Nonsinusoidal Circuits

Objectives

- Become familiar with the components of the Fourier series expansion for any sinusoidal or nonsinusoidal function.

- Understand how the appearance and time axis plot of a waveform can identify which terms of a Fourier series will be present.

- Be able to determine the response of a network to any input defined by a Fourier series expansion.

- Learn how to add two or more waveforms defined by Fourier series expansions.

25.1 INTRODUCTION

Any waveform that differs from the basic description of the sinusoidal waveform is referred to as **nonsinusoidal.** The most obvious and familiar are the dc, square-wave, triangular, sawtooth, and rectified waveforms in Fig. 25.1.
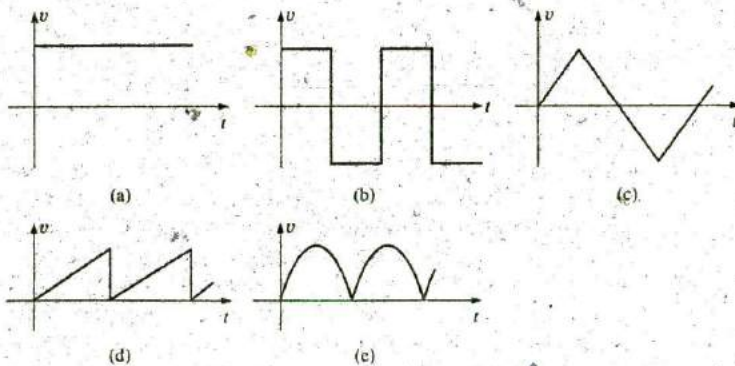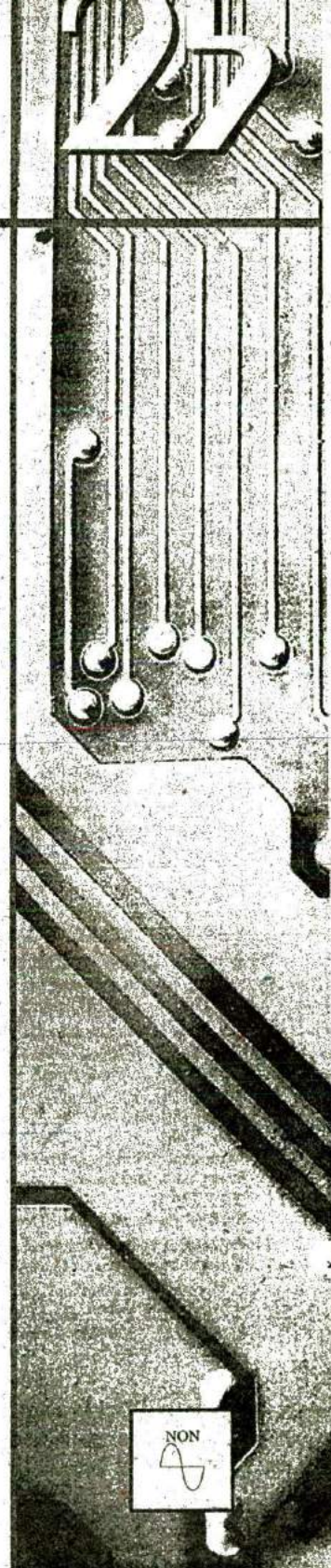

FIG. 25.1

Common nonsinusoidal waveforms: (a) dc; (b) square-wave; (c) triangular; (d) sawtooth; (e) rectified.

The output of many electrical and electronic devices are nonsinusoidal, even though the applied signal may be purely sinusoidal. For example, the network in Fig. 25.2 uses a diode to clip off the negative portion of the applied signal in a process called *half-wave rectification,* which is used in the development of dc levels from a sinusoidal input. You will find in your electronics courses that the diode is similar to a mechanical switch, but it is different because it can conduct current in only one direction. The output waveform is definitely nonsinusoidal, but note that it has the same period as the applied signal and matches the input for half the period.

This chapter demonstrates how a nonsinusoidal waveform like the output in Fig. 25.2 can be represented by a series of terms. It also explains how to determine the response of a network to such an input.
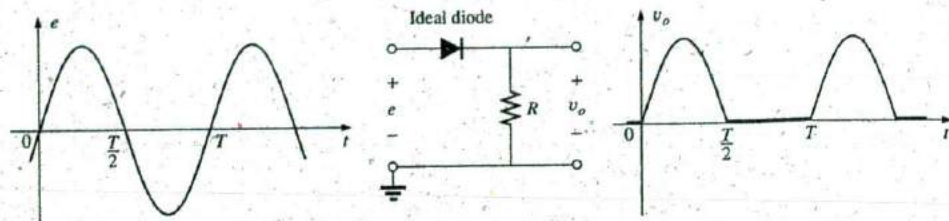
FIG. 25.2

Half-wave rectifier producing a nonsinusoidal waveform.

25.2 FOURIER SERIES

Fourier series refers to a series of terms, developed in 1822 by Baron Jean Fourier (Fig. 25.3), that can be used to represent a nonsinusoidal periodic waveform. In the analysis of these waveforms, we solve for each term in the Fourier series:

$$f(t) = \underbrace{A_0}_{\substack{\text{dc or}\\\text{average value}}} + \underbrace{A_1 \sin \omega t + A_2 \sin 2\omega t + A_3 \sin 3\omega t + \cdots + A_n \sin n\omega t}_{\text{sine terms}}$$
$$+ \underbrace{B_1 \cos \omega t + B_2 \cos 2\omega t + B_3 \cos 3\omega t + \cdots + B_n \cos n\omega t}_{\text{cosine terms}} \qquad (25.1)$$

Depending on the waveform, a large number of these terms may be required to approximate the waveform closely for the purpose of circuit analysis.

As shown in Eq. (25.1), the Fourier series has three basic parts. The first is the dc term A_0, which is the average value of the waveform over one full cycle. The second is a series of sine terms. There are no restrictions on the values or relative values of the amplitudes of these sine terms, but each will have a frequency that is an integer multiple of the frequency of the first sine term of the series. The third part is a series of cosine terms. There are again *no* restrictions on the values or relative values of the amplitudes of these cosine terms, but each will have a frequency that is an integer multiple of the frequency of the first cosine term of the series. For a particular waveform, it is quite possible that all of the sine *or* cosine terms are zero. Characteristics of this type can be determined by simply examining the nonsinusoidal waveform and its position on the horizontal axis.

The first term of the sine and cosine series is called the **fundamental component.** It represents the minimum frequency term required to represent a particular waveform, and it also has the same frequency as the waveform being represented. A fundamental term, therefore, must be present in any Fourier series representation. The other terms with higher-order frequencies (integer multiples of the fundamental) are called the **harmonic terms.** A term that has a frequency equal to twice the fundamental is the second harmonic; three times, the third harmonic; and so on.

FIG. 25.3

Baron Jean Fourier.
Courtesy of the Smithsonian Institution
Photo No. 56,822

French (Auxerre, Grenoble, Paris)
(1768–1830)
Mathematician, Egyptologist, and Administrator
Professor of Mathematics, École Polytechnique

Best known for an infinite mathematical series of sine and cosine terms called the *Fourier series,* which he used to show how the conduction of heat in solids can be analyzed and defined. Although he was primarily a mathematician, a great deal of Fourier's work revolved around real-world physical occurrences such as heat transfer, sunspots, and the weather. He joined the École Polytechnique in Paris as a faculty member when the institute first opened. Napoleon requested his aid in the research of Egyptian antiquities, resulting in a three-year stay in Egypt as Secretary of the Institut d'Égypte. Napoleon made him a baron in 1809, and he was elected to the Académie des Sciences in 1817.

Average Value: A_0

The dc term of the Fourier series is the average value of the waveform over one full cycle. If the net area above the horizontal axis equals that

below in one full period, $A_0 = 0$, and the dc term does not appear in the expansion. If the area above the axis is greater than that below over one full cycle, A_0 is positive and will appear in the Fourier series representation. If the area below the axis is greater, A_0 is negative and will appear with the negative sign in the expansion.

Odd Function (Point Symmetry)

If a waveform is such that its value for $+t$ is the negative of that for $-t$, it is called an odd function or is said to have point symmetry.

Fig. 25.4(a) is an example of a waveform with point symmetry. Note that the waveform has a peak value at t_1 that matches the magnitude (with the opposite sign) of the peak value at $-t_1$. For waveforms of this type, all the parameters $B_{1 \to \infty}$ of Eq. (25.1) will be zero. In fact,

waveforms with point symmetry can be fully described by just the dc and sine terms of the Fourier series.
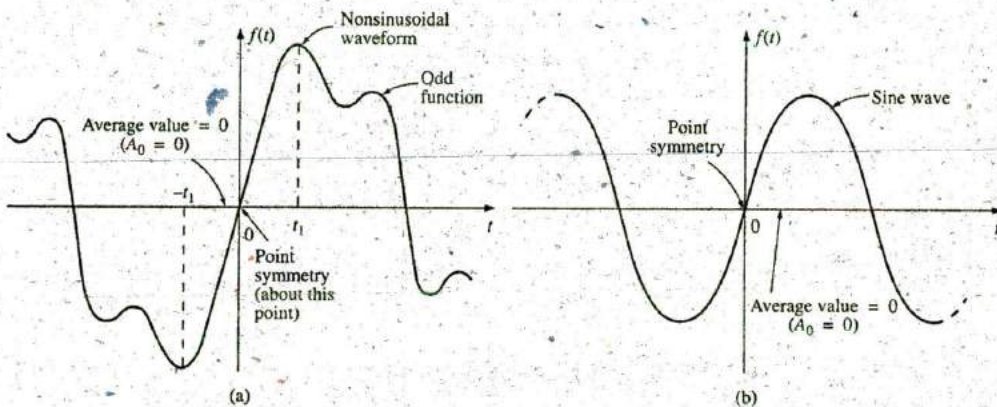

(a)   (b)

FIG. 25.4
Point symmetry.

Note in Fig. 25.4(b) that a sine wave is an odd function with point symmetry.

For both waveforms in Fig. 25.4, the following mathematical relationship is true:

$$\boxed{f(t) = -f(-t)} \qquad \text{(odd function)} \qquad (25.2)$$

In words, it states that the magnitude of the function at $+t$ is equal to the negative of the magnitude at $-t$ [t_1 in Fig. 25.4(a)].

Even Function (Axis Symmetry)

If a waveform is symmetric about the vertical axis, it is called an even function or is said to have axis symmetry.

Fig. 25.5(a) is an example of such a waveform. Note that the value of the function at t_1 is equal to the value at $-t_1$. For waveforms of this type, all the parameters $A_{1 \to \infty}$ will be zero. In fact,

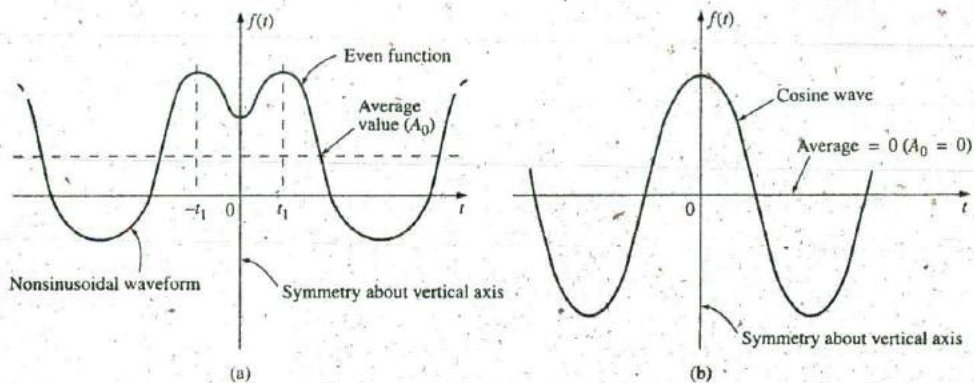**FIG. 25.5**

Axis symmetry.

waveforms with axis symmetry can be fully described by just the dc and cosine terms of the Fourier series.

Note in Fig. 25.5(b) that a cosine wave is an even function with axis symmetry.

For both waveforms in Fig. 25.5, the following mathematical relationship is true:

$$\boxed{f(t) = f(-t)} \qquad \text{(even function)} \qquad \text{(25.3)}$$

In words, it states that the magnitude of the function is the same at $+t_1$ as at $-t$ [t_1 in Fig. 25.5(a)].

Mirror or Half-Wave Symmetry

If a waveform has half-wave or mirror symmetry as demonstrated by the waveform of Fig. 25.6, the even harmonics of the series of sine and cosine terms will be zero.
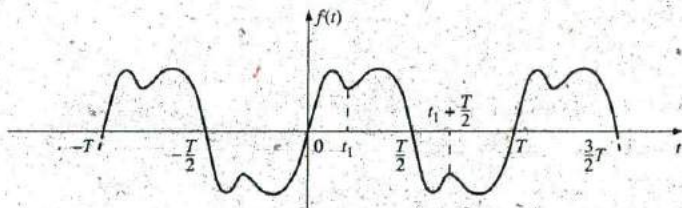

FIG. 25.6

Mirror symmetry.

In functional form, the waveform must satisfy the following relationship:

$$\boxed{f(t) = -f\left(t + \frac{T}{2}\right)} \qquad \text{(25.4)}$$

Eq. (25.4) states that the waveform encompassed in one time interval $T/2$ will repeat itself in the next $T/2$ time interval, but in the negative sense (t_1 in Fig. 25.6). For example, the waveform in Fig. 25.6 from zero to $T/2$ will repeat itself in the time interval $T/2$ to T but below the horizontal axis.

Repetitive on the Half-Cycle

The repetitive nature of a waveform can determine whether specific harmonics will be present in the Fourier series expansion. In particular,

if a waveform is repetitive on the half-cycle as demonstrated by the waveform in Fig. 25.7, the odd harmonics of the series of sine and cosine terms are zero.
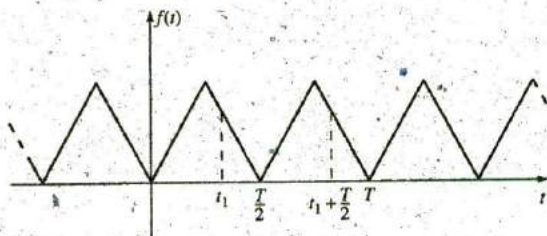

- **FIG. 25.7**
A waveform repetitive on the half-cycle.

In functional form, the waveform must satisfy the following relationship:

$$f(t) = f\left(t + \frac{T}{2}\right) \qquad (25.5)$$

Eq. (25.5) states that the function repeats itself after each $T/2$ time interval (t_1 in Fig. 25.7). The waveform, however, will also repeat itself after each period T. In general, therefore, for a function of this type, if the period T of the waveform is chosen to be twice that of the minimum period ($T/2$), the odd harmonics will all be zero.

Mathematical Approach

The constants A_0, $A_{1 \to n}$, and $B_{1 \to n}$ can be determined by using the following integral formulas:

$$A_0 = \frac{1}{T}\int_0^T f(t)\, dt \qquad (25.6)$$

$$A_n = \frac{2}{T}\int_0^T f(t)\, \sin n\omega t\, dt \qquad (25.7)$$

$$B_n = \frac{2}{T}\int_0^T f(t)\, \cos n\omega t\, dt \qquad (25.8)$$

These equations have been presented for recognition purposes only; they are not used in the following analysis.

Instrumentation

Three types of instrumentation are available that reveal the dc, fundamental, and harmonic content of a waveform: *the spectrum analyzer, wave analyzer,* and *Fourier analyzer.* The purpose of such instrumentation is not solely to determine the composition of a particular waveform, but also to reveal the level of distortion that may have been introduced by a system. For instance, an amplifier may be increasing the applied signal by a factor of 50, but in the process it may have distorted the waveform in a way that is quite unnoticeable from the oscilloscope display. The amount of distortion appears in the form of harmonics at frequencies that are multiples of the applied frequency. Each of the above instruments reveals which frequencies are having the most impact on the distortion, permitting their removal with properly designed filters.

The spectrum analyzer is shown in Fig. 25.8. It has the appearance of an oscilloscope, but rather than display a waveform that is voltage (vertical axis) versus time (horizontal axis), it generates a display scaled off in dB (vertical axis) versus frequency (horizontal axis). Such a display is said to be in the *frequency domain,* in contrast to the *time domain* of the standard oscilloscope. The height of the vertical line in the display of Fig. 25.8 reveals the impact of that frequency on the shape of the waveform. Spectrum analyzers are unable to provide the phase angle associated with each component.
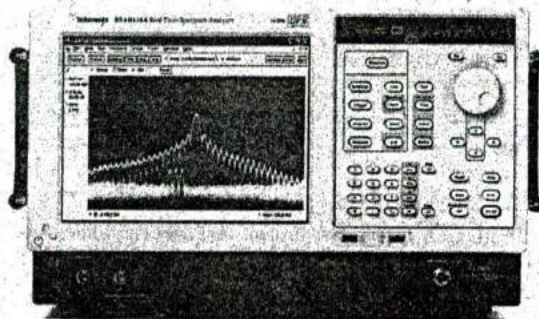

FIG. 25.8
Spectrum analyzer.
(Courtesy of Teletronix, Inc.)

EXAMPLE 25.1 Determine which components of the Fourier series are present in the waveforms in Fig. 25.9.

Solutions:

a. The waveform has a net area above the horizontal axis and therefore will have **a positive dc term A_0.**

   The waveform has axis symmetry, resulting in **only cosine terms** in the expansion.

   The waveform has half-cycle symmetry, resulting in **only even terms** in the cosine series.

b. The waveform has the same area above and below the horizontal axis within each period, resulting in $A_0 = 0$.

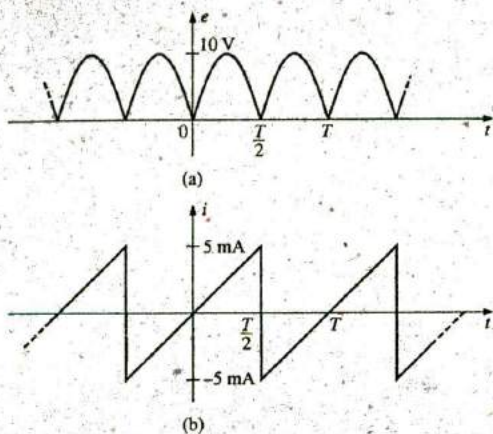   The waveform has point symmetry, resulting in **only sine terms** in the expansion.

FIG. 25.9
Example 25.1.

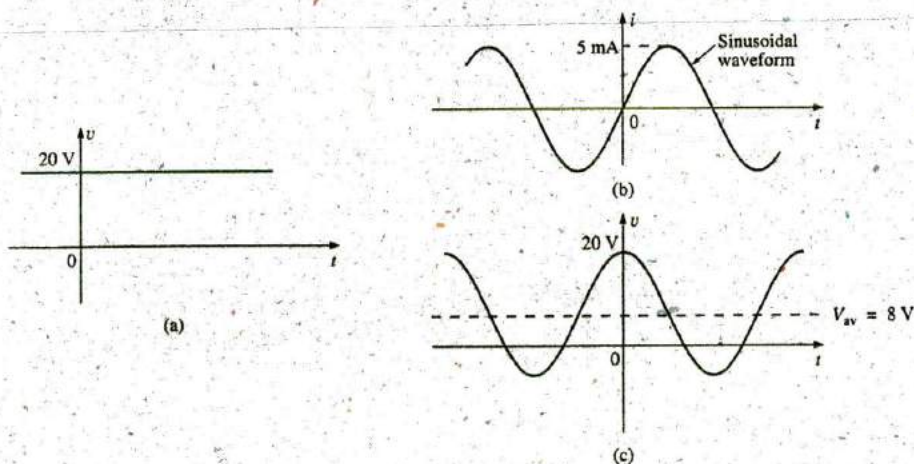**EXAMPLE 25.2** Write the Fourier series expansion for the waveforms in Fig. 25.10.

FIG. 25.10
Example 25.2.

Solutions:

a. $A_0 = 20$ $A_{1 \to n} = 0$ $B_{1 \to n} = 0$
   $v = 20$

b. $A_0 = 0$ $A_1 = 5 \times 10^{-3}$ $A_{2 \to n} = 0$ $B_{1 \to n} = 0$
   $i = 5 \times 10^{-3} \sin \omega t$

c. $A_0 = 8$ $A_{1 \to n} = 0$ $B_1 = 12$ $B_{2 \to n} = 0$
   $v = 8 + 12 \cos \omega t$

EXAMPLE 25.3 Sketch the following Fourier series expansion:

$$v = 2 + 1 \cos \alpha + 2 \sin \alpha$$

Solution: Note Fig. 25.11.
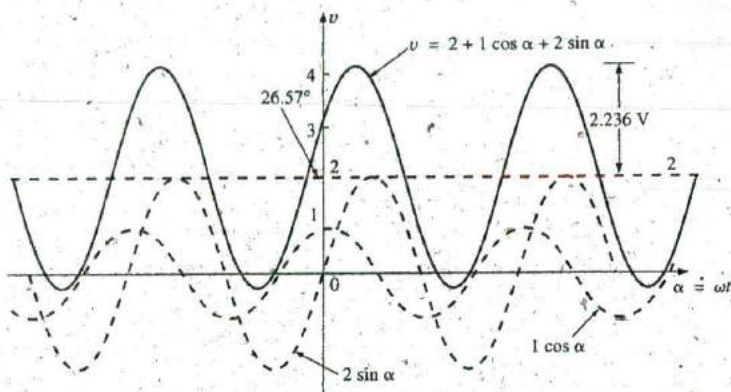
FIG. 25.11
Example 25.3.

The solution could be obtained graphically by first plotting all of the functions and then considering a sufficient number of points on the horizontal axis, or phasor algebra could be used as follows:

$$1 \cos \alpha + 2 \sin \alpha = 1 \text{ V } \angle 90° + 2 \text{ V } \angle 0° = j1 \text{ V} + 2 \text{ V}$$
$$= 2 \text{ V} + j1 \text{ V} = 2.236 \text{ V } \angle 26.57°$$
$$= 2.236 \sin(\alpha + 26.57°)$$

and
$$v = 2 + 2.236 \sin(\alpha + 26.57°)$$

which is simply the sine wave portion riding on a dc level of 2 V. That is, its positive maximum is $2 \text{ V} + 2.236 \text{ V} = 4.236 \text{ V}$, and its minimum is $2 \text{ V} - 2.236 \text{ V} = -0.236 \text{ V}$.

EXAMPLE 25.4 Sketch the following Fourier series expansion:

$$i = 1 \sin \omega t + 1 \sin 2\omega t$$

Solution: See Fig. 25.12. Note that in this case the sum of the two sinusoidal waveforms of different frequencies is *not* a sine wave. Recall that complex algebra can be applied only to waveforms having the *same* frequency. In this case, the solution is obtained graphically point by point, as shown for $t = t_1$.
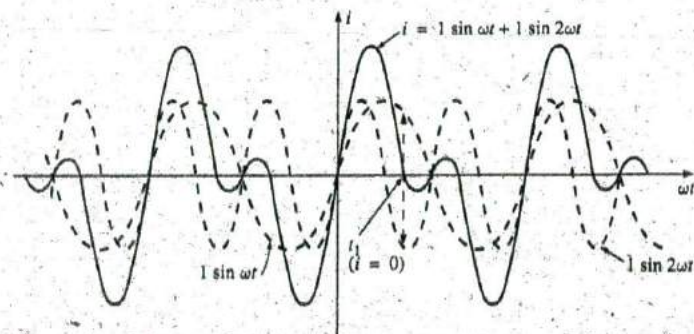

FIG. 25.12
Example 25.4.

As an additional example in the use of the Fourier series approach, consider the square wave shown in Fig. 25.13. The average value is zero, so $A_0 = 0$. It is an odd function, so all the constants $B_{1 \to n}$ equal zero; only sine terms are present in the series expansion. Since the waveform satisfies the criteria for $f(t) = -f(t + T/2)$, the even harmonics are also zero.
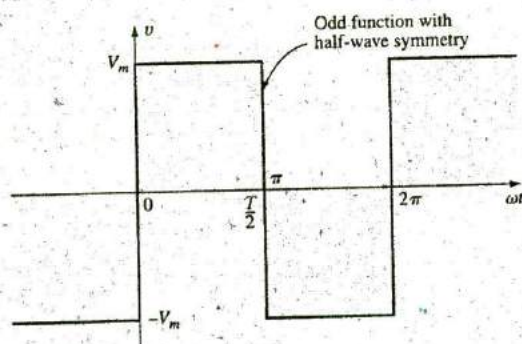

FIG. 25.13
Square wave.

The expression obtained after evaluating the various coefficients using Eq. (25.8) is

$$v = \frac{4}{\pi} V_m \left(\sin \omega t + \frac{1}{3} \sin 3\omega t + \frac{1}{5} \sin 5\omega t + \frac{1}{7} \sin 7\omega t + \cdots + \frac{1}{n} \sin n\omega t \right) \quad (25.9)$$

Note that the fundamental does indeed have the same frequency as that of the square wave. If we add the fundamental and third harmonics, we obtain the results shown in Fig. 25.14.

Even with only the first two terms, a few characteristics of the square wave are beginning to appear. If we add the next two terms (Fig. 25.15), the width of the pulse increases, and the number of peaks increases.
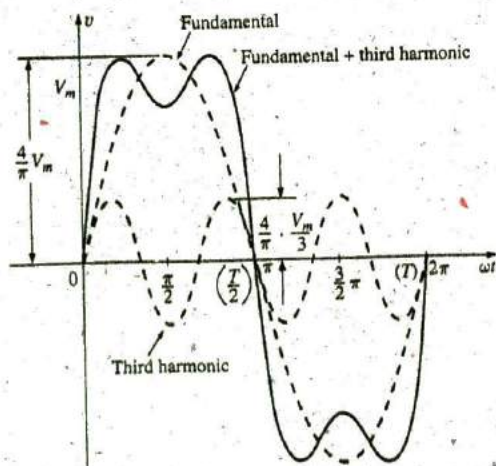

FIG. 25.14
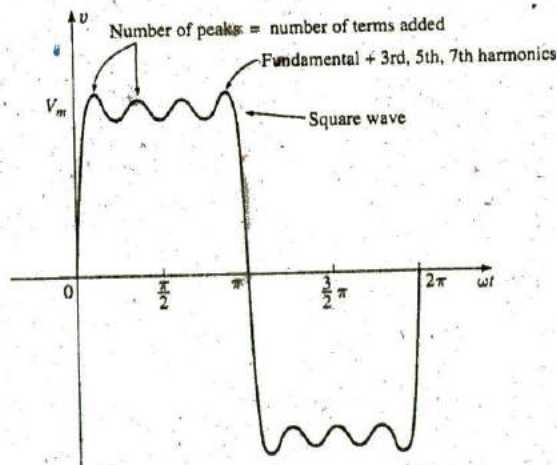*Fundamental plus third harmonic.*

FIG. 25.15
Fundamental plus third, fifth, and seventh harmonics.

As we continue to add terms, the series better approximate the square wave. Note, however, that the amplitude of each succeeding term diminishes to the point at which it is negligible compared with those of the first few terms. A good approximation is to assume that the waveform is composed of the harmonics up to and including the ninth. Any higher harmonics would be less than one-tenth the fundamental. If the waveform just described were shifted above or below the horizontal axis, the Fourier series would be altered only by a change in the dc term. Fig. 25.16(c), for example, is the sum of Fig. 25.16(a) and (b). The Fourier series for the complete waveform is, therefore,

$$v = v_1 + v_2 = V_m + \text{Eq. (25.9)}$$
$$= V_m + \frac{4}{\pi} V_m \left(\sin \omega t + \frac{1}{3} \sin 3\omega t + \frac{1}{5} \sin 5\omega t + \frac{1}{7} \sin 7\omega t + \cdots \right)$$

and $$v = V_m \left[1 + \frac{4}{\pi} \left(\sin \omega t + \frac{1}{3} \sin 3\omega t + \frac{1}{5} \sin 5\omega t + \frac{1}{7} \sin 7\omega t + \cdots \right) \right]$$
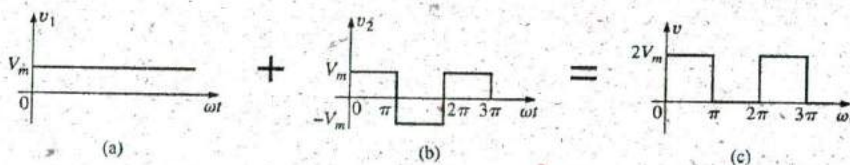

FIG. 25.16

Shifting a waveform vertically with the addition of a dc term.

The equation for the half-wave rectified pulsating waveform in Fig. 25.17(b) is

$$v_2 = 0.318V_m + 0.500V_m \sin \alpha - 0.212V_m \cos - 0.0424V_m \cos 4\alpha - \cdots \quad \textbf{(25.10)}$$

The waveform in Fig. 25.17(c) is the sum of the two in Fig. 25.17(a) and (b). The Fourier series for the waveform in Fig. 25.17(c) is, therefore,

$$v_T = v_1 + v_2 = -\frac{V_m}{2} + \text{Eq. (25.10)}$$
$$= -0.500V_m + 0.318V_m + 0.500V_m \sin \alpha - 0.212V_m \cos 2\alpha - 0.0424V_m \cos 4\alpha + \cdots$$

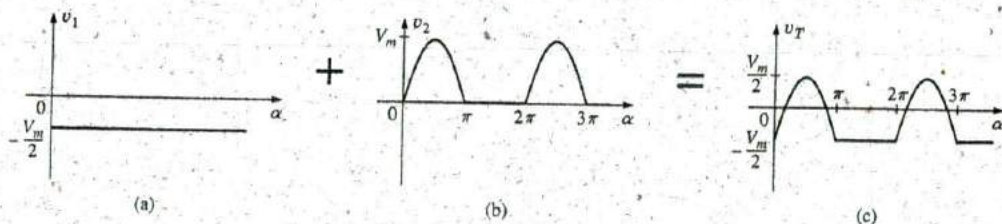and $$v_T = -0.182V_m + 0.5V_m \sin \alpha - 0.212V_m \cos 2\alpha - 0.0424V_m \cos 4\alpha + \cdots$$

FIG. 25.17

Lowering a waveform with the addition of a negative dc component.

If either waveform were shifted to the right or left, the phase shift would be subtracted from or added to, respectively, the sine and cosine terms. The dc term would not change with a shift to the right or left.

If the half-wave rectified signal is shifted 90° to the left, as in Fig. 25.18, the Fourier series becomes

$$v = 0.318V_m + 0.500V_m \underbrace{\sin(\alpha + 90°)}_{\cos \alpha} - 0.212V_m \cos 2(\alpha + 90°) - 0.0424V_m \cos 4(\alpha + 90°) + \cdots$$

$$= 0.318V_m + 0.500V_m \cos \alpha - 0.212V_m \cos(2\alpha + 180°) - 0.0424V_m \cos(4\alpha + 360°) + \cdots$$

and $v = 0.318V_m + 0.500V_m \cos \alpha + 0.212V_m \cos 2\alpha - 0.0424V_m \cos 4\alpha + \cdots$
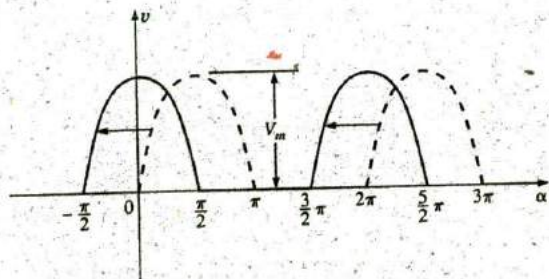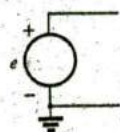

FIG. 25.18

Changing the phase angle of a waveform.

## 25.3 CIRCUIT RESPONSE TO A NONSINUSOIDAL INPUT

The Fourier series representation of a nonsinusoidal input can be applied to a linear network using the principle of superposition. Recall that this theorem allowed us to consider the effects of each source of a circuit independently. If we replace the nonsinusoidal input with the terms of the Fourier series deemed necessary for practical considerations, we can use superposition to find the response of the network to each term (Fig. 25.19).

$e = A_0 + A_1 \sin \alpha + \ldots + A_n \sin n\alpha + \ldots$
$+ B_1 \cos \alpha + \ldots + B_n \cos n\alpha + \ldots$

Linear network

A_0

$A_1 \sin \alpha$

$A_n \sin n\alpha$

$B_1 \cos \alpha$

$B_n \cos n\alpha$
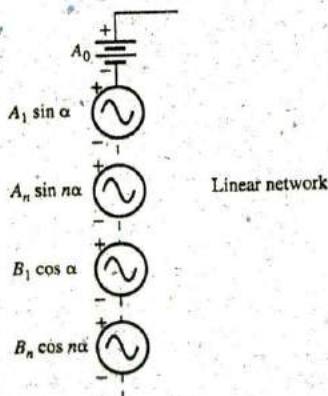
Linear network

FIG. 25.19

Setting up the application of a Fourier series of terms to a linear network.

The total response of the system is then the algebraic sum of the values obtained for each term. The major change between using this theorem for nonsinusoidal circuits and using it for the circuits previously described is that the frequency will be different for each term in the nonsinusoidal application. Therefore, the reactances

$$X_L = 2\pi f L \quad \text{and} \quad X_C = \frac{1}{2\pi f C}$$

will change for each term of the input voltage or current.

In Chapter 13, we found that the rms value of any waveform was given by

$$\sqrt{\frac{1}{T}\int_0^T f^2(t)\,dt}$$

If we apply this equation to the Fourier series

$$v(\alpha) = V_0 + V_{m_1}\sin\alpha + \cdots + V_{m_n}\sin n\alpha + V'_{m_1}\cos\alpha + \cdots + V'_{m_n}\cos n\alpha$$

then

$$V_{\text{rms}} = \sqrt{V_0^2 + \frac{V_{m_1}^2 + \cdots + V_{m_n}^2 + V_{m_1}'^2 + \cdots + V_{m_n}'^2}{2}} \qquad (25.11)$$

However, since

$$\frac{V_{m_1}^2}{2} = \left(\frac{V_{m_1}}{\sqrt{2}}\right)\left(\frac{V_{m_1}}{\sqrt{2}}\right) = (V_{1_{\text{rms}}})(V_{1_{\text{rms}}}) = V_{1_{\text{rms}}}^2$$

then

$$V_{\text{rms}} = \sqrt{V_0^2 + V_{1_{\text{rms}}}^2 + \cdots + V_{n_{\text{rms}}}^2 + V_{1_{\text{rms}}}'^2 + \cdots + V_{n_{\text{rms}}}'^2} \qquad (25.12)$$

Similarly, for

$$i(\alpha) = I_0 + I_{m_1}\sin\alpha + \cdots + I_{m_n}\sin n\alpha + I'_{m_1}\cos\alpha + \cdots + I'_{m_n}\cos n\alpha$$

we have

$$I_{\text{rms}} = \sqrt{I_0^2 + \frac{I_{m_1}^2 + \cdots + I_{m_n}^2 + I_{m_1}'^2 + \cdots + I_{m_n}'^2}{2}} \qquad (25.13)$$

and

$$I_{\text{rms}} = \sqrt{I_0^2 + I_{1_{\text{rms}}}^2 + \cdots + I_{n_{\text{rms}}}^2 + I_{1_{\text{rms}}}'^2 + \cdots + I_{n_{\text{rms}}}'^2} \qquad (25.14)$$

The total power delivered is the sum of that delivered by the corresponding terms of the voltage and current. In the following equations, all voltages and currents are rms values:

$$P_T = V_0 I_0 + V_1 I_1 \cos\theta_1 + \cdots + V_n I_n \cos\theta_n + \cdots \qquad (25.15)$$

$$P_T = I_0^2 R + I_1^2 R + \cdots + I_n^2 R + \cdots \qquad (25.16)$$

or

$$P_T = I_{rms}^2 R \qquad (25.17)$$

with I_{rms} as defined by Eq. (25.13), and, similarly,

$$P_T = \frac{V_{rms}^2}{R} \qquad (25.18)$$

with V_{rms} as defined by Eq. (25.11).

EXAMPLE 25.5

a. Sketch the input resulting from the combination of sources in Fig. 25.20.
b. Determine the rms value of the input in Fig. 25.20.

Solutions:

a. Note Fig. 25.21.
b. Eq. (25.12):

$$V_{rms} = \sqrt{V_0^2 + \frac{V_m^2}{2}}$$

$$= \sqrt{(4\ V)^2 + \frac{(6\ V)^2}{2}} = \sqrt{16 + \frac{36}{2}}\ V = \sqrt{34}\ V$$

$$= 5.83\ V$$

FIG. 25.20
Example 25.5.

It is particularly interesting to note from Example 25.5 that the rms value of a waveform having both dc and ac components is not simply the sum of the effective values of each. In other words, there is a temptation in the absence of Eq. (25.12) to state that $V_{rms} = 4\ V + 0.707\ (6\ V) = 8.24\ V$, which is incorrect and, in fact, exceeds the correct level by some 41%.

FIG. 25.21
Wave pattern generated by the source in Fig. 25.20.

Instrumentation

It is important to realize that not every DMM will read the rms value of nonsinusoidal waveforms such as the one appearing in Fig. 25.21. Many are designed to read the rms value of sinusoidal waveforms only. It is important to read the manual provided with the meter to see if it is a *true rms* meter that can read the rms value of any waveform.

We learned in Chapter 13 that the rms value of a square wave is the peak value of the waveform. Let us test this result using the Fourier expansion and Eq. (25.11).

EXAMPLE 25.6 Determine the rms value of the square wave of Fig. 25.13 with $V_m = 20\ V$ using the first six terms of the Fourier expansion, and compare the result to the actual rms value of 20 V.

Solution:

$$v = \frac{4}{\pi}(20\ V)\sin \omega t + \frac{4}{\pi}\left(\frac{1}{3}\right)(20\ V)\sin 3\omega t + \frac{4}{\pi}\left(\frac{1}{5}\right)(20\ V)\sin 5\omega t + \frac{4}{\pi}\left(\frac{1}{7}\right)(20\ V)\sin 7\omega t$$

$$+ \frac{4}{\pi}\left(\frac{1}{9}\right)(20\ V)\sin 9\omega t + \frac{4}{\pi}\left(\frac{1}{11}\right)(20\ V)\sin 11\omega t$$

$$v = 25.465 \sin \omega t + 8.488 \sin 3\omega t + 5.093 \sin 5\omega t + 3.638 \sin 7\omega t + 2.829 \sin 9\omega t + 2.315 \sin 11\omega t$$
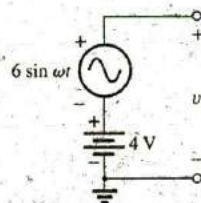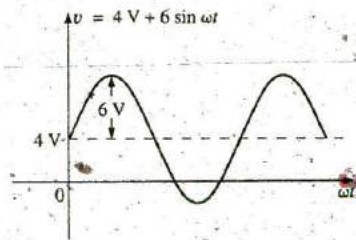
Eq. (25.11):

$$V_{rms} = \sqrt{V_0^2 + \frac{V_{m_1}^2 + V_{m_2}^2 + V_{m_3}^2 + V_{m_4}^2 + V_{m_5}^2 + V_{m_6}^2}{2}}$$

$$= \sqrt{(0\ V)^2 + \frac{(25.465\ V)^2 + (8.488\ V)^2 + (5.093\ V)^2 + (3.638\ V)^2 + (2.829\ V)^2 + (2.315\ V)^2}{2}}$$

$$= 19.66\ V$$

The solution differs less than 0.4 V from the correct answer of 20 V. However, each additional term in the Fourier series brings the result closer to the 20 V level. An infinite number results in an exact solution of 20 V.
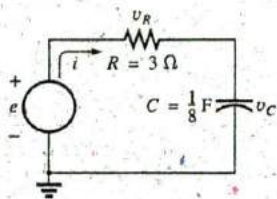

FIG. 25.22
Example 25.7.

EXAMPLE 25.7 The input to the circuit in Fig. 25.22 is

$$e = 12 + 10 \sin 2t$$

a. Find the current i and the voltages v_R and v_C.
b. Find the rms values of i, v_R, and v_C.
c. Find the power delivered to the circuit.

Solutions:

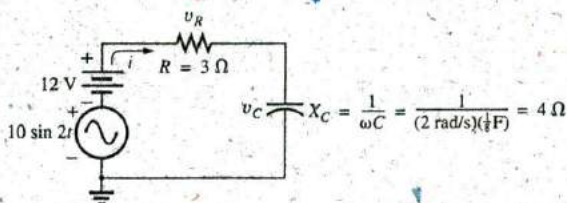a. Redraw the original circuit as shown in Fig. 25.23. Then apply superposition:

FIG. 25.23
Circuit in Fig. 25.22 with the components of the Fourier series input.

1. *For the 12 V dc supply portion of the input,* $I = 0$ since the capacitor is an open circuit to dc when v_C has reached its final (steady-state) value. Therefore,

$$V_R = IR = 0\ V \quad \text{and} \quad V_C = 12\ V$$

2. *For the ac supply,*

$$Z = 3\ \Omega - j4\ \Omega = 5\ \Omega\ \angle -53.13°$$

and

$$I = \frac{E}{Z} = \frac{\frac{10}{\sqrt{2}}V\ \angle 0°}{5\ \Omega\ \angle -53.13°} = \frac{2}{\sqrt{2}}\ A\ \angle +53.13°$$

$$V_R = (I\ \angle \theta)(R\ \angle 0°) = \left(\frac{2}{\sqrt{2}}\ A\ \angle +53.13°\right)(3\ \Omega\ \angle 0°)$$

$$= \frac{6}{\sqrt{2}}\ V\ \angle +53.13°$$

and

$$V_C = (I \angle \theta)(X_C \angle -90°) = \left(\frac{2}{\sqrt{2}} A \angle +53.13°\right)(4\,\Omega\,\angle -90°)$$

$$= \frac{8}{\sqrt{2}} V \angle -36.87°$$

In the time domain,

$$i = 0 + 2\sin(2t + 53.13°)$$

Note that even though the dc term was present in the expression for the input voltage, the dc term for the current in this circuit is zero:

$$v_R = 0 + 6\sin(2t + 53.13°)$$

and

$$v_C = 12 + 8\sin(2t - 36.87°)$$

b. Eq. (25.14): $I_{rms} = \sqrt{(0)^2 + \frac{(2\,A)^2}{2}} = \sqrt{2}\,A = 1.414\,A$

Eq. (25.12): $V_{R_{rms}} = \sqrt{(0)^2 + \frac{(6\,V)^2}{2}} = \sqrt{18}\,V = 4.243\,V$

Eq. (25.12): $V_{C_{rms}} = \sqrt{(12\,V)^2 + \frac{(8\,V)^2}{2}} = \sqrt{176}\,V = 13.267\,V$

c. $P = I_{rms}^2 R = \left(\frac{2}{\sqrt{2}}A\right)^2 (3\,\Omega) = 6\,W$

**EXAMPLE 25.8** Find the response of the circuit in Fig. 25.24 to the input shown.

$$e = 0.318E_m + 0.500E_m\sin\omega t - 0.212E_m\cos 2\omega t - 0.0424E_m\cos 4\omega t + \cdots$$
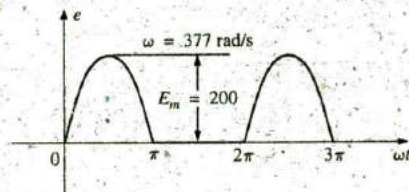
Solution: For discussion purposes, only the first three terms are used to represent e. Converting the cosine terms to sine terms and substituting for E_m gives us

$$e = 63.60 + 100.0\sin\omega t - 42.40\sin(2\omega t + 90°)$$

Using phasor notation, we see that the original circuit becomes like the one shown in Fig. 25.25.

(a)

$\omega = 377$ rad/s
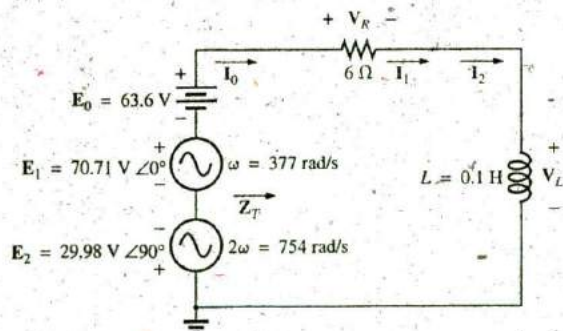
$E_m = 200$
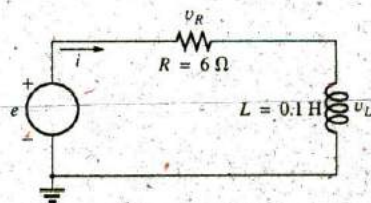
(b)

FIG. 25.24
Example 25.8.

FIG. 25.25
Circuit in Fig. 25.24 with the components of the Fourier series input.

Applying Superposition *For the dc term* ($E_0 = 63.6$ V):

$$X_L = 0 \quad \text{(short for dc)}$$
$$Z_T = R \angle 0° = 6 \, \Omega \angle 0°$$
$$I_0 = \frac{E_0}{R} = \frac{63.6 \text{ V}}{6 \, \Omega} = 10.60 \text{ A}$$
$$V_{R_0} = I_0 R = E_0 = 63.60 \text{ V}$$
$$V_{L_0} = 0$$

The average power is

$$P_0 = I_0^2 R = (10.60 \text{ A})^2 (6 \, \Omega) = 674.2 \text{ W}$$

For the fundamental term ($E_1 = 70.71$ V $\angle 0°$, $\omega = 377$):

$$X_{L_1} = \omega L = (377 \text{ rad/s})(0.1 \text{ H}) = 37.7 \, \Omega$$
$$Z_{T_1} = 6 \, \Omega + j \, 37.7 \, \Omega = 38.17 \, \Omega \angle 80.96°$$
$$I_1 = \frac{E_1}{Z_{T_1}} = \frac{70.71 \text{ V} \angle 0°}{38.17 \, \Omega \angle 80.96°} = 1.85 \text{ A} \angle -80.96°$$
$$V_{R_1} = (I_1 \angle \theta)(R \angle 0°) = (1.85 \text{ A} \angle -80.96°)(6 \, \Omega \angle 0°)$$
$$= 11.10 \text{ V} \angle -80.96°$$
$$V_{L_1} = (I_1 \angle \theta)(X_{L_1} \angle 90°) = (1.85 \text{ A} \angle -80.96°)(37.7 \, \Omega \angle 90°)$$
$$= 69.75 \text{ V} \angle 9.04°$$

The average power is

$$P_1 = I_1^2 R = (1.85 \text{ A})^2 (6 \, \Omega) = 20.54 \text{ W}$$

For the second harmonic ($E_2 = 29.98$ V $\angle -90°$, $\omega = 754$): The phase angle of E_2 was changed to $-90°$ to give it the same polarity as the input voltages E_0 and E_1. We have

$$X_{L_2} = \omega L = (754 \text{ rad/s})(0.1 \text{ H}) = 75.4 \, \Omega$$
$$Z_{T_2} = 6 \, \Omega + j \, 75.4 \, \Omega = 75.64 \, \Omega \angle 85.45°$$
$$I_2 = \frac{E_2}{Z_{T_2}} = \frac{29.98 \text{ V} \angle +90°}{75.64 \, \Omega \angle 85.45°} = 0.396 \text{ A} \angle -174.45°$$
$$V_{R_2} = (I_2 \angle \theta)(R \angle 0°) = (0.396 \text{ A} \angle -174.45°)(6 \, \Omega \angle 0°)$$
$$= 2.38 \text{ V} \angle -174.45°$$
$$V_{L_2} = (I_2 \angle \theta)(X_{L_2} \angle 90°) = (0.396 \text{ A} \angle -174.45°)(75.4 \, \Omega \angle 90°)$$
$$= 29.9 \text{ V} \angle -84.45°$$

The average power is

$$P_2 = I_2^2 R = (0.396 \text{ A})^2 (6 \, \Omega) = 0.941 \text{ W}$$

The Fourier series expansion for i is

$$i = 10.6 + \sqrt{2}(1.85) \sin(377t - 80.96°) + \sqrt{2}(0.396) \sin(754t - 174.45°)$$

and

$$I_{rms} = \sqrt{(10.6 \text{ A})^2 + (1.85 \text{ A})^2 + (0.396 \text{ A})^2} = 10.77 \text{ A}$$

The Fourier series expansion for v_R is

$$v_R = 63.6 + \sqrt{2}(11.10) \sin(377t - 80.96°) + \sqrt{2}(2.38) \sin(754t - 174.45°)$$

and

$$V_{R_{rms}} = \sqrt{(63.6 \text{ V})^2 + (11.10 \text{ V})^2 + (2.38 \text{ V})^2} = 64.61 \text{ V}$$

The Fourier series expansion for v_L is

$$v_L = \sqrt{2}(69.75) \sin(377t + 9.04°) + \sqrt{2}(29.93) \sin(754t - 84.45°)$$

and $\quad V_{L_{rms}} = \sqrt{(69.75\,V)^2 + (29.93\,V)^2} = 75.90\,V$

The total average power is

$$P_T = I_{rms}^2 R = (10.77\,A)^2(6\,\Omega) = 695.96\,W = P_0 + P_1 + P_2$$

25.4 ADDITION AND SUBTRACTION OF NONSINUSOIDAL WAVEFORMS

The Fourier series expression for the waveform resulting from the addition or subtraction of two nonsinusoidal waveforms can be found using phasor algebra if the terms having the same frequency are considered separately.

For example, the sum of the following two nonsinusoidal waveforms is found using this method:

$$v_1 = 30 + 20 \sin 20t + \cdots + 5 \sin(60t + 30°)$$
$$v_2 = 60 + 30 \sin 20t + 20 \sin 40t + 10 \cos 60t$$

1. dc terms:

$$V_{T_0} = 30\,V + 60\,V = 90\,V$$

2. $\omega = 20$:

$$V_{T_1(max)} = 30\,V + 20\,V = 50\,V$$
$$v_{T_1} = 50 \sin 20t$$

3. $\omega = 40$:

$$v_{T_2} = 20 \sin 40t$$

4. $\omega = 60$:

$$5 \sin(60t + 30°) = (0.707)(5)\,V\,\angle 30° = 3.54\,V\,\angle 30°$$
$$10 \cos 60t = 10 \sin(60t + 90°) \Rightarrow (0.707)(10)\,V\,\angle 90°$$
$$= 7.07\,V\,\angle 90°$$

$$V_{T_3} = 3.54\,V\,\angle 30° + 7.07\,V\,\angle 90°$$
$$= 3.07\,V + j\,1.77\,V + j\,7.07\,V = 3.07\,V + j\,8.84\,V$$
$$V_{T_3} = 9.36\,V\,\angle 70.85°$$

and $\quad v_{T_3} = 13.24 \sin(60t + 70.85°)$

with

$$v_T = v_1 + v_2 = 90 + 50 \sin 20t + 20 \sin 40t + 13.24 \sin(60t + 70.85°)$$

25.5 COMPUTER ANALYSIS

PSpice

Fourier Series The computer analysis begins with a verification of the waveform in Fig. 25.15, demonstrating that only four terms of a Fourier series can generate a waveform that has a number of characteristics of a square wave. The square wave has a peak value of 10 V at a frequency of 1 kHz, resulting in the following Fourier series using Eq. (25.9) (and recognizing that $\omega = 2\pi f = 6283.19$ rad/s):

$$v = \frac{4}{\pi}(10\,V)\left(\sin \omega t + \frac{1}{3} \sin 3\omega t + \frac{1}{5} \sin 5\omega t + \frac{1}{7} \sin 7\omega t\right)$$
$$= 12.732 \sin \omega t + 4.244 \sin 3\omega t + 2.546 \sin 5\omega t + 1.819 \sin 7\omega t$$

Each term of the Fourier series is treated as an independent ac source, as shown in Fig. 25.26 with its peak value and applicable frequency. The sum of the source voltages appears across the resistor R and generates the waveform in Fig. 25.27.
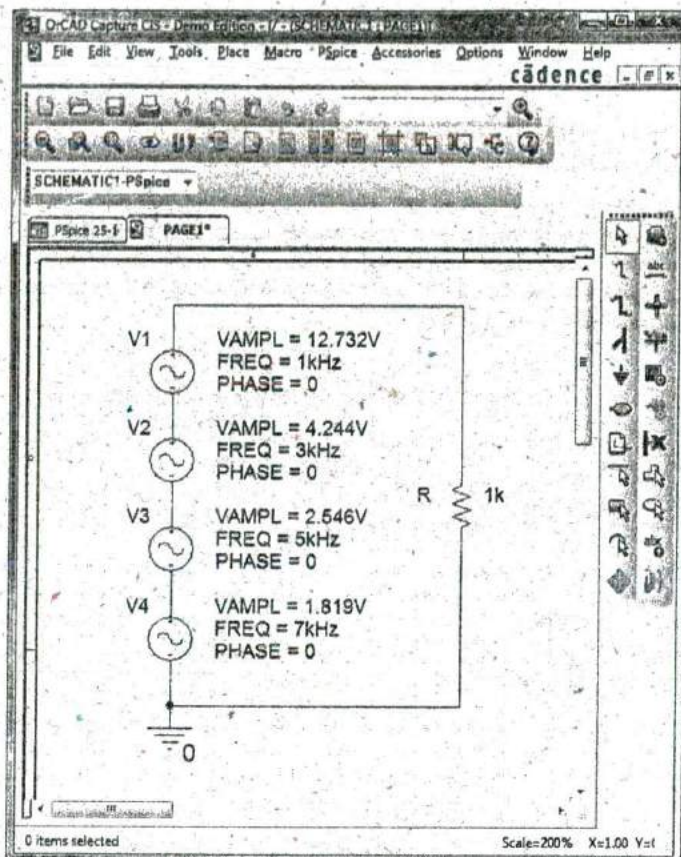

FIG. 25.26

Using PSpice to apply four terms of the Fourier expansion of a 10 V square wave to a load resistor of 1 kΩ.

Each source used **VSIN**, and since we want to display the result against time, choose **Time Domain(Transient)** in the **Simulation Settings**. For each source, select the **Property Editor** dialog box. Set **AC, FREQ, PHASE, VAMPL,** and **VOFF** (at 0 V). (Due to limited space, only **VAMPL, FREQ,** and **PHASE** are displayed in Fig. 25.26.) Under **Display,** set all of the remaining quantities on **Do Not Display.**

Set the **Run to time** at 2 ms, so that two cycles of the fundamental frequency of 1 kHz appear. The **Start saving data after** remains at the default value of 0 s, and the **Maximum step size** at 1 μs, even though 2 ms/1000 = 2 μs, because we want to have additional plot points for the complex waveform. Once the **SCHEMATIC1** window appears, **Trace-Add Trace-V(R:1)-OK** results in the waveform in Fig. 25.27. To make the horizontal line at 0 V heavier, right-click on the line, select **Properties,** and then choose the green color and wider line. Click **OK,** and the wider line in Fig. 25.27 results, making it a great deal clearer where the 0 V line is located. Through the same process, make the curve yellow and wider as shown in the same figure. Using the cursors, you find that the first peak

reaches 11.84 V at 0.063 ms and then drops to 8.920 V at 0.124 ms. The average value of the waveform is clearly +10 V in the positive region, as shown by the dashed line entered using **Plot-Label-Line**. In every respect, the waveform is beginning to have the characteristics of a periodic square wave with a peak value of 10 V and a frequency of 1 kHz.
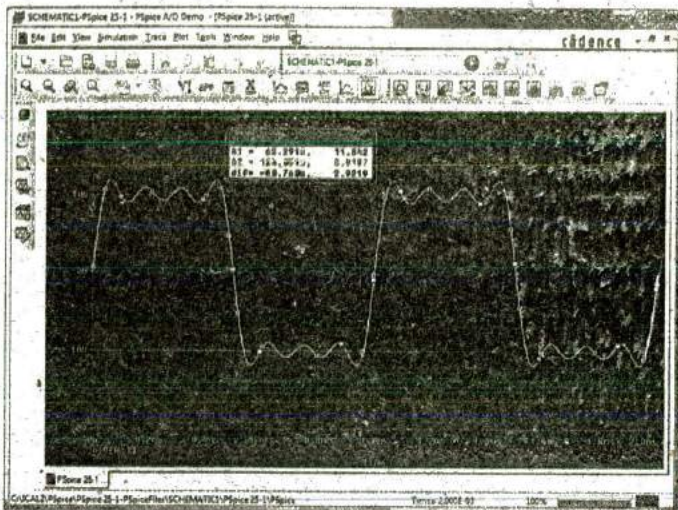

FIG. 25.27
The resulting waveform of the voltage across the resistor R in Fig. 25.26.

Fourier Components   A frequency spectrum plot revealing the magnitude and frequency of each component of a Fourier series can be obtained by returning to **Plot** and selecting **Axis Settings** followed by **X Axis** and then **Fourier** under **Processing Options**. Click **OK,** and a number of spikes appear on the far left of the screen, with a frequency spectrum that extends from 0 Hz to 600 kHz. Select **Plot-Axis Settings** again, go to **Data Range,** and select **User-Defined** to change the range to 0 Hz to 10 kHz since this is the range of interest for this waveform. Click **OK,** and the graph in Fig. 25.28 results,
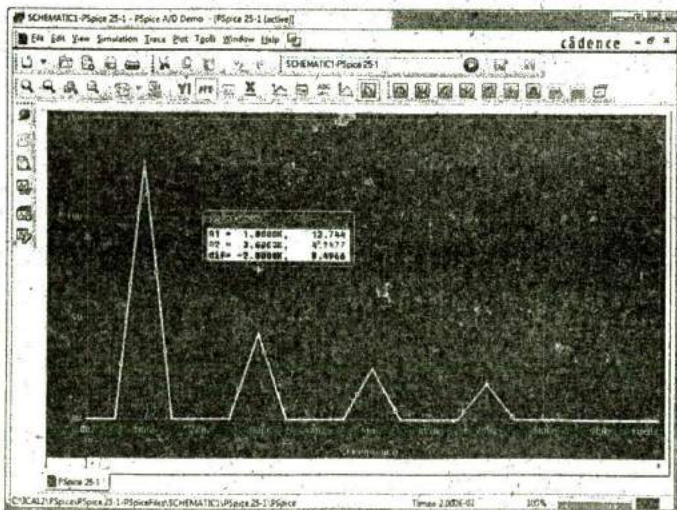

FIG. 25.28
The Fourier components of the waveform in Fig. 25.27.

giving the magnitude and frequency of the components of the waveform. Using the left cursor, you find that the highest peak is 12.74 V at 1 kHz, comparing very well with the source **VI** having a peak value of 12.732 V at 1 kHz. Using the right-click cursor, you can move over to 3 kHz and find a magnitude of 4.248 V, again comparing very well with source **V2** with a peak value of 4.244 V.

PROBLEMS

SECTION 25.2 Fourier Series

1. For the waveforms in Fig. 25.29, determine whether the following will be present in the Fourier series representation:
   a. dc term
   b. cosine terms
   c. sine terms
   d. even-ordered harmonics
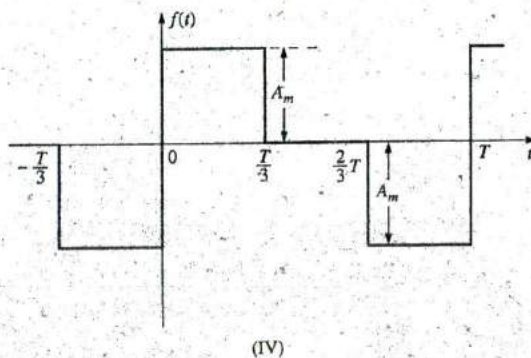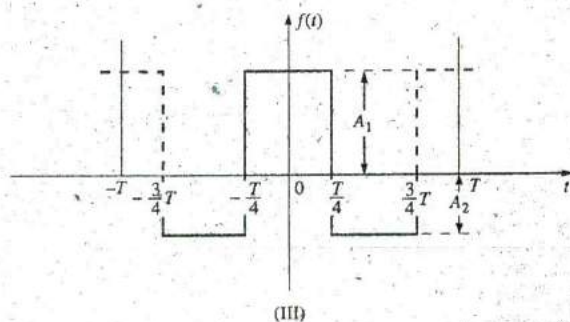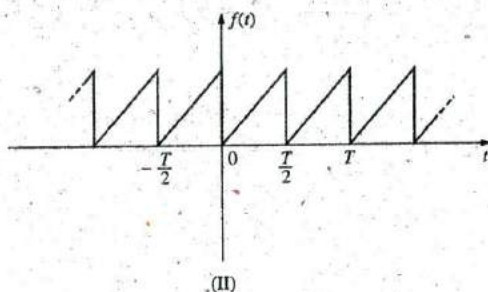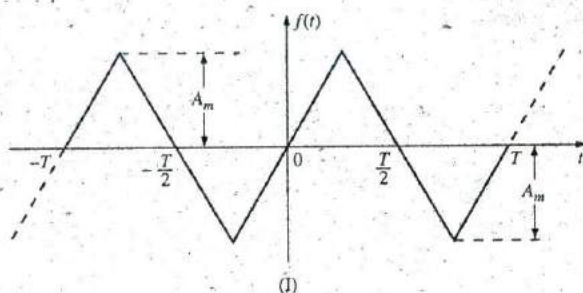   e. odd-ordered harmonics

(I)

(II)

(III)

(IV)

FIG. 25.29
Problem 1.

2. If the Fourier series for the waveform in Fig. 25.30(a) is

$$i = \frac{2I_m}{\pi}\left(1 + \frac{2}{3}\cos 2\omega t - \frac{2}{15}\cos 4\omega t + \frac{2}{35}\cos 6\omega t + \cdots\right)$$

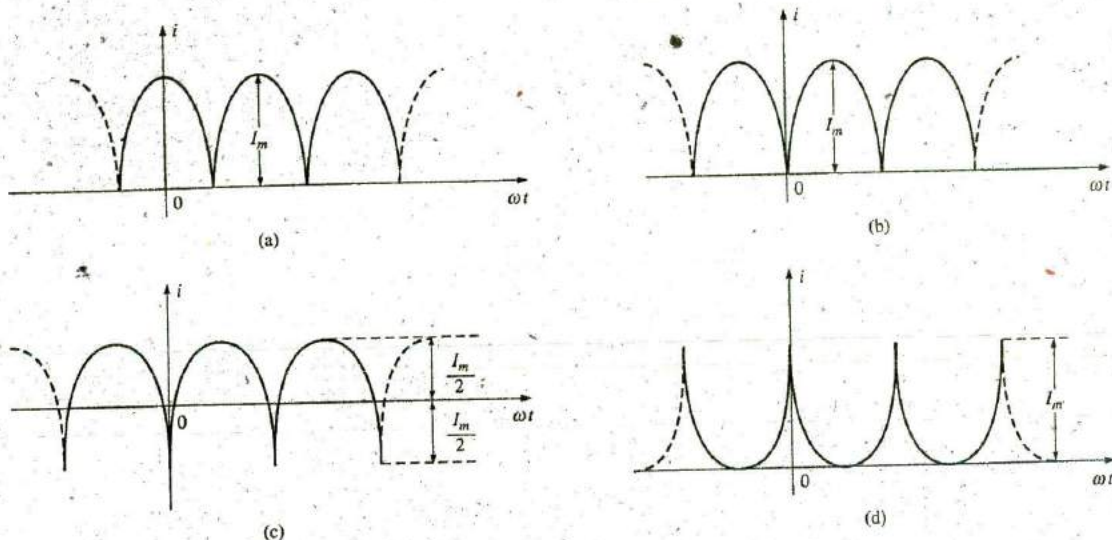find the Fourier series representation for waveforms (b) through (d).

(a)

(b)

(c)

(d)

FIG. 25.30

Problem 2.

3. Sketch the following nonsinusoidal waveforms with $\alpha = \omega t$ as the abscissa:
   a. $v = -4 + 2 \sin \alpha$
   b. $v = (\sin \alpha)^2$
   c. $i = 2 - 2 \cos \alpha$

4. Sketch the following nonsinusoidal waveforms with α as the abscissa:
   a. $i = 3 \sin \alpha - 6 \sin 2\alpha$
   b. $v = 2 \cos 2\alpha + \sin \alpha$

5. Sketch the following nonsinusoidal waveforms with ωt as the abscissa:
   a. $i = 50 \sin \omega t + 25 \sin 3\omega t$
   b. $i = 50 \sin \alpha - 25 \sin 3\alpha$
   c. $i = 4 + 3 \sin \omega t + 2 \sin 2\omega t - 1 \sin 3\omega t$

SECTION 25.3 Circuit Response
to a Nonsinusoidal Input

6. Find the average and effective values of the following non-sinusoidal waves:
   a. $v = 100 + 50 \sin \omega t + 25 \sin 2\omega t$
   b. $i = 3 + 2 \sin(\omega t - 53°) + 0.8 \sin(2\omega t - 70°)$

7. Find the rms value of the following nonsinusoidal waves:
   a. $v = 20 \sin \omega t + 15 \sin 2\omega t - 10 \sin 3\omega t$
   b. $i = 6 \sin(\omega t + 20°) + 2 \sin(2\omega t + 30°) - 1 \sin(3\omega t + 60°)$

8. Find the total average power to a circuit whose voltage and current are as indicated in Problem 6.

9. Find the total average power to a circuit whose voltage and current are as indicated in Problem 7.

10. The Fourier series representation for the input voltage to the circuit in Fig. 25.31 is

$$e = 18 + 30 \sin 400t$$

FIG. 25.31
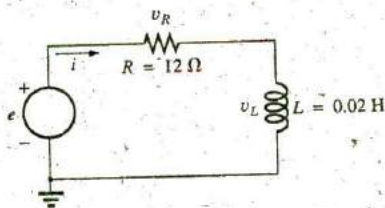
Problems 10, 11, and 12.

   a. Find the nonsinusoidal expression for the current i.
   b. Calculate the rms value of the current.
   c. Find the expression for the voltage across the resistor.
   d. Calculate the rms value of the voltage across the resistor.
   e. Find the expression for the voltage across the reactive element.
   f. Calculate the rms value of the voltage across the reactive element.
   g. Find the average power delivered to the resistor.

11. Repeat Problem 10 for

$$e = 24 + 30 \sin 400t + 10 \sin 800t$$

12. Repeat Problem 10 for the following input voltage:

$$e = -60 + 20 \sin 300t - 10 \sin 600t$$

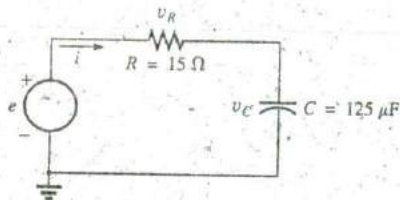13. Repeat Problem 10 for the circuit in Fig. 25.32.

FIG. 25.32
Problem 13.

*14. The input voltage in Fig. 25.33(a) to the circuit in Fig. 25.33(b) is a full-wave rectified signal having the following Fourier series expansion:
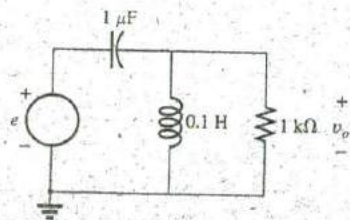
$$e = \frac{(2)(100\ \text{V})}{\pi}\left(1 + \frac{2}{3}\cos 2\omega t - \frac{2}{15}\cos 4\omega t + \frac{2}{53}\cos 6\omega t + \cdots\right)$$

where $\omega = 377$.

a. Find the Fourier series expression for the voltage v_o using only the first three terms of the expression.
b. Find the rms value of v_o.
c. Find the average power delivered to the 1 kΩ resistor.
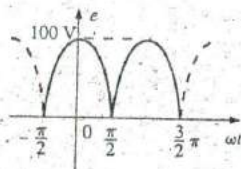
(a)
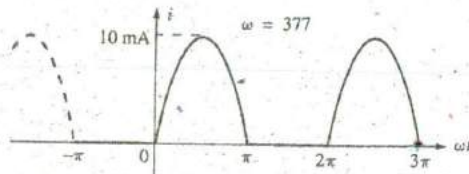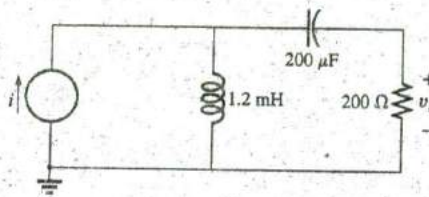
(b)

FIG. 25.33
Problem 14.

*15. Find the Fourier series expression for the voltage v_o in Fig. 25.34.

(a)

(b)

FIG. 25.34
Problem 15.

SECTION 25.4   Addition and Subtraction of Nonsinusoidal Waveforms

16. Perform the indicated operations on the following nonsinusoidal waveforms:
    a. [60 + 70 sin ωt + 20 sin(2ωt + 90°) + 10 sin(3ωt + 60°)] + [20 + 30 sin ωt − 20 cos 2ωt + 5 cos 3ωt]
    b. [20 + 60 sin α + 10 sin(2α − 180°) + 5 cos(3α + 90°)] − [5 − 10 sin α + 4 sin(3α − 30°)]

17. Find the nonsinusoidal expression for the current i_s of the diagram in Fig. 25.35.

$i_2 = 10 + 30 \sin 20t - 0.5 \sin(40t + 90°)$
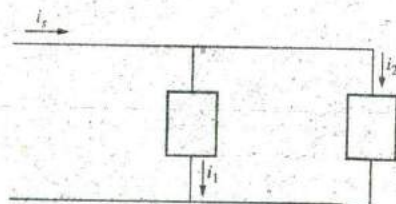$i_1 = 20 + 4 \sin(20t + 90°) + 0.5 \sin(40t + 30°)$

FIG. 25.35
Problem 17.

18. Find the nonsinusoidal expression for the voltage e of the diagram in Fig. 25.36.

$v_1 = 20 - 200 \sin 600t + 100 \cos 1200t + 75 \sin 1800t$
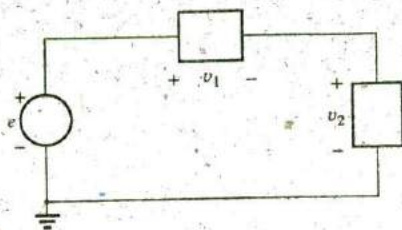$v_2 = -10 + 150 \sin(600t + 30°) + 50 \sin(1800t + 60°)$

NON

FIG. 25.36
Problem 18.

SECTION 25.5 Computer Analysis

PSpice

19. Plot the waveform in Fig. 25.11 for two or three cycles. Then obtain the Fourier components, and compare them to the applied signal.

20. Plot a half-rectified waveform with a peak value of 20 V using Eq. (25.10). Use the dc term, the fundamental term, and four harmonics. Compare the resulting waveform to the ideal half-rectified waveform.

21. Demonstrate the effect of adding two more terms to the waveform in Fig. 25.27, and generate the Fourier spectrum.

GLOSSARY

Axis symmetry A sinusoidal or nonsinusoidal function that has symmetry about the vertical axis.

Even harmonics The terms of the Fourier series expansion that have frequencies that are even multiples of the fundamental component.

Fourier series A series of terms, developed in 1826 by Baron Jean Fourier, that can be used to represent a nonsinusoidal function.

Fundamental component The minimum frequency term required to represent a particular waveform in the Fourier series expansion.

Half-wave (mirror) symmetry A sinusoidal or nonsinusoidal function that satisfies the relationship

$$f(t) = -f\left(t + \frac{T}{2}\right)$$

Harmonic terms The terms of the Fourier series expansion that have frequencies that are integer multiples of the fundamental component.

Nonsinusoidal waveform Any waveform that differs from the fundamental sinusoidal function.

Odd harmonics The terms of the Fourier series expansion that have frequencies that are odd multiples of the fundamental component.

Point symmetry A sinusoidal or nonsinusoidal function that satisfies the relationship $f(\alpha) = -f(-\alpha)$.

System Analysis: An Introduction

Objectives

- Develop an understanding and an appreciation of the system analysis of packaged systems.

- Learn how to determine the impedance, admittance, and hybrid parameters for any electrical/electronic system.

- Understand how to use the parameters of a system to determine the input and output impedance, the voltage gains, and the current gains.

- Understand the effect of one stage on another in a cascaded system.

26.1 INTRODUCTION

The growing number of packaged systems in the electrical, electronic, and computer fields now requires that some form of system analysis appear in the syllabus of the introductory course. Although the content of this chapter will be a surface treatment at best, the material will introduce a number of important terms and techniques employed in the system analysis approach. The increasing use of packaged systems is quite understandable when we consider the advantages associated with such structures: reduced size, sophisticated and tested design, reduced construction time, reduced cost compared to discrete designs, and so forth. The use of any packaged system is limited solely to the proper utilization of the provided terminals of the system. Entry into the internal structure is not permitted, which also eliminates the possibility of repair to such systems.

The LM386N unit appearing in Fig. 26.1 is a popular low-voltage audio amplifier manufactured by the National Semiconductor Corporation. The actual size appears in Fig. 26.1(a), an enlarged image in Fig. 26.1(b), and the internal construction in Fig. 26.1(c). Note that it contains quite a few electronic devices and a host of resistors with a very limited number of terminal connections. The result is that access to the internal elements is impossible, and control is only offered by the eight external pins, as shown in the package outline of Fig. 26.2. Terminals 1 and 8 are used to control the gain. If left open, the gain of the amplifier is 20, but by adding a capacitor or a series R-C combination between these two terminals, the gain can be varied up to a maximum of about 200. The supply voltage connected to pin V_s is typically between 6 V and 12 V, and the package dissipation is about 1.25 W. For the range of supply voltages V_s, the dc drain current from the dc source varies between 4 mA and 8 mA. The input impedance is about 50 kΩ, so that an applied signal of 12.5 mV will result in an input base current of about 0.25 μA. A rather simple audio design with a gain of 200 appears in Fig. 26.3, with the input applied to terminal 3 and the other input terminal grounded. The 10 kΩ potentiometer at the input controls the level of applied signal, and the output is connected to an 8 Ω speaker via a 250 μF capacitor. The 250 μF capacitor provides isolation from the load for the dc biasing conditions of the amplifier but is essentially a short-circuit for audio frequencies so that the desired ac signal can reach the speaker. The R-C combination of the 0.05 μF capacitor and the 10 Ω resistance is a protective path for undesired high-frequency spikes resulting from any switching action around the amplifier and undesireable high-frequency signals picked up by the amplifier. At frequencies
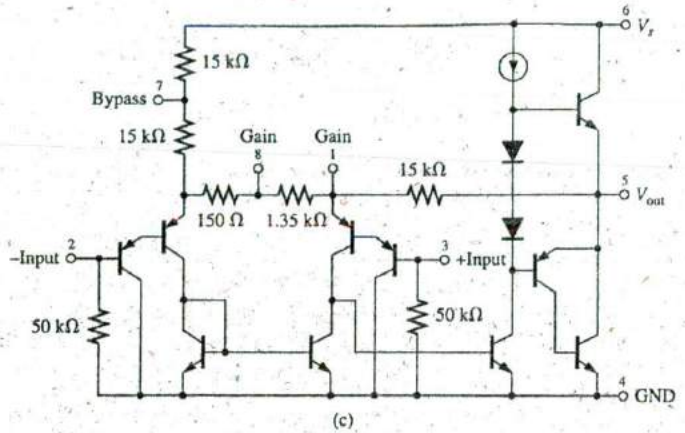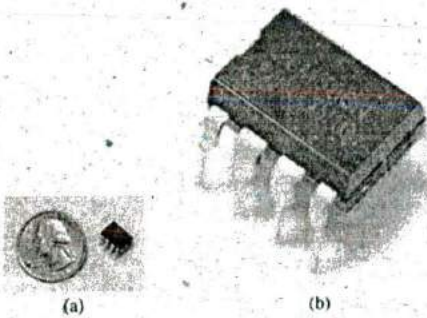
(a) (b) (c)
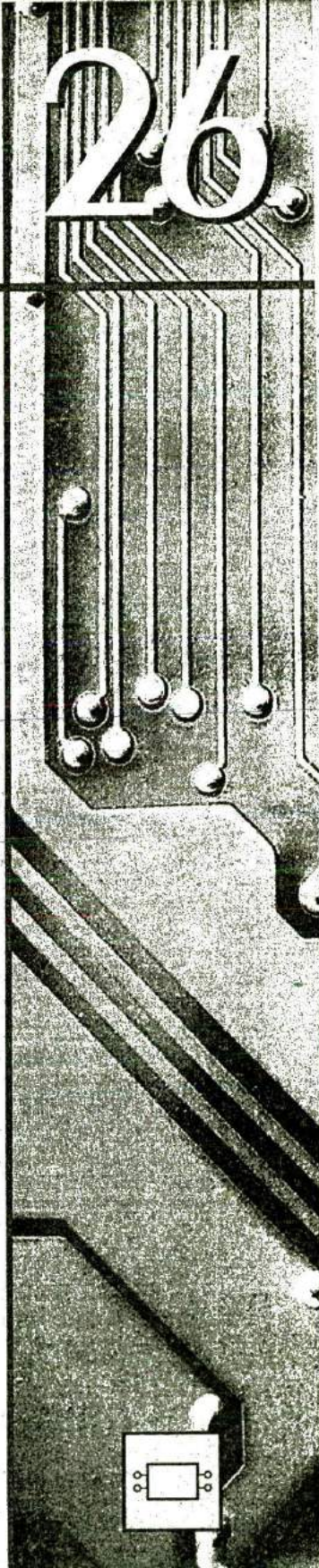
FIG. 26.1

LM386 low-voltage audio power amplifier. (a) Actual size; (b) enlarged photo;
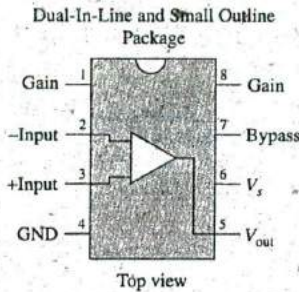*(c) internal construction.*
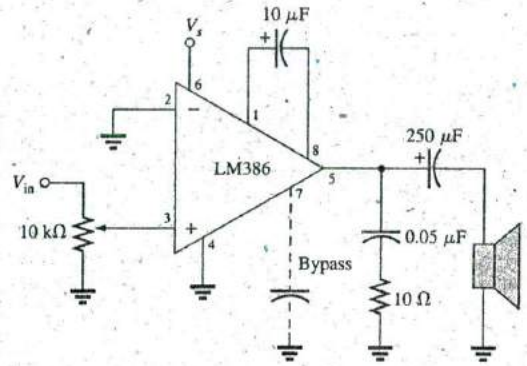

FIG. 26.2

LM386 terminal identification.

FIG. 26.3

LM386 low-voltage amplifier

higher than the normal audio range, the 0.05 μF capacitor can be approximated by a short providing a discharge path for the undesired signal. The 10 Ω resistor will ensure that the load is not shorted out at these high frequencies.

The basic operation of the amplifier will be left for the electronics courses. For the moment, however, take note of the fact that access to the internal components is quite limited and the use of the device must be designed using only the access terminals provided. Data sheets will provide a great deal of information on the internal structure and the voltage, current, resistance, and power levels associated with the typical use of the amplifier. This chapter will provide an introduction to how to work the systems where access to the internal structure is limited.

System analysis includes the development of two-, three-, or multi-port models of devices, systems, or structures. The emphasis in this

chapter will be on the configuration most frequently subject to modeling techniques: the two-port system of Fig. 26.4.

Note that in Fig. 26.4 there are two ports of entry or interest, each having a pair of terminals. For some devices, the **two-port network** of Fig. 26.4 may appear as shown in Fig. 26.5(a). The block diagram of Fig. 26.5(a) simply indicates that terminals 1′ and 2′ are in common, which is a particular case of the general two-port network. A **single-port network** and a **multiport network** appear in Fig. 26.5(b). The former has been analyzed throughout the text, while the characteristics of the latter will be touched on in this chapter, with a more extensive coverage left for a more advanced course.

The latter part of this chapter introduces a set of equations (and, subsequently, networks) that will allow us to model the device or system appearing within the enclosed structure of Fig. 26.4. That is, we will be able to establish a network that will display the same terminal characteristics as those of the original system, device, and so on. In Fig. 26.6, for example, a transistor appears between the four external terminals. Through the analysis to follow, we will find a combination of network elements that will allow us to replace the transistor with a network that will behave very much like the original device for a specific set of operating conditions. Methods such as mesh and nodal analysis can then be applied to determine any unknown quantities. The models, when reduced to their simplest forms as determined by the operating conditions, can also provide very quick estimates of network behavior without a lengthy mathematical derivation. In other words, someone well-versed in the use of models can analyze the operation of large, complex systems in short order. The results may be only approximate in most cases, but this quick return for a minimum of effort is often worthwhile.

The analysis of this chapter is limited to linear (fixed-value) systems with bilateral elements. Three sets of parameters are developed for the two-port configuration, referred to as the **impedance (z)**, **admittance (y)**, and **hybrid (h) parameters**. Table 26.1 at the end of the chapter relates the three sets of parameters.

26.2 THE IMPEDANCE PARAMETERS Z$_i$ AND Z$_o$

For the two-port system of Fig. 26.7, Z$_i$ is the **input impedance** between terminals 1 and 1′, and Z$_o$ is the **output impedance** between terminals 2 and 2′. For multiport networks, an impedance level can be defined between any two (adjacent or not) terminals of the network.
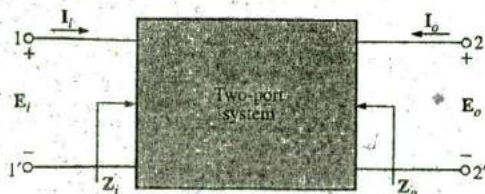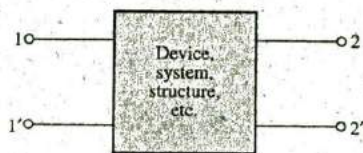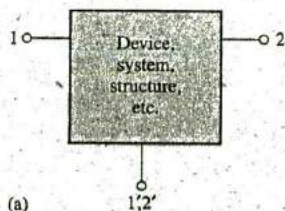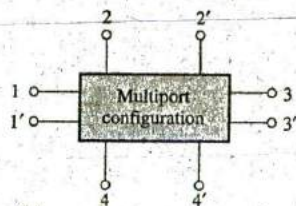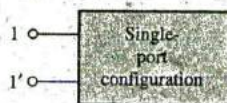

FIG. 26.4

Two-port system.

(a)

(b)

FIG. 26.5

(a) Two-port system; (b) single-port system and multiport system.
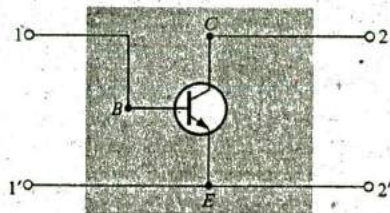
FIG. 26.6

Two-port transistor configuration.

FIG. 26.7

Defining Z$_i$ and Z$_o$.

The input impedance is defined by Ohm's law in the following form:

$$Z_i = \frac{E_i}{I_i} \qquad \text{(ohms, } \Omega) \qquad\qquad (26.1)$$

with I_i the current resulting from the application of a voltage E_i.
The output impedance Z_o is defined by

$$Z_o = \frac{E_o}{I_o}\Big|_{E_i\,=\,0\,\text{V}} \qquad \text{(ohms, } \Omega) \qquad\qquad (26.2)$$

with I_o the current resulting from the application of a voltage E_o to the output terminals, with E_i set to zero.

Note that both I_i and I_o are defined as entering the package. This is common practice for a number of system analysis methods to avoid concern about the actual direction for each current and also to define Z_i and Z_o as positive quantities in Eqs. (26.1) and (26.2), respectively. If I_o were chosen to be leaving the system, Z_o as defined in Eq. (26.2) would have to have a negative sign.

An experimental setup for determining Z_i for any two input terminals is provided in Fig. 26.8. The sensing resistor R_s is chosen small enough not to disturb the basic operation of the system or to require too large a voltage E_g to establish the desired level of E_i. Under operating conditions, the voltage across R_s is $E_g - E_i$, and the current through the sensing resistor is

$$I_{R_s} = \frac{V_{R_s}}{R_s} = \frac{E_g - E_i}{R_s}$$

But
$$I_i = I_{R_s} \qquad \text{and} \qquad Z_i = \frac{E_i}{I_i} = \frac{E_i}{I_{R_s}}$$

The sole purpose of the sensing resistor, therefore, was to determine I_i using purely voltage measurements.

As we progress through this chapter, keep in mind that we cannot use an ohmmeter to measure Z_i or Z_o since we are dealing with ac systems whose impedance may be sensitive to the applied frequency. Ohmmeters can be used to measure resistance in a dc or an ac network, but recall that ohmmeters are employed only on a de-energized network, and their internal source is a dc battery.

The output impedance Z_o can be determined experimentally using the setup of Fig. 26.9. Note that a sensing resistor is introduced again, with E_g being an applied voltage to establish typical operating conditions. In addition, note that the input signal must be set to zero, as defined by
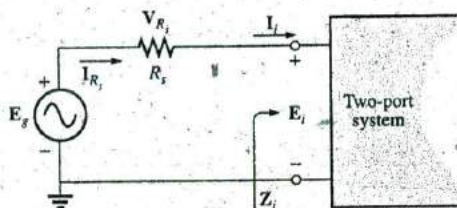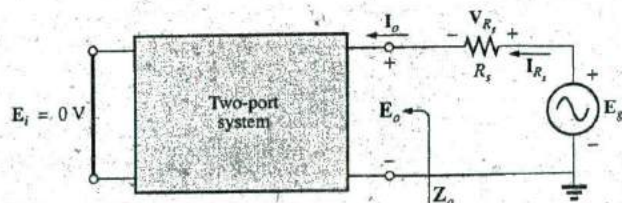

FIG. 26.8
Determining Z_i.

FIG. 26.9
Determining Z_o.

Eq. (26.2). The voltage across the sensing resistor is $E_g - E_o$, and the current through the sensing resistor is

$$I_{R_s} = \frac{V_{R_s}}{R_s} = \frac{E_g - E_o}{R_s}$$

but $I_o = I_{R_s}$ and $Z_o = \frac{E_o}{I_o} = \frac{E_o}{I_{R_s}}$

For the majority of situations, Z_i and Z_o will be purely resistive, resulting in an angle of zero degrees for each impedance. The result is that either a DMM or a scope can be used to find the required magnitude of the desired quantity. For instance, for both Z_i and Z_o, V_{R_s} can be measured directly with the DMM, as can the required levels of E_g, E_i, or E_o. The current for each case can then be determined using Ohm's law, and the impedance level can be determined using either Eq. (26.1) or Eq. (26.2).

If we use an oscilloscope, we must be more sensitive to the common ground requirement. For instance, in Fig. 26.7, E_g and E_i can be measured with the oscilloscope since they have a common ground. Trying to measure V_{R_s} directly with the ground of the oscilloscope at the top input terminal of E_i would result in a shorting effect across the input terminals of the system due to the common ground between the supply and oscilloscope. If the input impedance of the system is "shorted out," the current I_i can rise to dangerous levels because the only resistance in the input circuit is the relatively small sensing resistor R_s. If we use the DMM to avoid concern about the grounding situation, we must be sure that the meter is designed to operate properly at the frequency of interest. Many commercial units are limited to a few kilohertz.

If the input impedance has an angle other than zero degrees (purely resistive), a DMM cannot be used to find the reactive component at the input terminals. The magnitude of the total impedance will be correct if measured as described above, but the angle from which the resistive and reactive components can be determined will not be provided. If an oscilloscope is used, the network must be hooked up as shown in Fig. 26.10. Both the voltage E_g and V_{R_s} can be displayed on the oscilloscope at the same time, and the phase angle between E_g and V_{R_s} can be determined. Since V_{R_s} and I_i are in phase, the angle determined will also be the angle between E_g and I_i. The angle we are looking for is between E_i and I_i, not between E_g and I_i, but since R_s is usually chosen small enough, we can assume that the voltage drop across R_s is so small compared to E_g that $E_i \cong E_g$. Substituting the peak, peak-to-peak, or rms values from the
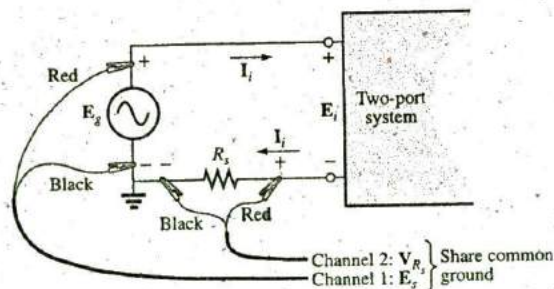

FIG. 26.10
Determining Z_i using an oscilloscope.

oscilloscope measurements, along with the angle just determined, will permit a determination of the magnitude and angle for Z_i, from which the resistive and reactive components can be determined using a few basic geometric relationships. The reactive nature (inductive or capacitive) of the input impedance can be determined when the angle between E_i and I_i is computed. For a dual-trace oscilloscope, if E_g leads V_{R_s} (E_i leads I_i), the network is inductive; if the reverse is true, the network is capacitive.

To determine the angle associated with Z_o, the sensing resistor must again be moved to the bottom to form a common ground with the supply E_g. Then, using the approximation $E_g \cong E_o$, we can determine the magnitude and angle of Z_o.
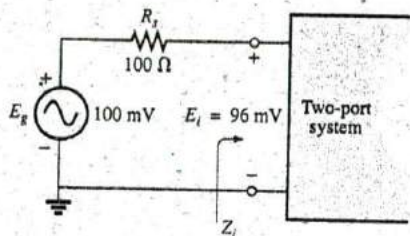

FIG. 26.11

Example 26.1.

EXAMPLE 26.1 Given the DMM measurements appearing in Fig. 26.11, determine the input impedance Z_i for the system if the input impedance is known to be purely resistive.

Solution:

$$V_{R_s} = E_g - E_i = 100 \text{ mV} - 96 \text{ mV} = 4 \text{ mV}$$

$$I_i = I_{R_s} = \frac{V_{R_s}}{R_s} = \frac{4 \text{ mV}}{100 \ \Omega} = 40 \ \mu\text{A}$$

$$Z_i = R_i = \frac{E_i}{I_i} = \frac{96 \text{ mV}}{40 \ \mu\text{A}} = 2.4 \text{ k}\Omega$$

EXAMPLE 26.2 Using the provided DMM measurements of Fig. 26.12, determine the output impedance Z_o for the system if the output impedance is known to be purely resistive.
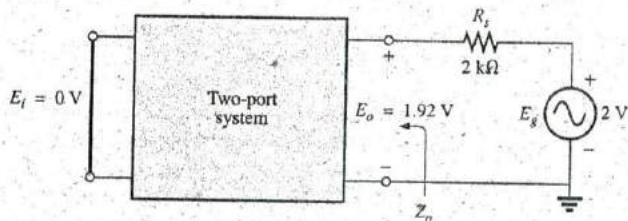

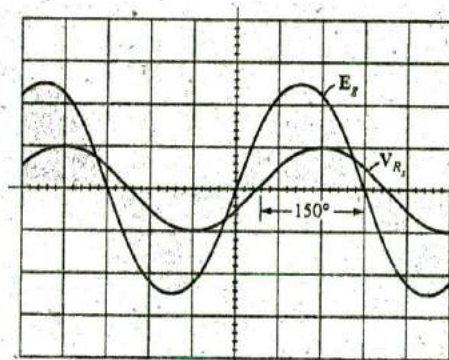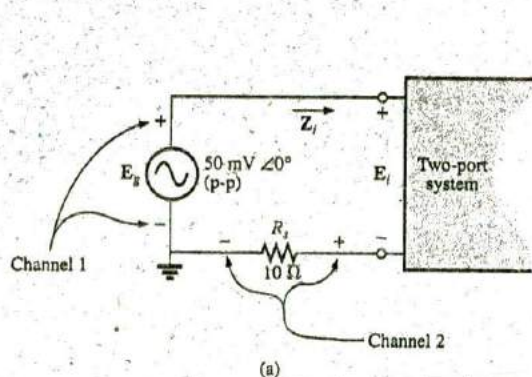
FIG. 26.12

Example 26.2.

Solution:

$$V_{R_s} = E_g - E_o = 2 \text{ V} - 1.92 \text{ V} = 0.08 \text{ V} = 80 \text{ mV}$$

$$I_o = I_{R_s} = \frac{V_{R_s}}{R_s} = \frac{80 \text{ mV}}{2 \text{ k}\Omega} = 40 \ \mu\text{A}$$

$$Z_o = \frac{E_o}{I_o} = \frac{1.92 \text{ V}}{40 \ \mu\text{A}} = 48 \text{ k}\Omega$$

EXAMPLE 26.3 The input characteristics for the system of Fig. 26.13(a) are unknown. Using the oscilloscope measurements of Fig. 26.13(b), determine the input impedance for the system. If a

FIG. 26.13

Example 26.3.

E_g: Vertical sensitivity = 10 mV/div.
V_{R_s}: Vertical sensitivity = 1 mV/div.

(b)

reactive component exists, determine its magnitude and whether it is inductive or capacitive.

Solution: The magnitude of Z_i:

$$I_{i(p-p)} = I_{R_s(p-p)} = \frac{V_{R_s(p-p)}}{R_s} = \frac{2 \text{ mV}}{10 \text{ } \Omega} = 200 \text{ } \mu\text{A}$$

$$Z_i = \frac{E_i}{I_i} \cong \frac{E_g}{I_i} = \frac{50 \text{ mV}}{200 \text{ } \mu\text{A}} = 250 \text{ } \Omega$$

The angle of Z_i: The phase angle between $\mathbf{E}_g$ and $\mathbf{V}_{R_s}$ (or $\mathbf{I}_{R_s} = \mathbf{I}_i$) is

$$180° - 150° = 30°$$

with $\mathbf{E}_g$ leading $\mathbf{I}_i$, so the system is inductive. Therefore,

$$Z_i = 250 \text{ } \Omega \angle 30°$$
$$= 216.51 \text{ } \Omega + j \text{ } 125 \text{ } \Omega = R + jX_L$$

26.3 THE VOLTAGE GAINS $A_{v_{NL}}$, A_v, AND A_{v_T}

The voltage gain for the two-port system of Fig. 26.14 is defined by

$$\boxed{A_{v_{NL}} = \frac{E_o}{E_i}} \qquad (26.3)$$

The capital letter **A** in the notation was chosen from the term amplification factor, with the subscript v selected to specify that voltage levels are involved. The subscript *NL* reveals that the ratio was determined under *no-load* conditions; that is, a load was not applied to the output terminals when the gain was determined. The no-load voltage gain is the gain typically provided with packaged systems since the applied load is a function of the application.

The magnitude of the ratio can be determined using a DMM or an oscilloscope. The oscilloscope, however, must be used to determine the phase shift between the two voltages.
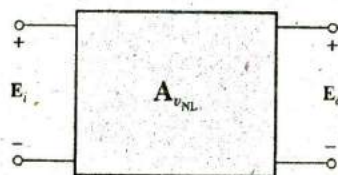

FIG. 26.14

Defining the no-load gain $A_{v_{NL}}$.

In Fig. 26.15 a load has been introduced to establish a loaded gain that will be denoted simply as A_v and defined by

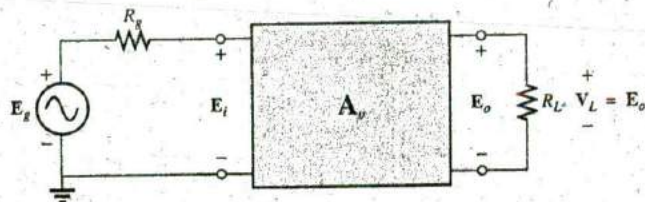$$A_v = \frac{E_o}{E_i}\bigg|_{\text{with } R_L} \tag{26.4}$$

FIG: 26.15
Defining the loaded voltage gain A_v (and A_{v_T}).

For all two-port systems the loaded gain A_v will always be less than the no-load gain.

In other words, the application of a load will always reduce the gain below the no-load level.

A third voltage gain can be defined using Fig. 26.15 since it has an applied voltage source with an associated internal resistance—a situation often encountered in electronic systems. The total voltage gain of the system is represented by A_{v_T} and is defined by

$$A_{v_T} = \frac{E_o}{E_g} \tag{26.5}$$

It is the voltage gain from the source E_g to the output terminals E_o. Due to loss of signal voltage across the source resistance,

the voltage gain A_{v_T} is always less than the loaded voltage gain A_v or unloaded gain $A_{v_{NL}}$.

If we expand Eq. (26.5) as

$$A_{v_T} = \frac{E_o}{E_g} = \frac{E_o}{E_g}(1) = \frac{E_o}{E_g}\left(\frac{E_i}{E_i}\right) = \frac{E_o}{E_i} \cdot \frac{E_i}{E_g}$$

then

$$A_{v_T} = A_v \frac{E_i}{E_g} \qquad \text{(if loaded)}$$

or

$$A_{v_T} = A_{v_{NL}} \frac{E_i}{E_g} \qquad \text{(if unloaded)}$$

The relationship between E_i and E_g can be determined from Fig. 26.15 if we recognize that E_i is across the input impedance Z_i and thus apply the voltage divider rule as follows:

$$E_i = \frac{Z_i(E_g)}{Z_i + R_g}$$

or

$$\frac{E_i}{E_g} = \frac{Z_i}{Z_i + R_g}$$

Substituting into these relationships results in

$$A_{v_T} = A_v \frac{Z_i}{Z_i + R_g} \qquad \text{(if loaded)} \qquad (26.6)$$

$$A_{v_T} = A_{v_{NL}} \frac{Z_i}{Z_i + R_g} \qquad \text{(if unloaded)} \qquad (26.7)$$

A two-port equivalent model for an unloaded system based on the definitions of Z_i, Z_o, and $A_{v_{NL}}$ is provided in Fig. 26.16. Both Z_i and Z_o appear as resistive values since this is typically the case for most electronic amplifiers. However, both Z_i and Z_o can have reactive components and not invalidate the equivalency of the model.
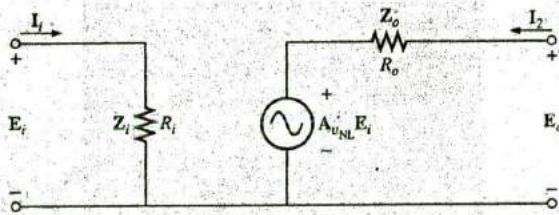

FIG. 26.16
Equivalent model for two-port amplifier.

The input impedance is defined by $Z_i = E_i/I_i$ and the voltage $E_o = A_{v_{NL}}E_i$ in the absence of a load, resulting in $A_{v_{NL}} = E_o/E_i$ as defined. The output impedance is defined with E_i set to zero volts, resulting in $A_{v_{NL}}E_i = 0$ V, which permits the use of a short-circuit equivalent for the controlled source. The result is $Z_o = E_o/I_o$, as defined, and the parameters and structure of the equivalent model are validated.

If a load is applied as in Fig. 26.17, an application of the voltage divider rule will result in

$$E_o = \frac{R_L(A_{v_{NL}}E_i)}{R_L + R_o}$$

and

$$A_v = \frac{E_o}{E_i} = A_{v_{NL}} \frac{R_L}{R_L + R_o} \qquad (26.8)$$

For any value of R_L or R_o, the ratio $R_L/(R_L + R_o)$ must be less than 1, mandating that A_v is always less than $A_{v_{NL}}$ as stated earlier. Further,

for a fixed output impedance (R_o), the larger the load resistance (R_L), the closer is the loaded gain to the no-load level.

An experimental procedure for determining R_o can be developed if we solve Eq. (26.8) for the output impedance R_o:
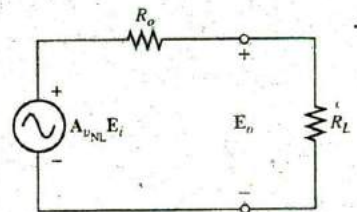
$$A_v = \frac{R_L}{R_L + R_o} A_{v_{NL}}$$

FIG. 26.17
Applying a load to the output of Fig. 26.16.

or
$$A_v(R_L + R_o) = R_L A_{v_{NL}}$$
$$A_v R_L + A_v R_o = R_L A_{v_{NL}}$$

and
$$A_v R_o = R_L A_{v_{NL}} - A_v R_L$$

with
$$R_o = \frac{R_L(A_{v_{NL}} - A_v)}{A_v}$$

or
$$\boxed{R_o = R_L\left(\frac{A_{v_{NL}}}{A_v} - 1\right)} \qquad (26.9)$$

Equation (26.9) reveals that the output impedance R_o of an amplifier can be determined by first measuring the voltage gain E_o/E_i without a load in place to find $A_{v_{NL}}$ and then measuring the gain with a load R_L to find A_v. Substitution of $A_{v_{NL}}$, A_v, and R_L into Eq. (26.9) will then provide the value for R_o.

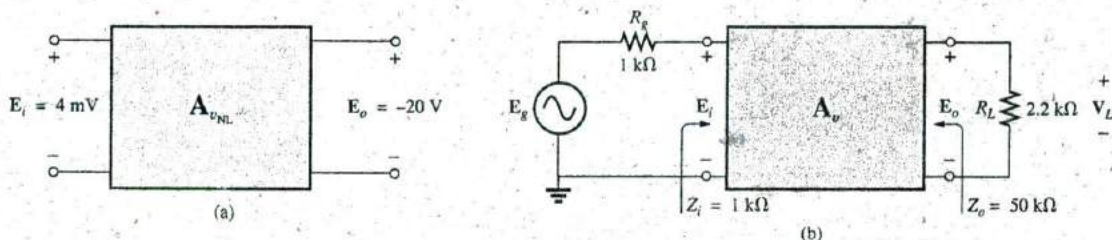**EXAMPLE 26.4** For the system of Fig. 26.18(a) employed in the loaded amplifier of Fig. 26.18(b):

FIG. 26.18
Example 26.4.

a. Determine the no-load voltage gain $A_{v_{NL}}$.
b. Find the loaded voltage gain A_v.
c. Calculate the loaded voltage gain A_{v_T}.
d. Determine R_o from Eq. (26.9), and compare it to the specified value of Fig. 26.18.

Solutions:

a. $A_{v_{NL}} = \dfrac{E_o}{E_i} = \dfrac{-20\ V}{4\ mV} = -5000$

b. $A_v = A_{v_{NL}} \dfrac{R_L}{R_L + R_o} = (-5000)\left(\dfrac{2.2\ k\Omega}{2.2\ k\Omega + 50\ k\Omega}\right)$
$= (-5000)(0.0421) = -210.73$

c. $A_{v_T} = A_v \dfrac{Z_i}{Z_i + R_g} = (-210.73)\left(\dfrac{1\ k\Omega}{1\ k\Omega + 1\ k\Omega}\right)$
$= (-210.73)\left(\dfrac{1}{2}\right) = -105.36$

d. $R_o = R_L\left(\dfrac{A_{v_{NL}}}{A_v} - 1\right) = 2.2\ k\Omega\left(\dfrac{-5000}{-210.73} - 1\right)$
$= 2.2\ k\Omega(23.727 - 1) = 2.2\ k\Omega(22.727)$
$= 50\ k\Omega$ as specified

26.4 THE CURRENT GAINS A_i AND A_{iT}, AND THE POWER GAIN A_G

The current gain of two-port systems is typically calculated from voltage levels. A no-load gain is not defined for current gain since the absence of R_L requires that $I_o = E_o/R_L = 0$ A and $A_i = I_o/I_i = 0$.

For the system of Fig. 26.19, however, a load has been applied, and

$$I_o = -\frac{E_o}{R_L}$$
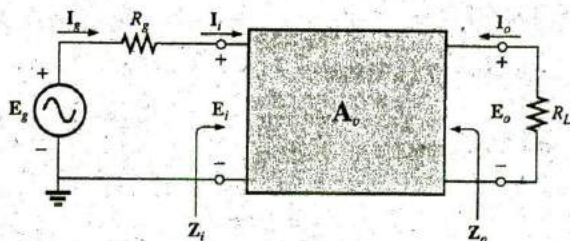
with

$$I_i = -\frac{E_i}{Z_i}$$

FIG. 26.19
Defining A_i and A_{iT}.

Note the need for a minus sign when I_o is defined, because the defined polarity of E_o would establish the opposite direction for I_o through R_L.

The loaded current gain is

$$A_i = \frac{I_o}{I_i} = \frac{-E_o/R_L}{E_i/Z_i} = -\frac{E_o}{E_i}\left(\frac{Z_i}{R_L}\right)$$

and

$$\boxed{A_i = -A_v\frac{Z_i}{R_L}} \qquad (26.10)$$

In general, therefore, the loaded current gain can be obtained directly from the loaded voltage gain and the ratio of impedance levels Z_i over R_L.

If the ratio $A_{iT} = I_o/I_g$ were required, we would proceed as follows:

$$I_o = -\frac{E_o}{R_L}$$

with

$$I_i = \frac{E_g}{R_g + Z_i}$$

and

$$A_{iT} = \frac{I_o}{I_g} = \frac{-E_o/R_L}{E_g/(R_g + Z_i)} = -\left(\frac{E_o}{E_g}\right)\left(\frac{R_g + Z_i}{R_L}\right)$$

or

$$\boxed{A_{iT} = \frac{I_o}{I_g} = -A_{vT}\left(\frac{R_g + Z_i}{R_L}\right)} \qquad (26.11)$$

The result obtained with Eq. (26.10) or (26.11) will be the same since $I_g = I_i$, but we now have the option of which gain is available or which one to use.

Returning to Fig. 26.16 (repeated in Fig. 26.20), we can determine an equation for the current gain in terms of the no-load voltage gain.
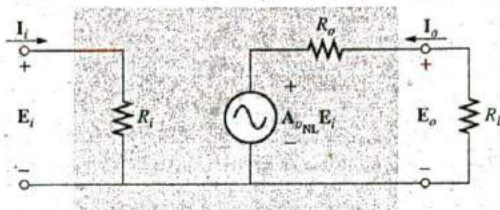

FIG. 26.20

Developing an equation for A_i in terms of $A_{v_{NL}}$.

Through Ohm's law,

$$I_o = -\frac{A_{v_{NL}}E_i}{R_L + R_o}$$

but

$$E_i = I_i R_i$$

and

$$I_o = -\frac{A_{v_{NL}}(I_i R_i)}{R_L + R_o}$$

so that

$$A_i = \frac{I_o}{I_i} = -A_{v_{NL}}\frac{R_i}{R_L + R_o} \qquad (26.12)$$

The result is an equation for the loaded current gain of an amplifier in terms of the nameplate no-load voltage gain and the resistive elements of the network.

Recall an earlier conclusion that the larger the value of R_L, the larger is the loaded voltage gain. For current levels, Equation (26.12) reveals that

the larger the level of R_L, the less is the current gain of a loaded amplifier.

In the design of an amplifier, therefore, one must balance the desired voltage gain with the current gain and the resulting ac output power level.

For the system of Fig. 26.20, the power delivered to the load is determined by E_o^2/R_L, whereas the power delivered at the input terminals is E_i^2/R_i. The power gain is therefore defined by

$$A_G = \frac{P_o}{P_i} = \frac{E_o^2/R_L}{E_i^2/R_i} = \frac{E_o^2}{E_i^2}\frac{R_i}{R_L} = \left(\frac{E_o}{E_i}\right)^2\frac{R_i}{R_L}$$

and

$$A_G = A_v^2\frac{R_i}{R_L} \qquad (26.13)$$

Expanding the conclusion gives

$$A_G = (A_v)\left(A_v\frac{R_i}{R_L}\right) = (A_v)(-A_i)$$

so
$$A_G = -A_v A_i \qquad (26.14)$$

Don't be concerned about the minus sign. A_v or A_i will be negative to ensure that the power gain is positive, as obtained from Eq. (26.13).

If we substitute $A_v = -A_i R_L/R_i$ [from Eq. (26.10)] into Eq. (26.14), we find

$$A_G = -A_v A_i = -\left(\frac{-A_i R_L}{R_i}\right) A_i$$

or
$$A_G = A_i^2 \frac{R_L}{R_i} \qquad (26.15)$$

which has a format similar to that of Eq. (26.13), but now A_G is given in terms of the current gain of the system.

The last power gain to be defined is

$$A_{G_T} = \frac{P_L}{P_g} = \frac{E_o^2/R_L}{E_g I_g} = \frac{E_o^2/R_L}{E_g^2/(R_g + R_i)} = \left(\frac{E_o}{E_g}\right)^2 \left(\frac{R_g + R_i}{R_L}\right)$$

or
$$A_{G_T} = A_{v_T}^2\left(\frac{R_g + R_i}{R_L}\right) \qquad (26.16)$$

Expanding gives

$$A_{G_T} = A_{v_T}\left(A_{v_T} \frac{R_g + R_i}{R_L}\right)$$

and
$$A_{G_T} = -A_{v_T} A_{i_T} \qquad (26.17)$$

EXAMPLE 26.5 Given the system of Fig. 26.21 with its nameplate data:
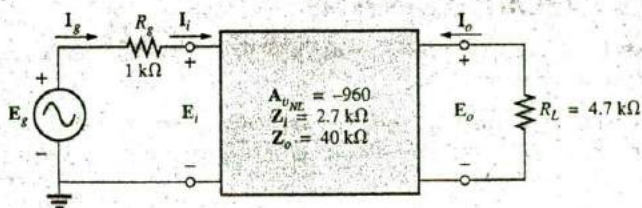

FIG. 26.21
Example 26.5.

a. Determine A_v.
b. Calculate A_i.
c. Increase R_L to double its current value, and note the effect on A_v and A_i.
d. Find A_{i_T}.
e. Calculate A_G.
f. Determine A_i from Eq. (26.1), and compare it to the value obtained in part (b).

Solutions:

a. $A_v = A_{v_{NL}} \dfrac{R_L}{R_L + R_o} = (-960)\left(\dfrac{4.7 \text{ k}\Omega}{4.7 \text{ k}\Omega + 40 \text{ k}\Omega}\right) = -100.94$

b. $A_i = -A_{v_{NL}} \dfrac{R_i}{R_L + R_o} = -(-960)\left(\dfrac{2.7 \text{ k}\Omega}{4.7 \text{ k}\Omega + 40 \text{ k}\Omega}\right) = 57.99$

c. $R_L = 2(4.7 \text{ k}\Omega) = 9.4 \text{ k}\Omega$

$A_v = A_{v_{NL}}\left(\dfrac{R_L}{R_L + R_o}\right) = (-960)\left(\dfrac{9.4 \text{ k}\Omega}{9.4 \text{ k}\Omega + 40 \text{ k}\Omega}\right)$

$= -182.67 \quad$ versus -100.94, which is a significant increase

$A_i = -A_{v_{NL}}\left(\dfrac{R_i}{R_L + R_o}\right) = -(-960)\left(\dfrac{2.7 \text{ k}\Omega}{40 \text{ k}\Omega + 9.4 \text{ k}\Omega}\right)$

$= 52.47 \quad$ versus 57.99

which is a drop in level but not as significant as the change in A_v.

d. $A_{i_T} = A_i = 57.99 \quad$ as obtained in part (b)

However, $A_{i_T} = -A_{v_T}\left(\dfrac{R_g + R_i}{R_L}\right)$

$= -\left[A_v \dfrac{R_i}{(R_i + R_g)}\right]\left[\dfrac{(R_g + R_i)}{R_L}\right]$

$= -A_v \dfrac{R_i}{R_L} = -(-100.94)\left(\dfrac{2.7 \text{ k}\Omega}{4.7 \text{ k}\Omega}\right)$

$= 57.99 \quad$ as well

e. $A_G = A_v^2 \dfrac{R_i}{R_L} = (100.94)^2\left(\dfrac{2.7 \text{ k}\Omega}{4.7 \text{ k}\Omega}\right) = 5853.19$

f. $A_G = -A_v A_i$

or $A_i = \dfrac{A_G}{A_v} = -\dfrac{(5853.19)}{(-100.94)}$

$= 57.99 \quad$ as found in part (b)

26.5 CASCADED SYSTEMS

When considering cascaded systems, as in Fig. 26.22, the most important fact to remember is that

the equations for cascaded systems employ the loaded voltage and current gains for each stage and not the nameplate unloaded levels.
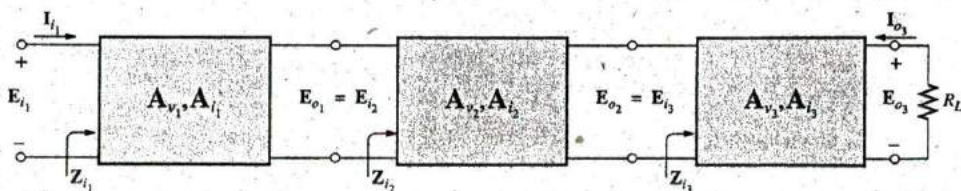

FIG. 26.22
Cascaded system.

Too often the labeled no-load gains are employed, resulting in enormous overall gains and unreasonably high expectations for the system. In addition, bear in mind that the input impedance of stage 3 may affect the input impedance of stage 2 and, therefore, the load on stage 1.

In general, therefore, the equations for cascaded systems initially appear to offer a high level of simplicity to the analysis. Be aware, however, that each term of the overall equations must be carefully evaluated before using the equation.

The total voltage gain for the system of Fig. 26.22 is

$$A_{v_T} = A_{v_1} \cdot A_{v_2} \cdot A_{v_3} \qquad (26.18)$$

where, as noted previously, the amplification factor of each stage is determined under loaded conditions.

The total current gain for the system of Fig. 26.22 is

$$A_{i_T} = A_{i_1} \cdot A_{i_2} \cdot A_{i_3} \qquad (26.19)$$

where, again, the gain of each stage is determined under loaded (connected) conditions.

The current gain between any two stages can also be determined using an equation developed earlier in the chapter. For cascaded systems, the equation has the following general format:

$$A_i = A_v \frac{Z_i}{R_L} \qquad (26.20)$$

where A_v is the loaded voltage gain corresponding to the desired loaded current gain. That is, if the gain is from the first to the third stages, then the voltage gain substituted is also from the first to third stages. The input impedance Z_i is for the first stage of interest, and R_L is the loading on the last stage of interest.

For example, for the three-stage amplifier of Fig. 26.22,

$$A_{i_T} = A_{v_T} \frac{Z_{i_1}}{R_L}$$

whereas for the first two stages,

$$A'_i = A'_v \frac{Z_{i_1}}{Z_{i_3}}$$

where

$$A'_i = \frac{I_{o_2}}{I_{i_1}} \quad \text{and} \quad A'_v = \frac{E_{o_2}}{E_{i_1}}$$

The total power gain is determined by

$$A_{G_T} = A_{v_T} A_{i_T} \qquad (26.21)$$

whereas the gain between specific stages is simply the product of the voltage and current gains for each section. For example, for the first two stages of Fig. 26.22,

$$A'_G = A'_{v_2} \cdot A'_{i_2}$$

where $\quad A'_{v_2} = A_{v_1} \cdot A_{v_2} \quad$ and $\quad A'_{i_2} = A_{i_1} \cdot A_{i_2}$

EXAMPLE 26.6 For the cascaded system of Fig. 26.23, with its nameplate no-load parameters:

a. Determine the load voltage and current gain for each stage, and redraw the system of Fig. 26.23 with the loaded parameters.
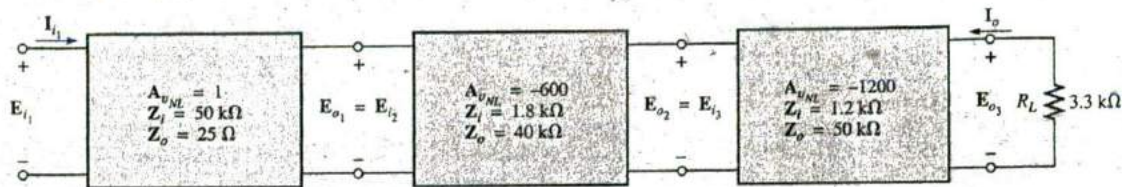
FIG. 26.23
Example 26.6.

b. Calculate the total voltage and current gain.
c. Find the total power gain of the system using Eq. (26.21).
d. Calculate the voltage and current gain for the first two stages using Eqs. (26.18) and (26.19).
e. Determine the current gain for the first two stages using Eq. (26.20), and compare your answer with the result of part (d).
f. Calculate the power gain for the first two stages using Eq. (26.21).
g. Determine the power gain for the first two stages using Eq. (26.13). Compare this answer with the result of part (f).
h. Calculate the incorrect voltage gain for the entire system using Eq. (26.18) and the no-load nameplate level for each stage. Compare this answer to the result of part (b).

Solutions:

a. $A_{v_1} = A_{v_{NL_1}} \dfrac{R_L}{R_L + R_o} = A_{v_{NL_1}} \dfrac{Z_{i_2}}{Z_{i_2} + R_{o_1}} = (1)\dfrac{1.8 \text{ k}\Omega}{1.8 \text{ k}\Omega + 25 \Omega}$
$= 0.986$

$A_{v_2} = A_{v_{NL_2}} \dfrac{Z_{i_3}}{Z_{i_3} + R_{o_2}} = (-600)\dfrac{1.2 \text{ k}\Omega}{1.2 \text{ k}\Omega + 40 \text{ k}\Omega} = -17.476$

$A_{v_3} = A_{v_{NL_3}} \dfrac{R_L}{R_L + R_{o_3}} = (-1200)\dfrac{3.3 \text{ k}\Omega}{3.3 \text{ k}\Omega + 50 \text{ k}\Omega} = -74.296$

$A_{i_1} = -A_{v_{NL_1}} \dfrac{R_i}{R_L + R_o} = -A_{v_{NL_1}} \dfrac{Z_{i_1}}{Z_{i_2} + R_{o_1}} = -(1)\dfrac{50 \text{ k}\Omega}{1.8 \text{ k}\Omega + 25 \Omega}$
$= -27.397$

$A_{i_2} = -A_{v_{NL_2}} \dfrac{Z_{i_2}}{Z_{i_3} + R_{o_2}} = -(-600)\dfrac{1.8 \text{ k}\Omega}{1.2 \text{ k}\Omega + 40 \text{ k}\Omega} = 26.214$

$A_{i_3} = -A_{v_{NL_3}} \dfrac{Z_{i_3}}{R_L + R_{o_3}} = -(-1200)\dfrac{1.2 \text{ k}\Omega}{3.3 \text{ k}\Omega + 50 \text{ k}\Omega} = 27.017$
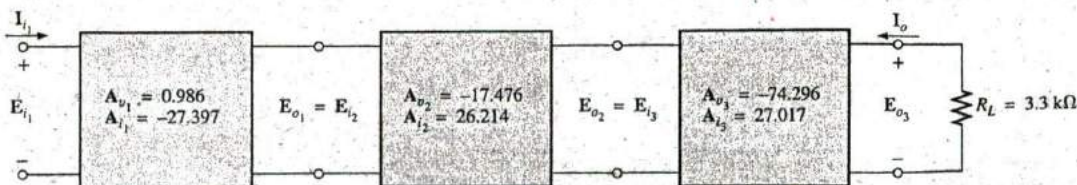
Note Fig. 26.24.

FIG. 26.24
Solution to Example 26.6.

b. $A_{v_T} = \dfrac{E_{o_3}}{E_{i_1}} = A_{v_1} \cdot A_{v_2} \cdot A_{v_3} = (0.986)(-17.476)(-74.296)$

$\qquad = 1280.22$

$A_{i_T} = \dfrac{I_{o_3}}{I_{i_1}} = A_{i_1} \cdot A_{i_2} \cdot A_{i_3} = (-27.397)(26.214)(27.017)$

$\qquad = -19{,}403.20$

c. $A_{G_T} = -A_{v_T} \cdot A_{i_T} = -(1280.22)(-19{,}403.20) = 24.84 \times 10^6$

d. $A'_{v_2} = A_{v_1} \cdot A_{v_2} = (0.986)(-17.476) = -17.231$

$\quad A'_{i_2} = A_{i_1} \cdot A_{i_2} = (-27.397)(26.214) = -718.185$

e. $A'_{i_2} = A_v \dfrac{Z_i}{R_L} = A'_{v_2} \dfrac{Z_{i_1}}{Z_{i_3}} = (-17.231)\dfrac{50 \text{ k}\Omega}{1.2 \text{ k}\Omega}$

$\qquad = -717.958 \qquad \text{versus} \ -718.185$

with the difference due to the level of accuracy carried through the calculations.

f. $A'_{G_2} = A'_{v_2} \cdot A'_{i_2} = (-17.231)(-718.185) = 12{,}375.05$

g. $A'_{G_2} = A_v^2 \dfrac{R_i}{R_L} = (A'_{v_2})^2 \dfrac{R_{i_1}}{Z_{i_3}} = (-17.231)^2 \dfrac{50 \text{ k}\Omega}{1.2 \text{ k}\Omega} = 12{,}371.14$

h. $A_{v_T} = A_{v_1} \cdot A_{v_2} \cdot A_{v_3} = (1)(-600)(-1200) = 7.2 \times 10^5$

$\quad 720{,}000 : 1280.22 = 562.40 : 1$

which is certainly a significant difference in results.

26.6 IMPEDANCE (z) PARAMETERS

For the two-port configuration of Fig. 26.25, four variables are specified. For most situations, if any two are specified, the remaining two variables can be determined. The four variables can be related by the following equations:

$$\boxed{E_1 = z_{11}I_1 + z_{12}I_2} \qquad (26.22a)$$

$$\boxed{E_2 = z_{21}I_1 + z_{22}I_2} \qquad (26.22b)$$

The **impedance parameters** z_{11}, z_{12}, and z_{22} are measured in ohms.

To model the system, each impedance parameter must be determined by setting a particular variable to zero.

z_{11}

For z_{11}, if I_2 is set to zero, as shown in Fig. 26.26, Eq. (26.22a) becomes

$$E_1 = z_{11}I_1 + z_{12}(0)$$

and

$$\boxed{z_{11} = \dfrac{E_1}{I_1}\Big|_{I_2 = 0}} \qquad \text{(ohms, } \Omega) \qquad (26.23)$$

Equation (26.23) reveals that with I_2 set to zero, the impedance parameter is determined by the resulting ratio of E_1 to I_1. Since E_1 and I are both input quantities, with I_2 set to zero, the parameter z_{11} is formally referred to in the following manner:
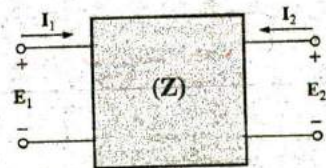
$z_{11} = $ **open-circuit, input-impedance parameter**

FIG. 26.25
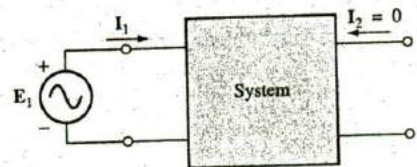
Two-port impedance parameter configuration.

FIG. 26.26

Determining z_{11}.

Z12

For z_{12}, I_1 is set to zero, and Eq. (26.22a) results in

$$z_{12} = \left. \frac{E_1}{I_2} \right|_{I_2 = 0} \qquad \text{(ohms, } \Omega) \qquad (26.24)$$
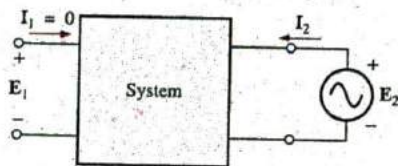
FIG. 26.27

Determining z_{12}.

For most systems where input and output quantities are to be compared, the ratio of interest is usually that of the output quantity divided by the input quantity. In this case, the *reverse* is true, resulting in the following:

z_{12} = *open-circuit, reverse-transfer impedance parameter*

The term *transfer* is included to indicate that z_{12} will relate an input and output quantity (for the condition $I_1 = 0$). The network configuration for determining z_{12} is shown in Fig. 26.27.

For an applied source E_2, the ratio E_1/I_2 will determine z_{12} with I_1 set to zero.

Z21

FIG. 26.28

Determining z_{21}.

To determine z_{21}, set I_2 to zero and find the ratio E_2/I_1 as determined by Eq. (26.22b). That is,

$$z_{21} = \left. \frac{E_2}{I_1} \right|_{I_2 = 0} \qquad \text{(ohms, } \Omega) \qquad (26.25)$$

In this case, input and output quantities are again the determining variables, requiring the term *transfer* in the nomenclature. However, the ratio is that of an output to an input quantity, so the descriptive term *forward* is applied, and

z_{21} = *open-circuit, forward-transfer impedance parameter*

The determining network is shown in Fig. 26.28. For an applied voltage E_1, it is determined by the ratio E_2/I_1 with I_2 set to zero.

Z22

FIG. 26.29

Determining z_{22}.

The remaining parameter, z_{22}, is determined by

$$z_{22} = \left. \frac{E_2}{I_2} \right|_{I_1 = 0} \qquad \text{(ohms, } \Omega) \qquad (26.26)$$

as derived from Eq. (26.22b) with I_1 set to zero. Since it is the ratio of the output voltage to the output current with I_1 set to zero, we have the terminology

z_{22} = *open-circuit, output-impedance parameter*

The required network is shown in Fig. 26.29. For an applied voltage E_2, it is determined by the resulting ratio E_2/I_2 with $I_1 = 0$.
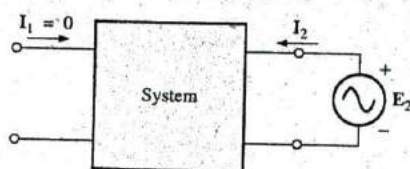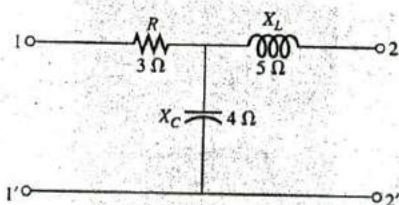
FIG. 26.30

T configuration.

EXAMPLE 26.7 Determine the impedance (z) parameters for the T network of Fig. 26.30.

Solution: For z_{11}, the network will appear as shown in Fig. 26.31, with $Z_1 = 3 \, \Omega \, \angle 0°$, $Z_2 = 5 \, \Omega \, \angle 90°$, and $Z_3 = 4 \, \Omega \, \angle -90°$:

$$I_1 = \frac{E_1}{Z_1 + Z_3}$$

Thus

$$z_{11} = \frac{E_1}{I_1}\bigg|_{I_2=0}$$

and

$$\boxed{z_{11} = Z_1 + Z_3} \tag{26.27}$$

For z_{12}, the network will appear as shown in Fig. 26.32, and

$$E_1 = I_2 Z_3$$

Thus

$$z_{12} = \frac{E_1}{I_2}\bigg|_{I_1=0} = \frac{I_2 Z_3}{I_2}$$

and

$$\boxed{z_{12} = Z_3} \tag{26.28}$$

For z_{21}, the required network appears in Fig. 26.33, and

$$E_2 = I_1 Z_3$$

Thus,

$$z_{21} = \frac{E_2}{I_1}\bigg|_{I_2=0} = \frac{I_1 Z_3}{I_1}$$

and

$$\boxed{z_{21} = Z_3} \tag{26.29}$$

For z_{22}, the determining configuration is shown in Fig. 26.34, and

$$I_2 = \frac{E_2}{Z_2 + Z_3}$$

Thus

$$z_{22} = \frac{E_2}{I_2}\bigg|_{I_1=0} = \frac{I_2(Z_2 + Z_3)}{I_2}$$

and

$$\boxed{z_{22} = Z_2 + Z_3} \tag{26.30}$$

Note that for the T configuration, $z_{12} = z_{21}$. For $Z_1 = 3 \, \Omega \, \angle 0°$, $Z_2 = 5 \, \Omega \, \angle 90°$, and $Z_3 = 4 \, \Omega \, \angle -90°$, we have

$$z_{11} = Z_1 + Z_3 = 3 \, \Omega - j \, 4 \, \Omega$$
$$z_{12} = z_{21} = Z_3 = 4 \, \Omega \, \angle -90° = -j \, 4 \, \Omega$$
$$z_{22} = Z_2 + Z_3 = 5 \, \Omega \, \angle 90° + 4 \, \Omega \, \angle -90° = 1 \, \Omega \, \angle 90° = j \, 1 \, \Omega$$

For a set of impedance parameters, the terminal (external) behavior of the device or network within the configuration of Fig. 26.25 is determined. An *equivalent circuit* for the system can be developed using the impedance parameters and Eqs. (26.22a) and (26.22b). Two possibilities for the impedance parameters appear in Fig. 26.35.

Applying Kirchhoff's voltage law to the input and output loops of the network of Fig. 26.35(a) results in

$$E_1 - z_{11}I_1 - z_{12}I_2 = 0$$

and

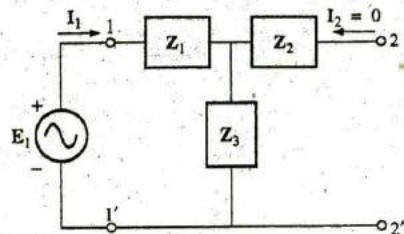$$E_2 - z_{22}I_2 - z_{21}I_1 = 0$$
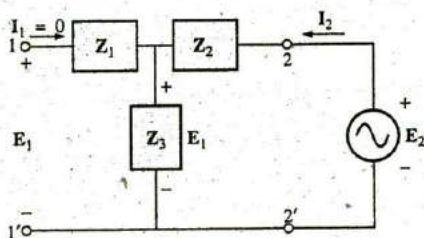

FIG. 26.31
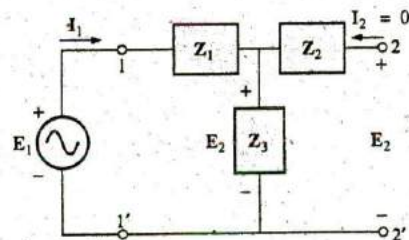*Determining z_{11}.*

FIG. 26.32
Determining z_{12}.

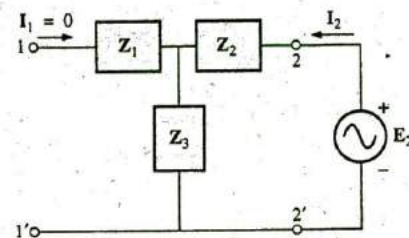
FIG. 26.33
Determining z_{21}.
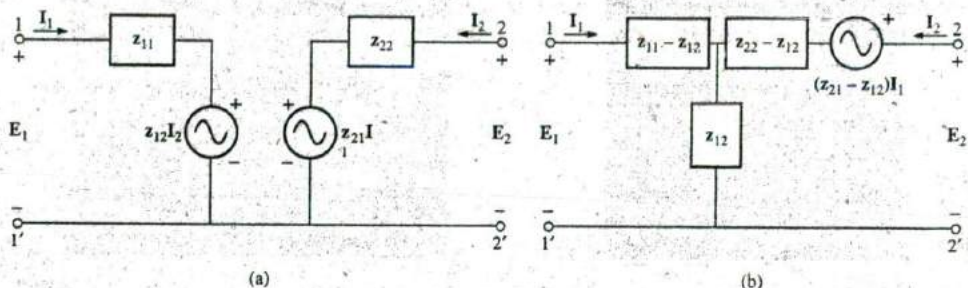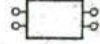

FIG. 26.34
Determining z_{22}.

FIG. 26.35

Two possible two-port, z-parameter equivalent networks.

which, when rearranged, become

$$E_1 = z_{11}I_1 + z_{12}I_2 \qquad E_2 = z_{21}I_1 + z_{22}I_2$$

matching Eqs. (26.22a) and (26.22b).

For the network of Fig. 26.35(b),

$$E_1 - I_1(z_{11} - z_{12}) - z_{12}(I_1 + I_2) = 0$$

and $\quad E_2 - I_1(z_{21} - z_{12}) - I_2(z_{22} - z_{12}) - z_{12}(I_1 + I_2) = 0$

which, when rearranged, are

$$E_1 = I_1(z_{11} - z_{12} + z_{12}) + I_2 z_{12}$$
$$E_2 = I_1(z_{21} - z_{12} + z_{12}) + I_2(z_{22} - z_{12} + z_{12})$$

and
$$E_1 = z_{11}I_1 + z_{12}I_2$$
$$E_2 = z_{21}I_1 + z_{22}I_2$$

Note in each network the necessity for a current-controlled voltage source, that is, a voltage source the magnitude of which is determined by a particular current of the network.

The usefulness of the impedance parameters and the resulting equivalent networks can best be described by considering the system of Fig. 26.36(a), which contains a device (or system) for which the impedance parameters have been determined. As shown in Fig. 26.36(b), the equivalent network
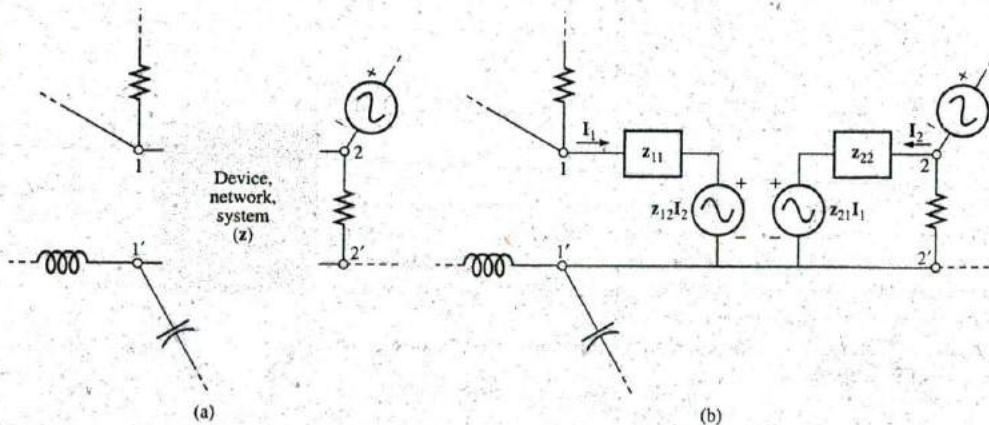

FIG. 26.36

Substitution of the z-parameter equivalent network into a complex system.

for the device (or system) can then be substituted, and methods such as mesh analysis, nodal analysis, and so on, can be employed to determine required unknown quantities. The device itself can then be replaced with an equivalent circuit and the desired solutions obtained more directly and with less effort than is required using only the characteristics of the device.

EXAMPLE 26.8 Draw the equivalent circuit in the form shown in Fig. 26.35(b) using the impedance parameters determined in Example 26.7.

Solution: The circuit appears in Fig. 26.37.
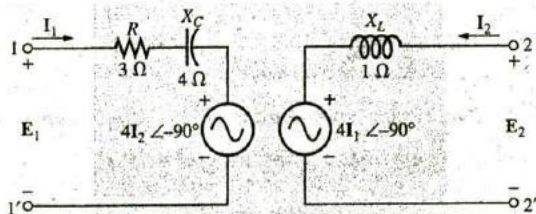
FIG. 26.37
Example 26.8.

26.7 ADMITTANCE (y) PARAMETERS

The equations relating the four terminal variables of Fig. 26.25 can also be written in the following form:

$$I_1 = y_{11}E_1 + y_{12}E_2 \qquad (26.31a)$$

$$I_2 = y_{21}E_1 + y_{22}E_2 \qquad (26.31b)$$

Note that in this case each term of each equation has the units of current, compared to voltage for each term of Eqs. (26.22a) and (26.22b). In addition, the unit of each coefficient is siemens, compared with ohms for the impedance parameters.

The impedance parameters were determined by setting a particular current to zero through an open-circuit condition. For the admittance (y) **parameters** of Eqs. (26.31a) and (26.31b), a voltage is set to zero through a short-circuit condition.

The terminology applied to each of the admittance parameters follows directly from the descriptive terms applied to each of the impedance parameters. The equations for each are determined directly from Eqs. (26.31a) and (26.31b) by setting a particular voltage to zero.

y₁₁

$$y_{11} = \frac{I_1}{E_1}\bigg|_{E_2 = 0} \qquad \text{(siemens, S)} \qquad (26.32)$$

y_{11} = short-circuit, input-admittance parameter
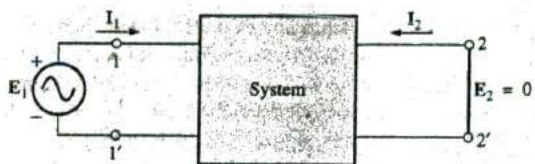
FIG. 26.38
y_{11} *determination.*

The determining network appears in Fig. 26.38.

y_{12}

$$y_{12} = \frac{I_1}{E_2}\bigg|_{E_1 = 0} \qquad \text{(siemens, S)} \qquad (26.33)$$

y_{12} = *short-circuit, reverse-transfer admittance parameter*

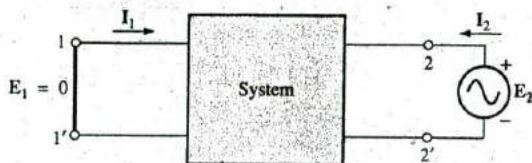The network for determining y_{12} appears in Fig. 26.39.

FIG. 26.39
y_{12} *determination.*

y_{21}

$$y_{21} = \frac{I_2}{E_1}\bigg|_{E_2 = 0} \qquad \text{(siemens, S)} \qquad (26.34)$$

y_{21} = *short-circuit, forward-transfer admittance parameter*

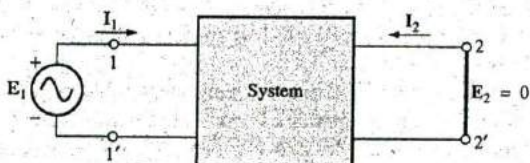The network for determining y_{21} appears in Fig. 26.40.

FIG. 26.40
y_{21} *determination.*

y_{22}

$$y_{22} = \frac{I_2}{E_2}\bigg|_{E_1 = 0} \qquad \text{(siemens, S)} \qquad (26.35)$$

y_{22} = *short-circuit, forward-transfer admittance parameter*
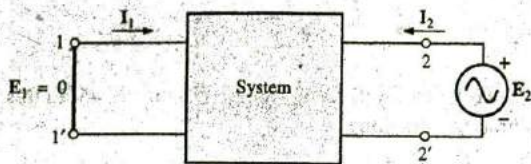
The required network appears in Fig. 26.41.

FIG. 26.41

y_{22} *determination.*

EXAMPLE 26.9 Determine the admittance parameters for the π network of Fig. 26.42.

Solution: The network for y_{11} will appear as shown in Fig. 26.43, with

$Y_1 = 0.2$ mS $\angle 0°$ $\qquad$ $Y_2 = 0.02$ mS $\angle -90°$ $\qquad$ $Y_3 = 0.25$ mS $\angle 90°$

We use $\qquad\qquad$ $I_1 = E_1 Y_T = E_1(Y_1 + Y_2)$

with $\qquad\qquad\qquad$ $y_{11} = \dfrac{I_1}{E_1}\Big|_{E_2 = 0}$

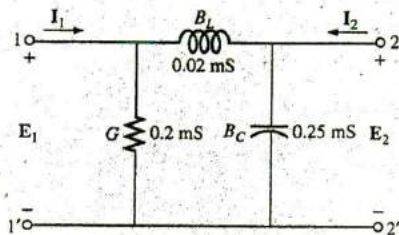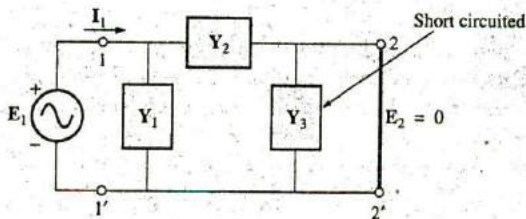and $\qquad\qquad\qquad$ $\boxed{y_{11} = Y_1 + Y_2}$ $\qquad\qquad$ (26.36)

FIG. 26.43

Determining y_{11}.

The determining network for y_{12} appears in Fig. 26.44. Y_1 is short circuited; so $I_{Y_2} = I_1$, and

$$I_{Y_2} = I_1 = -E_2 Y_2$$

The minus sign results because the defined direction of I_1 in Fig. 26.44 is opposite to the actual flow direction due to the applied source E_2; that is,
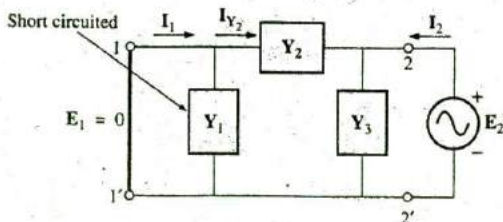
$$y_{12} = \frac{I_1}{E_2}\Big|_{E_1 = 0}$$

FIG. 26.44

Determining y_{12}.

FIG. 26.42

π network.

and $$y_{12} = -Y_2 \qquad\qquad (26.37)$$

The network employed for y_{21} appears in Fig. 26.45. In this case, Y_3 is short circuited, resulting in

$$I_{Y_2} = I_2 \qquad \text{and} \qquad I_{Y_2} = I_2 = -E_1Y_2$$

with $$y_{21} = \frac{I_2}{E_1}\bigg|_{E_2=0}$$
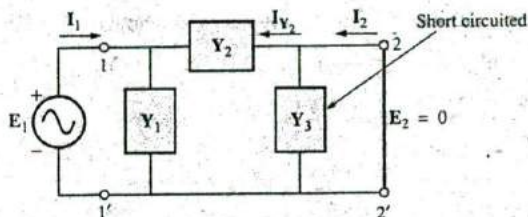
and $$y_{21} = -Y_2 \qquad\qquad (26.38)$$

FIG. 26.45
Determining y_{21}.

Note that for the π configuration, $y_{12} = y_{21}$, which was expected since the impedance parameters for the T network were such that $z_{12} = z_{21}$. A T network can be converted directly to a π network using the Y-Δ transformation.

The determining network for y_{22} appears in Fig. 26.46, and

$$Y_T = Y_2 + Y_3 \qquad \text{and} \qquad I_2 = E_2(Y_2 + Y_3)$$

Thus $$y_{22} = \frac{I_2}{E_2}\bigg|_{E_1=0}$$
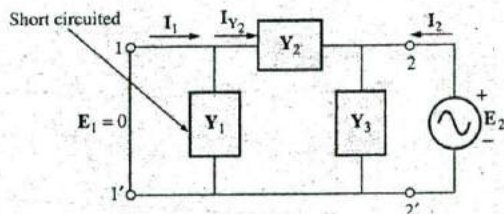
and $$y_{22} = Y_2 + Y_3 \qquad\qquad (26.39)$$

FIG. 26.46
Determining y_{22}.

Substituting values, we have

$$Y_1 = 0.2 \text{ mS } \angle 0°$$
$$Y_2 = 0.02 \text{ mS } \angle -90°$$
$$Y_3 = 0.25 \text{ mS } \angle 90°$$

$$y_{11} = Y_1 + Y_2$$
$$= 0.2 \text{ mS} - j\,0.02 \text{ mS } (L)$$

$$y_{12} = y_{21} = -Y_2 = -(-j\,0.02\text{ mS})$$
$$= j\,0.02\text{ mS }(C)$$
$$y_{22} = Y_2 + Y_3 = -j\,0.02\text{ mS} + j\,0.25\text{ mS}$$
$$= j\,0.23\text{ mS }(C)$$

Note the similarities between the results for y_{11} and y_{22} for the π network compared with z_{11} and z_{22} for the T network.

Two networks satisfying the terminal relationships of Eqs. (26.31a) and (26.31b) are shown in Fig. 26.47. Note the use of parallel branches since each term of Eqs. (26.31a) and (26.31b) has the units of current, and the most direct route to the equivalent circuit is an application of Kirchhoff's current law in reverse. That is, find the network that satisfies Kirchhoff's current law relationship. For the impedance parameters, each term had the units of volts, so Kirchhoff's voltage law was applied in reverse to determine the series combination of elements in the equivalent circuit of Fig. 26.47(a).
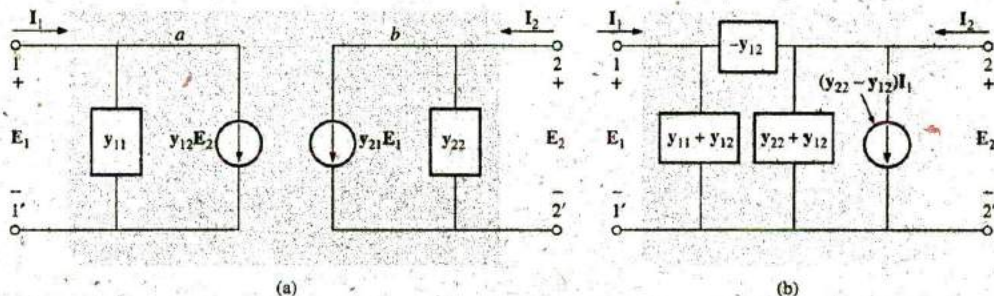

(a)   (b)

FIG. 26.47
Two possible two-port, y-parameter equivalent networks.

Applying Kirchhoff's current law to the network of Fig. 26.47(a), we have

$$
\begin{array}{cc}
\text{Entering} & \text{Leaving} \\
\end{array}
$$
$$\text{Node } a:\quad I_1 = y_{11}E_1 + y_{12}E_2$$
$$\text{Node } b:\quad I_2 = y_{22}E_2 + y_{21}E_1$$

which, when rearranged, are Eqs. (26.31a) and (26.31b).

For the results of Example 26.9, the network of Fig. 26.48 will result if the equivalent network of Fig. 26.47(a) is employed.
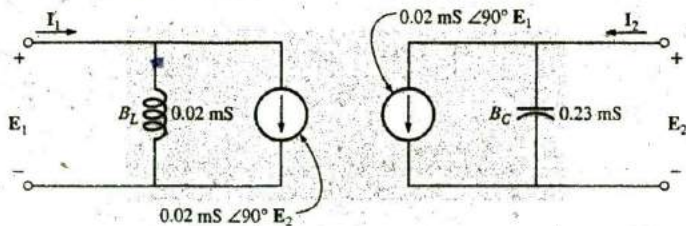

FIG. 26.48
Equivalent network for the results of Example 26.9.

26.8 HYBRID (h) PARAMETERS

The **hybrid (h) parameters** are employed extensively in the analysis of transistor networks. The term *hybrid* is derived from the fact that the parameters have a mixture of units (a hybrid set) rather than a single unit of measurement such as ohms or siemens used for the **z** and **y** parameters, respectively. The defining hybrid equations have a mixture of current *and* voltage variables on one side, as follows:

$$E_1 = h_{11}I_1 + h_{12}E_2 \qquad (26.40a)$$

$$I_2 = h_{21}I_2 + h_{22}E_2 \qquad (26.40b)$$

To determine the hybrid parameters, it will be necessary to establish both the short-circuit and the open-circuit conditions, depending on the parameter desired.

h_{11}

$$h_{11} = \frac{E_1}{I_1}\Big|_{E_2=0} \qquad \text{(ohms, } \Omega\text{)} \qquad (26.41)$$

h_{11} = *short-circuit, input-impedance parameter*
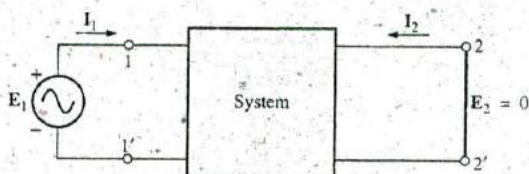
The determining network is shown in Fig. 26.49.

FIG. 26.49
h_{11} *determination.*

h_{12}

$$h_{12} = \frac{E_1}{E_2}\Big|_{I_1=0} \qquad \text{(dimensionless)} \qquad (26.42)$$

h_{12} = *open-circuit, reverse-transfer voltage ratio parameter*

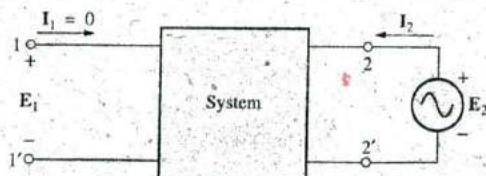The network employed in determining h_{12} is shown in Fig. 26.50.

FIG. 26.50
h_{12} *determination.*

h_{21}

$$h_{21} = \frac{I_2}{I_1} \Big|_{E_2 = 0} \qquad \text{(dimensionless)} \qquad (26.43)$$

h_{21} = *short-circuit, forward-transfer current ratio parameter*
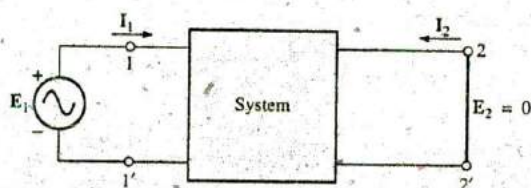
The determining network appears in Fig. 26.51.

FIG. 26.51
h_{21} *determination.*

h_{22}

$$h_{22} = \frac{I_2}{E_2} \Big|_{I_1 = 0.} \qquad \text{(siemens, S)} \qquad (26.44)$$

h_{22} = *open-circuit, output admittance parameter*

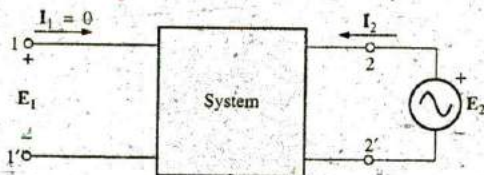The network employed to determine h_{22} is shown in Fig. 26.52.

FIG. 26.52
h_{22} *determination.*

The subscript notation for the hybrid parameters is reduced to the following for most applications. The letter chosen is that letter appearing in boldface in the preceding description of each parameter:

$$h_{11} = h_i \qquad h_{12} = h_r \qquad h_{21} = h_f \qquad h_{22} = h_o$$

The hybrid equivalent circuit appears in Fig. 26.53. Since the unit of measurement for each term of Eq. (26.40a) is the volt, Kirchhoff's voltage law was applied in reverse to obtain the series input circuit indicated. The unit of measurement of each term of Eq. (26.40b) has the units of current, resulting in the parallel elements of the output circuit as obtained by applying Kirchhoff's current law in reverse.

Note that the input circuit has a voltage-controlled voltage source whose controlling voltage is the output terminal voltage, while the output circuit has a current-controlled current source whose controlling current is the current of the input circuit.
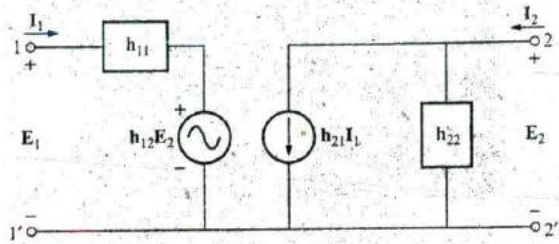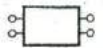
FIG. 26.53
Two-port, hybrid-parameter equivalent network.

EXAMPLE 26.10 For the hybrid equivalent circuit of Fig. 26.54:

a. Determine the current ratio (gain) $A_i = I_2/I_1$.
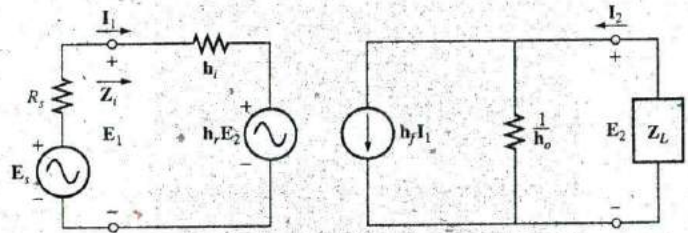b. Determine the voltage ratio (gain) $A_v = E_2/E_1$.

FIG. 26.54
Example 26.10.

Solutions:

a. Using the current divider rule, we have

$$I_2 = \frac{(1/h_o)h_f I_1}{(1/h_o) + Z_L} = \frac{h_f I_1}{1 + h_o Z_L}$$

and

$$\boxed{A_i = \frac{I_2}{I_1} = \frac{h_f}{1 + h_o Z_L}} \tag{26.45}$$

b. Applying Kirchhoff's voltage law to the input circuit gives

$$E_1 - h_i I_1 - h_r E_2 = 0 \quad \text{and} \quad I_1 = \frac{E_1 - h_r E_2}{h_i}$$

Apply Kirchhoff's current law to the output circuit gives

$$I_2 = h_f I_1 + h_o E_2$$

However,

$$I_2 = -\frac{E_2}{Z_L}$$

so

$$-\frac{E_2}{Z_L} = h_f I_1 + h_o E_2$$

Substituting for I_i gives

$$-\frac{E_2}{Z_L} = h_f\left(\frac{E_1 - h_r E_2}{h_i}\right) + h_o E_2$$

or

$$h_i E_2 = -h_f Z_L E_1 + h_r h_f Z_L E_2 - h_i h_o Z_L E_2$$

and

$$E_2(h_i - h_r h_f Z_L + h_i h_o Z_L) = h_f Z_L E_1$$

with the result that

$$A_v = \frac{E_2}{E_1} = \frac{-h_f Z_L}{h_i(1 + h_o Z_L) - h_r h_f Z_L} \qquad (26.46)$$

EXAMPLE 26.11 For a particular transistor, $h_i = 1\ k\Omega$, $h_r = 4 \times 10^{-4}$, $h_f = 50$, and $h_o = 25\ \mu s$. Determine the current and the voltage gain if Z_L is a 2-$k\Omega$ resistive load.

Solution:

$$A_i = \frac{h_f}{1 + h_o Z_L} = \frac{50}{1 + (25\ \mu S)(2\ k\Omega)}$$

$$= \frac{50}{1 + (50 \times 10^{-3})} = \frac{50}{1.050} = 47.62$$

$$A_v = \frac{-h_f Z_L}{h_i(1 + h_o Z_L) - h_r h_f Z_L}$$

$$= \frac{-(50)(2\ k\Omega)}{(1\ k\Omega)(1.050) - (4 \times 10^{-4})(50)(2\ k\Omega)}$$

$$= \frac{-100 \times 10^3}{(1.050 \times 10^3) - (0.04 \times 10^3)} = -\frac{100}{1.01} = -99$$

The minus sign simply indicates a phase shift of 180° between E_2 and E_1 for the defined polarities in Fig. 26.54.

26.9 INPUT AND OUTPUT IMPEDANCES

The input and output impedances will now be determined for the hybrid equivalent circuit and a z-parameter equivalent circuit. The input impedance can always be determined by the ratio of the input voltage to the input current with or without a load applied. The output impedance is always determined with the source voltage or current set to zero. We found in the previous section that for the hybrid equivalent circuit of Fig. 26.54,

$$E_1 = h_i I_1 + h_r E_2$$
$$E_2 = -I_2 Z_L$$

and

$$\frac{I_2}{I_1} = \frac{h_f}{1 + h_o Z_L}$$

By substituting for I_2 in the second equation (using the relationship of the last equation), we have

$$E_2 = -\left(\frac{h_f I_1}{1 + h_o Z_L}\right) Z_L$$

so the first equation becomes

$$E_1 = h_i I_1 + h_r \left(-\frac{h_f I_1 Z_L}{1 + h_o Z_L} \right)$$

and

$$E_1 = I_1 \left(h_i - \frac{h_r h_f Z_L}{1 + h_o Z_L} \right)$$

Thus,

$$Z_i = \frac{E_1}{I_1} = h_i - \frac{h_r h_f Z_L}{1 + h_o Z_L} \qquad (26.47)$$

For the output impedance, we will set the source voltage to zero but preserve its internal resistance R_s as shown in Fig. 26.55.
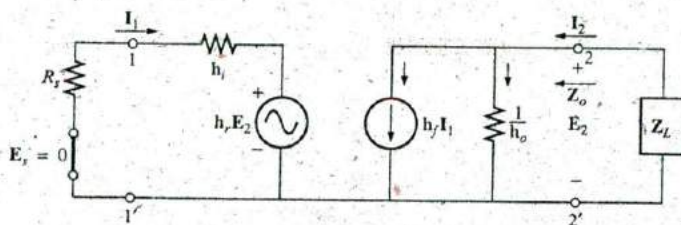

FIG. 26.55

Determining Z_o for the hybrid equivalent network.

Since

$$E_s = 0$$

then

$$I_1 = -\frac{h_r E_2}{h_i + R_s}$$

From the output circuit,

$$I_2 = h_f I_1 + h_o E_2$$

or

$$I_2 = h_f \left(-\frac{h_r E_2}{h_i + R_s} \right) + h_o E_2$$

and

$$I_2 = \left(-\frac{h_r h_f}{h_i + R_s} + h_o \right) E_2$$

Thus,

$$Z_o = \frac{E_2}{I_2} = \frac{1}{h_o - \dfrac{h_r h_f}{h_i + R_s}} \qquad (26.48)$$

EXAMPLE 26.12 Determine Z_i and Z_o for the transistor having the parameters of Example 26.11 if $R_s = 1\ k\Omega$.

Solution:

$$Z_i = h_i - \frac{h_r h_f Z_L}{1 + h_o Z_L} = 1\ k\Omega - \frac{0.04\ k\Omega}{1.050}$$
$$= 1 \times 10^3 - 0.0381 \times 10^3 = \mathbf{961.9\ \Omega}$$

$$Z_o = \frac{1}{h_o - \dfrac{h_r h_f}{h_i + R_s}} = \frac{1}{25\ \mu S - \dfrac{(4 \times 10^{-4})(50)}{1\ k\Omega + 1\ k\Omega}}$$

$$= \frac{1}{25 \times 10^{-6} - 10 \times 10^{-6}} = \frac{1}{15 \times 10^{-6}}$$

$$= 66.67\ k\Omega$$

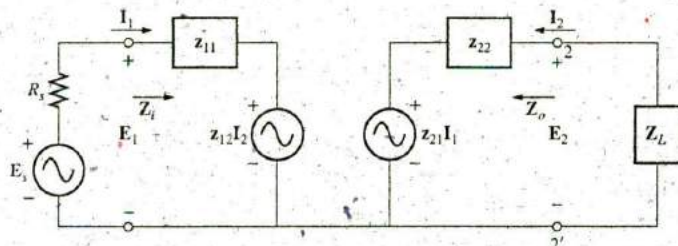For the z-parameter equivalent circuit of Fig. 26.56,

FIG. 26.56

Determining Z_i for the z-parameter equivalent network.

$$I_2 = -\frac{z_{21} I_1}{z_{22} + Z_L}$$

and

$$I_1 = \frac{E_1 - z_{12} I_2}{z_{11}}$$

or

$$E_1 = z_{11} I_1 + z_{12} I_2 = z_{11} I_1 + z_{12}\left(-\frac{z_{21} I_1}{z_{22} + Z_L}\right)$$

and

$$\boxed{Z_i = \frac{E_1}{I_1} = z_{11} - \frac{z_{12} z_{21}}{z_{22} + Z_L}} \qquad (26.49)$$

For the output impedance, $E_s = 0$, and

$$I_1 = -\frac{z_{12} I_2}{R_s + z_{11}} \qquad \text{and} \qquad I_2 = \frac{E_2 - z_{21} I_1}{z_{22}}$$

or

$$E_2 = z_{22} I_2' + z_{21} I_1 = z_{22} I_2 + z_{21}\left(-\frac{z_{12} I_2}{R_s + z_{11}}\right)$$

and

$$E_2 = z_{22} I_2 - \frac{z_{12} z_{21} I_2}{R_s + z_{11}}$$

Thus,

$$\boxed{Z_o = \frac{E_2}{I_2} = z_{22} - \frac{z_{12} z_{21}}{R_s + z_{11}}} \qquad (26.50)$$

26.10 CONVERSION BETWEEN PARAMETERS

The equations relating the z and y parameters can be determined directly from Eqs. (26.22) and (26.31). For Eqs. (26.31a) and (26.31b),

$$I_1 = y_{11} E_1 + y_{12} E_2$$
$$I_2 = y_{21} E_1 + y_{22} E_2$$

The use of determinants results in

$$E_1 = \frac{\begin{vmatrix} I_1 & y_{12} \\ I_2 & y_{22} \end{vmatrix}}{\begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix}} = \frac{y_{22}I_1 - y_{12}I_2}{y_{11}y_{22} - y_{12}y_{21}}$$

Substituting the notation

$$\Delta_y = y_{11}y_{22} - y_{12}y_{21}$$

we have

$$E_1 = \frac{y_{22}}{\Delta_y} I_1 - \frac{y_{12}}{\Delta_y} I_2$$

which, when related to Eq. (26.22a),

$$E_1 = z_{11}I_1 + z_{12}I_2$$

indicates that

$$z_{11} = \frac{y_{22}}{\Delta_y} \quad \text{and} \quad z_{12} = -\frac{y_{12}}{\Delta_y}$$

and, similarly,

$$z_{21} = -\frac{y_{21}}{\Delta_y} \quad \text{and} \quad z_{22} = \frac{y_{11}}{\Delta_y}$$

For the conversion of z parameters to the admittance domain, determinants are applied to Eqs. (26.22a) and (26.22b). The impedance parameters can be found in terms of the hybrid parameters by first forming the determinant for I_1 from the hybrid equations:

$$E_1 = h_{11}I_1 + h_{12}E_2$$
$$I_2 = h_{21}I_1 + h_{22}E_2$$

That is,

$$I_1 = \frac{\begin{vmatrix} E_1 & h_{12} \\ I_2 & h_{22} \end{vmatrix}}{\begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix}} = \frac{h_{22}}{\Delta_h} E_1 - \frac{h_{12}}{\Delta_h} I_2$$

and

$$\frac{h_{22}}{\Delta_h} E_1 = I_1 + \frac{h_{12}}{\Delta_h} I_2$$

or

$$E_1 = \frac{\Delta_h I_1}{h_{22}} + \frac{h_{12}}{h_{22}} I_2$$

which, when related to the impedance-parameter equation,

$$E_1 = z_{11}I_1 + z_{12}I_2$$

indicates that

$$z_{11} = \frac{\Delta_h}{h_{22}} \quad \text{and} \quad z_{12} = \frac{h_{12}}{h_{22}}$$

The remaining conversions are left as an exercise. A complete table of conversions appears in Table 26.1.

TABLE 26.1
Conversions between z, y, and h parameters.

To \ From	z		y		h	
z	z_{11}	z_{12}	$\dfrac{y_{22}}{\Delta_y}$	$\dfrac{-y_{12}}{\Delta_y}$	$\dfrac{\Delta_h}{h_{22}}$	$\dfrac{h_{22}}{h_{22}}$
	z_{21}	z_{22}	$\dfrac{-y_{21}}{\Delta_y}$	$\dfrac{y_{11}}{\Delta_y}$	$\dfrac{-h_{21}}{h_{22}}$	$\dfrac{1}{h_{22}}$
y	$\dfrac{z_{22}}{\Delta_z}$	$\dfrac{-z_{12}}{\Delta_z}$	y_{11}	y_{12}	$\dfrac{1}{h_{11}}$	$\dfrac{-h_{12}}{h_{11}}$
	$\dfrac{-z_{21}}{\Delta_z}$	$\dfrac{z_{11}}{\Delta_z}$	y_{21}	y_{22}	$\dfrac{h_{21}}{h_{11}}$	$\dfrac{\Delta_h}{h_{11}}$
h	$\dfrac{\Delta_z}{z_{22}}$	$\dfrac{z_{12}}{z_{22}}$	$\dfrac{1}{y_{11}}$	$\dfrac{-y_{12}}{y_{11}}$	h_{11}	h_{12}
	$\dfrac{-z_{21}}{z_{22}}$	$\dfrac{1}{z_{22}}$	$\dfrac{y_{21}}{y_{11}}$	$\dfrac{\Delta_y}{y_{11}}$	h_{21}	h_{22}

PROBLEMS

SECTION 26.2 The Impedance Parameters Z_i and Z_o

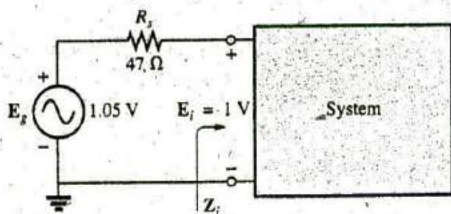1. Given the indicated voltage levels of Fig. 26.57, determine the magnitude of the input impedance Z_i.

FIG. 26.57
Problem 1.

2. For a system with

$E_i = 120 \text{ V} \angle 0°$ and $I_i = 6.2 \text{ A} \angle -10.8°$

determine the input impedance in rectangular form. At a frequency of 60 Hz, determine the nameplate values of the parameters.

3. For the multiport system of Fig. 26.58:
   a. Determine the magnitude of I_{i_1} if $E_{i_1} = 20$ mV.
   b. Find Z_{i_2} using the information provided.
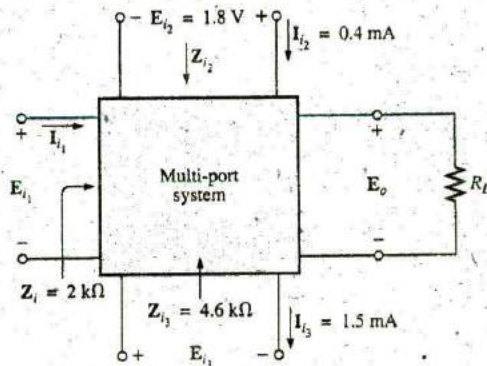   c. Calculate the magnitude of E_{i_3}.

FIG. 26.58
Problem 3.

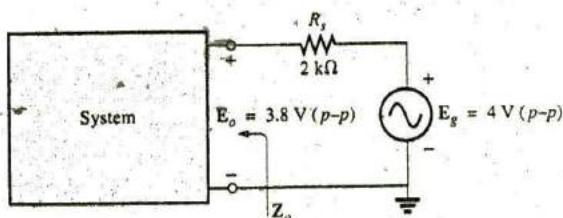4. Given the indicated voltage levels of Fig. 26.59, determine Z_o.

FIG. 26.59
Problems 4 through 6.

5. For the configuration of Fig. 26.59, determine Z_o if $e_g = 2 \sin 377t$ and $v_R = 40 \times 10^{-3} \sin 377t$, with $R_s = 0.91 \text{ k}\Omega$.

6. Determine Z_o for the system of Fig. 26.59 if $E_g = 1.8$ V (p-p) and $E_o = 0.6$ V rms.

7. Determine the output impedance for the system of Fig. 26.60, given the indicated scope measurements.

e_g: Vertical sensitivity — 0.2 V/div.
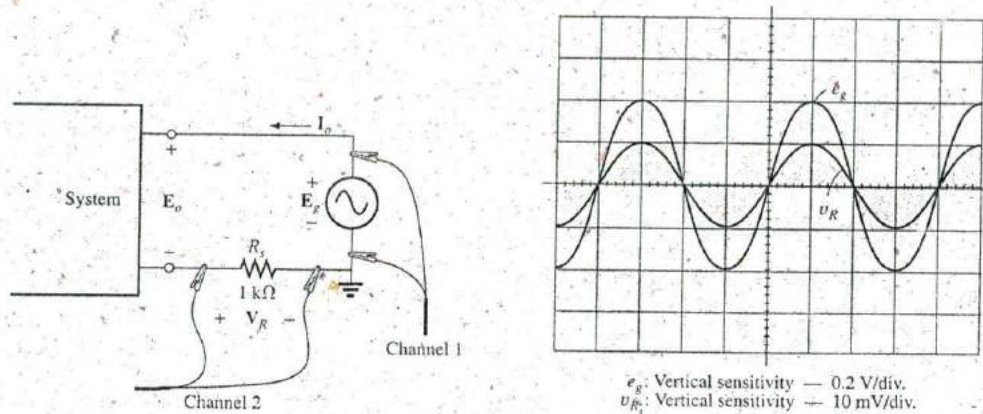v_R: Vertical sensitivity — 10 mV/div.

FIG. 26.60
Problem 7.

SECTION 26.3 The Voltage Gains $A_{v_{NL}}$, A_{v}, and A_{v_T}

8. Given the system of Fig. 26.61, determine the no-load voltage gain $A_{v_{NL}}$.
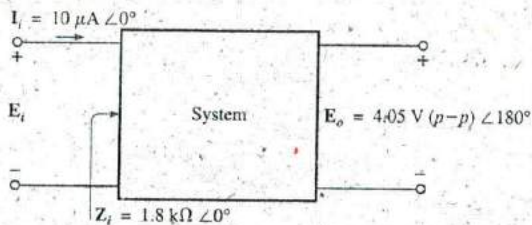

$I_i = 10 \,\mu\text{A} \angle 0°$

E_i System $E_o = 4.05$ V (p-p) $\angle 180°$

$Z_i = 1.8 \text{ k}\Omega \angle 0°$

FIG. 26.61
Problem 8.

9. For the system of Fig. 26.62:
   a. Determine $A_v = E_o/E_i$.
   b. Find $A_{v_T} = E_o/E_g$.

R_g 0.5 kΩ

E_g

E_i

$A_{v_{NL}} = -3200$
$Z_i = 2.2 \text{ k}\Omega$
$Z_o = 40 \text{ k}\Omega$
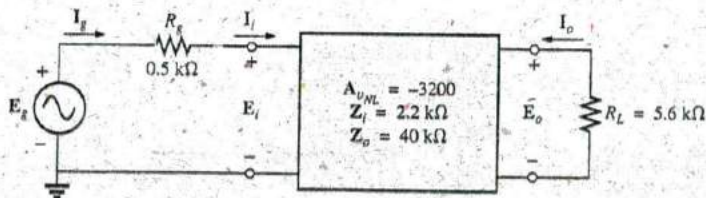
E_o $R_L = 5.6 \text{ k}\Omega$

FIG. 26.62
Problems 9, 12, and 13.

10. For the system of Fig. 26.63(a), the no-load output voltage is −1440 mV, with 1.2 mV applied at the input terminals. In Fig. 26.63(b), a 4.7-kΩ load is applied to the same system, and the output voltage drops to −192 mV, with the same applied input signal. What is the output impedance of the system?
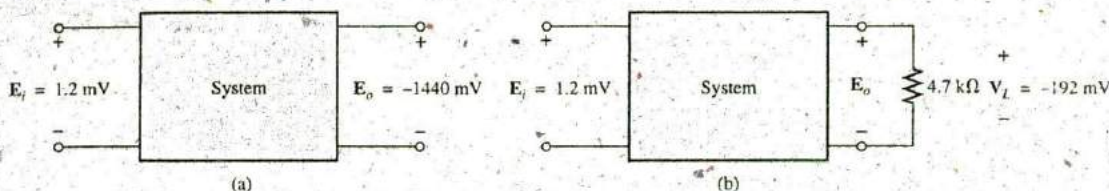

(a)   (b)

$E_i = 1.2$ mV   System $E_o = -1440$ mV $E_i = 1.2$ mV   System E_o 4.7 kΩ $V_L = -192$ mV

FIG. 26.63
Problem 10.

*11. For the system of Fig. 26.64, if

$$A_v = -160 \quad I_o = 4 \text{ mA } \angle 0° \quad E_g = 70 \text{ mV } \angle 0°$$

a. Determine the no-load voltage gain.
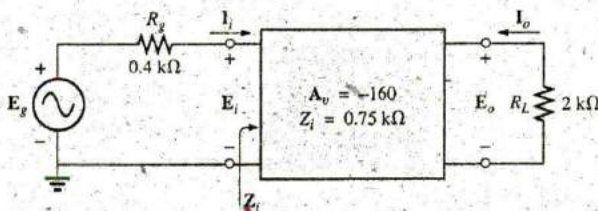b. Find the magnitude of E_i.
c. Determine Z_i.

R_g 0.4 kΩ I_i E_g E_i $A_v = -160$ $Z_i = 0.75$ kΩ I_o E_o R_L 2 kΩ Z_i

FIG. 26.64
Problems 11 and 14.

SECTION 26.4 The Current Gains A_i and A_{iT} and the Power Gain A_G

12. For the system of Fig. 26.62:
   a. Determine $A_i = I_o/I_i$.
   b. Find $A_{iT} = I_o/I_g$.
   c. Compare the results of parts (a) and (b), and explain why the results compare as they do.

13. For the system of Fig. 26.62:
   a. Determine A_G using Eq. (26.13), and compare the value with the result obtained using Eq. (26.14).
   b. Find A_{G_T} using Eq. (26.16), and compare the value to the result obtained using Eq. (26.17).

14. For the system of Fig. 26.64:
   a. Determine the magnitude of $A_i = I_o/I_i$.
   b. Find the power gain $A_{G_T} = P_L/P_g$.

SECTION 26.5 Cascaded Systems

15. For the two-stage system of Fig. 26.65:
   a. Determine the total voltage gain $A_{v_T} = V_L/E_i$.
   b. Find the total current gain $A_{i_T} = I_o/I_i$.
   c. Find the current gain of each stage A_{i_1} and A_{i_2}.
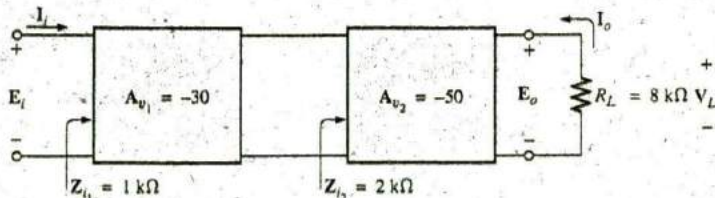   d. Determine the total current gain using the results of part (c), and compare it to the result obtained in part (b).

I_i E_i $A_{v_1} = -30$ $Z_{i_1} = 1$ kΩ $A_{v_2} = -50$ $Z_{i_2} = 2$ kΩ I_o E_o $R_L = 8$ kΩ V_L

FIG. 26.65
Problem 15.

*16. For the system of Fig. 26.66:

   a. Determine A_{v_2} if $A_{v_T} = -6912$.

   b. Determine Z_{i_2} using the information provided.

   c. Find A_{i_3} and A_{i_T} using the information provided in Fig. 26.68.
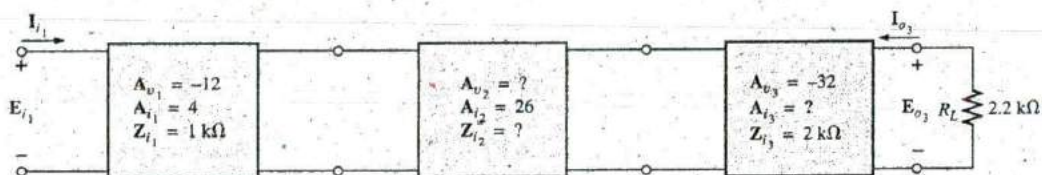

FIG. 26.66

Problem 16.

SECTION 26.6 Impedance (z) Parameters

17. a. Determine the impedance (z) parameters for the π network of Fig. 26.67.

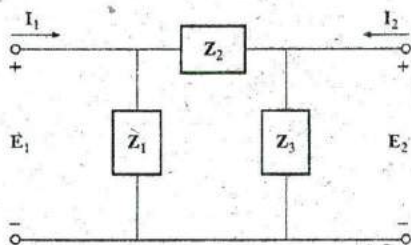   **b.** Sketch the z-parameter equivalent circuit (using either form of Fig. 26.35).

FIG. 26.67

Problems 17 and 21.

18. a. Determine the impedance (z) parameters for the network of Fig. 26.68.

   b. Sketch the z-parameter equivalent circuit (using either form of Fig. 26.35).

FIG. 26.68

Problems 18 and 22.

SECTION 26.7 Admittance (y) Parameters

19. a. Determine the admittance (y) parameters for the T network of Fig. 26.69.

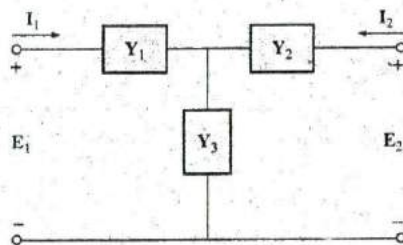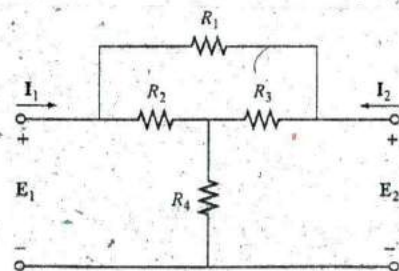   **b.** Sketch the y-parameter equivalent circuit (using either form of Fig. 26.47).

FIG. 26.69

Problems 19 and 23.

20. a. Determine the admittance (y) parameters for the network of Fig. 26.70.

   b. Sketch the y-parameter equivalent circuit (using either form of Fig. 26.47).
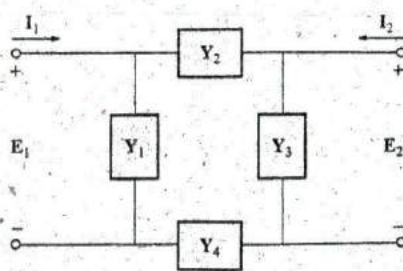

FIG. 26.70

Problems 20 and 24.

SECTION 26.8 Hybrid (h) Parameters

21. a. Determine the h parameters for the network of Fig.
    26.67.
    b. Sketch the hybrid equivalent circuit.
22. a. Determine the h parameters for the network of Fig.
    26.68.
    b. Sketch the hybrid equivalent circuit.
23. a. Determine the h parameters for the network of Fig.
    26.69.
    b. Sketch the hybrid equivalent circuit.
24. a. Determine the h parameters for the network of Fig.
    26.70.
    b. Sketch the hybrid equivalent circuit.
25. For the hybrid equivalent circuit of Fig. 26.71:
    a. Determine the current gain $A_i = I_2/I_1$.
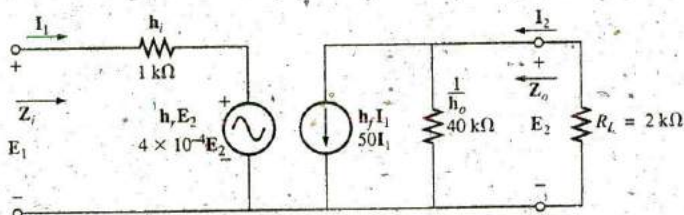   b. Determine the voltage gain $A_v = E_2/E_1$.

FIG. 26.71
Problems 25 and 26.

SECTION 26.9 Input and Output Impedances

26. For the hybrid equivalent circuit of Fig. 26.71:
    a. Determine the input impedance.
    b. Determine the output impedance.
27. Determine the input and output impedances for the z-parameter equivalent circuit of Fig. 26.72.
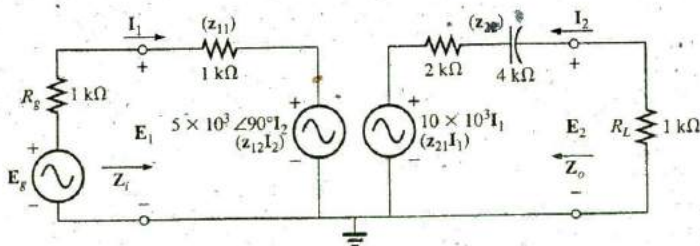28. Determine the expression for the input and output impedance of the y-parameter equivalent circuit.

FIG. 26.72
Problems 27, 32, and 34.

SECTION 26.10 Conversion between Parameters

29. Determine the **h** parameters for the following **z** parameters:

$$z_{11} = 4 \text{ k}\Omega$$
$$z_{12} = 2 \text{ k}\Omega$$
$$z_{21} = 3 \text{ k}\Omega$$
$$z_{22} = 4 \text{ k}\Omega$$

30. a. Determine the **z** parameters for the following **h** parameters:

$$h_{11} = 1 \text{ k}\Omega$$
$$h_{12} = 2 \times 10^{-4}$$
$$h_{21} = 100$$
$$h_{22} = 20 \times 10^{-6} \text{ S}$$

   b. Determine the **y** parameters for the hybrid parameters indicated in part (a).

SECTION 26.11 Computer Analysis

PSpice or Electronics Workbench

31. For $E_1 = 4 \text{ V} \angle 30°$, determine E_2 across a 2-kΩ resistive load between 2 and 2' for the network of Fig. 26.37.

32. For $E_g = 2 \text{ V} \angle 0°$, determine E_2 for the network of Fig. 26.72.

33. Determine Z_i for the network of Fig. 26.37 with a 2-kΩ resistive load from 2 to 2'.

34. Determine Z_i for the network of Fig. 26.72.

GLOSSARY

Admittance (y) parameters A set of parameters, having the units of siemens, that can be used to establish a two-port equivalent network for a system.

Hybrid (h) parameters A set of mixed parameters (ohms, siemens, some unitless) that can be used to establish a two-port equivalent network for a system.

Impedance (z) parameters A set of parameters, having the units of ohms, that can be used to establish a two-port equivalent network for a system.

Input impedance The impedance appearing at the input terminals of a system.

Output impedance The impedance appearing at the output terminals of a system with the energizing source set to zero.

Single-port network A network having a single set of access terminals.

Two-port network A network having two pairs of access terminals.