APPENDIX

T NORMS AND S NORMS

Fuzzy set operations such as the *complement* (*NOT*), *union* (*OR*), and *intersection* (*AND*) as well as operations associated with the evaluation of linguistic descriptions and neurofuzzy models can be parameterized through T norms and S norms. The concepts of T norm and S norm were originally developed by mathematicians in connection with probability theory, but they have found considerable application in fuzzy-neural approaches. The operators can be thought of as the extension of fuzzy operations. For example, the T norm (T stands for triangular) can be thought of as the extension of *AND*. Following Terano et al. (1994) we present here semi-formally the basic ideas involved.

A.1 T NORMS

A T norm is a two-place function T: $[0,1] \times [0,1] \rightarrow [0,1]$ satisfying the following four axioms:

$$x\,T\,1 = x, \quad x\,T\,0 = 0 \qquad \forall x \in [0,1] \qquad \text{(A-1)}$$

$$x_1\,T x_2 = x_2\,T x_1 \qquad \forall x_1, x_2 \in [0,1] \qquad \text{(A-2)}$$

$$x_1\,T(x_2\,T x_3) = (x_1\,T x_2) T x_3 \qquad \forall x_1, x_2, x_3 \in [0,1] \qquad \text{(A-3)}$$

$$\text{if } x_1 \leq x_2, \text{ then } x_1\,T x_3 \leq x_2\,T x_3 \qquad \forall x_1, x_2, x_3 \in [0,1] \qquad \text{(A-4)}$$

Equation (A-1) is a boundary condition refering to crisp *AND*, while equations (A-2) and (A-3) are *commutative* and *associative* laws; Equation (A-4)

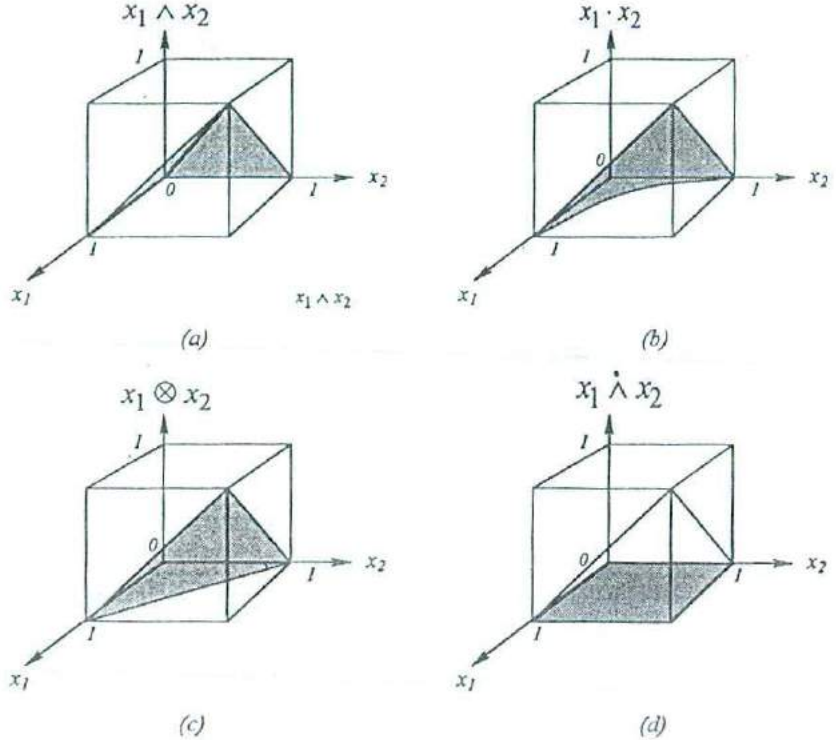

Figure A.1 Representative T norms: (a) Logical product, (b) algebraic product, (c) bounded product, and (d) drastic product.

requires the preservation of order, and it guarantees that the order of evaluation cannot be reversed at the third evaluation.

The *logical product* produced by a min operation is a representative T-norm operation.

$$x_1 \, T x_2 = x_1 \wedge x_2 \tag{A-5}$$

It corresponds to the intersection of fuzzy sets. It is easy to verify that equation (A-5) above satisfies equations (A-1)–(A-4). If (A-5) is expressed graphically, we obtain Figure A1. It is easy to see in Figure A.1*a* that the four corners of the $x_1 \wedge x_2$(shaded) square are the value of the crisp *AND* operation.

What other kinds of operations can the T norm produce? The most important ones for neurofuzzy applications are the *algebraic product* $x_1 \cdot x_2$, the bounded product $x_1 \otimes x_2$, and the drastic product $x_1 \dot{\wedge} x_2$ defined as follows

$$x_1 \cdot x_2 = x_1 x_2 \tag{A-6}$$

$$x_1 \otimes x_2 = (x_1 + x_2 - 1) \vee 0 \tag{A-7}$$

$$x_1 \dot{\wedge} x_2 = \begin{cases} x_2 & x_1 = 1 \\ x_1 & x_2 = 1 \\ 0 & \text{otherwise} \end{cases} \tag{A-8}$$

The graphs of these are given in Figure A.1*b*–*d*. Using these graphs it is easy to show that

$$0 \le x_1 \dot{\wedge} x_2 \le x_1 \otimes x_2 \le x_1 \cdot x_2 \le x_1 \wedge x_2 \tag{A-9}$$

The T norm can produce an infinite number of other operations which can be placed in order between the *drastic product* and the *logical product* as seen in Figure A.1.

A.2 S NORMS

The S norm, which is the extension of *OR*, is also called the T conorm.

An S-norm is a two-place function S: $[0, 1] \times [0, 1] \to [0, 1]$ satisfying the following four axioms:

$$x \, S \, 1 = 1, \quad x \, S \, 0 = x \qquad \forall x \in [0,1] \tag{A-10}$$

$$x_1 \, S \, x_2 = x_2 \, S \, x_1 \qquad \forall x_1, x_2 \in [0,1] \tag{A-11}$$

$$x_1 \, S(x_2 \, S \, x_3) = (x_1 \, S \, x_2) S \, x_3 \qquad \forall x_1, x_2, x_3 \in [0,1] \tag{A-12}$$

$$\text{if } x_1 \le x_2, \text{ then } x_1 \, S \, x_3 \le x_2 \, S \, x_3 \qquad \forall x_1, x_2, x_3 \in [0,1] \tag{A-13}$$

A representative S norm is the *logical sum* produced by a *max* operation.

$$x_1 \, S \, x_2 = x_1 \vee x_2 \tag{A-14}$$

Others include the algebraic sum $x_1 \dotplus x_2$, the bounded sum $x_1 \oplus x_2$, and the drastic sum $x_1 \dot{\vee} x_2$:

$$x_1 \dotplus x_2 = x_1 + x_2 - x_1 x_2 \tag{A-15}$$

$$x_1 \oplus x_2 = (x_1 + x_2) \wedge 1 \tag{A-16}$$

$$x_1 \dot{\vee} x_2 = \begin{cases} x_2 & x_1 = 0 \\ x_1 & x_2 = 0 \\ 1 & \text{otherwise} \end{cases} \tag{A-17}$$

The properties of these are given in Figure A.2 a–d. As is obvious from these figures, we have

$$x_1 \vee x_2 \leq x_1 \dotplus x_2 \leq x_1 \oplus x_2 \leq x_1 \dot{\vee} x_2 \tag{A-18}$$

It is easy to see by inspecting Figure A.2 and equations (A-18) that the order is the reverse of the T norm.

The T norm shown in Figure A.1, and S norm shown in Figure A.2 must meet the boundary conditions shown in Figure A.3. It is easy to show that the smallest S norm is the logical sum and the largest S norm is the drastic sum, as also seen in Figure A.3.

Various fuzzy *negations*, T norms and S norms have been proposed, but it is convenient to employ the ones that meet the following conditions:

$$\neg(x_1 \, S x_2) = \neg x_1 \, T \neg x_2 \tag{A-19}$$

$$\neg(x_1 \, T x_2) = \neg x_1 \, S \neg x_2 \tag{A-20}$$

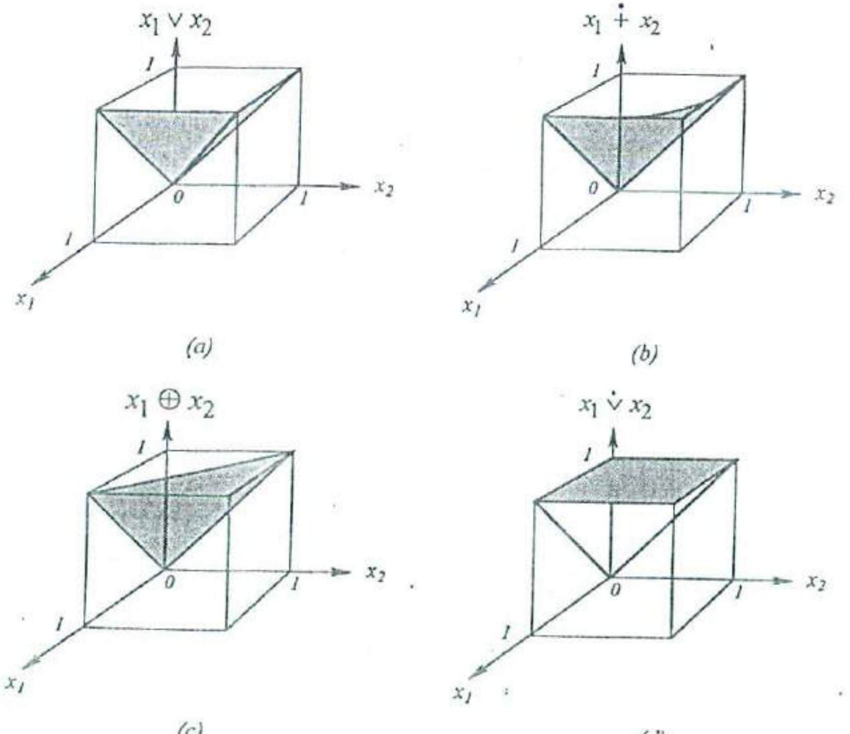

Figure A.2 Representative S norms: (*a*) Logical sum, (*b*) algebraic sum, (*c*) bounded sum, and (*d*) drastic sum.

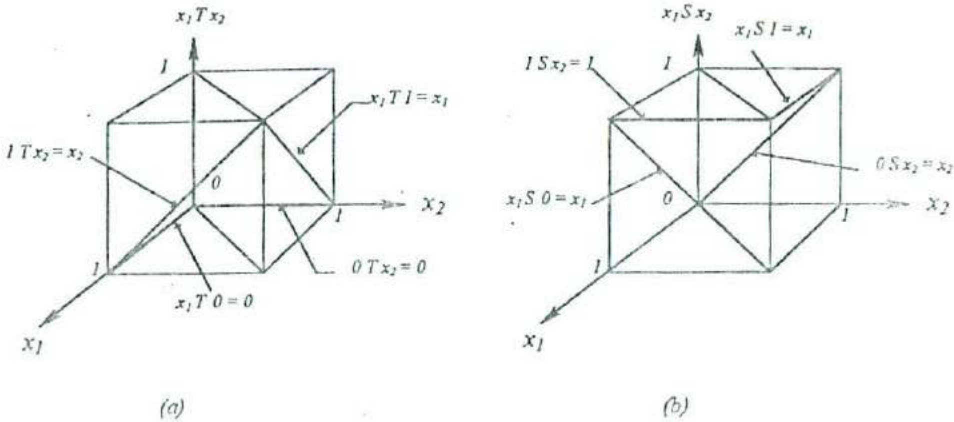

Figure A.3 Boundary conditions for (*a*) general *T* norm and (*b*) general *S* norm.

These correspond to de Morgan's laws for crisp operations and are called *fuzzy de Morgan's laws*. When these equations above arise, the T norm and S norm are dual with respect to fuzzy negation, and it can be shown that the logical product and logical sum, algebraic product and sum, bounded product and sum, and drastic product and sum all show duality of the variance from 1 in the fuzzy negation. In practical applications, the logical pair is standard, and the algebraic and bounded pairs are used occasionally. The drastic pair has the property of being discontinuous and is important in terms of the lower bound for T norms and the upper bound for S norms, but it is not often used in practical applications.

The logical pair of T norms and S norms is most often used due to its explicit physical meaning and the fact that the

$$([0,1], \leq, 1-, \wedge, \vee) \qquad \text{(A-21)}$$

system gives complete pseudo-Boolean algebra (and thus good mathematical characteristics). Only complements do not arise:

$$x \vee (1-x) \geq 0, \qquad x \vee \leq 1 \qquad \text{(A-22)}$$

and all the other properties found in crisp logic arise just the same. With crisp logic, the equality of the above equation arises. The complements are called the *laws of contradiction and exclusion*. The law of contradiction—that is, that no property and its negation can exist at the same time—and exclusion—that is, that both the property and its negation exist with no ambiguous intermediates—are properties unique to crisp logic and not found in fuzzy logic.

A.3 T NORMS AND FUZZY IMPLICATIONS

Let us look at the mathematical basis for modeling the fuzzy relations involved in fuzzy *if/then* rules, that is, expression of the form

$$\text{if } X \text{ is } A, \text{ then } Y \text{ is } B \tag{A-23}$$

with various implications. If the elements in the total space are fixed and we confine our discussion to evaluations within [0, 1], fuzzy implications are two-variable functions or two-item relations of [0, 1]:

$$\phi: [0,1] \times [0,1] \rightarrow [0,1] \tag{A-24}$$

The implication "if X is A, the n Y is B" is described by $(X$ *is* $(NOT\ A))$ $OR(Y$ is $B)$ in crisp cases. Therefore if we replace NOT with fuzzy negation and OR with max, the most standard fuzzy logic operations, we get the implication operator

$$\phi = (1 - \mu_A(x)) \vee \mu_B(y) \tag{A-25}$$

which we have called *Zadeh's implication operator* in Chapter 5.

In order to look into the meaning of the equation (A-25), let us graph the T norm and the S norm as shown in Figure A.4. The four crisp points (shaded areas in the figure) are preserved, and the figure is composed of two triangular planes. However, given just the coordinate axis and the four crisp points, imagine Figure A.4b for the same graph of two triangular planes if

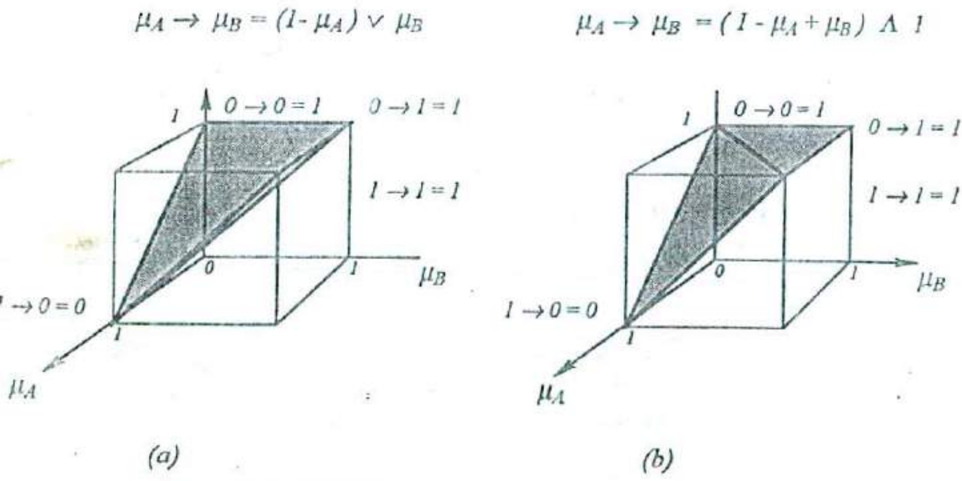

Figure A.4 Two fuzzy implications (a) Zadeh and (b) Lukasiewicz.

asked to interpolate. Expressing Figure A.4*b* as an equation, we get

$$\phi = (1 - \mu_A(x) + \mu_B(y)) \wedge 1 \tag{A-26}$$

This is the limited sum operation for fuzzy negation of the variance from 1 of x and y, and it is an equation known as the *Lukasiewicz implication in multiple logic*.

REFERENCE

Terano, T., Asai, K., and Sugeno, M., *Applied Fuzzy Systems*, Academic Press, Boston, 1994.

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