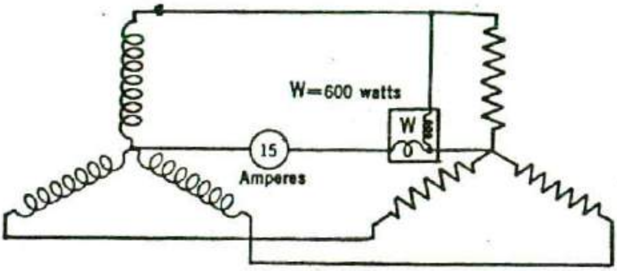


FIG. 57. See Problem 54.

watts. A voltmeter shows a balanced line voltage of 230 volts. Find the line currents to the load and the voltage from line to neutral at the load, assuming that the generated voltage contains only fundamental and third-harmonic components.

ɪapter IX Unbalanced Polyphase Circuits

Unbalanced Loads. The previous chapter developed the method of ;alculating the currents in the various branches of balanced polyphase oads when the impedances and impressed voltages are known. In the ɔresent chapter, methods of calculating the various branch currents will ɔe developed when known voltages are impressed upon unbalanced oads. Any polyphase load in which the impedance in one or more ɔhases differs from those of other phases is said to be unbalanced. Even :hough the load impedances of the various phases are identical, one of :he methods of calculating unbalanced loads must be employed if the voltages impressed on the load are unequal and differ in phase by angles which are not equal. Some of the simpler types of unbalanced loads which are solvable by rather simple direct methods will be considered first.

Unbalanced Delta Loads. If the three-phase line voltages across the terminals of an unbalanced delta load are fixed, the voltage drop across each phase impedance is known. The currents in each phase can therefore be determined directly. The line currents can be found by adding vectorially the two component currents coming toward or flowing away from the line terminal in question as was done in series-parallel circuit analysis. The following example will illustrate the procedure.

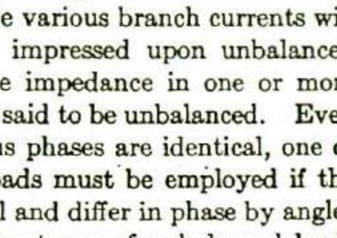


FIG. 1. Unbalanced delta load. See example 1.

Example 1. Given the unbalanced delta load shown in Fig. 1. Calculate all currents for the three-phase balanced voltages shown on the figure, if the voltage sequence is *ab–ca–bc*.

Since the voltages shown are assumed to be maintained at the terminals *a*, *b*, and *c*, the complex expressions for the phase voltages may be established. Take some phase voltage as a reference, say $\mathbf{V}_{ab}$ for this example. Therefore,

$$\mathbf{V}_{ab} = 100 + j0$$

$$\mathbf{V}_{bc} = 100 \underline{/120^\circ} = -50 + j86.6$$

$$\mathbf{V}_{ca} = 100 \underline{/-120^\circ} = -50 - j86.6 \text{ volts}$$

Then

$$\mathbf{I}_{ab} = \frac{\mathbf{V}_{ab}}{\mathbf{Z}_{ab}} = \frac{100 + j0}{6 + j8} = 6 - j8 = 10 \underline{/-53.1^\circ} \text{ amperes}$$

$$\mathbf{I}_{bc} = \frac{\mathbf{V}_{bc}}{\mathbf{Z}_{bc}} = \frac{-50 + j86.6}{4 - j3} = -18.39 + j7.856 = 20 \underline{/156.9^\circ} \text{ amperes}$$

$$\mathbf{I}_{ca} = \frac{\mathbf{V}_{ca}}{\mathbf{Z}_{ca}} = \frac{-50 - j86.6}{20 + j0} = -2.5 - j4.33 = 5 \underline{/-120^\circ} \text{ amperes}$$

The line currents are:

$$\mathbf{I}_{a'a} = \mathbf{I}_{ab} + \mathbf{I}_{ac} = 6 - j8 + 2.5 + j4.33 = 8.5 - j3.67$$
$$= 9.26 \underline{/-23.4^\circ} \text{ amperes}$$

$$\mathbf{I}_{b'b} = \mathbf{I}_{ba} + \mathbf{I}_{bc} = -6 + j8 - 18.39 + j7.856$$
$$= -24.39 + j15.856 = 29 \underline{/146.9^\circ} \text{ amperes}$$

$$\mathbf{I}_{c'c} = \mathbf{I}_{ca} + \mathbf{I}_{cb} = -2.5 - j4.33 + 18.39 - j7.856$$
$$= 15.89 - j12.186 = 20 \underline{/-37.3^\circ} \text{ amperes}$$

Unbalanced Wye Loads. If the load voltages at the terminals a, b, and c of an unbalanced wye load like that shown in Fig. 2 can be assumed to remain constant at their specified values, then the phase currents of an equivalent delta which replaces the wye can be found directly as shown in example 1. The line currents to this equivalent delta are obviously the currents in the phases of the wye load.

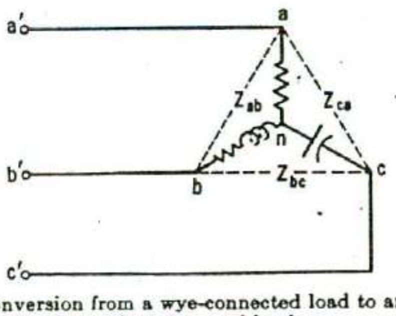


FIG. 2. Conversion from a wye-connected load to an equivalent delta-connected load.

Example 2. A balanced set of three-phase voltages is connected to an unbalanced set of wye-connected impedances as shown in Fig. 2. The following values are assumed to be known:

$\mathbf{V}_{ab} = 212 \underline{/90^\circ}$ volts $\qquad \mathbf{Z}_{an} = 10 + j0$ ohms

$\mathbf{V}_{bc} = 212 \underline{/-150^\circ}$ volts $\qquad \mathbf{Z}_{bn} = 10 + j10$ ohms

$\mathbf{V}_{ca} = 212 \underline{/-30^\circ}$ volts $\qquad \mathbf{Z}_{cn} = 0 - j20$ ohms

The line currents $\mathbf{I}_{a'a}$, $\mathbf{I}_{b'b}$, and $\mathbf{I}_{c'c}$ are to be determined by the wye to delta conversion method. (See Chapter V, page 210, for the general theory involved in making wye to delta conversions.)

In Fig. 2 the equivalent delta impedances may be expressed in terms of the wye impedances as follows:

$$Z_{ab} = \frac{(Z_{an}Z_{bn} + Z_{bn}Z_{cn} + Z_{cn}Z_{an})}{Z_{nc}} = \frac{S}{Z_{nc}}$$

$$Z_{bc} = \frac{S}{Z_{an}} \quad \text{and} \quad Z_{ca} = \frac{S}{Z_{bn}}$$

Numerically, the equivalent delta impedances are:

$$Z_{ab} = \frac{300 - j300}{0 - j20} = (15 + j15) = 21.2 \,/\underline{45^\circ} \text{ ohms}$$

$$Z_{bc} = \frac{300 - j300}{10 - j0} = (30 - j30) = 42.4 \,/\underline{-45^\circ} \text{ ohms}$$

$$Z_{ca} = \frac{300 - j300}{10 + j10} = (0 - j30) = 30.0 \,/\underline{-90^\circ} \text{ ohms}$$

The load currents in the equivalent delta are:

$$I_{ab} = \frac{V_{ab}}{Z_{ab}} = \frac{212 \,/\underline{90^\circ}}{21.2 \,/\underline{45^\circ}} = 10 \,/\underline{45^\circ} \text{ amperes}$$

$$I_{bc} = \frac{V_{bc}}{Z_{bc}} = \frac{212 \,/\underline{-150^\circ}}{42.4 \,/\underline{-45^\circ}} = 5.0 \,/\underline{-105^\circ} \text{ amperes}$$

$$I_{ca} = \frac{V_{ca}}{Z_{ca}} = \frac{212 \,/\underline{-30^\circ}}{30 \,/\underline{-90^\circ}} = 7.07 \,/\underline{60^\circ} \text{ amperes}$$

The actual line and load currents are:

$$\begin{aligned} I_{a'a} &= I_{ab} - I_{ca} \\ &= 10 \,/\underline{45^\circ} - 7.07 \,/\underline{60^\circ} = 3.66 \,/\underline{15^\circ} \text{ amperes} \\ I_{b'b} &= I_{bc} - I_{ab} \\ &= 5 \,/\underline{-105^\circ} - 10 \,/\underline{45^\circ} = 14.56 \,/\underline{-125.1^\circ} \text{ amperes} \\ I_{c'c} &= I_{ca} - I_{bc} \\ &= 7.07 \,/\underline{60^\circ} - 5 \,/\underline{-105^\circ} = 11.98 \,/\underline{66.2^\circ} \text{ amperes} \end{aligned}$$

As a single check on the above arithmetic let the calculated value of $[I_{a'a}Z_{an} - I_{b'b}Z_{bn}]$ be compared with the originally specified value of V_{ab}, which was $212 \,/\underline{90^\circ}$ volts.

$$\begin{aligned} [I_{a'a}Z_{an} - I_{b'b}Z_{bn}] &= (35.4 + j9.48) - (35.35 - j202.6) \\ &= (0.05 + j212.1) \text{ volts} \qquad (\textit{Check}) \end{aligned}$$

The conversion of a wye to its equivalent delta along with the solution of the delta as illustrated in the above example will usually require an equal or greater amount of work than the direct solution of the wye employing two simultaneous equations obtained by the application of Kirchhoff's laws.

Vector diagrams of the voltages and currents involved in the foregoing example are given in Fig. 3.

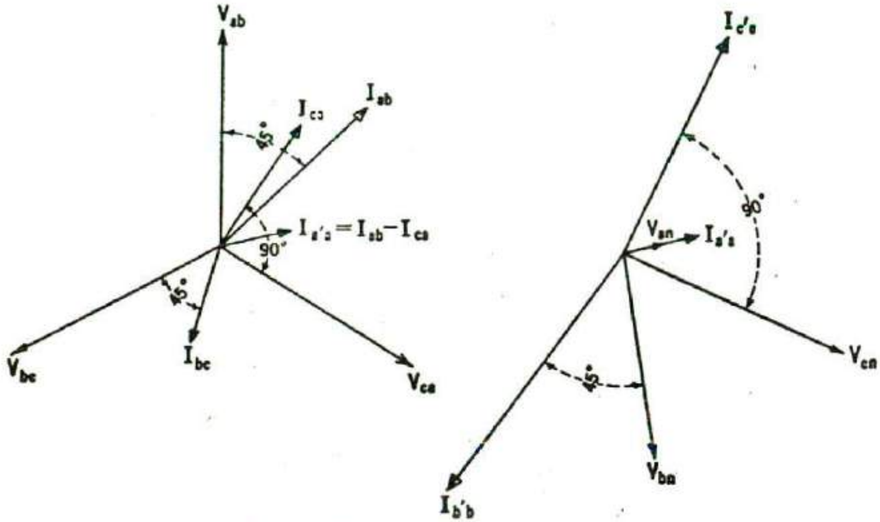


FIG. 3. Vector diagrams for example 2.

Problem 1. Determine the values of V_{an}, V_{bn}, and V_{cn} in example 2.

Ans.: $V_{an} = 36.6 \underline{/15°}$; $V_{bn} = 205.6 \underline{/-80.1°}$; $V_{cn} = 239.6 \underline{/-23.8°}$ volts.

Problem 2. Determine the power dissipated in each of the three phases (*an*, *bn*, and *cn*) of example 2.

Ans.: $P_{an} = 134$; $P_{bn} = 2120$; $P_{cn} = 0$ watts.

Problem 3. Find the magnitudes of $I_{a'a}$, $I_{b'b}$, and $I_{c'c}$ in Fig. 2 if $V_{ab} = 212 \underline{/90°}$, $V_{bc} = 212 \underline{/-30°}$, and $V_{ca} = 212 \underline{/-150°}$ volts. As in example 2, $Z_{an} = (10 + j0)$, $Z_{bn} = (10 + j10)$, and $Z_{cn} = (0 - j20)$ ohms.

Ans.: $I_{a'a} = 13.65$; $I_{b'b} = 6.20$; $I_{c'c} = 7.54$ amperes.

Combined Delta and Wye Loads. Delta-connected loads are sometimes operated in conjunction with wye-connected loads as shown in Fig. 4. If the three-phase, line-to-line voltages V_{ab}, V_{bc}, and V_{ca} remain sensibly constant irrespective of load conditions, a relatively simple solution may be effected by first converting the wye load to an equivalent delta load. The two parallel deltas may then be combined to form a single equivalent delta-connected load and the equivalent delta currents calculated directly as

$$I_{ab(eq)} = \frac{V_{ab}}{Z_{ab(eq)}} \quad I_{bc(eq)} = \frac{V_{bc}}{Z_{bc(eq)}} \quad I_{ca(eq)} = \frac{V_{ca}}{Z_{ca(eq)}}$$

The above currents may be combined in the usual manner to find the line currents $\mathbf{I}_{a'a}$, $\mathbf{I}_{b'b}$, and $\mathbf{I}_{c'c}$. The details are reserved for student analysis. (See Problem 15, page 404.)

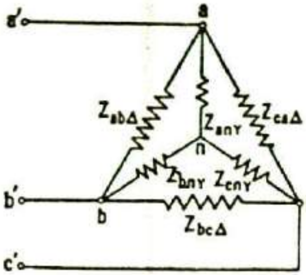


FIG. 4. Delta and wye loads on the same system of voltages.

Network Solutions. The solutions of unbalanced polyphase circuits are simply applications of Kirchhoff's laws. Some of the details are illustrated in the following example which refers to Fig. 5.

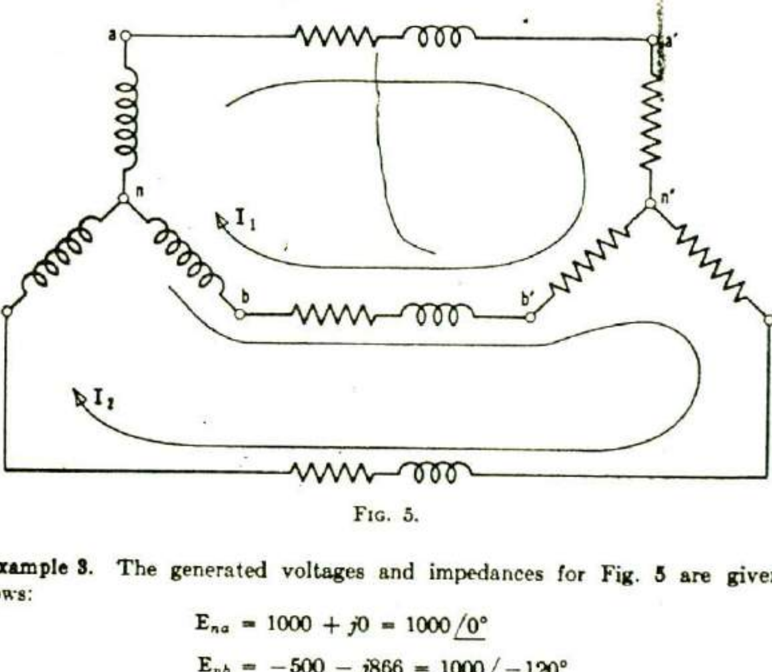


FIG. 5.

Example 3. The generated voltages and impedances for Fig. 5 are given as follows:

$$\mathbf{E}_{na} = 1000 + j0 = 1000\,\underline{/0^\circ}$$

$$\mathbf{E}_{nb} = -500 - j866 = 1000\,\underline{/-120^\circ}$$

$$\mathbf{E}_{nc} = -500 + j866 = 1000\,\underline{/-240^\circ}$$

$\mathbf{Z}_{na} = 2 + j8$, $\mathbf{Z}_{aa'} = 1 + j2$, $\mathbf{Z}_{a'n'} = 19 + j18 = 26.2\,\underline{/43.45^\circ}$, $\mathbf{Z}_{nb} = 2 + j8$, $\mathbf{Z}_{bb'} = 1 + j2$, $\mathbf{Z}_{b'n'} = 49 - j2 = 49.04\,\underline{/-2.34^\circ}$, $\mathbf{Z}_{nc} = 2 + j8$, $\mathbf{Z}_{cc'} = 1 + j2$, and $\mathbf{Z}_{c'n'} = 29 + j50 = 57.8\,\underline{/59.9^\circ}$.

In unbalanced polyphase circuits specification of the sequence employed is important because different solutions result from the two possible voltage sequences. For this example the sequence *abc* is assumed. This means that voltage of phase *b* lags that of phase *a* by 120°. All impedances in series are additive. Therefore the impedance of $naa'n'$ is $\mathbf{Z}_a = 2 + j8 + 1 + j2 + 19 + j18 = 22 + j28 = 35.6\,\underline{/51.8^\circ}$ ohms.

Likewise $\mathbf{Z}_b = 52 + j8 = 52.6\underline{/8.8^\circ}$ and $\mathbf{Z}_c = 32 + j60 = 68.0\underline{/61.9^\circ}$. The mesh-current solution will be illustrated first and for this solution the labeling of mesh currents is shown in Fig. 5. The equations are

$$(\mathbf{Z}_a + \mathbf{Z}_b)\mathbf{I}_1 - \mathbf{Z}_b\mathbf{I}_2 = \mathbf{E}_{na} + \mathbf{E}_{bn} = \mathbf{E}_{na} - \mathbf{E}_{nb} \quad (1)$$

$$(\mathbf{Z}_b + \mathbf{Z}_c)\mathbf{I}_2 - \mathbf{Z}_b\mathbf{I}_1 = \mathbf{E}_{nb} + \mathbf{E}_{cn} = \mathbf{E}_{nb} - \mathbf{E}_{nc} \quad (2)$$

Inserting the numerical values in the above two equations gives

$$(74 + j36)\mathbf{I}_1 - (52 + j8)\mathbf{I}_2 = 1500 + j866 \quad (3)$$

$$-(52 + j8)\mathbf{I}_1 + (84 + j68)\mathbf{I}_2 = -j1732 \quad (4)$$

$$\mathbf{I}_1 = \frac{\begin{vmatrix} (1500 + j866) & -(52 + j8) \\ -j1732 & (84 + j68) \end{vmatrix}}{\begin{vmatrix} (74 + j36) & -(52 + j8) \\ -(52 + j8) & (84 + j68) \end{vmatrix}} = 16.0\underline{/-34.9^\circ} \text{ amperes} = \mathbf{I}_{aa'}$$

$$\mathbf{I}_2 = \frac{\begin{vmatrix} (74 + j36) & (1500 + j866) \\ (52 + j8) & -j1732 \end{vmatrix}}{\begin{vmatrix} (74 + j36) & -(52 + j8) \\ (52 + j8) & (84 + j68) \end{vmatrix}} = 20.7\underline{/-109.2^\circ} \text{ amperes} = \mathbf{I}_{c'c}$$

$$\mathbf{I}_{bb'} = -\mathbf{I}_1 + \mathbf{I}_2 = -16\underline{/-34.9^\circ} + 20.7\underline{/-109.2^\circ} = 22.5\underline{/-152.5^\circ} \text{ amperes}$$

The voltage drops at the load may now be determined as

$$\mathbf{V}_{a'n'} = \mathbf{I}_{aa'}\mathbf{Z}_{a'n'} = 16\underline{/-34.9^\circ}\ \ 26.2\underline{/43.45^\circ} = 419\underline{/8.55^\circ} \text{ volts}$$

$$\mathbf{V}_{b'n'} = \mathbf{I}_{bb'}\mathbf{Z}_{b'n'} = 22.5\underline{/-152.5^\circ}\ \ 49.04\underline{/-2.34^\circ} = 1105\underline{/-154.84^\circ} \text{ volts}$$

$$\mathbf{V}_{c'n'} = \mathbf{I}_{cc'}\mathbf{Z}_{c'n'} = 20.7\underline{/-109.2^\circ}\ \ 57.8\underline{/59.9^\circ} = 1197\underline{/-49.3^\circ} \text{ volts}$$

The line-to-line voltages at the load are obtained by adding the voltages encountered in tracing through the load circuit from one line to the other as follows:

$$\begin{aligned}\mathbf{V}_{a'b'} &= \mathbf{V}_{a'n'} + \mathbf{V}_{n'b'} = \mathbf{V}_{a'n'} - \mathbf{V}_{b'n'} = 419\underline{/8.55^\circ} - 1105\underline{/-154.84^\circ} \\ &= 1512\underline{/20.6^\circ} \text{ volts}\end{aligned}$$

$$\mathbf{V}_{b'c'} = \mathbf{V}_{b'n'} + \mathbf{V}_{n'c'} = 1835\underline{/166.2^\circ} \text{ volts}$$

$$\mathbf{V}_{c'a'} = \mathbf{V}_{c'n'} + \mathbf{V}_{n'a'} = 1039\underline{/-69.3^\circ} \text{ volts}$$

The above line voltages could be calculated from the generated voltage and line drops Thus the application of Kirchhoff's voltage law gives

$$\mathbf{E}_{bn} + \mathbf{E}_{na} = \mathbf{I}_{aa'}(\mathbf{Z}_{na} + \mathbf{Z}_{aa'}) + \mathbf{V}_{a'b'} + \mathbf{I}_{b'b}(\mathbf{Z}_{bb'} + \mathbf{Z}_{nb})$$

or

$$\begin{aligned}\mathbf{V}_{a'b'} &= (\mathbf{E}_{bn} + \mathbf{E}_{na}) - \mathbf{I}_{aa'}(\mathbf{Z}_{na} + \mathbf{Z}_{aa'}) - \mathbf{I}_{b'b}(\mathbf{Z}_{bb'} + \mathbf{Z}_{nb}) \\ &= 1500 + j866 - 16.0\underline{/-34.9^\circ}\ (3 + j10) + 22.5\underline{/-152.5^\circ}\ (3 + j10) \\ &= 1413.2 + j531.6 = 1512\underline{/20.6^\circ} \text{ volts} \qquad \textit{(Check)}\end{aligned}$$

This calculation indicates that line and generator drops can be subtracted from the generated voltages to obtain the load voltages but the computation must be made with due regard to the proper phase of all quantities. Power in any branch is obtained in the usual way from the voltage and current in the particular branch.

The phasor diagrams of all voltages and currents may be obtained by plotting the complex quantities calculated for this example.

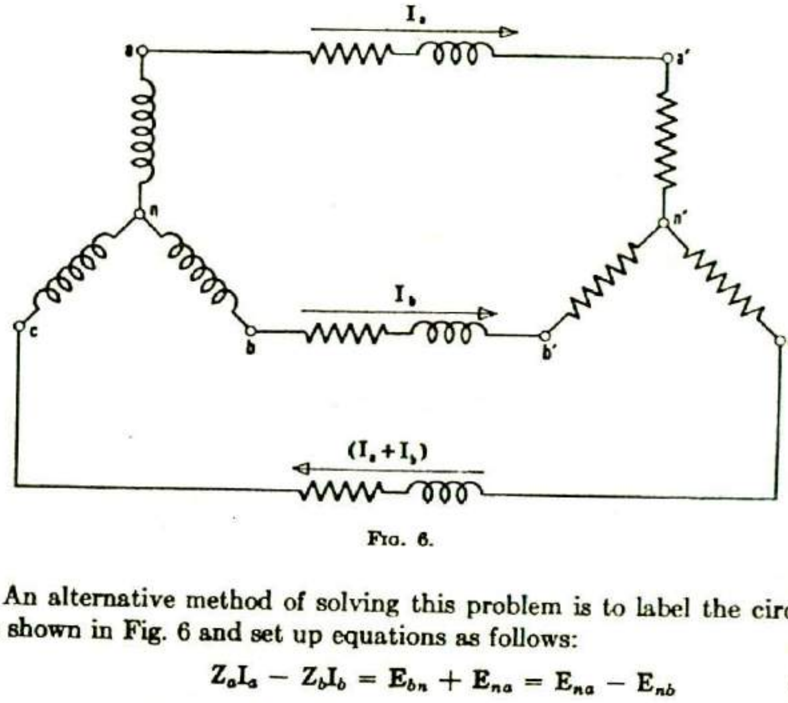


FIG. 6.

An alternative method of solving this problem is to label the circuit as shown in Fig. 6 and set up equations as follows:

$$\mathbf{Z}_a\mathbf{I}_a - \mathbf{Z}_b\mathbf{I}_b = \mathbf{E}_{bn} + \mathbf{E}_{na} = \mathbf{E}_{na} - \mathbf{E}_{nb} \tag{5}$$

$$\mathbf{Z}_b\mathbf{I}_b + \mathbf{Z}_c(\mathbf{I}_a + \mathbf{I}_b) = \mathbf{E}_{cn} + \mathbf{E}_{nb} = \mathbf{E}_{nb} - \mathbf{E}_{nc} \tag{6}$$

or

$$\mathbf{Z}_c\mathbf{I}_a + (\mathbf{Z}_b + \mathbf{Z}_c)\mathbf{I}_b = \mathbf{E}_{nb} - \mathbf{E}_{nc} \tag{6a}$$

Equations (5) and (6a) may be solved for the currents. This method is equivalent to the loop-current method, previously demonstrated. As a matter of fact if the current $\mathbf{I}_{aa'}$ in Fig. 6 were labeled $\mathbf{I}_1$ the current $\mathbf{I}_{c'c}$ labeled $\mathbf{I}_2$, and $\mathbf{I}_{b'b}$ labeled $(\mathbf{I}_1 - \mathbf{I}_2)$, equations identical with (1) and (2) would result if the same loops are employed.

Positive Circuit Directions. A great deal of needless confusion exists in the minds of many students relative to the correct positive circuit directions of the quantities involved in polyphase circuit analysis. The basic principles concerning circuit direction have been presented in the earlier chapters. (See pages 95–96, 284–285, and 327.) These principles are, of course, entirely applicable to polyphase circuits as well as to single-phase circuits.

In general, all generated emf's in polyphase systems have specified *lative* polarities and angular positions with respect to one another. his information must be known either directly or indirectly if the circuit investigation is to proceed. For example, if a three-phase alternator is connected in wye it may be assumed that the individual phases are connected subtractively at a common junction as shown in Fig. 7.

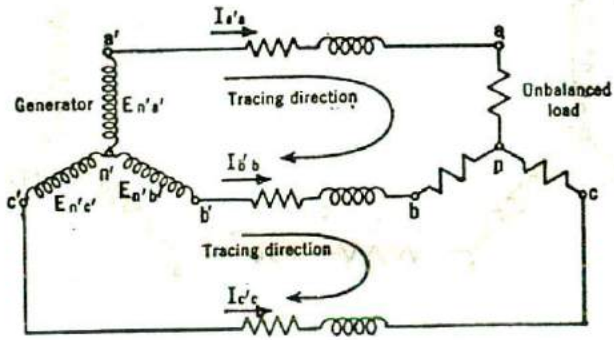


FIG. 7. A three-wire taree-phase network. (See pages 380–382.)

It is only by means of subtractive polarities that a three-phase, wye-connected machine can give balanced line-to-line voltages. Unless otherwise specified, the individual phase generated emf's of a three-phase machine may be assumed to be 120° apart in time phase. The foregoing facts are sufficient for a specification of the positive circuit directions in the network shown in Fig. 7.

A positive circuit direction may be arbitrarily assigned to any one generated emf. For example, if the *a* phase generated emf in Fig. 7 is considered, either $\mathbf{E}_{n'a'}$ or $\mathbf{E}_{a'n'}$ may be taken as positive. One of these having been selected as positive, the positive circuit directions of the other systematically labeled emf's are fixed because of the relatively fixed polarities that the generated emf's bear toward one another. If $\mathbf{E}_{n'a'}$ is taken as positive, then $\mathbf{E}_{n'b'}$ and $\mathbf{E}_{n'c'}$ are also taken as the positive circuit directions because only when all phase voltages are considered away from the neutral or when all are considered toward the neutral does the usual 120° phase angle between adjacent phase voltages in a three-phase system exist. Thus either of the two following systems of generated voltages may be employed in analyzing the network shown in Fig. 7.

(1) $$E_{n'a'}, E_{n'b'}, E_{n'c'}$$

or

(2) $$E_{a'n'}, E_{b'n'}, E_{c'n'}$$

With the generated voltage relations established the solution is effected by employing the same methods used to solve any network, two of which were illustrated in example 3.

The Wye-Wye System with Neutral Connection. Four-wire, three-phase systems similar to the one shown in Fig. 8 are sometimes employed in the transmission and distribution of electrical energy. The connection of the point n' of the wye-connected generator (or transformer bank) to the point n of the wye-connected load distinguishes Fig. 8 from the three-wire, three-phase system shown in Fig. 7.

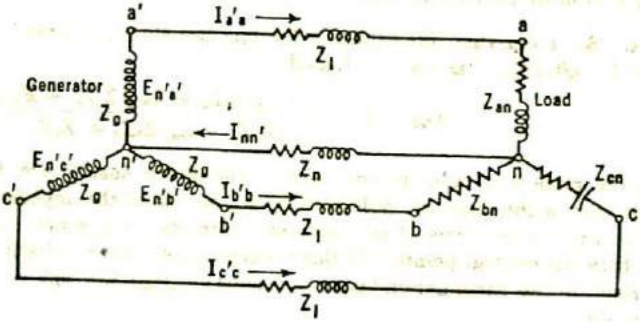


FIG. 8. A four-wire three-phase system.

In general, the details involved in solving for $I_{a'a}$, $I_{b'b}$, $I_{c'c}$, and $I_{nn'}$ of Fig. 8 are similar to those which have been presented for the wye-wye system without neutral connection. If the wye-wye system of Fig. 8 is solved straightforwardly by the determinant method, three-row, three-column matrices are encountered, and a considerable amount of labor is involved in effecting a complete solution in a perfectly general case. Because of the inherent symmetry of the basic voltage equations, however, several simplifications may be made. If, for example, Kirchhoff's emf law is applied to loops $n'a'ann'$, $n'b'bnn'$, and $n'c'cnn'$, it is plain that

$$I_{a'a} = \frac{E_{n'a'} - I_{nn'}Z_n}{(Z_g + Z_l + Z_{an})}; \quad I_{b'b} = \frac{E_{n'b'} - I_{nn'}Z_n}{(Z_g + Z_l + Z_{bn})};$$

$$I_{c'c} = \frac{E_{n'c'} - I_{nn'}Z_n}{(Z_g + Z_l + Z_{cn})}$$

Since

$$\mathbf{I}_{a'a} + \mathbf{I}_{b'b} + \mathbf{I}_{c'c} = \mathbf{I}_{nn'} \tag{7}$$

it follows that

$$\frac{\mathbf{E}_{n'a'} - \mathbf{I}_{nn'}\mathbf{Z}_n}{\mathbf{Z}_a} + \frac{\mathbf{E}_{n'b'} - \mathbf{I}_{nn'}\mathbf{Z}_n}{\mathbf{Z}_b} + \frac{\mathbf{E}_{n'c'} - \mathbf{I}_{nn'}\mathbf{Z}_n}{\mathbf{Z}_c} = \mathbf{I}_{nn'} \tag{8}$$

where, for simplicity in writing,

$$\mathbf{Z}_g + \mathbf{Z}_l + \mathbf{Z}_{an} = \mathbf{Z}_a \tag{9}$$

$$\mathbf{Z}_g + \mathbf{Z}_l + \mathbf{Z}_{bn} = \mathbf{Z}_b \tag{10}$$

$$\mathbf{Z}_g + \mathbf{Z}_l + \mathbf{Z}_{cn} = \mathbf{Z}_c \tag{11}$$

The remaining details are reserved for student analysis. (See Problem 4 below and Problem 16 at the close of the chapter.)

Problem 4. Solve equation (8) explicitly for $\mathbf{I}_{nn'}$ and state in words how to find $\mathbf{I}_{a'a}$, $\mathbf{I}_{b'b}$, and $\mathbf{I}_{c'c}$ after $\mathbf{I}_{nn'}$ has been evaluated.

$$Ans.: \quad \mathbf{I}_{nn'} = \frac{\mathbf{E}_{n'a'}\mathbf{Z}_b\mathbf{Z}_c + \mathbf{E}_{n'b'}\mathbf{Z}_c\mathbf{Z}_a + \mathbf{E}_{n'c'}\mathbf{Z}_a\mathbf{Z}_b}{\mathbf{Z}_a\mathbf{Z}_b\mathbf{Z}_c + \mathbf{Z}_n(\mathbf{Z}_b\mathbf{Z}_c + \mathbf{Z}_c\mathbf{Z}_a + \mathbf{Z}_a\mathbf{Z}_b)}.$$

Note: If the numerator and denominator of the above answer are divided by $\mathbf{Z}_a\mathbf{Z}_b\mathbf{Z}_c$, both sides of the equation multiplied by $\mathbf{Z}_n$, and all of the impedances of the right member written in terms of admittances, there results a simple formula for the voltage between neutral points. If this voltage is solved for initially, substitution of the result in the three unnumbered equations on page 381 will yield the line currents directly.

The Wye-Delta System. A three-phase, wye-connected generator is shown connected to a delta load in Fig. 9. The solution of this system

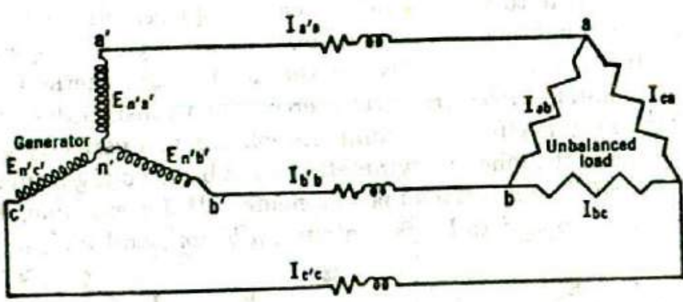


FIG. 9. A wye-delta circuit arrangement.

for currents in all branches may be effected by application of conventional Kirchhoff's laws which would require establishment of three emf equations and three current equations. Another method would consist of first converting the delta to an equivalent wye-connected load and then solving by employing two equations. However the least amount of work will usually be encountered if the loop-current method or the Kirchhoff's law equivalent employing three unknown currents is applied directly to the original circuit. This solution requires only three equations which are readily solved by determinants.

Phase-Sequence Effects. The direction of rotation of polyphase induction motors is dependent upon the phase sequence of the applied voltages. Also, the two wattmeters in the two-wattmeter method of measuring three-phase power interchange their readings when subjected to a reversal of phase sequence even though the system is balanced. But the magnitudes of the various currents and component voltages in balanced systems are not affected by a reversal of phase sequence.

In an unbalanced polyphase system, a reversal of voltage phase sequence will, in general, cause certain branch currents to change in magnitude as well as in time-phase position, although the total watts and vars generated remain the same. (See example following.)

Unless otherwise stated, the term " phase sequence " refers to voltage phase sequence. It should be recognized that, in unbalanced systems, the line currents and phase currents have their own phase sequence which may or may not be the same as the voltage sequence.

Example 4. The effects of reversal of voltage sequence upon the magnitudes of the currents in the wye-connected load of Fig. 2 are illustrated by the results of example 2 and of Problem 3.

For the *ab–ca–bc* voltage sequence of example 2, page 374,

$$I_{a'a} = 3.66,\ I_{b'b} = 14.56,\ \text{and}\ I_{c'c} = 11.98\ \text{amperes}$$

For the *ab–bc–ca* voltage sequence of Problem 3, page 376,

$$I_{a'a} = 13.65,\ I_{b'b} = 6.20,\ \text{and}\ I_{c'c} = 7.54\ \text{amperes}$$

Methods of Checking Voltage Phase Sequence. Sometimes in practice it becomes desirable and even necessary to know the phase sequence of a particular polyphase system. There are two general methods for checking voltage phase sequence: one based on direction of rotation of induction motors; the other, on unbalanced polyphase circuit phenomena.

Method One. Small polyphase induction motors which have previously been checked against a known phase sequence can be employed to test the phase sequence of a given system. In two- and three-phase systems, only two different phase sequences are possible, and consequently the direction in which the motor rotates can be used as an indicator of phase sequence. The principle of operation involves rotating magnetic field theory which rightfully belongs in the domain of a-c machinery.

Method Two. In general, any unbalanced set of load impedances can be employed as a voltage phase sequence checker. The different effects produced by changes in phase sequence can be determined theoretically, and when an effect peculiar to one sequence is noted in the actual installation, that effect can be used to designate the phase sequence of the system.

One of the most common devices for checking phase sequence in three-phase systems is the unbalanced circuit arrangement shown in Fig. 10. The three line wires, the voltage phase sequence of which is to be tested, are arbitrarily labeled. The free end of one lamp is connected to the line marked a. The other lamp is connected to line c, and the inductance coil is connected to line b as shown in Fig. 10. *If lamp 'a' is brighter than lamp 'c,' the phase sequence of the line-to-line voltages is ab–bc–ca. If lamp 'c' is brighter than lamp 'a,' the phase sequence is ab–ca–bc.*

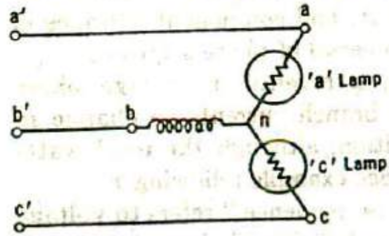


FIG. 10. A two-lamp method for checking phase sequence in three-phase systems. Lamp 'a' is brighter for ab–bc–ca sequence, lamp 'c' is brighter for ab–ca–bc sequence.

The foregoing statements are based upon the results of theoretical analyses, the details of which are outlined below. Assuming that the lamps are similar, their brightnesses will depend upon the voltages $Z_{an}I_{an}$ and $Z_{cn}I_{cn}$. These voltages may be determined by the Kirchhoff equation method as shown below:

$$\mathbf{I}_{an} + \mathbf{I}_{bn} + \mathbf{I}_{cn} = 0 \tag{12}$$

$$\mathbf{Z}_{an}\mathbf{I}_{an} - \mathbf{Z}_{bn}\mathbf{I}_{bn} = \mathbf{V}_{ab} \tag{13}$$

$$\mathbf{Z}_{bn}\mathbf{I}_{bn} - \mathbf{Z}_{cn}\mathbf{I}_{cn} = \mathbf{V}_{bc} \tag{14}$$

Upon the elimination of $\mathbf{I}_{cn}$ from equation (14), there results

$$\mathbf{Z}_{cn}\mathbf{I}_{an} + (\mathbf{Z}_{bn} + \mathbf{Z}_{cn})\mathbf{I}_{bn} = \mathbf{V}_{bc} \tag{15}$$

Equations (13) and (15) can now be solved by inspection for $\mathbf{I}_{an}$ and the result multiplied by $\mathbf{Z}_{an}$. The voltage across the a lamp is

$$\mathbf{Z}_{an}\mathbf{I}_{an} = \mathbf{Z}_{an}\left[\frac{\mathbf{V}_{ab}(\mathbf{Z}_{bn} + \mathbf{Z}_{cn}) + \mathbf{V}_{bc}\mathbf{Z}_{bn}}{\mathbf{Z}_{an}(\mathbf{Z}_{bn} + \mathbf{Z}_{cn}) + \mathbf{Z}_{cn}\mathbf{Z}_{bn}}\right] \tag{16}$$

The voltage across the *c* lamp is

$$\mathbf{Z}_{cn}\mathbf{I}_{cn} = \mathbf{V}_{ca} + \mathbf{Z}_{an}\mathbf{I}_{an} \tag{17}$$

Example 5. For the sake of illustrating the effect of reversal of phase sequence upon the magnitudes of $\mathbf{Z}_{an}\mathbf{I}_{an}$ and $\mathbf{Z}_{cn}\mathbf{I}_{cn}$, a numerical case will be considered. The lamps $\mathbf{Z}_{an}$ and $\mathbf{Z}_{cn}$ of Fig. 10 will be assumed to be pure resistances each of 100 ohms magnitude. $\mathbf{Z}_{bn}$ will be assumed equal to $100\,/\underline{90°}$ ohms, that is, a hypothetically pure inductance. The magnitude of the line-to-line voltages will be taken as 100 volts each and will first be assigned the following vector positions:

$$\mathbf{V}_{ab} = 100\,/\underline{0°} \text{ volts}$$

$$\mathbf{V}_{bc} = 100\,/\underline{-120°} \text{ volts}$$

$$\mathbf{V}_{ca} = 100\,/\underline{-240°} \text{ volts}$$

Under these conditions

$$\mathbf{Z}_{an}\mathbf{I}_{an} = 100\,/\underline{0°}\left[\frac{(100\,/\underline{0°})\,(141.4\,/\underline{45°}) + (100\,/\underline{-120°})\,(100\,/\underline{90°})}{22{,}380\,/\underline{63.45°}}\right]$$

$$= 86.4\,/\underline{-48.45°} \text{ volts} \tag{18}$$

$$\begin{aligned}\mathbf{Z}_{cn}\mathbf{I}_{cn} &= (100\,/\underline{-240°}) + (86.3\,/\underline{-48.45°})\\ &= 23.2\,/\underline{71.55°} \text{ volts}\end{aligned} \tag{19}$$

The *a* lamp is therefore brighter than the *c* lamp for phase sequence *ab–bc–ca*.

Now let the line-to-line voltages be assigned vector positions which represent a reversal of phase sequence, namely,

$$\mathbf{V}_{ab} = 100\,/\underline{0°} \text{ volts}$$

$$\mathbf{V}_{bc} = 100\,/\underline{-240°} \text{ volts}$$

$$\mathbf{V}_{ca} = 100\,/\underline{-120°} \text{ volts}$$

For *ab–ca–bc* phase sequence

$$\mathbf{Z}_{an}\mathbf{I}_{an} = 100\,/\underline{0°}\left[\frac{(100\,/\underline{0°})\,(141.1\,/\underline{45°}) + (100\,/\underline{-240°})\,(100\,/\underline{90°})}{22{,}380\,/\underline{63.45°}}\right]$$

$$= 23.2\,/\underline{11.55°} \text{ volts} \tag{20}$$

$$\begin{aligned}\mathbf{Z}_{cn}\mathbf{I}_{cn} &= 100\,/\underline{-120°} + 23.2\,/\underline{11.55°}\\ &= 86.4\,/\underline{-108.45°} \text{ volts}\end{aligned} \tag{21}$$

The *c* lamp is therefore brighter than the *a* lamp for phase sequence *ab–ca–bc*. The above numerical results would be somewhat different if the resistance of the inductance coil had been considered. However, if the ratio (X_L/R) of the coil is relatively high, the difference between the lamp voltages is easily discernible.

Example 6. Another convenient form of voltage sequence checker is shown in Fig. 11*a*. It consists of a condenser (X_C), a resistor (R), and a voltmeter (Vm).

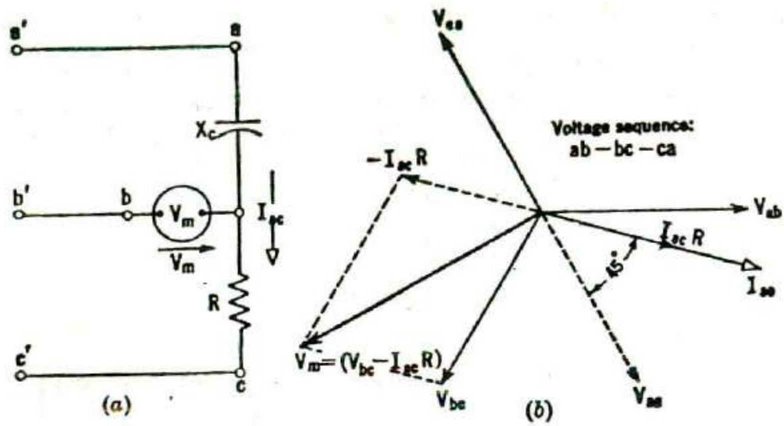


FIG. 11. A voltmeter method of checking phase sequence in three-phase systems. See example 6 and Problems 5 and 6.

The voltmeter (whose current consumption is negligibly small compared with the current through X_C and R) is connected between the line labeled b and the junction between X_C and R. X_C and R are connected in series across the voltage $\mathbf{V}_{ac}$ (or $\mathbf{V}_{ca}$) with the condenser connected to the a line and the resistor to the c line. If $X_C = 100$ ohms, $R = 100$ ohms, and $V_{ab} = V_{bc} = V_{ca} = 141.4$ volts,

$$\mathbf{I}_{ac} = \frac{141.4\underline{/-60^\circ}}{141.4\underline{/-45^\circ}} = 1\underline{/-15^\circ} \text{ amperes} \begin{cases}\text{for sequence } ab\text{–}bc\text{–}ca \text{ as shown} \\ \text{in Fig. 11}b.\end{cases}$$

$$\mathbf{V}_{bc} = \mathbf{Vm} + \mathbf{I}_{ac}R \quad \text{or} \quad \mathbf{Vm} = \mathbf{V}_{bc} - \mathbf{I}_{ac}R$$

$$\begin{aligned}\mathbf{Vm} &= (141.4\underline{/-120^\circ}) - (1\underline{/-15^\circ})\,(100\underline{/0^\circ}) \\ &= -167.3 - j96.6 = 193\underline{/-150^\circ} \text{ volts}\end{aligned}$$

The above result shows that the voltmeter (Vm) reads above the line voltage (in the ratio of 193 to 141 in this case) for voltage sequence ab–bc–ca. The same general result is obtained with any combination of X_C and R provided X_C is roughly equal in ohmic value to R or greater in ohmic value than R.

Problem 5. Show by means of a qualitative vector diagram that the voltmeter (Vm) of Fig. 11*a* reads below line voltage for voltage sequence ab–ca–bc.

Problem 6. What is the magnitude of the voltmeter reading in Fig. 11*a* if $X_C = 100$ ohms, $R = 100$ ohms, and $V_{ab} = V_{bc} = V_{ca} = 141.4$ volts if the voltage sequence is ab–ca–bc?

Ans.: 51.8 volts.

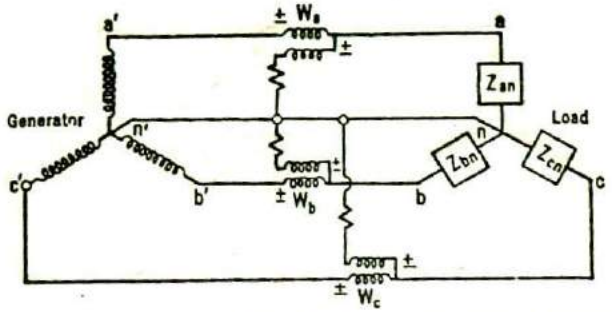


FIG. 12. The three-wattmeter method of measuring four-wire, three-phase power.

The Three-Wattmeter Method of Measuring Three-Phase Power. The total power delivered to a three-phase, wye-connected load with neutral connection can obviously be measured with three wattmeters connected as shown in Fig. 12. W_a measures the *an* phase power, W_b measures the *bn* phase power, and W_c measures the *cn* phase power. The sum of the three wattmeter readings therefore equals the total power consumed by the load. It is plain that if each individual phase of the wye-connected load is dissipative in character all the wattmeters shown in Fig. 12 will indicate positive power.

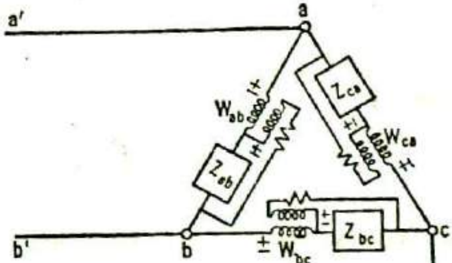


FIG. 13. The three-wattmeter method of measuring individual phase powers in a delta-connected load.

The total power absorbed by an unbalanced delta-connected load can be measured with the aid of three wattmeters as shown in Fig. 13. Individual phase powers are measured by the wattmeters. This method of measuring power would not, in general, be used unless the individual phase powers were desired.

The Two-Wattmeter Method of Measuring Three-Wire, Three-Phase Power. Except for inherent meter losses and errors, the three wattmeters connected as shown in Fig. 14 will measure accurately the power consumed by the three-phase load *abc*. A general proof of the foregoing statement will be given, and then certain important deductions will be made therefrom.

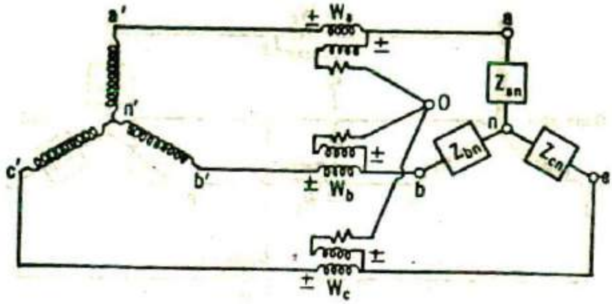


FIG. 14. A three-wattmeter method of measuring three-phase power which is independent of the potential and hence of the physical position of the point O.

The total average power delivered to the three-phase load shown in Fig. 14 over a time interval T is

$$P_{abc} = \frac{1}{T}\int_0^T (v_{an}i_{a'a} + v_{bn}i_{b'b} + v_{cn}i_{c'c})\,dt \tag{22}$$

The total average power measured by the three wattmeters shown in Fig. 14 is

$$P_{\text{meters}} = \frac{1}{T}\int_0^T (v_{a0}i_{a'a} + v_{b0}i_{b'b} + v_{c0}i_{c'c})\,dt \tag{23}$$

Under any condition it is plain that

$$v_{a0} = v_{an} - v_{0n} \tag{24}$$

$$v_{b0} = v_{bn} - v_{0n} \tag{25}$$

$$v_{c0} = v_{cn} - v_{0n} \tag{26}$$

Equation (23) may therefore be written as

$$P_{\text{meters}} = \frac{1}{T}\int_0^T (v_{an}i_{a'a} + v_{bn}i_{b'b} + v_{cn}i_{c'c})\,dt - \frac{1}{T}\int_0^T v_{0n}(i_{a'a} + i_{b'b} + i_{c'c})\,dt \tag{27}$$

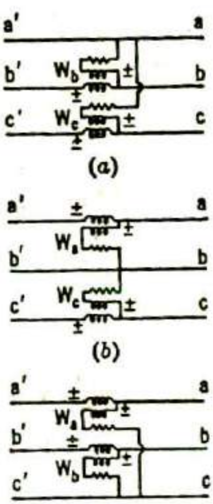
Since $(i_{a'a} + i_{b'b} + i_{c'c}) = 0$, it follows that

$$P_{\text{meters}} = \frac{1}{T}\int_0^T (v_{an}i_{a'a} + v_{bn}i_{b'b} + v_{cn}i_{c'c})\,dt \tag{28}$$

It is thus shown that the three wattmeters in Fig. 14 measure the load power irrespective of voltage or current balance, of wave form, and of the potential of the point O. The last fact is highly significant. It indicates that the wattmeter potential coils need not have equal resistances when employed as shown in Fig. 14. It also indicates that the point O can be placed on any one of the three lines, thereby reducing one wattmeter reading to zero. Although the proof was based on a wye-connected load, the entire proof holds equally well for delta-connected loads. A simple way of extending the proof to cover delta loads is to recognize the fact that any delta load can be reduced to an equivalent wye-connected load. (See Chapter V, pages 206–209.)

The practical significance of placing point O on one of the three lines is that only two wattmeters are required to measure the total three-phase power. This expedient is widely utilized in measuring three-wire, three-phase power because it possesses no inherent limitations as regards balance or wave form.

The two wattmeters used to measure three-phase power may be placed in the circuit as shown in Fig. 15a, b, or c. The three combina-



W_b reads $V_{ba} I_{b'b} \cos\theta \Big]_{I_{b'b}}^{V_{ba}}$

W_c reads $V_{ca} I_{c'c} \cos\theta \Big]_{I_{c'c}}^{V_{ca}}$

W_a reads $V_{ab} I_{a'a} \cos\theta \Big]_{I_{a'a}}^{V_{ab}}$

W_c reads $V_{cb} I_{c'c} \cos\theta \Big]_{I_{c'c}}^{V_{cb}}$

W_a reads $V_{ac} I_{a'a} \cos\theta \Big]_{I_{a'a}}^{V_{ac}}$

W_b reads $V_{bc} I_{b'b} \cos\theta \Big]_{I_{b'b}}^{V_{bc}}$

FIG. 15. Different circuit positions that the two wattmeters employed to measure three-phase power can take.

tions are obtained by placing the point O of Fig. 14 on lines a, b, and c, respectively.

For the relative polarities of the wattmeter coils shown in Figs. 14

and 15 the instruments will read up-scale if positive power is being metered. Under the condition of sinusoidal wave form of current and voltage, positive power is indicated if the current through the current coil in the ± direction is less than 90° out of phase with the voltage which is across the potential circuit in the ± direction. If one of the meters reads down-scale when connected as shown in Fig. 15, the relative polarity of the coils is changed to obtain up-scale reading and this reading is reckoned as negative power in finding the algebraic sum of the wattmeter readings.

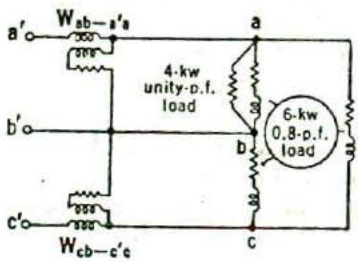


FIG. 16. A particular unbalanced three-phase load.

Example 7. In Fig. 16, *abc* represents a balanced three-phase system of voltages. The magnitude of each voltage is 200 volts, and the phase sequence is *ab–ca–bc*.

A balanced, 0.8-power-factor, induction motor load of 6 kw is connected across *abc* and a 4-kw, unity-power-factor load is connected across *ab* as shown in the diagram.

Let it be required to find the individual readings of the wattmeters, $W_{ab-a'a}$ and $W_{cb-c'c}$, which are connected to measure the total load power. The subscripts designate the voltage and current which are operative in a given meter in producing positive up-scale deflection. Obviously, the meter will read down-scale, thus indicating negative power if the operative voltage and current are separated by more than 90° in time phase.

Let $\mathbf{V}_{ab}$ be selected as reference. Then

$$\mathbf{V}_{ab} = 200\angle 0^\circ,\ \mathbf{V}_{bc} = 200\angle -240^\circ, \quad \text{and} \quad \mathbf{V}_{ca} = 200\angle -120^\circ \text{ volts}$$

The current in each phase of the induction motor is

$$I_\phi = \frac{2000}{200 \times 0.8} = 12.5 \text{ ampéres}$$

and these phase currents lag the applied phase voltages by $\cos^{-1} 0.8$ or 36.9°. The unity-power-factor load current is, of course, in phase with $\mathbf{V}_{ab}$. Therefore

$$\mathbf{I}_{ab} = \frac{4000}{200}\angle 0^\circ + 12.5\angle -36.9^\circ$$
$$= (20 + j0) + (10 - j7.5)$$
$$= (30 - j7.5) \text{ amperes}$$

$$\mathbf{I}_{bc} = 12.5\angle -240^\circ - 36.9^\circ = 12.5\angle 83.1^\circ$$
$$= (1.5 + j12.4) \text{ amperes}$$

$$\mathbf{I}_{ca} = 12.5\angle -120^\circ - 36.9^\circ = 12.5\angle -156.9^\circ$$
$$= (-11.5 - j4.90) \text{ amperes}$$

The line currents are

$$\mathbf{I}_{a'a} = (30 - j7.5) - (-11.5 - j4.90)$$
$$= 41.5 - j2.60 = 41.6\underline{/-3.58} \text{ amperes}$$
$$\mathbf{I}_{b'b} = (1.5 + j12.4) - (30 - j7.5)$$
$$= -28.5 + j19.9 = 34.7\underline{/145°} \text{ amperes}$$
$$\mathbf{I}_{c'c} = (-11.5 - j4.90) - (1.5 + j12.4)$$
$$= -13.0 - j17.3 = 21.7\underline{/-127°} \text{ amperes}$$

A vector diagram of the voltages and currents is shown in Fig. 17. Since the magnitudes and relative time-phase positions of the line-to-line voltages and the line currents are known, the wattmeter readings can be determined.

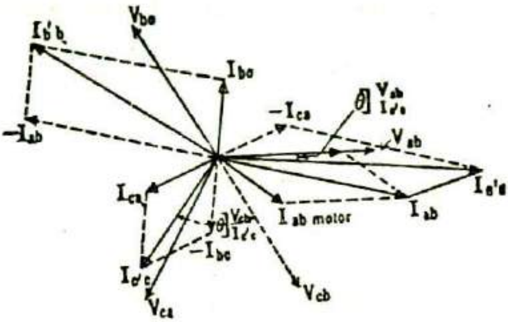


FIG. 17. Vector or phasor diagram of voltages and currents in a particular unbalanced three-phase circuit. (See Fig. 16.)

$$W_{ab-a'a} = V_{ab}I_{a'a}\cos\theta\Big]_{\mathbf{I}_{a'a}}^{\mathbf{V}_{ab}}$$
$$= 200 \times 41.6 \cos 3.58° = 8300 \text{ watts}$$
$$W_{cb-c'c} = V_{cb}I_{c'c}\cos\theta\Big]_{\mathbf{I}_{c'c}}^{\mathbf{V}_{cb}}$$
$$= 200 \times 21.7 \cos 67° = 1700 \text{ watts}$$

The other wattmeter combinations which will correctly measure the three-phase power are

(1) $W_{ac-a'a}$ together with $W_{bc-b'b}$,

(2) $W_{ba-b'b}$ together with $W_{ca-c'c}$.

In the present example

$$W_{ac-a'a} = V_{ac}I_{a'a}\cos\theta\Big]_{\mathbf{I}_{a'a}}^{\mathbf{V}_{ac}}$$
$$= 200 \times 41.6 \times \cos 63.58° = 3705 \text{ watts}$$
$$W_{bc-b'b} = V_{bc}I_{b'b}\cos\theta\Big]_{\mathbf{I}_{b'b}}^{\mathbf{V}_{bc}}$$
$$= 200 \times 34.7 \times \cos 25° = 6295 \text{ watts}$$

Problem 7. Calculate the readings of $W_{ba-b'b}$ and $W_{ca-c'c}$ in the above example and compare the sum of the wattmeter readings thus found with the total connected load.

Ans.: $W_{ba-b'b} = 5685$, $W_{ca-c'c} = 4315$ watts.

The Use of $n - 1$ Wattmeters to Measure n-Wire Power. In general, $n - 1$ wattmeter elements can be employed to measure n-wire power. The wattmeter elements may take the form of individual wattmeters, in which case the total power is equal to the algebraic sum of the wattmeter readings; or all movable members may be connected to a common shaft in which case the total power is indicated directly on one scale. The latter type of instrument is called a polyphase wattmeter.

Reactive Volt-Amperes in Unbalanced Four-Wire, Three-Phase Systems. The reactive volt-amperes of each individual phase of the load shown in Fig. 18 can be measured with three reactive volt-ampere

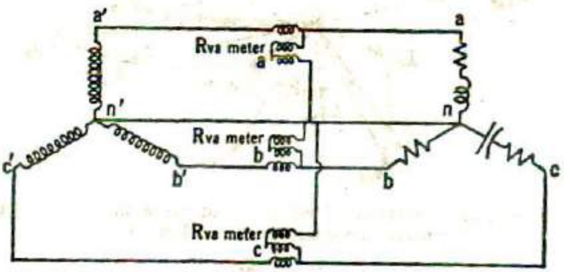


FIG. 18. Measurement of total reactive volt-amperes in a four-wire, three-phase system with three reactive volt-ampere meters.

meters. Sinusoidal wave forms of currents and voltages are assumed since the term " reactive volt-amperes " as well as any measurements of that quantity are ambiguous when other than sinusoidal wave forms are encountered.

In Fig. 18

$$\text{Meter } a \text{ reads } V_{an}I_{an} \sin \theta \Big]_{I_{an}}^{V_{an}} \text{ vars}$$

$$\text{Meter } b \text{ reads } V_{bn}I_{bn} \sin \theta \Big]_{I_{bn}}^{V_{bn}} \text{ vars}$$

$$\text{Meter } c \text{ reads } V_{cn}I_{cn} \sin \theta \Big]_{I_{cn}}^{V_{cn}} \text{ vars}$$

The algebraic sum of the above readings is of practical importance.

Assume the phase angle to be positive if the current lags the voltage and negative if the current leads the voltage. These conventions are merely matters of definition. (See page 97.) A meter properly connected to give up-scale readings for lagging-current reactive volt-amperes will read down-scale when subjected to leading-current reactive volt-amperes. If then in a particular case a meter reads down-scale, the relative polarities of the current and potential circuits of the meter are reversed. The resulting up-scale reading is considered as negative reactive volt-amperes in finding the total reactive volt-amperes of the system. With negative reactive volt-amperes defined as it is, the total vars of a system may, of course, be negative.

Example 8. In Fig. 18 let

$$V_{an} = 100\underline{/0^\circ} \text{ volts} \qquad Z_{an} = 25\underline{/45^\circ} \text{ ohms}$$

$$V_{bn} = 100\underline{/-120^\circ} \text{ volts} \qquad Z_{bn} = 50\underline{/0^\circ} \text{ ohms}$$

$$V_{cn} = 100\underline{/-240^\circ} \text{ volts} \qquad Z_{cn} = 20\underline{/-60^\circ} \text{ ohms}$$

The individual readings of the three reactive volt-ampere meters and the algebraic sum of the readings are to be determined.

$$I_{an} = \frac{100\underline{/0^\circ}}{25\underline{/45^\circ}} = 4.0\underline{/-45^\circ} \text{ amperes}.$$

$$I_{bn} = \frac{100\underline{/-120^\circ}}{50\underline{/0^\circ}} = 2.0\underline{/-120^\circ} \text{ amperes}$$

$$I_{cn} = \frac{100\underline{/-240^\circ}}{20\underline{/-60^\circ}} = 5.0\underline{/180^\circ} \text{ amperes}$$

The relative vector positions of the phase voltages and phase currents which actuate the meters are shown in Fig. 19.

Reactive volt-ampere meter a reads

$(100 \times 4 \times 0.707) = 283$ vars

Reactive volt-ampere meter b reads

$(100 \times 2 \times 0.0) = 0$ var

Reactive volt-ampere meter c reads

$(100 \times 5 \times -0.866) = -433$ vars

The algebraic sum of the meter readings or the "total" number of vars is -150.

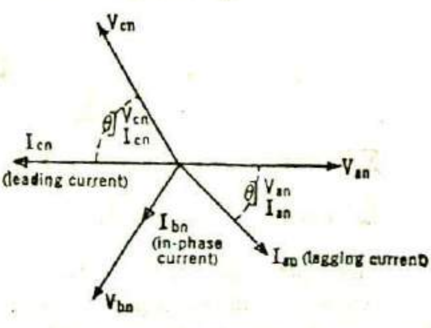


FIG. 19. Phasor diagram of the phase voltages and phase currents of the four-wire, three-phase load shown in Fig. 18 for a particular set of load impedances.

If wattmeters were to replace the reactive volt-ampere meters shown in Fig. 18, their readings would be as shown below:

$$W_a = 100 \times 4 \times 0.707 = 283 \text{ watts}$$

$$W_b = 100 \times 2 \times 1.000 = 200 \text{ watts}$$

$$W_c = 100 \times 5 \times 0.500 = 250 \text{ watts}$$

The total number of watts is 733.

Power Factor in Unbalanced Three-Phase Systems. Power factor in a single-phase system or in a balanced polyphase system has a definite physical significance. It is the ratio of the phase watts to the phase volt-amperes. Under conditions of sinusoidal wave form, power factor is equivalent to the cosine of the time-phase angular displacement between phase voltage and phase current.

In an unbalanced polyphase system each phase has its own particular power factor. The result is that the term " power factor " as applied to the combined unbalanced polyphase system can have only such meaning as is given to it by definition. The average of the individual phase power factors is a good general indication of the ratio of total watts to total volt-amperes in certain cases where the phase loads are all inductive or all capacitive. Where both capacitive and inductive phase loads are encountered, the compensating effect of capacitive reactive volt-amperes and inductive reactive volt-amperes is not taken into account. Another serious limitation to " average " power factor concept is that the individual phase power factors are not easily determined in many practical installations. " Average " power factor is generally not considered when specifying the power factor of an unbalanced polyphase system.

One recognized definition called vector power factor of an unbalanced polyphase system is

$$\text{Vector p.f.} = \frac{\sum VI \cos\theta}{\sqrt{(\sum VI \sin\theta)^2 + (\sum VI \cos\theta)^2}} \tag{29}$$

$$\sum VI \cos\theta = V_a I_a \cos\theta_a + V_b I_b \cos\theta_b + V_c I_c \cos\theta_c + \cdots \tag{30}$$

$$\sum VI \sin\theta = V_a I_a \sin\theta_a + V_b I_b \sin\theta_b + V_c I_c \sin\theta_c + \cdots \tag{31}$$

The subscripts employed in the above equations refer to individual phase values. For example θ_a is the angular displacement between phase voltage and phase current in the a phase of the system. $\sum VI \cos\theta$ is the total power consumed by the polyphase load, the power factor of which is under investigation. $\sum VI \sin\theta$ is the algebraic sum of the individual phase reactive volt-amperes. In evaluating $\sum VI \sin\theta$ in any particular case due regard must be given to the sign of each component.

It is evident that the denominator of equation (29) can be evaluated as if it were the magnitude of a resultant vector, the right-angle components of which are $(\sum VI \cos\theta)$ and $(\sum VI \sin\theta)$. This fact is illustrated graphically in Fig. 20 for the particular three-phase system

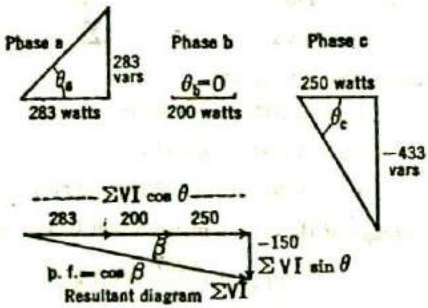


FIG. 20. Illustrating the concept of vector volt-amperes in a particular case.

discussed on pages 392–394. Considering watts and vars as the right-angle components which go to form " vector volt-amperes " it is plain that

$$\sum \mathbf{VI} = \sqrt{(\sum VI \sin\theta)^2 + (\sum VI \cos\theta)^2}\,\underline{/\beta} \tag{32}$$

or

$$\sum \mathbf{VI} = V_aI_a\underline{/\theta_a} + V_bI_b\underline{/\theta_b} + V_cI_c\underline{/\theta_c} \tag{33}$$

Power factor, as defined by equation (29), can now be written in any one of several different ways.

$$\text{Vector p.f.} = \cos\tan^{-1}\frac{(\sum VI \sin\theta)}{(\sum VI \cos\theta)} = \cos\beta \tag{34}$$

or

$$\text{Vector p.f.} = \frac{\sum VI \cos\theta}{\text{magnitude of } \sum \mathbf{VI}} \tag{35}$$

Example 9. The " average " power factor of the unbalanced load described on pages 392–394 is to be compared with the power factor as defined by equations (29), (34), or (35). The circuit arrangement is shown in Fig. 18, and the previously determined values are indicated below.

$V_{an} = 100\angle 0^\circ$ volts $\qquad I_{an} = 4.0\angle -45^\circ$ amperes

$V_{bn} = 100\angle -120^\circ$ volts $\qquad I_{bn} = 2.0\angle -120^\circ$ amperes

$V_{cn} = 100\angle -240^\circ$ volts $\qquad I_{cn} = 5.0\angle 180^\circ$ amperes

a-phase vars = 283 $\qquad a$-phase watts = 283

b-phase vars = 000 $\qquad b$-phase watts = 200

c-phase vars = -433 $\qquad c$-phase watts = 250

$\sum VI \sin\theta = -150$ vars $\qquad \sum VI \cos\theta = 733$ watts

The individual phase power factors are

$$\text{P.f.}_a = 0.707 \text{ (result of lagging current)}$$

$$\text{P.f.}_b = 1.000 \text{ (result of in-phase current)}$$

$$\text{P.f.}_c = 0.500 \text{ (result of leading current)}$$

The arithmetical average of the above phase power factors is

$$\text{P.f.}_{av} = \frac{2.207}{3} = 0.736$$

The power factor of the unbalanced load as defined by equation (29) is

$$\text{Vector p.f.} = \frac{733}{\sqrt{(-150)^2 + (733)^2}} = \frac{733}{748} = 0.98$$

Inasmuch as the latter determination of power factor recognizes the compensating effect of "leading" and "lagging" reactive volt-amperes it is somewhat more significant than the "average" power factor.

Measurement of $\sum VI \sin\theta$ in a Three-Wire, Three-Phase Circuit. Power factors in three-wire, three-phase systems are very often measured in terms of $\sum VI\cos\theta$ and $\sum VI \sin\theta$. $\sum VI\cos\theta$ can be measured with the aid of either two or three wattmeters as shown in previous articles. It may be shown that $\sum VI \sin\theta$ can also be measured in a three-wire, three-phase system with either two or three reactive volt-ampere meters. Only the two-meter method of measuring $\sum VI \sin\theta$ will be considered.

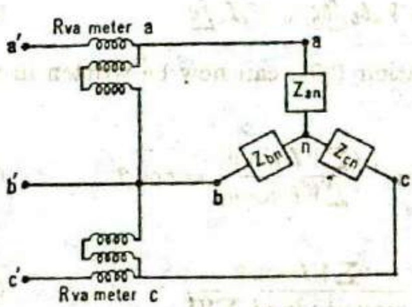


FIG. 21. The two reactive volt-ampere meter method of measuring $\Sigma VI \sin\theta$ in a three-wire, three-phase system.

The two meters shown in Fig. 21 are assumed to be reactive volt-ampere meters which are capable of reading $VI \sin \theta \Big]_{\mathbf{I}}^{\mathbf{V}}$. These meters are connected into the circuit in a manner which is exactly like two wattmeters in the two-wattmeter method of measuring three-phase power. It will be shown presently that, when they are connected in this fashion, the algebraic sum of the two reactive volt-ampere meter readings is equal to $\sum VI \sin \theta$ of the three-phase circuit. $\sum VI \sin \theta$ for a polyphase system has been defined in equation (31) of the present chapter.

Connected as shown in Fig. 21

$$\text{Reactive volt-ampere meter } a \text{ reads } \left\{ V_{ab} I_{a'a} \sin \theta \Big]_{\mathbf{I}_{a'a}}^{\mathbf{V}_{ab}} \right\}$$

$$\text{Reactive volt-ampere meter } c \text{ reads } \left\{ V_{cb} I_{c'c} \sin \theta \Big]_{\mathbf{I}_{c'c}}^{\mathbf{V}_{cb}} \right\}$$

For the sake of analysis, the above readings will be expressed temporarily in terms of the complex components of the voltages and currents. In Chapter IV it was shown that under the conditions of sinusoidal wave form

$$VI \sin \theta \Big]_{\mathbf{I}}^{\mathbf{V}} = v'i - vi' \tag{36}$$

where

$$\mathbf{V} = v + jv' \quad \text{and} \quad \mathbf{I} = i + ji'$$

Reference to Fig. 21 will show that $\mathbf{I}_{a'a} = \mathbf{I}_{an}$ and that $\mathbf{I}_{c'c} = \mathbf{I}_{cn}$. Also $\mathbf{V}_{ab} = \mathbf{V}_{an} - \mathbf{V}_{bn}$ and $\mathbf{V}_{cb} = \mathbf{V}_{cn} - \mathbf{V}_{bn}$.

$$\begin{aligned} V_{ab} I_{a'a} \sin \theta \Big]_{\mathbf{I}_{a'a}}^{\mathbf{V}_{ab}} &= V_{ab} I_{an} \sin \theta \Big]_{\mathbf{I}_{an}}^{\mathbf{V}_{ab}} \\ &= (v'_{ab} i_{an} - v_{ab} i'_{an}) \\ &= (v'_{an} i_{an} - v'_{bn} i_{an} - v_{an} i'_{an} + v_{bn} i'_{an}) \\ &= (v'_{an} i_{an} - v_{an} i'_{an}) + (v_{bn} i'_{an} - v'_{bn} i_{an}) \end{aligned} \tag{37}$$

$$\begin{aligned} V_{cb} I_{c'c} \sin \theta \Big]_{\mathbf{I}_{c'c}}^{\mathbf{V}_{cb}} &= V_{cb} I_{cn} \sin \theta \Big]_{\mathbf{I}_{cn}}^{\mathbf{V}_{cb}} \\ &= (v'_{cb} i_{cn} - v_{cb} i'_{cn}) \\ &= (v'_{cn} i_{cn} - v'_{bn} i_{cn} - v_{cn} i'_{cn} + v_{bn} i'_{cn}) \\ &= (v'_{cn} i_{cn} - v_{cn} i'_{cn}) + (v_{bn} i'_{cn} - v'_{bn} i_{cn}) \end{aligned} \tag{38}$$

It will be noticed that $(v_{bn}i'_{an} - v'_{bn}i_{an})$ of equation (37) and $(v_{bn}i'_{cn} - v'_{bn}i_{cn})$ of equation (38) can be added so as to yield

$$v_{bn}(i'_{an} + i'_{cn}) - v'_{bn}(i_{an} + i_{cn}) = (v'_{bn}i_{bn} - v_{bn}i'_{bn}) \tag{39}$$

Therefore the sum of equations (37) and (38) reduces to

$$(v'_{an}i_{an} - v_{an}i'_{an}) + (v'_{bn}i_{bn} - v_{bn}i'_{bn}) + (v'_{cn}i_{cn} - v_{cn}i'_{cn})$$

which in turn is easily recognized as the total reactive volt-amperes of the three-phase load or $\sum VI \sin \theta$.

No restrictions as to the balance of either voltage or current have been imposed upon the foregoing derivation. Two reactive volt-ampere meters connected into a three-wire, three-phase circuit as shown in Fig. 21 will, therefore, measure $\sum VI \sin \theta$ regardless of the condition of balance. Although the generality is rather difficult to incorporate into the derivation, the algebraic sum of the readings will be equal to $\sum VI \sin \theta$ whenever the reactive volt-amperes are restricted to those cases where both voltages and current wave forms are sinusoidal, provided the reactive volt-ampere meters are connected into the three-wire, three-phase line in a manner similar to the wattmeters shown in Fig. 15*a*, *b*, or *c*.

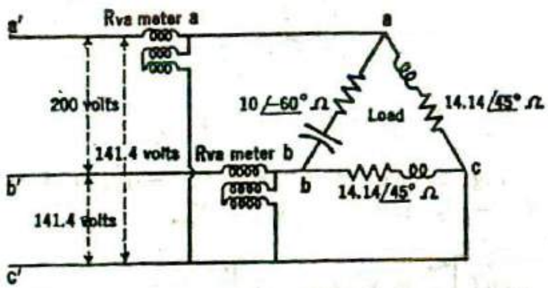


FIG. 22. A particular unbalanced three-phase load.

Example 10. In Fig. 22, *abc* represents an unbalanced three-phase system of voltages, the phase sequence of which is *ab–bc–ca*. In magnitude

$$V_{ab} = 200,\ V_{bc} = 141.4 \quad \text{and} \quad V_{ca} = 141.4 \text{ volts}$$

If $\mathbf{V}_{ab}$ is assumed to occupy the reference axis position, then

$$\mathbf{V}_{ab} = 200\underline{/0^\circ},\ \mathbf{V}_{bc} = 141.4\underline{/-135^\circ},\ \mathbf{V}_{ca} = 141.4\underline{/-225^\circ} \text{ volts}$$

It will be assumed that the load impedances have the values shown on the circuit diagram, namely,

$$\mathbf{Z}_{ab} = 10\underline{/-60^\circ} \text{ ohms}$$
$$\mathbf{Z}_{bc} = 14.14\underline{/45^\circ} \text{ ohms}$$
$$\mathbf{Z}_{ca} = 14.14\underline{/45^\circ} \text{ ohms}$$

Assuming that the line-to-line voltages remain fixed at the values given above, the delta-phase currents are

$$\mathbf{I}_{ab} = \frac{200\underline{/0^\circ}}{10\underline{/-60^\circ}} = 20\underline{/60^\circ} \text{ amperes}$$

$$\mathbf{I}_{bc} = \frac{141.4\underline{/-135^\circ}}{14.14\underline{/45^\circ}} = 10\underline{/180^\circ} \text{ amperes}$$

$$\mathbf{I}_{ca} = \frac{141.4\underline{/-225^\circ}}{14.14\underline{/45^\circ}} = 10\underline{/90^\circ} \text{ amperes}$$

From which

$$\mathbf{I}_{a'a} = \mathbf{I}_{ab} - \mathbf{I}_{ca} = 10 + j7.32 = 12.4\underline{/36.2^\circ} \text{ amperes}$$

$$\mathbf{I}_{b'b} = \mathbf{I}_{bc} - \mathbf{I}_{ab} = -20 - j17.32 = 26.45\underline{/-139.1^\circ} \text{ amperes}$$

$$\mathbf{I}_{c'c} = \mathbf{I}_{ca} - \mathbf{I}_{bc} = 10 + j10 = 14.14\underline{/45^\circ} \text{ amperes}$$

The voltages and currents are represented graphically in Fig. 23.

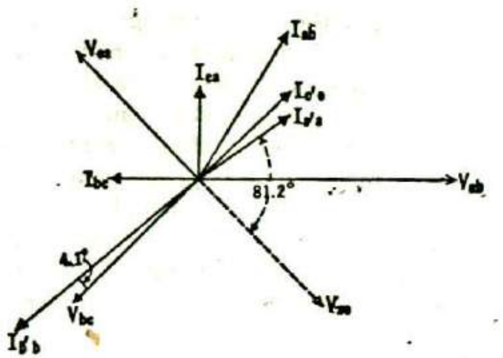


FIG. 23. Phasor voltages and currents in the three-phase circuit shown in Fig. 22.

The meters shown in Fig. 22 are assumed to be reactive volt-ampere meters, and the present example concerns itself with the predetermination of their readings.

Reactive volt-ampere meter a reads

$$V_{ac}I_{a'a}\sin\theta\Big]_{\mathbf{I}_{a'a}}^{\mathbf{V}_{ac}} = 141.4 \times 12.4 \times \sin -81.2^\circ = -1732 \text{ vars}$$

Reactive volt-ampere meter b reads

$$V_{bc}I_{b'b}\sin\theta\Big]_{\mathbf{I}_{b'b}}^{\mathbf{V}_{bc}} = 141.4 \times 26.45 \sin 4.1^\circ = 268 \text{ vars}$$

The algebraic sum of the meter readings is

$$-1732 + 268 = -1464 \text{ vars}$$

The actual value of $\sum VI \sin\theta$ as determined from the individual phase voltages and currents is

$$\sum VI \sin\theta = -(200 \times 20 \times 0.866) + (141.4 \times 10 \times 0.707) + (141.4 \times 10 \times 0.707) = -1464 \text{ vars}$$

Problem 8. If the reactive volt-ampere meters shown in Fig. 22 are placed so that the current coils carry $I_{a'a}$ and $I_{c'c}$, what will be the individual meter readings in vars? It is assumed that the potential circuits of the meters are connected in such a manner that the algebraic sum of the readings will be equal to $\sum VI \sin\theta$.

Ans.: Meter *a* reads -1464 vars; meter *c* reads zero.

Problem 9. What is the power factor of the unbalanced load shown in Fig. 22 as determined from $\sum VI \sin\theta$ and $\sum VI \cos\theta$?

Ans.: 0.939.

Phasor Relations as Found from Experimentally Determined Magnitudes of Current and Voltage. Phasor diagrams of the voltages of polyphase loads may be formed from measurements of the voltages by forming in a closed polygon those line voltages which according to Kirchhoff's laws add to zero when tracing from one line in a continuous direction to each adjacent line in sequence until the starting point is reached. Line-to-neutral voltages in a star connection may then be inscribed in the polygon so that they combine according to Kirchhoff's laws to form the line voltages. The principle of duality indicates a similar procedure may be followed to establish phasor diagrams of line and phase currents in a mesh connection. The phase relations may then be found by solving the diagrams either graphically or analytically and the solutions adapted to any desired sequence. See Problems 31 and 32.

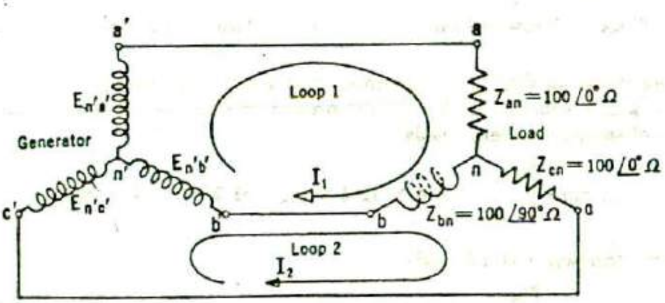


Fig. 24. Loop-current method of labeling. See example 11.

Example 11. Let it be required to find the branch currents $\mathbf{I}_{an}$, $\mathbf{I}_{bn}$, and $\mathbf{I}_{cn}$ of Fig. 24 by the loop-current method if

$$\mathbf{E}_{n'a'} = 57.7\angle{-30°},\ \mathbf{E}_{n'b'} = 57.7\angle{-150°},\ \text{and}\ \mathbf{E}_{n'c'} = 57.7\angle{90°} \text{ volts}$$

Since only two loop currents are required to traverse all the branches,

$$\mathbf{Z}_{11}\mathbf{I}_1 - \mathbf{Z}_{12}\mathbf{I}_2 = \mathbf{E}_1 = \mathbf{E}_{b'n'} + \mathbf{E}_{n'a'} = 100\underline{/0^\circ} \text{ volts}$$

$$-\mathbf{Z}_{21}\mathbf{I}_1 + \mathbf{Z}_{22}\mathbf{I}_2 = \mathbf{E}_2 = \mathbf{E}_{c'n'} + \mathbf{E}_{n'b'} = 100\underline{/-120^\circ} \text{ volts}$$

where the minus signs account for the opposite directions of $\mathbf{I}_1$ and $\mathbf{I}_2$ through $\mathbf{Z}_{n'b'bn}$. If the generator impedances of Fig. 24 are neglected,

$$\mathbf{Z}_{11} = 100\underline{/0^\circ} + 100\underline{/90^\circ} = 141.4\underline{/45^\circ} \text{ ohms}$$

$$\mathbf{Z}_{22} = 100\underline{/90^\circ} + 100\underline{/0^\circ} = 141.4\underline{/45^\circ} \text{ ohms}$$

Without regard for sign, which has been taken care of in the above voltage equations

$$\mathbf{Z}_{12} = \mathbf{Z}_{21} = 100\underline{/90^\circ} \text{ ohms}$$

The voltage equations may be solved directly for $\mathbf{I}_1$ and $\mathbf{I}_2$ as shown below:

$$\mathbf{I}_1 = \mathbf{I}_{an} = \frac{\begin{vmatrix} 100\underline{/0^\circ} & -100\underline{/90^\circ} \\ 100\underline{/-120^\circ} & 141.4\underline{/45^\circ} \end{vmatrix}}{\begin{vmatrix} 141.4\underline{/45^\circ} & -100\underline{/90^\circ} \\ -100\underline{/90^\circ} & 141.4\underline{/45^\circ} \end{vmatrix}} = \frac{19{,}320\underline{/15^\circ}}{22{,}380\underline{/63.45^\circ}} = 0.864\underline{/-48.45^\circ} \text{ ampere}$$

$$\mathbf{I}_2 = \mathbf{I}_{nc} = \frac{\begin{vmatrix} 141.4\underline{/45^\circ} & 100\underline{/0^\circ} \\ -100\underline{/90^\circ} & 100\underline{/-120^\circ} \end{vmatrix}}{22{,}380\underline{/63.45^\circ}} = \frac{5185\underline{/-45^\circ}}{22{,}380\underline{/63.45^\circ}} = 0.232\underline{/-108.45^\circ} \text{ ampere}$$

$\mathbf{I}_{cn} = -\mathbf{I}_{nc} = 0.232\underline{/71.55^\circ}$, and $\mathbf{I}_{bn} = \mathbf{I}_2 - \mathbf{I}_1$

Example 12. In Fig. 25 are shown three load impedances $\mathbf{Z}_{an}$, $\mathbf{Z}_{bn}$, and $\mathbf{Z}_{cn}$ which are energized by $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$ (and, of course, $\mathbf{V}_{ca}$). The *an* coil is assumed to be coupled magnetically to the *cn* coil and, as shown in Fig. 25, the coefficient of coupling between

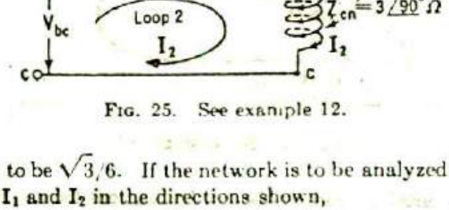


FIG. 25. See example 12.

the coils is assumed to be $\sqrt{3}/6$. If the network is to be analyzed by the loop-current method employing $\mathbf{I}_1$ and $\mathbf{I}_2$ in the directions shown,

$$\omega M_{ac} = \omega M_{ca} = \frac{\sqrt{3}}{6}\sqrt{\omega L_{an} \times \omega L_{cn}} = \frac{\sqrt{3}}{6}\sqrt{1 \times 3} = 0.5 \text{ ohm}$$

The positive sign of M is used here because the coils magnetize along a common axis in the same direction if wound as shown and if positive values of $\mathbf{I}_1$ and $\mathbf{I}_2$ are present. (See page 284.) Assume $\mathbf{V}_{ab} = 100\underline{/0^\circ}$ volts and $\mathbf{V}_{bc} = 100\underline{/-120^\circ}$ volts.

For the network shown in Fig. 25, the basic voltage equations become

$$\mathbf{Z}_{11}\mathbf{I}_1 + \mathbf{Z}_{12}\mathbf{I}_2 = \mathbf{V}_{ab} = 100\underline{/0^\circ} \text{ volts}$$

$$\mathbf{Z}_{21}\mathbf{I}_1 + \mathbf{Z}_{22}\mathbf{I}_2 = \mathbf{V}_{bc} = 100\underline{/-120^\circ} \text{ volts}$$

$$\mathbf{Z}_{11} = (2 + j1),\ \mathbf{Z}_{22} = (2 + j3), \text{ and } \mathbf{Z}_{12} = \mathbf{Z}_{21} = (-2 + j0.5) \text{ ohms}$$

Note: The minus sign in $\mathbf{Z}_{12}$ accounts for the fact that $\mathbf{I}_2$ flows through Z_{bn} opposite to $\mathbf{I}_1$ and $+j0.5$ in $\mathbf{Z}_{12}$ accounts for the fact that the $(j\omega M\mathbf{I}_2)$ voltage drop acts in the same direction in loop 1 as the $(j\omega L\mathbf{I}_1)$ voltage drop.

$$\mathbf{I}_1 = \frac{\begin{vmatrix} (100 + j0) & (-2 + j0.5) \\ (-50 - j86.6) & (2 + j3) \end{vmatrix}}{\begin{vmatrix} (2 + j1) & (-2 + j0.5) \\ (-2 + j0.5) & (2 + j3) \end{vmatrix}} = \frac{56.7 + j152}{-2.75 + j10} = 12.68 - j9.15$$

$$= 15.6\underline{/-35.8^\circ} \text{ amperes}$$

$$\mathbf{I}_2 = \frac{\begin{vmatrix} (2 + j1) & (100 + j0) \\ (-2 + j0.5) & (-50 - j86.6) \end{vmatrix}}{(-2.75 + j10)} = \frac{186.6 - j273}{-2.75 + j10} = -30.15 - j10.36$$

$$= 31.8\underline{/-161^\circ} \text{ amperes}$$

The branch currents follow directly from $\mathbf{I}_1$ and $\mathbf{I}_2$ as shown in example 11.

Example 13. The network shown in Fig. 26 represents two generators operating

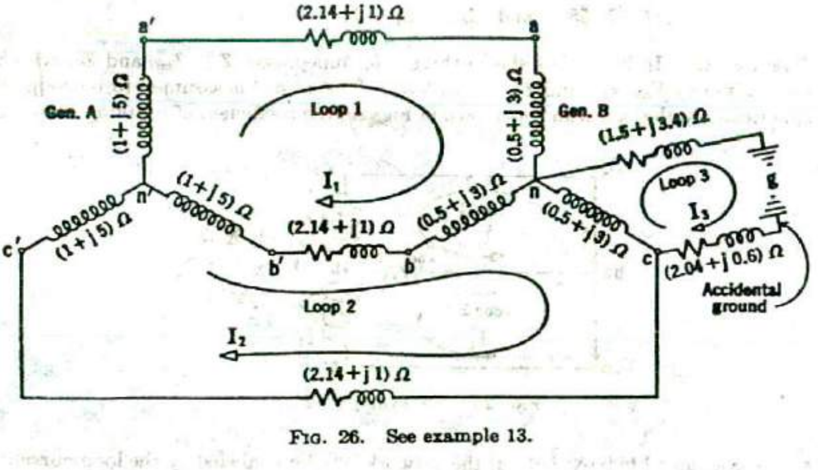


FIG. 26. See example 13.

in parallel. An accidental ground on the line leading out from terminal c is assumed to exist as shown and the problem is that of determining the short-circuit current $\mathbf{I}_{ngc}$ or loop current $\mathbf{I}_3$ in Fig. 26.

A study of Fig. 26 will show that the self-impedances of loops 1, 2, and 3 are, respectively,

$$\mathbf{Z}_{11} = (7.28 + j18) = 19.4\underline{/68^\circ} \text{ ohms}$$

$$\mathbf{Z}_{22} = (7.28 + j18) = 19.4\underline{/68^\circ} \text{ ohms}$$

$$\mathbf{Z}_{33} = (4.04 + j7.0) = 8.08\underline{/60^\circ} \text{ ohms}$$

Next, the mutual impedances will be obtained from an inspection of Fig. 26 and minus signs will be affixed to those mutual impedances that carry loop currents of opposite directions.

$$Z_{12} = Z_{21} = -(3.64 + j9.0) = -9.7\underline{/68^\circ} \text{ ohms}$$

$$Z_{23} = Z_{32} = -(0.50 + j3.0) = -3.04\underline{/80.5^\circ} \text{ ohms}$$

$$Z_{13} = Z_{31} = 0 \text{ (Since loops 1 and 3 have no common path.)}$$

Assume the generated phase voltages are

$$E_{n'a'} = E_{na} = 4000\underline{/0^\circ} \text{ volts}$$

$$E_{n'b'} = E_{nb} = 4000\underline{/-120^\circ} \text{ volts}$$

$$E_{n'c'} = E_{nc} = 4000\underline{/-240^\circ} \text{ volts}$$

The resultant voltages which exist in the three loops of Fig. 26 are

$$E_1 = E_{n'a'} - E_{na} + E_{nb} - E_{n'b'} = 0$$

$$E_2 = E_{n'b'} - E_{nb} + E_{nc} - E_{n'c'} = 0$$

$$E_3 = -E_{nc}$$

$$= -4000\underline{/-240^\circ} = 4000\underline{/-60^\circ} \text{ volts}$$

The equations for voltage equilibrium in the three meshes of Fig. 26 are

$$\begin{aligned}
(19.4\underline{/68^\circ})I_1 - (9.7\underline{/68^\circ})I_2 \quad + 0 \qquad\qquad &= 0\\
-(9.7\underline{/68^\circ})I_1 + (19.4\underline{/68^\circ})I_2 - (3.04\underline{/80.5^\circ})I_3 &= 0\\
0 - (3.04\underline{/80.5^\circ})I_2 + (8.08\underline{/60^\circ})I_3 &= 4000\underline{/-60^\circ}
\end{aligned}$$

The above equations will be solved simultaneously for I_1, I_2, and I_3 with the aid of elementary determinant theory. The common denominator of each current solution is

$$D = \begin{vmatrix} (19.4\underline{/68^\circ}) & -(9.7\underline{/68^\circ}) & 0 \\ -(9.7\underline{/68^\circ}) & (19.4\underline{/68^\circ}) & -(3.04\underline{/80.5^\circ}) \\ 0 & -(3.04\underline{/80.5^\circ}) & (8.08\underline{/60^\circ}) \end{vmatrix}$$

$$D = [-2920 - j837] - [(-117.8 - j135.4) + (-733 - j210)]$$

$$= (-2068 - j492) = 2122\underline{/193.4^\circ} \text{ ohms}^3$$

The desired current in the present instance is I_{ngc} or I_3.

$$I_3 = \frac{\begin{vmatrix} (19.4\underline{/68^\circ}) & -(9.7\underline{/68^\circ}) & 0 \\ -(9.7\underline{/68^\circ}) & (19.4\underline{/68^\circ}) & 0 \\ 0 & -(3.04\underline{/80.5^\circ}) & (4000\underline{/-60^\circ}) \end{vmatrix}}{2122\underline{/193.4^\circ}}$$

$$I_3 = \frac{1{,}131{,}000\underline{/76^\circ}}{2122\underline{/193.4^\circ}} = 533\underline{/-117.4^\circ} \text{ amperes}$$

Problem 10. Find the magnitudes of $I_{a'a}$, $I_{b'b}$, and $I_{c'c}$ in Fig. 26 utilizing the calculations of example 13 in so far as they are helpful.

Ans.: $I_{a'a} = 55.6$, $I_{b'b} = 55.6$; and $I_{c'c} = 111.2$ amperes.

PROBLEMS

11. An unbalanced delta system labeled *abc* at the corners consists of $Z_{ab} = 10\underline{/-60^\circ}$, $Z_{bc} = 5\underline{/0^\circ}$, and $Z_{ca} = 10\underline{/60^\circ}$ ohms. If $V_{cb} = 100\underline{/0^\circ}$ and the voltage sequence is *cb-ba-ac*, find the vector expressions for the currents entering the terminals *a*, *b*, and *c*. The three-phase supply voltages are balanced. Also solve for the opposite sequence.

12. An unbalanced load labeled *abc* at the corners consists of $Z_{ab} = 5\underline{/40^\circ}$, $Z_{bc} = 10\underline{/-30^\circ}$, and $Z_{ca} = 8\underline{/45^\circ}$ ohms. Three-phase balanced line voltages of 115 volts each are applied. If the sequence is *cb-ac-ba*, calculate the complex expressions for the line currents leaving terminals *a*, *b*, and *c* for $V_{cb} = 115\underline{/0^\circ}$ volts.

13. Refer to Fig. 27. V_{AB} and V_{CB} represent a balanced two-phase system of voltage drops, the magnitude of each being 115 volts. The voltage phase sequence

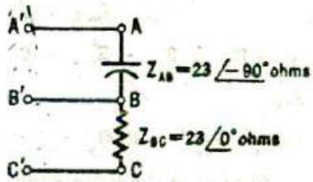


FIG. 27. See Problem 13.

is *AB-CB*. V'_{AB} is to be used as reference. Find I_{AB}, I_{CB}, $I_{BB'}$ and draw a vector diagram of the voltages and currents.

14. A wye-connected set of impedances consists of $Z_{an} = 5\underline{/0^\circ}$, $Z_{bn} = 5\underline{/60^\circ}$, and $Z_{cn} = 5\underline{/-60^\circ}$ ohms. Find the equivalent delta-connected impedances Z_{ab}, Z_{bc}, and Z_{ca} which can be used to replace the wye-connected set of impedances.

15. Refer to Fig. 28. The terminals $a'b'c'$ represent a balanced three-phase system of voltages the sequence of which is $b'c'$-$a'b'$-$c'a'$. The magnitude of each

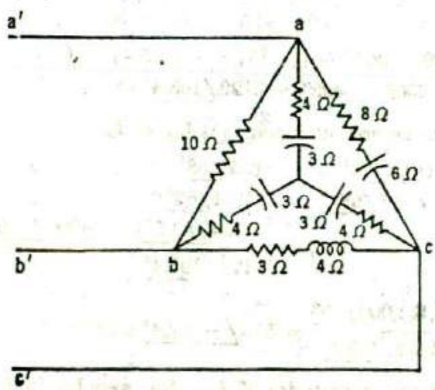


FIG. 28. See Problem 15.

line-to-line voltage is 230 volts. Find the readings of ammeters placed in the $a'a$, $b'b$, and $c'c$ lines.

16. In Fig. 7, page 380, it will be assumed that the generated voltages are

$\mathbf{E}_{n'a'} = 100\underline{/0^\circ}$, $\mathbf{E}_{n'b'} = 100\underline{/-120^\circ}$, $\mathbf{E}_{n'c'} = 100\underline{/-240^\circ}$ volts and that

$$Z_{n'a'an} = (2 - j1) \text{ ohms}$$

$$Z_{n'b'bn} = (1 - j3) \text{ ohms}$$

$$Z_{n'c'cn} = (3 + j4) \text{ ohms}$$

Find the line currents $\mathbf{I}_{a'a}$, $\mathbf{I}_{b'b}$, and $\mathbf{I}_{c'c}$. Draw a vector diagram of the line-to-line voltages and the line currents.

17. Refer to Fig. 8, page 381. Let it be assumed that the following quantities are known:

$$\mathbf{E}_{n'a'} = 1000 + j0 = 1000\underline{/0^\circ} \text{ volts}$$

$$\mathbf{E}_{n'b'} = -500 - j866 = 1000\underline{/-120^\circ} \text{ volts}$$

$$\mathbf{E}_{n'c'} = -500 + j866 = 1000\underline{/120^\circ} \text{ volts}$$

$$Z_{an} = 20 - j20 = 28.28\underline{/-45^\circ} \text{ ohms}$$

$$Z_{bn} = 50 + j0 = 50.0\underline{/0^\circ} \text{ ohms}$$

$$Z_{cn} = 30 + j52 = 60.0\underline{/60^\circ} \text{ ohms}$$

$$Z_g = 2 + j8 = 8.25\underline{/76^\circ} \text{ ohms}$$

$$Z_l = 1 + j1 = 1.41\underline{/45^\circ} \text{ ohms}$$

$$Z_n = 2.5 + j1 = 2.70\underline{/21.8^\circ} \text{ ohms}$$

Write the expressions for $\mathbf{I}_{aa'}$, $\mathbf{I}_{bb'}$, and $\mathbf{I}_{cc'}$, employing determinants and the numerical values of the **E**'s and **Z**'s specified above. Use loop currents $\mathbf{I}_1 = \mathbf{I}_{a'a}$, $\mathbf{I}_2 = \mathbf{I}_{b'b}$, and $\mathbf{I}_3 = \mathbf{I}_{c'c}$ all returning through line nn'. (Results may be left in the form of the ratio of two matrices.)

18. A delta-connected set of impedances consists of $Z_{ab} = 5\underline{/0^\circ}$, $Z_{bc} = 5\underline{/60^\circ}$, and $Z_{ca} = 5\underline{/-60^\circ}$ ohms. Find the equivalent wye-connected impedances Z_{an}, Z_{bn}, and Z_{cn} which can be employed to replace the above delta-connected impedances.

19. Refer to Fig. 9, page 382. Assume that the generator is capable of maintaining a balanced three-phase system of voltages $\mathbf{E}_{b'a'}$, $\mathbf{E}_{a'c'}$, $\mathbf{E}_{c'b'}$, the sequence of which is $b'a'$-$a'c'$-$c'b'$. The magnitude of each line voltage is 100 volts. $Z_{a'a} = Z_{b'b} = Z_{c'c} = 0.5 + j0.5$ ohm. $Z_{ab} = 10\underline{/0^\circ}$, $Z_{bc} = 10\underline{/60^\circ}$, and $Z_{ca} = 10\underline{/-60^\circ}$ ohms. Find $\mathbf{I}_{a'a}$, $\mathbf{I}_{b'b}$, $\mathbf{I}_{ab}$, $\mathbf{I}_{bc}$, and $\mathbf{I}_{ca}$ with respect to $\mathbf{V}_{a'b'}$ as a reference.

20. Explain, by means of qualitative vector diagrams, the operation of a three-phase-sequence indicator that employs an inductance coil in place of the condenser shown in Fig. 11*a*, page 386. Does the voltmeter read above or below line voltage for sequence *ab-ca-bc*?

21. Devise some scheme for checking the phase sequence of two-phase voltages.

22. Find the reading of a wattmeter which has its current coil in the $A'A$ line and its potential coil across the voltage V_{AC} in Problem 13 and Fig. 27.

23. Refer to Fig. 13, page 387. $V_{ab} = 200$, $V_{bc} = 141.4$, and $V_{ca} = 141.4$ volts. Sequence *ab-bc-ca*. $Z_{ab} = Z_{bc} = Z_{ca} = (8 - j6)$ ohms. Find the reading of each of the wattmeters. Find reading of a wattmeter with its current coil in line a and potential coil from a to b; also one with current coil in line c and potential coil from c to b.

24. (a) If a wattmeter W_a has its current coil in line a and its potential coil from line a to c of Fig. 1, page 373, what will it read for a sequence V_{ab}-V_{ca}-V_{bc}? If another wattmeter W_b has its current coil in line b and its potential coil connected from line b to c, what will it read?

(b) If W_a and W_b were varmeters what would they read?

25. (a) Find readings of wattmeters W_a and W_b with their current coils in lines a and b, respectively, supplying the load of Problem 11 if the potential coils are properly connected so that the sum of the readings will give the total power consumed by the load.

(b) Find readings if W_a and W_b are varmeters.

26. Refer to Fig. 29. $\mathbf{V}_{a'b'}$, $\mathbf{V}_{b'c'}$, and $\mathbf{V}_{c'a'}$ represent a balanced three-phase system of voltage drops, the magnitude of each being 200 volts. The voltage

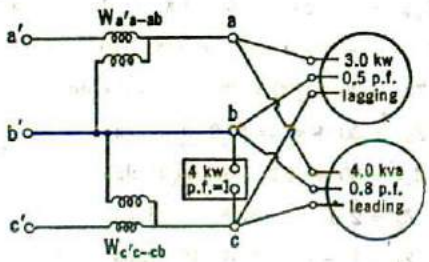


FIG. 29. See Problem 26.

sequence is $a'b'$-$b'c'$-$c'a'$. Two balanced three-phase loads indicated by the circles are connected to the terminals abc as shown in Fig. 29. In addition to the two balanced loads, a single-phase, 4-kw, unity-power-factor load is placed across the bc terminals as indicated.

(a) Find the reading of $W_{a'a\text{-}ab}$ and $W_{c'c\text{-}cb}$.

(b) If reactive volt-ampere meters replaced $W_{a'a\text{-}ab}$ and $W_{c'c\text{-}cb}$, find their respective readings.

(c) Find the combined vector power factor of the composite load.

27. In Fig. 21, page 396, it will be assumed that $\mathbf{V}_{a'b'}$, $\mathbf{V}_{b'c'}$, and $\mathbf{V}_{c'a'}$ represent a balanced three-phase system of voltages the sequence of which is $a'b'$-$c'a'$-$b'c'$. $\mathbf{Z}_{an} = 10\underline{/0^\circ}$, $\mathbf{Z}_{bn} = 10\underline{/-60^\circ}$, and $\mathbf{Z}_{cn} = 10\underline{/90^\circ}$ ohms. Assume line-to-line voltage of 100 volts.

(a) Find the readings of the two reactive volt-ampere meters shown in Fig. 21.

(b) Find the readings of wattmeters placed at similar positions in the circuit, namely, at the $a'a$-ab and the $c'c$-cb positions.

(c) Find the vector power factor of the unbalanced load as recognized by the A.I.E.E.

28. In Fig. 30, $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, $\mathbf{V}_{ca}$ are balanced three-phase voltages each having a magnitude of 200 volts and a phase sequence of ab-bc-ca. Determine the readings of the two wattmeters shown in the figure.

29. In Fig. 31, $\mathbf{E}_{n'a'}\mathbf{E}_{n'b'}\mathbf{E}_{n'c'}$ are balanced three-phase voltages with magnitudes

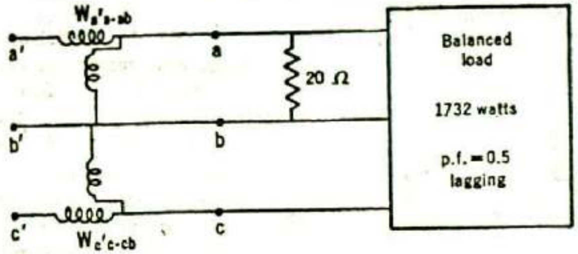


Fig. 30. See Problem 28.

of 115.4 volts and a phase sequence of $n'a'$-$n'b'$-$n'c'$. Find the following quantities and express all complex quantities with reference to $\mathbf{V}_{ab}$.

(a) $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, $\mathbf{V}_{ca}$.

(b) $\mathbf{I}_{ab}$, $\mathbf{I}_{bc}$, $\mathbf{I}_{ca}$.

(c) $\mathbf{I}_{a'a}$, $\mathbf{I}_{b'b}$, $\mathbf{I}_{c'c}$.

(d) The sum of the readings of the wattmeters W_a, W_b, W_c when they are connected as shown.

(e) The individual readings of wattmeters W_a, W_b, W_c if the common point O is connected to line $b'b$.

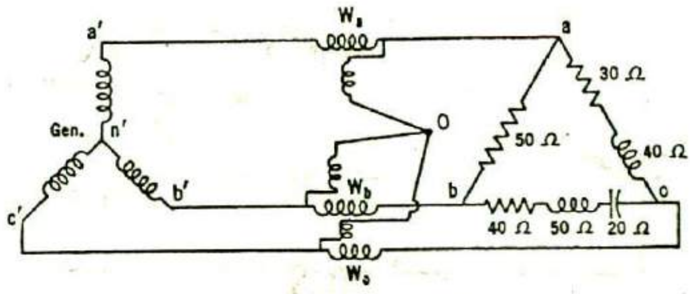


Fig. 31. See Problem 29.

30. The line-to-line voltages of a three-phase system are $V_{ab} = 200$, $V_{bc} = 150$, and $V_{ca} = 120$ volts. Write the polar expressions for $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{ca}$ with respect to $\mathbf{V}_{ab}$ as reference for both phase sequences.

31. Refer to Fig. 2. In a particular case measurements yield $V_{ab} = 140$, $V_{bc} = 120$, $V_{ca} = 150$, $V_{an} = 200$, $V_{bn} = 80$, and $V_{cn} = 104.2$ volts. Draw the qualitative phasor diagram of the voltages for sequence abc, and determine analytically the complex expressions for each of the voltages with respect to V_{ab} as a reference.

32. Refer to Fig. 1. In a particular case measurements yield $I_{a'a} = 20$, $I_{b'b} = 14$, $I_{c'c} = 15$, $I_{ab} = 12$, $I_{bc} = 2$, and $I_{ca} = 15$ amperes. For the line-current sequence of $a'a$–$c'c$–$b'b$ solve the qualitative phasor diagram analytically, and determine the complex expressions for each of the currents with respect to I_{ab} as a reference.

33. Calculate the line currents in Problem 16 by the loop-current method.

34. Refer to example 13, pages 402–403, including Fig. 26. Solve for $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$ by the loop-current method, neglecting the resistive components of *all* branch impedances for a voltage sequence E_{na}–E_{nc}–E_{nb}. (Results may be left in the form of the ratio of two matrices.)

35. In Fig. 32, $L_{ab} = L_{cb} = 0.01$ henry and the coefficient of coupling is 0.5. Assume no resistances or inductances except as indicated on the figure. The sequence of the balanced driving voltages is $n'a'$–$n'b'$–$n'c'$, and $E_{n'a'} = 57.7\underline{/90°}$ volts. For $\omega = 1000$ radians per second calculate the line and phase currents for the load. Use Maxwell's cyclic-current method.

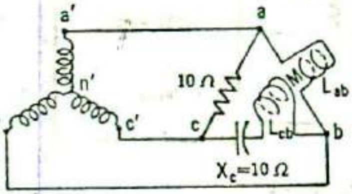


FIG. 32. See Problems 35 and 36.

36. Set up the determinant form of the solution for $I_{aa'}$ in Problem 35 if 3 ohms pure resistance is inserted in each line to the load and the same sequence and reference as specified in Problem 35 are employed. For uniformity in checking results, use loop currents as follows:

$$\text{Loop current } I_1 = I_{a'acc'}$$

$$\text{Loop current } I_2 = I_{c'cbb'}$$

$$\text{Loop current } I_3 = I_{a'n'b'ba}$$

37. Solve for $I_{a'a}$, $I_{b'b}$, and $I_{c'c}$ in Fig. 33 if $E_{n'a'} = 1350 + j0$ volts, $E_{n'b'} = -675 - j1170$ volts, and $E_{n'c'} = -675 + j1170$ volts.

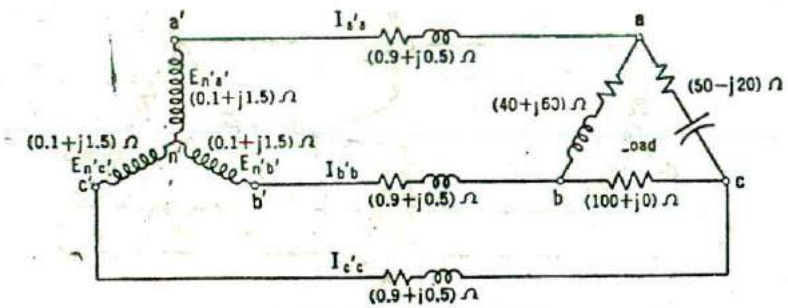


FIG. 33. See Problem 37.

chapter X Transmission Line Calculations

Line Constituents. A transmission line consists of the equivalent of two or more electrical conductors for the purpose of transmitting electric energy. For single-phase transmission the line may consist of a single conductor with a ground return or of two ordinary wires. For three-phase transmission, three wires are generally used although in some installations a neutral wire or its equivalent is employed. The wires of a transmission line are separated by some dielectric as air for overhead transmission, or by other insulating materials as in cables. Since the two conductors are separated by a dielectric, they form a condenser, the capacitance of which is uniformly distributed along the wires. When a difference of potential is applied between the wires, charging current flows. This effect could be simulated by a large number of condensers connected between the two wires as shown in Fig. 1. V_s denotes the sending-end voltage, and V_r represents receiver-end voltage. A representation of this kind is approximate because it shows the shunted capacitance lumped at certain points instead of being uniformly distributed. Within reasonable limits of accuracy it is permissible to make line calculations on the basis of lumped capacitance. Under the conditions of relatively low voltage and relatively short distances the shunted capacitance can even be neglected without seriously affecting the accuracy.

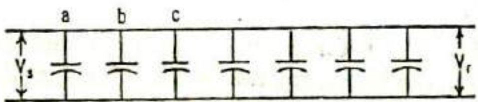


FIG. 1. Distributed shunted capacitance of a transmission line simulated by a large number of shunted condensers.

In addition to shunted capacitance the line has series resistance and inductance or inductive reactance. Thus the sections between con-

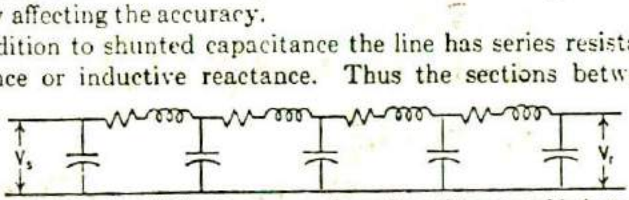


FIG. 2. Modification of Fig. 1 to account for series resistance and inductance of a transmission line.

densers, like ab and $a'b'$, bc and $b'c'$, etc., form loops through which flux will be set up by the mmf of the current flowing in the wires. These sections also have resistance. Hence, to account for these parameters, Fig. 1 should be modified to appear as shown in Fig. 2. Strictly speaking, each condenser should be shunted by a non-inductive resistance to account for any leakage of current from conductor to conductor because of imperfect insulation, moisture content of the air, and other factors. On a clear dry day the leakage is so small that it may usually be neglected. The greater the number of sections, like those shown in Fig. 2, into which the line is divided, the more nearly it will simulate the actual line which has uniformly distributed parameters. If more than two or three shunted condensers are used, it is just about as simple to calculate the line by assuming uniformly distributed parameters instead of concentrated quantities. Three of the usual arrangements of concentrated parameters will be considered.

The T Line. The T representation of a line is shown in Fig. 3. When all the shunted capacitance, C, of the line is concentrated in one condenser and half of the total series impedance, Z, is placed in each arm as indicated in Fig. 3, the circuit is known as the nominal T line. It is called nominal because the representation is only approximately correct. When the circuit parameters indicated in Fig. 3 are multiplied by certain hyperbolic correction factors,[1] the T thus formed represents the line exactly between terminals (V_s and V_r) and it then becomes the exact equivalent T. Calling Y the admittance due to the shunted capacitance C and using the quantities as labeled in Fig. 3, the determination of V_s in terms of the receiver voltage and current is made as follows.

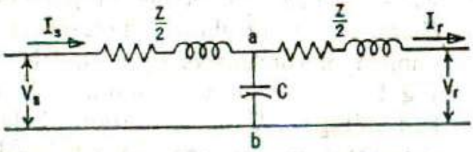


FIG. 3. T representation of a transmission line.

$$\mathbf{V}_{ab} = \mathbf{V}_r + \mathbf{I}_r\frac{\mathbf{Z}}{2}$$

$$\mathbf{I}_{ab} = \mathbf{V}_{ab}\mathbf{Y}$$

$$\mathbf{I}_s = \mathbf{I}_r + \mathbf{I}_{ab} = \mathbf{I}_r + \mathbf{Y}\left(\mathbf{V}_r + \mathbf{I}_r\frac{\mathbf{Z}}{2}\right) \quad (1)$$

$$\mathbf{V}_s = \mathbf{V}_{ab} + \mathbf{I}_s\frac{\mathbf{Z}}{2}$$

$$= \left(\mathbf{V}_r + \mathbf{I}_r\frac{\mathbf{Z}}{2}\right) + \left[\mathbf{I}_r + \mathbf{Y}\left(\mathbf{V}_r + \mathbf{I}_r\frac{\mathbf{Z}}{2}\right)\right]\frac{\mathbf{Z}}{2}$$

$$= \mathbf{V}_r\left(1 + \frac{\mathbf{YZ}}{2}\right) + \mathbf{I}_r\left(\mathbf{Z} + \frac{\mathbf{YZ}^2}{4}\right) \quad (2)$$

[1] See "Hyperbolic Functions Applied to Electrical Engineering," by A. E. Kennelly or "Electric Circuits Theory and Applications," by O. G. C. Dahl.

From equation (1),

$$\mathbf{I}_s = \mathbf{I}_r\left(1 + \frac{\mathbf{YZ}}{2}\right) + \mathbf{YV}_r \tag{3}$$

Equations (2) and (3) give the sending-end voltage and current in complex form. As indicated, all quantities in the equations must be expressed in vector form. The receiver current must be properly related in complex form to the receiver voltage. The power factor of the load determines the angle between $\mathbf{V}_r$ and $\mathbf{I}_r$. $\mathbf{V}_s$ and $\mathbf{I}_s$ being in complex form, power input may be determined in the usual way. The vector diagram of the T circuit of Fig. 3 is shown in Fig. 4. This diagram follows the above equations for calculating $\mathbf{V}_s$ and $\mathbf{I}_s$.

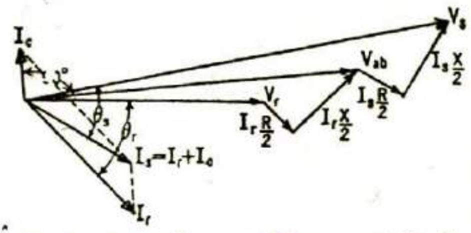


FIG. 4. Vector diagram of T representation in FIG. 3.

Problem 1. A 60-cycle, 3-phase line 200 miles long has a shunted capacitance to neutral per mile of 146×10^{-4} μf, an inductive reactance of 0.78 ohm per wire per mile, and a resistance of 0.42 ohm per wire per mile. The receiver voltage is 100,000 volts between lines. Use the nominal T line, and find the sending voltage and current for an 0.8 power-factor lagging load requiring 75 amperes per line at the receiver. *Ans.* 64,600 $\underline{/7.4°}$ volts, 62.3 $\underline{/24°}$ amperes.

The π Line. If one-half of the total line capacitance is concentrated at each end of the line and all the series resistance and reactance are concentrated at the center as shown in Fig. 5, the resultant configuration portrays the nominal π representation of a transmission line. Like the T line it is possible to alter the parameters by applying hyperbolic correction factors to obtain a π circuit which yields the correct relations between terminals. A π circuit thus corrected is called an exact equivalent π.

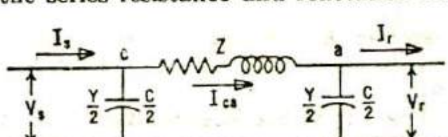


FIG. 5. π representation of a transmission line.

The π circuit is easily solved through a procedure similar to that employed for the T circuit.

$$\mathbf{I}_{ab} = \mathbf{V}_r\frac{\mathbf{Y}}{2}$$

$$\mathbf{I}_{ca} = \mathbf{I}_r + \mathbf{I}_{ab} = \mathbf{I}_r + \mathbf{V}_r\frac{\mathbf{Y}}{2}$$

$$\mathbf{V}_s = \mathbf{V}_r + \mathbf{I}_{ca}\mathbf{Z} = \mathbf{V}_r + \left(\mathbf{I}_r + \mathbf{V}_r\frac{\mathbf{Y}}{2}\right)\mathbf{Z}$$

$$= \mathbf{V}_r\left(1 + \frac{\mathbf{ZY}}{2}\right) + \mathbf{I}_r\mathbf{Z} \tag{4}$$

$$\mathbf{I}_s = \mathbf{I}_{ca} + \mathbf{I}_{cd}$$

$$\mathbf{I}_{cd} = \mathbf{V}_s\frac{\mathbf{Y}}{2} = \left[\mathbf{V}_r\left(1 + \frac{\mathbf{ZY}}{2}\right) + \mathbf{I}_r\mathbf{Z}\right]\frac{\mathbf{Y}}{2}$$

$$\mathbf{I}_s = \mathbf{I}_r + \mathbf{V}_r\frac{\mathbf{Y}}{2} + \left[\mathbf{V}_r\left(1 + \frac{\mathbf{ZY}}{2}\right) + \mathbf{I}_r\mathbf{Z}\right]\frac{\mathbf{Y}}{2}$$

$$= \mathbf{I}_r\left(1 + \frac{\mathbf{ZY}}{2}\right) + \mathbf{V}_r\mathbf{Y}\left(1 + \frac{\mathbf{ZY}}{4}\right) \tag{5}$$

Equations (4) and (5) are the solution of the π representation of a transmission line. The vector diagram of the π circuit is shown in Fig. 6, and the above calculations follow this diagram.

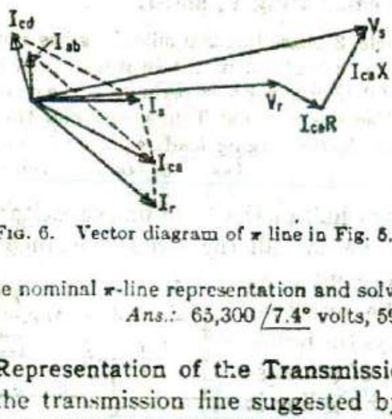


FIG. 6. Vector diagram of π line in Fig. 5.

Problem 2. Use the nominal π-line representation and solve Problem 1.

Ans.: 65,300 $\underline{/7.4°}$ volts, 59.75 $\underline{/22.2°}$ amperes.

The Steinmetz Representation of the Transmission Line. Another representation of the transmission line suggested by C. P. Steinmetz which yields approximately correct results is shown in Fig. 7. In the

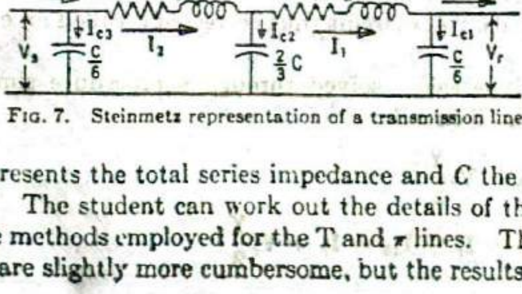


FIG. 7. Steinmetz representation of a transmission line.

figure, Z represents the total series impedance and C the total shunted capacitance. The student can work out the details of the solution by following the methods employed for the T and π lines. This circuit and the solution are slightly more cumbersome, but the results are generally somewhat closer to the theoretically correct values than those obtained from the use of the nominal T or π sections. The calculations must follow the vector diagram shown in Fig. 8.

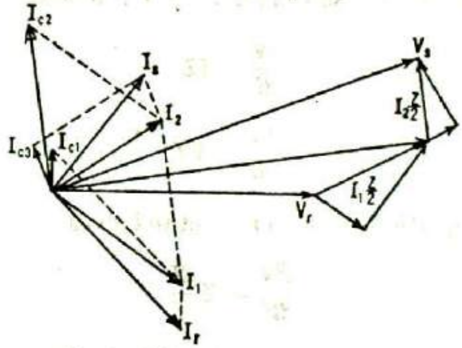


FIG. 8. Vector diagram of Fig. 7.

Problem 3. Derive the equations for the sending-end voltage and current in terms of the receiver quantities for the Steinmetz representation of a transmission line.

$$Ans.:\ \mathbf{V}_s = \left(1 + \frac{\mathbf{ZY}}{2} + \frac{\mathbf{Z}^2\mathbf{Y}^2}{36}\right)\mathbf{V}_r + \mathbf{Z}\left(1 + \frac{\mathbf{ZY}}{6}\right)\mathbf{I}_r$$

$$\mathbf{I}_s = \left(1 + \frac{5\mathbf{ZY}}{36} + \frac{\mathbf{Z}^2\mathbf{Y}^2}{216}\right)\mathbf{YV}_r + \left(1 + \frac{\mathbf{ZY}}{2} + \frac{\mathbf{Z}^2\mathbf{Y}^2}{36}\right)\mathbf{I}_r$$

Problem 4. Solve Problem 1 according to the Steinmetz representation of the line.

Ans.: 64,900 $\angle 7.3°$ volts, 60.9 $\angle 22.9°$ amperes.

Exact Solution of a Long Line with Uniformly Distributed Parameters. In the line shown in Fig. 9 let the series impedance per mile be **Z**, the shunted admittance per mile **Y**, and the length of the line considered *l*.

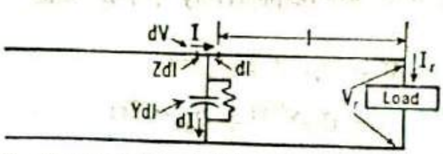


FIG. 9. Circuit used for deriving the exact solution of a transmission line.

The elementary voltage drop in the element *dl* is the current **I** in the element times the impedance **Z** *dl* of the element. Considering only the space variation of **V** and **I**,

$$d\mathbf{V} = \mathbf{IZ}\,dl \tag{6}$$

The current leaving the line over the length dl is the voltage V times the shunted admittance $\mathbf{Y}\,dl$ for the element. Thus

$$d\mathbf{I} = \mathbf{VY}\,dl \tag{7}$$

Equations (6) and (7) are solved for V and I as follows:

$$\frac{d\mathbf{V}}{dl} = \mathbf{I\dot{Z}} \tag{8}$$

$$\frac{d\mathbf{I}}{dl} = \mathbf{VY} \tag{9}$$

Differentiating equation (8) with respect to l gives

$$\frac{d^2\mathbf{V}}{dl^2} = \mathbf{Z}\frac{d\mathbf{I}}{dl} \tag{10}$$

Substituting equation (9) in equation (10),

$$\frac{d^2\mathbf{V}}{dl^2} = \mathbf{ZYV} \tag{11}$$

This is a linear differential equation of the second order, the solution of which can be shown[2] to be of the form

$$\mathbf{V} = C_1\epsilon^{\mathbf{m}_1 l} + C_2\epsilon^{\mathbf{m}_2 l}$$

where C_1 and C_2 are constants of integration and m_1 and m_2 are roots of the auxiliary equation, namely,

$$\mathbf{m}^2 = \mathbf{ZY}$$

$$\mathbf{m} = +\sqrt{\mathbf{ZY}} \quad \text{or} \quad -\sqrt{\mathbf{ZY}} \tag{12}$$

The two roots $\mathbf{m}_1$ and $\mathbf{m}_2$ are respectively $+\sqrt{\mathbf{ZY}}$ and $-\sqrt{\mathbf{ZY}}$. Therefore

$$\begin{aligned}\mathbf{V} &= C_1\epsilon^{\mathbf{m}_1 l} + C_2\epsilon^{\mathbf{m}_2 l} \\ &= C_1\epsilon^{\sqrt{\mathbf{ZY}}l} + C_2\epsilon^{-\sqrt{\mathbf{ZY}}l}\end{aligned} \tag{13}$$

From equation (8),

$$\mathbf{I} = \frac{1}{\mathbf{Z}}\frac{d\mathbf{V}}{dl} \tag{14}$$

[2] See any book on differential equations, such as "Differential Equations," by D. A. Murray, p. 63.

Differentiating equation (13) and substituting the result in (14) gives

$$\mathbf{I} = C_1 \frac{\sqrt{\mathbf{ZY}}}{\mathbf{Z}} \epsilon^{\sqrt{\mathbf{ZY}}l} - \frac{C_2\sqrt{\mathbf{ZY}}}{\mathbf{Z}} \epsilon^{-\sqrt{\mathbf{ZY}}l}$$

$$= C_1\sqrt{\mathbf{Y}/\mathbf{Z}}\, \epsilon^{\sqrt{\mathbf{ZY}}l} - C_2\sqrt{\mathbf{Y}/\mathbf{Z}}\, \epsilon^{-\sqrt{\mathbf{ZY}}l} \quad (15)$$

The constants of integration C_1 and C_2 in equations (13) and (15) can be evaluated from known boundary conditions. In this case the boundary conditions at the receiver are assumed to be known. Thus in Fig. 9 when

$$l = 0 \quad (16)$$

$$\mathbf{I} = \mathbf{I}_r \quad (17)$$

and

$$\mathbf{V} = \mathbf{V}_r \,. \quad (18)$$

Substituting equations (16), (17), and (18) in equations (13) and (15),

$$\mathbf{V}_r = C_1 + C_2 \quad (19)$$

$$\mathbf{I}_r = C_1 \sqrt{\mathbf{Y}/\mathbf{Z}} - C_2 \sqrt{\mathbf{Y}/\mathbf{Z}} \quad (20)$$

Equations (19) and (20) are now solved simultaneously for C_1 and C_2. Multiplying equation (19) by $\sqrt{\mathbf{Y}/\mathbf{Z}}$ gives

$$\mathbf{V}_r\sqrt{\mathbf{Y}/\mathbf{Z}} = C_1 \sqrt{\mathbf{Y}/\mathbf{Z}} + C_2 \sqrt{\mathbf{Y}/\mathbf{Z}} \quad (21)$$

Adding equations (20) and (21),

$$\mathbf{I}_r + \sqrt{\mathbf{Y}/\mathbf{Z}}\, \mathbf{V}_r = 2C_1 \sqrt{\mathbf{Y}/\mathbf{Z}}$$

$$C_1 = \frac{\mathbf{V}_r + \mathbf{I}_r\sqrt{\mathbf{Z}/\mathbf{Y}}}{2} \quad (22)$$

Subtracting equation (20) from equation (21),

$$\mathbf{V}_r \sqrt{\mathbf{Y}/\mathbf{Z}} - \mathbf{I}_r = 2C_2 \sqrt{\mathbf{Y}/\mathbf{Z}}$$

$$C_2 = \frac{\mathbf{V}_r - \mathbf{I}_r \sqrt{\mathbf{Z}/\mathbf{Y}}}{2} \quad (23)$$

It is apparent that C_1 and C_2 in the above equations are complex coefficients and might have been written in bold-face type. The expressions for voltage and current at any distance l from the receiver are obtained by substituting equations (22) and (23) in equations (13) and (15). Then

$$\mathbf{V} = \left(\frac{\mathbf{V}_r + \mathbf{I}_r\sqrt{\mathbf{Z}/\mathbf{Y}}}{2}\right)\epsilon^{\sqrt{\mathbf{ZY}}l} + \left(\frac{\mathbf{V}_r - \mathbf{I}_r\sqrt{\mathbf{Z}/\mathbf{Y}}}{2}\right)\epsilon^{-\sqrt{\mathbf{ZY}}l} \qquad (24)$$

$$\mathbf{I} = \left(\frac{\mathbf{I}_r + \mathbf{V}_r\sqrt{\mathbf{Y}/\mathbf{Z}}}{2}\right)\epsilon^{\sqrt{\mathbf{ZY}}l} + \left(\frac{\mathbf{I}_r - \mathbf{V}_r\sqrt{\mathbf{Y}/\mathbf{Z}}}{2}\right)\epsilon^{-\sqrt{\mathbf{ZY}}l} \qquad (25)$$

Equations (24) and (25) may be used as the working equations for the exact solution of long lines. Under certain condition it is convenient to have equations (24) and (25) expressed in terms of hyperbolic functions. This is done as follows.

From equation (24),

$$\mathbf{V} = \frac{\mathbf{V}_r}{2}\epsilon^{\sqrt{\mathbf{ZY}}l} + \frac{\mathbf{I}_r\sqrt{\mathbf{Z}/\mathbf{Y}}}{2}\epsilon^{\sqrt{\mathbf{ZY}}l} + \frac{\mathbf{V}_r}{2}\epsilon^{-\sqrt{\mathbf{ZY}}l} - \frac{\mathbf{I}_r\sqrt{\mathbf{Z}/\mathbf{Y}}}{2}\epsilon^{-\sqrt{\mathbf{ZY}}l}$$

$$= \mathbf{V}_r\left(\frac{\epsilon^{\sqrt{\mathbf{ZY}}l} + \epsilon^{-\sqrt{\mathbf{ZY}}l}}{2}\right) + \mathbf{I}_r\sqrt{\mathbf{Z}/\mathbf{Y}}\left(\frac{\epsilon^{\sqrt{\mathbf{ZY}}l} - \epsilon^{-\sqrt{\mathbf{ZY}}l}}{2}\right)$$

Since the analytic definition of

$$\sinh\theta = \frac{\epsilon^{\theta} - \epsilon^{-\theta}}{2}$$

and

$$\cosh\theta = \frac{\epsilon^{\theta} + \epsilon^{-\theta}}{2}$$

$$\mathbf{V} = \mathbf{V}_r\cosh\sqrt{\mathbf{ZY}}l + \mathbf{I}_r\sqrt{\mathbf{Z}/\mathbf{Y}}\sinh\sqrt{\mathbf{ZY}}l \qquad (26)$$

Similarly

$$\mathbf{I} = \mathbf{I}_r\cosh\sqrt{\mathbf{ZY}}l + \mathbf{V}_r\sqrt{\mathbf{Y}/\mathbf{Z}}\sinh\sqrt{\mathbf{ZY}}l \qquad (27)$$

Equations (26) and (27) are particularly convenient to use if tables of complex hyperbolic functions are available; otherwise, equations (24) and (25) may be more convenient.[3]

Physical Interpretation of Equations for Exact Solution. Equations (24) and (25) may be modified somewhat to make their physical significance more apparent. Since $\sqrt{\mathbf{ZY}}$ is a complex expression, we may substitute an expression such as $(\alpha + j\beta)$ for it. Also, letting

[3] See "Tables" or "Charts of Complex Hyperbolic Functions," by A. E. Kennelly, Harvard University Press.

$Z_0 = \sqrt{Z/Y}$ and $Y_0 = \sqrt{Y/Z}$, equations (24) and (25) may be written:

$$V = \left(\frac{V_r + I_r Z_0}{2}\right)\epsilon^{(\alpha+j\beta)l} + \left(\frac{V_r - I_r Z_0}{2}\right)\epsilon^{-(\alpha+j\beta)l} \quad (28)$$

$$I = \left(\frac{I_r + V_r Y_0}{2}\right)\epsilon^{(\alpha+j\beta)l} + \left(\frac{I_r - V_r Y_0}{2}\right)\epsilon^{-(\alpha+j\beta)l} \quad (29)$$

Recognizing that $\epsilon^{(\alpha+j\beta)l} = \epsilon^{\alpha l}\epsilon^{j\beta l}$ and that $\epsilon^{-(\alpha+j\beta)l} = \epsilon^{-\alpha l}\epsilon^{-j\beta l}$, we may write equations (28) and (29) as follows:

$$V = \left(\frac{V_r + I_r Z_0}{2}\right)\epsilon^{\alpha l}\epsilon^{j\beta l} + \left(\frac{V_r - I_r Z_0}{2}\right)\epsilon^{-\alpha l}\epsilon^{-j\beta l} \quad (30)$$

$$I = \left(\frac{I_r + V_r Y_0}{2}\right)\epsilon^{\alpha l}\epsilon^{j\beta l} + \left(\frac{I_r - V_r Y_0}{2}\right)\epsilon^{-\alpha l}\epsilon^{-j\beta l} \quad (31)$$

The quantity $\sqrt{ZY} = (\alpha + j\beta)$ is called the propagation constant. It determines how the wave is propagated with reference to change in magnitude and phase along the line. Equation (30) consists of two parts. The first, $\left(\frac{V_r + I_r Z_0}{2}\right)\epsilon^{\alpha l}\epsilon^{j\beta l}$, represents a quantity that increases in magnitude ($\epsilon^{\alpha l}$ increases) as we go from the receiver to the sending end or it becomes smaller as we proceed from the sending to the receiver end. This term must therefore represent a voltage wave which is being propagated from the sending to the receiver end. Hence it is called the direct wave or direct component. The direct component is analogous to a wave started in a body of water. As the wave leaves the source it becomes smaller and smaller. The second part of equation (30) is $\left(\frac{V_r - I_r Z_0}{2}\right)\epsilon^{-\alpha l}\epsilon^{-j\beta l}$. As we proceed from the load to the generator this component becomes smaller, since l increases and $\epsilon^{-\alpha l}$ decreases. Hence this wave must be originating at the receiver, and it is therefore called the reflected wave. It is analogous to the phenomena in a body of water as a wave strikes a bank. A reflection occurs, and a wave is then seen traveling away from the bank with gradually diminishing magnitude. Since, for a given distance of travel, α determines the magnitude of the wave, it is a measure of how much the wave is increased or decreased in magnitude, or, in other words, attenuated. For this reason it is called the attenuation constant or factor. The attenuation factor is the real part of the propagation constant. The factors $\epsilon^{j\beta l}$ and $\epsilon^{-j\beta l}$ will be recognized as operators which produce opposite rotations. The operator $\epsilon^{j\beta l}$ causes the direct component to advance in phase from its position at the load as we proceed from the receiving to the sending end, while $\epsilon^{-j\beta l}$ causes the reflected wave to fall behind its phase position at the receiver. Since β determines the change in phase for a given distance l along the line, it is called the phase constant. It is sometimes called the wave length constant because it determines the distance along the line over which a complete wave is subtended. This will be explained in more detail later. The loci of the variation of the direct and reflected waves can be represented as spirals, as shown in Fig. 10. The sum *od* of the direct and reflected waves of voltage at any point along the line such as at βl gives the resultant voltage at that point. When βl is 90°, the direct component of voltage *oa* is opposite to the reflected component *ob*. The resultant *oc*, which is the voltage of the line at this 90° or quarter-wave-length point, may be very small because of the cancellation effect of the two waves. A generator producing a low voltage, if connected at this point, could subtend a comparatively high voltage at the receiver. This is essentially a resonance phenomenon and is called quarter-wave resonance. As βl increases from this 90° point the voltage of the line increases until βl becomes 180°.

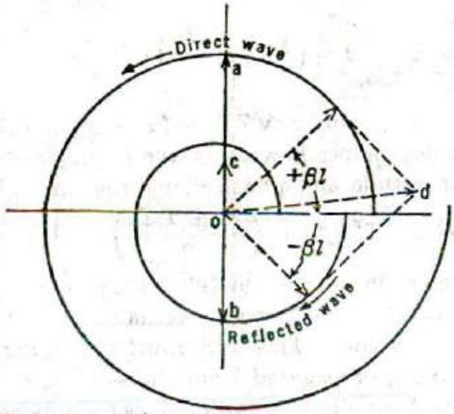


Fig. 10. Variation of direct and reflected waves of voltage with respect to the line angle βl for a particular case.

Here the direct and reflected waves add. This phenomenon is called half-wave resonance. As βl increases to 270° the direct and reflected waves are again opposite (as at quarter-wave) and we then have three-quarter-wave resonance.

Surge Impedance. Inspection of equation (28) makes it apparent that dimensionally $I_r Z_0$ must be a voltage. Hence Z_0 must be an impedance. Further evidence of this fact is obtained when it is remembered that $Z_0 = \sqrt{Z/Y}$. The reciprocal of Y is dimensionally an impedance, and the $\sqrt{Z/Y}$ then becomes $\sqrt{\text{impedance}^2}$ which is an impedance. Hence the quantity $Z_0 = \sqrt{Z/Y}$ is called the surge impedance of the line. The reciprocal, $\sqrt{Y/Z}$, is called surge admittance. The surge impedance is the impedance offered to the propagation of a wave along a line. In effect it is the impedance an advancing wave of voltage or current encounters as it travels along the line.

Terminal Reflections. The receiver voltage $\mathbf{V}_r$ is $\mathbf{I}_r\mathbf{Z}_r$ where $\mathbf{Z}_r$ is the impedance of the load. If $\mathbf{Z}_r$ is made equal to $\mathbf{Z}_0$ the receiver voltage $\mathbf{V}_r$ would equal $\mathbf{I}_r\mathbf{Z}_0$. Then the reflected wave in equation (30) is zero and the equation of the voltage along the line becomes

$$\mathbf{V} = \left(\frac{\mathbf{V}_r + \mathbf{V}_r}{2}\right)\epsilon^{\alpha l}\epsilon^{j\beta l} = \mathbf{V}_r\epsilon^{\alpha l}\epsilon^{j\beta l} \tag{32}$$

This variation is exponential in character, and no terminal reflections exist. The voltage, V, increases exponentially in magnitude as we proceed from the receiving end to the sending end. Simultaneously with the increase in magnitude there is a uniform advance in phase of V with respect to the load voltage $\mathbf{V}_r$. The wave encounters the same impedance (surge impedance) at the load as it did while advancing along the line. This termination makes the line behave as if it were infinitely long. Hence a line terminated in its surge impedance is sometimes called an infinite line. In communication work, terminating a line in an impedance equal to the surge impedance is sometimes called matching.

If a long line is short-circuited at the receiver $V_r = 0$ and equation (30) becomes

$$\mathbf{V} = \frac{\mathbf{I}_r\mathbf{Z}_0}{2}\epsilon^{\alpha l}\epsilon^{j\beta l} - \frac{\mathbf{I}_r\mathbf{Z}_0}{2}\epsilon^{-\alpha l}\epsilon^{-j\beta l} \tag{33}$$

Where l is 0,

$$\mathbf{V}_{l=0} = \frac{\mathbf{I}_r\mathbf{Z}_0}{2} - \frac{\mathbf{I}_r\mathbf{Z}_0}{2}$$

$$= \text{direct wave} - \text{reflected wave}$$

Thus it may be said that the voltage is reflected with a change in sign. The current wave under the same conditions becomes

$$\mathbf{I}_{l=0} = \frac{\mathbf{I}_r}{2} + \frac{\mathbf{I}_r}{2}$$

$$= \text{direct wave} + \text{reflected wave}$$

It follows, then, that the current wave is reflected with the same sign or the direct and reflected waves of current add arithmetically at the receiver.

If the line is open-circuited at the receiver, $I_r = 0$. Imposing this condition on equations (30) and (31) shows the voltage wave to be reflected with the same sign and the current with a change in sign.

Velocity of Propagation. In the foregoing equations, distance along the line, namely l, has been considered the independent variable. The other independent variable, time, has been tacitly taken into account by the use of vector quantities. In the evaluation of the velocity of propagation the interrelation of time-phase and space-phase effects must be recognized.

It is evident from the use of β in the foregoing equations that this quantity determines the phase shift of **V** or **I** per unit length of line, and as such it represents a number of radians per unit length of line. The length of line required to effect a complete cycle or 2π radians of phase shift is

$$\lambda = \frac{2\pi}{\beta} \text{ units} \tag{34}$$

where λ and β are expressed in any consistent set of units. To simplify visualizing the phenomenon, consider only the voltage wave.

Since λ is the distance for a phase shift of 2π radians, it is the distance along the line (see Fig. 11) from one zero value say at a on the voltage wave to a corresponding zero value at b, 2π radians or 360° from the first zero point. The distance λ thus represents the length of line over which a complete space wave or cycle of voltage is subtended, and in consequence λ is called the wave length of the propagated wave. As time

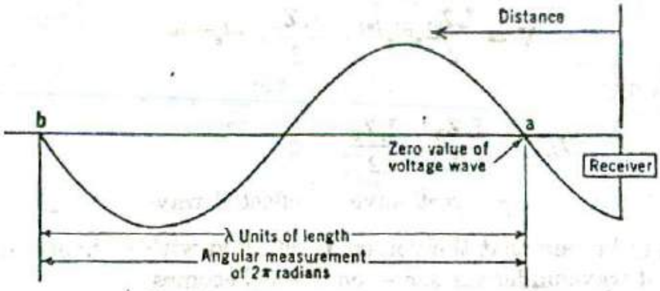


FIG. 11. A space wave or cycle from a to b.

elapses, the alternating voltage at point a will rise to a positive maximum, decrease to zero, then increase to a negative maximum, thence to zero. In this length of time, point b on the wave will have arrived at a in Fig. 11. In other words, during this length of time, the time for one cycle, $1/f$, all points on the wave will have traveled a distance of λ. The velocity of travel or propagation must then be $\frac{\lambda}{1/f}$, or λf units of length per second. Hence the velocity of propagation is

$$v = \lambda f = \frac{2\pi}{\beta} f \tag{35}$$

Equation (34) shows that the wave length for any line is determined by the quantity β. Hence β is often termed the wave-length constant, and it may be evaluated in terms of the circuit parameters from the original substitution, namely, $\sqrt{\mathbf{ZY}} = \alpha + j\beta$. Since $\mathbf{Z} = R + jX$ and $\mathbf{Y} = g - jb$, it follows that

$$\alpha + j\beta = \sqrt{(R + jX)(g - jb)} \tag{36}$$

$$\alpha^2 + j2\alpha\beta - \beta^2 = Rg - jRb + jgX + bX$$

$$\alpha^2 - \beta^2 = Rg + bX \tag{37}$$

$$2\alpha\beta = gX - Rb \tag{38}$$

Solving equations (37) and (38) simultaneously for β gives

$$\beta = \sqrt{\frac{\pm ZY - (Rg + bX)}{2}} \tag{39}$$

The preceding derivation shows that all terms in equation (39) are expressed algebraically and not in complex form. All the quantities are per unit values, that is, per centimeter, per mile, etc.

It is interesting to find the velocity of propagation under the conditions of zero series resistance and a negligibly small value of g. Imposing these conditions gives

$$\beta = \sqrt{\frac{\pm bX - bX}{2}} = \sqrt{-bX}$$

The two signs before ZY in equation (39) and before bX above resulted from the solution of a quadratic equation. As often occurs, one of such solutions has no physical reality. If the plus sign were used in the algebraic manipulation above, β would be zero, which would in turn give an infinite velocity of propagation. Obviously, this is impossible. When making arithmetic computations the proper sign to employ is that which will give a physically possible and plausible result. Had the equations been based on $g + jb$, it would have been necessary to use the plus sign before the ZY and bX above. Since b is the shunted susceptance due to the line capacitance, it must carry a negative sign upon substitution of a numerical value for it in accordance with the conventions employed in this book. Substituting the value of β above in equation (35) gives

$$v = \frac{2\pi f}{\sqrt{-bX}} = \frac{2\pi f}{\sqrt{2\pi fC \times 2\pi fL}} = \frac{1}{\sqrt{LC}} \text{ (for } r = 0 \text{ and } g = 0) \tag{40}$$

In equation (40) v is in miles per second if L is expressed in henrys per mile and C in farads per mile. If one further assumption is made in equation (40), namely, that the inductance due to the flux within the conductor is negligible, the velocity will be the same as that of light. This is illustrated by example 2, pages 429–433.

Example 1. An open-wire telephone line has a resistance of 10.26 ohms, an inductance of 0.00366 henry, and a capacitance of 0.00822 μf per loop mile (one mile of outgoing plus one mile of return conductor). Calculate the velocity of propagation for a 200-cycle and also for a 2000-cycle frequency, assuming that the values of R, L, and C are the same at both frequencies. Assume $g = 0$ in both cases.

At 200 cycles

$$X = 2\pi\, 200 \times 0.00366 = 4.6 \text{ ohms per loop mile}$$

$$Z = \sqrt{10.26^2 + 4.6^2} = 11.22 \text{ ohms per loop mile}$$

$$b = -2\pi f C = -2\pi\, 200 \times 0.00822 \times 10^{-6} = -10.32 \times 10^{-6} \text{ mho per loop mile}$$

$$Y = 10.32 \times 10^{-6} \text{ mho per loop mile}$$

$$\beta = \sqrt{\frac{\pm 11.22 \times 10.32 \times 10^{-6} - (-10.32 \times 10^{-6} \times 4.6)}{2}}$$

$$= \sqrt{\frac{163.5 \times 10^{-6}}{2}} = 9.05 \times 10^{-3}$$

$$v = \frac{2\pi f}{\beta} = \frac{2\pi\, 200}{9.05 \times 10^{-3}} = 139 \times 10^3 = 139{,}000 \text{ miles per second}$$

At 2000 cycles

$$X = 2\pi\, 2000 \times 0.00366 = 46 \text{ ohms per loop mile}$$

$$Z = \sqrt{10.26^2 + 46^2} = 47.1 \text{ ohms per loop mile}$$

$$b = -2\pi f C = -2\pi\, 2000 \times 0.00822 \times 10^{-6}$$

$$= -103.2 \times 10^{-6} \text{ mho per loop mile}$$

$$Y = 103.2 \times 10^{-6} \text{ mho per loop mile}$$

$$\beta = \sqrt{\frac{\pm 47.1 \times 103.2 \times 10^{-6} - (-103.2 \times 10^{-6} \times 46)}{2}}$$

$$= \sqrt{\frac{9610 \times 10^{-6}}{2}} = 69.3 \times 10^{-3}$$

$$v = \frac{2\pi f}{\beta} = \frac{2\pi\, 2000}{69.3 \times 10^{-3}} = 181{,}400 \text{ miles per second}$$

If parameters per mile to ground or neutral were used, Z would be halved, Y and b doubled and β would be the same.

Confusion sometimes arises as to what the velocity of propagation refers to physically. The velocity of propagation of a voltage or current wave is the velocity at which the impulse or pressure travels. It is not the velocity of current flow. The velocity of current flow for normal current densities in copper is very low, although the velocity of the pressure wave is high. The phenomenon is somewhat analogous to the application of pressure at one end of a long pipe filled with water. The pressure appears at the far end of the pipe very soon after it is applied at the near end. However, the actual rate of flow of water in the pipe may be very low, especially if only a comparatively small stream is permitted to emerge at the far end.

Determination of Transmission Line Parameters. 1. *Inductance.* The inductance per wire is used in transmission line calculations. It may be derived as follows. Consider two parallel conductors as shown in

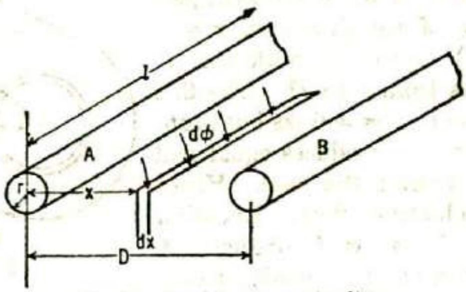


FIG. 12. Part of a two-wire line.

Fig. 12, each having a radius r and separated a distance D between centers. The fundamental equation of inductance when permeability is constant is

$$L = \frac{N\phi}{I} 10^{-9} \text{ henry}$$

where I is in abamperes and ϕ is in maxwells. The field intensity based on the law of Biot-Savart, at a distance of x centimeters from a long straight conductor carrying a current is $2I/x$ gilberts per centimeter, which in air is numerically equal to the flux density. Referring now to the open-wire line shown in Fig. 12,

$$d\phi = \left(\frac{2I}{x}\right)(l\,dx)$$

The total flux that exists outside of conductor A which causes an inductive effect on conductor A is

$$\phi_1 = \int_r^D \frac{2I}{x} l\,dx = 2lI \log_e \frac{D}{r}$$

$$L_1 = \frac{N\phi_1}{I} 10^{-9} = 2l \log_e \frac{D}{r} 10^{-9}$$

$$= 2 \times 2.3026 l \log_{10} \frac{D}{r} 10^{-9} \text{ henry} \quad (41)$$

where l is expressed in centimeters.

The flux included from $x = (D - r)$ to $x = (D + r)$ has some effect in inducing a net emf in the two conductors connected in series to form a coil. The effect is due to this flux linking all of conductor A and only part of conductor B. Integrating between the limits $x = r$ and $x = D$ includes the full effectiveness of the flux from $x = (D - r)$ to $x = D$ in causing the inductance. This balances the partial effectiveness of the flux from $x = D$ to $x = (D + r)$ which is neglected in taking the limits from r to D. The flux from $x = (D + r)$ to $x = \infty$ links both conductors and therefore produces equal and opposing emf's around the loop. Hence it has no net inductive effect. Equation (41) gives the inductance of conductor A due to all the flux on the outside of conductor A which is effective. The flux within the conductor causes some inductance which may be calculated as follows.

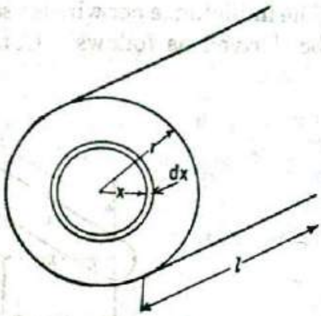


FIG. 13.

Assume that the current in conductor A is uniformly distributed across the cross-section. Let I' be the current per unit area. Refer to the cross-section of conductor A shown in Fig. 13. The total current responsible for the mmf causing flux through the element dx is $\pi x^2 I'$.

$$\text{mmf} = 4\pi NI = 4\pi\,(\pi x^2 I')$$

If the permeability of the conductor material is unity, the reluctance of the flux path formed by the element dx and a length of conductor l is

$$\mathcal{R} = \frac{2\pi x}{l\,dx} \text{ cgs units when } x \text{ is in centimeters}$$

$$d\phi = \frac{4\pi^2 x^2 I'}{\dfrac{2\pi x}{(l\,dx)}} = 2\pi x I' l\,dx \text{ maxwells}$$

The flux $d\phi$ links only the fibers of the conductor from the center to a distance x or $K\pi x^2$ fibers. To obtain the flux which links the whole conductor that produces the same effect as the actual flux which links $K\pi x^2$ fibers, it is only necessary to find the flux linking $K\pi r^2$ fibers (the entire conductor), which is equivalent to the flux $d\phi$ linking $K\pi x^2$ fibers. Calling the flux in question $d\phi_e$, we have for equivalent linkages

$$d\phi_e K\pi r^2 = d\phi K\pi x^2$$

or

$$d\phi_e = \frac{x^2}{r^2} d\phi$$

Hence

$$d\phi_e = \frac{x^2}{r^2}(2\pi x I' l\, dx)$$

$$\phi_e = \int_0^r \frac{2\pi x^3 I' l\, dx}{r^2} = \left(\frac{2\pi I' l}{r^2}\right)\left(\frac{r^4}{4}\right)$$

$$= \frac{\pi r^2 I' l}{2}$$

But $\pi r^2 I'$ is the total current I. Therefore

$$\phi_e = \frac{Il}{2}$$

The inductance due to this flux is

$$L_2 = \frac{N\phi_e}{I} 10^{-9} = \frac{1 \times Il\, 10^{-9}}{2I} = \frac{l\, 10^{-9}}{2} \text{ henry} \qquad (42)$$

The total inductance of conductor A is the sum of equations (41) and (42).

$$L = L_1 + L_2 = \left[\frac{l}{2} + 4.6052 l \log_{10}\frac{D}{r}\right] 10^{-9} \text{ henry} \qquad (43)$$

The inductance per mile is

$$L_{\text{mile}} = 0.5 \times 5280 \times 30.48\, 10^{-9}$$
$$+ 4.6052 \times 5280 \times 30.48 \times 10^{-9} \log_{10}\frac{D}{r}$$
$$= 0.805\, 10^{-4} + 0.741 \times 10^{-3} \log_{10}\frac{D}{r} \text{ henry} \qquad (44)$$

Equation (44) is the working equation. Usually the reactance is desired. It is found by multiplying the values obtained from equation (44) by the angular velocity $2\pi f$.

2. *Capacitance between Wires and to Ground.* The defining equation for capacitance is $C = Q/V$. The defining equation for difference of potential V is

$$V = \frac{W}{Q} = \frac{\text{work}}{\text{charge}}$$

The difference of electrostatic potential between two conductors is the work done in carrying a unit charge from the surface of one conductor to the other. Work is the product of force and the distance through which the force acts. By definition, if all quantities are expressed in the cgs electrostatic system of units, the force on a unit charge is numerically equal to the electrostatic field intensity. The electrostatic field intensity at point p, Fig. 14, at a perpendicular distance of r centimeters from a long straight wire is found as follows.

Let all quantities be expressed in the cgs electrostatic system of units, and let σ be the charge per unit length of wire. From Coulomb's law $f = QQ'/d^2$ dynes. Hence the force on a unit charge at point p due to a length of conductor dl is

$$df' = \frac{1 \times \sigma\, dl}{\rho^2}$$

where ρ is the distance in centimeters from p to dl. As θ varies between minus and plus 90° (on the basis of an infinite length of wire), it is

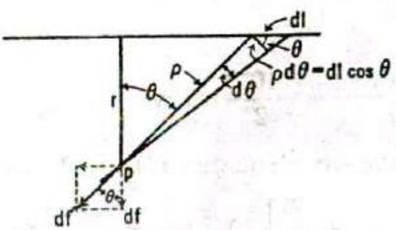


Fig. 14.

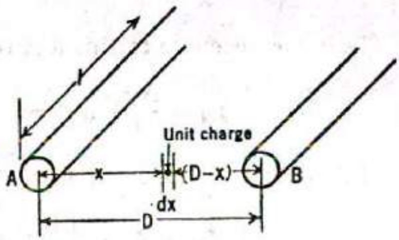


Fig. 15.

apparent that all the components of df' parallel to the wire add to zero. Therefore only the components perpendicular to the conductor need be added to obtain the resultant force on the unit charge.

$$df = df' \cos\theta$$
$$= \frac{\sigma\, dl}{\rho^2}\cos\theta = \frac{\sigma\rho\, d\theta}{\rho^2} = \frac{\sigma\, d\theta}{\rho}$$
$$= \frac{\sigma\, d\theta}{r/\cos\theta} = \frac{\sigma}{r}\cos\theta\, d\theta$$
$$f = \int_{-\pi/2}^{\pi/2} \frac{\sigma}{r}\cos\theta\, d\theta = \frac{2\sigma}{r}\text{ dynes} \tag{45}$$

The force on the unit charge in Fig. 15 is due to the effect of conductor A (say + charge) and that of conductor B (negative charge if A is positive).

$$f_A = \frac{2\sigma}{x}$$
$$f_B = \frac{2\sigma}{D - x}$$
$$f = f_A + f_B = \frac{2\sigma}{x} + \frac{2\sigma}{D - x} \tag{46}$$

$$dW = f\,dx = \left(\frac{2\sigma}{x} + \frac{2\sigma}{D - x}\right) dx$$

$$W = V = \int_{r}^{D-r} \left(\frac{2\sigma}{x} + \frac{2\sigma}{D - x}\right) dx = 4\sigma \log_e \frac{D - r}{r} \qquad (47)$$

The charge on the line for a length l is σl. Therefore

$$C = \frac{Q}{V} = \frac{\sigma l}{4\sigma \log_e \dfrac{D - r}{r}} = \frac{l}{4 \log_e \dfrac{D - r}{r}} \text{ cgs esu} \qquad (48)$$

where r now represents the radius of the conductor and is not the same as in the derivation of equation (45). All quantities in equation (48) are in the cgs or absolute electrostatic system of units, giving C in esu or statfarads.

Equation (48) gives the capacitance between two parallel wires. The capacitance to ground or neutral is usually desired in the calculation of transmission lines. Since the plane of neutral potential is symmetrically located between positive and negative charges (assuming a uniform dielectric such as air), the potential between one wire and neutral,[4] or what is also ground potential, is one-half of the potential

[4] The preceding and following equations of capacitance are only approximately correct because they are based on several assumptions which are only partially fulfilled. First, the charge on the conductor is assumed uniform. This assumption requires in part that the conductors be removed an infinite distance from all charged bodies and that the conductors are circular in shape. Under such conditions the distribution of the electrostatic field is pictured in Fig. 16. Equipotential surfaces are those in which all electrostatic lines of force enter and leave perpendicularly. One equipotential surface XX' is shown in Fig. 16. This surface is at a distance halfway between the two conductors and is therefore at a potential midway between the positively and negatively charged conductors. Such a surface is said to be at zero potential, and it is sometimes called the neutral plane between conductors, or simply the neutral. If the earth is considered a conductor and to be at zero potential, it may be assumed to be the same as the equipotential plane XX'. Hence the potential and capacitance to earth or ground may be taken the same as that to the equipotential surface XX' in Fig. 16 provided $D/2$ is relatively small compared with the physical height of the conductor above actual ground. Even though all the above assumptions are not completely fulfilled, the equations given yield results which are sufficiently accurate for most work concerning transmission lines. For more accurate derivations of capacitance the reader is referred to works on electrostatics and electrodynamics.

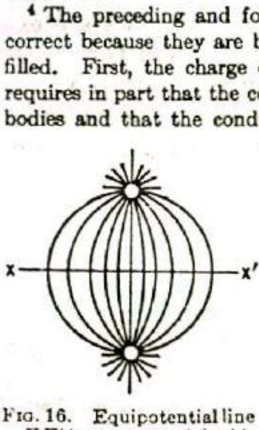


FIG. 16. Equipotential line XX' is at a potential midway between the positively charged top and negatively charged lower conductors.

[given in equation (47)] between wires. Hence

$$V_g = \frac{1}{2}\left(4\sigma \log_e \frac{D-r}{r}\right) = 2\sigma \log_e\left(\frac{D-r}{r}\right)$$

and

$$C_g = \frac{\sigma l}{2\sigma \log_e \dfrac{D-r}{r}} = \frac{l}{2 \log_e \dfrac{D-r}{r}} \text{esu} \tag{49}$$

Expressed in farads per mile, equations (48) and (49) for the capacitance between conductors and between one conductor and ground become:

$$C_{\text{farads per mile}} = \frac{1940 \times 10^{-11}}{\log_{10} \dfrac{D-r}{r}} \tag{50}$$

$$C_{g\ \text{farads per mile}} = \frac{3880 \times 10^{-11}}{\log_{10} \dfrac{D-r}{r}} \tag{51}$$

Equations (50) and (51) are the working equations. As long as D and r are expressed in the same units, the actual units are immaterial.

Equations (44), (50), and (51) form the basis of tables wherein values of L or C may be immediately determined when the size of wire and spacings are known. Samples of tables where the quantities are expressed in units per thousand feet are shown in Tables I and II.[5]

When equations (44), (50), and (51) are applied to three-phase transmission the distance D is that for equilateral spacing, as shown in Fig. 17. These equations are often applied to plane spacings, as shown

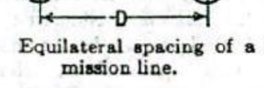


FIG. 17. Equilateral spacing of a transmission line.

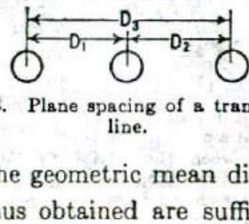


FIG. 18. Plane spacing of a transmission line.

in Fig. 18, in which cases D is taken as the geometric mean distance, that is, $D = \sqrt[3]{D_1 D_2 D_3}$. The results thus obtained are sufficiently accurate for most computations.

[5] Reprinted by permission from "Electrical Engineers' Handbook: Electric Power," fourth edition, edited by Pender and Del Mar, pp. 14–39 and 14–34, John Wiley & Sons, Inc., 1949.

TABLE I

SELF-INDUCTANCE OF SOLID NON-MAGNETIC WIRES*

Millihenrys per 1000 FEET of each wire of a single-phase or of a symmetrical three-phase line

Size of Wire, cir mils or A.W.G.	Diam. of Wire, inches	Inches between Wires, center to center							
		1	3	6	9	12	18	24	30
1,000,000	1.0000	0.05750	0.1245	0.1667	0.1915	0.2090	0.2337	0.2512	0.2648
750,000	0.8660	0.06627	0.1332	0.1755	0.2002	0.2178	0.2425	0.2600	0.2736
500,000	0.7071	0.07863	0.1456	0.1879	0.2126	0.2301	0.2548	0.2724	0.2860
350,000	0.5916	0.08950	0.1565	0.1987	0.2235	0.2410	0.2657	0.2832	0.2968
250,000	0.5000	0.09976	0.1667	0.2090	0.2337	0.2512	0.2760	0.2935	0.3071
0000	0.4600	0.1048	0.1718	0.2141	0.2388	0.2563	0.2810	0.2986	0.3122
000	0.4096	0.1119	0.1789	0.2211	0.2459	0.2634	0.2881	0.3057	0.3193
00	0.3648	0.1190	0.1860	0.2282	0.2529	0.2705	0.2952	0.3127	0.3263
0	0.3249	0.1260	0.1930	0.2353	0.2600	0.2775	0.3022	0.3198	0.3334
1	0.2893	0.1331	0.2001	0.2423	0.2671	0.2846	0.3093	0.3269	0.3405
2	0.2576	0.1402	0.2072	0.2494	0.2741	0.2917	0.3164	0.3339	0.3475
4	0.2043	0.1543	0.2213	0.2635	0.2883	0.3058	0.3305	0.3481	0.3617
6	0.1620	0.1685	0.2354	0.2777	0.3024	0.3199	0.3447	0.3622	0.3758
8	0.1285	0.1826	0.2496	0.2918	0.3165	0.3341	0.3588	0.3763	0.3899
10	0.1019	0.1967	0.2637	0.3060	0.3307	0.3482	0.3729	0.3905	0.4041
12	0.08081	0.2109	0.2778	0.3201	0.3448	0.3623	0.3871	0.4046	0.4182
14	0.06408	0.2250	0.2920	0.3342	0.3590	0.3765	0.4012	0.4187	0.4323
16	0.05082	0.2391	0.3061	0.3484	0.3731	0.3906	0.4153	0.4329	0.4465

Size of Wire, cir mils or A.W.G.	Feet between Wires, center to center								
	3	4	5	6	8	10	15	20	25
1,000,000	0.2760	0.2935	0.3071	0.3182	0.3358	0.3494	0.3741	0.3916	0.4052
750,000	0.2847	0.3023	0.3159	0.3270	0.3445	0.3581	0.3828	0.4004	0.4140
500,000	0.2971	0.3146	0.3282	0.3393	0.3569	0.3705	0.3952	0.4127	0.4263
350,000	0.3080	0.3255	0.3391	0.3502	0.3678	0.3814	0.4061	0.4236	0.4372
250,000	0.3182	0.3358	0.3494	0.3605	0.3780	0.3916	0.4163	0.4339	0.4475
0000	0.3233	0.3408	0.3544	0.3656	0.3831	0.3967	0.4214	0.4390	0.4526
000	0.3304	0.3479	0.3615	0.3726	0.3902	0.4038	0.4285	0.4460	0.4596
00	0.3374	0.3550	0.3686	0.3797	0.3972	0.4108	0.4356	0.4531	0.4667
0	0.3445	0.3620	0.3756	0.3867	0.4043	0.4179	0.4426	0.4601	0.4737
1	0.3516	0.3691	0.3827	0.3938	0.4114	0.4250	0.4497	0.4672	0.4808
2	0.3586	0.3762	0.3898	0.4009	0.4184	0.4320	0.4568	0.4743	0.4879
4	0.3728	0.3903	0.4039	0.4150	0.4326	0.4462	0.4709	0.4884	0.5020
6	0.3869	0.4045	0.4181	0.4292	0.4467	0.4603	0.4850	0.5026	0.5162
8	0.4011	0.4186	0.4322	0.4433	0.4608	0.4744	0.4992	0.5167	0.5303
10	0.4152	0.4327	0.4463	0.4574	0.4750	0.4886	0.5133	0.5308	0.5444
12	0.4293	0.4469	0.4605	0.4716	0.4891	0.5027	0.5274	0.5450	0.5586
14	0.4435	0.4610	0.4746	0.4857	0.5033	0.5169	0.5416	0.5591	0.5727
16	0.4576	0.4751	0.4887	0.4998	0.5174	0.5310	0.5557	0.5732	0.586[illegible]

* The inductances given in this table also apply, with a practically negligible error (about 1 per cent), to ordinary stranded wires of the *same cross-section.*

Example 2. Exact Solution of a Transmission Line. A 60-cycle transmission line 200 miles long consists of three No. 0000 solid conductors with 10-ft equilateral spacing. Calculate the sending voltage when the receiver voltage is 110 kv between lines and when the line is supplying a balanced load of 18,000 kw at 0.8 power-factor lag. Also calculate the sending-end current and the efficiency of the line at 25° C. Assume that the conductance to ground is negligible.

TABLE II

CAPACITANCE TO NEUTRAL* OF SMOOTH ROUND WIRES

Microfarads per 1000 FEET of each wire of a single-phase or of a symmetrical three-phase line

Size of Wire, A.W.G.	Diam. of Wire, inches	Inches between Wires, center to center							
		1	3	6	9	12	18	24	30
0000	0.4600	0.01199	0.006608	0.005192	0.004618	0.004282	0.003884	0.003643	0.003477
000	0.4096	0.01099	0.006317	0.005013	0.004477	0.004161	0.003783	0.003555	0.003396
00	0.3648	0.01016	0.006055	0.004847	0.004344	0.004045	0.003688	0.003470	0.003319
0	0.3249	0.009458	0.005812	0.004692	0.004218	0.003936	0.003597	0.003390	0.003245
1	0.2893	0.008855	0.005587	0.004546	0.004100	0.003833	0.003511	0.003313	0.003174
2	0.2576	0.008332	6.005381	0.004408	0.003988	0.003735	0.003428	0.003239	0.003107
4	0.2043	0.007455	0.005010	0.004157	0.003781	0.003553	0.003274	0.003102	0.002980
6	0.1620	0.006753	0.004688	0.003933	0.003595	0.003388	0.003134	0.002975	0.002863
8	0.1285	0.006177	0.004406	0.003732	0.003426	0.003238	0.003005	0.002859	0.002755
10	0.1019	0.005693	0.004155	0.003551	0.003273	0.003100	0.002886	0.002751	0.002655
12	0.08081	0.005277	0.003931	0.003386	0.003132	0.002974	0.002776	0.002651	0.002562
14	0.06408	0.004921	0.003730	0.003235	0.003003	0.002858	0.002675	0.002558	0.002475

Size of Wire, A.W.G.	Feet between Wires, center to center								
	3	4	5	6	8	10	15	20	25
0000	0.003351	0.003171	0.003043	0.002947	0.002806	0.002706	0.002542	0.002436	0.002361
000	0.003276	0.003103	0.002981	0.002839	0.002753	0.002657	0.002498	0.002396	0.802323
00	0.003204	0.003039	0.002922	0.002833	0.002702	0.002610	0.002456	0.002358	0.002287
0	0.003135	0.002977	0.002864	0.002779	0.002653	0.002564	0.002416	0.002320	0.002251
1	0.003069	0.002917	0.002809	0.002727	0.002606	0.002520	0.002376	0.002284	0.002217
2	0.003006	0.002860	0.002756	0.002677	0.002560	0.002477	0.002338	0.002249	0.002184
4	0.002887	0.002752	0.002656	0.002582	0.002474	0.002396	0.002266	0.002182	0.002121
6	0.002777	0.002652	0.002563	0.002494	0.002392	0.002319	0.002197	0.002118	0.002061
8	0.002676	0.002559	0.002476	0.002412	0.002317	0.002248	0.002133	0.002059	0.002004
10	0.002581	0.002473	0.002395	0.002335	0.002245	0.002181	0.002073	0.002002	0.001951
12	0.002493	0.002392	0.002319	0.002262	0.002178	0.902118	0.002016	0.001949	0.001900
14	0.002411	0.002316	0.002247	0.002194	0.002115	0.002058	0.001961	0.001898	0.001852

* The capacitance *between* wires equals one-half the values given in this table.

All calculations will be made per phase to neutral or ground.

$$V_n = \frac{110{,}000}{\sqrt{3}} = 63{,}500 \text{ volts}$$

$$I_r = \frac{18{,}000{,}000}{\sqrt{3}\,110{,}000 \times 0.8} = 118 \text{ amperes}$$

From wire tables the 60-cycle resistance per mile of No. 0000 at 25° C is 0.271 ohm. The diameter of No. 0000 wire is 460 mils.

$$L_{\text{mile}} = 0.805 \times 10^{-4} + 0.741 \times 10^{-3} \log_{10} \frac{120}{0.23}$$

$$= 0.805 \times 10^{-4} + 0.741 \times 10^{-3} \times 2.718$$

$$= 0.805 \times 10^{-4} + 2.01 \times 10^{-3}$$

$$= 20.9 \times 10^{-4} \text{ henry per mile}$$

Reactance per mile = $2\pi\, 60 \times 20.9 \times 10^{-4} = 0.788$ ohm

$$b = \frac{-1}{X_c} = -2\pi f C$$

$$C_{\text{mile}} = \frac{3880 \times 10^{-11}}{\log_{10} \dfrac{120 - 0.23}{0.23}} = 1430 \times 10^{-11} \text{ farad}$$

Susceptance per mile = $-2\pi\, 60 \times 1430 \times 10^{-11} = -0.538 \times 10^{-5}$ mho

$$\mathbf{Y} = g - jb = +j0.538 \times 10^{-5} = 0.538 \times 10^{-5} \underline{/90^\circ} \text{ mho}$$

$$\mathbf{Z} = r + jX = 0.271 + j0.788 = 0.834 \underline{/71.05^\circ} \text{ ohms}$$

$$\sqrt{\mathbf{ZY}} = \sqrt{0.834 \underline{/71.05^\circ}\, 0.538 \times 10^{-5} \underline{/90^\circ}} = 2.12 \times 10^{-3} \underline{/80.5^\circ}$$

$$\sqrt{\mathbf{Z}/\mathbf{Y}} = \sqrt{\frac{0.834 \underline{/71.05^\circ}}{0.538 \times 10^{-5} \underline{/90^\circ}}} = 3.94 \times 10^{2} \underline{/-9.48^\circ} \text{ ohms}$$

$$\sqrt{\mathbf{Y}/\mathbf{Z}} = 0.254 \times 10^{-2} \underline{/9.48^\circ} \text{ mhos}$$

For $l = 200$ miles,

$$\epsilon^{\sqrt{\mathbf{ZY}}l} = \epsilon^{0.424 \underline{/80.5^\circ}} = \epsilon^{0.07 + j0.418} = \epsilon^{0.07} \epsilon^{j23.9^\circ}$$

$$\mathbf{V}_r = 63{,}500 + j0 \text{ volts}$$

$$\mathbf{I}_r = 118 \underline{/-36.9^\circ} \text{ amperes}$$

$$\left(\frac{\mathbf{V}_r + \mathbf{I}_r \sqrt{\mathbf{Z}/\mathbf{Y}}}{2}\right) = \left(\frac{63{,}500 + 118 \underline{/-36.9^\circ} \times 3.94 \times 10^{2} \underline{/-9.48^\circ}}{2}\right)$$

$$= 47{,}800 - j16{,}800 \text{ volts}$$

$$\left(\frac{\mathbf{V}_r + \mathbf{I}_r \sqrt{\mathbf{Z}/\mathbf{Y}}}{2}\right)\epsilon^{\sqrt{\mathbf{ZY}}l} = (47{,}800 - j16{,}800)\epsilon^{0.07}\epsilon^{j23.9^\circ}$$

$$= (51{,}300 - j18{,}050)\epsilon^{j23.9^\circ}$$

$$= 54{,}400 \underline{/-19.4^\circ} \underline{/23.9^\circ}$$

$$= 54{,}400 \underline{/4.5^\circ} \text{ volts}$$

$$\left(\frac{\mathbf{V}_r - \mathbf{I}_r \sqrt{\mathbf{Z}/\mathbf{Y}}}{2}\right)\epsilon^{\sqrt{\mathbf{ZY}}l} = (15{,}700 + j16{,}800)\epsilon^{-0.07}\epsilon^{-j23.9^\circ}$$

$$= (14{,}610 + j15{,}630)\epsilon^{-j23.9^\circ}$$

$$= 21{,}400 \underline{/46.9^\circ} \underline{/-23.9^\circ}$$

$$= 21{,}400 \underline{/23^\circ} \text{ volts}$$

$$\mathbf{V}_s = \left(\frac{\mathbf{V}_r + \mathbf{I}_r \sqrt{\mathbf{Z}/\mathbf{Y}}}{2}\right)\epsilon^{\sqrt{\mathbf{ZY}}l} + \left(\frac{\mathbf{V}_r - \mathbf{I}_r \sqrt{\mathbf{Z}/\mathbf{Y}}}{2}\right)\epsilon^{-\sqrt{\mathbf{ZY}}l}$$

$$= 54{,}400 \underline{/4.5^\circ} + 21{,}400 \underline{/23^\circ}$$

$$= 54{,}200 + j4270 + 19{,}680 + j8355$$

$$= 73{,}880 + j12{,}625 = 74{,}970 \underline{/9.7^\circ} \text{ volts}$$

The current at the sending end could be calculated in a similar way. However, for illustrative purposes it will be calculated from equation (27).

$$\mathbf{I}_s = \mathbf{I}_r \cosh \sqrt{\mathbf{ZY}}l + \mathbf{V}_r \sqrt{\mathbf{Y}/\mathbf{Z}} \sinh \sqrt{\mathbf{ZY}}l$$

The following relations are convenient to use when dealing with hyperbolic functions of complex angles:

$$\sinh (x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$
$$\cosh (x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$
$$\sinh jx = j \sin x$$
$$\cosh jx = \cos x$$

$$\sqrt{\mathbf{ZY}}l = 2.12 \times 10^{-3} \underline{/80.5^\circ} \times 200 = 0.424 \underline{/80.5^\circ} = 0.07 + j0.418$$

$$\begin{aligned}\cosh (0.07 + j0.418) &= \cosh 0.07 \cosh j0.418 + \sinh 0.07 \sinh j0.418\\ &= \cosh 0.07 \cos 0.418 + j \sinh 0.07 \sin 0.418\\ &= \cosh 0.07 \cos 23.9^\circ + j \sinh 0.07 \sin 23.9^\circ\\ &= 1.00245 \times 0.9143 + j0.07 \times 0.4051\\ &= 0.915 + j0.02835\end{aligned}$$

$$\begin{aligned}\sinh (0.07 + j0.418) &= \sinh 0.07 \cosh j0.418 + \cosh 0.07 \sinh j0.418\\ &= \sinh 0.07 \cos 23.9^\circ + j \cosh 0.07 \sin 23.9^\circ\\ &= 0.07 \times 0.9143 + j1.00245 \times 0.4051\\ &= 0.0639 + j0.406\end{aligned}$$

$$\mathbf{V}_r \sqrt{\mathbf{Y}/\mathbf{Z}} = 63{,}500 \times 0.254 \times 10^{-2} \underline{/9.48^\circ} = 161.30 \underline{/9.48^\circ} \text{ amperes}$$

$$\begin{aligned}\mathbf{V}_r \sqrt{\mathbf{Y}/\mathbf{Z}} \sinh \sqrt{\mathbf{ZY}}l &= 161.3 \underline{/9.48^\circ} (0.0639 + j0.406)\\ &= -0.66 + j66.3 \text{ amperes}\end{aligned}$$

$$\mathbf{I}_r \cosh \sqrt{\mathbf{ZY}}l = (118 \underline{/-36.9^\circ})(0.915 + j0.0284) = 88.4 - j62.1 \text{ amperes}$$

$$\begin{aligned}\mathbf{I}_s &= 88.4 - j62.1 - 0.66 + j66.3\\ &= 87.8 + j4.2 = 87.9 \underline{/2.8^\circ} \text{ amperes}\end{aligned}$$

As a check on the sending voltage, V_s will be calculated by the hyperbolic equation

$$\mathbf{V}_r \cosh \sqrt{\mathbf{ZY}}l + \mathbf{I}_r \sqrt{\mathbf{Z}/\mathbf{Y}} \sinh \sqrt{\mathbf{ZY}}l$$

$$\begin{aligned}\mathbf{V}_s &= 63{,}500 \times 0.915 \underline{/1.75^\circ} + (118 \underline{/-36.9^\circ} \times 3.94 \times 10^2 \underline{/-9.48^\circ})(0.0639 + j0.406)\\ &= 58{,}100 + j1770 + 15{,}700 + j10{,}880\\ &= 73{,}800 + j12{,}652 = 74{,}850 \underline{/9.7^\circ} \text{ volts}\end{aligned}$$

$$\begin{aligned}P_s = vi + v'i' &= 73{,}800 \times 87.8 + 12{,}652 \times 4.2\\ &= 6{,}490{,}000 + 53{,}100\\ &= 6{,}543{,}000 \text{ watts per phase}\end{aligned}$$

$$\text{Efficiency} = \frac{6000}{6543} = 0.917$$

If tables of complex hyperbolic functions are available, the hyperbolic solution is greatly simplified.

Calculation of Velocity of Propagation. From equations (35) and (39).

$$v = \frac{2\pi f}{\beta}$$

$$\beta = \sqrt{\frac{\pm ZY - (Rg + bX)}{2}}$$

$$ZY = [2.12 \times 10^{-3}]^2 = 4.5 \times 10^{-6}$$

$$Rg = 0$$

$$bX = -0.538 \times 10^{-5} \times 0.788 = -0.424 \times 10^{-5}$$

$$\beta = \sqrt{\frac{\pm 4.5 \times 10^{-6} + 4.24 \times 10^{-6}}{2}} = 2.09 \times 10^{-3}$$

$$v = \frac{377}{2.09 \times 10^{-3}} = 180{,}300 \text{ miles per second}$$

If the resistance and the inductance due to the flux within the conductor are neglected, the velocity from equation (40) is

$$\frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{2.01 \times 10^{-3} \times 1430 \times 10^{-11}}}$$

$$= 186{,}400 \text{ miles per second, or the velocity of light}$$

PROBLEMS

5. Solve Problem 1, page 411, by the exact method of calculating transmission lines.

6. Points A and B are 150 miles apart and are connected by a parallel-wire line having parameters as follows:

> Effective resistance per loop mile at 1000 cycles, 60 ohms
> Effective inductance per loop mile at 1000 cycles, 0.0042 henry
> Effective capacitance per loop mile at 1000 cycles, 0.00755 μf
> Shunted conductance is negligible.

The line is assumed to be terminated at point B with an impedance equal to its surge impedance. Find the voltage, current, and power received at point B when 50 volts at 1000 cycles are impressed at A. (A loop mile consists of one mile of outgoing plus one mile of return conductor.) Use $\mathbf{V}_A$ as reference.

7. Calculate by means of the formula the inductance in henrys per mile of No. 0000 wire with an equilateral spacing of 6 feet.

8. Calculate the capacitance per mile between wires and between one wire and neutral or ground for the line in Problem 7.

9. A 3-phase 60-cycle transmission line is 150 miles long and consists of three No. 0000 wires spaced at corners of an equilateral triangle which are 15 feet apart. The line is to deliver 138,000 line-to-line volts and 45,000 kw total power at 0.8 p.f. lagging at the receiver. Calculate the required sending-end voltage, current, power factor, and efficiency of transmission if the nominal T line is used. See bottom of page 430 for resistance of No. 0000 wire. Use $\mathbf{V}_{\text{line-to-neutral}}$ as reference.

10. Work Problem 9 if the nominal π line is employed.

11. Work Problem 9 if the Steinmetz three-condenser method of representing the line is used.

12. Work Problem 9 if the exact method of calculating long lines is employed.

13. Calculate the velocity of propagation of the wave in Problem 12.

14. (*a*) If 138,000 line-to-line volts were maintained at the sending end of the line in Problem 9, what would be the receiver-end voltage with the receiver end open? Employ the exact method of solution. (*b*) What is the magnitude of the direct wave at the receiver? (*c*) of the reflected wave?

15. What is the velocity of propagation of the wave in Problem 6?

16. What is the attenuation in decibels per mile of the transmission line described in Problem 6?