chapter XI

Electric Wave Filters

The frequency characteristics of certain types of networks can be employed to separate waves of different frequencies. The separation may be effected primarily for the purpose of selecting a desired band of frequencies or for the purpose of rejecting an undesired band. Selected bands are called pass or transmission bands, and rejected bands are called stop or attenuation bands. Any network which possesses definite properties of frequency discrimination and which is capable of separating electric waves of different frequencies is called an *electric wave filter* or, simply, a *filter*.

Selective Properties of Circuit Elements and Elementary Circuits. Single reactive circuit elements are sometimes employed to pass or reject frequency bands when only broad discrimination is to be made. Thus *blocking* condensers in many vacuum tube circuits discriminate very satisfactorily between waves of zero frequency (direct current) and high-frequency waves. Inductance coils can be employed to pass direct current and practically eliminate frequencies which are of the order of 1000 kilocycles.

High-Frequency Line Drain. A high-frequency disturbance can be drained from a low-frequency, two-wire line with a condenser arrangement similar to that shown in Fig. 1*a*. The condensers constitute a

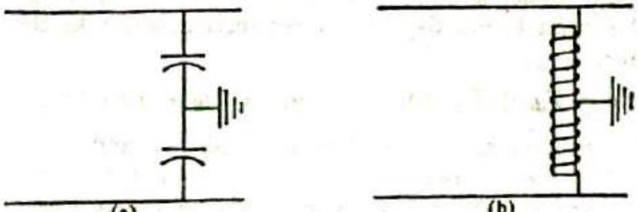


FIG. 1. Devices for draining induced disturbances from two-wire lines.

relatively high impedance to the low-frequency line voltage, both line-to-line and line-to-ground. At the same time a relatively low line-to-ground impedance is presented to the high-frequency variation which in the present case is assumed to be the result of an induced disturbance.

Low-Frequency Line Drain. A method sometimes used to drain a low-frequency induced disturbance from a two-wire line is shown diagrammatically in Fig. 1*b*. The drain coil is ironclad and offers a relatively high impedance to current which tends to flow from line-to-line. If, however, both lines are raised simultaneously above (or below) ground potential by induction, the currents which flow from the lines to ground magnetize the core in opposite directions. With respect to the induced currents, the two halves of the coil are in series opposition with the result that the impedances offered to these currents to ground are relatively very low. The device can be used to drain charges from telephone lines which are electrostatically induced from neighboring power lines.

Typical Smoothing Network. A very common form of filter is the elementary π-section shown in Fig. 2. This particular type of filter section is widely used to give d-c output from rectified a-c wave forms.

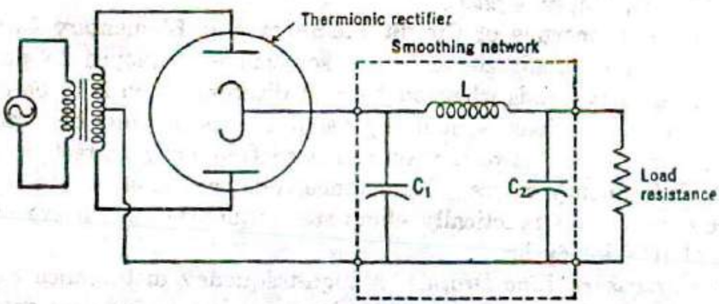


FIG. 2. A commonly used filter section.

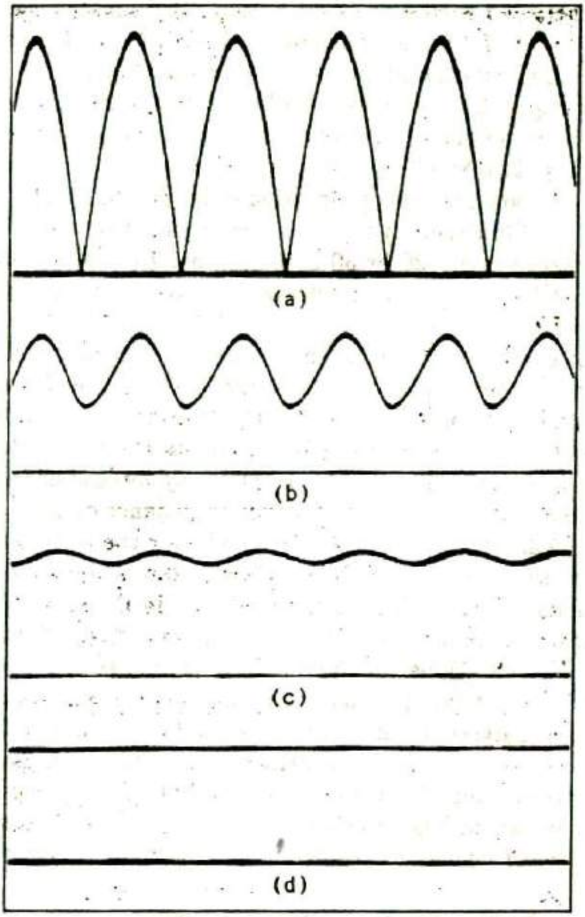
The output voltage of the rectifying device, namely, that which appears across the input terminals of the filter section, will take the following general form:

$$v = V_{dc} + V_{m1} \sin (\omega_1 t + \alpha_1) + \text{higher harmonics}$$

where V_{dc} is the average value of the rectified wave and ω_1 is the angular velocity of the lowest-frequency component present in the voltage variation. A typical voltage input variation is shown in Oscillogram 1*a*.

If, for example, both halves of 60-cycle wave are rectified symmetrically, the lowest frequency component in the rectified voltage wave will be that of 120 cycles, in which case $\omega_1 = 754$ radians per second. In unsymmetrical rectification ω_1 is generally equal to the fundamental angular velocity of the alternating variation which is being rectified.

Under ideal conditions the filter section shown in Fig. 2 should pass waves of zero frequency with no attenuation and absolutely stop waves which are of other than zero frequency. Obviously, these ideal conditions of operation can only be approached in practice, but the



OSCILLOGRAM 1.

(a) Rectified a-c wave, no filtering.
(b) Rectified a-c wave, choke filtering only.
(c) Rectified a-c wave, choke and input condenser filtering.
(d) Rectified a-c wave, complete π-section filtering. (See Fig. 2.)

difference between ideal operation and actual operation can be made exceedingly small by proper design. See Oscillogram 1*d*.

For full-wave, 60-cycle rectification satisfactory filtering can usually be obtained if C_1 and C_2 of Fig. 2 are about 4 or 5 μf each and L is about 30 or 40 henrys. The permissible voltage regulation will, to a large extent, determine the amount of resistance that can be present in the inductance coil in any particular instance. In any case R is very small as compared with $\omega_1 L$. The result is that, when the π-section is energized with a rectified voltage, it presents a relatively low impedance to zero-frequency current. The impedances offered to other than zero-frequency currents are relatively very high.

If, for example, $L = 30$ henrys and ω_1 is 754 radians per second, the series impedance of the filter section to the ω_1 component of current is approximately 22,600 ohms. The series impedances to the higher-frequency components are proportionately greater. The series impedance of the filter section to the d-c component of current may, in a particular case, be only 20 or 30 ohms. Therefore, the inductance coil acting by itself will tend to smooth out the rectified wave as shown in Oscillogram 1*b*.

The input condenser, C_1, is an important member of the filter section, although it is entirely possible to design a smoothing network which does not employ a condenser at the C_1 position shown in Fig. 2. It will be noted that C_1 is placed directly across the output terminals of the rectifying device. It provides a relatively low-impedance path for all a-c components. The anode-cathode impedance of the tube may be 10 or 20 times greater than $1/\omega_1 C_1$, in which case the voltage drop across C_1 is only a small fraction of the total drop due to the a-c components of the rectified voltage. This aids materially in the smoothing process but at the same time increases the actual plate current of the rectifying elements. Filter sections which employ a condenser directly across the terminals of the rectifying device are called condenser input sections.[1]

A complete analysis of the composite circuit shown in Fig. 2 is complicated by the presence of the transformer, tube, and load impedances and will not be undertaken at this time. Actually the smoothing network or ripple filter shown in Fig. 2 is a particular form of low-pass filter, the general theory of which is considered on pages 464–468 of the present chapter.

Image Impedances of Four-Terminal Networks. Most filter sections take the form of a four-terminal network, and as such they possess one pair of input terminals and one pair of output terminals. With this arrangement of terminals, a filter section can be inserted directly into a two-wire line.

General four-terminal network theory is rather elaborate and is not

[1] For details see "Electrical Engineers' Handbook: Electric Communication and Electronics," fourth edition, edited by Pender and McIlwain, pp. 7–106, 7–108, John Wiley & Sons, Inc., 1950.

considered to be suitable first-course material. There are certain aspects of the subject, however, that are essential to a proper understanding of elementary filter theory. One of these is the concept of image impedances.

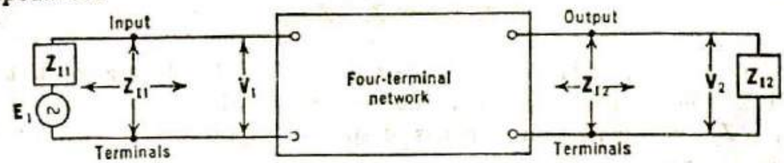


FIG. 3. Four-terminal network terminated on the image impedance basis.

The rectangle shown in Fig. 3 is assumed to be any form of four-terminal network, the internal circuit elements of which may or may not be accessible. It is also assumed that the individual circuit elements are linear. Circuit elements are linear if effects are proportional to causes, for example, if currents are proportional to applied voltages.

The image impedances of a four-terminal network are called $\mathbf{Z}_{I1}$ and $\mathbf{Z}_{I2}$ and are defined in the following manner. (Refer to Fig. 3.) If the impedance across the input terminals (looking into the network) is $\mathbf{Z}_{I1}$ when the output terminals are closed through $\mathbf{Z}_{I2}$, and if the impedance across the output terminals (looking into the network) is $\mathbf{Z}_{I2}$ when the input terminals are closed through $\mathbf{Z}_{I1}$, then $\mathbf{Z}_{I1}$ and $\mathbf{Z}_{I2}$ are called the image impedances of the network. If a four-terminal network is terminated in its image impedances, $\mathbf{Z}_{I1}$ and $\mathbf{Z}_{I2}$, the impedance looking either way from the input terminals is $\mathbf{Z}_{I1}$ and the impedance looking either direction from the output terminals is $\mathbf{Z}_{I2}$. The network is correctly matched when the input impedance is $\mathbf{Z}_{I1}$ and the output impedance is $\mathbf{Z}_{I2}$ and under these conditions the network is said to be terminated on the image basis.

A special case of image impedance termination is employed in elementary filter theory. The assumption is made that $\mathbf{Z}_{I1} = \mathbf{Z}_{I2}$, and this particular value of impedance is called the characteristic impedance of the filter section.

The image impedance at either end of a given network can be determined from the open-circuit and short-circuit impedances. By open-circuit impedance, $\mathbf{Z}_{o\text{-}c}$, is meant the impedance looking into one set of terminals when the other set of terminals is open-circuited. By short-circuit impedance, $\mathbf{Z}_{s\text{-}c}$, is meant the impedance looking into one set of terminals when the other set of terminals is short-circuited. It can be shown that image impedance at either end of a four-terminal network is the geometric mean of the open-circuit and short-circuit impedances.

Thus in Fig. 3:

$$Z_{I1} = \sqrt{Z_{o\text{-}c1} Z_{s\text{-}c1}} \tag{1}$$

and

$$Z_{I2} = \sqrt{Z_{o\text{-}c2} Z_{s\text{-}c2}} \tag{2}$$

Generalized proofs of the above equations will not be given but it will be shown presently that the relations stated are correct when $Z_{I1} = Z_{I2}$, the condition which is of special importance in elementary filter theory.

Characteristic Impedances of T- and π-Sections. The basic units of elementary filter theory are the symmetrical T- and π-sections shown in Fig. 4. Although both of these sections are essentially three-terminal networks, they are usually inserted into a two-wire line in the same manner as a four-terminal network. Viewed as three-terminal networks, the T-section is a wye-connected set of impedances and the π-section is a delta-connected set of impedances. It should not be supposed that the Z_1 and Z_2 values given in Fig. 4*a* and Fig. 4*b* are, in

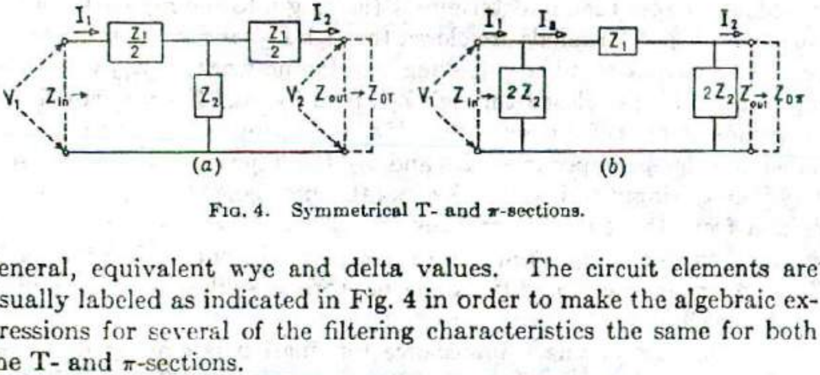


Fig. 4. Symmetrical T- and π-sections.

general, equivalent wye and delta values. The circuit elements are usually labeled as indicated in Fig. 4 in order to make the algebraic expressions for several of the filtering characteristics the same for both the T- and π-sections.

The series impedance of a symmetrical T-section is composed of two similar units, each of which is labeled $Z_1/2$ in Fig. 4*a*. The impedance labeled Z_2 in Fig. 4*a* is called the shunt impedance of the T-section. The shunt impedance of a symmetrical π-section is composed of two equal branches, each labeled $2Z_2$ in Fig. 4*b*, and these shunt branches are located on either side of the series impedance Z_1. If the series and shunt impedances are designated in accordance with Fig. 4, ladder structures formed by the cascade arrangement of successive sections are generally similar. (See Fig. 10 and Fig. 11.)

If the output terminals of the T-section shown in Fig. 4*a* are closed through an impedance Z_{oT}, the impedance across the input terminals (looking into the network) is:

$$\mathbf{Z}_{in} = \frac{\mathbf{Z}_1}{2} + \frac{\mathbf{Z}_2\left(\dfrac{\mathbf{Z}_1}{2} + \mathbf{Z}_{oT}\right)}{\mathbf{Z}_2 + \dfrac{\mathbf{Z}_1}{2} + \mathbf{Z}_{oT}} \tag{3}$$

In order for $\mathbf{Z}_{in}$ to equal $\mathbf{Z}_{oT}$, it follows that:

$$\mathbf{Z}_{oT} = \frac{\mathbf{Z}_1}{2} + \frac{\dfrac{\mathbf{Z}_1\mathbf{Z}_2}{2} + \mathbf{Z}_2\mathbf{Z}_{oT}}{\dfrac{\mathbf{Z}_1}{2} + \mathbf{Z}_2 + \mathbf{Z}_{oT}} \tag{4}$$

The above equation may be solved for $\mathbf{Z}_{oT}$ and the result stated in terms of $\mathbf{Z}_1$ and $\mathbf{Z}_2$. Thus it can be shown that *the characteristic impedance of the* T-*section* is:

$$\mathbf{Z}_{oT} = \sqrt{\mathbf{Z}_1\mathbf{Z}_2 + \frac{\mathbf{Z}_1^2}{4}} = \sqrt{\mathbf{Z}_1\mathbf{Z}_2\left(1 + \frac{\mathbf{Z}_1}{4\mathbf{Z}_2}\right)} \tag{5}$$

If the output terminals of the π-section shown in Fig. 4*b* are closed through an impedance $\mathbf{Z}_{o\pi}$, the impedance across the input terminals (looking into the network) is:

$$\mathbf{Z}_{in} = \frac{2\mathbf{Z}_2\left(\mathbf{Z}_1 + \dfrac{2\mathbf{Z}_2\mathbf{Z}_{o\pi}}{2\mathbf{Z}_2 + \mathbf{Z}_{o\pi}}\right)}{2\mathbf{Z}_2 + \mathbf{Z}_1 + \dfrac{2\mathbf{Z}_2\mathbf{Z}_{o\pi}}{2\mathbf{Z}_2 + \mathbf{Z}_{o\pi}}} \tag{6}$$

In order to determine the conditions under which $\mathbf{Z}_{in}$ is equal to $\mathbf{Z}_{o\pi}$ it is simply necessary to set $\mathbf{Z}_{in} = \mathbf{Z}_{o\pi}$ in the above equation and solve the resulting equation for $\mathbf{Z}_{o\pi}$. After $\mathbf{Z}_{in}$ has been set equal to $\mathbf{Z}_{o\pi}$ and all fractions cleared, it will be found that:

$$\mathbf{Z}_{o\pi}^2\,(\mathbf{Z}_1 + 4\mathbf{Z}_2) = 4\mathbf{Z}_1\mathbf{Z}_2^2$$

From which *the characteristic impedance of the π-section* is

$$\mathbf{Z}_{o\pi} = \sqrt{\frac{4\mathbf{Z}_1\mathbf{Z}_2^2}{\mathbf{Z}_1 + 4\mathbf{Z}_2}} = \sqrt{\frac{\mathbf{Z}_1\mathbf{Z}_2}{\left(1 + \dfrac{\mathbf{Z}_1}{4\mathbf{Z}_2}\right)}} \tag{7}$$

Equations (5) and (7) are important relations in filter theory because they define the characteristic impedances $\mathbf{Z}_{oT}$ and $\mathbf{Z}_{o\pi}$ in terms of the series and shunt elements out of which the T- and π-sections are composed. If a filter section is terminated in its characteristic impedance, the impedance across the input terminals (looking into the network) is the same as the receiving-end impedance. (The importance of designing filter sections to have particular characteristic impedances will become more evident after composite filter sections are studied.) It should be noted that a given filter section terminated at both ends in its characteristic impedance is terminated on the image basis and that in this particular case $\mathbf{Z}_{I1}$ and $\mathbf{Z}_{I2}$ are equal. (See Fig. 3.) Reference to equations (5) and (7) will show that:

$$\mathbf{Z}_{oT}\mathbf{Z}_{o\pi} = \mathbf{Z}_1\mathbf{Z}_2 \tag{8}$$

and

$$\mathbf{Z}_{o\pi} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_{oT}} \tag{9}$$

Equations (8) and (9) define a rather important relationship that exists between the characteristic impedances of T- and π-sections, the $\mathbf{Z}_1$'s and $\mathbf{Z}_2$'s of which are equal.

Filter theory is based upon $\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}_{oT}$, and $\mathbf{Z}_{o\pi}$ to such an extent that the physical significance of each of these four impedances should be clearly understood. The reader who is unfamiliar with filter theory nomenclature should at this stage review the definitions which have been given for $\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}_{oT}$, and $\mathbf{Z}_{o\pi}$. [See Fig. 4 and equations (5) and (7).]

Example 1. In Fig. 4*a*, let each $\mathbf{Z}_1/2$ take the form of an inductance coil, the inductance of which is 0.047 henry and the resistance of which is 1 ohm. The shunt arm, namely, $\mathbf{Z}_2$, is to take the form of a 300-μf condenser. (*Note:* This is an unconventional set of parameters for this type of filter section but since some of the experimental results which follow are based upon these particular values they will be used here to illustrate the calculation of $\mathbf{Z}_{oT}$.)

The method of calculating $\mathbf{Z}_{oT}$ at 50 cycles is as follows:

$$\frac{\mathbf{Z}_1}{2} = \frac{R_1}{2} + j\omega\frac{L_1}{2} = 1 + j14.77 = 14.8\,\underline{/86.1^\circ} \text{ ohms}$$

$$\mathbf{Z}_1 = 29.6\,\underline{/86.1^\circ} \text{ ohms} \quad \text{(Full series arm impedance.)}$$

$$\mathbf{Z}_2 = 0 - j\frac{1}{\omega C_2} = 0 - j10.61 = 10.61\,\underline{/-90^\circ} \text{ ohms}$$

$$\mathbf{Z}_{oT} = \sqrt{\mathbf{Z}_1\mathbf{Z}_2 + \frac{\mathbf{Z}_1^{\,2}}{4}}$$

$$= \sqrt{(29.6\,\underline{/86.1^\circ})(10.61\,\underline{/-90^\circ}) + \frac{(29.6\,\underline{/86.1^\circ})^2}{4}}$$

$$= 9.83\,\underline{/2.5^\circ} = 9.81 + j0.43 \text{ ohms}$$

The physical significance of the above value of $\mathbf{Z}_{oT}$ is that, if an impedance of 9.83 $\underline{/2.5^\circ}$ ohms is placed across the output terminals of this symmetrical T-section, *the impedance looking into the input terminals is also* 9.83 $\underline{/2.5^\circ}$ *ohms.*

Problem 1. Neglect the resistances of the two inductance coils that form the series impedance of the filter section in the illustrative example given above and find $\mathbf{Z}_{oT}$ at 50 cycles and at 100 cycles. (It may be of interest to know that this symmetrical T-section forms a low-pass filter that passes all frequencies up to 60 cycles and attenuates those above 60 cycles.)

Ans.: At 50 cycles, $\mathbf{Z}_{oT} = 9.76 \underline{/0^\circ}$ ohms.
At 100 cycles, $\mathbf{Z}_{oT} = 23.65 \underline{/90^\circ}$ ohms.

Problem 2. The series impedance, $\mathbf{Z}_1$, of a symmetrical π-section (like that shown in Fig. 4*b*) consists of a 0.02-henry inductance coil, the resistance of which is assumed to be negligibly small. Each of the shunt arms, namely, $2\mathbf{Z}_2$, is composed of a 2.0-μf condenser. (This symmetrical π-section forms a low-pass filter which passes all frequencies below 900 cycles without attenuation as will be shown later.)

Find the characteristic impedance of this section at 200 cycles and at 2000 cycles. Use equation (7) and recognize that

$$\mathbf{Z}_1 = 0.02\omega \underline{/90^\circ} \quad \text{and} \quad \mathbf{Z}_2 = \frac{10^6}{4\omega} \underline{/-90^\circ} \text{ ohms}$$

Ans.: At 200 cycles, $\mathbf{Z}_{o\pi} = 71.8 \underline{/0^\circ}$ ohms.
At 2000 cycles, $\mathbf{Z}_{o\pi} = 48 \underline{/-90^\circ}$ ohms.

Characteristic Impedance as a Function of Open-Circuit and Short-Circuit Impedances. Reference to Fig. 5*a* will show that the open-circuit impedance of a T-section (looking into the section) is:

$$\mathbf{Z}_{o\text{-}c} = \frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2 \tag{10}$$

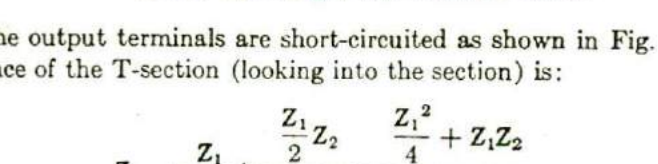


FIG. 5. $\mathbf{Z}_{oc}$ and $\mathbf{Z}_{sc}$ of a symmetrical T-section.

When the output terminals are short-circuited as shown in Fig. 5*b* the impedance of the T-section (looking into the section) is:

$$\mathbf{Z}_{s\text{-}c} = \frac{\mathbf{Z}_1}{2} + \frac{\frac{\mathbf{Z}_1}{2}\mathbf{Z}_2}{\frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2} = \frac{\frac{\mathbf{Z}_1^{\,2}}{4} + \mathbf{Z}_1\mathbf{Z}_2}{\frac{\mathbf{Z}_1}{2} + \mathbf{Z}_2} \tag{11}$$

The geometric mean of $\mathbf{Z}_{o\text{-}c}$ and $\mathbf{Z}_{s\text{-}c}$ is:

$$\sqrt{\mathbf{Z}_{o\text{-}c}\mathbf{Z}_{s\text{-}c}} = \sqrt{\mathbf{Z}_1\mathbf{Z}_2 + \frac{\mathbf{Z}_1^{\,2}}{4}} \tag{12}$$

It has already been shown that

$$Z_{oT} = \sqrt{Z_1Z_2 + \frac{Z_1^2}{4}} \quad \text{[See equation (5).]}$$

Therefore,

$$Z_{oT} = \sqrt{Z_{o\text{-}c}Z_{s\text{-}c}} \tag{13}$$

The fact that Z_{oT} is equivalent to the geometric mean of $Z_{o\text{-}c}$ and $Z_{s\text{-}c}$ provides the basis for a simple experimental method of determining the characteristic impedance of a given section.

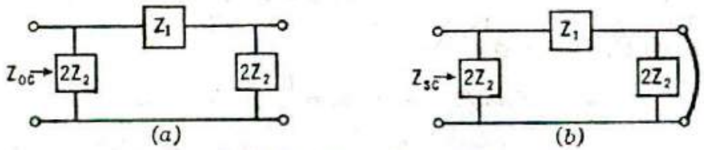


FIG. 6. Z_{oc} and Z_{sc} of a symmetrical π-section.

Reference to Fig. 6*a* will show that the open-circuit impedance of a symmetrical π-section (looking into the section) is:

$$Z_{o\text{-}c} = \frac{2Z_2(Z_1 + 2Z_2)}{Z_1 + 4Z_2} \tag{14}$$

If the output terminals of the π-section are short-circuited as shown in Fig. 6*b*, the input impedance is:

$$Z_{s\text{-}c} = \frac{2Z_2Z_1}{Z_1 + 2Z_2} \tag{15}$$

$$\sqrt{Z_{o\text{-}c}Z_{s\text{-}c}} = \sqrt{\frac{4Z_1Z_2^2}{Z_1 + 4Z_2}} \tag{16}$$

Comparison of the above relation with equation (7) will show that:

$$Z_{o\pi} = \sqrt{Z_{o\text{-}c}Z_{s\text{-}c}} \tag{17}$$

Equations (13) and (17) indicate that the characteristic impedance of either the T- or π-section is equal to the geometric mean of their respective open- and short-circuit impedances. It should be evident that the symbols $Z_{o\text{-}c}$ and $Z_{s\text{-}c}$ in equations (13) and (17) refer to open- and short-circuit impedances of the particular section that is under investigation. In general $Z_{oT} \neq Z_{o\pi}$.

Problem 3. Referring to Fig. 7 find (*a*) $Z_{o\text{-}c}$, (*b*) $Z_{s\text{-}c}$, and (*c*) Z_{oT} at 200 cycles.
Ans.: (*a*) $186.2 \angle{-90°}$, (*b*) $26.0 \angle{90°}$, and (*c*) $69.5 \angle{0°}$ ohms.

Problem 4. Referring to Fig. 8, find (*a*) $\mathbf{Z}_{o\text{-}c}$, (*b*) $\mathbf{Z}_{s\text{-}c}$, and (*c*) $\mathbf{Z}_{o\pi}$ at 200 cycles.
Ans.: (*a*) $192.5\,\underline{/-90°}$, (*b*) $26.8\,\underline{/90°}$, and (*c*) $71.8\,\underline{/0°}$ ohms.

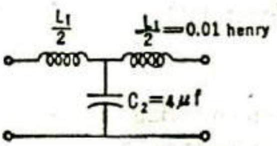


Fig. 7. A particular symmetrical T-section for use with Problem 3

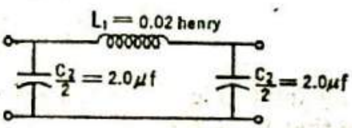


Fig. 8. A particular symmetrical π-section for use in connection with Problem 4.

Physical Operation of Symmetrical T- and π-Sections. The T- and π-sections shown in Fig. 4 possess some remarkable properties when their output terminals are connected to the characteristic impedances $\mathbf{Z}_{oT}$ and $\mathbf{Z}_{o\pi}$ respectively. Before considering the filtering properties of these sections, some of the basic relationships that follow directly from elementary circuit theory will be established.

The conditions imposed on equations (4) and (6), page 441, make $\mathbf{Z}_{\text{in}} = \mathbf{Z}_{\text{out}}$ for either type of section. Hence $\mathbf{I}_1 = \mathbf{V}_1/\mathbf{Z}_o$ and $\mathbf{I}_2 = \mathbf{V}_2/\mathbf{Z}_o$, where $\mathbf{Z}_o$ symbolizes the characteristic impedance of the particular type of section considered. It follows directly that

$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = \frac{\mathbf{V}_1}{\mathbf{V}_2} \quad \text{and} \quad \frac{W_1}{W_2} = \frac{V_1 I_1 \cos\theta}{V_2 I_2 \cos\theta} = \frac{I_1^{\,2}}{I_2^{\,2}} \tag{18}$$

where the subscripts 1 refer to input quantities and the subscripts 2 refer to output quantities. Since the impedance looking into the input terminals is the same as the terminating impedance, the angle between $\mathbf{V}_1$ and $\mathbf{I}_1$ is equal to the angle between $\mathbf{V}_2$ and $\mathbf{I}_2$. This angle is symbolized as θ in equation (18) and is equal to $\tan^{-1}(X_o/R_o)$, where X_o and R_o are the reactive and resistive components of the characteristic impedance $\mathbf{Z}_o$. The basic relationships contained in equation (18) are illustrated photographically for a particular T-section in Oscillogram 2, page 446. These relationships will be used later in defining the attenuation of filter sections.

The next basic relationship to be established is that the ratio of input current to output current, namely, $\mathbf{I}_1/\mathbf{I}_2$, is completely defined by the series arm impedance ($\mathbf{Z}_1$) and the shunt arm impedance ($\mathbf{Z}_2$) out of which the symmetrical T- or π-section is composed. For the T-section shown in Fig. 4*a* it is plain from Kirchhoff's emf law that

$$\frac{\mathbf{Z}_1}{2}\mathbf{I}_1 + \frac{\mathbf{Z}_1}{2}\mathbf{I}_2 + \mathbf{Z}_{oT}\mathbf{I}_2 = \mathbf{V}_1 = \mathbf{Z}_{oT}\mathbf{I}_1 \tag{19}$$

Whence

$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = \frac{\mathbf{Z}_{oT} + \dfrac{\mathbf{Z}_1}{2}}{\mathbf{Z}_{oT} - \dfrac{\mathbf{Z}_1}{2}} \quad \text{(for T-sections)} \tag{20}$$

Referring to Fig. 4*b* for the π-section and remembering that $\mathbf{V}_1 = \mathbf{I}_1\mathbf{Z}_{o\pi}$ and that $\mathbf{V}_2 = \mathbf{I}_2\mathbf{Z}_{o\pi}$, the current $\mathbf{I}_{\text{series}}$ in the series arm is:

$$\mathbf{I}_{\text{series}} = \mathbf{I}_1 - \frac{\mathbf{I}_1\mathbf{Z}_{o\pi}}{2\mathbf{Z}_2} = \mathbf{I}_2 + \frac{\mathbf{I}_2\mathbf{Z}_{o\pi}}{2\mathbf{Z}_2} \tag{21}$$

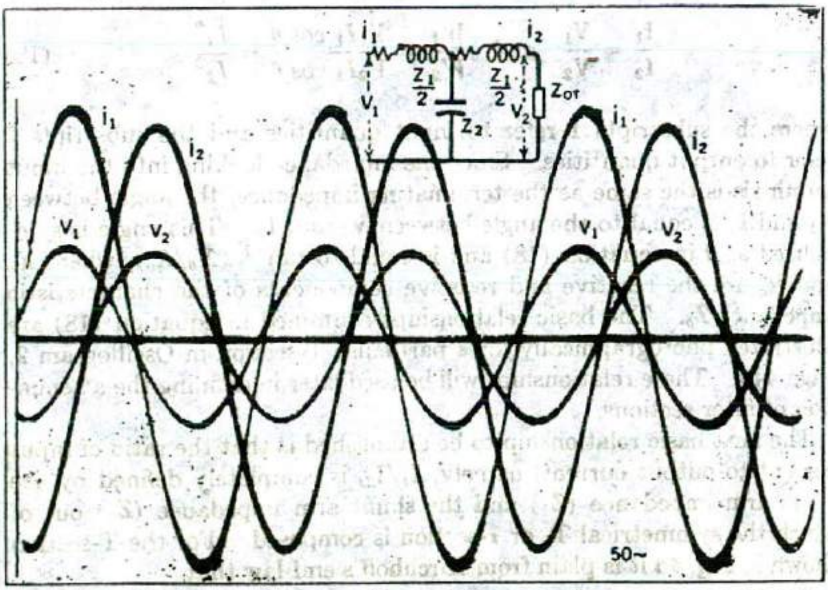
from which

$$\mathbf{I}_1 \frac{(2\mathbf{Z}_2 - \mathbf{Z}_{o\pi})}{2\mathbf{Z}_2} = \mathbf{I}_2 \frac{(2\mathbf{Z}_2 + \mathbf{Z}_{o\pi})}{2\mathbf{Z}_2} \tag{22}$$

and

$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = \frac{2\mathbf{Z}_2 + \mathbf{Z}_{o\pi}}{2\mathbf{Z}_2 - \mathbf{Z}_{o\pi}} \tag{23}$$

Reference to equations (20) and (23) above and to equations (5) and (7), page 441, will show that the ratio $\mathbf{I}_1/\mathbf{I}_2$ is defined wholly in terms of $\mathbf{Z}_1$ and $\mathbf{Z}_2$ for either T- or π-sections. It will be shown later that the



Oscillogram 2. Illustrating attenuation and phase shift in a symmetrical T-section. v_1 and i_1 are input voltage and current respectively. v_2 and i_2 are output voltage and current respectively.

right members of equations (20) and (23) are identically equal when written wholly in terms of Z_1 and Z_2. For the present, equation (20) will be used to define the ratio I_1/I_2 in T-sections and equation (23) will be used to define this ratio in π-sections.

Example 2. Refer to the symmetrical T-section shown in Fig. 9*a*. Let it be required to evaluate the ratio I_1/I_2 at $f = 50$ cycles. Since this is the same T-section as described in example 1, page 442, the results of example 1 may be used here to define Z_1, Z_2, and Z_{oT}.

$$\frac{Z_1}{2} = (1 + j14.77),\ Z_2 = (0 - j10.61), \quad \text{and} \quad Z_{oT} = (9.81 + j0.43) \text{ ohms}$$

$$\frac{I_1}{I_2} = \frac{Z_{oT} + \frac{Z_1}{2}}{Z_{oT} - \frac{Z_1}{2}} = \frac{(9.81 + j0.43) + (1 + j14.77)}{(9.81 + j0.43) - (1 + j14.77)}$$

$$= \frac{(10.81 + j15.20)}{(8.81 - j14.34)} = \frac{18.7 \underline{/54.6^\circ}}{16.8 \underline{/-58.4^\circ}} = 1.11 \underline{/113^\circ}$$

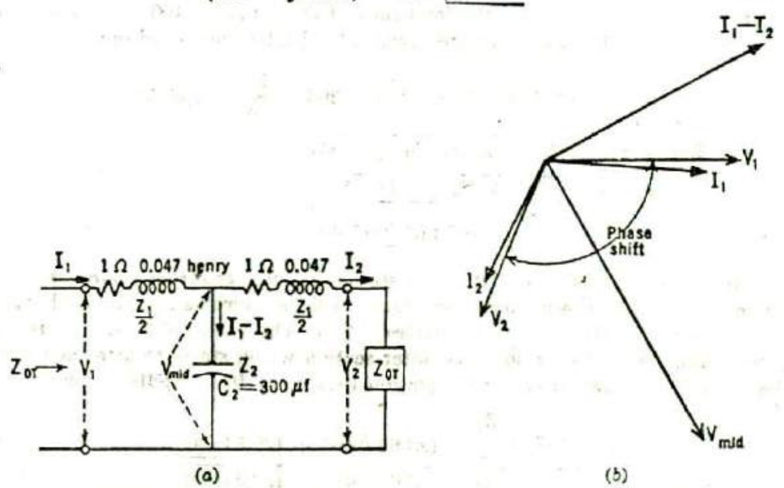


Fig. 9. A symmetrical T-section terminated in its characteristic impedance, together with a vector diagram of the currents and voltages in a particular case.

The physical significance of the above complex number is that the magnitude of I_1 is 1.11 times as great as the magnitude of I_2 and that I_1 *leads* I_2 by 113°. (See Fig. 9*b*.) It will be shown presently that the ratio I_1/I_2 defines the *attenuation* of the filter section and that the associated angle of I_1/I_2 defines the *phase shift* of the section.

A worthwhile exercise for the student at this stage is that of correlating the results given above with those determined by elementary circuit analysis. Let V_1 of Fig. 9*a* $= 100 \underline{/0^\circ}$ volts and solve for I_1 and I_2 by ordinary methods. The results are illustrated in Fig. 9*b* and in Oscillogram 2 which is a photographic record of v_1, i_1, v_2, and i_2 for the particular T-section shown in Fig. 9*a*.

Example 3. Let it be required to find the ratio I_1/I_2 in Fig. 9a if the resistances of the inductance coils are neglected, assuming that the frequency of the supply voltage is 50 cycles.

$$\frac{Z_1}{2} = (0 + j14.77), \quad Z_2 = (0 - j10.61), \quad \frac{Z_1^2}{4} = (j14.77)^2$$

$$Z_{oT} = \sqrt{Z_1Z_2 + (Z_1/2)^2} = \sqrt{(j29.54)(-j10.61) + (j14.77)^2}$$
$$= \sqrt{313.4 - 218.2} = \sqrt{95.2} = 9.76\underline{/0^\circ} \text{ ohms}$$

Employing equation (20):

$$\frac{I_1}{I_2} = \frac{9.76 + j14.77}{9.76 - j14.77} = \frac{17.7\underline{/56.5^\circ}}{17.7\underline{/-56.5^\circ}} = 1\underline{/113^\circ}$$

Thus the output current I_2 is shown to be as great in magnitude as the input current I_1. This condition exists generally in symmetrical T- and π-sections when the resistances are negligibly small provided the characteristic impedance for the frequency considered is a pure ohmic resistance.

Example 4. Let it be required to find the characteristic impedance and the current ratio I_1/I_2 in Fig. 9a if the frequency of the supply is 100 cycles and if the resistances of the inductance coils are neglected. Under these conditions:

$$\frac{Z_1}{2} = (0 + j29.54), \quad Z_2 = (0 - j5.305), \quad \frac{Z_1^2}{4} = (j29.54)^2$$

$$Z_{oT} = \sqrt{(j59.08)(-j5.305) + (j29.54)^2}$$
$$= \sqrt{(313\underline{/0^\circ}) + (873\underline{/+180^\circ})}$$
$$= \sqrt{560\underline{/+180^\circ}} = 23.66\underline{/90^\circ} \text{ ohms}$$

The characteristic impedance of the filter section has changed from a pure resistance (of 9.76 ohms) to a pure inductive reactance of 23.66 ohms as a result of changing the frequency from 50 cycles to 100 cycles. *Note:* The values of L_1 and C_2 used in Fig. 9a make this section a low-pass filter section which starts to attenuate at 60 cycles, as will be shown later. See equation (55), page 465. At 100 cycles:

$$\frac{I_1}{I_2} = \frac{Z_{oT} + \frac{Z_1}{2}}{Z_{oT} - \frac{Z_1}{2}} = \frac{(23.66\underline{/90^\circ}) + (29.54\underline{/90^\circ})}{(23.66\underline{/90^\circ}) - (29.54\underline{/90^\circ})}$$

$$= \frac{53.2\underline{/90^\circ}}{5.88\underline{/-90^\circ}} = 9.04\underline{/+180^\circ}$$

It will be observed that, at 100 cycles, I_1 is 9.04 times as great as I_2 which indicates that marked attenuation is taking place. It will also be observed that the phase shift is 180°, a condition that always obtains in a resistanceless filter section which is operating in the attenuation band and which is terminated in its characteristic impedance.

The importance of the ratio I_1/I_2 has been emphasized in the foregoing examples because the physical operation of a filter section is concisely defined by this ratio.

Problem 5. Find the ratio I_1/I_2 of the symmetrical π-section shown in Fig. 8 page 445, at 200 cycles and at 2000 cycles. Neglect the resistance of the inductance coil and recognize that $Z_1 = (0 + j0.02\omega)$ is the full series arm and that $Z_2 = \left(0 - j\frac{10^6}{4\omega}\right)$ is the combined shunt arm since the total series inductance (L_1) is 0.02 henry and the combined shunt capacitance (C_2) is 4 μf. (See Fig. 4*b* and Fig. 8.) Note also that $2Z_2 = \left(0 - j\frac{10^6}{2\omega}\right)$ ohms.

Ans.: At 200 cycles $I_1/I_2 = 1\underline{/+20.5°}$.
At 2000 cycles $I_1/I_2 = 10.6\underline{/-180°}$.

Problem 6. Find the current ratio I_1/I_2 of the symmetrical T-section shown in Fig. 7, page 445, at 200 cycles and at 2000 cycles. Neglect the resistances of the inductance coils.

Ans.: At 200 cycles $I_1/I_2 = 1\underline{/20.5°}$.
At 2000 cycles $I_1/I_2 = 10.6\underline{/+180°}$.

Transmission Constant of a Filter Section. A transmission constant which applies to a generator feeding a resistance load has been defined in equation (80), page 136. It will be remembered that the reference used in that case was selected with a view toward including the effects of a possible mismatch between the resistance of the generator and the resistance of the load. Another transmission constant which applies to long lines was used in Chapter X. In this case it was called the *propagation* constant, the term usually employed for the transmission constant of long lines.

Where a filter section or other four-terminal network is terminated on an image impedance basis as shown in Fig. 3, the impedance match between the generator and load is already effected and the definition of the transmission constant is somewhat different from that given in equation (80), page 136. Assuming that the filter section is terminated on an image impedance basis and that we wish to specify a measure of the attenuation and phase shift of the filter itself, we employ the following definition of the transmission constant:

$$\gamma = \alpha + j\beta = \log_\epsilon \frac{Z_T'}{Z_{I1}} = \log_\epsilon \frac{V_1/I_2}{V_1/I_1} = \log_\epsilon \frac{I_1}{I_2} \tag{24}$$

where Z_T' is the transfer impedance from the input terminals of the filter section to the output terminals, namely, V_1/I_2

Z_{I1} is the image impedance seen looking to the right of the input terminals, namely, V_1/I_1

α is called the attenuation of the filter section

β is called the phase-shift constant of the filter section.

Actually the α and β defined in equation (24) apply to any four-terminal network which is terminated on an image impedance basis as shown in Fig. 3. As such they apply directly to a filter section which is terminated in its characteristic impedance, since characteristic impedance termination is but a special case of image impedance termination where $\mathbf{Z}_{I1} = \mathbf{Z}_{I2}$.

The attenuation, α, is a measure of the ratio of the power input to the power output of a filter section which is terminated in its characteristic impedance, since under these conditions the real part of equation (24) may be written as:

$$\alpha = \log_\epsilon \frac{\sqrt{I_1^2 R_0}}{\sqrt{I_2^2 R_0}} = \frac{1}{2} \log_\epsilon \frac{I_1^2 R_0}{I_2^2 R_0} = \frac{1}{2} \log_\epsilon \frac{W_1}{W_2} \tag{25}$$

where R_0 is the resistive component of $\mathbf{Z}_0$
W_1 is the power entering the input terminals
W_2 is the power leaving the output terminals.

From equation (24) it is plain that

$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = \epsilon^{(\alpha + j\beta)} = \epsilon^{\alpha}\epsilon^{j\beta} = K \underline{/\beta} \tag{26}$$

where $K = \epsilon^{\alpha} = I_1 / I_2$
β = angle of lead of $\mathbf{I}_1$ with respect to $\mathbf{I}_2$.

As applied to a series or cascade arrangement of filter sections like those shown in Fig. 10, page 452:

$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = \frac{\mathbf{I}_2}{\mathbf{I}_3} = \frac{\mathbf{I}_3}{\mathbf{I}_4} \tag{27}$$

and the transmission constant (together with the attenuation and phase-shift) may be reckoned on a per section (or $\mathbf{I}_1 / \mathbf{I}_2$) basis or on a combined basis of $\mathbf{I}_1 / \mathbf{I}_4$, since both arrangements are presumably terminated on a characteristic impedance basis.

Units of Attenuation or Transmission Loss. Filter section attenuation is usually expressed in either nepers or decibels. (See pages 136–137.) These units of transmission loss are both defined on a logarithmic basis, since their greatest field of application is in the transmission of sound, the loudness of which is a logarithmic function of the sound energy.

The Neper. The general definition of attenuation expressed in nepers is:

$$(\text{Attenuation in nepers}) = \tfrac{1}{2} \log_\epsilon \frac{W_{(\text{general})}}{W_{(\text{reference})}} \tag{28}$$

or

$$(\text{Attenuation in nepers}) = 1.151 \log_{10} \frac{W_{(\text{general})}}{W_{(\text{reference})}} \tag{29}$$

where $W_{(\text{general})}$ is any particular power level which might be under discussion

$W_{(\text{reference})}$ is the power level employed as reference from which $W_{(\text{general})}$ is to be measured.

Reference to equation (18) or to equation (25) will show that, for a filter section which is terminated in its characteristic impedance, the output power W_2 is employed as the reference power level and

$$(\text{Attenuation in nepers}) = \tfrac{1}{2} \log_\epsilon \frac{W_1}{W_2} = \tfrac{1}{2} \log_\epsilon \frac{I_1{}^2 R_0}{I_2{}^2 R_0} = \log_\epsilon \epsilon^\alpha = \alpha \tag{30}$$

If the filter section is not terminated in its characteristic impedance, equation (28) is employed and W_1 is used for $W_{(\text{general})}$ and W_2 is used for $W_{(\text{reference})}$.

The Decibel. The decibel is an arbitrarily defined unit of transmission loss (or gain) which has come into general use since about 1925.[2] The customary abbreviation is db. The general definition of attenuation expressed in decibels is

$$(\text{Attenuation in decibels}) = 10 \log_{10} \frac{W_{(\text{general})}}{W_{(\text{reference})}} \tag{31}$$

where $W_{(\text{general})}$ and $W_{(\text{reference})}$ have the same meanings as employed in connection with equation (28).

If the filter section is terminated on a characteristic impedance basis, reference to equation (18) or to equation (25) will show that

$$\begin{aligned}(\text{Attenuation in decibels}) &= 10 \log_{10} \left[\frac{I_1}{I_2}\right]^2 = 10 \log_{10} \epsilon^{2\alpha} \\ &= 20\alpha \log_{10} \epsilon = 8.686\alpha \end{aligned} \tag{32}$$

Comparison of equations (30) and (32) will show that the decibel is a transmission unit which is 1/8.686 times as large as the neper (or napier). In practice the decibel is used almost exclusively in the United States. Because of its rationality, the neper is widely used in theoretical derivations.

It should be noted that transmission loss (or attenuation) units define power ratios and under special conditions define current and voltage

[2] Originally the decibel was called the "transmission unit" (abbreviated TU). See "The Transmission Unit and Telephone Transmission Reference Systems," by W. H. Martin, *Bell System Technical Journal*, Vol. 3, p. 400.

ratios. These units do not specify the actual loss (or gain) in either watts, amperes, or volts. If, for example, it is known that the ratio of power input to power output in a particular case is 3, the transmission loss or attenuation is:

$$\tfrac{1}{2}\log_\epsilon 3 = 0.55 \text{ neper} \quad \text{or} \quad 10\log_{10} 3 = 4.77 \text{ decibels}$$

If the current ratio is 3 and the input and output impedances are equal, the transmission loss is:

$$\tfrac{1}{2}\log_\epsilon 3^2 = 1.1 \text{ nepers} \quad \text{or} \quad 10\log_{10} 3^2 = 9.54 \text{ decibels}$$

The actual values of power or current are not specified in the statements given above, only logarithmic functions of the ratios.

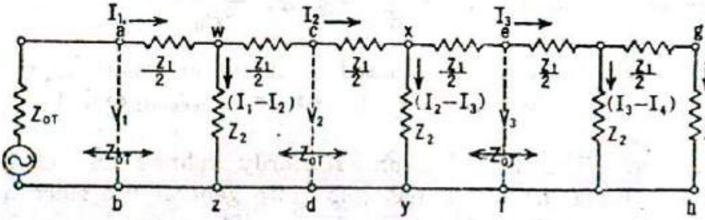


FIG. 10. Three symmetrical T-sections terminated on a characteristic impedance basis.

Example 5. If the vector current ratio per section of each of the three T-sections shown in Fig. 10 is $3\underline{/30°}$ or $3\underline{/\pi/6}$ radian:

$$\frac{I_1}{I_2} = \frac{I_2}{I_3} = \frac{I_3}{I_4} = \epsilon^{\gamma_1} = \epsilon^{\alpha_1}\epsilon^{j\beta_1} = 3\underline{/30°}$$

from which

$$\epsilon^{\alpha_1} = 3 \quad \text{or} \quad \alpha_1 = \log_\epsilon 3 = 1.1 \text{ neper per section}$$
$$\beta_1 = 30° \quad \text{or} \quad \pi/6 \text{ radian, phase shift of } I_2 \text{ behind } I_1$$

On a three-section basis:

$$\frac{I_1}{I_4} = \epsilon^{\gamma_3} = \epsilon^{\alpha_3}\epsilon^{j\beta_3} = 27\underline{/90°}$$

From which the attenuation and phase shift of the three sections may be calculated as

$$\epsilon^{\alpha_3} = 27 \quad \text{or} \quad \alpha_3 = \log_\epsilon 27 = 3.3 \text{ nepers}$$
$$= 28.6 \text{ decibels}$$

$\beta_3 = 90°$ or $\pi/2$ radians, phase shift of I_4 behind I_1.

Problem 7. The current ratio in a particular filter section is known to be $1.11\underline{/113°}$ as in example 2, page 448. If the section is terminated in its characteristic impedance, find the attenuation in nepers and in decibels.

Ans.: 0.1043 neper, 0.905 decibel.

Problem 8. Calculate the attenuation in decibels and in nepers for the various power and current ratios indicated below. In the case of the current ratios, it is assumed that the filter sections to which they apply are terminated on a characteristic impedance basis. The few calculated values that appear in the table may be used as guide.

W_1/W_2	db	nepers	I_1/I_2	db	nepers
1	0	0	1	0	0
10	10	1.15	10	20	2.3
100			100		
1,000			1,000		
5,000			10,000		

With respect to a specified reference power level, any particular circuit power may be measured in plus or minus decibels, depending on whether the circuit power is greater or less than the reference power level. Several reference power levels have been used in sound engineering, namely, 6 milliwatts in telephone circuits, 12.5 milliwatts in public address systems, and a relatively new reference level which is designed to be generally applicable and which is specified as "1 milliwatt in 600 ohms." Thus, 6 milliwatts might be reckoned as $10 \log_{10} (6/1) = +7.78$ db with respect to a 1-milliwatt reference or as $10 \log_{10} (6/12.5) = -3.19$ db with respect to a 12.5-milliwatt reference.

General Considerations. Elementary filter theory concerns itself with uniform ladder structures which are composed of either conventional T- or π-sections. With the definitions which have been given to $\mathbf{Z}_1$ and $\mathbf{Z}_2$ in T- and π-sections, the ladder structures formed by cascade arrangements of these sections are equivalent except for their terminations.

Figure 10 illustrates a ladder structure composed of symmetrical T-sections which is midseries terminated. A ladder structure is said to be midseries terminated when it is terminated at the midpoint of a series arm such as wx. It will be noted that g is the midpoint of such a series arm. Under ideal conditions the structure is terminated at both sending and receiving ends in impedances which are equal to the characteristic impedance of a T-section, namely, $\mathbf{Z}_{oT}$. (Methods will be considered later whereby generating devices of one impedance can be properly matched to a load device of a different impedance.)

Figure 11 illustrates a ladder structure composed of symmetrical π-sections. Comparison of Fig. 10 and Fig. 11 will show the general circuit equivalence of T- and π-sections except for the terminations. The arrangement shown in Fig. 11 may be thought of as symmetrical T-sections which are terminated at planes such that the shunt arm $\mathbf{Z}_2$ is bisected longitudinally, leaving $2\mathbf{Z}_2$ directly across the input and output terminals. This form of termination is called midshunt termination. It has a certain practical significance which will be discussed in a later article.

A *low-pass* filter is a network designed to pass currents of all frequencies below a critical or cut-off frequency and materially to reduce the amplitude of currents of all frequencies above this critical frequency. Under certain ideal conditions which will be considered, a low-pass filter will pass all frequencies from zero up to a predetermined number of cycles with theoretical zero attenuation. The frequency at which the theoretical attenuation takes on a finite value is called the cut-off frequency.

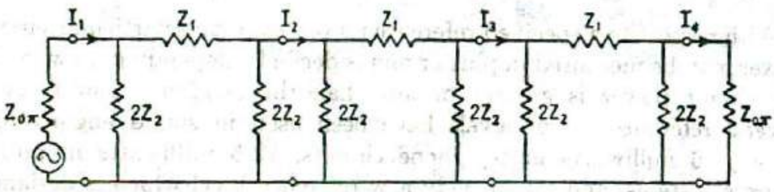


FIG. 11. Three symmetrical π-sections terminated on a characteristic impedance basis.

The general arrangements of circuit elements for elementary low-pass filter sections are illustrated in Fig. 14, page 465.

A *high-pass* filter is a network designed to pass currents of all frequencies above a critical or cut-off frequency and materially to reduce the amplitude of currents of all frequencies below this critical frequency. Under ideal conditions, a high-pass filter attenuates all frequencies from zero up to a predetermined number of cycles and transmits higher frequencies with theoretical zero attenuation. In a high-pass filter the lowest frequency at which theoretical zero attenuation obtains is called cut-off frequency. Elementary high-pass filter sections are shown in Fig. 16, page 468.

A Fundamental Filter Equation. An equation which defines the propagation constant of a filter section wholly in terms of an arbitrarily selected series arm ($\mathbf{Z}_1$) and an arbitrarily selected shunt arm ($\mathbf{Z}_2$) is necessary in the design of filter sections.

Reference to equations (20) and (23), page 446, and to equation (24), page 449, shows that

$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = \epsilon^{\gamma} = \frac{\mathbf{Z}_{oT} + \dfrac{\mathbf{Z}_1}{2}}{\mathbf{Z}_{oT} - \dfrac{\mathbf{Z}_1}{2}} \quad \text{(for T-sections)} \qquad (33)$$

$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = \epsilon^{\gamma} = \frac{2\mathbf{Z}_2 + \mathbf{Z}_{o\pi}}{2\mathbf{Z}_2 - \mathbf{Z}_{o\pi}} \quad \text{(for } \pi\text{-sections)} \qquad (34)$$

After the value of Z_{oT} as given in equation (5), page 441, is substituted in equation (33), the following form may be obtained:

$$\frac{I_1}{I_2} = \epsilon^{\gamma} = \frac{\sqrt{1 + \dfrac{Z_1}{4Z_2}} + \sqrt{\dfrac{Z_1}{4Z_2}}}{\sqrt{1 + \dfrac{Z_1}{4Z_2}} - \sqrt{\dfrac{Z_1}{4Z_2}}} \quad \text{(for T-sections)} \tag{35}$$

After substituting the value of $Z_{o\pi}$ as given in equation (7) into equation (34), the following form may be obtained:

$$\frac{I_1}{I_2} = \epsilon^{\gamma} = \frac{\sqrt{1 + \dfrac{Z_1}{4Z_2}} + \sqrt{\dfrac{Z_1}{4Z_2}}}{\sqrt{1 + \dfrac{Z_1}{4Z_2}} - \sqrt{\dfrac{Z_1}{4Z_2}}} \quad \text{(for } \pi\text{-sections)} \tag{36}$$

Hence, for like values of Z_1 and Z_2;

$$\frac{I_1}{I_2} \text{ (for T-sections)} = \frac{I_1}{I_2} \text{ (for } \pi\text{-sections)} \tag{37}$$

Since $\epsilon^{\gamma} = \epsilon^{(\alpha + j\beta)}$, it follows that

$$\epsilon^{(\alpha+j\beta)} = \frac{\left(\sqrt{1 + \dfrac{Z_1}{4Z_2}} + \sqrt{\dfrac{Z_1}{4Z_2}}\right)^2}{1} \tag{38}$$

Although equation (38) defines both α and β in terms of Z_1 and Z_2, a different form is usually employed in the actual evaluation process.[3] An algebraic rearrangement of the quantities involved in equation (38) will show that:

$$\alpha + j\beta = 2 \log_{\epsilon} \left(\sqrt{1 + \frac{Z_1}{4Z_2}} + \sqrt{\frac{Z_1}{4Z_2}}\right) \tag{39}$$

The above relation is one form of fundamental filter equation, since the

[3] A fundamental filter equation which is sometimes called Campbell's equation (after G. A. Campbell who discovered the filtering properties of various lumped impedance networks) is:

$$\cosh \gamma = 1 + \frac{Z_1}{2Z_2} = (\cosh \alpha \cos \beta + j \sinh \alpha \sin \beta)$$

The above form need not be used here but, for the reader who is familiar with the manipulation of complex hyperbolic functions, Campbell's equation is much more elegant than is equation (39). See "Physical Theory of the Wave-Filter," by G. A. Campbell, *Bell System Technical Journal*, Vol. I, November, 1922.

attenuation constant and the phase-shift constant are defined wholly in terms of the *full series arm impedance* ($\mathbf{Z}_1$) and the *full shunt arm impedance* ($\mathbf{Z}_2$). The analysis of any symmetrical T- or π-section composed of series and shunt arms of $\mathbf{Z}_1$ and $\mathbf{Z}_2$, respectively, may be carried through with the aid of equation (39).

Since the right-hand member of equation (39) is, in general, a complex number, it is capable of defining both α and β of either T- or π-sections which are terminated on a characteristic impedance basis. In the manipulation of the factor $\mathbf{Z}_1/4\mathbf{Z}_2$ in equation (39), care should be exercised in determining the correct sign of the associated angle if the correct sign of β is desired.

Example 6. Let it be required to determine the attenuation and phase shift of a filter section whose full series arm is $565.6\underline{/60^\circ}$ ohms (at a particular frequency) and whose full shunt arm is $200\underline{/-90^\circ}$ ohms. *Note:* Characteristic impedance termination is implied in a case of this kind unless otherwise stated.

$$\mathbf{Z}_1 = 565.6\underline{/60^\circ} \quad \text{and} \quad \mathbf{Z}_2 = 200\underline{/-90^\circ}\text{ ohms}$$

$$\sqrt{\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}} = \sqrt{\frac{565.6\underline{/60^\circ}}{800\underline{/-90^\circ}}} = \sqrt{0.707\underline{/150^\circ}} = 0.841\underline{/75^\circ} = (0.2175 + j0.812)$$

$$\begin{aligned}
\sqrt{1+\frac{\mathbf{Z}_1}{4\mathbf{Z}_2}} &= \sqrt{1\underline{/0^\circ} + 0.707\underline{/150^\circ}} = \sqrt{0.525\underline{/42.4^\circ}} \\
&= 0.725\underline{/21.2^\circ} = (0.676 + j0.262) \\
\alpha + j\beta &= 2\log_\epsilon[(0.676 + j0.262) + (0.2175 + j0.812)] \\
&= 2\log_\epsilon(0.893 + j1.074) \\
&= 2\log_\epsilon(1.396\underline{/50.25^\circ}) \\
&= (2\log_\epsilon 1.396) + j\frac{100.5}{57.3} = (0.668 + j1.76)
\end{aligned}$$

The attenuation of the filter section is 0.668 neper or 5.80 decibels. The vector input current is 1.76 radians or 100.5° ahead of the vector output current since $\alpha = 0.668$ neper and $\beta = 1.76$ radians.

In this example the resistance of the series arm is relatively high (565.6/2 ohms) and yet the attenuation is relatively low because the filter section is operating in its pass band.

Example 7. Let it be required to find the attenuation and phase shift of the π-section shown in Fig. 8, page 445, by means of equation (39). The resistances of the circuit elements are to be neglected and the frequency is assumed to be 200 cycles. At 200 cycles, $\omega = 1257$ radians per second and

$$\mathbf{Z}_1 = 0 + j\omega L_1 = 25.14\underline{/90^\circ}\text{ ohms}$$

$$2\mathbf{Z}_2 = 0 - j\frac{1}{\frac{\omega C_2}{2}} = 397.5\underline{/-90^\circ}\text{ ohms}$$

$$4\mathbf{Z}_2 = 795\,\underline{/-90^\circ}\text{ ohms}$$

$$\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = \frac{25.14\,\underline{/90^\circ}}{795\,\underline{/-90^\circ}} = 0.0316\,\underline{/+180^\circ}$$

$$\begin{aligned}\alpha + j\beta &= 2\log_\epsilon\left[\sqrt{1\,\underline{/0^\circ} + 0.0316\,\underline{/+180^\circ}} + \sqrt{0.0316\,\underline{/+180^\circ}}\right] \\ &= 2\log_\epsilon(1.0\,\underline{/10.25^\circ}) = (2\log_\epsilon 1.0) + \left(2j\frac{10.25}{57.3}\right) \\ &= 0 + j0.358\end{aligned}$$

Therefore $\alpha = 0$ and $\beta = 0.358$ radian or 20.5°. It will be noted that, as a result of neglecting the resistances of the circuit elements, the theoretical attenuation is zero.

Problem 9. A high-pass filter section is composed of two 7.96-μf condensers and a coil having an inductance of 0.0159 henry in the form of a T. The resistance of the inductance coil is assumed to be 4 ohms. (A condenser occupies each of the $\mathbf{Z}_1/2$ positions in Fig. 4a, page 440, and the inductance coil occupies the $\mathbf{Z}_2$ position in this T-section.) Find the attenuation and phase shift of this filter section at 200 cycles employing equation (39). At 200 cycles:

$$\omega = 1257 \text{ radians per second } \frac{\mathbf{Z}_1}{2} = 100\,\underline{/-90^\circ} \quad \mathbf{Z}_2 = 20.4\,\underline{/78.7^\circ}\text{ ohms}$$

Ans.: $\alpha = 17.8$ db; $\beta = -165^\circ$.

Problem 10. Evaluate α and β in equation (39) if $\mathbf{Z}_1 = 200\,\underline{/90^\circ}$ ohms and $\mathbf{Z}_2 = 50\,\underline{/-90^\circ}$ ohms. *Ans.:* $\alpha = 0$; $\beta = \pi$ radians.

Filter Section Analysis Assuming Zero Resistance. It is quite customary to neglect the resistive components of $\mathbf{Z}_1$ and $\mathbf{Z}_2$ in filter section analysis because the attenuation produced by these resistive components is incidental to the predominant filtering action that takes place. The discrepancy between theoretical results based on zero resistance and actual results will not be great if the resistances are relatively small compared with the reactances. Also the algebraic manipulations involved in filter design are greatly simplified by neglecting the resistive components of $\mathbf{Z}_1$ and $\mathbf{Z}_2$.

If the above resistances are neglected and if the filter sections are properly terminated, the pass bands are transmitted with zero attenuation while the stop bands experience certain varying degrees of attenuation. It will also be shown that the phase shift is 180° throughout the stop band under the conditions stated above. Before elaborating upon these customary generalizations, two examples based entirely upon equation (39) will be presented.

Example 8. Consider a symmetrical T-section in which $\mathbf{Z}_1 = j\omega L_1$ and in which $\mathbf{Z}_2 = -j\dfrac{1}{\omega C_2}$. Let it be required to predict the behavior of the filter section wholly in terms of the relationship stated in equation (39).

$$\frac{Z_1}{4Z_2} = \frac{-\omega^2 L_1 C_2}{4} = \frac{\omega^2 L_1 C_2}{4} \angle +180^\circ$$

Since $Z_1/4Z_2$ possesses the general form given above, it will be convenient to reckon ω in $1/\sqrt{L_1C_2}$ units, thereby giving $Z_1/4Z_2$ definite numerical values for various different frequency units. The evaluation of the right-hand member of equation (39) for various frequencies is shown in tabular form in Table I.

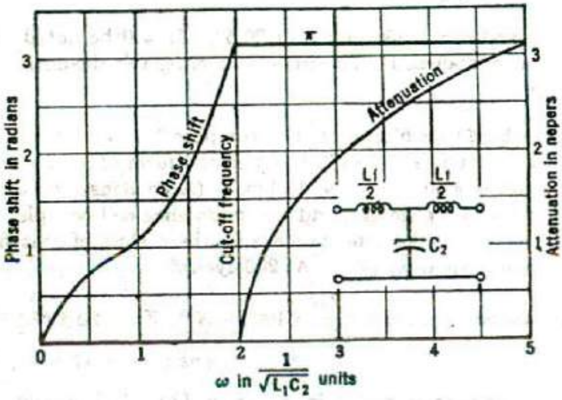


FIG. 12. Variations of phase shift and attenuation in a prototype low-pass filter section. (See Table I, page 459.)

The variations of attenuation and phase shift can readily be determined from an examination of columns (8) and (9) of the table. It will be observed that the filter section which is under discussion has theoretical zero attenuation between the limits of $\omega = 0$ and $\omega = 2/\sqrt{L_1C_2}$ radians per second. The section obviously operates as a low-pass filter. The arrangement of the series and shunt arms of this low-pass filter together with the general trends in the variations of attenuation and phase shift are shown in Fig. 12. The fact that the cut-off point occurs at $\omega = 2/\sqrt{L_1C_2}$ radians per second will be given more attention in a later article. The present example concerns itself primarily with the development of equation (39) in a particular case.

Example 9. Consider a symmetrical T-section in which $Z_1 = -j\,\frac{1}{\omega C_1}$ and $Z_2 = j\omega L_2$. Let it be required to predict the behavior of the filter section wholly in terms of equation (39). In the present case:

$$\frac{Z_1}{4Z_2} = -\frac{1}{4\omega^2 L_2 C_1} = \frac{1}{4\omega^2 L_2 C_1} \angle -180^\circ$$

The same units of angular velocity as employed in example 8 are convenient units to employ in the present analysis. Also the evaluation of the right-hand member of equation (39) can be conveniently presented in tabular form. The calculations are indicated in Table II, and results are shown graphically in Fig. 13. T-sections

TABLE I

$$\left[Z_1 = j\omega L_1 \quad \text{and} \quad Z_2 = -j\frac{1}{\omega C_2}\right] \qquad \frac{Z_1}{4Z_2} = \frac{\omega^2 L_1 C_2}{4}\underline{/+180^\circ}$$

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
ω	$\frac{Z_1}{4Z_2}$	$\sqrt{\frac{Z_1}{4Z_2}}$	$\sqrt{\frac{Z_1}{4Z_2}+1}$	(3) + (4) Cartesian Form	(3) + (4) Polar Form	$2\log_e$ (6)	α nepers	β radians
0	0	0	1	$1 + j0$	$1\underline{/0^\circ}$	$0 + j0$	0	0
$1/\sqrt{L_1C_2}$	−0.25	$j0.50$	0.866	$0.866 + j0.5$	$1\underline{/30^\circ}$	$0 + j1.05$	0	1.05
$1.5/\sqrt{L_1C_2}$	−0.563	$j0.75$	0.662	$0.662 + j0.75$	$1\underline{/48.6^\circ}$	$0 + j1.70$	0	1.70
$2/\sqrt{L_1C_2}$	−1.0	$j1.00$	0.000	$0 + j1.0$	$1\underline{/90^\circ}$	$0 + j3.14$	0	3.14
$2.5/\sqrt{L_1C_2}$	−1.562	$j1.25$	$j0.75$	$0 + j2.0$	$2\underline{/90^\circ}$	$1.38 + j3.14$	1.38	3.14
$3/\sqrt{L_1C_2}$	−2.25	$j1.50$	$j1.13$	$0 + j2.63$	$2.63\underline{/90^\circ}$	$1.93 + j3.14$	1.93	3.14
$4/\sqrt{L_1C_2}$	−4.0	$j2.00$	$j1.73$	$0 + j3.73$	$3.73\underline{/90^\circ}$	$2.63 + j3.14$	2.63	3.14
$6/\sqrt{L_1C_2}$	−9.0	$j3.00$	$j2.83$	$0 + j5.83$	$5.83\underline{/90^\circ}$	$3.52 + j3.14$	3.52	3.14
$8/\sqrt{L_1C_2}$	−16.0	$j4.00$	$j3.87$	$0 + j7.87$	$7.87\underline{/90^\circ}$	$4.12 + j3.14$	4.12	3.14
∞	$-\infty$	$j\infty$	$j\infty$	$0 + j\infty$	$\infty\underline{/90^\circ}$	$\infty + j3.14$	∞	3.14

TABLE II

$$\left[Z_1 = -j\frac{1}{\omega C_1} \quad \text{and} \quad Z_2 = j\omega L_2 \right] \qquad \frac{Z_1}{4Z_2} = \frac{1}{4\omega^2 L_2 C_1} \angle -180^\circ$$

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
ω	$\frac{Z_1}{4Z_2}$	$\sqrt{\frac{Z_1}{4Z_2}}$	$\sqrt{\frac{Z_1}{4Z_2}+1}$	(3) + (4) Cartesian Form	(3) + (4) Polar Form	$2\log_e$ (6)	α nepers	β radians
0	$-\infty$	$-j\infty$	$-j\infty$	$0 - j\infty$	$\infty \angle -90^\circ$	$\infty - j3.14$	∞	-3.14
$\frac{0.1}{\sqrt{C_1L_2}}$	-25	$-j5.0$	$-j4.9$	$0 - j9.9$	$9.9 \angle -90^\circ$	$4.58 - j3.14$	4.58	-3.14
$\frac{0.25}{\sqrt{C_1L_2}}$	-4.0	$-j2.0$	$-j1.73$	$0 - j3.73$	$3.73 \angle -90^\circ$	$2.63 - j3.14$	2.63	-3.14
$\frac{0.40}{\sqrt{C_1L_2}}$	-1.563	$-j1.25$	$-j0.75$	$0 - j2.00$	$2.0 \angle -90^\circ$	$1.38 - j3.14$	1.38	-3.14
$\frac{0.50}{\sqrt{C_1L_2}}$	-1.00	$-j1.0$	0.00	$0 - j1.00$	$1.0 \angle -90^\circ$	$0 - j3.14$	0	-3.14
$\frac{0.60}{\sqrt{C_1L_2}}$	-0.694	$-j0.833$	0.554	$0.554 - j0.833$	$1.0 \angle -56.4^\circ$	$0 - j1.97$	0	-1.97
$\frac{1.0}{\sqrt{C_1L_2}}$	-0.250	$-j0.500$	0.866	$0.866 - j0.500$	$1.0 \angle -30^\circ$	$0 - j1.05$	0	-1.05
$\frac{2.0}{\sqrt{C_1L_2}}$	-0.0625	$-j0.250$	0.968	$0.968 - j0.250$	$1.0 \angle -14.5^\circ$	$0 - j0.506$	0	-0.506
$\frac{4.0}{\sqrt{C_1L_2}}$	-0.0156	$-j0.125$	0.992	$0.992 - j0.125$	$1.0 \angle -7.2^\circ$	$0 - j0.246$	0	-0.246
∞	0	0	1.00	$1.00 - j0$	$1.0 \angle 0^\circ$	$0 - j0$	0	0

consisting of series-arm condensers and shunt-arm inductances are thus shown to operate effectively as high-pass filters.

The phase shift constant, β, in Table I represents a lag of section output voltage and current with respect to section input voltage and current. In Table II, β represents a lead of section output voltage and current with respect to section input voltage.

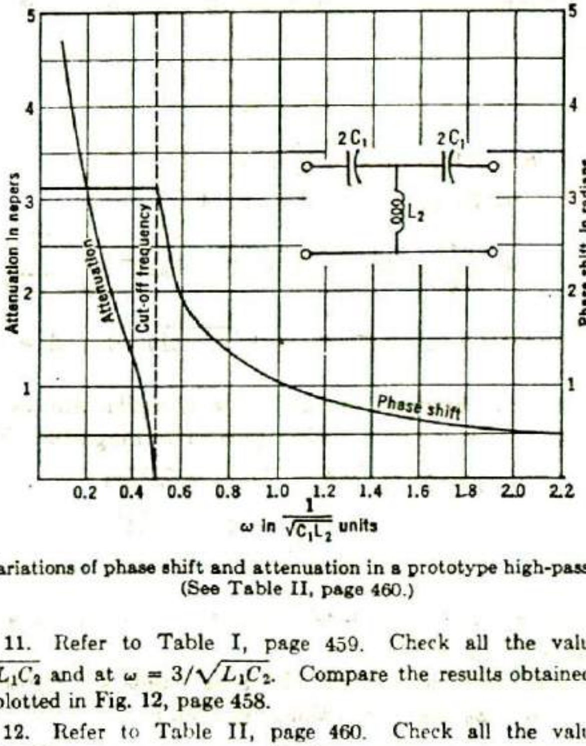


FIG. 13. Variations of phase shift and attenuation in a prototype high-pass filter section. (See Table II, page 460.)

Problem 11. Refer to Table I, page 459. Check all the values listed at $\omega = 1.5/\sqrt{L_1C_2}$ and at $\omega = 3/\sqrt{L_1C_2}$. Compare the results obtained for α and β with those plotted in Fig. 12, page 458.

Problem 12. Refer to Table II, page 460. Check all the values listed at $\omega = 0.25/\sqrt{C_1L_2}$ and at $\omega = 2.0/\sqrt{C_1L_2}$. Compare the results obtained for α and β with those plotted in Fig. 13, page 461.

The chief facts to be gained from the foregoing analyses are:

(1) α is equal to zero within the pass-band region.

(2) β is equal to $\pm\pi$ within the stop-band region.

A study of Tables I and II will show that the pass bands are limited to those regions where $\mathbf{Z}_1/4\mathbf{Z}_2$ possesses values between 0 and -1. These results might have been anticipated mathematically by investigating the possible values of α and β when $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are reactances of opposite types. Let

$$\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = \mathbf{A}$$

It is plain that $\mathbf{A} = A\underline{/\pm\pi}$ since

$$\frac{X_L\underline{/90^\circ}}{4X_C\underline{/-90^\circ}} \quad \text{or} \quad \frac{X_C\underline{/-90^\circ}}{4X_L\underline{/90^\circ}}$$

are complex numbers which have associated angles of $+\pi$ or $-\pi$ radians, respectively.

If

$$-1 \leqq A \leqq 0$$

$$\begin{aligned}\alpha + j\beta &= 2\log_e(\sqrt{1-A} + \sqrt{-A}) \\ &= 2\log_e(\sqrt{1-A} + j\sqrt{A}) \\ &= 2\left(\log_e\sqrt{1-A+A} + j\tan^{-1}\frac{\sqrt{A}}{\sqrt{1-A}}\right)\end{aligned}$$

Hence $\alpha = 0$ and $\beta = 2\tan^{-1}(\sqrt{A}/\sqrt{1-A})$ when $\mathbf{A} = \mathbf{Z}_1/4\mathbf{Z}_2$ lies between 0 and -1.

When $\mathbf{Z}_1/4\mathbf{Z}_2$ lies between -1 and $-\infty$ a similar analysis will show that for $\mathbf{Z}_1/4\mathbf{Z}_2 = A'\underline{/\pm\pi}$, A' being greater in magnitude than unity.

$$\begin{aligned}\alpha + j\beta &= 2\log_e(\sqrt{1-A'} + \sqrt{-A'}) \\ &= 2\log_e(j\sqrt{A'-1} + j\sqrt{A'}) \\ &= 2\log_e(\sqrt{A'-1} + \sqrt{A'}) + j(\pm\pi)\end{aligned}$$

Hence $\alpha = 2\log_e(\sqrt{A'-1} + \sqrt{A'})$ and $\beta = \pm\pi$ when $\mathbf{A}' = \mathbf{Z}_1/4\mathbf{Z}_2$ lies between -1 and $-\infty$.

The above analysis shows that the pass bands are limited to those regions where $\mathbf{Z}_1/4\mathbf{Z}_2$ takes on values between and including 0 and -1. Hence:

$$-1 \leqq \frac{\mathbf{Z}_1}{4\mathbf{Z}_2} \leqq 0 \tag{40}$$

defines the pass-band regions in terms of $\mathbf{Z}_1$ and $\mathbf{Z}_2$. The boundaries of a pass band in a particular case may be obtained by setting:

$$\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = 0 \quad \text{and} \quad \frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = -1 \tag{41}$$

or by setting

$$\frac{\mathbf{Z}_1}{\mathbf{Z}_2} = 0 \quad \text{and} \quad \frac{\mathbf{Z}_1}{\mathbf{Z}_2} = -4 \tag{42}$$

Reference to equation (39) will show that $\alpha = 0$ when $\mathbf{Z}_1/4\mathbf{Z}_2 = 0$ and when $\mathbf{Z}_1/4\mathbf{Z}_2 = -1$.

Example 10. Refer to the symmetrical π-section shown in Fig. 8, page 445. Let it be required to predict the pass-band boundaries in terms of the relationships stated in (42). The full series arm of Fig. 8 is $L_1 = 0.02$ henry and $\mathbf{Z}_1 = 0.02\omega\underline{/90^\circ}$ ohms. The full shunt arm is $C_2 = 4.0\ \mu$f and $\mathbf{Z}_2 = (10^6/4\omega)\underline{/-90^\circ}$ ohms.

Setting $\mathbf{Z}_1/\mathbf{Z}_2 = 0$ yields

$$\frac{0.02\omega\underline{/90^\circ}}{\frac{10^6}{4\omega}\underline{/-90^\circ}} = 0 \quad \text{or} \quad \omega = 0 \quad \text{(one boundary)}$$

Setting $\mathbf{Z}_1/\mathbf{Z}_2 = -4$ yields

$$\frac{0.02\omega\underline{/90^\circ}}{\frac{10^6}{4\omega}\underline{/-90^\circ}} = \frac{-0.08\omega^2}{10^6} = -4$$

from which

$$\omega^2 = \sqrt{50 \times 10^6}$$

$$\omega = 7070 \text{ radians per second} \quad \text{(one boundary)}$$

The value of ω given above represents the cut-off angular velocity of this particular low-pass filter section and corresponds to a frequency of $7070/2\pi$ or 1125 cycles.

Cut-Off Frequencies of Elementary Low- and High-Pass Sections. The frequency limits of the pass band for an elementary low-pass filter without resistance may be obtained from equation (38). For a low-pass filter $\mathbf{Z}_1 = j\omega L_1$ and $\mathbf{Z}_2 = -j\dfrac{1}{\omega C_2}\cdot$ If these values are substituted in equation (38), the result, after a little algebraic simplification, is:

$$\epsilon^{\alpha+j\beta} = \epsilon^{\alpha}\epsilon^{j\beta} = 1 - \frac{2\omega^2 L_1 C_2}{4} + 2\sqrt{\frac{\omega^4 L_1{}^2 C_2{}^2}{16} - \frac{\omega^2 L_1 C_2}{4}} \tag{43}$$

For no attenuation $\alpha = 0$, and

$$\epsilon^{j\beta} = \cos\beta + j\sin\beta = 1 - \frac{2\omega^2 L_1 C_2}{4} + 2\sqrt{\frac{\omega^4 L_1{}^2 C_2{}^2}{16} - \frac{\omega^2 L_1 C_2}{4}} \tag{44}$$

Since the last term of equation (44) is the only one that may become imaginary, it follows that the real part must be $\cos\beta$. Therefore

$$\cos\beta = 1 - \frac{2\omega^2 L_1 C_2}{4} \tag{45}$$

Since $\cos\beta$ can vary from 1 to -1, the limits for ω may be obtained. Hence

$$\pm 1 = 1 - \frac{2\omega^2 L_1 C_2}{4} \tag{46}$$

and

$$\omega = 0 \quad \text{or} \quad \frac{2}{\sqrt{L_1C_2}} \tag{47}$$

or

$$f_c = \frac{1}{\pi\sqrt{L_1C_2}} \quad \text{(for low-pass)} \tag{48}$$

Equation (48) gives the upper or cut-off frequency for an elementary low-pass filter. In other words, any wave of frequency between zero and f_c is passed without attenuation provided that the filter section is terminated in the characteristic impedance for that frequency.

For a high-pass filter, $\mathbf{Z}_1 = -j\dfrac{1}{\omega C_1}$ and $\mathbf{Z}_2 = j\omega L_2$ If these values are substituted in equation (38), a similar method of analysis as used in obtaining equation (45) gives

$$\cos\beta = 1 - \frac{2}{4\omega^2C_1L_2} \tag{49}$$

Substituting the limits of ± 1 for $\cos\beta$, the upper limit of frequency is found to be ∞ while the lower limit or cut-off frequency is:

$$f_c = \frac{1}{4\pi\sqrt{C_1L_2}} \quad \text{for high-pass} \tag{50}$$

Equation (50) gives the cut-off frequency for an elementary high-pass filter. This means that any frequency above the cut-off frequency f_c is passed with no attenuation if the filter section is terminated in the characteristic impedance for the particular frequency.

Constant-*k* Low-Pass Filter. Filter sections in which the series and shunt arms are inverse impedance functions possess a peculiar property. The product of $\mathbf{Z}_1$ and $\mathbf{Z}_2$ is independent of frequency. Reference to either the T- or π-section of Fig. 14 will show that

$$\mathbf{Z}_{1k}\mathbf{Z}_{2k} = (j\omega L_{1k})\left(-j\frac{1}{\omega C_{2k}}\right) = \frac{L_{1k}}{C_{2k}} = R_k^2 \tag{51}$$

$\sqrt{L_{1k}/C_{2k}}$ is an important characteristic of the filter section, and inasmuch as

$$\sqrt{\frac{L_{1k}}{C_{2k}}} = R_k = \text{a constant}$$

filter sections of this type are called constant-k sections. There are many other types of filter sections, several of which are derived in one way or another from constant-k sections. For this reason the parameters of constant-k sections usually carry the subscript k in order to designate properly the type of filter section that is under discussion. The parameters of some of the more elaborate filter sections are specified directly in terms of L_k and C_k.

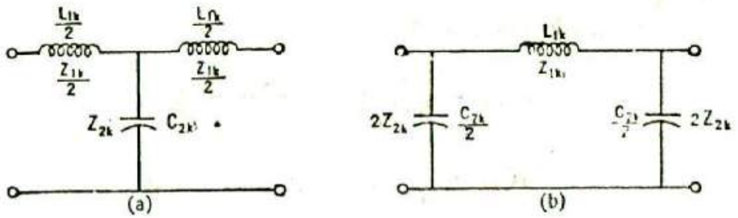


FIG. 14. Prototype or constant-k low-pass filter sections.

The general theory of the constant-k low-pass filter has already been presented. It remains only to develop the design equations for this type of filter.

$$\frac{Z_{1k}}{Z_{2k}} = \frac{j\omega L_{1k}}{-j\dfrac{1}{\omega C_{2k}}} = -\omega^2 L_{1k} C_{2k} \tag{52}$$

The boundaries of the pass band are determined by setting Z_{1k}/Z_{2k} equal to -4 and equal to zero. [See equation (42), page 462.]

$$-\omega^2 L_{1k} C_{2k} = 0 \quad \text{yields} \quad \omega = 0 \tag{53}$$

$$-\omega^2 L_{1k} C_{2k} = -4 \quad \text{yields} \quad \omega_c = \frac{2}{\sqrt{L_{1k} C_{2k}}} \tag{54}$$

ω_c is the angular velocity at which cut-off takes place and as such forms the upper boundary of the pass band. The *cut-off frequency of a low-pass, constant-k-type* filter is:

$$f_c = \frac{\omega_c}{2\pi} = \frac{1}{\pi\sqrt{L_{1k} C_{2k}}} \tag{55}$$

It will be observed that f_c is governed wholly by the magnitude of the $L_{1k}C_{2k}$ product. The lower the cut-off frequency, the higher is the $L_{1k}C_{2k}$ product, and vice versa.

Another important consideration in either the theory or design of a filter section is the matter of correct terminating impedances. A single section can be properly matched to its sending and receiving ends if terminated on an image basis, as explained on page 439. If more than one filter section is to be employed between sending and receiving ends, it is desirable to design each section to have the same characteristic impedance. Under these conditions minimum reflection loss results when the various sections are arranged as shown in Fig. 10 or Fig. 11. A detailed analysis of these losses will not be given here since they are similar in nature to reflection losses on long lines. (See Chapter X.)

For a constant-k, low-pass T-section:

$$\mathbf{Z}_{oTk} = \sqrt{\frac{L_{1k}}{C_{2k}}\left(1 - \frac{\omega^2 L_{1k} C_{2k}}{4}\right)} \tag{56}$$

$$L_{1k}C_{2k} = \frac{4}{\omega_c^{\,2}} \quad \text{[See equation (54).]}$$

Therefore, for a constant-k, low-pass T-section:

$$\begin{aligned} \mathbf{Z}_{oTk} &= \sqrt{\frac{L_{1k}}{C_{2k}}}\sqrt{1 - \frac{f^2}{f_c^{\,2}}} \\ &= R_k\sqrt{1 - \frac{f^2}{f_c^{\,2}}} \end{aligned} \tag{57}$$

For a constant-k, low-pass π-section:

$$\mathbf{Z}_{o\pi k} = \frac{\sqrt{L_{1k}/C_{2k}}}{\sqrt{1 - \dfrac{f^2}{f_c^{\,2}}}} = \frac{R_k}{\sqrt{1 - \dfrac{f^2}{f_c^{\,2}}}} \tag{58}$$

The variations of Z_{oTk} and $Z_{o\pi k}$ from $f = 0$ to $f = f_c$ are illustrated in Fig. 15. The fact that the correct terminating impedance of a constant-k section varies over such wide limits is a very serious limitation in certain communication circuits. For a fixed receiving impedance it is plain that either the T- or π-section is correctly terminated at only one frequency. The opposite trends in Z_{oTk} and $Z_{o\pi k}$ are combined in one form of filter section to obtain a characteristic impedance which is reasonably constant over the frequency range of the pass band. (See m-derived filter sections, pages 480–484.)

The zero-frequency value of either Z_{oTk} or $Z_{o\pi k}$ is:

$$R_k = \sqrt{\frac{L_{1k}}{C_{2k}}} \quad \text{[See equations (57) and (58).]} \tag{59}$$

L_{1k} and C_{2k} can be related to one another through the value of R_k^2. [See equation (51).]

$$L_{1k} = R_k^2 C_{2k} \tag{60}$$

$$C_{2k} = \frac{L_{1k}}{R_k^2} \tag{61}$$

The design values of L_{1k} and C_{2k} are usually specified in terms of cut-off frequency, f_c, and the zero-frequency value of the characteristic

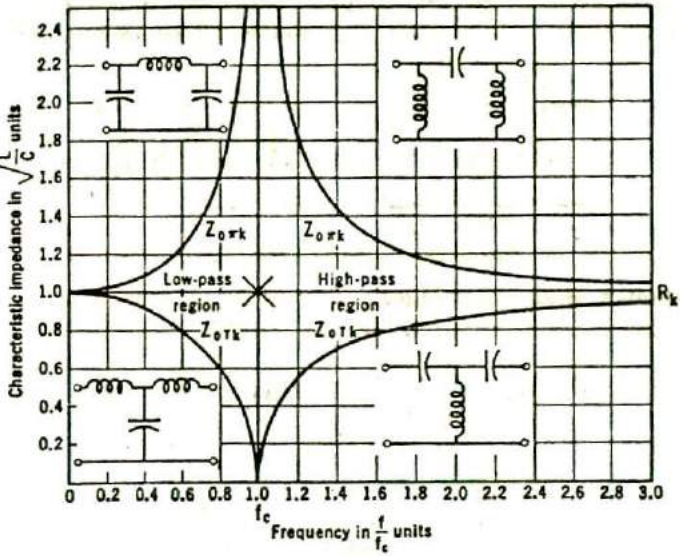


FIG. 15. Variations of the characteristic impedances of low-pass and high-pass constant-k filter sections.

impedance, R_k. It has been shown that:

$$f_c = \frac{1}{\pi\sqrt{L_{1k}C_{2k}}} \quad \text{[See equation (55).]}$$

Eliminating C_{2k} as given in equation (61) from the above equation yields:

$$f_c = \frac{1}{\pi\sqrt{L_{1k}^2/R_k^2}}$$

or

$$L_{1k} = \frac{R_k}{\pi f_c} \quad \text{(for low-pass filter)} \tag{62}$$

From equations (61) and (62) it is plain that:

$$C_{2k} = \frac{L_{1k}}{R_k^2} = \frac{1}{\pi R_k f_c} \quad \text{(for low-pass filter)} \tag{63}$$

Equations (62) and (63) specify the values of L and C to employ in a constant-k, low-pass filter section in terms of f_c and R_k.

Problem 13. Design both T- and π-section, low-pass filters of the constant-k type which will have a zero-frequency characteristic impedance of 600 ohms and a cut-off frequency of 940 cycles. Draw the circuit arrangement in each case, indicating the particular values (in henrys or microfarads) of each circuit element. *Ans.:* The *full* series arm $L_{1k} = 0.203$ henry; and the *full* shunt arm $C_{2k} = 0.565$ μf.

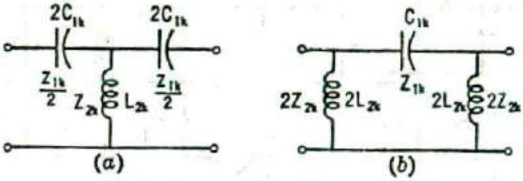


FIG. 16. Prototype or constant-k high-pass filter sections.

Constant-k High-Pass Filter. Prototype or constant-k, high-pass filter sections are illustrated in Fig. 16. In the present case:

$$\mathbf{Z}_{1k}\mathbf{Z}_{2k} = \left(-j\frac{1}{\omega C_{1k}}\right)(j\omega L_{2k}) = \frac{L_{2k}}{C_{1k}} = R_k^2 \tag{64}$$

and

$$\frac{\mathbf{Z}_{1k}}{\mathbf{Z}_{2k}} = \frac{-j\dfrac{1}{\omega C_{1k}}}{j\omega L_{2k}} = -\frac{1}{\omega^2 C_{1k}L_{2k}} \tag{65}$$

The boundaries of the pass band are again determined by setting $\mathbf{Z}_{1k}/\mathbf{Z}_{2k}$ equal to -4 and equal to zero. [See equation (42), page 462.]

$$-\frac{1}{\omega^2 C_{1k}L_{2k}} = 0 \quad \text{yields} \quad \omega = \infty \tag{66}$$

$$-\frac{1}{\omega^2 C_{1k}L_{2k}} = -4 \quad \text{yields} \quad \omega_c = \frac{1}{2\sqrt{C_{1k}L_{2k}}} \tag{67}$$

The cut-off frequency of a high-pass, constant-k filter is

$$f_c = \frac{\omega_c}{2\pi} = \frac{1}{4\pi\sqrt{C_{1k}L_{2k}}} \tag{68}$$

$\mathbf{Z}_{oT}$ and $\mathbf{Z}_{o\pi}$ may be expressed in terms of f_c, f, and $\sqrt{L_{2k}/C_{1k}}$ For a constant-k, high-pass T-section:

$$\mathbf{Z}_{oTk} = \sqrt{\frac{L_{2k}}{C_{1k}}} \times \sqrt{1 - \frac{f_c^2}{f^2}} \tag{69}$$

For a constant-k, high-pass π-section:

$$\mathbf{Z}_{o\pi k} = \sqrt{\frac{L_{2k}}{C_{1k}}} \div \sqrt{1 - \frac{f_c^2}{f^2}} \tag{70}$$

General trends in Z_{oTk} and $Z_{o\pi k}$ in constant-k, high-pass filter sections are illustrated in Fig. 15. Both Z_{oTk} and $Z_{o\pi k}$ approach the common value $\sqrt{L_{2k}/C_{1k}}$ at $f = \infty$. Because it is a useful common base from which to work, $\sqrt{L_{2k}/C_{1k}}$ is given special designation, namely R_k. R_k is known as the infinite-frequency characteristic impedance. Since

$$\sqrt{\frac{L_{2k}}{C_{1k}}} = R_k \tag{71}$$

$$L_{2k} = R_k^2 C_{1k} \quad \text{and} \quad C_{1k} = \frac{L_{2k}}{R_k^2} \tag{72}$$

If the above values are substituted separately in equation (68), the following relationships are obtained:

$$C_{1k} = \frac{1}{4\pi R_k f_c} \quad \text{(for high-pass filter)} \tag{73}$$

$$L_{2k} = \frac{R_k}{4\pi f_c} \quad \text{(for high-pass filter)} \tag{74}$$

Equations (73) and (74) may be employed in the design of constant-k, high-pass filter sections which are to have a particular cut-off frequency and which are to have infinite-frequency characteristic impedances equal to R_k.

Problem 14. What are the cut-off frequency and infinite-frequency characteristic impedance of the high-pass filter section that can be constructed from two 1-μf condensers and one 15-millihenry inductance coil?

Ans.: $f_c = 919$ cycles; $R_k = 173$ ohms.

Tabulation and Review of Constant-k Filter Theory. The important features contained in equations (51) to (74) inclusive are summarized concisely in Table III, pages 471–472. The attenuation and phase shift in Table III are expressed in forms which derive directly from "Campbell's" equation. (See footnote 3 on page 455.) It has been shown in examples 8 and 9, pages 457–458, how the attenuation and phase shift may be calculated from equation (39), page 455, without the aid of hyperbolic functions. For the reader who is familiar with complex hyperbolic functions the following derivation and application of "Campbell's" equation may be of interest.

Derivation and Application of Campbell's Equation. The application of Kirchhoff's emf law to the $wxyz$ loop of the filter sections shown in Fig. 10, page 452, yields

$$\mathbf{Z}_1\mathbf{I}_2 + \mathbf{Z}_2(\mathbf{I}_2 - \mathbf{I}_3) - \mathbf{Z}_2(\mathbf{I}_1 - \mathbf{I}_2) = 0 \tag{75}$$

or

$$\mathbf{Z}_1\mathbf{I}_2 + 2\mathbf{Z}_2\mathbf{I}_2 - \mathbf{Z}_2\mathbf{I}_3 - \mathbf{Z}_2\mathbf{I}_1 = 0 \tag{76}$$

Dividing the above equation through by $\mathbf{Z}_2\mathbf{I}_2$ and transposing results in

$$\frac{\mathbf{I}_1}{\mathbf{I}_2} + \frac{\mathbf{I}_3}{\mathbf{I}_2} = 2 + \frac{\mathbf{Z}_1}{\mathbf{Z}_2} \tag{77}$$

Since

$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = \epsilon^{\gamma} = \frac{\mathbf{I}_2}{\mathbf{I}_3}$$

it follows that

$$\frac{\mathbf{I}_3}{\mathbf{I}_2} = \epsilon^{-\gamma}$$

$$\epsilon^{\gamma} + \epsilon^{-\gamma} = 2 + \frac{\mathbf{Z}_1}{\mathbf{Z}_2} \tag{78}$$

$$\frac{\epsilon^{\gamma} + \epsilon^{-\gamma}}{2} = \cosh \gamma = 1 + \frac{\mathbf{Z}_1}{2\mathbf{Z}_2} \quad \text{(Campbell's equation)} \tag{79}$$

A more useful form for the purposes at hand may be derived as follows:

$$\begin{aligned} \cosh \gamma = \cosh (\alpha + j\beta) &= \frac{\epsilon^{(\alpha+j\beta)} + \epsilon^{-(\alpha+j\beta)}}{2} \\ &= \frac{\epsilon^{\alpha}\epsilon^{j\beta} + \epsilon^{-\alpha}\epsilon^{-j\beta}}{2} \end{aligned} \tag{80}$$

Converting the $\epsilon^{j\beta}$ terms into their rectangular forms results in:

$$\begin{aligned} \cosh (\alpha + j\beta) &= \frac{\epsilon^{\alpha} (\cos \beta + j \sin \beta) + \epsilon^{-\alpha} (\cos \beta - j \sin \beta)}{2} \\ &= \frac{(\epsilon^{\alpha} + \epsilon^{-\alpha})}{2} \cos \beta + j \frac{(\epsilon^{\alpha} - \epsilon^{-\alpha})}{2} \sin \beta \end{aligned}$$

From the analytical definitions of hyperbolic cosine and hyperbolic sine, it follows that

$$\cosh (\alpha + j\beta) = \cosh \alpha \cos \beta + j \sinh \alpha \sin \beta$$

Therefore

$$\cosh \alpha \cos \beta + j \sinh \alpha \sin \beta = 1 + \frac{\mathbf{Z}_1}{2\mathbf{Z}_2} \tag{81}$$

The above form may be used directly to derive the attenuation and phase-shift expressions given in Table III, page 472.

In the stop band, $\beta = \pi$. Since $\cos \beta$ then becomes -1 in equation

TABLE III

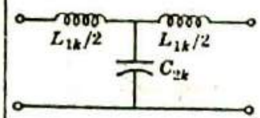
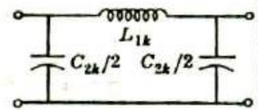
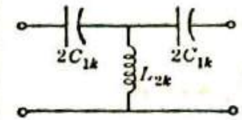
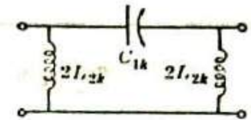
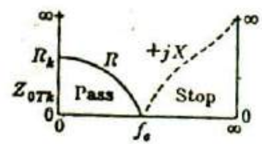
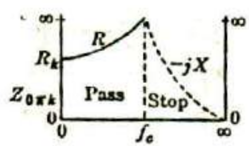
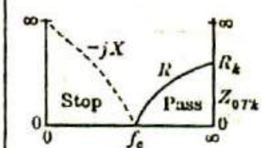
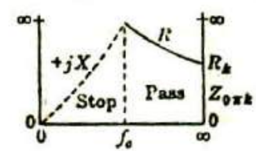
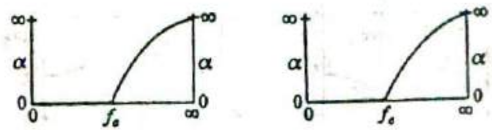
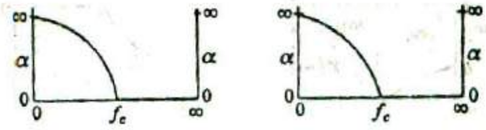
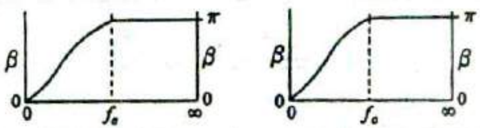
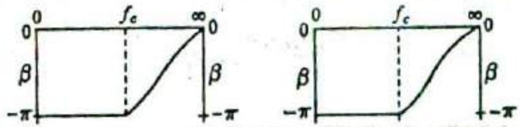
	Constant-k Low Pass		Constant-k High Pass	
	T-Section	π-Section	T-Section	π-Section
Circuit Configuration				
Cut-Off Frequency	$f_c = \dfrac{1}{\pi\sqrt{L_{1k}C_{2k}}}$		$f_c = \dfrac{1}{4\pi\sqrt{C_{1k}L_{2k}}}$	
Z_1Z_2 Base Characteristic Impedance	Zero-Frequency Characteristic Z $R_k = \sqrt{L_{1k}/C_{2k}}$		Infinite-Frequency Characteristic Z $R_k = \sqrt{L_{2k}/C_{1k}}$	
Z_0 With Respect To f	 $Z_{0Tk} = R_k \times \sqrt{1 - f^2/f_c^2}$	 $Z_{0\pi k} = R_k/\sqrt{1 - f^2/f_c^2}$	 $Z_{0Tk} = R_k \times \sqrt{1 - f_c^2/f^2}$	 $Z_{0\pi k} = R_k/\sqrt{1 - f_c^2/f^2}$

TABLE III (Continued)

α Attenuation (quantitative values given in Fig. 12, page 458.)	 In the stop-band: $\alpha = \cosh^{-1}\left(\frac{\omega^2 L_{1k} C_{2k}}{2} - 1\right)$	 $\alpha = \cosh^{-1}\left(\frac{1}{2\omega^2 C_{1k} L_{2k}} - 1\right)$
β Phase Shift (quantitative values given in Fig. 13, page 461.)	 In the pass-band: $\beta = \cos^{-1}\left(1 - \frac{\omega^2 L_{1k} C_{2k}}{2}\right)$	 $\beta = \cos^{-1}\left(1 - \frac{1}{2\omega^2 C_{1k} L_{2k}}\right)$
Values of Circuit Elements in Terms of R_k and f_c	$L_{1k} = \frac{R_k}{\pi f_c}$ $C_{2k} = \frac{1}{\pi R_k f_c}$	$C_{1k} = \frac{1}{4\pi R_k f_c}$ $L_{2k} = \frac{R_k}{4\pi f_c}$

(81) and $\sin \beta = 0$:

$$-\cosh \alpha = 1 + \frac{\mathbf{Z}_1}{2\mathbf{Z}_2}$$

$$\alpha = \cosh^{-1}\left(-\frac{\mathbf{Z}_1}{2\mathbf{Z}_2} - 1\right) \quad \text{(in stop band)} \qquad (82)$$

In the pass band, $\alpha = 0$. Since $\cosh 0 = 1$ and $\sinh 0 = 0$, equation (81) becomes:

$$\cos \beta = \left(1 + \frac{\mathbf{Z}_1}{2\mathbf{Z}_2}\right)$$

$$\beta = \cos^{-1}\left(1 + \frac{\mathbf{Z}_1}{2\mathbf{Z}_2}\right) \quad \text{(in pass band)} \qquad (83)$$

As applied to a constant-k low-pass filter section:

$$\mathbf{Z}_1 = \omega L_{1k}\underline{/90^\circ} \quad \mathbf{Z}_2 = \frac{1}{\omega C_{2k}}\underline{/-90^\circ} \quad \frac{\mathbf{Z}_1}{\mathbf{Z}_2} = -\omega^2 L_{1k} C_{2k}$$

Equation (82) then takes the form:

$$\alpha = \cosh^{-1}\left(\frac{\omega^2 L_{1k} C_{2k}}{2} - 1\right)$$

as shown in Table III. Equation (83) takes the form:

$$\beta = \cos^{-1}\left(1 - \frac{\omega^2 L_{1k} C_{2k}}{2}\right)$$

as shown in Table III. Corresponding expressions for α and β may be derived for the constant-k high-pass filter section. The results are shown in Table III.

Band-Pass and Band-Elimination Filters. Band-pass filters are networks which are designed to attenuate all frequencies except those in a specified band. A band-pass filter may be formed by placing a low-pass filter section (having a cut-off frequency of f_{cl}) in series with a high-pass filter section (having a cut-off frequency of f_{ch}). Then f_{cl} is made higher than f_{ch} by the specified band width, which is $f_{cl} - f_{ch}$. A study of the attenuation graphs shown in Table III will show how f_{cl} and f_{ch} should be adjusted to give a zero-attenuation band.

A band-pass filter may take the form of a single section as shown in Fig. 17. The section shown in Fig. 17 is called a constant-k band-pass filter when $L_2C_2 = L_1C_1$ because under these conditions:

$$\mathbf{Z}_1\mathbf{Z}_2 = \frac{L_2}{C_1} = \frac{L_1}{C_2} = \text{a constant}$$

An analysis of the band-pass filter will not be given here, although such an analysis may be carried through in a manner similar to those given for the low-pass and high-pass sections.

Band-elimination filters are networks which are designed to pass all frequencies except those in a specified band. A band-elimination filter may be formed by placing a low-pass section (having a cut-off frequency of f_{cl}) in parallel with a high-pass section (having a cut-off frequency of f_{ch}). Then f_{cl} is made lower than f_{ch} by the specified band width, which is $f_{ch} - f_{cl}$. All frequencies have a pass band (through one of the parallel sections) except where the two attenuation graphs overlap. (See attenuation graphs in Table III.)

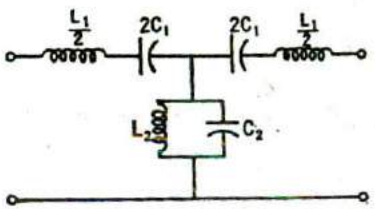


FIG. 17. Band-pass filter contained in a single section.

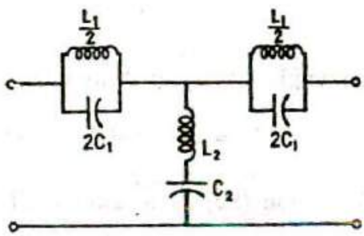


FIG. 18. Band-elimination filter contained in a single section.

A band-elimination filter may take the form of a single section as shown in Fig. 18. The section shown in Fig. 18 is called a constant-k band-elimination filter when $L_2C_2 = L_1C_1$ because under these conditions $\mathbf{Z}_1\mathbf{Z}_2$ is a constant. It will be observed that the arms of Fig. 18 are the reverse of those in Fig. 17.

Two Limitations of Constant-*k* Sections. The constant-k type of filter section has two rather serious shortcomings. First, its characteristic impedance is not sufficiently constant over the transmission band for certain classes of work. (See Fig. 15.) Second, the attenuation does not rise very abruptly at the boundary of the transmission band. (See Figs. 12 and 13.)

In order to overcome the inherent limitations of the constant-k type, Zobel[4] devised a filter section which he called the m-derived type. The m-derived half section may be employed to give practically uniform characteristic impedance over a large part of the pass band and at the same time increase the abruptness with which cut-off occurs. Full m-derived sections may be employed to give further increased attenuation near the cut-off point, and by proper adjustment of the parameter

[4] "Theory and Design of Uniform and Composite Electric Wave Filters," by O. J. Zobel, *Bell System Technical Journal*, January, 1923.

m they can be made to meet any practical attenuation requirement in this region. When worked in conjunction with constant-*k* sections, the *m*-derived sections overcome both the aforementioned shortcomings of the constant-*k* sections. However, *m*-derived sections by them-

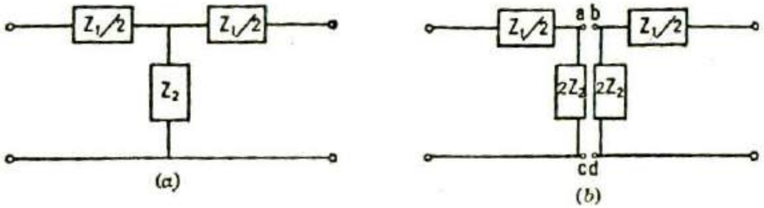


FIG. 19. Illustrating the circuit configuration of half sections formed by longitudinal bisection of shunt arm of a prototype T-section.

selves have certain limitations which will become apparent after the attenuation characteristics of these sections have been studied.

***m*-Derived Half Sections.** If the full shunt arm of Fig. 19*a* is separated into two parallel paths of $2Z_2$ ohms each, the original T-section may

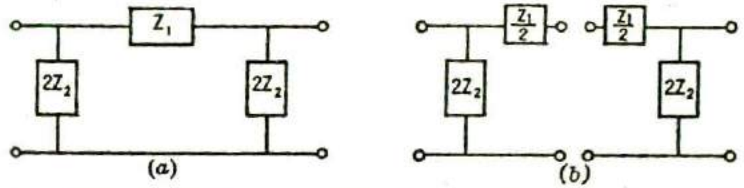


FIG. 20. Illustrating the circuit configuration of half sections formed by longitudinal bisection of the series arm of a prototype π-section.

be separated into two similar parts as shown in Fig. 19*b*. Each of these parts is known as a half section or as an L-type section. If the full series arm of the π-section shown in Fig. 20*a* is separated into two series elements of $Z_1/2$ ohms each, the original π-section can be separated into two half sections as shown in Fig. 20*b*. A comparison of Fig. 20*b* with Fig. 19*b* will show the equivalence of half sections formed by "halving" π-sections and those formed by "halving" T-sections.

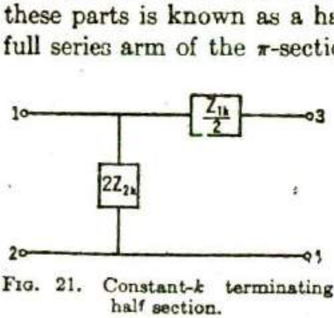


FIG. 21. Constant-*k* terminating half section.

The image impedances of the half section shown in Fig. 21 may be found from open-circuit and short-circuit conditions. Let the open-circuit and short-circuit impedances be known as $Z_{o\text{-}c}$ and $Z_{s\text{-}c}$, respectively.

-31

The impedance looking into terminals 1 and 2 is:

$$Z_{12} = \sqrt{Z_{o\text{-}c}Z_{s\text{-}c}} = \sqrt{\frac{2Z_{1k}Z_{2k}^2}{\dfrac{Z_{1k}}{2} + 2Z_{2k}}}$$

from which

$$Z_{12} = \sqrt{\frac{Z_{1k}Z_{2k}}{1 + \dfrac{Z_{1k}}{4Z_{2k}}}} = Z_{o\pi k} \tag{84}$$

The impedance looking into terminals 3 and 4 is:

$$Z_{34} = \sqrt{Z_{o\text{-}c}Z_{s\text{-}c}} = \sqrt{\left(\frac{Z_{1k}}{2} + 2Z_{2k}\right)\frac{Z_{1k}}{2}}$$

or

$$Z_{34} = \sqrt{Z_{1k}Z_{2k}\left(1 + \frac{Z_{1k}}{4Z_{2k}}\right)} = Z_{oTk} \tag{85}$$

The half section shown in Fig. 21 has the impedance characteristics of a π-section between terminals 1 and 2 and the impedance characteristics of a T-section between terminals 3 and 4. It may, therefore, be used to match a π-section to a T-section. Also it may be used to match a filter section to a terminating impedance which differs from the characteristic impedance of the filter section or to change the impedance level at any point in a two-wire line. The proper values of $Z_{1k}/2$ and $2Z_{2k}$ to be employed in effecting any desired impedance transformation may be determined by solving equations (84) and (85) simultaneously for Z_{1k} and Z_{2k} in terms of Z_{12} and Z_{34}.

Some little difficulty is usually encountered in presenting m-derived filter theory to beginning students because certain anticipations have to be made at the outset of the investigation. Inasmuch as anticipations must be indulged in in any event, the actual circuit configuration of the m-derived half section will be accepted and its operating characteristics studied.

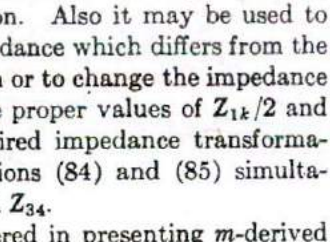


FIG. 22. m-derived terminating half section.

It will now be assumed that the half section shown in Fig. 21 takes the particular form shown in Fig. 22. A new parameter, m, has been arbitrarily introduced. It is simply a numeric which may, for the purposes at hand, range in value from zero to unity. The change in circuit configuration from Fig. 21 to Fig. 22

may be interpreted as follows:

(a) $\frac{Z_{1k}}{2}$ of the constant-k half section is changed to some fractional part of $\frac{Z_{1k}}{2}$ in Fig. 22.

(b) $2Z_{2k}$ of Fig. 21 is changed to $\frac{2Z_{2k}}{m}$ in Fig. 22.

(c) In series with $\frac{2Z_{2k}}{m}$ in Fig. 22 is placed an impedance $\frac{1-m^2}{4m}2Z_{1k}$.

It may be shown that, if the change in (a) is made, the changes in (b) and (c) must be made if the two half sections shown in Figs. 21 and 22 are to have the same characteristic impedance looking into the 3–4 terminals.

The half section shown in Fig. 22 has some very desirable characteristics. Its characteristic impedance looking into terminals 3 and 4 is:

$$
\begin{aligned}
Z_{34m} = \sqrt{Z_{o\text{-}c}Z_{s\text{-}c}} &= \sqrt{\left(m\frac{Z_{1k}}{2} + \frac{1-m^2}{2m}Z_{1k} + \frac{2Z_{2k}}{m}\right)m\frac{Z_{1k}}{2}} \\
&= \sqrt{\frac{m^2Z_{1k}^2}{4} + \frac{mZ_{1k}^2}{4m} - \frac{m^2Z_{1k}^2}{4} + \frac{2mZ_{1k}Z_{2k}}{2m}} \\
&= \sqrt{Z_{1k}Z_{2k} + \frac{Z_{1k}^2}{4}} = Z_{oTk} \qquad (86)
\end{aligned}
$$

The equation above shows that terminals 3 and 4 of the m-derived half section can be used to match the impedance of a constant-k T-section or any other equivalent impedance including the 3–4 terminal characteristic impedance of Fig. 21.

The characteristic impedance of the m-derived terminating half section looking into terminals 1 and 2 is:

$$Z_{12m} = \sqrt{Z_{o\text{-}c}Z_{s\text{-}c}}$$

where

$$Z_{o\text{-}c} = \left[\frac{(1-m^2)}{2m}Z_{1k} + 2\frac{Z_{2k}}{m}\right]$$

$$Z_{s\text{-}c} = \frac{\left[\frac{(1-m^2)}{2m}Z_{1k} + 2\frac{Z_{2k}}{m}\right]\left[m\frac{Z_{1k}}{2}\right]}{\left[\frac{(1-m^2)}{2m}Z_{1k} + 2\frac{Z_{2k}}{m} + m\frac{Z_{1k}}{2}\right]}$$

$$Z_{12m} = \sqrt{\frac{\left(\frac{1-m^2}{2m} Z_{1k} + 2\frac{Z_{2k}}{m}\right)^2 \left(\frac{m^2 Z_{1k}^2}{4}\right)}{\left(\frac{1-m^2}{2m} Z_{1k} + 2\frac{Z_{2k}}{m} + m\frac{Z_{1k}}{2}\right)\left(\frac{mZ_{1k}}{2}\right)}}$$

$$= \sqrt{\frac{\left(\frac{1-m^2}{4} Z_{1k}^2 + Z_{1k}Z_{2k}\right)^2}{\frac{Z_{1k}^2}{4} - \frac{m^2 Z_{1k}^2}{4} + Z_{1k}Z_{2k} + \frac{m^2 Z_{1k}^2}{4}}}$$

$$= \frac{Z_{1k}Z_{2k} + \frac{Z_{1k}^2}{4}(1-m^2)}{\sqrt{Z_{1k}Z_{2k} + \frac{Z_{1k}^2}{4}}}$$

$$= \frac{Z_{1k}Z_{2k}}{Z_{oTk}}\left[1 + \frac{Z_{1k}}{4Z_{2k}}(1-m^2)\right] \tag{87}$$

or remembering (9):

$$Z_{12m} = Z_{o\pi k}\left[1 + \frac{Z_{1k}}{4Z_{2k}}(1-m^2)\right] \tag{88}$$

In addition to being a function of Z_{1k} and Z_{2k}, Z_{12m} is a function of m. With the proper choice of m, Z_{12m} can be made reasonably constant over about 90 per cent of the transmission band. The changes of $Z_{o\pi k}$ and the modifying factor $\left[1 + \frac{Z_{1k}}{4Z_{2k}}(1-m^2)\right]$ with respect to frequency combine in such a manner as to make Z_{12m} approximately constant over wide ranges of frequency.

Example 11. Consider the general trend of $Z_{o\pi}$ for the constant-k, low-pass section shown in Fig. 15. Instead of this rapidly rising curve, the change in the output characteristic impedance of a low-pass, m-derived half section at the 1–2 terminals is:

$$Z_{12m} = Z_{o\pi k}\left[1 + \frac{j\omega L_{1k}}{-j4\frac{1}{\omega C_{2k}}}(1-m^2)\right]$$

or

$$Z_{12m} = Z_{o\pi k}\left[1 - \frac{\omega^2 L_{1k}C_{2k}}{4}(1-m^2)\right]$$

Physically, m may be equal to any value between zero and unity. Mathematical experimentation shows that good results are obtained when $m = 0.60$. The calculated values of $Z_{o\pi k}$ and the modifying factor are shown in Table IV, and a graph of Z_{12m} for $m = 0.6$ is contained in Fig. 23. It will be remembered that f_c for a

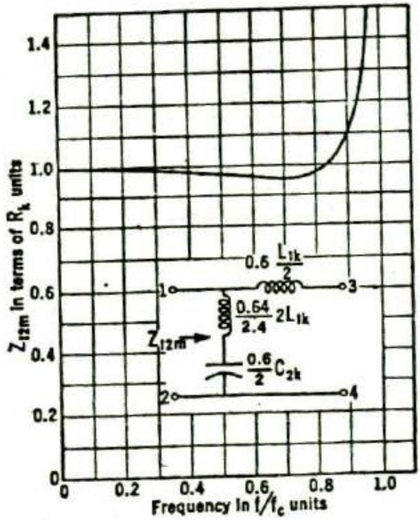


FIG. 23. Variation of Z_{12m} for $m = 0.6$.

TABLE IV

$$Z_{12m} = Z_{o\pi k}\left[1 - \frac{f^2}{f_c^2}(1 - m^2)\right] \text{ for } m = 0.6$$

$$R_k = \sqrt{L_{1k}/C_{2k}}$$

$\frac{f}{f_c}$	$\left[1 - \frac{f^2}{f_c^2}(0.64)\right]$	$Z_{o\pi k}$	Z_{12m}
0	1.000	R_k	R_k
0.10	0.994	1.005 R_k	0.999 R_k
0.20	0.974	1.02 R_k	0.993 R_k
0.40	0.898	1.09 R_k	0.979 R_k
0.60	0.770	1.25 R_k	0.963 R_k
0.80	0.590	1.67 R_k	0.963 R_k
0.90	0.482	2.30 R_k	1.108 R_k
0.95	0.424	3.16 R_k	1.34 R_k
1.00	0.360	∞	∞

low-pass filter section is $1/\pi\sqrt{L_{1k}C_{2k}}$ and that $Z_{o\pi k} = R_k/\sqrt{1 - (f^2/f_c^2)}$. The expression for Z_{12m} in this particular case is, therefore, reducible to

$$Z_{12m} = \frac{R_k}{\sqrt{1 - \frac{f^2}{f_c^2}}}\left[1 - \frac{f^2}{f_c^2}(1 - m^2)\right]$$

If it is necessary to work closer to the cut-off frequency than a value of $m = 0.6$ will permit, m may be made somewhat less than 0.60. However, these slightly lower values of m cause the Z_{12m} variation to be more irregular throughout the first 90 per cent of the transmission band. Numerical experimentation will show the effects caused by different values of m.

Problem 15. Plot, with respect to frequency, the variation of the characteristic output impedance of a low-pass, m-derived terminating half section (Z_{12m}) for $m = 0.55$. Reckon frequency in f/f_c units. (See Table IV and Fig. 23.)

Full m-Derived Sections. Full m-derived T-sections are shown in Fig. 24. As in the m-derived half section, the series and shunt arms are specified in terms of the constant-k impedances Z_{1k} and Z_{2k}. Any constant-k-type section may be altered to yield what is known as an m-derived section. Only the low-pass and high-pass, m-derived T-sections will be considered in detail. These are shown in Fig. 24b and 24c.

The variations of the characteristic impedance of full m-derived low-pass π-sections are generally similar to the curve shown in Fig. 23. A comparison of the characteristic impedance curves of different m-derived filter sections is shown in Fig. 25.

In establishing an m-derived T-section the parameters are so readjusted from the constant-k values that the m-derived section characteristic impedance is identical with the constant-k section characteristic impedance. This requires that

$$Z_{2m} = \left[\frac{1 - m^2}{4m} Z_{1k} + \frac{Z_{2k}}{m}\right] \quad \text{if} \quad Z_{1m} = mZ_{1k}$$

as may be seen from the following algebraic steps:

$$Z_{oTm} = Z_{oTk} \quad \text{(imposed condition)} \tag{89}$$

Reference to equation (5) will show that, if $Z_{1m} = mZ_{1k}$:

$$\sqrt{(mZ_{1k})Z_{2m} + \frac{(mZ_{1k})^2}{4}} = \sqrt{Z_{1k}Z_{2k} + \frac{Z_{1k}^2}{4}} \tag{90}$$

Squaring both sides of the above equation and solving for Z_{2m}:

$$Z_{2m} = \frac{1 - m^2}{4m} Z_{1k} + \frac{Z_{2k}}{m} \tag{91}$$

One of the most important characteristics of a full m-derived section is its theoretical infinite attenuation near the point of cut-off.

Frequencies of Infinite Attenuation. Since Z_{1k} and Z_{2k} are different types of reactances, the shunt arm of Fig. 24*a* will, at some frequency, become resonant. If the shunt arm is in resonance, its impedance is theoretically equal to zero and the attenuation becomes infinitely large. The frequency at which these phenomena occur is know as f_∞,

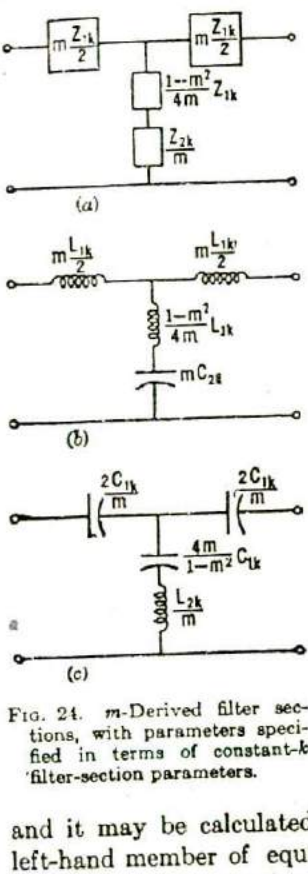


FIG. 24. m-Derived filter sections, with parameters specified in terms of constant-k filter-section parameters.

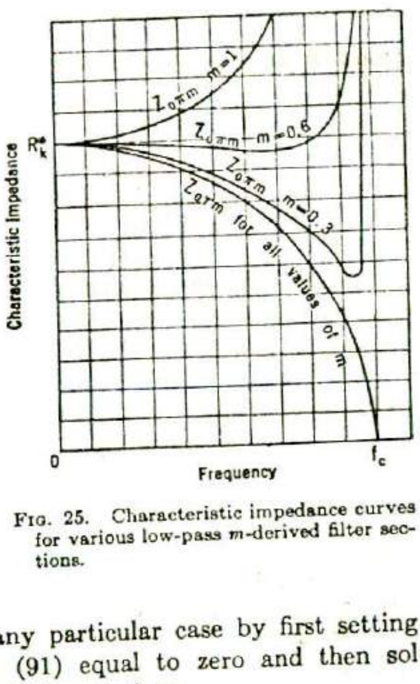


FIG. 25. Characteristic impedance curves for various low-pass m-derived filter sections.

and it may be calculated in any particular case by first setting the left-hand member of equation (91) equal to zero and then solving for f. In a low-pass, m-derived filter section:

$$f_\infty = \frac{1}{2\pi\sqrt{\frac{(1-m^2)}{4}L_{1k}C_{2k}}} = \frac{1}{\pi\sqrt{L_{1k}C_{2k}}\sqrt{1-m^2}} \tag{92}$$

The cut-off frequency of the m-derived section is equal to the cut-off frequency of the constant-k section from which it is derived. (See

Table V, page 485.) In the constant-k, low-pass section:

$$f_c = \frac{1}{\pi\sqrt{L_{1k}C_{2k}}} \quad \text{[See equation (55).]}$$

Therefore

$$f_\infty = \frac{f_c}{\sqrt{1 - m^2}} \tag{93}$$

from which

$$m = \sqrt{1 - \frac{f_c^{\,2}}{f_\infty^{\,2}}} \quad \text{(for low-pass section)} \tag{94}$$

In a similar manner it may be shown that for a high-pass, m-derived filter section:

$$f_\infty = f_c\sqrt{1 - m^2} \tag{95}$$

and

$$m = \sqrt{1 - \frac{f_\infty^{\,2}}{f_c^{\,2}}} \quad \text{(for high-pass section)} \tag{96}$$

Equations (94) and (96) illustrate the manner in which f_c and f_∞ determine the value of m that should be employed if theoretical infinite attenuation is to obtain at a specified f_∞. If, for example, a 1000-cycle cut-off frequency, low-pass filter is to have infinite attenuation at 1050 cycles, m is evaluated in accordance with equation (94). Thus:

$$m = \sqrt{1 - \frac{(1000)^2}{(1050)^2}} = 0.307 \quad \text{approximately}$$

The nearer f_∞ is to f_c, the lower will be the value of m. The reverse order of reasoning indicates that the lower the value of m, the sharper will be the cut-off. These facts are illustrated graphically in Fig. 26.

General Method of Analyzing m-Derived Filter Section Operation. Certain aspects of m-derived filter section operation may not be apparent from the cursory treatment that has been presented. The exact manner in which the phase shift and attenuation vary with respect to frequency can be obtained by subjecting the filter section to the "general" method of analysis. This method is summed up in equation (39), which, for convenience, is restated below.

$$\alpha + j\beta = 2\log_e\left[\sqrt{\frac{Z_1}{4Z_2}} + \sqrt{\frac{Z_1}{4Z_2} + 1}\right] \tag{39}$$

For the sake of illustration a low-pass, m-derived, T-section will be analyzed. From Fig. 24*b* it is evident that

$$\mathbf{Z}_{1m} = j\omega m L_{1k} = \mathbf{Z}_1$$

and

$$\mathbf{Z}_{2m} = j\left[\omega \frac{(1 - m^2)}{4m} L_{1k} - \frac{1}{\omega m C_{2k}}\right] = \mathbf{Z}_2$$

Therefore, in the present case,

$$\frac{\mathbf{Z}_1}{4\mathbf{Z}_2} = \frac{\omega m L_{1k}}{4\left[\omega \frac{(1 - m^2)}{4m} L_{1k} - \frac{1}{\omega m C_{2k}}\right]}$$

$$= \frac{\omega^2 m^2 L_{1k} C_{2k}}{\omega^2 (1 - m^2) L_{1k} C_{2k} - 4} \tag{97}$$

The above expression is actually a complex number, the associated angle of which is 180° or 0°, depending upon whether $[\omega^2 (1 - m^2) L_{1k} C_{2k}]$ is less than or greater than 4. The foregoing statement follows directly

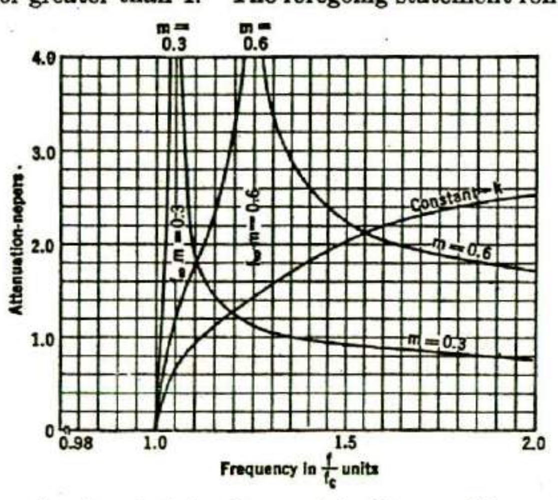


FIG. 26. Attenuation characteristics of two *m*-derived low-pass filter sections compared with those of a constant-*k* low-pass filter section.

from an inspection of $\mathbf{Z}_1/4\mathbf{Z}_2$ wherein all the factors are expressed in polar form. Let ω be arbitrarily reckoned in $1/\sqrt{L_{1k}C_{2k}}$ units. It should be observed that in this method of analysis the cut-off angular velocity or frequency is not necessarily anticipated by the choice of this convenient unit. Thus, for $\omega = 1/\sqrt{L_{1k}C_{2k}}$ radians per second, equation (39) reduces to

$$\alpha + j\beta = 2 \log_e \left[\sqrt{\frac{m^2}{(1 - m^2) - 4}} + \sqrt{\frac{m^2}{(1 - m^2) - 4} + 1}\right]$$

For a particular value of m it becomes a simple matter to evaluate α and β at any desired frequency. The calculations for $m = 0.6$ at various frequencies are shown in Table V. The variations of attenuation are represented graphically in Fig. 26 together with certain other attenuation curves. An inspection of column (9), Table V, will reveal the irregular manner in which the phase shift varies with frequency.

Problem 16. Graph the variation of attenuation with respect to frequency of a low-pass, m-derived T-section in which $m = 0.40$. The frequency may be indicated in terms of $1/\sqrt{L_{1k}C_{2k}}$ units of angular velocity. (See Table V, page 485.)

Comparison of Attenuation Characteristics. Constant-k and m-derived filter sections are sometimes worked in cascade because of the complementary nature of their respective attenuation characteristics. It has been shown that the attenuation of a constant-k, low-pass section is zero at cut-off frequency and that it increases gradually with increases of frequency above cut-off frequency. (See Fig. 12.) A similar situation holds for the constant-k, high-pass section except, of course, for the fact that the attenuation increases as the frequency decreases from the cut-off frequency. The attenuation characteristics of m-derived sections are radically different in character from those of constant-k sections. The differences are shown graphically in Fig. 26 for low-pass sections. Similar curves can be determined for high-pass sections.

It is plain from an inspection of Fig. 26 that a constant-k section can be combined with one or more m-derived sections to give high attenuation near cut-off as well as high attenuation in other regions of the stop band. In general, an m-derived section by itself will not give high attenuation in regions which are too widely removed from the point of theoretical infinite attenuation. (See Fig. 26.)

General Design Procedure. Filter sections are usually designed for a particular characteristic impedance and a particular cut-off frequency (or frequencies). Theoretically, at least, these conditions can be met accurately and straightforwardly. Usually certain attenuation requirements must also be met. These attenuation requirements are generally met by a method of successive approximations.

The *first step* in elementary filter design is the determination of the inductances and capacitances to be employed in a constant-k section. These values are found from the basic design equations.

The *second step* is the evaluation of the m-derived, terminating half-section inductances and capacitances. These values follow directly from the parameters of the constant-k section and the selected value of m. It is assumed here that the terminating half sections are required primarily for impedance-matching purposes, in which case the value of m will generally be 0.6.

TABLE V

Evaluation of $\left[\alpha + j\beta = 2 \log_e \left(\sqrt{\frac{Z_1}{4Z_2}} + \sqrt{\frac{Z_1}{4Z_2} + 1}\right)\right]$ where $\frac{Z_1}{4Z_2} = \frac{\omega^2 m^2 L_{1k} C_{2k}}{\omega^2(1 - m^2) L_{1k} C_{2k} - 4}$

$m = 0.6$ and ω is reckoned in $\left[\frac{1}{\sqrt{L_{1k} C_{2k}}}\right]$ units which are called U. Under these conditions $\frac{Z_1}{4Z_2} = \frac{0.36U^2}{0.64U^2 - 4}$

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
ω U units	$\frac{Z_1}{4Z_2}$	$\sqrt{\frac{Z_1}{4Z_2}}$	$\sqrt{\frac{Z_1}{4Z_2} + 1}$	(3) + (4) Cartesian Form	(3) + (4) Polar Form	$2 \log_e$ (6)	α nepers	β radians
0	0	0	1	$1 + j0$	$1 \angle 0^\circ$	$0 + j0$	0	0
1.0	$+0.107 \angle 180^\circ$	$j0.327$	0.945	$0.945 + j0.327$	$1 \angle 19.2^\circ$	$0 + j0.67$	0	0.67
1.5	$+0.316 \angle 180^\circ$	$j0.564$	0.826	$0.826 + j0.564$	$1 \angle 34.3^\circ$	$0 + j1.197$	0	1.20
2.0	$+1.00 \angle 180^\circ$	$j1.0$	0	$0 + j1.0$	$1 \angle 90^\circ$	$0 + j3.14$	0	3.14
2.1	$+1.35 \angle 180^\circ$	$j1.16$	$j0.59$	$0 + j1.75$	$1.75 \angle 90^\circ$	$1.12 + j3.14$	1.12	3.14
2.25	$+2.40 \angle 180^\circ$	$j1.55$	$j1.18$	$0 + j2.73$	$2.73 \angle 90^\circ$	$2.01 + j3.14$	2.01	3.14
2.50	∞	∞	∞	∞	∞	∞	∞	3.14 0.00
2.75	$+3.24 \angle 0^\circ$	1.80	2.06	$3.86 + j0$	$3.86 \angle 0^\circ$	$2.70 + j0$	2.70	0
3.0	$+1.84 \angle 0^\circ$	1.36	1.68	$3.04 + j0$	$3.04 \angle 0^\circ$	$2.22 + j0$	2.22	0
4.0	$+0.923 \angle 0^\circ$	0.961	1.39	$2.35 + j0$	$2.35 \angle 0^\circ$	$1.71 + j0$	1.71	0
6.0	$+0.683 \angle 0^\circ$	0.826	1.30	$2.13 + j0$	$2.13 \angle 0^\circ$	$1.51 + j0$	1.51	0

If a sharp cut-off section is required, a full m-derived section, wherein m is about 0.2 or 0.3, can be employed. The evaluation of the inductances and capacitances to use in the full m-derived section constitutes the *third step* in the general design procedure.

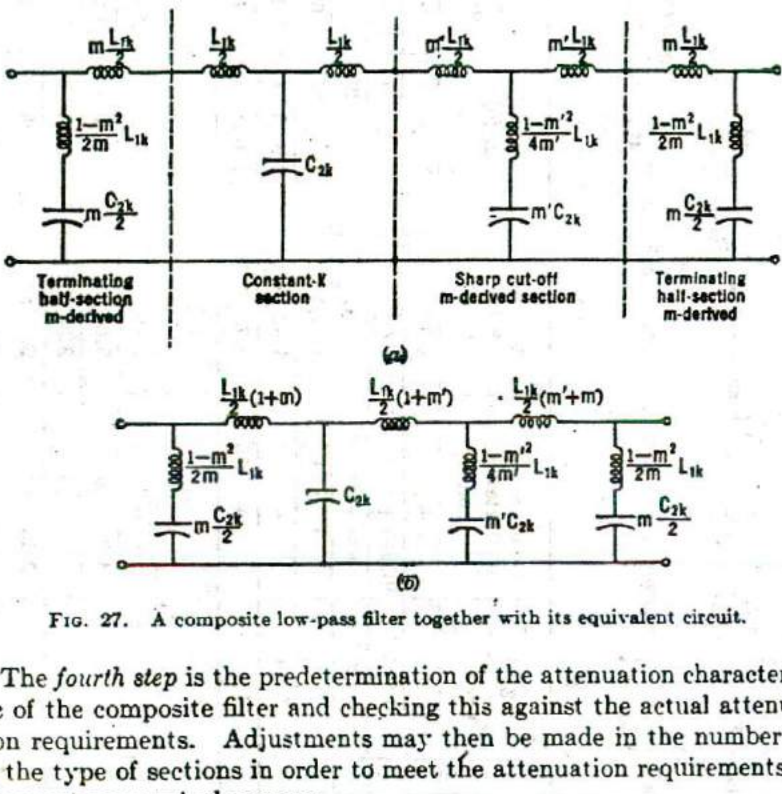


FIG. 27. A composite low-pass filter together with its equivalent circuit.

The *fourth step* is the predetermination of the attenuation characteristic of the composite filter and checking this against the actual attenuation requirements. Adjustments may then be made in the number or in the type of sections in order to meet the attenuation requirements in the most economical manner.

The method of combining a constant-k section, a full m-derived section, and m-derived terminating half sections to form a low-pass filter is illustrated in Fig. 27a. It will be noted that the assembly shown in Fig. 27a is reducible to that shown in Fig. 27b.

PROBLEMS

17. Consider a π-type filter section in which the full series arm, Z_1, consists of a 100-millihenry inductance coil the resistance of which is 50 ohms. Each of the two shunt arms consists of a 0.3-μf condenser the resistance of which is negligibly small.

(a) Find the open-circuit impedance, Z_{o-c}, and the short-circuit impedance, Z_{s-c}, of the section at 500 cycles.

(b) Find the characteristic impedance at 500 cycles, at 1300 cycles, and at 2000 cycles.

18. Each of the series arms $(Z_1/2)$ of a symmetrical T-section consists of a condenser the capacitance of which is 0.6 μf and the resistance of which is negligibly small. The shunt arm (Z_2) is a 200-millihenry inductance coil the resistance of which is 60 ohms.

(a) Find the characteristic impedance and the propagation constant of the section at 200 cycles.

(b) Find the characteristic impedance and the propagation constant of the section at 600 cycles.

19. The characteristic impedance of a filter section is to be measured. The measuring device is a 1-B Western Electric impedance bridge which indicates the R component of the impedance directly and the X component in terms of $+L$ or $-L$. Plus L readings indicate that $X = X_L = 2\pi fL$, and negative L readings indicate that $X = X_C = 2\pi f(-L)$. With the output terminals of the section open-circuited the bridge readings are: $R = 10$ ohms and $L = -190$ millihenrys at 400 cycles. With the output terminals of the section short-circuited the bridge readings are: $R = 20$ ohms and $L = +250$ millihenrys at 400 cycles. Find the characteristic impedance of the filter section at 400 cycles.

20. The series arms of a T-section are each of 100 ohms capacitance. The shunt arm is a 100-ohm inductive reactance. (a) Determine the characteristic impedance of this section for the constants given. (b) Also calculate Z_{oT} for half the frequency at which the constants are given. (c) Is the frequency for the reactances given within the pass or stop band? (d) Answer for one-half the frequency at which the reactances are given. (e) Calculate the attenuation in nepers for the two frequencies. (f) What can you say about the characteristic impedance in the pass band as compared with the attenuation band for an ideal prototype section? (g) Is this also true of ideal prototype π-sections?

21. A resistanceless, constant-k, low-pass T-section has a cut-off frequency of 10,000 cycles and a zero-frequency characteristic impedance of 800 ohms. Evaluate the phase shift at 1000, 4000, 7000, and 10,000 cycles. Evaluate the attenuation at 11,000, 15,000, 20,000, and 25,000 cycles. Plot phase shift in degrees and attenuation in decibels against cycles per second.

22. Consider a symmetrical π-type section in which the inductance of the full series arm is 0.10 henry and the capacitance of each of the two condensers which go to form the π-section is 0.3 μf.

(a) Neglecting the resistive components of the circuit elements, find the propagation constant at 500 cycles, at 1300 cycles, and at 2000 cycles.

(b) What is the attenuation in decibels at each of the three frequencies referred to above?

23. (a) What is the decibel level of 0.00001 watt with respect to a 1-milliwatt reference power level?

(b) What is the decibel level of 6 watts with respect to a 1-milliwatt reference power level?

24. What is the cut-off frequency of a low-pass, constant-k, π-type filter section in which the inductance of the full series arm is 20 henrys and the capacitance of each condenser is 8.0 μf? What is the characteristic impedance of the section at 200 cycles?

25. A T-section filter has series arms $Z_1/2 = j100$ ohms and its shunt arm $Z_2 = -j1000$ ohms.

(a) Calculate the characteristic impedance.

(b) Calculate the attenuation in decibels and the phase shift in degrees.

(c) Are the reactances of the section for a frequency within the pass or stop band?

(d) Calculate the characteristic impedance of the section for 5 times the frequency for which the constants are given.

(e) Calculate the attenuation in decibels and phase shift for part (d).

26. A π-section filter has its series arm $Z_1 = -j100$ ohms and its shunt arms $2Z_2 = j500$ ohms.

(a) Calculate the characteristic impedance.

(b) Calculate the attenuation in decibels and the phase shift in degrees.

(c) Are the reactances given for a frequency within the pass band or stop band?

(d) Repeat parts (a), (b), and (c) for a frequency of one-fifth of that for which the impedances are given.

27. Nine T-sections each having series arms of $Z_1/2 = j500$ ohms and shunt arms $Z_2 = -j200$ ohms are connected in series or cascade. If the input voltage is 100, find the output voltage of the ninth section and the output current, assuming characteristic termination.

28. Find the circuit element values of a high-pass, constant-k, T-type filter section which is to have a cut-off frequency of 5000 cycles and an infinite-frequency characteristic impedance of 600 ohms. Repeat for a π-type section. Draw circuit diagrams showing the configurations of the circuit elements and the values of each in millihenrys and microfarads.

29. A generator having an impedance of $800\underline{/0^\circ}$ ohms is to be connected to a load impedance of $100\underline{/0^\circ}$ through a half-section of the kind shown in Fig. 21, page 475. Find the value of $Z_{1k}/2$ (the series arm impedance) and of $2Z_{2k}$ (the shunt arm impedance) which will properly match the generator to the load. $Z_{1k}/2$ is arbitrarily taken as inductive.

30. Design a high-pass, m-derived, T-type filter section which will have a cut-off frequency of 5000 cycles, an infinite-frequency characteristic impedance of 600 ohms, and an infinite-attenuation frequency of 4500 cycles.

31. Design m-derived half sections which will properly match, at 800 cycles, a low-pass, constant-k, T-type section the full series arm of which is 0.30 henry and the full shunt arm of which is 0.03 μf. The value of m is to be taken as 0.60.

32. Consider an m-derived, low-pass, T-section in which Z_{1m} is mL_{1k} and Z_{2m} consists of $(1 - m^2/4m)L_{1k}$ in series with mC_{2k}. Let mL_{1k} be known as L_{1m}, $(1 - m^2/4m)L_{1k}$ be known as L_{2m}, and mC_{2k} be known as C_{2m}. Show that the cut-off frequency, namely, $1/\pi\sqrt{L_{1k}C_{2k}}$, can be written as $1/[\pi\sqrt{(L_{1m} + 4L_{2m})(C_{2m})}]$.

33. Refer to the composite low-pass filter shown in Fig. 27. The requirements to be met are: (1) zero-frequency characteristic impedance of 600 ohms, (2) cut-off frequency of 5000 cycles, (3) variation in characteristic impedance of not more than 30 ohms over the lower 80 per cent of the pass band, (4) attenuation of 40 decibels between the limits of 5242 and 10,000 cycles.

(a) Calculate the values of L_{1k} and C_{2k}.

(b) Design terminating half sections on the basis of $m = 0.60$.

(c) Design the full m-derived section to have theoretical infinite attenuation at 5242 cycles.

(d) Make a graph of the attenuation of the composite filter between the limits of 5242 and 10,000 cycles and compare the results with the attenuation requirements. Use the three attenuation graphs shown in Fig. 26, page 493, at $f/f_c = 1.05$, 1.10, 1.15, 1.20, 1.25, 1.30, 1.35, 1.40, 1.45, 1.50, 1.75, and 2 to obtain the composite attenuation graph.

chapter XII Symmetrical Components

Symmetrical components furnish a tool of great power for analytically determining the performance of certain types of unbalanced electrical circuits involving rotating electrical machines. It is particularly useful in analyzing the performance of polyphase electrical machinery when operated from systems of unbalanced voltages. Although it can be used to solve unbalanced static networks like those in Chapter IX, such application will in general be more cumbersome and laborious than the methods already considered. For unbalanced networks containing rotating machines, however, the method of symmetrical components provides the only practicable method of accounting for the unbalanced effects of these machines and is widely used in practice. It is also convenient for analyzing some types of polyphase transformer problems.

The method of "symmetrical components," in its most useful form, is founded upon Fortescue's[1] theorem regarding the resolution of unbalanced systems into symmetrical components. Although the present discussion will confine itself to three-phase systems, any unbalanced polyphase system of vectors can be resolved into balanced systems of vectors called "symmetrical components."

Fortescue's theorem, applied to a general three-phase system of vectors, is that any unbalanced three-phase system of vectors can be resolved into three balanced systems of vectors, namely:

1. A balanced system of three-phase vectors having the same phase sequence as the original unbalanced system of vectors. This balanced system is called the "positive-sequence system."

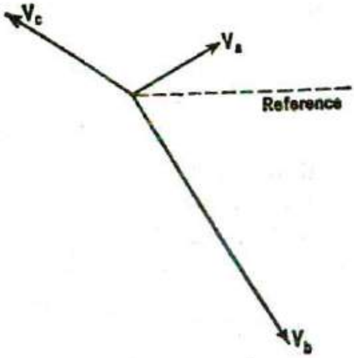
2. A balanced system of three-phase vectors having a phase sequence which is opposite to that of the original unbalanced system of vectors. This balanced system is called the "negative-sequence system."

3. A system of three single-phase vectors which are equal in magnitude and which have exactly the same time-phase position with respect to any given reference axis. This system of single-phase vectors is known as the zero-sequence or uniphase system.

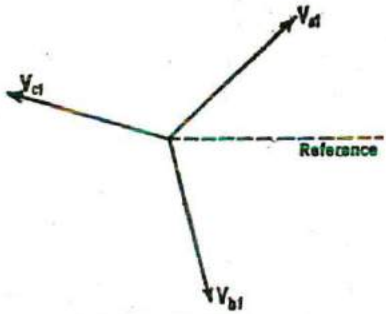
A general proof of the resolution theorem will not be given because a little experience with the method will soon convince the reader that

[1] Fortescue, "Method of Symmetrical Co-ordinates Applied to the Solution of Polyphase Networks," *Transactions, A.I.E.E.*, Vol. 37, 1918.

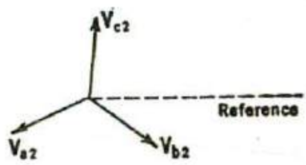
the theorem as stated is correct. In this respect Fortescue's theorem is similar to Fourier's theorem regarding complex waves. In Chapter VI it is shown that any complex wave may be resolved into definite harmonic components by the Fourier method. The ultimate proof of the theorem rests upon the fact that the components thus determined can be synthesized to form the original complex wave. In a similar manner it will be shown that any given unbalanced three-phase system of vectors may be resolved into the three balanced systems outlined above and that the composition of these balanced systems yields the original unbalanced system of vectors.



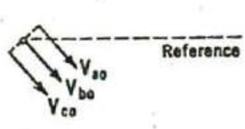
(a) Original vectors.



(b) Positive-sequence vectors.



(c) Negative-sequence vectors.



(d) Zero-sequence vectors.

FIG. 1. Original set of three-phase vectors together with their symmetrical components.

The Original Unbalanced System of Vectors. Any number of vectors up to and including three may be considered as an unbalanced system of three-phase vectors. The vectors that form the unbalanced system may have any specified magnitude (including zero) and may possess any specified phase positions with respect to one another. In Fig. 1*a* is shown a set of three unbalanced vectors that will later be resolved into their symmetrical components. If the vectors that form the original unbalanced set come to us merely as three specified vectors, they can arbitrarily be assigned subscripts *a*, *b*, and *c* in the order shown in Fig. 1*a*. Thus the original vectors $\mathbf{V}_a$, $\mathbf{V}_b$, and $\mathbf{V}_c$ are arbitrarily assigned the *abc* phase sequence. (See Chapter IX, pages

383–384.) Although the vectors shown in Fig. 1*a* are labeled as voltages, the proposed resolution applies equally well to a system of current vectors.

Owing to the fact that the symmetrical components will have to carry an additional subscript to designate the system to which they belong, single-subscript notation will be employed in connection with the original vectors wherever this can be done without loss of clarity. For complete specification, the positive circuit directions of the original three-phase voltages or currents must be indicated on a separate circuit diagram. The importance of complete specification will become apparent when numerical problems are considered.

The Positive-Phase Sequence System. As previously stated, the original unbalanced system of vectors is to be resolved into two balanced three-phase systems and one uniphase system. It will be shown presently that the balanced three-phase systems must be of opposite phase sequence. Therefore one balanced system has the same phase sequence as the original three-phase system and the other has a phase sequence opposite to that of the original system.

The balanced system of three-phase vectors that has the same phase sequence as the original system is called the positive-sequence system. If the original vectors are assigned the phase sequence of *abc*, then the phase sequence of the positive-sequence vectors is *abc* as shown in Fig. 1*b*. The positive-sequence vectors are completely determined when the magnitude and phase position of any one of them is known. A method of evaluating any one of the positive-sequence vectors in terms of the original vector values will be given presently. The positive-sequence vectors are designated as

$$\mathbf{V}_{a1}, \quad \mathbf{V}_{b1}, \quad \text{and} \quad \mathbf{V}_{c1}$$

The subscript 1 indicates that the vector thus labeled belongs to the positive-sequence system. The letters refer to the original vector of which the positive-sequence vector is a component part.

The vectors of any balanced three-phase system may be conveniently related to one another with the aid of the operator **a**. The general properties of this operator are considered in Chapter IV, page 121–122.

a1 is a unit vector 120° ahead of the reference axis. $\mathbf{a}^2 1$ is a unit vector 240° ahead of the reference axis. Thus:

$$\left.\begin{aligned} \mathbf{a}1 &= \epsilon^{j120^\circ} = -0.5 + j0.866 \\ \mathbf{a}^2 1 &= \epsilon^{j240^\circ} = -0.5 - j0.866 \end{aligned}\right\} \qquad (1)$$

The operator **a** applied to any vector rotates that vector through 120° in the positive or counterclockwise direction. The operator $\mathbf{a}^2$ applied to any vector rotates that vector through 240° in the positive direction, which is, of course, equivalent to a rotation of 120° in the negative direction.

If, for example, $\mathbf{V}_{a1}$ has been determined, the positive-sequence system may be written simply as

$$\left.\begin{aligned} \mathbf{V}_{a1} &= \mathbf{V}_{a1} \\ \mathbf{V}_{b1} &= \mathbf{a}^2\mathbf{V}_{a1} = \mathbf{V}_{a1}\,\underline{/-120^\circ} \\ \mathbf{V}_{c1} &= \mathbf{a}\mathbf{V}_{a1} = \mathbf{V}_{a1}\,\underline{/-240^\circ} \end{aligned}\right\} \begin{array}{c}\text{the positive-sequence}\\ \text{system of vectors}\end{array} \tag{2}$$

The Negative-Phase Sequence System. The balanced system of three-phase vectors which is opposite in phase sequence to that of the original vectors is called the negative-sequence system. If the original vectors have a phase sequence of *abc* the negative-sequence vectors have a phase sequence of *acb* as shown in Fig. 1*c*. Since the negative-sequence system is balanced, it is completely determined when one of the voltages is known. The negative-sequence vectors are designated as

$$\mathbf{V}_{a2}, \quad \mathbf{V}_{b2}, \quad \text{and} \quad \mathbf{V}_{c2}$$

Subscript 2 indicates that the vectors belong to the negative-sequence system. The a, b, and c subscripts indicate components of $\mathbf{V}_a$, $\mathbf{V}_b$, and $\mathbf{V}_c$ respectively. If $\mathbf{V}_{a2}$ is known, the negative-sequence system can be written in the following form:

$$\left.\begin{aligned} \mathbf{V}_{a2} &= \mathbf{V}_{a2} \\ \mathbf{V}_{b2} &= \mathbf{a}\mathbf{V}_{a2} = \mathbf{V}_{a2}\,\underline{/-240^\circ} \\ \mathbf{V}_{c2} &= \mathbf{a}^2\mathbf{V}_{a2} = \mathbf{V}_{a2}\,\underline{/-120^\circ} \end{aligned}\right\} \begin{array}{c}\text{the negative-sequence}\\ \text{system of vectors}\end{array} \tag{3}$$

$\mathbf{V}_{a2}$, $\mathbf{V}_{b2}$, and $\mathbf{V}_{c2}$ are shown graphically in Fig. 1*c*.

The Zero-Phase Sequence System. The remaining system consists of three vectors, identical in magnitude and in time phase, as shown in Fig. 1*d*. These vectors form what is known as the uniphase or the zero-sequence system, and have special significance in certain physical problems. For the present it will be sufficient to think of the zero-sequence vectors as components of the original vectors $\mathbf{V}_a$, $\mathbf{V}_b$, and $\mathbf{V}_c$. The zero-sequence vectors are designated as

$$\mathbf{V}_{a0}, \quad \mathbf{V}_{b0}, \quad \text{and} \quad \mathbf{V}_{c0}$$

Since the above voltages are equal:

$$\left.\begin{aligned} \mathbf{V}_{a0} &= \mathbf{V}_{a0} \\ \mathbf{V}_{b0} &= \mathbf{V}_{a0} \\ \mathbf{V}_{c0} &= \mathbf{V}_{a0} \end{aligned}\right\} \begin{array}{c}\text{the zero-sequence}\\ \text{system of vectors}\end{array} \tag{4}$$

Graphical Composition of Sequence Vectors. It is evident that

$$(\mathbf{V}_{a1} + \mathbf{V}_{a2} + \mathbf{V}_{a0})$$
$$(\mathbf{V}_{b1} + \mathbf{V}_{b2} + \mathbf{V}_{b0})$$

and

$$(\mathbf{V}_{c1} + \mathbf{V}_{c2} + \mathbf{V}_{c0})$$

form a three-phase system of voltages which, in general, is unbalanced. The above-indicated compositions are carried out graphically in Fig. 2, employing the individual voltages contained in Figs. 1*b*, 1*c*, and 1*d*.

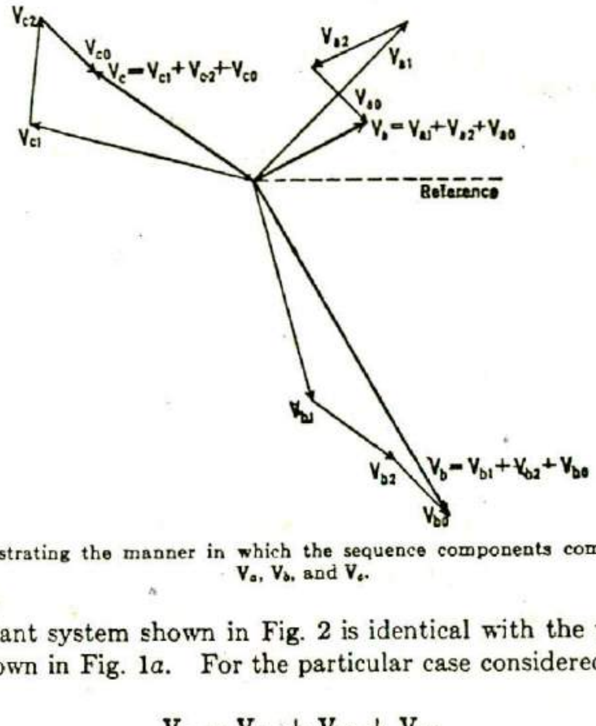


FIG. 2. Illustrating the manner in which the sequence components combine to form $\mathbf{V}_a$, $\mathbf{V}_b$, and $\mathbf{V}_c$.

The resultant system shown in Fig. 2 is identical with the unbalanced system shown in Fig. 1*a*. For the particular case considered it is plain that

$$\mathbf{V}_a = \mathbf{V}_{a1} + \mathbf{V}_{a2} + \mathbf{V}_{a0} \tag{5}$$

$$\mathbf{V}_b = \mathbf{V}_{b1} + \mathbf{V}_{b2} + \mathbf{V}_{b0} \tag{6}$$

$$\mathbf{V}_c = \mathbf{V}_{c1} + \mathbf{V}_{c2} + \mathbf{V}_{c0} \tag{7}$$

In terms of the operator **a**, the above relations may be stated as

$$\mathbf{V}_a = \mathbf{V}_{a1} + \mathbf{V}_{a2} + \mathbf{V}_{a0} \tag{8}$$

$$\mathbf{V}_b = \mathbf{a}^2\mathbf{V}_{a1} + \mathbf{a}\mathbf{V}_{a2} + \mathbf{V}_{a0} \tag{9}$$

$$\mathbf{V}_c = \mathbf{a}\mathbf{V}_{a1} + \mathbf{a}^2\mathbf{V}_{a2} + \mathbf{V}_{a0} \tag{10}$$

An inspection of equations (8), (9), and (10) will show that the original system of vectors can be completely specified in terms of $\mathbf{V}_{a1}$, $\mathbf{V}_{a2}$, $\mathbf{V}_{a0}$, and the operator $\mathbf{a}$. The next step in the study of symmetrical components is the evaluation of $\mathbf{V}_{a1}$, $\mathbf{V}_{a2}$, and $\mathbf{V}_{a0}$ in terms of the original vectors $\mathbf{V}_a$, $\mathbf{V}_b$, and $\mathbf{V}_c$.

Evaluation of $\mathbf{V}_{a1}$. The resolution of an unbalanced system of vectors into its symmetrical components is essentially a geometric process, and different geometric methods have been devised whereby the resolution can be effected. However, none of the geometric methods thus far devised possesses the neat simplicity of the complex algebra method given below.

Before proceeding with the algebraic method it is well to understand that certain operations are performed solely for the purpose of obtaining the combination $(1 + \mathbf{a} + \mathbf{a}^2)$ which is equal to zero. Various simplifications may thus be made when quantities can be so collected as to possess the coefficient $(1 + \mathbf{a} + \mathbf{a}^2)$.

If equation (9) is multiplied by $\mathbf{a}$ the result is:

$$\mathbf{a}\mathbf{V}_b = \mathbf{a}^3\mathbf{V}_{a1} + \mathbf{a}^2\mathbf{V}_{a2} + \mathbf{a}\mathbf{V}_{a0}$$

or, since $\mathbf{a}^3 = 1$,

$$\mathbf{a}\mathbf{V}_b = \mathbf{V}_{a1} + \mathbf{a}^2\mathbf{V}_{a2} + \mathbf{a}\mathbf{V}_{a0} \tag{11}$$

If equation (10) is multiplied by $\mathbf{a}^2$, the result is:

$$\mathbf{a}^2\mathbf{V}_c = \mathbf{a}^3\mathbf{V}_{a1} + \mathbf{a}^4\mathbf{V}_{a2} + \mathbf{a}^2\mathbf{V}_{a0}$$

or, since $\mathbf{a}^4 = \mathbf{a}$,

$$\mathbf{a}^2\mathbf{V}_c = \mathbf{V}_{a1} + \mathbf{a}\mathbf{V}_{a2} + \mathbf{a}^2\mathbf{V}_{a0} \tag{12}$$

Adding equations (8), (11), and (12) yields

$$\mathbf{V}_a + \mathbf{a}\mathbf{V}_b + \mathbf{a}^2\mathbf{V}_c = 3\mathbf{V}_{a1} + (1 + \mathbf{a} + \mathbf{a}^2)(\mathbf{V}_{a2} + \mathbf{V}_{a0})$$

whence:

$$\mathbf{V}_{a1} = \tfrac{1}{3}(\mathbf{V}_a + \mathbf{a}\mathbf{V}_b + \mathbf{a}^2\mathbf{V}_c) = \tfrac{1}{3}(\mathbf{V}_a + \mathbf{V}_b\,\underline{/120^\circ} + \mathbf{V}_c\underline{/240^\circ}) \tag{13}$$

Geometrically speaking, the above equation means that $\mathbf{V}_{a1}$ is a vector one-third as large as the vector which results from the addition of the three vectors $\mathbf{V}_a$, $\mathbf{V}_b\,\underline{/120^\circ}$, and $\mathbf{V}_c\,\underline{/240^\circ}$.

Example 1. If the vectors shown in Fig. 1*a* are:

$$\mathbf{V}_a = 10\,\underline{/30^\circ},\quad \mathbf{V}_b = 30\,\underline{/-60^\circ},\quad \text{and}\quad \mathbf{V}_c = 15\,\underline{/145^\circ}\text{ units}$$

$$\begin{aligned}\mathbf{V}_{a1} &= \tfrac{1}{3}(10\,\underline{/30^\circ} + \mathbf{a}30\,\underline{/-60^\circ} + \mathbf{a}^2 15\,\underline{/145^\circ}) \\ &= \tfrac{1}{3}(10\,\underline{/30^\circ} + 30\,\underline{/60^\circ} + 15\,\underline{/25^\circ}) \\ &= 12.42 + j12.45 = 17.6\,\underline{/45^\circ.0}\text{ units}\end{aligned}$$

Since $V_{b1} = V_{a1} \underline{/-120^\circ}$ and $V_{c1} = V_{a1} \underline{/+120^\circ}$, the positive-sequence system of vectors becomes:

$$V_{a1} = 17.6 \underline{/45^\circ},\ V_{b1} = 17.6 \underline{/-75^\circ},\ \text{and}\ V_{c1} = 17.6 \underline{/165^\circ}\ \text{units}$$

The above results are indicated graphically in Fig. 1*b*.

Evaluation of V_{a2}. The negative-sequence component of V_{a2} can be evaluated in a manner almost identical with that given above for the evaluation of V_{a1}. It is simply necessary to study equations (8), (9), and (10) with a view toward eliminating the V_{a1} and V_{a0} terms and at the same time retain the V_{a2} terms. The desired results can be obtained by multiplying equation (9) through by a^2 and equation (10) through by a. Equation (9) multiplied by a^2 reduces to

$$a^2V_b = aV_{a1} + V_{a2} + a^2V_{a0} \tag{14}$$

Equation (10) multiplied by a reduces to

$$aV_c = a^2V_{a1} + V_{a2} + aV_{a0} \tag{15}$$

Adding equations (8), (14), and (15) yields

$$V_a + a^2V_b + aV_c = 3V_{a2} + (1 + a + a^2)(V_{a1} + V_{a0})$$

Since $(1 + a + a^2) = 0$,

$$V_{a2} = \tfrac{1}{3}(V_a + a^2V_b + aV_c) = \tfrac{1}{3}(V_a + V_b \underline{/240^\circ} + V_c \underline{/120^\circ}) \tag{16}$$

V_{a2} is therefore a vector one-third the magnitude of $[V_a + (V_b$ rotated through $+240^\circ) + (V_c$ rotated through $+120^\circ)]$.

Example 2. If $V_a = 10 \underline{/30^\circ}$, $V_b = 30 \underline{/-60^\circ}$, and $V_c = 15 \underline{/145^\circ}$ units:

$$\begin{aligned} V_{a2} &= \tfrac{1}{3}(10 \underline{/30^\circ} + a^2 30 \underline{/-60^\circ} + a15 \underline{/145^\circ}) \\ &= \tfrac{1}{3}(10 \underline{/30^\circ} + 30 \underline{/180^\circ} + 15 \underline{/265^\circ}) \\ &= -7.55 - j3.32 = 8.25 \underline{/-156.2^\circ}\ \text{units} \end{aligned}$$

V_{a2} for this particular case is shown in Fig. 1*c* together with V_{b2} and V_{c2}. $V_{b2} = V_{a2} \underline{/120^\circ}$ and $V_{c2} = V_{a2} \underline{/-120^\circ}$.

Evaluation of V_{a0}. The direct addition of equations (8), (9), and (10) will show that:

$$V_a + V_b + V_c = V_{a1}(1 + a^2 + a) + V_{a2}(1 + a + a^2) + 3V_{a0}$$

or

$$V_{a0} = \tfrac{1}{3}(V_a + V_b + V_c) \tag{17}$$

The zero-sequence component is simply a vector one-third as large as the vector obtained by adding V_a, V_b, and V_c.

Example 3. If $\mathbf{V}_a = 10 \underline{/30^\circ}$, $\mathbf{V}_b = 30 \underline{/-60^\circ}$, and $\mathbf{V}_c = 15 \underline{/145^\circ}$ units:

$$\mathbf{V}_{a0} = \tfrac{1}{3}(10 \underline{/30^\circ} + 30 \underline{/-60^\circ} + 15 \underline{/145^\circ})$$
$$= 3.79 - j4.13 = 5.60 \underline{/-47.4^\circ} \text{ units}$$

The above value of $\mathbf{V}_{a0}$ together with corresponding values of $\mathbf{V}_{b0}$ and $\mathbf{V}_{c0}$ are shown in Fig. 1*d*.

Example 4. (*a*) The results obtained in the foregoing examples can be checked by comparing the complex expression for $(\mathbf{V}_{a1} + \mathbf{V}_{a2} + \mathbf{V}_{a0})$ with the complex expression of the original vector $\mathbf{V}_a$. The results of the foregoing examples are tabulated below.

$$\mathbf{V}_{a1} = 12.42 + j12.45 = 17.6 \underline{/45^\circ} \text{ units}$$
$$\mathbf{V}_{a2} = -7.55 - j3.32 = 8.25 \underline{/-156.2^\circ} \text{ units}$$
$$\mathbf{V}_{a0} = 3.79 - j4.13 = 5.60 \underline{/-47.4^\circ} \text{ units}$$
$$(\mathbf{V}_{a1} + \mathbf{V}_{a2} + \mathbf{V}_{a0}) = 8.66 + j5.00 = 10 \underline{/30^\circ} = \mathbf{V}_a$$

(*b*)
$$\mathbf{V}_{b1} = \mathbf{a}^2 17.6 \underline{/45^\circ} = 17.6 \underline{/-75^\circ} = 4.56 - j17.0 \text{ units}$$
$$\mathbf{V}_{b2} = \mathbf{a}8.25 \underline{/-156.2^\circ} = 8.25 \underline{/-36.2^\circ} = 6.66 - j4.87 \text{ units}$$
$$\mathbf{V}_{b0} = 5.60 \underline{/-47.4^\circ} = 3.79 - j4.13 \text{ units}$$
$$(\mathbf{V}_{b1} + \mathbf{V}_{b2} + \mathbf{V}_{b0}) = 15.0 - j26.0 = 30.0 \underline{/-60^\circ} = \mathbf{V}_b$$

(*c*)
$$\mathbf{V}_{c1} = \mathbf{a}17.6 \underline{/45^\circ} = 17.6 \underline{/165^\circ} = -17.0 + j4.56 \text{ units}$$
$$\mathbf{V}_{c2} = \mathbf{a}^2 8.25 \underline{/-156.2^\circ} = 8.25 \underline{/83.8^\circ} = 0.89 + j8.20 \text{ units}$$
$$\mathbf{V}_{c0} = 5.60 \underline{/-47.4^\circ} = 3.79 - j4.13 \text{ units}$$
$$(\mathbf{V}_{c1} + \mathbf{V}_{c2} + \mathbf{V}_{c0}) = -12.32 + j8.63 = 15 \underline{/145^\circ} = \mathbf{V}_c$$

Problem 1. Given the following three vector voltages: $\mathbf{V}_a = 150 \underline{/0^\circ}$, $\mathbf{V}_b = 86.6 \underline{/-90^\circ}$, and $\mathbf{V}_c = 86.6 \underline{/90^\circ}$ volts.

(*a*) Find the symmetrical components of $\mathbf{V}_a$ and check the results by adding $\mathbf{V}_{a1}$, $\mathbf{V}_{a2}$, and $\mathbf{V}_{a0}$.

(*b*) Evaluate $\mathbf{V}_b$ and $\mathbf{V}_c$ in terms of the symmetrical components of $\mathbf{V}_a$ found in part (*a*).

(*c*) Draw a vector diagram illustrating all symmetrical components.

Ans.: (*a*) $\mathbf{V}_{a1} = 100 \underline{/0^\circ}$, $\mathbf{V}_{a2} = 0$, $\mathbf{V}_{a0} = 50 \underline{/0^\circ}$ volts.

Absence of Zero-Sequence Components. The zero-sequence components are non-existent in any system of voltages (or currents) if the vector sum of the original vectors is equal to zero. [See equation (17).] This fact may often be used advantageously in making numerical calculations because the original system of vectors is then directly reducible to two balanced three-phase systems of opposite phase sequence. An absence of zero-sequence components may have important physical significance in the analysis of practical problems. Some of the practical problems in which symmetrical-component analyses are used successfully will be referred to briefly in the following paragraphs and one of these problems will be treated in detail in the next chapter.

Three-Phase, Line-to-Line Voltages. The line-to-line voltages shown in Fig. 3 for either the wye or delta are:

$$\mathbf{V}_{ab} = (\mathbf{V}_{an} - \mathbf{V}_{bn}) \tag{18}$$

$$\mathbf{V}_{bc} = (\mathbf{V}_{bn} - \mathbf{V}_{cn}) \tag{19}$$

$$\mathbf{V}_{ca} = (\mathbf{V}_{cn} - \mathbf{V}_{an}) \tag{20}$$

For the delta the voltages to neutral are those of an equivalent wye. Regardless of the degree of unbalance in the line-to-line voltages

$$\mathbf{V}_{ab} + \mathbf{V}_{bc} + \mathbf{V}_{ca} = (\mathbf{V}_{an} - \mathbf{V}_{bn}) + (\mathbf{V}_{bn} - \mathbf{V}_{cn}) + (\mathbf{V}_{cn} - \mathbf{V}_{an}) = 0 \tag{21}$$

The zero-sequence components of the line-to-line voltages are non-existent because

$$\mathbf{V}_{ab0} = \mathbf{V}_{bc0} = \mathbf{V}_{ca0} = \tfrac{1}{3}(\mathbf{V}_{ab} + \mathbf{V}_{bc} + \mathbf{V}_{ca}) = 0 \tag{22}$$

Therefore three-phase, line-to-line voltages may be represented by a positive-sequence system and a negative-sequence system of voltages as represented by the voltage vector diagrams of Fig. 3. It should be realized that Fig. 3 shows a specific case. As has been previously stated, the relative magnitude of the positive- and negative-sequence voltages and the angle between $\mathbf{V}_{an1}$ and $\mathbf{V}_{an2}$ may take on an infinite number of different values in the most general case. The fact that unbalanced line-to-line voltages may be resolved into two balanced systems of opposite sequence is of considerable importance in the analyses of three-phase rotating machinery. When unbalanced voltages are applied to a three-phase induction motor, for example, the operation of the motor may be analyzed on the basis of balanced systems of voltages of opposite phase sequence.

The positive-sequence voltages and negative-sequence voltages shown in Fig. 3 are obtained in any particular case in terms of the vector values of $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{ca}$ as outlined in equations (13) and (16). In terms of the present notation

$$\mathbf{V}_{ab1} = \tfrac{1}{3}(\mathbf{V}_{ab} + \mathbf{V}_{bc}\underline{/120^\circ} + \mathbf{V}_{ca}\underline{/-120^\circ}) \tag{23}$$

$$\mathbf{V}_{ab2} = \tfrac{1}{3}(\mathbf{V}_{ab} + \mathbf{V}_{bc}\underline{/-120^\circ} + \mathbf{V}_{ca}\underline{/120^\circ}) \tag{24}$$

It will be observed from equation (23) that the positive-sequence component of the base vector ($\mathbf{V}_{ab}$ in this case) is obtained by advancing (through 120°) the vector which lags the base vector and retarding (through 120°) the vector which leads the base vector. Reversed operations are employed to secure the negative-sequence components as

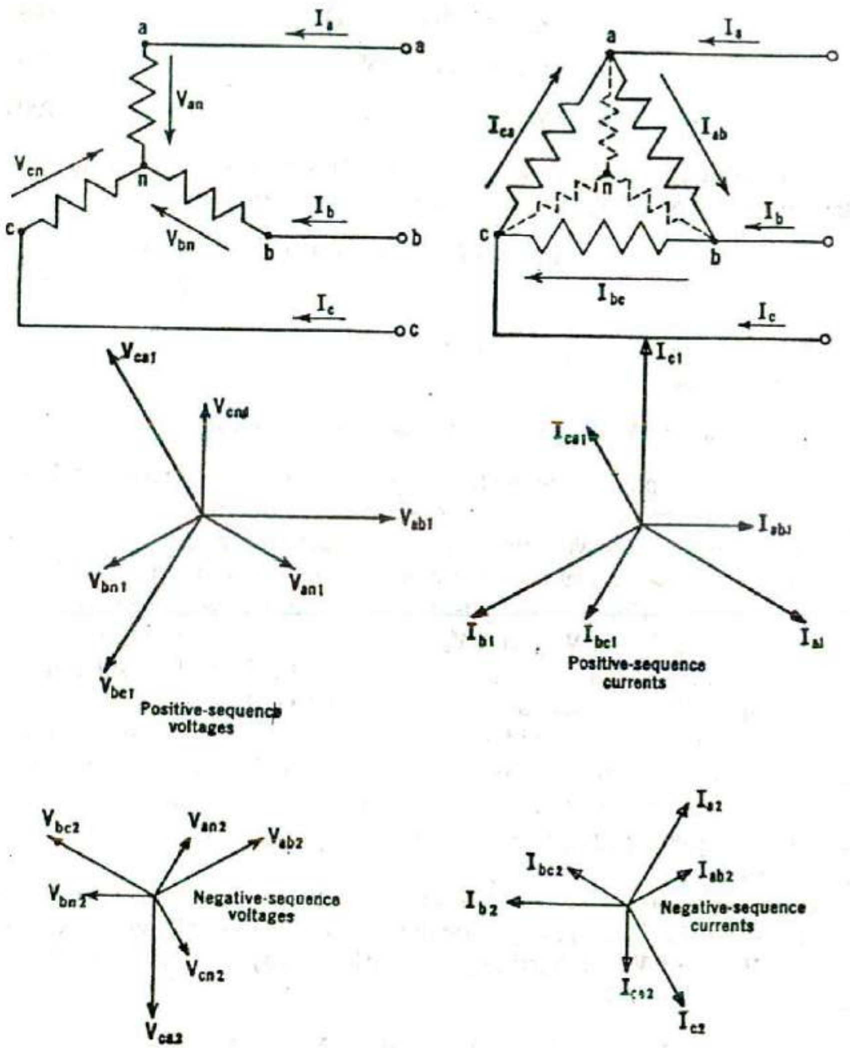


FIG. 3. Positive and negative systems of voltages and currents for a specific three-phase system.

shown in equation (24). If the general scheme is understood, neither changes in notation nor reversals of phase sequence (of the original vectors) can cause confusion.

The statement following equations (23) and (24) and the equations themselves are based upon a line-voltage sequence of *ab–bc–ca* where $\mathbf{V}_{bc}$ actually lags the base vector $\mathbf{V}_{ab}$. Occasions arise where the formulas as given by equations (13), (16) and (17) should be applied as labeled even though the vector of phase *b* does not lag but actually leads the base vector. An illustration involving currents follows. Assume the impedances for the wye load shown in Fig. 3 are $\mathbf{Z}_{na} = 5.77\underline{/0°}$, $\mathbf{Z}_{nb} = 10\underline{/90°}$, and $\mathbf{Z}_{nc} = 10\underline{/-90°}$. If the applied line voltages are $\mathbf{V}_{ba} = 100\underline{/30°}$, $\mathbf{V}_{cb} = 100\underline{/-90°}$, and $\mathbf{V}_{ac} = 100\underline{/150°}$ solution will yield voltages and currents as follows:

$$\mathbf{V}_{na} = 57.7\underline{/0°} \qquad \mathbf{V}_{nb} = 57.7\underline{/-120°} \qquad \mathbf{V}_{nc} = 57.7\underline{/120°}$$

$$\mathbf{I}_{na} = 10\underline{/0°} \qquad \mathbf{I}_{nb} = 5.77\underline{/150°} \qquad \mathbf{I}_{nc} = 5.77\underline{/-150°}$$

Inspection of these results shows the voltage sequence to be *a–b–c*, and this might be the starting point and called the positive sequence system. The actual current sequence is *a–c–b*. If the currents in this case are resolved into their symmetrical components, that in phase *b* should be advanced 120° as equation (13) would indicate even though $\mathbf{I}_{nb}$ actually leads $\mathbf{I}_{na}$ which might be taken as the base vector. Otherwise the system of positive sequence currents would not correspond to the positive sequence system of voltages. In general it is customary at the start to assume a positive sequence of *a–b–c* and initially label the vector which lags the reference vector so the sequence is *a–b–c*. Then the positive sequence voltage or current in any subsequent calculations will be obtained by advancing, that is, rotating counterclockwise 120°, the *b* phase voltage or current regardless of whether it actually lags or leads the base vector. This is necessary to make all positive sequence systems of voltages and currents correspond. Otherwise a negative sequence system of currents may be the one to correspond to a positive sequence system of voltages, and this would lead to confusion.

Problem 2. A three-phase system of line voltages, $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{ca}$, are unbalanced to the extent that $\mathbf{V}_{ab1} = 4000\underline{/-60°}$ and $\mathbf{V}_{ab2} = 2000\underline{/180°}$ volts. ($\mathbf{V}_{ab0}$ is, of course, equal to zero.)

(*a*) Draw a common-origin vector diagram illustrating the positive-sequence voltages and the negative-sequence voltages of $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{ca}$.

(*b*) Find the magnitudes of the three voltages $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{ca}$.

Ans.: (*b*) $V_{ab} = 3464$, $V_{bc} = 3464$, $V_{ca} = 6000$ volts.

Phase Voltages of Wye-Connected Loads. Reference to equation (21) will show that the phase voltages, $\mathbf{V}_{an}$, $\mathbf{V}_{bn}$, and $\mathbf{V}_{cn}$, may possess any vector values whatsoever and yet the vector sum of the line-to-line voltages is zero. In general, however,

$$\mathbf{V}_{an} + \mathbf{V}_{bn} + \mathbf{V}_{cn} \neq 0$$

The individual phase voltages will, therefore, generally possess zero-sequence components even though these components are absent in the line-to-line voltages. Under balanced conditions the phase voltages will, of course, possess no zero-sequence components.

Example 5. In Fig. 3, let

$$\mathbf{V}_{an} = 10\underline{/0°} \quad \mathbf{V}_{bn} = 20\underline{/-90°} \quad \mathbf{V}_{cn} = 10\underline{/135°} \text{ volts}$$

Under these conditions

$$\mathbf{V}_{ab} = (10 + j0) - (0 - j20) = 10 + j20$$

$$\mathbf{V}_{bc} = (0 - j20) - (-7.07 + j7.07) = 7.07 - j27.07$$

$$\mathbf{V}_{ca} = (-7.07 + j7.07) - (10 + j0) = -17.07 + j7.07$$

$$\mathbf{V}_{ab0} = \tfrac{1}{3}[(10 + j20) + (7.07 - j27.07) + (-17.07 + j7.07)] = 0$$

$$\mathbf{V}_{an0} = \tfrac{1}{3}[(10 + j0) + (0 - j20) + (-7.07 + j7.07)]$$
$$= \tfrac{1}{3}(2.93 - j12.93) = 0.98 - j4.31 \text{ volts}$$

It will be noted that triple subscripts have been used in the above example in connection with the component voltages $\mathbf{V}_{ab0}$ and $\mathbf{V}_{an0}$. Where both line-to-line and phase voltages are involved in the same discussion, triple subscripts of this kind may be used advantageously. These subscripts tell whether line-to-line voltages or phase voltages are being considered, they specify the positive circuit direction of the voltages, and they designate the order of the system to which the component voltage belongs.

Delta-Wye Voltage Transformations. In symmetrical-component analyses it is very often particularly advantageous to consider delta-connected systems on an equivalent wye basis. If the delta-connected load shown in Fig. 3 is to be analyzed on an equivalent wye basis, the load impedances are first converted to their equivalent wye values in the conventional manner and then the line-to-line voltages are resolved into their symmetrical components as shown in equations (23) and (24). The remaining problem is that of finding the equivalent wye voltages in terms of the line-to-line voltages.

For a–b–c sequence

$$\mathbf{V}_{bn1} = \mathbf{V}_{an1}\underline{/-120°} \quad \text{and} \quad \mathbf{V}_{an1} - \mathbf{V}_{bn1} = \mathbf{V}_{ab1}$$

It follows that

$$\mathbf{V}_{an1} - \mathbf{V}_{an1}\underline{/-120°} = \mathbf{V}_{ab1}$$

$$\mathbf{V}_{an1}[1 - (-0.5 - j0.866)] = \mathbf{V}_{ab1}$$

Hence

$$\mathbf{V}_{an1} = \frac{\mathbf{V}_{ab1}}{\sqrt{3}\underline{/30°}} = \frac{\mathbf{V}_{ab1}}{\sqrt{3}}\underline{/-30°} \tag{25}$$

The complete positive-sequence system of voltages is shown in Fig. 3.

In a corresponding manner it may be shown that

$$\mathbf{V}_{an2} = \frac{\mathbf{V}_{ab2}}{\sqrt{3}\,/\!-30^\circ} = \frac{\mathbf{V}_{ab2}}{\sqrt{3}}\,/30^\circ \tag{26}$$

The complete negative-sequence system of voltages is shown in Fig. 3.

Equations (25) and (26) are useful in the analysis of either wye- or delta-connected loads where the line-to-line voltages are specified. They are also important in the analysis of delta-wye transformer banks like that shown in Fig. 4.

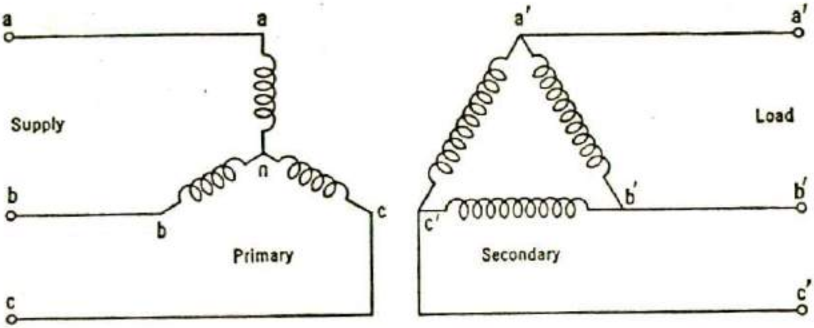


Fig. 4. Wye-delta transformer bank. The windings of transformer a are $a'b'$ and an, transformer b, $b'c'$ and bn, and transformer c, $c'a'$ and cn.

It should be noted in passing that $\mathbf{V}_{an0}$ may possess a finite value even though the zero-sequence components of the line-to-line voltages are of zero value. The fact that $\mathbf{V}_{an0}$ cannot be evaluated in terms of the line-to-line voltages presents no serious handicap as will be shown later, but it does preclude the possibility of immediately evaluating the voltage to neutral ($\mathbf{V}_{an} = \mathbf{V}_{an1} + \mathbf{V}_{an2} + \mathbf{V}_{an0}$).

Problem 3. In the wye-delta transformer bank shown in Fig. 4, the operation of the three transformers, and the polarities of the windings are such that

$$\mathbf{V}_{a'b'} = n\mathbf{V}_{an} \qquad \text{(transformer } a\text{)}$$

$$\mathbf{V}_{b'c'} = n\mathbf{V}_{bn} \qquad \text{(transformer } b\text{)}$$

$$\mathbf{V}_{c'a'} = n\mathbf{V}_{cn} \qquad \text{(transformer } c\text{)}$$

where n is the voltage transformation ratio of the transformers. The primary line-to-line voltages are unbalanced in magnitude to the extent that $\mathbf{V}_{ab1} = 4000\,/\!-60^\circ$ and $\mathbf{V}_{ab2} = 1000\,/\!-90^\circ$ volts. ($\mathbf{V}_{ab1}$ and $\mathbf{V}_{ab2}$ are, of course, written with respect to a common reference axis.) The sequence of the primary line-to-line voltages is assumed to be ab–ba–ca, and $\mathbf{V}_{an0}$ is to be taken as zero.

(a) Find the magnitude and vector position of $\mathbf{V}_{ab}$ and of $\mathbf{V}_{bc}$.

(b) If the transformation ratio of the transformers is 10, find the magnitude and vector position of $V_{a'b'}$ and of $V_{b'c'}$.

Ans.: $V_{ab} = 4890\underline{/-65.85^\circ}$, $V_{bc} = 3173\underline{/170.94^\circ}$ volts.
$V_{a'b'} = 28{,}230\underline{/-84.14^\circ}$, $V_{b'c'} = 23{,}800\underline{/135.95^\circ}$ volts.

Problem 4. Find the relative vector positions of V_{ab} and $V_{a'b'}$ of the wye-delta transformer bank of Fig. 4 if $V_{ab2} = 0$ and $V_{an0} = 0$. Find the relative vector positions of V_{bc} and $V_{b'c'}$ under the same conditions. (The sequence of the supply voltages V_{ab}, V_{bc}, and V_{ca} is assumed to be *ab–bc–ca*.)

Ans.: $V_{a'b'}$ lags V_{ab} by 30°; $V_{b'c'}$ lags V_{bc} by 30°.
The supply voltages are balanced and the positive-sequence voltage vector diagram of Fig. 3 applies directly since $V_{a'b'} = nV_{an}$, and $V_{b'c'} = nV_{bn}$.

Three-Phase, Three-Wire Line Currents and Associated Delta-Phase Currents. The line currents of a three-phase, three-wire system can contain no zero-sequence components regardless of whether the system is wye- or delta-connected. Reference to the wye-connected load given in Fig. 3 will show that at the junction n

$$I_a + I_b + I_c = 0$$

Therefore,

$$I_{a0} = \tfrac{1}{3}(I_a + I_b + I_c) = 0 \tag{27}$$

Reference to the delta-connected load given in Fig. 3 will show that

$$I_a = I_{ab} - I_{ca} \tag{28}$$

$$I_b = I_{bc} - I_{ab} \tag{29}$$

$$I_c = I_{ca} - I_{bc} \tag{30}$$

Hence

$$I_a + I_b + I_c = (I_{ab} - I_{ca}) + (I_{bc} - I_{ab}) + (I_{ca} - I_{bc}) = 0 \tag{31}$$

Regardless of the degree of unbalance of the individual phase currents, I_{ab}, I_{bc}, and I_{ca}, the vector sum of the line currents, I_a, I_b, and I_c, is equal to zero and therefore no zero-sequence components are present in the line currents.

The individual delta-phase currents will, in general, possess zero-sequence components since $(I_{ab} + I_{bc} + I_{ca})$ is, in general, not equal to zero. The zero-sequence components of the phase currents in a delta-connected system cannot be evaluated in terms of the line currents.

For a–b–c sequence of line currents,

$$I_{ab1} - I_{ca1} = I_{a1} \quad \text{and} \quad I_{ca1} = I_{ab1}\underline{/120^\circ}$$

Employing the same type of derivation as that employed in the derivation of equation (25), it is easy to show that

$$\mathbf{I}_{ab1}[1 - (-0.5 + j0.866)] = \mathbf{I}_{a1}$$

$$\mathbf{I}_{ab1} = \frac{\mathbf{I}_{a1}}{\sqrt{3}}\underline{/30^\circ} \qquad (32)$$

A complete positive-sequence system of currents is shown in Fig. 3. The vector diagram of the positive-sequence currents shows that $\mathbf{I}_{ab1}$ is $1/\sqrt{3}$ as large as $\mathbf{I}_{a1}$ and 30° in advance of $\mathbf{I}_{a1}$.

In a corresponding manner it may be shown that

$$\mathbf{I}_{ab2} = \frac{\mathbf{I}_{a2}}{\sqrt{3}}\underline{/-30^\circ} \qquad (33)$$

In a wye-delta transformer bank like that shown in Fig. 4 where no zero-sequence components of current can exist in the wye primary windings, no zero-sequence currents will be present in the delta secondary windings since $N_pI_p = N_sI_s$. In this connection, N_p represents the primary turns and N_s the secondary turns of one transformer. (The magnetizing current is neglected in the statement $N_pI_p = N_sI_s$ or else I_p represents simply the load component of the primary current.) The fact that a transformer bank like that shown in Fig. 4 eliminates zero-sequence currents is of importance in power network short-circuit studies.

Problem 5. Find the line current, $\mathbf{I}_a$, in the delta-connected system shown in Fig. 3 if

$$\mathbf{I}_{ab1} = 10\underline{/0^\circ}, \quad \mathbf{I}_{ab2} = 5\underline{/60^\circ}, \quad \text{and} \quad \mathbf{I}_{ab0} = 7\underline{/19.5^\circ} \text{ amperes}$$

Ans.: $\mathbf{I}_a = 15\underline{/0^\circ}$ amperes.

Three-Phase Line Currents Associated with a Neutral Return. If a wye-wye system operates with grounded neutrals or with a connecting wire between neutrals, the vector sum of the line currents will not, in general, be equal to zero. In this case:

$$\mathbf{I}_{a0} = \mathbf{I}_{b0} = \mathbf{I}_{c0} = \tfrac{1}{3}(\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c) \qquad (34)$$

It will be noted that the ground or neutral return current, namely, $(\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c)$, is three times as large as the individual zero-sequence components of the line currents. Each line wire carries a component of current which is equal in magnitude and in time phase with similar components in the other two lines. These zero-sequence components are sometimes called *uniphase* components and have important physical significance in connection with the inductive interference between three-phase power lines and paralleling telephone lines.

Where the line currents possess uniphase components, no manner of transposition of the power system line wires will prevent these components from establishing inductive interference in paralleling telephone lines, the reason being that the uniphase components in the three line wires establish similarly directed magnetic interference. In a case of this kind, transposition of the telephone wires themselves is required to balance out the undesirable emf's that are induced by the power system currents. Inductive interference studies usually refer to the uniphase or zero-sequence currents as *residuals* since they represent the component currents that remain after the positive- and negative-sequence components have been taken from the original unbalanced system of currents. The fact that the *residuals* can be separated from the two balanced systems of currents is an important feature in interference problems.

The zero-sequence components of the line currents of grounded or four-wire wye systems are also of importance in the evaluation of the short-circuit currents in power systems.

Example 6. A line-to-ground short circuit on a grounded wye-connected alternator is shown in Fig. 5. Let it be required to find the three-phase symmetrical

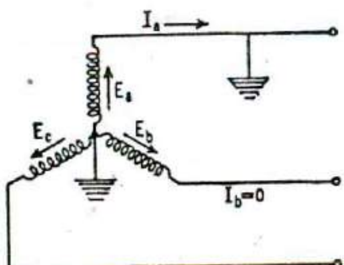


FIG. 5. A particular case of unbalanced three-phase line currents.

components of the line currents $\mathbf{I}_a$, $\mathbf{I}_b$, and $\mathbf{I}_c$, where $\mathbf{I}_a = I\underline{/\alpha}$, $I_b = 0$, and $I_c = 0$. I is the magnitude of the short-circuit current, $\mathbf{I}_a$, and α is the angular displacement of this current from any arbitrary reference axis. The three line currents may be considered as an unbalanced three-phase system of currents even though two of the currents are equal to zero.

The original system of currents is represented by

$$\mathbf{I}_a = I\underline{/\alpha} \quad \mathbf{I}_b = 0 \quad \mathbf{I}_c = 0$$

The positive-sequence components of the above currents are

$$\mathbf{I}_{a1} = \tfrac{1}{3}I\underline{/\alpha} \quad \mathbf{I}_{b1} = \tfrac{1}{3}I\underline{/\alpha - 120^\circ} \quad \mathbf{I}_{c1} = \tfrac{1}{3}I\underline{/\alpha + 120^\circ}$$

The negative-sequence components are

$$\mathbf{I}_{a2} = \tfrac{1}{3}I\underline{/\alpha} \quad \mathbf{I}_{b2} = \tfrac{1}{3}I\underline{/\alpha + 120^\circ} \quad \mathbf{I}_{c2} = \tfrac{1}{3}I\underline{/\alpha - 120^\circ}$$

The zero-sequence components are

$$\mathbf{I}_{a0} = \mathbf{I}_{b0} = \mathbf{I}_{c0} = \tfrac{1}{3}I\underline{/\alpha}$$

Graphical representations of the above results are shown in Fig. 6. It will be observed that

$$\mathbf{I}_{a1} + \mathbf{I}_{a2} + \mathbf{I}_{a0} = \mathbf{I}_a = I\underline{/\alpha}$$

$$\mathbf{I}_{b1} + \mathbf{I}_{b2} + \mathbf{I}_{b0} = \mathbf{I}_b = 0$$

$$\mathbf{I}_{c1} + \mathbf{I}_{c2} + \mathbf{I}_{c0} = \mathbf{I}_c = 0$$

Symmetrical components of the kind given above are used in single line-to-ground

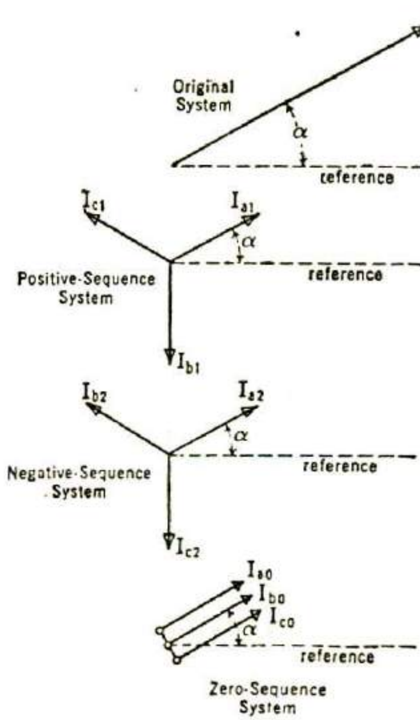


FIG. 6. The resolution of a single current $I\underline{/\alpha}$ into its three-phase symmetrical components.

short-circuit current analyses, and although this type of problem is not considered in the present chapter a study of Fig. 6 at this stage will prove to be instructive.

Problem 6. The three line currents in a four-wire wye system like that shown in Fig. 7, are:

$\mathbf{I}_{a'a} = \mathbf{I}_a = 20\underline{/-60°}$, $\mathbf{I}_{b'b} = \mathbf{I}_b = 12\underline{/-100°}$, and $\mathbf{I}_{c'c} = \mathbf{I}_c = 10\underline{/75°}$ amperes

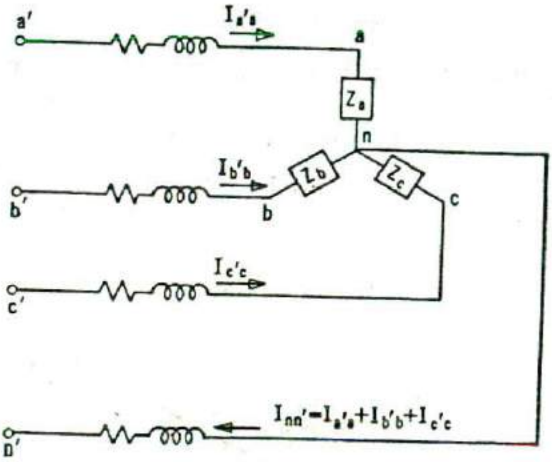


FIG. 7. Three-phase four-wire system for Problem 6.

Find the positive-, negative-, and zero-sequence components of the above line currents and check the results either graphically or by the vector addition of the symmetrical components.

Ans.:
$$\begin{aligned}
\mathbf{I}_{a1} &= 9.45 - j6.76 = 11.62\underline{/-35.6°} \text{ amperes}\\
\mathbf{I}_{b1} &= -10.58 - j4.80 = 11.62\underline{/-155.6°} \text{ amperes}\\
\mathbf{I}_{c1} &= 1.136 + j11.58 = 11.62\underline{/84.4°} \text{ amperes}\\
\mathbf{I}_{a2} &= -2.95 - j4.07 = 5.03\underline{/-125.9°} \text{ amperes}\\
\mathbf{I}_{b2} &= 5.0 - j0.517 = 5.03\underline{/-5.9°} \text{ amperes}\\
\mathbf{I}_{c2} &= -2.05 + j4.59 = 5.03\underline{/114.1°} \text{ amperes}\\
\mathbf{I}_{a0} &= 3.503 - j6.49 = 7.375\underline{/-61.65°} \text{ amperes}
\end{aligned}$$

Power from Symmetrical Components. For any unbalanced three-phase system the total power consumed is the sum of the powers absorbed in each phase. Thus

$$P = P_a + P_b + P_c = V_a I_a \cos \theta_{\mathbf{V}_a}^{\mathbf{I}_a} + V_b I_b \cos \theta_{\mathbf{V}_b}^{\mathbf{I}_b} + V_c I_c \cos \theta_{\mathbf{V}_c}^{\mathbf{I}_c}$$

If the voltage of a given phase, say $\mathbf{V}_a$, is resolved into several components, the power for that phase may be obtained by adding the products of each component of voltage by the current times the cosine of the angle between the particular voltage component and the current. Reference to Fig. 8 will make this evident. Here

$$P_a = I_a(V_a \cos\theta) = I_a(V_1 \cos\theta_1 + V_2 \cos\theta_2 + V_0 \cos\theta_0)$$
$$= I_a V_1 \cos\theta_{I_a}^{V_1} + I_a V_2 \cos\theta_{I_a}^{V_2} + I_a V_0 \cos\theta_{I_a}^{V_0}$$

Similarly, if the current is divided into components, the power is the sum of the products of voltage by the current times the cosine of the phase angle between the respective components of current and the voltage. From these facts it should be apparent that if both voltage and current are resolved into components, the power will be the sum of the products of each component of voltage by each component of current times the cosine of the angle between the particular component of voltage and current appearing in each of the products.

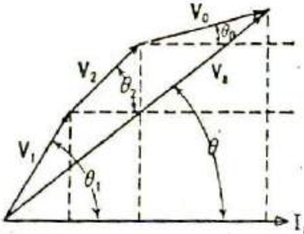


FIG. 8. In-phase component of $\mathbf{V}_a$ with respect to $\mathbf{I}_a$ is the sum of the in-phase components of each of the component voltages of $\mathbf{V}_a$.

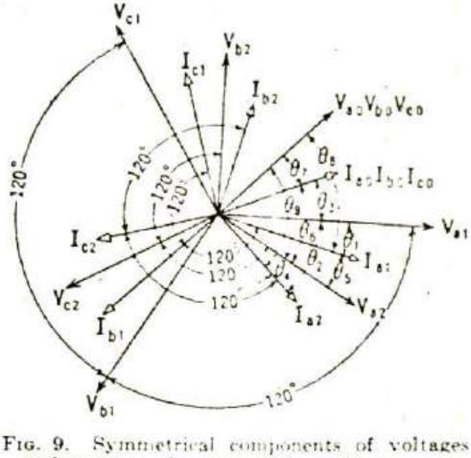


FIG. 9. Symmetrical components of voltages and currents of a general three-phase system.

Figure 9 shows the symmetrical components of currents and voltages for any three-phase system. The subscripts a, b, and c denote the phase while 0, 1, and 2 are the usual symbols denoting the sequence components. In terms of the components shown for phase a, the power is

$$\begin{aligned} P_a = {} & V_{a1}I_{a1}\cos\theta_1 + V_{a1}I_{a2}\cos\theta_2 + V_{a1}I_{a0}\cos\theta_3 + V_{a2}I_{a2}\cos\theta_4 \\ & + V_{a2}I_{a1}\cos\theta_5 + V_{a2}I_{a0}\cos\theta_6 + V_{a0}I_{a0}\cos\theta_7 + V_{a0}I_{a1}\cos\theta_8 \\ & + V_{a0}I_{a2}\cos\theta_9 \end{aligned} \tag{35}$$

For phase b

$$\begin{aligned} P_b = {} & V_{b1}I_{b1}\cos\theta_1 + V_{b1}I_{b2}\cos(120^\circ + \theta_2) + V_{b1}I_{b0}\cos(120^\circ + \theta_3) \\ & + V_{b2}I_{b2}\cos\theta_4 + V_{b2}I_{b1}\cos(120^\circ + \theta_5) + V_{b2}I_{b0}\cos(120^\circ - \theta_6) \\ & + V_{b0}I_{b0}\cos\theta_7 + V_{b0}I_{b1}\cos(120^\circ + \theta_8) \\ & + V_{b0}I_{b2}\cos(120^\circ - \theta_9) \end{aligned} \tag{36}$$

For phase *c*

$$\begin{aligned} P_c = {} & V_{c1}I_{c1}\cos\theta_1 + V_{c1}I_{c2}\cos(240^\circ + \theta_2) + V_{c1}I_{c0}\cos(240^\circ + \theta_3) \\ & + V_{c2}I_{c2}\cos\theta_4 + V_{c2}I_{c1}\cos(240^\circ + \theta_5) + V_{c2}I_{c0}\cos(240^\circ - \theta_6) \\ & + V_{c0}I_{c0}\cos\theta_7 + V_{c0}I_{c1}\cos(240^\circ + \theta_8) \\ & + V_{c0}I_{c2}\cos(240^\circ - \theta_9) \end{aligned} \quad (37)$$

It should be remembered that only magnitudes of voltages and currents appear in equations (35), (36), and (37), and that $V_{a1} = V_{b1} = V_{c1}$, $V_{a2} = V_{b2} = V_{c2}$, $V_{a0} = V_{b0} = V_{c0}$, $I_{a1} = I_{b1} = I_{c1}$, $I_{a2} = I_{b2} = I_{c2}$, and $I_{a0} = I_{b0} = I_{c0}$. Under these conditions if equations (35), (36), and (37) are added, the terms containing θ_2 add to zero because they represent three equal quantities at 120-degree angles. Similarly, the terms containing θ_3, θ_5, θ_6, θ_8, and θ_9 add to zero. Dropping reference to particular phases, this leaves

$$P = P_a + P_b + P_c = 3V_1I_1\cos\theta_1 + 3V_2I_2\cos\theta_4 + 3V_0I_0\cos\theta_7 \quad (38)$$

It will be noted that $\cos\theta_1 = \cos\theta_{\mathbf{V}_1}^{\mathbf{I}_1}$, $\cos\theta_4 = \cos\theta_{\mathbf{V}_2}^{\mathbf{I}_2}$, and $\cos\theta_7 = \cos\theta_{\mathbf{V}_0}^{\mathbf{I}_0}$. Hence

$$\therefore P = 3V_1I_1\cos\theta_{\mathbf{V}_1}^{\mathbf{I}_1} + 3V_2I_2\cos\theta_{\mathbf{V}_2}^{\mathbf{I}_2} + 3V_0I_0\cos\theta_{\mathbf{V}_0}^{\mathbf{I}_0} \quad (39)$$

Equation (39) shows that the total power consumed by an unbalanced three-phase system is the sum of the powers represented by each of the symmetrical component systems. Hence, to obtain total power the algebraic sum of the total positive-, total negative-, and total zero-sequence powers may be calculated.

Copper Losses in Terms of Symmetrical Components. The copper loss for any unbalanced three-phase system is

$$P = I_a^2R_a + I_b^2R_b + I_c^2R_c \quad (40)$$

where phase currents and the corresponding phase resistances are used.

By referring to Fig. 10 and by remembering that

$$\mathbf{I}_a = \mathbf{I}_{a1} + \mathbf{I}_{a2} + \mathbf{I}_{a0}$$

it follows that

$$I_a^2 = (I_{a1} + I_{a2}\cos\beta + I_{a0}\cos\alpha)^2 + (I_{a2}\sin\beta + I_{a0}\sin\alpha)^2 \quad (41)$$

Similarly,

$$\begin{aligned} I_b^2 = {} & [I_{b1}\cos 240^\circ + I_{b2}\cos(120^\circ - \beta) + I_{b0}\cos\alpha]^2 \\ & + [I_{b1}\sin 240^\circ + I_{b2}\sin(120^\circ - \beta) + I_{b0}\sin\alpha]^2 \end{aligned} \quad (42)$$

$$\begin{aligned} I_c^2 = {} & [I_{c1}\cos 120^\circ + I_{c2}\cos(240^\circ - \beta) + I_{c0}\cos\alpha]^2 \\ & + [I_{c1}\sin 120^\circ + I_{c2}\sin(240^\circ - \beta) + I_{c0}\sin\alpha]^2 \end{aligned} \quad (43)$$

When R_a, R_b, and R_c are different the sequence components of current should be combined to obtain I_a, I_b, and I_c, and equation (40) used to calculate the copper loss. If, however, $R_a = R_b = R_c = R$, substitution of equations (41), (42), and (43) in equation (40), dropping reference to phase, and expanding and combining terms algebraically give

$$P = 3I_1{}^2R + 3I_2{}^2R + 3I_0{}^2R$$
$$= 3(I_1{}^2 + I_2{}^2 + I_0{}^2)R \qquad (44)$$

Equation (44) shows that the total copper loss due to the resultant currents is the same as the sum of the copper losses due to the sequence components calculated separately.

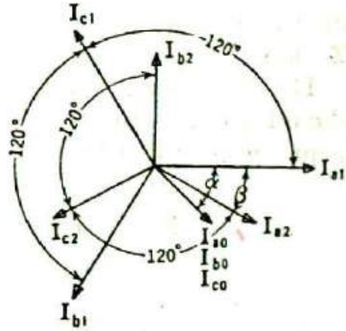


FIG. 10. Symmetrical components of currents in a general three-phase system.

If the resistances to the positive-, negative-, and zero-sequence currents are different, the copper loss may be determined from

$$P = 3I_1{}^2R_1 + 3I_2{}^2R_2 + 3I_0{}^2R_0 \quad (45)$$

where R_1, R_2, and R_0 are respectively the *resistance to* the positive-, negative-, and zero-sequence components of current. In using equation (45) it must be remembered that each of the sequence resistances must be the same for all three phases, since equality of phase resistances was assumed in obtaining equation (44), of which (45) is a modification.

Positive-, Negative-, and Zero-Sequence Impedance Components. For purposes of some analyses, three self-impedances may be separated or resolved into their symmetrical components exactly like three voltages or currents. If the voltages or currents which are to be associated with these component impedances are resolved in the order a–b–c, then the impedances should be resolved in the same order. [See equations (13), (16), and (17).] The term self-impedance implies that no mutual coupling exists between the individual impedances. In order to distinguish the components of self-impedance from the components of mutual impedance which are considered later, double subscripts of the kind given below will be used.

The symmetrical components of three self-impedances, $\mathbf{Z}_{aa}$, $\mathbf{Z}_{bb}$, and $\mathbf{Z}_{cc}$ are

$$\mathbf{Z}_{aa1} = \tfrac{1}{3}(\mathbf{Z}_{aa} + \mathbf{Z}_{bb}\underline{/120^\circ} + \mathbf{Z}_{cc}\underline{/-120^\circ}) \qquad (46)$$

$$\mathbf{Z}_{aa2} = \tfrac{1}{3}(\mathbf{Z}_{aa} + \mathbf{Z}_{bb}\underline{/-120^\circ} + \mathbf{Z}_{cc}\underline{/120^\circ}) \qquad (47)$$

$$\mathbf{Z}_{aa0} = \tfrac{1}{3}(\mathbf{Z}_{aa} + \mathbf{Z}_{bb} + \mathbf{Z}_{cc}) \qquad (48)$$

As above defined $\mathbf{Z}_{aa1}$, $\mathbf{Z}_{aa2}$, and $\mathbf{Z}_{aa0}$ are called positive-sequence impedance, negative-sequence impedance, and zero-sequence impedance respectively. These component impedances have little physical significance but they are useful in a general mathematical formulation of symmetrical-component theory. It should be pointed out at this stage that the resistance (or in-phase) parts of the component impedances may possess negative signs even though the real parts of $\mathbf{Z}_{aa}$, $\mathbf{Z}_{bb}$, and $\mathbf{Z}_{cc}$ are all positive.

The above symmetrical components of an unbalanced *set of impedances* should not be confused with impedance to positive-, negative-, and zero-sequence currents which are defined as follows:

$$\text{Impedance to positive-sequence,}\quad \mathbf{Z}_{a1} = \frac{\mathbf{V}_{a1}}{\mathbf{I}_{a1}}$$

$$\text{Impedance to negative-sequence,}\quad \mathbf{Z}_{a2} = \frac{\mathbf{V}_{a2}}{\mathbf{I}_{a2}}$$

$$\text{Impedance to zero-sequence,}\quad \mathbf{Z}_{a0} = \frac{\mathbf{V}_{a0}}{\mathbf{I}_{a0}}$$

These impedances to sequence component currents are usually applied to systems where the impedances of all phases are the same or balanced. In order to avoid confusion a *double-letter subscript* will be used on positive-, negative-, and zero-sequence *components of impedance.* For *impedance* to positive-, negative-, and zero-sequence currents a *single-letter subscript* will be used. In both cases the figure subscripts 1, 2, and 0 will denote positive, negative, and zero sequence, respectively.

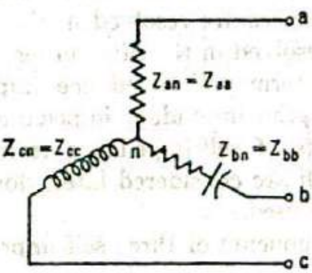


FIG. 11. See example 7.

Example 7. Let the wye-connected impedances of Fig. 11 be

$$\mathbf{Z}_{aa} = (6 + j0) \quad \mathbf{Z}_{bb} = (5.2 - j3) \quad \mathbf{Z}_{cc} = (0 + j12) \text{ ohms}$$

Employing equations (46), (47), and (48), the component impedances are

$$\begin{aligned}\mathbf{Z}_{aa1} &= \tfrac{1}{3}[(6+j0) + (5.2-j3)(-0.5+j0.866) + (0+j12)(-0.5-j0.866)] \\ &= \tfrac{1}{3}[(6+j0) + (0+j6) + (10.4-j6)] \\ &= \tfrac{1}{3}(16.4+j0) = 5.47 + j0 \text{ ohms}\end{aligned}$$

$$\begin{aligned}\mathbf{Z}_{aa2} &= \tfrac{1}{3}[(6+j0) + (5.2-j3)(-0.5-j0.866) + (0+j12)(-0.5+j0.866)] \\ &= \tfrac{1}{3}[(6+j0) + (-5.2-j3) + (-10.4-j6)] \\ &= \tfrac{1}{3}(-9.6-j9) = -3.2 - j3 \text{ ohms}\end{aligned}$$

$$\begin{aligned}\mathbf{Z}_{aa0} &= \tfrac{1}{3}[(6+j0) + (5.2-j3) + (0+j12)] \\ &= \tfrac{1}{3}(11.2+j9) = 3.73 + j3 \text{ ohms}\end{aligned}$$

In accordance with previous considerations, it follows that

$$\begin{aligned}\mathbf{Z}_{bb1} &= \mathbf{Z}_{aa1}\underline{/-120^\circ} & \mathbf{Z}_{cc1} &= \mathbf{Z}_{aa1}\underline{/120^\circ} \\ \mathbf{Z}_{bb2} &= \mathbf{Z}_{aa2}\underline{/120^\circ} & \mathbf{Z}_{cc2} &= \mathbf{Z}_{aa2}\underline{/-120^\circ} \\ \mathbf{Z}_{bb0} &= \mathbf{Z}_{aa0} & \mathbf{Z}_{cc0} &= \mathbf{Z}_{aa0}\end{aligned}$$

The sum of the impedance components of one phase equals the actual impedance of that phase. For example,

$$\mathbf{Z}_{aa} = (5.47+j0) + (-3.2-j3) + (3.73+j3) = 6 + j0 \text{ ohms}$$

Problem 7. Find $\mathbf{Z}_{bb1}$, $\mathbf{Z}_{bb2}$, and $\mathbf{Z}_{bb0}$ in the above example, employing the values of $\mathbf{Z}_{aa1}$, $\mathbf{Z}_{aa2}$, and $\mathbf{Z}_{aa0}$ which have been evaluated. Repeat for $\mathbf{Z}_{cc1}$, $\mathbf{Z}_{cc2}$, and $\mathbf{Z}_{cc0}$.

Ans.: $\mathbf{Z}_{bb} = \mathbf{Z}_{bb1} + \mathbf{Z}_{bb2} + \mathbf{Z}_{bb0}$
$= (-2.73 - j4.73) + (4.20 - j1.27) + (3.73 + j3.0)$
$= (5.2 - j3.0)$ ohms.

Problem 8. Given three wye-connected impedances:

$$\mathbf{Z}_{an} = (15+j0) \quad \mathbf{Z}_{bn} = (6-j3.464) \quad \mathbf{Z}_{cn} = (6+j3.464) \text{ ohms}$$

(a) Find the symmetrical components of $\mathbf{Z}_{an}$ in accordance with the resolutions given in equations (46), (47), and (48).

(b) Find $\mathbf{Z}_{bn1}$, $\mathbf{Z}_{bn2}$, and $\mathbf{Z}_{bn0}$ in terms of the symmetrical components of $\mathbf{Z}_{an}$ and check $(\mathbf{Z}_{bn1} + \mathbf{Z}_{bn2} + \mathbf{Z}_{bn0})$ with the given value of $(6 - j3.464)$ ohms.

Ans.: (a) $\mathbf{Z}_{an1} = 5\underline{/0^\circ}$; $\mathbf{Z}_{an2} = 1\underline{/0^\circ}$; $\mathbf{Z}_{an0} = 9\underline{/0^\circ}$ ohms.

Sequence Rule as Applied to Component Voltages. If the voltage drop across one phase, say phase a, is written in terms of the symmetrical components of both current and impedance, nine component voltages appear. That is,

$$\begin{aligned}\mathbf{V}_a = \mathbf{I}_a\mathbf{Z}_a &= (\mathbf{I}_{a1} + \mathbf{I}_{a2} + \mathbf{I}_{a0})(\mathbf{Z}_{aa1} + \mathbf{Z}_{aa2} + \mathbf{Z}_{aa0}) \\ &= \mathbf{I}_{a1}\mathbf{Z}_{aa1} + \mathbf{I}_{a1}\mathbf{Z}_{aa2} + \mathbf{I}_{a1}\mathbf{Z}_{aa0} + \mathbf{I}_{a2}\mathbf{Z}_{aa1} + \mathbf{I}_{a2}\mathbf{Z}_{aa2} + \mathbf{I}_{a2}\mathbf{Z}_{aa0} \\ &\quad + \mathbf{I}_{a0}\mathbf{Z}_{aa1} + \mathbf{I}_{a0}\mathbf{Z}_{aa2} + \mathbf{I}_{a0}\mathbf{Z}_{aa0}\end{aligned} \tag{49}$$

These nine component voltages may be grouped in such a manner as to form the positive-, negative-, and zero-sequence components of $\mathbf{V}_a$, and this grouping may be made in accordance with an easily remembered rule.

The Sequence Rule

The order of the voltage system to which an IZ drop belongs is equal to the sum of the orders of the systems to which I and Z belong individually.

In the application of the sequence rule, positive-sequence terms are of first order, negative-sequence terms are of second order, and zero-sequence terms are of zero or third order. In summing the orders both $(1 + 0)$ and $(2 + 2)$ are considered as belonging to the first order, since order 4 is considered as order 1, there being only three orders. In this connection, the zero in $(1 + 0)$ may be reckoned either as zero or three. Also $(1 + 2)$ is of order 3, or a zero-sequence term. As applied to the component voltages of equation (49), the sequence rule states

$$\mathbf{V}_{a1} = \mathbf{I}_{a1}\mathbf{Z}_{aa0} + \mathbf{I}_{a2}\mathbf{Z}_{aa2} + \mathbf{I}_{a0}\mathbf{Z}_{aa1} \tag{50}$$

$$\mathbf{V}_{a2} = \mathbf{I}_{a1}\mathbf{Z}_{aa1} + \mathbf{I}_{a2}\mathbf{Z}_{aa0} + \mathbf{I}_{a0}\mathbf{Z}_{aa2} \tag{51}$$

$$\mathbf{V}_{a0} = \mathbf{I}_{a1}\mathbf{Z}_{aa2} + \mathbf{I}_{a2}\mathbf{Z}_{aa1} + \mathbf{I}_{a0}\mathbf{Z}_{aa0} \tag{52}$$

Obviously the real basis upon which the above equations are written is that, as written, they satisfy the definitions which were originally attached to $\mathbf{V}_{a1}$, $\mathbf{V}_{a2}$, and $\mathbf{V}_{a0}$. To satisfy these definitions, $\mathbf{V}_{a1}$ must be the positive-sequence component of the base vector $\mathbf{V}_a$, $\mathbf{V}_{a2}$ must be the negative-sequence component of the base vector $\mathbf{V}_a$, and $\mathbf{V}_{a0}$ must be the zero-sequence component. The proof that $\mathbf{V}_{a1}$, as written in equation (50), satisfies the definition of a positive-sequence voltage is outlined below.

Applying equation (50) to the b phase and making appropriate substitutions,

$$\begin{aligned}\mathbf{V}_{b1} &= \mathbf{I}_{b1}\mathbf{Z}_{bb0} + \mathbf{I}_{b2}\mathbf{Z}_{bb2} + \mathbf{I}_{b0}\mathbf{Z}_{bb1} \\ &= (\mathbf{I}_{a1}\underline{/-120^\circ})\mathbf{Z}_{aa0} + (\mathbf{I}_{a2}\underline{/120^\circ})\mathbf{Z}_{aa2}\underline{/120^\circ} + \mathbf{I}_{a0}\mathbf{Z}_{aa1}\underline{/-120^\circ} \\ &= \mathbf{I}_{a1}\mathbf{Z}_{aa0}\underline{/-120^\circ} + \mathbf{I}_{a2}\mathbf{Z}_{aa2}\underline{/-120^\circ} + \mathbf{I}_{a0}\mathbf{Z}_{aa1}\underline{/-120^\circ} \end{aligned} \tag{50a}$$

Comparison of equations (50a) and (50) will show that $\mathbf{V}_{b1}$ is equal in magnitude to $\mathbf{V}_{a1}$ and 120° behind $\mathbf{V}_{a1}$, as, of course, it should be if $\mathbf{V}_{a1}$, $\mathbf{V}_{b1}$, and $\mathbf{V}_{c1}$ are to form a positive-sequence system.

Applying equation (50) to the c phase and making appropriate substitutions,

$$\begin{aligned}\mathbf{V}_{c1} &= \mathbf{I}_{c1}\mathbf{Z}_{cc0} + \mathbf{I}_{c2}\mathbf{Z}_{cc2} + \mathbf{I}_{c0}\mathbf{Z}_{cc1} \\ &= (\mathbf{I}_{a1}\underline{/120^\circ})\mathbf{Z}_{aa0} + (\mathbf{I}_{a2}\underline{/-120^\circ})\mathbf{Z}_{aa2}\underline{/-120^\circ} + \mathbf{I}_{a0}\mathbf{Z}_{aa1}\underline{/120^\circ} \\ &= \mathbf{I}_{a1}\mathbf{Z}_{aa0}\underline{/120^\circ} + \mathbf{I}_{a2}\mathbf{Z}_{aa2}\underline{/120^\circ} + \mathbf{I}_{a0}\mathbf{Z}_{aa1}\underline{/120^\circ} \qquad (50b)\end{aligned}$$

Comparison of equations ($50b$) and (50) will show that $\mathbf{V}_{c1}$ is equal in magnitude to $\mathbf{V}_{a1}$ and 120° ahead of $\mathbf{V}_{a1}$, which is the necessary requirement that $\mathbf{V}_{a1}$, $\mathbf{V}_{b1}$, and $\mathbf{V}_{c1}$ form a positive-sequence system of voltages.

In a manner similar to that outlined above, $\mathbf{V}_{a2}$ of equation (51) may be shown to be a member of a balanced negative-sequence system of voltages $\mathbf{V}_{a2}$, $\mathbf{V}_{b2}$, and $\mathbf{V}_{c2}$.

In the following problem the reader is asked to analyze equation (52) with a view toward showing that the **IZ** components of that equation are correctly chosen to form a zero-sequence system of voltages.

Problem 9. Prove that $\mathbf{V}_{a0}$ (equal to $\mathbf{I}_{a1}\mathbf{Z}_{aa2} + \mathbf{I}_{a2}\mathbf{Z}_{aa1} + \mathbf{I}_{a0}\mathbf{Z}_{aa0}$) is equal in magnitude and in time phase, with

$$\mathbf{V}_{b0} = \mathbf{I}_{b1}\mathbf{Z}_{bb2} + \mathbf{I}_{b2}\mathbf{Z}_{bb1} + \mathbf{I}_{b0}\mathbf{Z}_{bb0}$$

and with

$$\mathbf{V}_{c0} = \mathbf{I}_{c1}\mathbf{Z}_{cc2} + \mathbf{I}_{c2}\mathbf{Z}_{cc1} + \mathbf{I}_{c0}\mathbf{Z}_{cc0}$$

Application of the Sequence Rule to Unbalanced Three-Wire Loads. The foregoing theory may be applied to any three-wire load which consists of individual or non-coupled phase impedances. Since the individual phases of three-phase rotating equipment are closely coupled magnetically, the present method of analysis does not apply directly to rotating equipment. (A method of accounting for the mutual impedances of rotating equipment is given in Chapter XIII, and a general method of accounting for mutual impedance effects is given later in the present chapter.)

In applying equations (50), (51), and (52) to the a phase of a wye-connected load like that shown in Fig. 11, it is noted that, since $\mathbf{I}_{a0} = 0$,

$$\mathbf{V}_{an1} = \mathbf{I}_{a1}\mathbf{Z}_{an0} + \mathbf{I}_{a2}\mathbf{Z}_{an2} \qquad (53)$$

$$\mathbf{V}_{an2} = \mathbf{I}_{a1}\mathbf{Z}_{an1} + \mathbf{I}_{a2}\mathbf{Z}_{an0} \qquad (54)$$

$$\mathbf{V}_{an0} = \mathbf{I}_{a1}\mathbf{Z}_{an2} + \mathbf{I}_{a2}\mathbf{Z}_{an1} \qquad (55)$$

If the line-to-line voltages, namely, $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{ca}$, are known, $\mathbf{V}_{an1}$ and $\mathbf{V}_{an2}$ may be evaluated directly from equations (25) and (26). (See page 500.) If $\mathbf{V}_{an1}$ and $\mathbf{V}_{an2}$ are known, $\mathbf{I}_{a1}$ and $\mathbf{I}_{a2}$ may be determined directly from equations (53) and (54), provided that $\mathbf{Z}_{an1}$, $\mathbf{Z}_{an2}$, and $\mathbf{Z}_{an0}$ are known.

Since $\mathbf{I}_{a0} = 0$,

$$\mathbf{I}_{an} = \mathbf{I}_a = \mathbf{I}_{a1} + \mathbf{I}_{a2} \tag{56}$$

$$\mathbf{I}_{bn} = \mathbf{I}_b = \mathbf{I}_{a1}\underline{/-120^\circ} + \mathbf{I}_{a2}\underline{/120^\circ} \tag{57}$$

$$\mathbf{I}_{cn} = \mathbf{I}_c = \mathbf{I}_{a1}\underline{/120^\circ} + \mathbf{I}_{a2}\underline{/-120^\circ} \tag{58}$$

Even though $\mathbf{I}_{a0} = 0$, $\mathbf{V}_{an0}$ will, in general, possess a finite value since by equation (55) $\mathbf{V}_{an0} = \mathbf{I}_{a1}\mathbf{Z}_{an2} + \mathbf{I}_{a2}\mathbf{Z}_{an1}$.

Example 8. Let the line-to-line voltages and the phase impedances of the wye-connected load shown in Fig. 11 be as follows:

$$V_{ab} = 200 \quad V_{bc} = 141.4 \quad V_{ca} = 141.4 \text{ volts}$$

$$\mathbf{Z}_{an} = (6 + j0) \quad \mathbf{Z}_{bn} = (5.2 - j3) \quad \mathbf{Z}_{cn} = (0 + j12) \text{ ohms}$$

If the voltage sequence is *ab-bc-ca* and if $\mathbf{V}_{ab}$ is taken as reference,

$$\mathbf{V}_{ab} = 200\underline{/0^\circ} \quad \mathbf{V}_{bc} = 141.4\underline{/-135^\circ} \quad \mathbf{V}_{ca} = 141.4\underline{/135^\circ} \text{ volts}$$

Resolution of the above line-to-line voltages into symmetrical components yields

$$\mathbf{V}_{ab1} = \tfrac{1}{3}[200\underline{/0^\circ} + 141.4\underline{/-15^\circ} + 141.4\underline{/15^\circ}] = 157.8\underline{/0^\circ} \text{ volts}$$

$$\mathbf{V}_{ab2} = \tfrac{1}{3}[200\underline{/0^\circ} + 141.4\underline{/105^\circ} + 141.4\underline{/-105^\circ}] = 42.3\underline{/0^\circ} \text{ volts}$$

$$\mathbf{V}_{ab0} = \tfrac{1}{3}[200\underline{/0^\circ} + 141.4\underline{/-135^\circ} + 141.4\underline{/135^\circ}] = 0$$

From equations (25) and (26)

$$\mathbf{V}_{an1} = \frac{157.8\underline{/0^\circ}}{\sqrt{3}}\underline{/-30^\circ} = 91\underline{/-30^\circ} \text{ volts}$$

$$\mathbf{V}_{an2} = \frac{42.3\underline{/0^\circ}}{\sqrt{3}}\underline{/30^\circ} = 24.4\underline{/30^\circ} \text{ volts}$$

The symmetrical components of the phase impedances are

$$\mathbf{Z}_{an1} = 5.47\underline{/0^\circ}, \quad \mathbf{Z}_{an2} = (-3.2 - j3) = 4.38\underline{/-136.8^\circ}$$

and

$$\mathbf{Z}_{an0} = (3.73 + j3) = 4.78\underline{/38.8^\circ} \text{ ohms} \quad \text{(See example 7, page 510.)}$$

From equations (53) and (54)

$$\mathbf{I}_{a1} = \frac{\begin{vmatrix} \mathbf{V}_{an1} & \mathbf{Z}_{an2} \\ \mathbf{V}_{an2} & \mathbf{Z}_{an0} \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_{an0} & \mathbf{Z}_{an2} \\ \mathbf{Z}_{an1} & \mathbf{Z}_{an0} \end{vmatrix}} = \frac{\begin{vmatrix} 91\,\underline{/-30^\circ} & 4.38\,\underline{/-136.8^\circ} \\ 24.4\,\underline{/30^\circ} & 4.78\,\underline{/38.8^\circ} \end{vmatrix}}{\begin{vmatrix} 4.78\,\underline{/38.8^\circ} & 4.38\,\underline{/-136.8^\circ} \\ 5.47\,\underline{/0^\circ} & 4.78\,\underline{/38.8^\circ} \end{vmatrix}} = \frac{491\,\underline{/20.15^\circ}}{44.8\,\underline{/59.95^\circ}}$$

$$\mathbf{I}_{a1} = 10.95\,\underline{/-39.8^\circ} = 8.42 - j7.02 \text{ amperes}$$

$$\mathbf{I}_{a2} = \frac{\begin{vmatrix} \mathbf{Z}_{an0} & \mathbf{V}_{an1} \\ \mathbf{Z}_{an1} & \mathbf{V}_{an2} \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_{an0} & \mathbf{Z}_{an2} \\ \mathbf{Z}_{an1} & \mathbf{Z}_{an0} \end{vmatrix}} = \frac{\begin{vmatrix} 4.78\,\underline{/38.8^\circ} & 91\,\underline{/-30^\circ} \\ 5.47\,\underline{/0^\circ} & 24.4\,\underline{/30^\circ} \end{vmatrix}}{44.8\,\underline{/59.95^\circ}} = \frac{528\,\underline{/137.5^\circ}}{44.8\,\underline{/59.95^\circ}}$$

$$\mathbf{I}_{a2} = 11.8\,\underline{/77.45^\circ} = 2.56 + j11.5 \text{ amperes}$$

$$\mathbf{I}_{an} = \mathbf{I}_a = \mathbf{I}_{a1} + \mathbf{I}_{a2} = (8.42 - j7.02) + (2.56 + j11.5)$$
$$= 10.98 + j4.48 \text{ amperes}$$

In polar form

$$\mathbf{I}_a = 11.83\,\underline{/22.2^\circ} \text{ amperes}$$

After $\mathbf{I}_{a1}$ and $\mathbf{I}_{a2}$ have been evaluated, $\mathbf{I}_{b1}$, $\mathbf{I}_{b2}$, $\mathbf{I}_{c1}$, and $\mathbf{I}_{c2}$ follow directly, and hence $\mathbf{I}_{bn}$ and $\mathbf{I}_{cn}$ may be determined from the values of $\mathbf{I}_{a1}$ and $\mathbf{I}_{a2}$.

If the value of $\mathbf{V}_{an}$ is to be determined by the method of symmetrical components,

$$\mathbf{V}_{an} = \mathbf{V}_{an1} + \mathbf{V}_{an2} + \mathbf{V}_{an0}$$

where, from equation (55), $\mathbf{V}_{an0} = \mathbf{I}_{a1}\mathbf{Z}_{an2} + \mathbf{I}_{a2}\mathbf{Z}_{an1}$. In this case

$$\mathbf{V}_{an0} = (10.95\,\underline{/-39.8^\circ})\,(4.38\,\underline{/-136.8^\circ}) + (11.8\,\underline{/77.45^\circ})\,(5.47\,\underline{/0^\circ})$$
$$= -34 + j60.2 \text{ volts}$$

$$\mathbf{V}_{an} = (78.85 - j45.5) + (21.15 + j12.2) + (-34 + j60.2)$$
$$= 66 + j26.9 \text{ volts}$$

Problem 10. Study through the details of the above example and evaluate $\mathbf{I}_b$, $\mathbf{I}_c$, $\mathbf{V}_{bn}$, and $\mathbf{V}_{cn}$ by the method of symmetrical components. Check $\mathbf{V}_{an} - \mathbf{V}_{bn}$ against the given value of $\mathbf{V}_{ab} = 200\,\underline{/0^\circ}$ volts, recognizing that slide-rule calculations were employed in the evaluations of $\mathbf{I}_{a1}$, $\mathbf{I}_{a2}$, and $\mathbf{V}_{an}$.

Ans.: $\mathbf{I}_b = -21.53 - j7.31 = 22.7\,\underline{/-161.2^\circ}$ amperes.

Magnetic Coupling between Phases. If the three phases (including the line wires) possess mutual coupling of the kind shown in Fig. 12, the voltage drop in phase *an* due to its mutual coupling with phases *bn* and *cn* is:

$$\mathbf{V}_{am} = \mathbf{I}_b\mathbf{Z}_{ab} + \mathbf{I}_c\mathbf{Z}_{ac} \tag{59}$$

where subscript m designates the fact that this voltage drop excludes the self-impedance voltage drop, namely, $\mathbf{I}_a\mathbf{Z}_{aa}$. If simple magnetic coupling is involved,

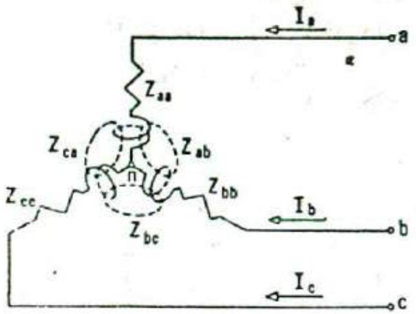


FIG. 12. Impedance in wye with mutual coupling between phases.

$$\mathbf{Z}_{ab} = jX_{ab} = \pm j\omega M_{ab} \quad (60)$$

$$\mathbf{Z}_{ac} = jX_{ac} = \pm j\omega M_{ac} \quad (61)$$

(See Chapter VII.)

The signs of the mutual reactances are defined by the assigned directions of current flow and the modes of winding the mutually coupled coils.

The impedance drop in phase an due to the self-impedance of that phase will be called $\mathbf{V}_{aa}$, and the total voltage drop in phase an then becomes:

$$\mathbf{V}_{an} = \mathbf{V}_{aa} + \mathbf{V}_{am} = \mathbf{I}_a\mathbf{Z}_{aa} + \mathbf{I}_b\mathbf{Z}_{ab} + \mathbf{I}_c\mathbf{Z}_{ac} \quad (62)$$

The problem of expressing the impedance drops of equation (62) in terms of symmetrical components will now be undertaken. Obviously $\mathbf{I}_a$, $\mathbf{I}_b$, and $\mathbf{I}_c$ may be expressed in terms of the symmetrical components of any one of these currents and $\mathbf{Z}_{aa}$ may be resolved into symmetrical components if the other self-impedances $\mathbf{Z}_{bb}$ and $\mathbf{Z}_{cc}$ are known. In this connection:

$$\mathbf{Z}_{aa1} = \tfrac{1}{3}(\mathbf{Z}_{aa} + \mathbf{Z}_{bb}\underline{/120^\circ} + \mathbf{Z}_{cc}\underline{/-120^\circ}), \quad \text{etc.}$$

if the other resolutions are effected in the a–b–c order.

The self-impedance voltage drop in phase an may be written in terms of symmetrical components in accordance with the sequence rule.

$$\mathbf{V}_{aa} = \mathbf{V}_{aa1} + \mathbf{V}_{aa2} + \mathbf{V}_{aa0} \quad (63)$$

where

$$\mathbf{V}_{aa1} = \mathbf{I}_{a1}\mathbf{Z}_{aa0} + \mathbf{I}_{a2}\mathbf{Z}_{aa2} + \mathbf{I}_{a0}\mathbf{Z}_{aa1} \quad (64)$$

$$\mathbf{V}_{aa2} = \mathbf{I}_{a1}\mathbf{Z}_{aa1} + \mathbf{I}_{a2}\mathbf{Z}_{aa0} + \mathbf{I}_{a0}\mathbf{Z}_{aa2} \quad (65)$$

$$\mathbf{V}_{aa0} = \mathbf{I}_{a1}\mathbf{Z}_{aa2} + \mathbf{I}_{a2}\mathbf{Z}_{aa1} + \mathbf{I}_{a0}\mathbf{Z}_{aa0} \quad (66)$$

There remains the problem of resolving the mutual impedances $\mathbf{Z}_{ab} = \mathbf{Z}_{ba}$, $\mathbf{Z}_{bc} = \mathbf{Z}_{cb}$, and $\mathbf{Z}_{ca} = \mathbf{Z}_{ac}$ into symmetrical components that can be advantageously associated with $\mathbf{I}_{a1}$, $\mathbf{I}_{a2}$, and $\mathbf{I}_{a0}$ to account for the presence of $\mathbf{I}_b\mathbf{Z}_{ab}$ and $\mathbf{I}_c\mathbf{Z}_{ac}$ in equation (62). At this stage of the development it is rather difficult to say which of the three mutual impedances should be considered as the base mutual impedance. It turns out that the symmetrical components of $\mathbf{Z}_{bc}$ can best be associated with $\mathbf{I}_{a1}$, $\mathbf{I}_{a2}$, and $\mathbf{I}_{a0}$. See equation (72).

Resolving the mutual impedances into symmetrical components with $\mathbf{Z}_{bc}$ as base yields

$$\mathbf{Z}_{bc1} = \tfrac{1}{3}(\mathbf{Z}_{bc} + \mathbf{Z}_{ca}\underline{/120^\circ} + \mathbf{Z}_{ab}\underline{/-120^\circ}) \tag{67}$$

$$\mathbf{Z}_{bc2} = \tfrac{1}{3}(\mathbf{Z}_{bc} + \mathbf{Z}_{ca}\underline{/-120^\circ} + \mathbf{Z}_{ab}\underline{/120^\circ}) \tag{68}$$

$$\mathbf{Z}_{bc0} = \tfrac{1}{3}(\mathbf{Z}_{bc} + \mathbf{Z}_{ca} + \mathbf{Z}_{ab}) = \mathbf{Z}_{ca0} = \mathbf{Z}_{ab0} \tag{69}$$

$$\mathbf{Z}_{ab1} = \mathbf{Z}_{bc1}\underline{/120^\circ} \quad \mathbf{Z}_{ca1} = \mathbf{Z}_{bc1}\underline{/-120^\circ} \tag{70}$$

$$\mathbf{Z}_{ab2} = \mathbf{Z}_{bc2}\underline{/-120^\circ} \quad \mathbf{Z}_{ca2} = \mathbf{Z}_{bc2}\underline{/120^\circ} \tag{71}$$

In terms of symmetrical components,

$$\begin{aligned}\mathbf{V}_{am} &= \mathbf{I}_b\mathbf{Z}_{ab} + \mathbf{I}_c\mathbf{Z}_{ac} \\ &= (\mathbf{I}_{a1}\underline{/-120^\circ} + \mathbf{I}_{a2}\underline{/120^\circ} + \mathbf{I}_{a0})(\mathbf{Z}_{bc1}\underline{/120^\circ} + \mathbf{Z}_{bc2}\underline{/-120^\circ} \\ &+ \mathbf{Z}_{bc0}) + (\mathbf{I}_{a1}\underline{/120^\circ} + \mathbf{I}_{a2}\underline{/-120^\circ} + \mathbf{I}_{a0})(\mathbf{Z}_{bc1}\underline{/-120^\circ} \\ &+ \mathbf{Z}_{bc2}\underline{/120^\circ} + \mathbf{Z}_{bc0})\end{aligned} \tag{72}$$

Eighteen component voltages appear if the multiplications indicated in equation (72) are carried out. These components may be grouped into positive-, negative-, and zero-sequence terms in accordance with the sequence rule. For example, the component voltages of the first order are

$$\begin{aligned}\mathbf{I}_{a1}\mathbf{Z}_{bc0}\underline{/-120^\circ} + \mathbf{I}_{a2}\mathbf{Z}_{bc2} + \mathbf{I}_{a0}\mathbf{Z}_{bc1}\underline{/120^\circ} + \mathbf{I}_{a1}\mathbf{Z}_{bc0}\underline{/120^\circ} + \mathbf{I}_{a2}\mathbf{Z}_{bc2} \\ + \mathbf{I}_{a0}\mathbf{Z}_{bc1}\underline{/-120^\circ} = \mathbf{V}_{am1}\end{aligned} \tag{73}$$

If the like terms in the above equations are further grouped, the following form results:

$$\mathbf{V}_{am1} = -\mathbf{I}_{a1}\mathbf{Z}_{bc0} + 2\mathbf{I}_{a2}\mathbf{Z}_{bc2} - \mathbf{I}_{a0}\mathbf{Z}_{bc1} \tag{74}$$

The negative-sequence or second-order terms of equation (72) may be combined to form

$$\mathbf{V}_{am2} = 2\mathbf{I}_{a1}\mathbf{Z}_{bc1} - \mathbf{I}_{a2}\mathbf{Z}_{bc0} - \mathbf{I}_{a0}\mathbf{Z}_{bc2} \tag{75}$$

The zero-sequence terms of equation (72) may be combined to form

$$\mathbf{V}_{am0} = -\mathbf{I}_{a1}\mathbf{Z}_{bc2} - \mathbf{I}_{a2}\mathbf{Z}_{bc1} + 2\mathbf{I}_{a0}\mathbf{Z}_{bc0} \tag{76}$$

Equations (74), (75), and (76) contain all eighteen component voltages represented in equation (72), and these equations may be combined systematically with equations (64), (65), and (66) to yield the positive-, negative-, and zero-sequence components of the complete phase voltage, namely, $\mathbf{V}_{an} = \mathbf{V}_{aa} + \mathbf{V}_{am}$.

Adding equations (64) and (74), equations (65) and (75), and equations (66) and (76) results in

$$\mathbf{V}_{an1} = \mathbf{I}_{a1}(\mathbf{Z}_{aa0} - \mathbf{Z}_{bc0}) + \mathbf{I}_{a2}(\mathbf{Z}_{aa2} + 2\mathbf{Z}_{bc2}) + \mathbf{I}_{a0}(\mathbf{Z}_{aa1} - \mathbf{Z}_{bc1}) \quad (77)$$

$$\mathbf{V}_{an2} = \mathbf{I}_{a1}(\mathbf{Z}_{aa1} + 2\mathbf{Z}_{bc1}) + \mathbf{I}_{a2}(\mathbf{Z}_{aa0} - \mathbf{Z}_{bc0}) + \mathbf{I}_{a0}(\mathbf{Z}_{aa2} - \mathbf{Z}_{bc2}) \quad (78)$$

$$\mathbf{V}_{an0} = \mathbf{I}_{a1}(\mathbf{Z}_{aa2} - \mathbf{Z}_{bc2}) + \mathbf{I}_{a2}(\mathbf{Z}_{aa1} - \mathbf{Z}_{bc1}) + \mathbf{I}_{a0}(\mathbf{Z}_{aa0} + 2\mathbf{Z}_{bc0}) \quad (79)$$

The above set of equations represents a powerful tool in the field of circuit analysis because with the aid of this set of equations any degree of unbalance and any degree of magnetic or capacitive coupling may be handled on a symmetrical-component basis. Equations (77), (78), and (79) are particularly useful in accounting for transmission line reactance voltage drops because these voltage drops result from mutual coupling between the line wires. These equations are also useful in accounting for the mutual impedance of the fourth wire of a four-wire, three-phase system.

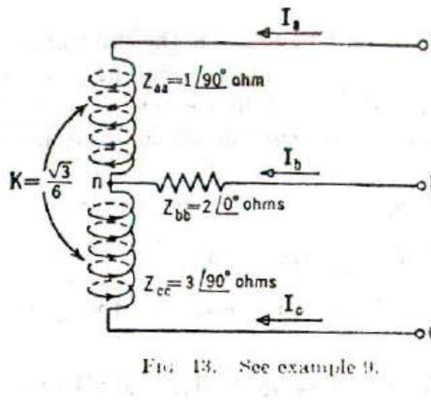


FIG. 13. See example 9.

Example 9. Let it be required to find the current $\mathbf{I}_a$ in Fig. 13 by the method of symmetrical components, if $\mathbf{V}_{ab} = \mathbf{V}_{bc} = \mathbf{V}_{ca} = 100$ volts and the sequence of these voltages is *ab-bc-ca*. From previous considerations, it is plain that

$$\mathbf{V}_{an1} = \frac{\mathbf{V}_{ab1}}{\sqrt{3}}\underline{/-30°} \quad \text{and} \quad \mathbf{V}_{an2} = 0$$

If $\mathbf{V}_{ab}$ is chosen as the reference vector,

$$\mathbf{V}_{an1} = \frac{100}{\sqrt{3}}\underline{/-30°} = (50 - j28.9) \text{ volts}$$

The self-impedances $[\mathbf{Z}_{aa} = (0 + j1)$, $\mathbf{Z}_{bb} = (2 + j0)$, and $\mathbf{Z}_{cc} = (0 + j3)]$ may be resolved into symmetrical components in the usual manner.

$$\mathbf{Z}_{aa1} = \tfrac{1}{3}(\mathbf{Z}_{aa} + \mathbf{Z}_{bb}\underline{/120^\circ} + \mathbf{Z}_{cc}\underline{/-120^\circ}) = (0.533 + j0.411) \text{ ohms}$$

$$\mathbf{Z}_{aa2} = \tfrac{1}{3}(\mathbf{Z}_{aa} + \mathbf{Z}_{bb}\underline{/-120^\circ} + \mathbf{Z}_{cc}\underline{/120^\circ}) = (-1.199 - j0.744) \text{ ohms}$$

$$\mathbf{Z}_{aa0} = \tfrac{1}{3}(\mathbf{Z}_{aa} + \mathbf{Z}_{bb} + \mathbf{Z}_{cc}) = (0.667 + j1.33) \text{ ohms}$$

As indicated on the circuit diagram (Fig. 13), the coefficient of coupling between the two inductance coils is $\sqrt{3}/6$. This coefficient is interpreted to mean that

$$\omega M_{ca} = \omega M_{ac} = \frac{\sqrt{3}}{6}\sqrt{\omega L_{aa} \times \omega L_{cc}} = \frac{\sqrt{3}}{6}\sqrt{1 \times 3} = 0.5 \text{ ohm}$$

If the modes of winding and the space positions of the coils are as represented in Fig. 13,

$$\mathbf{Z}_{ca} = (0 - j\omega M_{ca}) = (0 - j0.5) \text{ ohm}$$

$\mathbf{Z}_{ab}$ and $\mathbf{Z}_{bc}$ are both zero because no coupling exists between phases a and b or between phases b and c under the specified conditions.

In accordance with equations (67), (68), and (69),

$$\mathbf{Z}_{bc1} = \tfrac{1}{3}(0.5\underline{/-90^\circ + 120^\circ}) = 0.144 + j0.083 \text{ ohm}$$

$$\mathbf{Z}_{bc2} = \tfrac{1}{3}(0.5\underline{/-90^\circ - 120^\circ}) = -0.144 + j0.083 \text{ ohm}$$

$$\mathbf{Z}_{bc0} = \tfrac{1}{3}(0.5\underline{/-90^\circ}) = 0 - j0.167 \text{ ohm}$$

Since $\mathbf{I}_{a0}$ is equal to zero, it follows from equations (77) and (78) that

$$\mathbf{V}_{an1} = 50 - j28.9 = \mathbf{I}_{a1}(0.667 + j1.50) + \mathbf{I}_{a2}(-1.487 - j0.578)$$

$$\mathbf{V}_{an2} = 0 \qquad = \mathbf{I}_{a1}(0.821 + j0.578) + \mathbf{I}_{a2}(0.667 + j1.50)$$

The above equations may be solved simultaneously for $\mathbf{I}_{a1}$ and $\mathbf{I}_{a2}$

$$\mathbf{I}_{a1} = \frac{\begin{vmatrix} (50 - j28.9) & (-1.487 - j0.578) \\ 0 & (0.667 + j1.50) \end{vmatrix}}{\begin{vmatrix} (0.667 + j1.50) & (-1.487 - j0.578) \\ (0.821 + j0.578) & (0.667 + j1.50) \end{vmatrix}} = \frac{76.6 + j55.7}{-0.918 + j3.33}$$

$$= (9.63 - j25.6) \text{ amperes}$$

$$\mathbf{I}_{a2} = \frac{\begin{vmatrix} (0.667 + j1.50) & (50 - j28.9) \\ (0.821 + j0.578) & 0 \end{vmatrix}}{(-0.918 + j3.33)} = \frac{-57.7 - j5.1}{-0.918 + j3.33}$$

$$= (3.01 + j16.46) \text{ amperes}$$

$$\mathbf{I}_a = \mathbf{I}_{a1} + \mathbf{I}_{a2} = (9.63 - j25.6) + (3.01 + j16.46)$$
$$= (12.64 - j9.14) = 15.6\underline{/-35.85^\circ} \text{ amperes}$$

PROBLEMS

11. The line-to-neutral voltages of a four-wire, three-phase system are represented by the following vector expressions: $\mathbf{V}_a = 200\underline{/0^\circ}$, $\mathbf{V}_b = 100\underline{/-75^\circ}$, $\mathbf{V}_c = 150\underline{/-150^\circ}$.

Find the positive-, negative-, and zero-sequence components of the above voltages, and check the results obtained by graphical additions of the symmetrical components.

12. The three line currents of a four-wire wye load (like that shown in Fig. 7, page 506) directed to the common junction are $\mathbf{I}_{an} = 15 - j20$, $\mathbf{I}_{bn} = -8 + j15$, and $\mathbf{I}_{cn} = 8 - j25$ amperes. Find $\mathbf{I}_{an1}$, $\mathbf{I}_{an2}$, and $\mathbf{I}_{an0}$ assuming these currents were calculated from a voltage system where the actual voltage sequence was a-b-c.

13. Voltages to neutral on a four-wire Y-load are maintained at $\mathbf{V}_{na} = 100\underline{/0^\circ}$, $\mathbf{V}_{nb} = 100\underline{/-120^\circ}$, and $\mathbf{V}_{nc} = 100\underline{/120^\circ}$ volts. Impedances are $\mathbf{Z}_{na} = 10\underline{/0^\circ}$, $\mathbf{Z}_{nb} = 10\underline{/90^\circ}$, and $\mathbf{Z}_{nc} = 10\underline{/-90^\circ}$.

(*a*) Find the positive-, negative-, and zero-sequence line currents, if the positive-sequence voltage system is a-b-c.

(*b*) Find the power due to each of the sequences, positive, negative, and zero.

(*c*) Should the phasor which lags the base phasor be rotated forward 120° to obtain the positive-sequence current? Why?

14. (*a*) Three-phase voltages are supplied by lines a, b, and c. If a short circuit is placed from line a to line b, find positive-, negative-, and zero-sequence components of the line voltages at the short circuit in terms of a line voltage of V.

(*b*) If the short-circuit current is I find the symmetrical components of the current at the short circuit.

15. The three wye-connected impedances through which the currents of Problem 12 flow are, respectively,

$$\mathbf{Z}_{an} = 20 - j20 \text{ ohms}$$

$$\mathbf{Z}_{bn} = 30 + j10 \text{ ohms}$$

$$\mathbf{Z}_{cn} = 10 - j20 \text{ ohms}$$

Find $\mathbf{Z}_{an1}$, $\mathbf{Z}_{an2}$, and $\mathbf{Z}_{an0}$.

16. Employing the symmetrical components $\mathbf{I}_{an1}$, $\mathbf{I}_{an2}$, $\mathbf{I}_{an0}$, $\mathbf{Z}_{an1}$, $\mathbf{Z}_{an2}$, and $\mathbf{Z}_{an0}$ determined in Problems 12 and 15, evaluate $\mathbf{V}_{an} = \mathbf{I}_{an}\mathbf{Z}_{an}$ in terms of symmetrical components and check the result against the known value of $\mathbf{I}_{an}\mathbf{Z}_{an}$.

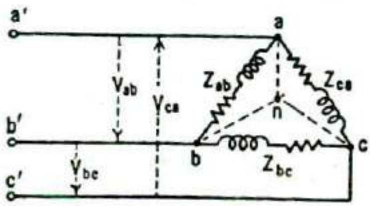


FIG. 14. See Problem 17.

17. Assume that the three-phase line voltages shown in Fig. 14 are

$$V_{bc} = 200\underline{/0^\circ}, \quad V_{ca} = 100\underline{/120^\circ}, \quad V_{ab} = 173.2\underline{/210^\circ}$$

(*a*) Find $\mathbf{V}_{bc1}$, $\mathbf{V}_{bc2}$, and $\mathbf{V}_{bc0}$.

(*b*) Find $\mathbf{V}_{nc1}$, $\mathbf{V}_{nc2}$, and $\mathbf{V}_{nc0}$. Employ phase sequence bc, ab, ca.

18. The three line-to-line voltages shown in Fig. 14 are

$$V_{ab} = 100, \quad V_{bc} = 150, \quad V_{ca} = 175 \text{ volts}$$

Sequence ab-bc-ca.

(*a*) Find $\mathbf{V}_{ab1}$, $\mathbf{V}_{ab2}$, and $\mathbf{V}_{ab0}$.

(b) Find $\mathbf{V}_{an1}$ and $\mathbf{V}_{an2}$, the equivalent wye voltages of the delta load shown in Fig. 14.

(c) Explain how the line currents may be determined from $\mathbf{V}_{an1}$, $\mathbf{V}_{an2}$, and the delta load impedances.

19. The line-to-line voltages of a three-wire, three-phase system are $V_{ab} = 200$ volts, $V_{bc} = 141.4$ volts, and $V_{ca} = 141.4$ volts. The sequence of the voltages is *ab–ca–bc*. A wye-connected set of static impedances ($\mathbf{Z}_{an} = 20\underline{/0^\circ}$ ohms, $\mathbf{Z}_{bn} = 30\underline{/60^\circ}$ ohms, and $\mathbf{Z}_{cn} = 20\underline{/0^\circ}$ ohms) is connected to the three lines a, b, and c in the order indicated by the subscripts. Find the line currents $\mathbf{I}_{an}$, $\mathbf{I}_{bn}$, and $\mathbf{I}_{cn}$ by the method of symmetrical components.

20. Solve for $\mathbf{I}_a$ in Fig. 13 by the method of symmetrical components if $V_{ab} = 200$, $V_{bc} = 173.2$, and $V_{ca} = 100$ volts. The sequence of the line-to-line voltages is *ab–bc–ca*.

chapter XIII Power System Short-Circuit Calculations

Power systems are subject to three kinds of short circuits. First, all three lines of a three-phase system may become electrically connected. This is known as a three-phase short circuit. Second, only two lines may be electrically connected, which constitutes a line-to-line short circuit. Third, a single wire may be electrically connected to ground. This is called a line-to-ground short circuit. Although the electrical connections referred to may be of varying impedance, short-circuit calculations are based upon zero impedance at the point of short circuit. In other words, a perfect short circuit is assumed. Short circuits on systems are usually called faults.

A distribution system should be protected in such a way that a faulty or short-circuited section will be isolated from the rest of the system. This is accomplished through the use of relays which operate circuit breakers. To protect a system, relays are set to trip in a certain length of time after the fault occurs. By varying the amount of time required for a relay to operate, certain selective operation of circuit breakers may be obtained. After proper adjustments are made, this selective operation causes only the faulty section of the line to be isolated. In order to determine the proper time settings of these relays and in order to determine the sizes of circuit breakers necessary, the magnitudes of the short-circuit currents that these devices are to handle must be known. In general, different values of short-circuit current occur for the three-phase symmetrical, line-to-line, and line-to-ground short circuits. Usually the three-phase symmetrical short circuit yields the lowest value of short-circuit current (except when the system has practically no grounds). Hence relay settings are usually based upon three-phase symmetrical faults because it is desirable to protect a system for the minimum fault current. If the relay trips a circuit breaker for minimum fault current, it will obviously open the breaker for the highest fault current, but the converse is not true. Since a breaker must interrupt the largest short-circuit current that can possibly exist, the size of a circuit breaker is determined by the largest possible fault current. The greatest current usually occurs for either the line-to-line or line-to-ground fault. Obviously, the determination of short-circuit currents in power systems is required if the proper settings of relays and proper selection of circuit-breaker sizes are to be made.

Bases for Short-Circuit Calculations. A distribution network consists of many lines which may be connected by transformers and which, in general, operate at different nominal voltages. To establish a simple network for purposes of calculation, the impedances of all lines and transformers are expressed in ohms referred to a common voltage base or in percentage referred to a common kilovolt-ampere base. The former generally appears simpler to the beginner, but the latter method is actually the better and is to be preferred. The two methods yield identical results.

Method Using Ohms on a Kilovolt Base. In general, various branches of an electrical distribution system operate at different potentials. In representing such a system by a system of impedances, it is desirable to employ a scheme which permits the combination of the different impedances so that the network can be represented by a single impedance between the source and the fault. This requires the determination of an impedance, Z_2, which may be used with an arbitrarily selected voltage, V_2, such that the same kva will be taken as when the actual impedance, Z_1, is used with the actual voltage V_1. Stated algebraically,

$$\left(\frac{V_2}{Z_2}\right) V_2 = \frac{V_1}{Z_1} V_1$$

or

$$Z_2 = Z_1 \left(\frac{V_2}{V_1}\right)^2 \tag{1}$$

Equation (1) shows that the original impedance must be multiplied by the square of the ratio of voltage to be used to the nominal operating voltage for the impedance. To illustrate, suppose that 1000 volts are impressed on an impedance of 100 ohms and that it is desired to find the current and kva taken.

$$I_1 = \frac{1000}{100} = 10 \text{ amperes}$$

$$va = 1000 \times 10 = 10{,}000$$

Now assume that it is desired to work the same problem when all values are referred to a 2000-volt base. Then

$$Z_2 = \left(\frac{2000}{1000}\right)^2 \times 100 = 400 \text{ ohms}$$

$$I_2 = \frac{2000}{400} = 5 \text{ amperes}$$

$$va = 2000 \times 5 = 10{,}000$$

The foregoing example shows that there is no difference between calculating the volt-amperes for the actual voltage and impedance and for some other selected voltage and an equivalent impedance found by multiplying the original impedance by the square of the ratio of the selected voltage to the original. The current on the actual voltage base is then found by multiplying the result calculated on the selected voltage base by the ratio of the voltages. Thus the actual current at 1000 volts is:

$$I_1 = 5 \times \frac{2000}{1000} = 10 \text{ amperes}$$

This procedure is evident from the following relationship.

$$V_1 I_1 = V_2 I_2$$

or

$$I_1 = \frac{V_2}{V_1} I_2$$

Example 1. Calculate the short-circuit current for the system shown in Fig. 1. A 10 to 1 ratio wye-wye connected transformer bank is represented at A. A transformer has resistance and leakage reactance which may be referred to either side as

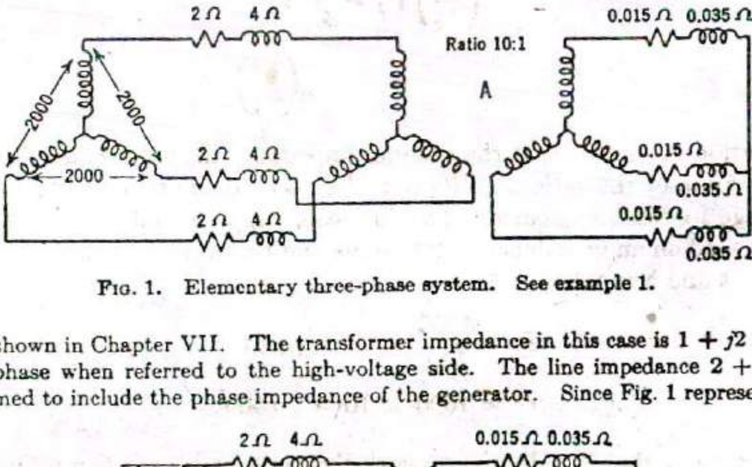


FIG. 1. Elementary three-phase system. See example 1.

was shown in Chapter VII. The transformer impedance in this case is $1 + j2$ ohms per phase when referred to the high-voltage side. The line impedance $2 + j4$ is assumed to include the phase impedance of the generator. Since Fig. 1 represents a

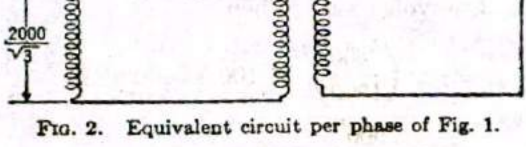


FIG. 2. Equivalent circuit per phase of Fig. 1.

balanced circuit, all calculations will be made per phase. The equivalent circuit for one phase to neutral is shown in Fig. 2, and the corresponding one-line diagram is shown in Fig. 3. A short line at the generator neutral is used to represent the neutral bus, and a cross at the end of the line denotes the point of short circuit. The per phase voltage is impressed between the neutral bus and the point X. The trans-

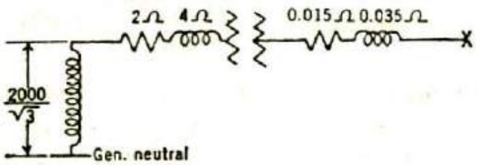


FIG. 3. One-line diagram of Fig. 2 and Fig. 1.

former impedance causes a drop in voltage from its primary to its secondary side and therefore acts like a series impedance. Transferring the impedance of the secondary line to its equivalent value on a 2000-volt base (the primary line-to-line voltage), or

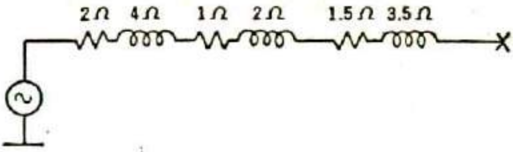


FIG. 4. Reduction of Fig. 3 to a series of impedances.

to a $2000/\sqrt{3}$ volts to neutral base which is the same, and inserting the transformer equivalent impedance, reduces the one-line diagram to the equivalent circuit shown in Fig. 4. Then

$$\mathbf{I} = \frac{2000/\sqrt{3}}{(2 + j4) + (1 + j2) + (1.5 + j3.5)}$$
$$= 47 - j99.2 \quad \text{or} \quad 109.8 \text{ amperes}$$

The actual current at the fault is found by referring the current to the voltage of the faulty line.

$$\text{Fault current} = 109.8 \times 10 = 1098 \text{ amperes}$$

Problem 1. A wye-connected generator rated at 2200 terminal volts has 0.2 ohm resistance and 2 ohms reactance per phase. The generator is connected by lines each having an impedance of $2.06\ \underline{/29.05°}$ ohms to a wye-wye transformer bank. Each transformer has a total equivalent impedance referred to the high side of $100\ \underline{/60°}$ ohms, and the transformer bank is connected to a load through lines each of which has a resistance of 50 ohms and an inductive reactance of 100 ohms. If the ratio of transformation is 6 and the low-voltage side is connected to the generator lines, calculate the actual fault current for a three-phase symmetrical short circuit at the load.

Ans.: 22.3 amperes.

Percentage Method. In general, short-circuit calculations are made through the use of percentage resistances and reactances. Percentage reactance is defined as the percentage of the rated voltage which is consumed in the reactance drop when rated current flows. Expressed algebraically,

$$\% \text{ reactance} = \frac{I_{\text{rated}} \times \text{ohms}}{V_{\text{rated}}} \times 100 \tag{2}$$

Percentage resistance is similarly defined. Percentage values are manipulated like ohmic values. When percentage values are employed, a common kva base is used instead of a common voltage base as employed in the ohmic method. The derivation of the method for determining the percentage reactance on different kva bases follows. Three-phase will be assumed since it is the most common.

Let p be the percentage reactance based on a particular 3-phase kva.

kv = the voltage between the three-phase lines in kilovolts.
X = the reactance in ohms.

Then

$$IX = \frac{X \text{ kva } 10^{-3}}{\sqrt{3} \text{ kv}} \text{ kilovolts}$$

$$p = \frac{100IX}{\text{kv}/\sqrt{3}} = \frac{\dfrac{100X \text{ kva } 10^{-3}}{\sqrt{3} \text{ kv}}}{\text{kv}/\sqrt{3}} = \frac{X \text{ kva}}{\overline{\text{kv}}^2\, 10} \tag{3}$$

Equation (3) shows that percentage reactance varies directly with the kva when the rest of the factors remain constant. A similar relation holds true for percentage resistance. Although equation (3) was derived on the assumption of three-phase it is equally applicable to single-phase.

Example 2. By way of illustrating the use of percentage resistance and reactance, example 1, which was worked on the ohmic basis, will be reworked employing the percentage method. Ordinarily, much of the data on a system is expressed in percentage and no transformation from ohmic to percentage impedance is necessary. Since the parameters in the previous example are given in ohms, the transformation to percentage will be shown. Also, to illustrate changing to a common base, the percentage impedance of the lines on the generator side of the transformer and the transformer will be found on a 10,000-kva base, while that on the secondary side will be found on a 100-kva base.

For the lines on generator side of transformer:

$$\text{Base current } I = \frac{10{,}000{,}000}{\sqrt{3}\, 2000} = 2885 \text{ amperes}$$

$$\% \, IX \text{ drop due to base current} = 100 \times \frac{2885 \times 4}{2000/\sqrt{3}} = 1000, \text{ or } 1000\% \text{ reactance}$$

$$\% \, IR \text{ drop due to base current} = 100 \times \frac{2885 \times 2}{2000/\sqrt{3}} = 500, \text{ or } 500\% \text{ resistance}$$

Transformer impedance on 10,000-kva base:

$$\% \, IR \text{ drop} = \frac{100 \times 2885 \times 1}{2000/\sqrt{3}} = 250$$

$$\% \, IX \text{ drop} = \frac{100 \times 2885 \times 2}{2000/\sqrt{3}} = 500$$

The line impedance on the secondary side of the transformer based on 100 kva is determined as follows:

$$\text{Nominal rated voltage on secondary } \frac{2000}{10} = 200 \text{ volts}$$

$$\text{Base current } I = \frac{100{,}000}{\sqrt{3}\,200} = 288.5 \text{ amperes}$$

$$\% \, IX \text{ drop} = \frac{100 \times 288.5 \times 0.035}{200/\sqrt{3}} = 8.75$$

$$\% \, IR \text{ drop} = \frac{100 \times 288.5 \times 0.015}{200/\sqrt{3}} = 3.75$$

The circuit of Fig. 1 with parameters expressed in percentage is shown in Fig. 5. It is common to receive data on distribution networks expressed like those in Fig. 5.

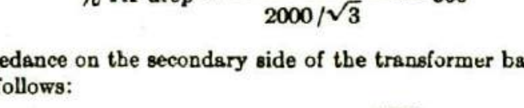


FIG. 5. One-line diagram of Fig. 1 with parameters expressed on a percentage basis.

Before simplifying, a common kva base is chosen to which all constants are referred. This base may be any arbitrarily selected. A 1000-kva base is chosen for this example because it yields convenient numerical quantities.

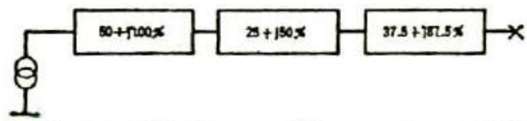


FIG. 6. Impedances of Fig. 5 expressed in per cent on a 1000-kva base.

It was shown that percentage reactance and resistance, and hence impedance, vary directly with the kva base. Employing this principle yields the circuit shown in Fig. 6. The combined impedance to the fault is

$$50 + j100 + 25 + j50 + 37.5 + j87.5 = 112.5 + j237.5\%$$

or

$$\sqrt{112.5^2 + 237.5^2} = 263\%$$

This result indicates that 263 per cent of the rated voltage is necessary to cause 1000 kva to be delivered by the generator. Since only rated voltage, or 100 per cent voltage, is available, the total short-circuit kva must be $\frac{100}{263} \times 1000 = 380.5$ kva. If the fault current is desired at the actual voltage of the faulty line, namely, 200 volts, it is found as follows:

$$I_{\text{fault}} = \frac{380.5 \times 1000}{\sqrt{3} \times 200} = 1098 \text{ amperes}$$

Problem 2. Rework Problem 1, page 525, employing percentage values.

Per Unit Method. A study of the percentage method will show that problems could be worked by using percentage values expressed in hundredths, which would be equivalent to moving the decimal point two places to the left in the calculations shown in example 2. In other words, quantities could be expressed on a per unit basis instead of on a per hundred basis as in the percentage method. Thus instead of a reactance of 15 per cent a value of 0.15 would be used. A little experience with both schemes shows relatively little difference in the methods. Both methods are used according to personal preferences.

Accuracy of Short-Circuit Calculations. In general, extreme accuracy in the determination of short-circuit currents in distribution systems is not required. Because the resistance of most synchronous apparatus is low compared to the reactance, the final impedance to the fault in many cases is about the same as the reactance. For this reason, and because of the resulting simplification of the calculations, only reactances are generally used. An exception to these statements occurs when stability studies of systems are made. It then becomes necessary to consider phase angles, and then both resistance and reactance must be considered.

When several sources of current are in parallel, it is customary to assume that all the generated voltages are in phase and equal in magnitude at the time of short circuit. Load currents on the system are neglected. All synchronous apparatus like generators, synchronous motors, and rotary converters are considered as sources of short-circuit current. The kinetic energy of these rotating machines causes them to act like generators during the first few cycles of short circuit. In spite of all these approximations, tests have shown that calculations based upon these assumptions are usually within about 5 per cent of the correct values. From 5 to 10 per cent error in the values of short-circuit currents is usually tolerable in the determination of circuit-breaker sizes and relay settings.

Three-Phase Short Circuits. Three-phase short-circuit currents are determined by means of the same principles employed in the analysis of balanced three-phase systems. The method is best shown by an example.

Example 3. It is desired to find the short-circuit current for the system shown in Fig. 7. The data for the system are shown in Table I. A symmetrical three-phase

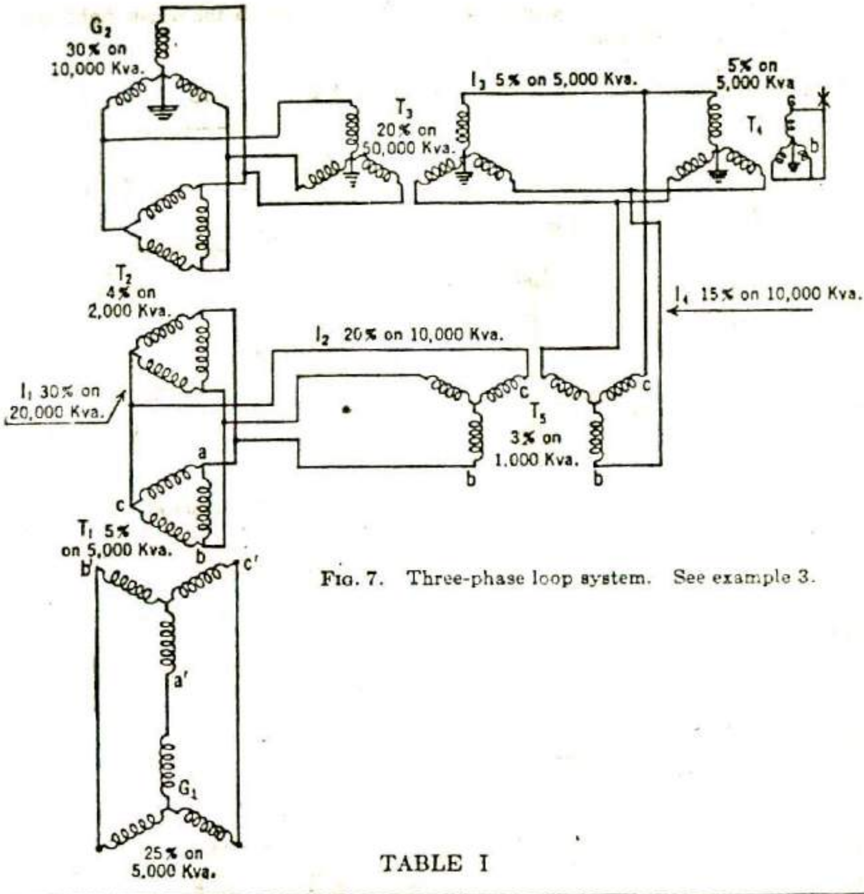


FIG. 7. Three-phase loop system. See example 3.

TABLE I

Apparatus	Rating kva	% Reactance	% Reactance based on kva
Generator 1	5,000	25	5,000
Generator 2	10,000	30	10,000
Transformer 1	4,000	5	5,000
Transformer 2	2,000	4	2,000
Transformer 3	5,000	20	50,000
Transformer 4	5,000	5	5,000
Transformer 5	1,000	3	1,000
Line 1		30	20,000
Line 2		20	10,000
Line 3		5	5,000
Line 4		15	10,000

short circuit is assumed at the point denoted by the cross in the upper right-hand corner of the circuit diagram.

The following represents a satisfactory procedure.

1. A one-line diagram of the system as shown in Fig. 8 is drawn.

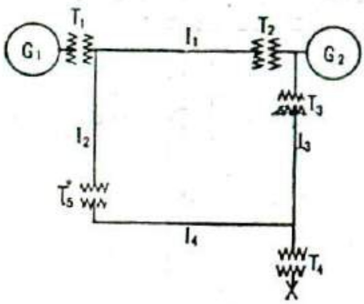


FIG. 8. One-line diagram of Fig. 7.

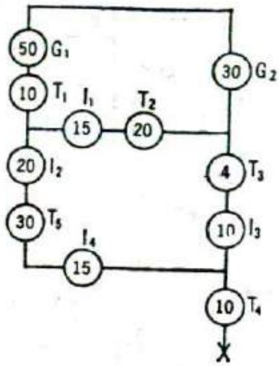


FIG. 9. One-line diagram of Fig. 8 where G_1 and G_2 are connected to a common neutral bus and all reactances are shown on a 10,000-kva base.

2. A common kva base upon which all reactances are based is chosen. Any convenient base may be used; here a 10,000-kva base is selected.

3. A one-line diagram is drawn in which all sources of current are connected to a so-called neutral bus. Circles represent reactances, and the value of the various reactances referred to the selected common kva base is placed in the circle as shown in Fig. 9.

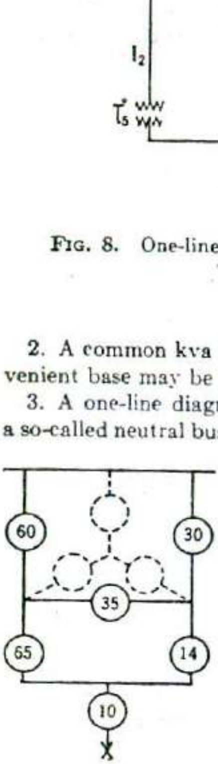


FIG. 10.

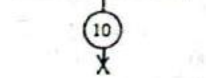


FIG. 11.

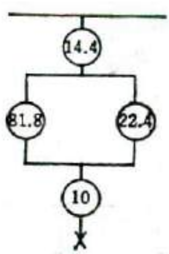


FIG. 12.

FIG. 13.

4. Reactances are combined according to laws of series or parallel circuits, and substitution of wyes for deltas or the reverse are made so as to obtain a single reactance between the neutral bus and the point of short circuit. These steps are illustrated in the successive Figs. 10, 11, 12, 13, and 14. The dotted lines and circles indicate the circuit arrangement to be employed in replacing an existing circuit arrangement. The resultant reactance to the fault based on

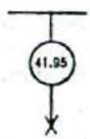


FIG. 14. Resultant percentage of reactance on a 10,000-kva base of Fig. 7 to the point of short circuit.

10,000 kva is 41.95 per cent.

$$\text{Short-circuit kva} = \frac{100}{41.95} \times 10{,}000 = 23{,}800$$

If the nominal voltage of the line at the short circuit is 12,000 volts, the current at the fault is

$$\frac{23{,}800 \times 1000}{\sqrt{3}\ 12{,}000} = 1144 \text{ amperes}$$

The distribution of currents throughout the network may be determined by retracing the steps and using the percentage values just exactly as though they were ohmic quantities. For example, the currents in the divided circuit of Fig. 12 may be determined as follows. To indicate the branch under consideration, a subscript which is the same as the branch impedance is used.

$$V_{17.55} = 17.55 \times 1144 \text{ volts}^1$$

$$I_{81.8} = \frac{17.55 \times 1144}{81.8} = 246 \text{ amperes}$$

$$I_{22.4} = \frac{17.55 \times 1144}{22.4} = 898 \text{ amperes}$$

If the nominal voltage of any line differs from the 12,000-volt base used above, the actual current is determined by multiplying the current calculated on the 12,000-volt base by the ratio of 12,000 to the nominal voltage for the line in question.

Problem 3. Find the actual currents delivered by generators G_1 and G_2.
Ans.: $I_{G1} = 344$ amperes, $I_{G2} = 800$ amperes.

Line-to-Line Short Circuits. Line-to-line short-circuit currents may be determined in accordance with the principles set forth in Chapter IX, or they may be calculated by the method of symmetrical components. The method of symmetrical components possesses the advantage of accounting in a measure for the change in the impedance of synchronous machines when the loading is changed from balanced three-phase to single-phase line-to-line loading. Furthermore the method of symmetrical components reduces the calculations to the solution of balanced three-phase systems. Certain modifications of the network parameters are necessary in employing the method of symmetrical components, and in addition the combination of the balanced systems solutions must be properly made to obtain the final result.

The method of symmetrical components for effecting a solution of the line-to-line short-circuit problem will be developed with reference to Fig. 15. The fundamental objective is to determine the positive- and negative-sequence components of current in terms of (the known quanti-

[1] This number of volts is only proportional to the actual voltage and is used merely as a convenient means to determine the distribution of currents.

ties) the induced voltage and impedance. The following symbols are used:

E, generated voltage per phase
n', electrical neutral at the point of short circuit
V_1, positive-sequence voltage to neutral at the short circuit
V_2, negative-sequence voltage to neutral at the short circuit
V_0, zero-sequence voltage to neutral at the short circuit
Z_1, impedance to positive sequence
Z_2, impedance to negative sequence

According to Kirchhoff's voltage law, the positive-sequence voltage to neutral at the short circuit must be the positive-sequence generated voltage minus the positive-sequence drop. A similar relation obtains

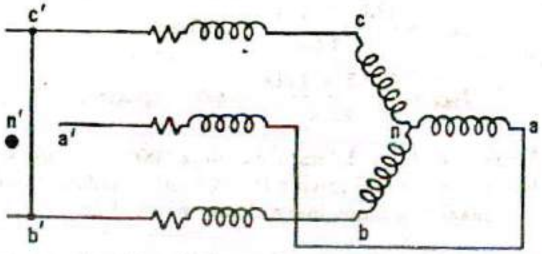


FIG. 15. Line-to-line short circuit on a three-phase system.

for the negative sequence. Since all generated voltages at the generator are assumed to be balanced, the positive-sequence generated voltage is E. The negative-sequence generated voltage is zero. Hence for any particular phase

$$\mathbf{V}_1 = \mathbf{E} - \mathbf{I}_1\mathbf{Z}_1 \tag{4}$$

$$\mathbf{V}_2 = 0 - \mathbf{I}_2\mathbf{Z}_2 \tag{5}$$

Since there is no ground return or fourth wire in Fig. 15, there can be no zero-sequence current in this system. At the short circuit

$$\mathbf{V}_{b'c'} = \mathbf{V}_{b'n'} + \mathbf{V}_{n'c'} = 0 \tag{6}$$

or

$$\mathbf{V}_{n'b'} = \mathbf{V}_{n'c'} \tag{7}$$

The three voltages to neutral at the short circuit in terms of their symmetrical components are (assuming *ab-bc-ca* sequence)

$$\mathbf{V}_{n'a'} = \mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_0 \tag{8}$$

$$\mathbf{V}_{n'b'} = \mathbf{V}_1\underline{/-120^\circ} + \mathbf{V}_2\underline{/120^\circ} + \mathbf{V}_0 \tag{9}$$

$$\mathbf{V}_{n'c'} = \mathbf{V}_1\underline{/120^\circ} + \mathbf{V}_2\underline{/-120^\circ} + \mathbf{V}_0 \tag{10}$$

Substituting equations (9) and (10) in equation (7),

$$\mathbf{V}_1\underline{/-120^\circ} + \mathbf{V}_2\underline{/120^\circ} + \mathbf{V}_0 = \mathbf{V}_1\underline{/120^\circ} + \mathbf{V}_2\underline{/-120^\circ} + \mathbf{V}_0$$

$$\mathbf{V}_1(\underline{/-120^\circ} - \underline{/120^\circ}) = \mathbf{V}_2(\underline{/-120^\circ} - \underline{/120^\circ})$$

or

$$\mathbf{V}_1 = \mathbf{V}_2 \tag{11}$$

Equation (11) shows that equations (4) and (5) are equal. Therefore

$$\mathbf{E} - \mathbf{I}_1\mathbf{Z}_1 = -\mathbf{I}_2\mathbf{Z}_2 \tag{12}$$

If $\mathbf{I}_2$ can be expressed in terms of $\mathbf{I}_1$, the sequence components of currents can be found. Since no zero-sequence current can exist in the circuit of Fig. 15, $\mathbf{I}_1$ and $\mathbf{I}_2$ are found as shown below.

$$\mathbf{I}_{na} = \mathbf{I}_1 + \mathbf{I}_2 = 0 \quad \text{(Line } na \text{ is open.)} \tag{13}$$

$$\mathbf{I}_{nb} = \mathbf{I}_1\underline{/-120^\circ} + \mathbf{I}_2\underline{/120^\circ} \tag{14}$$

$$\mathbf{I}_{nc} = \mathbf{I}_1\underline{/120^\circ} + \mathbf{I}_2\underline{/-120^\circ} \tag{15}$$

Because of the short circuit,

$$\mathbf{I}_{nb} = \mathbf{I}_{cn} = -\mathbf{I}_{nc} \tag{16}$$

Substituting equations (14) and (15) in equation (16),

$$\mathbf{I}_1\underline{/-120^\circ} + \mathbf{I}_2\underline{/120^\circ} = -\mathbf{I}_1\underline{/120^\circ} - \mathbf{I}_2\underline{/-120^\circ}$$

$$\mathbf{I}_1(\underline{/-120^\circ} + \underline{/120^\circ}) + \mathbf{I}_2(\underline{/-120^\circ} + \underline{/120^\circ}) = 0$$

$$\mathbf{I}_1 = -\mathbf{I}_2 \tag{17}$$

Substituting equation (17) in equation (12) yields

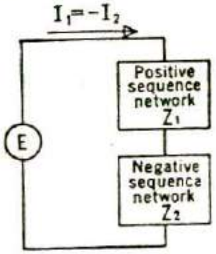


FIG. 16. Arrangement of sequence networks for determination of positive- and negative-sequence currents for a line-to-line short circuit.

$$\mathbf{E} - \mathbf{I}_1\mathbf{Z}_1 - \mathbf{I}_1\mathbf{Z}_2 = 0$$

$$\mathbf{I}_1 = \frac{\mathbf{E}}{\mathbf{Z}_1 + \mathbf{Z}_2} \tag{18}$$

Equations 17 and 18 show that the arrangement illustrated in Fig. 16 may be used to calculate the positive- and negative-sequence currents at the fault for a line-to-line short circuit.

Impedances to Positive and Negative Sequence. Before equation (18) can be applied, the values of the impedances to positive and negative sequence must be known. The impedance to positive sequence is the impedance offered to a system whose voltages a, b, and c, respectively, lag the one preceding it by 120°. The impedance to negative sequence is the impedance offered to a system whose voltages *a*, *b*, and *c*, respectively, lead the one preceding it by 120°. It should be apparent, and it can be demonstrated by test, that the impedances of lines and transformers are no different for a polyphase system of voltages when two lines are interchanged (opposite sequence). Hence impedances to positive and negative sequence for all lines and static machinery like transformers are the same. For a synchronous generator it would seem that these impedances are different since one system causes a reaction from the armature that rotates in the same direction as the rotating field structure, whereas the other causes an armature reaction that rotates in a direction opposite to the field structure. The values of Z_1 and Z_2 may be obtained from a three-phase and a line-to-line short-circuit test. The relation between the line-to-line short-circuit current designated by I' and the three-phase short-circuit current represented by I''' is established for an alternator of voltage E_n to neutral as follows:

$$I''' = \frac{E_n}{Z_1} \tag{19}$$

For a line-to-line short circuit between terminals *b* and *c* at the generator, Fig. 15, a combination of equations (15), (17), and (18) gives the current:

$$\begin{aligned} \mathbf{I}' = \mathbf{I}_{nc} &= \frac{\mathbf{E}_n}{\mathbf{Z}_1 + \mathbf{Z}_2}\underline{/120^\circ} - \frac{\mathbf{E}_n}{\mathbf{Z}_1 + \mathbf{Z}_2}\underline{/-120^\circ} \\ &= \frac{\mathbf{E}_n}{\mathbf{Z}_1 + \mathbf{Z}_2}(\cos 120^\circ + j \sin 120^\circ - \cos 120^\circ + j \sin 120^\circ) \\ &= +j\sqrt{3}\,\frac{\mathbf{E}_n}{\mathbf{Z}_1 + \mathbf{Z}_2} \end{aligned} \tag{20}$$

Since $\mathbf{Z}_1 + \mathbf{Z}_2$ is practically $Z_1 + Z_2$ owing to the resistances in both cases being small compared to the reactance,[2] the magnitude of $\mathbf{I}'$ is:

$$I' = \frac{\sqrt{3}E_n}{Z_1 + Z_2} \tag{21}$$

Let $k = I'/I'''$. Then

$$k\,\frac{E_n}{Z_1} = \frac{\sqrt{3}E_n}{Z_1 + Z_2}$$

[2] In general the resistances of generators and transformers are sufficiently low in comparison with the corresponding reactances that it is customary to neglect resistances in making short-circuit calculations. For this reason reactances only are used in many of the subsequent computations even though the formulas are written in terms of impedances. If these facts are not kept in mind the rather loose use of the terms reactance and impedance may become confusing.

or

$$Z_2 = \frac{\sqrt{3}Z_1}{k} - Z_1 \tag{22}$$

Equation (22) shows that Z_2 depends upon the ratio, k, of the line-to-line and three-phase short-circuit currents. When this ratio is known and the impedance to positive sequence is determined by the ordinary methods, Z_2 can be determined. One salient-pole machine with an amortisseur winding tested by one of the authors gave a value of 1.44 for k, while another non-salient pole machine without an amortisseur winding yielded a value of 1.46.[3]

TABLE II

IMPEDANCES AND REACTANCES TO DIFFERENT SEQUENCES OF SALIENT-POLE SYNCHRONOUS GENERATORS WITH DAMPER WINDINGS

Name of Reactance	Synchronous X_s	Positive-Sequence X_1	Negative-Sequence X_2	Zero-Sequence X_0
Per Cent Reactances	100	100	Approximate Range 25–50	Approximate Range 2–20
Name of Impedance	Synchronous Z_s	Positive-Sequence Z_1	Negative-Sequence Z_2	Zero-Sequence Z_0
Approximate Per Cent Impedances	100	100	Approximate Range 25–50	Approximate Range 3–20

Table II shows approximate ranges of impedances to positive-, negative-, and zero-sequence currents of one class of generators with reference to the synchronous impedance taken as 100.

Example 4. Each of the line reactances in Fig. 15 is 10 per cent based on 1000 kva, and the positive-sequence impedance of the alternator is 25 per cent based on 1000 kva. A value of 1.45 is assumed for k. The short-circuit currents in the three lines for a short circuit between lines b and c are to be determined. The nominal rated

[3] The values of reactances to negative sequence depend upon the size and the design of the machines and vary over rather wide limits for special cases. The reader is referred to Wagner and Evans, "Symmetrical Components," p. 99, McGraw-Hill Book Company, for extensive data on synchronous generator reactances to the different sequences.

line voltage of the system is 2200 volts. For the generator

$$Z_2 = \frac{\sqrt{3}}{1.45} Z_1 - Z_1 = 1.2Z_1 - Z_1 = 0.2Z_1$$
$$= 0.2 \times 25 = 5\%$$

The positive- and negative-sequence circuits are shown in Figs. 17 and 18, respectively. The resultant impedances to positive and negative sequence are 35 per cent and 15 per cent, respectively. From equation (18) and Fig. 16,

$$\mathbf{I}_1 = \frac{1{,}000{,}000}{\sqrt{3} \times 2200} \times \frac{100}{(35 + 15)} = 525 \text{ amperes}$$
$$\mathbf{I}_2 = -\mathbf{I}_1 = -525\underline{/0^\circ} \text{ amperes}$$
$$\mathbf{I}_{na} = \mathbf{I}_1 + \mathbf{I}_2 = 525\underline{/0^\circ} - 525\underline{/0^\circ} = 0$$
$$\mathbf{I}_{nb} = \mathbf{I}_1\underline{/-120^\circ} + \mathbf{I}_2\underline{/120^\circ} = 525\underline{/-120^\circ} - 525\underline{/120^\circ} = -j910 \text{ amperes}$$
$$\mathbf{I}_{nc} = \mathbf{I}_1\underline{/120^\circ} + \mathbf{I}_2\underline{/-120^\circ} = 525\underline{/120^\circ} - 525\underline{/-120^\circ} = +j910 \text{ amperes}$$

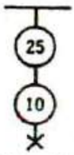


FIG. 17. Positive-sequence system of Fig. 15. See example 4.

FIG. 18. Negative-sequence system of Fig. 15. See example 4.

Example 5. The short-circuit current for the system shown in Fig. 7 for a line-to-line short circuit is to be determined. The ratio k will be used as 1.45. Nominal line voltage at short circuit is 12,000 volts. The lines shorted are designated as b and c, and the fault is again assumed at the upper right-hand corner of the diagram.

Solution. A 10,000-kva base will be used. The positive-sequence network is the same as that shown in Fig. 9. The negative-sequence network shown in Fig. 19 is similar to the positive-sequence system except for the values of the generator reactances. For the generators

$$Z_2 = \frac{\sqrt{3}Z_1}{1.45} - Z_1 = 0.2Z_1$$

The resultant Z_1 (Fig. 14) is 41.95 per cent.

The resultant Z_2 (Fig. 23) is 26.17 per cent as obtained from the reductions indicated by Figs. 20, 21, 22, and 23.

$$\mathbf{I}_1 = -\mathbf{I}_2 = \frac{10{,}000{,}000}{\sqrt{3}\ 12{,}000} \times \frac{100}{41.95 + 26.17} + 706 \text{ amperes}$$

At the short circuit where currents in all three lines are considered in the same direction, that is, either to or from the short circuit,

$$\mathbf{I}_a = \mathbf{I}_1 + \mathbf{I}_2 = 706\underline{/0^\circ} - 706\underline{/0^\circ} = 0$$
$$\mathbf{I}_b = 706\underline{/-120^\circ} - 706\underline{/120^\circ} = -j1223 \text{ amperes}$$
$$\mathbf{I}_c = 706\underline{/120^\circ} - 706\underline{/-120^\circ} = +j1223 \text{ amperes}$$

To obtain the currents in the other lines, the positive- and negative-sequence currents should first be found by retracing the steps in each system as outlined for the three-phase short circuit. The current in the lines from the secondaries of transformer T_1 will be found in order to illustrate the procedure. The distribution of positive-

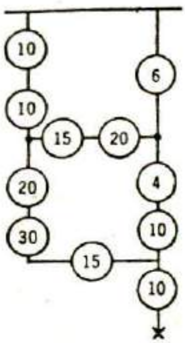


FIG. 19. Negative-sequence network of FIG. 7 for a line-to-line short circuit at the point indicated by the cross.

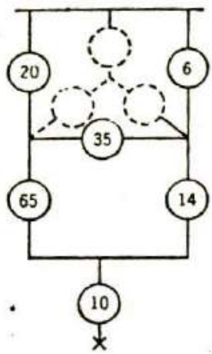


FIG. 20.

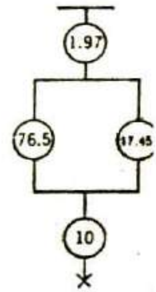


FIG. 21.

and negative-sequence components of current as shown in Figs. 24 and 25 are first found by retracing previous steps. If, when retracing the network from the short circuit, only transformers with both primary and secondary windings similarly connected are encountered, the actual current may be found by combining the sequence components as determined for Figs. 24 and 25. When a transformer like T_1 which is connected differently on primary and secondary is encountered, the symmetrical components in the lines on the primary side are no longer the same as those in the secondary lines. Failure to recognize this fact will introduce large errors in the short-circuit calculations. The short-circuit currents in the secondary lines from transformer T_1 are found from the sequence components shown in Figs. 24 and 25, as follows:

FIG. 22.

FIG. 23. Resultant percentage of reactance to negative sequence for a line-to-line short circuit at point indicated on Fig. 7

$$\mathbf{I}_a = \mathbf{I}_1 + \mathbf{I}_2 = 212\underline{/0^\circ} - 144.7\underline{/0^\circ} = 67.3 \text{ amperes}$$

$$\mathbf{I}_b = 212\underline{/-120^\circ} - 144.7\underline{/120^\circ} = -33.65 - j308.8 \text{ amperes}$$

$$\mathbf{I}_c = 212\underline{/120^\circ} - 144.7\underline{/-120^\circ} = -33.65 + j308.8 \text{ amperes}$$

The currents in the lines on the primary of T_1, Fig. 7, are determined from the phase currents in the delta and are obviously equal to them if the ratio of each transformer is 1 to 1 and the magnetizing currents are neglected. If the impedances of all phases of a delta-connected bank of transformers like that shown in Fig. 26 are equal, and if the sum of the generated voltages of the three phases is zero, application of Kirchhoff's laws will yield the following equations:

$$\mathbf{I}_{aa'} + \mathbf{I}_{bb'} + \mathbf{I}_{cc'} = 0 \tag{23}$$

$$\mathbf{I}_{ba}\mathbf{Z}_{ba} + \mathbf{I}_{ac}\mathbf{Z}_{ac} + \mathbf{I}_{cb}\mathbf{Z}_{cb} = \mathbf{E}_{ba} + \mathbf{E}_{ac} + \mathbf{E}_{cb} = 0 \tag{24}$$

Since $\mathbf{Z}_{ba} = \mathbf{Z}_{ac} = \mathbf{Z}_{cb}$, equation (24) becomes

$$\mathbf{I}_{ba} + \mathbf{I}_{ac} + \mathbf{I}_{cb} = 0 \tag{25}$$

Further application of Kirchhoff's current law gives

$$\mathbf{I}_{aa'} = \mathbf{I}_{ba} - \mathbf{I}_{ac} \tag{26}$$

$$\mathbf{I}_{bb'} = \mathbf{I}_{cb} - \mathbf{I}_{ba} \tag{27}$$

$$\mathbf{I}_{cc'} = \mathbf{I}_{ac} - \mathbf{I}_{cb} \tag{28}$$

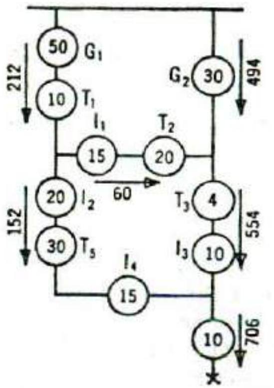


FIG. 24. Distribution of positive-sequence component currents for example 5.

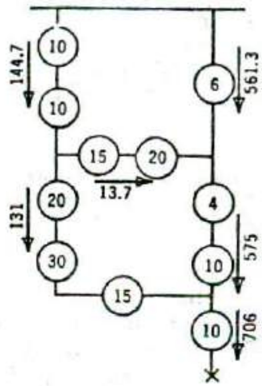


FIG. 25. Distribution of negative-sequence component currents for example 5.

Substituting $\mathbf{I}_{ac}$ from equation (25) in equation (26), then eliminating $\mathbf{I}_{cb}$ between this result and equation (27), and finally substituting the value of $\mathbf{I}_{bb'}$ from equation (23), the following expression for $\mathbf{I}_{ba}$ results:

$$\mathbf{I}_{ba} = \tfrac{2}{3}\mathbf{I}_{aa'} + \tfrac{1}{3}\mathbf{I}_{cc'} \tag{29}$$

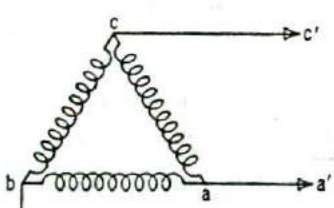


FIG. 26.

Similarly $\mathbf{I}_{ac}$ and $\mathbf{I}_{cb}$ are found to be, respectively,

$$\mathbf{I}_{ac} = \tfrac{2}{3}\mathbf{I}_{cc'} + \tfrac{1}{3}\mathbf{I}_{bb'} \tag{30}$$

$$\mathbf{I}_{cb} = \tfrac{2}{3}\mathbf{I}_{bb'} + \tfrac{1}{3}\mathbf{I}_{aa'} \tag{31}$$

The currents $\mathbf{I}_a$, $\mathbf{I}_b$, and $\mathbf{I}_c$ in the secondary lines of T_1 of Fig. 7 correspond to $\mathbf{I}_{aa'}$,

$I_{bb'}$, and $I_{cc'}$, respectively, in equations (29), (30), and (31).[4] Hence

$$
\begin{aligned}
I_{ba} &= \tfrac{2}{3}(67.3) + \tfrac{1}{3}(-33.65 + j308.8) \\
&= 33.7 + j102.9 = 108.2\underline{/71.8^\circ} \text{ amperes} \\
I_{ac} &= \tfrac{2}{3}(-33.65 + j308.8) + \tfrac{1}{3}(-33.65 - j308.8) \\
&= -33.7 + j102.9 = 108.2\underline{/108.2^\circ} \text{ amperes} \\
I_{cb} &= \tfrac{2}{3}(-33.65 - j308.8) + \tfrac{1}{3}(67.3) \\
&= -j205.8 = 205.8\underline{/-90^\circ} \text{ amperes}
\end{aligned}
$$

On a 1 to 1 ratio, I_{ba}, I_{ca}, and I_{cb} above are the line currents from generator G_1, Fig. 7, or, in other words, the above currents are on a $\sqrt{3}$ 12,000-volt base. If the nominal voltage of the generator is 6600 volts, the currents in the three lines from the generator are

$$108.2 \times \frac{\sqrt{3}\ 12{,}000}{6600} = 341 \text{ amperes}$$

$$108.2 \times \frac{\sqrt{3}\ 12{,}000}{6600} = 341 \text{ amperes}$$

$$205.8 \times \frac{\sqrt{3}\ 12{,}000}{6600} = 648 \text{ amperes}$$

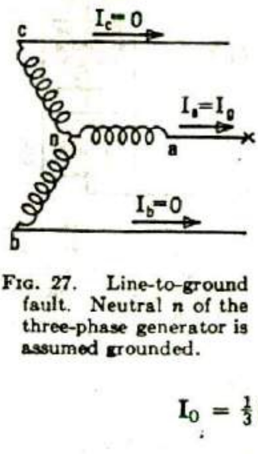


Fig. 27. Line-to-ground fault. Neutral n of the three-phase generator is assumed grounded.

Line-to-Ground Short Circuits. If a system has a number of wye-connected generators and transformers with grounded neutrals, there is a possibility of having a large short-circuit current for a line-to-ground fault. Such fault currents are most conveniently calculated with the aid of symmetrical components. An elementary circuit illustrating a line-to-ground fault is shown in Fig. 27. Application of equations (13), (16), and (17) of Chapter XII gives the symmetrical components of the currents as

$$I_0 = \tfrac{1}{3}(I_a + I_b + I_c) = \frac{I_a}{3} \tag{32}$$

$$I_1 = \tfrac{1}{3}(I_a + I_b\underline{/120^\circ} + I_c\underline{/-120^\circ}) = \frac{I_a}{3} \tag{33}$$

$$I_2 = \tfrac{1}{3}(I_a + I_b\underline{/-120^\circ} + I_c\underline{/120^\circ}) = \frac{I_a}{3} \tag{34}$$

Therefore

$$I_0 = I_1 = I_2 = \frac{I_a}{3} \tag{35}$$

[4] Equations other than (29), (30), and (31) for the currents in the transformer windings can be derived from the basic equations given.

-35

Let $\mathbf{E}$ be the induced voltage in phase a of the generator. According to Kirchhoff's emf law, the sum of all the drops must be equal to the sum of the emf's around a closed loop. Then

$$\mathbf{E} = \mathbf{I}_0\mathbf{Z}_0 + \mathbf{I}_1\mathbf{Z}_1 + \mathbf{I}_2\mathbf{Z}_2$$

Substituting equation (35) gives

$$\mathbf{E} = \mathbf{I}_0 (\mathbf{Z}_0 + \mathbf{Z}_1 + \mathbf{Z}_2) \tag{36}$$

Combining equations (35) and (36),

$$\mathbf{I}_0 = \frac{\mathbf{I}_a}{3} = \frac{\mathbf{E}}{\mathbf{Z}_0 + \mathbf{Z}_1 + \mathbf{Z}_2} \tag{37}$$

Equation (37) is the working equation for the line-to-ground fault.

Equations 35 and 37 show that the arrangement illustrated in Fig. 28 may be used to calculate the positive-, negative-, and zero-sequence currents at the fault for a line-to-ground short circuit.

The impedances to positive and negative sequence $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are exactly the same as those used for the line-to-line fault. The impedance to zero sequence, however, is different. Whereas the positive- and negative-sequence networks were alike in the number and arrangement of circuit elements, the zero-sequence network is radically different and usually much simpler.

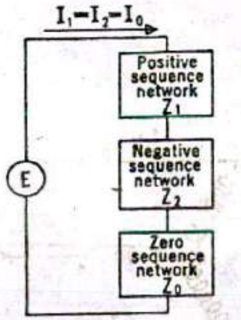


FIG. 28. Arrangement of sequence networks for calculating positive-, negative-, and zero-sequence currents for a line-to-ground short circuit.

Impedance to Zero Sequence for Generators. The determination of the impedance to zero sequence for generators is analogous to the determination of the impedance to negative sequence.

Let I_n represent the line-to-ground short-circuit current for a generator.

Let I''' represent the short-circuit current for a three-phase symmetrical short circuit.

Also let

$$\frac{I_n}{I'''} = k_n \tag{38}$$

From equation (37), if the ratio of X/R for all impedances is the same or if R is negligible compared to X, as is usual,

$$I_n = 3I_0 = \frac{3E}{Z_1 + Z_2 + Z_0} \tag{39}$$

Also

$$I''' = \frac{E}{Z_1} \quad \text{and} \quad k_n I''' = I_n = k_n \frac{E}{Z_1}$$

Therefore

$$k_n \frac{E}{Z_1} = \frac{3E}{Z_1 + Z_2 + Z_0} \tag{40}$$

Solving equation (40) for Z_0 gives

$$Z_0 = Z_1 \left(\frac{3}{k_n} - 1 \right) - Z_2 \tag{41}$$

The value of Z_0 thus depends upon the values of the impedances to positive and negative sequence and also upon the ratio of the line-to-ground and three-phase short-circuit currents. For example, k_n for the nonsalient-pole machine used in the previous example was shown by test to be about 2.4. For this machine

$$Z_0 = Z_1 \left(\frac{3}{2.4} - 1 \right) - 0.2Z_1 = 0.05Z_1$$

The approximate range of impedance to zero sequence for one class of generators is shown in Table II on page 535. The values are given relative to the synchronous impedance taken as 100.

Impedance to Zero Sequence for Transformers. The impedance to zero sequence for transformers is either infinite or the ordinary leakage impedance,[5] depending upon the connection. Where the connection permits zero-sequence currents to flow, the impedance to zero sequence is the ordinary impedance of the transformer; otherwise it is infinite. Since the zero-sequence currents in the three lines of a three-phase system are all in phase, a fourth wire or ground connection on the neutral of transformers connected in wye is required to furnish a complete circuit for the return of the zero-sequence line currents. In addition, there must be another winding on the transformer to permit current to flow so that the resultant magnetomotive force acting upon the transformer core due to the zero-sequence current is zero (exciting current neglected). If these compensating currents are not permitted to exist, the inductive reactance of a single winding to the zero-sequence current is so high that the amount of this current which can flow is entirely negligible. The corresponding impedance may then be considered

[5] If the transformers have more than two windings which carry zero-sequence current, reactance due to certain mutual-inductance effects of the several windings should be included. For a discussion of the reactance of multiwinding transformers, see O. G. C. Dahl, "Electric Circuits," McGraw-Hill Book Company.

infinite. A few examples as shown in Fig. 29 will illustrate these principles.

Transformer Bank A. No zero-sequence currents can flow since there is no return path. Therefore the impedance to zero sequence is infinite.

Transformer Bank B. Zero-sequence currents can flow. Winding *p* furnishes a path for the compensating currents of those in winding *S*. Hence the impedance to zero sequence is the ordinary leakage impedance.

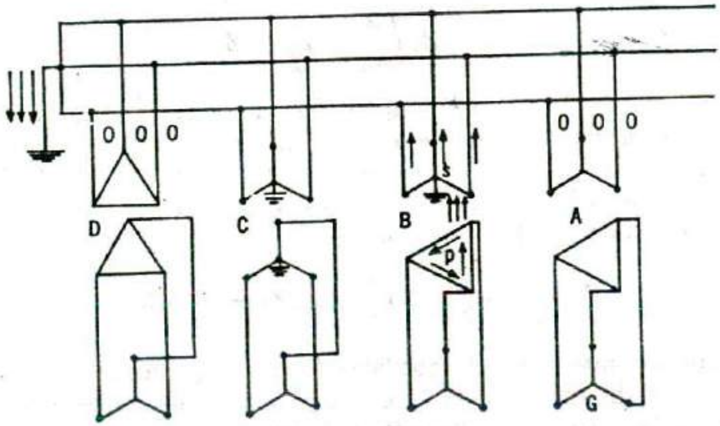


FIG. 29. Zero-sequence currents can flow in *B* but not in any of the other transformers.

Transformer Banks C and D. No zero-sequence currents can flow. The impedances to zero sequence are infinite. If the neutral of the wye-connected generator supplying transformer bank *C* were grounded, zero-sequence currents could flow in both primary and secondary of *C*. Under these conditions the impedance to zero sequence of transformer bank *C* would be the ordinary leakage impedance.

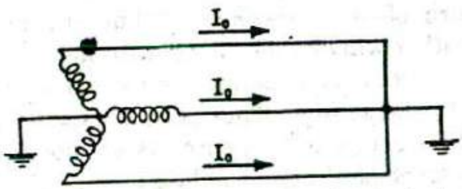


FIG. 30. Zero-sequence impedance of a transmission line is the impedance of the three conductors in parallel in series with a ground return.

Impedance to Zero Sequence of Transmission Lines. The impedance to zero sequence of a transmission line, Fig. 30, is the impedance of the three conductors in parallel with a ground return. The reactance depends upon the depth at which the return current appears to flow. A discussion sufficiently adequate to yield a working knowledge of the determination of reactance to zero sequence of transmission lines is somewhat involved and beyond the scope of this book. Those interested are referred to other works on the subject.[6] For purposes of illustration of the method of calculating line-to-ground fault currents in this book, certain values of reactance to zero sequence of a line are assumed.

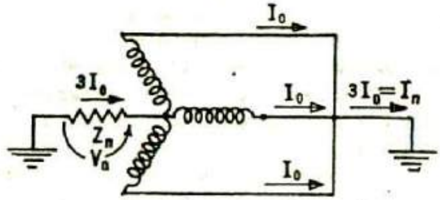


FIG. 31. Flow of zero-sequence currents through an impedance in the neutral.

If an impedance Z_n as shown in the neutral of the generator of Fig. 31 is encountered, it should be entered into the zero-sequence networks as $3Z_n$. This may be shown as follows. The ordinary impedance $\mathbf{Z}_n$ is defined as the drop $\mathbf{V}_n$ across the impedance divided by the current through it. Hence

$$\mathbf{Z}_n = \frac{\mathbf{V}_n}{\mathbf{I}_n} \tag{42}$$

Since

$$\mathbf{I}_n = 3\mathbf{I}_0$$

$$\mathbf{Z}_n = \frac{\mathbf{V}_n}{3\mathbf{I}_0} \tag{43}$$

Since there are no positive- or negative-sequence currents in the neutral, $\mathbf{V}_n$ for this case is considered the zero-sequence voltage which is due to the zero-sequence current $\mathbf{I}_0$. Hence

$$\mathbf{Z}_0 = \frac{\mathbf{V}_0}{\mathbf{I}_0} = \frac{\mathbf{V}_n}{\mathbf{I}_0} \tag{44}$$

Substitution of $\mathbf{V}_n/\mathbf{I}_0$ from equation (43) in equation (44) gives

$$\mathbf{Z}_0 = 3\mathbf{Z}_n \tag{45}$$

Thus the impedance to zero sequence as defined in equation (45) is three times as large as the actual impedance in the conductor. Since the only zero-sequence current flowing in the zero-sequence network is $\mathbf{I}_0$, the

[6] See "Symmetrical Components" by Wagner and Evans and "Applications of the Method of Symmetrical Components" by Lyon, McGraw-Hill Book Company

value $Z_0 = 3Z_n$ should be entered into the zero-sequence network to yield the correct voltage drop.

Calculation of Line-to-Ground Fault Current. The system shown in Fig. 7, which was previously employed for three-phase and line-to-line short circuits, will be calculated for a line-to-ground fault on one of the secondary lines of transformer T_4. A determination of the reactance to zero-sequence of line l_3 is assumed to yield 20 per cent reactance on a 5000-kva base. The problem will be worked on a 10,000-kva base as before.

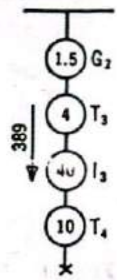


FIG. 32. Zero-sequence network for a line-to-ground fault on Fig. 7 at the point indicated by the cross.

Solution. The positive- and negative-sequence networks are the same as those previously employed. They are shown in Figs. 24 and 25. The impedances to positive and negative sequence are the same for the line-to-ground fault solution and the general distribution of the positive- and negative-sequence currents is the same, but the actual magnitudes of the positive- and negative-sequence currents will be different because of the effect of the impedance to zero sequence in reducing the magnitude of the resultant positive- and negative-sequence currents. The resultant impedances to positive and negative sequence of 41.95 and 26.17 per cent, respectively, are still valid. An inspection of Fig. 7 shows that no zero-sequence current can exist in transformers T_5, T_2, T_1, or generator G_1. Therefore the zero-sequence network consists of G_2, T_3, and T_4 along with line l_3. The zero-sequence network is shown in Fig. 32. If $k_n = 2.4$ and $k_1 = 1.45$, substitution in equation (41) gives $Z_0 = 0.05Z_1$. For generator G_2

$$Z_0 = 0.05 \times 30 = 1.5 \text{ per cent}$$

Resultant Z_0 for the zero-sequence network $= 1.5 + 4 + 40 + 10 = 55.5$ per cent.

$$\mathbf{I}_0 = \mathbf{I}_1 = \mathbf{I}_2 = \frac{\mathbf{I}_a}{3} = \frac{E}{Z_0 + Z_1 + Z_2}$$

In terms of percentage impedances,

$$\mathbf{I}_0 = \mathbf{I}_1 = \mathbf{I}_2 = \frac{\mathbf{I}_a}{3} = \frac{10{,}000{,}000}{\sqrt{3}\,12{,}000} \times \frac{100}{55.5 + 41.95 + 26.17}$$
$$= 389\,\underline{/0^\circ} \text{ amperes}$$

For a positive-sequence current of 389 amperes the distribution is shown in Fig. 33. These values are determined by multiplying the currents in Fig. 24 by 389/706. Similarly the negative-sequence current distribution is determined and shown in Fig. 34.

The currents on a 12,000-volt base are now found by combining the symmetrical components.

Fault current:

$$\mathbf{I}_a = \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_0 = 3 \times 389\,\underline{/0^\circ} = 1167\,\underline{/0^\circ} \text{ amperes}$$

$$\mathbf{I}_b = 389\,\underline{/-120^\circ} + 389\,\underline{/120^\circ} + 389 = 0$$

$$\mathbf{I}_c = 389\,\underline{/120^\circ} + 389\,\underline{/-120^\circ} + 389 = 0$$

Transformer T_3 and line l_3:

$\mathbf{I}_a = 305.2 + 316.8 + 389 = 1011$ $\quad I_a = 1011$ amperes

$\mathbf{I}_b = 305.2\underline{/-120°} + 316.8\underline{/120°} + 389 = 78 + j10.5$ $\quad I_b = 78.7$ amperes

$\mathbf{I}_c = 305.2\underline{/120°} + 316.8\underline{/-120°} + 389 = 78 - j10.5$ $\quad I_c = 78.7$ amperes

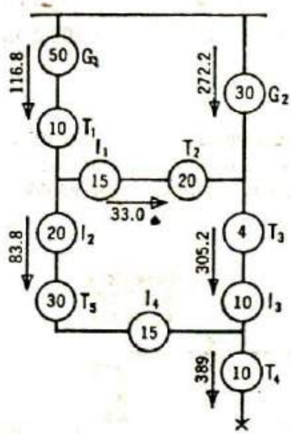


FIG. 33. Positive-sequence current distribution for line-to-ground fault on Fig. 7.

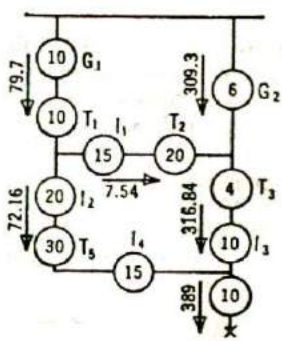


FIG. 34. Negative-sequence current distribution for line-to-ground fault on Fig. 7.

Line l_2, transformer T_5, and l_4:

$\mathbf{I}_a = 83.8 + 72.16 = 155.9$ $\quad I_a = 155.9$ amperes

$\mathbf{I}_b = 83.8\underline{/-120°} + 72.16\underline{/120°} = -77.95 - j10$ $\quad I_b = 78.6$ amperes

$\mathbf{I}_c = 83.8\underline{/120°} + 72.16\underline{/-120°} = -77.95 + j10$ $\quad I_c = 78.6$ amperes

Line l_1 and transformer T_2:

$\mathbf{I}_a = 33.0 + 7.54 = 40.54$ $\quad I_a = 40.5$ amperes

$\mathbf{I}_b = 33.0\underline{/-120°} + 7.54\underline{/120°} = -20.3 - j22.0$ $\quad I_b = 30.0$ amperes

$\mathbf{I}_c = 33.0\underline{/120°} + 7.54\underline{/-120°} = -20.3 + j22.0$ $\quad I_c = 30.0$ amperes

Secondary side of transformer T_1:

$\mathbf{I}_a = 116.8 + 79.7 = 196.5$ $\quad I_a = 196.5$ amperes

$\mathbf{I}_b = 116.8\underline{/-120°} + 79.7\underline{/120°} = -98.2 - j32$ $\quad I_b = 103.3$ amperes

$\mathbf{I}_c = 116.8\underline{/120°} + 79.7\underline{/-120°} = -98.2 + j32$ $\quad I_c = 103.3$ amperes

Current in windings of transformer T_1 (see Fig. 7):

$\mathbf{I}_{ba} = \frac{2}{3}(196.5) + \frac{1}{3}(-98.25 + j32) = 98.3 + j10.7$ $\quad I_{ba} = 98.9$ amperes

$\mathbf{I}_{ac} = \frac{2}{3}(-98.25 + j32) + \frac{1}{3}(-98.25 - j32)$

$= -98.3 + j10.7$ $\quad I_{ac} = 98.9$ amperes

$\mathbf{I}_{cb} = \frac{2}{3}(-98.25 - j32) + \frac{1}{3}(196.5) = -j21.4$ $\quad I_{cb} = 21.4$ amperes

Since these are the delta transformer currents, on a 1 to 1 ratio they are also the currents in the phases of the wye primary, and therefore the currents in the lines from generator G_1 on a $\sqrt{3}$ 12,000 line voltage base.

Current in G_2:

$$\mathbf{I}_a = 272.2 + 309.3 + 389 = 970.5 \qquad I_a = 970.5 \text{ amperes}$$

$$\mathbf{I}_b = 272.2\,\underline{/-120^\circ} + 309.3\,\underline{/120^\circ} + 389 = 98.25 + j32 \qquad I_b = 103.3 \text{ amperes}$$

$$\mathbf{I}_c = 272.2\,\underline{/120^\circ} + 309.3\,\underline{/-120^\circ} + 389 = 98.25 - j32 \qquad I_c = 103.3 \text{ amperes}$$

PROBLEMS

4. Refer to Fig. 35. All circuit elements are assumed to have zero resistance. The reactances to positive sequence are the numbers preceded by j on the diagram. Generator A is a 3000-kva machine having a rated terminal voltage of 6600-volts. Generator B is a 6600-volt, 5000-kva machine.

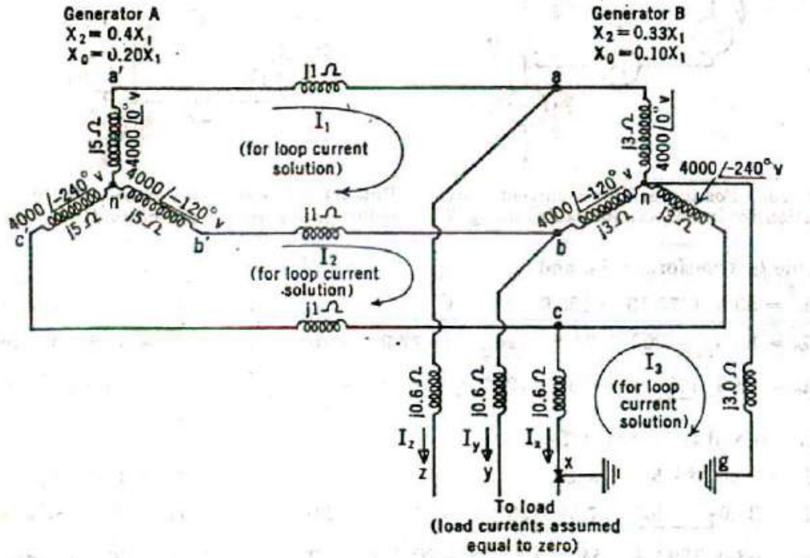


FIG. 35. See Problem 4.

(a) Solve for the currents in all branches by one of the methods considered in Chapter IX, assuming that the impedances shown on the diagram hold for any kind of unbalance.

(b) Solve for the currents in all branches by the method of symmetrical components, taking into account the difference in impedance to the positive, negative, and zero sequences. Impedances to positive sequence for the generators are those shown on the diagram.

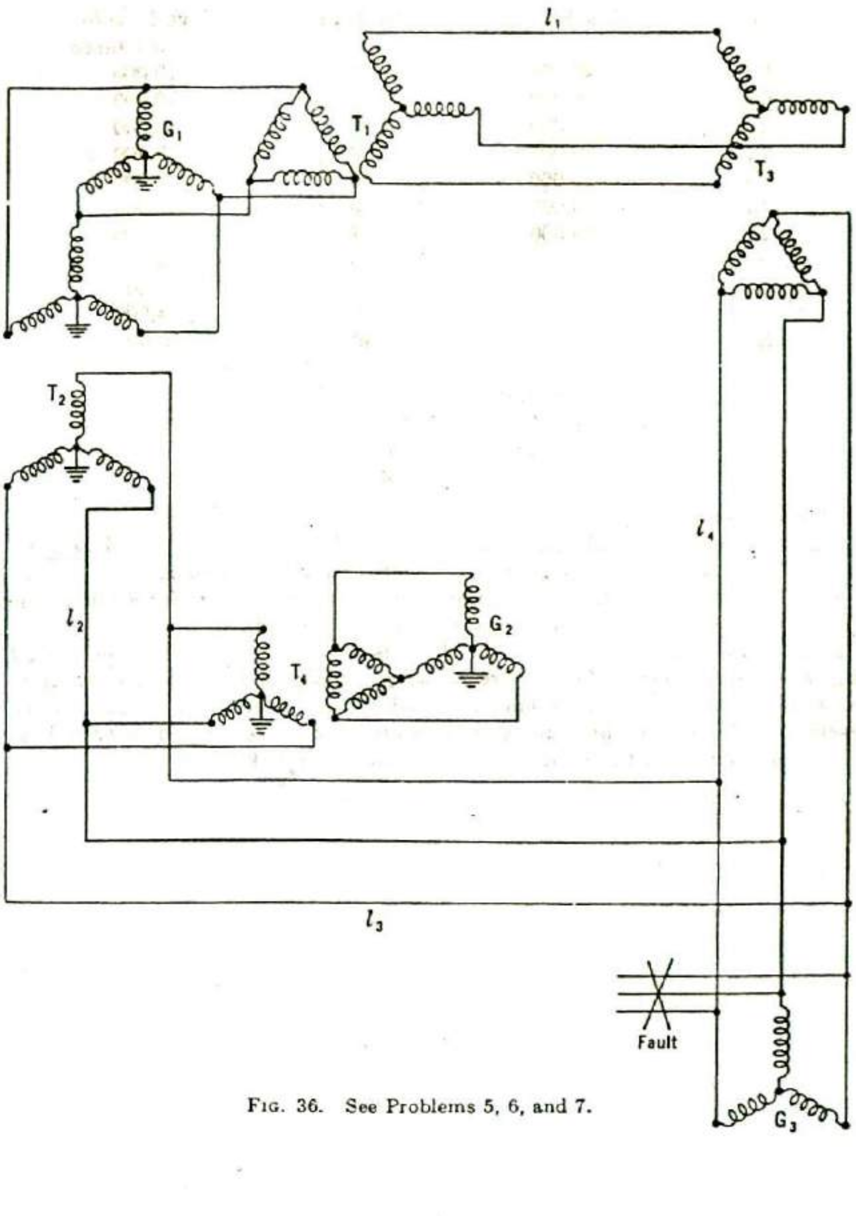


FIG. 36. See Problems 5, 6, and 7.

5. The following data refer to Fig. 36.

Apparatus	Kva Rating	% Reactance	Kva Base for % Reactance
G_1	20,000	30	10,000
G_2	10,000	50	10,000
G_3	20,000	20	10,000
T_1	10,000	2	2,500
T_2	10,000	20	30,000
T_3	10,000	6	10,000
T_4	10,000	7	10,000
l_1		30	20,000
l_2		20	10,000
l_3		10	4,000
l_4		40	30,000

Generator	k	k_n
G_1	1.4	2.1
G_2	1.5	2.3
G_3	1.3	

All resistances are assumed negligible.

Calculate currents in all lines, transformers, and generators for a 3-phase symmetrical short circuit at the point marked fault. Express currents on a 33-kv base.

6. Calculate currents in all lines, transformers, and generators for a line-to-line fault at the point marked fault. Express currents on a 33-kv base.

7. Calculate currents in all lines, transformers, and generators for a line-to-ground fault at the point marked fault. Use 25 per cent based on 10,000 kva as the zero-sequence reactance of l_2 including lines and ground return. The zero-sequence reactance of l_3 including lines and ground return is 12 per cent based on 4,000 kva. Assume negligible resistance, and express currents on a 33-kv base.

chapter XIV Transient Conditions

The expressions which have thus far been derived for currents and voltages have carried with them certain tacit assumptions. All the alternating currents and voltages in any particular circuit have been assumed to be recurring, periodic functions of time; in other words, the circuit in question has been assumed to be in a steady-state condition.

Before a circuit (or machine) can arrive at a steady-state condition of operation which is different from some previous state, the circuit (or machine) passes through a transition period in which the currents and voltages are not recurring periodic functions of time. For example, immediately after the establishment of a circuit the currents and voltages have not, in general, settled into their steady-state conditions. The period required for the currents and voltages to adjust themselves to their steady-state modes of variation is called the transient period. During transient periods the mathematical expressions for the currents and voltages contain certain terms other than the steady-state terms. These additional terms are called transient terms, and they are usually of short duration, being damped out by certain damping factors which depend for their values upon the circuit parameters.

In general, any switching operation within the circuit itself or any voltage which is suddenly induced from an outside source will cause transient conditions to exist in the circuit. Although transient periods are generally of short duration, it is during these periods that some of the most serious and involved operating problems are encountered.

It should not be inferred that transient variations are always violent or that they always represent undesirable circuit conditions. Various devices actually operate by virtue of recurring transient phenomena. Notable among these devices are: (1) certain classes of sweep circuits, and (2) certain types of tube inverters. Sweep circuits are employed extensively to produce linear time axes in cathode-ray oscillographs and cathode-ray television tubes. Inverters are employed to convert direct to alternating current.

Examples of Elementary Transient Conditions. Example 1. In Fig. 1 it is assumed that the RL branch is suddenly energized with a constant potential difference by closing the switch S at $t = 0$. The general equation for voltage equilibrium in the resulting series circuit is:

$$L\frac{di}{dt} + Ri = E \tag{1}$$

If L, R, and E are constant the above equation may be solved explicitly for i in any one of several different ways. One of the most direct methods of solution in a simple case of this kind is to separate variables and integrate. Thus:

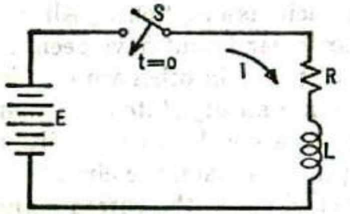


FIG. 1. A series RL branch which is suddenly energized by a constant potential difference E at $t = 0$.

$$L\frac{di}{(E - Ri)} = dt$$

or

$$L\int\frac{di}{(E - Ri)} = \int dt \tag{2}$$

Whence:

$$-\frac{L}{R}\log_\epsilon (E - Ri) = t + c$$

or

$$\log_\epsilon (E - Ri) = -\frac{Rt}{L} + c_1 \tag{3}$$

where ϵ is the base of the natural logarithms, namely, 2.71828 · · ·, and c_1 is a constant of integration. From the definition of a logarithm it follows that:

$$E - Ri = \epsilon^{(-Rt/L)+c_1}$$

Therefore:

$$E - Ri = c_2\epsilon^{-Rt/L} \tag{4}$$

Solving the above equation for i yields:

$$i = \frac{E}{R} - c_3\epsilon^{-Rt/L} \tag{5}$$

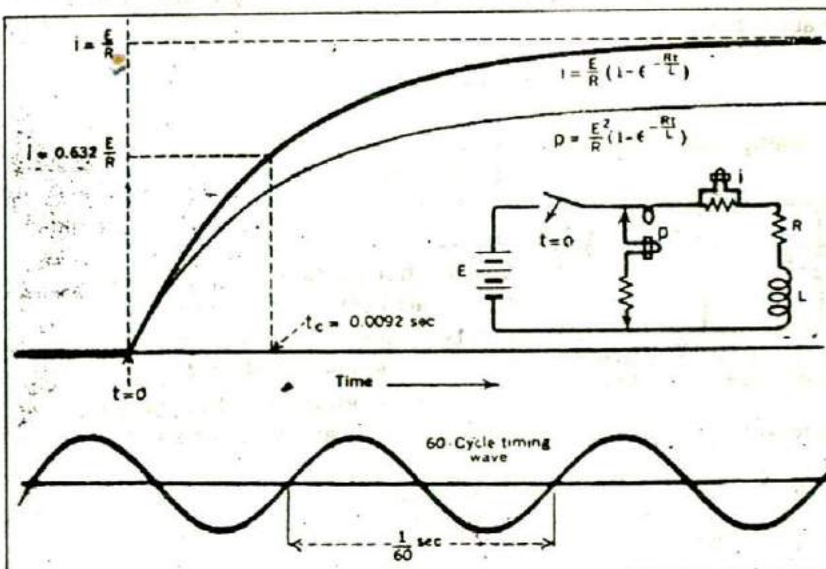
The constant of integration c_3 must be evaluated in terms of the boundary conditions that surround the switching operation. Boundary conditions are usually specified in terms of the circuit currents and the condenser voltages that exist at the instant a given switching operation is performed. In general, the specification and incorporation of boundary conditions require an understanding of the natural characteristics of the circuit parameters involved. For example, if a circuit possesses inductance the current cannot change abruptly, that is, cannot become discontinuous with respect to time. Therefore the current in an inductive branch at the instant a given switching operation is performed is equal to the current that exists in the branch just prior to switching operation. In the present case: $i = 0$ at $t = 0$, and this physical fact can be employed to determine the value of c_3 in equation (5). Imposing the boundary condition on equation (5) results in:

$$0 = \frac{E}{R} - c_3 \quad \text{or} \quad c_3 = \frac{E}{R} \tag{6}$$

The general expression for current becomes:

$$i = \underbrace{\frac{E}{R}}_{\text{steady-state term}} - \underbrace{\frac{E}{R}\epsilon^{-Rt/L}}_{\text{transient term}} \tag{7}$$

It will be noted that the complete expression for i consists of two terms: a steady-state term and a transient term. In general this distinct division of terms is present in complete current solutions. Under cer-



OSCILLOGRAM 1. Growth of current in an RL circuit which is suddenly energized with a constant potential difference, E. The instantaneous power delivered to the circuit is also shown.

tain conditions one or the other of the terms may be zero. The fact that the complete expression for current can be divided distinctly into a steady-state term and a transient term is of considerable importance. Under ordinary conditions the steady-state term can be evaluated in terms of elementary circuit concepts rather than by involved processes of integration. The transient term can usually be found in terms of simple exponential components if the circuit parameters are constant.

The time variations of the two terms of the current solution given in equation (7) can easily be visualized. The steady-state term, E/R, is independent of time; the transient term has a value of $(-E/R)$ at $t = 0$ and approaches zero exponentially as time increases. The two terms combine to form the current that actually flows in the RL circuit during the transient period. Oscillogram 1 illustrates the actual growth of current in an RL circuit when it is suddenly energized with a constant potential difference. It will be noted that the transition in current in this case is from zero to a steady d-c value equal to E/R.

In certain elementary types of circuits the length of time required for the current to make 63.2 per cent of its total transition is called the time constant of the circuit. The time constant of the RL circuit is L/R, as may be shown by direct substitution in equation (7). Thus if t is set equal to L/R in equation (7) it is simply a matter of algebra to show that:

$$[i]_{\text{at } t = L/R} = 0.632\frac{E}{R}$$

Example 2. The circuit shown in Fig. 2 is assumed to be carrying a steady current equal to E/R at $t = 0$. At $t = 0$, either the switch S is assumed to change from point a to point b in an infinitely short period of time or it is assumed that a dead short circuit occurs between the points a and b. In either event the RL branch is de-energized at $t = 0$ and left to subside through the short-circuit path. The basic voltage equation for the RL branch at and after $t = 0$ is:

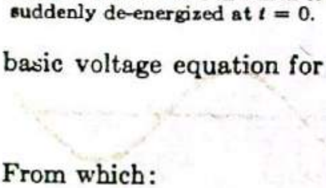


FIG. 2. An RL branch which is suddenly de-energized at $t = 0$.

$$L\frac{di}{dt} + Ri = 0 \tag{8}$$

From which:

$$i = \underbrace{0}_{\text{steady-state term}} + \underbrace{c_1 \epsilon^{-Rt/L}}_{\text{transient term}} \tag{9}$$

As previously mentioned, a current flowing through a circuit which has an appreciable amount of inductance cannot change its value instantaneously. Since $i = E/R$ just prior to $t = 0$, i is also equal to E/R at $t = 0$. Therefore:

$$\frac{E}{R} = c_1 \tag{10}$$

and

$$i = \underset{\text{steady-state term}}{0} + \underset{\text{transient term}}{\frac{E}{R}\epsilon^{-Rt/L}} \tag{11}$$

It will be observed that the transition in current is from (E/R) at $t = 0$ to zero current at $t = \infty$ and that the rate of subsidence is governed by the ratio of R to L. The current actually comes to zero in a relatively short period of time because the driving voltage, $L\,di/dt$ or $N\,d\phi/dt$, becomes so small that it can no longer maintain a net movement of electrons in one direction. Thus when the energy of the collapsing magnetic field becomes so small that it cannot overcome the internal atomic forces that tend to prevent net drifts of electrons, the current actually becomes zero. The failure of theoretical equations to account for exceedingly minute effects of this kind is of no practical importance.

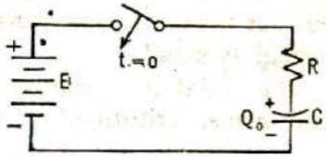


FIG. 3. A series RC circuit suddenly energized with a constant potential difference of E volts.

Example 3. If the condenser shown in Fig. 3 has a charge of Q_0 units of electrical charge at $t = 0$, the basic voltage equation at and after $t = 0$ is:

$$Ri + \frac{q}{C} = E \tag{12}$$

where

$$q = \int_0^t i\,dt + Q_0 \tag{13}$$

Differentiating equation (12) with respect to t and substituting i for dq/dt yields:

$$R\frac{di}{dt} - \frac{i}{C} = 0 \tag{14}$$

From which:

$$= c_1\epsilon^{-t/RC} \tag{15}$$

The resultant voltage causing current to flow in the circuit at the instant of closing the switch is $(E - Q_0/C)$. Therefore the current instantly acquires a value $\frac{E - Q_0/C}{R}$ at $t = 0$ since the self-inductance is assumed to be negligibly small. In this connection it should be noted that the initial Q_0/C voltage of the condenser may possess either polarity with respect to the applied voltage E. For the case shown in Fig. 3

the polarity of Q_0/C is opposite to that of the applied voltage E. Since

$$i = \frac{(E - Q_0/C)}{R} \quad \text{at} \quad t = 0$$

it follows that

$$c_1 = \frac{(E - Q_0/C)}{R} \tag{16}$$

and

$$i = \frac{(E - Q_0/C)}{R}\epsilon^{-t/RC} \tag{17}$$

Equation (17) is the mathematical expression for the direct charging current taken by a condenser when the self-inductance of the circuit is negligibly small.

The variation of charge can be found by solving equation (12) for q and then substituting for i its value from equation (17). Thus

$$\begin{aligned} q &= CE - CRi \\ &= CE - (CE - Q_0)\epsilon^{-t/RC} \end{aligned} \tag{18}$$

If the initial charge $Q_0 = 0$, the variations of current and charge as given by equations (17) and (18) are shown in Fig. 4.

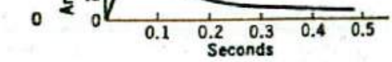


FIG. 4. Charging a condenser $C = 100\mu f$ through a resistance $R = 1000$ ohms from a d-c source of 1000 volts.

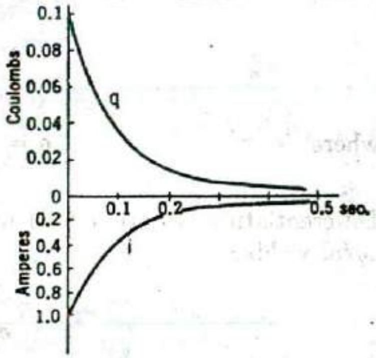


FIG. 5. Discharge of a condenser $C = 100\mu f$ through a resistance $R = 1000$ ohms. Initial charge at a potential of 1000 volts.

If a condenser of C units capacitance replaces the inductance L of Fig. 2, it is a simple matter to show that:

$$i = -\frac{E}{R}\epsilon^{-t/RC} \tag{19}$$

and

$$q = CE\epsilon^{-t/RC} \tag{20}$$

Equation (19) is the expression for the discharge current in an RC circuit which contains a condenser initially charged to a potential difference of E volts. Equation (20) is the expression for the decay of charge under the same conditions. The variations of current and charge as given by equations (19) and (20) are shown in Fig. 5. Condenser charge and discharge currents are similar except for sign and are simple exponential variations. The steady-state current in either of the two cases is obviously equal to zero.

The time constants of the above RC circuits are both equal to RC since it is at this value of time that the current has made 63.2 per cent of its total change.

Sawtooth Wave Form Produced by Simple Transient Effects. Various forms of circuits have been devised to produce sawtooth wave forms or approximations thereto. One of the most elementary is

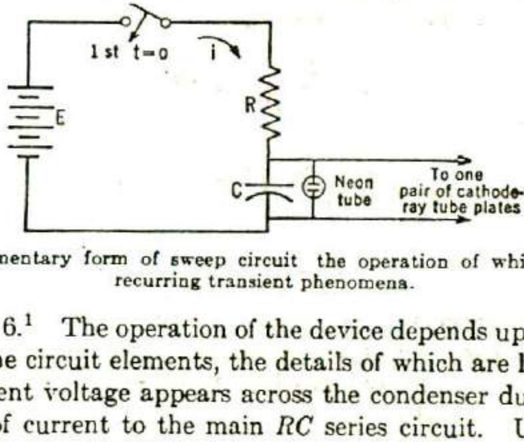


FIG. 6. An elementary form of sweep circuit the operation of which depends upon recurring transient phenomena.

shown in Fig. 6.[1] The operation of the device depends upon the natural behavior of the circuit elements, the details of which are listed below.

1. A transient voltage appears across the condenser due to the transient inrush of current to the main RC series circuit. Until a certain critical voltage is established across the condenser, the neon discharge tube remains un-ionized and acts practically as an open circuit.

2. When the condenser voltage has built up to a certain critical value, say E_1, the neon tube ionizes and suddenly places a low-resistance path across the condenser. The ionized tube thus provides a means of discharging the condenser because the time constant of the discharge path is relatively very small as compared with the time constant of the main RC series circuit. The voltage across the condenser drops from the value E_1 to some lower value, say E_2, in a very small fraction of the time required for the establishment of E_1.

[1] In practice the neon tube of Fig. 6 would probably be replaced by a gas triode which has an extremely low de-ionization time, for example, a type 885 tube. In this case the anode-cathode path of the triode replaces the neon tube of Fig. 6 and the grid of the triode can be used to control the starting of the discharge current.

3. After the condenser has been discharged to the voltage E_2, the neon tube ceases to be a conducting path (becomes de-ionized) and permits the applied potential difference to recharge the condenser. The cycle of transient phenomena thus repeats itself, and the voltage e_c takes on an approximate sawtooth wave form.

During the charging period the condenser voltage is:

$$e_{c1} = \frac{q}{C} = \frac{\int_0^t i\,dt + Q_0}{C} \tag{21}$$

or

$$e_{c1} = \frac{\int_0^t i\,dt}{C} + E_2 \tag{22}$$

E_2 is the voltage left on the condenser from the previous cycle due to the discharge tube de-ionizing before zero condenser voltage is reached.

From equation (17) it is evident that

$$i = \frac{E - E_2}{R}\epsilon^{-t/RC} \tag{23}$$

Therefore,

$$e_{c1} = \frac{E - E_2}{RC}\int_0^t \epsilon^{-t/RC}\,dt + E_2 \tag{24}$$

or

$$e_{c1} = E - (E - E_2)\epsilon^{-t/RC} \tag{25}$$

The rising condenser voltage is in this case exponential in character rather than linear. However, when the actual change in voltage, $(E_1 - E_2)$, is small as compared with $(E - E_2)$ fairly satisfactory results can be obtained.

The condenser voltage continues to build up in accordance with equation (25) until the voltage E_1 is attained, at which time the neon tube discharges the condenser in the manner previously described. Obviously E must be greater than E_1.

A mathematical analysis of the conditions during the discharge period is complicated by the variability of the resistance of the discharge path. The exact behavior of the circuit during the discharge period is usually unimportant because the discharge period is of relatively short duration and does not represent the "working" part of the cycle. It should be recognized that the series resistance, R, is generally of the order of 10,000 times the value of the tube resistance when the tube is ionized. Therefore during the discharge period the tube cannot receive any appreciable percentage of the applied voltage. It is plain that the device would cease to function as a sawtooth-wave-form generator, if, during an ionized period, the tube received a voltage sufficient to sustain ionization.

The general nature of the approximate sawtooth wave form produced is shown in Fig. 7. An obvious place for improvement is in the rising or building-up portion of the curve. The rising part of the curve can be made practically linear by replacing the constant resistance, R,

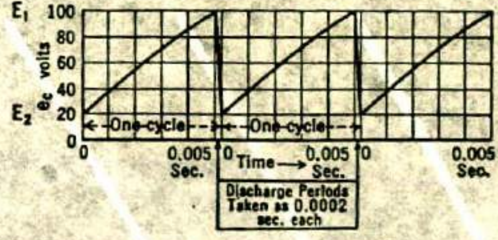


FIG. 7. Approximate sawtooth wave form as determined from equation (25) for the particular case of $E = 220$ volts, $E_1 = 100$ volts, $E_2 = 20$ volts, $R = 100{,}000$ ohms, and $C = 0.1\ \mu f$. The overall time of one cycle under these conditions is approximately 0.0052 second.

with a resistance that varies inversely as the amount of current passing through it. Many of the modern vacuum tubes, particularly the pentodes, possess this resistance characteristic from plate to cathode, provided they are worked between certain limits as regards plate-to-cathode voltage.

If the transient current inrush is maintained constant at I amperes by means of a variable resistance, then

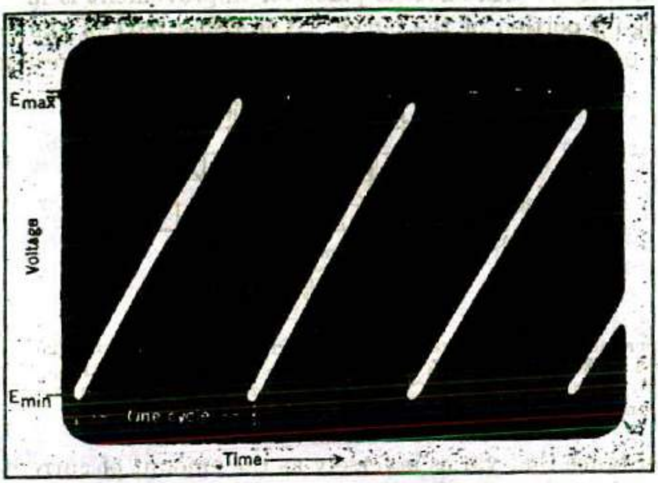
$$e_{c1} = K\int_0^t I\,dt + E_2$$

$$= KIt + E_2$$

Under the conditions stated above, the rising part of the voltage curve shown in Fig. 7 would become linear with respect to time.

In addition to the use of a pentode type tube for maintaining constant charging or discharging current, some sweep circuits employ a grid-controlled mercury-vapor discharge tube as a starting and stopping valve. Various other combinations of electron tubes are also employed to produce sawtooth wave forms.

Oscillogram 2 is a photographic record of the wave form produced by a modern sweep circuit which employs a series of transient conditions to effect the desired result. In obtaining the photographic record one pair of plates of a cathode-ray tube are energized with one sweep-circuit potential difference and another pair of plates are energized with the potential difference developed by an identical sweep circuit. The linearity of the sweep-circuit voltage is clearly shown.



OSCILLOGRAM 2. Illustrating the linearity of the potential difference developed by a modern sweep circuit. In this particular case the return time, that is, the time required for the voltage to return from E_{max} to E_{min}, is so short that the trace is not discernible on the photographic record.

The *RL* Circuit Energized with an Alternating Potential Difference. If an alternating potential difference replaces the battery shown in Fig. 1, the expression for dynamic equilibrium is:

$$L\frac{di}{dt} + Ri = E_m \sin(\omega t + \lambda) \tag{26}$$

or

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E_m}{L}\sin(\omega t + \lambda) \tag{27}$$

The symbol λ represents the phase of the voltage wave at which the switch of Fig. 1 is closed. Reference to Fig. 8 will show more clearly the exact meaning of λ. It is the angular displacement expressed in degrees or radians between the point $e = 0$ and the point $t = 0$ measured positively from the point where $e = 0$ and de/dt is positive.

The factor λ provides a convenient means of examining a-c transient conditions. In general, the magnitude of an a-c transient depends upon the time of the cycle at which the switching operation is performed. Most switching operations are performed with no regard for, or rather no knowledge of, the point on the voltage wave at which the transient period begins. Under these circumstances the investigator analyzes the effect of starting the transient disturbance at different points along the voltage wave. This is done by assigning different values to λ. In the case of surges or inrushes most attention is paid to those values of λ that produce the greatest currents or voltages.

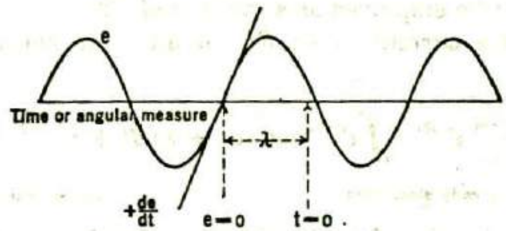


FIG. 8. Illustrating the physical significance of the symbol λ.

Equation (27) is representative of a general class of differential equations. The derivative of the dependent variable, (i), with respect to the independent variable, (t), added to the dependent variable, times some coefficient, equals some function of time. This form of equation defines the basic relationships involved in many physical problems, being particularly prevalent among the problems of electric circuit theory. The equation admits of relatively simple solution if all coefficients are constant and the right member is an exponential or sinusoidal function of time.

Let equation (27) be written as

$$\frac{di}{dt} + ai = h \sin (\omega t + \lambda) \tag{28}$$

where $a = R/L$ and $h = E_m/L$.

The solution of equation (28) takes the following form:

$$i = h\epsilon^{-at} \int \epsilon^{at} \sin (\omega t + \lambda)\, dt + c_1\epsilon^{-at} \tag{29}$$

The proof of the solution stated above rests in its ability to satisfy the original equation, namely, equation (28). In terms of the above solution:

$$\frac{di}{dt} = h\epsilon^{-at}\epsilon^{at} \sin (\omega t + \lambda) - ah\epsilon^{-at} \int \epsilon^{at} \sin (\omega t + \lambda)\, dt - ac_1\epsilon^{-at} \tag{30}$$

and

$$ai = ah\epsilon^{-at} \int \epsilon^{at} \sin (\omega t + \lambda)\, dt + ac_1\epsilon^{-at} \tag{31}$$

Adding equations (30) and (31) will show that equation (29) is a general solution of equation (28). The solution stated in equation (29) is limited to those cases where a and h are constant. For the particular problem at hand this means that R, L, and E_m must be constant before equation (29) can be employed as a solution of (28).

The solution for current in an RL circuit with sinusoidal voltage applied is:

$$i = \underbrace{\frac{E_m}{L} \epsilon^{-Rt/L} \int \epsilon^{Rt/L} \sin (\omega t + \lambda)\, dt}_{\text{steady-state term}} + \underbrace{c_1 \epsilon^{-Rt/L}}_{\text{transient term}} \tag{32}$$

The relative complexities of the two terms in the above solution should be noted. Mathematically, the steady-state term is known as the "particular integral," and the transient term as the "complementary function." The integration involved in the evaluation of the steady-state term can be carried out by the method of successive parts, but the algebraic simplification of the results is a tedious process.

With sinusoidal applied voltages, familiar algebraic methods may be employed to find the steady-state terms of general current solutions. Many of the disagreeable details connected with the evaluation of complete current solutions are thus avoided. For example, several lengthy mathematical relations are involved in the integration method of finding the steady-state term of equation (32) which is simply:

$$i_s = \frac{E_m}{Z} \sin (\omega t + \lambda - \theta) \tag{33}$$

where

$$Z = \sqrt{R^2 + \omega^2 L^2} \quad \text{and} \quad \theta = \tan^{-1} \omega L/R$$

Actually equation (33) can be thought of as following from two physical facts. The maximum value of the steady-state current is E_m/Z where $Z = \sqrt{R^2 + \omega^2 L^2}$, and the steady-state current wave lags the applied voltage wave by the angle whose tangent is $\omega L/R$. The complete expression for current becomes:

$$i = \frac{E_m}{Z} \sin (\omega t + \lambda - \theta) + c_1 \epsilon^{-Rt/L} \tag{34}$$

The constant of integration c_1 must be found from the initial conditions — those existing at the time of closing the switch. If the circuit current is zero just prior to closing the switch, then,

$$i = 0 \quad \text{at} \quad t = 0$$ (See page 550.)

Imposing the above condition on equation (34) yields

$$0 = \frac{E_m}{Z} \sin(\lambda - \theta) + c_1$$

From which:

$$c_1 = \frac{-E_m}{Z} \sin(\lambda - \theta) \tag{35}$$

and

$$i = \underbrace{\frac{E_m}{Z} \sin(\omega t + \lambda - \theta)}_{\text{steady-state term}} - \underbrace{\frac{E_m}{Z} \sin(\lambda - \theta)\epsilon^{-Rt/L}}_{\text{transient term}} \tag{36}$$

It will be noted from the above equation that the transient term is equal to zero when $(\lambda - \theta) = 0, \pi, 2\pi$, etc. If the RL branch is highly inductive the ratio of ωL to R is large, thereby causing θ to approach $\pi/2$ as an upper limit. In cases of this kind the transient term is zero when λ is approximately equal to $\pi/2, 3\pi/2, 5\pi/2$, etc. Physically this means that zero transient effects take place in highly inductive circuits when the circuit is energized at points of approximately maximum voltage on the voltage wave.

The transient term of equation (36) is maximum (for given values of R, L, ω, and E_m) when $(\lambda - \theta) = \pi/2, 3\pi/2, 5\pi/2$, etc. When θ is approximately equal to $\pi/2$ it is plain that the transient term is a maximum when λ is approximately equal to $0, \pi, 2\pi$, etc. Therefore in a highly inductive circuit the transient term is maximum when the switch is closed at points of approximately zero voltage on the voltage wave. A detailed study of equation (36) will show that conditions which make for the maximum possible transient terms do not necessarily make for the maximum possible values of i. In highly inductive circuits the difference between the two sets of conditions is not large and maximum transient disturbance is usually assumed to be the result of those conditions that make $\sin(\lambda - \theta) = 1$ or $\sin(\lambda - \theta) = -1$.

The steady-state term and the transient term, together with the resultant current, are illustrated in Fig. 9 for the case of $\theta = 85°$ and for $(\lambda - \theta) = 3\pi/2$. Under these conditions:

$$\lambda = 270° + 85° = 355° = -5°$$

It will be noted that the switch is closed when the steady-state term is at a maximum (negative) value and that the transient term is at its maximum (positive) value. The transient term and the steady-state term combine at $t = 0$ to make the resultant current equal to zero, which of course must be the case in an inductive circuit which is at rest just prior to the application of a potential difference.

Under the condition of constant R and L, the maximum value of the resultant current i is less than $2I_m$, where $I_m = E_m/Z$, the maximum value of the steady-state term. This fact may be easily substantiated

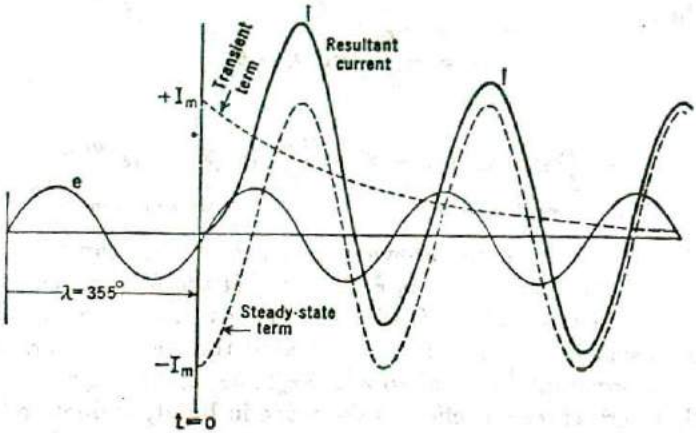


FIG. 9. Illustrating the manner in which the steady-state term and the transient term of equation (36) combine to form the resultant current. For the case shown, $\theta = 85°$ and $\sin(\lambda - \theta) = -1$.

from the graphs shown in Fig. 9. The effective value of the current during the early transient period is somewhat less than

$$\sqrt{I_{dc}^2 + I^2} = \sqrt{2I^2 + I^2} = \sqrt{3}I \quad \text{[See equation (28), page 252.]}$$

where $I_{dc} = I_m = \sqrt{2}I$ and I is the effective value of the steady-state term.

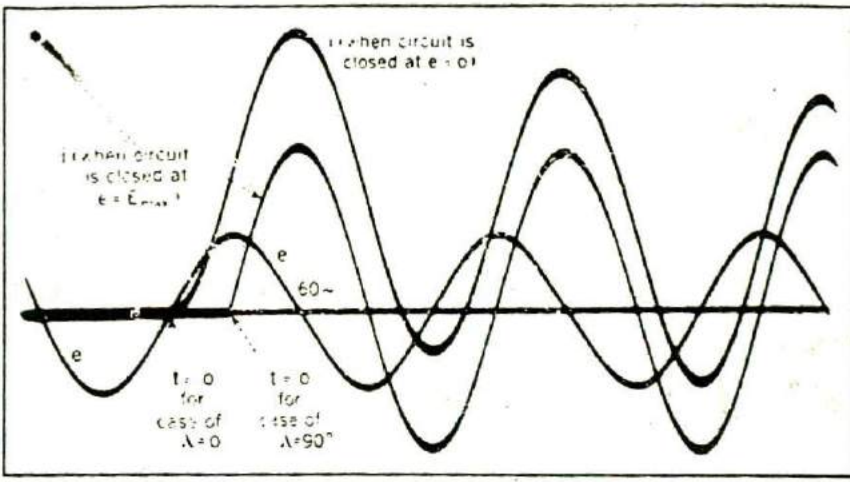
The transient term in an RL circuit is often referred to as the d-c component since it is unidirectional. This subsiding unidirectional component of current is of theoretical interest because it is partly responsible for the radical changes that take place in synchronous generator impedances during transient periods.

Oscillogram 3 illustrates the resultant current in a highly inductive circuit when $\lambda = 0$ and $\lambda = \pi/2$. The two current records are placed on the same oscillogram by means of superimposed exposures. In taking oscillograms of this kind it is necessary to employ some device for closing the circuit at the desired point on the voltage wave.

Problem 1. Plot the steady-state term and the transient term of equation (36) for two cycles of the steady-state variation under the following conditions:

(*a*) The applied voltage is a 60-cycle sinusoidal variation, the maximum value of which is 311 volts.

(*b*) $R = \omega L = 4$ ohms.



OSCILLOGRAM 3. Illustrating the current variations in an RL circuit which is suddenly energized with a p.d. of sinusoidal wave form. R and L, in this particular case, are sensibly constant. Two cases, namely, $\lambda = 0$ and $\lambda = 90°$, are shown.

(c) The switch is closed at such a time as to make the transient term acquire a negative maximum value.

Graph the resultant current i on the same plot.

Ans.: $i = 55 \sin (377t + 90°) - 55\epsilon^{-377t}$ amperes.

Problem 2. Analyze equation (36) for the case in which L is negligibly small.

Ans.: $i = \dfrac{E_m}{R} \sin (\omega t + \lambda)$.

The *RC* Circuit Energized with an Alternating Potential Difference. If an alternating potential difference replaces the battery shown in Fig. 3, the expression for dynamic equilibrium is:

$$Ri + \frac{q}{C} = E_m \sin (\omega t + \lambda) \tag{37}$$

Since $i = \dfrac{dq}{dt}$

$$R\frac{dq}{dt} + \frac{q}{C} = E_m \sin (\omega t + \lambda) \tag{38}$$

or

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{E_m}{R} \sin (\omega t + \lambda) \tag{39}$$

Since equation (39) is a linear differential equation of the first order and first degree, the integrating factor[2] which makes the left-hand side

[2] Consult any standard book on differential equations.

an exact derivative is:

$$\epsilon^{\int \frac{dt}{RC}} = \epsilon^{t/RC} \tag{40}$$

Multiplying equation (39) by $\epsilon^{t/RC}$ gives

$$\epsilon^{t/RC}\frac{dq}{dt} + \epsilon^{t/RC}\frac{q}{CR} = \epsilon^{t/RC}\frac{E_m}{R}\sin(\omega t + \lambda) \tag{41}$$

or

$$\epsilon^{t/RC}\,dq + \epsilon^{t/RC}\frac{q}{CR}\,dt = \epsilon^{t/RC}\frac{E_m}{R}\sin(\omega t + \lambda)\,dt \tag{42}$$

Integrating gives

$$q\epsilon^{t/RC} = \int \epsilon^{t/RC}\frac{E_m}{R}\sin(\omega t + \lambda)\,dt + K$$

or

$$q\epsilon^{t/RC} = \frac{E_m}{R}\left\{\frac{\epsilon^{t/RC}\left[\frac{1}{RC}\sin(\omega t + \lambda) - \omega\cos(\omega t + \lambda)\right]}{\frac{1}{R^2C^2} + \omega^2}\right\} + K \tag{43}$$

Dividing equation (43) through by $\epsilon^{t/RC}$, expressing the difference of the sine and cosine terms as a single cosine function, and making a few algebraic transformations give

$$q = -\frac{E_m}{\omega\sqrt{R^2 + \frac{1}{\omega^2C^2}}}\cos(\omega t + \lambda + \theta) + K\epsilon^{-t/RC} \tag{44}$$

where $\theta = \tan^{-1}\frac{1}{\omega CR} = \tan^{-1}\frac{X_c}{R}$.

Imposing the initial condition, namely, $q = Q_0$ when $t = 0$, and solving for K give

$$K = Q_0 + \frac{E_m}{\omega\sqrt{R^2 + \frac{1}{\omega^2C^2}}}\cos(\lambda + \theta) \tag{45}$$

Substituting (45) in (44) and replacing $\frac{1}{\omega C}$ by X_c

$$q = -\frac{E_m}{\omega\sqrt{R^2 + X_c^2}}\cos(\omega t + \lambda + \theta) + \left[Q_0 + \frac{E_m}{\omega\sqrt{R^2 + X_c^2}}\cos(\lambda + \theta)\right]\epsilon^{-t/RC} \tag{46}$$

Equation (46) is the general equation for the charge on the condenser. If the initial charge is zero,

$$q = -\frac{E_m}{\omega\sqrt{R^2 + X_c^2}}\cos(\omega t + \lambda + \theta) + \frac{E_m \epsilon^{-t/RC}}{\omega\sqrt{R^2 + X_c^2}}\cos(\lambda + \theta) \quad (47)$$

The first term of the right-hand member of equation (47) is the steady-state term whereas the last term is the transient. It should be noted that at the time $t = 0$, the transient is always exactly equal and opposite

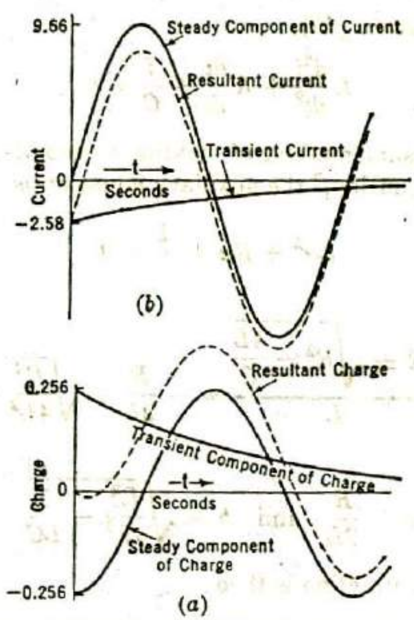


FIG. 10. Circuit containing $R = 100$ ohms, $C = 100\ \mu f$ when $e = 1000 \sin(377t - 14.95°)$ volts is impressed. Initial charge on condenser $= 0$.

to the steady-state component. These results are shown in Fig. 10*a*. This is the same relation that exists between the steady-state term and transient of *current* in the *RL* circuit.

The current in the *RC* circuit is obtained by differentiation of equation (47). Thus

$$i = \frac{dq}{dt} = \frac{E_m}{\sqrt{R^2 + X_c^2}}\sin(\omega t + \lambda + \theta) - \frac{E_m \epsilon^{-t/RC}}{RC\omega\sqrt{R^2 + X_c^2}}\cos(\lambda + \theta) \quad (48)$$

A study of equation (48) and the corresponding graph, Fig. 10*b*, reveals that there is no fixed relation between the transient and the steady-state component of current at the time $t = 0$. The relative magnitudes are dependent upon the ratio of $\frac{X_c}{R} = \frac{1}{RC\omega}$ and the time angle λ at which the switch is closed.

The *RLC* Series Circuit with a Constant Direct Voltage Suddenly Applied. Since the emf applied to the circuit must equal the sum of all the drops at every instant, the condition for dynamic equilibrium is:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = E \tag{49}$$

Differentiating equation (49),

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = 0 \tag{50}$$

Employing the usual method of solving a second-order, first-degree linear differential equation,[3] the auxiliary equation is:

$$L\alpha^2 + R\alpha + \frac{1}{C} = 0$$

Hence,

$$\alpha = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L} = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

Let

$$a = \frac{R}{2L} \quad \text{and} \quad b = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

The complementary function is then

$$i = k_1\epsilon^{(-a+b)t} + k_2\epsilon^{(-a-b)t}$$

The complete solution is the sum of the complementary function and the particular integral, the latter being the steady-state current. Since this case involves a constant direct voltage on a condenser, the steady-state current is 0.

Hence the complete solution is:

$$i = k_1\epsilon^{(-a+b)t} + k_2\epsilon^{(-a-b)t} + 0 \tag{51}$$

The constants k_1 and k_2 must be evaluated by imposing certain known conditions. In this case when $t = 0$, $i = 0$, and $q = Q_0$, the latter being the initial charge on the condenser before closing the switch.

[3] See any standard book on differential equations.

For $i = 0$ and $t = 0$ in equation (51)

$$0 = k_1 + k_2 \quad \text{or} \quad k_2 = -k_1 \tag{52}$$

From equation (49)

$$\frac{1}{C}\int i\,dt = \frac{q}{C} = E - L\frac{di}{dt} - Ri$$

and

$$q = CE - CL\frac{di}{dt} - CRi \tag{53}$$

Substituting (51) in (53) gives

$$q = CE - CL[k_1(-a+b)\epsilon^{(-a+b)t} + k_2(-a-b)\epsilon^{(-a-b)t}] - CR\,k_1\epsilon^{(-a+b)t} - CR\,k_2\epsilon^{(-a-b)t} \tag{54}$$

Imposing[4] the condition that $q = Q_0$ when $t = 0$ on equation (54), substituting equation (52), and solving for k_1 give

$$k_1 = \frac{CE - Q_0}{2CLb} \tag{55}$$

From equation (52)

$$k_2 = -k_1 = -\frac{CE - Q_0}{2CLb} \tag{56}$$

The final equation for current is now obtained by substituting equations (55) and (56) in equation (51) and replac. ıg b by its equal. Hence,

Since $b = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$ [in equations (57) and (5] may be real, imaginary or zero, there are three cases to be considerec

[4] It is important that initial conditions be imposed on the inal equation rather than on one of the differentiated forms. Note that equation comes from equation (49) without any differentiation of the original voltage eq (49).

$$i = \frac{CE - Q_0}{\sqrt{R^2C^2 - 4LC}}[\epsilon^{(-a+b)t} - \epsilon^{(-a-b)t}] \tag{57}$$

If the values of k_1, k_2, a and b are substituted in equation the expression for charge becomes

$$q = CE - (CE - Q_0)\left[\frac{RC + \sqrt{R^2C^2 - 4LC}}{2\sqrt{R^2C^2 - 4LC}}\epsilon^{(-a+b)t} - \frac{RC - \sqrt{R^2C^2 - 4LC}}{2\sqrt{R^2C^2 - 4LC}}\epsilon^{(-a-b)t}\right] \tag{58}$$

Case I. When $\frac{R^2}{4L^2} > \frac{1}{LC}$, the exponents of ϵ in equations (57) and (58) are real. When $t = 0$, the current is zero, and the quantity of electricity on the condenser is the initial charge before the switch was closed. Since $a = \frac{R}{2L}$ while $b = \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$, $-a + b$ will be negative as long as $\left(\frac{R}{2L}\right)^2 > \frac{1}{LC}$. Hence as t becomes infinite, the exponential terms become zero. The current therefore becomes zero and the charge on the condenser becomes CE. A graphical representation of the variation of current and charge is shown in Fig. 11. Both the current and charge are unidirectional and the phenomena are non-oscillatory.

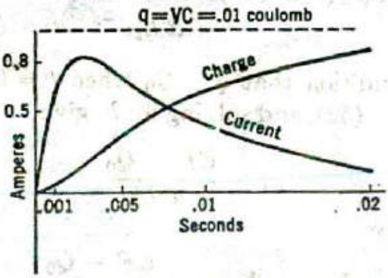


Fig. 11. Circuit containing R = 100 ohms, C = 100 μf, L = 0.1 henry when a d-c voltage V = 100 volts is impressed. Initial charge = 0.

Case II. When $\frac{R^2}{4L^2} < \frac{1}{LC}$, b becomes imaginary. To evaluate the expression for b it may be written as

$$\sqrt{(-1)\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)} = \sqrt{-1}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = j\beta$$

where

$$\beta = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

Equation (57) then becomes:

$$\begin{aligned} i &= \frac{CE - Q_0}{\sqrt{R^2C^2 - 4LC}}[\epsilon^{(-a+j\beta)t} - \epsilon^{(-a-j\beta)t}] \\ &= \frac{CE - Q_0}{\sqrt{R^2C^2 - 4LC}}[\epsilon^{-at}\epsilon^{j\beta t} - \epsilon^{-at}\epsilon^{-j\beta t}] \\ &= \frac{(CE - Q_0)\epsilon^{-at}}{\sqrt{R^2C^2 - 4LC}}[\cos\beta t + j\sin\beta t - \cos\beta t + j\sin\beta t] \\ &= \frac{(CE - Q_0)\epsilon^{-at}}{\sqrt{R^2C^2 - 4LC}}[2j\sin\beta t] \end{aligned} \tag{59}$$

For $\frac{R^2}{4L^2} < \frac{1}{LC}$, $R^2C^2 < 4LC$ and the denominator of equation (59) may be written as $j\sqrt{4LC - R^2C^2}$. Substituting $j\sqrt{4LC - R^2C^2}$ for $\sqrt{R^2C^2 - 4LC}$ in equation (59) gives the final expression for current in terms of all real quantities, as

$$i = \frac{2(CE - Q_0)\epsilon^{-\alpha t}}{\sqrt{4LC - R^2C^2}} \sin \beta t \tag{60}$$

Through a similar series of substitutions in and algebraic transformations of equation (58), the charge is found to be

$$q = CE - \frac{2(CE - Q_0)\sqrt{LC}\,\epsilon^{-\alpha t}}{\sqrt{4LC - R^2C^2}} \sin(\beta t + \theta) \tag{61}$$

where

$$\theta = \tan^{-1} \frac{\sqrt{4LC - R^2C^2}}{RC}$$

If the initial charge on the condenser is zero, the expressions for current and charge respectively are:

$$i = \frac{2CE\epsilon^{-\alpha t}}{\sqrt{4LC - R^2C^2}} \sin \beta t \tag{62}$$

$$q = CE - \frac{2CE\sqrt{LC}\epsilon^{-\alpha t}}{\sqrt{4LC - R^2C^2}} \sin(\beta t + \theta) \tag{63}$$

A graphical representation of equations (62) and (63) is shown in Fig. 12. Oscillogram 4 also shows the variation of current with time in another RLC circuit. It should be noted that the current is proportional to the slope dq/dt of the curve of charge variation at every instant. An examination of equation (62) shows that after an infinite time the current becomes zero which is the steady stat . Also equation (63) reveals that the charge becomes CE after an infinite time has elapsed. For all practical purposes, however, these final or steady states are sensibly reached after a few seconds; in some cases in a few microseconds. (See page 553 for explanation.) From the time of closing the switch to the time of reaching the final state the current and quantity oscillate about their final values. Case II is therefore called the oscilla-

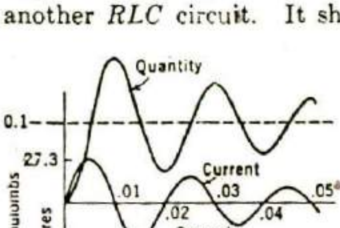
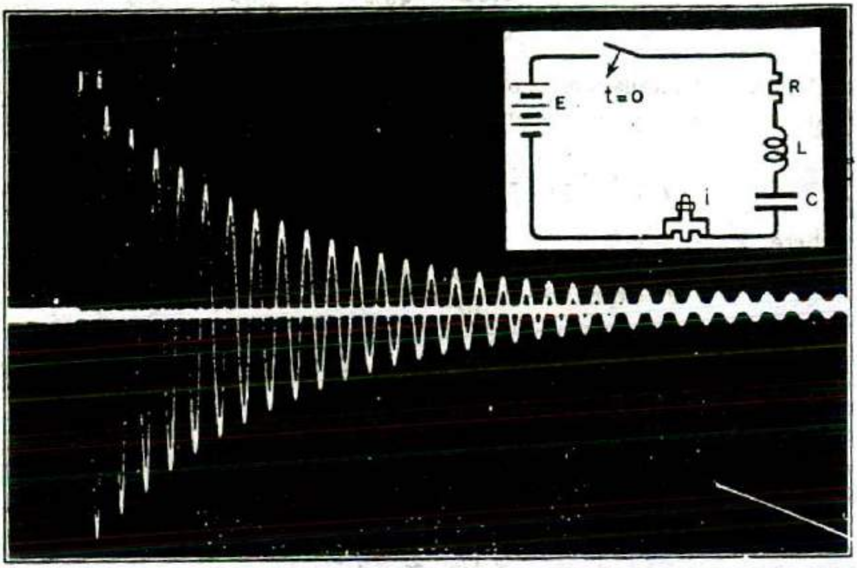


FIG. 12. Circuit containing $R = 5$ ohms, $C = 100$ μf, $L = 0.1$ henry, when a d-c voltage $V = 1000$ volts is impressed. Initial condenser charge = 0.

tory case. It is sometimes called the trigonometric case. Physically the current starts to flow and charges the condenser. Because of the low resistance compared with the inductance, the current continues to flow into the condenser when the magnetic field of the inductance collapses. The condenser charge thus overruns its final value and the potential drop across the condenser becomes higher than the impressed



OSCILLOGRAM 4. Photographic record of the current variation in a particular *RLC* series circuit which is suddenly energized with a constant potential difference.

voltage. The condenser then begins to discharge. These oscillations continue until the excess energy is dissipated in the resistance. The phenomenon is analogous to the case of a weight suspended from a spring with a low value of mechanical damping.

The frequency of the oscillation f_0 is obtained from equation (62) or (63). For a complete cycle βt must be 2π radians and since the time for a complete cycle is defined as the period T, we may write

$$\beta T = 2\pi$$

or

$$T = \frac{1}{f_0} = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}} \tag{64}$$

Hence

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \tag{65}$$

A comparison of the above equation with equation (10) on page 145 shows that the oscillatory frequency of the series RLC circuit when the resistance is zero is the same as the resonant frequency. Practically, they become the same when $R^2/4L^2$ is negligibly small compared with $1/LC$.

Case III. When $R^2/4L^2 = 1/LC$, $b = 0$ and the exponents of ϵ in equations (57) and (58) are real and negative as in case I. Hence the variations of current and charge are similar to those in case I. Case III is called the critical or limiting case and like case I is non-oscillatory.

Decay of Current and Charge in an RLC Circuit. The basic equation for this condition is:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = 0 \tag{66}$$

Equation (66) is obviously a special case of equation (49) where $E = 0$. Since equation (49) was solved in detail, the results of equation (66) will be found as special cases of equations (57), (58), (60), and (61) by making $E = 0$. It is plain that there will be three cases for the condition of zero voltage on (or short circuit of) the RLC circuit. These, as before, are the non-oscillatory case I where $R^2/4L^2 > 1/LC$, the oscillatory case 2 where $R^2/4L^2 < 1/LC$, and the critical case III, also non-oscillatory, where $R^2/4L^2 = 1/LC$.

Non-Oscillatory Case. The equations for current and charge for the non-oscillatory case are obtained from equations (57) and (58) respectively by setting $E = 0$. Thus

$$i = \frac{-Q_0}{\sqrt{R^2C^2 - 4LC}}\left[\epsilon^{(-a+b)t} - \epsilon^{(-a-b)t}\right] \tag{67}$$

and

$$q = Q_0\left[\frac{RC + \sqrt{R^2C^2 - 4LC}}{2\sqrt{R^2C^2 - 4LC}}\,\epsilon^{(-a+b)t} - \frac{RC - \sqrt{R^2C^2 - 4LC}}{2\sqrt{R^2C^2 - 4LC}}\,\epsilon^{(-a-b)t}\right] \tag{68}$$

A graphical representation of equations (67) and (68) is shown in Fig. 13. If desired, Q_0 can be replaced in the above equations by CV where V is the voltage drop across the condenser for the charge Q_0.

Oscillatory Case. If E is made equal to zero in equations (60) and (61), the equations for the decay of current and charge respectively are obtained as follows:

$$i = \frac{-2Q_0\epsilon^{-\alpha t}}{\sqrt{4LC - R^2C^2}} \sin \beta t \tag{69}$$

$$q = \frac{2Q_0\sqrt{LC}\epsilon^{-\alpha t}}{\sqrt{4LC - R^2C^2}} \sin (\beta t + \theta) \tag{70}$$

The variation of i and q as given by these equations is shown in Fig. 14. A comparison of equations (69) and (70) with equations (62) and (63) will show that the frequencies of oscillation for all of them are identical and are therefore given by equation (65).

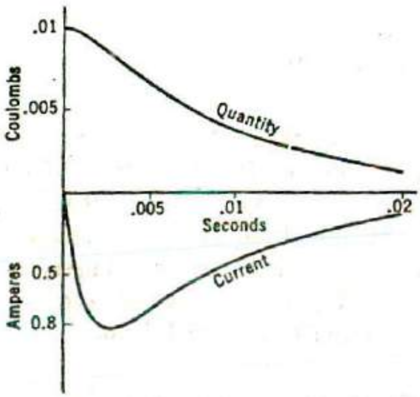


Fig. 13. Decay of current and charge or quantity in a circuit containing $R = 100$ ohms, $C = 100$ μf, and $L = 0.1$ henry when the initial charge on the condenser is 0.01 coulomb at a potential of 100 volts.

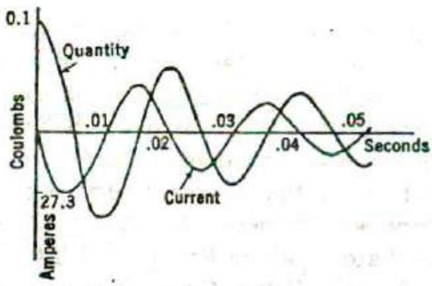


Fig. 14. Decay of current and charge or quantity in a circuit containing $R = 5$ ohms, $C = 100$ μf, and $L = 0.1$ henry when the initial charge on the condenser is 0.1 coulomb at a potential of 1000 volts.

Critical Case. Qualitatively this case is no different from the non-oscillatory case previously discussed. If b in equations (67) and (68) is made zero, the equations for the critical case result. Obviously Fig. 13 represents the general type of variation of current and charge for this condition.

Natural Circuit Behavior in Terms of Poles and Zeros. The concept of complex frequency was introduced in Chapter V in order to illustrate how steady-state circuit behavior could be obtained from the s-plane poles and zeros which characterized the network function which was under discussion. In actual practice, complex frequency probably finds a greater field of usefulness in transient analysis than it does in the analysis of the steady state. If, for example, the LRC series circuit is energized at $t = 0$ with the voltage having an angular frequency of ω_d

radians/second

$$e = E_m \sin(\omega_d t + 90°) = E_m \cos \omega_d t$$

the analysis may be carried forward with the aid of a complex exponential voltage excitation of the form

$$\mathbf{e} = \mathbf{E}\epsilon^{\mathbf{s}_d t} = \mathbf{E}\epsilon^{\alpha_d t}\epsilon^{j\omega_d t} \tag{71}$$

For this case we recognize that $\mathbf{E} = E_m \underline{/0°}$ and $\alpha_d = 0$. Since

$$\epsilon^{j\omega_d t} = \cos \omega_d t + j \sin \omega_d t \tag{72}$$

it is apparent that the *real* part of $\mathbf{e}$ corresponds to the desired excitation. We may carry through the solution for i and at the end retain only the real part of i as the real $i(t)$. In other words, $\epsilon^{j\omega_d t} = \cos \omega_d t + j \sin \omega_d t$ corresponds to two voltage excitations, only one of which is actually used to energize the circuit. As long as the circuit is linear, each exciting voltage develops its own current, and in the present instance we are interested in the current associated with the real part of $\mathbf{e}$. If $e = E_m \sin \omega_d t$ had been the specified driving voltage, it might have been more convenient to employ the imaginary part of $\mathbf{e}$. The fact that the $\mathbf{E}$ of $\mathbf{e} = \mathbf{E}\epsilon^{\mathbf{s}_d t}$ is a complex number actually allows us to use either the real or the imaginary part of the final solution depending largely upon the manner in which the actual driving voltage $e(t)$ is specified.

In solving for $i(t)$ by way of the complex exponential i, we let

$$\mathrm{i} = \mathrm{i}_s + \mathrm{i}_t \tag{73}$$

in

$$L\frac{d\mathrm{i}}{dt} + R\mathrm{i} + \frac{1}{C}\int \mathrm{i}\, dt = \mathbf{e} = \mathbf{E}\epsilon^{\mathbf{s}_d t} \tag{74}$$

where i_s is the steady-state component of the current

i_t is the transient component of the current

The fact that

$$\mathrm{i}_s = \mathbf{I}_s \epsilon^{\mathbf{s}_d t} \tag{75}$$

may be verified by direct substitution in equation (74). This substitution will also show that

$$\mathbf{I}_s = \frac{\mathbf{E}}{L\mathbf{s}_d + R + \dfrac{1}{C\mathbf{s}_d}} = \frac{\mathbf{E}}{L}\left(\frac{\mathbf{s}_d}{\mathbf{s}_d^2 + \dfrac{R}{L}\mathbf{s}_d + \dfrac{1}{LC}}\right) \tag{76}$$

It follows that

$$L\frac{d\mathrm{i}_t}{dt} + R\mathrm{i}_t + \frac{1}{C}\int \mathrm{i}_t\, dt = 0 \tag{77}$$

This homogeneous equation is evidently satisfied by

$$\mathbf{i}_t = \mathbf{A}\epsilon^{\mathbf{s}t} \tag{78}$$

provided that

$$\left(L\mathbf{s} + R + \frac{1}{C\mathbf{s}}\right) = 0 \tag{79}$$

Solving this equation for s yields

$$\mathbf{s} = \mathbf{s}_1 = -\frac{R}{2L} + j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \alpha_n + j\omega_n \tag{80}$$

$$\mathbf{s} = \mathbf{s}_2 = -\frac{R}{2L} - j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \alpha_n - j\omega_n \tag{81}$$

ω_n is the natural angular frequency of the circuit, and $\alpha_n = -R/2L$ is the natural damping factor. Since two values of s satisfy equation (79), we have

$$\mathbf{i}_t = \mathbf{A}_1\epsilon^{\mathbf{s}_1 t} + \mathbf{A}_2\epsilon^{\mathbf{s}_2 t} \tag{82}$$

In complex exponential form, the complete circuit current is found by substituting equation (82) in equation (73). Thus

$$\mathbf{i} = \mathbf{I}_s\epsilon^{\mathbf{s}_d t} + \mathbf{A}_1\epsilon^{\mathbf{s}_1 t} + \mathbf{A}_2\epsilon^{\mathbf{s}_2 t} \tag{83}$$

and the corresponding capacitor charge is

$$\mathbf{q} = \int \mathbf{i}\, dt = \frac{\mathbf{I}_s}{\mathbf{s}_d}\epsilon^{\mathbf{s}_d t} + \frac{\mathbf{A}_1}{\mathbf{s}_1}\epsilon^{\mathbf{s}_1 t} + \frac{\mathbf{A}_2}{\mathbf{s}_2}\epsilon^{\mathbf{s}_2 t} \tag{84}$$

The A's depend for their values upon the initial circuit conditions, for example, the values of i and q at $t = 0$. If $\mathbf{i} = 0$ and $\mathbf{q} = 0$ at $t = 0$, substitution in equation (83) gives

$$\mathbf{A}_1 + \mathbf{A}_2 = -\mathbf{I}_s = -\frac{\mathbf{E}}{L}\frac{\mathbf{s}_d}{(\mathbf{s}_d - \mathbf{s}_1)(\mathbf{s}_d - \mathbf{s}_2)} \quad \text{[See equation (76)]} \tag{85}$$

Similarly, substitution in equation (84) yields

$$\frac{\mathbf{A}_1}{\mathbf{s}_1} + \frac{\mathbf{A}_2}{\mathbf{s}_2} = -\frac{\mathbf{I}_s}{\mathbf{s}_d} = -\frac{\mathbf{E}}{L}\frac{1}{(\mathbf{s}_d - \mathbf{s}_1)(\mathbf{s}_d - \mathbf{s}_2)} \tag{86}$$

Solving equations (85) and (86) for $\mathbf{A}_1$ and $\mathbf{A}_2$ yields

$$\mathbf{A}_1 = \frac{\mathbf{E}}{L}\frac{\mathbf{s}_1}{(\mathbf{s}_d - \mathbf{s}_1)(\mathbf{s}_2 - \mathbf{s}_1)} \tag{87}$$

$$\mathbf{A}_2 = -\frac{\mathbf{E}}{L}\frac{\mathbf{s}_2}{(\mathbf{s}_d - \mathbf{s}_2)(\mathbf{s}_2 - \mathbf{s}_1)} \tag{88}$$

If these values of $\mathbf{A}_1$ and $\mathbf{A}_2$ along with $\mathbf{I}_s$ from equation (76) are substituted in equation (83), the expression for i is obtained. The actual circuit current is the real part of i. It may be expressed as

$$i(t) = \mathcal{R}\left[\frac{\mathbf{E}}{L}\left(\frac{\mathbf{s}_d\epsilon^{\mathbf{s}_d t}}{(\mathbf{s}_d - \mathbf{s}_1)(\mathbf{s}_d - \mathbf{s}_2)} + \frac{\mathbf{s}_1\epsilon^{\mathbf{s}_1 t}}{(\mathbf{s}_d - \mathbf{s}_1)(\mathbf{s}_2 - \mathbf{s}_1)} - \frac{\mathbf{s}_2\epsilon^{\mathbf{s}_2 t}}{(\mathbf{s}_d - \mathbf{s}_2)(\mathbf{s}_2 - \mathbf{s}_1)}\right)\right] \quad (89)$$

The price paid for using complex exponential forms of current and voltage is the transformation back to real current or voltage at the close of the solution. If, for example, we want the actual *steady-state component* of current in equation (89), we evaluate the real part of the steady-state component of this current. This is

$$\mathcal{R}\left[\frac{\mathbf{E}}{L}\frac{\mathbf{s}_d\epsilon^{\mathbf{s}_d t}}{(\mathbf{s}_d - \mathbf{s}_1)(\mathbf{s}_d - \mathbf{s}_2)}\right] = \mathcal{R}\left[\frac{\mathbf{E}}{L}\frac{\mathbf{s}_d\epsilon^{\mathbf{s}_d t}}{\mathbf{s}_d^2 + \dfrac{R}{L}\mathbf{s}_d + \dfrac{1}{LC}}\right]$$

For $\mathbf{s}_d = j\omega_d$ and $\mathbf{E} = E_m\underline{/0^\circ}$ the expression within the bracket becomes

$$\frac{E_m}{(L/j\omega_d)}\frac{\epsilon^{j\omega_d t}}{\left(\dfrac{1}{LC} - \omega_d^2\right) + j\dfrac{R}{L}\omega_d} = \frac{E_m(\cos\omega_d t + j\sin\omega_d t)}{R + j\left(\omega_d L - \dfrac{1}{\omega_d C}\right)}$$

The real part of the above expression is

$$\mathcal{R}\left[\frac{E_m}{(L/j\omega_d)}\frac{\epsilon^{j\omega_d t}}{\left(\dfrac{1}{LC} - \omega_d^2\right) + j\dfrac{R}{L}\omega_d}\right]$$

$$= \frac{E_m\left[R\cos\omega_d t + \left(\omega_d L - \dfrac{1}{\omega_d C}\right)\sin\omega_d t\right]}{R^2 + \left(\omega_d L - \dfrac{1}{\omega_d C}\right)^2}$$

$$= [E_m\cos(\omega_d t - \theta)]/Z \quad (90)$$

where $Z = \sqrt{R^2 + \left(\omega_d L - \dfrac{1}{\omega_d C}\right)^2}$ and $\theta = \tan^{-1}\dfrac{\left(\omega_d L - \dfrac{1}{\omega_d C}\right)}{R}$

Transformations from the *s* plane to the *t* plane are often accomplished by means of Laplace transforms, a technique which the reader will encounter repeatedly in later courses.

The *RLC* Series Circuit with Alternating Voltage Suddenly Applied. The basic voltage equation of the *RLC* circuit shown in Fig. 15 is

$$L\frac{di}{dt} + Ri + \frac{q}{C} = E_m \sin(\omega t + \lambda) \tag{91}$$

The above equation can be put in terms of one dependent variable by differentiating the entire equation with respect to the independent variable, t. Differentiating as indicated above,

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}\frac{dq}{dt} = E_m\omega \cos(\omega t + \lambda) \tag{92}$$

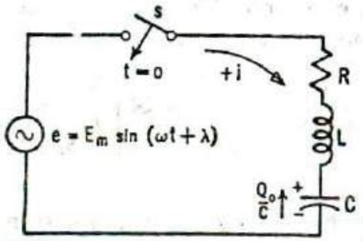


FIG. 15. An *RLC* series circuit energized with an alternating voltage at $t = 0$.

Dividing through by L and substituting i for dq/dt results in

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = \frac{E_m\omega}{L}\cos(\omega t + \lambda) \tag{93}$$

Equation (93) is a linear differential equation of the second order, first degree, the solution of which consists of the sum of a complementary function or transient term and the particular integral or steady-state term. The former is obtained as indicated previously. The auxiliary equation is

$$\alpha^2 + \frac{R}{L}\alpha + \frac{1}{LC} = 0 \tag{94}$$

and

$$\alpha = \frac{-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2} = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \tag{95}$$

Let

$$a = \frac{R}{2L} \quad \text{and} \quad b = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \tag{96}$$

By definition

$$\alpha_1 = -a + b \quad \text{and} \quad \alpha_2 = -a - b \tag{97}$$

The transient term of the complete solution is

$$i_t = c_1\epsilon^{(-a+b)t} + c_2\epsilon^{(-a-b)t} \tag{98}$$

The steady-state term of the complete solution is

$$i_s = \frac{E_m}{Z}\sin(\omega t + \lambda - \theta) \tag{99}$$

where

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \quad \text{and} \quad \theta = \tan^{-1}\frac{\left(\omega L - \frac{1}{\omega C}\right)}{R}$$

The complete expression for current becomes

$$i = \frac{E_m}{Z}\sin(\omega t + \lambda - \theta) + c_1\epsilon^{(-a+b)t} + c_2\epsilon^{(-a-b)t} \tag{100}$$

The two physical facts from which c_1 and c_2 can be evaluated are the state of current and the state of charge that exist in the circuit at the instant of closing the switch. Let it be assumed that

$$\left.\begin{aligned} i &= 0 \\ q &= Q_0 \end{aligned}\right\} \quad \text{at} \quad t = 0 \tag{101}$$

If the original voltage equation has been differentiated it is important that the initial conditions be imposed upon the original voltage equation rather than upon one of the differentiated forms. In the present case the initial conditions can be imposed upon equations (91) and (100).

Imposing the initial conditions upon equation (91) yields

$$L\left[\frac{di}{dt}\right]_{t=0} + \frac{Q_0}{C} = E_m \sin\lambda$$

or

$$L\left[\frac{E_m\omega}{Z}\cos(\lambda - \theta) + c_1\alpha_1 + c_2\alpha_2\right] + \frac{Q_0}{C} = E_m \sin\lambda$$

From which

$$c_1\alpha_1 + c_2\alpha_2 = \frac{E_m}{L}\sin\lambda - \frac{Q_0}{LC} - \frac{E_m\omega}{Z}\cos(\lambda - \theta) \tag{102}$$

Imposing the initial conditions on equation (100) results in

$$0 = \frac{E_m}{Z}\sin(\lambda - \theta) + c_1 + c_2$$

or

$$c_1 + c_2 = -\frac{E_m}{Z}\sin(\lambda - \theta) \tag{103}$$

Equations (102) and (103) may be solved simultaneously for c_1 and c_2. From equation (103)

$$c_2 = -\frac{E_m}{Z}\sin(\lambda - \theta) - c_1 \tag{104}$$

Substituting the above value of c_2 into equation (102) yields

$$c_1\alpha_1 - \left[\frac{E_m}{Z}\sin(\lambda - \theta)\right]\alpha_2 - \epsilon_1\alpha_2 = \frac{E_m}{L}\sin\lambda - \frac{Q_0}{LC} - \frac{E_m\omega}{Z}\cos(\lambda - \theta)$$

Whence

$$c_1(\alpha_1 - \alpha_2) = \frac{1}{L}\left[E_m\sin\lambda - \frac{Q_0}{C} - \frac{E_m\omega L}{Z}\cos(\lambda - \theta)\right] + \alpha_2\frac{E_m}{Z}\sin(\lambda - \theta) \tag{105}$$

It will be remembered that

$$\alpha_1 = (-a + b) \quad \text{and} \quad \alpha_2 = (-a - b)$$

Therefore

$$\alpha_1 - \alpha_2 = 2b$$

and

$$\frac{\alpha_2}{\alpha_1 - \alpha_2} = -\frac{a}{2b} - \frac{b}{2b} = -\frac{R}{4Lb} - \frac{1}{2}$$

Dividing equation (105) through by $(\alpha_1 - \alpha_2)$ and making substitutions for $(\alpha_1 - \alpha_2)$ and α_2,

$$c_1 = \frac{1}{2bL}\left[E_m\sin\lambda - \frac{Q_0}{C} - \frac{E_m\omega L}{Z}\cos(\lambda - \theta)\right] - \frac{R}{4bL}\frac{E_m}{Z}\sin(\lambda - \theta) - \frac{E_m}{2Z}\sin(\lambda - \theta)$$

Collecting the b terms in the above equation,

$$c_1 = \frac{1}{2bL}\left[E_m\sin\lambda - \frac{Q_0}{C} - \frac{E_m\omega L}{Z}\cos(\lambda - \theta) - \frac{E_m R}{2Z}\sin(\lambda - \theta)\right] - \frac{E_m}{2Z}\sin(\lambda - \theta) \tag{106}$$

From equation (103) it is evident that

$$c_2 = -c_1 - \frac{E_m}{Z}\sin(\lambda - \theta) \tag{107}$$

Therefore,

$$c_2 = -\frac{1}{2bL}\left[E_m \sin\lambda - \frac{Q_0}{C} - \frac{E_m\omega L}{Z}\cos(\lambda - \theta) - \frac{E_m R}{2Z}\sin(\lambda - \theta)\right] - \frac{E_m}{2Z}\sin(\lambda - \theta) \quad (108)$$

For the sake of simplicity in writing, the following abbreviation will be adopted:

$$\left[E_m \sin\lambda - \frac{Q_0}{C} - \frac{E_m\omega L}{Z}\cos(\lambda - \theta) - \frac{E_m R}{2Z}\sin(\lambda - \theta)\right] = E_d \quad (109)$$

It will be observed that E_d is a voltage which is governed in magnitude by E_m, λ, Q_0, and the circuit parameters. The complete expression for current can now be written in terms of the applied voltage, the initial condenser charge, and the circuit parameters.

$$i = \frac{E_m}{Z}\sin(\omega t + \lambda - \theta) + \frac{E_d\epsilon^{-at}}{bL}\left[\frac{\epsilon^{bt} - \epsilon^{-bt}}{2}\right] - \frac{E_m}{Z}\sin(\lambda - \theta)\,\epsilon^{-at}\left[\frac{\epsilon^{bt} + \epsilon^{-bt}}{2}\right] \quad (110)$$

The transient component of the current consists of two terms, each of which is damped out with the damping factor ϵ^{-at} or $\epsilon^{-Rt/2L}$. The transient terms may be given different mathematical forms depending upon the nature of the symbol b. Since b is equal to $\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$ it is evident that b may be either real or imaginary. A singular condition exists when b is equal to zero.

Case I. If $R^2/4L^2$ is greater than $1/LC$, b is a real number and the complete expression for current in the RLC series circuit may be written as

$$i = \underbrace{\frac{E_m}{Z}\sin(\omega t + \lambda - \theta)}_{\text{steady-state term}} + \underbrace{\frac{E_d}{bL}\epsilon^{-at}\sinh bt - \frac{E_m}{Z}\sin(\lambda - \theta)\epsilon^{-at}\cosh bt}_{\text{transient terms}} \quad (111)$$

The above expression follows directly from equation (110) since, if b is real,

$$\frac{\epsilon^{bt} - \epsilon^{-bt}}{2} = \sinh bt \quad \text{and} \quad \frac{\epsilon^{bt} + \epsilon^{-bt}}{2} = \cosh bt$$

Both transient terms are damped out by $\epsilon^{-Rt/2L}$. The damping factor

$Rt/2L$ is relatively large when $\frac{R^2}{4L^2} > \frac{1}{LC}$ because of the relatively large value of $R/2L$. In general, the transient terms in this case are not predominantly large as compared with the steady-state term.

Case II. If $R^2/4L^2$ is less than $1/LC$, b takes the form of an imaginary number and a change in notation becomes desirable. Let

$$b = j\beta \quad \text{where} \quad \beta = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

If $\frac{R^2}{4L^2} < \frac{1}{LC}$, β is a real number and b in equation (110) can be replaced by its equivalent, $j\beta$. In this connection j has its customary significance, namely, $\sqrt{-1}$.

$$i = \underbrace{\frac{E_m}{Z}\sin(\omega t + \lambda - \theta)}_{\text{steady-state term}} + \underbrace{\frac{E_d}{\beta L}\epsilon^{-at}\sin\beta t - \frac{E_m}{Z}\sin(\lambda - \theta)\epsilon^{-at}\cos\beta t}_{\text{transient terms}} \quad (112)$$

The above equation comes directly from equation (110) if it is recognized that the analytical expressions for $\sin\beta t$ and $\cos\beta t$ are

$$\frac{\epsilon^{j\beta t} - \epsilon^{-j\beta t}}{2j} = \sin\beta t \quad \text{and} \quad \frac{\epsilon^{j\beta t} + \epsilon^{-j\beta t}}{2} = \cos\beta t$$

The two transient terms of equation (112) are exponentially damped sine and cosine terms of like frequency. Since the damping factors are identical, the sine and cosine terms can be combined by the method outlined on page 241. If the two transient terms are combined, equation (112) takes the following form:

$$i = \underbrace{\frac{E_m}{Z}\sin(\omega t + \lambda - \theta)}_{\text{steady-state term}} + \underbrace{I_t\epsilon^{-at}\sin(\beta t - \sigma)}_{\text{transient term}} \quad (113)$$

where

$$I_t = \sqrt{\left[\frac{E_d}{\beta L}\right]^2 + \left[\frac{E_m}{Z}\sin(\lambda - \theta)\right]^2}$$

and

$$\sigma = \tan^{-1}\frac{E_m\beta L\sin(\lambda - \theta)}{E_d Z}$$

In the present case the complete expression for current consists of two sinusoidal terms. The frequency of the steady-state term, $\omega/2\pi$, is determined solely by the frequency of the applied voltage; that of the transient term, $\beta/2\pi$, is governed entirely by the circuit parameters, R, L, and C. The frequency of the transient term may be less than, equal to, or greater than that of the applied voltage. In any event the transient oscillation disappears as soon as the damping factor, $\epsilon^{-Rt/2L}$, causes the transient term to become sensibly equal to zero.

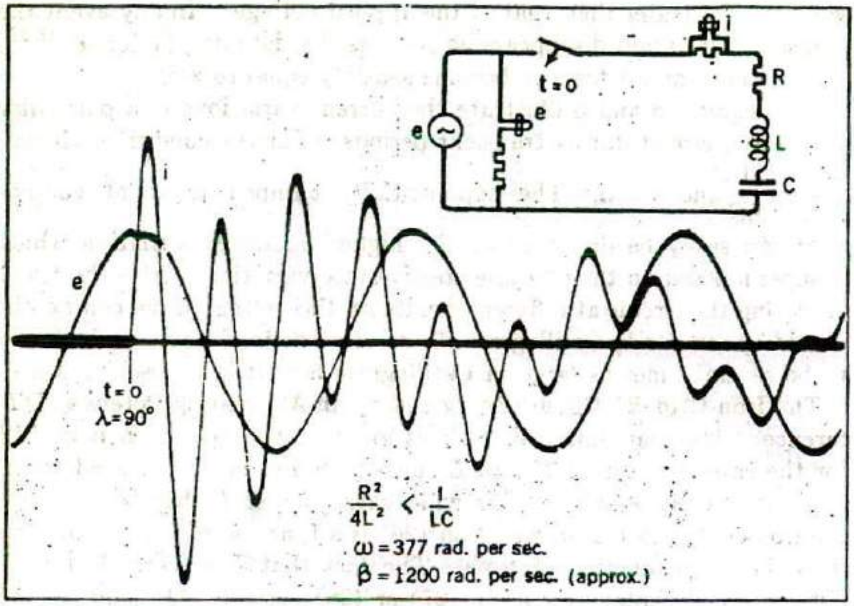
Oscillograms 5 and 6 illustrate the current variations in a particular RLC series circuit during transient periods. For the conditions shown, $\frac{R^2}{4L^2} < \frac{1}{LC}$ and $\beta > \omega$. The exponentially damped transient component can easily be discerned as the higher frequency variation which is superimposed on the 60-cycle steady-state variation. Also the effect of closing the circuit at different points on the voltage wave can be observed by comparing Oscillograms 5 and 6. The transient term is shown to be several times as large in Oscillogram 5 as it is in Oscillogram 6.

The Iron-Clad RL Circuit Energized by an Alternating Potential Difference. The mathematical analysis given in the article on page 558 for the case of constant R and L cannot, in general, be applied to an iron-clad circuit because of the wide variations of L that occur. For the iron-clad circuit, L in equation (26) is a function of i which in turn is an intricate function of time. The fact that L is variable makes both the coefficients of equation (27) or (28) variable. In general, the solution of differential equations with variable coefficients is a difficult task. It is plain that no general solution can be obtained because the variation of L in any particular case must necessarily be defined in terms of particular constants rather than in terms of arbitrary constants. Although the variation of L can sometimes be approximated with the aid of simple functions, the actual variation in many cases of importance cannot be expressed in terms of practical mathematical functions.

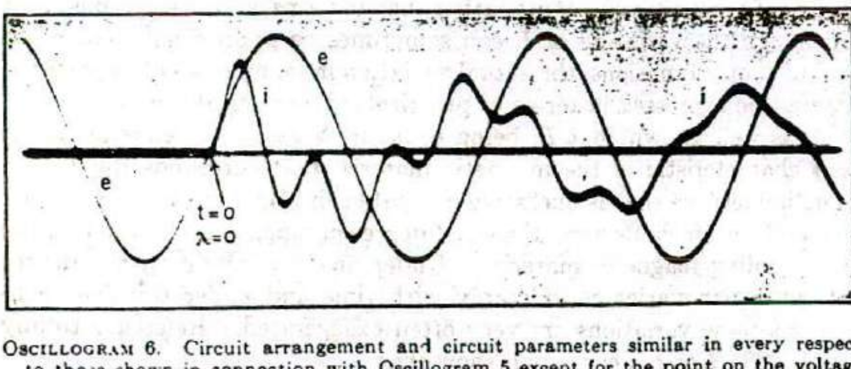
It is well known that L, being equal to $N\,d\phi/di$, depends upon the ϕ–i characteristic of the magnetic material that surrounds the L coil. The inductance that is operative in establishing an $L\,di/dt$ voltage drop depends for its value upon the exact degree of magnetic saturation of the surrounding magnetic material. Under any a-c condition the degree of saturation varies considerably with time and under transient conditions these variations are very often exaggerated. Reference to any typical B–H or ϕ–i curve will show that

$$L = N\frac{d\phi}{di}$$

is much greater over the straight portion of the curve than it is after the upper bend is reached. This fact plays an important role in determining the current inrush to iron-clad circuits, because, in general, circuits of this character are highly inductive and the variable L becomes an extremely influential parameter.



Oscillogram 5. Photographic record of the current variation in a particular *RLC* series circuit which is suddenly energized with an alternating potential difference. *R*, *L*, and *C* are sensibly constant.



Oscillogram 6. Circuit arrangement and circuit parameters similar in every respect to those shown in connection with Oscillogram 5 except for the point on the voltage wave at which the circuit is energized. In the present case $\lambda = 0°$.

Circuit problems involving variable parameters can be solved by step-by-step methods provided the exact variation of the parameters is known. In the present case the variation of L is known if the $N\phi/i$ characteristic of the surrounding magnetic material is known. The data usually take the form of either the ϕ–i characteristic and the number of turns or the B–H characteristic, the dimensions of the magnetic circuit, and the number of turns. In any event it is somewhat more direct to substitute for $L\ di/dt$ [in equation (26)] its equivalent $N\ d\phi/dt$ value. The basic equation then becomes

$$N \frac{d\phi}{dt} + Ri = E_m \sin (\omega t + \lambda) \tag{114}$$

where ϕ is expressed in webers if the other quantities are expressed in practical units.

In many iron-clad circuits the maximum magnitude of the Ri term is of the order of 1 per cent of the maximum magnitude of the applied voltage. Under these conditions the $N\ d\phi/dt$ component of equation (114) is very nearly equal to the applied voltage and in approximate steady-state solutions the Ri drop can be neglected. The Ri drop cannot be entirely neglected in the transient solution of the problem because it is instrumental in helping to govern the maximum value of the initial current inrush. The resistance is also an important factor in governing the length of time required for the iron-clad circuit to adjust itself to steady-state operating conditions.

If the Ri drop is neglected and if it is assumed that $\lambda = 0$, equation (114) reduces to

$$N \frac{d\phi}{dt} = E_m \sin \omega t \tag{115}$$

from which

$$\phi = \frac{E_m}{N} \int \sin \omega t\, dt = - \frac{E_m}{\omega N} \cos \omega t + c_1 \tag{116}$$

The constant of integration c_1 may be evaluated in terms of the residual magnetism. ϕ may be at either positive or negative residual values at $t = 0$, and in general the exact state of residual magnetism is unknown. A compromise may be made by assuming that $\phi = 0$ at $t = 0$ unless the maximum possible current inrush is to be determined. In this case a maximum value of positive residual magnetism is assumed if the applied voltage is taken as $E_m \sin \omega t$. The manner in which residual magnetism helps to determine the initial current inrush will soon be apparent.

Assuming that $\phi = 0$ at $t = 0$, c_1 of equation (116) becomes

$$c_1 = \frac{E_m}{\omega N} \tag{117}$$

Under these conditions

$$\phi = \frac{E_m}{\omega N} (1 - \cos \omega t) \tag{118}$$

or

$$\phi = \phi_m\,(1 - \cos \omega t) \tag{119}$$

where $\phi_m = E_m/\omega N$, the approximate maximum value of the magnetic flux under steady-state operating conditions. Since $(\cos \omega t)$ varies between $+1$ and -1, it is plain that the flux variation as defined by

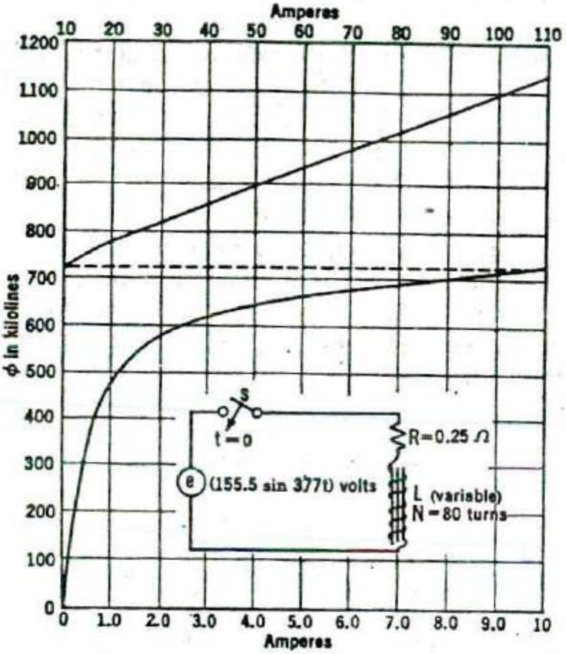


FIG. 16. Magnetization curve of a particular iron-clad *RL* circuit.

equation (119) varies from zero at $t = 0$ to $2\phi_m$ at $t = T/2$. In order to produce a flux equal to $2\phi_m$, the iron-clad inductance coil must draw a particular value of magnetizing current as defined by the ϕ–i characteristic of the magnetic circuit. For example, in the circuit shown in Fig. 16

$$\phi_m = \frac{155.5}{377 \times 80} = 0.00516 \text{ weber}$$

or

$$\phi_m = 0.00516 \times 10^5 = 516 \text{ kilolines}$$

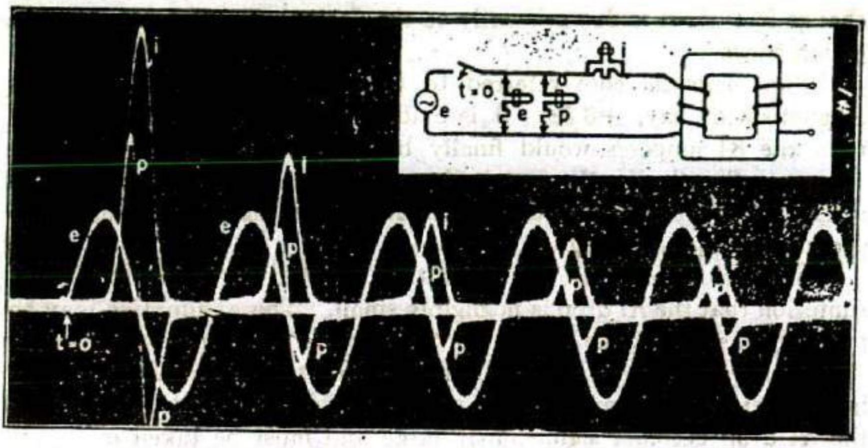
Reference to the magnetization curve will show that the current required to establish ϕ_m is approximately 1.2 amperes, whereas current required to establish $2\phi_m$ is approximately 84 amperes. This great change in current is due primarily to the flattening out of the magnetization curve.

If the magnetic core referred to above had possessed a residual magnetism of, say, $+0.5\phi_m$, it is evident that a much larger current than the 84 amperes would finally be required to produce the $2\phi_m$ change in flux during the first half cycle. Actually the initial current inrush to an iron-clad circuit is somewhat less than that required to produce a $2\phi_m$ flux change.

It will be remembered that equation (119) carries with it the assumption that the Ri drop is negligibly small. This assumption may be perfectly justified if the flux is worked between its normal steady-state values of $+\phi_m$ and $-\phi_m$. But in attempting to produce a $2\phi_m$ change in flux starting with zero flux, the circuit draws such a large current that the Ri drop becomes significantly large and must be taken into consideration. Under the above conditions the Ri drop consumes an appreciable portion of the applied voltage during the second quarter cycle after the switch is closed, thereby reducing the magnitude of the $N\ d\phi/dt$ component in this region. As a result, ϕ reaches a maximum value of something less than $2\phi_m$ shortly before $t = T/2$, and it is at this point that the maximum instantaneous current occurs.

The ordinary iron-core transformer with open secondary operates as a simple iron-core RL circuit. Oscillogram 7 illustrates the nature of the starting current taken by the primary winding of an iron-core transformer when the secondary is open-circuited. In this particular case the initial peak current is considerably more than 100 times the steady-state maximum value of primary current when the secondary is open-circuited. However, the initial current inrush reaches a peak value which is only about 4.5 times the value of the maximum full-load current of the transformer. For the case shown in Oscillogram 7 the actual transient period is of approximately 0.5-second duration. Only the early part of the transient period is shown in the oscillogram.

The Method of Finite Differences. Although it involves step-by-step calculations, the "method of finite differences" is very often employed in circuit analysis when variable parameters are encountered. The step-by-step calculations are based upon the assumption that the parameters remain sensibly constant over small finite intervals of time. Usually the basic voltage equation is rewritten so that all differentials take the form of finite increments. The circuit voltage and current are then assumed to remain constant over an arbitrarily assigned increment of time, Δt. As a first approximation the applied voltage and current are assumed to be constant at their "start-of-period" values. If, then, after assigning a particular value to Δt, only a single unknown incre-



OSCILLOGRAM 7. Iron-core transformer current and power inrushes when the primary is energized at the $e = 0$ point on the voltage wave.

e = 60-cycle applied emf $\quad$ E (eff.) = 117 volts
i = instantaneous current $\quad$ Peak i = 174 amperes
p = instantaneous power $\quad$ Peak p = 10.5 kw

Steady-state conditions: P_{av} = 30 watts, I_{eff} = 0.825 ampere.
Transformer rating: 115 volts, 3 kva, 26.1 amperes, 60 cycles.

mental quantity remains in the equation, it can be solved for by methods of elementary algebra. The process can best be illustrated by means of an example.

The predetermination of the initial current inrush to an iron-clad circuit will serve to illustrate the details of the method of finite differences. If finite differences of ϕ and t are employed, equation (114) takes the following form:

$$N \frac{\Delta\phi}{\Delta t} + Ri = E_m \sin\left(\sum \Delta\rho + \lambda\right) \tag{120}$$

where $\sum \Delta\rho = \sum \omega\, \Delta t$, the angular displacement along the voltage wave of the point under investigation from the point of $t = 0$.

Judgment must be exercised in the choice of Δt in any particular case. The selection of the size of Δt in a-c circuits is governed largely by the magnitude of ω. If points every 10° along the voltage wave are desired, then each Δt is taken as $\frac{1}{18}$ of π/ω second. The choice of smaller increments will, of course, make for more accurate solutions. Δt should never be chosen so large that significant changes in the parameters take place within the time interval represented by Δt.

At the beginning of a period i and $E_m \sin(\sum \Delta\rho + \lambda)$ have particular values. Letting $E_m \sin(\sum \Delta\rho + \lambda)$ be written as e and solving equation (120) for $\Delta\phi$ results in

$$\Delta\phi = \frac{(e - Ri)\,\Delta t}{N} \tag{121}$$

If practical units of e, R, i, and t are employed in the above equation, $\Delta\phi$ is given in webers.

Various refinements can be employed to improve the accuracy of the method of finite differences as outlined above. Very often, however, the improved accuracy is not warranted because of the uncertainties that surround the initial conditions and other experimental data.

Numerical Example. (1) The emf applied to the iron-clad RL circuit shown in Fig. 16 is

$$e = \sqrt{2} \times 110 \sin 377t \text{ volts}$$

This signifies that a 60-cycle voltage, the effective value of which is 110 volts, is applied to the circuit at the point of zero voltage where de/dt is positive. A simpler way of expressing the same thing is to say that a 110-volt 60-cycle voltage is applied at $\lambda = 0$.

(2) $N = 80$ turns and $R = 0.25$ ohm as indicated in the circuit diagram of Fig. 16.

(3) The residual magnetism is zero, and the flux varies in accordance with the ϕ–i curve given in Fig. 16 for the first half cycle of the applied emf. [redacted] Therefore the hysteresis effects which occur after the first half cycle and which complicate the determination of succeeding maxima can be neglected. Let the numerical coefficients enumerated above be inserted into equation (120).

$$80\,\frac{\Delta\phi}{\Delta t} + 0.25i = 155.5 \sin \sum \Delta\rho$$

or

$$\Delta\phi = \frac{(155.5 \sin \sum \Delta\rho - 0.25i)}{80}\,\Delta t \text{ webers}$$

It will be somewhat more convenient in the present example if $\Delta\phi$ is reckoned in kilolines.

$$\Delta\phi = \frac{(e - 0.25i)\,\Delta t}{80} \times 10^5 \text{ kilolines}$$

where $e = 155.5 \sin \sum \Delta\rho$.

Each time increment will be taken as 0.0005 second, a value which corresponds to an angular displacement along a 60-cycle wave of 10.8°.

The initial conditions are such as to make both e and i zero at $t = 0$. Assuming that both e and i maintain zero value throughout the first time interval, the change in flux during this period, $\Delta\phi_1$, is equal to zero.

At the beginning of the second period, $\sum \Delta t = 0.0005$ second and $e = 155.5 \sin 10.8°$ volts. For each interval i is assumed to have its "start-of-period" value, which in this case is zero.

$$\Delta\phi_2 = \frac{(29.1 - 0)\,0.0005}{80} \times 10^5$$
$$= 18.2 \text{ kilolines}$$

At the close of the second or the beginning of the third period the current is assumed to have acquired the value required for the establishment of $\Delta\phi_2$. Reference to the magnetization curve will show that the establishment of 18.2 kilolines requires approximately 0.03 ampere.

At the beginning of the third period, $\sum \Delta t = 0.001$ second and $e = 155.5 \sin 21.6°$ volts.

$$\Delta\phi_3 = \frac{(57.2 - 0.25 \times 0.03)\, 0.0005}{80} \times 10^5$$

$$= 35.7 \text{ kilolines}$$

The current required to establish $\sum \Delta\phi$ [(18.2 + 35.7) kilolines] is approximately 0.09 ampere. Other $\Delta\phi$'s can be added by the step-by-step method outlined above. The results of a series of such calculations are shown in Table I.

TABLE I

Period	$\Sigma\, \Delta t$ seconds	$\Sigma\, \Delta\rho$ degrees	$E_m \sin \Sigma\, \Delta\rho$ volts	Ri volts	$\Delta\phi$ kilolines	$\Sigma\, \Delta\phi$ kilolines	i amperes
1	0	0	0	0	0	0	0
2	0.0005	10.8	29.1	0	18.2	18.2	0.03
3	0.0010	21.6	55.7	Negligible	35.7	53.9	0.09
4	0.0015	32.4	83.3	Negligible	52.1	106.0	0.18
5	0.0020	43.2	106.0	Negligible	66.0	172.0	0.29
6	0.0025	54.0	126.0	Negligible	79.0	251.0	0.43
7	0.0030	64.8	141.0	Negligible	88.0	339.0	0.58
8	0.0035	75.6	151.0	Negligible	94.0	433.0	0.75
9	0.0040	86.4	155.0	Negligible	97.0	530.0	1.4
10	0.0045	97.2	154.0	Negligible	96.0	626.0	3.1
11	0.0050	108.0	148.0	0.78	92.0	718.0	9.0
12	0.0055	118.8	136.0	2.25	84.0	802.0	25.0
13	0.0060	129.6	120.0	6.25	71.0	873.0	44.5
14	0.0065	140.4	99.2	11.1	55.0	928.0	58.0
15	0.0070	151.2	74.9	14.5	38.0	966.0	66.5
16	0.0075	162.0	48.1	16.6	20.0	986.0	72.0
17	0.0080	172.8	19.5	18.0	1.0	987.0	72.0
18	0.0085	183.6	−9.8	18.0	−17.0	970.0	67.0
19	0.0090	194.4	−38.7	16.7	−35.0	935.0	59.0
20	0.0095	205.2	−66.2	14.7	−51.0	884.0	47.0

It will be noted that the current reaches a maximum value of approximately 72 amperes at $t = 0.008$ second. This corresponds to a point approximately 173° out along the voltage wave from the point at which the switch is closed, namely, the $e = 0$ point.

The general trend of the current variation is similar to that shown in Oscillogram 7. It will be observed that the current values are relatively very small during the first quarter cycle after the switch is closed. It is during this period that the Ri drop is negligibly small.

The change of flux that occurs during the period of negligible Ri drop can be calculated straightforwardly, and it may be of interest to compare the step-by-step results with a result which is very nearly accurate from a theoretical point of view. From equation (118)

$$\phi = \frac{155.5}{377 \times 80} (1 - \cos 377t) \text{ webers}$$

If t is taken as 0.0045 second, $(377t)$ is equal to approximately 1.7 radians or 97.2°. At $t = 0.0045$ second,

$$\phi = \frac{155.5}{377 \times 80} (1 - \cos 97.2°) \times 10^5 \text{ kilolines}$$

or

$$\phi = 579 \text{ kilolines} \quad \text{at} \quad t = 0.0045 \text{ second}$$

The value ϕ at $t = 0.0045$ second as determined by the step-by-step method is 626 kilolines. (See Table I.)

PROBLEMS

3. (*a*) Find the current in a coil containing $L = 1$ henry and $R = 0.4$ ohm one second after applying a d-c voltage of 10 volts.

(*b*) What will the current be after 2.5 seconds?

(*c*) What is the value of the voltage accelerating [redacted] 2.5 seconds?

4. A coil has 0.1 henry and 1 ohm resistance and carries 10 amperes. If its terminals are suddenly short-circuited, what will be the value of current 0.1 second later? How long will it take the current to fall to 0.1 ampere?

5. Find the number of ohms resistance which may be placed in series with an inductance of 0.1 henry so as to permit the current in the circuit to reach 63.2 per cent of its final value in 2 seconds after the voltage is applied.

6. Ten volts direct current are applied to a 0.1-ohm resistance in series with a 1-henry inductance.

(*a*) Calculate the energy stored in the inductance 10 seconds after the voltage is applied. State units.

(*b*) Derive the expression for the energy dissipated in the resistance in the time t after the voltage is applied.

7. A 50-μf condenser with no initial charge is in series with a 1-megohm resistor. How long will it take to attain 63.2 per cent of its final charge?

8. A 50-μf condenser has stored 0.1 coulomb.

(*a*) If it is discharged through a 1000-ohm resistor, how long will it take until it has 0.001 coulomb remaining?

(*b*) What will be the initial value of current?

(*c*) What will be the value of current when 0.001 coulomb remains on the condenser?

9. A 100-μf condenser has a charge of 0.1 coulomb. If it is discharged through a 10,000-ohm resistance, what will be the amount of energy in joules remaining in the condenser 1 second after the discharge is started?

10. A d-c voltage was applied to a resistance of 10,000 ohms in series with a 10C-μf condenser. After 1 second there were 19.98 joules stored in the condenser which had no initial charge. How many volts were applied to the circuit?

11. A 1-megohm resistance is in series with a 1-μf condenser. A d-c voltage of 100 volts is suddenly applied to the circuit.

(*a*) Calculate the energy stored in the condenser 1 second after the voltage is applied.

(*b*) Derive the expression for the energy dissipated in the resistance during the first second after the voltage is applied.

(*c*) How much energy will be dissipated in the resistance in charging the condenser to full charge?

12. What fraction of total charge will the condenser in Problem 11 have after 2 seconds?

13. A voltage $e = 100 \sin [377t + (\pi/4)]$ is impressed on a 1-henry inductance coil containing 1 ohm resistance. What are the values of the steady and the transient components of current at $t = 0$?

14. A voltage $e = 100 \sin (377t + 30°)$ is impressed on a 100-μf condenser having no initial charge and containing 1 ohm resistance.

(*a*) What are the values of the steady and transient components of charge at $t = 0$?

(*b*) What are the corresponding values of current?

15. A circuit contains $R = 100$ ohms, $C = 200$ μf, and $L = 0.1$ henry in series. If a d-c voltage of 50 volts is impressed, calculate the current and charge after 0.01 second, assuming no initial charge on the condenser.

16. A circuit contains $R = 5$ ohms, $L = 0.1$ henry, and $C = 200$ μf in series.

(*a*) Calculate the current and charge 0.01 second after 1000 volts are impressed if there was no initial charge on the condenser.

(*b*) Is the circuit oscillatory?

(*c*) If so, what is its frequency?

17. The condenser in the circuit of Problem 16 is charged to a potential of 1000 volts. If the circuit is connected upon itself, what will be the value of current and charge after 0.0125 second has elapsed?

18. Given an *RLC* series circuit which is suddenly energized with an alternating potential difference which is equal to

$$e = 141 \sin (377t - 45°) \text{ volts}$$

$R = 1.0$ ohm $\quad L = 0.041$ henry $\quad C = 18.7$ μf $\quad Q_0 = 0$

(*a*) Write equation (113) for this particular case, employing numerical coefficients. The result is to be in the form:

$$i = k_1 \sin (k_2t + k_3) + k_4\epsilon^{-k_5t} \sin (k_6t - k_7) \text{ amperes}$$

where all k's are expressed numerically.

(*b*) Make sketches of the steady-state term, the transient term, and the resultant current for the first three or four cycles of steady-state phenomena on the same plot. Show also the e variation.