Chapter 5

Fourier Integrals and Applications

THE NEED FOR FOURIER INTEGRALS

In Chapter 2 we considered the theory and applications involving the expansion of a function $f(x)$ of period $2L$ into a Fourier series. One question which arises quite naturally is: what happens in the case where $L \to \infty$? We shall find that in such case the Fourier series becomes a *Fourier integral*. We shall discuss Fourier integrals and their applications in this chapter.

THE FOURIER INTEGRAL

Let us assume the following conditions on $f(x)$:

1. $f(x)$ and $f'(x)$ are piecewise continuous in *every* finite interval.
2. $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges, i.e. $f(x)$ is absolutely integrable in $(-\infty, \infty)$.

Then *Fourier's integral theorem* states that

$$f(x) = \int_0^\infty \{A(\alpha)\cos \alpha x + B(\alpha)\sin \alpha x\}\, d\alpha \qquad (1)$$

where

$$\left.\begin{aligned} A(\alpha) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos \alpha x\, dx \\ B(\alpha) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin \alpha x\, dx \end{aligned}\right\} \qquad (2)$$

The result (1) holds if x is a point of continuity of $f(x)$. If x is a point of discontinuity, we must replace $f(x)$ by $\dfrac{f(x+0)+f(x-0)}{2}$ as in the case of Fourier series. Note that the above conditions are sufficient but not necessary.

The similarity of (1) and (2) with corresponding results for Fourier series is apparent. The right-hand side of (1) is sometimes called a *Fourier integral expansion* of $f(x)$.

EQUIVALENT FORMS OF FOURIER'S INTEGRAL THEOREM

Fourier's integral theorem can also be written in the forms

$$f(x) = \frac{1}{\pi}\int_{\alpha=0}^{\infty}\int_{u=-\infty}^{\infty} f(u)\cos \alpha(x-u)\, du\, d\alpha \qquad (3)$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\, e^{i\alpha(x-u)}\, du\, d\alpha$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha x}\, d\alpha \int_{-\infty}^{\infty} f(u)\, e^{-i\alpha u}\, du \qquad (4)$$

where it is understood that if $f(x)$ is not continuous at x the left side must be replaced by $\dfrac{f(x+0)+f(x-0)}{2}$.

These results can be simplified somewhat if $f(x)$ is either an odd or an even function, and we have

$$f(x) \;=\; \frac{2}{\pi}\int_0^\infty \sin \alpha x\, d\alpha \int_0^\infty f(u) \sin \alpha u\, du \qquad \text{if } f(x) \text{ is odd} \tag{5}$$

$$f(x) \;=\; \frac{2}{\pi}\int_0^\infty \cos \alpha x\, d\alpha \int_0^\infty f(u) \cos \alpha u\, du \qquad \text{if } f(x) \text{ is even} \tag{6}$$

FOURIER TRANSFORMS

From (4) it follows that if

$$F(\alpha) \;=\; \int_{-\infty}^{\infty} f(u)\, e^{-i\alpha u}\, du \tag{7}$$

then

$$f(x) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\, e^{i\alpha x}\, d\alpha \tag{8}$$

The function $F(\alpha)$ is called the *Fourier transform* of $f(x)$ and is sometimes written $F(\alpha) = \mathcal{F}\{f(x)\}$. The function $f(x)$ is the *inverse Fourier transform* of $F(\alpha)$ and is written $f(x) = \mathcal{F}^{-1}\{F(\alpha)\}$.

Note: The constants 1 and $1/2\pi$ preceding the integral signs in (7) and (8) could be replaced by any two constants whose product is $1/2\pi$. In this book, however, we shall keep to the above choice.

FOURIER SINE AND COSINE TRANSFORMS

If $f(x)$ is an odd function, then Fourier's integral theorem reduces to (5). If we let

$$F_S(\alpha) \;=\; \int_0^\infty f(u) \sin \alpha u\, du \tag{9}$$

then it follows from (5) that

$$f(x) \;=\; \frac{2}{\pi}\int_0^\infty F_S(\alpha) \sin \alpha x\, d\alpha \tag{10}$$

We call $F_S(\alpha)$ the *Fourier sine transform* of $f(x)$, while $f(x)$ is the *inverse Fourier sine transform* of $F_S(\alpha)$.

Similarly, if $f(x)$ is an even function, Fourier's integral theorem reduces to (6). Thus if we let

$$F_C(\alpha) \;=\; \int_0^\infty f(u) \cos \alpha u\, du \tag{11}$$

then it follows from (6) that

$$f(x) \;=\; \frac{2}{\pi}\int_0^\infty F_C(\alpha) \cos \alpha x\, d\alpha \tag{12}$$

We call $F_C(\alpha)$ the *Fourier cosine transform* of $f(x)$, while $f(x)$ is the *inverse Fourier cosine transform* of $F_C(\alpha)$.

PARSEVAL'S IDENTITIES FOR FOURIER INTEGRALS

In Chapter 2, page 23, we arrived at Parseval's identity for Fourier series. An analogy exists for Fourier integrals.

If $F(\alpha)$ and $G(\alpha)$ are Fourier transforms of $f(x)$ and $g(x)$ respectively we can show that

$$\int_{-\infty}^{\infty} f(x)\, g(x)\, dx \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\,\overline{G(\alpha)}\, d\alpha \tag{13}$$

where the bar signifies the complex conjugate obtained on replacing i by $-i$. In particular, if $f(x) = g(x)$ and hence $F(\alpha) = G(\alpha)$, then we have

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\alpha)|^2\, d\alpha \tag{14}$$

We can refer to (14), or to the more general (13), as *Parseval's identity* for Fourier integrals.

Corresponding results can be written involving sine and cosine transforms. If $F_S(\alpha)$ and $G_S(\alpha)$ are the Fourier sine transforms of $f(x)$ and $g(x)$, respectively, then

$$\int_0^{\infty} f(x)\, g(x)\, dx \quad = \quad \frac{2}{\pi}\int_0^{\infty} F_S(\alpha)\, G_S(\alpha)\, d\alpha \tag{15}$$

Similarly, if $F_C(\alpha)$ and $G_C(\alpha)$ are the Fourier cosine transforms of $f(x)$ and $g(x)$, respectively, then

$$\int_0^{\infty} f(x)\, g(x)\, dx \quad = \quad \frac{2}{\pi}\int_0^{\infty} F_C(\alpha)\, G_C(\alpha)\, d\alpha \tag{16}$$

In the special case where $f(x) = g(x)$, (15) and (16) become respectively

$$\int_0^{\infty} \{f(x)\}^2\, dx \quad = \quad \frac{2}{\pi}\int_0^{\infty} \{F_S(\alpha)\}^2\, d\alpha \tag{17}$$

$$\int_0^{\infty} \{f(x)\}^2\, dx \quad = \quad \frac{2}{\pi}\int_0^{\infty} \{F_C(\alpha)\}^2\, d\alpha \tag{18}$$

THE CONVOLUTION THEOREM FOR FOURIER TRANSFORMS

The *convolution* of the functions $f(x)$ and $g(x)$ is defined by

$$f * g \quad = \quad \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du \tag{19}$$

An important theorem, often referred to as the *convolution theorem*, states that the Fourier transform of the convolution of $f(x)$ and $g(x)$ is equal to the product of the Fourier transforms of $f(x)$ and $g(x)$. In symbols,

$$\mathcal{F}\{f * g\} \quad = \quad \mathcal{F}\{f\}\,\mathcal{F}\{g\} \tag{20}$$

The convolution has other important properties. For example, we have for functions f, g, and h:

$$f * g \;=\; g * f, \quad f*(g*h) \;=\; (f*g)*h, \quad f*(g+h) \;=\; f*g + f*h \tag{21}$$

i.e., the convolution obeys the commutative, associative and distributive laws of algebra.

APPLICATIONS OF FOURIER INTEGRALS AND TRANSFORMS

Fourier integrals and transforms can be used in solving a variety of boundary value problems arising in science and engineering. See Problems 5.20–5.22.

Solved Problems

THE FOURIER INTEGRAL AND FOURIER TRANSFORMS

5.1. Show that (*1*) and (*3*), page 80, are equivalent forms of Fourier's integral theorem.

Let us start with the form

$$f(x) = \frac{1}{\pi}\int_{\alpha=0}^{\infty}\int_{u=-\infty}^{\infty} f(u)\cos\alpha(x-u)\,du\,d\alpha \tag{1}$$

which is proved later (see Problems 5.10–5.14). The result (*1*) can be written as

$$f(x) = \frac{1}{\pi}\int_{\alpha=0}^{\infty}\int_{u=-\infty}^{\infty} f(u)[\cos\alpha x\cos\alpha u + \sin\alpha x\sin\alpha u]\,du\,d\alpha$$

or

$$f(x) = \int_{\alpha=0}^{\infty} \{A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x\}\,d\alpha \tag{2}$$

where we let

$$\left.\begin{aligned} A(\alpha) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\alpha u\,du \\ B(\alpha) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\sin\alpha u\,du \end{aligned}\right\} \tag{3}$$

Conversely, by substituting (*3*) into (*2*) we obtain (*1*). Thus the two forms are equivalent.

5.2. Show that (*3*) and (*4*), page 80, are equivalent.

We have from (*3*), page 80, and the fact that $\cos\alpha(x-u)$ is an even function of α:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\cos\alpha(x-u)\,du\,d\alpha \tag{1}$$

Then, using the fact that $\sin\alpha(x-u)$ is an odd function of α, we have

$$0 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\sin\alpha(x-u)\,du\,d\alpha \tag{2}$$

Multiplying (*2*) by i and adding to (*1*) we then have

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)[\cos\alpha(x-u) + i\sin\alpha(x-u)]\,du\,d\alpha \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)e^{i\alpha(x-u)}\,du\,d\alpha \end{aligned}$$

Similarly we can deduce that (*3*), page 80, follows from (*4*).

5.3. (*a*) Find the Fourier transform of $f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$.

(*b*) Graph $f(x)$ and its Fourier transform for $a = 3$.

(*a*) The Fourier transform of $f(x)$ is

$$\begin{aligned} F(\alpha) &= \int_{-\infty}^{\infty} f(u)e^{-i\alpha u}\,du = \int_{-a}^{a}(1)e^{-i\alpha u}\,du = \left.\frac{e^{-i\alpha u}}{-i\alpha}\right|_{-a}^{a} \\ &= \frac{e^{i\alpha a} - e^{-i\alpha a}}{i\alpha} = 2\frac{\sin\alpha a}{\alpha}, \quad \alpha \neq 0 \end{aligned}$$

For $\alpha = 0$, we obtain $F(\alpha) = 2a$.

(b) The graphs of $f(x)$ and $F(\alpha)$ for $a = 3$ are shown in Figs. 5-1 and 5-2 respectively.

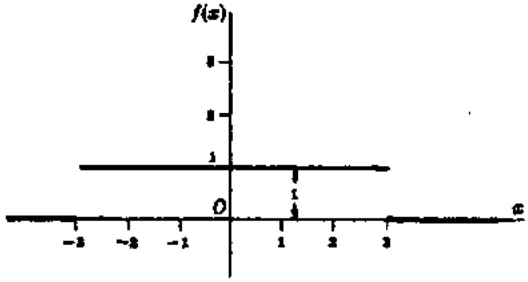

Fig. 5-1

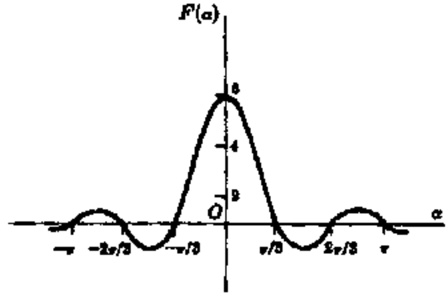

Fig. 5-2

5.4. (a) Use the result of Problem 5.3 to evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\sin \alpha a \cos \alpha x}{\alpha}\, d\alpha$.

(b) Deduce the value of $\displaystyle\int_0^{\infty} \frac{\sin u}{u}\, du$.

(a) From Fourier's integral theorem, if

$$F(\alpha) = \int_{-\infty}^{\infty} f(u) e^{-i\alpha u}\, du \qquad \text{then} \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha) e^{i\alpha x}\, d\alpha$$

Then from Problem 5.3,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} 2\frac{\sin \alpha a}{\alpha} e^{i\alpha x}\, d\alpha = \begin{cases} 1 & |x| < a \\ 1/2 & |x| = a \\ 0 & |x| > a \end{cases} \tag{1}$$

The left side of (1) is equal to

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin \alpha a \cos \alpha x}{\alpha}\, d\alpha + \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{\sin \alpha a \sin \alpha x}{\alpha}\, d\alpha \tag{2}$$

The integrand in the second integral of (2) is odd and so the integral is zero. Then from (1) and (2), we have

$$\int_{-\infty}^{\infty} \frac{\sin \alpha a \cos \alpha x}{\alpha}\, d\alpha = \begin{cases} \pi & |x| < a \\ \pi/2 & |x| = a \\ 0 & |x| > a \end{cases} \tag{3}$$

(b) If $x = 0$ and $a = 1$ in the result of (a), we have

$$\int_{-\infty}^{\infty} \frac{\sin \alpha}{\alpha}\, d\alpha = \pi \qquad \text{or} \qquad \int_0^{\infty} \frac{\sin \alpha}{\alpha}\, d\alpha = \frac{\pi}{2}$$

since the integrand is even.

5.5. (a) Find the Fourier cosine transform of $f(x) = e^{-mx}$, $m > 0$.

(b) Use the result in (a) to show that

$$\int_0^{\infty} \frac{\cos pv}{v^2 + \beta^2}\, dv = \frac{\pi}{2\beta} e^{-p\beta} \qquad (p > 0,\ \beta > 0)$$

(a) The Fourier cosine transform of $f(x) = e^{-mx}$ is by definition

$$F_C(\alpha) = \int_0^\infty e^{-mu} \cos \alpha u \, du$$

$$= \frac{e^{-mu}(-m \cos \alpha u + \alpha \sin \alpha u)}{m^2 + \alpha^2}\Bigg|_0^\infty$$

$$= \frac{m}{m^2 + \alpha^2}$$

(b) From (12), page 81, we have

$$f(x) = \frac{2}{\pi}\int_0^\infty F_C(\alpha) \cos \alpha x \, d\alpha$$

or

$$e^{-mx} = \frac{2}{\pi}\int_0^\infty \frac{m \cos \alpha x}{m^2 + \alpha^2}\, d\alpha$$

i.e.

$$\int_0^\infty \frac{\cos \alpha x}{m^2 + \alpha^2}\, d\alpha = \frac{\pi}{2m} e^{-mx}$$

Replacing α by v, x by p, and m by β, we have

$$\int_0^\infty \frac{\cos pv}{v^2 + \beta^2}\, dv = \frac{\pi}{2\beta} e^{-p\beta}, \qquad p > 0,\ \beta > 0$$

5.6. **Solve the integral equation**

$$\int_0^\infty f(x) \sin \alpha x \, dx = \begin{cases} 1 - \alpha & 0 \leqq \alpha \leqq 1 \\ 0 & \alpha > 1 \end{cases}$$

If we write

$$F_S(\alpha) = \int_0^\infty f(x) \sin \alpha x \, dx = \begin{cases} 1 - \alpha & 0 \leqq \alpha \leqq 1 \\ 0 & \alpha > 1 \end{cases}$$

then, by (10), page 81,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_S(\alpha) \sin \alpha x \, d\alpha$$

$$= \frac{2}{\pi}\int_0^1 (1 - \alpha) \sin \alpha x \, d\alpha$$

$$= \frac{2(x - \sin x)}{\pi x^2}$$

THE CONVOLUTION THEOREM

5.7. **Prove the convolution theorem on page 82.**

We have by definition of the Fourier transform

$$F(\alpha) = \int_{-\infty}^\infty f(u) e^{-i\alpha u}\, du, \qquad G(\alpha) = \int_{-\infty}^\infty g(v) e^{-i\alpha v}\, dv \tag{1}$$

Then

$$F(\alpha)\, G(\alpha) = \int_{-\infty}^\infty \int_{-\infty}^\infty f(u)\, g(v) e^{-i\alpha(u+v)}\, du\, dv \tag{2}$$

Let $u + v = x$ in the double integral (2) which we wish to transform from the variables (u, v) to the variables (u, x). From advanced calculus we know that

$$du\, dv = \frac{\partial(u, v)}{\partial(u, x)}\, du\, dx \tag{3}$$

where the Jacobian of the transformation is given by

$$\frac{\partial(u,v)}{\partial(u,x)} = \begin{vmatrix} \frac{\partial u}{\partial u} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial u} & \frac{\partial v}{\partial x} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Thus (2) becomes

$$\begin{aligned} F(\alpha)\,G(\alpha) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\,g(x-u)e^{-i\alpha x}\,du\,dx \\ &= \int_{-\infty}^{\infty} e^{-i\alpha x}\left[\int_{-\infty}^{\infty} f(u)\,g(x-u)\,du\right]dx \\ &= \mathcal{F}\left\{\int_{-\infty}^{\infty} f(u)\,g(x-u)\,du\right\} \\ &= \mathcal{F}\{f * g\} \end{aligned}$$

where $f * g = \int_{-\infty}^{\infty} f(u)\,g(x-u)\,du$ is the convolution of f and g.

From this we have equivalently

$$\begin{aligned} f * g &= \mathcal{F}^{-1}\{F(\alpha)\,G(\alpha)\} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha x}F(\alpha)\,G(\alpha)\,d\alpha \end{aligned}$$

5.8. Show that $f * g = g * f$.

Let $x - u = v$. Then

$$\begin{aligned} f * g &= \int_{-\infty}^{\infty} f(u)\,g(x-u)\,du = \int_{-\infty}^{\infty} f(x-v)\,g(v)\,dv \\ &= \int_{-\infty}^{\infty} g(v)\,f(x-v)\,dv = g * f \end{aligned}$$

5.9. Solve the integral equation

$$y(x) = g(x) + \int_{-\infty}^{\infty} y(u)r(x-u)\,du$$

where $g(x)$ and $r(x)$ are given.

Suppose that the Fourier transforms of $y(x)$, $g(x)$ and $r(x)$ exist, and denote them by $Y(\alpha)$, $G(\alpha)$ and $R(\alpha)$ respectively. Then, taking the Fourier transform of both sides of the given integral equation, we have by the convolution theorem

$$Y(\alpha) = G(\alpha) + Y(\alpha)\,R(\alpha) \quad \text{or} \quad Y(\alpha) = \frac{G(\alpha)}{1 - R(\alpha)}$$

Then $$y(x) = \mathcal{F}^{-1}\left\{\frac{G(\alpha)}{1-R(\alpha)}\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{G(\alpha)}{1-R(\alpha)}\right\}e^{i\alpha x}\,d\alpha$$

assuming this integral exists.

5.10. Solve for $y(x)$ the integral equation

$$\int_{-\infty}^{\infty}\frac{y(u)\,du}{(x-u)^2+a^2} = \frac{1}{x^2+b^2} \qquad 0 < a < b$$

We have

$$\mathcal{F}\left\{\frac{1}{x^2+b^2}\right\} = \int_{-\infty}^{\infty} \frac{e^{-i\alpha x}}{x^2+b^2}\,dx = 2\int_0^{\infty} \frac{\cos \alpha x}{x^2+b^2}\,dx = \frac{\pi}{b}e^{-b\alpha}$$

on making use of Problem 5.5(*b*). Then, taking the Fourier transform of both sides of the integral equation, we find

$$\mathcal{F}(y)\mathcal{F}\left\{\frac{1}{x^2+a^2}\right\} = \mathcal{F}\left\{\frac{1}{x^2+b^2}\right\}$$

i.e.
$$Y(\alpha)\frac{\pi}{a}e^{-a\alpha} = \frac{\pi}{b}e^{-b\alpha} \qquad \text{or} \qquad Y(\alpha) = \frac{a}{b}e^{-(b-a)\alpha}$$

Thus
$$y(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha x}Y(\alpha)\,d\alpha = \frac{a}{b\pi}\int_0^{\infty} e^{-(b-a)\alpha}\cos \alpha x\,d\alpha = \frac{(b-a)a}{b\pi[x^2+(b-a)^2]}$$

PROOF OF THE FOURIER INTEGRAL THEOREM

5.11. Present a heuristic demonstration of Fourier's integral theorem by use of a limiting form of Fourier series.

Let
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \tag{1}$$

where $a_n = \frac{1}{L}\int_{-L}^{L} f(u)\cos\frac{n\pi u}{L}\,du$ and $b_n = \frac{1}{L}\int_{-L}^{L} f(u)\sin\frac{n\pi u}{L}\,du$.

Then by substitution of these coefficients into (*1*) we find

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(u)\,du + \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(u)\cos\frac{n\pi}{L}(u-x)\,du \tag{2}$$

If we assume that $\int_{-\infty}^{\infty}|f(u)|\,du$ converges, the first term on the right of (*2*) approaches zero as $L \to \infty$, while the remaining part appears to approach

$$\lim_{L\to\infty}\frac{1}{L}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f(u)\cos\frac{n\pi}{L}(u-x)\,du \tag{3}$$

This last step is not rigorous and makes the demonstration heuristic.

Calling $\Delta\alpha = \pi/L$, (*3*) can be written

$$f(x) = \lim_{\Delta\alpha\to 0}\sum_{n=1}^{\infty}\Delta\alpha\, F(n\,\Delta\alpha) \tag{4}$$

where we have written

$$F(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\cos\alpha(u-x)\,du \tag{5}$$

But the limit (*4*) is equal to

$$f(x) = \int_0^{\infty} F(\alpha)\,d\alpha = \frac{1}{\pi}\int_0^{\infty} d\alpha\int_{-\infty}^{\infty} f(u)\cos\alpha(u-x)\,du \tag{6}$$

which is Fourier's integral formula.

This demonstration merely provides a possible result. To be rigorous, we start with the double integral in (*6*) and examine the convergence. This method is considered in Problems 5.12–5.15.

5.12. Prove that: (*a*) $\lim_{\alpha\to\infty}\int_0^L \frac{\sin \alpha v}{v}\,dv = \frac{\pi}{2}$, (*b*) $\lim_{\alpha\to\infty}\int_{-L}^0 \frac{\sin \alpha v}{v}\,dv = \frac{\pi}{2}$.

(*a*) Let $\alpha v = y$. Then $\lim_{\alpha\to\infty}\int_0^L \frac{\sin \alpha v}{v}\,dv = \lim_{\alpha\to\infty}\int_0^{\alpha L}\frac{\sin y}{y}\,dy = \int_0^{\infty}\frac{\sin y}{y}\,dy = \frac{\pi}{2}$, as can be shown by using Problem 5.40.

(b) Let $\alpha v = -y$. Then $\lim_{\alpha\to\infty}\int_{-L}^{0}\frac{\sin\alpha v}{v}\,dv = \lim_{\alpha\to\infty}\int_{0}^{\alpha L}\frac{\sin y}{y}\,dy = \frac{\pi}{2}$.

5.13. Riemann's theorem states that if $F(x)$ is piecewise continuous in (a,b), then

$$\lim_{\alpha\to\infty}\int_a^b F(x)\sin\alpha x\,dx = 0$$

with a similar result for the cosine (see Problem 5.41). Use this to prove that

$$(a)\quad \lim_{\alpha\to\infty}\int_0^L f(x+v)\frac{\sin\alpha v}{v}\,dv = \frac{\pi}{2}f(x+0)$$

$$(b)\quad \lim_{\alpha\to\infty}\int_{-L}^0 f(x+v)\frac{\sin\alpha v}{v}\,dv = \frac{\pi}{2}f(x-0)$$

where $f(x)$ and $f'(x)$ are assumed piecewise continuous [see condition 1. on page 80].

(a) Using Problem 5.12(a), it is seen that a proof of the given result amounts to proving that

$$\lim_{\alpha\to\infty}\int_0^L \{f(x+v) - f(x+0)\}\frac{\sin\alpha v}{v}\,dv = 0$$

This follows at once from Riemann's theorem, because $F(v) = \frac{f(x+v)-f(x+0)}{v}$ is piecewise continuous in $(0,L)$ since $\lim_{v\to 0+}F(v)$ exists and $f(x)$ is piecewise continuous.

(b) A proof of this is analogous to that in part (a) if we make use of Problem 5.12(b).

5.14. If $f(x)$ satisfies the additional condition that $\int_{-\infty}^{\infty}|f(x)|\,dx$ converges, prove that

(a) $\lim_{\alpha\to\infty}\int_0^{\infty} f(x+v)\frac{\sin\alpha v}{v}\,dv = \frac{\pi}{2}f(x+0)$, (b) $\lim_{\alpha\to\infty}\int_{-\infty}^{0} f(x+v)\frac{\sin\alpha v}{v}\,dv = \frac{\pi}{2}f(x-0)$.

(a) We have

$$\int_0^{\infty} f(x+v)\frac{\sin\alpha v}{v}\,dv = \int_0^L f(x+v)\frac{\sin\alpha v}{v}\,dv + \int_L^{\infty} f(x+v)\frac{\sin\alpha v}{v}\,dv \qquad (1)$$

$$\int_0^{\infty} f(x+0)\frac{\sin\alpha v}{v}\,dv = \int_0^L f(x+0)\frac{\sin\alpha v}{v}\,dv + \int_L^{\infty} f(x+0)\frac{\sin\alpha v}{v}\,dv \qquad (2)$$

Subtracting,

$$\int_0^{\infty}\{f(x+v)-f(x+0)\}\frac{\sin\alpha v}{v}\,dv \qquad (3)$$
$$= \int_0^L \{f(x+v)-f(x+0)\}\frac{\sin\alpha v}{v}\,dv + \int_L^{\infty} f(x+v)\frac{\sin\alpha v}{v}\,dv - \int_L^{\infty} f(x+0)\frac{\sin\alpha v}{v}\,dv$$

Denoting the integrals in (3) by I, I_1, I_2 and I_3 respectively, we have $I = I_1+I_2+I_3$ so that

$$|I| \le |I_1| + |I_2| + |I_3| \qquad (4)$$

Now $$|I_2| \le \int_L^{\infty}\left|f(x+v)\frac{\sin\alpha v}{v}\right|dv \le \frac{1}{L}\int_L^{\infty}|f(x+v)|\,dv$$

Also $$|I_3| \le |f(x+0)|\left|\int_L^{\infty}\frac{\sin\alpha v}{v}\,dv\right|$$

Since $\int_0^{\infty}|f(x)|\,dx$ and $\int_0^{\infty}\frac{\sin\alpha v}{v}\,dv$ both converge, we can choose L so large that $|I_2| \le \epsilon/3$, $|I_3| \le \epsilon/3$. Also, we can choose α so large that $|I_1| \le \epsilon/3$. Then from (4) we have $|I| \le \epsilon$ for α and L sufficiently large, so that the required result follows.

(b) This result follows by reasoning exactly analogous to that in part (a).

5.15. Prove Fourier's integral formula if $f(x)$ satisfies the conditions stated on page 80.

We must prove that $\lim_{L\to\infty} \frac{1}{\pi}\int_{\alpha=0}^{L}\int_{u=-\infty}^{\infty} f(u)\cos\alpha(x-u)\,du\,d\alpha = \frac{f(x+0)+f(x-0)}{2}$.

Since $\left|\int_{-\infty}^{\infty} f(u)\cos\alpha(x-u)\,du\right| \leq \int_{-\infty}^{\infty}|f(u)|\,du$, which converges, it follows by the Weierstrass M test for integrals that $\int_{-\infty}^{\infty} f(u)\cos\alpha(x-u)\,du$ converges absolutely and uniformly for all α. We can show from this that the order of integration can be reversed to obtain

$$\begin{aligned}\frac{1}{\pi}\int_{\alpha=0}^{L} d\alpha\int_{u=-\infty}^{\infty} f(u)\cos\alpha(x-u)\,du &= \frac{1}{\pi}\int_{u=-\infty}^{\infty} f(u)\,du\int_{\alpha=0}^{L}\cos\alpha(x-u)\,d\alpha \\ &= \frac{1}{\pi}\int_{u=-\infty}^{\infty} f(u)\frac{\sin L(u-x)}{u-x}\,du \\ &= \frac{1}{\pi}\int_{v=-\infty}^{\infty} f(x+v)\frac{\sin Lv}{v}\,dv \\ &= \frac{1}{\pi}\int_{-\infty}^{0} f(x+v)\frac{\sin Lv}{v}\,dv + \frac{1}{\pi}\int_{0}^{\infty} f(x+v)\frac{\sin Lv}{v}\,dv\end{aligned}$$

where we have let $u = x + v$.

Letting $L\to\infty$, we see by Problem 5.14 that the given integral converges to $\frac{f(x+0)+f(x-0)}{2}$ as required.

SOLUTIONS USING FOURIER INTEGRALS

5.16. A semi-infinite thin bar ($x \geqq 0$) whose surface is insulated has an initial temperature equal to $f(x)$. A temperature of zero is suddenly applied to the end $x=0$ and maintained. (a) Set up the boundary value problem for the temperature $u(x,t)$ at any point x at time t. (b) Show that

$$u(x,t) = \frac{1}{\pi}\int_0^{\infty}\int_0^{\infty} f(v)e^{-\kappa\lambda^2 t}\sin\lambda v\,\sin\lambda x\,d\lambda\,dv$$

(a) The boundary value problem is

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2} \qquad x>0,\ t>0 \tag{1}$$

$$u(x,0) = f(x), \quad u(0,t) = 0, \quad |u(x,t)| < M \tag{2}$$

where the last condition is used since the temperature must be bounded for physical reasons.

(b) A solution of (1) obtained by separation of variables is

$$u(x,t) = e^{-\kappa\lambda^2 t}(A\cos\lambda x + B\sin\lambda x)$$

From the second of boundary conditions (2) we find $A=0$ so that

$$u(x,t) = Be^{-\kappa\lambda^2 t}\sin\lambda x \tag{3}$$

Now since there is no restriction on λ we can replace B in (3) by a function $B(\lambda)$ and still have a solution. Furthermore we can integrate over λ from 0 to ∞ and still have a solution. This is the analog of the superposition theorem for discrete values of λ used in connection with Fourier series. We thus arrive at the possible solution

$$u(x,t) = \int_0^{\infty} B(\lambda)e^{-\kappa\lambda^2 t}\sin\lambda x\,d\lambda \tag{4}$$

From the first of boundary conditions (3) we find

$$f(x) = \int_0^\infty B(\lambda) \sin \lambda x \, d\lambda$$

which is an integral equation for the determination of $B(\lambda)$. From page 81, we see that since $f(x)$ must be an odd function, we have

$$B(\lambda) = \frac{2}{\pi}\int_0^\infty f(x) \sin \lambda x \, dx = \frac{2}{\pi}\int_0^\infty f(v) \sin \lambda v \, dv$$

Using this in (4) we find

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \int_0^\infty f(v) e^{-\kappa\lambda^2 t} \sin \lambda v \sin \lambda x \, d\lambda \, dv$$

5.17. Show that the result of Problem 5.16 can be written

$$u(x,t) = \frac{1}{\sqrt{\pi}}\left[\int_{-x/2\sqrt{\kappa t}}^\infty e^{-w^2} f(2w\sqrt{\kappa t}+x)\, dw - \int_{x/2\sqrt{\kappa t}}^\infty e^{-w^2} f(2w\sqrt{\kappa t}-x)\, dw\right]$$

Since $\sin \lambda v \sin \lambda x = \frac{1}{2}[\cos \lambda(v-x) - \cos \lambda(v+x)]$, the result of Problem 5.16 can be written

$$u(x,t) = \frac{1}{\pi}\int_0^\infty \int_0^\infty f(v) e^{-\kappa\lambda^2 t} [\cos \lambda(v-x) - \cos \lambda(v+x)] \, d\lambda \, dv$$

$$= \frac{1}{\pi}\int_0^\infty f(v)\left[\int_0^\infty e^{-\kappa\lambda^2 t} \cos \lambda(v-x)\, d\lambda - \int_0^\infty e^{-\kappa\lambda^2 t} \cos \lambda(v+x)\, d\lambda\right] dv$$

From the integral

$$\int_0^\infty e^{-\alpha\lambda^2} \cos \beta\lambda \, d\lambda = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\, e^{-\beta^2/4\alpha}$$

(see Problem 4.9, page 72) we find

$$u(x,t) = \frac{1}{2\sqrt{\pi\kappa t}}\left[\int_0^\infty f(v) e^{-(v-x)^2/4\kappa t}\, dv - \int_0^\infty f(v) e^{-(v+x)^2/4\kappa t}\, dv\right]$$

Letting $(v-x)/2\sqrt{\kappa t} = w$ in the first integral and $(v+x)/2\sqrt{\kappa t} = w$ in the second integral, we find that

$$u(x,t) = \frac{1}{\sqrt{\pi}}\left[\int_{-x/2\sqrt{\kappa t}}^\infty e^{-w^2} f(2w\sqrt{\kappa t}+x)\, dw - \int_{x/2\sqrt{\kappa t}}^\infty e^{-w^2} f(2w\sqrt{\kappa t}-x)\, dw\right]$$

5.18. In case the initial temperature $f(x)$ in Problem 5.16 is the constant u_0, show that

$$u(x,t) = \frac{2u_0}{\sqrt{\pi}}\int_0^{x/2\sqrt{\kappa t}} e^{-w^2}\, dw = u_0 \operatorname{erf}(x/2\sqrt{\kappa t})$$

where $\operatorname{erf}(x/2\sqrt{\kappa t})$ is the error function (see page 69).

If $f(x,t) = u_0$, we obtain from Problem 5.17

$$u(x,t) = \frac{u_0}{\sqrt{\pi}}\left[\int_{-x/2\sqrt{\kappa t}}^\infty e^{-w^2}\, dw - \int_{x/2\sqrt{\kappa t}}^\infty e^{-w^2}\, dw\right]$$

$$= \frac{u_0}{\sqrt{\pi}}\int_{-x/2\sqrt{\kappa t}}^{x/2\sqrt{\kappa t}} e^{-w^2}\, dw = \frac{2u_0}{\sqrt{\pi}}\int_0^{x/2\sqrt{\kappa t}} e^{-w^2}\, dw = u_0 \operatorname{erf}(x/2\sqrt{\kappa t})$$

We can show that this actually is a solution of the corresponding boundary value problem (see Problem 5.48).

5.19. **Find a bounded solution to Laplace's equation $\nabla^2 v = 0$ for the half plane $y > 0$ (Fig. 5-3) if v takes on the value $f(x)$ on the x-axis.**

The boundary value problem for the determination of $v(x,y)$ is given by

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$$v(x,0) = f(x), \qquad |v(x,y)| < M$$

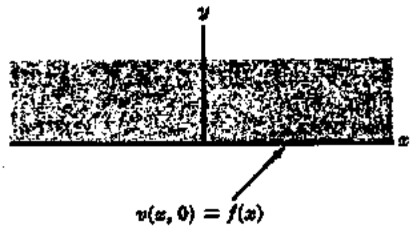

Fig. 5-3

To solve this, let $v = XY$ in the partial differential equation, where X depends only on x and Y depends only on y. Then, on separating the variables, we have

$$\frac{X''}{X} = -\frac{Y''}{Y}$$

Setting each side equal to $-\lambda^2$ we find

$$X'' + \lambda^2 X = 0, \qquad Y'' - \lambda^2 Y = 0$$

so that
$$X = a_1 \cos \lambda x + b_1 \sin \lambda x, \qquad Y = a_2 e^{\lambda y} + b_2 e^{-\lambda y}$$

Then the solution is
$$v(x,y) = (a_1 \cos \lambda x + b_1 \sin \lambda x)(a_2 e^{\lambda y} + b_2 e^{-\lambda y})$$

If $\lambda > 0$ the term in $e^{\lambda y}$ is unbounded as $y \to \infty$; so that to keep $v(x,y)$ bounded we must have $a_2 = 0$. This leads to the solution

$$v(x,y) = e^{-\lambda y}[A \cos \lambda x + B \sin \lambda x]$$

Since there is no restriction on λ, we can replace A by $A(\lambda)$, B by $B(\lambda)$ and integrate over λ to obtain

$$v(x,y) = \int_0^\infty e^{-\lambda y}[A(\lambda) \cos \lambda x + B(\lambda) \sin \lambda x]\, d\lambda \tag{1}$$

The boundary condition $v(x,0) = f(x)$ yields

$$\int_0^\infty [A(\lambda) \cos \lambda x + B(\lambda) \sin \lambda x]\, d\lambda = f(x)$$

Thus, from Fourier's integral theorem we find

$$A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u) \cos \lambda u\, du, \qquad B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u) \sin \lambda u\, du$$

Putting these in (1) we have finally:

$$v(x,y) = \frac{1}{\pi}\int_{\lambda=0}^{\infty}\int_{u=-\infty}^{\infty} e^{-\lambda y} f(u) \cos \lambda(u-x)\, du\, d\lambda \tag{2}$$

5.20. **Show that the solution to Problem 5.19 can be written in the form**

$$v(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y\, f(u)}{y^2 + (u-x)^2}\, du$$

Write the result (2) of Problem 5.19 as

$$v(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u)\left[\int_0^\infty e^{-\lambda y} \cos \lambda(u-x)\, d\lambda\right] du \tag{1}$$

Then by elementary integration we have

$$\int_0^\infty e^{-\lambda y} \cos \lambda(u-x)\, d\lambda = \frac{y}{y^2 + (u-x)^2} \tag{2}$$

so that (1) becomes
$$v(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y\, f(u)}{y^2 + (u-x)^2}\, du \tag{3}$$

SOLUTIONS BY USE OF FOURIER TRANSFORMS

5.21. By taking the Fourier transform with respect to the variable x, show that

$$(a)\ \mathcal{F}\left(\frac{\partial v}{\partial x}\right) = i\alpha\mathcal{F}(v), \quad (b)\ \mathcal{F}\left(\frac{\partial^2 v}{\partial x^2}\right) = -\alpha^2\mathcal{F}(v), \quad (c)\ \mathcal{F}\left(\frac{\partial v}{\partial t}\right) = \frac{\partial}{\partial t}\mathcal{F}(v)$$

(a) By definition we have on using integration by parts:

$$\begin{aligned}\mathcal{F}\left(\frac{\partial v}{\partial x}\right) &= \int_{-\infty}^{\infty}\frac{\partial v}{\partial x}e^{-i\alpha x}\,dx \\ &= e^{-i\alpha x}v\Big|_{-\infty}^{\infty} + i\alpha\int_{-\infty}^{\infty}ve^{-i\alpha x}\,dx \\ &= i\alpha\int_{-\infty}^{\infty}ve^{-i\alpha x}\,dx \\ &= i\alpha\,\mathcal{F}(v)\end{aligned}$$

where we suppose that $v \to 0$ as $x \to \pm\infty$.

(b) Let $v = \partial w/\partial x$ in part (a) then

$$\mathcal{F}\left(\frac{\partial^2 w}{\partial x^2}\right) = i\alpha\mathcal{F}\left(\frac{\partial w}{\partial x}\right) = (i\alpha)^2\,\mathcal{F}(w)$$

Then if we formally replace w by v we have

$$\mathcal{F}\left(\frac{\partial^2 v}{\partial x^2}\right) = (i\alpha)^2\,\mathcal{F}(v) = -\alpha^2\,\mathcal{F}(v)$$

provided that v and $\dfrac{\partial v}{\partial x} \to 0$ as $x \to \pm\infty$.

In general we can show that

$$\mathcal{F}\left(\frac{\partial^n v}{\partial x^n}\right) = (i\alpha)^n\,\mathcal{F}(v)$$

if $v, \dfrac{\partial v}{\partial x}, \ldots, \dfrac{\partial^{n-1}v}{\partial x^{n-1}} \to 0$ as $x \to \pm\infty$.

(c) By definition

$$\mathcal{F}\left(\frac{\partial v}{\partial t}\right) = \int_{-\infty}^{\infty}\frac{\partial v}{\partial t}e^{-i\alpha x}\,dx = \frac{\partial}{\partial t}\int_{-\infty}^{\infty}ve^{-i\alpha x}\,dx = \frac{\partial}{\partial t}\mathcal{F}(v)$$

5.22. (a) Use Fourier transforms to solve the boundary value problem

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2}, \quad u(x,0) = f(x), \quad |u(x,t)| < M$$

where $-\infty < x < \infty$, $t > 0$. (b) Give a physical interpretation.

(a) Taking the Fourier transform with respect to x of both sides of the given partial differential equation and using results (b) and (c) of Problem 5.21, we have

$$\frac{d}{dt}\mathcal{F}(u) = -\kappa\lambda^2\,\mathcal{F}(u) \tag{1}$$

where we have written the total derivative since $\mathcal{F}(u)$ depends only on t and not on x. Solving the ordinary differential equation (1) for $\mathcal{F}(u)$, we obtain

$$\mathcal{F}(u) = Ce^{-\kappa\alpha^2 t} \tag{2}$$

or more explicitly

$$\mathcal{F}\{u(x,t)\} = Ce^{-\kappa\alpha^2 t} \tag{3}$$

Putting $t = 0$ in (3) we see that

$$\mathcal{F}\{u(x,0)\} = \mathcal{F}\{f(x)\} = C \tag{4}$$

so that (2) becomes

$$\mathcal{F}\{u\} = \mathcal{F}\{f\}e^{-\kappa\alpha^2 t} \tag{5}$$

We can now apply the convolution theorem. By Problem 4.9, page 72,

$$e^{-\kappa\alpha^2 t} = \mathcal{F}\left\{\sqrt{\frac{1}{4\pi\kappa t}}\,e^{-(x^2/4\kappa t)}\right\} \tag{6}$$

Hence $$u(x,t) = f(x) * \sqrt{\frac{1}{4\pi\kappa t}}\,e^{-(x^2/4\kappa t)} = \int_{-\infty}^{\infty} f(w)\sqrt{\frac{1}{4\pi\kappa t}}\,e^{-\{(x-w)^2/4\kappa t\}}\,dw \tag{7}$$

If we now change variables from w to z according to the transformation $(x-w)^2/4\kappa t = z^2$ or $(x-w)/2\sqrt{\kappa t} = z$, (7) becomes

$$u(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2} f(x - 2z\sqrt{\kappa t})\,dz \tag{8}$$

(b) The problem is that of determining the temperature in a thin infinite bar whose surface is insulated and whose initial temperature is $f(x)$.

5.23. An infinite string is given an initial displacement $y(x,0) = f(x)$ and then released. Determine its displacement at any later time t.

The boundary value problem is

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} \tag{1}$$

$$y(x,0) = f(x), \quad y_t(x,0) = 0, \quad |y(x,t)| < M \tag{2}$$

where $-\infty < x < \infty$, $t > 0$.

Letting $y = XT$ in (1) we find in the usual manner that a solution satisfying the second boundary condition in (2) is given by

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)\cos\lambda a t$$

By assuming that A and B are functions of λ and integrating from $\lambda = 0$ to ∞ we then arrive at the possible solution

$$y(x,t) = \int_0^{\infty} [A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]\cos\lambda a t\,d\lambda \tag{3}$$

Putting $t = 0$ in (3), we see from the first boundary condition in (2) that we must have

$$f(x) = \int_0^{\infty} [A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]\,d\lambda$$

Then it follows from (1) and (2), page 80, that

$$A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\lambda v\,dv, \qquad B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\lambda v\,dv \tag{4}$$

where we have changed the dummy variable from x to v.

Substitution of (4) into (3) yields

$$\begin{aligned} y(x,t) &= \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(v)[\cos\lambda x\cos\lambda v + \sin\lambda x\sin\lambda v]\cos\lambda a t\,dv\,d\lambda \\ &= \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(v)\cos\lambda(x-v)\cos\lambda a t\,dv\,d\lambda \\ &= \frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(v)[\cos\lambda(x+at-v) + \cos\lambda(x-at-v)]\,dv\,d\lambda \end{aligned}$$

where in the last step we have used the trigonometric identity

$$\cos A \cos B \quad = \quad \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

with $A = \lambda(x-v)$ and $B = \lambda at$.

By interchanging the order of integration, the result can be written

$$y(x,t) \quad = \quad \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^\infty f(v)\cos\lambda(x+at-v)\,dv\,d\lambda + \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^\infty f(v)\cos\lambda(x-at-v)\,dv\,d\lambda \qquad (5)$$

But we know from Fourier's integral theorem [equation (3), page 80] that

$$f(x) \quad = \quad \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(v)\cos\lambda(x-v)\,dv\,d\lambda$$

Then, replacing x by $x+at$ and $x-at$ respectively, we see that (5) can be written

$$y(x,t) \quad = \quad \frac{1}{2}[f(x+at) + f(x-at)] \qquad (6)$$

which is the required solution.

Supplementary Problems

THE FOURIER INTEGRAL AND FOURIER TRANSFORMS

5.24. (a) Find the Fourier transform of $f(x) = \begin{cases} 1/2\epsilon & |x| < \epsilon \\ 0 & |x| > \epsilon \end{cases}$.

(b) Determine the limit of this transform as $\epsilon \to 0+$ and discuss the result.

5.25. (a) Find the Fourier transform of $f(x) = \begin{cases} 1-x^2 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$.

(b) Evaluate $\displaystyle\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2}\,dx$.

5.26. If $f(x) = \begin{cases} 1 & 0 \leqq x < 1 \\ 0 & x \geqq 1 \end{cases}$ find (a) the Fourier sine transform, (b) the Fourier cosine transform of $f(x)$. In each case obtain the graphs of $f(x)$ and its transform.

5.27. (a) Find the Fourier sine transform of e^{-x}, $x \geqq 0$.

(b) Show that $\displaystyle\int_0^\infty \frac{x\sin mx}{x^2+1}\,dx = \frac{\pi}{2}e^{-m}$, $m > 0$ by using the result in (a).

(c) Explain from the viewpoint of Fourier's integral theorem why the result in (b) does not hold for $m = 0$.

5.28. Solve for $y(x)$ the integral equation

$$\int_0^\infty y(x)\sin xt\,dx \quad = \quad \begin{cases} 1 & 0 \leqq t < 1 \\ 2 & 1 \leqq t < 2 \\ 0 & t \geqq 2 \end{cases}$$

and verify the solution by direct substitution.

5.29. If $F(\alpha)$ is the Fourier transform of $f(x)$ show that it is possible to find a constant c so that $F(x) = f(x) = ce^{-x^2}$.

PARSEVAL'S IDENTITY

5.30. Evaluate (a) $\int_0^\infty \frac{dx}{(x^2+1)^2}$, (b) $\int_0^\infty \frac{x^2\,dx}{(x^2+1)^2}$ by use of Parseval's identity.
[*Hint.* Use the Fourier sine and cosine transforms of e^{-x}, $x > 0$.]

5.31. Use Problem 5.26 to show that (a) $\int_0^\infty \left(\frac{1-\cos x}{x}\right)^2 dx = \frac{\pi}{2}$, (b) $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$.

5.32. Show that $\int_0^\infty \frac{(x\cos x - \sin x)^2}{x^6}\,dx = \frac{\pi}{15}$.

5.33. Prove the results given by (a) equation *(13)*, page 82; (b) equation *(14)*, page 82.

5.34. Establish the results of equations *(15)*, *(16)*, *(17)* and *(18)* on page 82.

CONVOLUTION THEOREM

5.35. Verify the convolution theorem for the functions $f(x) = g(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$.

5.36. Verify the convolution theorem for the functions $f(x) = g(x) = e^{-x^2}$.

5.37. Solve the integral equation $\int_{-\infty}^{\infty} y(u)\,y(x-u)\,du = e^{-x^2}$.

5.38. Prove that $f * (g+h) = f*g + f*h$.

5.39. Prove that $f * (g * h) = (f * g) * h$.

PROOF OF FOURIER INTEGRAL THEOREM

5.40. By interchanging the order of integration in $\int_{y=0}^{\infty}\int_{x=0}^{\infty} e^{-xy}\sin y\,dx\,dy$, prove that

$$\int_0^\infty \frac{\sin y}{y}\,dy = \frac{\pi}{2}$$

and thus complete the proof in Problem 5.12.

5.41. Let n be any real number. Is Fourier's integral theorem valid for $f(x) = e^{-x^n}$? Explain.

SOLUTIONS USING FOURIER INTEGRALS

5.42. An infinite thin bar ($-\infty < x < \infty$) whose surface is insulated has an initial temperature given by

$$f(x) = \begin{cases} u_0 & |x| < a \\ 0 & |x| \geqq a \end{cases}$$

Show that the temperature at any point x at any time t is

$$u(x,t) = \frac{u_0}{2}\left[\operatorname{erf}\left(\frac{x+a}{2\sqrt{\kappa t}}\right) - \operatorname{erf}\left(\frac{x-a}{2\sqrt{\kappa t}}\right)\right]$$

5.43. A semi-infinite solid ($x > 0$) has an initial temperature given by $f(x) = u_0 e^{-bx^2}$. If the plane face ($x = 0$) is insulated show that the temperature at any point x at any time t is

$$u(x,t) = \frac{u_0}{\sqrt{1+4\kappa bt}} e^{-bx^2/(1+4\kappa bt)}$$

5.44. Solve and physically interpret the following boundary value problem:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad y > 0$$

$$u(x,0) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases} \qquad |u(x,y)| < M$$

5.45. Show that if $u(x,0) = \begin{cases} 0 & x < 0 \\ u_0 & x > 0 \end{cases}$ in Problem 5.44, then

$$u(x,y) = \frac{u_0}{2} + \frac{u_0}{\pi}\tan^{-1}\frac{x}{y}$$

5.46. Work Problem 5.44 if $u(x,0) = \begin{cases} 0 & x < -1 \\ 1 & -1 < x < 1 \\ 0 & x > 1 \end{cases}$.

5.47. The region bounded by $x > 0$, $y > 0$ has one edge $x = 0$ kept at potential zero and the other edge $y = 0$ kept at potential $f(x)$. (a) Show that the potential at any point (x, y) is given by

$$v(x,y) = \frac{1}{\pi}\int_0^\infty y f(v)\left[\frac{1}{(v-x)^2+y^2} - \frac{1}{(v+x)^2+y^2}\right]dv$$

(b) If $f(x) = 1$, show that $v(x,y) = \frac{2}{\pi}\tan^{-1}\frac{x}{y}$.

5.48. Verify that the result obtained in Problem 5.18 is actually a solution of the corresponding boundary value problem.

5.49. The lines $y = 0$ and $y = a$ in the xy-plane (see Fig. 5-4) are kept at potentials 0 and $f(x)$ respectively. Show that the potential at points (x, y) between these lines is given by

$$v(x,y) = \frac{1}{\pi}\int_{\lambda=0}^\infty \int_{u=-\infty}^\infty f(u)\frac{\sinh \lambda y}{\sinh \lambda a}\cos\lambda(u-x)\,du\,d\lambda$$

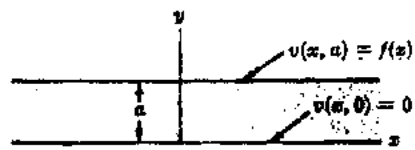

Fig. 5-4

5.50. An infinite string coinciding with the x-axis is given an initial shape $f(x)$ and an initial velocity $g(x)$. Assuming that gravity is neglected, show that the displacement of any point x of the string at time t is given by

$$y(x,t) = \frac{1}{2}[f(x+at) + f(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(u)\,du$$

5.51. Work Problem 5.50 if gravity is taken into account.

5.52. A semi-infinite cantilever beam ($x > 0$) clamped at $x = 0$ is given an initial shape $f(x)$ and released. Find the resulting displacement at any later time t.

Chapter 6

Bessel Functions and Applications

BESSEL'S DIFFERENTIAL EQUATION

Bessel functions arise as solutions of the differential equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0 \qquad n \geq 0 \tag{1}$$

which is called *Bessel's differential equation*. The general solution of (*1*) is given by

$$y = c_1 J_n(x) + c_2 Y_n(x) \tag{2}$$

The solution $J_n(x)$, which has a finite limit as x approaches zero, is called a *Bessel function of the first kind of order n*. The solution $Y_n(x)$, which has no finite limit (i.e. is unbounded) as x approaches zero, is called a *Bessel function of the second kind of order n* or a *Neumann function*.

If the independent variable x in (*1*) is changed to λx, where λ is a constant, the resulting equation is

$$x^2y'' + xy' + (\lambda^2x^2 - n^2)y = 0 \tag{3}$$

with general solution

$$y = c_1 J_n(\lambda x) + c_2 Y_n(\lambda x) \tag{4}$$

The differential equation (*1*) or (*3*) is obtained, for example, from Laplace's equation $\nabla^2 u = 0$ expressed in cylindrical coordinates (ρ, ϕ, z). See Problem 6.1.

THE METHOD OF FROBENIUS

An important method for obtaining solutions of differential equations such as Bessel's equation is known as the *method of Frobenius*. In this method we assume a solution of the form

$$y = \sum_{k=-\infty}^{\infty} c_k x^{k+\beta} \tag{5}$$

where $c_k = 0$ for $k < 0$, so that (*5*) actually begins with the term involving c_0 which is assumed different from zero.

By substituting (*5*) into a given differential equation we can obtain an equation for the constant β (called an *indicial equation*), as well as equations which can be used to determine the constants c_k. The process is illustrated in Problem 6.3.

BESSEL FUNCTIONS OF THE FIRST KIND

We define the Bessel function of the first kind of order n as

$$J_n(x) = \frac{x^n}{2^n\Gamma(n+1)}\left\{1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2\cdot 4(2n+2)(2n+4)} - \cdots\right\} \tag{6}$$

or

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{r!\,\Gamma(n+r+1)} \tag{7}$$

where $\Gamma(n+1)$ is the gamma function (Chapter 4). If n is a positive integer, $\Gamma(n+1) = n!$, $\Gamma(1) = 1$. For $n = 0$, (6) becomes

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + \cdots \tag{8}$$

The series (6) or (7) converges for all x. Graphs of $J_0(x)$ and $J_1(x)$ are shown in Fig. 6-1.

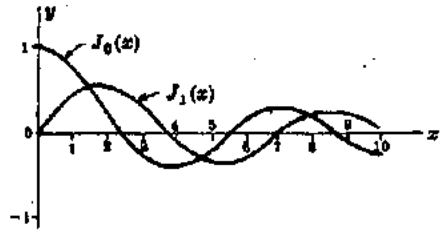

Fig. 6-1

If n is half an odd integer, $J_n(x)$ can be expressed in terms of sines and cosines. See Problems 6.6 and 6.9.

A function $J_{-n}(x)$, $n > 0$, can be defined by replacing n by $-n$ in (6) or (7). If n is an integer, then we can show that (see Problem 6.5)

$$J_{-n}(x) = (-1)^n J_n(x) \tag{9}$$

If n is not an integer, $J_n(x)$ and $J_{-n}(x)$ are linearly independent, and for this case the general solution of (1) is

$$y = AJ_n(x) + BJ_{-n}(x) \qquad n \neq 0, 1, 2, 3, \ldots \tag{10}$$

BESSEL FUNCTIONS OF THE SECOND KIND

We shall define the Bessel function of the second kind of order n as

$$Y_n(x) = \begin{cases} \dfrac{J_n(x)\cos n\pi - J_{-n}(x)}{\sin n\pi} & n \neq 0, 1, 2, 3, \ldots \\[2ex] \lim\limits_{p \to n} \dfrac{J_p(x)\cos p\pi - J_{-p}(x)}{\sin p\pi} & n = 0, 1, 2, 3, \ldots \end{cases} \tag{11}$$

For the case where $n = 0, 1, 2, 3, \ldots$ we obtain the following series expansion for $Y_n(x)$:

$$Y_n(x) = \frac{2}{\pi}\{\ln(x/2) + \gamma\} J_n(x) - \frac{1}{\pi}\sum_{k=0}^{n-1} \frac{(n-k-1)!\,(x/2)^{2k-n}}{k!} - \frac{1}{\pi}\sum_{k=0}^{\infty} (-1)^k \{\Phi(k) + \Phi(n+k)\} \frac{(x/2)^{2k+n}}{k!\,(n+k)!} \tag{12}$$

where $\gamma = 0.5772156\ldots$ is Euler's constant (page 68) and

$$\Phi(p) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p}, \qquad \Phi(0) = 0 \tag{13}$$

Graphs of the functions $Y_0(x)$ and $Y_1(x)$ are shown in Fig. 6-2. Note that these functions, as well as all the functions $Y_n(x)$ where $n > 0$, are unbounded at $x = 0$.

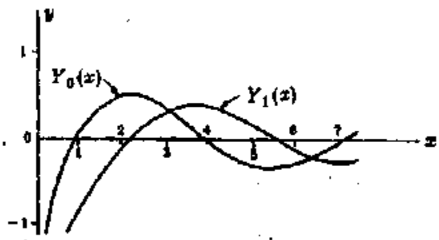

Fig. 6-2

If n is half an odd integer $Y_n(x)$ can be expressed in terms of trigonometric functions.

GENERATING FUNCTION FOR $J_n(x)$

The function

$$e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(x)t^n \tag{14}$$

is called the *generating function* for Bessel functions of the first kind of integral order. It is very useful in obtaining properties of these functions for integer values of n – properties which can then often be proved for all values of n.

RECURRENCE FORMULAS

The following results are valid for all values of n.

1. $J_{n+1}(x) = \dfrac{2n}{x}J_n(x) - J_{n-1}(x)$

2. $J_n'(x) = \dfrac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$

3. $xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$

4. $xJ_n'(x) = xJ_{n-1}(x) - nJ_n(x)$

5. $\dfrac{d}{dx}[x^nJ_n(x)] = x^nJ_{n-1}(x)$

6. $\dfrac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$

If n is an integer these can be proved by using the generating function. Note that results 3. and 4. are respectively equivalent to 5. and 6.

The functions $Y_n(x)$ satisfy exactly the same formulas, where $Y_n(x)$ replaces $J_n(x)$.

FUNCTIONS RELATED TO BESSEL FUNCTIONS

1. **Hankel functions of the first and second kinds are defined respectively by**

$$H_n^{(1)}(x) = J_n(x) + iY_n(x), \qquad H_n^{(2)}(x) = J_n(x) - iY_n(x) \tag{15}$$

2. **Modified Bessel functions.** The *modified Bessel function of the first kind of order n* is defined as

$$I_n(x) = i^{-n}J_n(ix) = e^{-n\pi i/2}J_n(ix) \tag{16}$$

If n is an integer,

$$I_{-n}(x) = I_n(x) \tag{17}$$

but if n is not an integer, $I_n(x)$ and $I_{-n}(x)$ are linearly independent.

The *modified Bessel function of the second kind of order n* is defined as

$$K_n(x) = \begin{cases} \dfrac{\pi}{2}\left[\dfrac{I_{-n}(x) - I_n(x)}{\sin n\pi}\right] & n \neq 0,1,2,3,\ldots \\ \lim\limits_{p\to n}\dfrac{\pi}{2}\left[\dfrac{I_{-p}(x) - I_p(x)}{\sin p\pi}\right] & n = 0,1,2,3,\ldots \end{cases} \tag{18}$$

These functions satisfy the differential equation

$$x^2y'' + xy' - (x^2+n^2)y = 0 \tag{19}$$

and the general solution of this equation is

$$y = c_1 I_n(x) + c_2 K_n(x) \tag{20}$$

or, if $n \neq 0, 1, 2, 3, \ldots,$

$$y = A I_n(x) + B I_{-n}(x) \tag{21}$$

Graphs of the functions $I_0(x)$, $I_1(x)$, $K_0(x)$, $K_1(x)$ are shown in Figs. 6-3 and 6-4.

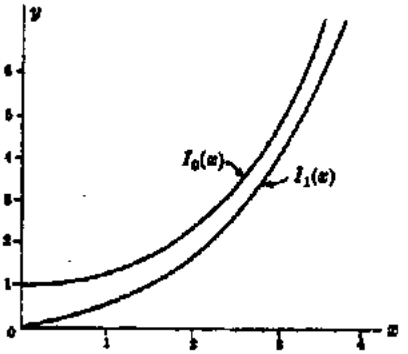

Fig. 6-3

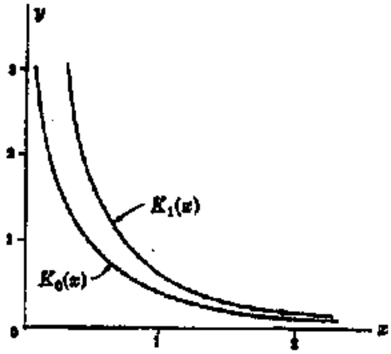

Fig. 6-4

3. **Ber, Bei, Ker, Kei functions.** The functions $\text{Ber}_n(x)$ and $\text{Bei}_n(x)$ are respectively the real and imaginary parts of $J_n(i^{3/2}x)$, where $i^{3/2} = e^{3\pi i/4} = (\sqrt{2}/2)(-1+i)$, i.e.

$$J_n(i^{3/2}x) = \text{Ber}_n(x) + i\,\text{Bei}_n(x) \tag{22}$$

The functions $\text{Ker}_n(x)$ and $\text{Kei}_n(x)$ are respectively the real and imaginary parts of $e^{-n\pi i/2}K_n(i^{1/2}x)$, where $i^{1/2} = e^{\pi i/4} = (\sqrt{2}/2)(1+i)$, i.e.

$$e^{-n\pi i/2}K_n(i^{1/2}x) = \text{Ker}_n(x) + i\,\text{Kei}_n(x) \tag{23}$$

The functions are useful in connection with the equation

$$x^2y'' + xy' - (ix^2 + n^2)y = 0 \tag{24}$$

which arises in electrical engineering and other fields. The general solution of this equation is

$$y = c_1 J_n(i^{3/2}x) + c_2 K_n(i^{1/2}x) \tag{25}$$

If $n = 0$ we often denote $\text{Ber}_n(x)$, $\text{Bei}_n(x)$, $\text{Ker}_n(x)$, $\text{Kei}_n(x)$ by $\text{Ber}(x)$, $\text{Bei}(x)$, $\text{Ker}(x)$, $\text{Kei}(x)$, respectively. The graphs of these functions are shown in Figs. 6-5 and 6-6.

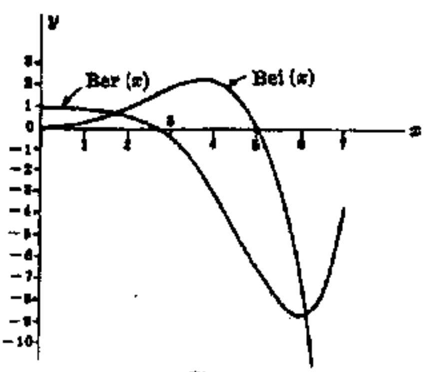

Fig. 6-5

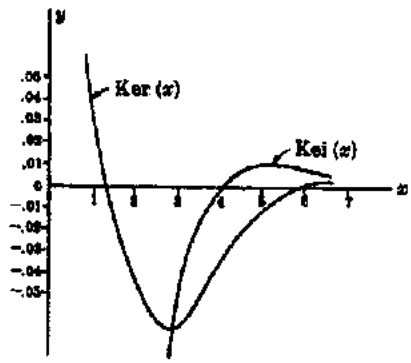

Fig. 6-6

EQUATIONS TRANSFORMABLE INTO BESSEL'S EQUATION

The equation

$$x^2y'' + (2k+1)xy' + (\alpha^2x^{2r} + \beta^2)y = 0 \qquad (26)$$

where k, α, r, β are constants, has the general solution

$$y = x^{-k}[c_1J_{\kappa/r}(\alpha x^r/r) + c_2Y_{\kappa/r}(\alpha x^r/r)] \qquad (27)$$

where $\kappa = \sqrt{k^2-\beta^2}$. If $\alpha = 0$ the equation is an *Euler* or *Cauchy equation* (see Problem 6.79) and has solution

$$y = x^{-k}(c_3x^{\kappa} + c_4x^{-\kappa}) \qquad (28)$$

ASYMPTOTIC FORMULAS FOR BESSEL FUNCTIONS

For large values of x we have the following asymptotic formulas:

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right), \qquad Y_n(x) \sim \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right) \qquad (29)$$

ZEROS OF BESSEL FUNCTIONS

We can show that if n is any real number, $J_n(x) = 0$ has an infinite number of roots which are all real. The difference between successive roots approaches π as the roots increase in value. This can be seen from (29). We can also show that the roots of $J_n(x) = 0$ [the *zeros* of $J_n(x)$] lie between those of $J_{n-1}(x) = 0$ and $J_{n+1}(x) = 0$. Similar remarks can be made for $Y_n(x)$. For a table giving zeros of Bessel functions see Appendix E, page 177.

ORTHOGONALITY OF BESSEL FUNCTIONS OF THE FIRST KIND

If λ and μ are two different constants, we can show (see Problem 6.23) that

$$\int_0^1 xJ_n(\lambda x)\,J_n(\mu x)\,dx = \frac{\mu J_n(\lambda)\,J_n'(\mu) - \lambda J_n(\mu)J_n'(\lambda)}{\lambda^2 - \mu^2} \qquad (30)$$

while (see Problem 6.24)

$$\int_0^1 xJ_n^2(\lambda x)\,dx = \frac{1}{2}\left[J_n'^2(\lambda) + \left(1 - \frac{n^2}{\lambda^2}\right)J_n^2(\lambda)\right] \qquad (31)$$

From (30) we can see that if λ and μ are any two different roots of the equation

$$RJ_n(x) + SxJ_n'(x) = 0 \qquad (32)$$

where R and S are constants, then

$$\int_0^1 xJ_n(\lambda x)\,J_n(\mu x)\,dx = 0 \qquad (33)$$

which states that the functions $\sqrt{x}\,J_n(\lambda x)$ and $\sqrt{x}\,J_n(\mu x)$ are orthogonal in $(0,1)$. Note that as special cases of (32) λ and μ can be any two different roots of $J_n(x) = 0$ or of $J_n'(x) = 0$. We can also say that the functions $J_n(\lambda x)$, $J_n(\mu x)$ are orthogonal with respect to the density or weight function x.

SERIES OF BESSEL FUNCTIONS OF THE FIRST KIND

As in the case of Fourier series, we can show that if $f(x)$ and $f'(x)$ are piecewise continuous then at every point of continuity of $f(x)$ in the interval of $0 < x < 1$ there will exist a Bessel series expansion having the form

$$f(x) = A_1J_n(\lambda_1 x) + A_2J_n(\lambda_2 x) + \cdots = \sum_{p=1}^{\infty} A_pJ_n(\lambda_p x) \tag{34}$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive roots of (32) with $R/S \geqq 0$, $S \neq 0$ and

$$A_p = \frac{2\lambda_p^2}{(\lambda_p^2 - n^2 + R^2/S^2)J_n^2(\lambda_p)} \int_0^1 xJ_n(\lambda_p x)\,f(x)\,dx \tag{35}$$

At any point of discontinuity the series on the right in (34) converges to $\frac{1}{2}[f(x+0) + f(x-0)]$, which can be used in place of the left side of (34).

In case $S = 0$, so that $\lambda_1, \lambda_2, \ldots$ are the roots of $J_n(x) = 0$,

$$A_p = \frac{2}{J_{n+1}^2(\lambda_p)} \int_0^1 xJ_n(\lambda_p x)\,f(x)\,dx \tag{36}$$

If $R = 0$ and $n = 0$, then the series (34) starts out with the constant term

$$A_1 = 2\int_0^1 x\,f(x)\,dx \tag{37}$$

In this case the positive roots are those of $J_n'(x) = 0$.

ORTHOGONALITY AND SERIES OF BESSEL FUNCTIONS OF THE SECOND KIND

The above results for Bessel functions of the first kind can be extended to Bessel functions of the second kind. See Problems 6.32 and 6.33.

SOLUTIONS TO BOUNDARY VALUE PROBLEMS USING BESSEL FUNCTIONS

The expansion of functions into Bessel series enables us to solve various boundary value problems arising in science and engineering. See Problems 6.28, 6.29, 6.31, 6.34, 6.35.

Solved Problems

BESSEL'S DIFFERENTIAL EQUATION

6.1. Show how Bessel's differential equation (3), page 97, is obtained from Laplace's equation $\nabla^2 u = 0$ expressed in cylindrical coordinates (ρ, ϕ, z).

Laplace's equation in cylindrical coordinates is given by

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{1}$$

If we assume a solution of the form $u = P\Phi Z$, where P is a function of ρ, Φ is a function of ϕ and Z is a function of z, then (1) becomes

$$P''\Phi Z + \frac{1}{\rho}P'\Phi Z + \frac{1}{\rho^2}P\Phi''Z + P\Phi Z'' = 0 \tag{2}$$

where the primes denote derivatives with respect to the particular independent variable involved. Dividing (2) by $P\Phi Z$ yields

$$\frac{P''}{P} + \frac{1}{\rho}\frac{P'}{P} + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} + \frac{Z''}{Z} = 0 \tag{3}$$

Equation (3) can be written as

$$\frac{P''}{P} + \frac{1}{\rho}\frac{P'}{P} + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} = -\frac{Z''}{Z} \tag{4}$$

Since the right side depends only on z while the left side depends only on ρ and φ, it follows that each side must be a constant, say $-\lambda^2$. Thus we have

$$\frac{P''}{P} + \frac{1}{\rho}\frac{P'}{P} + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} = -\lambda^2 \tag{5}$$

and

$$Z'' - \lambda^2 Z = 0 \tag{6}$$

If we now multiply both sides of (5) by ρ^2 it becomes

$$\rho^2\frac{P''}{P} + \rho\frac{P'}{P} + \frac{\Phi''}{\Phi} = -\lambda^2\rho^2 \tag{7}$$

which can be written as

$$\rho^2\frac{P''}{P} + \rho\frac{P'}{P} + \lambda^2\rho^2 = -\frac{\Phi''}{\Phi} \tag{8}$$

Since the right side depends only on ϕ, while the left side depends only on ρ, it follows that each side must be a constant, say μ^2. Thus we have

$$\rho^2\frac{P''}{P} + \rho\frac{P'}{P} + \lambda^2\rho^2 = \mu^2 \tag{9}$$

and

$$\Phi'' + \mu^2\Phi = 0 \tag{10}$$

The equation (9) can be written as

$$\rho^2 P'' + \rho P' + (\lambda^2\rho^2 - \mu^2)P = 0 \tag{11}$$

which is Bessel's differential equation (1) on page 97 with P instead of y, ρ instead of x and μ instead of n.

6.2. Show that if we let $\lambda\rho = x$ in equation (11) of Problem 6.1, then it becomes

$$x^2y'' + xy' + (x^2 - \mu^2)y = 0$$

We have

$$\frac{dP}{d\rho} = \frac{dP}{dx}\frac{dx}{d\rho} = \frac{dP}{dx}\lambda = \lambda\frac{dy}{dx}$$

where $y(x)$, or briefly y, represents that function of x which $P(\rho)$ becomes when $\rho = x/\lambda$.

Similarly

$$\frac{d^2P}{d\rho^2} = \frac{d}{d\rho}\left(\frac{dP}{d\rho}\right) = \frac{d}{dx}\left(\lambda\frac{dy}{dx}\right)\frac{dx}{d\rho} = \frac{d}{dx}\left(\lambda\frac{dy}{dx}\right)\lambda = \lambda^2\frac{d^2y}{dx^2}$$

Then equation (11) of Problem 6.1 which can be written

$$\rho^2\frac{d^2P}{d\rho^2} + \rho\frac{dP}{d\rho} + (\lambda^2\rho^2 - \mu^2)P = 0$$

becomes

$$\left(\frac{x}{\lambda}\right)^2\lambda^2\frac{d^2y}{dx^2} + \left(\frac{x}{\lambda}\right)\lambda\frac{dy}{dx} + (x^2 - \mu^2)y = 0$$

or

$$x^2y'' + xy' + (x^2 - \mu^2)y = 0$$

as required.

6.3. **Use the method of Frobenius to find series solutions of Bessel's differential equation $x^2y'' + xy' + (x^2 - n^2)y = 0$.**

Assuming a solution of the form $y = \sum c_k x^{k+\beta}$ where k goes from $-\infty$ to ∞ and $c_k = 0$ for $k < 0$, we have

$$(x^2 - n^2)y = \sum c_k x^{k+\beta+2} - \sum n^2 c_k x^{k+\beta} = \sum c_{k-2} x^{k+\beta} - \sum n^2 c_k x^{k+\beta}$$

$$xy' = \sum (k+\beta) c_k x^{k+\beta}$$

$$x^2 y'' = \sum (k+\beta)(k+\beta-1) c_k x^{k+\beta}$$

Then by addition,

$$\sum [(k+\beta)(k+\beta-1)c_k + (k+\beta)c_k + c_{k-2} - n^2 c_k] x^{k+\beta} = 0$$

and since the coefficients of the $x^{k+\beta}$ must be zero, we find

$$[(k+\beta)^2 - n^2]c_k + c_{k-2} = 0 \qquad (1)$$

Letting $k = 0$ in (1) we obtain, since $c_{-2} = 0$, the indicial equation $(\beta^2 - n^2)c_0 = 0$; or assuming $c_0 \neq 0$, $\beta^2 = n^2$. Then there are two cases, given by $\beta = -n$ and $\beta = n$. We shall consider first the case $\beta = n$ and obtain the second case by replacing n by $-n$.

Case 1: $\beta = n$.

In this case (1) becomes

$$k(2n + k)c_k + c_{k-2} = 0 \qquad (2)$$

Putting $k = 1, 2, 3, 4, \ldots$ successively in (2), we have

$$c_1 = 0, \quad c_2 = \frac{-c_0}{2(2n+2)}, \quad c_3 = 0, \quad c_4 = \frac{-c_2}{4(2n+4)} = \frac{c_0}{2 \cdot 4(2n+2)(2n+4)}, \quad \ldots$$

Thus the required series is

$$y = c_0 x^n + c_2 x^{n+2} + c_4 x^{n+4} + \cdots$$

$$= c_0 x^n \left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2 \cdot 4(2n+2)(2n+4)} - \cdots\right] \qquad (3)$$

Case 2: $\beta = -n$.

On replacing n by $-n$ in Case 1, we find

$$y = c_0 x^{-n} \left[1 - \frac{x^2}{2(2-2n)} + \frac{x^4}{2 \cdot 4(2-2n)(4-2n)} - \cdots\right] \qquad (4)$$

Now if $n = 0$, both of these series are identical. If $n = 1, 2, \ldots$ the second series fails to exist. However, if $n \neq 0, 1, 2, \ldots$ the two series can be shown to be linearly independent, and so for this case the general solution is

$$y = Cx^n \left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2 \cdot 4(2n+2)(2n+4)} - \cdots\right]$$

$$+ Dx^{-n} \left[1 - \frac{x^2}{2(2-2n)} + \frac{x^4}{2 \cdot 4(2-2n)(4-2n)} - \cdots\right] \qquad (5)$$

The cases where $n = 0, 1, 2, 3, \ldots$ are treated later (see Problems 6.17 and 6.18).

The first series in (5), with suitable choice of multiplicative constant, provides the definition of $J_n(x)$ given by (6), page 97.

BESSEL FUNCTIONS OF THE FIRST KIND

6.4. **Using the definition (6) of $J_n(x)$ given on page 97, show that if $n \neq 0, 1, 2, \ldots$, then the general solution of Bessel's equation is $y = AJ_n(x) + BJ_{-n}(x)$.**

Note that the definition of $J_n(x)$ on page 97 agrees with the series of Case 1 in Problem 6.3, apart from a constant factor depending only on n. It follows that the result (5) can be written $y = AJ_n(x) + BJ_{-n}(x)$ for the cases $n \neq 0, 1, 2, \ldots$.

6.5. (a) Prove that $J_{-n}(x) = (-1)^n J_n(x)$ for $n = 1, 2, 3, \ldots$.

(b) Use (a) to explain why $AJ_n(x) + BJ_{-n}(x)$ is not the general solution of Bessel's equation for integer values of n.

(a) Replacing n by $-n$ in (6) or the equivalent (7) on page 98, we have

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{-n+2r}}{r!\,\Gamma(-n+r+1)}$$
$$= \sum_{r=0}^{n-1} \frac{(-1)^r (x/2)^{-n+2r}}{r!\,\Gamma(-n+r+1)} + \sum_{r=n}^{\infty} \frac{(-1)^r (x/2)^{-n+2r}}{r!\,\Gamma(-n+r+1)}$$

Now since $\Gamma(-n+r+1)$ is infinite for $r = 0, 1, \ldots, n-1$, the first sum on the right is zero. Letting $r = n+k$ in the second sum, it becomes

$$\sum_{k=0}^{\infty} \frac{(-1)^{n+k} (x/2)^{n+2k}}{(n+k)!\,\Gamma(k+1)} = (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{\Gamma(n+k+1)k!} = (-1)^n J_n(x)$$

(b) From (a) it follows that for integer values of n, $J_{-n}(x)$ and $J_n(x)$ are linearly dependent and so $AJ_n(x) + BJ_{-n}(x)$ cannot be a general solution of Bessel's equation. If n is not an integer, then we can show that $J_{-n}(x)$ and $J_n(x)$ are linearly independent, so that $AJ_n(x) + BJ_{-n}(x)$ is a general solution (see Problem 6.12).

6.6. Prove (a) $J_{1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\sin x$, (b) $J_{-1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\cos x$.

(a)

$$J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{1/2+2r}}{r!\,\Gamma(r+3/2)} = \frac{(x/2)^{1/2}}{\Gamma(3/2)} - \frac{(x/2)^{5/2}}{1!\,\Gamma(5/2)} + \frac{(x/2)^{9/2}}{2!\,\Gamma(7/2)} - \cdots$$
$$= \frac{(x/2)^{1/2}}{(1/2)\sqrt{\pi}} - \frac{(x/2)^{5/2}}{1!\,(3/2)(1/2)\sqrt{\pi}} + \frac{(x/2)^{7/2}}{2!\,(5/2)(3/2)(1/2)\sqrt{\pi}} - \cdots$$
$$= \frac{(x/2)^{1/2}}{(1/2)\sqrt{\pi}}\left\{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right\} = \frac{(x/2)^{1/2}}{(1/2)\sqrt{\pi}}\,\frac{\sin x}{x} = \sqrt{\frac{2}{\pi x}}\sin x$$

(b)

$$J_{-1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{-1/2+2r}}{r!\,\Gamma(r+1/2)} = \frac{(x/2)^{-1/2}}{\Gamma(1/2)} - \frac{(x/2)^{3/2}}{1!\,\Gamma(3/2)} + \frac{(x/2)^{7/2}}{2!\,\Gamma(5/2)} - \cdots$$
$$= \frac{(x/2)^{-1/2}}{\sqrt{\pi}}\left\{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right\} = \sqrt{\frac{2}{\pi x}}\cos x$$

6.7. Prove that for all n:

(a) $\dfrac{d}{dx}\{x^n J_n(x)\} = x^n J_{n-1}(x)$, (b) $\dfrac{d}{dx}\{x^{-n} J_n(x)\} = -x^{-n} J_{n+1}(x)$.

(a)

$$\frac{d}{dx}\{x^n J_n(x)\} = \frac{d}{dx}\sum_{r=0}^{\infty} \frac{(-1)^r x^{2n+2r}}{2^{n+2r} r!\,\Gamma(n+r+1)} = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2n+2r-1}}{2^{n+2r-1} r!\,\Gamma(n+r)}$$
$$= x^n \sum_{r=0}^{\infty} \frac{(-1)^r x^{(n-1)+2r}}{2^{(n-1)+2r} r!\,\Gamma[(n-1)+r+1]} = x^n J_{n-1}(x)$$

(b)
$$\begin{aligned}\frac{d}{dx}\{x^{-n}J_n(x)\} &= \frac{d}{dx}\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{2^{n+2r}r!\,\Gamma(n+r+1)}\\ &= x^{-n}\sum_{r=1}^{\infty}\frac{(-1)^r x^{n+2r-1}}{2^{n+2r-1}(r-1)!\,\Gamma(n+r+1)}\\ &= x^{-n}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}x^{n+2k+1}}{2^{n+2k+1}k!\,\Gamma(n+k+2)} \quad = \quad {}_{+1}(x)\end{aligned}$$

6.8. **Prove that for all n:**

(a) $J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$, (b) $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x)$.

From Problem 6.7(a), $x^nJ_n'(x) + nx^{n-1}J_n(x) = x^nJ_{n-1}(x)$

or
$$xJ_n'(x) + nJ_n(x) = xJ_{n-1}(x) \tag{1}$$

From Problem 6.7(b), $x^{-n}J_n'(x) - nx^{-n-1}J_n(x) = -x^{-n}J_{n+1}(x)$

or
$$xJ_n'(x) - nJ_n(x) = -xJ_{n+1}(x) \tag{2}$$

(a) Adding (1) and (2) and dividing by $2x$ gives

$$J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$$

(b) Subtracting (2) from (1) and dividing by x gives

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x)$$

6.9. **Show that** (a) $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x - x\cos x}{x}\right)$

(b) $J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{x\sin x + \cos x}{x}\right)$

(a) From Problems 6.8(b) and 6.6 we have on letting $n = 1/2$,

$$J_{3/2}(x) = \frac{1}{x}J_{1/2}(x) - J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x - x\cos x}{x}\right)$$

(b) From Problems 6.8(b) and 6.6 we have on letting $n = -\frac{1}{2}$,

$$J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{x\sin x + \cos x}{x}\right)$$

6.10. **Evaluate the integrals** (a) $\int x^nJ_{n-1}(x)\,dx$, (b) $\int \frac{J_{n+1}(x)}{x^n}\,dx$.

From Problem 6.7,

(a) $\frac{d}{dx}\{x^nJ_n(x)\} = x^nJ_{n-1}(x)$. Then $\int x^nJ_{n-1}(x)\,dx = x^nJ_n(x) + c$.

(b) $\frac{d}{dx}\{x^{-n}J_n(x)\} = -x^{-n}J_{n+1}(x)$. Then $\int \frac{J_{n+1}(x)}{x^n}\,dx = -x^{-n}J_n(x) + c$.

6.11. Evaluate (a) $\int x^4 J_1(x)\,dx$, (b) $\int x^3 J_3(x)\,dx$.

(a) **Method 1.** Integration by parts gives

$$\begin{aligned}\int x^4 J_1(x)\,dx &= \int (x^2)[x^2 J_1(x)\,dx] \\ &= x^2[x^2 J_2(x)] - \int [x^2 J_2(x)][2x\,dx] \\ &= x^4 J_2(x) - 2\int x^3 J_2(x)\,dx \\ &= x^4 J_2(x) - 2x^3 J_3(x) + c\end{aligned}$$

Method 2. We have, using $J_1(x) = -J_0'(x)$ [Problem 6.7(b)],

$$\int x^4 J_1(x)\,dx = -\int x^4 J_0'(x)\,dx = -\left\{x^4 J_0(x) - \int 4x^3 J_0(x)\,dx\right\}$$

$$\int x^3 J_0(x)\,dx = \int x^2[xJ_0(x)\,dx] = x^2[xJ_1(x)] - \int [xJ_1(x)][2x\,dx]$$

$$\begin{aligned}\int x^2 J_1(x)\,dx &= -\int x^2 J_0'(x)\,dx = -\left\{x^2 J_0(x) - \int 2xJ_0(x)\,dx\right\} \\ &= -x^2 J_0(x) + 2xJ_1(x)\end{aligned}$$

Then
$$\begin{aligned}\int x^4 J_1(x)\,dx &= -x^4 J_0(x) + 4[x^3 J_1(x) - 2\{-x^2 J_0(x) + 2xJ_1(x)\}] + c \\ &= (8x^2 - x^4)J_0(x) + (4x^3 - 16x)J_1(x)\end{aligned}$$

(b)
$$\begin{aligned}\int x^3 J_3(x)\,dx &= \int x^5[x^{-2} J_3(x)\,dx] \\ &= x^5[-x^{-2} J_2(x)] - \int [-x^{-2} J_2(x)]5x^4\,dx \\ &= -x^3 J_2(x) + 5\int x^2 J_2(x)\,dx\end{aligned}$$

$$\begin{aligned}\int x^2 J_2(x)\,dx &= \int x^3[x^{-1} J_2(x)]\,dx \\ &= x^3[-x^{-1} J_1(x)] - \int [-x^{-1} J_1(x)]3x^2\,dx \\ &= -x^2 J_1(x) + 3\int xJ_1(x)\,dx\end{aligned}$$

$$\begin{aligned}\int xJ_1(x)\,dx &= -\int xJ_0'(x)\,dx = -\left[xJ_0(x) - \int J_0(x)\,dx\right] \\ &= -xJ_0(x) + \int J_0(x)\,dx\end{aligned}$$

Then
$$\begin{aligned}\int x^3 J_3(x)\,dx &= -x^3 J_2(x) + 5\left\{-x^2 J_1(x) + 3\left[-xJ_0(x) + \int J_0(x)\,dx\right]\right\} \\ &= -x^3 J_2(x) - 5x^2 J_1(x) - 15xJ_0(x) + 15\int J_0(x)\,dx\end{aligned}$$

The integral $\int J_0(x)\,dx$ cannot be obtained in closed form. In general, $\int x^p J_q(x)\,dx$ can be obtained in closed form if $p + q \geqq 0$ and $p + q$ is odd, where p and q are integers. If, however, $p + q$ is even, the result can be obtained in terms of $\int J_0(x)\,dx$.

6.12. (a) Prove that $J_n'(x)J_{-n}(x) - J_{-n}'(x)J_n(x) = \dfrac{2\sin n\pi}{\pi x}$.

(b) Discuss the significance of the result of (a) with regard to the linear dependence of $J_n(x)$ and $J_{-n}(x)$.

(a) Since $J_n(x)$ and $J_{-n}(x)$, abbreviated J_n, J_{-n} respectively, satisfy Bessel's equation, we have

$$x^2J_n'' + xJ_n' + (x^2-n^2)J_n = 0, \qquad x^2J_{-n}'' + xJ_{-n}' + (x^2-n^2)J_{-n} = 0$$

Multiply the first equation by J_{-n}, the second by J_n and subtract. Then

$$x^2[J_n''J_{-n} - J_{-n}''J_n] + x[J_n'J_{-n} - J_{-n}'J_n] = 0$$

which can be written

$$x\frac{d}{dx}[J_n'J_{-n} - J_{-n}'J_n] + [J_n'J_{-n} - J_{-n}'J_n] = 0$$

or

$$\frac{d}{dx}(x[J_n'J_{-n} - J_{-n}'J_n]) = 0$$

Integrating, we find

$$J_n'J_{-n} - J_{-n}'J_n = c/x \tag{1}$$

To determine c use the series expansions for J_n and J_{-n} to obtain

$$J_n = \frac{x^n}{2^n\Gamma(n+1)} - \cdots, \quad J_n' = \frac{x^{n-1}}{2^n\Gamma(n)} - \cdots, \quad J_{-n} = \frac{x^{-n}}{2^{-n}\Gamma(-n+1)} - \cdots,$$

$$J_{-n}' = \frac{x^{-n-1}}{2^{-n}\Gamma(-n)} - \cdots$$

and then substitute in (1). We find

$$c = \frac{1}{\Gamma(n)\Gamma(1-n)} - \frac{1}{\Gamma(n+1)\Gamma(-n)} = \frac{2}{\Gamma(n)\Gamma(1-n)} = \frac{2\sin n\pi}{\pi}$$

using the result 1, page 68. This gives the required result.

(b) The expression $J_n'J_{-n} - J_{-n}'J_n$ in (a) is the *Wronskian* of J_n and J_{-n}. If n is an integer, we see from (a) that the Wronskian is zero, so that J_n and J_{-n} are linearly dependent, as is also clear from Problem 6.5(a). On the other hand, if n is not an integer, they are linearly independent, since in such case the Wronskian differs from zero.

GENERATING FUNCTION AND MISCELLANEOUS RESULTS

6.13. Prove that $e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(x)t^n$.

We have

$$e^{\frac{x}{2}(t-\frac{1}{t})} = e^{xt/2}e^{-x/2t} = \left\{\sum_{r=0}^{\infty}\frac{(xt/2)^r}{r!}\right\}\left\{\sum_{k=0}^{\infty}\frac{(-x/2t)^k}{k!}\right\} = \sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{r+k}t^{r-k}}{r!\,k!}$$

Let $r-k=n$ so that n varies from $-\infty$ to ∞. Then the sum becomes

$$\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{n+2k}t^n}{(n+k)!\,k!} = \sum_{n=-\infty}^{\infty}\left\{\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{n+2k}}{k!\,(n+k)!}\right\}t^n = \sum_{n=-\infty}^{\infty}J_n(x)t^n$$

6.14. Prove (a) $\cos(x\sin\theta) = J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \cdots$

(b) $\sin(x\sin\theta) = 2J_1(x)\sin\theta + 2J_3(x)\sin 3\theta + 2J_5(x)\sin 5\theta + \cdots$

Let $t = e^{i\theta}$ in Problem 6.13. Then

$$\begin{aligned}
e^{\frac{1}{2}x(e^{i\theta}-e^{-i\theta})} &= e^{ix\sin\theta} = \sum_{-\infty}^{\infty}J_n(x)e^{in\theta} = \sum_{-\infty}^{\infty}J_n(x)\{\cos n\theta + i\sin n\theta\} \\
&= \{J_0(x) + [J_{-1}(x)+J_1(x)]\cos\theta + [J_{-2}(x)+J_2(x)]\cos 2\theta + \cdots\} \\
&\qquad + i\{[J_1(x)-J_{-1}(x)]\sin\theta + [J_2(x)-J_{-2}(x)]\sin 2\theta + \cdots\} \\
&= \{J_0(x) + 2J_2(x)\cos 2\theta + \cdots\} + i\{2J_1(x)\sin\theta + 2J_3(x)\sin 3\theta + \cdots\}
\end{aligned}$$

where we have used Problem 6.5(a). Equating real and imaginary parts gives the required results.

6.15. Prove $J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\,d\theta \qquad n = 0,1,2,\ldots$

Multiply the first and second results of Problem 6.14 by $\cos n\theta$ and $\sin n\theta$ respectively and integrate from 0 to π using

$$\int_0^\pi \cos m\theta \cos n\theta\, d\theta = \begin{cases} 0 & m \neq n \\ \pi/2 & m = n \end{cases}, \qquad \int_0^\pi \sin m\theta \sin n\theta\, d\theta = \begin{cases} 0 & m \neq n \\ \pi/2 & m = n \neq 0 \end{cases}$$

Then if n is even or zero, we have

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta)\cos n\theta\, d\theta, \qquad 0 = \frac{1}{\pi}\int_0^\pi \sin(x\sin\theta)\sin n\theta\, d\theta$$

or on adding,

$$J_n(x) = \frac{1}{\pi}\int_0^\pi [\cos(x\sin\theta)\cos n\theta + \sin(x\sin\theta)\sin n\theta]\,d\theta = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\,d\theta$$

Similarly, if n is odd,

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \sin(x\sin\theta)\sin n\theta\, d\theta, \qquad 0 = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta)\cos n\theta\, d\theta$$

and by adding,

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\,d\theta$$

Thus we have the required result whether n is even or odd, i.e. $n = 0,1,2,\ldots$.

6.16. Prove the result of Problem 6.8(*b*) for integer values of n by using the generating function.

Differentiating both sides of the generating function with respect to t, we have, omitting the limits $-\infty$ to ∞ for n,

$$e^{\frac{x}{2}(t-\frac{1}{t})}\,\frac{x}{2}\left(1+\frac{1}{t^2}\right) = \sum nJ_n(x)t^{n-1}$$

or

$$\frac{x}{2}\left(1+\frac{1}{t^2}\right)\sum J_n(x)t^n = \sum nJ_n(x)t^{n-1}$$

i.e.

$$\sum \frac{x}{2}\left(1+\frac{1}{t^2}\right)J_n(x)t^n = \sum nJ_n(x)t^{n-1}$$

This can be written as

$$\sum \frac{x}{2}J_n(x)t^n + \sum \frac{x}{2}J_n(x)t^{n-2} = \sum nJ_n(x)t^{n-1}$$

or

$$\sum \frac{x}{2}J_n(x)t^n + \sum \frac{x}{2}J_{n+2}(x)t^n = \sum (n+1)J_{n+1}(x)t^n$$

i.e.

$$\sum\left[\frac{x}{2}J_n(x) + \frac{x}{2}J_{n+2}(x)\right]t^n = \sum (n+1)J_{n+1}(x)t^n$$

Since coefficients of t^n must be equal, we have

$$\frac{x}{2}J_n(x) + \frac{x}{2}J_{n+2}(x) = (n+1)J_n(x)$$

from which the required result is obtained on replacing n by $n-1$.

BESSEL FUNCTIONS OF THE SECOND KIND

6.17. (*a*) Show that if n is not an integer, the general solution of Bessel's equation is

$$y = EJ_n(x) + F\left[\frac{J_n(x)\cos n\pi - J_{-n}(x)}{\sin n\pi}\right]$$

where E and F are arbitrary constants.

(b) Explain how to use part (a) to obtain the general solution of Bessel's equation in case n is an integer.

(a) Since J_{-n} and J_n are linearly independent, the general solution of Bessel's equation can be written

$$y = c_1 J_n(x) + c_2 J_{-n}(x)$$

and the required result follows on replacing the arbitrary constants c_1, c_2 by E, F, where

$$c_1 = E + \frac{F \cos n\pi}{\sin n\pi}, \qquad c_2 = \frac{-F}{\sin n\pi}$$

Note that we define the Bessel function of the second kind if n is not an ineger by

$$Y_n(x) = \frac{J_n(x) \cos n\pi - J_{-n}(x)}{\sin n\pi}$$

(b) The expression

$$\frac{J_n(x) \cos n\pi - J_{-n}(x)}{\sin n\pi}$$

becomes an "indeterminate" of the form 0/0 for the case when n is an integer. This is because for an integer n we have $\cos n\pi = (-1)^n$ and $J_{-n}(x) = (-1)^n J_n(x)$ [see Problem 6.5]. This "indeterminate form" can be evaluated by using L'Hospital's rule, i.e.

$$\lim_{p \to n} \left[\frac{J_p(x) \cos p\pi - J_{-p}(x)}{\sin p\pi}\right] = \lim_{p \to n} \frac{\dfrac{\partial}{\partial p}[J_p(x) \cos p\pi - J_{-p}(x)]}{\dfrac{\partial}{\partial p}[\sin p\pi]}$$

This motivates the definition (11) on page 98.

6.18. Use Problem 6.17 to obtain the general solution of Bessel's equation for $n = 0$.

In this case we must evaluate

$$\lim_{p \to 0} \left[\frac{J_p(x) \cos p\pi - J_{-p}(x)}{\sin p\pi}\right] \tag{1}$$

Using L'Hospital's rule (differentiating the numerator and denominator with respect to p), we find for the limit in (1)

$$\lim_{p \to 0} \left[\frac{(\partial J_p/\partial p) \cos p\pi - (\partial J_{-p}/\partial p)}{\pi \cos p\pi}\right] = \frac{1}{\pi}\left[\frac{\partial J_p}{\partial p} - \frac{\partial J_{-p}}{\partial p}\right]_{p=0}$$

where the notation indicates that we are to take the partial derivatives of $J_p(x)$ and $J_{-p}(x)$ with respect to p and then put $p = 0$. Since $\partial J_{-p}/\partial(-p) = -\partial J_{-p}/\partial p$, the required limit is also equal to

$$\frac{2}{\pi} \frac{\partial J_p}{\partial p}\bigg|_{p=0}$$

To obtain $\partial J_p/\partial p$ we differentiate the series

$$J_p(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{p+2r}}{r!\,\Gamma(p+r+1)}$$

with respect to p and obtain

$$\frac{\partial J_p}{\partial p} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \frac{\partial}{\partial p}\left\{\frac{(x/2)^{p+2r}}{\Gamma(p+r+1)}\right\} \tag{2}$$

Now if we let $\dfrac{(x/2)^{p+2r}}{\Gamma(p+r+1)} = G$, then $\ln G = (p+2r)\ln(x/2) - \ln\Gamma(p+r+1)$ so that differentiation with respect to p gives

$$\frac{1}{G}\frac{\partial G}{\partial p} = \ln(x/2) - \frac{\Gamma'(p+r+1)}{\Gamma(p+r+1)}$$

Then for $p = 0$, we have

$$\frac{\partial G}{\partial p}\bigg|_{p=0} = \frac{(x/2)^{2r}}{\Gamma(r+1)}\left[\ln(x/2) - \frac{\Gamma'(r+1)}{\Gamma(r+1)}\right] \tag{3}$$

Using (2) and (3), we have

$$\frac{2}{\pi}\frac{\partial J_p}{\partial p}\bigg|_{p=0} = \frac{2}{\pi}\sum_{r=0}^{\infty}\frac{(-1)^r(x/2)^{2r}}{r!\,\Gamma(r+1)}\left[\ln(x/2) - \frac{\Gamma'(r+1)}{\Gamma(r+1)}\right]$$

$$= \frac{2}{\pi}(\ln(x/2)+\gamma)J_0(x) + \frac{2}{\pi}\left[\frac{x^2}{2^2} - \frac{x^4}{2^2 4^2}(1+\tfrac{1}{2}) + \frac{x^6}{2^2 4^2 6^2}(1+\tfrac{1}{2}+\tfrac{1}{3}) - \cdots\right]$$

where the last series is obtained on using the result 6, on page 69. This last series is the series for $Y_0(x)$. We can in a similar manner obtain the series (12), page 98, for $Y_n(x)$ where n is an integer. The general solution if n is an integer is then given by $y = c_1 J_n(x) + c_2 Y_n(x)$.

FUNCTIONS RELATED TO BESSEL FUNCTIONS

6.19. Prove that the recurrence formula for the modified Bessel function of the first kind $I_n(x)$ is

$$I_{n+1}(x) = I_{n-1}(x) - \frac{2n}{x}I_n(x)$$

From Problem 6.8(b) we have

$$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x) \tag{1}$$

Replace x by ix to obtain

$$J_{n+1}(ix) = -\frac{2in}{x}J_n(ix) - J_{n-1}(ix) \tag{2}$$

Now by definition $I_n(x) = i^{-n}J_n(ix)$ or $J_n(ix) = i^n I_n(x)$, so that (2) becomes

$$i^{n+1}I_{n+1}(x) = -\frac{2in}{x}i^n I_n(x) - i^{n-1}I_n(x)$$

Dividing by i^{n+1} then gives the required result.

6.20. If n is not an integer, show that

(a) $H_n^{(1)}(x) = \dfrac{J_{-n}(x) - e^{-in\pi}J_n(x)}{i\sin n\pi}$, **(b)** $H_n^{(2)}(x) = \dfrac{e^{in\pi}J_n(x) - J_{-n}(x)}{i\sin n\pi}$

(a) By definition of $H_n^{(1)}(x)$ and $Y_n(x)$ (see pages 99 and 98 respectively) we have

$$H_n^{(1)}(x) = J_n(x) + iY_n(x) = J_n(x) + i\left[\frac{J_n(x)\cos n\pi - J_{-n}(x)}{\sin n\pi}\right]$$

$$= \frac{J_n(x)\sin n\pi + iJ_n(x)\cos n\pi - iJ_{-n}(x)}{\sin n\pi}$$

$$= i\left[\frac{J_n(x)(\cos n\pi - i\sin n\pi) - J_{-n}(x)}{\sin n\pi}\right]$$

$$= i\left[\frac{J_n(x)e^{-in\pi} - J_{-n}(x)}{\sin n\pi}\right] = \frac{J_{-n}(x) - e^{-in\pi}J_n(x)}{i\sin n\pi}$$

(b) Since $H_n^{(2)}(x) = J_n(x) - iY_n(x)$, we find on replacing i by $-i$ in the result of part (a),

$$H_n^{(2)}(x) = \frac{J_{-n}(x) - e^{in\pi}J_n(x)}{-i\sin n\pi} = \frac{e^{in\pi}J_n(x) - J_{-n}(x)}{i\sin n\pi}$$

6.21. Show that **(a)** $\operatorname{Ber}(x) = 1 - \dfrac{x^4}{2^2 4^2} + \dfrac{x^8}{2^2 4^2 6^2 8^2} - \cdots$

(b) $\operatorname{Bei}(x) = \dfrac{x^2}{2^2} - \dfrac{x^6}{2^2 4^2 6^2} + \dfrac{x^{10}}{2^2 4^2 6^2 8^2 10^2} - \cdots$

We have

$$\begin{aligned} J_0(i^{3/2}x) &= 1 - \frac{(i^{3/2}x)^2}{2^2} + \frac{(i^{3/2}x)^4}{2^2 4^2} - \frac{(i^{3/2}x)^6}{2^2 4^2 6^2} + \frac{(i^{3/2}x)^8}{2^2 4^2 6^2 8^2} - \cdots \\ &= 1 - \frac{i^3 x^2}{2^2} + \frac{i^6 x^4}{2^2 4^2} - \frac{i^9 x^6}{2^2 4^2 6^2} + \frac{i^{12} x^8}{2^2 4^2 6^2 8^2} - \cdots \\ &= 1 + \frac{ix^2}{2^2} - \frac{x^4}{2^2 4^2} - \frac{ix^6}{2^2 4^2 6^2} + \frac{x^8}{2^2 4^2 6^2 8^2} - \cdots \\ &= \left(1 - \frac{x^4}{2^2 4^2} + \frac{x^8}{2^2 4^2 6^2 8^2} - \cdots\right) + i\left(\frac{x^2}{2^2} - \frac{x^6}{2^2 4^2 6^2} + \cdots\right) \end{aligned}$$

and the required result follows on noting that $J_0(i^{3/2}x) = \text{Ber}(x) + i\,\text{Bei}(x)$ and equating real and imaginary parts. Note that the subscript zero has been omitted from $\text{Ber}_0(x)$ and $\text{Bei}_0(x)$.

EQUATIONS TRANSFORMABLE INTO BESSEL'S EQUATION

6.22. Find the general solution of the equation $xy'' + y' + ay = 0$.

The equation can be written as $x^2y'' + xy' + axy = 0$ and is a special case of equation (26), page 101, where $k = 0$, $\alpha = \sqrt{a}$, $r = 1/2$, $\beta = 0$. Then the solution as given by (27), page 101, is

$$y = c_1 J_0(2\sqrt{ax}) + c_2 Y_0(2\sqrt{ax})$$

ORTHOGONALITY OF BESSEL FUNCTIONS

6.23. Prove that $\displaystyle\int_0^1 xJ_n(\lambda x)\,J_n(\mu x)\,dx = \frac{\mu J_n(\lambda)\,J_n'(\mu) - \lambda J_n(\mu)\,J_n'(\lambda)}{\lambda^2 - \mu^2}$ if $\lambda \neq \mu$.

From (3) and (4), page 97, we see that $y_1 = J_n(\lambda x)$ and $y_2 = J_n(\mu x)$ are solutions of the equations

$$x^2y_1'' + xy_1' + (\lambda^2x^2 - n^2)y_1 = 0, \qquad x^2y_2'' + xy_2' + (\mu^2x^2 - n^2)y_2 = 0$$

Multiplying the first equation by y_2, the second by y_1 and subtracting, we find

$$x^2[y_2y_1'' - y_1y_2''] + x[y_2y_1' - y_1y_2'] = (\mu^2 - \lambda^2)x^2y_1y_2$$

which on division by x can be written as

$$x\frac{d}{dx}[y_2y_1' - y_1y_2'] + [y_2y_1' - y_1y_2'] = (\mu^2 - \lambda^2)xy_1y_2$$

or

$$\frac{d}{dx}\{x[y_2y_1' - y_1y_2']\} = (\mu^2 - \lambda^2)xy_1y_2$$

Then by integrating and omitting the constant of integration,

$$(\mu^2 - \lambda^2)\int xy_1y_2\,dx = x[y_2y_1' - y_1y_2']$$

or, using $y_1 = J_n(\lambda x)$, $y_2 = J_n(\mu x)$ and dividing by $\mu^2 - \lambda^2 \neq 0$,

$$\int xJ_n(\lambda x)\,J_n(\mu x)\,dx = \frac{x[\lambda J_n(\mu x)\,J_n'(\lambda x) - \mu J_n(\lambda x)\,J_n'(\mu x)]}{\mu^2 - \lambda^2}$$

Thus

$$\int_0^1 xJ_n(\lambda x)\,J_n(\mu x)\,dx = \frac{\lambda J_n(\mu)\,J_n'(\lambda) - \mu J_n(\lambda)\,J_n'(\mu)}{\mu^2 - \lambda^2}$$

which is equivalent to the required result.

6.24. Prove that $\displaystyle\int_0^1 xJ_n^2(\lambda x)\,dx = \frac{1}{2}\left[J_n'^2(\lambda) + \left(1 - \frac{n^2}{\lambda^2}\right)J_n^2(\lambda)\right]$.

Let $\mu \to \lambda$ in the result of Problem 6.23. Then, using L'Hospital's rule, we find

$$\begin{aligned} \int_0^1 xJ_n^2(\lambda x)\,dx &= \lim_{\mu\to\lambda} \frac{\lambda J_n'(\mu)\,J_n'(\lambda) - J_n(\lambda)\,J_n'(\mu) - \mu J_n(\lambda)\,J_n''(\mu)}{2\mu} \\ &= \frac{\lambda J_n'^2(\lambda) - J_n(\lambda)\,J_n'(\lambda) - \lambda J_n(\lambda)\,J_n''(\lambda)}{2\lambda} \end{aligned}$$

But since $\lambda^2 J_n''(\lambda) + \lambda J_n'(\lambda) + (\lambda^2 - n^2) J_n(\lambda) = 0$, we find on solving for $J_n''(\lambda)$ and substituting,

$$\int_0^1 x J_n^2(\lambda x)\, dx = \frac{1}{2}\left[J_n'^2(\lambda) + \left(1 - \frac{n^2}{\lambda^2}\right) J_n^2(\lambda)\right]$$

6.25. Prove that if λ and μ are any two different roots of the equation $RJ_n(x) + SxJ_n'(x) = 0$, where R and S are constants, then

$$\int_0^1 x J_n(\lambda x)\, J_n(\mu x)\, dx = 0$$

i.e. $\sqrt{x}\, J_n(\lambda x)$ and $\sqrt{x}\, J_n(\mu x)$ are orthogonal in $(0, 1)$.

Since λ and μ are roots of $RJ_n(x) + SxJ_n'(x) = 0$, we have

$$RJ_n(\lambda) + S\lambda J_n'(\lambda) = 0, \qquad RJ_n(\mu) + S\mu J_n'(\mu) = 0 \qquad (1)$$

Then since R and S are not both zero we find from (1),

$$\mu J_n(\lambda)\, J_n'(\mu) - \lambda J_n(\mu)\, J_n'(\lambda) = 0$$

and so from Problem 6.23 we have the required result

$$\int_0^1 x J_n(\lambda x)\, J_n(\mu x)\, dx = 0$$

SERIES OF BESSEL FUNCTIONS OF THE FIRST KIND

6.26. If $f(x) = \sum_{p=1}^{\infty} A_p J_n(\lambda_p x)$, $0 < x < 1$, where λ_p, $p = 1, 2, 3, \ldots$, are the positive roots of $J_n(x) = 0$, show that

$$A_p = \frac{2}{J_{n+1}^2(\lambda_p)} \int_0^1 x J_n(\lambda_p x)\, f(x)\, dx$$

Multiply the series for $f(x)$ by $xJ_n(\lambda_k x)$ and integrate term by term from 0 to 1. Then

$$\begin{aligned} \int_0^1 x J_n(\lambda_k x)\, f(x)\, dx &= \sum_{p=1}^{\infty} A_p \int_0^1 x J_n(\lambda_k x)\, J_n(\lambda_p x)\, dx \\ &= A_k \int_0^1 x J_n^2(\lambda_k x)\, dx \\ &= \frac{1}{2} A_k J_n'^2(\lambda_k) \end{aligned}$$

where we have used Problems 6.24 and 6.25 together with the fact that $J_n(\lambda_k) = 0$. It follows that

$$A_k = \frac{2}{J_n'^2(\lambda_k)} \int_0^1 x J_n(\lambda_k x)\, f(x)\, dx$$

To obtain the required result from this, we note that from the recurrence formula 3, page 99, which is equivalent to the formula 6 on that page, we have

$$\lambda_k J_n'(\lambda_k) = nJ_n(\lambda_k) - \lambda_k J_{n+1}(\lambda_k)$$

or since $J_n(\lambda_k) = 0$,

$$J_n'(\lambda_k) = -J_{n+1}(\lambda_k)$$

6.27. Expand $f(x) = 1$ in a series of the form

$$\sum_{p=1}^{\infty} A_p J_0(\lambda_p x)$$

for $0 < x < 1$, if λ_p, $p = 1, 2, 3, \ldots$, are the positive roots of $J_0(x) = 0$.

From Problem 6.26 we have

$$A_p = \frac{2}{J_1^2(\lambda_p)}\int_0^1 xJ_0(\lambda_p x)\,dx = \frac{2}{\lambda_p^2 J_1^2(\lambda_p)}\int_0^{\lambda_p} vJ_0(v)\,dv$$

$$= \frac{2}{\lambda_p^2 J_1^2(\lambda_p)}\,vJ_1(v)\Big|_0^{\lambda_p} = \frac{2}{\lambda_p J_1(\lambda_p)}$$

where we have made the substitution $v = \lambda_p x$ in the integral and used the result of Problem 6.10(a) with $n = 1$.

Thus we have the required series

$$f(x) = 1 = \sum_{p=1}^{\infty} \frac{2}{\lambda_p J_1(\lambda_p)} J_0(\lambda_p x)$$

which can be written

$$\frac{J_0(\lambda_1 x)}{\lambda_1 J_1(\lambda_1)} + \frac{J_0(\lambda_2 x)}{\lambda_2 J_1(\lambda_2)} + \cdots = \frac{1}{2}$$

SOLUTIONS USING BESSEL FUNCTIONS OF THE FIRST KIND

6.28. A circular plate of unit radius (see Fig. 6-7) has its plane faces insulated. If the initial temperature is $F(\rho)$ and if the rim is kept at temperature zero, find the temperature of the plate at any time.

Since the temperature is independent of ϕ, the boundary value problem for determining $u(\rho, t)$ is

$$\frac{\partial u}{\partial t} = \kappa\left(\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho}\right) \qquad (1)$$

$$u(1,t) = 0, \quad u(\rho, 0) = F(\rho), \quad |u(\rho,t)| < M$$

Let $u = P(\rho)\,T(t) = PT$ in equation (1). Then

$$PT' = \kappa\left(P''T + \frac{1}{\rho}P'T\right)$$

or dividing by κPT,

$$\frac{T'}{\kappa T} = \frac{P''}{P} + \frac{1}{\rho}\frac{P'}{P} = -\lambda^2$$

from which

$$T' + \kappa\lambda^2 T = 0, \qquad P'' + \frac{1}{\rho}P' + \lambda^2 P = 0$$

These have general solutions

$$T = c_1 e^{-\kappa\lambda^2 t}, \qquad P = A_1 J_0(\lambda\rho) + B_1 Y_0(\lambda\rho)$$

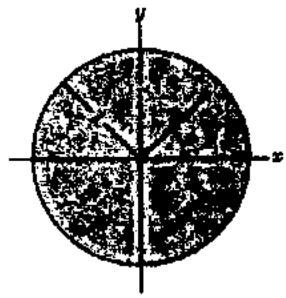

Fig. 6-7

Since $u = PT$ is bounded at $\rho = 0$, $B_1 = 0$. Then

$$u(\rho, t) = Ae^{-\kappa\lambda^2 t}J_0(\lambda\rho)$$

where $A = A_1 c_1$.

From the first boundary condition,

$$u(1,t) = Ae^{-\kappa\lambda^2 t}J_0(\lambda) = 0$$

from which $J_0(\lambda) = 0$ and $\lambda = \lambda_1, \lambda_2, \ldots$ are the positive roots.

Thus a solution is

$$u(\rho, t) = Ae^{-\kappa\lambda_m^2 t}J_0(\lambda_m\rho) \qquad m = 1, 2, 3, \ldots$$

By superposition, a solution is

$$u(\rho, t) = \sum_{m=1}^{\infty} A_m e^{-\kappa\lambda_m^2 t}J_0(\lambda_m\rho)$$

From the second boundary condition,

$$u(\rho, 0) = F(\rho) = \sum_{m=1}^{\infty} A_m J_0(\lambda_m \rho)$$

Then from Problem 6.26 with $n = 0$ we have

$$A_m = \frac{2}{J_1^2(\lambda_m)} \int_0^1 \rho F(\rho)\, J_0(\lambda_m \rho)\, d\rho$$

and so

$$u(\rho, t) = \sum_{m=1}^{\infty} \left\{ \left[\frac{2}{J_1^2(\lambda_m)} \int_0^1 \rho F(\rho)\, J_0(\lambda_m \rho)\, d\rho \right] e^{-\kappa \lambda_m^2 t} J_0(\lambda_m \rho) \right\} \tag{2}$$

which can be established as the required solution.

Note that this solution also gives the temperature of an infinitely long solid cylinder whose convex surface is kept at temperature zero and whose initial temperature is $F(\rho)$.

6.29. A solid conducting cylinder of unit height and radius and with diffusivity κ is initially at temperature $f(\rho, z)$ (see Fig. 6-8). The entire surface is suddenly lowered to temperature zero and kept at this temperature. Find the temperature at any point of the cylinder at any subsequent time.

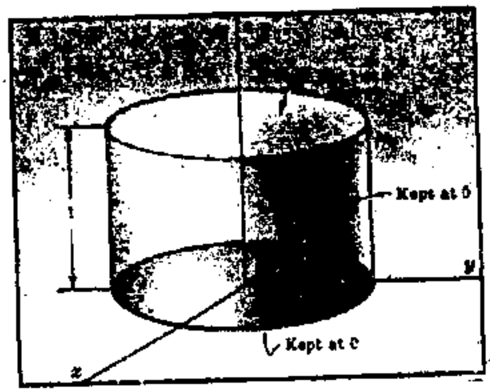

Fig. 6-8

Since there is no ϕ-dependence, as is evident from symmetry, the heat conduction equation is

$$\frac{\partial u}{\partial t} = \kappa\left(\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{\partial^2 u}{\partial z^2}\right) \tag{1}$$

where $u = u(\rho, z, t)$. The boundary conditions are given by

$$u(\rho, z, 0) = f(\rho, z), \quad u(\rho, 0, t) = 0, \quad u(\rho, 1, t) = 0, \quad u(1, z, t) = 0, \quad |u(\rho, z, t)| < M \tag{2}$$

where $0 \leq \rho < 1$, $0 < z < 1$, $t > 0$.

To solve this boundary value problem let $U = PZT = P(\rho)Z(z)T(t)$ in (1) to obtain

$$PZT' = \kappa\left(P''ZT + \frac{1}{\rho}P'ZT + PZ''T\right)$$

Then dividing by κPZT we have

$$\frac{T'}{\kappa T} = \frac{P''}{P} + \frac{1}{\rho}\frac{P'}{P} + \frac{Z''}{Z}$$

Since the left side depends only on t while the right side depends only on ρ and z, each side must be a constant, say $-\lambda^2$. Thus

$$T' + \kappa\lambda^2 T = 0$$

$$\frac{P''}{P} + \frac{1}{\rho}\frac{P'}{P} + \frac{Z''}{Z} = -\lambda^2 \tag{3}$$

The last equation can be written as

$$\frac{P''}{P} + \frac{1}{\rho}\frac{P'}{P} = -\lambda^2 - \frac{Z''}{Z}$$

from which we see that each side must be a constant, say $-\mu^2$. From this we obtain the two equations

$$\rho P'' + P' + \mu^2 \rho P = 0 \tag{4}$$

$$Z'' - \nu^2 Z = 0 \tag{5}$$

where we have written

$$\nu^2 = \mu^2 - \lambda^2 \tag{6}$$

The solutions of (3), (4) and (5) are given by

$$T = c_1 e^{-\kappa\lambda^2 t}, \quad P = c_2 J_0(\mu\rho) + c_3 Y_0(\mu\rho), \quad Z = c_4 e^{\nu z} + c_5 e^{-\nu z}$$

Thus a solution to (1) is given by the product of these, i.e.

$$u(\rho, z, t) = [c_1 e^{-\kappa\lambda^2 t}][c_2 J_0(\mu\rho) + c_3 Y_0(\mu\rho)][c_4 e^{\nu z} + c_5 e^{-\nu z}]$$

Now from the boundedness condition at $\rho = 0$ we must have $c_3 = 0$. Thus the solution becomes

$$u(\rho, z, t) = e^{-\kappa\lambda^2 t} J_0(\mu\rho)[Ae^{\nu z} + Be^{-\nu z}] \tag{7}$$

From the second boundary condition in (2) we see that

$$u(\rho, 0, t) = e^{-\kappa\lambda^2 t} J_0(\mu\rho)(A + B) = 0$$

so that we must have $A + B = 0$ or $B = -A$. Then (7) becomes

$$u(\rho, z, t) = Ae^{-\kappa\lambda^2 t} J_0(\mu\rho)[e^{\nu z} - e^{-\nu z}]$$

From the third condition we have

$$u(\rho, 1, t) = Ae^{-\kappa\lambda^2 t} J_0(\mu\rho)[e^{\nu} - e^{-\nu}] = 0$$

which can be satisfied only if $e^{\nu} - e^{-\nu} = 0$ or

$$e^{2\nu} = 1 = e^{2k\pi i} \qquad k = 0, 1, 2, \ldots$$

It follows that we must have $2\nu = 2k\pi i$ or

$$\nu = k\pi i \qquad k = 0, 1, 2, \ldots \tag{8}$$

Using this in (7), it becomes

$$u(\rho, z, t) = Ce^{-\kappa\lambda^2 t} J_0(\mu\rho) \sin k\pi z$$

where C is a new constant.

From the fourth condition in (2) we obtain

$$u(1, z, t) = Ce^{-\kappa\lambda^2 t} J_0(\mu) \sin k\pi z = 0$$

which can be satisfied only if $J_0(\mu) = 0$ so that

$$\mu = r_1, r_2, \ldots \tag{9}$$

where r_m $(m = 1, 2, \ldots)$ is the mth positive root of $J_0(x) = 0$. Now from (6), (8) and (9) it follows that

$$\lambda^2 = \mu^2 - \nu^2 = r_m^2 + k^2\pi^2$$

so that a solution satisfying all conditions in (2) but the first is given by

$$u(\rho, z, t) = Ce^{-\kappa(r_m^2 + k^2\pi^2)t} J_0(r_m\rho) \sin k\pi z \tag{10}$$

where $k = 1, 2, 3, \ldots$, $m = 1, 2, 3, \ldots$. Replacing C by C_{km} and summing over k and m we obtain by the superposition principle the solution

$$u(\rho, z, t) = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} C_{km} e^{-\kappa(r_m^2 + k^2\pi^2)t} J_0(r_m\rho) \sin k\pi z \tag{11}$$

The first condition in (2) now leads to

$$f(\rho, z) = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} C_{km} J_0(r_m\rho) \sin k\pi z$$

This can be written as

$$f(\rho, z) = \sum_{k=1}^{\infty} \left\{ \sum_{m=1}^{\infty} C_{km} J_0(r_m\rho) \right\} \sin k\pi z = \sum_{k=1}^{\infty} b_k \sin k\pi z$$

where

$$b_k = \sum_{m=1}^{\infty} C_{km} J_0(r_m\rho) \tag{12}$$

It follows from this that b_k are the Fourier coefficients obtained when $f(\rho, z)$ is expanded into a Fourier sine series in z [we think of ρ as kept constant in this case]. Thus by the methods of Chapter 2 we have

$$b_k = \frac{2}{1}\int_0^1 f(\rho, z) \sin k\pi z \, dz \tag{13}$$

We now must find C_{km} from the expansion (12). Since b_k is a function of ρ, this is simply the expansion of b_k into a Bessel series as in Problem 6.26, and we find

$$C_{km} = \frac{2}{J_1^2(r_m)}\int_0^1 \rho b_k J_0(r_m\rho)\, d\rho \tag{14}$$

This becomes on using (13)

$$C_{km} = \frac{4u}{J_1^2(r_m)}\int_0^1\int_0^1 \rho f(\rho, z)\, J_0(r_m\rho) \sin k\pi z \, d\rho\, dz \tag{15}$$

The required solution is thus given by (11) with the coefficients (15).

6.30. Work Problem 6.29 if $f(\rho, z) = u_0$, a constant.

In this case we find from (15) of Problem 6.29

$$\begin{aligned} C_{km} &= \frac{4u_0}{J_1^2(r_m)}\int_0^1\int_0^1 \rho\, J_0(r_m\rho) \sin k\pi z \, d\rho\, dz \\ &= \frac{4u_0}{J_1^2(r_m)}\left\{\int_0^1 \rho\, J_0(r_m\rho)\, d\rho\right\}\left\{\int_0^1 \sin k\pi z\, dz\right\} \\ &= \frac{4u_0}{J_1^2(r_m)}\left\{\frac{J_1(r_m)}{r_m}\right\}\left\{\frac{1-\cos k\pi}{k\pi}\right\} \\ &= \frac{4u_0(1-\cos k\pi)}{k\pi r_m J_1(r_m)} \end{aligned}$$

on using the same procedure as in Problem 6.27. The required solution is thus

$$u(\rho, z, t) = \frac{4u_0}{\pi}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{1-\cos k\pi}{k r_m J_1(r_m)}\, e^{-\kappa(r_m^2 + k^2\pi^2)t} J_0(r_m\rho) \sin k\pi z$$

6.31. A drum consists of a stretched circular membrane of unit radius whose rim, represented by the circle of Fig. 6-7, is fixed. If the membrane is struck so that its initial displacement is $F(\rho, \phi)$ and is then released, find the displacement at any time.

The boundary value problem for the displacement $z(\rho, \phi, t)$ from the equilibrium or rest position (the xy-plane) is

$$\frac{\partial^2 z}{\partial t^2} = a^2\left(\frac{\partial^2 z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial z}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 z}{\partial \phi^2}\right)$$

$$z(1, \phi, t) = 0, \quad z(\rho, \phi, 0) = 0, \quad z_t(\rho, \phi, 0) = 0, \quad z(\rho, \phi, 0) = F(\rho, \phi)$$

Let $z = P(\rho)\Phi(\phi)T(t) = P\Phi T$. Then

$$P\Phi T'' = a^2\left(P''\Phi T + \frac{1}{\rho}P'\Phi T + \frac{1}{\rho^2}P\Phi'' T\right)$$

Dividing by $a^2 P\Phi T$,

$$\frac{T''}{a^2 T} = \frac{P''}{P} + \frac{1}{\rho}\frac{P'}{P} + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} = -\lambda^2$$

and so
$$T'' + \lambda^2 a^2 T = 0 \tag{1}$$
$$\frac{P''}{P} + \frac{1}{\rho}\frac{P'}{P} + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} = -\lambda^2 \tag{2}$$

Multiplying (2) by ρ^2, the variables can be separated to yield
$$\frac{\rho^2 P''}{P} + \frac{\rho P'}{P} + \lambda^2\rho^2 = -\frac{\Phi''}{\Phi} = \mu^2$$

so that
$$\Phi'' + \mu^2\Phi = 0 \tag{3}$$
$$\rho^2 P'' + \rho P' + (\lambda^2\rho^2 - \mu^2)P = 0 \tag{4}$$

General solutions of (1), (3) and (4) are
$$T = A_1 \cos \lambda at + B_1 \sin \lambda at \tag{5}$$
$$\Phi = A_2 \cos \mu\phi + B_2 \sin \mu\phi \tag{6}$$
$$P = A_3 J_\mu(\lambda\rho) + B_3 Y_\mu(\lambda\rho) \tag{7}$$

A solution $z(\rho, \phi, t)$ is given by the product of these.

Since z must have period 2π in the variable ϕ, we must have $\mu = m$ where $m = 0, 1, 2, 3, \ldots$ from (6).

Also, since z is bounded at $\rho = 0$ we must take $B_3 = 0$.

Furthermore, to satisfy $z_t(\rho, \phi, 0) = 0$ we must choose $B_1 = 0$.

Then a solution is
$$u(\rho, \phi, t) = J_m(\lambda\rho) \cos \lambda at\,(A \cos m\phi + B \sin m\phi)$$

Since $z(1, \phi, t) = 0$, $J_m(\lambda) = 0$ so that $\lambda = \lambda_{mk}$, $k = 1, 2, 3, \ldots$, are the positive roots.

By superposition (summing over both m and k),
$$\begin{aligned} z(\rho, \phi, t) &= \sum_{m=0}^{\infty}\sum_{k=1}^{\infty} J_m(\lambda_{mk}\rho) \cos(\lambda_{mk}at)(A_{mk} \cos m\phi + B_{mk} \sin m\phi) \\ &= \sum_{m=0}^{\infty}\left\{\left[\sum_{k=1}^{\infty} A_{mk} J_m(\lambda_{mk}\rho)\right] \cos m\phi \right. \\ &\qquad \left. + \left[\sum_{k=1}^{\infty} B_{mk} J_m(\lambda_{mk}\rho)\right] \sin m\phi\right\} \cos \lambda_{mk}at \end{aligned} \tag{8}$$

Putting $t = 0$, we have
$$z(\rho, \phi, 0) = F(\rho, \phi) = \sum_{m=0}^{\infty} (C_m \cos m\phi + D_m \sin m\phi) \tag{9}$$

where
$$\begin{aligned} C_m &= \sum_{k=1}^{\infty} A_{mk} J_m(\lambda_{mk}\rho) \\ D_m &= \sum_{k=1}^{\infty} B_{mk} J_m(\lambda_{mk}\rho) \end{aligned} \tag{10}$$

But (9) is simply a Fourier series and we can determine C_m and D_m by the usual methods. We find
$$C_m = \begin{cases} \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} F(\rho, \phi) \cos m\phi \, d\phi & m = 1, 2, 3, \ldots \\ \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} F(\rho, \phi)\, d\phi & m = 0 \end{cases}$$
$$D_m = \frac{1}{\pi}\int_0^{2\pi} F(\rho, \phi) \sin m\phi \, d\phi \qquad m = 0, 1, 2, 3, \ldots$$

From (10), using the results of Bessel series expansions, we have

$$A_{mk} = \frac{2}{[J_{m+1}(\lambda_{mk})]^2}\int_0^1 \rho J_m(\lambda_{mk}\rho)C_m\,d\rho$$

$$= \begin{cases} \dfrac{2}{\pi[J_{m+1}(\lambda_{mk})]^2}\displaystyle\int_0^1\int_0^{2\pi}\rho F(\rho,\phi)\,J_m(\lambda_{mk}\rho)\cos m\phi\,d\rho\,d\phi & \text{if } m=1,2,3,\ldots \\ \dfrac{1}{\pi[J_1(\lambda_{0k})]^2}\displaystyle\int_0^1\int_0^{2\pi}\rho F(\rho,\phi)\,J_0(\lambda_{0k}\rho)\,d\rho\,d\phi & \text{if } m=0 \end{cases}$$

$$B_{mk} = \frac{2}{[J_{m+1}(\lambda_{mk})]^2}\int_0^1 \rho J_m(\lambda_{mk}\rho)\,D_m\,d\rho$$

$$= \frac{2}{\pi[J_{m+1}(\lambda_{mk})]^2}\int_0^1\int_0^{2\pi}\rho F(\rho,\phi)\,J_m(\lambda_{mk}\rho)\sin m\phi\,d\rho\,d\phi \qquad \text{if } m=0,1,2,\ldots$$

Using these values of A_{mk} and B_{mk} in (8) yields the required solution.

Note that the various modes of vibration of the drum are obtained by specifying particular values of m and k. The frequencies of vibration are then given by

$$f_{mk} = \frac{\lambda_{mk}}{2\pi}a$$

Because these are not integer multiples of the lowest frequency, we would expect noise rather than a musical tone.

SERIES USING BESSEL FUNCTIONS OF THE SECOND KIND

6.32. **Let $u_0(\lambda_m\rho) = Y_0(\lambda_m a)\,J_0(\lambda_m\rho) - J_0(\lambda_m a)\,Y_0(\lambda_m\rho)$ where λ_m, $m=1,2,3,\ldots$, are the positive roots of $Y_0(\lambda a)\,J_0(\lambda b) - J_0(\lambda a)\,Y_0(\lambda b) = 0$. Show that**

$$\int_a^b \rho u_0(\lambda_m\rho)u_0(\lambda_n\rho)\,d\rho = 0 \qquad m\neq n$$

The functions $P_m = u_0(\lambda_m\rho)$ and $P_n = u_0(\lambda_n\rho)$ satisfy the equations

$$\rho P_m'' + P_m' + \lambda_m^2\rho P_m = 0 \tag{1}$$

$$\rho P_n'' + P_n' + \lambda_n^2\rho P_n = 0 \tag{2}$$

Multiplying (1) by P_n, (2) by P_m, and subtracting, we find

$$\rho(P_nP_m'' - P_mP_n'') + P_nP_m' - P_mP_n' = (\lambda_n^2-\lambda_m^2)\rho P_mP_n$$

which can be written

$$\rho\frac{d}{d\rho}(P_nP_m' - P_mP_n') + P_nP_m' - P_mP_n' = (\lambda_n^2-\lambda_m^2)\rho P_mP_n$$

or

$$\frac{d}{d\rho}[\rho(P_nP_m' - P_mP_n')] = (\lambda_n^2-\lambda_m^2)\rho P_mP_n$$

Then by integrating both sides from a to b we have

$$(\lambda_n^2-\lambda_m^2)\int_a^b \rho P_mP_n\,d\rho = \rho(P_nP_m' - P_mP_n')\Big|_a^b$$

$$= \rho[\lambda_m u_0(\lambda_n\rho)\,u_0'(\lambda_m\rho) - \lambda_n u_0(\lambda_m\rho)\,u_0'(\lambda_n\rho)]\Big|_a^b$$

$$= 0$$

on using the facts $u_0(\lambda_m a)=0$, $u_0(\lambda_n a)=0$, $u_0(\lambda_m b)=0$, $u_0(\lambda_n b)=0$. Then since $\lambda_m\neq\lambda_n$ we have

$$\int_a^b \rho P_mP_n\,d\rho = \int_a^b \rho u_0(\lambda_m\rho)\,u_0(\lambda_n\rho)\,d\rho = 0$$

6.33. **Show how to expand a function $F(\rho)$ into a series of the form $\sum_{m=1}^{\infty} A_m u_0(\lambda_m\rho)$ where the functions $u_0(\lambda_m\rho)$ are given in Problem 6.32.**

Suppose that

$$F(\rho) = \sum_{m=1}^{\infty} A_m u_0(\lambda_m\rho) \tag{1}$$

Then on multiplying both sides by $\rho u_0(\lambda_n\rho)$ and integrating from a to b we find

$$\int_a^b \rho F(\rho)\, u_0(\lambda_n\rho)\, d\rho = \sum_{m=1}^{\infty} A_m \int_a^b \rho u_0(\lambda_m\rho)\, u_0(\lambda_n\rho)\, d\rho$$

$$= A_n \int_a^b \rho[u_0(\lambda_n\rho)]^2\, d\rho$$

on making use of Problem 6.32.

Thus

$$A_n = \frac{\int_a^b \rho F(\rho)\, u_0(\lambda_n\rho)\, d\rho}{\int_a^b \rho[u_0(\lambda_n\rho)]^2\, d\rho} \tag{2}$$

Although these coefficients have been obtained formally, we can show that when these coefficients are used in the right side of (1) it does converge to $F(\rho)$ at points of continuity, assuming that $F(\rho)$ and $F'(\rho)$ are piecewise continuous, while at points of discontinuity it converges to $\frac{1}{2}[F(\rho+0) + F(\rho-0)]$.

6.34. **A very long hollow cylinder of inner radius a and outer radius b (whose cross section is indicated in Fig. 6-9) is made of conducting material of diffusivity κ. If the inner and outer surfaces are kept at temperature zero while the initial temperature is a given function $f(\rho)$, where ρ is the distance from the axis, find the temperature at any point at any later time t.**

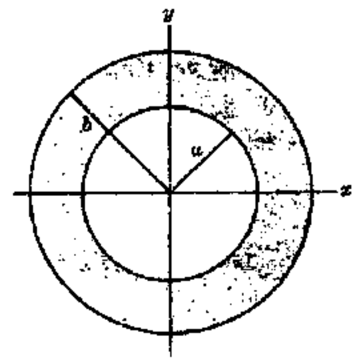

Fig. 6-9

Since symmetry shows that there is no ϕ- or z-dependence, the boundary value problem which we must solve for $u = u(\rho, t)$ is

$$\frac{\partial u}{\partial t} = \kappa\left(\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho}\right) \tag{1}$$

$$u(a,t) = 0,\ u(b,t) = 0,\ u(\rho,0) = f(\rho),\ |u(\rho,t)| < M \tag{2}$$

By separation of variables we have as in Problem 6.28

$$u(\rho,t) = e^{-\kappa\lambda^2 t}[a_1 J_0(\lambda\rho) + b_1 Y_0(\lambda\rho)] \tag{3}$$

From $u(a,t) = 0$ and $u(b,t) = 0$ we find

$$a_1 J_0(\lambda a) + b_1 Y_0(\lambda a) = 0, \qquad a_1 J_0(\lambda b) + b_1 Y_0(\lambda b) = 0 \tag{4}$$

These equations lead to the equation

$$Y_0(\lambda a)\, J_0(\lambda b) - J_0(\lambda a)\, Y_0(\lambda b) = 0 \tag{5}$$

for determining λ. The equation (5) has infinitely many positive roots $\lambda_1, \lambda_2, \ldots$.

From the first equation in (4) we find

$$b_1 = -\frac{a_1 J_0(\lambda a)}{Y_0(\lambda a)}$$

so that (3) can be written

$$u(\rho,t) = A e^{-\kappa\lambda^2 t}[Y_0(\lambda a)\, J_0(\lambda\rho) - J_0(\lambda a)\, Y_0(\lambda\rho)] \tag{6}$$

where A is a constant.

Using the fact that for $\lambda = \lambda_m$ (6) is a solution, together with the principle of superposition, we obtain the solution

$$u(\rho, t) = \sum_{m=1}^{\infty} A_m e^{-\kappa\lambda_m^2 t} u_0(\lambda_m \rho) \tag{7}$$

where
$$u_0(\lambda_m \rho) = Y_0(\lambda_m a) J_0(\lambda_m \rho) - J_0(\lambda_m a) Y_0(\lambda_m \rho) \tag{8}$$

From the condition $u(\rho, 0) = f(\rho)$ we now obtain from (7)

$$f(\rho) = \sum_{m=1}^{\infty} A_m u_0(\lambda_m \rho) \tag{9}$$

Then
$$A_m = \frac{\int_a^b \rho f(\rho)\, u_0(\lambda_m \rho)\, d\rho}{\int_a^b \rho [u_0(\lambda_m \rho)]^2\, d\rho} \tag{10}$$

Substitution of these coefficients into (7) gives the required solution.

6.25. A simple pendulum initially has a length of l_0 and makes an angle θ_0 with the vertical. It is then released from this position. If the length l of the pendulum increases with time t according to $l = l_0 + \epsilon t$ where ϵ is a constant, find the position of the pendulum at any time assuming the oscillations to be small.

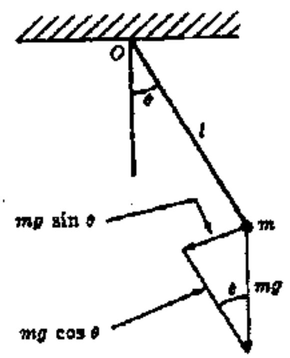

Fig. 6-10

Let m be the mass of the bob and θ the angle which the pendulum makes with the vertical at any time t. The weight mg can be resolved into two components, one tangential to the path and given by $mg \sin\theta$ and the other perpendicular to it and given by $mg\cos\theta$, as shown in Fig. 6-10. From mechanics we know that

$$\text{Torque about } O = \frac{d}{dt}(\text{Angular momentum about } O)$$

or
$$(-mg \sin\theta)l = \frac{d}{dt}(ml^2\dot{\theta}) \tag{1}$$

where $\dot{\theta} = d\theta/dt$. This equation can be written as

$$l\ddot{\theta} + 2\dot{l}\dot{\theta} + g\sin\theta = 0$$

or since $l = l_0 + \epsilon t$,
$$(l_0 + \epsilon t)\ddot{\theta} + 2\epsilon\dot{\theta} + g\theta = 0$$

Letting $x = l_0 + \epsilon t$ in this equation it becomes

$$x\frac{d^2\theta}{dx^2} + 2\frac{d\theta}{dx} + \frac{g}{\epsilon^2}\theta = 0 \tag{2}$$

Multiplying by x and comparing with equations (26) and (27), page 101, we find that the solution is

$$\theta = \frac{1}{\sqrt{l_0 + \epsilon t}}\left[AJ_1\left(\frac{2\sqrt{g}}{\epsilon}\sqrt{l_0 + \epsilon t}\right) + BY_1\left(\frac{2\sqrt{g}}{\epsilon}\sqrt{l_0 + \epsilon t}\right)\right] \tag{3}$$

Since $\theta = \theta_0$ at $t = 0$ we have

$$\theta_0 = \frac{1}{\sqrt{l_0}}\left[AJ_1\left(\frac{2\sqrt{gl_0}}{\epsilon}\right) + BY_1\left(\frac{2\sqrt{gl_0}}{\epsilon}\right)\right] \tag{4}$$

To satisfy $\dot{\theta} = 0$ at $t = 0$ we must first obtain $\dot{\theta} = d\theta/dt$. We find

$$\dot{\theta} = \frac{d\theta}{dt} = -\frac{\epsilon}{2(l_0+\epsilon t)^{3/2}}\left[AJ_1\left(\frac{2\sqrt{g}}{\epsilon}\sqrt{l_0+\epsilon t}\right) + BY_1\left(\frac{2\sqrt{g}}{\epsilon}\sqrt{l_0+\epsilon t}\right)\right]$$
$$+ \frac{\sqrt{g}}{l_0+\epsilon t}\left[AJ_1'\left(\frac{2\sqrt{g}}{\epsilon}\sqrt{l_0+\epsilon t}\right) + BY_1'\left(\frac{2\sqrt{g}}{\epsilon}\sqrt{l_0+\epsilon t}\right)\right]$$

Now since $\dot{\theta} = 0$ for $t = 0$ we find

$$0 = -\frac{\epsilon}{2l_0^{3/2}}\left[AJ_1\left(\frac{2\sqrt{gl_0}}{\epsilon}\right) + BY_1\left(\frac{2\sqrt{gl_0}}{\epsilon}\right)\right]$$
$$+ \frac{\sqrt{g}}{l_0}\left[AJ_1'\left(\frac{2\sqrt{gl_0}}{\epsilon}\right) + BY_1'\left(\frac{2\sqrt{gl_0}}{\epsilon}\right)\right]$$

or using (4)

$$AJ_1'\left(\frac{2\sqrt{gl_0}}{\epsilon}\right) + BY_1'\left(\frac{2\sqrt{gl_0}}{\epsilon}\right) = \frac{\epsilon\theta_0}{2\sqrt{g}} \tag{5}$$

Solving for A and B from (4) and (5) we find

$$A = \frac{\sqrt{l_0}\,Y_1' - (\epsilon/2\sqrt{g})Y_1}{J_1Y_1' - Y_1J_1'}\,\theta_0$$
$$B = \frac{(\epsilon/2\sqrt{g})J_1 - \sqrt{l_0}\,J_1'}{J_1Y_1' - Y_1J_1'}\,\theta_0 \tag{6}$$

where the argument $2\sqrt{gl_0}/\epsilon$ in J_1, J_1', Y_1, Y_1' has been omitted.

Now from Problem 6.58 with $n = 1$ we know that

$$J_1(x)\,Y_1'(x) - Y_1(x)\,J_1'(x) = \frac{2}{\pi x}$$

so that
$$J_1\left(\frac{2\sqrt{gl_0}}{\epsilon}\right)Y_1'\left(\frac{2\sqrt{gl_0}}{\epsilon}\right) - Y_1\left(\frac{2\sqrt{gl_0}}{\epsilon}\right)J_1'\left(\frac{2\sqrt{gl_0}}{\epsilon}\right) = \frac{\epsilon}{\pi\sqrt{gl_0}}$$

Thus (6) becomes

$$A = \frac{\pi\sqrt{g}\,l_0\theta_0}{\epsilon}\,Y_1'\left(\frac{2\sqrt{gl_0}}{\epsilon}\right) - \frac{\pi\sqrt{l_0}\,\theta_0}{2}\,Y_1\left(\frac{2\sqrt{gl_0}}{\epsilon}\right)$$
$$B = \frac{\pi\sqrt{l_0}\,\theta_0}{2}\,J_1\left(\frac{2\sqrt{gl_0}}{\epsilon}\right) - \frac{\pi\sqrt{g}\,l_0\theta_0}{\epsilon}\,J_1'\left(\frac{2\sqrt{gl_0}}{\epsilon}\right) \tag{7}$$

Now from formula 3, page 99, with $n = 1$ and the corresponding formula involving Y_n for $n = 1$, we have from (7)

$$A = -\frac{\pi\sqrt{l_0}\,\theta_0}{2}\,Y_2\left(\frac{2\sqrt{gl_0}}{\epsilon}\right)$$
$$B = \frac{\pi\sqrt{l_0}\,\theta_0}{2}\,J_2\left(\frac{2\sqrt{gl_0}}{\epsilon}\right) \tag{8}$$

Using these in (3) we thus find

$$\theta = \frac{\pi\sqrt{l_0}\,\theta_0}{2\sqrt{l_0+\epsilon t}}\left[J_2\left(\frac{2\sqrt{gl_0}}{\epsilon}\right)Y_1\left(\frac{2\sqrt{g}}{\epsilon}\sqrt{l_0+\epsilon t}\right) - Y_2\left(\frac{2\sqrt{gl_0}}{\epsilon}\right)J_1\left(\frac{2\sqrt{g}}{\epsilon}\sqrt{l_0+\epsilon t}\right)\right] \tag{9}$$

Supplementary Problems

BESSEL FUNCTIONS OF THE FIRST KIND

6.36. (a) Show that $J_1(x) = \frac{x}{2} - \frac{x^3}{2^2 4} + \frac{x^5}{2^2 4^2 6} - \frac{x^7}{2^2 4^2 6^2 8} + \cdots$ and verify that the interval of convergence is $-\infty < x < \infty$.

(b) Show that $J_0'(x) = -J_1(x)$.

(c) Show that $\frac{d}{dx}[xJ_1(x)] = xJ_0(x)$.

6.37. Evaluate (a) $J_{5/2}(x)$ and (b) $J_{-5/2}(x)$ in terms of sines and cosines.

6.38. Find $J_3(x)$ in terms of $J_0(x)$ and $J_1(x)$.

6.39. Prove that (a) $J_n''(x) = \frac{1}{4}[J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)]$

(b) $J_n'''(x) = \frac{1}{8}[J_{n-3}(x) - 3J_{n-1}(x) + 3J_{n+1}(x) - J_{n+3}(x)]$

and generalize these results.

6.40. Evaluate (a) $\int x^3 J_2(x)\,dx$, (b) $\int_0^1 x^3 J_0(x)\,dx$, (c) $\int x^2 J_0(x)\,dx$.

6.41. Evaluate (a) $\int J_1(\sqrt[3]{x})\,dx$, (b) $\int \frac{J_2(x)}{x^2}\,dx$.

6.42. Evaluate $\int J_0(x)\sin x\,dx$.

6.43. Verify directly the result $J_n'(x)J_{-n}(x) - J_{-n}'(x)J_n(x) = \frac{2\sin n\pi}{\pi x}$ for (a) $n = \frac{1}{2}$ and (b) $n = \frac{3}{2}$.

GENERATING FUNCTION AND MISCELLANEOUS RESULTS

6.44. Use the generating function to prove that $J_n'(x) = \frac{1}{2}[J_{n-1}(x) + J_{n+1}(x)]$ for the case where n is an integer.

6.45. Use the generating function to work Problem 6.39 for the case where n is an integer.

6.46. Show that (a) $1 = J_0(x) + 2J_2(x) + 2J_4(x) + \cdots$

(b) $J_1(x) - J_3(x) + J_5(x) - J_7(x) + \cdots = \frac{1}{2}\sin x$

6.47. Show that $\frac{x}{4}J_1(x) = J_2(x) - 2J_4(x) + 3J_6(x) - \cdots$.

6.48. Show that $J_0(x) = \frac{2}{\pi}\int_0^{\pi/2} \cos(x \sin\theta)\,d\theta$.

6.49. Show that (a) $\int_0^{\pi/2} J_1(x\cos\theta)\,d\theta = \frac{1 - \cos x}{x}$

(b) $\int_0^{\pi/2} J_0(x\sin\theta)\cos\theta\sin\theta\,d\theta = \frac{J_1(x)}{x}$.

6.50. Show that $\int_0^x J_0(t)\,dt = 2\sum_{k=0}^{\infty} J_{2k+1}(x)$.

6.51. Show that (a) $\int_0^\infty e^{-ax} J_0(bx)\,dx = \dfrac{1}{\sqrt{a^2+b^2}}$

(b) $\int_0^\infty e^{-ax} J_n(bx)\,dx = \dfrac{(\sqrt{a^2+b^2}-a)^n}{\sqrt{a^2+b^2}}$, $n > -1$

6.52. Show that $\int_0^\infty J_0(x)\,dx = 1$.

6.53. Prove that $|J_n(x)| \leqq 1$ for all integers n. Is the result true if n is not an integer?

BESSEL FUNCTIONS OF THE SECOND KIND

6.54. Show that (a) $Y_{n+1}(x) = \dfrac{2n}{x} Y_n(x) - Y_{n-1}(x)$, (b) $Y_n'(x) = \dfrac{1}{2}[Y_{n-1}(x) - Y_{n+1}(x)]$.

6.55. Explain why the recurrence formulas for $J_n(x)$ on page 99 hold if $J_n(x)$ is replaced by $Y_n(x)$.

6.56. Prove that $Y_0'(x) = -Y_1(x)$.

6.57. Evaluate (a) $Y_{1/2}(x)$, (b) $Y_{-1/2}(x)$, (c) $Y_{3/2}(x)$, (d) $Y_{-3/2}(x)$.

6.58. Prove that $J_n(x)\,Y_n'(x) - J_n'(x)\,Y_n(x) = \dfrac{2}{\pi x}$.

6.59. Evaluate (a) $\int x^3 Y_2(x)\,dx$, (b) $\int Y_3(x)\,dx$, (c) $\int \dfrac{Y_3(x)}{x^3}\,dx$.

6.60. Prove the result (11), page 98.

FUNCTIONS RELATED TO BESSEL FUNCTIONS

6.61. Show that $I_0(x) = 1 + \dfrac{x^2}{2^2} + \dfrac{x^4}{2^2 4^2} + \dfrac{x^6}{2^2 4^2 6^2} + \cdots$.

6.62. Show that (a) $I_n'(x) = \frac{1}{2}\{I_{n-1}(x) + I_{n+1}(x)\}$, (b) $xI_n'(x) = xI_{n-1}(x) - nI_n(x)$.

6.63. Show that $e^{\frac{x}{2}(t+\frac{1}{t})} = \sum_{-\infty}^{\infty} I_n(x)t^n$ is the generating function for $I_n(x)$.

6.64. Show that $I_0(x) = \dfrac{2}{\pi}\int_0^{\pi/2} \cosh(x \sin\theta)\,d\theta$.

6.65. Show that (a) $\sinh x = 2[I_1(x) + I_3(x) + \cdots]$

(b) $\cosh x = I_0(x) + 2[I_2(x) + I_4(x) + \cdots]$.

6.66. Show that (a) $I_{3/2}(x) = \sqrt{\dfrac{2}{\pi x}}\left(\cosh x - \dfrac{\sinh x}{x}\right)$, (b) $I_{-3/2}(x) = \sqrt{\dfrac{2}{\pi x}}\left(\sinh x - \dfrac{\cosh x}{x}\right)$

6.67. (a) Show that $K_{n+1}(x) = K_{n-1}(x) + \dfrac{2n}{x}K_n(x)$. (b) Explain why the functions $K_n(x)$ satisfy the same recurrence formulas as $I_n(x)$.

6.68. Give asymptotic formulas for (a) $H_n^{(1)}(x)$, (b) $H_n^{(2)}(x)$.

6.69. Show that (a) $\operatorname{Ber}_n(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+n}}{k!\,\Gamma(n+k+1)} \cos\left(\frac{3n+2k}{4}\right)\pi.$

(b) $\operatorname{Bei}_n(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+n}}{k!\,\Gamma(n+k+1)} \sin\left(\frac{3n+2k}{4}\right)\pi.$

6.70. Show that

$$\operatorname{Ker}(x) = -\{\ln(x/2)+\gamma\}\operatorname{Ber}(x) + \frac{\pi}{4}\operatorname{Bei}(x) + 1 - \frac{(x/2)^4}{2!^2}(1+\tfrac{1}{2}) + \frac{(x/2)^8}{4!^2}(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}) - \cdots$$

EQUATIONS TRANSFORMABLE INTO BESSEL'S EQUATION

6.71. Prove that (27), page 101, is a solution of (26).

6.72. Solve $4xy'' + 4y' + y = 0$.

6.73. Solve (a) $xy'' + 2y' + xy = 0$, (b) $y'' + x^2y = 0$.

6.74. Solve $y'' + e^{2x}y = 0$. [*Hint.* Let $e^x = u$].

6.75. (a) Show by direct substitution that $y = J_0(2\sqrt{x})$ is a solution of $xy'' + y' + y = 0$ and (b) write the general solution.

6.76. (a) Show by direct substitution that $y = \sqrt{x}\,J_{1/3}(\frac{2}{3}x^{3/2})$ is a solution of $y'' + xy = 0$ and (b) write the general solution.

6.77. (a) Show that Bessel's equation $x^2y'' + xy' + (x^2 - n^2)y = 0$ can be transformed into

$$\frac{d^2u}{dx^2} + \left(1 - \frac{n^2 - 1/4}{x^2}\right)u = 0$$

where $y = u/\sqrt{x}$. (b) Discuss the case where $n = \pm 1/2$.

(b) Discuss the case where x is large and explain the connection with the asymptotic formulas on page 101.

6.78. Solve $x^2y'' - xy' + x^2y = 0$.

6.79. Show that the equation (26) on page 101 has the solution (28) if $a = 0$. [*Hint.* Let $y = x^p$ and choose p appropriately, or make the transformation $x = e^t$.]

ORTHOGONAL SERIES OF BESSEL FUNCTIONS

6.80. Is the result of Problem 6.27, page 113, valid for $-1 \leq x \leq 1$? Justify your answer.

6.81. Show that $\int xJ_n^2(\lambda x)\,dx = \frac{x^2}{2}[J_n^2(\lambda x) + J_{n+1}^2(\lambda x)] - \frac{nx}{\lambda}J_n(\lambda x)\,J_{n+1}(\lambda x) + c$

6.82. Prove the results (34) and (35), page 102.

6.83. Show that $$\frac{1-x^2}{8} = \sum_{p=1}^{\infty} \frac{J_0(\lambda_p x)}{\lambda_p^3 J_1(\lambda_p)} \qquad -1 < x < 1$$

where λ_p are the positive roots of $J_0(\lambda) = 0$.

6.84. Show that $$x = 2\sum_{p=1}^{\infty} \frac{J_1(\lambda_p x)}{\lambda J_2(\lambda_p)} \qquad -1 < x < 1$$

where λ_p are the positive roots of $J_1(\lambda) = 0$.

6.85. Show that
$$x^3 = \sum_{p=1}^{\infty} \frac{2(8-\lambda_p^2)\,J_1(\lambda_p x)}{\lambda_p^3 J_1'(\lambda_p)} \qquad -1 < x < 1$$
where λ_p are the positive roots of $J_1'(\lambda) = 0$.

6.86. Show that
$$x^2 = \sum_{p=1}^{\infty} \frac{2(\lambda_p^2-4)\,J_0(\lambda_p x)}{\lambda_p^3 J_1(\lambda_p)} \qquad -1 < x < 1$$
where λ_p are the positive roots of $J_0(\lambda) = 0$.

6.87. Show that
$$\frac{J_0(ax)}{2J_0(a)} = \sum_{p=1}^{\infty} \frac{\lambda_p J_0(\lambda_p x)}{(\lambda_p^2 - a^2)\,J_1(\lambda_p)} \qquad -1 < x < 1$$
where λ_p are the positive roots of $J_0(\lambda) = 0$.

6.88. If $f(x) = \sum_{p=1}^{\infty} A_p J_0(\lambda_p x)$ where $J_0(\lambda_p) = 0$, $p = 1, 2, 3, \ldots$, show that
$$\int_0^1 x[f(x)]^2\,dx = \frac{1}{2}\sum_{p=1}^{\infty} A_p^2 J_1^2(\lambda_p)$$
Compare with Parseval's identity for Fourier series.

6.89. Use Problems 6.84 and 6.88 to show that
$$\sum_{p=1}^{\infty} \frac{1}{\lambda_p^2} = \frac{1}{4}$$
where λ_p are the positive roots of $J_0(\lambda) = 0$.

6.90. Derive the results (a) (35) on page 102, (b) (36) on page 102, and (c) (37) on page 102.

SOLUTIONS USING BESSEL FUNCTIONS

6.91. The temperature of a long solid circular cylinder of unit radius is initially zero. At $t = 0$ the surface is given a constant temperature u_0 which is then maintained. Show that the temperature of the cylinder is given by
$$u(\rho, t) = u_0\left\{1 - 2\sum_{n=1}^{\infty} \frac{J_0(\lambda_n \rho)}{\lambda_n J_1(\lambda_n)}\,e^{-\kappa\lambda_n^2 t}\right\}$$
where λ_n, $n = 1, 2, 3, \ldots$, are the positive roots of $J_0(\lambda) = 0$ and κ is the diffusivity.

6.92. Show that if $F(\rho) = u_0(1-\rho^2)$, then the temperature of the plate of Problem 6.28 is given by
$$u(\rho, t) = 4u_0 \sum_{n=1}^{\infty} \frac{J_0(\lambda_n\rho)\,J_2(\lambda_n)}{\lambda_n^2 J_1^2(\lambda_n)}\,e^{-\kappa\lambda_n^2 t}$$

6.93. A cylinder $0 < \rho < a$, $0 < z < l$ has the end $z = 0$ at temperature $f(\rho)$ while the other surfaces are kept at temperature zero. Show that the steady-state temperature at any point is given by
$$u(\rho, z) = \frac{2}{a^2}\sum_{n=1}^{\infty} \frac{J_0(\lambda_n\rho)\sinh\lambda_n(l-z)}{J_1^2(\lambda_n a)\sinh\lambda_n l}\int_0^a \rho f(\rho)\,J_0(\lambda_n\rho)\,d\rho$$
where $J_0(\lambda_n a) = 0$, $n = 1, 2, 3, \ldots$.

6.94. A circular membrane of unit radius lies in the xy-plane with its center at the origin. Its edge $\rho = 1$ is fixed in the xy-plane and it is set into vibration by displacing it an amount $f(\rho)$ and then releasing it. Show that the displacement is given by
$$z(\rho, t) = 2\sum_{n=1}^{\infty} \frac{J_0(\lambda_n\rho)\cos\lambda_n t}{J_1^2(\lambda_n)}\int_0^1 \rho f(\rho)\,J_0(\lambda_n\rho)\,d\rho$$
where λ_n are the roots of $J_0(\lambda) = 0$.

6.95. (a) Solve the boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2}$$

where $0 < \rho < 1$, $0 < \phi < 2\pi$, $t > 0$, u is bounded, and

$$u(1, \phi, t) = 0, \quad u(\rho, \phi, 0) = \rho \cos 3\phi, \quad u_t(\rho, \phi, 0) = 0$$

(b) Give a physical interpretation to the solution.

6.96. Solve and interpret the boundary value problem

$$\frac{\partial}{\partial x}\left(x\frac{\partial y}{\partial x}\right) = \frac{\partial^2 y}{\partial t^2}$$

given that $y(x, 0) = f(x)$, $y_t(x, 0) = 0$, $y(1, t) = 0$ and $y(x, t)$ is bounded for $0 \leqq x \leqq 1$, $t > 0$.

6.97. (a) Work Problem 6.93 if the end $z = 0$ is kept at temperature $f(\rho, \phi)$. (b) Determine the temperature in the special case where $f(\rho, \phi) = \rho^3 \cos \phi$.

6.98. (a) Work Problem 6.93 if there is radiation obeying Newton's law of cooling at the end $z = 0$.

6.99. A chain of constant mass per unit length is suspended vertically from one end O as indicated in Fig. 6-11. If the chain is displaced slightly at time $t = 0$ so that its shape is given by $f(x)$, $0 < x < L$, and then released, show that the displacement of any point x at time t is given by

$$y(x, t) = \sum_{n=1}^{\infty} A_n J_0\left(2\lambda_n \sqrt{\frac{L - x}{g}}\right) \cos \lambda_n t$$

where λ_n are the roots of $J_0(2\lambda\sqrt{L/g}) = 0$ and

$$A_n = \frac{2}{J_1^2(\lambda_n)} \int_0^1 v J_0(\lambda_n v)\, f(L - \tfrac{1}{4}gv^2)\, dv$$

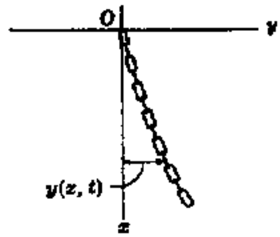

Fig. 6-11

6.100. Determine the frequencies of the normal modes for the vibrating chain of Problem 6.99 and indicate whether you would expect music or noise from the vibrations.

6.101. A solid circular cylinder $0 < \rho < a$, $0 < z < L$ has its bases kept at temperature zero and the convex surface at constant temperature u_0. Show that the steady-state temperature at any point of the cylinder is

$$u(\rho, z) = \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{I_0[(2n-1)\pi\rho/L] \sin[(2n-1)\pi z/L]}{(2n-1) I_0[(2n-1)\pi a/L]}$$

where I_0 is the modified Bessel function of order zero.

6.102. Suppose that the chain in Problem 6.99, which is initially at rest, is given an initial velocity distribution defined by $h(x)$, $0 < x < L$. Show that the displacement of any point x of the string at any time t is given by

$$y(x, t) = \sum_{n=1}^{\infty} B_n J_0\left(2\lambda_n \sqrt{\frac{L - x}{g}}\right) \sin \lambda_n t$$

where λ_n are the roots of $J_0(2\lambda\sqrt{L/g}) = 0$ and

$$B_n = \frac{2}{\lambda_n J_1^2(\lambda_n)} \int_0^1 v J_0(\lambda_n v)\, h(L - \tfrac{1}{4}gv^2)\, dv$$

6.103. Work Problem 6.99 if the chain is given both an initial shape [illegible] initial velocity distribution $h(x)$.

6.104. The surface $\rho = 1$ of an infinite cylinder is kept at temperature $f(z)$. Show that the steady-state temperature everywhere in the cylinder is given by

$$u(\rho, z) = \frac{1}{\pi}\int_{\lambda=0}^{\infty}\int_{v=-\infty}^{\infty} \frac{f(v)\cos\lambda(v-z)I_0(\lambda\rho)}{I_0(\lambda)}\,d\lambda\,dv$$

6.105. A string stretched between $x = 0$ and $x = L$ has a variable density given by $\sigma = \sigma_0 + \epsilon x$ where σ_0 and ϵ are constants. The string is given an initial shape $f(x)$ and then released.

(a) Show that if the tension τ is constant the boundary value problem is given by

$$\tau\frac{\partial^2 y}{\partial x^2} = (\sigma_0 + \epsilon x)\frac{\partial^2 y}{\partial t^2} \qquad 0 < x < L,\ t > 0$$

$$y(0,t) = 0, \quad y(L,t) = 0, \quad y(x,0) = f(x), \quad y_t(x,0) = 0, \quad |y(x,t)| < M$$

(b) Show that the frequencies of the normal modes of vibration are given by $f_n = \omega_n/2\pi$ where the ω_n $(n = 1, 2, 3, \ldots)$ are the positive roots of the equation

$$J_{1/3}(\alpha\omega)J_{-1/3}(\beta\omega) = J_{1/3}(\beta\omega)J_{-1/3}(\alpha\omega)$$

in which $$\alpha = \frac{2\sigma_0}{3\epsilon}\sqrt{\frac{\sigma_0}{\tau}}, \qquad \beta = \frac{2(\sigma_0 + \epsilon L)}{3\epsilon}\sqrt{\frac{\sigma_0 + \epsilon L}{\tau}}$$

MISCELLANEOUS PROBLEMS

6.106. A particle moves along the positive x-axis with a force of repulsion per unit mass equal to a constant α^2 times the instantaneous distance from the origin. If the mass m increases with time according to $m = m_0 + \epsilon t$, where m_0 and ϵ are constants, and if initially the particle is located at the origin and traveling with speed v_0, show that the position x at any time $t > 0$ is given by

$$x = \frac{m_0 v_0}{\epsilon}\left\{K_0\!\left(\frac{\alpha m_0}{\epsilon}\right)I_0\!\left(\frac{\alpha m_0}{\epsilon} + \alpha t\right) - I_0\!\left(\frac{\alpha m_0}{\epsilon}\right)K_0\!\left(\frac{\alpha m_0}{\epsilon} + \alpha t\right)\right\}$$

6.107. Show that if $m \neq n$

$$\int \frac{J_m(\lambda x)J_n(\lambda x)}{x}\,dx = \frac{\lambda x}{m^2 - n^2}\left(J_m'(\lambda x)J_n(\lambda x) - J_m(\lambda x)J_n'(\lambda x)\right) + c$$

6.108. Deduce the integral $\int \frac{J_m^2(\lambda x)}{x}\,dx$ by using a limiting procedure in the result of Problem 6.107.

6.109. Show that $$\int_0^{\infty}\frac{J_n(x)}{x^{n-1}}\,dx = \frac{1}{2^{n-1}\Gamma(n)} \qquad n > 0$$

6.110. Explain how the Sturm-Liouville theory of Chapter 3 can be used to arrive at various results involving Bessel functions obtained in this chapter.

6.111. A cylinder of unit height and radius (see Fig. 6-3, page 115) has its top surface kept at temperature u_0 and the other surfaces at temperature zero. Show that the steady-state temperature at any point is given by

$$u(\rho, z) = 2u_0\sum_{n=1}^{\infty}\frac{(\sinh\lambda_n z)J_0(\lambda_n\rho)}{(\lambda_n\sinh\lambda_n)J_1(\lambda_n)}$$

where λ_n are the positive roots of $J_0(\lambda) = 0$.

6.112. Work Problem 6.29 if the base $z = 1$ is insulated.

6.113. Work Problem 6.29 if the convex surface is insulated.

6.114. Work Problem 6.29 if the bases $z = 0$ and $z = 1$ are kept at constant temperatures u_1 and u_2 respectively. [*Hint.* Let $u(\rho, z, t) = v(\rho, z, t) + w(\rho, z)$ and choose $w(\rho, z)$ appropriately, noting that physically it represents the steady-state solution.]

6.115. Show how Problem 6.29 can be solved if the radius of the cylinder is a while the height is h.

6.116. Work Problem 6.29 if the initial temperature is $f(\rho, \phi, z)$.

6.117. A membrane has the form of the region bounded by two concentric circles of radii a and b as shown in Fig. 6-12.

(a) Show that the frequencies of the various modes of vibration are given by

$$f_{mn} = \frac{\lambda_{mn}}{2\pi}\sqrt{\frac{\tau}{\mu}}$$

where τ is the tension per unit length, μ is the mass per unit area, and λ_{mn} are roots of the equation

$$J_m(\lambda_m a)\, Y_m(\lambda b) - J_m(\lambda b)\, Y_m(\lambda a) = 0$$

(b) Find the displacement at any time of any point of the membrane if the membrane is given an initial shape and then released.

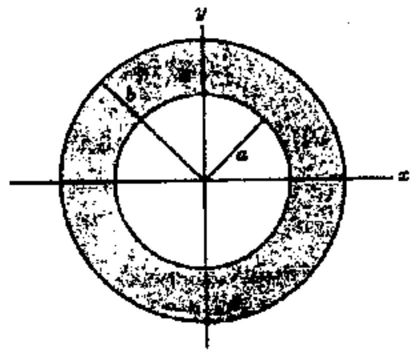

Fig. 6-12

6.118. A metal conducting pipe of diffusivity κ has inner radius a, outer radius b and height h. A coordinate system is chosen so that one of the bases lies in the xy-plane and the axis of the pipe is chosen to be the z-axis. If the initial temperature of the pipe is $f(\rho, z)$, $a < \rho < b$, $0 < z < h$, while the surface is kept at temperature zero, find the temperature at any point at any time.

6.119. Work Problem 6.118 if the initial temperature is $f(\rho, \phi, z)$.

6.120. Work Problem 6.118 if (a) the bases are insulated, (b) the convex surfaces are insulated, (c) the entire surface is insulated.

Chapter 7

Legendre Functions and Applications

LEGENDRE'S DIFFERENTIAL EQUATION

Legendre functions arise as solutions of the differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \tag{1}$$

which is called *Legendre's differential equation*. The general solution of (*1*) in the case where $n = 0, 1, 2, 3, \ldots$ is given by

$$y = c_1 P_n(x) + c_2 Q_n(x) \tag{2}$$

where $P_n(x)$ are polynomials called *Legendre polynomials* and $Q_n(x)$ are called *Legendre functions of the second kind*. The $Q_n(x)$ are unbounded at $x = \pm 1$.

The differential equation (*1*) is obtained, for example, from Laplace's equation $\nabla^2 u = 0$ expressed in spherical coordinates (r, θ, ϕ), when it is assumed that u is independent of ϕ. See Problem 7.1.

LEGENDRE POLYNOMIALS

The Legendre polynomials are defined by

$$P_n(x) = \frac{(2n-1)(2n-3)\cdots 1}{n!}\left\{x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}x^{n-4} - \cdots\right\} \tag{3}$$

Note that $P_n(x)$ is a polynomial of degree n. The first few Legendre polynomials are as follows:

$$P_0(x) = 1 \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_1(x) = x \qquad P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \qquad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

In all cases $P_n(1) = 1$, $P_n(-1) = (-1)^n$.

The Legendre polynomials can also be expressed by *Rodrigue's formula*:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \tag{4}$$

GENERATING FUNCTION FOR LEGENDRE POLYNOMIALS

The function

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n \tag{5}$$

is called the *generating function* for Legendre polynomials and is useful in obtaining their properties.

RECURRENCE FORMULAS

1. $P_{n+1}(x) = \dfrac{2n+1}{n+1}xP_n(x) - \dfrac{n}{n+1}P_{n-1}(x)$

2. $P'_{n+1}(x) - P'_{n-1}(x) = (2n+2)P_n(x)$

LEGENDRE FUNCTIONS OF THE SECOND KIND

If $|x| < 1$, the Legendre functions of the second kind are given by the following, according as n is even or odd respectively:

$$Q_n(x) = \frac{(-1)^{n/2}2^n[(n/2)!]^2}{n!}\left\{x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \cdots\right\} \tag{6}$$

$$Q_n(x) = \frac{(-1)^{(n+1)/2}2^{n-1}\{[(n-1)/2]!\}^2}{1\cdot3\cdot5\cdots n}\left\{1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \cdots\right\} \tag{7}$$

For $n > 1$, the leading coefficients are taken so that the recurrence formulas for $P_n(x)$ above apply also $Q_n(x)$.

ORTHOGONALITY OF LEGENDRE POLYNOMIALS

The following results are fundamental:

$$\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0 \qquad \text{if } m \neq n \tag{8}$$

$$\int_{-1}^{1} [P_n(x)]^2\,dx = \frac{2}{2n+1} \tag{9}$$

The first shows that any two different Legendre polynomials are orthogonal in the interval $-1 < x < 1$.

SERIES OF LEGENDRE POLYNOMIALS

If $f(x)$ and $f'(x)$ are piecewise continuous then at every point of continuity of $f(x)$ in the interval $-1 < x < 1$ there will exist a Legendre series expansion having the form

$$f(x) = A_0P_0(x) + A_1P_1(x) + A_2P_2(x) + \cdots = \sum_{k=0}^{\infty} A_kP_k(x) \tag{10}$$

where
$$A_k = \frac{2k+1}{2}\int_{-1}^{1} f(x)\,P_k(x)\,dx \tag{11}$$

At any point of discontinuity the series on the right in (10) converges to $\frac{1}{2}[f(x+0)+f(x-0)]$, which can be used to replace the left side of (10).

ASSOCIATED LEGENDRE FUNCTIONS

The differential equation

$$(1-x^2)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0 \tag{12}$$

is called *Legendre's associated differential equation*. If $m=0$ this reduces to Legendre's equation (1). Solutions to (12) are called *associated Legendre functions*. We consider the case where m and n are non-negative integers. In this case the general solution of (12) is given by

$$y = c_1 P_n^m(x) + c_2 Q_n^m(x) \tag{13}$$

where $P_n^m(x)$ and $Q_n^m(x)$ are called *associated Legendre functions of the first and second kinds* respectively. They are given in terms of the ordinary Legendre functions by

$$P_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x) \tag{14}$$

$$Q_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}Q_n(x) \tag{15}$$

Note that if $m>n$, $P_n^m(x)=0$. The functions $Q_n^m(x)$ are unbounded for $x=\pm 1$.

The differential equation (12) is obtained from Laplace's equation $\nabla^2 u = 0$ expressed in spherical coordinates (r,θ,ϕ). See Problem 7.21.

ORTHOGONALITY OF ASSOCIATED LEGENDRE FUNCTIONS

As in the case of Legendre polynomials, the Legendre functions $P_n^m(x)$ are orthogonal in $-1<x<1$, i.e.

$$\int_{-1}^{1} P_n^m(x)\,P_k^m(x)\,dx = 0 \qquad n \neq k \tag{16}$$

We also have

$$\int_{-1}^{1} [P_n^m(x)]^2\,dx = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!} \tag{17}$$

Using these, we can expand a function $f(x)$ in a series of the form

$$f(x) = \sum_{k=0}^{\infty} A_k P_k^m(x) \tag{18}$$

SOLUTIONS TO BOUNDARY VALUE PROBLEMS USING LEGENDRE FUNCTIONS

Various boundary value problems can be solved by use of Legendre functions. See Problems 7.18–7.20 and 7.28–7.30.

Solved Problems

LEGENDRE'S DIFFERENTIAL EQUATION

7.1. By letting $u = R\Theta$, where R depends only on r and Θ depends only on θ, in Laplace's equation $\nabla^2 u = 0$ expressed in spherical coordinates, show that R and Θ satisfy the equations

$$r^2 \frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + \lambda^2 R = 0 \qquad \frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) - \lambda^2(\sin\theta)\Theta = 0$$

Laplace's equation in spherical coordinates is given by

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0 \tag{1}$$

See (4), page 5. If u is independent of ϕ, then the equation can be written

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial u}{\partial\theta}\right) = 0 \tag{2}$$

Letting $u = R\Theta$ in this equation, where it is supposed that R depends only on r while Θ depends only on θ, we have

$$\frac{\Theta}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) = 0$$

Multiplying by r^2, dividing by $R\Theta$ and rearranging, we find

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right)$$

Since one side depends only on r while the other depends only on θ, it follows that each side must be a constant, say $-\lambda^2$. Then we have

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\lambda^2 \tag{3}$$

and

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) = \lambda^2 \tag{4}$$

which can be rewritten respectively as

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + \lambda^2 R = 0 \tag{5}$$

and

$$\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) - \lambda^2(\sin\theta)\Theta = 0 \tag{6}$$

as required.

7.2. Show that the solution for the R-equation in Problem 7.1 can be written as

$$R = Ar^n + \frac{B}{r^{n+1}}$$

where $\lambda^2 = -n(n+1)$.

The R-equation of Problem 7.1 is

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + \lambda^2 R = 0$$

This is an Euler or Cauchy equation and can be solved by letting $R = r^p$ and determining p. Alternatively, comparison with (26) and (28), page 101, for the case where $x = r$, $y = R$, $k = \frac{1}{2}$, $\alpha = 0$, $\beta = \lambda$ shows that the general solution is

$$R = r^{-1/2}[Ar^{\sqrt{1/4-\lambda^2}} + Br^{-\sqrt{1/4-\lambda^2}}]$$

or

$$R = Ar^{-1/2+\sqrt{1/4-\lambda^2}} + Br^{-1/2-\sqrt{1/4-\lambda^2}} \tag{1}$$

This solution can be simplified if we write

$$-\frac{1}{2} + \sqrt{\frac{1}{4} - \lambda^2} = n \tag{2}$$

so that

$$-\frac{1}{2} - \sqrt{\frac{1}{4} - \lambda^2} = -n - 1 \tag{3}$$

In such case (1) becomes

$$R = Ar^n + \frac{B}{r^{n+1}} \tag{4}$$

Multiplying equations (2) and (3) together leads to

$$\lambda^2 = -n(n+1) \tag{5}$$

7.3. Show that the Θ-equation (6) of Problem 7.1 becomes Legendre's differential equation (1), page 130, on making the transformation $\xi = \cos\theta$.

Using the value $\lambda^2 = -n(n+1)$ from (5) of Problem 7.2 in the Θ-equation (6) of Problem 7.1, it becomes

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + n(n+1)(\sin\theta)\Theta = 0 \tag{1}$$

We now let $\xi = \cos\theta$ in this equation. Then

$$\frac{d\Theta}{d\theta} = \frac{d\Theta}{d\xi}\frac{d\xi}{d\theta} = -\sin\theta\frac{d\Theta}{d\xi}$$

Thus

$$\sin\theta\frac{d\Theta}{d\theta} = -\sin^2\theta\frac{d\Theta}{d\xi} = (\xi^2-1)\frac{d\Theta}{d\xi}$$

since $\sin^2\theta = 1 - \cos^2\theta = 1 - \xi^2$. It follows that

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = \frac{d}{d\theta}\left[(\xi^2-1)\frac{d\Theta}{d\xi}\right]$$

$$= \frac{d}{d\xi}\left[(\xi^2-1)\frac{d\Theta}{d\xi}\right]\frac{d\xi}{d\theta} = \frac{d}{d\xi}\left[(1-\xi^2)\frac{d\Theta}{d\xi}\right]\sin\theta \tag{2}$$

Using this in (1) and canceling the factor $\sin\theta$, we obtain

$$\frac{d}{d\xi}\left[(1-\xi^2)\frac{d\Theta}{d\xi}\right] + n(n+1)\Theta = 0 \tag{3}$$

Replacing Θ by y and ξ by x, and carrying out the indicated differentiation, yields the required Legendre equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \tag{4}$$

7.4. Use the method of Frobenius to find series solutions of Legendre's differential equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$.

Assuming a solution of the form $y = \sum c_k x^{k+\beta}$ where the summation index k goes from $-\infty$ to ∞ and $c_k = 0$ for $k < 0$, we have

$$n(n+1)y = \sum n(n+1)c_k x^{k+\beta}$$

$$-2xy' = \sum -2(k+\beta)c_k x^{k+\beta}$$

$$(1-x^2)y'' = \sum (k+\beta)(k+\beta-1)c_k x^{k+\beta-2} - \sum (k+\beta)(k+\beta-1)c_k x^{k+\beta}$$

$$= \sum (k+\beta+2)(k+\beta+1)c_{k+2}x^{k+\beta} - \sum (k+\beta)(k+\beta-1)c_k x^{k+\beta}$$

Then by addition,

$$\sum [(k+\beta+2)(k+\beta+1)c_{k+2} - (k+\beta)(k+\beta-1)c_k - 2(k+\beta)c_k + n(n+1)c_k]x^{k+\beta} = 0$$

and since the coefficient of $x^{k+\beta}$ must be zero, we find

$$(k+\beta+2)(k+\beta+1)c_{k+2} + [n(n+1) - (k+\beta)(k+\beta+1)]c_k = 0 \tag{1}$$

Letting $k = -2$ we obtain, since $c_{-2} = 0$, the indicial equation $\beta(\beta-1)c_0 = 0$ or, assuming $c_0 \neq 0$, $\beta = 0$ or 1.

Case 1: $\beta = 0$.

In this case (*1*) becomes

$$(k+2)(k+1)c_{k+2} + [n(n+1) - k(k+1)]c_k = 0 \tag{2}$$

Putting $k = -1, 0, 1, 2, 3, \ldots$ in succession, we find that c_1 is arbitrary while

$$c_2 = -\frac{n(n+1)}{2!}c_0, \quad c_3 = \frac{1\cdot 2 - n(n+1)}{3!}c_1, \quad c_4 = \frac{[2\cdot 3 - n(n+1)]}{4!}c_2, \quad \ldots$$

and so we obtain

$$y = c_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \cdots\right]$$
$$+ c_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \cdots\right] \tag{3}$$

Since we have a solution with two arbitrary constants, we need not consider Case 2: $\beta = 1$.

For an even integer $n \geqq 0$, the first of the above series terminates and gives a polynomial solution. For an odd integer $n > 0$, the second series terminates and gives a polynomial solution. Thus for any integer $n \geqq 0$ the equation has polynomial solutions. If $n = 0, 1, 2, 3$, for example, we obtain from (*3*) the polynomials

$$c_0, \quad c_1 x, \quad c_0(1-3x^2), \quad c_1\left(\frac{3x-5x^3}{2}\right)$$

which are, apart from a multiplicative constant, the Legendre polynomials $P_n(x)$. This multiplicative constant is chosen so that $P_n(1) = 1$.

The series solution in (*3*) which does not terminate can be shown to diverge for $x = \pm 1$. This second solution, which is unbounded for $x = \pm 1$ or equivalently for $\theta = 0, \pi$, is called a *Legendre function of the second kind* and is denoted by $Q_n(x)$. It follows that the general solution of Legendre's differential equation can be written as

$$y = c_1 P_n(x) + c_2 Q_n(x)$$

In case n is not an integer both series solutions are unbounded for $x = \pm 1$.

7.5. Show that a solution of Laplace's equation $\nabla^2 u = 0$ which is independent of ϕ is given by

$$u = \left(A_1 r^n + \frac{B_1}{r^{n+1}}\right)[A_2 P_n(\xi) + B_2 Q_n(\xi)]$$

where $\xi = \cos\theta$.

This result follows at once from Problems 7.1 through 7.4 since $u = R\Theta$ where

$$R = A_1 r^n + \frac{B_1}{r^{n+1}}$$

and the general solution of the Θ-equation (Legendre's equation) is written in terms of two linearly independent solutions $P_n(\xi)$ and $Q_n(\xi)$ as

$$\Theta = A_2 P_n(\xi) + B_2 Q_n(\xi)$$

The functions $P_n(\xi)$ and $Q_n(\xi)$ are the Legendre functions of the first and second kinds respectively.

LEGENDRE POLYNOMIALS

7.6. Derive formula (3), page 130, for the Legendre polynomials.

From (2) of Problem 7.4 we see that if $k = n$ then $c_{n+2} = 0$ and thus $c_{n+4} = 0$, $c_{n+6} = 0, \ldots$. Then letting $k = n-2, n-4, \ldots$ we find from (2) of Problem 7.4,

$$c_{n-2} = -\frac{n(n-1)}{2(2n-1)}c_n, \qquad c_{n-4} = -\frac{(n-2)(n-3)}{4(2n-3)}c_{n-2} = \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}c_n, \qquad \ldots$$

This leads to the polynomial solutions

$$y = c_n\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}x^{n-4} - \cdots\right]$$

The Legendre polynomials $P_n(x)$ are defined by choosing

$$c_n = \frac{(2n-1)(2n-3)\cdots 3\cdot 1}{n!}$$

This choice is made in order that $P_n(1) = 1$.

7.7. Derive Rodrigue's formula $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}(x^2-1)^n$.

By Problem 7.6 the Legendre polynomials are given by

$$P_n(x) = \frac{(2n-1)(2n-3)\cdots 3\cdot 1}{n!}\left\{x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}x^{n-4} - \cdots\right\}$$

Now integrating this n times from 0 to x, we obtain

$$\frac{(2n-1)(2n-3)\cdots 3\cdot 1}{(2n)!}\left\{x^{2n} - nx^{2n-2} + \frac{n(n-1)}{2!}x^{2n-4} - \cdots\right\}$$

which can be written

$$\frac{(2n-1)(2n-3)\cdots 3\cdot 1}{(2n)(2n-1)(2n-2)\cdots 2\cdot 1}(x^2-1)^n \qquad \text{or} \qquad \frac{1}{2^n n!}(x^2-1)^n$$

which proves that

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

GENERATING FUNCTION

7.8. Prove that $\dfrac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$.

Using the binomial theorem

$$(1+v)^p = 1 + pv + \frac{p(p-1)}{2!}v^2 + \frac{p(p-1)(p-2)}{3!}v^3 + \cdots$$

we have

$$\frac{1}{\sqrt{1-2xt+t^2}} = [1-t(2x-t)]^{-1/2}$$
$$= 1 + \frac{1}{2}t(2x-t) + \frac{1\cdot 3}{2\cdot 4}t^2(2x-t)^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}t^3(2x-t)^3 + \cdots$$

and the coefficient of t^n in this expansion is

$$\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}(2x)^n - \frac{1\cdot 3\cdot 5\cdots(2n-3)}{2\cdot 4\cdot 6\cdots(2n-2)}\cdot\frac{(n-1)}{1!}(2x)^{n-2}$$
$$+ \frac{1\cdot 3\cdot 5\cdots 2n-5}{2\cdot 4\cdot 6\cdots 2n-4}\cdot\frac{(n-2)(n-3)}{2!}(2x)^{n-4} - \cdots$$

which can be written as

$$\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}\left\{x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}x^{n-4} - \cdots\right\}$$

i.e. $P_n(x)$. The required result thus follows.

RECURRENCE FORMULAS FOR LEGENDRE POLYNOMIALS

7.9. Prove that $P_{n+1}(x) = \frac{2n+1}{n+1}xP_n(x) - \frac{n}{n+1}P_{n-1}(x)$.

From the generating function of Problem 7.8 we have

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n \qquad (1)$$

Differentiating with respect to t,

$$\frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} nP_n(x)t^{n-1}$$

Multiplying by $1-2xt+t^2$,

$$\frac{x-t}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty}(1-2xt+t^2)nP_n(x)t^{n-1} \qquad (2)$$

Now the left side of (2) can be written in terms of (1) and we have

$$\sum_{n=0}^{\infty}(x-t)P_n(x)t^n = \sum_{n=0}^{\infty}(1-2xt+t^2)nP_n(x)t^{n-1}$$

i.e.

$$\sum_{n=0}^{\infty} xP_n(x)t^n - \sum_{n=0}^{\infty} P_n(x)t^{n+1} = \sum_{n=0}^{\infty} nP_n(x)t^{n-1} - \sum_{n=0}^{\infty} 2nxP_n(x)t^n + \sum_{n=0}^{\infty} nP_n(x)t^{n+1}$$

Equating the coefficients of t^n on each side, we find

$$xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2nxP_n(x) + (n-1)P_{n-1}(x)$$

which yields the required result.

7.10. Given that $P_0(x) = 1$, $P_1(x) = x$, find (a) $P_2(x)$ and (b) $P_3(x)$.

Using the recurrence formula of Problem 7.9, we have on letting $n=1$,

$$P_2(x) = \frac{3}{2}xP_1(x) - \frac{1}{2}P_0(x) = \frac{3}{2}x^2 - \frac{1}{2} = \frac{1}{2}(3x^2-1)$$

Similarly letting $n=2$,

$$P_3(x) = \frac{5}{3}xP_2(x) - \frac{2}{3}P_1(x) = \frac{5}{3}x\left(\frac{3x^2-1}{2}\right) - \frac{2}{3}x = \frac{1}{2}(5x^3-3x)$$

LEGENDRE FUNCTIONS OF THE SECOND KIND

7.11. Obtain the results (6) and (7), page 131, for the Legendre functions of the second kind in the case where n is a non-negative integer.

The Legendre functions of the second kind are the series solutions of Legendre's equation which do not terminate. From (3) of Problem 7.4 we see that if n is even the series which does not terminate is

$$x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \cdots$$

while if n is odd the series which does not terminate is

$$1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \cdots$$

These series solutions, apart from multiplicative constants, provide definitions for Legendre functions of the second kind and are given by (6) and (7) on page 131. The multiplicative constants are chosen so that the Legendre functions of the second kind will satisfy the same recurrence formulas (page 131) as the Legendre polynomials.

7.12. Obtain the Legendre functions of the second kind (a) $Q_0(x)$, (b) $Q_1(x)$, and (c) $Q_2(x)$.

(a) From (6), page 131, we have if $n = 0$,

$$\begin{aligned} Q_0(x) &= x + \frac{2}{3!}x^3 + \frac{1\cdot 3\cdot 2\cdot 4}{5!}x^5 + \frac{1\cdot 3\cdot 5\cdot 2\cdot 4\cdot 6}{7!}x^7 + \cdots \\ &= x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \end{aligned}$$

where we have used the expansion $\ln(1+u) = u - u^2/2 + u^3/3 - u^4/4 + \cdots$.

(b) From (7), page 131, we have if $n = 1$,

$$\begin{aligned} Q_1(x) &= -\left\{1 - \frac{(1)(2)}{2!}x^2 + \frac{(1)(-1)(2)(4)}{4!}x^4 - \frac{(1)(-1)(-3)(2)(4)(6)}{6!}x^6 + \cdots\right\} \\ &= x\left\{x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right\} - 1 = \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1 \end{aligned}$$

(c) The recurrence formulas for $Q_n(x)$ are identical with those of $P_n(x)$. Then from Problem 7.9,

$$Q_{n+1}(x) = \frac{2n+1}{n+1}xQ_n(x) - \frac{n}{n+1}Q_{n-1}(x)$$

Putting $n = 1$, we have on using parts (a) and (b),

$$Q_2(x) = \frac{3}{2}xQ_1(x) - \frac{1}{2}Q_0(x) = \left(\frac{3x^2-1}{4}\right)\ln\left(\frac{1+x}{1-x}\right) - \frac{3x}{2}$$

ORTHOGONALITY OF LEGENDRE POLYNOMIALS

7.13. Prove that $\displaystyle\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0$ if $m \neq n$.

Since $P_m(x)$, $P_n(x)$ satisfy Legendre's equation,

$$\begin{aligned} (1-x^2)P_m'' - 2xP_m' + m(m+1)P_m &= 0 \\ (1-x^2)P_n'' - 2xP_n' + n(n+1)P_n &= 0 \end{aligned}$$

Then multiplying the first equation by P_n, the second equation by P_m and subtracting, we find

$$(1-x^2)[P_nP_m'' - P_mP_n''] - 2x[P_nP_m' - P_mP_n'] = [n(n+1) - m(m+1)]P_mP_n$$

which can be written

$$(1-x^2)\frac{d}{dx}[P_nP_m' - P_mP_n'] - 2x[P_nP_m' - P_mP_n'] = [n(n+1) - m(m+1)]P_mP_n$$

or

$$\frac{d}{dx}\{(1-x^2)[P_nP_m' - P_mP_n']\} = [n(n+1) - m(m+1)]P_mP_n$$

Thus by integrating we have

$$[n(n+1) - m(m+1)]\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = (1-x^2)[P_nP_m' - P_mP_n']\Big|_{-1}^{1} = 0$$

Then since $m \neq n$,

$$\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0$$

7.14. Prove that $\displaystyle\int_{-1}^{1}[P_n(x)]^2\,dx = \frac{2}{2n+1}$.

From the generating function

$$\frac{1}{\sqrt{1-2tx+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

we have on squaring both sides,

$$\frac{1}{1-2tx+t^2} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P_m(x)\,P_n(x)t^{m+n}$$

Then by integrating from -1 to 1 we have

$$\int_{-1}^{1}\frac{dx}{1-2tx+t^2} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{\int_{-1}^{1} P_m(x)\,P_n(x)\,dx\right\}t^{m+n}$$

Using the result of Problem 7.13 on the right side and performing the integration on the left side,

$$-\frac{1}{2t}\ln(1-2tx+t^2)\Big|_{-1}^{1} = \sum_{n=0}^{\infty}\left\{\int_{-1}^{1}[P_n(x)]^2\,dx\right\}t^{2n}$$

or

$$\frac{1}{t}\ln\left(\frac{1+t}{1-t}\right) = \sum_{n=0}^{\infty}\left\{\int_{-1}^{1}[P_n(x)]^2\,dx\right\}t^{2n}$$

i.e.

$$\sum_{n=0}^{\infty}\frac{2t^{2n}}{2n+1} = \sum_{n=0}^{\infty}\left\{\int_{-1}^{1}[P_n(x)]^2\,dx\right\}t^{2n}$$

Equating coefficients of t^{2n}, it follows that

$$\int_{-1}^{1}[P_n(x)]^2\,dx = \frac{2}{2n+1}$$

SERIES OF LEGENDRE POLYNOMIALS

7.15. If $f(x) = \sum_{k=0}^{\infty} A_kP_k(x)$, $-1 < x < 1$, **show that**

$$A_k = \frac{2k+1}{2}\int_{-1}^{1} P_k(x)\,f(x)\,dx$$

Multiplying the given series by $P_m(x)$ and integrating from -1 to 1, we have on using Problems 7.13 and 7.14,

$$\int_{-1}^{1} P_m(x) f(x)\,dx = \sum_{k=0}^{\infty} A_k \int_{-1}^{1} P_m(x) P_k(x)\,dx$$

$$= A_m \int_{-1}^{1} [P_m(x)]^2\,dx = \frac{2A_m}{2m+1}$$

Then as required,

$$A_m = \frac{2m+1}{2} \int_{-1}^{1} P_m(x) f(x)\,dx$$

7.16. Expand the function $f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & -1 < x < 0 \end{cases}$ in a series of the form $\sum_{k=0}^{\infty} A_k P_k(x)$.

By Problem 7.15

$$A_k = \frac{2k+1}{2} \int_{-1}^{1} P_k(x) f(x)\,dx = \frac{2k+1}{2} \int_{-1}^{0} P_k(x)(0)\,dx + \frac{2k+1}{2} \int_{0}^{1} P_k(x)(1)\,dx$$

$$= \frac{2k+1}{2} \int_{0}^{1} P_k(x)\,dx$$

Then

$$A_0 = \frac{1}{2} \int_0^1 P_0(x)\,dx = \frac{1}{2} \int_0^1 (1)\,dx = \frac{1}{2}$$

$$A_1 = \frac{3}{2} \int_0^1 P_1(x)\,dx = \frac{3}{2} \int_0^1 x\,dx = \frac{3}{4}$$

$$A_2 = \frac{5}{2} \int_0^1 P_2(x)\,dx = \frac{5}{2} \int_0^1 \frac{3x^2-1}{2}\,dx = 0$$

$$A_3 = \frac{7}{2} \int_0^1 P_3(x)\,dx = \frac{7}{2} \int_0^1 \frac{5x^3-3x}{2}\,dx = -\frac{7}{16}$$

$$A_4 = \frac{9}{2} \int_0^1 P_4(x)\,dx = \frac{9}{2} \int_0^1 \frac{35x^4-30x^2+3}{8}\,dx = 0$$

$$A_5 = \frac{11}{2} \int_0^1 P_5(x)\,dx = \frac{11}{2} \int_0^1 \frac{63x^5-70x^3+15x}{8}\,dx = \frac{11}{32}$$

etc. Thus

$$f(x) = \frac{1}{2} P_0(x) + \frac{3}{4} P_1(x) - \frac{7}{16} P_3(x) + \frac{11}{32} P_5(x) - \cdots$$

The general term for the coefficients in this series can be obtained by using the recurrence formula 2 on page 131 and the results of Problem 7.34. We find

$$A_n = \frac{2n+1}{2} \int_0^1 P_n(x)\,dx = \frac{1}{2} \int_0^1 [P'_{n+1}(x) - P'_{n-1}(x)]\,dx = \frac{1}{2} [P_{n-1}(0) - P_{n+1}(0)]$$

For n even $A_n = 0$, while for n odd we can use Problem 7.34(c).

7.17. Expand $f(x) = x^2$ in a series of the form $\sum_{k=0}^{\infty} A_k P_k(x)$.

Method 1.

We must find A_k, $k = 0, 1, 2, 3, \ldots$, such that

$$x^2 = A_0 P_0(x) + A_1 P_1(x) + A_2 P_2(x) + A_3 P_3(x) + \cdots$$

$$= A_0(1) + A_1(x) + A_2\left(\frac{3x^2-1}{2}\right) + A_3\left(\frac{5x^3-3x}{2}\right) + \cdots$$

Since the left side is a polynomial of degree 2 we must have $A_3 = 0,\ A_4 = 0,\ A_5 = 0,\ \ldots$. Thus

$$x^2 = A_0 - \frac{A_2}{2} + A_1 x + \frac{3}{2}A_2 x^2$$

from which
$$A_0 - \frac{A_2}{2} = 0, \quad A_1 = 0, \quad \frac{3}{2}A_2 = 1$$

Then
$$A_0 = \frac{1}{3}, \quad A_1 = 0, \quad A_2 = \frac{2}{3}$$

i.e.
$$x^2 = \frac{1}{3}P_0(x) + \frac{2}{3}P_2(x) .$$

Method 2.

Using the method of Problem 7.15 we see that if

$$x^2 = \sum_{k=0}^{\infty} A_k P_k(x)$$

then
$$A_k = \frac{2k+1}{2}\int_{-1}^{1} x^2 P_k(x)\,dx$$

Putting $k = 0, 1, 2, \ldots,$ we find as before $A_0 = \frac{1}{3},\ A_1 = 0,\ A_2 = \frac{2}{3},\ A_3 = 0,\ A_4 = 0,\ \ldots$ so that

$$x^2 = \frac{1}{3}P_0(x) + \frac{2}{3}P_2(x)$$

In general when we expand a polynomial in a series of Legendre polynomials, the series, which terminates, can most easily be found by using Method 1.

SOLUTIONS USING LEGENDRE FUNCTIONS

7.18. Find the potential v (a) interior to and (b) exterior to a hollow sphere of unit radius if half of its surface is charged to potential v_0 and the other half to potential zero.

Choose the sphere in the position shown in Fig. 7-1. Then v is independent of ϕ and we can use the results of Problem 7.5. A solution is

$$v(r,\theta) = \left(A_1 r^n + \frac{B_1}{r^{n+1}}\right)[A_2 P_n(\xi) + B_2 Q_n(\xi)]$$

where $\xi = \cos\theta$. Since v must be bounded at $\theta = 0$ and π, i.e. $\xi = \pm 1$, we must choose $B_2 = 0$. Then

$$v(r,\theta) = \left(Ar^n + \frac{B}{r^{n+1}}\right)P_n(\xi) \qquad (1)$$

The boundary conditions are

$$v(1,\theta) = \begin{cases} v_0 & \text{if } 0 < \theta < \frac{\pi}{2} \quad \text{i.e.} \quad 0 < \xi < 1 \\ 0 & \text{if } \frac{\pi}{2} < \theta < \pi \quad \text{i.e.} \quad -1 < \xi < 0 \end{cases}$$

and v is bounded.

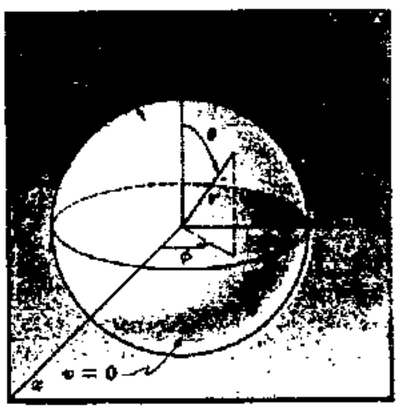

Fig. 7-1

(a) Interior Potential, $0 \leq r < 1$.

Since v is bounded at $r = 0$, choose $B = 0$ in (1). Then a solution is

$$Ar^n P_n(\xi) = Ar^n P_n(\cos\theta)$$

By superposition,
$$v(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\xi)$$

When $r = 1$,
$$v(1,\theta) = \sum_{n=0}^{\infty} A_n P_n(\xi)$$

Then as in Problem 7.15,
$$A_n = \frac{2n+1}{2}\int_{-1}^{1} v(1,\theta)P_n(\xi)\,d\xi = \left(\frac{2n+1}{2}\right)v_0\int_0^1 P_n(\xi)\,d\xi$$

from which
$$A_0 = \frac{1}{2}v_0,\quad A_1 = \frac{3}{4}v_0,\quad A_2 = 0,\quad A_3 = -\frac{7}{16}v_0,\quad A_4 = 0,\quad A_5 = \frac{11}{32}v_0,\quad \ldots$$

Thus
$$v(r,\theta) = \frac{v_0}{2}\left[1 + \frac{3}{2}rP_1(\cos\theta) - \frac{7}{8}r^3P_3(\cos\theta) + \frac{11}{16}r^5P_5(\cos\theta) + \cdots\right] \qquad (2)$$

(b) **Exterior Potential,** $1 < r < \infty$.

Since v is bounded as $r \to \infty$, choose $A = 0$ in (1). Then a solution is
$$\frac{B}{r^{n+1}}P_n(\xi) = \frac{B}{r^{n+1}}P_n(\cos\theta)$$

By superposition,
$$v(r,\theta) = \sum_{n=0}^{\infty}\frac{B_n}{r^{n+1}}P_n(\xi)$$

When $r = 1$,
$$v(1,\theta) = \sum_{n=0}^{\infty} B_n P_n(\xi)$$

Then $B_n = A_n$ of part (a) and so
$$v(r,\theta) = \frac{v_0}{2r}\left[1 + \frac{3}{2r}P_1(\cos\theta) - \frac{7}{8r^3}P_3(\cos\theta) + \frac{11}{16r^5}P_5(\cos\theta) + \cdots\right] \qquad (3)$$

7.19. A uniform hemisphere (see Fig. 7-2) has its convex surface kept at temperature u_0 while its base is kept at temperature zero. Find the steady-state temperature inside.

The boundary value problem in this case is
$$\nabla^2 u = 0$$
where
$$u = u_0 \quad \text{on the convex surface}$$
$$u = 0 \quad \text{on the base}$$

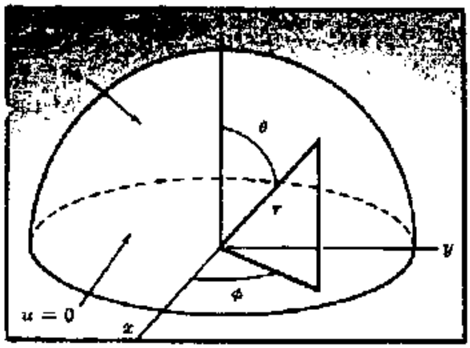

Fig. 7-2

The solution can be obtained from the results of Problem 7.18. To see this we note that the present problem is equivalent to the problem of solving Laplace's equation inside a sphere of which the top half surface is kept at temperature u_0 and the bottom half surface is kept at temperature $-u_0$. By symmetry, the plane of separation will then automatically be at temperature zero as required in this problem.

We can then obtain the required solution by first subtracting $v_0/2$ from the solution in Problem 7.18 and then replacing $v_0/2$ by u_0. The result is
$$u(r,\theta) = u_0\left[\frac{3}{2}rP_1(\cos\theta) - \frac{7}{8}r^3P_3(\cos\theta) + \frac{11}{16}r^5P_5(\cos\theta) + \cdots\right]$$

The problem can also, of course, be solved directly without use of the results in Problem 7.18.

7.20. (a) Find the gravitational potential at any point on the axis of a thin uniform ring of radius a. (b) Find the potential of the ring in part (a) at any point in space.

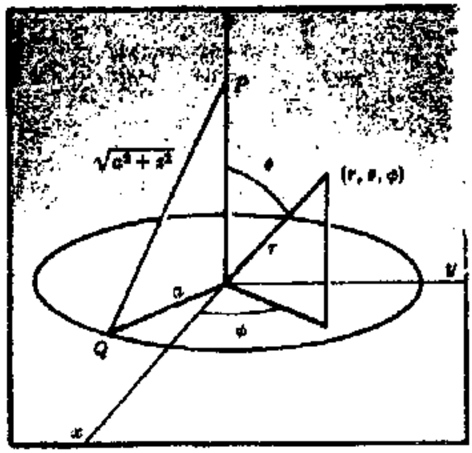

Fig. 7-3

(a) Choose the ring to be in the xy-plane so that the axis is the z-axis as indicated in Fig. 7-3. Then the potential at any point P on the z-axis is seen to be the mass of the ring divided by the distance $\sqrt{a^2+z^2}$ from any point Q on the ring to the point P. Letting σ denote the mass per unit length of the ring it follows that the potential at P is

$$v_P = \frac{2\pi a\sigma}{\sqrt{a^2+z^2}} \tag{1}$$

(b) In this case we must solve Laplace's equation $\nabla^2 v = 0$ where v reduces to v_P for points P on the z-axis. Now we know that because of the manner in which the ring has been located that v is independent of ϕ. We thus have as a solution to Laplace's equation

$$v = \left(A_1 r^n + \frac{B_1}{r^{n+1}}\right)[A_2 P_n(\xi) + B_2 Q_n(\xi)]$$

where $\xi = \cos\theta$. Since v must be bounded at $\theta = 0$ and π, i.e. $\xi = \pm 1$, we must choose $B_2 = 0$. Then

$$v = \left(Ar^n + \frac{B}{r^{n+1}}\right)P_n(\xi) \tag{2}$$

There are two cases to be considered, corresponding to the regions $0 \leq r < a$ and $r > a$.

Case 1: $0 \leq r < a$.

In this case we must choose $B = 0$ in (2) since otherwise the solution is unbounded at $r = 0$. Then $v = Ar^n P_n(\xi)$. By superposition we are led to consider the solution

$$v = \sum_{n=0}^{\infty} A_n r^n P_n(\xi) \tag{3}$$

Now when $\theta = 0$, i.e. $\xi = 1$, this must reduce to the potential on the z-axis, in which case $r = z$. Then we must have

$$\frac{2\pi a\sigma}{\sqrt{a^2+z^2}} = \sum_{n=0}^{\infty} A_n z^n \tag{4}$$

In order to obtain A_n we must expand the left side as a power series in z. We use the binomial theorem to obtain

$$\frac{2\pi a\sigma}{\sqrt{a^2+z^2}} = 2\pi\sigma\left(1+\frac{z^2}{a^2}\right)^{-1/2}$$

$$= 2\pi\sigma\left[1 - \frac{1}{2}\left(\frac{z}{a}\right)^2 + \frac{1\cdot 3}{2\cdot 4}\left(\frac{z}{a}\right)^4 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{z}{a}\right)^6 + \cdots\right] \tag{5}$$

Comparison of (4) and (5) leads to

$$A_0 = 2\pi\sigma, \quad A_1 = 0, \quad A_2 = -\frac{2\pi\sigma}{2a^2}, \quad A_3 = 0, \quad A_4 = \frac{2\pi\sigma\cdot 1\cdot 3}{a^4\cdot 2\cdot 4}, \quad \ldots$$

Using these in (3) we then find

$$v = 2\pi\sigma\left[P_0(\cos\theta) - \frac{1}{2}\left(\frac{r}{a}\right)^2 P_2(\cos\theta) + \frac{1\cdot 3}{2\cdot 4}\left(\frac{r}{a}\right)^4 P_4(\cos\theta) - \cdots\right] \tag{6}$$

where $0 \leq r < a$.

Case 2: $r > a$.

In this case we must choose $A = 0$ in (2) since otherwise the solution becomes unbounded as $r \to \infty$. Then $v = BP_n(t)/r^{n+1}$ and by superposition we are led to consider the solution

$$v = \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} P_n(t) \tag{7}$$

As in Case 1, this must reduce to the potential on the z-axis for $\theta = 0$ and $r = z$, i.e.

$$\frac{2\pi a\sigma}{\sqrt{a^2+z^2}} = \sum_{n=0}^{\infty} \frac{B_n}{z^{n+1}} \tag{8}$$

Thus, to find B_n we must expand the left side in inverse powers of z. Again we use the binomial theorem to obtain

$$\frac{2\pi a\sigma}{\sqrt{a^2+z^2}} = \frac{2\pi a\sigma}{z}\left(1+\frac{a^2}{z^2}\right)^{-1/2}$$

$$= \frac{2\pi a\sigma}{z}\left[1 - \frac{1}{2}\left(\frac{a}{z}\right)^2 + \frac{1\cdot 3}{2\cdot 4}\left(\frac{a}{z}\right)^4 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{a}{z}\right)^6 + \cdots\right] \tag{9}$$

Comparison of (8) and (9) leads to

$$B_0 = 2\pi a\sigma, \quad B_1 = 0, \quad B_2 = -2\pi a\sigma\left(\frac{1}{2}a^2\right), \quad B_3 = 0, \quad B_4 = 2\pi a\sigma\left(\frac{1\cdot 3}{2\cdot 4}a^4\right), \quad \ldots$$

Using these in (7) we then find

$$v = \frac{2\pi a\sigma}{r}\left[P_0(\cos\theta) - \frac{1}{2}\left(\frac{a}{r}\right)^2 P_2(\cos\theta) + \frac{1\cdot 3}{2\cdot 4}\left(\frac{a}{r}\right)^4 P_4(\cos\theta) - \cdots\right] \tag{10}$$

where $r > a$.

ASSOCIATED LEGENDRE FUNCTIONS

7.21. Show how Legendre's associated differential equation (12), page 132, is obtained from Laplace's equation $\nabla^2 u = 0$ expressed in spherical coordinates (r, θ, ϕ).

In this case we must modify the results obtained in Problem 7.1 by including the ϕ-dependence. Then letting $u = R\Theta\Phi$ in (1) of Problem 7.1 we obtain

$$\frac{\Theta\Phi}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R\Phi}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{R\Theta}{r^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = 0 \tag{1}$$

Multiplying by r^2, dividing by $R\Theta\Phi$ and rearranging, we find

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\phi^2}$$

Since one side depends only on r, while the other depends only on θ and ϕ, it follows that each side must be a constant, say $-\lambda^2$. Then we have

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\lambda^2 \tag{2}$$

and

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = \lambda^2 \tag{3}$$

The equation (2) is identical with (2) in Problem 7.1, so that we have as solution according to Problem 7.2

$$R = A_1 r^n + \frac{B_1}{r^{n+1}} \tag{4}$$

where we use $\lambda^2 = -n(n+1)$.

If now we multiply equation (3) by $\sin^2\theta$ and rearrange, it can be written as

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - n(n+1)\sin^2\theta$$

Since one side depends only on ϕ while the other side depends only on θ each side must be a constant, say $-m^2$. Then we have

$$\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + [n(n+1)\sin^2\theta - m^2]\Theta = 0 \tag{5}$$

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0 \tag{6}$$

If we now make the transformation $\xi = \cos\theta$ in equation (5) we find as in Problem 7.3 that it can be written as

$$(1-\xi^2)\frac{d}{d\xi}\left[(1-\xi^2)\frac{d\Theta}{d\xi}\right] + [n(n+1)(1-\xi^2) - m^2]\Theta = 0$$

Dividing by $1-\xi^2$ the equation becomes

$$(1-\xi^2)\frac{d^2\Theta}{d\xi^2} - 2\xi\frac{d\Theta}{d\xi} + \left[n(n+1) - \frac{m^2}{1-\xi^2}\right]\Theta = 0 \tag{7}$$

which is Legendre's associated differential equation (12) on page 132 if we replace Θ by y and ξ by x.

The general solution of (7) is shown in Problem 7.22 to be

$$\Theta = A_2 P_n^m(\xi) + B_2 Q_n^m(\xi) \tag{8}$$

where $\xi = \cos\theta$ and

$$P_n^m(\xi) = (1-\xi^2)^{m/2}\frac{d^m}{d\xi^m}P_n(\xi) \tag{9}$$

$$Q_n^m(\xi) = (1-\xi^2)^{m/2}\frac{d^m}{d\xi^m}Q_n(\xi) \tag{10}$$

We call $P_n^m(\xi)$ and $Q_n^m(\xi)$ *associated Legendre functions of the first and second kinds* respectively.

The general solution of (6) is

$$\Phi = A_3\cos m\phi + B_3\sin m\phi \tag{11}$$

If the function $u(r,\theta,\phi)$ is to be periodic of period 2π in ϕ, we must have m equal to an integer, which we take as positive. For the case $m=0$ the solution $u(r,\theta,\phi)$ is independent of ϕ and reduces to that given in Problem 7.5.

7.22. (a) Show that if m is a positive integer and u_n is any solution of Legendre's differential equation, then $d^m u_n/dx^m$ is a solution of Legendre's associated differential equation.

(b) Obtain the general solution of Legendre's associated equation.

(a) If Legendre's differential equation has the solution u_n then we must have

$$(1-x^2)u_n'' - 2xu_n' + n(n+1)u_n = 0$$

By differentiating this equation m times and letting $v_n^m = d^m u_n/dx^m$ we obtain

$$(1-x^2)\frac{d^2v_n^m}{dx^2} - 2(m+1)x\frac{dv_n^m}{dx} + [n(n+1) - m(m+1)]v_n^m = 0$$

In this equation we now let $v_n^m = (1-x^2)^p y$. Then it becomes

$$(1-x^2)^2y'' - [2(m+1)x(1-x^2) + 4px(1-x^2)]y'$$
$$+ \{4(m+1)px^2 + (4p^2-2p)x^2 - 2p + [n(n+1) - m(m+1)](1-x^2)\}y = 0$$

If we now choose $p = -m/2$, this equation becomes after dividing by $1 - x^2$

$$(1-x^2)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0 \tag{1}$$

which is Legendre's associated differential equation. Since $v_n^m = (1-x^2)^{-m/2}y$, it follows that $y = (1-x^2)^{m/2}v_n^m$, or

$$y = (1-x^2)^{m/2}\frac{d^m u_n}{dx^m} \tag{2}$$

is a solution of (1).

(b) Since the general solution of Legendre's equation is $c_1P_n(x) + c_2Q_n(x)$, we can show that the general solution of Legendre's associated differential equation is

$$y = c_1P_n^m(x) + c_2Q_n^m(x) \tag{3}$$

where $$P_n^m(x) = (1-x^2)^{m/2}\frac{d^mP_n}{dx^m}, \qquad Q_n^m(x) = (1-x^2)^{m/2}\frac{d^mQ_n}{dx^m} \tag{4}$$

7.23. Obtain the associated Legendre functions (a) $P_2^1(x)$, (b) $P_3^2(x)$, (c) $P_2^3(x)$, (d) $Q_2^1(x)$.

(a) $P_2^1(x) = (1-x^2)^{1/2}\dfrac{d}{dx}P_2(x) = (1-x^2)^{1/2}\dfrac{d}{dx}\left(\dfrac{3x^2-1}{2}\right) = 3x(1-x^2)^{1/2}$

(b) $P_3^2(x) = (1-x^2)^{2/2}\dfrac{d^2}{dx^2}P_3(x) = (1-x^2)\dfrac{d^2}{dx^2}\left(\dfrac{5x^3-3x}{2}\right) = 15x - 15x^3$

(c) $P_2^3(x) = (1-x^2)^{3/2}\dfrac{d^3}{dx^3}P_2(x) = 0.$ Note that in general $P_n^m(x) = 0$ if $m > n$.

(d) Using Problem 7.12(c) we find

$$Q_2^1(x) = (1-x^2)^{1/2}\frac{d}{dx}Q_2(x) = (1-x^2)^{1/2}\frac{d}{dx}\left\{\frac{3x^2-1}{4}\ln\left(\frac{1+x}{1-x}\right) - \frac{3x}{2}\right\}$$

$$= (1-x^2)^{1/2}\left[\frac{3x}{2}\ln\left(\frac{1+x}{1-x}\right) + \frac{3x^2-2}{1-x^2}\right]$$

7.24. Verify that $P_3^2(x)$ is a solution of Legendre's associated equation (12), page 132, for $m=2$, $n=3$.

By Problem 7.23, $P_3^2(x) = 15x - 15x^3$. Substituting this in the equation

$$(1-x^2)y'' - 2xy' + \left[3\cdot 4 - \frac{4}{1-x^2}\right]y = 0$$

we find after simplifying,

$$(1-x^2)(-90x) - 2x(15-45x^2) + \left[12 - \frac{4}{1-x^2}\right][15x - 15x^3] = 0$$

and so $P_3^2(x)$ is a solution.

7.25. Verify the result (16), page 132, for the functions $P_2^1(x)$ and $P_3^1(x)$.

We have from Problem 7.23(a), $P_2^1(x) = 3x(1-x^2)^{1/2}$. Also,

$$P_3^1(x) = (1-x^2)^{3/2}\frac{d}{dx}P_3(x) = (1-x^2)^{3/2}\frac{d}{dx}\left(\frac{5x^3-3x}{2}\right) = \frac{15x^2}{2}(1-x^2)^{3/2}$$

Then $$\int_{-1}^{1} P_2^1(x)P_3^1(x)\,dx = \int_{-1}^{1}\frac{45x^3}{2}(1-x^2)^2\,dx = 0$$

7.26. Verify the result (17), page 132, for the function $P_2^1(x)$.

Since $P_2^1(x) = 3x(1-x^2)^{1/2}$,

$$\int_{-1}^{1} [P_2^1(x)]^2\,dx \quad = \quad 9\int_{-1}^{1} x^2(1-x^2)\,dx \quad = \quad 9\left[\frac{x^3}{3} - \frac{x^5}{5}\right]\Bigg|_{-1}^{1} \quad = \quad \frac{36}{15} \quad = \quad \frac{12}{5}$$

Now according to (17), page 132, the required result should be

$$\frac{2}{2(2)+1}\,\frac{(2+1)!}{(2-1)!} \quad = \quad \frac{2}{5}\cdot\frac{3!}{1!} \quad = \quad \frac{12}{5}$$

so that the verification is achieved.

7.27. Expand $v_0(1-x^2)$ in a series of the form $\sum_{k=0}^{\infty} A_k P_k^m(x)$ where v_0 is a constant and $m = 2$.

We must find A_k, $k = 0, 1, 2, \ldots$, so that

$$v_0(1-x^2) \quad = \quad A_0 P_0^2(x) + A_1 P_1^2(x) + A_2 P_2^2(x) + \cdots \tag{1}$$

Method 1.

Since

$$P_k^2(x) \quad = \quad (1-x^2)\frac{d^2}{dx^2}P_k(x)$$

we have

$$P_0^2(x) = 0, \quad P_1^2(x) = 0, \quad P_2^2(x) = (1-x^2)\frac{d^2}{dx^2}\left(\frac{3x^2-1}{2}\right) = 3(1-x^2),$$

$$P_3^2(x) = (1-x^2)\frac{d^2}{dx^2}\left(\frac{5x^3-3x}{2}\right) = 15x(1-x^2), \quad \ldots$$

Then (1) becomes

$$v_0(1-x^2) \quad = \quad 3A_2(1-x^2) + 15A_3x(1-x^2) + \cdots$$

By comparing coefficients on each side we see that this can be satisfied if $3A_2 = v_0$, $15A_3 = 0$ and $A_k = 0$ for $k > 3$. Thus we have

$$v_0(1-x^2) \quad = \quad \frac{v_0}{3}P_2^2(x) \tag{2}$$

so that the required expansion consists of only one term.

Method 2.

If $f(x) = \sum_{k=0}^{\infty} A_k P_k^m(x)$, then on multiplying by $P_n^m(x)$ and integrating from -1 to 1 we obtain

$$\int_{-1}^{1} f(x)\,P_n^m(x)\,dx \quad = \quad \sum_{k=0}^{\infty} A_k \int_{-1}^{1} P_n^m(x)\,P_k^m(x)\,dx$$

Using (16) and (17), page 132, we see that the right side reduces to the single term

$$\frac{2}{2n+1}\,\frac{(n+m)!}{(n-m)!}A_n$$

so that

$$A_n \quad = \quad \frac{(2n+1)(n-m)!}{2(n+m)!}\int_{-1}^{1} f(x)\,P_n^m(x)\,dx$$

If $f(x) = v_0(1-x^2)$ and $m = 2$, then

$$A_n \quad = \quad \frac{(2n+1)(n-2)!}{2(n+2)!}\int_{-1}^{1} v_0(1-x^2)\,P_n^2(x)\,dx$$

Using this we can show that $A_2 = v_0/3$, $A_4 = 0$, $A_5 = 0$, $\ldots$ and so we obtain the result (2) as in Method 1.

7.28. Show that a solution to Laplace's equation $\nabla^2 v = 0$ in spherical coordinates is given by

$$v = \left(A_1 r^n + \frac{B_1}{r^{n+1}}\right)[A_2 P_n^m(\cos\theta) + B_2 Q_n^m(\cos\theta)][A_3 \cos m\phi + B_3 \sin m\phi]$$

This follows at once from Problems 7.21 and 7.22 since $u = R\Theta\Phi$ where

$$R = A_1 r^n + \frac{B_1}{r^{n+1}}$$

$$\Theta = A_2 P_n^m(\cos\theta) + B_2 Q_n^m(\cos\theta)$$

$$\Phi = A_3 \cos m\phi + B_3 \sin m\phi$$

7.29. Suppose that the surface of the sphere of Problem 7.18 is kept at potential $v_0 \sin^2\theta \cos 2\phi$. Determine the potential (*a*) inside and (*b*) outside the surface.

(*a*) **Interior Potential.** $0 \leqq r < 1$.

Since v is bounded at $r = 0$ we must choose $B_1 = 0$ in the solution as given in Problem 7.28. Also since v is bounded at $\theta = 0$ and π, we must choose $B_2 = 0$. Then a bounded solution is given by

$$v(r, \theta, \phi) = r^n P_n^m(\cos\theta)(A \cos m\phi + B \sin m\phi)$$

Since m and n can be any non-negative integers we can replace A by A_{mn}, B by B_{mn} and then, using the superposition principle, sum over m and n to obtain the solution

$$v(r, \theta, \phi) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} r^n P_n^m(\cos\theta)(A_{mn} \cos m\phi + B_{mn} \sin m\phi) \tag{1}$$

Now the boundary potential is given by

$$v(1, \theta, \phi) = v_0 \sin^2\theta \cos 2\phi \tag{2}$$

By comparison of (*2*) with

$$v(1, \theta, \phi) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P_n^m(\cos\theta)(A_{mn} \cos m\phi + B_{mn} \sin m\phi) \tag{3}$$

obtained from (*1*) with $r = 1$ it is seen that we must have $B_{mn} = 0$ for all m and $A_{mn} = 0$ for $m \neq 2$. Hence, (*3*) becomes

$$v(1, \theta, \phi) = \sum_{n=0}^{\infty} A_{2n} P_n^2(\cos\theta) \cos 2\phi$$

Comparison with (*2*) then shows that we must have

$$v_0 \sin^2\theta = \sum_{n=0}^{\infty} A_{2n} P_n^2(\cos\theta)$$

or using $\cos\theta = \xi$

$$v_0(1 - \xi^2) = \sum_{n=0}^{\infty} A_{2n} P_n^2(\xi)$$

$$= A_{20}P_0^2(\xi) + A_{21}P_1^2(\xi) + A_{22}P_2^2(\xi) + \cdots \tag{4}$$

We have already obtained this expansion in Problem 7.27, from which we see that $A_{22} = v_0/3$, while all other coefficients are zero. It thus follows from (*1*) that

$$v(r, \theta, \phi) = \frac{v_0}{3} r^2 P_2^2(\cos\theta) \cos 2\phi = v_0 r^2 \sin^2\theta \cos 2\phi \tag{5}$$

(*b*) **Exterior Potential.** $r > 1$.

Since v must be bounded as $r \to \infty$ in this case and is also bounded at $\theta = 0$ and π, we choose $A_1 = 0$, $B_2 = 0$ in the solution of Problem 7.28. Thus a solution is

$$v(r, \theta, \phi) = \frac{P_n^m(\cos\theta)}{r^{n+1}}(A \cos m\phi + B \sin m\phi)$$

or by superposition

$$v(r,\theta,\phi) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{P_n^m(\cos\theta)}{r^{n+1}}(A_{mn}\cos m\phi + B_{mn}\sin m\phi) \tag{6}$$

Using the fact that $v(1,\theta,\phi) = v_0 \sin^2\theta \cos 2\phi$ we again find $m = 2$, $B_{mn} = 0$ which leads to equation (4) of part (a). As before we then find $A_{22} = v_0/3$, while all other coefficients are zero, leading to the required solution

$$v(r,\theta,\phi) = \frac{v_0}{3r^3}P_2^2(\cos\theta)\cos 2\phi$$

$$= \frac{v_0}{r^3}\sin^2\theta\cos 2\phi \tag{7}$$

It is easy to check that the above are the required solutions by direct substitution.

7.30. Solve Problem 7.18 if the surface potential is $f(\theta,\phi)$.

As in Problem 7.29 we are led to the following solutions inside and outside the sphere:

Inside the sphere, $0 \leq r < 1$

$$v(r,\theta,\phi) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} r^n P_n^m(\cos\theta)(A_{mn}\cos m\phi + B_{mn}\sin m\phi) \tag{1}$$

Outside the sphere, $r > 1$

$$v(r,\theta,\phi) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{P_n^m(\cos\theta)}{r^{n+1}}(A_{mn}\cos m\phi + B_{mn}\sin m\phi) \tag{2}$$

For the case $r = 1$ both of these lead to

$$f(\theta,\phi) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P_n^m(\cos\theta)(A_{mn}\cos m\phi + B_{mn}\sin m\phi)$$

This is equivalent to the expansion

$$F(\xi,\phi) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P_n^m(\xi)(A_{mn}\cos m\phi + B_{mn}\sin m\phi) \tag{3}$$

where $\xi = \cos\theta$. Let us write this as

$$F(\xi,\phi) = \sum_{n=0}^{\infty} C_n P_n^m(\xi) \tag{4}$$

where

$$C_n = \sum_{m=0}^{\infty}(A_{mn}\cos m\phi + B_{mn}\sin m\phi) \tag{5}$$

As in Method 2 of Problem 7.27 we find from (4)

$$C_n = \frac{(2n+1)(n-m)!}{2(n+m)!}\int_{-1}^{1} F(\xi,\phi)\,P_n^m(\xi)\,d\xi \tag{6}$$

We also see from (5) that A_{mn} and B_{mn} are simply the Fourier coefficients obtained by expansion of C_n (which is a function of ϕ) in a Fourier series. Using the methods of Fourier series it follows that

$$A_{0n} = \frac{1}{2\pi}\int_0^{2\pi} C_n\,d\phi$$

$$A_{mn} = \frac{1}{\pi}\int_0^{2\pi} C_n \cos m\phi\,d\phi \qquad m = 1,2,3,\ldots$$

$$B_{mn} = \frac{1}{\pi}\int_0^{2\pi} C_n \sin m\phi\,d\phi \qquad m = 1,2,3,\ldots$$

Combining these results we see that

$$A_{0n} = \frac{(2n+1)(n-m)!}{4\pi(n+m)!}\int_{-1}^{1}\int_0^{2\pi} F(\xi,\phi)\,P_n^m(\xi)\,d\xi\,d\phi$$

while for $m = 1, 2, 3, \ldots$

$$A_{mn} = \frac{(2n+1)(n-m)!}{2\pi(n+m)!}\int_{-1}^{1}\int_{0}^{2\pi} F(\xi,\phi)\,P_n^m(\xi)\cos m\phi\,d\xi\,d\phi$$

$$B_{mn} = \frac{(2n+1)(n-m)!}{2\pi(n+m)!}\int_{-1}^{1}\int_{0}^{2\pi} F(\xi,\phi)\,P_n^m(\xi)\sin m\phi\,d\xi\,d\phi$$

Using these results in (1) and (2) we obtain the required solutions.

Supplementary Problems

LEGENDRE POLYNOMIALS

7.31. Use Rodrigue's formula (4), page 130, to verify the formulas for $P_0(x), P_1(x), \ldots, P_5(x)$, on page 130.

7.32. Obtain the formulas for $P_6(x)$ and $P_8(x)$ using a recurrence formula.

7.33. Evaluate (a) $\int_0^1 xP_5(x)\,dx$, (b) $\int_{-1}^{1} [P_2(x)]^2\,dx$, (c) $\int_{-1}^{1} P_2(x)\,P_4(x)\,dx$.

7.34. Show that (a) $P_n(1) = 1$ (c) $P_{2n-1}(0) = 0$

(b) $P_n(-1) = (-1)^n$ (d) $P_{2n}(0) = (-1)^n \dfrac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}$

for $n = 1, 2, 3, \ldots$.

7.35. Use the generating function to prove that $P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)\,P_n(x)$.

7.36. Prove that (a) $P'_{n+1}(x) - xP'_n(x) = (n+1)\,P_n(x)$, (b) $xP'_n(x) - P'_{n-1}(x) = nP_n(x)$.

7.37. Show that $\displaystyle\sum_{n=0}^{\infty} P_n(\cos\theta) = \frac{1}{2}\csc\frac{\theta}{2}$.

7.38. Show that (a) $P_2(\cos\theta) = \frac{1}{4}(1 + 3\cos 2\theta)$, (b) $P_3(\cos\theta) = \frac{1}{8}(3\cos\theta + 5\cos 3\theta)$.

7.39. Show that $P_7(x) = \frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x)$.

7.40. Show from the generating function that (a) $P_n(1) = 1$, (b) $P_n(-1) = (-1)^n$.

7.41. Show that $\displaystyle\sum_{k=1}^{\infty} \frac{x^k P_{k-1}(x)}{k} = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$, $-1 < x < 1$.

LEGENDRE FUNCTIONS OF THE SECOND KIND

7.42. Prove that the series (6) and (7) on page 131 which are nonterminating are convergent for $-1 < x < 1$ but divergent for $x = \pm 1$.

7.43. Find $Q_3(x)$.

7.44. Write the general solution of $(1 - x^2)y'' - 2xy' + 2y = 0$.

SERIES OF LEGENDRE POLYNOMIALS

7.45. Expand $x^4 - 3x^2 + x$ in a series of the form $\sum_{k=0}^{\infty} A_k P_k(x)$.

7.46. Expand $f(x) = \begin{cases} 2x+1 & 0 < x \leq 1 \\ 0 & -1 \leq x < 0 \end{cases}$ in a series of the form $\sum_{k=0}^{\infty} A_k P_k(x)$, writing the first four nonzero terms.

7.47. If $f(x) = \sum_{k=0}^{\infty} A_k P_k(x)$, obtain *Parseval's identity*

$$\int_{-1}^{1} [f(x)]^2\,dx = 2\sum_{k=0}^{\infty} \frac{A_k^2}{2k+1}$$

and illustrate by using the function of Problem 7.45.

SOLUTIONS USING LEGENDRE FUNCTIONS

7.48. Find the potential v (*a*) interior and (*b*) exterior to a hollow sphere of unit radius with center at the origin if the surface is charged to potential $v_0(1 + 3\cos\theta)$ where v_0 is constant.

7.49. Solve Problem 7.48 if the surface potential is $v_0 \sin^2\theta$.

7.50. Find the steady-state temperature within the region bounded by two concentric spheres of radii a and $2a$ if the temperatures of the outer and inner spheres are u_0 and 0 respectively.

7.51. Find the gravitational potential at any point outside a solid uniform sphere of radius a of mass m.

7.52. Is there a solution to Problem 7.51, if the point is inside the sphere? Explain.

7.53. Interpret Problem 7.48 as a temperature problem.

7.54. Show that the potential due to a uniform spherical shell of inner radius a and outer radius b is given by

$$v = \begin{cases} 2\pi\sigma(b^2 - a^2) & r < a \\ 2\pi\sigma(3b^2r - 2a^3 - r^3)/3r & a < r < b \\ 4\pi\sigma(b^3 - a^3)/3r & r > b \end{cases}$$

7.55. A solid uniform circular disc of radius a and mass M is located in the xy-plane with center at the origin. Show that the gravitational potential at any point of the plane is given by

$$v = \frac{2M}{a}\left[1 - \frac{r}{a}P_1(\cos\theta) + \frac{1}{2}\left(\frac{r}{a}\right)^2 P_2(\cos\theta) - \frac{1}{2\cdot 4}\left(\frac{r}{a}\right)^4 P_4(\cos\theta) + \frac{1\cdot 3}{2\cdot 4\cdot 6}\left(\frac{r}{a}\right)^6 P_6(\cos\theta) - \cdots\right]$$

if $r < a$ and

$$v = \frac{M}{r}\left[1 - \frac{1}{4}\left(\frac{a}{r}\right)^2 P_2(\cos\theta) + \frac{1\cdot 3}{4\cdot 6}\left(\frac{a}{r}\right)^4 P_4(\cos\theta) - \frac{1\cdot 3\cdot 5}{4\cdot 6\cdot 8}\left(\frac{a}{r}\right)^6 P_6(\cos\theta) + \cdots\right]$$

if $r > a$.

ASSOCIATED LEGENDRE FUNCTIONS

7.56. Find (*a*) $P_3^2(x)$, (*b*) $P_4^2(x)$, (*c*) $P_4^3(x)$.

7.57. Find (*a*) $Q_1^1(x)$, (*b*) $Q_1^2(x)$.

7.58. Verify that the expressions for $P_2^1(x)$ and $Q_2^1(x)$ are solutions of the corresponding differential equation and thus write the general solution.

7.59. Verify formulas *(16)* and *(17)*, page 132, for the case where (*a*) $m = 1$, $n = 1$, $l = 2$, (*b*) $m = 1$, $n = 1$, $l = 1$.

7.60. Obtain a generating function for $P_n^m(x)$.

7.61. Use the generating function to obtain results *(16)* and *(17)* on page 132.

7.62. Show how to expand $f(x)$ in a series of the form $\sum_{k=0}^{\infty} A_k P_k^m(x)$ and illustrate by using the cases (*a*) $f(x) = x^2$, $m = 2$ and (*b*) $f(x) = x(1-x)$, $m = 1$. Verify the corresponding Parseval's identity in each case.

7.63. Work Problem 7.18 if the potential on the surface is $v_0 \sin^3\theta \cos\theta \cos 3\phi$.

MISCELLANEOUS PROBLEMS

7.64. Show that
$$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{[n/2]} \frac{(-1)^k(2n-2k)!}{k!\,(n-k)!\,(n-2k)!}x^{n-2k}$$
where $[n/2]$ is the largest integer $\leq n/2$.

7.65. Show that
$$P_n(x) = \frac{1}{\pi}\int_0^{\pi}(x + \sqrt{x^2-1}\cos u)^n\,du$$
Use the result to find $P_2(x)$ and $P_3(x)$.

7.66. Show that
$$\int_{-1}^{1}(1-x^2)P_m'(x)\,P_n'(x)\,dx = \begin{cases} 0 & m \neq n \\ \dfrac{2n(n+1)}{2n+1} & m = n \end{cases}$$

7.67. Show that
$$\int_{-1}^{1}P_n(x)\ln(1-x)\,dx = \begin{cases} -2/n(n+1) & n \neq 0 \\ 2(\ln 2 - 1) & n = 0 \end{cases}$$

7.68. (*a*) Show that $\int_{-1}^{1} x^m P_n(x)\,dx = 0$ if $m < n$ or if $m - n$ is an odd positive integer.

(*b*) Show that
$$\int_{-1}^{1} x^{n+2p}P_n(x)\,dx = \frac{(n+2p)!\,\Gamma(p+\frac{1}{2})}{2^n(2p)!\,\Gamma(p+n+\frac{3}{2})}$$
for any non-negative integers n and p.

7.69. Show that a solution of the wave equation
$$\nabla^2 V = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}$$
depending on r, θ, and t, but not on ϕ, is given by
$$V = [A_1 J_{n+1/2}(\omega r/c) + B_1 J_{-n-1/2}(\omega r/c)][A_2 P_n(\cos\theta) + B_2 Q_n(\cos\theta)][A_3\cos\omega t + B_3 \sin\omega t]$$

7.70. Work Problem 7.69 if there is also ϕ-dependence.

7.71. A heat-conducting region is bounded by two concentric spheres of radii a and b ($a < b$) which have their surfaces maintained at constant temperatures u_1 and u_2 respectively. Find the steady-state temperature at any point of the region.

7.72. Interpret Problem 7.18 as a temperature problem.

7.73. Obtain a solution similar to that given in Problem 7.69 for the heat conduction equation

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u$$

where u depends on r, θ, and t but not on ϕ.

Chapter 8

Hermite, Laguerre and Other Orthogonal Polynomials

HERMITE'S DIFFERENTIAL EQUATION. HERMITE POLYNOMIALS

An important equation which arises in problems of physics is called *Hermite's differential equation*; it is given by

$$y'' - 2xy' + 2ny = 0 \tag{1}$$

where $n = 0, 1, 2, 3, \ldots$.

The equation (*1*) has polynomial solutions called *Hermite polynomials* given by *Rodrigue's formula*

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}) \tag{2}$$

The first few Hermite polynomials are

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2, \quad H_3(x) = 8x^3 - 12x \tag{3}$$

Note that $H_n(x)$ is a polynomial of degree n.

GENERATING FUNCTION FOR HERMITE POLYNOMIALS

The generating function for Hermite polynomials is given by

$$e^{2tx - t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n \tag{4}$$

This result is useful in obtaining many properties of $H_n(x)$.

RECURRENCE FORMULAS FOR HERMITE POLYNOMIALS

We can show (see Problems 8.2 and 8.20) that the Hermite polynomials satisfy the recurrence formulas

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \tag{5}$$

$$H_n'(x) = 2nH_{n-1}(x) \tag{6}$$

Starting with $H_0(x) = 1$, $H_1(x) = 2x$, we can use (*5*) to obtain higher-degree Hermite polynomials.

ORTHOGONALITY OF HERMITE POLYNOMIALS

We can show (see Problem 8.4) that

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x)\, H_n(x)\, dx = 0 \qquad m \neq n \tag{7}$$

so that the Hermite polynomials are mutually orthogonal with respect to the weight or density function e^{-x^2}.

In the case where $m = n$ we can show (see Problem 8.4) that the left side of (7) becomes

$$\int_{-\infty}^{\infty} e^{-x^2} H_n^2(x)\,dx = 2^n n!\sqrt{\pi} \tag{8}$$

From this we can normalize the Hermite polynomials so as to obtain an orthonormal set.

SERIES OF HERMITE POLYNOMIALS

Using the orthogonality of the Hermite polynomials it is possible to expand a function in a series having the form

$$f(x) = A_0H_0(x) + A_1H_1(x) + A_2H_2(x) + \cdots \tag{9}$$

where
$$A_n = \frac{1}{2^n n!\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} f(x)\, H_n(x)\,dx \tag{10}$$

See Problem 8.6.

In general such series expansions are possible when $f(x)$ and $f'(x)$ are piecewise continuous.

LAGUERRE'S DIFFERENTIAL EQUATION. LAGUERRE POLYNOMIALS

Another differential equation of importance in physics is *Laguerre's differential equation* given by

$$xy'' + (1-x)y' + ny = 0 \tag{11}$$

where $n = 0, 1, 2, 3, \ldots$.

This equation has polynomial solutions called *Laguerre polynomials* given by

$$L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x}) \tag{12}$$

which is also referred to as *Rodrigue's formula* for the Laguerre polynomials.

The first few Laguerre polynomials are

$$L_0(x) = 1, \quad L_1(x) = 1 - x, \quad L_2(x) = x^2 - 4x + 2, \quad L_3(x) = 6 - 18x + 9x^2 - x^3 \tag{13}$$

Note that $L_n(x)$ is a polynomial of degree n.

SOME IMPORTANT PROPERTIES OF LAGUERRE POLYNOMIALS

In the following we list some properties of the Laguerre polynomials.

1. Generating function.

$$\frac{e^{-xt/(1-t)}}{1-t} = \sum_{n=0}^{\infty} \frac{L_n(x)}{n!} t^n \tag{14}$$

2. Recurrence formulas.

$$L_{n+1}(x) = (2n+1-x)L_n(x) - n^2L_{n-1}(x) \tag{15}$$

$$L_n'(x) - nL_{n-1}'(x) + nL_{n-1}(x) = 0 \tag{16}$$

$$xL_n'(x) = nL_n(x) - n^2L_{n-1}(x) \tag{17}$$

3. Orthogonality.

$$\int_0^\infty e^{-x} L_m(x)\, L_n(x)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ (n!)^2 & \text{if } m = n \end{cases} \tag{18}$$

4. Series expansions.

If
$$f(x) = A_0 L_0(x) + A_1 L_1(x) + A_2 L_2(x) + \cdots \tag{19}$$

then
$$A_n = \frac{1}{(n!)^2} \int_0^\infty e^{-x} f(x)\, L_n(x)\, dx \tag{20}$$

MISCELLANEOUS ORTHOGONAL POLYNOMIALS AND THEIR PROPERTIES

There are many other examples of orthogonal polynomials. Some of the more important ones, together with their properties, are given in the following list.

1. Associated Laguerre polynomials $L_n^m(x)$.

These are polynomials defined by

$$L_n^m(x) = \frac{d^m}{dx^m} L_n(x) \tag{21}$$

and satisfying the equation

$$xy'' + (m+1-x)y' + (n-m)y = 0 \tag{22}$$

If $m > n$ then $L_n^m(x) = 0$.

We have
$$\int_0^\infty x^m e^{-x} L_n^m(x)\, L_p^m(x)\, dx = 0 \qquad p \neq n \tag{23}$$

$$\int_0^\infty x^m e^{-x} \{L_n^m(x)\}^2\, dx = \frac{(n!)^3}{(n-m)!} \tag{24}$$

2. Chebyshev polynomials $T_n(x)$.

These are polynomials defined by

$$T_n(x) = \cos(n \cos^{-1} x) = x^n - \binom{n}{2} x^{n-2}(1-x^2) + \binom{n}{4} x^{n-4}(1-x^2)^2 - \cdots \tag{25}$$

and satisfying the differential equation

$$(1-x^2)y'' - xy' + n^2 y = 0 \tag{26}$$

where $n = 0, 1, 2, \ldots$.

A recurrence formula for $T_n(x)$ is given by

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \tag{27}$$

and the generating function is

$$\frac{1 - tx}{1 - 2tx + t^2} = \sum_{n=0}^\infty T_n(x) t^n \tag{28}$$

We also have

$$\int_{-1}^1 \frac{T_m(x)\, T_n(x)}{\sqrt{1-x^2}}\, dx = 0 \qquad m \neq n \tag{29}$$

$$\int_{-1}^1 \frac{\{T_n(x)\}^2}{\sqrt{1-x^2}}\, dx = \begin{cases} \pi & n = 0 \\ \pi/2 & n = 1, 2, \ldots \end{cases} \tag{30}$$

Solved Problems

HERMITE POLYNOMIALS

8.1. Use the generating function for the Hermite polynomials to find (a) $H_0(x)$, (b) $H_1(x)$, (c) $H_2(x)$, (d) $H_3(x)$.

We have

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!} = H_0(x) + H_1(x)t + \frac{H_2(x)}{2!}t^2 + \frac{H_3(x)}{3!}t^3 + \cdots$$

Now

$$e^{2tx-t^2} = 1 + (2tx - t^2) + \frac{(2tx-t^2)^2}{2!} + \frac{(2tx-t^2)^3}{3!} + \cdots$$

$$= 1 + (2x)t + (2x^2-1)t^2 + \left(\frac{4x^3-6x}{3}\right)t^3 + \cdots$$

Comparing the two series, we have

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2, \quad H_3(x) = 8x^3 - 12x$$

8.2. Prove that $H_n'(x) = 2nH_{n-1}(x)$.

Differentiating $e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$ with respect to x,

$$2te^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n'(x)}{n!} t^n$$

or

$$\sum_{n=0}^{\infty} \frac{2H_n(x)}{n!} t^{n+1} = \sum_{n=0}^{\infty} \frac{H_n'(x)}{n!} t^n$$

Equating coefficients of t^n on both sides,

$$\frac{2H_{n-1}(x)}{(n-1)!} = \frac{H_n'(x)}{n!} \quad \text{or} \quad H_n'(x) = 2nH_{n-1}(x)$$

8.3. Prove that $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})$.

We have

$$e^{2tx-t^2} = e^{x^2-(t-x)^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

Then

$$\frac{\partial^n}{\partial t^n}(e^{2tx-t^2})\bigg|_{t=0} = H_n(x)$$

But

$$\frac{\partial^n}{\partial t^n}(e^{2tx-t^2})\bigg|_{t=0} = e^{x^2}\frac{\partial^n}{\partial t^n}\left[e^{-(t-x)^2}\right]\bigg|_{t=0}$$

$$= e^{x^2}\frac{\partial^n}{\partial(-x)^n}\left[e^{-(t-x)^2}\right]\bigg|_{t=0} = (-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2})$$

8.4. Prove that $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\, dx = \begin{cases} 0 & m \neq n \\ 2^n n! \sqrt{\pi} & m = n \end{cases}$

We have

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!}, \qquad e^{2sx-s^2} = \sum_{m=0}^{\infty} \frac{H_m(x)s^m}{m!}$$

Multiplying these,

$$e^{2tx-t^2+2sx-s^2} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{H_m(x)H_n(x)s^m t^n}{m!\,n!}$$

Multiplying by e^{-x^2} and integrating from $-\infty$ to ∞,

$$\int_{-\infty}^{\infty} e^{-[(x-s-t)^2-2st]}\,dx \quad = \quad \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{s^m t^n}{m!\,n!}\int_{-\infty}^{\infty} e^{-x^2}H_m(x)\,H_n(x)\,dx$$

Now the left side is equal to

$$e^{2st}\int_{-\infty}^{\infty} e^{-(x-s-t)^2}\,dx \quad = \quad e^{2st}\int_{-\infty}^{\infty} e^{-u^2}\,du \quad = \quad e^{2st}\sqrt{\pi} \quad = \quad \sqrt{\pi}\sum_{m=0}^{\infty}\frac{2^m s^m t^m}{m!}$$

By equating coefficients the required result follows.

The result

$$\int_{-\infty}^{\infty} e^{-x^2}H_m(x)\,H_n(x)\,dx \quad = \quad 0 \qquad m \neq n$$

can also be proved by using a method similar to that of Problem 7.13, page 138 (see Problem 8.24).

8.5. Show that the Hermite polynomials satisfy the differential equation

$$y'' - 2xy' + 2ny \; = \; 0$$

From (5) and (6), page 154, we have on eliminating $H_{n-1}(x)$:

$$H_{n+1}(x) \quad = \quad 2xH_n(x) - H_n'(x) \tag{1}$$

Differentiating both sides we have

$$H_{n+1}'(x) \quad = \quad 2xH_n'(x) + 2H_n(x) - H_n''(x) \tag{2}$$

But from (6), page 154, we have on replacing n by $n+1$:

$$H_{n+1}'(x) \quad = \quad 2(n+1)\,H_n(x) \tag{3}$$

Using (3) in (2) we then find on simplifying:

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) \quad = \quad 0$$

which is the required result.

We can also proceed as in Problem 8.25.

8.6. (a) If $f(x) = \sum_{k=0}^{\infty} A_k H_k(x)$ show that $A_k = \dfrac{1}{2^k k!\sqrt{\pi}}\displaystyle\int_{-\infty}^{\infty} e^{-x^2} f(x)\,H_k(x)\,dx$.

(b) Expand x^3 in a series of Hermite polynomials.

(a) If $f(x) = \sum_{k=0}^{\infty} A_k H_k(x)$ then on multiplying both sides by $e^{-x^2}H_n(x)$ and integrating term by term from $-\infty$ to ∞ (assuming this to be possible) we arrive at

$$\int_{-\infty}^{\infty} e^{-x^2}f(x)\,H_n(x)\,dx \quad = \quad \sum_{k=0}^{\infty} A_k\int_{-\infty}^{\infty} e^{-x^2}H_k(x)\,H_n(x)\,dx \tag{1}$$

But from Problem 8.4

$$\int_{-\infty}^{\infty} e^{-x^2}H_k(x)\,H_n(x)\,dx \quad = \quad \begin{cases} 0 & k \neq n \\ 2^n n!\sqrt{\pi} & k = n \end{cases}$$

Thus (1) becomes

$$\int_{-\infty}^{\infty} e^{-x^2}f(x)\,H_n(x)\,dx \quad = \quad A_n 2^n n!\sqrt{\pi}$$

or

$$A_n \quad = \quad \frac{1}{2^n n!\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2}f(x)\,H_n(x)\,dx \tag{2}$$

which yields the required result on replacing n by k.

(b) We must find coefficients A_k, $k = 1, 2, 3, \ldots$, such that

$$x^3 = \sum_{k=0}^{\infty} A_k H_k(x) \tag{3}$$

Method 1.

The expansion (3) can be written

$$x^3 = A_0 H_0(x) + A_1 H_1(x) + A_2 H_2(x) + A_3 H_3(x) + \cdots \tag{4}$$

or

$$x^3 = A_0(1) + A_1(2x) + A_2(4x^2 - 2) + A_3(8x^3 - 12x) + \cdots \tag{5}$$

Since $H_k(x)$ is a polynomial of degree k we see that we must have $A_4 = 0$, $A_5 = 0$, $A_6 = 0$, $\ldots$; otherwise the left side of (5) is a polynomial of degree 3 while the right side would be a polynomial of degree greater than 3. Thus we have from (5)

$$x^3 = (A_0 - 2A_2) + (2A_1 - 12A_3)x + 4A_2 x^2 + 8A_3 x^3$$

Then equating coefficients of like powers of x on both sides we find

$$8A_3 = 1, \quad 4A_2 = 0, \quad 2A_1 - 12A_3 = 0, \quad A_0 - 2A_2 = 0$$

from which

$$A_0 = 0, \quad A_1 = \frac{3}{4}, \quad A_2 = 0, \quad A_3 = \frac{1}{8}$$

Thus (3) becomes

$$x^3 = \frac{3}{4}H_1(x) + \frac{1}{8}H_3(x)$$

which is the required expansion.

Check.

$$\frac{3}{4}H_1(x) + \frac{1}{8}H_3(x) = \frac{3}{4}(2x) + \frac{1}{8}(8x^3 - 12x) = x^3$$

Method 2.

The coefficients A_k in (1) are given by

$$A_k = \frac{1}{2^k k! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} x^3 H_k(x)\, dx$$

as obtained in part (a) with $f(x) = x^3$.

Putting $k = 0, 1, 2, 3, 4, \ldots$ and integrating we then find

$$A_0 = 0, \quad A_1 = \frac{3}{4}, \quad A_2 = 0, \quad A_3 = \frac{1}{8}, \quad A_4 = 0, \quad A_5 = 0, \quad \ldots$$

and we are led to the same result as in Method 1.

In general, for expansion of polynomials the first of the above methods will be easier and faster.

8.7. (a) Write Parseval's identity corresponding to the series expansion $f(x) = \sum_{k=0}^{\infty} A_k H_k(x)$.
(b) Verify the result of part (a) for the case where $f(x) = x^3$.

(a) We can obtain Parseval's identity formally by first squaring both sides of $f(x) = \sum_{k=0}^{\infty} A_k H_k(x)$ to obtain

$$\{f(x)\}^2 = \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} A_k A_p H_k(x) H_p(x)$$

Then multiplying by e^{-x^2} and integrating from $-\infty$ to ∞ we find

$$\int_{-\infty}^{\infty} e^{-x^2}\{f(x)\}^2\, dx = \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} A_k A_p \int_{-\infty}^{\infty} e^{-x^2} H_k(x) H_p(x)\, dx$$

Making use of the results of Problem 8.4 this can be written as

$$\int_{-\infty}^{\infty} e^{-x^2}\{f(x)\}^2\,dx \quad = \quad \sqrt{\pi}\sum_{k=0}^{\infty} 2^k k!\,A_k^2$$

which is Parseval's identity for the Hermite polynomials.

(b) From Problem 8.6 it follows that if $f(x) = x^3$ then $A_0 = 0,\ A_1 = \frac{3}{4},\ A_2 = 0,\ A_3 = \frac{1}{8},\ A_4 = 0,$ $A_5 = 0,\ \ldots$. Thus Parseval's identity becomes

$$\int_{-\infty}^{\infty} e^{-x^2}(x^3)^2\,dx \quad = \quad \sqrt{\pi}\,[2(1!)(\tfrac{3}{4})^2 + 2^3(3!)(\tfrac{1}{8})^2]$$

Now the right side reduces to $15\sqrt{\pi}/8$. The left side is

$$\int_{-\infty}^{\infty} x^6 e^{-x^2}\,dx \quad = \quad 2\int_0^{\infty} x^6 e^{-x^2}\,dx \quad = \quad \int_0^{\infty} u^{5/2}e^{-u}\,du$$

$$= \quad \Gamma(\tfrac{7}{2}) \quad = \quad (\tfrac{5}{2})(\tfrac{3}{2})(\tfrac{1}{2})\sqrt{\pi}$$

$$= \quad \frac{15}{8}\sqrt{\pi}$$

where we have made the transformation $x = \sqrt{u}$. Thus Parseval's identity is verified.

LAGUERRE POLYNOMIALS

8.8. Determine the Laguerre polynomials (a) $L_0(x)$, (b) $L_1(x)$, (c) $L_2(x)$, (d) $L_3(x)$.

We have $L_n(x) = e^x \dfrac{d^n}{dx^n}(x^n e^{-x})$. Then

(a) $L_0(x) \;=\; 1$

(b) $L_1(x) \;=\; e^x \dfrac{d}{dx}(xe^{-x}) \;=\; 1 - x$

(c) $L_2(x) \;=\; e^x \dfrac{d^2}{dx^2}(x^2e^{-x}) \;=\; 2 - 4x + x^2$

(d) $L_3(x) \;=\; e^x \dfrac{d^3}{dx^3}(x^3e^{-x}) \;=\; 6 - 18x + 9x^2 - x^3$

8.9. Prove that the Laguerre polynomials $L_n(x)$ are orthogonal in $(0, \infty)$ with respect to the weight function e^{-x}.

From Laguerre's differential equation we have for any two Laguerre polynomials $L_m(x)$ and $L_n(x)$,

$$xL_m'' + (1-x)L_m' + mL_m \;=\; 0$$
$$xL_n'' + (1-x)L_n' + nL_n \;=\; 0$$

Multiplying these equations by L_n and L_m respectively and subtracting, we find

$$x[L_nL_m'' - L_mL_n''] + (1-x)[L_nL_m' - L_m'L_n'] \quad = \quad (n-m)L_mL_n$$

or

$$\frac{d}{dx}[L_nL_m' - L_mL_n'] + \frac{1-x}{x}[L_nL_m' - L_mL_n'] \quad = \quad \frac{(n-m)L_mL_n}{x}$$

Multiplying by the integrating factor

$$e^{\int(1-x)/x\,dx} \;=\; e^{\ln x - x} \;=\; xe^{-x}$$

this can be written as

$$\frac{d}{dx}\{xe^{-x}[L_nL_m' - L_mL_n']\} \quad = \quad (n-m)e^{-x}L_mL_n$$

so that by integrating from 0 to ∞,

$$(n-m)\int_0^\infty e^{-x}L_m(x)\,L_n(x)\,dx \quad = \quad xe^{-x}[L_nL_m' - L_mL_n']\Big|_0^\infty \quad = \quad 0$$

Thus if $m \neq n$,

$$\int_0^\infty e^{-x}L_m(x)\,L_n(x)\,dx \quad = \quad 0$$

which proves the required result.

8.10. **Prove that** $L_{n+1}(x) = (2n+1-x)\,L_n(x) - n^2\,L_{n-1}(x)$.

The generating function for the Laguerre polynomials is

$$\frac{e^{-xt/(1-t)}}{1-t} \quad = \quad \sum_{n=0}^{\infty} \frac{L_n(x)}{n!}\,t^n \tag{1}$$

Differentiating both sides with respect to t yields

$$\frac{e^{-xt/(1-t)}}{(1-t)^2} - \frac{xe^{-xt/(1-t)}}{(1-t)^3} \quad = \quad \sum_{n=0}^{\infty} \frac{nL_n(x)}{n!}\,t^{n-1} \tag{2}$$

Multiplying both sides by $(1-t)^2$ and using (1) on the left side we find

$$\sum_{n=0}^{\infty} (1-t)\frac{L_n(x)}{n!}\,t^n - \sum_{n=0}^{\infty} \frac{xL_n(x)}{n!}\,t^n \quad = \quad \sum_{n=0}^{\infty} (1-t)^2\frac{nL_n(x)}{n!}\,t^{n-1}$$

which can be written as

$$\sum_{n=0}^{\infty} \frac{L_n(x)}{n!}\,t^n - \sum_{n=0}^{\infty} \frac{L_n(x)}{n!}\,t^{n+1} - \sum_{n=0}^{\infty} \frac{xL_n(x)}{n!}\,t^n$$

$$= \quad \sum_{n=0}^{\infty} \frac{nL_n(x)}{n!}\,t^{n-1} - \sum_{n=0}^{\infty} \frac{2nL_n(x)}{n!}\,t^n + \sum_{n=0}^{\infty} \frac{nL_n(x)}{n!}\,t^{n+1}$$

If we now equate coefficients of t^n on both sides of this equation we find

$$\frac{L_n(x)}{n!} - \frac{L_{n-1}(x)}{(n-1)!} - \frac{xL_n(x)}{n!} \quad = \quad \frac{(n+1)L_{n+1}(x)}{(n+1)!} - \frac{2nL_n(x)}{n!} + \frac{(n-1)L_{n-1}(x)}{(n-1)!}$$

Multiplying by $n!$ and simplifying we then obtain, as required,

$$L_{n+1}(x) \quad = \quad (2n+1-x)\,L_n(x) - n^2L_{n-1}(x)$$

8.11. **Expand** $x^3 + x^2 - 3x + 2$ **in a series of Laguerre polynomials, i.e.** $\sum_{k=0}^{\infty} A_kL_k(x)$.

We shall use a method similar to Method 1 of Problem 8.6(b). Since we must expand a polynomial of degree 3 we need only take terms up to $L_3(x)$. Thus

$$x^3 + x^2 - 3x + 2 \;=\; A_0L_0(x) + A_1L_1(x) + A_2L_2(x) + A_3L_3(x)$$

Using the results of Problem 8.8 this can be written

$$x^3 + x^2 - 3x + 2 \;=\; (A_0 + A_1 + 2A_2 + 6A_3) - (A_1 + 4A_2 + 18A_3)x + (A_2 + 9A_3)x^2 - A_3x^3$$

Then, equating like powers of x on both sides we have

$$A_0 + A_1 + 2A_2 + 6A_3 \;=\; 2, \quad A_1 + 4A_2 + 18A_3 \;=\; 3, \quad A_2 + 9A_3 \;=\; 1, \quad -A_3 \;=\; 1$$

Solving these we find

$$A_0 \;=\; 7, \quad A_1 \;=\; -19, \quad A_2 \;=\; 10, \quad A_3 \;=\; -1$$

Then the required expansion is

$$x^3 + x^2 - 3x + 2 \;=\; 7L_0(x) - 19L_1(x) + 10L_2(x) - L_3(x)$$

We can also work the problem by using (19) and (20), page 156.

MISCELLANEOUS ORTHOGONAL POLYNOMIALS

8.12. Obtain the associated Laguerre polynomials (a) $L_2^1(x)$, (b) $L_2^2(x)$, (c) $L_3^2(x)$, (d) $L_3^4(x)$.

(a) $L_2^1(x) = \dfrac{d}{dx}L_2(x) = \dfrac{d}{dx}(2-4x+x^2) = 2x - 4$

(b) $L_2^2(x) = \dfrac{d^2}{dx^2}L_2(x) = \dfrac{d^2}{dx^2}(2-4x+x^2) = 2$

(c) $L_3^2(x) = \dfrac{d^2}{dx^2}L_3(x) = \dfrac{d^2}{dx^2}(6-18x+9x^2-x^3) = 18 - 6x$

(d) $L_3^4(x) = \dfrac{d^4}{dx^4}L_3(x) = 0.$ In general $L_n^m(x) = 0$ if $m > n$.

8.13. Verify the result (*24*), page 156, for $m = 1$, $n = 2$.

We must show that

$$\int_0^\infty xe^{-x}\{L_2^1(x)\}^2\,dx = \frac{(2!)^3}{1!} = 8$$

Now since $L_2^1(x) = 2x - 4$ by Problem 8.12(a) we have

$$\begin{aligned}\int_0^\infty xe^{-x}(2x-4)^2\,dx &= 4\int_0^\infty x^3e^{-x}\,dx - 16\int_0^\infty x^2e^{-x}\,dx + 16\int_0^\infty xe^{-x}\,dx \\ &= 4\,\Gamma(4) - 16\,\Gamma(3) + 16\,\Gamma(2) \\ &= 4(3!) - 16(2!) + 16(1!) \\ &= 8\end{aligned}$$

so that the result is verified.

8.14. Verify the result (*23*), page 156, with $m = 2$, $n = 2$, $p = 3$.

We must show that

$$\int_0^\infty x^2e^{-x}L_2^2(x)\,L_3^2(x)\,dx = 0$$

Since $L_2^2(x) = 2$, $L_3^2(x) = 18 - 6x$ by Problem 8.12(a) and (b) respectively the integral is

$$\begin{aligned}\int_0^\infty x^2e^{-x}(2)(18-6x)\,dx &= 36\int_0^\infty x^2e^{-x}\,dx - 12\int_0^\infty x^3e^{-x}\,dx \\ &= 36\,\Gamma(3) - 12\,\Gamma(4) \\ &= 36(2!) - 12(3!) = 0\end{aligned}$$

as required.

8.15. Verify that $L_3^2(x)$ satisfies the differential equation (*22*), page 156, in the special case $m = 2$, $n = 3$.

From Problem 8.12(c) we have $L_3^2(x) = 18 - 6x$. The differential equation (*22*), page 156, with $m = 2$, $n = 3$ is

$$xy'' + (3-x)y' + y = 0$$

Substituting $y = 18 - 6x$ in this equation we have

$$x(0) + (3-x)(-6) + 18 - 6x = 0$$

which is an identity. Thus $L_3^2(x)$ satisfies the differential equation.

8.16. Show that the Chebyshev polynomial $T_n(x)$ is given by

$$T_n(x) \quad = \quad x^n - \binom{n}{2}x^{n-2}(1-x^2) + \binom{n}{4}x^{n-4}(1-x^2)^2 - \binom{n}{6}x^{n-6}(1-x^2)^3 + \cdots$$

We have by definition

$$T_n(x) \quad = \quad \cos(n\cos^{-1}x)$$

Let $u = \cos^{-1}x$ so that $x = \cos u$. Then $T_n(x) = \cos nu$. Now by De Moivre's theorem

$$(\cos u + i\sin u)^n \quad = \quad \cos nu + i\sin nu$$

Thus $\cos nu$ is the real part of $(\cos u + i\sin u)^n$. But this expansion is, by the binomial theorem,

$$(\cos u)^n + \binom{n}{1}(\cos u)^{n-1}(i\sin u) + \binom{n}{2}(\cos u)^{n-2}(i\sin u)^2 + \binom{n}{3}(\cos u)^{n-3}(i\sin u)^3 + \cdots$$

and the real part of this is given by

$$\cos^n u - \binom{n}{2}\cos^{n-2}u\sin^2 u + \binom{n}{4}\cos^{n-4}u\sin^4 u - \cdots$$

Then since $\cos u = x$ and $\sin^2 u = 1-x^2$, this becomes

$$x^n - \binom{n}{2}x^{n-2}(1-x^2) + \binom{n}{4}x^{n-4}(1-x^2)^2 - \cdots$$

8.17. Find (a) $T_2(x)$ and (b) $T_3(x)$.

Using Problem 8.16 we find for $n=2$ and $n=3$ respectively:

(a) $T_2(x) \quad = \quad x^2 - \binom{2}{2}x^0(1-x^2) \quad = \quad x^2-(1-x^2) \quad = \quad 2x^2-1$

(b) $T_3(x) \quad = \quad x^3 - \binom{3}{2}x^1(1-x^2) \quad = \quad x^3 - 3x(1-x^2) \quad = \quad 4x^3-3x$

Another method.

Since $T_0(x) = \cos 0 = 1$, $T_1(x) = \cos(\cos^{-1}x) = x$ we have from the recurrence formula (87), page 156, on putting $n=1$ and $n=2$ respectively,

$$T_2(x) \quad = \quad 2xT_1(x) - T_0(x) \quad = \quad 2x^2-1$$

$$T_3(x) \quad = \quad 2xT_2(x) - T_1(x) \quad = \quad 2x(2x^2-1) - x \quad = \quad 4x^3-3x$$

8.18. Verify that $T_n(x) = \cos(n\cos^{-1}x)$ satisfies the differential equation

$$(1-x^2)y'' - xy' + n^2y \quad = \quad 0$$

for the case $n=3$.

From Problem 8.17(b), $T_3(x) = 4x^3-3x$ and the differential equation for $n=3$ is

$$(1-x^2)y'' - xy' + 9y \quad = \quad 0$$

Then if $y = 4x^3-3x$ the left side becomes

$$(1-x^2)(24x) - x(12x^2-3) + 9(4x^3-3x) \quad = \quad 0$$

so that the differential equation reduces to an identity.

Supplementary Problems

HERMITE POLYNOMIALS

8.19. Use Rodrigue's formula (*2*), page 154, to obtain the Hermite polynomials $H_0(x)$, $H_1(x)$, $H_2(x)$, $H_3(x)$.

8.20. Use the generating function to obtain the recurrence formula (*5*) on page 154, and obtain $H_2(x)$, $H_3(x)$ given that $H_0(x) = 1$, $H_1(x) = 2x$.

8.21. Show directly that (*a*) $\int_{-\infty}^{\infty} e^{-x^2} H_2(x) H_3(x)\,dx = 0$, (*b*) $\int_{-\infty}^{\infty} e^{-x^2}[H_2(x)]^2\,dx = 8\sqrt{\pi}$.

8.22. Evaluate $\int_{-\infty}^{\infty} x^2 e^{-x^2} H_n(x)\,dx$.

8.23. Show that $H_{2n}(0) = \dfrac{(-1)^n(2n)!}{n!}$.

8.24. Prove the result (*7*), page 154, by using a method similar to that in Problem 7.13, pages 138 and 139.

8.25. Work Problem 8.5, page 158, by using (*a*) Rodrigue's formula, (*b*) the method of Frobenius.

8.26. (*a*) Expand $f(x) = x^3 - 3x^2 + 2x$ in a series of the form $\sum_{k=0}^{\infty} A_k H_k(x)$. (*b*) Verify Parseval's identity for the function in part (*a*).

8.27. Find the general solution of Hermite's differential equation for the cases (*a*) $n = 0$ and (*b*) $n = 1$.

LAGUERRE POLYNOMIALS

8.28. Find $L_4(x)$ and show that it satisfies Laguerre's equation (*11*), page 155, for $n = 4$.

8.29. Use the generating function to obtain the recurrence formula (*16*) on page 155.

8.30. Use formula (*15*) to determine $L_2(x)$, $L_3(x)$ and $L_4(x)$ if we define $L_n(x) = 0$ when $n = -1$ and $L_n(x) = 1$ when $n = 0$.

8.31. Show that $nL_{n-1}(x) = nL'_{n-1}(x) - L'_n(x)$.

8.32. Prove that $\int_0^{\infty} e^{-x}\{L_n(x)\}^2\,dx = (n!)^2$.

8.33. Prove the results (*19*) and (*20*), page 156.

8.34. Expand $f(x) = x^3 - 3x^2 + 2x$ in a series of the form $\sum_{k=0}^{\infty} A_k L_k(x)$.

8.35. Illustrate Parseval's identity for Problem 8.34.

8.36. Find the general solution of Laguerre's differential equation for $n = 0$.

8.37. Obtain Laguerre's differential equation (*11*), page 155, from the generating function (*14*), page 155.

MISCELLANEOUS ORTHOGONAL POLYNOMIALS

8.38. Find (*a*) $L_4^3(x)$, (*b*) $L_5^3(x)$.

8.39. Verify the results (*23*) and (*24*), page 156, for $m = 2$, $n = 3$.

8.40. Verify that $L_4^2(x)$ satisfies the differential equation (*22*), page 156, in the special case $m = 2$, $n = 4$.

8.41. Evaluate $\int_0^\infty x^2 e^{-x} L_4^2(x)\,dx$.

8.42. Show that a generating function for the associated Laguerre polynomials is given by

$$\frac{(-t)^m e^{-xt/(1-t)}}{(1-t)^{m+1}} = \sum_{k=m}^{\infty} \frac{L_k^m(x)}{k!} t^k$$

8.43. Solve Chebyshev's differential equation (*26*), page 156, for the case where $n = 0$.

8.44. Find (*a*) $T_4(x)$ and (*b*) $T_5(x)$.

8.45. Expand $f(x) = x^3 + x^2 - 4x + 2$ in a series of Chebyshev polynomials $\sum_{k=0}^{\infty} A_k T_k(x)$.

8.46. (*a*) Write Parseval's identity corresponding to the expansion of $f(x)$ in a series of Chebyshev polynomials and (*b*) verify the identity by using the function of Problem 8.45.

8.47. Prove the recurrence formula (*27*), page 156.

8.48. Prove the results (*29*) and (*30*) on page 156.

MISCELLANEOUS PROBLEMS

8.49. (*a*) Find the general solution of Hermite's differential equation. (*b*) Write the general solution for the cases where $n = 1$ and $n = 2$. [*Hint*: Let $y = vH_n(x)$ and determine v so that Hermite's equation is satisfied.]

8.50. In *quantum mechanics* the Schroedinger equation for a harmonic oscillator is given by

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}(E - \tfrac{1}{2}\kappa x^2)\psi = 0$$

where E, m, h, κ are constants. Show that solutions of this equation are given by

$$\psi = C_n H_n(x/a) e^{-x^2/4a^2}$$

where $n = 0, 1, 2, 3, \ldots$ and

$$a = \sqrt[4]{\frac{h^2}{16\pi^2 \kappa m}} \qquad E = (n + \tfrac{1}{2})\frac{h}{2\pi}\sqrt{\frac{\kappa}{m}}$$

The differential equation is a Sturm-Liouville differential equation whose eigenvalues and eigenfunctions are given by E and ψ respectively.

8.51. (*a*) Find the general solution of Laguerre's differential equation. (*b*) Write the general solution for the cases $n = 1$ and $n = 2$. [*Hint*: Let $y = vL_n(x)$. See also Problem 8.49.]

8.52. Prove the results (*18*) on page 156 by using the generating function.

8.53. (*a*) Show that Laguerre's associated differential equation (*22*), page 156, is obtained by differentiating Laguerre's equation (*11*) m times with respect to x, and thus (*b*) show that a solution is $d^m L_n/dx^m$.

8.54. Prove the results (*23*) and (*24*) on page 156.

8.55. (*a*) Find the general solution of Chebyshev's differential equation. (*b*) Write the general solution for the cases $n = 1$ and $n = 2$. [*Hint*: Let $y = vT_n(x)$.]

8.56. Discuss the theory of (*a*) Hermite polynomials, (*b*) Laguerre polynomials, (*c*) associated Laguerre polynomials, and (*d*) Chebyshev polynomials from the viewpoint of Sturm-Liouville theory.

8.57. Discuss the relationship between the expansion of a function in Fourier series and in Chebyshev polynomials.