Chapter 1

Boundary Value Problems

MATHEMATICAL FORMULATION AND SOLUTION OF PHYSICAL PROBLEMS

In solving problems of science and engineering the following steps are generally taken.

1. **Mathematical formulation.** To achieve such formulation we usually adopt *mathematical models* which serve to approximate the real objects under investigation.

Example 1.

To investigate the motion of the earth or other planet about the sun we can choose *points* as mathematical models of the sun and earth. On the other hand, if we wish to investigate the motion of the earth about its axis, the mathematical model cannot be a point but might be a sphere or even more accurately an ellipsoid.

In the mathematical formulation we use known *physical laws* to set up *equations* describing the problem. If the laws are unknown we may even be led to set up *experiments* in order to discover them.

Example 2.

In describing the motion of a planet about the sun we use *Newton's laws* to arrive at a *differential equation* involving the distance of the planet from the sun at any time.

2. **Mathematical solution.** Once a problem has been successfully formulated in terms of equations, we need to solve them for the unknowns involved, subject to the various conditions which are given or implied in the physical problem. One important consideration is whether such solutions actually *exist* and, if they do exist, whether they are *unique*.

   In the attempt to find solutions, the need for new kinds of mathematical analysis – leading to new mathematical problems – may arise.

Example 3.

J.B.J. Fourier, in attempting to solve a problem in heat flow which he had formulated in terms of partial differential equations, was led to the mathematical problem of expansion of functions into series involving sines and cosines. Such series, now called *Fourier series*, are of interest from the point of view of mathematical theory and in physical applications, as we shall see in Chapter 2.

3. **Physical interpretation.** After a solution has been obtained, it is useful to interpret it physically. Such interpretations may be of value in suggesting other kinds of problems, which could lead to new knowledge of a mathematical or physical nature.

In this book we shall be mainly concerned with the mathematical formulation of physical problems in terms of *partial differential equations* and with the solution of such equations by methods commonly called *Fourier methods.*

DEFINITIONS PERTAINING TO PARTIAL DIFFERENTIAL EQUATIONS

A *partial differential equation* is an equation containing an unknown function of two or more variables and its partial derivatives with respect to these variables.

The *order* of a partial differential equation is the order of the highest derivative present.

Example 4.

$\frac{\partial^2 u}{\partial x\,\partial y} = 2x - y$ is a partial differential equation of order two, or a second-order partial differential equation. Here u is the *dependent variable* while x and y are *independent variables*.

A *solution* of a partial differential equation is any function which satisfies the equation identically.

The *general solution* is a solution which contains a number of arbitrary independent functions equal to the order of the equation.

A *particular solution* is one which can be obtained from the general solution by particular choice of the arbitrary functions.

Example 5.

As seen by substitution, $u = x^2y - \frac{1}{2}xy^2 + F(x) + G(y)$ is a *solution* of the partial differential equation of Example 4. Because it contains two arbitrary independent functions $F(x)$ and $G(y)$, it is the *general solution*. If in particular $F(x) = 2\sin x$, $G(y) = 3y^4 - 5$, we obtain the *particular solution*

$$u = x^2y - \tfrac{1}{2}xy^2 + 2\sin x + 3y^4 - 5$$

A *singular solution* is one which cannot be obtained from the general solution by particular choice of the arbitrary functions.

Example 6.

If $u = x\frac{\partial u}{\partial x} - \left(\frac{\partial u}{\partial x}\right)^2$, where u is a function of x and y, we see by substitution that both $u = xF(y) - [F(y)]^2$ and $u = x^2/4$ are solutions. The first is the general solution involving one arbitrary function $F(y)$. The second, which cannot be obtained from the general solution by any choice of $F(y)$, is a singular solution.

A *boundary value problem* involving a partial differential equation seeks all solutions of the equation which satisfy conditions called *boundary conditions*. Theorems relating to the existence and uniqueness of such solutions are called *existence* and *uniqueness theorems*.

LINEAR PARTIAL DIFFERENTIAL EQUATIONS

The general *linear partial differential equation* of order two in two independent variables has the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \qquad (1)$$

where $A, B, \ldots, G$ may depend on x and y but not on u. A second-order equation with independent variables x and y which does not have the form *(1)* is called *nonlinear*.

If $G = 0$ identically the equation is called *homogeneous*, while if $G \neq 0$ it is called *nonhomogeneous*. Generalizations to higher-order equations are easily made.

Because of the nature of the solutions of *(1)*, the equation is often classified as *elliptic*, *hyperbolic*, or *parabolic* according as $B^2 - 4AC$ is less than, greater than, or equal to zero, respectively.

SOME IMPORTANT PARTIAL DIFFERENTIAL EQUATIONS

1. Vibrating string equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

This equation is applicable to the small transverse vibrations of a taut, flexible string, such as a violin string, initially located on the x-axis and set into motion (see Fig. 1-1). The function $y(x,t)$ is the displacement of any point x of the string at time t. The constant $a^2 = \tau/\mu$, where τ is the (constant) tension in the string and μ is the (constant) mass per unit length of the string. It is assumed that no external forces act on the string and that it vibrates only due to its elasticity.

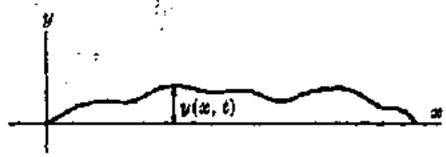

Fig. 1-1

The equation can easily be generalized to higher dimensions, as for example the vibrations of a membrane or drumhead in two dimensions. In two dimensions, the equation is

$$\frac{\partial^2 z}{\partial t^2} = a^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$$

2. Heat conduction equation

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u$$

Here $u(x,y,z,t)$ is the temperature at position (x,y,z) in a solid at time t. The constant κ, called the *diffusivity*, is equal to $K/\sigma\mu$, where the *thermal conductivity* K, the *specific heat* σ and the density (mass per unit volume) μ are assumed constant. We call $\nabla^2 u$ the *Laplacian* of u; it is given in three-dimensional rectangular coordinates (x,y,z) by

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

3. Laplace's equation

$$\nabla^2 v = 0$$

This equation occurs in many fields. In the theory of heat conduction, for example, v is the *steady-state temperature*, i.e. the temperature after a long time has elapsed, whose equation is obtained by putting $\partial u/\partial t = 0$ in the heat conduction equation above. In the theory of gravitation or electricity v represents the *gravitational* or *electric potential* respectively. For this reason the equation is often called the *potential equation*.

The problem of solving $\nabla^2 v = 0$ inside a region $\mathcal{R}$ when v is some given function on the boundary of $\mathcal{R}$ is often called a *Dirichlet problem*.

4. Longitudinal vibrations of a beam

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

This equation describes the motion of a beam (Fig. 1-2, page 4) which can vibrate longitudinally (i.e. in the x-direction) the vibrations being assumed small. The variable $u(x,t)$ is the longitudinal displacement from the equilibrium position of the cross section at x. The constant $c^2 = E/\mu$, where E is the modulus of elasticity (stress divided by strain) and depends on the properties of the beam, μ is the density (mass per unit volume).

Note that this equation is the same as that for a vibrating string.

5. Transverse vibrations of a beam $$\frac{\partial^2 y}{\partial t^2} + b^2 \frac{\partial^4 y}{\partial x^4} = 0$$

This equation describes the motion of a beam (initially located on the x-axis, see Fig. 1-3) which is vibrating transversely (i.e. perpendicular to the x-direction) assuming small vibrations. In this case $y(x,t)$ is the transverse displacement or deflection at any time t of any point x. The constant $b^2 = EI/A\mu$, where E is the modulus of elasticity, I is the moment of inertia of any cross section about the x-axis, A is the area of cross section and μ is the mass per unit length. In case an external transverse force $F(x,t)$ is applied, the right-hand side of the equation is replaced by $b^2F(x,t)/EI$.

Fig. 1-2

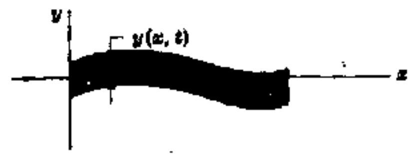

Fig. 1-3

THE LAPLACIAN IN DIFFERENT COORDINATE SYSTEMS

The Laplacian $\nabla^2 u$ often arises in partial differential equations of science and engineering. Depending on the type of problem involved, the choice of coordinate system may be important in obtaining solutions. For example, if the problem involves a cylinder, it will often be convenient to use *cylindrical coordinates*; while if it involves a sphere, it will be convenient to use *spherical coordinates*.

The Laplacian in cylindrical coordinates (ρ, ϕ, z) (see Fig. 1-4) is given by

$$\nabla^2 u = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} \tag{2}$$

The transformation equations between rectangular and cylindrical coordinates are

$$x = \rho \cos\phi, \quad y = \rho \sin\phi, \quad z = z \tag{3}$$

where $\rho \geqq 0$, $0 \leqq \phi < 2\pi$, $-\infty < z < \infty$.

The Laplacian in spherical coordinates (r, θ, ϕ) (see Fig. 1-5) is given by

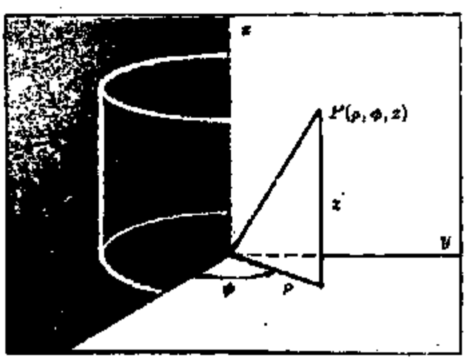

Fig. 1-4

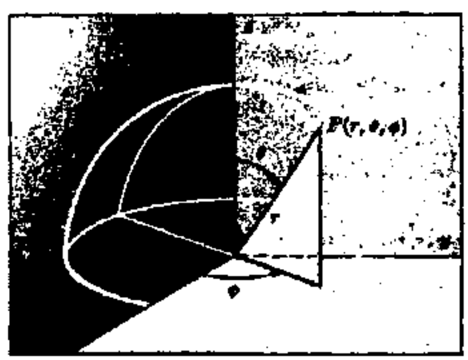

Fig. 1-5

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} \qquad (4)$$

The transformation equations between rectangular and spherical coordinates are

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta \qquad (5)$$

where $r \geqq 0,\ 0 \leqq \theta \leqq \pi,\ 0 \leqq \phi < 2\pi$.

METHODS OF SOLVING BOUNDARY VALUE PROBLEMS

There are many methods by which boundary value problems involving linear partial differential equations can be solved. In this book we shall be concerned with two methods which represent somewhat opposing points of view.

In the first method we seek to find the general solution of the partial differential equation and then particularize it to obtain the actual solution by using the boundary conditions. In the second method we first find particular solutions of the partial differential equation and then build up the actual solution by use of these particular solutions. Of the two methods the second will be found to be of far greater applicability than the first.

1. **General solutions.** In this method we first find the general solution and then that particular solution which satisfies the boundary conditions. The following theorems are of fundamental importance.

   Theorem 1-1 (**Superposition principle**): If $u_1, u_2, \ldots, u_n$ are solutions of a linear homogeneous partial differential equation, then $c_1u_1 + c_2u_2 + \cdots + c_nu_n$, where $c_1, c_2, \ldots, c_n$ are constants, is also a solution.

   Theorem 1-2: The general solution of a linear nonhomogeneous partial differential equation is obtained by adding a particular solution of the nonhomogeneous equation to the general solution of the homogeneous equation.

   We can sometimes find general solutions by using the methods of ordinary differential equations. See Problems 1.15 and 1.16.

   If $A, B, \ldots, F$ in (*1*) are constants, then the general solution of the homogeneous equation can be found by assuming that $u = e^{ax+by}$, where a and b are constants to be determined. See Problems 1.17–1.20.

2. **Particular solutions by separation of variables.** In this method, which is simple but powerful, it is assumed that a solution can be expressed as a product of unknown functions each of which depends on only one of the independent variables. The success of the method hinges on being able to write the resulting equation so that one side depends on only one variable while the other side depends on the remaining variables—from which it is concluded that each side must be a constant. By repetition of this, the unknown functions can be determined. Superposition of these solutions can then be used to find the actual solution. See Problems 1.21–1.25.

Solved Problems

MATHEMATICAL FORMULATION OF PHYSICAL PROBLEMS

1.1. Derive the vibrating string equation on page 3.

Referring to Fig. 1-6, assume that Δs represents an element of arc of the string. Since the tension is assumed constant, the net upward vertical force acting on Δs is given by

$$\tau \sin\theta_2 - \tau \sin\theta_1 \qquad (1)$$

Since $\sin\theta = \tan\theta$, approximately, for small angles, this force is

$$\tau \frac{\partial y}{\partial x}\bigg|_{x+\Delta x} - \tau \frac{\partial y}{\partial x}\bigg|_{x} \qquad (2)$$

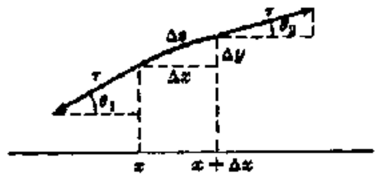

Fig. 1-6

using the fact that the slope is $\tan\theta = \frac{\partial y}{\partial x}$. We use here the notation $\frac{\partial y}{\partial x}\Big|_x$ and $\frac{\partial y}{\partial x}\Big|_{x+\Delta x}$ for the partial derivatives of y with respect to x evaluated at x and $x+\Delta x$, respectively. By Newton's law this net force is equal to the mass of the string ($\mu\,\Delta s$) times the acceleration of Δs, which is given by $\frac{\partial^2 y}{\partial t^2} + \epsilon$ where $\epsilon \to 0$ as $\Delta s \to 0$. Thus we have approximately

$$\tau\left[\frac{\partial y}{\partial x}\bigg|_{x+\Delta x} - \frac{\partial y}{\partial x}\bigg|_{x}\right] = (\mu\,\Delta s)\left(\frac{\partial^2 y}{\partial t^2} + \epsilon\right) \qquad (3)$$

If the vibrations are small, then $\Delta s = \Delta x$ approximately, so that (3) becomes on division by $\mu\,\Delta x$:

$$\frac{\tau}{\mu}\,\frac{\frac{\partial y}{\partial x}\Big|_{x+\Delta x} - \frac{\partial y}{\partial x}\Big|_{x}}{\Delta x} = \frac{\partial^2 y}{\partial t^2} + \epsilon \qquad (4)$$

Taking the limit as $\Delta x \to 0$ (in which case $\epsilon \to 0$ also), we have

$$\frac{\tau}{\mu}\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial^2 y}{\partial t^2} \quad \text{or} \quad \frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}, \quad \text{where } a^2 = \tau/\mu$$

1.2. Write the boundary conditions for a vibrating string of length L for which (a) the ends $x=0$ and $x=L$ are fixed, (b) the initial shape is given by $f(x)$, (c) the initial velocity distribution is given by $g(x)$, (d) the displacement at any point x at time t is bounded.

(a) If the string is fixed at $x = 0$ and $x = L$, then the displacement $y(x,t)$ at $x=0$ and $x=L$ must be zero for all times $t > 0$, i.e.

$$y(0,t) = 0, \quad y(L,t) = 0 \qquad t > 0$$

(b) Since the string has an initial shape given by $f(x)$, we must have

$$y(x,0) = f(x) \qquad 0 < x < L$$

(c) Since the initial velocity of the string at any point x is $g(x)$, we must have

$$y_t(x,0) = g(x) \qquad 0 < x < L$$

Note that $y_t(x,0)$ is the same as $\partial y/\partial t$ evaluated at $t=0$.

(d) Since $y(x,t)$ is bounded, we can find a constant M independent of x and t such that

$$|y(x,t)| < M \qquad 0 < x < L,\ t > 0$$

1.3. Write boundary conditions for a vibrating string for which (a) the end $x=0$ is moving so that its displacement is given in terms of time by $G(t)$, (b) the end $x=L$ is not fixed but is free to move.

(a) The displacement at $x=0$ is given by $y(0,t)$. Thus we have

$$y(0,t) = G(t) \qquad t > 0$$

(b) If τ is the tension, the transverse force acting at any point x is

$$\tau \frac{\partial y}{\partial x} = \tau y_x(x,t)$$

Since the end $x = L$ is free to move so that there is no force acting on it, the boundary condition is given by

$$\tau y_x(L,t) = 0 \quad \text{or} \quad y_x(L,t) = 0 \qquad t > 0$$

1.4. **Suppose that in Problem 1.1 the tension in the string is variable, i.e. depends on the particular point taken. Denoting this tension by $\tau(x)$, show that the equation for the vibrating string is**

$$\frac{\partial}{\partial x}\left[\tau(x)\frac{\partial y}{\partial x}\right] = \mu \frac{\partial^2 y}{\partial t^2}$$

In this case we write (2) of Problem 1.1 as

$$\tau(x)\frac{\partial y}{\partial x}\bigg|_{x+\Delta x} - \tau(x)\frac{\partial y}{\partial x}\bigg|_x$$

so that the corresponding equation (4) is

$$\frac{\tau(x)\frac{\partial y}{\partial x}\Big|_{x+\Delta x} - \tau(x)\frac{\partial y}{\partial x}\Big|_x}{\mu\,\Delta x} = \frac{\partial^2 y}{\partial t^2} + \epsilon$$

Thus, taking the limit as $\Delta x \to 0$ (in which case $\epsilon \to 0$), we obtain

$$\frac{\partial}{\partial x}\left[\tau(x)\frac{\partial y}{\partial x}\right] = \mu \frac{\partial^2 y}{\partial t^2}$$

after multiplying by μ.

1.5. **Show that the heat flux across a plane in a conducting medium is given by $-K\frac{\partial u}{\partial n}$, where u is the temperature, n is a normal in a direction perpendicular to the plane and K is the thermal conductivity of the medium.**

Suppose we have two parallel planes I and II a distance Δn apart (Fig. 1-7), having temperatures u and $u + \Delta u$, respectively. Then the heat flows from the plane of higher temperature to the plane of lower temperature. Also, the amount of heat per unit area per unit time, called the *heat flux*, is directly proportional to the difference in temperature Δu and inversely proportional to the distance Δn. Thus we have

$$\text{Heat flux from I to II} = -K\frac{\Delta u}{\Delta n} \qquad (1)$$

where K is the constant of proportionality, called the *thermal conductivity*. The minus sign occurs in (1) since if $\Delta u > 0$ the heat flow actually takes place from II to I.

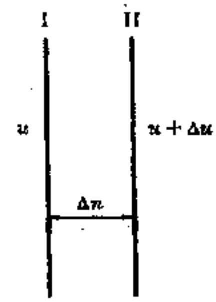

Fig. 1-7

By taking the limit of (1) as Δn and thus Δu approaches zero, we have as required:

$$\text{Heat flux across plane I} = -K\frac{\partial u}{\partial n} \qquad (2)$$

We sometimes call $\frac{\partial u}{\partial n}$ the *gradient* of u which in vector form is ∇u, so that (2) can be written

$$\text{Heat flux across plane I} = -K\,\nabla u \qquad (3)$$

1.6. **If the temperature at any point (x,y,z) of a solid at time t is $u(x,y,z,t)$ and if K, σ and μ are respectively the thermal conductivity, specific heat and density of the solid, all assumed constant, show that**

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u \quad \text{where} \quad \kappa = K/\sigma\mu$$

Consider a small volume element of the solid V, as indicated in Fig. 1-8 and greatly enlarged in Fig. 1-9. By Problem 1.5 the amount of heat per unit area per unit time entering the element through face $PQRS$ is $-K\frac{\partial u}{\partial x}\Big|_x$, where $\frac{\partial u}{\partial x}\Big|_x$ indicates the derivative of u with respect to x evaluated at the position x. Since the area of face $PQRS$ is $\Delta y\,\Delta z$, the total amount of heat entering the element through face $PQRS$ in time Δt is

$$-K\frac{\partial u}{\partial x}\Big|_x \Delta y\,\Delta z\,\Delta t \tag{1}$$

Similarly, the amount of heat leaving the element through face $NWZT$ is

$$-K\frac{\partial u}{\partial x}\Big|_{x+\Delta x} \Delta y\,\Delta z\,\Delta t \tag{2}$$

where $\frac{\partial u}{\partial x}\Big|_{x+\Delta x}$ indicates the derivative of u with respect to x evaluated at $x+\Delta x$.

The amount of heat which remains in the element is given by the amount entering minus the amount leaving, which is, from (1) and (2),

$$\left\{K\frac{\partial u}{\partial x}\Big|_{x+\Delta x} - K\frac{\partial u}{\partial x}\Big|_x\right\}\Delta y\,\Delta z\,\Delta t \tag{3}$$

In a similar way we can show that the amounts of heat remaining in the element due to heat transfer taking place in the y- and z-directions are given by

$$\left\{K\frac{\partial u}{\partial y}\Big|_{y+\Delta y} - K\frac{\partial u}{\partial y}\Big|_y\right\}\Delta x\,\Delta z\,\Delta t \tag{4}$$

and

$$\left\{K\frac{\partial u}{\partial z}\Big|_{z+\Delta z} - K\frac{\partial u}{\partial z}\Big|_z\right\}\Delta x\,\Delta y\,\Delta t \tag{5}$$

respectively.

The total amount of heat gained by the element is given by the sum of (3), (4) and (5). This amount of heat serves to raise its temperature by the amount Δu. Now, we know that the heat needed to raise the temperature of a mass m by Δu is given by $m\sigma\,\Delta u$, where σ is the specific heat. If the density of the solid is μ, the mass is $m = \mu\,\Delta x\,\Delta y\,\Delta z$. Thus the quantity of heat given by the sum of (3), (4) and (5) is equal to

$$\sigma\mu\,\Delta x\,\Delta y\,\Delta z\,\Delta u \tag{6}$$

If we now equate the sum of (3), (4) and (5) to (6), and divide by $\Delta x\,\Delta y\,\Delta z\,\Delta t$, we find

$$\left\{\frac{K\frac{\partial u}{\partial x}\Big|_{x+\Delta x} - K\frac{\partial u}{\partial x}\Big|_x}{\Delta x}\right\} + \left\{\frac{K\frac{\partial u}{\partial y}\Big|_{y+\Delta y} - K\frac{\partial u}{\partial y}\Big|_y}{\Delta y}\right\} + \left\{\frac{K\frac{\partial u}{\partial z}\Big|_{z+\Delta z} - K\frac{\partial u}{\partial z}\Big|_z}{\Delta z}\right\} = \sigma\mu\frac{\Delta u}{\Delta t}$$

In the limit as $\Delta x, \Delta y, \Delta z$ and Δt all approach zero the above equation becomes

$$\frac{\partial}{\partial x}\left(K\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(K\frac{\partial u}{\partial z}\right) = \sigma\mu\frac{\partial u}{\partial t} \tag{7}$$

or, as K is a constant,

$$K\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \sigma\mu\frac{\partial u}{\partial t} \tag{8}$$

This can be rewritten as

$$\frac{\partial u}{\partial t} = \kappa\nabla^2 u \tag{9}$$

where $\kappa = \dfrac{K}{\sigma\mu}$ is called the *diffusivity*.

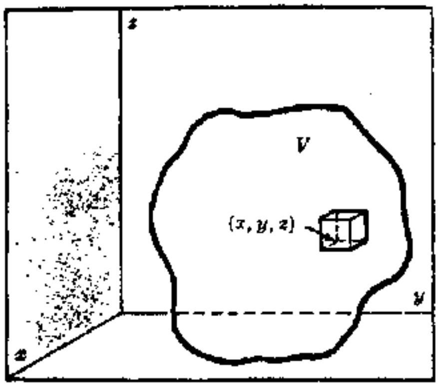

Fig. 1-8

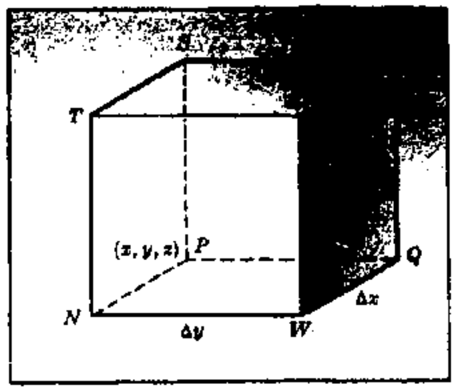

Fig. 1-9

1.7. Work Problem 1.6 by using vector methods.

Let V be an arbitrary volume lying within the solid, and let S denote its surface (see Fig. 1-8). The total flux of heat across S, or the quantity of heat leaving S per unit time, is

$$\iint_S (-K\nabla u)\cdot \mathbf{n}\, dS$$

where $\mathbf{n}$ is an outward-drawn unit normal to S. Thus the quantity of heat entering S per unit time is

$$\iint_S (K\nabla u)\cdot \mathbf{n}\, dS \quad = \quad \iiint_V \nabla\cdot(K\nabla u)\, dV \tag{1}$$

by the divergence theorem. The heat contained in a volume V is given by

$$\iiint_V \sigma\mu u\, dV$$

Then the time rate of increase of heat is

$$\frac{\partial}{\partial t}\iiint_V \sigma\mu u\, dV \quad = \quad \iiint_V \sigma\mu\frac{\partial u}{\partial t}\, dV \tag{2}$$

Equating the right-hand sides of (1) and (2),

$$\iiint_V \left[\sigma\mu\frac{\partial u}{\partial t} - \nabla\cdot(K\nabla u)\right] dV \quad = \quad 0$$

and since V is arbitrary, the integrand, assumed continuous, must be identically zero, so that

$$\sigma\mu\frac{\partial u}{\partial t} \quad = \quad \nabla\cdot(K\nabla u)$$

or if K, σ, μ are constants,

$$\frac{\partial u}{\partial t} \quad = \quad \frac{K}{\sigma\mu}\nabla\cdot\nabla u \quad = \quad \kappa\nabla^2 u \tag{3}$$

1.8. Show that for steady-state heat flow the heat conduction equation of Problem 1.6 or 1.7 reduces to Laplace's equation, $\nabla^2 u = 0$.

In the case of steady-state heat flow the temperature u does not depend on time t, so that $\frac{\partial u}{\partial t} = 0$. Thus the equation $\frac{\partial u}{\partial t} = \kappa\nabla^2 u$ becomes $\nabla^2 u = 0$.

1.9. A thin bar of diffusivity κ has its ends at $x = 0$ and $x = L$ on the x-axis (see Fig. 1-10). Its lateral surface is insulated so that heat cannot enter or escape.

(a) If the initial temperature is $f(x)$ and the ends are kept at temperature zero, set up the boundary value problem. (b) Work part (a) if the end $x = L$ is insulated. (c) Work part (a) if the end $x = L$ radiates into the surrounding medium, which is assumed to be at temperature u_0.

This is a problem in *one-dimensional* heat conduction since the temperature can only depend on the position x at any time t and can thus be denoted by $u(x,t)$. The heat conduction equation is thus given by

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < L,\ t > 0 \qquad (1)$$

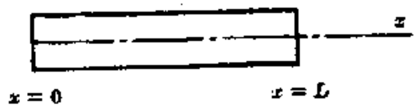

Fig. 1-10

(a) Since the ends are kept at temperature zero, we have

$$u(0,t) = 0, \quad u(L,t) = 0 \qquad t > 0 \qquad (2)$$

Since the initial temperature is $f(x)$, we have

$$u(x,0) = f(x) \qquad 0 < x < L \qquad (3)$$

Also, from physical considerations the temperature must be bounded; hence

$$|u(x,t)| < M \qquad 0 < x < L,\ t > 0 \qquad (4)$$

The problem of solving (1) subject to conditions (2), (3) and (4) is the required boundary value problem. A problem exactly equivalent to that considered above is that of an *infinite slab* of conducting material bounded by the planes $x = 0$ and $x = L$, where the planes are kept at temperature zero and where the temperature distribution initially is $f(x)$.

(b) If the end $x = L$ is insulated instead of being at temperature zero, then we must find a replacement for the condition $u(L,t) = 0$ in (2). To do this we note that if the end $x = L$ is insulated then the flux at $x = L$ is zero. Thus we have

$$-K\frac{\partial u}{\partial x}\bigg|_{x=L} = 0 \qquad \text{or equivalently} \qquad u_x(L,t) = 0 \qquad (5)$$

which is the required boundary condition.

(c) It is known from physical laws of heat transfer that the heat flux of radiation from one object at temperature U_1 to another object at temperature U_2 is given by $\alpha(U_1^4 - U_2^4)$, where α is a constant and the temperatures U_1 and U_2 are given in *absolute* or *Kelvin* temperature which is the number of Celsius (centigrade) degrees plus 273. This law is often called *Stefan's radiation law*. From this we obtain the boundary condition

$$-Ku_x(L,t) = \alpha(u_1^4 - u_0^4) \qquad \text{where} \quad u_1 = u(L,t) \qquad (6)$$

If u_1 and u_0 do not differ too greatly from each other, we can write

$$\begin{aligned} u_1^4 - u_0^4 &= (u_1 - u_0)(u_1^3 + u_1^2 u_0 + u_1 u_0^2 + u_0^3) \\ &= (u_1 - u_0)u_0^3\left[\left(\frac{u_1}{u_0}\right)^3 + \left(\frac{u_1}{u_0}\right)^2 + \frac{u_1}{u_0} + 1\right] \\ &\sim 4u_0^3(u_1 - u_0) \end{aligned}$$

since $(u_1/u_0)^3$, $(u_1/u_0)^2$, (u_1/u_0) *are* approximately equal to 1. Using this approximation, which is often referred to as *Newton's law of cooling*, we can write (6) as

$$-Ku_x(L,t) = \beta(u_1 - u_0) \qquad (7)$$

where β is a constant.

CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS

1.10. Determine whether each of the following partial differential equations is linear or nonlinear, state the order of each equation, and name the dependent and independent variables.

(a) $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$ linear, order 2, dep. var. u, ind. var. x, t

(b) $x^2\frac{\partial^3 R}{\partial y^3} = y^3\frac{\partial^2 R}{\partial x^2}$ linear, order 3, dep. var. R, ind. var. x, y

(c) $W\frac{\partial^2 W}{\partial r^2} = rst$ nonlinear, order 2, dep. var. W, ind. var. r, s, t

(d) $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$ linear, order 2, dep. var ϕ, ind. var. x, y, z

(e) $\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = 1$ nonlinear, order 1, dep. var. z, ind. var. u, v

1.11. Classify each of the following equations as elliptic, hyperbolic or parabolic.

(a) $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$

$u = \phi,\ A = 1,\ B = 0,\ C = 1;\quad B^2 - 4AC = -4 < 0$ and the equation is elliptic.

(b) $\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2}$

$y = t,\ A = \kappa,\ B = 0,\ C = 0;\quad B^2 - 4AC = 0$ and the equation is parabolic.

(c) $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$

$y = t,\ u = y,\ A = a^2,\ B = 0,\ C = -1;\quad B^2 - 4AC = 4a^2 > 0$ and the equation is hyperbolic.

(d) $\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial x\,\partial y} + 4\frac{\partial^2 u}{\partial y^2} + 5\frac{\partial u}{\partial x} - 2\frac{\partial u}{\partial y} + 4u = 2x - 3y$

$A = 1,\ B = 3,\ C = 4;\quad B^2 - 4AC = -7 < 0$ and the equation is elliptic.

(e) $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} + 3y^2\frac{\partial u}{\partial x} = 0$

$A = x,\ B = 0,\ C = y;\quad B^2 - 4AC = -4xy$. Hence, in the region $xy > 0$ the equation is elliptic; in the region $xy < 0$ the equation is hyperbolic; if $xy = 0$, the equation is parabolic.

SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

1.12. Show that $u(x,t) = e^{-8t}\sin 2x$ is a solution to the boundary value problem

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(\pi,t) = 0, \qquad u(x,0) = \sin 2x$$

From $u(x,t) = e^{-8t}\sin 2x$ we have

$$u(0,t) = e^{-8t}\sin 0 = 0, \qquad u(\pi,t) = e^{-8t}\sin 2\pi = 0, \qquad u(x,0) = e^{-0}\sin 2x = \sin 2x$$

and the boundary conditions are satisfied.

Also $\frac{\partial u}{\partial t} = -8e^{-8t}\sin 2x, \quad \frac{\partial u}{\partial x} = 2e^{-8t}\cos 2x, \quad \frac{\partial^2 u}{\partial x^2} = -4e^{-8t}\sin 2x$

Then substituting into the differential equation, we have

$$-8e^{-8t}\sin 2x = 2(-4e^{-8t}\sin 2x)$$

which is an identity.

1.13. (*a*) Show that $v = F(y - 3x)$, where F is an arbitrary differentiable function, is a general solution of the equation

$$\frac{\partial v}{\partial x} + 3\frac{\partial v}{\partial y} = 0$$

(*b*) Find the particular solution which satisfies the condition $v(0, y) = 4 \sin y$.

(*a*) Let $y - 3x = u$. Then $v = F(u)$ and

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial u}\frac{\partial u}{\partial x} = F'(u)(-3) = -3F'(u)$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial u}\frac{\partial u}{\partial y} = F'(u)(1) = F'(u)$$

Thus

$$\frac{\partial v}{\partial x} + 3\frac{\partial v}{\partial y} = 0$$

Since the equation is of order one, the solution $v = F(u) = F(y - 3x)$, which involves one arbitrary function, is a general solution.

(*b*) $v(x, y) = F(y - 3x)$. Then $v(0, y) = F(y) = 4 \sin y$. But if $F(y) = 4 \sin y$, then $v(x, y) = F(y - 3x) = 4 \sin(y - 3x)$ is the required solution.

1.14. (*a*) Show that $y(x, t) = F(2x + 5t) + G(2x - 5t)$ is a general solution of

$$4\frac{\partial^2 y}{\partial t^2} = 25\frac{\partial^2 y}{\partial x^2}$$

(*b*) Find a particular solution satisfying the conditions

$$y(0, t) = y(\pi, t) = 0, \quad y(x, 0) = \sin 2x, \quad y_t(x, 0) = 0$$

(*a*) Let $2x + 5t = u$, $2x - 5t = v$. Then $y = F(u) + G(v)$.

$$\frac{\partial y}{\partial t} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial G}{\partial v}\frac{\partial v}{\partial t} = F'(u)(5) + G'(v)(-5) = 5F'(u) - 5G'(v) \qquad (1)$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}(5F'(u) - 5G'(v)) = 5\frac{\partial F'}{\partial u}\frac{\partial u}{\partial t} - 5\frac{\partial G'}{\partial v}\frac{\partial v}{\partial t} = 25F''(u) + 25G''(v) \qquad (2)$$

$$\frac{\partial y}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial G}{\partial v}\frac{\partial v}{\partial x} = F'(u)(2) + G'(v)(2) = 2F'(u) + 2G'(v) \qquad (3)$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}[2F'(u) + 2G'(v)] = 2\frac{\partial F'}{\partial u}\frac{\partial u}{\partial x} + 2\frac{\partial G'}{\partial v}\frac{\partial v}{\partial x} = 4F''(u) + 4G''(v) \qquad (4)$$

From (*2*) and (*4*), $4\frac{\partial^2 y}{\partial t^2} = 25\frac{\partial^2 y}{\partial x^2}$ and the equation is satisfied. Since the equation is of order 2 and the solution involves two arbitrary functions, it is a general solution.

(*b*) We have from $y(x, t) = F(2x + 5t) + G(2x - 5t)$,

$$y(x, 0) = F(2x) + G(2x) = \sin 2x \qquad (5)$$

Also

$$y_t(x, t) = \frac{\partial y}{\partial t} = 5F'(2x + 5t) - 5G'(2x - 5t)$$

so that

$$y_t(x, 0) = 5F'(2x) - 5G'(2x) = 0 \qquad (6)$$

Differentiating (*5*),

$$2F'(2x) + 2G'(2x) = 2 \cos 2x \qquad (7)$$

From (*6*),

$$F'(2x) = G'(2x) \qquad (8)$$

Then from (*7*), and (*8*),

$$F'(2x) = G'(2x) = \tfrac{1}{2} \cos 2x$$

from which $F(2x) = \frac{1}{2}\sin 2x + c_1, \quad G(2x) = \frac{1}{2}\sin 2x + c_2$

i.e. $y(x,t) = \frac{1}{2}\sin(2x+5t) + \frac{1}{2}\sin(2x-5t) + c_1 + c_2$

Using $y(0,t) = 0$ or $y(\pi,t) = 0$, $c_1 + c_2 = 0$ so that

$$y(x,t) = \frac{1}{2}\sin(2x+5t) + \frac{1}{2}\sin(2x-5t) = \sin 2x \cos 5t$$

which can be checked as the required solution.

METHODS OF FINDING SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

1.15. (*a*) Solve the equation $\dfrac{\partial^2 z}{\partial x\,\partial y} = x^2 y$.

(*b*) Find the particular solution for which $z(x,0) = x^2$, $z(1,y) = \cos y$.

(*a*) Write the equation as $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = x^2 y$. Then integrating with respect to x, we find

$$\frac{\partial z}{\partial y} = \tfrac{1}{3}x^3 y + F(y) \tag{1}$$

where $F(y)$ is arbitrary.

Integrating (*1*) with respect to y,

$$z = \tfrac{1}{6}x^3y^2 + \int F(y)\,dy + G(x) \tag{2}$$

where $G(x)$ is arbitrary. The result (*2*) can be written

$$z = z(x,y) = \tfrac{1}{6}x^3y^2 + H(y) + G(x) \tag{3}$$

which has two arbitrary (independent) functions and is therefore a general solution.

(*b*) Since $z(x,0) = x^2$, we have from (*3*)

$$x^2 = H(0) + G(x) \quad \text{or} \quad G(x) = x^2 - H(0) \tag{4}$$

Thus $$z = \tfrac{1}{6}x^3y^2 + H(y) + x^2 - H(0) \tag{5}$$

Since $z(1,y) = \cos y$, we have from (*5*)

$$\cos y = \tfrac{1}{6}y^2 + H(y) + 1 - H(0) \quad \text{or} \quad H(y) = \cos y - \tfrac{1}{6}y^2 - 1 + H(0) \tag{6}$$

Using (*6*) in (*5*), we find the required solution

$$z = \tfrac{1}{6}x^3y^2 + \cos y - \tfrac{1}{6}y^2 + x^2 - 1$$

1.16. Solve $t\dfrac{\partial^2 u}{\partial x\,\partial t} + 2\dfrac{\partial u}{\partial x} = x^2$.

Write the equation as $\dfrac{\partial}{\partial x}\left[t\dfrac{\partial u}{\partial t} + 2u\right] = x^2$. Integrating with respect to x,

$$t\frac{\partial u}{\partial t} + 2u = \frac{x^3}{3} + F(t) \quad \text{or} \quad \frac{\partial u}{\partial t} + \frac{2}{t}u = \frac{x^3}{3t} + \frac{F(t)}{t}$$

This is a linear equation having integrating factor $e^{\int (2/t)\,dt} = e^{2\ln t} = e^{\ln t^2} = t^2$. Then

$$\frac{\partial}{\partial t}(t^2 u) = \frac{x^3 t}{3} + tF(t)$$

Integrating, $$t^2 u = \frac{x^3t^2}{6} + \int tF(t)\,dt + H(x) = \frac{x^3t^2}{6} + G(t) + H(x)$$

and this is the required general solution.

1.17. Find solutions of $\dfrac{\partial^2 u}{\partial x^2} + 3\dfrac{\partial^2 u}{\partial x\,\partial y} + 2\dfrac{\partial^2 u}{\partial y^2} = 0$.

Assume $u = e^{ax+by}$. Substituting in the given equation, we find

$$(a^2 + 3ab + 2b^2)e^{ax+by} = 0 \quad \text{or} \quad a^2 + 3ab + 2b^2 = 0$$

Then $(a+b)(a+2b) = 0$ and $a = -b$, $a = -2b$. If $a = -b$, $e^{-bx+by} = e^{b(y-x)}$ is a solution for any value of b. If $a = -2b$, $e^{-2bx+by} = e^{b(y-2x)}$ is a solution for any value of b.

Since the equation is linear and homogeneous, sums of these solutions are solutions (Theorem 1-1). For example, $3e^{2(y-x)} - 2e^{3(y-x)} + 5e^{\pi(y-x)}$ is a solution (among many others), and one is thus led to $F(y-x)$ where F is arbitrary, which can be verified as a solution. Similarly, $G(y-2x)$, where G is arbitrary, is a solution. The general solution found by addition is then given by

$$u = F(y-x) + G(y-2x)$$

1.18. Find a general solution of (a) $2\dfrac{\partial u}{\partial x} + 3\dfrac{\partial u}{\partial y} = 2u$, (b) $4\dfrac{\partial^2 u}{\partial x^2} - 4\dfrac{\partial^2 u}{\partial x\,\partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

(a) Let $u = e^{ax+by}$. Then $2a + 3b = 2$, $a = \dfrac{2-3b}{2}$, and $e^{((2-3b)/2)x+by} = e^x e^{(b/2)(2y-3x)}$ is a solution.

Thus $u = e^x F(2y - 3x)$ is a general solution.

(b) Let $u = e^{ax+by}$. Then $4a^2 - 4ab + b^2 = 0$ and $b = 2a, 2a$. From this $u = e^{a(x+2y)}$ and so $F(x+2y)$ is a solution.

By analogy with repeated roots for ordinary differential equations we might be led to believe $xG(x+2y)$ or $yG(x+2y)$ to be another solution, and that this is in fact true is easy to verify. Thus a general solution is

$$u = F(x+2y) + xG(x+2y) \quad \text{or} \quad u = F(x+2y) + yG(x+2y)$$

1.19. Solve $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 10e^{2x+y}$.

The homogeneous equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ has general solution $u = F(x+iy) + G(x-iy)$ by Problem 1.39(c).

To find a particular solution of the given equation assume $u = ae^{2x+y}$ where a is an unknown constant. This is the *method of undetermined coefficients* as in ordinary differential equations. We find $a = 2$, so that the required general solution is

$$u = F(x+iy) + G(x-iy) + 2e^{2x+y}$$

1.20. Solve $\dfrac{\partial^2 u}{\partial x^2} - 4\dfrac{\partial^2 u}{\partial y^2} = e^{2x+y}$.

The homogeneous equation has general solution

$$u = F(2x+y) + G(2x-y)$$

To find a particular solution, we would normally assume $u = ae^{2x+y}$ as in Problem 1.19 but this assumed solution is already included in $F(2x+y)$. Hence we assume as in ordinary differential equations that $u = axe^{2x+y}$ (or $u = aye^{2x+y}$). Substituting, we find $a = \frac{1}{4}$.

Then a general solution is

$$u = F(2x+y) + G(2x-y) + \tfrac{1}{4}xe^{2x+y}$$

SEPARATION OF VARIABLES

1.21. Solve the boundary value problem

$$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}, \qquad u(0,y) = 8e^{-3y}$$

by the method of separation of variables.

Let $u = XY$ in the given equation, where X depends only on x and Y depends only on y.

Then $$X'Y = 4XY' \quad \text{or} \quad X'/4X = Y'/Y$$

where $X' = dX/dx$ and $Y' = dY/dy$.

Since X depends only on x and Y depends only on y and since x and y are independent variables, each side must be a constant, say c.

Then $X' - 4cX = 0$, $Y' - cY = 0$, whose solutions are $X = Ae^{4cx}$, $Y = Be^{cy}$.

A solution is thus given by

$$u(x,y) = XY = ABe^{c(4x+y)} = Ke^{c(4x+y)}$$

From the boundary condition,

$$u(0,y) = Ke^{cy} = 8e^{-3y}$$

which is possible if and only if $K = 8$ and $c = -3$. Then $u(x,y) = 8e^{-3(4x+y)} = 8e^{-12x-3y}$ is the required solution.

1.22. Solve Problem 1.21 if $u(0,y) = 8e^{-3y} + 4e^{-5y}$.

As before a solution is $Ke^{c(4x+y)}$. Then $K_1e^{c_1(4x+y)}$ and $K_2e^{c_2(4x+y)}$ are solutions and by the principle of superposition so also is their sum; i.e. a solution is

$$u(x,y) = K_1e^{c_1(4x+y)} + K_2e^{c_2(4x+y)}$$

From the boundary condition,

$$u(0,y) = K_1e^{c_1y} + K_2e^{c_2y} = 8e^{-3y} + 4e^{-5y}$$

which is possible if and only if $K_1 = 8$, $K_2 = 4$, $c_1 = -3$, $c_2 = -5$.

Then $u(x,y) = 8e^{-3(4x+y)} + 4e^{-5(4x+y)} = 8e^{-12x-3y} + 4e^{-20x-5y}$ is the required solution.

1.23. Solve $\dfrac{\partial u}{\partial t} = 2\dfrac{\partial^2 u}{\partial x^2}$, $0 < x < 3$, $t > 0$, given that $u(0,t) = u(3,t) = 0$,

$$u(x,0) = 5\sin 4\pi x - 3\sin 8\pi x + 2\sin 10\pi x, \qquad |u(x,t)| < M$$

where the last condition states that u is bounded for $0 < x < 3$, $t > 0$.

Let $u = XT$. Then $XT' = X''T$ and $X''/X = T'/2T$. Each side must be a constant, which we call $-\lambda^2$. (If we use $+\lambda^2$, the resulting solution obtained does not satisfy the boundedness condition for real values of λ.) Then

$$X'' + \lambda^2 X = 0, \qquad T' + 2\lambda^2 T = 0$$

with solutions $$X = A_1\cos\lambda x + B_1\sin\lambda x, \qquad T = c_1e^{-2\lambda^2 t}$$

A solution of the partial differential equation is thus given by

$$u(x,t) = XT = c_1e^{-2\lambda^2 t}(A_1\cos\lambda x + B_1\sin\lambda x) = e^{-2\lambda^2 t}(A\cos\lambda x + B\sin\lambda x)$$

Since $u(0,t) = 0$, $e^{-2\lambda^2 t}(A) = 0$ or $A = 0$. Then

$$u(x,t) = Be^{-2\lambda^2 t}\sin\lambda x$$

Since $u(3,t) = 0$, $Be^{-2\lambda^2 t}\sin 3\lambda = 0$. If $B = 0$, the solution is identically zero, so we must have $\sin 3\lambda = 0$ or $3\lambda = m\pi$, $\lambda = m\pi/3$, where $m = 0, \pm 1, \pm 2, \ldots$. Thus a solution is

$$u(x,t) = Be^{-2m^2\pi^2 t/9}\sin\frac{m\pi x}{3}$$

Also, by the principle of superposition,

$$u(x,t) = B_1e^{-2m_1^2\pi^2 t/9}\sin\frac{m_1\pi x}{3} + B_2e^{-2m_2^2\pi^2 t/9}\sin\frac{m_2\pi x}{3} + B_3e^{-2m_3^2\pi^2 t/9}\sin\frac{m_3\pi x}{3} \qquad (1)$$

is a solution. By the last boundary condition,

$$u(x,0) = B_1 \sin\frac{m_1\pi x}{3} + B_2 \sin\frac{m_2\pi x}{3} + B_3 \sin\frac{m_3\pi x}{3}$$
$$= 5\sin 4\pi x - 3\sin 8\pi x + 2\sin 10\pi x$$

which is possible if and only if $B_1 = 5,\ m_1 = 12,\ B_2 = -3,\ m_2 = 24,\ B_3 = 2,\ m_3 = 30.$

Substituting these in (1), the required solution is

$$u(x,t) = 5e^{-32\pi^2 t}\sin 4\pi x - 3e^{-128\pi^2 t}\sin 8\pi x + 2e^{-200\pi^2 t}\sin 10\pi x \quad (2)$$

This boundary value problem has the following interpretation as a heat flow problem. A bar whose surface is insulated (Fig. 1-11) has a length of 3 units and a diffusivity of 2 units. If its ends are kept at temperature zero units and its initial temperature $u(x,0) = 5\sin 4\pi x - 3\sin 8\pi x + 2\sin 10\pi x$, find the temperature at position x at time t, i.e. find $u(x,t)$.

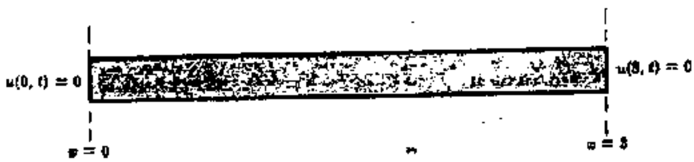

Fig. 1-11

1.24. Solve $\dfrac{\partial^2 y}{\partial t^2} = 16\dfrac{\partial^2 y}{\partial x^2},\ 0 < x < 2,\ t > 0,$ subject to the conditions $y(0,t) = 0,\ y(2,t) = 0,$ $y(x,0) = 6\sin\pi x - 3\sin 4\pi x,\ y_t(x,0) = 0,\ |y(x,t)| < M.$

Let $y = XT$, where X depends only on x, T depends only on t. Then substitution in the differential equation yields

$$XT'' = 16X''T \quad \text{or} \quad X''/X = T''/16T$$

on separating the variables. Since each side must be a constant, say $-\lambda^2$, we have

$$X'' + \lambda^2 X = 0, \quad T'' + 16\lambda^2 T = 0$$

Solving these we find

$$X = a_1\cos\lambda x + b_1\sin\lambda x, \quad T = a_2\cos 4\lambda t + b_2\sin 4\lambda t$$

Thus a solution is

$$y(x,t) = (a_1\cos\lambda x + b_1\sin\lambda x)(a_2\cos 4\lambda t + b_2\sin 4\lambda t) \quad (1)$$

To find the constants it is simpler to proceed by using first those boundary conditions involving two zeros, such as $y(0,t) = 0,\ y_t(x,0) = 0$. From $y(0,t) = 0$ we see from (1) that

$$a_1(a_2\cos 4\lambda t + b_2\sin 4\lambda t) = 0$$

so that to obtain a non zero solution (1) we must have $a_1 = 0$. Thus (1) becomes

$$y(x,t) = (b_1\sin\lambda x)(a_2\cos 4\lambda t + b_2\sin 4\lambda t) \quad (2)$$

Differentiation of (2) with respect to t yields

$$y_t(x,t) = (b_1\sin\lambda x)(-4\lambda a_2\sin 4\lambda t + 4\lambda b_2\cos 4\lambda t)$$

so that we have on putting $t = 0$ and using the condition $y_t(x,0) = 0$

$$y_t(x,0) = (b_1\sin\lambda x)(4\lambda b_2) = 0 \quad (3)$$

In order to obtain a solution (2) which is not zero we see from (3) that we must have $b_2 = 0$. Thus (2) becomes

$$y(x,t) = B\sin\lambda x\cos 4\lambda t$$

on putting $b_2 = 0$ and writing $B = b_1 a_2$.

From $y(2,t) = 0$ we now find

$$B\sin 2\lambda\cos 4\lambda t = 0$$

and we see that we must have $\sin 2\lambda = 0$, i.e. $2\lambda = m\pi$ or $\lambda = m\pi/2$ where $m = 0, \pm1, \pm2, \ldots$.

Thus
$$y(x,t) = B\sin\frac{m\pi x}{2}\cos 2m\pi t \qquad (4)$$

is a solution. Since this solution is bounded, the condition $|y(x,t)| < M$ is automatically satisfied.

In order to satisfy the last condition, $y(x,0) = 6\sin\pi x - 3\sin 4\pi x$, we first use the principle of superposition to obtain the solution

$$y(x,t) = B_1\sin\frac{m_1\pi x}{2}\cos 2m_1\pi t + B_2\sin\frac{m_2\pi x}{2}\cos 2m_2\pi t \qquad (5)$$

Then putting $t = 0$ we arrive at

$$y(x,0) = B_1\sin\frac{m_1\pi x}{2} + B_2\sin\frac{m_2\pi x}{2}$$
$$= 6\sin\pi x - 3\sin 4\pi x$$

This is possible if and only if $B_1 = 6$, $m_1 = 2$, $B_2 = -3$, $m_2 = 8$. Thus the required solution (5) is

$$y(x,t) = 6\sin\pi x\cos 4\pi t - 3\sin 4\pi x\cos 16\pi t \qquad (6)$$

This boundary value problem can be interpreted physically in terms of the vibrations of a string. The string has its ends fixed at $x = 0$ and $x = 2$ and is given an initial shape $f(x) = 6\sin\pi x - 3\sin 4\pi x$. It is then released so that its initial velocity is zero. Then (6) gives the displacement of any point x of the string at any later time t.

1.25. Solve $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$, $0 < x < 3$, $t > 0$, given that $u(0,t) = u(3,t) = 0$, $u(x,0) = f(x)$, $|u(x,t)| < M$.

This problem differs from Problem 1.23 only in the condition $u(x,0) = f(x)$. In seeking to satisfy this last condition we see that taking a finite number of terms, as in (1) of Problem 1.23, will be insufficient for arbitrary $f(x)$. Thus we are led to assume that infinitely many terms are taken, i.e.

$$u(x,t) = \sum_{m=1}^{\infty} B_m e^{-2m^2\pi^2 t/9}\sin\frac{m\pi x}{3}$$

The condition $u(x,0) = f(x)$ then leads to

$$f(x) = \sum_{m=1}^{\infty} B_m \sin\frac{m\pi x}{3}$$

or the problem of expansion of a function into a sine series. Such trigonometric expansions, or *Fourier series*, will be considered in detail in the next chapter.

Supplementary Problems

MATHEMATICAL FORMULATION OF PHYSICAL PROBLEMS

1.26. If a taut, horizontal string with fixed ends vibrates in a vertical plane under the influence of gravity, show that its equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} - g$$

where g is the acceleration due to gravity.

1.27. A thin bar located on the x-axis has its ends at $x = 0$ and $x = L$. The initial temperature of the bar is $f(x)$, $0 < x < L$, and the ends $x = 0$, $x = L$ are maintained at constant temperatures T_1, T_2 respectively. Assuming the surrounding medium is at temperature u_0 and that Newton's law of cooling applies, show that the partial differential equation for the temperature of the bar at any point at any time is given by

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2} - \beta(u - u_0)$$

and write the corresponding boundary conditions.

8. Write the boundary conditions in Problem 1.27 if (a) the ends $x = 0$ and $x = L$ are insulated, (b) the ends $x = 0$ and $x = L$ radiate into the surrounding medium according to Newton's law of cooling.

29. The gravitational potential v at any point (x, y, z) outside of a mass m located at the point (X, Y, Z) is defined as the mass m divided by the distance of the point (x, y, z) from (X, Y, Z). Show that v satisfies Laplace's equation $\nabla^2 v = 0$.

1.30. Extend the result of Problem 1.29 to a solid body.

1.31. A string has its ends fixed at $x = 0$ and $x = L$. It is displaced a distance h at its midpoint and then released. Formulate a boundary value problem for the displacement $y(x, t)$ of any point x of the string at time t.

CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS

1.32. Determine whether each of the following partial differential equations is linear or nonlinear, state the order of each equation, and name the dependent and independent variables.

(a) $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = 0$ (c) $\phi\frac{\partial \phi}{\partial x} = \frac{\partial^3 \phi}{\partial y^3}$ (e) $\frac{\partial z}{\partial r} + \frac{\partial z}{\partial s} = \frac{1}{z^2}$

(b) $(x^2+y^2)\frac{\partial^4 T}{\partial z^4} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$ (d) $\frac{\partial^2 y}{\partial t^2} - 4\frac{\partial^2 y}{\partial x^2} = x^2$

1.33. Classify each of the following equations as elliptic, hyperbolic or parabolic.

(a) $\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} = 0$

(b) $\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x\,\partial y} = 4$

(c) $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x\,\partial y} + 2\frac{\partial^2 z}{\partial y^2} = x + 3y$

(d) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$

(e) $(x^2-1)\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + (y^2-1)\frac{\partial^2 u}{\partial y^2} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

(f) $(M^2-1)\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0, \quad M > 0$

SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

1.34. Show that $z(x, y) = 4e^{-3x}\cos 3y$ is a solution to the boundary value problem

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0, \quad z(x, \pi/2) = 0, \quad z(x, 0) = 4e^{-3x}$$

1.35. (a) Show that $v(x, y) = xF(2x + y)$ is a general solution of $x\frac{\partial v}{\partial x} - 2x\frac{\partial v}{\partial y} = v$.

(b) Find a particular solution satisfying $v(1, y) = y^2$.

?. Find a partial differential equation having general solution $u = F(x - 3y) + G(2x + y)$.

Find a partial differential equation having general solution

$$(a)\quad z = e^x f(2y - 3x), \qquad (b)\quad z = f(2x + y) + g(x - 2y)$$

NERAL SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

.8. (a) Solve $x\frac{\partial^2 z}{\partial x\,\partial y} + \frac{\partial z}{\partial y} = 0$.

(b) Find the particular solution for which $z(x, 0) = x^5 + x - \frac{68}{x}$, $z(2, y) = 3y^4$.

1.39. Find general solutions of each of the following.

(a) $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$ (b) $\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 3u$ (c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(d) $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x\,\partial y} - 3\frac{\partial^2 z}{\partial y^2} = 0$ (e) $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x\,\partial y} + \frac{\partial^2 z}{\partial y^2} = 0$

1.40. Find general solutions of each of the following.

(a) $\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = x$ (c) $\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^3\,\partial y} = 4$

(b) $\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} + 12t^2$ (d) $\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x\,\partial y} + 2\frac{\partial^2 z}{\partial y^2} = x \sin y$

1.41. Solve $\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2\,\partial y^2} + \frac{\partial^4 u}{\partial y^4} = 16.$

1.42. Show that a general solution of $\frac{\partial^2 v}{\partial r^2} + \frac{2}{r}\frac{\partial v}{\partial r} = \frac{1}{c^2}\frac{\partial^2 v}{\partial t^2}$ is $v = \frac{F(r-ct) + G(r+ct)}{r}$.

SEPARATION OF VARIABLES

1.43. Solve each of the following boundary value problems by the method of separation of variables.

(a) $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0, \quad u(x,0) = 4e^{-x}$

(b) $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial y} + u, \quad u(x,0) = 3e^{-5x} + 2e^{-3x}$

(c) $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = 0, \quad u(\pi,t) = 0, \quad u(x,0) = 2\sin 3x - 4\sin 5x$

(d) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u_x(0,t) = 0, \quad u(2,t) = 0, \quad u(x,0) = 8\cos\frac{3\pi x}{4} - 6\cos\frac{9\pi x}{4}$

(e) $\frac{\partial u}{\partial t} = 3\frac{\partial u}{\partial x}, \quad u(x,0) = 8e^{-2x}$

(f) $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} - 2u, \quad u(x,0) = 10e^{-x} - 6e^{-4x}$

(g) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = 0, \quad u(4,t) = 0, \quad u(x,0) = 6\sin\frac{\pi x}{2} + 3\sin \pi x$

1.44. Solve and give a physical interpretation to the boundary value problem

$$\frac{\partial^2 y}{\partial t^2} = 4\frac{\partial^2 y}{\partial x^2}, \quad y(0,t) = y(5,t) = 0, \quad y(x,0) = 0, \quad y_t(x,0) = f(x) \qquad (0 < x < 5,\ t > 0)$$

if (a) $f(x) = 5\sin \pi x$, (b) $f(x) = 3\sin 2\pi x - 2\sin 5\pi x$.

1.45. Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 2u$ if $u(0,t) = 0, \quad u(3,t) = 0, \quad u(x,0) = 2\sin \pi x - \sin 4\pi x.$

1.46. Suppose that in Problem 1.24 we have $y(x,0) = f(x)$, where $0 < x < 2$. Show how the problem can be solved if we know how to expand $f(x)$ in a series of sines.

1.47. Suppose that in Problem 1.25 the boundary conditions are $u_x(0,t) = 0$, $u(3,t) = 0$, $u(x,0) = f(x)$. Show how the problem can be solved if we know how to expand $f(x)$ in a series of cosines. Give a physical interpretation of this problem.

Chapter 2

Fourier Series and Applications

THE NEED FOR FOURIER SERIES

In Problem 1.25, page 17, we saw that to obtain a solution to a particular boundary value problem we should need to know how to expand a function into a trigonometric series. In this chapter we shall investigate the theory of such series and shall use the theory to solve many boundary value problems.

Since each term of the trigonometric series considered in Problem 1.25 is periodic, it is clear that if we are to expand functions in such series, the functions should also be periodic. We therefore turn now to the consideration of periodic functions.

PERIODIC FUNCTIONS

A function $f(x)$ is said to have a *period* P or to be *periodic* with period P if for all x, $f(x+P) = f(x)$, where P is a positive constant. The least value of $P > 0$ is called the *least period* or simply *the period* of $f(x)$.

Example 1.

The function $\sin x$ has periods $2\pi, 4\pi, 6\pi, \ldots$, since $\sin(x+2\pi)$, $\sin(x+4\pi)$, $\sin(x+6\pi)$, ... all equal $\sin x$. However, 2π is the *least period* or *the period* of $\sin x$.

Example 2.

The period of $\sin nx$ or $\cos nx$, where n is a positive integer, is $2\pi/n$.

Example 3.

The period of $\tan x$ is π.

Example 4.

A constant has any positive number as a period.

Other examples of periodic functions are shown in the graphs of Fig. 2-1.

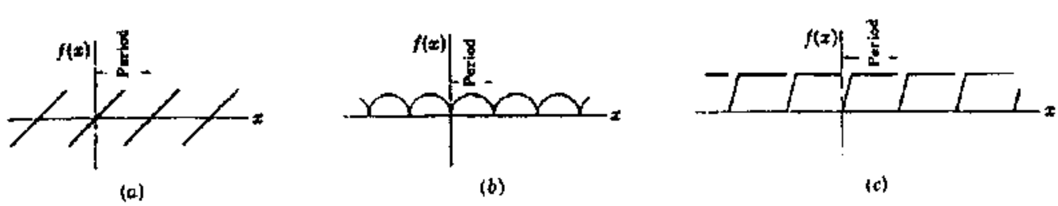

Fig. 2-1

PIECEWISE CONTINUOUS FUNCTIONS

A function $f(x)$ is said to be *piecewise continuous* in an interval if (i) the interval can be divided into a finite number of subintervals in each of which $f(x)$ is continuous and (ii) the limits of $f(x)$ as x approaches the endpoints of each subinterval are finite. Another way of stating this is to say that a piecewise continuous function is one that has at most a finite number of finite discontinuities. An example of a piecewise continuous function is shown in Fig. 2-2. The functions of Fig. 2-1(*a*) and (*c*) are piecewise continuous. The function of Fig. 2-1(*b*) is continuous.

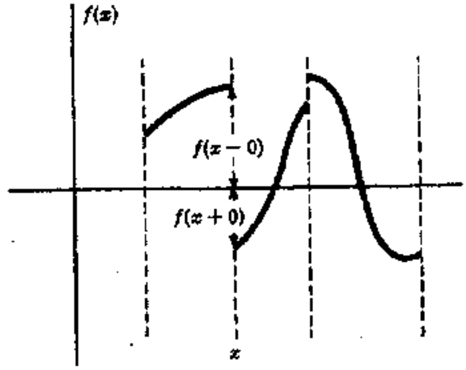

Fig. 2-2

The *limit of* $f(x)$ *from the right* or the *right-hand limit* of $f(x)$ is often denoted by $\lim_{\epsilon \to 0} f(x+\epsilon) = f(x+0)$, where $\epsilon > 0$. Similarly, the *limit of* $f(x)$ *from the left* or the *left-hand limit* of $f(x)$ is denoted by $\lim_{\epsilon \to 0} f(x-\epsilon) = f(x-0)$, where $\epsilon > 0$. The values $f(x+0)$ and $f(x-0)$ at the point x in Fig. 2-2 are as indicated. The fact that $\epsilon \to 0$ and $\epsilon > 0$ is sometimes indicated briefly by $\epsilon \to 0+$. Thus, for example, $\lim_{\epsilon \to 0+} f(x+\epsilon) = f(x+0)$, $\lim_{\epsilon \to 0+} f(x-\epsilon) = f(x-0)$.

DEFINITION OF FOURIER SERIES

Let $f(x)$ be defined in the interval $(-L, L)$ and determined outside of this interval by $f(x+2L) = f(x)$, i.e. assume that $f(x)$ has the period $2L$. The *Fourier series* or *Fourier expansion* corresponding to $f(x)$ is defined to be

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \tag{1}$$

where the *Fourier coefficients* a_n and b_n are

$$\begin{cases} a_n = \dfrac{1}{L} \displaystyle\int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx \\ b_n = \dfrac{1}{L} \displaystyle\int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx \end{cases} \qquad n = 0, 1, 2, \ldots \tag{2}$$

Motivation for this definition is supplied in Problem 2.4.

If $f(x)$ has the period $2L$, the coefficients a_n and b_n can be determined equivalently from

$$\begin{cases} a_n = \dfrac{1}{L} \displaystyle\int_{c}^{c+2L} f(x) \cos \frac{n\pi x}{L}\, dx \\ b_n = \dfrac{1}{L} \displaystyle\int_{c}^{c+2L} f(x) \sin \frac{n\pi x}{L}\, dx \end{cases} \qquad n = 0, 1, 2, \ldots \tag{3}$$

where c is any real number. In the special case $c = -L$, (*3*) becomes (*2*). Note that the constant term in (*1*) is equal to $\dfrac{a_0}{2} = \dfrac{1}{2L} \displaystyle\int_{-L}^{L} f(x)\, dx$, which is the *mean* of $f(x)$ over a period.

If $L = \pi$, the series (*1*) and the coefficients (*2*) or (*3*) are particularly simple. The function in this case has the period 2π.

It should be emphasized that the series (*1*) is only the series which *corresponds* to $f(x)$. We do not know whether this series converges or even, if it does converge, whether it converges to $f(x)$. This problem of convergence was examined by *Dirichlet*, who developed conditions for convergence of Fourier series which we now consider.

DIRICHLET CONDITIONS

Theorem 2-1: Suppose that

(i) $f(x)$ is defined and single-valued except possibly at a finite number of points in $(-L, L)$

(ii) $f(x)$ is periodic with period $2L$

(iii) $f(x)$ and $f'(x)$ are piecewise continuous in $(-L, L)$

Then the series (*1*) with coefficients (*2*) or (*3*) converges to

(*a*) $f(x)$ if x is a point of continuity

(*b*) $\dfrac{f(x+0)+f(x-0)}{2}$ if x is a point of discontinuity

For a proof see Problems 2.18–2.23.

According to this result we can write

$$f(x) \quad = \quad \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} \; + \; b_n \sin\frac{n\pi x}{L}\right) \tag{4}$$

at any point of continuity x. However, if x is a point of discontinuity, then the left side is replaced by $\frac{1}{2}[f(x+0)+f(x-0)]$, so that the series converges to the mean value of $f(x+0)$ and $f(x-0)$.

The conditions (i), (ii) and (iii) imposed on $f(x)$ are *sufficient* but not *necessary*, i.e. if the conditions are satisfied the convergence is guaranteed. However, if they are not satisfied the series may or may not converge. The conditions above are generally satisfied in cases which arise in science or engineering.

There are at present no known necessary and sufficient conditions for convergence of Fourier series. It is of interest that continuity of $f(x)$ does not *alone* insure convergence of a Fourier series.

ODD AND EVEN FUNCTIONS

A function $f(x)$ is called *odd* if $f(-x) = -f(x)$. Thus x^3, $x^5 - 3x^3 + 2x$, $\sin x$, $\tan 3x$ are odd functions.

A function $f(x)$ is called *even* if $f(-x) = f(x)$. Thus x^4, $2x^6 - 4x^2 + 5$, $\cos x$, $e^x + e^{-x}$ are even functions.

The functions portrayed graphically in Fig. 2-1(*a*) and 2-1(*b*) are odd and even respectively, but that of Fig. 2-1(*c*) is neither odd nor even.

In the Fourier series corresponding to an odd function, only sine terms can be present. In the Fourier series corresponding to an even function, only cosine terms (and possibly a constant, which we shall consider to be a cosine term) can be present.

HALF-RANGE FOURIER SINE OR COSINE SERIES

A half-range Fourier sine or cosine series is a series in which only sine terms or only cosine terms are present, respectively. When a half-range series corresponding to a given function is desired, the function is generally defined in the interval $(0, L)$ [which is half of the interval $(-L, L)$, thus accounting for the name *half-range*] and then the function is specified as odd or even, so that it is clearly defined in the other half of the interval, namely $(-L, 0)$. In such case, we have

$$\begin{cases} a_n = 0, \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx & \text{for } \textit{half-range sine series} \\ b_n = 0, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx & \text{for } \textit{half-range cosine series} \end{cases} \tag{5}$$

PARSEVAL'S IDENTITY states that

$$\frac{1}{L}\int_{-L}^{L} \{f(x)\}^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \tag{6}$$

if a_n and b_n are the Fourier coefficients corresponding to $f(x)$ and if $f(x)$ satisfies the Dirichlet conditions.

UNIFORM CONVERGENCE

Suppose that we have an infinite series $\sum_{n=1}^{\infty} u_n(x)$. We define the Rth *partial sum* of the series to be the sum of the first R terms of the series, i.e.

$$S_R(x) = \sum_{n=1}^{R} u_n(x) \tag{7}$$

Now by definition the infinite series is said to *converge* to $f(x)$ in some interval if given any positive number ϵ, there exists for each x in the interval a positive number N such that

$$|S_R(x) - f(x)| < \epsilon \quad \text{whenever } R > N \tag{8}$$

The number N depends in general not only on ϵ but also on x. We call $f(x)$ the *sum* of the series.

An important case occurs when N depends on ϵ but *not* on the value of x in the interval. In such case we say that the series converges *uniformly* or is *uniformly convergent* to $f(x)$.

Two very important properties of uniformly convergent series are summarized in the following two theorems.

***Theorem 2-2*:** If each term of an infinite series is continuous in an interval (a, b) and the series is uniformly convergent to the sum $f(x)$ in this interval, then

1. $f(x)$ is also continuous in the interval
2. the series can be integrated term by term, i.e.

$$\int_a^b \left\{\sum_{n=1}^{\infty} u_n(x)\right\} dx = \sum_{n=1}^{\infty}\int_a^b u_n(x)\,dx \tag{9}$$

***Theorem 2-3*:** If each term of an infinite series has a derivative and the series of derivatives is uniformly convergent, then the series can be differentiated term by term, i.e.

$$\frac{d}{dx}\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty}\frac{d}{dx}u_n(x) \tag{10}$$

There are various ways of proving the uniform convergence of a series. The most obvious way is to actually find the sum $S_R(x)$ in closed form and then apply the definition directly. A second and most powerful way is to use a theorem called the ***Weierstrass M-test***.

Theorem 2-4 **(*Weierstrass M test*):** If there exists a set of constants M_n, $n = 1, 2, \ldots$, such that for all x in an interval $|u_n(x)| \leqq M_n$, and if furthermore $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly in the interval. Incidently, the series is also *absolutely convergent*, i.e. $\sum_{n=1}^{\infty} |u_n(x)|$ converges, under these conditions.

Example 5.

The series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ converges uniformly in the interval $(-\pi, \pi)$ [or, in fact, in any interval], since a set of constants $M_n = 1/n^2$ can be found such that

$$\left|\frac{\sin nx}{n^2}\right| \leqq \frac{1}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges}$$

INTEGRATION AND DIFFERENTIATION OF FOURIER SERIES

Integration and differentiation of Fourier series can be justified by using Theorems 2-2 and 2-3, which hold for series in general. It must be emphasized, however, that those theorems provide sufficient conditions and are not necessary. The following theorem for integration is especially useful.

***Theorem 2-5*:** The Fourier series corresponding to $f(x)$ may be integrated term by term from a to x, and the resulting series will converge uniformly to $\int_a^x f(u)\,du$, provided that $f(x)$ is piecewise continuous in $-L \leqq x \leqq L$ and both a and x are in this interval.

COMPLEX NOTATION FOR FOURIER SERIES

Using Euler's identities,

$$e^{i\theta} = \cos\theta + i\sin\theta, \qquad e^{-i\theta} = \cos\theta - i\sin\theta \tag{11}$$

where i is the imaginary unit such that $i^2 = -1$, the Fourier series for $f(x)$ can be written in complex form as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \tag{12}$$

where

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-in\pi x/L}\,dx \tag{13}$$

In writing the equality (*12*), we are supposing that the Dirichlet conditions are satisfied and further that $f(x)$ is continuous at x. If $f(x)$ is discontinuous at x, the left side of (*12*) should be replaced by $\frac{f(x+0) + f(x-0)}{2}$.

DOUBLE FOURIER SERIES

The idea of a Fourier series expansion for a function of a single variable x can be extended to the case of functions of two variables x and y, i.e. $f(x, y)$. For example, we can expand $f(x, y)$ into a *double Fourier sine series*

$$f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} \sin\frac{m\pi x}{L_1} \sin\frac{n\pi y}{L_2} \tag{14}$$

where

$$B_{mn} = \frac{4}{L_1 L_2}\int_0^{L_1}\int_0^{L_2} f(x, y) \sin\frac{m\pi x}{L_1} \sin\frac{n\pi y}{L_2}\,dx\,dy \tag{15}$$

Similar results can be obtained for cosine series or for series having both sines and cosines.

These ideas can be generalized to *triple Fourier series*, etc.

APPLICATIONS OF FOURIER SERIES

There are numerous applications of Fourier series to solutions of boundary value problems. For example:

1. **Heat flow.** See Problems 2.25–2.29.
2. **Laplace's equation.** See Problems 2.30, 2.31.
3. **Vibrating systems.** See Problems 2.32, 2.33.

Solved Problems

FOURIER SERIES

2.1. Graph each of the following functions.

(a) $$f(x) = \begin{cases} 3 & 0 < x < 5 \\ -3 & -5 < x < 0 \end{cases} \qquad \text{Period} = 10$$

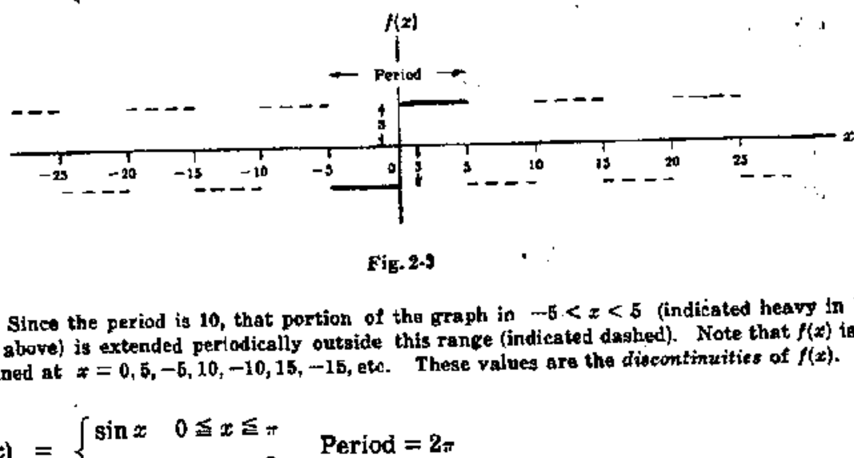

Fig. 2-3

Since the period is 10, that portion of the graph in $-5 < x < 5$ (indicated heavy in Fig. 2-3 above) is extended periodically outside this range (indicated dashed). Note that $f(x)$ is not defined at $x = 0, 5, -5, 10, -10, 15, -15$, etc. These values are the *discontinuities* of $f(x)$.

(b) $$f(x) = \begin{cases} \sin x & 0 \leqq x \leqq \pi \\ 0 & \pi < x < 2\pi \end{cases} \qquad \text{Period} = 2\pi$$

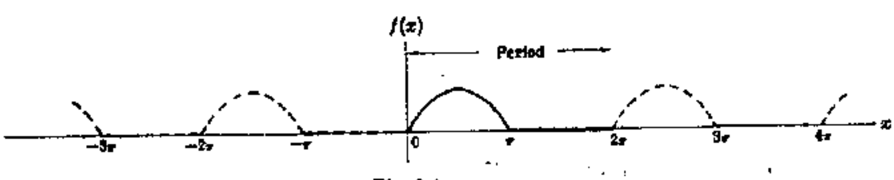

Fig. 2-4

Refer to Fig. 2-4 above. Note that $f(x)$ is defined for all x and is continuous everywhere.

(c) $f(x) = \begin{cases} 0 & 0 \leqq x < 2 \\ 1 & 2 \leqq x < 4 \\ 0 & 4 \leqq x < 6 \end{cases}$ Period = 6

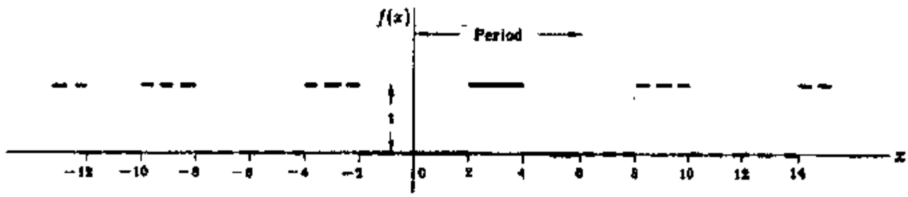

Fig. 2-5

Refer to Fig. 2-5 above. Note that $f(x)$ is defined for all x and is discontinuous at $x = \pm 2, \pm 4, \pm 8, \pm 10, \pm 14, \ldots$.

2.2. Prove $\displaystyle\int_{-L}^{L} \sin\frac{k\pi x}{L}\,dx = \int_{-L}^{L} \cos\frac{k\pi x}{L}\,dx = 0$ if $k = 1, 2, 3, \ldots$.

$$\int_{-L}^{L} \sin\frac{k\pi x}{L}\,dx = -\frac{L}{k\pi}\cos\frac{k\pi x}{L}\Big|_{-L}^{L} = -\frac{L}{k\pi}\cos k\pi + \frac{L}{k\pi}\cos(-k\pi) = 0$$

$$\int_{-L}^{L} \cos\frac{k\pi x}{L}\,dx = \frac{L}{k\pi}\sin\frac{k\pi x}{L}\Big|_{-L}^{L} = \frac{L}{k\pi}\sin k\pi - \frac{L}{k\pi}\sin(-k\pi) = 0$$

2.3. Prove (a) $\displaystyle\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$

(b) $\displaystyle\int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = 0$

where m and n can assume any of the values $1, 2, 3, \ldots$.

(a) From trigonometry:

$$\cos A \cos B = \tfrac{1}{2}\{\cos(A-B) + \cos(A+B)\}, \qquad \sin A \sin B = \tfrac{1}{2}\{\cos(A-B) - \cos(A+B)\}$$

Then, if $m \neq n$, we have by Problem 2.2,

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L}\left\{\cos\frac{(m-n)\pi x}{L} + \cos\frac{(m+n)\pi x}{L}\right\}dx = 0$$

Similarly, if $m \neq n$,

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L}\left\{\cos\frac{(m-n)\pi x}{L} - \cos\frac{(m+n)\pi x}{L}\right\}dx = 0$$

If $m = n$, we have

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L}\left(1 + \cos\frac{2n\pi x}{L}\right)dx = L$$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L}\left(1 - \cos\frac{2n\pi x}{L}\right)dx = L$$

Note that if $m = n = 0$ these integrals are equal to $2L$ and 0 respectively.

(b) We have $\sin A \cos B = \frac{1}{2}\{\sin(A-B) + \sin(A+B)\}$. Then by Problem 2.2, if $m \neq n$,

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L}\left\{\sin\frac{(m-n)\pi x}{L} + \sin\frac{(m+n)\pi x}{L}\right\}dx = 0$$

If $m = n$,

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L}\sin\frac{2n\pi x}{L}\,dx = 0$$

The results of parts (a) and (b) remain valid when the limits of integration $-L, L$ are replaced by $c, c+2L$ respectively.

2.4. If the series $A + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$ converges uniformly to $f(x)$ in $(-L, L)$, show that for $n = 1, 2, 3, \ldots$,

(a) $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$, (b) $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$, (c) $A = \frac{a_0}{2}$.

(a) Multiplying

$$f(x) = A + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right) \tag{1}$$

by $\cos\frac{m\pi x}{L}$ and integrating from $-L$ to L, using Problem 2.3, we have

$$\begin{aligned}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx &= A\int_{-L}^{L}\cos\frac{m\pi x}{L}\,dx \\ &\quad + \sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx + b_n\int_{-L}^{L}\cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx\right\} \\ &= a_m L \quad \text{if } m \neq 0\end{aligned} \tag{2}$$

Thus

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx \qquad \text{if } m = 1, 2, 3, \ldots$$

(b) Multiplying (1) by $\sin\frac{m\pi x}{L}$ and integrating from $-L$ to L, using Problem 2.3, we have

$$\begin{aligned}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx &= A\int_{-L}^{L}\sin\frac{m\pi x}{L}\,dx \\ &\quad + \sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L}\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx + b_n\int_{-L}^{L}\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx\right\} \\ &= b_m L\end{aligned}$$

Thus

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx \qquad \text{if } m = 1, 2, 3, \ldots$$

(c) Integration of (1) from $-L$ to L, using Problem 2.2, gives

$$\int_{-L}^{L} f(x)\,dx = 2AL \qquad \text{or} \qquad A = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

Putting $m = 0$ in the result of part (a), we find $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$ and so $A = \frac{a_0}{2}$.

The above results also hold when the integration limits $-L, L$ are replaced by $c, c+2L$.

Note that in all parts above, interchange of summation and integration is valid because the series is *assumed* to converge uniformly to $f(x)$ in $(-L, L)$. Even when this assumption is not warranted, the coefficients a_m and b_m as obtained above are called *Fourier coefficients* corresponding to $f(x)$, and the corresponding series with these values of a_m and b_m is called the *Fourier series* corresponding to $f(x)$. An important problem in this case is to investigate conditions under which this series actually converges to $f(x)$. Sufficient conditions for this convergence are the *Dirichlet conditions* established below in Problems 2.18–2.23.

2.5. (a) Find the Fourier coefficients corresponding to the function

$$f(x) = \begin{cases} 0 & -5 < x < 0 \\ 3 & 0 < x < 5 \end{cases} \qquad \text{Period} = 10$$

(b) Write the corresponding Fourier series.

(c) How should $f(x)$ be defined at $x=-5$, $x=0$ and $x=5$ in order that the Fourier series will converge to $f(x)$ for $-5 \leqq x \leqq 5$?

The graph of $f(x)$ is shown in Fig. 2-6 below.

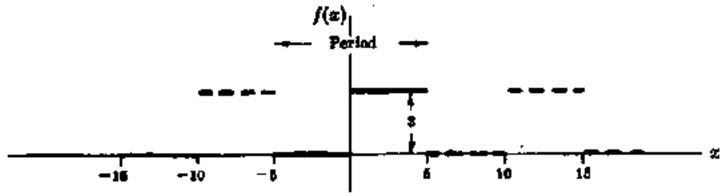

Fig. 2-6

(a) Period $= 2L = 10$ and $L=5$. Choose the interval c to $c+2L$ as -5 to 5, so that $c=-5$. Then

$$\begin{aligned} a_n &= \frac{1}{L}\int_c^{c+2L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{5}\int_{-5}^{5} f(x)\cos\frac{n\pi x}{5}\,dx \\ &= \frac{1}{5}\left\{\int_{-5}^{0}(0)\cos\frac{n\pi x}{5}\,dx + \int_0^5 (3)\cos\frac{n\pi x}{5}\,dx\right\} = \frac{3}{5}\int_0^5 \cos\frac{n\pi x}{5}\,dx \\ &= \frac{3}{5}\left(\frac{5}{n\pi}\sin\frac{n\pi x}{5}\right)\Bigg|_0^5 = 0 \qquad \text{if } n \neq 0 \end{aligned}$$

If $n=0$, $a_n = a_0 = \frac{3}{5}\int_0^5 \cos\frac{0\pi x}{5}\,dx = \frac{3}{5}\int_0^5 dx = 3.$

$$\begin{aligned} b_n &= \frac{1}{L}\int_c^{c+2L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{5}\int_{-5}^{5} f(x)\sin\frac{n\pi x}{5}\,dx \\ &= \frac{1}{5}\left\{\int_{-5}^{0}(0)\sin\frac{n\pi x}{5}\,dx + \int_0^5 (3)\sin\frac{n\pi x}{5}\,dx\right\} = \frac{3}{5}\int_0^5 \sin\frac{n\pi x}{5}\,dx \\ &= \frac{3}{5}\left(-\frac{5}{n\pi}\cos\frac{n\pi x}{5}\right)\Bigg|_0^5 = \frac{3(1-\cos n\pi)}{n\pi} \end{aligned}$$

(b) The corresponding Fourier series is

$$\begin{aligned} \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right) &= \frac{3}{2} + \sum_{n=1}^{\infty}\frac{3(1-\cos n\pi)}{n\pi}\sin\frac{n\pi x}{5} \\ &= \frac{3}{2} + \frac{6}{\pi}\left(\sin\frac{\pi x}{5} + \frac{1}{3}\sin\frac{3\pi x}{5} + \frac{1}{5}\sin\frac{5\pi x}{5} + \cdots\right) \end{aligned}$$

(c) Since $f(x)$ satisfies the Dirichlet conditions, we can say that the series converges to $f(x)$ at all points of continuity and to $\frac{f(x+0)+f(x-0)}{2}$ at points of discontinuity. At $x=-5$, 0 and 5, which are points of discontinuity, the series converges to $(3+0)/2 = 3/2$, as seen from the graph. The series will converge to $f(x)$ for $-5 \leqq x \leqq 5$ if we redefine $f(x)$ as follows:

$$f(x) = \begin{cases} 3/2 & x=-5 \\ 0 & -5<x<0 \\ 3/2 & x=0 \\ 3 & 0<x<5 \\ 3/2 & x=5 \end{cases} \qquad \text{Period} = 10$$

2.6. **Expand $f(x)=x^2$, $0<x<2\pi$, in a Fourier series if the period is 2π.**

The graph of $f(x)$ with period 2π is shown in Fig. 2-7.

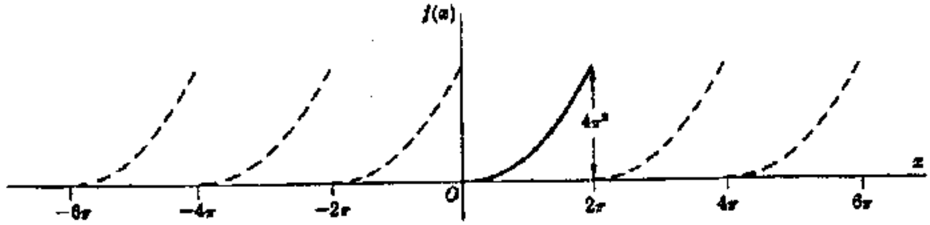

Fig. 2-7

Period $= 2L = 2\pi$ and $L = \pi$. Choosing $c = 0$, we have

$$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\,dx$$

$$= \frac{1}{\pi}\left\{(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right\}\Bigg|_0^{2\pi} = \frac{4}{n^2}, \qquad n \neq 0$$

If $n = 0$, $a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx = \frac{8\pi^2}{3}$.

$$b_n = \frac{1}{L}\int_c^{c+2L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,dx$$

$$= \frac{1}{\pi}\left\{(x^2)\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right\}\Bigg|_0^{2\pi} = \frac{-4\pi}{n}$$

Then $f(x) = x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$ for $0 < x < 2\pi$.

2.7. Using the results of Problem 2.6, prove that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$.

At $x = 0$ the Fourier series of Problem 2.6 reduces to $\frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}$.

But by the Dirichlet conditions, the series converges at $x = 0$ to $\frac{1}{2}(0 + 4\pi^2) = 2\pi^2$.

Hence the desired result.

ODD AND EVEN FUNCTIONS. HALF-RANGE FOURIER SERIES

2.8. Classify each of the following functions according as they are even, odd, or neither even nor odd.

(a) $f(x) = \begin{cases} 2 & 0 < x < 3 \\ -2 & -3 < x < 0 \end{cases}$ Period $= 6$

From Fig. 2-8 below it is seen that $f(-x) = -f(x)$, so that the function is odd.

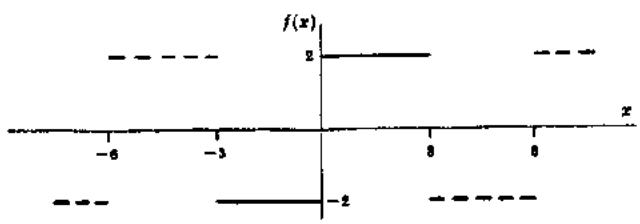

Fig. 2-8

(b) $f(x) = \begin{cases} \cos x & 0 < x < \pi \\ 0 & \pi < x < 2\pi \end{cases}$ Period $= 2\pi$

From Fig. 2-9 below it is seen that the function is neither even nor odd.

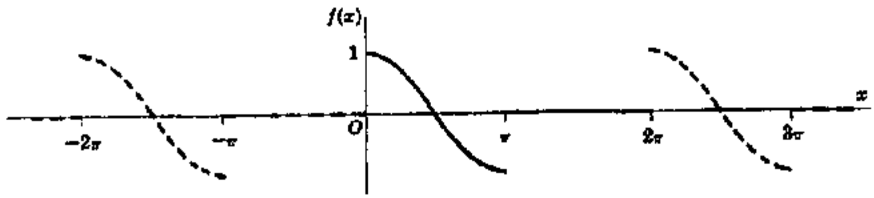

Fig. 2-9

(c) $f(x) = x(10-x)$, $0 < x < 10$, Period $= 10$.

From Fig. 2-10 below the function is seen to be even.

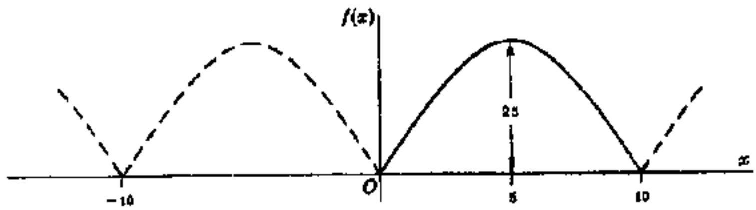

Fig. 2-10

2.9. **Show that an even function can have no sine terms in its Fourier expansion.**

Method 1.

No sine terms appear if $b_n = 0$, $n = 1, 2, 3, \ldots$. To show this, let us write

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{-L}^{0} f(x)\sin\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad (1)$$

If we make the transformation $x = -u$ in the first integral on the right of (1), we obtain

$$\frac{1}{L}\int_{-L}^{0} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{0}^{L} f(-u)\sin\left(-\frac{n\pi u}{L}\right)du = -\frac{1}{L}\int_{0}^{L} f(-u)\sin\frac{n\pi u}{L}\,du$$

$$= -\frac{1}{L}\int_{0}^{L} f(u)\sin\frac{n\pi u}{L}\,du = -\frac{1}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad (2)$$

where we have used the fact that for an even function $f(-u) = f(u)$ and in the last step that the dummy variable of integration u can be replaced by any other symbol, in particular x. Thus from (1), using (2), we have

$$b_n = -\frac{1}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = 0$$

Method 2.

Assuming convergence
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$$

Then
$$f(-x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} - b_n\sin\frac{n\pi x}{L}\right)$$

If $f(x)$ is even, $f(-x) = f(x)$. Hence

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} - b_n\sin\frac{n\pi x}{L}\right)$$

and so $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} = 0$, i.e. $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$

and no sine terms appear. This method is weaker than Method 1 since convergence is assumed.

In a similar manner we can show that an odd function has no cosine terms (or constant term) in its Fourier expansion.

2.10. If $f(x)$ is even, show that (a) $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$, (b) $b_n = 0$.

(a) $$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{-L}^{0} f(x)\cos\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

Letting $x = -u$,

$$\frac{1}{L}\int_{-L}^{0} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_0^L f(-u)\cos\left(\frac{-n\pi u}{L}\right)du = \frac{1}{L}\int_0^L f(u)\cos\frac{n\pi u}{L}\,du$$

since by definition of an even function $f(-u) = f(u)$. Then

$$a_n = \frac{1}{L}\int_0^L f(u)\cos\frac{n\pi u}{L}\,du + \frac{1}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

(b) This follows by Method 1 of Problem 2.9.

2.11. Expand $f(x) = \sin x$, $0 < x < \pi$, in a Fourier cosine series.

A Fourier series consisting of cosine terms alone is obtained only for an even function. Hence we extend the definition of $f(x)$ so that it becomes even (dashed part of Fig. 2-11). With this extension, $f(x)$ is defined in an interval of length 2π. Taking the period as 2π, we have $2L = 2\pi$, so that $L = \pi$.

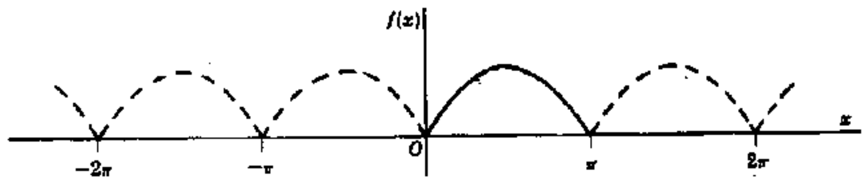

Fig. 2-11

By Problem 2.10, $b_n = 0$ and

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{\pi}\int_0^\pi \sin x\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^\pi \{\sin(x+nx) + \sin(x-nx)\}\,dx = \frac{1}{\pi}\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\}\Bigg|_0^\pi$$

$$= \frac{1}{\pi}\left\{\frac{1-\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi - 1}{n-1}\right\} = \frac{1}{\pi}\left\{-\frac{1+\cos n\pi}{n+1} - \frac{1+\cos n\pi}{n-1}\right\}$$

$$= \frac{-2(1+\cos n\pi)}{\pi(n^2-1)} \quad \text{if } n \neq 1$$

For $n = 1$, $a_1 = \frac{2}{\pi}\int_0^\pi \sin x\cos x\,dx = \frac{2}{\pi}\frac{\sin^2 x}{2}\Bigg|_0^\pi = 0.$

Then
$$f(x) = \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(1+\cos n\pi)}{n^2-1}\cos nx$$
$$= \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \cdots\right)$$

2.12. Expand $f(x) = x$, $0 < x < 2$, in a half-range (*a*) sine series, (*b*) cosine series.

(*a*) Extend the definition of the given function to that of the odd function of period 4 shown in Fig. 2-12 below. This is sometimes called the *odd extension* of $f(x)$. Then $2L = 4$, $L = 2$.

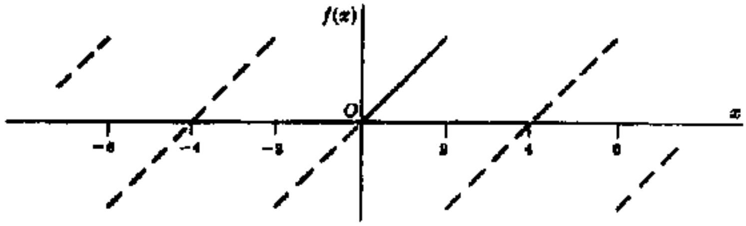

Fig. 2-12

Thus $a_n = 0$ and
$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{2}\int_0^2 x\sin\frac{n\pi x}{2}\,dx$$
$$= \left\{(x)\left(\frac{-2}{n\pi}\cos\frac{n\pi x}{2}\right) - (1)\left(\frac{-4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right\}\Big|_0^2 = \frac{-4}{n\pi}\cos n\pi$$

Then
$$f(x) = \sum_{n=1}^{\infty}\frac{-4}{n\pi}\cos n\pi\sin\frac{n\pi x}{2}$$
$$= \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \cdots\right)$$

(*b*) Extend the definition of $f(x)$ to that of the even function of period 4 shown in Fig. 2-13 below. This is the *even extension* of $f(x)$. Then $2L = 4$, $L = 2$.

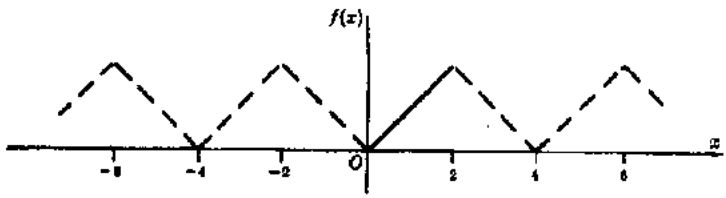

Fig. 2-13

Thus $b_n = 0$,
$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx$$
$$= \left\{(x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(\frac{-4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right\}\Big|_0^2$$
$$= \frac{-4}{n^2\pi^2}(\cos n\pi - 1) \qquad \text{if } n \neq 0$$

If $n = 0$, $a_0 = \int_0^2 x\,dx = 2$.

Then
$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}(\cos n\pi - 1)\cos\frac{n\pi x}{2}$$
$$= 1 - \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right)$$

It should be noted that although both series of (a) and (b) represent $f(x)$ in the interval $0 < x < 2$, the second series converges more rapidly.

PARSEVAL'S IDENTITY

2.13. Assuming that the Fourier series corresponding to $f(x)$ converges uniformly to $f(x)$ in $(-L, L)$, prove Parseval's identity

$$\frac{1}{L}\int_{-L}^{L} \{f(x)\}^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

where the integral is assumed to exist.

If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$, then multiplying by $f(x)$ and integrating term by term from $-L$ to L (which is justified since the series is uniformly convergent), we obtain

$$\int_{-L}^{L}\{f(x)\}^2\,dx = \frac{a_0}{2}\int_{-L}^{L} f(x)\,dx + \sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx + b_n\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx\right\}$$
$$= \frac{a_0^2}{2}L + L\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \qquad (1)$$

where we have used the results

$$\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = La_n, \quad \int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = Lb_n, \quad \int_{-L}^{L} f(x)\,dx = La_0 \qquad (2)$$

obtained from the Fourier coefficients.

The required result follows on dividing both sides of (1) by L. Parseval's identity is valid under less restrictive conditions than imposed here. In Chapter 8 we shall discuss the significance of Parseval's identity in connection with generalizations of Fourier series known as *orthonormal series*.

2.14. (a) Write Parseval's identity corresponding to the Fourier series of Problem 2.12(b).
(b) Determine from (a) the sum S of the series $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots + \frac{1}{n^4} + \cdots$.

(a) Here $L = 2$; $a_0 = 2$; $a_n = \frac{4}{n^2\pi^2}(\cos n\pi - 1)$, $n \neq 0$; $b_n = 0$.

Then Parseval's identity becomes

$$\frac{1}{2}\int_{-2}^{2}\{f(x)\}^2\,dx = \frac{1}{2}\int_{-2}^{2} x^2\,dx = \frac{(2)^2}{2} + \sum_{n=1}^{\infty}\frac{16}{n^4\pi^4}(\cos n\pi - 1)^2$$

or $\frac{8}{3} = 2 + \frac{64}{\pi^4}\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right)$, i.e. $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96}$.

(b)
$$S = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right) + \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \cdots\right)$$
$$= \left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right) + \frac{1}{2^4}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right)$$
$$= \frac{\pi^4}{96} + \frac{S}{16}, \quad \text{from which } S = \frac{\pi^4}{90}$$

2.15. **Prove that for all positive integers M,**

$$\frac{a_0^2}{2} + \sum_{n=1}^{M}(a_n^2 + b_n^2) \leqq \frac{1}{L}\int_{-L}^{L}\{f(x)\}^2\,dx$$

where a_n and b_n are the Fourier coefficients corresponding to $f(x)$, and $f(x)$ is assumed piecewise continuous in $(-L, L)$.

Let $$S_M(x) = \frac{a_0}{2} + \sum_{n=1}^{M}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right) \qquad (1)$$

For $M = 1, 2, 3, \ldots$ this is the sequence of partial sums of the Fourier series corresponding to $f(x)$.

We have $$\int_{-L}^{L}\{f(x) - S_M(x)\}^2\,dx \geqq 0 \qquad (2)$$

since the integrand is non-negative. Expanding the integrand, we obtain

$$2\int_{-L}^{L} f(x)\,S_M(x)\,dx - \int_{-L}^{L} S_M^2(x)\,dx \leqq \int_{-L}^{L}\{f(x)\}^2\,dx \qquad (3)$$

Multiplying both sides of (1) by $2f(x)$ and integrating from $-L$ to L, using equations (2) of Problem 2.13, gives

$$2\int_{-L}^{L} f(x)\,S_M(x)\,dx = 2L\left\{\frac{a_0^2}{2} + \sum_{n=1}^{M}(a_n^2 + b_n^2)\right\} \qquad (4)$$

Also, squaring (1) and integrating from $-L$ to L, using Problem 2.3, we find

$$\int_{-L}^{L} S_M^2(x)\,dx = L\left\{\frac{a_0^2}{2} + \sum_{n=1}^{M}(a_n^2 + b_n^2)\right\} \qquad (5)$$

Substitution of (4) and (5) into (3) and dividing by L yields the required result.

Taking the limit as $M \to \infty$, we obtain *Bessel's inequality*

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \leqq \frac{1}{L}\int_{-L}^{L}\{f(x)\}^2\,dx \qquad (6)$$

If the equality holds, we have Parseval's identity (Problem 2.13).

We can think of $S_M(x)$ as representing an *approximation* to $f(x)$, while the left hand side of (2), divided by $2L$, represents the *mean square error* of the approximation. Parseval's identity indicates that as $M \to \infty$ the mean square error approaches zero, while Bessel's inequality indicates the possibility that this mean square error does not approach zero.

The results are connected with the idea of *completeness*. If, for example, we were to leave out one or more terms in a Fourier series ($\cos 4\pi x/L$, say), we could never get the mean square error to approach zero, no matter how many terms we took. We shall return to these ideas from a generalized viewpoint in Chapter 3.

INTEGRATION AND DIFFERENTIATION OF FOURIER SERIES

2.16. (*a*) Find a Fourier series for $f(x) = x^2$, $0 < x < 2$, by integrating the series of Problem 2.12(*a*). (*b*) Use (*a*) to evaluate the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$.

(*a*) From Problem 2.12(*a*),

$$x = \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \cdots\right) \qquad (1)$$

Integrating both sides from 0 to x (applying Theorem 2-5, page 24) and multiplying by 2, we find

$$x^2 = C - \frac{16}{\pi^2}\left(\cos\frac{\pi x}{2} - \frac{1}{2^2}\cos\frac{2\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} - \cdots\right) \qquad (2)$$

where $C = \frac{16}{\pi^2}\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right)$.

(b) To determine C in another way, note that (2) represents the Fourier cosine series for x^2 in $0 < x < 2$. Then since $L = 2$ in this case,

$$C = \frac{a_0}{2} = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{2}\int_0^2 x^2\,dx = \frac{4}{3}$$

Then from the value of C in (a), we have

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{16}\cdot\frac{4}{3} = \frac{\pi^2}{12}$$

2.17. Show that term by term differentiation of the series in Problem 2.12(a) is not valid.

Term by term differentiation yields $2\left(\cos\frac{\pi x}{2} - \cos\frac{2\pi x}{2} + \cos\frac{3\pi x}{2} - \cdots\right)$. Since the nth term of this series does not approach 0, the series does not converge for any value of x.

CONVERGENCE OF FOURIER SERIES

2.18. Prove that (a) $\frac{1}{2} + \cos t + \cos 2t + \cdots + \cos Mt = \frac{\sin(M+\frac{1}{2})t}{2\sin\frac{1}{2}t}$

(b) $\frac{1}{\pi}\int_0^{\pi}\frac{\sin(M+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt = \frac{1}{2}$, $\quad \frac{1}{\pi}\int_{-\pi}^{0}\frac{\sin(M+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt = \frac{1}{2}$.

(a) We have $\cos nt \sin\frac{1}{2}t = \frac{1}{2}\{\sin(n+\frac{1}{2})t - \sin(n-\frac{1}{2})t\}$. Then summing from $n = 1$ to M,

$$\begin{aligned}\sin\tfrac{1}{2}t\{\cos t + \cos 2t + \cdots + \cos Mt\} &= (\sin\tfrac{3}{2}t - \sin\tfrac{1}{2}t) + (\sin\tfrac{5}{2}t - \sin\tfrac{3}{2}t) \\ &\quad + \cdots + [\sin(M+\tfrac{1}{2})t - \sin(M-\tfrac{1}{2})t] \\ &= \tfrac{1}{2}\{\sin(M+\tfrac{1}{2})t - \sin\tfrac{1}{2}t\}\end{aligned}$$

On dividing by $\sin\frac{1}{2}t$ and adding $\frac{1}{2}$, the required result follows.

(b) Integrate the result in (a) from 0 to π and $-\pi$ to 0 respectively. This gives the required results, since the integrals of all the cosine terms are zero.

2.19. Prove that $\lim\limits_{n\to\infty}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \lim\limits_{n\to\infty}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$ if $f(x)$ is piecewise continuous.

This follows at once from Problem 2.15, since if the series $\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$ is convergent, $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 0$.

The result is sometimes called *Riemann's theorem*.

2.20. Prove that $\lim\limits_{M\to\infty}\int_{-\pi}^{\pi} f(x)\sin(M+\frac{1}{2})x\,dx = 0$ if $f(x)$ is piecewise continuous.

We have

$$\int_{-\pi}^{\pi} f(x)\sin(M+\tfrac{1}{2})x\,dx = \int_{-\pi}^{\pi}\{f(x)\sin\tfrac{1}{2}x\}\cos Mx\,dx + \int_{-\pi}^{\pi}\{f(x)\cos\tfrac{1}{2}x\}\sin Mx\,dx$$

Then the required result follows at once by using the result of Problem 2.19, with $f(x)$ replaced by $f(x)\sin\frac{1}{2}x$ and $f(x)\cos\frac{1}{2}x$, respectively, which are piecewise continuous if $f(x)$ is.

The result can also be proved when the integration limits are a and b instead of $-\pi$ and π.

2.21. Assuming that $L = \pi$, i.e. that the Fourier series corresponding to $f(x)$ has period $2L = 2\pi$, show that

$$S_M(x) \quad = \quad \frac{a_0}{2} + \sum_{n=1}^{M}(a_n \cos nx + b_n \sin nx) \quad = \quad \frac{1}{\pi}\int_{-\pi}^{\pi} f(t+x)\,\frac{\sin(M+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt$$

Using the formulas for the Fourier coefficients with $L = \pi$, we have

$$\begin{aligned} a_n \cos nx + b_n \sin nx &= \left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\cos nu\,du\right)\cos nx + \left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\sin nu\,du\right)\sin nx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)(\cos nu\cos nx + \sin nu \sin nx)\,du \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\cos n(u-x)\,du \end{aligned}$$

Also,
$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)\,du$$

Then
$$\begin{aligned} S_M(x) &= \frac{a_0}{2} + \sum_{n=1}^{M}(a_n \cos nx + b_n \sin nx) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)\,du + \frac{1}{\pi}\sum_{n=1}^{M}\int_{-\pi}^{\pi} f(u)\cos n(u-x)\,du \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\left\{\frac{1}{2} + \sum_{n=1}^{M}\cos n(u-x)\right\}du \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\,\frac{\sin(M+\frac{1}{2})(u-x)}{2\sin\frac{1}{2}(u-x)}\,du \end{aligned}$$

using Problem 2.18. Letting $u - x = t$, we have

$$S_M(x) = \frac{1}{\pi}\int_{-\pi-x}^{\pi-x} f(t+x)\,\frac{\sin(M+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt$$

Since the integrand has period 2π, we can replace the interval $-\pi-x, \pi-x$ by any other interval of length 2π, in particular $-\pi, \pi$. Thus we obtain the required result.

2.22. Prove that

$$S_M(x) - \left(\frac{f(x+0)+f(x-0)}{2}\right) = \frac{1}{\pi}\int_{-\pi}^{0} \frac{f(t+x)-f(x-0)}{2\sin\frac{1}{2}t}\sin(M+\tfrac{1}{2})t\,dt + \frac{1}{\pi}\int_{0}^{\pi} \frac{f(t+x)-f(x+0)}{2\sin\frac{1}{2}t}\sin(M+\tfrac{1}{2})t\,dt$$

From Problem 2.21,

$$S_M(x) = \frac{1}{\pi}\int_{-\pi}^{0} f(t+x)\,\frac{\sin(M+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt + \frac{1}{\pi}\int_{0}^{\pi} f(t+x)\,\frac{\sin(M+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt \qquad (1)$$

Multiplying the integrals of Problem 2.18(b) by $f(x-0)$ and $f(x+0)$ respectively,

$$\frac{f(x+0)+f(x-0)}{2} = \frac{1}{\pi}\int_{-\pi}^{0} f(x-0)\,\frac{\sin(M+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt + \frac{1}{\pi}\int_{0}^{\pi} f(x+0)\,\frac{\sin(M+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt \qquad (2)$$

Subtracting (2) from (1) yields the required result.

2.23. If $f(x)$ and $f'(x)$ are piecewise continuous in $(-\pi, \pi)$, prove that

$$\lim_{M\to\infty} S_M(x) = \frac{f(x+0)+f(x-0)}{2}$$

The function $\dfrac{f(t+x)-f(x+0)}{2\sin\frac{1}{2}t}$ is piecewise continuous in $0 < t \leq \pi$ because $f(x)$ is piecewise continuous.

Also,

$$\lim_{t\to 0+}\frac{f(t+x)-f(x+0)}{2\sin\frac{1}{2}t} = \lim_{t\to 0+}\frac{f(x+t)-f(x+0)}{t}\cdot\frac{\frac{1}{2}t}{\sin\frac{1}{2}t} = \lim_{t\to 0+}\frac{f(x+t)-f(x+0)}{t}$$

exists, since by hypothesis $f'(x)$ is piecewise continuous, so that the right-hand derivative of $f(x)$ at each x exists.

Thus $\dfrac{f(t+x)-f(x+0)}{2\sin\frac{1}{2}t}$ is piecewise continuous in $0 \leq t \leq \pi$.

Similarly, $\dfrac{f(t+x)-f(x-0)}{2\sin\frac{1}{2}t}$ is piecewise continuous in $-\pi \leq t \leq 0$.

Then from Problems 2.20 and 2.22, we have

$$\lim_{M\to\infty}\left\{S_M(x) - \frac{f(x+0)+f(x-0)}{2}\right\} = 0 \quad \text{or} \quad \lim_{M\to\infty} S_M(x) = \frac{f(x+0)+f(x-0)}{2}$$

DOUBLE FOURIER SERIES

2.24. Obtain formally the Fourier coefficients (*15*), page 24, for the double Fourier sine series (*14*).

Suppose that
$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2} \tag{1}$$

We can write this as
$$f(x,y) = \sum_{m=1}^{\infty} C_m \sin\frac{m\pi x}{L_1} \tag{2}$$

where
$$C_m = \sum_{n=1}^{\infty} B_{mn}\sin\frac{n\pi y}{L_2} \tag{3}$$

Now we can consider (*2*) as a Fourier series in which y is kept constant so that the Fourier coefficients C_m are given by

$$C_m = \frac{2}{L_1}\int_0^{L_1} f(x,y)\sin\frac{m\pi x}{L_1}\,dx \tag{4}$$

On noting that C_m is a function of y, we see that (*3*) can be considered as a Fourier series for which the coefficients B_{mn} are given by

$$B_{mn} = \frac{2}{L_2}\int_0^{L_2} C_m \sin\frac{n\pi y}{L_2}\,dy \tag{5}$$

If we now use (*4*) in (*5*), we see that

$$B_{mn} = \frac{4}{L_1L_2}\int_0^{L_1}\int_0^{L_2} f(x,y)\sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}\,dx\,dy \tag{6}$$

APPLICATIONS TO HEAT CONDUCTION

2.25. Find the temperature of the bar in Problem 1.23, page 15, if the initial temperature is 25°C.

This problem is identical with Problem 1.23, except that to satisfy the initial condition $u(x,0) = 25$ it is necessary to superimpose an *infinite number of solutions*, i.e. we must replace equation (*1*) of that problem by

$$u(x,t) = \sum_{m=1}^{\infty} B_m e^{-2m^2\pi^2 t/9}\sin\frac{m\pi x}{3}$$

which for $t=0$ yields

$$25 = \sum_{m=1}^{\infty} B_m \sin\frac{m\pi x}{3} \qquad 0 < x < 3$$

This amounts to expanding 25 in a *Fourier sine series*. By the methods of this chapter we then find

$$B_m = \frac{2}{L}\int_0^L f(x)\sin\frac{m\pi x}{L}\,dx = \frac{2}{3}\int_0^3 25\sin\frac{m\pi x}{3}\,dx = \frac{50(1-\cos m\pi)}{m\pi}$$

The result can be written

$$u(x,t) \;=\; \sum_{m=1}^{\infty} \frac{50(1-\cos m\pi)}{m\pi} e^{-2m^2\pi^2 t/9} \sin\frac{m\pi x}{3}$$

$$= \; \frac{100}{\pi}\left\{ e^{-2\pi^2 t/9} \sin\frac{\pi x}{3} + \frac{1}{3} e^{-2\pi^2 t} \sin \pi x + \cdots \right\}$$

which can be verified as the required solution.

This problem illustrates the importance of Fourier series in solving boundary value problems.

2.26. Solve the boundary value problem

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = 10, \quad u(3,t) = 40, \quad u(x,0) = 25, \quad |u(x,t)| < M$$

This is the same as Problem 1.23, page 15, except that the ends of the bar are at temperatures 10°C and 40°C instead of 0°C. As far as the solution goes, this makes quite a difference since we can no longer conclude that $A = 0$ and $\lambda = m\pi/3$ as in that problem.

To solve the present problem assume that $u(x,t) = v(x,t) + \psi(x)$ where $\psi(x)$ is to be suitably determined. In terms of $v(x,t)$ the boundary value problem becomes

$$\frac{\partial v}{\partial t} = 2\frac{\partial^2 v}{\partial x^2} + 2\psi''(x), \;\; v(0,t) + \psi(0) = 10, \;\; v(3,t) + \psi(3) = 40, \;\; v(x,0) + \psi(x) = 25, \;\; |v(x,t)| < M$$

This can be simplified by choosing

$$\psi''(x) = 0, \quad \psi(0) = 10, \quad \psi(3) = 40$$

from which we find $\psi(x) = 10x + 10$, so that the resulting boundary value problem is

$$\frac{\partial v}{\partial t} = 2\frac{\partial^2 v}{\partial x^2}, \quad v(0,t) = 0, \quad v(3,t) = 0, \quad v(x,0) = 15 - 10x$$

As in Problem 1.23 we find from the first three of these,

$$v(x,t) = \sum_{m=1}^{\infty} B_m e^{-2m^2\pi^2 t/9} \sin\frac{m\pi x}{3}$$

The last condition yields

$$15 - 10x = \sum_{m=1}^{\infty} B_m \sin\frac{m\pi x}{3}$$

from which

$$B_m = \frac{2}{3}\int_0^3 (15-10x)\sin\frac{m\pi x}{3}\,dx = \frac{30}{m\pi}(\cos m\pi - 1)$$

Since $u(x,t) = v(x,t) + \psi(x)$, we have finally

$$u(x,t) = 10x + 10 + \sum_{m=1}^{\infty} \frac{30}{m\pi}(\cos m\pi - 1) e^{-2m^2\pi^2 t/9} \sin\frac{m\pi x}{3}$$

as the required solution.

The term $10x + 10$ is the *steady-state temperature*, i.e. the temperature after a long time has elapsed.

2.27. A bar of length L whose entire surface is insulated including its ends at $x = 0$ and $x = L$ has initial temperature $f(x)$. Determine the subsequent temperature of the bar.

In this case, the boundary value problem is

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2} \tag{1}$$

$$|u(x,t)| < M, \quad u_x(0,t) = 0, \quad u_x(L,t) = 0, \quad u(x,0) = f(x) \tag{2}$$

Letting $u = XT$ in (1) and separating the variables, we find

$$XT' = \kappa X''T \qquad \text{or} \qquad \frac{T'}{\kappa T} = \frac{X''}{X}$$

Setting each side equal to the constant $-\lambda^2$, we find

$$T' + \kappa\lambda^2 T = 0, \qquad X'' + \lambda^2 X = 0$$

so that
$$X = a\cos\lambda x + b\sin\lambda x, \qquad T = ce^{-\kappa\lambda^2 t}$$

A solution is thus given by

$$u(x,t) = e^{-\kappa\lambda^2 t}(A\cos\lambda x + B\sin\lambda x)$$

where $A = ac$, $B = bc$.

From $u_x(0,t) = 0$ we have $B = 0$ so that

$$u(x,t) = Ae^{-\kappa\lambda^2 t}\cos\lambda x$$

Then from $u_x(L,t) = 0$ we have

$$\sin\lambda L = 0 \qquad \text{or} \qquad \lambda L = m\pi, \qquad m = 0,1,2,3,\ldots$$

Thus
$$u(x,t) = Ae^{-\kappa m^2\pi^2 t/L^2}\cos\frac{m\pi x}{L} \qquad m = 0,1,2,\ldots$$

To satisfy the last condition, $u(x,0) = f(x)$, we use the superposition principle to obtain

$$u(x,t) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m e^{-\kappa m^2\pi^2 t/L^2}\cos\frac{m\pi x}{L}$$

Then from $u(x,0) = f(x)$ we see that

$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m e^{-\kappa m^2\pi^2 t/L^2}\cos\frac{m\pi x}{L}$$

Thus, from Fourier series we find

$$A_m = \frac{2}{L}\int_0^L f(x)\cos\frac{m\pi x}{L}\,dx$$

and
$$u(x,t) = \frac{1}{L}\int_0^L f(x)\,dx + \frac{2}{L}\sum_{m=1}^{\infty}\left(e^{-\kappa m^2\pi^2 t/L^2}\cos\frac{m\pi x}{L}\right)\int_0^L f(x)\cos\frac{m\pi x}{L}\,dx$$

2.28. A circular plate of unit radius, whose faces are insulated, has half of its boundary kept at constant temperature u_1 and the other half at constant temperature u_2 (see Fig. 2-14). Find the steady-state temperature of the plate.

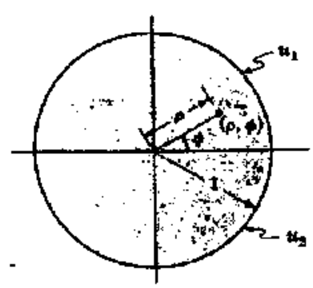

Fig. 2-14

In polar coordinates (ρ,ϕ) the partial differential equation for steady-state heat flow is

$$\frac{\partial^2 u}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2} = 0 \qquad (1)$$

The boundary conditions are

$$u(1,\phi) = \begin{cases} u_1 & 0 < \phi < \pi \\ u_2 & \pi < \phi < 2\pi \end{cases} \tag{2}$$

$$|u(\rho,\phi)| < M, \quad \text{i.e. } u \text{ is bounded in the region} \tag{3}$$

Let $u(\rho,\phi) = P\Phi$ where P is a function of ρ and Φ is a function of ϕ. Then (1) becomes

$$P''\Phi + \frac{1}{\rho}P'\Phi + \frac{1}{\rho^2}P\Phi'' = 0$$

Dividing by $P\Phi$, multiplying by ρ^2 and rearranging terms,

$$\frac{\rho^2 P''}{P} + \frac{\rho P'}{P} = -\frac{\Phi''}{\Phi}$$

Setting each side equal to λ^2,

$$\Phi'' + \lambda^2\Phi = 0 \qquad \rho^2 P'' + \rho P' - \lambda^2 P = 0 \tag{4}$$

The first equation in (4) has general solution

$$\Phi = A_1 \cos\lambda\phi + B_1 \sin\lambda\phi$$

By letting $P = \rho^k$ in the second equation of (4), which is a *Cauchy* or *Euler differential equation*, we find $k = \pm\lambda$; so that ρ^λ and $\rho^{-\lambda}$ are solutions. Thus we obtain the general solution

$$P = A_2\rho^\lambda + B_2\rho^{-\lambda} \tag{5}$$

Since $u(\rho,\phi)$ must have period 2π in ϕ, we must have $\lambda = m = 0,1,2,3,\ldots$.

Also, since u must be bounded at $\rho = 0$, we must have $B_2 = 0$. Thus

$$u = P\Phi = A_2\rho^m(A_1 \cos m\phi + B_1 \sin m\phi) = \rho^m(A \cos m\phi + B \sin m\phi)$$

By superposition, a solution is

$$u(\rho,\phi) = \frac{A_0}{2} + \sum_{m=1}^{\infty} \rho^m(A_m \cos m\phi + B_m \sin m\phi)$$

from which

$$u(1,\phi) = \frac{A_0}{2} + \sum_{m=1}^{\infty} (A_m \cos m\phi + B_m \sin m\phi)$$

Then from the theory of Fourier series,

$$A_m = \frac{1}{\pi}\int_0^{2\pi} u(1,\phi)\cos m\phi\, d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi} u_1 \cos m\phi\, d\phi + \frac{1}{\pi}\int_\pi^{2\pi} u_2 \cos m\phi\, d\phi = \begin{cases} 0 & \text{if } m > 0 \\ u_1 + u_2 & \text{if } m = 0 \end{cases}$$

$$B_m = \frac{1}{\pi}\int_0^{2\pi} u(1,\phi)\sin m\phi\, d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi} u_1 \sin m\phi\, d\phi + \frac{1}{\pi}\int_\pi^{2\pi} u_2 \sin m\phi\, d\phi = \frac{(u_1 - u_2)}{m\pi}(1 - \cos m\pi)$$

Then:

$$u(\rho,\phi) = \frac{u_1 + u_2}{2} + \sum_{m=1}^{\infty} \frac{(u_1 - u_2)(1 - \cos m\pi)}{m\pi}\rho^m \sin m\phi$$

$$= \frac{u_1 + u_2}{2} + \frac{2(u_1 - u_2)}{\pi}\left(\rho \sin\phi + \tfrac{1}{3}\rho^3 \sin 3\phi + \tfrac{1}{5}\rho^5 \sin 5\phi + \cdots\right)$$

$$= \frac{u_1 + u_2}{2} + \frac{u_1 - u_2}{\pi}\tan^{-1}\left(\frac{2\rho \sin\phi}{1 - \rho^2}\right)$$

on making use of Problem 2.54.

2.29. A square plate with sides of unit length has its faces insulated and its sides kept at 0°C. If the initial temperature is specified, determine the subsequent temperature at any point of the plate.

Choose a coordinate system as shown in Fig. 2-15. Then the equation for the temperature $u(x, y, t)$ at any point (x, y) at time t is

$$\frac{\partial u}{\partial t} = \kappa\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad (1)$$

The boundary conditions are given by

$$|u(x, y, t)| < M$$

$$u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0$$

$$u(x, y, 0) = f(x, y)$$

where $0 < x < 1,\ 0 < y < 1,\ t > 0$.

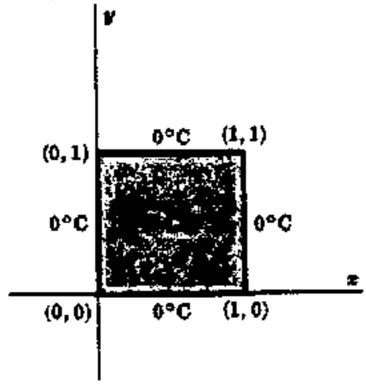

Fig. 2-15

To solve the boundary value problem let $u = XYT$, where X, Y, T are functions of x, y, t respectively. Then (1) becomes

$$XYT' = \kappa(X''YT + XY''T)$$

Dividing by κXYT yields

$$\frac{T'}{\kappa T} = \frac{X''}{X} + \frac{Y''}{Y}$$

Since the left side is a function of t alone, while the right side is a function of x and y, we see that each side must be a constant, say $-\lambda^2$ (which is needed for boundedness). Thus

$$T' + \kappa\lambda^2 T = 0 \qquad \frac{X''}{X} + \frac{Y''}{Y} = -\lambda^2 \qquad (2)$$

The second equation can be written as

$$\frac{X''}{X} = -\frac{Y''}{Y} - \lambda^2$$

and since the left side depends only on x while the right side depends only on y each side must be a constant, say $-\mu^2$. Thus

$$X'' + \mu^2 X = 0 \qquad Y'' + (\lambda^2 - \mu^2)Y = 0 \qquad (3)$$

Solutions to the two equations in (3) and the first equation in (2) are given by

$$X = a_1 \cos \mu x + b_1 \sin \mu x, \quad Y = a_2 \cos \sqrt{\lambda^2 - \mu^2}\, y + b_2 \sin \sqrt{\lambda^2 - \mu^2}\, y, \quad T = a_3 e^{-\kappa\lambda^2 t}$$

It follows that a solution to (1) is given by

$$u(x, y, t) = (a_1 \cos \mu x + b_1 \sin \mu x)(a_2 \cos \sqrt{\lambda^2 - \mu^2}\, y + b_2 \sin \sqrt{\lambda^2 - \mu^2}\, y)(a_3 e^{-\kappa\lambda^2 t})$$

From the boundary condition $u(0, y, t) = 0$ we see that $a_1 = 0$. From $u(x, 0, t) = 0$ we see that $a_2 = 0$. Thus the solution satisfying these two conditions is

$$u(x, y, t) = Be^{-\kappa\lambda^2 t} \sin \mu x \sin \sqrt{\lambda^2 - \mu^2}\, y$$

where we have written $B = b_1 b_2 a_3$.

From the boundary condition $u(1, y, t) = 0$ we see that $\mu = m\pi,\ m = 1, 2, 3, \ldots$. From $u(x, 1, t) = 0$ we see that $\sqrt{\lambda^2 - \mu^2} = n\pi,\ n = 1, 2, 3, \ldots$, or $\lambda = \sqrt{m^2 + n^2}\,\pi$.

It follows that a solution satisfying all the conditions except $u(x, y, 0) = f(x, y)$ is given by

$$u(x, y, t) = Be^{-\kappa(m^2+n^2)\pi^2 t} \sin m\pi x \sin n\pi y$$

Now, by the superposition theorem we can arrive at the possible solution

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}e^{-\kappa(m^2+n^2)\pi^2 t}\sin m\pi x \sin n\pi y \tag{4}$$

Letting $t = 0$ and using the condition $u(x,y,0) = f(x,y)$, we arrive at

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\sin m\pi x \sin n\pi y$$

As in Problem 2.24 we then find that

$$B_{mn} = 4\int_0^1\int_0^1 f(x,y)\sin m\pi x \sin n\pi y\, dx\, dy \tag{5}$$

Thus the formal solution to our problem is given by (4), where the B_{mn} are determined from (5).

LAPLACE'S EQUATION

2.30. **Suppose that the square plate of Problem 2.29 has three sides kept at temperature zero, while the fourth side is kept at temperature u_1. Determine the steady-state temperature everywhere in the plate.**

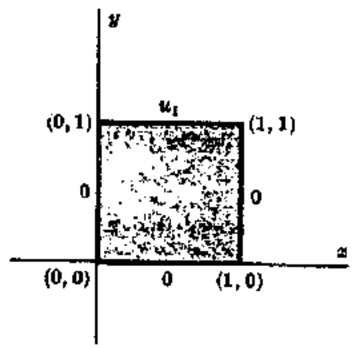

Fig. 2-16

Choose the side having temperature u_1 to be the one where $y = 1$, as shown in Fig. 2-16. Since we wish the steady-state temperature u, which does not depend on time t, the equation is obtained from (1) of Problem 2.29 by setting $\partial u/\partial t = 0$; i.e. Laplace's equation in two dimensions:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

The boundary conditions are

$$u(0,y) = u(1,y) = u(x,0) = 0, \quad u(x,1) = u_1$$

and $|u(x,y)| < M$.

To solve this boundary value problem let $u = XY$ in (1) to obtain

$$X''Y + XY'' = 0 \quad \text{or} \quad \frac{X''}{X} = -\frac{Y''}{Y}$$

Setting each side equal to $-\lambda^2$ yields

$$X'' + \lambda^2 X = 0 \qquad Y'' - \lambda^2 Y = 0$$

from which

$$X = a_1\cos\lambda x + b_1\sin\lambda x \qquad Y = a_2\cosh\lambda y + b_2\sinh\lambda y$$

Then a possible solution is

$$u(x,y) = (a_1\cos\lambda x + b_1\sin\lambda x)(a_2\cosh\lambda y + b_2\sinh\lambda y)$$

From $u(0,y) = 0$ we find $a_1 = 0$. From $u(x,0) = 0$ we find $a_2 = 0$. From $u(1,y) = 0$ we find $\lambda = m\pi$, $m = 1,2,3,\ldots$. Thus a solution satisfying all these conditions is

$$u(x,y) = B\sin m\pi x \sinh m\pi y$$

To satisfy the last condition, $u(x,1) = u_1$, we must first use the principle of superposition to obtain the solution

$$u(x,y) = \sum_{m=1}^{\infty} B_m\sin m\pi x \sinh m\pi y \tag{2}$$

Then from $u(x,1) = u_1$ we must have

$$u_1 = \sum_{m=1}^{\infty}(B_m\sinh m\pi)\sin m\pi x$$

Thus, using the theory of Fourier series,

$$B_m \sinh m\pi = 2\int_0^1 u_1 \sin m\pi x = \frac{2u_1(1-\cos m\pi)}{m\pi}$$

from which
$$B_m = \frac{2u_1(1-\cos m\pi)}{m\pi \sinh m\pi} \tag{3}$$

From (2) and (3) we obtain

$$u(x,y) = \frac{2u_1}{\pi}\sum_{m=1}^{\infty} \frac{1-\cos m\pi}{m \sinh m\pi} \sin m\pi x \sinh m\pi y$$

Note that this is a *Dirichlet problem*, since we are solving Laplace's equation $\nabla^2 u = 0$ for u inside a region $\mathcal{R}$ when u is specified on the boundary of $\mathcal{R}$.

2.31. If the square plate of Problem 2.29 has its sides kept at constant temperatures u_1, u_2, u_3, u_4, respectively, show how to determine the steady-state temperature.

The temperatures at which the sides are kept are indicated in Fig. 2-17. The fact that most of these temperatures are nonzero makes for the same type of difficulty considered in Problem 2.26. To overcome this difficulty we break the problem up into four problems of the type of Problem 2.30, where three of the four sides have temperature zero. We can then show that the solution to the given problem is the sum of solutions to the problems indicated by Figs. 2-18 to 2-21 below.

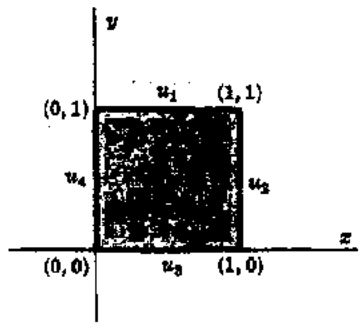

Fig. 2-17

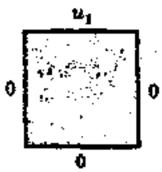

Fig. 2-18

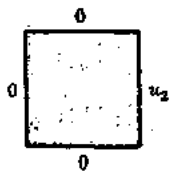

Fig. 2-19

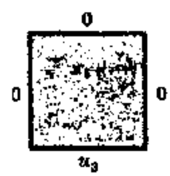

Fig. 2-20

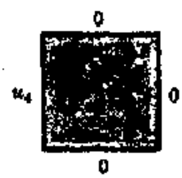

Fig. 2-21

The details are left to Problem 2.57 which provides a generalization to the case where the side temperatures may vary.

APPLICATIONS TO VIBRATING STRINGS AND MEMBRANES

2.32. A string of length L is stretched between points $(0,0)$ and $(L,0)$ on the x-axis. At time $t=0$ it has a shape given by $f(x)$, $0<x<L$, and it is released from rest. Find the displacement of the string at any later time.

The equation of the vibrating string is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \qquad 0<x<L,\ t>0$$

where $y(x,t)$ = displacement from x-axis at time t (Fig. 2-22).

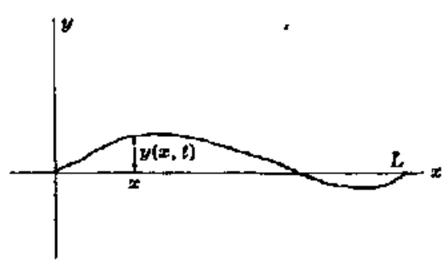

Fig. 2-22

Since the ends of the string are fixed at $x = 0$ and $x = L$,

$$y(0,t) = y(L,t) = 0 \qquad t > 0$$

Since the initial shape of the string is given by $f(x)$,

$$y(x,0) = f(x) \qquad 0 < x < L$$

Since the initial velocity of the string is zero,

$$y_t(x,0) = 0 \qquad 0 < x < L$$

To solve this boundary value problem, let $y = XT$ as usual.

Then $$XT'' = a^2X''T \qquad \text{or} \qquad T''/a^2T = X''/X$$

Calling the separation constant $-\lambda^2$, we have

$$T'' + \lambda^2a^2T = 0 \qquad X'' + \lambda^2X = 0$$

and $$T = A_1 \sin \lambda at + B_1 \cos \lambda at \qquad X = A_2 \sin \lambda x + B_2 \cos \lambda x$$

A solution is thus given by

$$y(x,t) = XT = (A_2 \sin \lambda x + B_2 \cos \lambda x)(A_1 \sin \lambda at + B_1 \cos \lambda at)$$

From $y(0,t) = 0$, $A_2 = 0$. Then

$$y(x,t) = B_2 \sin \lambda x(A_1 \sin \lambda at + B_1 \cos \lambda at) = \sin \lambda x(A \sin \lambda at + B \cos \lambda at)$$

From $y(L,t) = 0$, we have $\sin \lambda L(A \sin \lambda at + B \cos \lambda at) = 0$, so that $\sin \lambda L = 0$, $\lambda L = m\pi$ or $\lambda = m\pi/L$, since the second factor must not be equal to zero. Now,

$$y_t(x,t) = \sin \lambda x(A\lambda a \cos \lambda at - B\lambda a \sin \lambda at)$$

and $y_t(x,0) = (\sin \lambda x)(A\lambda a) = 0$, from which $A = 0$. Thus

$$y(x,t) = B \sin \frac{m\pi x}{L} \cos \frac{m\pi at}{L}$$

To satisfy the condition $y(x,0) = f(x)$, it will be necessary to superpose solutions. This yields

$$y(x,t) = \sum_{m=1}^{\infty} B_m \sin \frac{m\pi x}{L} \cos \frac{m\pi at}{L}$$

Then $$y(x,0) = f(x) = \sum_{m=1}^{\infty} B_m \sin \frac{m\pi x}{L}$$

and from the theory of Fourier series,

$$B_m = \frac{2}{L}\int_0^L f(x) \sin \frac{m\pi x}{L}\, dx$$

The final result is

$$y(x,t) = \sum_{m=1}^{\infty} \left(\frac{2}{L}\int_0^L f(x) \sin \frac{m\pi x}{L}\, dx\right) \sin \frac{m\pi x}{L} \cos \frac{m\pi at}{L}$$

which can be verified as the solution.

The terms in this series represent the *natural* or *normal modes of vibration*. The frequency of the mth normal mode f_m is obtained from the term involving $\cos \frac{m\pi at}{L}$ and is given by

$$2\pi f_m = \frac{m\pi a}{L} \qquad \text{or} \qquad f_m = \frac{ma}{2L} = \frac{m}{2L}\sqrt{\frac{\tau}{\mu}}$$

Since all the frequencies are integer multiples of the lowest frequency f_1, the vibrations of the string will yield a musical tone, as in the case of a violin or piano string. The first three normal modes are illustrated in Fig. 2-23. As time increases the shapes of these modes vary from curves shown solid to curves shown dashed and then back again, the time for a complete cycle being the

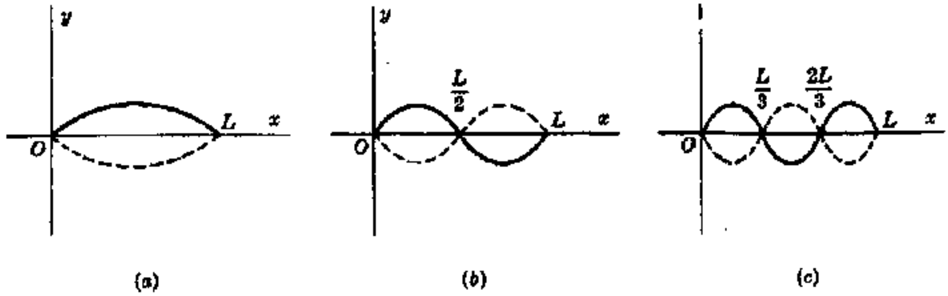

Fig. 2-23

period and the reciprocal of this period being the frequency. We call the mode (a) the *fundamental mode* or *first harmonic*, while (b) and (c) are called the *second* and *third harmonic* (or *first* and *second overtone*), respectively.

2.33. A square drumhead or membrane has edges which are fixed and of unit length. If the drumhead is given an initial transverse displacement and then released, determine the subsequent motion.

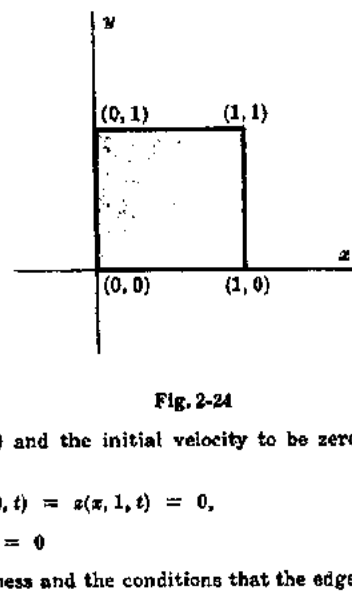

Fig. 2-24

Assume a coordinate system as in Fig. 2-24 and suppose that the transverse displacement from the equilibrium position (i.e. the perpendicular distance from the xy-plane) of any point (x, y) at time t is given by $z(x, y, t)$.

Then the equation for the transverse motion is

$$\frac{\partial^2 z}{\partial t^2} = a^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) \qquad (1)$$

where $a^2 = \tau/\mu$, the quantity τ being the tension per unit length along any line drawn in the drumhead, and μ is the mass per unit area.

Assuming the initial transverse displacement to be $f(x,y)$ and the initial velocity to be zero, we have the conditions

$$|z(x,y,t)| < M, \quad z(0,y,t) = z(1,y,t) = z(x,0,t) = z(x,1,t) = 0,$$

$$z(x,y,0) = f(x,y), \quad z_t(x,y,0) = 0$$

where we have in addition expressed the condition for boundedness and the conditions that the edges do not move.

To solve the boundary value problem we let $z = XYT$ in (1), where X, Y, T are functions of x, y, and t respectively. Then, proceeding as in Problem 2.29, we find

$$\frac{T''}{a^2 T} = \frac{X''}{X} + \frac{Y''}{Y}$$

and we are led exactly as in Problem 2.29 to the equation

$$T'' + \lambda^2 a^2 T = 0, \quad X'' + \mu^2 X = 0, \quad Y'' + (\lambda^2 - \mu^2)Y = 0$$

Solutions of these equations are

$$X = a_1 \cos \mu x + b_1 \sin \mu x, \quad Y = a_2 \cos \sqrt{\lambda^2 - \mu^2}\, y + b_2 \sin \sqrt{\lambda^2 - \mu^2}\, y$$

$$T = a_3 \cos \lambda a t + b_3 \sin \lambda a t$$

A solution of (1) is thus given by

$$z(x,y,t) = (a_1 \cos \mu x + b_1 \sin \mu x)(a_2 \cos \sqrt{\lambda^2 - \mu^2}\, y + b_2 \sin \sqrt{\lambda^2 - \mu^2}\, y)(a_3 \cos \lambda a t + b_3 \sin \lambda a t)$$

From $z(0,y,t)=0$ we find $a_1=0$. From $z(x,0,t)=0$ we find $a_2=0$. From $z_t(x,y,0)=0$ we find $b_3=0$. Thus the solution satisfying these conditions (and the boundedness condition) is

$$z(x,y,t) = B \sin \mu x \sin \sqrt{\lambda^2-\mu^2}\, y \cos \lambda a t$$

From $z(1,y,t)=0$ we see that $\mu = m\pi$, $m=1,2,3,\ldots$. From $z(x,1,t)=0$ we see that $\sqrt{\lambda^2-\mu^2}=n\pi$, $n=1,2,3,\ldots$, i.e. $\lambda=\sqrt{m^2+n^2}\,\pi$.

Thus a solution satisfying all conditions but $z(x,y,0)=f(x,y)$ is given by

$$z(x,y,t) = B \sin m\pi x \sin n\pi y \cos \sqrt{m^2+n^2}\,\pi a t$$

By the superposition theorem we can arrive at the possible solution

$$z(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} \sin m\pi x \sin n\pi y \cos \sqrt{m^2+n^2}\,\pi a t \qquad (2)$$

Then, letting $t=0$ and using $z(x,y,0)=f(x,y)$, we arrive at

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} \sin m\pi x \sin n\pi y$$

from which we are led as in Problem 2.24 to

$$B_{mn} = 4\int_0^1\int_0^1 f(x,y) \sin m\pi x \sin n\pi y \, dx\, dy \qquad (3)$$

Thus the formal solution to our problem is given by (2), where the coefficients B_{mn} are determined from (3).

In this problem the natural modes have frequencies f_{mn} given by $2\pi f_{mn} = \sqrt{m^2+n^2}\,\pi a$, i.e.

$$f_{mn} = \frac{1}{2}\sqrt{m^2+n^2}\sqrt{\frac{\tau}{\mu}} \qquad (4)$$

The lowest mode, $m=0,\ n=1$ or $m=1,\ n=0$, has frequency $\frac{1}{2}\sqrt{\tau/\mu}$. The next higher one has $m=1,\ n=1$ with frequency $\frac{1}{2}\sqrt{2\tau/\mu}$, which is not an integer multiple of the lowest (i.e. fundamental) frequency. Similarly, higher modes do not in general have frequencies which are integer multiples of the fundamental frequency. In such case we do not get music.

Supplementary Problems

FOURIER SERIES

2.34. Graph each of the following functions and find its corresponding Fourier series, using properties of even and odd functions wherever applicable.

(a) $f(x) = \begin{cases} 8 & 0<x<2 \\ -8 & 2<x<4 \end{cases}$ Period 4

(b) $f(x) = \begin{cases} -x & -4 \leqq x \leqq 0 \\ x & 0 \leqq x \leqq 4 \end{cases}$ Period 8

(c) $f(x) = 4x,\ 0<x<10$, Period 10

(d) $f(x) = \begin{cases} 2x & 0 \leqq x \leqq 3 \\ 0 & -3<x<0 \end{cases}$ Period 6

2.35. In each part of Problem 2.34, tell where the discontinuities of $f(x)$ are located and to what value the series converges at these discontinuities.

2.36. Expand $f(x) = \begin{cases} 2-x & 0<x<4 \\ x-6 & 4<x<8 \end{cases}$ in a Fourier series of period 8.

2.37. (*a*) Expand $f(x) = \cos x,\ 0 < x < \pi,$ in a Fourier sine series.

(*b*) How should $f(x)$ be defined at $x = 0$ and $x = \pi$ so that the series will converge to $f(x)$ for $0 \leqq x \leqq \pi$?

2.38. (*a*) Expand in a Fourier series $f(x) = \cos x,\ 0 < x < \pi,$ if the period is π; and (*b*) compare with the result of Problem 2.37, explaining the similarities and differences if any.

2.39. Expand $f(x) = \begin{cases} x & 0 < x < 4 \\ 8 - x & 4 < x < 8 \end{cases}$ in a series of (*a*) sines, (*b*) cosines.

2.40. Prove that for $0 \leqq x \leqq \pi$,

(*a*) $x(\pi - x) = \dfrac{\pi^2}{6} - \left(\dfrac{\cos 2x}{1^2} + \dfrac{\cos 4x}{2^2} + \dfrac{\cos 6x}{3^2} + \cdots\right)$

(*b*) $x(\pi - x) = \dfrac{8}{\pi}\left(\dfrac{\sin x}{1^3} + \dfrac{\sin 3x}{3^3} + \dfrac{\sin 5x}{5^3} + \cdots\right)$

2.41. Use Problem 2.40 to show that

(*a*) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, (*b*) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$, (*c*) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3} = \frac{\pi^3}{32}$.

2.42. Show that $\dfrac{1}{1^3} + \dfrac{1}{3^3} - \dfrac{1}{5^3} - \dfrac{1}{7^3} + \dfrac{1}{9^3} + \dfrac{1}{11^3} - \cdots = \dfrac{3\pi^3\sqrt{2}}{128}$.

INTEGRATION AND DIFFERENTIATION OF FOURIER SERIES

2.43. (*a*) Show that for $-\pi < x < \pi$,

$$x = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right)$$

(*b*) By integrating the result of (*a*), show that for $-\pi \leqq x \leqq \pi$,

$$x^2 = \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right)$$

(*c*) By integrating the result of (*b*), show that for $-\pi \leqq x \leqq \pi$,

$$x(\pi - x)(\pi + x) = 12\left(\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \cdots\right)$$

(*d*) Show that the series on the right in parts (*b*) and (*c*) converge uniformly to the functions on the left.

2.44. (*a*) Show that for $-\pi < x < \pi$,

$$x \cos x = -\frac{1}{2}\sin x + 2\left(\frac{2}{1 \cdot 3}\sin 2x - \frac{3}{2 \cdot 4}\sin 3x + \frac{4}{3 \cdot 5}\sin 4x - \cdots\right)$$

(*b*) Use (*a*) to show that for $-\pi \leqq x \leqq \pi$,

$$x \sin x = 1 - \frac{1}{2}\cos x - 2\left(\frac{\cos 2x}{1 \cdot 3} - \frac{\cos 3x}{2 \cdot 4} + \frac{\cos 4x}{3 \cdot 5} - \cdots\right)$$

2.45. By differentiating the result of Problem 2.40(*b*), prove that for $0 \leqq x \leqq \pi$,

$$x = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$$

PARSEVAL'S IDENTITY

2.46. By using Problem 2.40 and Parseval's identity, show that

(*a*) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$ (*b*) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$

2.47. Show that $\frac{1}{1^2 \cdot 3^2} + \frac{1}{3^2 \cdot 5^2} + \frac{1}{5^2 \cdot 7^2} + \cdots = \frac{\pi^2 - 8}{16}$. [*Hint.* Use Problem 2.11.]

2.48. Show that (*a*) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$, (*b*) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = \frac{\pi^6}{960}$.

2.49. Show that $\frac{1}{1^2 \cdot 2^2 \cdot 3^2} + \frac{1}{2^2 \cdot 3^2 \cdot 4^2} + \frac{1}{3^2 \cdot 4^2 \cdot 5^2} + \cdots = \frac{4\pi^2 - 39}{16}$.

SOLUTIONS USING FOURIER SERIES

2.50. (*a*) Solve the boundary value problem

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} \qquad u(0,t) = u(4,t) = 0 \qquad u(x,0) = 25x$$

where $0 < x < 4$, $t > 0$.

(*b*) Interpret physically the boundary value problem in (*a*).

2.51. (*a*) Show that the solution of the boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad u_x(0,t) = u_x(\pi,t) = 0 \qquad u(x,0) = f(x)$$

where $0 < x < \pi$, $t > 0$, is given by

$$u(x,t) = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx + \frac{2}{\pi}\sum_{m=1}^{\infty} e^{-m^2 t}\cos mx \int_0^{\pi} f(x)\cos mx\,dx$$

(*b*) Interpret physically the boundary value problem in (*a*).

2.52. Find the steady-state temperature in a bar whose ends are located at $x = 0$ and $x = 10$, if these ends are kept at 150°C and 100°C respectively.

2.53. A circular plate of unit radius (see Fig. 2-14, page 39) whose faces are insulated has its boundary kept at temperature $120 + 60\cos 2\phi$. Find the steady-state temperature of the plate.

2.54. Show that $\rho\sin\phi + \frac{1}{3}\rho^3\sin 3\phi + \frac{1}{5}\rho^5\sin 5\phi + \cdots = \frac{1}{2}\tan^{-1}\left(\frac{2\rho\sin\phi}{1-\rho^2}\right)$

and thus complete Problem 2.28.

2.55. A string 2 ft long is stretched between two fixed points $x = 0$ and $x = 2$. If the displacement of the string from the x-axis at $t = 0$ is given by $f(x) = 0.03\,x(2-x)$ and if the initial velocity is zero, find the displacement at any later time.

2.56. A square plate of side a has one side maintained at temperature $f(x)$ and the others at zero, as indicated in Fig. 2-25. Show that the steady-state temperature at any point of the plate is given by

$$u(x,y) = \sum_{k=1}^{\infty}\left[\frac{2}{a\sinh(k\pi)}\int_0^a f(x)\sin\frac{k\pi x}{a}\,dx\right]\sin\frac{k\pi x}{a}\sinh\frac{k\pi y}{a}$$

2.57. Work Problem 2.56 if the sides are maintained at temperatures $f_1(x), g_1(y), f_2(x), g_2(y)$, respectively. [*Hint.* Use the principle of superposition and the result of Problem 2.56.]

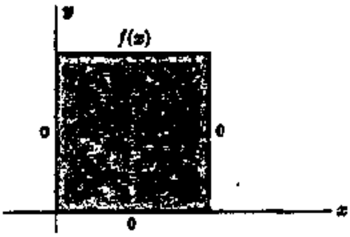

Fig. 2-25

Fig. 2-26

2.58. An infinitely long plate of width a (indicated by the shaded region of Fig. 2-25) has its two parallel sides maintained at temperature 0 and its other side at constant temperature u_0. (a) Show that the steady-state temperature is given by

$$u(x,y) = \frac{4u_0}{\pi}\left(e^{-y}\sin\frac{\pi x}{a} + \frac{1}{3}e^{-3y}\sin\frac{3\pi x}{a} + \frac{1}{5}e^{-5y}\sin\frac{5\pi x}{a} + \cdots\right)$$

(b) Use Problem 2.54 to show that

$$u(x,y) = \frac{2u_0}{\pi}\tan^{-1}\left[\frac{\sin(\pi x/a)}{\sinh y}\right]$$

2.59. Solve Problem 1.26 if the string has its ends fixed at $x = 0$ and $x = L$ and if its initial displacement and velocity are given by $f(x)$ and $g(x)$ respectively.

2.60. A square plate (Fig. 2-27) having sides of unit length has its edges fixed in the xy-plane and is set into transverse vibration.

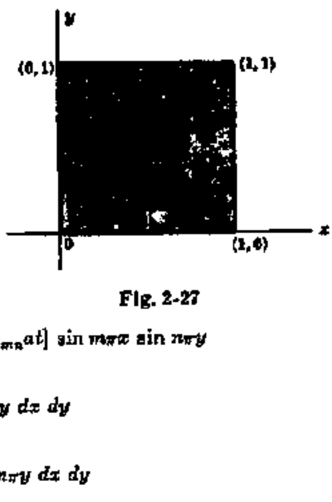

Fig. 2-27

(a) Show that the transverse displacement $z(x,y,t)$ of any point (x,y) at time t is given by

$$\frac{\partial^2 z}{\partial t^2} = a^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$$

where a^2 is a constant.

(b) Show that if the plate is given an initial shape $f(x,y)$ and released with velocity $g(x,y)$, then the displacement is given by

$$z(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[A_{mn}\cos\lambda_{mn}at + B_{mn}\sin\lambda_{mn}at\right]\sin m\pi x\sin n\pi y$$

where

$$A_{mn} = 4\int_0^1\int_0^1 f(x,y)\sin m\pi x\sin n\pi y\,dx\,dy$$

$$B_{mn} = \frac{4}{a\lambda_{mn}}\int_0^1\int_0^1 g(x,y)\sin m\pi x\sin n\pi y\,dx\,dy$$

and $\lambda_{mn} = \pi\sqrt{m^2+n^2}$.

2.61. Work Problem 2.60 for a rectangular plate of sides b and c.

2.62. Prove that the result for $u(x,t)$ obtained in Problem 2.25 actually satisfies the partial differential equation and the boundary conditions.

2.63. Solve the boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \alpha^2 u \qquad 0 < x < L,\ t > 0$$

$$u(0,t) = u_1, \quad u(L,t) = u_2, \quad u(x,0) = 0$$

where α and L are constants, and interpret physically.

2.64. Work Problem 2.63 if $u(x,0) = f(x)$.

2.65. Solve and interpret physically the boundary value problem

$$\frac{\partial^2 y}{\partial t^2} + b^2 \frac{\partial^4 y}{\partial x^4} = 0$$

where $y(0,t) = 0$, $y(L,t) = 0$, $y(x,0) = f(x)$, $y_t(x,0) = 0$, $y_{xx}(0,t) = 0$, $y_{xx}(L,t) = 0$, $|y(x,t)| < M$.

2.66. Work Problem 2.65 if $y_t(x,0) = g(x)$.

2.67. A plate is bounded by two concentric circles of radius a and b, as shown in Fig. 2-28. The faces are insulated and the boundaries are kept at temperatures $f(\theta)$ and $g(\theta)$ respectively. Show that the steady-state temperature at any point (r,θ) is given by

$$u(r,\theta) = A_0 + B_0 \ln r + \sum_{n=1}^{\infty} \left\{ \left(A_n r^n + \frac{B_n}{r^n} \right) \cos n\theta + \left(C_n r^n + \frac{D_n}{r^n} \right) \sin n\theta \right\}$$

where A_0 and B_0 are determined from

$$A_0 + B_0 \ln a = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta$$

$$A_0 + B_0 \ln b = \frac{1}{2\pi} \int_0^{2\pi} g(\theta)\, d\theta$$

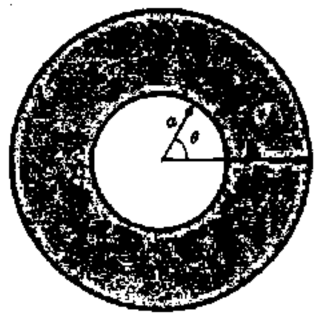
Fig. 2-28

A_n, B_n are determined from

$$A_n a^n + B_n a^{-n} = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta, \qquad A_n b^n + B_n b^{-n} = \frac{1}{\pi} \int_0^{2\pi} g(\theta) \cos n\theta \, d\theta$$

and C_n, D_n are determined from

$$C_n a^n + D_n a^{-n} = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta, \qquad C_n b^n + D_n b^{-n} = \frac{1}{\pi} \int_0^{2\pi} g(\theta) \sin n\theta \, d\theta$$

2.68. Investigate the limiting cases of Problem 2.67 as (a) $a \to 0$, (b) $b \to \infty$, and give physical interpretations.

2.69. (a) Solve the boundary value problem

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} + \beta e^{-\gamma x}$$

where $u(0,t) = 0$, $u(L,t) = 0$, $u(x,0) = f(x)$, $|u(x,t)| < M$, and (b) give a physical interpretation.

2.70. Work Problem 2.69 if $\beta e^{-\gamma x}$ is replaced by $u_0 \sin \alpha x$, where u_0 and α are constants.

2.71. Solve $\dfrac{\partial^2 y}{\partial t^2} = a^2 \dfrac{\partial^2 y}{\partial x^2} - g$ where $y(0,t) = 0$, $y(L,t) = 0$, $y(x,0) = f(x)$, $y_t(x,0) = 0$, $|y(x,t)| < M$, and give a physical interpretation.

2.72. Find the steady-state temperature in a solid cube of unit side (Fig. 2-29) if the face in the xy-plane is kept at the prescribed temperature $F(x,y)$, while all other faces are kept at temperature zero.

2.73. How would you solve Problem 2.72 if temperatures were prescribed on the other faces also?

2.74. How would you solve Problem 2.72 if the initial temperature inside the cube was given and you wished to find the temperature inside the cube at any later time?

2.75. Generalize the result of Problem 2.72 to any rectangular parallelepiped.

2.76. A plate in the form of a sector of a circle of radius a has central angle β, as shown in Fig. 2-30. If the circular part is maintained at a temperature $f(\theta)$, $0 < \theta < \beta$, while the bounding radii are maintained at temperature zero, find the steady-state temperature everywhere in the sector.

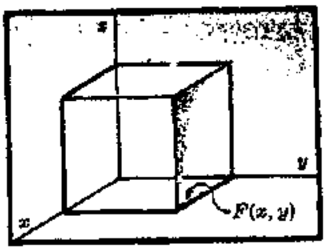

Fig. 2-29

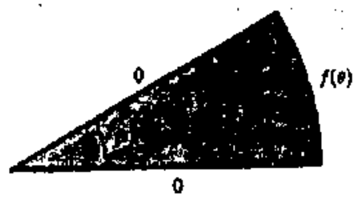

Fig. 2-30

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Chapter 3

Orthogonal Functions

DEFINITIONS INVOLVING ORTHOGONAL FUNCTIONS. ORTHONORMAL SETS

Many properties of Fourier series considered in Chapter 2 depended on such results as

$$\int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx \;=\; 0, \qquad \int_0^L \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx \;=\; 0 \qquad (m \neq n) \tag{1}$$

In this chapter we shall seek to generalize some ideas of Chapter 2. To do this we first recall some elementary properties of *vectors.*

Two vectors $\mathbf{A}$ and $\mathbf{B}$ are called *orthogonal* (perpendicular) if $\mathbf{A}\cdot\mathbf{B} = 0$ or $A_1B_1 + A_2B_2 + A_3B_3 = 0$, where $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$. Although not geometrically or physically obvious, these ideas can be generalized to include vectors with more than three components. In particular we can think of a function, say $A(x)$, as being a vector with an *infinity of components* (i.e. an *infinite-dimensional vector*), the value of each component being specified by substituting a particular value of x taken from some interval (a, b). It is natural in such case to define two functions, $A(x)$ and $B(x)$, as *orthogonal* in (a, b) if

$$\int_a^b A(x)\,B(x)\,dx \;=\; 0 \tag{2}$$

The left side of *(2)* is often called the *scalar product* of $A(x)$ and $B(x)$.

A vector $\mathbf{A}$ is called a *unit vector* or *normalized vector* if its magnitude is unity, i.e. if $\mathbf{A}\cdot\mathbf{A} = A^2 = 1$. Extending the concept, we say that the function $A(x)$ is *normal* or *normalized* in (a, b) if

$$\int_a^b \{A(x)\}^2\,dx \;=\; 1 \tag{3}$$

From the above it is clear that we can consider a set of functions $\{\phi_k(x)\}$, $k = 1, 2, 3, \ldots$, having the properties

$$\int_a^b \phi_m(x)\,\phi_n(x)\,dx \;=\; 0 \qquad m \neq n \tag{4}$$

$$\int_a^b \{\phi_m(x)\}^2\,dx \;=\; 1 \qquad m = 1, 2, 3, \ldots \tag{5}$$

Each member of the set is orthogonal to every other member of the set and is also normalized. We call such a set of functions an *orthonormal set* in (a, b).

The equations *(4)* and *(5)* can be summarized by writing

$$\int_a^b \phi_m(x)\,\phi_n(x)\,dx \;=\; \delta_{mn} \tag{6}$$

where δ_{mn}, called *Kronecker's symbol*, is defined as 0 if $m \neq n$ and 1 if $m = n$.

Example 1.

The set of functions

$$\phi_m(x) = \sqrt{\frac{2}{\pi}} \sin mx \qquad m = 1, 2, 3, \ldots$$

is an orthonormal set in the interval $0 \leqq x \leqq \pi$.

ORTHOGONALITY WITH RESPECT TO A WEIGHT FUNCTION

If

$$\int_a^b \psi_m(x)\,\psi_n(x)\,w(x)\,dx = \delta_{mn} \tag{7}$$

where $w(x) \geqq 0$, we often say that the set $\{\psi_k(x)\}$ is orthonormal with respect to the *density function* or *weight function* $w(x)$. In such case the set $\phi_m(x) = \sqrt{w(x)}\,\psi_m(x)$, $m = 1, 2, 3, \ldots$, is an orthonormal set in (a, b).

EXPANSION OF FUNCTIONS IN ORTHONORMAL SERIES

Just as any vector $\mathbf{r}$ in 3 dimensions can be expanded in a set of mutually orthogonal unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in the form $\mathbf{r} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, so we consider the possibility of expanding a function $f(x)$ in a set of orthonormal functions, i.e.

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x) \qquad a \leqq x \leqq b \tag{8}$$

Such series, called *orthonormal series*, are generalizations of Fourier series and are of great interest and utility both from theoretical and applied viewpoints.

Assuming that the series on the right of *(8)* converges to $f(x)$, we can formally multiply both sides by $\phi_m(x)$ and integrate both sides from a to b to obtain

$$c_m = \int_a^b f(x)\,\phi_m(x)\,dx \tag{9}$$

which are called the *generalized Fourier coefficients*. As in the case of Fourier series, an investigation should be made to determine whether the series on the right of *(8)* with coefficients *(9)* actually converges to $f(x)$. In practice, if $f(x)$ and $f'(x)$ are piecewise continuous in (a, b), then the series on the right of *(8)* with coefficients given by *(9)* converges to $\frac{1}{2}[f(x+0) + f(x-0)]$ as in the case of Fourier series.

APPROXIMATIONS IN THE LEAST-SQUARES SENSE

Let $f(x)$ and $f'(x)$ be piecewise continuous in (a, b). Let $\phi_m(x)$, $m = 1, 2, \ldots$, be orthonormal in (a, b). Suppose now that we consider the finite sum

$$S_M(x) = \sum_{n=1}^{M} \alpha_n \phi_n(x) \tag{10}$$

as an approximation to $f(x)$, where α_n, $n = 1, 2, 3, \ldots$, are constants presently unknown. Then the *mean square error* of this approximation is given by

$$\text{Mean square error} = \frac{\int_a^b [f(x) - S_M(x)]^2\,dx}{b-a} \tag{11}$$

and the *root mean square error* E_{rms} is given by the square root of *(11)*, i.e.

$$E_{rms} = \sqrt{\frac{1}{b-a}\int_a^b [f(x) - S_M(x)]^2\,dx} \tag{12}$$

We now seek to determine the constants α_n which will produce the *least* root mean square error. The result is supplied in the following theorem which is proved in Problem 3.5.

Theorem 3-1: The root mean square error *(12)* is least (i.e. a minimum) when the coefficients are equal to the generalized Fourier coefficients *(9)*, i.e. when

$$\alpha_n = c_n = \int_a^b f(x)\,\phi_n(x)\,dx \tag{13}$$

We often say that $S_M(x)$ with coefficients c_n is an *approximation to $f(x)$ in the least-squares sense* or a *least-squares approximation to $f(x)$*.

It is of interest to note that once we have worked out an approximation to $f(x)$ in the least-squares sense by using the coefficients c_n, we do not have to recompute these coefficients if we wish to have a better approximation. This is sometimes referred to as the *principle of finality*.

PARSEVAL'S IDENTITY FOR ORTHONORMAL SERIES. COMPLETENESS

For the case where $\alpha_n = c_n$ we can show (see Problem 3.5) that the root mean square error is given by

$$E_{\text{rms}} = \frac{1}{\sqrt{b-a}}\left[\int_a^b [f(x)]^2\,dx - \sum_{n=1}^{M} c_n^2\right]^{1/2} \tag{14}$$

It is seen that E_{rms} depends on M. As $M \to \infty$ we would expect that $E_{\text{rms}} \to 0$, in which case we would have

$$\int_a^b [f(x)]^2\,dx = \sum_{n=1}^{\infty} c_n^2 \tag{15}$$

Now, *(15)* could certainly not be true if, for example, we left out certain functions $\phi_n(x)$ in the series approximation, i.e. if the set of functions were incomplete. We are therefore led to define a set of functions $\phi_n(x)$ to be *complete* if and only if $E_{\text{rms}} \to 0$ as $M \to \infty$, so that *(15)* is valid. We refer to *(15)* as *Parseval's identity for orthonormal series of functions*. In *(6)* of Chapter 2, page 23, we have obtained Parseval's identity for the special case of Fourier series.

In the case where $E_{\text{rms}} \to 0$ as $M \to \infty$, i.e.

$$\lim_{M\to\infty} \int_a^b [f(x) - S_M(x)]^2\,dx = 0 \tag{16}$$

we sometimes write

$$\underset{M\to\infty}{\text{l.i.m.}}\; S_M(x) = f(x) \tag{17}$$

This is read *the limit in mean of $S_M(x)$ as $M \to \infty$ equals $f(x)$* or *$S_M(x)$ converges in the mean to $f(x)$ as $M \to \infty$* and is equivalent to *(16)*.

STURM-LIOUVILLE SYSTEMS. EIGENVALUES AND EIGENFUNCTIONS

A boundary value problem having the form

$$\left.\begin{aligned} &\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]y = 0 \qquad a \le x \le b \\ &\alpha_1 y(a) + \alpha_2 y'(a) = 0, \qquad \beta_1 y(b) + \beta_2 y'(b) = 0 \end{aligned}\right\} \tag{18}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are given constants; $p(x), q(x), r(x)$ are given functions which we shall assume to be differentiable and λ is an unspecified parameter independent of x, is called a *Sturm-Liouville boundary value problem* or *Sturm-Liouville system*. Such systems arise in practice on using the separation of variables method in solution of partial differential equations. In such case λ is the "separation constant." See Problem 3.14.

A nontrivial solution of this system, i.e. one which is not identically zero, exists in general only for a particular set of values of the parameter λ. These values are called the *characteristic values*, or more often *eigenvalues*, of the system. The corresponding solutions are called *characteristic functions* or *eigenfunctions* of the system. In general to each eigenvalue there is one eigenfunction, although exceptions can occur.

If $p(x)$ and $q(x)$ are real, then the eigenvalues are real. Also, the eigenfunctions form an orthogonal set with respect to the weight function $r(x)$, which is generally taken as nonnegative, i.e. $r(x) \geqq 0$. It follows that by suitable normalization the set of functions can be made an orthonormal set with respect to $r(x)$ in $a \leqq x \leqq b$. See Problems 3.8–3.11.

THE GRAM-SCHMIDT ORTHONORMALIZATION PROCESS

Given a finite or infinite set of linearly independent functions $\psi_1(x), \psi_2(x), \psi_3(x), \ldots$ defined in an interval (a, b) it is possible to generate from these functions a set of orthonormal functions in (a, b). To do this we first consider a new set of functions obtained from the $\psi_k(x)$ and given by

$$c_{11}\psi_1(x), \quad c_{21}\psi_1(x) + c_{22}\psi_2(x), \quad c_{31}\psi_1(x) + c_{32}\psi_2(x) + c_{33}\psi_3(x), \quad \ldots \tag{19}$$

where the c's are constants to be determined. We shall designate the functions in (*19*) by $\phi_1(x), \phi_2(x), \phi_3(x), \ldots$.

We now choose the constants $c_{11}, c_{21}, c_{22}, \ldots$ so that the functions $\phi_1(x), \phi_2(x), \phi_3(x), \ldots$ are mutually orthogonal and also normalized in (a, b). The process, known as the *Gram-Schmidt orthonormalization process*, is illustrated in Problem 3.12.

An extension to the case where orthonormalization is with respect to a given weight function is easily made.

APPLICATIONS TO BOUNDARY VALUE PROBLEMS

In the course of solving boundary value problems using separation of variables we often arrive at Sturm-Liouville differential equations (see Problem 3.15, for example). The parameter λ in these equations is the separation constant, and the values of λ which are obtained represent the real eigenvalues. The solution of the boundary value problem is then obtained in terms of the corresponding mutually orthogonal eigenfunctions.

For an illustration which does not involve Fourier series, see Problem 3.13. Other illustrations involving this general procedure will be given in later chapters.

Solved Problems

ORTHOGONAL FUNCTIONS AND ORTHONORMAL SERIES

3.1. (*a*) Show that the set of functions

$$1, \quad \sin\frac{\pi x}{L}, \quad \cos\frac{\pi x}{L}, \quad \sin\frac{2\pi x}{L}, \quad \cos\frac{2\pi x}{L}, \quad \sin\frac{3\pi x}{L}, \quad \cos\frac{3\pi x}{L}, \quad \ldots$$

form an orthogonal set in the interval $(-L, L)$.

(*b*) Determine the corresponding normalizing constants for the set in (*a*) so that the set is orthonormal in $(-L, L)$.

(*a*) This follows at once from the results of Problems 2.2 and 2.3, page 26.

(*b*) By Problem 2.3,

$$\int_{-L}^{L} \sin^2\frac{m\pi x}{L}\,dx = L, \qquad \int_{-L}^{L} \cos^2\frac{m\pi x}{L}\,dx = L$$

Then

$$\int_{-L}^{L} \left(\sqrt{\frac{1}{L}}\sin\frac{m\pi x}{L}\right)^2 dx = 1, \qquad \int_{-L}^{L} \left(\sqrt{\frac{1}{L}}\cos\frac{m\pi x}{L}\right)^2 dx = 1$$

Also,

$$\int_{-L}^{L} (1)^2\,dx = 2L \quad \text{or} \quad \int_{-L}^{L} \left(\frac{1}{\sqrt{2L}}\right)^2 dx = 1$$

Thus the required orthonormal set is given by

$$\frac{1}{\sqrt{2L}}, \quad \frac{1}{\sqrt{L}}\sin\frac{\pi x}{L}, \quad \frac{1}{\sqrt{L}}\cos\frac{\pi x}{L}, \quad \frac{1}{\sqrt{L}}\sin\frac{2\pi x}{L}, \quad \frac{1}{\sqrt{L}}\cos\frac{2\pi x}{L}, \quad \ldots$$

3.2. Let $\{\phi_n(x)\}$ be a set of functions which are mutually orthonormal in (a, b). Prove that if $\sum_{n=1}^{\infty} c_n \phi_n(x)$ converges uniformly to $f(x)$ in (a, b), then

$$c_n = \int_a^b f(x)\,\phi_n(x)\,dx$$

Multiplying both sides of

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x) \tag{1}$$

by $\phi_m(x)$ and integrating from a to b, we have

$$\int_a^b f(x)\,\phi_m(x)\,dx = \sum_{n=1}^{\infty} c_n \int_a^b \phi_m(x)\,\phi_n(x)\,dx \tag{2}$$

where the interchange of integration and summation is justified by the fact that the series converges uniformly to $f(x)$. Now since the functions $\{\phi_n(x)\}$ are mutually orthonormal in (a, b), we have

$$\int_a^b \phi_m(x)\,\phi_n(x)\,dx = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

so that (*2*) becomes

$$\int_a^b f(x)\,\phi_m(x)\,dx = c_m \tag{3}$$

as required.

We call the coefficients c_m given by (*3*) the *generalized Fourier coefficients* corresponding to $f(x)$ even though nothing may be known about the convergence of the series in (*1*). As in the case of Fourier series, convergence of $\sum_{n=1}^{\infty} c_n \phi_n(x)$ is then investigated using the coefficients (*3*). The conditions of convergence depend of course on the types of orthonormal functions used. In the remainder of this book we shall be concerned with many examples of orthonormal functions and series.

LEAST-SQUARES APPROXIMATIONS. PARSEVAL'S IDENTITY AND COMPLETENESS

3.3. If $S_M(x) = \sum_{n=1}^{M} \alpha_n \phi_n(x)$, where $\phi_n(x)$, $n = 1, 2, \ldots$, is orthonormal in (a, b), prove that

$$\int_a^b [f(x) - S_M(x)]^2 \, dx = \int_a^b [f(x)]^2 \, dx - 2 \sum_{n=1}^{M} \alpha_n c_n + \sum_{n=1}^{M} \alpha_n^2$$

where $c_n = \int_a^b f(x)\, \phi_n(x) \, dx$ are the generalized Fourier coefficients corresponding to $f(x)$.

We have

$$f(x) - S_M(x) = f(x) - \sum_{n=1}^{M} \alpha_n \phi_n(x)$$

By squaring we obtain

$$[f(x) - S_M(x)]^2 = [f(x)]^2 - 2 \sum_{n=1}^{M} \alpha_n f(x)\, \phi_n(x) + \sum_{m=1}^{M} \sum_{n=1}^{M} \alpha_m \alpha_n \phi_m(x)\, \phi_n(x)$$

Integrating both sides from a to b using

$$c_n = \int_a^b f(x)\, \phi_n(x) \, dx, \qquad \int_a^b \phi_m(x)\, \phi_n(x) \, dx = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

we obtain

$$\int_a^b [f(x) - S_M(x)]^2 \, dx = \int_a^b [f(x)]^2 \, dx - 2 \sum_{n=1}^{M} \alpha_n c_n + \sum_{n=1}^{M} \alpha_n^2$$

We have assumed that $f(x)$ is such that all the above integrals exist.

3.4. Show that

$$\int_a^b [f(x) - S_M(x)]^2 \, dx = \int_a^b [f(x)]^2 \, dx + \sum_{n=1}^{M} (\alpha_n - c_n)^2 - \sum_{n=1}^{M} c_n^2$$

This follows from Problem 3.3 by noting that

$$\begin{aligned} \int_a^b [f(x)]^2 \, dx - 2 \sum_{n=1}^{M} \alpha_n c_n + \sum_{n=1}^{M} \alpha_n^2 &= \int_a^b [f(x)]^2 \, dx + \sum_{n=1}^{M} (\alpha_n^2 - 2\alpha_n c_n) \\ &= \int_a^b [f(x)]^2 \, dx + \sum_{n=1}^{M} [(\alpha_n - c_n)^2 - c_n^2] \\ &= \int_a^b [f(x)]^2 \, dx + \sum_{n=1}^{M} (\alpha_n - c_n)^2 - \sum_{n=1}^{M} c_n^2 \end{aligned}$$

3.5. (*a*) Prove Theorem 3.1, page 54: The root mean square error is a minimum when the coefficients α_n equal the Fourier coefficients c_n.

(*b*) What is the value of the root mean square error in this case?

(*a*) From Problem 3.4 we have

$$\int_a^b [f(x) - S_M(x)]^2 \, dx = \int_a^b [f(x)]^2 \, dx + \sum_{n=1}^{M} (\alpha_n - c_n)^2 - \sum_{n=1}^{M} c_n^2$$

Now the root mean square error will be a minimum when the above is a minimum. However, it is clear that the right-hand side is a minimum when $\sum_{n=1}^{M} (\alpha_n - c_n)^2 = 0$, i.e. when $\alpha_n = c_n$ for all n.

(b) From part (a) we see that the minimum value of the root mean square error is given by

$$E_{rms} = \left[\frac{1}{b-a}\int_a^b [f(x)-S_M(x)]^2\,dx\right]^{1/2}$$

$$= \frac{1}{\sqrt{b-a}}\left[\int_a^b [f(x)]^2\,dx - \sum_{n=1}^{\infty} c_n^2\right]^{1/2}$$

3.6. Prove that if c_n, $n = 1, 2, 3, \ldots$, denote the generalized Fourier coefficients corresponding to $f(x)$, then

$$\sum_{n=1}^{\infty} c_n^2 \leqq \int_a^b [f(x)]^2\,dx$$

From Problem 3.5 we see that, since the root mean square error must be nonnegative,

$$\sum_{n=1}^{M} c_n^2 \leqq \int_a^b [f(x)]^2\,dx \tag{1}$$

Then, taking the limit as $M \to \infty$ and noting that the right side does not depend on M, it follows that

$$\sum_{n=1}^{\infty} c_n^2 \leqq \int_a^b [f(x)]^2\,dx \tag{2}$$

This inequality is often called *Bessel's inequality*.

As a consequence of (2) we see that if the right side of (2) exists, then the series on the left must converge. In the special case where the equality holds in (2) we obtain *Parseval's identity*.

3.7. Show that $\lim_{n\to\infty} \int_a^b f(x)\,\phi_n(x)\,dx = 0$.

By definition we have $c_n = \int_a^b f(x)\,\phi_n(x)\,dx$. But since $\sum_{n=1}^{\infty} c_n^2$ converges by Problem 3.6, the nth term c_n^2, and with it c_n, must approach zero as $n \to \infty$, which is the required result. Note that this result for the special case of Fourier series is *Riemann's theorem* (see Problem 2.19, page 35).

STURM-LIOUVILLE SYSTEMS. EIGENVALUES AND EIGENFUNCTIONS

3.8. (a) Verify that the system $y'' + \lambda y = 0$, $y(0) = 0$, $y(1) = 0$ is a Sturm-Liouville system. (b) Find the eigenvalues and eigenfunctions of the system. (c) Prove that the eigenfunctions are orthogonal in $(0, 1)$. (d) Find the corresponding set of normalized eigenfunctions. (e) Expand $f(x) = 1$ in a series of these orthonormal functions.

(a) The system is a special case of (18), page 54, with $p(x) = 1$, $q(x) = 0$, $r(x) = 1$, $a = 0$, $b = 1$, $\alpha_1 = 1$, $\alpha_2 = 0$, $\beta_1 = 1$, $\beta_2 = 0$ and thus is a Sturm-Liouville system.

(b) The general solution of $y'' + \lambda y = 0$ is $y = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$. From the boundary condition $y(0) = 0$ we have $A = 0$, i.e. $y = B\sin\sqrt{\lambda}\,x$. From the boundary condition $y(1) = 0$ we have $B\sin\sqrt{\lambda} = 0$; since B cannot be zero (otherwise the solution will be identically zero, i.e. trivial), we must have $\sin\sqrt{\lambda} = 0$. Then $\sqrt{\lambda} = m\pi$, $\lambda = m^2\pi^2$, where $m = 1, 2, 3, \ldots$ are the required eigenvalues.

The eigenfunctions belonging to the eigenvalues $\lambda = m^2\pi^2$ can be designated by $B_m \sin m\pi x$, $m = 1, 2, 3, \ldots$. Note that we exclude the value $m = 0$ or $\lambda = 0$ as an eigenvalue, since the corresponding eigenfunction is zero.

(c) The eigenfunctions are orthogonal since

$$\int_0^1 (B_m \sin m\pi x)(B_n \sin n\pi x)\,dx = B_m B_n \int_0^1 \sin m\pi x \sin n\pi x\,dx$$

$$= \frac{B_m B_n}{2}\int_0^1 [\cos(m-n)\pi x - \cos(m+n)\pi x]\,dx$$

$$= \frac{B_m B_n}{2}\left[\frac{\sin(m-n)\pi x}{(m-n)\pi} - \frac{\sin(m+n)\pi x}{(m+n)\pi}\right]\Bigg|_0^1 = 0, \quad m \neq n$$

(*d*) The eigenfunctions will be orthonormal if

$$\int_0^1 (B_m \sin m\pi x)^2\,dx = 1$$

i.e. if $B_m^2 \int_0^1 \sin^2 m\pi x\,dx = \frac{B_m^2}{2}\int_0^1 (1-\cos 2m\pi x)\,dx = \frac{B_m^2}{2} = 1$, or $B_m = \sqrt{2}$, taking the positive square root. Thus the set $\sqrt{2}\sin m\pi x$, $m = 1, 2, \ldots$, is an orthonormal set.

(*e*) We must find constants $c_1, c_2, \ldots$ such that

$$f(x) = \sum_{m=1}^{\infty} c_m \phi_m(x)$$

where $f(x) = 1$, $\phi_m(x) = \sqrt{2}\sin m\pi x$. By the methods of Fourier series,

$$c_m = \int_0^1 f(x)\,\phi_m(x)\,dx = \sqrt{2}\int_0^1 \sin m\pi x\,dx = \frac{\sqrt{2}\,(1-\cos m\pi)}{m\pi}$$

Then the required series [Fourier series] is, assuming $0 < x < 1$,

$$1 = \sum_{m=1}^{\infty} \frac{2(1-\cos m\pi)}{m\pi}\sin m\pi x$$

3.9. Show that the eigenvalues of a Sturm-Liouville system are real.

We have
$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x)+\lambda r(x)]y = 0 \qquad (1)$$

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \qquad \beta_1 y(b) + \beta_2 y'(b) = 0 \qquad (2)$$

Then assuming $p(x), q(x), r(x), \alpha_1, \alpha_2, \beta_1, \beta_2$ are real, while λ and y may be complex, we have on taking the complex conjugate (represented by using a bar, as in $\bar{y}, \bar{\lambda}$):

$$\frac{d}{dx}\left[p(x)\frac{d\bar{y}}{dx}\right] + [q(x)+\bar{\lambda}r(x)]\bar{y} = 0 \qquad (3)$$

$$\alpha_1 \bar{y}(a) + \alpha_2 \bar{y}'(a) = 0, \qquad \beta_1 \bar{y}(b) + \beta_2 \bar{y}'(b) = 0 \qquad (4)$$

Multiplying equation (*1*) by $\bar{y}$, (*3*) by y and subtracting, we find after simplifying,

$$\frac{d}{dx}\left[p(x)(y\bar{y}' - \bar{y}y')\right] = (\lambda - \bar{\lambda})r(x)y\bar{y}$$

Then integrating from a to b, we have

$$(\lambda-\bar{\lambda})\int_a^b r(x)|y|^2\,dx = p(x)(y\bar{y}' - \bar{y}y')\Big|_a^b = 0 \qquad (5)$$

on using the conditions (*2*) and (*4*). Since $r(x) \geqq 0$ and is not identically zero in (a, b), the integral on the left of (*5*) is positive and so $\lambda - \bar{\lambda} = 0$ or $\lambda = \bar{\lambda}$, so that λ is real.

3.10. Show that the eigenfunctions belonging to two different eigenvalues are orthogonal with respect to $r(x)$ in (a, b).

If y_1 and y_2 are eigenfunctions belonging to the eigenvalues λ_1 and λ_2 respectively,

$$\frac{d}{dx}\left[p(x)\frac{dy_1}{dx}\right] + [q(x)+\lambda_1 r(x)]y_1 = 0 \qquad (1)$$

$$\alpha_1 y_1(a) + \alpha_2 y_1'(a) = 0, \qquad \beta_1 y_1(b) + \beta_2 y_1'(b) = 0 \qquad (2)$$

$$\frac{d}{dx}\left[p(x)\frac{dy_2}{dx}\right] + [q(x)+\lambda_2 r(x)]y_2 = 0 \qquad (3)$$

$$\alpha_1 y_2(a) + \alpha_2 y_2'(a) = 0, \qquad \beta_1 y_2(b) + \beta_2 y_2'(b) = 0 \qquad (4)$$

Then multiplying (1) by y_2, (2) by y_1 and subtracting, we find as in Problem 3.9,

$$\frac{d}{dx}[p(x)(y_1y_2' - y_2y_1')] \;=\; (\lambda_1 - \lambda_2)r(x)y_1y_2$$

Integrating from a to b, we have on using (3) and (4),

$$(\lambda_1 - \lambda_2)\int_a^b r(x)y_1y_2\,dx \;=\; p(x)(y_1y_2' - y_2y_1')\Big|_a^b \;=\; 0$$

and since $\lambda_1 \neq \lambda_2$ we have the required result

$$\int_a^b r(x)y_1y_2\,dx \;=\; 0$$

3.11. **Given the Sturm-Liouville system $y'' + \lambda y = 0$, $y(0) = 0$, $y'(L) + \beta y(L) = 0$, where β and L are given constants. Find (a) the eigenvalues and (b) the normalized eigenfunctions of the system. (c) Expand $f(x)$, $0 < x < L$, in a series of these normalized eigenfunctions.**

(a) The general solution of $y'' + \lambda y = 0$ is

$$y \;=\; A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$$

Then from the condition $y(0) = 0$ we find $A = 0$, so that

$$y \;=\; B\sin\sqrt{\lambda}\,x$$

The condition $y'(L) + \beta y(L) = 0$ gives

$$B\sqrt{\lambda}\cos\sqrt{\lambda}\,L + \beta B\sin\sqrt{\lambda}\,L \;=\; 0 \qquad \text{or} \qquad \tan\sqrt{\lambda}\,L \;=\; -\frac{\sqrt{\lambda}}{\beta} \tag{1}$$

which is the equation for determining the eigenvalues λ. This equation cannot be solved exactly; however we can obtain approximate values graphically. To do this we let $v = \sqrt{\lambda}\,L$ so that the equation becomes

$$\tan v \;=\; -\frac{v}{\beta L} \tag{2}$$

The values of v, and from these the values of λ, can be obtained from the intersection points $v_1, v_2, v_3, \ldots$ of the graphs of $w = \tan v$ and $w = -v/\beta L$, as indicated in Fig. 3-1. In construction of these we have assumed that β and L are positive. We also note that we need only find the positive roots of (1).

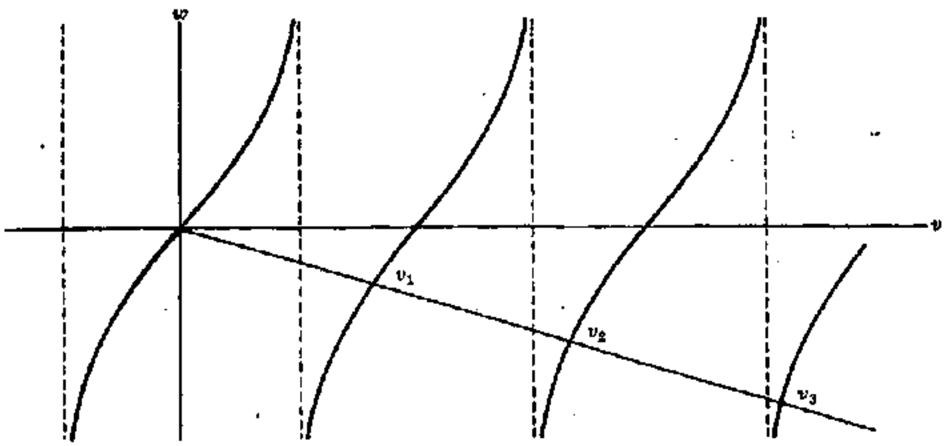


Fig. 3-1

(b) The eigenfunctions are given by

$$\phi_n(x) = B_n \sin\sqrt{\lambda_n}\,x \tag{3}$$

where λ_n, $n = 1, 2, 3, \ldots$, represent the eigenvalues obtained in part (a). To normalize these we require

$$\int_0^L B_n^2 \sin^2\sqrt{\lambda_n}\,x\,dx = 1$$

i.e.

$$\frac{B_n^2}{2}\int_0^L (1 - \cos 2\sqrt{\lambda_n}\,x)\,dx = 1$$

or

$$B_n^2 = \frac{4\sqrt{\lambda_n}}{2\sqrt{\lambda_n}\,L - \sin 2\sqrt{\lambda_n}\,L} \tag{4}$$

Thus a set of normalized eigenfunctions is given by

$$\phi_n(x) = \sqrt{\frac{4\sqrt{\lambda_n}}{2\sqrt{\lambda_n}\,L - \sin 2\sqrt{\lambda_n}\,L}}\,\sin\sqrt{\lambda_n}\,x \qquad n = 1, 2, \ldots \tag{5}$$

(c) If $f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$, then

$$c_n = \int_0^L f(x)\,\phi_n(x)\,dx = \sqrt{\frac{4\sqrt{\lambda_n}}{2\sqrt{\lambda_n}\,L - \sin 2\sqrt{\lambda_n}\,L}}\int_0^L f(x)\sin\sqrt{\lambda_n}\,x\,dx \tag{6}$$

Thus the required expansion is that with coefficients given by (6). The expansion for $f(x)$ can equivalently be written as

$$f(x) = \sum_{n=1}^{\infty}\frac{4\sqrt{\lambda_n}}{2\sqrt{\lambda_n}\,L - \sin 2\sqrt{\lambda_n}\,L}\left\{\int_0^L f(x)\sin\sqrt{\lambda_n}\,x\,dx\right\}\sin\sqrt{\lambda_n}\,x \tag{7}$$

GRAM-SCHMIDT ORTHONORMALIZATION PROCESS

3.12. Generate a set of polynomials orthonormal in the interval $(-1, 1)$ from the sequence $1, x, x^2, x^3, \ldots$.

According to the Gram-Schmidt process we consider the functions

$$\phi_1(x) = c_{11}, \quad \phi_2(x) = c_{21} + c_{22}x, \quad \phi_3(x) = c_{31} + c_{32}x + c_{33}x^2, \quad \ldots$$

Since $\phi_2(x)$ must be orthogonal to $\phi_1(x)$ in $(-1, 1)$, we have

$$\int_{-1}^{1}(c_{11})(c_{21} + c_{22}x)\,dx = 0 \qquad \text{i.e.} \quad c_{11}(2c_{21}) = 0$$

from which $c_{21} = 0$, because $c_{11} \neq 0$. Thus we have

$$\phi_1(x) = c_{11} \qquad \phi_2(x) = c_{22}x$$

In order that $\phi_1(x)$ and $\phi_2(x)$ be normalized in $(-1, 1)$ we must have

$$\int_{-1}^{1}(c_{11})^2\,dx = 1 \qquad \int_{-1}^{1}(c_{22}x)^2\,dx = 1$$

from which

$$c_{11} = \pm\sqrt{\frac{1}{2}} \qquad c_{22} = \pm\sqrt{\frac{3}{2}}$$

Since $\phi_3(x)$ must be orthogonal to $\phi_1(x)$ and $\phi_2(x)$ in $(-1, 1)$, we have

$$\int_{-1}^{1}(c_{11})(c_{31} + c_{32}x + c_{33}x^2)\,dx = 0, \qquad \int_{-1}^{1}(c_{22}x)(c_{31} + c_{32}x + c_{33}x^2)\,dx = 0$$

from which

$$2c_{31} + \tfrac{2}{3}c_{33} = 0 \qquad \text{or} \qquad c_{33} = -3c_{31}, \; c_{32} = 0$$

Thus

$$\phi_3(x) = c_{31}(1 - 3x^2)$$

In order that $\phi_3(x)$ be normalized in $(-1,1)$ we must have

$$\int_{-1}^{1} [c_{31}(1-3x^2)]^2\,dx = 1 \qquad \text{whence} \qquad c_{31} = \pm\frac{1}{2}\sqrt{\frac{5}{2}}$$

The orthonormal functions thus far are given by

$$\phi_1(x) = \pm\sqrt{\frac{1}{2}}, \quad \phi_2(x) = \pm\sqrt{\frac{3}{2}}\,x, \quad \phi_3(x) = \pm\sqrt{\frac{5}{2}}\left(\frac{3x^2-1}{2}\right)$$

By continuing the process (see Problem 3.29) we find

$$\phi_4(x) = \pm\sqrt{\frac{7}{2}}\left(\frac{5x^3-3x}{2}\right), \quad \phi_5(x) = \pm\sqrt{\frac{9}{2}}\left(\frac{35x^4-30x^2+3}{8}\right), \quad \ldots$$

From these we obtain the *Legendre polynomials*

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{3x^2-1}{2}, \quad P_3(x) = \frac{5x^3-3x}{2},$$

$$P_4(x) = \frac{35x^4-30x^2+3}{8}, \quad \ldots$$

The polynomials are such that $P_n(1)=1$, $n=0,1,2,3,\ldots$. We shall investigate Legendre polynomials and applications in Chapter 7.

APPLICATIONS TO BOUNDARY VALUE PROBLEMS

3.13. A thin conducting bar whose ends are at $x=0$ and $x=L$ has the end $x=0$ at temperature zero, while at the end $x=L$ radiation takes place into a medium of temperature zero. Assuming that the surface is insulated and that the initial temperature is $f(x)$, $0<x<L$, find the temperature at any point x of the bar at any time t.

The heat conduction equation for the temperature in a bar whose surface is insulated is

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2} \tag{1}$$

Assuming Newton's law of cooling applies at the end $x=L$, we obtain the condition

$$-Ku_x(L,t) = h[u(L,t)-0]$$

or

$$u_x(L,t) = -\beta u(L,t) \tag{2}$$

where $\beta = K/h$, K being the thermal conductivity and h a constant of proportionality. The remaining boundary conditions are given by

$$u(0,t) = 0, \quad u(x,0) = f(x), \quad |u(x,t)| < M$$

To solve this boundary value problem we let $u=XT$ in (1) to obtain the solution

$$u = e^{-\kappa\lambda^2 t}(A\cos\lambda x + B\sin\lambda x)$$

From $u(0,t)=0$ we find $A=0$, so that

$$u(x,t) = Be^{-\kappa\lambda^2 t}\sin\lambda x$$

The boundary condition (2) yields

$$\tan\lambda L = -\frac{\lambda}{\beta}. \tag{3}$$

This equation is exactly the same as (1) on page 60 with λ replaced by λ^2. Denoting the nth positive root of (3) by λ_n, $n=1,2,3,\ldots$, we see that solutions are

$$u(x,t) = B_n e^{-\kappa\lambda_n^2 t}\sin\lambda_n x$$

Using the principle of superposition we then arrive at a solution

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\kappa\lambda_n^2 t}\sin\lambda_n x$$

The last boundary condition, $u(x,0) = f(x)$, now leads to

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \lambda_n x$$

We can find B_m by multiplying both sides by $\sin \lambda_m x$ and then integrating, using the fact that

$$\int_0^L \sin \lambda_m x \sin \lambda_n x \, dx = 0 \qquad m \neq n$$

However the result is already available to us from (6) of Problem 3.11 if we replace λ_n by λ_n^2. Thus the solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4\lambda_n e^{-\kappa\lambda_n^2 t} \sin \lambda_n x}{2\lambda_n L - \sin 2\lambda_n L} \left\{ \int_0^L f(x) \sin \lambda_n x \, dx \right\}$$

3.14. (a) Show that separation of variables in the boundary value problem

$$g(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[K(x)\frac{\partial u}{\partial x}\right] + h(x)u \qquad 0 < x < L,\ t > 0$$

$$u(0,t) = 0, \quad u(L,t) = 0, \quad u(x,0) = f(x), \quad |u(x,t)| < M$$

leads to a Sturm-Liouville system. (b) Give a physical interpretation of the equation in (a). (c) How would you proceed to solve the boundary value problem in (a)?

(a) Letting $u = XT$ in the given equation, we find

$$g(x)XT' = T\frac{d}{dx}\left[K(x)\frac{dX}{dx}\right] + h(x)XT$$

Then dividing by $g(x)XT$ yields

$$\frac{T'}{T} = \frac{1}{g(x)X}\frac{d}{dx}\left[K(x)\frac{dX}{dx}\right] + h(x)$$

Setting each side equal to $-\lambda$, we find

$$T' + \lambda T = 0 \tag{1}$$

$$\frac{d}{dx}\left[K(x)\frac{dX}{dx}\right] + [h(x) + \lambda g(x)]X = 0 \tag{2}$$

Also, from the conditions $u(0,t) = 0$ and $u(L,t) = 0$ we are led to the conditions

$$X(0) = 0 \qquad X(L) = 0 \tag{3}$$

The required Sturm-Liouville system is given by (2) and (3). Note that the Sturm-Liouville differential equation (2) corresponds to that of (18), page 54, if we choose $y = X$, $p(x) = K(x)$, $q(x) = h(x)$, $r(x) = g(x)$.

(b) By comparison with the derivation of the heat conduction equation on page 9 we see that $u(x,t)$ can be interpreted as the temperature at any point x at time t. In such case $K(x)$ is the (nonconstant) thermal conductivity and $g(x)$ is the specific heat multiplied by the density. The term $h(x)u$ can represent the fact that a Newton's law of cooling type radiation into a medium of temperature zero is taking place at the surface of the bar, with a proportionality factor that depends on position.

(c) From equation (2) subject to boundary conditions (3) we can find eigenvalues λ_n and normalized eigenfunctions $X_n(x)$, where $n = 1, 2, 3, \ldots$. Equation (1) gives $T = ce^{-\lambda t}$. Thus a solution obtained by superposition is

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} X_n(x)$$

From the boundary condition $u(x,0) = f(x)$ we have

$$f(x) = \sum_{n=1}^{\infty} c_n X_n(x)$$

which leads to

$$c_n = \int_0^L f(x)\,X_n(x)\,dx$$

Thus we obtain the solution

$$u(x,t) = \sum_{n=1}^{\infty}\left\{\int_0^L f(x)\,X_n(x)\,dx\right\} e^{-\lambda_n t} X_n(x)$$

Supplementary Problems

ORTHOGONAL FUNCTIONS AND ORTHONORMAL SERIES

3.15. Given the functions a_0, $a_1 + a_2x$, $a_3 + a_4x + a_5x^2$ where $a_0, \ldots, a_5$ are constants. Determine the constants so that these functions are mutually orthonormal in the interval $(0,1)$.

3.16. Generalize Problem 3.15 to arbitrary finite intervals.

3.17. (a) Show that the functions $1, 1-x, 2-4x+x^2$ are mutually orthogonal in $(0,\infty)$ with respect to the density function e^{-x}. (b) Obtain a mutually orthonormal set.

3.18. Give a vector interpretation to functions which are orthonormal with respect to a density or weight function.

3.19. (a) Show that the functions $\cos(n\cos^{-1}x)$, $n = 0,1,2,3,\ldots$, are mutually orthogonal in $(-1,1)$ with respect to the weight function $(1-x^2)^{-1/2}$. (b) Obtain a mutually orthonormal set of these functions.

3.20. Show how to expand $f(x)$ into a series $\sum_{n=1}^{\infty} c_n\phi_n(x)$, where $\phi_n(x)$ are mutually orthonormal in (a,b) with respect to the weight function $w(x)$.

3.21. (a) Expand $f(x)$ into a series having the form $\sum_{n=0}^{\infty} c_n\phi_n(x)$, where $\phi_n(x)$ are the mutually orthonormal functions of Problem 3.19. (b) Discuss the relationship of the series in (a) to Fourier series.

APPROXIMATIONS IN THE LEAST-SQUARES SENSE. PARSEVAL'S IDENTITY AND COMPLETENESS

3.22. Let $\mathbf{r}$ be any three-dimensional vector. Show that

(a) $(\mathbf{r}\cdot\mathbf{i})^2 + (\mathbf{r}\cdot\mathbf{j})^2 \leq r^2$ (b) $(\mathbf{r}\cdot\mathbf{i})^2 + (\mathbf{r}\cdot\mathbf{j})^2 + (\mathbf{r}\cdot\mathbf{k})^2 = r^2$

where $r^2 = \mathbf{r}\cdot\mathbf{r}$ and discuss these with reference to Bessel's inequality and Parseval's identity. Compare with Problem 3.6.

3.23. Suppose that one term in any orthonormal series (such as a Fourier series) is omitted. (a) Can we expand an arbitrary function $f(x)$ in the series? (b) Can Parseval's identity be satisfied? (c) Can Bessel's inequality be satisfied? Justify your answers.

3.24. (a) Find c_1, c_2, c_3 such that $\int_{-\pi}^{\pi} [x - (c_1\sin x + c_2\sin 2x + c_3\sin 3x)]^2\,dx$ is a minimum.

(b) What is the mean square error and root mean square error in approximating x by $c_1\sin x + c_2\sin 2x + c_3\sin 3x$, where c_1, c_2, c_3 are the values obtained in (a)?

(c) Suppose that it is desired to approximate x by $\alpha_1 \sin x + \alpha_2 \sin 2x + \alpha_3 \sin 3x + \alpha_4 \sin 4x$ in the least-squares sense in the interval $(-\pi, \pi)$. Are the values $\alpha_1, \alpha_2, \alpha_3$ the same as c_1, c_2, c_3 of part (a)? Explain and discuss the significance of this.

3.25. Verify that Bessel's inequality holds in Problem 3.24.

3.26. Discuss the relationship of Problem 3.24 with the expansion of $f(x) = x$ in a Fourier series in the interval $(-\pi, \pi)$.

3.27. Prove that the set of orthonormal functions $\phi_n(x)$, $n = 1, 2, 3, \ldots$, cannot be complete in (a, b) if there exists some function $f(x)$ different from zero which is orthogonal to all members of the set, i.e. if

$$\int_a^b f(x)\,\phi_n(x)\,dx = 0 \qquad n = 1, 2, 3, \ldots$$

3.28. Is the converse of Problem 3.27 true? Explain.

GRAM-SCHMIDT ORTHONORMALIZATION PROCESS

3.29. Verify that a continuation of the process in Problem 3.12 produces the indicated results for $\phi_4(x)$ and $\phi_5(x)$.

3.30. Given the set of functions $1, x, x^2, x^3, \ldots$, obtain from these a set of functions which are mutually orthonormal in $(-1, 1)$ with respect to the weight function x.

3.31. Work Problem 3.30 if the interval is $(0, \infty)$ and the weight function is e^{-x}. The polynomials thus obtained are *Laguerre polynomials.*

3.32. Is it possible to use the Gram-Schmidt process to obtain from x, $1 - x$, $3 + 2x$ a set of functions orthonormal in $(0, 1)$? Explain.

STURM-LIOUVILLE SYSTEMS. EIGENVALUES AND EIGENFUNCTIONS

3.33. (a) Verify that the system $y'' + \lambda y = 0$, $y'(0) = 0$, $y(1) = 0$ is a Sturm-Liouville system.

(b) Find the eigenvalues and eigenfunctions of the system.

(c) Prove that the eigenfunctions are orthogonal and determine the corresponding orthonormal functions.

3.34. Work Problem 3.33, if the boundary conditions are (a) $y(0) = 0$, $y'(1) = 0$; (b) $y'(0) = 0$, $y'(1) = 0$.

3.35. Show that in Problem 3.11 we have

$$B_n^2 = \frac{2(\lambda_n + \beta^2)}{L\lambda_n + L\beta^2 + \beta}$$

3.36. Show that any equation having the form $a_0(x)y'' + a_1(x)y' + [a_2(x) + \lambda a_3(x)]y = 0$ can be written in Sturm-Liouville form as

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]y = 0$$

with $$p(x) = e^{\int (a_1/a_0)\,dx}; \qquad q(x) = \frac{a_2}{a_0}p(x), \qquad r(x) = \frac{a_3}{a_0}p(x)$$

3.37. Discuss Problem 3.13 if the boundary conditions are replaced by $u_x(0, t) = h_1 u(0, t)$, $u_x(L, t) = h_2 u(L, t)$.

3.38. (a) Show that separation of variables in the boundary value problem

$$\rho(x)\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left[r(x)\frac{\partial y}{\partial x}\right] + h(x)y$$

$$y(0,t) = 0, \quad y(L,t) = 0, \quad y(x,0) = f(x), \quad y_t(x,0) = 0, \quad |y(x,t)| < M$$

leads to a Sturm-Liouville system. (b) Give a physical interpretation of the equations in (a). (c) How would you solve the boundary value problem?

3.39. Discuss Problem 3.38 if the boundary conditions $y(0,t) = 0$, $y(L,t) = 0$ are replaced by $y_x(0,t) = h_1 y(0,t)$, $y_x(L,t) = h_2 y(L,t)$, respectively.

APPLICATIONS TO BOUNDARY VALUE PROBLEMS

3.40. (a) Solve the boundary value problem

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2} \qquad 0 < x < L,\ t > 0$$

$$u(0,t) = 0, \quad u_x(L,t) = 0, \quad u(x,0) = f(x), \quad |u(x,t)| < M$$

and (b) interpret physically.

3.41. (a) Solve the boundary value problem

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$$

$$y(0,t) = 0, \quad y_x(L,t) = 0, \quad y(x,0) = f(x), \quad y_t(x,0) = 0, \quad |y(x,t)| < M$$

and (b) interpret physically.

3.42. (a) Solve the boundary value problem

$$\frac{\partial^2 y}{\partial t^2} + b^2\frac{\partial^4 y}{\partial x^4} = 0 \qquad 0 < x < L,\ t > 0$$

$$y(0,t) = 0, \quad y_x(0,t) = 0, \quad y(L,t) = 0, \quad y_x(L,t) = 0, \quad y(x,0) = f(x), \quad |y(x,t)| < M$$

and (b) interpret physically.

3.43. Show that the solution of the boundary value problem

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2} \qquad 0 < x < l,\ t > 0$$

$$u_x(0,t) = hu(0,t), \quad u_x(l,t) = -hu(l,t), \quad u(x,0) = f(x)$$

where κ, h and l are constants, is

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\kappa\lambda_n^2 t}\,\frac{\lambda_n\cos\lambda_n x + h\sin\lambda_n x}{(\lambda_n^2 + h^2)l + 2h}\int_0^l f(x)(\lambda_n\cos\lambda_n x + h\sin\lambda_n x)\,dx$$

where λ_n are solutions of the equation $\tan\lambda l = \dfrac{2h\lambda}{\lambda^2 - h^2}$. Give a physical interpretation.

Chapter 4

Gamma, Beta and Other Special Functions

SPECIAL FUNCTIONS

In the process of obtaining solutions to boundary value problems we often arrive at *special functions*. In this chapter we shall survey some special functions that will be employed in later chapters. If desired, the student may skip this chapter, returning to it should the need arise.

THE GAMMA FUNCTION

The *gamma function*, denoted by $\Gamma(n)$, is defined by

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx \qquad (1)$$

which is convergent for $n > 0$.

A recurrence formula for the gamma function is

$$\Gamma(n+1) = n\Gamma(n) \qquad (2)$$

where $\Gamma(1) = 1$ (see Problem 4.1). From (2), $\Gamma(n)$ can be determined for all $n > 0$ when the values for $1 \leq n < 2$ (or any other interval of unit length) are known (see table on page 68). In particular if n is a positive integer, then

$$\Gamma(n+1) = n! \qquad n = 1, 2, 3, \ldots \qquad (3)$$

For this reason $\Gamma(n)$ is sometimes called the *factorial function*.

Examples. $\Gamma(2) = 1! = 1, \quad \Gamma(6) = 5! = 120, \quad \dfrac{\Gamma(5)}{\Gamma(3)} = \dfrac{4!}{2!} = 12$

It can be shown (Problem 4.4) that

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi} \qquad (4)$$

The recurrence relation (2) is a difference equation which has (1) as a solution. By taking (1) as the definition of $\Gamma(n)$ for $n > 0$, we can generalize the gamma function to $n < 0$ by use of (2) in the form

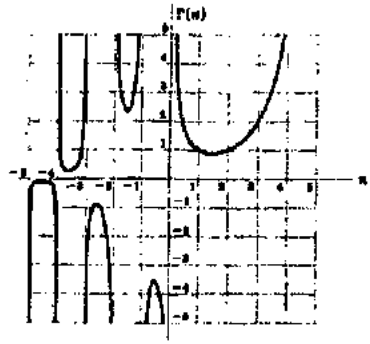
$$\Gamma(n) = \frac{\Gamma(n+1)}{n} \qquad (5)$$

See Problem 4.7, for example. The process is called *analytic continuation*.

TABLE OF VALUES AND GRAPH OF THE GAMMA FUNCTION

n	$\Gamma(n)$
1.00	1.0000
1.10	0.9514
1.20	0.9182
1.30	0.8975
1.40	0.8873
1.50	0.8862
1.60	0.8935
1.70	0.9086
1.80	0.9314
1.90	0.9618
2.00	1.0000

Fig. 4-1



ASYMPTOTIC FORMULA FOR $\Gamma(n)$

If n is large, the computational difficulties inherent in a direct calculation of $\Gamma(n)$ are apparent. A useful result in such case is supplied by the relation

$$\Gamma(n+1) = \sqrt{2\pi n}\, n^n e^{-n} e^{\theta/12(n+1)} \qquad 0 < \theta < 1 \tag{6}$$

For most practical purposes the last factor, which is very close to 1 for large n, can be omitted. If n is an integer, we can write

$$n! \sim \sqrt{2\pi n}\, n^n e^{-n} \tag{7}$$

where $\sim$ means "is approximately equal to for large n". This is sometimes called *Stirling's factorial approximation* (or *asymptotic formula*) *for* $n!$.

MISCELLANEOUS RESULTS INVOLVING THE GAMMA FUNCTION

1. $$\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin x\pi}$$

   In particular if $x = \frac{1}{2}$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ as in (4).

2. $$2^{2x-1}\,\Gamma(x)\,\Gamma(x+\tfrac{1}{2}) = \sqrt{\pi}\,\Gamma(2x)$$

   This is called the *duplication formula* for the gamma function.

3. $$\Gamma(x)\,\Gamma\left(x+\frac{1}{m}\right)\Gamma\left(x+\frac{2}{m}\right)\cdots\Gamma\left(x+\frac{m-1}{m}\right) = m^{(1/2)-mx}\,(2\pi)^{(m-1)/2}\,\Gamma(mx)$$

   The duplication formula is a special case of this with $m = 2$.

4. $$\Gamma(x+1) \sim \sqrt{2\pi x}\, x^x e^{-x}\left\{1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51{,}840x^3} + \cdots\right\}$$

   This is called *Stirling's asymptotic series* for the gamma function. The series in braces is an asymptotic series as defined on page 70.

5. $$\Gamma'(1) = \int_0^\infty e^{-x}\ln x\,dx = -\gamma$$

   where γ is *Euler's constant* and is defined as

   $$\lim_{M\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{M} - \ln M\right) = 0.5772156\ldots$$

6. $$\frac{\Gamma'(p+1)}{\Gamma(p+1)} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p} - \gamma$$

THE BETA FUNCTION

The *beta function*, denoted by $B(m,n)$, is defined by

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \tag{8}$$

which is convergent for $m > 0,\ n > 0$.

The beta function is connected with the gamma function according to the relation

$$B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \tag{9}$$

See Problem 4.12. Using (*4*) we can define $B(m,n)$ for $m < 0,\ n < 0$.

Many integrals can be evaluated in terms of beta or gamma functions. Two useful results are

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta = \tfrac{1}{2}B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{2\,\Gamma(m+n)} \tag{10}$$

valid for $m > 0$ and $n > 0$ (see Problems 4.11 and 4.14) and

$$\int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \Gamma(p)\,\Gamma(1-p) = \frac{\pi}{\sin p\pi} \qquad 0 < p < 1 \tag{11}$$

See Problem 4.18.

OTHER SPECIAL FUNCTIONS

Many other special functions are of importance in science and engineering. Some of these are given in the following list. Others will be considered in later chapters.

1. **Error function.** $$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du = 1 - \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}\,du$$

2. **Complementary error function.** $$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}\,du = 1 - \operatorname{erf}(x)$$

3. **Exponential integral.** $$Ei(x) = \int_x^\infty \frac{e^{-u}}{u}\,du$$

4. **Sine integral.** $$Si(x) = \int_0^x \frac{\sin u}{u}\,du = \frac{\pi}{2} - \int_x^\infty \frac{\sin u}{u}\,du$$

5. **Cosine integral.** $$Ci(x) = \int_x^\infty \frac{\cos u}{u}\,du$$

6. **Fresnel sine integral.** $$S(x) = \sqrt{\frac{2}{\pi}}\int_0^x \sin u^2\,du = 1 - \sqrt{\frac{2}{\pi}}\int_x^\infty \sin u^2\,du$$

7. **Fresnel cosine integral.** $$C(x) = \sqrt{\frac{2}{\pi}}\int_0^x \cos u^2\,du = 1 - \sqrt{\frac{2}{\pi}}\int_x^\infty \cos u^2\,du$$

ASYMPTOTIC SERIES OR EXPANSIONS

Consider the series

$$S(x) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n} + \cdots \qquad (12)$$

and suppose that

$$S_n(x) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n} \qquad (13)$$

are the partial sums of the series.

If $R_n(x) = f(x) - S_n(x)$, where $f(x)$ is given, is such that for every n

$$\lim_{x\to\infty} x^n |R_n(x)| = 0 \qquad (14)$$

then $S(x)$ is called an *asymptotic series* or *expansion* of $f(x)$ and we denote this by writing $f(x) \sim S(x)$.

In practice the series *(12)* diverges. However, by taking the sum of successive terms of the series, stopping just before the terms begin to increase, we may obtain a useful approximation for $f(x)$. The approximation becomes better the larger the value of x.

Various operations with asymptotic series are permissible. For example, asymptotic series may be multiplied together or integrated term by term to yield another asymptotic series.

Solved Problems

THE GAMMA FUNCTION

4.1. **Prove:** (*a*) $\Gamma(n+1) = n\Gamma(n)$, $n > 0$; (*b*) $\Gamma(n+1) = n!$, $n = 1, 2, 3, \ldots$.

(*a*)
$$\Gamma(n+1) = \int_0^\infty x^n e^{-x}\,dx = \lim_{M\to\infty}\int_0^M x^n e^{-x}\,dx$$
$$= \lim_{M\to\infty}\left\{(x^n)(-e^{-x})\Big|_0^M - \int_0^M (-e^{-x})(nx^{n-1})\,dx\right\}$$
$$= \lim_{M\to\infty}\left\{-M^n e^{-M} + n\int_0^M x^{n-1}e^{-x}\,dx\right\} = n\,\Gamma(n) \qquad \text{if } n > 0$$

(*b*)
$$\Gamma(1) = \int_0^\infty e^{-x}\,dx = \lim_{M\to\infty}\int_0^M e^{-x}\,dx = \lim_{M\to\infty}(1 - e^{-M}) = 1$$

Put $n = 1, 2, 3, \ldots$ in $\Gamma(n+1) = n\,\Gamma(n)$. Then

$$\Gamma(2) = 1\,\Gamma(1) = 1, \quad \Gamma(3) = 2\,\Gamma(2) = 2\cdot 1 = 2!, \quad \Gamma(4) = 3\,\Gamma(3) = 3\cdot 2! = 3!$$

In general, $\Gamma(n+1) = n!$ if n is a positive integer.

4.2. Evaluate (*a*) $\dfrac{\Gamma(6)}{2\,\Gamma(3)}$, (*b*) $\dfrac{\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})}$, (*c*) $\dfrac{\Gamma(3)\,\Gamma(2.5)}{\Gamma(5.5)}$, (*d*) $\dfrac{6\,\Gamma(\frac{8}{3})}{5\,\Gamma(\frac{2}{3})}$.

(*a*) $$\frac{\Gamma(6)}{2\,\Gamma(3)} = \frac{5!}{2\cdot 2!} = \frac{5\cdot 4\cdot 3\cdot 2}{2\cdot 2} = 30$$

(*b*) $$\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})} = \frac{\frac{3}{2}\,\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2})} = \frac{\frac{3}{2}\cdot\frac{1}{2}\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2})} = \frac{3}{4}$$

(*c*) $$\frac{\Gamma(3)\,\Gamma(2.5)}{\Gamma(5.5)} = \frac{2!\,(1.5)(0.5)\,\Gamma(0.5)}{(4.5)(3.5)(2.5)(1.5)(0.5)\,\Gamma(0.5)} = \frac{16}{315}$$

(d) $\dfrac{6\,\Gamma(\frac{8}{3})}{5\,\Gamma(\frac{2}{3})} = \dfrac{6(\frac{5}{3})(\frac{2}{3})\,\Gamma(\frac{2}{3})}{5\,\Gamma(\frac{2}{3})} = \dfrac{4}{3}$

4.3. Evaluate (a) $\int_0^\infty x^3 e^{-x}\,dx$, (b) $\int_0^\infty x^6 e^{-2x}\,dx$.

(a) $\int_0^\infty x^3 e^{-x}\,dx = \Gamma(4) = 3! = 6$

(b) Let $2x = y$. Then the integral becomes

$$\int_0^\infty \left(\frac{y}{2}\right)^6 e^{-y}\frac{dy}{2} = \frac{1}{2^7}\int_0^\infty y^6 e^{-y}\,dy = \frac{\Gamma(7)}{2^7} = \frac{6!}{2^7} = \frac{45}{8}$$

4.4. Prove that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

We have $\Gamma(\frac{1}{2}) = \int_0^\infty x^{-1/2}e^{-x}\,dx = 2\int_0^\infty e^{-u^2}\,du$, on letting $x = u^2$. It follows that

$$\{\Gamma(\tfrac{1}{2})\}^2 = \left\{2\int_0^\infty e^{-u^2}\,du\right\}\left\{2\int_0^\infty e^{-v^2}\,dv\right\} = 4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}\,du\,dv$$

Changing to polar coordinates (ρ, ϕ), where $u = \rho\cos\phi$, $v = \rho\sin\phi$, the last integral becomes

$$4\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{\infty} e^{-\rho^2}\rho\,d\rho\,d\phi = 4\int_{\phi=0}^{\pi/2} -\tfrac{1}{2}e^{-\rho^2}\Big|_{\rho=0}^{\infty}\,d\phi = \pi$$

and so $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

4.5. Evaluate (a) $\int_0^\infty \sqrt{y}\,e^{-y^3}\,dy$, (b) $\int_0^\infty 3^{-4z^2}\,dz$, (c) $\int_0^1 \dfrac{dx}{\sqrt{-\ln x}}$.

(a) Letting $y^3 = x$, the integral becomes

$$\int_0^\infty \sqrt{x^{1/3}}\,e^{-x}\cdot\tfrac{1}{3}x^{-2/3}\,dx = \frac{1}{3}\int_0^\infty x^{-1/2}e^{-x}\,dx = \frac{1}{3}\Gamma(\tfrac{1}{2}) = \frac{\sqrt{\pi}}{3}$$

(b) $\int_0^\infty 3^{-4z^2}\,dz = \int_0^\infty (e^{\ln 3})^{(-4z^2)}\,dz = \int_0^\infty e^{-(4\ln 3)z^2}\,dz$. Let $(4\ln 3)z^2 = x$ and the integral becomes

$$\int_0^\infty e^{-x}\,d\left(\frac{x^{1/2}}{\sqrt{4\ln 3}}\right) = \frac{1}{2\sqrt{4\ln 3}}\int_0^\infty x^{-1/2}e^{-x}\,dx = \frac{\Gamma(\frac{1}{2})}{2\sqrt{4\ln 3}} = \frac{\sqrt{\pi}}{4\sqrt{\ln 3}}$$

(c) Let $-\ln x = u$. Then $x = e^{-u}$. When $x = 1$, $u = 0$; when $x = 0$, $u = \infty$. The integral becomes

$$\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\,du = \int_0^\infty u^{-1/2}e^{-u}\,du = \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

4.6. Evaluate $\int_0^\infty x^m e^{-ax^n}\,dx$, **where** m, n, a **are positive constants.**

Letting $ax^n = y$, the integral becomes

$$\int_0^\infty \left\{\left(\frac{y}{a}\right)^{1/n}\right\}^m e^{-y}\,d\left\{\left(\frac{y}{a}\right)^{1/n}\right\} = \frac{1}{na^{(m+1)/n}}\int_0^\infty y^{(m+1)/n-1}e^{-y}\,dy = \frac{1}{na^{(m+1)/n}}\Gamma\left(\frac{m+1}{n}\right)$$

4.7. Evaluate (a) $\Gamma(-1/2)$, (b) $\Gamma(-5/2)$.

We use the generalization to negative values defined by $\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$.

(a) Letting $n=-\frac{1}{2}$, $\Gamma(-1/2)=\dfrac{\Gamma(1/2)}{-1/2}=-2\sqrt{\pi}$.

(b) Letting $n=-3/2$, $\Gamma(-3/2)=\dfrac{\Gamma(-1/2)}{-3/2}=\dfrac{-2\sqrt{\pi}}{-3/2}=\dfrac{4\sqrt{\pi}}{3}$, using (a).

Then $\Gamma(-5/2)=\dfrac{\Gamma(-3/2)}{-5/2}=-\dfrac{8}{15}\sqrt{\pi}$.

4.8. Prove that $\displaystyle\int_0^1 x^m(\ln x)^n\,dx=\frac{(-1)^n n!}{(m+1)^{n+1}}$, where n is a positive integer and $m>-1$.

Letting $x=e^{-y}$, the integral becomes $(-1)^n\displaystyle\int_0^\infty y^n e^{-(m+1)y}\,dy$. If $(m+1)y=u$, this last integral becomes

$$(-1)^n\int_0^\infty \frac{u^n}{(m+1)^n}e^{-u}\frac{du}{m+1} \;=\; \frac{(-1)^n}{(m+1)^{n+1}}\int_0^\infty u^n e^{-u}\,du \;=\; \frac{(-1)^n}{(m+1)^{n+1}}\Gamma(n+1) \;=\; \frac{(-1)^n n!}{(m+1)^{n+1}}$$

4.9. Prove that $\displaystyle\int_0^\infty e^{-\alpha\lambda^2}\cos\beta\lambda\,d\lambda \;=\; \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\,e^{-\beta^2/4\alpha}$.

Let $I=I(\alpha,\beta)=\displaystyle\int_0^\infty e^{-\alpha\lambda^2}\cos\beta\lambda\,d\lambda$. Then

$$\frac{\partial I}{\partial\beta} \;=\; \int_0^\infty(-\lambda e^{-\alpha\lambda^2})\sin\beta\lambda\,d\lambda$$

$$= \frac{e^{-\alpha\lambda^2}}{2\alpha}\sin\beta\lambda\Big|_0^\infty-\frac{\beta}{2\alpha}\int_0^\infty e^{-\alpha\lambda^2}\cos\beta\lambda\,d\lambda \;=\; -\frac{\beta}{2\alpha}I$$

Thus
$$\frac{1}{I}\frac{\partial I}{\partial\beta} \;=\; -\frac{\beta}{2\alpha} \qquad\text{or}\qquad \frac{\partial}{\partial\beta}\ln I \;=\; -\frac{\beta}{2\alpha} \tag{1}$$

Integration with respect to β yields

$$\ln I \;=\; -\frac{\beta^2}{4\alpha}+c_1$$

or
$$I \;=\; I(\alpha,\beta) \;=\; Ce^{-\beta^2/4\alpha} \tag{2}$$

But $C=I(\alpha,0)=\displaystyle\int_0^\infty e^{-\alpha\lambda^2}d\lambda=\frac{1}{2\sqrt{\alpha}}\int_0^\infty x^{-1/2}e^{-x}\,dx=\frac{\Gamma(\frac{1}{2})}{2\sqrt{\alpha}}=\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$, on letting $x=\alpha\lambda^2$.

Thus, as required,

$$I \;=\; \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\,e^{-\beta^2/4\alpha}$$

4.10. A particle is attracted toward a fixed point O with a force inversely proportional to its instantaneous distance from O. If the particle is released from rest, find the time for it to reach O.

At time $t=0$ let the particle be located on the x-axis at $x=a>0$ and let O be the origin. Then by Newton's law

$$m\frac{d^2x}{dt^2} \;=\; -\frac{k}{x} \tag{1}$$

where m is the mass of the particle and $k>0$ is a constant of proportionality.

Let $\dfrac{dx}{dt}=v$, the velocity of the particle. Then $\dfrac{d^2x}{dt^2}=\dfrac{dv}{dt}=\dfrac{dv}{dx}\dfrac{dx}{dt}=v\dfrac{dv}{dx}$ and (1) becomes

$$mv\frac{dv}{dx} \;=\; \frac{k}{x} \qquad\text{or}\qquad \frac{mv^2}{2} \;=\; -k\ln x+c \tag{2}$$

upon integrating. Since $v=0$ at $x=a$, we find $c=k\ln a$. Then

$$\frac{mv^2}{2} \;=\; k\ln\frac{a}{x} \qquad\text{or}\qquad v \;=\; \frac{dx}{dt} \;=\; -\sqrt{\frac{2k}{m}}\sqrt{\ln\frac{a}{x}} \tag{3}$$

where the negative sign is chosen since x is decreasing as t increases. We thus find that the time T taken for the particle to go from $x = a$ to $x = 0$ is given by

$$T \;=\; \sqrt{\frac{m}{2k}}\int_0^a \frac{dx}{\sqrt{\ln a/x}} \tag{4}$$

Letting $\ln a/x = u$ or $x = ae^{-u}$, this becomes

$$T \;=\; a\sqrt{\frac{m}{2k}}\int_0^\infty u^{-1/2}e^{-u}\,du \;=\; a\sqrt{\frac{m}{2k}}\,\Gamma(\tfrac{1}{2}) \;=\; a\sqrt{\frac{\pi m}{2k}}$$

THE BETA FUNCTION

4.11. Prove that (a) $B(m,n) = B(n,m)$, (b) $B(m,n) = 2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$.

(a) Using the transformation $x = 1-y$, we have

$$B(m,n) \;=\; \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \;=\; \int_0^1 (1-y)^{m-1}y^{n-1}\,dy$$

$$=\; \int_0^1 y^{n-1}(1-y)^{m-1}\,dy \;=\; B(n,m)$$

(b) Using the transformation $x = \sin^2\theta$, we have

$$B(m,n) \;=\; \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \;=\; \int_0^{\pi/2}(\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1}\,2\sin\theta\cos\theta\,d\theta$$

$$=\; 2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$$

4.12. Prove that $\quad B(m,n) \;=\; \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \quad m,n > 0.$

Letting $z = x^2$, we have $\Gamma(m) \;=\; \int_0^\infty z^{m-1}e^{-z}\,dz \;=\; 2\int_0^\infty x^{2m-1}e^{-x^2}\,dx.$

Similarly, $\Gamma(n) \;=\; 2\int_0^\infty y^{2n-1}e^{-y^2}\,dy.$ Then

$$\Gamma(m)\,\Gamma(n) \;=\; 4\left(\int_0^\infty x^{2m-1}e^{-x^2}\,dx\right)\left(\int_0^\infty y^{2n-1}e^{-y^2}\,dy\right)$$

$$=\; 4\int_0^\infty\int_0^\infty x^{2m-1}y^{2n-1}e^{-(x^2+y^2)}\,dx\,dy$$

Transforming to polar coordinates, $x = \rho\cos\phi$, $y = \rho\sin\phi$,

$$\Gamma(m)\,\Gamma(n) \;=\; 4\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{\infty}\rho^{2(m+n)-1}e^{-\rho^2}\cos^{2m-1}\phi\,\sin^{2n-1}\phi\,d\rho\,d\phi$$

$$=\; 4\left(\int_{\rho=0}^{\infty}\rho^{2(m+n)-1}e^{-\rho^2}\,d\rho\right)\left(\int_{\phi=0}^{\pi/2}\cos^{2m-1}\phi\,\sin^{2n-1}\phi\,d\phi\right)$$

$$=\; 2\,\Gamma(m+n)\int_0^{\pi/2}\cos^{2m-1}\phi\,\sin^{2n-1}\phi\,d\phi \;=\; \Gamma(m+n)\,B(n,m)$$

$$=\; \Gamma(m+n)\,B(m,n)$$

using the results of Problem 4.11. Hence the required result follows.

The above argument can be made rigorous by using a limiting procedure.

4.13. Evaluate (a) $\int_0^1 x^4(1-x)^3\,dx$, (b) $\int_0^2 \frac{x^2\,dx}{\sqrt{2-x}}$, (c) $\int_0^a y^4\sqrt{a^2-y^2}\,dy$.

(a) $$\int_0^1 x^4(1-x)^3\,dx = B(5,4) = \frac{\Gamma(5)\,\Gamma(4)}{\Gamma(9)} = \frac{4!\,3!}{8!} = \frac{1}{280}$$

(b) Letting $x = 2v$, the integral becomes

$$4\sqrt{2}\int_0^1 \frac{v^2}{\sqrt{1-v}}\,dv = 4\sqrt{2}\int_0^1 v^2(1-v)^{-1/2}\,dv = 4\sqrt{2}\,B(3,\tfrac{1}{2}) = \frac{4\sqrt{2}\,\Gamma(3)\,\Gamma(1/2)}{\Gamma(7/2)} = \frac{64\sqrt{2}}{15}$$

(c) Letting $y^2 = a^2x$ or $y = a\sqrt{x}$, the integral becomes

$$\frac{a^6}{2}\int_0^1 x^{3/2}(1-x)^{1/2}\,dx = \frac{a^6}{2}B(5/2,3/2) = \frac{a^6\,\Gamma(5/2)\,\Gamma(3/2)}{2\,\Gamma(4)} = \frac{\pi a^6}{32}$$

4.14. Show that $\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta = \frac{\Gamma(m)\,\Gamma(n)}{2\,\Gamma(m+n)}$ $m,n > 0$.

This follows at once from Problems 4.11 and 4.12.

4.15. Evaluate (a) $\int_0^{\pi/2} \sin^6\theta\,d\theta$, (b) $\int_0^{\pi/2} \sin^4\theta\cos^5\theta\,d\theta$, (c) $\int_0^{\pi} \cos^4\theta\,d\theta$.

(a) Let $2m-1 = 6$, $2n-1 = 0$, i.e. $m = 7/2$, $n = 1/2$, in Problem 4.14.

Then the required integral has the value $\frac{\Gamma(7/2)\,\Gamma(1/2)}{2\,\Gamma(4)} = \frac{5\pi}{32}$.

(b) Letting $2m-1 = 4$, $2n-1 = 5$, the required integral has the value $\frac{\Gamma(5/2)\,\Gamma(3)}{2\,\Gamma(11/2)} = \frac{8}{315}$.

(c) $\int_0^{\pi} \cos^4\theta\,d\theta = 2\int_0^{\pi/2} \cos^4\theta\,d\theta$. Thus, letting $2m-1 = 0$, $2n-1 = 4$ in Problem 4.14, the value is $\frac{2\,\Gamma(1/2)\,\Gamma(5/2)}{2\,\Gamma(3)} = \frac{3\pi}{8}$.

4.16. Prove $\int_0^{\pi/2} \sin^p\theta\,d\theta = \int_0^{\pi/2} \cos^p\theta\,d\theta =$ (a) $\frac{1\cdot 3\cdot 5\cdots(p-1)}{2\cdot 4\cdot 6\cdots p}\,\frac{\pi}{2}$ if p is an even positive integer, (b) $\frac{2\cdot 4\cdot 6\cdots(p-1)}{1\cdot 3\cdot 5\cdots p}$ if p is an odd positive integer.

From Problem 4.14 with $2m-1 = p$, $2n-1 = 0$, we have

$$\int_0^{\pi/2} \sin^p\theta\,d\theta = \frac{\Gamma[\frac{1}{2}(p+1)]\,\Gamma(\frac{1}{2})}{2\,\Gamma[\frac{1}{2}(p+2)]}$$

(a) If $p = 2r$, the integral equals

$$\frac{\Gamma(r+\frac{1}{2})\,\Gamma(\frac{1}{2})}{2\,\Gamma(r+1)} = \frac{(r-\frac{1}{2})(r-\frac{3}{2})\cdots\frac{1}{2}\,\Gamma(\frac{1}{2})\cdot\Gamma(\frac{1}{2})}{2r(r-1)\cdots 1}$$

$$= \frac{(2r-1)(2r-3)\cdots 1}{2r(2r-2)\cdots 2}\,\frac{\pi}{2} = \frac{1\cdot 3\cdot 5\cdots(2r-1)}{2\cdot 4\cdot 6\cdots 2r}\,\frac{\pi}{2}$$

(b) If $p = 2r+1$, the integral equals

$$\frac{\Gamma(r+1)\,\Gamma(\frac{1}{2})}{2\,\Gamma(r+\frac{3}{2})} = \frac{r(r-1)\cdots 1\cdot\sqrt{\pi}}{2(r+\frac{1}{2})(r-\frac{1}{2})\cdots\frac{1}{2}\sqrt{\pi}} = \frac{2\cdot 4\cdot 6\cdots 2r}{1\cdot 3\cdot 5\cdots(2r+1)}$$

In both cases $\int_0^{\pi/2} \sin^p\theta\,d\theta = \int_0^{\pi/2} \cos^p\theta\,d\theta$, as seen by letting $\theta = \frac{\pi}{2} - \phi$.

4.17. Evaluate (a) $\int_0^{\pi/2} \cos^6\theta\,d\theta$, (b) $\int_0^{\pi/2} \sin^3\theta\cos^2\theta\,d\theta$, (c) $\int_0^{2\pi} \sin^8\theta\,d\theta$.

(a) From Problem 4.16 the integral equals $\dfrac{1\cdot3\cdot5}{2\cdot4\cdot6}\dfrac{\pi}{2} = \dfrac{5\pi}{32}$ [compare Problem 4.15(a)].

(b) The integral equals

$$\int_0^{\pi/2} \sin^3\theta(1-\sin^2\theta)\,d\theta = \int_0^{\pi/2}\sin^3\theta\,d\theta - \int_0^{\pi/2}\sin^5\theta\,d\theta = \frac{2}{1\cdot3} - \frac{2\cdot4}{1\cdot3\cdot5} = \frac{2}{15}$$

The method of Problem 4.15(b) can also be used.

(c) The given integral equals $4\int_0^{\pi/2}\sin^8\theta\,d\theta = 4\left(\dfrac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\dfrac{\pi}{2}\right) = \dfrac{35\pi}{64}$.

4.18. Given $\int_0^\infty \dfrac{x^{p-1}}{1+x}dx = \dfrac{\pi}{\sin p\pi}$, show that $\Gamma(p)\Gamma(1-p) = \dfrac{\pi}{\sin p\pi}$ where $0 < p < 1$.

Letting $\dfrac{x}{1+x} = y$ or $x = \dfrac{y}{1-y}$, the given integral becomes

$$\int_0^1 y^{p-1}(1-y)^{-p}\,dy = B(p, 1-p) = \Gamma(p)\Gamma(1-p)$$

and the result follows.

4.19. Evaluate $\int_0^\infty \dfrac{dy}{1+y^4}$.

Let $y^4 = x$. Then the integral becomes $\dfrac{1}{4}\int_0^\infty \dfrac{x^{-3/4}}{1+x}dx = \dfrac{\pi}{4\sin(\pi/4)} = \dfrac{\pi\sqrt{2}}{4}$ by Problem 4.18 with $p = \frac{1}{4}$.

The result can also be obtained by letting $y^2 = \tan\theta$.

4.20. Show that $\int_0^2 x\sqrt[3]{8-x^3}\,dx = \dfrac{16\pi}{9\sqrt{3}}$.

Letting $x^3 = 8y$ or $x = 2y^{1/3}$, the integral becomes

$$\int_0^1 2y^{1/3}\cdot\sqrt[3]{8(1-y)}\cdot\tfrac{2}{3}y^{-2/3}\,dy = \frac{8}{3}\int_0^1 y^{-1/3}(1-y)^{1/3}\,dy = \frac{8}{3}B(\tfrac{2}{3},\tfrac{4}{3})$$

$$= \frac{8}{3}\frac{\Gamma(\frac{2}{3})\Gamma(\frac{4}{3})}{\Gamma(2)} = \frac{8}{9}\Gamma(\tfrac{1}{3})\Gamma(\tfrac{2}{3}) = \frac{8}{9}\cdot\frac{\pi}{\sin\pi/3} = \frac{16\pi}{9\sqrt{3}}$$

4.21. Prove the *duplication formula*: $2^{2p-1}\Gamma(p)\Gamma(p+\frac{1}{2}) = \sqrt{\pi}\,\Gamma(2p)$.

Let $I = \int_0^{\pi/2}\sin^{2p}x\,dx$, $J = \int_0^{\pi/2}\sin^{2p}2x\,dx$.

Then $I = \frac{1}{2}B(p+\frac{1}{2},\frac{1}{2}) = \dfrac{\Gamma(p+\frac{1}{2})\sqrt{\pi}}{2\,\Gamma(p+1)}$

Letting $2x = u$, we find

$$J = \frac{1}{2}\int_0^\pi \sin^{2p}u\,du = \int_0^{\pi/2}\sin^{2p}u\,du = I$$

But

$$J = \int_0^{\pi/2}(2\sin x\cos x)^{2p}\,dx = 2^{2p}\int_0^{\pi/2}\sin^{2p}x\cos^{2p}x\,dx$$

$$= 2^{2p-1}B(p+\tfrac{1}{2}, p+\tfrac{1}{2}) = \frac{2^{2p-1}\{\Gamma(p+\frac{1}{2})\}^2}{\Gamma(2p+1)}$$

Then since $I = J$,

$$\frac{\Gamma(p+\frac{1}{2})\sqrt{\pi}}{2p\,\Gamma(p)} \;=\; \frac{2^{2p-1}\{\Gamma(p+\frac{1}{2})\}^2}{2p\,\Gamma(2p)}$$

and the required result follows.

4.22. Prove that $\displaystyle\int_0^\infty \frac{\cos x}{x^p}\,dx \;=\; \frac{\pi}{2\,\Gamma(p)\cos(p\pi/2)}, \quad 0 < p < 1.$

We have $\dfrac{1}{x^p} = \dfrac{1}{\Gamma(p)}\displaystyle\int_0^\infty u^{p-1}e^{-xu}\,du.$ Then

$$\int_0^\infty \frac{\cos x}{x^p}\,dx \;=\; \frac{1}{\Gamma(p)}\int_0^\infty\int_0^\infty u^{p-1}e^{-xu}\cos x\,du\,dx \;=\; \frac{1}{\Gamma(p)}\int_0^\infty \frac{u^p}{1+u^2}\,du \tag{1}$$

where we have reversed the order of integration and used the fact that

$$\int_0^\infty e^{-xu}\cos x\,dx \;=\; \frac{u}{u^2+1} \tag{2}$$

Letting $u^2 = v$ in the last integral in (1), we have by Problem 4.18

$$\int_0^\infty \frac{u^p}{1+u^2}\,du \;=\; \frac{1}{2}\int_0^\infty \frac{v^{(p-1)/2}}{1+v}\,dv \;=\; \frac{\pi}{2\sin(p+1)\pi/2} \;=\; \frac{\pi}{2\cos p\pi/2} \tag{3}$$

Substitution of (3) in (1) yields the required result.

STIRLING'S FORMULA

4.23. Show that for large n, $n! = \sqrt{2\pi n}\,n^n e^{-n}$ approximately.

We have

$$\Gamma(n+1) \;=\; \int_0^\infty x^n e^{-x}\,dx \;=\; \int_0^\infty e^{n\ln x - x}\,dx \tag{1}$$

The function $n\ln x - x$ has a relative maximum for $x = n$, as is easily shown by elementary calculus. This leads us to the substitution $x = n + y$. Then (1) becomes

$$\Gamma(n+1) \;=\; e^{-n}\int_{-n}^\infty e^{n\ln(n+y) - y}\,dy \;=\; e^{-n}\int_{-n}^\infty e^{n\ln n + n\ln(1+y/n) - y}\,dy \tag{2}$$

$$=\; n^n e^{-n}\int_{-n}^\infty e^{n\ln(1+y/n) - y}\,dy$$

Up to now the analysis is rigorous. The formal procedures which follow can be made rigorous by suitable limiting processes, but the proofs become involved and we shall omit them.

In (2) use the result

$$\ln(1+x) \;=\; x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \tag{3}$$

with $x = y/n$. Then on letting $y = \sqrt{n}\,v$, we find

$$\Gamma(n+1) \;=\; n^n e^{-n}\int_{-n}^\infty e^{-y^2/2n + y^3/3n^2 - \cdots}\,dy \;=\; n^n e^{-n}\sqrt{n}\int_{-\sqrt{n}}^\infty e^{-v^2/2 + v^3/3\sqrt{n} - \cdots}\,dv \tag{4}$$

When n is large a close approximation is

$$\Gamma(n+1) \;=\; n^n e^{-n}\sqrt{n}\int_{-\infty}^\infty e^{-v^2/2}\,dv \;=\; \sqrt{2\pi n}\,n^n e^{-n} \tag{5}$$

It is of interest that from (4) we can obtain the entire asymptotic series for the gamma function (result 4. on page 68). See Problem 4.36.

SPECIAL FUNCTIONS AND ASYMPTOTIC EXPANSIONS

4.24. (*a*) Prove that if $x > 0$, $p > 0$, then

$$I_p = \int_x^\infty \frac{e^{-u}}{u^p}du = S_n(x) + R_n(x)$$

where

$$S_n(x) = e^{-x}\left\{\frac{1}{x^p} - \frac{p}{x^{p+1}} + \frac{p(p+1)}{x^{p+2}} - \cdots + (-1)^n \frac{p(p+1)\cdots(p+n)}{x^{p+n}}\right\}$$

$$R_n(x) = (-1)^{n+1}p(p+1)\cdots(p+n)\int_x^\infty \frac{e^{-u}}{u^{p+n+1}}\,du$$

(*b*) Prove that $\lim_{x\to\infty} x^n\left|\int_x^\infty \frac{e^{-u}}{u^p}du - S_n(x)\right| = \lim_{x\to\infty} x^n|R_n(x)| = 0.$

(*c*) Explain the significance of the results in (*b*).

(*a*) Integrating by parts, we have

$$I_p = \int_x^\infty \frac{e^{-u}}{u^p}du = \frac{e^{-x}}{x^p} - p\int_x^\infty \frac{e^{-u}}{u^{p+1}}du = \frac{e^{-x}}{x^p} - pI_{p+1}$$

Similarly $I_{p+1} = \dfrac{e^{-x}}{x^{p+1}} - (p+1)I_{p+2}$ so that

$$I_p = \frac{e^{-x}}{x^p} - p\left\{\frac{e^{-x}}{x^{p+1}} - (p+1)I_{p+2}\right\} = \frac{e^{-x}}{x^p} - \frac{pe^{-x}}{x^{p+1}} + p(p+1)I_{p+2}$$

By continuing in this manner the required result follows.

(*b*)
$$|R_n(x)| = p(p+1)\cdots(p+n)\int_x^\infty \frac{e^{-u}}{u^{p+n+1}}du \leqq p(p+1)\cdots(p+n)\int_x^\infty \frac{e^{-u}}{x^{p+n+1}}du$$
$$\leqq \frac{p(p+1)\cdots(p+n)}{x^{p+n+1}}$$

since $\int_x^\infty e^{-u}\,du \leqq \int_0^\infty e^{-u}\,du = 1.$ Thus

$$\lim_{x\to\infty} x^n|R_n(x)| \leqq \lim_{x\to\infty} \frac{p(p+1)\cdots(p+n)}{x^{p+1}} = 0$$

(*c*) Because of the results in (*b*), we can say that

$$\int_x^\infty \frac{e^{-u}}{u^p}du \sim e^{-x}\left\{\frac{1}{x^p} - \frac{p}{x^{p+1}} + \frac{p(p+1)}{x^{p+2}} - \cdots\right\} \tag{1}$$

i.e. the series on the right is the asymptotic expansion of the function on the left.

4.25. Show that

$$\operatorname{erf}(x) \sim 1 - \frac{e^{-x^2}}{\sqrt{\pi}}\left(\frac{1}{x} - \frac{1}{2x^3} + \frac{1\cdot 3}{2^2x^5} - \frac{1\cdot 3\cdot 5}{2^3x^7} + \cdots\right)$$

We have
$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-v^2}\,dv = \frac{1}{\sqrt{\pi}}\int_0^{x^2} u^{-1/2}e^{-u}\,du$$
$$= 1 - \frac{1}{\sqrt{\pi}}\int_{x^2}^\infty u^{-1/2}e^{-u}\,du$$

Now from the result (*1*) of Problem 4.24 we have, on letting $p = 1/2$ and replacing x by x^2,

$$\int_{x^2}^\infty u^{-1/2}e^{-u}\,du \sim e^{-x^2}\left(\frac{1}{x} - \frac{1}{2x^3} + \frac{1\cdot 3}{2^2x^5} - \frac{1\cdot 3\cdot 5}{2^3x^7} + \cdots\right)$$

which gives the required result.

Supplementary Problems

THE GAMMA FUNCTION

4.26. Evaluate (a) $\dfrac{\Gamma(7)}{2\,\Gamma(4)\,\Gamma(3)}$, (b) $\dfrac{\Gamma(3)\,\Gamma(3/2)}{\Gamma(9/2)}$, (c) $\Gamma(1/2)\,\Gamma(3/2)\,\Gamma(5/2)$.

4.27. Evaluate (a) $\int_0^\infty x^4 e^{-x}\,dx$, (b) $\int_0^\infty x^6 e^{-3x}\,dx$, (c) $\int_0^\infty x^2 e^{-2x^2}\,dx$.

4.28. Find (a) $\int_0^\infty e^{-x^3}\,dx$, (b) $\int_0^\infty \sqrt[4]{x}\,e^{-\sqrt{x}}\,dx$, (c) $\int_0^\infty y^3 e^{-2y^5}\,dy$.

4.29. Show that $\int_0^\infty \dfrac{e^{-st}}{\sqrt{t}}\,dt = \sqrt{\dfrac{\pi}{s}}$, $s > 0$.

4.30. Prove that (a) $\Gamma(n) = \int_0^1 \left(\ln\dfrac{1}{x}\right)^{n-1} dx$, $n > 0$

(b) $\int_0^1 x^p\left(\ln\dfrac{1}{x}\right)^q dx = \dfrac{\Gamma(q+1)}{(p+1)^{q+1}}$, $p > -1$, $q > -1$

4.31. Evaluate (a) $\int_0^1 (\ln x)^4\,dx$, (b) $\int_0^1 (x\ln x)^3\,dx$, (c) $\int_0^1 \sqrt[3]{\ln(1/x)}\,dx$.

4.32. Evaluate (a) $\Gamma(-7/2)$, (b) $\Gamma(-1/3)$.

4.33. Prove that $\lim\limits_{x\to -m} |\Gamma(x)| = \infty$ where $m = 0, 1, 2, 3, \ldots$.

4.34. Prove that if m is a positive integer, $\Gamma(-m+\frac{1}{2}) = \dfrac{(-1)^m\,2^m\sqrt{\pi}}{1\cdot 3\cdot 5\cdots(2m-1)}$

4.35. Prove that $\Gamma'(1) = \int_0^\infty e^{-x}\ln x\,dx$ is a negative number. (It is equal to $-\gamma$, where $\gamma = 0.577215\ldots$ is called *Euler's constant*.)

4.36. Obtain the miscellaneous result 4. on page 68 from the result (4) of Problem 4.23.
[*Hint:* Expand $e^{(v^3/3\sqrt{n})-\cdots}$ in a power series and replace the lower limit of the integral by $-\infty$.]

THE BETA FUNCTION

4.37. Evaluate (a) $B(3,5)$, (b) $B(3/2,2)$, (c) $B(1/3,2/3)$.

4.38. Find (a) $\int_0^1 x^2(1-x)^3\,dx$, (b) $\int_0^1 \sqrt{(1-x)/x}\,dx$, (c) $\int_0^2 (4-x^2)^{3/2}\,dx$.

4.39. Evaluate (a) $\int_0^4 u^{3/2}(4-u)^{5/2}\,du$, (b) $\int_0^3 \dfrac{dx}{\sqrt{3x-x^2}}$.

4.40. Prove that $\int_0^a \dfrac{dy}{\sqrt{a^4-y^4}} = \dfrac{\{\Gamma(1/4)\}^2}{4a\sqrt{2\pi}}$.

4.41. Evaluate (a) $\int_0^{\pi/2} \sin^4\theta\cos^4\theta\,d\theta$, (b) $\int_0^{2\pi} \cos^6\theta\,d\theta$.

4.42. Evaluate (a) $\int_0^\pi \sin^5\theta\,d\theta$, (b) $\int_0^{\pi/2} \cos^5\theta\sin^2\theta\,d\theta$.

4.43. Prove that (a) $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \pi/\sqrt{2}$; (b) $\int_0^{\pi/2} \tan^p\theta\, d\theta = \frac{\pi}{2}\sec\frac{p\pi}{2}$, $0 < p < 1$.

4.44. Prove that (a) $\int_0^\infty \frac{x\,dx}{1+x^6} = \frac{\pi}{3\sqrt{3}}$, (b) $\int_0^\infty \frac{y^2\,dy}{1+y^4} = \frac{\pi}{2\sqrt{2}}$.

4.45. Prove that $\int_{-\infty}^\infty \frac{e^{2x}}{ae^{3x}+b}\,dx = \frac{2\pi}{3\sqrt{3}\,a^{2/3}b^{1/3}}$, where $a, b > 0$.

4.46. Prove that $\int_{-\infty}^\infty \frac{e^{2x}}{(e^{3x}+1)^2}\,dx = \frac{2\pi}{9\sqrt{3}}$. [*Hint:* Differentiate with respect to b in Problem 4.45.]

SPECIAL FUNCTIONS AND ASYMPTOTIC EXPANSIONS

4.47. Show that $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3\cdot 1!} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots\right)$.

4.48. Obtain the asymptotic expansion $Ei(x) \sim \frac{e^{-x}}{x}\left(1 - \frac{1!}{x} + \frac{2!}{x^2} - \frac{3!}{x^3} + \cdots\right)$.

4.49. Show that (a) $Si(-x) = -Si(x)$, (b) $Si(\infty) = \pi/2$.

4.50. Obtain the asymptotic expansions

$$Si(x) \sim \frac{\pi}{2} - \frac{\sin x}{x}\left(\frac{1}{x} - \frac{3!}{x^3} + \frac{5!}{x^5} - \cdots\right) - \frac{\cos x}{x}\left(1 - \frac{2!}{x^2} + \frac{4!}{x^4} - \cdots\right)$$

$$Ci(x) \sim \frac{\cos x}{x}\left(\frac{1}{x} - \frac{3!}{x^3} + \frac{5!}{x^5} - \cdots\right) - \frac{\sin x}{x}\left(1 - \frac{2!}{x^2} + \frac{4!}{x^4} - \cdots\right)$$

4.51. Show that $\int_0^\infty \frac{\sin x}{x^p}\,dx = \frac{\pi}{2\,\Gamma(p)\sin(p\pi/2)}$, $0 < p < 1$.

4.52. Show that $\int_0^\infty \sin x^2\,dx = \int_0^\infty \cos x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.

4.53. Prove that $\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e$