7.1 MAPPINGS

Let X and Y be two sets not necessarily distinct. A mapping of X into Y is a correspondence that associates with each element of X with unique element of Y.

A mapping is usually denoted by a single letter such as f, g, a, etc. The fact that f is a mapping of X into Y is often indicated by the symbol $f: X \to Y$.

If $f: X \to Y$, then for each $x \in X$, the corresponding element of Y is called the image of x under the mapping f and is denoted by (x). The set X is called the domain of the mapping f and the set Y is called its image space. The subset of Y consisting of those elements of Y which are image of some $x \in X$ *i e.* $\{ y/y \in Y$ and $y = f(x)$ for some $x \in X\}$ is called the range of the mapping f.

If may be noted that under a mapping f of X into Y, every $x \in X$ has one and exactly one image in Y. Where as the same $y \in Y$ may be the image of more than one $x \in X$ and there may be some $y \in Y$ which is not the image of any $x \in X$.

A mapping of X into Y is defined if we know the image of of each $x \in X$. The notation $f: x \to f(x)$ is used to indicate that under the mapping of X into Y, x is mapped into f(x), *i. e.* $f(x)$ is the image of x.

A mapping $f: X \to Y$ can be pictorially represented by listing the elements of X and Y inside two closed curves and drawing arrows from each $x \in X$ the conesponking image $y \in Y$.

Example : Let a mapping $f: Y \to Y$ be defined as follows. $1 \to 3$, $2 \to 2$, $3 \to 1$, $4 \to 1$, $3 \to 3$, $7 \to 2$, $9 \to 5$.

Here the domain of f is $x = \{1, 2, 4, 7, 9\}$ and the range of f is $\{1, 2, 3, 5\}$.

Example : Let $x = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $f: x \to X$ be a mapping defined as follows :

$f(1) = 1$, $f(2) = 5$, $f(3)=4$, $f(4)=8$,
$f(5)=6$, $f(6)=3$, $f(7)=7$, $f(8)=2$.

Hence the domain as range of f is the x.

Example : Let $X= \{1, 2, 3\}$ and $Y = \{a, b, c\}$.

The correspondence defined as.

$1 \to a$, $1 \to b$, $2 \to b$, $3 \to c$, is not a mapping because under the correspondence, two distinct elements Y correspond to the elements 1 of X.

CHAPTER–1

NUMBERS

1.1 The set of Real Numbers

(a) **Integers** (পূর্ণসংখ্যা) **:** The numbers 1, 2, 3, ………… are known as the ***natural*** or *counting* numbers. The natural numbers, their negatives and zero form the set of integers Z. Thus

$$Z = \{ \ldots\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\ldots \}$$

(b) **Rational numbers** (আনুপাতিক সংখ্যা) **:** Any number which can be expressed in the form $\frac{p}{q}$, where p and q are integers with $q \neq 0$, is called a rational number. Clearly any integer is a rational number (correspo nding to $q = 1$).

Examples of rational numbers are 2, $\frac{5}{7}$, $\frac{-3}{8}$, 1.2, etc.

In *decimal representation* of a rational number, the steps will either *terminate* or *a certain part of the steps will repeat.* For example,

$\frac{1}{8} = 0{\cdot}125$; here the steps terminate.

$\frac{1}{3} = 0{\cdot}3333 \ldots\ldots\ldots = 0.\dot{3}$; here the steps do not terminate, but 3 is repeated which is indicated by putting a dot over 3·

$\frac{1}{7} = 0.142857\ 142857\ 142857 \ldots\ldots = 0{\cdot}\dot{1}4285\dot{7}$; the dots over 1 and 7 show that the part '142857' is repeated.

(c) **Irrational numbers** (অমেয় সংখ্যা) **:** A number which represents a certain length on a straight line but cannot be represented in the form $\frac{p}{q}$ (p, q being integers $q \neq 0$) is called an irrational number.

In decimal representation of an irrational number the steps involved one non-terminating and non-recurring.

$\sqrt{2}$, $\sqrt{3}$, π, e, etc. are irrational numbers.

All the rational and irrational number together are said to form the **continuum of Real numbers** or **the Set of real numbers**, denoted by $\mathbb{R}$.

All the real numbers, positive or negative, rational or irrational can be represented on a straight line, say, the axis of X which is called the *real axis.*

Let XOX′ be the axis of x (or the real axis), the point O being the origin or the point of reference.

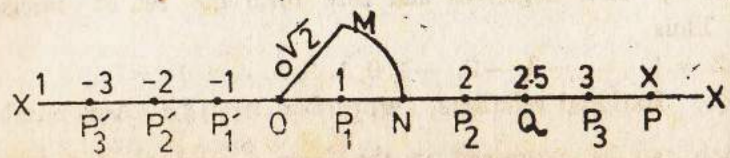


Fig. 1

Let P be any point on the real line. Let, on certain scale, the algebraic distance OP $= x$; that is, $x > 0$ if P lies to the right of O (on OX side), while $x < 0$, if is to the left of O (or OX′ side). We say that the point P represents the real number x. Evidently O represents the number 0 (zero). For different positions of P, x takes different values and we call x to be a real variable. If the actual distances of P_1 or P'_1, P_2 or P'_2, P_3 or P'_3, ... measured from O be 1, 2, 3, ... respectively, then P_1, P_2, P_3,...... represent the numbers 1, 2, 3, ... while P'_1, P'_2, P'_3, ... represent the numbers $-1, -2, -3$, ... as shown in Fig. 1. The point Q which is the mid point of the lines segment $P_2 P_3$ denotes the number 2·5. In the triangle OP_1M right-angled at P_1, $P_1M = OP_1 = 1$, $OM = \sqrt{2}$. A circle drawn with OM as radius cuts the x-axis at N which is to the right of O. Hence N represents the number $\sqrt{2}$.

1.2 Abosolute value of a real number.

The *absolute value* or *modulus* of a real number x is the magnitude or the numerical value of the number with positive sign. It is denoted by $|x|$. Thus

$$|x| = x, \text{ if } x \geqslant 0,$$

and $$|x| = -x, \text{ if } x < 0.$$

Since $(-x)^2 = x^2 = |x|^2 \geqslant 0$, we see that *the square of a real number is never negative.*

It is to be noted that

(i) $\sqrt{x^2} = |x|$

and (ii) $|x| \geqslant x$.

Ex. $|5| = 5$, $|0| = 0$, $|-5| = -(-5) = 5$;

$|5| = 5$, but $|-5| = 5 > -5$.

$\sqrt{9} = \sqrt{(\pm 3)^2} = |\pm 3| = 3.$

If the negative square root of 9 or any other positive real number is wanted then the radical sign should be preceeded by a '$-$' (minus) sign ; e.g. $-\sqrt{9} = -3$.

1.3 Imaginary and complex numbers.

Since the square of a real number is never negative, we introduce the number i, such that

$$i^2 = -1 \quad \text{or} \quad i = \sqrt{-1}$$

with $|i| = 1$.

"i" is called the *unit imaginary number.* Any number expressed as $z = x + iy$ (where x and y are real) is called a *complex number.* If $y = 0$, the number z is purely real ; if $x = 0$, then z is purely imaginary. $-1 + i$, $2 - \sqrt{3}\,i$, $\sqrt{-7}$ or $\sqrt{7}\,i$ are all complex numbers.

Art 1.3 : Properties of Absolute values :

(a) If x and y are any two real numbers, prove that

$$|x + y| \leq |x| + |y|$$

Proof : Let $x + y \geqslant 0$. Then

$$|x+y| = x+y \leqslant |x| + |y| \quad [\because x \leqslant |x|,\ y \leqslant |y|]$$

If $x + y < 0$, then

$$|x+y| = -(x+y) = (-x) + (-y) \leq |x| + |y|$$

$$[\because (-x) \leq |-x| = |x|,\ (-y) \leqslant |-y| = |y|]$$

Hence, in any case,

$$|x+y| \leqslant |x| + |y|$$

By repeated application of the above result, we can prove that

$$|x \pm y \pm z + \dots\dots| \leq |x| + |y| + |z| + \dots\dots$$

Ex (i) $|-5+7| = |2| = 2$; $|-5| = 5$, $|7| = 7$; $5+7=12$.

Now $2<12 \Rightarrow |-5+7| < |-5| + |7|$.

(ii) $|-5-7| = |-12| = 12$;

$|-5| + |-7| = 5+7 = 12$

$\therefore |-5-7| = 12 = |-5| + |-7|$.

(b) **Prove that**

$|x-y| \geq |x| - |y|$

Proof : We have

$|x| = |(x-y)+y| \leq |x-y| + |y|$ [*by (a)*]

or $|x| - |y| \leq |x-y|$

$\Rightarrow |x-y| \geq |x| - |y|$

Ex. (i) $|5-7| = 2$; $|5| - |7| = 5-7 = -2$

$\therefore |5-7| > |5| - |7|$ $(\because 2 > -2)$

(ii) $|7-5| = 2$; $|7| - |5| = 7-5 = 2$

$\therefore |7-5| = |7| - |5| = 2$.

(iii) Prove that $|xy| = |x| \, |y|$.

Proof : Let x, y be both positive. Then $xy > 0$.

So $|xy| = xy = |x| \, |y|$ $(\because |x| = x$, when $x>0$,

$|y| = y$, when $y>0$.)

If x, y are both negative, then $xy > 0$ and

So $|xy| = xy = (-x)(-y) = |x| \, |y|$.

If $x>0$, $y<0$, then $xy<0$ and

$|xy| = -xy = x(-y) = |x| \, |y|$

Simillarly, when $x<0$ and $y>0$

$|xy| = -xy = (-x)(y) = |x| \, |y|$.

Hence in any case, $|xy| = |x| \, |y|$.

Art. 1·4. Meaning of $|x-\alpha| < \delta$.

(a) Let $x-\alpha>0$.

Then $|x-\alpha| = x-\alpha$.

$\therefore |x-\alpha| < \delta \Rightarrow x-\alpha < \delta$, or $x < \alpha + \delta \cdots (i)$

(ii) If $x-\alpha<0$, *then*

$|x-\alpha| < \delta \Rightarrow -(x-\alpha) < \delta$

or $-x+\alpha<\delta$

or $-x < -\alpha+\delta$ *or* $x > \alpha-\delta \cdots (ii)$

Hence form (i) and (ii), We have,

$$\left.\begin{array}{c}\alpha-\delta < x < \alpha+\delta \\ \text{if } |x-\alpha| < \delta\end{array}\right\}$$

Cor. $\left.\begin{array}{l}|x-\alpha| \leq \delta \\ \Rightarrow \alpha-\delta \leq x \leq \alpha+\delta\end{array}\right\}$

While $|x| \leq \delta \Rightarrow -\delta \leq x \leq \delta$.

Ex. Give the equivalent of $|x+1| \leqq 2$.

By removing the absolute notation.

$|x+1| \leqq 2$ is written as $-2 \leqq x+1 \leqq 2$

or : $-2-1 \leqq x+1-1 \leqq 2-1$ or $-3 \leqq x \leqq 1$

Ex. Give the equivalents of the statement $-3 \leqq x \leqq 7$ in the terms of the the absolute notation.

Let the relation be

$|x-\alpha| \leqq \delta$ or $\alpha-\delta \leqq x \leqq \alpha+\delta$

If we compare it with $-3 \leqq x \leqq 7$, then we have

$\alpha-\delta = -3$ and $\alpha+\delta = 7$.

Solving these $\alpha = 2$ $\delta = 5$

Hence the expression $-3 \leqq x \leqq 7$ is equivalent to $|x-2| \leqq 5$.

1. 5. Draw the graph of $y = |x|$

$y = |x|$ means

$y = x$ if $x \geqslant 0$ and $y = -x$ if $x<0$

So, we are to draw two graphs for the two equations. Let us restrict our attention to the graph corress ponding to the interval $-5 \leqq x \leqq 5$.

x	−5	−3	0	3	5
y	5	3	0	3	5

The graph of $y = |x|$ is given by the Fig. (1) The graph of $y = -|x|$ is given in Fig. 2.

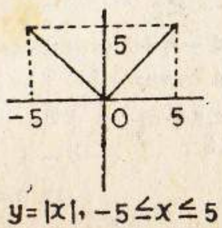


Fig. (1)

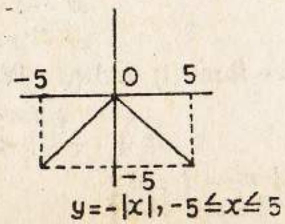


Fig. (2)

Ex. 1 Draw the graph of $y=\frac{1}{2}(x+|x|)$

Let us restrict our attention to the graph corressponding to the interval $-5\leq x\leq 5$.

For $x\geq 0$, $|x|=x$ and so $y=\frac{1}{2}(x+x)=x$ (1)

But for $x<0$, $|x|=-x$, and $y=\frac{1}{2}(x-x)=0$ (2)

This can be written as

$$y=\begin{cases} x, & x\geq 0 \\ 0, & x<0 \end{cases}$$

The graph of $y=\frac{1}{2}(x+|x|)$ for $-5\leq x\leq 5$ is shown in fig (3).

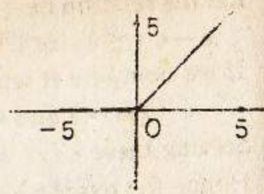


Fig. (3)

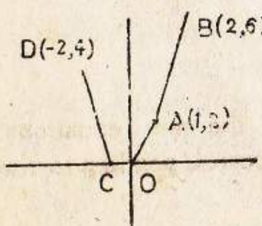


Fig. (4)

Ex 2 (a) Draw the graph of $y=2|x+1|+|x|+|x-1|-3$ in the interval $-2\leq x\leq 2$

For $x\geq 1$,

$2|x+1|=2x+2$, $|x|=x$, $|x-1|=x-1$,

$y=2x+2+x+x-1-3=4x-2$

or, $y=4x-2$ represents segment AB in Fig. (4)

For $0<x<1$,

$y=2|x+1|+|x|+|x-1|-3$

$=2x+2+x-(x-1)-3=2x$ or $y=2x$ represents OA Fig (2)

For $-1\leq x\leq 0$,

$y=2|x+1|+|x|+|x-1|-3=2x+2-x-(x-1)-3$
$=0$

or, $y=0$ reprsents OC (3)

For $x<-1$,

$y=2|x+1|+|x|+|x-1|-3$

$=-2(x+1)-x-(x-1)-3=-4x-4$

or, $y=-4x-4$ represents CD (4)

We draw four graphs for four equations (1)–(4). The four graphs are all straight lines. The combined graph is continuous and is shown in fig. (4).

Ex2 (c) show that Ex2 (b) may be expressed as

$$y\begin{cases} 4x-2, & x\geq 1 \\ 2x, & 0<x<1 \\ 0, & 1\leq x\leq 0 \\ -4x-4, & x<-1 \end{cases}$$

Art. 1.6. Graphs of Inequalities (অসমতার লেখচিত্র)

Let us consider the following cases involving absolute values,

For the equation $|x|=1$

x has only two solutions viz, $x=+1$ and $x=-1$

What will happen to $|x|\leq 1$

In this inequality x has solutions in the entire interval $-1\leq x\leq 1$.

For the equation $|x-3|=5$,

x has only two solutions x such as $x-3=5$ and $x-3=-5$ or $x=8$ and $x=-2$

Ex. Solve the inequality $|x|+|y|\geq 1$... ... ...(1)

Let us first consider the values of x and y in the first quadrant so that $x \geqq 0$ and $y \geqq 0$.

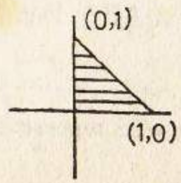


Fig (7)

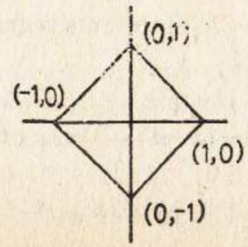


Fig (8)

The inequality (1) becomes $x+y \leqq 1$

For $x+y=1$, we get the line segment in the first quadrant, joining the points (1, 0) and (0, 1).

Hence $x+y \leqq 1$, $x \geqslant 0$, $y \geqslant 0$, $|x|+|y| \leqq 1$ is a graph consisting of all points in and on the triangle with vertices (0,0), (1, 0) and (0, 1) [Fig. (7)]

Considering the valus of (x, y) in the other quadrants, we see that the solution of the inequality

$$|x|+|y| \leq 1$$

is the set of all points lying in and on the quadrilateral with vertices at (1, 0), (0, 1), (−1, 0), (0, −1), [Fig. (8))

Exercise 1

1. If $|a-b|<l$, $|b-c|<m$, show that $|a-c|<l+m$.

2. Give the equivalents of the following terms in the absolute notation.

(i) $-7 \lesseqgtr x \lesseqgtr 13$ (ii) $-3<x<7$

Ans. $|x-3| \lesseqgtr 10$ Ans. $|x-2|<5$

(iii) $l-\varepsilon \lesseqgtr x \lesseqgtr l+\varepsilon$ Ans. $|x-\varepsilon| \lesseqgtr l$.

3. Find the equivalents of the following by removing the absolute notations.

(i) $|x-5|<7$ (ii) $|x+2| \lesseqgtr 5$ (iii) $0<|x-2|<3$

Ans. $-2<x<12$ Ans. $-7 \lesseqgtr x \lesseqgtr 3$ Ans. $-1<x<5, x \neq 2$

4. Draw the graphs of the following equations and express them free from modulus system.

(নিম্নলিখিত সমীকরণগুলির লেখচিত্র অঙ্কন কর এবং পরমমান বর্জন করে y এর ফাংশানের আকারে প্রকাশ কর)

(i) $y=\frac{1}{2}(x-|x|)$ in $-5 \lesseqgtr x \lesseqgtr 5$

(ii) $y=-\frac{1}{2}(x+|x|)$ in $-3 \lesseqgtr x \lesseqgtr 3$

(iii) $y=\frac{1}{2}(|x|-x)$ in $-5 \leqq x \leqslant 5$

(iv) $y=2|(x-1)|-|x|+2|x+1|-5$ in $-3 \lesseqgtr x \lesseqgtr 3$.

5. Draw the graphs of the following inequalities. (অসমতার)

(i) $|x|-|y| \geqq 1$ for $-2 \lesseqgtr x \lesseqgtr 2$

(ii) $|x|+2|y| \leqslant 1$

(iii) $2x+y \lesseqgtr 5$

$x-y \lesseqgtr 1$ for $-3 \lesseqgtr x \lesseqgtr 3$

$x+2y \lesseqgtr 7$.

5 (iv) Show that areas shown in the figure are represented by the following equation.

(a) $x \lesseqgtr 0, y \lesseqgtr 0, x+y \lesseqgtr 1$

(b) $x \leqq 0, y \geqslant 0, x+y \geqq 1$

(c) $x \geqq 0, y \geqslant 0, x+y \geqq 1$

(d) $x \geqslant 0, y \leqq 0, x+y \lesseqgtr 1$ Ans. (i)

(e) $x \geqq 0, y \lesseqgtr 0, x+y \lesseqgtr 1$

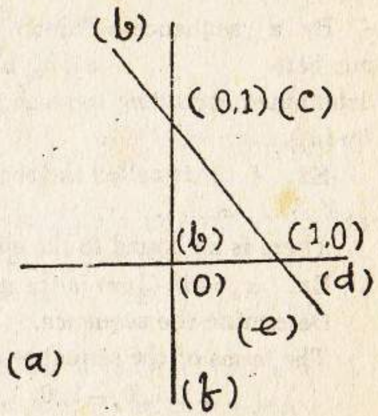
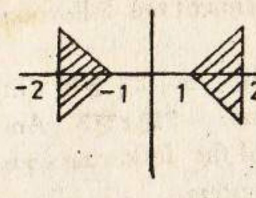
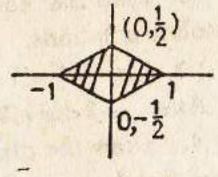


Fig. (9)

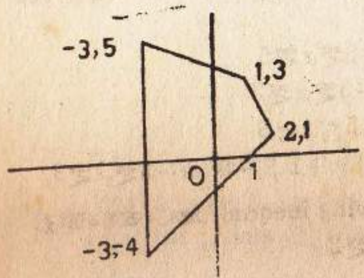
Ans. 5 (i)



5 (i)



5 (ii)



5 (iii)

6. If $y = |x| + |x-1|$, then show that

$$y = \begin{cases} 1-2x, & \text{for } x \leqslant 0 \\ 1, & \text{for } 0 < x < 1 \\ 2x-1 & \text{for } x \leqslant 1 \end{cases}$$

Sequence (ক্রম)

1.7. Definition : *A sequence is a set of numbers arranged in order so that there is a first one, a second one, third one and so on.*

By a sequence we mean an ordered infinite succession of numbers $a_1, a_2, a_3 \ldots \ldots a_m, \ldots \ldots$ determined according to some rule. Symbolically it is represented by $\{a_n\}$

Ex. $\{n\}$ is called the sequence of all positive integers *i, e* ; 1, 2, 3,……*n*……

There is no bound to the size of the numbers ; $\{n\}$ is endless.

Ex. $x_n = \sin(\frac{1}{2}n\pi)$ is the general term of a sequence.

Determine the sequence.

The terms of the sequence are as follows.

$$\ldots\ldots 1, 0, -1, 0, 1, 0, -1, \ldots \ldots \ldots$$

i. e. : the numbers are repeated in pairs with signs changed.

1. 8. Constant (ধ্রুবক) :—A constant is a quantity which remains unchanged during any mathematical operations or any investigation.

There are two types of constants.

(*i*) *Arbitrary constants*

(*ii*) *Absolute constants*

Thus quantities which have the same value under all circumstences are called **absolute constants.**

e. g, 1, 5, 9, 10, π, e

Again quantities which have the same value under one investigation but are different for different investigation are called **arbitrary constant**, *e. g.* in the equation of the straight line.

$$x \cos \alpha + y \sin \alpha = p$$

α and p are same for the same straight line. but they will be different for different straight lines. So α and p are arbitrary constants.

Art. 1.9 Variable (চলক) : A changing entity is called a variable. Generally we use the letters $x, y, z, u, t, \alpha, \beta$, etc for variables. There are two types of variables.

(*i*) *Independent variables* (স্বাধীন চলক)

(*ii*) *Dependent variables* (অধীন চলক)

1.10 Independent variable : A variable which may take any arbitrary value assigned to it is called an independent variable.

1.11. Dependent variable : A variable whose value depends on the values of second variable or on the values of a system of variables is called a dependent variable.

Ex. (i) $y = \sin x$.

For each value of x, there exists a value of $\sin x$ or y. Here y is the dependent variable and x is the independent variable.

(ii) $u=xyz$.

For every set of values x, y, z, there is a definite value of u. Hence in this case u is the dependent variable and x, y, z are independent variables.

1.12 Domain or Interval of a variable.

The set of all permissible values of a variable x is called the domain of the variable x.

Suppose the variable x assumes all values between two given numbers a and b including the values a and b (with $b>a$). Then the domain of x is called an **closed domain** or **closed interval** denoted by

$$a \le x \le b \quad \text{or} \quad x \in [a, b].$$

If one of the end points, say a, is not included in the interval $[a, b]$, then we say that the domain of x is open at the left and closed at the right ; we denote it by $a < x \le b$ or $x \in (a, b]$. Similarly, when $a \le x < b$, we write $x \in [a, b.)$

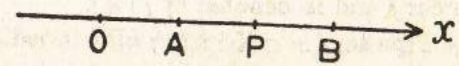
If both a and b are excluded from the interval $[a, b]$, then the domain or interval is called an **open domain** or **open interval.** It is dedoted by

$$a<x<b, \quad \text{or} \quad x \in (a, b)'$$

We can respresent an interval geometrically on the real line.

Let A, B represent the given numbers a, b respectively on the real line OX and let the point P representing the variable x can take any position on the line segment AB.

The interval $[a, b]$ is represented by the whole segment AB and P can take any position from the end A to the end B ; in the intervals $(a, b]$, $[a, b)$ and (a, b) the point A, the point B both of the points A and B are respectively excluded from the segment AB. The length of each of thise intervals is



Neighbourhood of a point : Let δ be an infinitesimally small orbitrary positive number. Then for a given number a the interval $(a-\delta, a+\delta)$ is called a *neighbourhood* of a for the variable x. If we say that 'x tends to a' denoted by '$x \to a$', we mean that *x is in the neighbourhood of a with $x \neq a$.*

Art. 2. : Function : Suppose that we have two non-empty sets X, Y and a rule f establishing a correspondence between the members of X and Y. If the rule f is such that it assigns to each element $x \in X$ a unique element $y \in Y$, then f is called a **function.** This is denoted by $f: X \to Y$ and read as 'f is a function of X to Y' or f is a mapping from X to Y'. We also express this as

$$f: x \to y \text{ or } y=f(x).$$

2.1. Function defined as sets of ordered pairs.

Let X and Y be two nonempty ordered sets. A subset f of $X \times Y$ is called a function from X to Y if and only if to each $x \in X$, there exists a unique y in Y such that $(x,y) \in f$.

i. e : (i) $\mathbf{x \in X, (x, y) \in f}$. **for some** $\mathbf{y \in Y}$

(ii) $(\mathbf{x}, \mathbf{y}_1) \in \mathbf{f}$ **and** $(\mathbf{x}, \mathbf{y}_2) \in \mathbf{f}$ $\to$**(implies that),** $y_1 = y_2$

The first condition exerts that a rule f which gives a image for each element of X and the second condition states the existence of unique image.

The graph of f is the subset of $X \times Y$ defined by $\{x, f(x) : x \in X\}$. The range of f is the set of all images under f and is denoted by $f[X]$

$\therefore f[X]=\{y \in Y : y=f(x) \text{ for some } x \in X\}$
$=\{f(x) ; x \in X\}$

Similarly if $A \subset X$, then the set $\{f(x) ; x \in A\}$ is called the image of A under f and is denoted by $f[A]$.

If $B \subset Y$, then the set $\{x \in X : f(x) \in B\}$ is called the inverse image of B under f and is denoted by $f^{-1}[B]$.

Art. 3. Equivalence relation : A relation R in a set is an equivalence relation if

(i) R is reflexive i. e ; for every $a \in A$, $(a, a) \in R$

(ii) R is symmetric i. e ; for every $(a, b) \in R$ implies $(b, a) \in R$

(iii) R is transitive i. e ; for every $(a, b) \in R$. and $(b, c) \in R$ implies that $(a, c) \in R$

Art. 4 Equivalence set : Set A is equivalent to set B represented by $A \sim B$ if there exists a fnnction $f : A \to B$ whieh is one—one and onto.

Art. 5. Types of functions :—

Let A and B be two sets and $f : A \to B$.

(i) f is one-to-one if $(x_1, y) \in f$ and $(x_2, y) \in f$, then $x_1 = x_2$. We express it by $f : A \xrightarrow[1-1]{} B$, (fig-1)

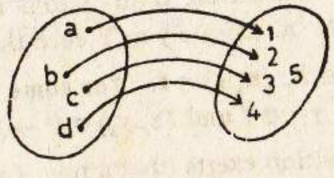


Fig. 1

One-one functinn from A **into** B

(2) If f is not one-one, f is called many-to-one.

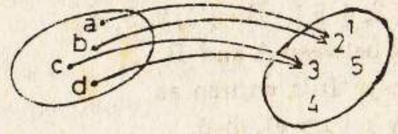
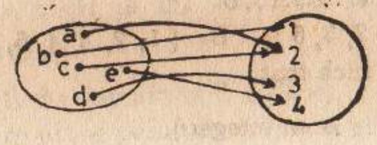


Fig. 2

A many-to-one function from **A into B** (Fig-2)

(3) f is said to be a function from **A onto B** if $R(f)=B$ i.e. if the range of f contains all the elements of B (fig-3)



$f : A \xrightarrow{\text{onto}} B$

Fig. 3

A function from **A onto B**

(4) f is one-one and onto B, then f is said to be a one-one correspondence between A and B (fig-4)

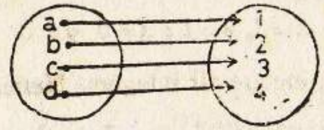


Fig. 4

Relation : Any nonempty subset R of a cartesian product $A \times B$ is called **a relation** from A to B. If $(x, y) \in R$, it is often written as $_xR_y$ read. "x is related to y."

In general a relation R from A to B, between two sets A and B is a subset of $A \times B$ i e., $R \subset A \times B$.

A relation from A to A i. e : a given subset of A × A, is called **a relation on A.**

Ex. 1 Let A={ 3, 4, 5 }, B={ 1, 2, 3, 4 }

Then a relation R between A and B.

exists in such that x>y. It is written as

Define R={ (x, y) | x>y}, then.

R={(3, 1,) (3, 2), (4, 3), (4, 1)}......(1)

Here each Ist element for x is greater than the 2nd element.

It is the relation R. But $A \times B$={ 3, 4, 5 } × { 1, 2, 3, 4 }

={(3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4), (5, 1), (5, 2), (5, 3), (5, 4)} ... ... (2)

From (1) and (2), we see that R is the subset of $A \times B$ *i e.* $R \subset A \times B$.

Ex. 2. Let A={ 1, 2, 3, 5 }, B={ 1, 2, 3, 4, 5, 6 }

Then a relation R is such that

$R = \{ x, y) \mid \frac{3x+1}{y+1}$ is an integer }.

Select the values of x from A

and y from B such that (3x+1) / (y+1)

is an integer. If x=1, y=1, then

(3x+1) / (y+1)=(3.1+1) / (1+1)=4/2=2 is an integer.

so the pair is (1, 1)

For x=3, y=4 ; x=3, y=1 : x=5, y=3

$\frac{3x+1}{y+1}$=2, 5, 4, etc are all integers. Hence

R={(1, 1), (3, 4), (3, 1), (5, 3}. It is the subset of $A \times B$ i. e, $R \subset A \times B$

Ex. 3. Let A={2, 3, 4}

such that R={(x, y) $\in A \times A \mid x+2y=10$ }

$A \times A$={ 2, 3, 4} × { 2, 3, 4}

$x=2, y=4$, ; x=4, $y=3$

2+2.4=10 ; 4+2.3=10.

∴ R={(2,4), (4, 3)}. $\subset A \times A$.

Domain and Range of f**:** For $f: X \to Y$, the permissible values which x can take from the set x is called the **domain** of f, while the set of corresponding values of $y \in Y$ is called the **range** of f. We will denote the domain and the range of f by D_f and R_f respectively.

Domian and Range of a Relation.

If A and B be sets and R is a relation from A to B, the **domain of** R is the set of all first elements (or first co-ordinates) of the pairs (x, y). which belongs to R.

The Range of R is the set of all second co-ordinates of of the pair (x, y).

From Ex. 1, the domain of R is {3, 4}, if an element is present more than once in the pair, take only one element for its domain The Range is {1, 2, 3}

From Ex. 2, the domain of R is { 1, 3, 5 } and the range is {1, 4, 3}

From Ex. 3. ; the domain of R is {2, 4}

and the range of is {4, 3}

Art. 6. Inverse function.

Let $f: A \to B$ be any function. Then $f^{-1}(B) = A$, since every element in A has its image in B. If $f(A)$ denotes the range of f, then $f^{-1}(f(A)) = A$

If $b \in B$, then $f^{-1}(b) = f^{-1}(\{b\})$

Here f^{-1} has two meaning as the inverse of an element of B and as the inverse of a subset of B-

2—

Def.: Let f be a function A into B and let $b \in B$.
If $f: A \to B$, then
$f^{-1}(b) = \{x \mid x \in A, (x) = b\}$

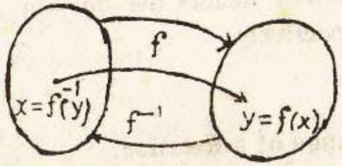


Fig. 5

Inverse of a Relation:

If R be a relation from a set A to another set B, then the inverse relation of R denoted R^{-1} is defined as the inverse relation i. e, R^{-1} from B to A if and only if

$$R^{-1} = \{(x), x\} : (x, y) \in R\}$$

It is clear that domain of R^{-1} is equal to the range of R and range of R^{-1} is equal to the domain of R

i. e; $D(R^{-1}) = R(R)$
$R(R^{-1}) = D(R)$

Ex. If $A = \{a, b, c,\}$, $B = \{1, 2\}$

$R = \{(a, 1), (a, 1), (b, 2) (b, 2\}$ is a relation from A to B. so that $R^{-1} = \{(1, a), (1, a), (2, b), (2, b)\}$

will be a relation from B to A i. e, R^{-1} is the inverse relation on. R.

Difference between Function and Relation:

Any non-empty subset R of a cartesian product $A \times B$ is called a relation from A to B.

If $(x, y) \in R$, then Ex 1, Ex 2, Ex 3 are all the examples of relations. There is no restriction on the elements of Domain of Relation. Same element of A may occur any number of times as first element in (x, y). But in Function it cannot be. A relation is then a function if all elements of the set A are the fi

elements of (x, y) without Repeatation. *i. e.* **a function is a relation but a relation may not be a function.**

A relation from A to B is a function if

(*i*) Domain $= A$. (*ii*) $(x, y) \in R$, and $(x, z) \in R$, then $y = z$.

or, it may be stated that 'A function from A to B is a relation which associates each element of A with one and only one element of B''

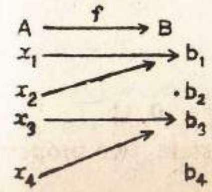


fig (6)
It is a function

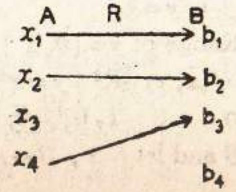


fig (7)
It is a relation as x_3 has no relation in B.

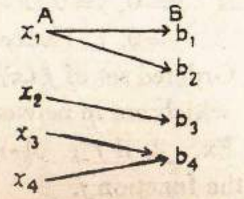


fig (8)
It is not a function as x_1 has images in B.

Different types of function will be discussed in the next Article.

Types of Relations

1. **Reflexive:** A relation R on a set A is known as reflexive if and only if each member of A is R related to itself.
$(x, x) \in R$ for each $x \in A$.

or, ${}_xR_x$ for each $x \in A$.

2. **Symmetric:** A relation R on a set A is known as Symmetric if $(x, y) \in R \to (y, x) \in R$

or, ${}_xR_y \to {}_yR_x$; $x, y \in A$.

If is also known as if $R^{-1} = R$, then a relation R is symmetric.

3. **Transitive:** A relation R on a set A is known as transitive if and only if $(x, y) \in R$ and $(y, z) \in R \to (x, z) \in R$

or, ${}_xR_y$ and ${}_yR_z \to {}_xR_z$, $x, y, z \in A$.

Ex. 1. If $f(x)$ is a function whose domain is the interval the set of all x, $-1 \leqq x \leqq 1$ and a rule of f is given by the equation $f(x) = x^2$, what set of ordered pairs is f?

Let $y = f(x)$ $\therefore$ $-1 \leqq x \leqq 1$

Domain of x is $[-1, 1]$

Range of $y = f(x)$ is obtained from the rule $f(x) = x^2$

or, $y = x^2$, for all value of x in $[-1, 1] \to [-1, 0] \cup [0, 1]$

If $x = 0, y = 0, x = \pm 1, y = 1$

$\therefore$ $y = 0, 1$. Hence Range of $y = [0, 1]$

Ordered set of $f(x) = \{(x, y) \in R^2 \mid y = x^2\}$

which are in between $= \{[-1, 0] \cup [0, 1]\} \times \{0, 1\}$

Ex. 2. If f : $A \to B$ and let f^{-1} ; $B \to A$. state two properties of the function f.

Ans. The function f must be both one—one and onto.

Ex. 3. Let the function $f = \{(x, y) \in R \mid y = x^2\}$

Find $f^{-1}(36), f^{-1}(-16), f^{-1}([-1, 1]), f[(-\infty, 0)], f^{-1}(9, 36)]$

$f^{-1}([-\infty, 0]) = \{x \mid -\infty \leqq f(x) \leqq 0\} = \{x \mid -\infty < x^2 \leqq 0\}$

$= \{0\}$ since no other number squared belongs to $[-\infty, 0]$

$f^{-1}([9, 36]) = \{x \mid 9 \leqq f(x) \leqq 36\} = \{x \mid 9 \mid \leqq x^2 \leqq 36\}$

$= \{x \mid 3 \leqq x \leqq 6, -6 \leqq x \leqq -3\} = [3, 6] \cup [-6, -3]$

$f^{-1}(-16) = \{x \mid f(x) = -16\} = \{x \mid x^2 = -16\} = \emptyset$ as there is no number whose squared is negative.

$f^{-1}(36) = \{x \mid f(x) = 36\} = \{x \mid x^2 = 36\}$

$= \{x \mid x = \pm 6\}$

$= \{6, -6\}$ only two values of x.

$f(x) = x^2$

$f^{-1}([-1, 1])$

$= \{x \mid -1 \leqq f(x) \leqq 1\} = \{x \mid -1 \leqq x^2 \leqq 1\}$

$= \{x \mid x^2 \leqq 1\} = \{x \mid x \mid \leqq 1\}$

$= [-1, 1]$

Ex. 4. If $f = \{(x, y) \in R \times R \mid y = 2x - 3\}$. Is it a function? Has its inverse? Ans. yes.

Ex. 5. Let $A = \{x \mid \in R$ and $\mid x - 1 \mid \leqq 1\}$ and $B = \{y \mid \in R$ and $\mid y \mid \leqq 1\}$.

Describe $A \times B$ without absolute value sign and sketch $A \times B$ in the Cartesian plane. [D. U. 1984]

Ans $A = \{x \mid x \in R$ and $-1 \leqq x - 1 \leqq 1$ or, $0 \leqq x \leqq 2\}$

$B = \{y \mid y \in R$ and $\mid y \mid \leqq 1$ or, $-1 \leqq y \leqq 1\}$

Draw lines $x = 0$ and $x = 2$

$y = -1$ and $y = 1$

The four lines form a rectangle. Thus $A \times B$ represents the area of $PNML$ together with bordersides as Product set and sketch $A \times B$ is the shaded area.

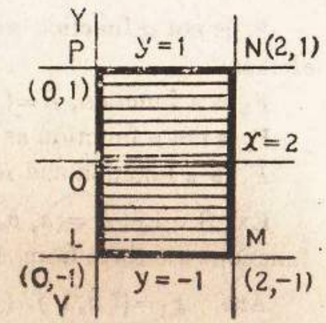


Fig. 9.

Ex. 6. Let $A = \{x \mid x \in R, -3 < x < 3\}$, Find the product set $A \times A$. such that $R = \{(x, y) \in A \times A \mid y = x^2\}$ Is it a function? The table of ordered pairs in $A \times A$ are obtained from $y = x^2$ is

x	0	1	-1	2	-2
y	0	1	1	4	4

Since x lies between 3 and—3, so these values are neglected.

Now the product set $A \times A$

$= [(0, 0), (1, 1), (-1, 1), (2, 4), (-2, 4)\}$

It is a function as no two pairs contain the same first element.

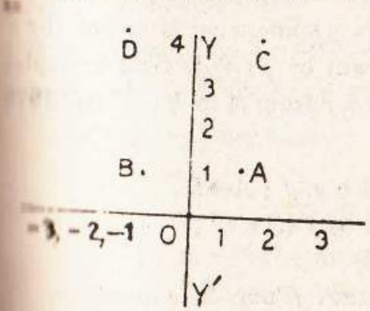


Fig. 10.

Ex. 7. Which are functions ?

Let $X=\{1, 2, 3\}$ and $Y=\{a, b, c, d\}$

If $F_1=(1, a), (2, a), (3,a)\}$, $F_2=\{(1, a), (1, b), (2, c), (3, d)\}$
$F_3=\{(1, a), (2, b), (3, d)\}$. $F_4=\{(1, b), (2, a)\}$
$F_5=\{(1, a), (2, b)\ (3, c)\}$ [R. U. 1978]

Solve F_1 is a function, Range, $R=\{a\}$

F_2 is not *a* function as $(1, a), (1, b)$ contain the same first element.

F_3 is a function, $R=\{a, b, d\}$

F_4 is not a function as element 3 of X is not present.

F_5 is a function and $R=\{a, b, c\}$

Ex. 8. Let $A=\{a, b, c\}$ and $B=\{1, 0\}$. How many different functions are there from A to B and what are they ? R. U. 1979

Ans. $F_1=\{(a,1), (b,1).(c,1)\}$, $F_4(a, 0), (b, 0), (c, 0)\}$,
$F_2=\{(a, 1), (b, 1)\}, (c, 0)\}$, $F_5=\{(a, 1), (b, 0), (c, 0)\}$
$F_4=\{a, 0), (b, 1), (c, 1)\}$, $F_6=\{(a, 1), (b, 0), (c, 0)\}$
$F_3=\{(a, 1), (b, 0), (c, 1)\}$, $F_7=\{(a, 0), (b, 0), (c, 1)\}$

Ranges $R_1=\{1\}$, $R_2=\{0\}$, $R_3=\{1, 0\}$, $R_4=\{0, 1\}$, $R_5=\{0\ 1\}$,
$R_6=\{0, 1\}$, $R_7=\{0, 1\}$

From the ranges, F_1, F_2 are constant functions and the remaining are onto functions.

Ex. 9. Let A and B be two sets. Define what is meant by a function f from A to B. What is meant by $f(A)$ }. Give examples of two sets A and B and of a function f from A to B. R. U. 1975

Solve. Fuuctions :—

Let f be a functions of A into B *i, e.* $f: A \to B$.

The renge of : $A \to B$ is denoted by $f(A)$

If $A=\{a, b, c, d\}$, $B=\{1, 2, 3\}$

The range of $f(A)$ of the function f may be a subset of B *i e*, $f(A) \subset B$ or, $f(A)=B$. Ranges of A

are $f(A)=\{1\}$, $f(A)=\{1, 2\}$, $f(A)=\{1, 2, 3\}$

Domain of f in each range $f(A)$ is $D=\{a, b, c, d\}$

So all of them are functions f from A to B.

Ex. 10. Give an example in each of the following function f.

i) f is many—one into
(ii) f is many—one onto
(iii) f is one—one into
(iv) f is one—one onto
(v) f is neither one—one nor onto

Ans. R is the set of real numbers.

$f=\{(x, y) \in R \times R,\ x \in R,\ y \in R \mid y=x^2\}$
Range of $f=R^+$, Domain of $f=R$

f—image is the subset of its domain

i. e., $\{f(x)\} \subset R$, $x \in R$

f is a mapping of R into R

Again $a, b \in R$,

$a \neq b \to f(a) \neq f(b)$ i, e, $a^2 \neq b^2$

f is many—one into.

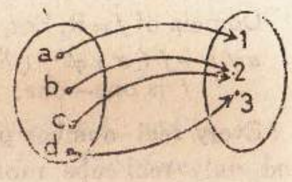


Fig. 11

(ii) Let $f: A \to B$ where $A=\{a, b, c, d\}$. $B=\{1, 2, 3\}$

$\therefore$ $f(a)=1$, $f(b)=1$, $2=f(c)$, $f(d)=3$.

Since element a, b have the same image 1 of B, hence the mapping is many—one. Again there is no element in B which is not an image of A. Thus f is onto

$\therefore$ f is many—one onto

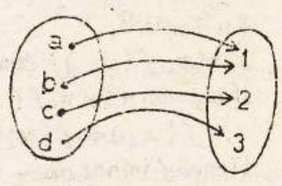


Fig. 12

(iii) $f\{(x, y) \in I \times I,\ x \in I,\ y \in I \mid f(x)=x^2\}$

I is set of $+ve$ integers.

Domain of $f=\{1, 2, 3, \dots\dots\}$,
Range of $f=\{1, 4, 9, \dots\dots\}$,

f image is the subset of its domain i. e, $\{ f(x) \subset I, x \in I \}$

Thus f is into-mapping since all elements of the range are not the image of Domain such as 4 of range is not the f image of any element of the domain.

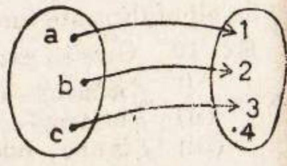


Fig. 13

For, $a, b \in I$, $a \neq b \rightarrow f(a) \neq f(b)$ $i, e, a^2 \neq b^2$

It is one—one.

Hence f is one—one and into

(iv) f $(x, y) \in R \times R, x \in R, y \in R \mid f(x) = x^3\}$

Domain of $f = R$, For, $a, b, \in R$

$a \neq b \rightarrow f(a) \neq f(b) \rightarrow a^3 \neq b^3$

f is one—one.

Every real number possesses one and only real cube root, all the elements in the range set R are the f-image of any element in the domain set R. Thus f is onto f : R→R.

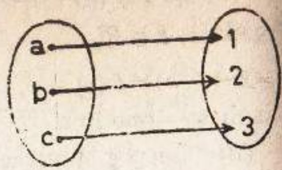


Fig. 14

∴ f is one—one and onto

(v) $f: \{(x, y) \in R \mid f(x) = \cos x\}$

$x_1, x_2, \in R$,

$x_1 \neq x_2, f(x_1) = \cos x_1, f(x_2) = \cos x_2$

If $x_2 = x_1 + 2n\pi, f(x_1 + n\pi) = \cos(x_1 + 2n\pi) = \cos x_1$

$\therefore f(x_1) = f(x_2)$

Hence f is not one—one

Again any element $x \in R$, say $f(x) = 2$

$f(x) = 2 \neq \cos x$, since $|\cos x| \leq 1$

Thus all the element of R in the range are not the f image of elements of the Domain so f is not onto

Hence f is neither one—one nor onto

Ex. 11. In each of the following cases, decide whether the given relation f is a function. If f is a function, determine its domain and range.

Which are onto and which are one—one

(i) $f = \{(x, y) \in R \times R \mid x^2 + y^2 = 1\}$ Ans. It is not a function.

(ii) $f = \{(x, y) \in R^2 : x - y = 2\}$ Ans 1—1 and onto

(iii) $f = (x, y) \in R^2 \mid y = 1\}$ Ans a function but not 1—1 and onto

(iv) $f = \{(x, y) \in A \times B \mid y = \sqrt{(1 - x^2)}$

$A = \{x \mid x$ real number $-1 \leq x \leq 1$ and B denotes all real numbers.

Ex. 12. Prove that the identity relation I_A is a function, hence called the identity function.

Proof : I_A is a relation by definition.

For each $x \in A, (x\ x) \in I_A$

So, domain of $I_A = A$. Finally if $(x, y) \in I_A$, and $(x, z) \in I_A$, then $x = y$ and $x = z$ by definition of I_A. Therefore, $y = z$ and I_A is a function.

Ex. 13. Show that f: $\{(x, y) \in R \times R \mid y = 2x\}$ is an 1—1 function but not onto function, R is the set integers.

Sol. : $x \in R, y \in R, (x, y) \in f$. Hence $D(f) = R$.

i. e. domain of f is R.

If $(x, y) \in f, (x, z) \in f$, then $y = 2x, z = 2x$, i. e, $y = z$.

Hence f is a function.

f is not onto R since $y = 3 \in R$ but $3 \neq 2x$ as no integer x such that $3 = 2x$.

Function of many (more than one) variables

Let $(x_1, x_2 \ldots\ldots, x_n)$ be an ordered n—tuple of real mumbers belonging to the set E_n and y ∈R. If under a rule f, there exists a unique value of y corresponding to each n— tuple $(x_1, x_2 \cdot x_n)$ belonging to the set X_n, where $X_n \subset E_n$, then f is a function of n independend variables $x_1, x_2, \ldots\ldots, x_n$; we express this as,

$f: (x_1, x_2, \ldots \ldots, x_n) \rightarrow y$ or $y = f(x_1, x_2, \ldots, x_n)$.

The set X_n is the domain of f and the set y consisting of the corresponding values of y obtained by the rule f is the range of f.

Ex 13. Let $f : (x, y) \to \sqrt{x^2+y^2-1}$

or, $z = f(x, y) = \sqrt{x^2+y^2-1}$; $x \in \mathbb{R}$, $y \in \mathbb{R}$, $z \in \mathbb{R}$.

Ans. Since z is real, $z^2 \geqslant 0$ or $x^2+y^2-1 \geqslant 0$ or $x^2+y^2 \geqslant 1$.

Thus z is real for all points (x, y) lying on or outside the circle $x^2+y^2=1$ in the xy-plane. So

$D_f : \{(x, y) : x^2+y^2 \geqslant 1\}$
For $= (x, y) \in D_f$, $z \geqslant 0$.
Hence R_f = range of $f = \{ z : z \in R^+\} = [0, \infty)$.

Inverse function : If a function f given by $y = f(x)$ is such that for any two elements x_1 and x_2 belonging to the domain of f, $f(x_1) \neq f(x_2)$ when $x_1 \neq x_2$,

Then there exists a unique function f^{-1} called the inverse of f with the properties

$f^{-1}(y) = x$ or $f^{-1} f(x) = x$

and $f f^{-1}(y) = f(x) = y \Rightarrow f f^{-1}(x) = x$ (interchanging the roles of x and y)

Note : For the function $y = f(x)$ to have its inverse f^{-1}, there exists a one-to-one correspodence between x and y under the rule f. We sometimes express this by saying that f is a **Single-valued function** of x.

A function f does not have an inverse, if there exist elements $x_1, x_2 \dots \dots$ belonging to Df such that

$f(x_1) = f(x_2) = \dots \dots$
even when $x_1 \neq x_2 \neq \dots \dots$

In such a case $f(x)$ is termed as a **many-valued function.**

It is clear from the definition that
$Df^{-1} = Rf$ and $Rf^{-1} = D_f$

Ex 14 Find the inverse of the function
$y = f(x) = 2x+3$,

Ans. Df = Domain of $f = \mathbb{R} = (-\infty, \infty)$
Rf = range of $f = \mathbb{R} = (-\infty, \infty)$

If $x_1, x_2 \in Df$ and $x_1 \neq x_2$, then
$2x_1+3 \neq 2x_2+3$ in $f(x_1) \neq f(x_2)$.

Hence f^{-1}, the inverse of f exists.

Solving for x, we get

$2x = y-3$ or $x = \dfrac{y-3}{2}$

Hence $f^{-1}(x) = \dfrac{x-3}{2}$.

It can be seen that

$$f f^{-1}(x) = f\left(\frac{x-3}{2}\right) = 2\left(\frac{x-3}{2}\right) + 3 = x,$$

also $f^{-1} f(x) = f^{-1}(2x+3) = \dfrac{(2x+3)-3}{2} = x$.

$D_f^{-1} = Rf = (-\infty, \infty)$, $R_f^{-1} = D_f = (-\infty, \infty)$.

Ex. 15. Does f^{-1} exist for the function $y = f(x) = x^2, x \in \mathbb{R}$.
We have, $D_f = (-\infty, \infty)$.

Let us take two numbers x_1 and x_2, such that $x_2 = -x_1$.
Then $x_1 \neq x_2$, but
$f(x_2) = x_2^2 = (-x_1)^2 = x_1^2 = f(x_1)$.

Hence the function $y = x^2$, does not have its inverse.

Inverse of trigonometric functions : These one periodic circular functions. So their inverses can be defined only on their principal parts.

(i) The principal part of $y = f(x) = \sin x$ has domain $D_f = [-\frac{1}{2}\pi, \frac{1}{2}\pi]$ in $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi$
and range $R_f = [-1, 1]$ in $-1 \leqslant y \leqslant 1$
$\therefore f^{-1}(x) = \sin^{-1} x$

has domain $D_f^{-1} = R_f = [-1, 1]$ and $R_f^{-1} = [-\frac{1}{2}\pi, \frac{1}{2}\pi]$

(ii) The principal part of $y = f(x) = \cos x$
has domain $D_f = [0, \pi]$ and range $R_f = [-1, 1]$,
so $f^{-1}(x) = \cos^{-1} x$
has domain $Df^{-1} = Rf = [-1, 1]$ and range $Rf^{-1} = [0, \pi]$

(iii) The Principal part of $y = f(x) = \tan x$
has domain $D_f = (-\frac{1}{2}\pi, \frac{1}{2}\pi)$ i. e ; $-\frac{1}{2}\pi, < x < \frac{1}{2}\pi$
and range $Rf = (\infty, \infty)$.
Hence $f^{-1}(x) = \tan^{-1} x$
has domain $D_f^{-1} = R_f = (-\infty, \infty)$
and range $R_f^{-1} = D_f = (-\frac{1}{2}\pi, \frac{1}{2}\pi)$

Inverse of the exponential function or e^x ;

Since $e^{x_1} \neq e^{x_2}$ when $x_1 \neq x_2$

So e^x has a unique inverse.

Let $y=f(x)=e^x$

Then $x=\log_e y=\ln y$ ($\because$ log is written as l_n)

Hence $f^{-1}(x)=$ inverse of $(e^x)=l_{nx}$.

Similarly, we can show that

the inverse of $(\log_e x)=e^x$, $x>0$.

$\therefore$ $\log(e^x)=x$; $e^{\log x}=x$

Note : (I) : If a function $y=f(x)$ is of the form

$$f(x)=\frac{f_1(x)}{f_2(x)}$$

Then $f(x)$ is not defined for values of x for which $f_2(x)=0$,

Since $\frac{1}{0}$ is undefined.

Again, if for some values of x, $f_1(x)=0$ as well as $f_2(x)=0$, then $f(x)=\frac{0}{0}$ for those values of x.

Now $\frac{0}{0}$ has no definite value ; in such cases, we say that $f(x)=\frac{0}{0}$ **is indeterminate and so undefined.**

(II) : **Reflecetion of a point about the line y = x**

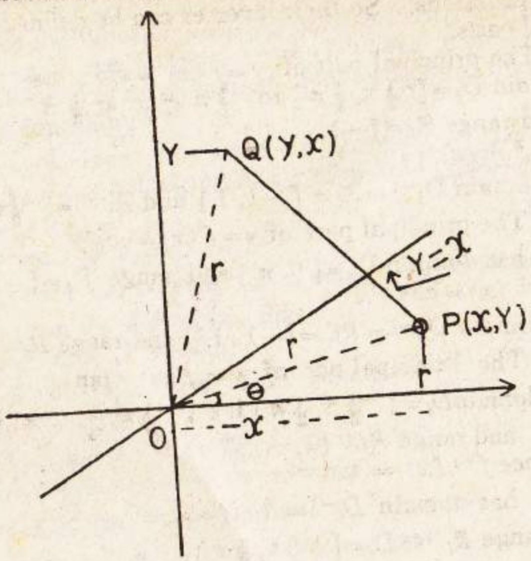


Fig 15

Let P(x, y) be any point on the xy plane. Then its reflection on the line $y=x$ is the point Q(y, x) which is evident from the geometry of the Fig. 15.

If $y=f(x)$, then $x=f^{-1}(y)$, whenever f^{-1} exists. Hence when (x, y) is a point on the graph of $y=f(x)$, (y, x) will be a point on the graph of $y=f^{-1}(x)$. Thus the graph of $y=f^{-1}(x)$ is obtained by reflecting the graph of $y=f(x)$ about the line $y=x$.

Ex. (i) $y=\frac{x^3-a^3}{x-a}$. If $x=a$, then

$y=\frac{0}{0}$ which is indeterminate and undefined.

(ii) $y=(x-a)^2$ is defined for all values of x, while $y=\frac{1}{(x-a)^2}$ is undefined for $x=a$, since

then $y=\frac{1}{0}$

(iii) $y=\frac{\sin x}{x}$. If $x\neq 0$, then y has a definite value but when $x=0$, then $y=0/0$ which is indeterminate and so y is not defined at $x=0$.

Similarly there are many more examples of these types.

1. 13. Classification of Functions (ফাংশানের বিভিন্ন রূপ)

Any given function is either **Algebraic or Transcendental**

(a) **Algebraic Function (বীজগণিতীয় ফাংশান)**

A function is said to be an algebraic functions which consists of a definite number of terms involving only the operations of addition, subtraction, multiplication, division, root extraction and raising to powers of one or more variables

Examples are

$y=x^3+3x^2+x+5$, $y=\frac{2x+1}{3x-1}$

$y=\sqrt{(x+3)}$ and $y=x+5$

There are two types of Algebraical functions. Such as

(i) Polynomial functions
(ii) Rational functions

(a) Polynomials (বহুপদী) : A function of the type

$y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, where a_0, a_1, $a_2 \cdots a_n$ are all constants and n is a positive integer, is called a polynomials in x of degree n.

(ii) Rational (আনুপাতিক) functions

A function which appears as a quotient of two polynomials.

Such as $\dfrac{a_0 + a_1x + \cdots + a_nx^n}{b_0 + b_1x + \ldots + b_nx^m}$ is called a rational function

If P (x) and Q (x) be two polynomials, the ratio $y = \dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ have no common factor, is said to be a rational function.

Irrational (অমেয়) function is an algebrical function of x when it involves root extraction of terms involving x.

e. g. $\sqrt{(x+2)}$, $\sqrt{(x^2+2)}+3$ are the examples of irrational functions.

(b Transcendental (তুরীয়) Fnnctions.

Functions which are not algebraic are said to be transcendental. The following are transcendental functions.

(i) *Trigonometric functions*
(ii) *Inverse Trigonometric functions*
(iii) *Exponential functions*
(iv) *Logarithmic functions*

(i) Trigonometric Functions :—

$y = \sin x$, $y = \cos x$, $y = \tan x$, $y = \cot x$, $y = \sec x$ etc, are all Trigonometric functions of x.

(ii) Inveres (বিপরীত) Trigonometric Functions.

$y = \sin^{-1} x$, $y = \cos^{-1} x$, $y = \tan^{-1} x$, $y = \cot^{-1} x$, $y = \sec^{-1} x$ etc are all inverse trigonometrical functions of x, or, inverse circular functions of x.

(iii) Exponential functions, (সূচকীয় ফাংশন)

$y = 2^x$, 10^x, a^x, x^x, e^x, etc are all exponential functions of x.

(iv) Logarithmic Functions

$y = \log_e x$, $\log_{10} x$, $\log_a x$, $\log\sin x$ etc. are all logarithmic functions of x.

(c) Explicit (ব্যক্ত) Functions : A function which is directly expressed in terms of the independent variable is called an explicit function e. g. $y = x \sin x$, $y = a \cos\theta$, $y = e^{ax} \cos bx$, etc are explicit functions.

(d) Implicit (অব্যক্ত) Function : A function which is not expressed directly in terms of the independent variable is called an implicit function e. g.

$$x^2 + y^2 = a^2,\ ax^2 + 2hxy + by^2 = 0.$$
$$3x^2y + 2xy^2 + 4xy + 5x + 7y + 3 = 0$$

In these examples, y is marked as explicit functions of x.

(e) Periodic (পিরিয়ডযুক্ত) Functions

If $f(x) = f(x+d)$ for all values of x for which the function is defined, $f(x)$ is called a periodic function with the period d, where d is minimum positive change in x for which $f(x) = f(x+d)$ holds.

Trigonometrical functions such as $\sin x$ and $\cos x$ are periodic with 2π as period, while $\tan x$ is periodic with period π

(f) Monotone (মানের ক্রম অনুসারী) function : A function $f(x)$ is said to be monotonic in a given interval $(a, b.)$ such that for any two values x_1 and x_2 $(x_1 < x_2)$ of the variable x in the interval either (i) $f(x_2) \leq f(x_1)$ or, (ii) $f(x_2) \geq f(x_1)$

If $f(x_2) \leq f(x_1)$, then the functions $f(x)$ is continually decreasing and so $f(x)$ is called a **monotonically decreasing function.**

Again if $f(x_2) \geq f(x_1)$, the function $f(x)$ is continually increasing In this case, $f(x)$, is called a monotonically increasing function.

(g) Odd or even Function : (অযুগ্ম ও যুগ্ম)

A function $f(x)$ is said to be **odd** if it changes sign with the change of sign of the variable x. That is $f(x)$ is odd, if

$$f(-x) = -f(x).$$

e. g. $f(x) = \sin x = -\sin(-x) = -f(-x) \therefore f(-x) = -f(x)$

Again let $f(x) = \sin^5 x \cos^2 x$, then $f(-x) = \sin^5(-x) \cos^2(-x)$
$= -\sin^5 x \cos^2 x \quad \therefore \quad f(-x) = -f(x)$

So $\sin x$ and $\sin^5 x \cos^2 x$ are both odd function.

A function $f(x)$ is called an **even** function if it does not change sign with the change of sign of x. If $f(x)$ is even, then

$$f(-x) = f(x).$$

e. g., $f(x) = \cos x = \cos(-x) = f(-x)$ i. e., $f(x) = f(-x)$.

cos x is an even function.

(i) $f(x) = \sin^6 x \tan x$

$f(-x) = \sin^6(-x) \tan(-x) = \sin^6 x \tan x$

$\therefore \quad f(-x) = f(x)$

(ii) $f(x) = ax^4 + bx^2 + c$

$f(-x) = a(-x)^4 + b(-x)^2 + c = ax^4 + bx^2 + c = f(x)$

$\therefore \quad f(-x) = f(x)$ These are the examples of even functions.

(j) Continuous and discontinuous (অবিচ্ছিন্ন ও বিচ্ছিন্ন) functions.

A function of x is said to be continuous in an interval, if it has a definite value for every value of x in the given interval e. g., $y = \sin x$; $\quad y = x^2$

If the function is undefined for any value of the variable in the interval then the function is said to be discontinuous for that value of the variable in the given domain or interval.

e. g; $y = \dfrac{x^2 - a^2}{x - a}$ ....(1), $\quad y = \dfrac{\sin x}{x}$ ....(2)

In the example (1), the function y is not defined for $x = a$ and from the example (2) we see that the function x is not defined for $x = 0$. Hence the functions are discontinuous at $x = a$ and $x = 0$ respectively.

For detail discussion see chapter on the continuity of a function.

1 14. Graphical Representation of Functions (ফাংশানের লেখচিত্র)

If y is a function of x i. e; $y = f(x)$ then we can represent the function graphically using cartesian co-ordinates. Generally we use the independent variable as a abscissa and dependent variable as ordinate. Thus each number pair (x, y), for $x \in D_f$, is a point on the xy plane and the collection of all points so obtained represents the graph of the function. From the graph of the function we can understand the nature of the function, that is how it changes with the change of the variable x.

1. 1'. **Graphs of the function.**

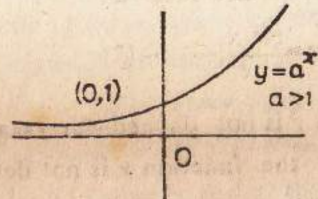
(a) Graph of $y=a^x$

Case 1 When $a>1$ and x is any positive or negative integer, the function $y=a^x$ is always positive.

For positive values of x, y increases indefinitely with the increasing values of x

When $x=0$, $y=a^0=1$, thus (0, 1 is on the curve.

For negative values of x, y will gradually decrease with the increasing values of $|x|$ and ultimately y tends to zero as $|x|$ tends to infinity. i. e. negative x axis is the assmptote of the curve $y=a^x$.



The curve is continuous.

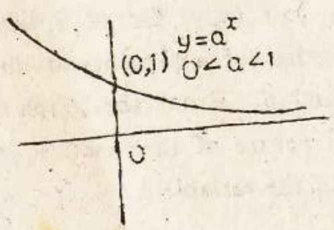
Fig 16

Now the graph of $y=a^x$, $a>1$ is shown in fig. 13

Case 11 When $0<a<1$, x is positive or, negative the function, $y=a^x$ is always positive.

We observe that for x, positive and $x\to\infty$ from the right hand sides of the origin. $y\to0$, i. e., positive x axis is the asymptote of the curve.

When $x=0$, $y=a^0=1$, Thus (0, 1) lies on the curve.



For negative value of x as $a<1$, y will increase indefinitely with the increasing values of $|x|$ *i. e.* y will increase without any limi . Let $a=\frac{1}{2}<1$, $x=-2,-3\ldots-n$ then $=a^x=(\frac{1}{2})^{-2}=2^2$.

Fig 17

$y=2^3,\ldots\ldots y^n=2^n$. If $n\to\infty$ then $y\to\infty$. Hence the case The curve is **monotonically** decreasing.

Now we can draw the graph. see fig. 17.

(b) Draw the graph of $y=e^x$

In this function e is positive and greater than 2. So y is positive for all values of x positive or negative. The graph is similar to that at $y=e^x$ with $a=e>1$ (Fig 16)

(c) Graph of the function $y=\log_a x$, $x>0$, $a>0$.

we can write $y=\log_a x$ as $x=a^y$

Case I When $a>1$, x is positive for all values of y.

x is monotonically increasing with the increase of y, conversely y increases monotonically with $y\to\infty$ as $x\to\infty$.

When $y=0$, then $x=a^0=1$, the point (1, 0) is on the curve. For negative values of y, x is decreasing with the increasing values of $|y|$ *i. e.* $y\to\infty$ as $x\to0$. *i. e.* negative y axis is the asymptote of the curve.

The curve is monotonically increasing from $-\infty$ to $+\infty$ when $a>1$, the graph of the function is shown below in fig. 18

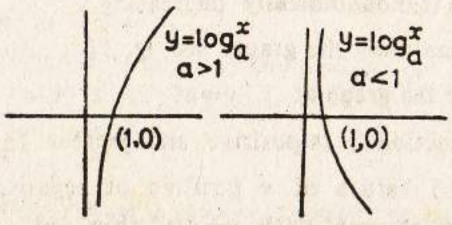
Case II. When $0<a<1$. the function $x=a^y$ is monotonically decreasing, and $x\to0$ as $y\to\infty$

The curve passes through (1, 0)

For $a<1$, x is monotonically increasing when y is negative *i. e.* $x\to\infty$, is $y\to-\infty$

Foot Note : Asymptote—If a straight line cuts a curve in two coincident points at an infinite distance from the origin and yet is not itself wholly at infinity is called an asymptote to the curve. See Chapter on Asymptotes of this book.

The curve is shown in fig. 19



Fig—18 Fig—19

(d) Graph of the Function

$y = \log_{10} x$

Form a table, plot the points in the graph papers and draw the graph through these points.

x	—.5	1	2	3	4	5	10	12
y	—.3	0	0.3	0.5	.6	.7	1	1·07

when x →0, then y tends to—∞ *i e.* negative y axis will be the asymptote of the cu ve. The graph of the function can be easily drawn. The graph is similar to Fig. 18 with $a=10$.

(e) Draw the graph of $y=\log_e x$

Form a table of logarithm to the base *e*. The table of *x* and *y* is

x	.5	1	2	.3	4	5	etc.
y	6	0	.7	1.1	1.4	1.9	ete.

From the tables we see that $y \to -\infty$ when $x \to 0$ i e.. negative y axis will be the asymptote of the curve.

$x \to \infty$ with the increasing value of y, plotting the above points on a graph paper. The graph of the function $y = \log_{10} x$ is similar to Fig. 18 with $a=e>1$

(f) Graphs of $y=x^n$, n being any positive or negative integer.

Case 1. When n is positive even integer. Let $n=2m$

Then $y=x^n=x^{2m}$

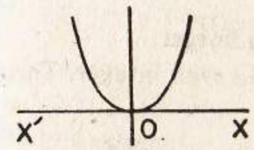


Fig 20

$y=x^{2m}$ is an even function of x. So its graph is symmetrical about the y=axis

Put $x=0$ then $y=0$.

$x=1$. then $y=1$.

$x=-1$, then $y=1$.

thus we see that the curve passes through (0, 0), (1, 1) and (—1, 1); y is always positive whether x is positive or negative.

So, there will be no branch of the curve in the 3rd or 4th quadrants.

Again $y=x^{2m}=(-x)^{2m}$

i. e. the curve is symmetrical about y axis.

y tends to infinity with the increasing values of $|x|$ with $-\infty < x < \infty$.

The graph of the function $y=x^n=x^{2m}$ is shown above (some) particular value of(n).

Case 11 When n is a positive odd integer.

Let $n=2m+1$, then $y=x^n=x^{2m+1}=x.(x)^{2m}$

Put $x=0$. then $y=0$

$x=1$. then $y=1$

$x=-1$. then $y=-1$

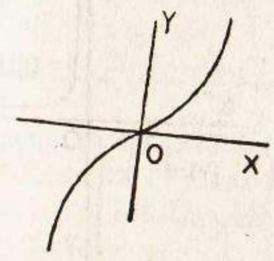


Fig 21

The graph passes through the points (0,0), (1,1) ; (—1, —1) Moreover y is positive or negative according as x is positive or negative. So there will be no branch of the curve in the 2nd and 4th quadrants.

The function $y=x^{2m+1}$ is odd and so it is symmetrical about the origin. That is, if (x,y) is a point on the graph $(-x,-y)$ is also a point on it. The graph of the function is shown above.

Case. III. When n is a negative even integer.

Let $n=-2m$, where m is any positive even integer. Then $y=x^n=x^{-2m}=1/x^{2m}$

The function is defined for all valuse of x except at $x=0$,

The function $y={}^1/x^{2m}$ even and so its graph is symmetrieal about y-axis.

when $x=1$, then $y=1$,

$x=-1$, then $y=1$,

The curve passes through $(1,1)$ and $(-1,1)$

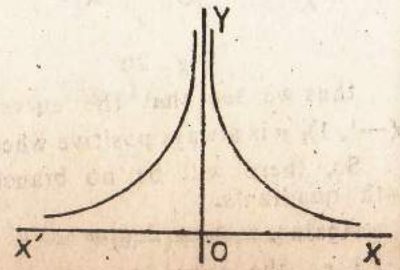


Fig. 22

y is always positive whether x is positive or negative as m is an even integer.

So, there is no part of the curve in the 3rd or 4th quadrants.

Again $y\to 0$ as $|x|\to\infty$ and $y\to\infty$ as $|x|\to o$ Hence the axes of coordinates are asymptotes of the curve.

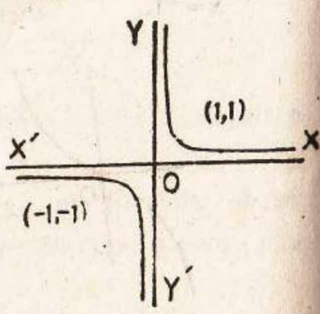
The graph of the function is shown in Fig. 22

Case. IV. When n is a negative odd integer.

Let $n=-(2m+1)$.

Then $y=x^n=\dfrac{1}{x^{2m+1}}$

The graph is not defined at $x=0$, Since $y=\dfrac{1}{x^{2m+1}}$ an odd function, the graph is symmetrieal about the origin, The graph passes through $(1,1)$ and $(-1,-1)$.



Fig—23

y is positive or negative according as x is positive or negative

So, there is no branch of the graph in the 2nd and 4th quadrants.

In $y=\dfrac{1}{x^{2m+1}}$ if x increases from 1 to $+\infty$, then y decreases from 1 to zero. Again if x decreases from 1 to zero, then y increases from 1 to ∞ .

Similarly for negative value of x, y decreases from $-\infty$ to zero (numerically) for x lying between 0 and $-\infty$ also x decreases from $-\infty$ to zero (numerically for y lying between 0 and ∞ ,)

The graph of the function is shown in fig. 23 ,

(g) Draw the graphs of $y=x^n$ when n is fractional

A few cases are given below .

Let $n=2/3,\ 1/3$, etc.

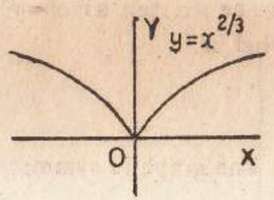


Fig. 24

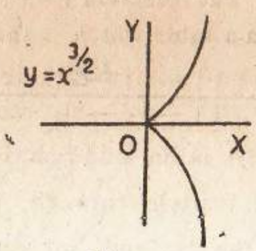


Fig. 25

[i] For the function $y=x^{2/3}=(x^{1/3})^2$ an even function.

we can form the table

x	0	1	-1	$2\sqrt{2}$	$-2\sqrt{2}$	ect.
y	0	1	1	2	2	etc.

The graph passes through (0,0), (1,1),(−1,1) etc. No branch of the graph lies in 3rd and 4th quadrants. With the increase of $|x|$,

y will also tends to $+\infty$,

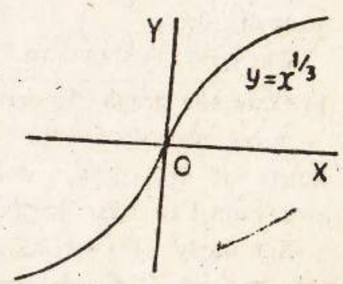


Fig. 26

The graph is shown in Fig 24

(ii) **For the function $y=x^{3/2}$**

we can form the following table.

x	0	1	2	3	4	etc.
y	0	1	$2\sqrt{2}$	$3\sqrt{3}$	8	etc.

$y=x^{3/2}=\sqrt{x^3}$, The function is defined for $x\geq 0$ with $y\geq 0$. So the graph lies entirely in the first quadrants and it passes through the points (0,0), (1,1) $(2,2\sqrt{2})$, $(3,3\sqrt{3})$, (4,8).

The graph is the part above x axis in fig. 25

The graph of $y^2=x^3$ or $y=\pm x^{3/2}$ consists of both the parts above and below the x axis in fig. 25. One part is the reflection of the other about the x—axis.

(iii) **The function $y=x^{1/3}$** can be written as $x=y^3$. Now we can form a table which is shown below.

y	0	1	−1	−2	etc.
x	0	1	−1	−8	etc.

$y=x^{1/3}$ is an odd function. The graph is symmetrical about the orgin it passes through, (0,0),(1,1),(—1,—1) etc. The graph lies in the 1st and 3rd quadrants, no part of it lies in the 2nd and 4th quadrants. The graph extends from—∞ to+∞ through (0,0),

The curve is shown in Fig 26.

[h] Draw the graph of y = sin x

y increases from 0 to 1 for values of $0\leq x\leq\frac{1}{2}\pi$, y decreases from 1 to 0 for $\frac{1}{2}\pi\leq x\leq\pi$

Similarly for $-\pi/2\leq x\leq 0$ and $-\pi\leq x\leq-\frac{1}{2}\pi$, y decreases.

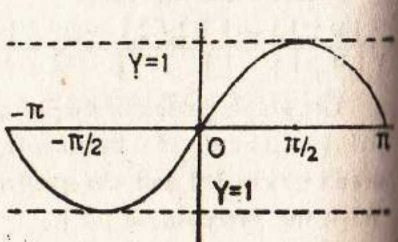


Fig. 27

from 0 to—1 and then increases from—1 to 0 again. The graph is shown in Fig. 27

(i) **Draw the graph of $y=\sin^{-1}x$**

Let us consider the equation $x=\sin y$

Since $\sin y$ takes all values from—1 to 1 for real values of y. the inverse function $y=\sin^{-1}x$ is defined over the principal part of the graph $x=\sin y$ given by

$y\in[-\frac{1}{2}\pi, \frac{1}{2}\pi]$

with $x\in[-1, 1]$

The graph of $y=\sin^{-1}x$ passes through points $(-1,-\frac{1}{2}\pi)$ (0, 0) and $(1, \frac{1}{2})$

The graph of $y=\sin^{-1}x$ can also be obtained by reflecting the graph of $y=\sin x$ drawn for $-\frac{1}{2}\pi\leq x\leq\frac{1}{2}\pi$ about the line $y=x$ [Fig 28]

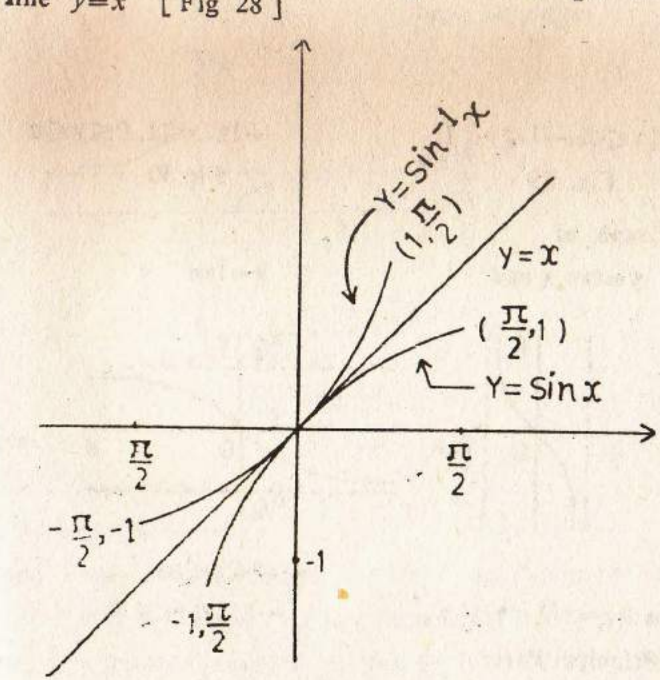
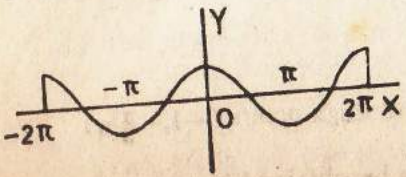


Fig 28

(j) **Graphs of** $y=\cos x$ and $y=\cos^{-1} x$

$y=\cos x$

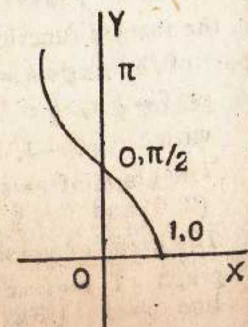


$-\infty \leqslant x \leqslant \infty, -1 \leqslant y \leqslant 1.$

Fig. 29

$(-1, \pi)$

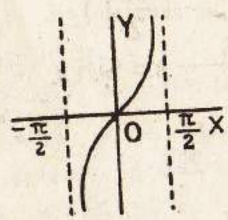
$y=\cos^{-1} x$



$-1 \leq x \leqslant 1, 0 \leq y \leqslant \pi$

Fig 30

(k) **Graph of** $y=\tan x$ **and**

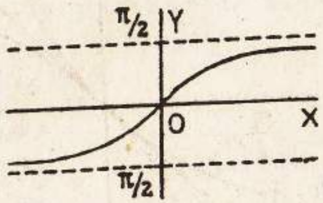


$y=\tan x; -\pi/2<x<\frac{1}{2}\pi;$

(Principal Part)

Fig. 31

$y=\tan^{-1} x$



$-\infty<x<\infty$

$-\pi/2<y<\pi/2$

Fig. 32

(l) **Graphs of** $y=\sec x$ and $y=\sec^{-1} x$.

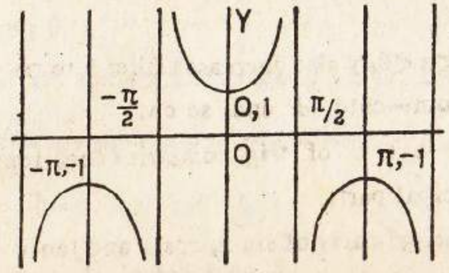


Fig 33

$y=\sec x;$ $-\infty<x<\infty; -\infty<y\leqslant -1$

or, $1 \leqslant y<\infty$

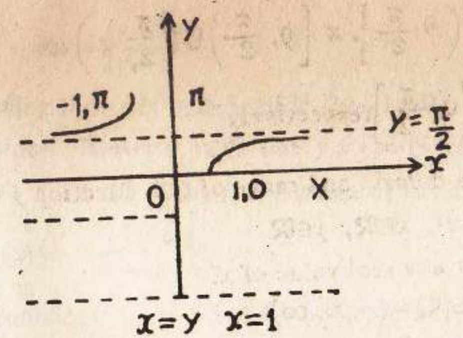


Fig. 34

$y=\sec^{-1} x,$

$-\infty<x\leqslant -1$ or $1 \leqslant x<\infty$

and $0 \leqslant y \leqslant \pi$ with $y \neq \frac{1}{2}\pi$

$y=\sec^{-1} x$; y increases from 0 to $\pi/2$ as x increases from 1 to ∞, y increases from $\pi/2$ to π as x increases from $-\infty$ to -1.

$y=\sec^{-1} x$ passes through (1, 0) ond $(-1, \pi)$.

$y=\sec x$ the function is discontinuous at $x=\pi/2, 3\pi/2, \ldots \ldots$

$-\pi/_2, -3\pi/2 \ldots\ldots$; y increases from 1 to ∞ for the values x in $0<x<\pi/2$.

For $-\pi/2<x<0$; y also increases from 1 to ∞. For $\frac{1}{2}\pi \leq x \leq \pi$ y increases from $-\infty$ to -1 and so on.

Note : The inverse of trigonometric functions one defined over their principal parts.

For the principal parts of $\sin x$, $\cos x$ and $\tan x$ x lies respectively in the intervals

$$[-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi], [0, \pi] \text{ and } \left(-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi\right)$$

The principal parts of $\operatorname{cosec} x$, $\sec x$ and $\cot x$ are defined

for $x \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, $x\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$

and $x \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ respectively.

Ex. 1 : Find the domain and range of the function f where

$y=f(x)=1-x^2$, $x \in \mathbb{R}$, $y \in \mathbb{R}$.

Ans. y is real for any real value of x.

$D_f=\{x : x \in \mathbb{R}\}=(-\infty, \infty)$.

Since $x^2 \geq 0$, therefore the maximum value of y is 1. Hence

$R_f=$ range of $f=\{y : y \leq 1\}=(-\infty, 1]$.

Ex 2. Find the domain and range of f where

$y=\{f(x)=\sqrt{1-x^2}$; $x \in \mathbb{R}$, $y \in \mathbb{R}$.$\}$

Since $y \in \mathbb{R}$, we have $y^2 \geq 0$

or, $1-x^2 \geq 0$ or $x^2 \geq 1$ or $|x|^2 \leq 1$

or $|x| \leq 1$ $\Rightarrow -1 \leq x \leq 1$

$\therefore$ $D_f=[-1, 1]$.

For $x \in D_f$, $0 \leq y \leq 1$ and so

$R_f=[0, 1]$

Ex. 3. Find the domain and range of f given by

$$y=f(x)=\frac{x-1}{2x-3} \text{ where } x \in \mathbb{R}, y \in \mathbb{R}$$

Ans. If $2x-3=0$ or $x=3/2$

then $y=\dfrac{1/2}{0}$ is not defined.

$\therefore$ $D_f=x : \{x \in \mathbb{R}$ but $x \neq 3/2\}=(-\infty, 3/2) \cup (3/2, \infty)$

Again solving for x,

$$y(2x-3)=x-1$$

$$\Rightarrow x=\frac{3y-1}{2y-1}$$

Showing that x is undefined if $2y-1=0$ or $y=\frac{1}{2}$

Hence $R_f=\{y : y \in \mathbb{R}$ but $y \neq \frac{1}{2}\}=(-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$.

Ex. 4 Find the domain and ranga of the function

$$f(x)=\frac{x^2+1}{x^2-5x+6}$$

The denominator $x^2-5x+6=(x-2)(x-3)$

which is zero when $x=2$ or $x=3$

Therefore $f(x)$ is not defined for $x=2$ and $x=3$.

Hence $D_f=\mathbb{R}-\{2, 3\}=(-\infty, 2) \cup (2, 3) \cup (3, \infty)$.

Let $y=f(x)=\dfrac{x^2+1}{x^2-5x+6}$

$\Rightarrow$ $(y-1)x^2-5y.x+(6y-1)=0$.

Treating this as a quadratic equation in x for a given y, we ... real solutions for x, if the discriminant

$$(-5y)^2-4(y-1)(6y-1)\geq 0$$

or, $y^2+28y-4\geq 0$

Now the roots of $y^2-28y-4=0$

are given by $y=\dfrac{-28\pm\sqrt{28^2+16}}{2}$

or $y=-14\pm 10\sqrt{2}$.

Hence $y^2+28y-4\geq 0$

When $y\leqslant -14-10\sqrt{2}$ or $y\geq -14+10\sqrt{2}$.

$\therefore\ R_f=(-\infty,-14-10\sqrt{2}\,]\cup[\,-14+10\sqrt{2},\infty)$.

Ex 5 Find the domain and range of the function

$$f(x)=\begin{cases}\dfrac{|x|^2}{e}, & -1<x<0\\ =x^2, & 0\leqslant x<2\end{cases}$$ (D. U. 1987)

If $-1<x<0$, x is negative, $|x|=-x$

$\therefore\ f(x)=y=-e^{-x}/_2=e^{-x}/_2$

when $x=0$, then $y=1$, and $x=-1$, then $y=1/\sqrt{e}$

Domain is the subset of D_f *i. e.* $(-1, 0)\subset D_f$...(1)

and range, $(1, 1/\sqrt{e})\subset R_f$... ...(2)

For $y=x^2$, $0\leqslant x<2$

x	0	1	2
y	0	1	4

Domain, $[0, 2)\subset D_f$, range, $[0, 4)\subset R_f$.

Hence the domain of $f(x)$, $D_f=[-1, 0)\cup[0, 2)=[-1, 2)$

Range of $f(x)$, $R_f=(1, 1/\sqrt{e})\cup[0, 4)=[0, 4)$

Ex 6 Find the domain and range of $f(x)=\dfrac{x^2-9}{x-3}$

Draw the graph of the function.

[$f(x)$ এর চারণ ও ব্যপ্তি নির্ণয় কর এবং লেখ চিত্রটি অঙ্কন কর ।]

Let $f(x)=\dfrac{x^2-9}{x-3}=x+3$

(If $x-3\neq 0$, or, $x\neq 3$)

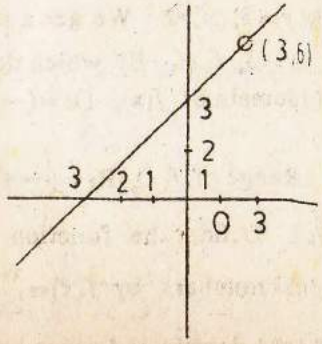


Fig 35

The function is undefined for $x=3$, *i. e*, the point (3, 6) is missing in the graph.

Domain, $D_f=(-\infty,3)\cup(3,\infty)$

Range, $R_f=(-\infty, 6)\cup(6,\infty)=\mathbb{R}-\{6\}$

Ex 7 Find the domain and range of

$$f(x)=\begin{cases}-4; & x<-2\\ -1; & -2\leq x\leq 2\\ 3; & x>2\end{cases}$$

Draw the graph.

when $y=-4$, $x<-2$

$\lvert x$	-3	-2
$\lvert y$	-1	-4

The straight line AC which does not contain $(-2,-4)$

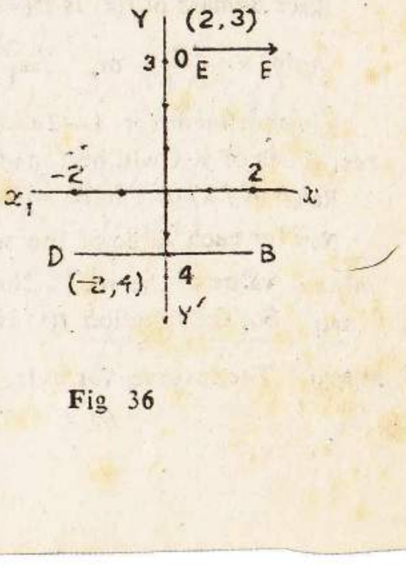


Fig 36

For. $y=-1$, $-2\leq x\leq 2$, BD is the straight line $y=-1$ including the point $B(2,-1)$, $D(-2,-1)$

For $y=3$, $x>2$ We get a straight line,

$y=3$, *i. e.*, EF which doess not contain the point E (2,3).

The domain of $f(x)$, $D_f=(-\infty,-2) \cup [-2,2] \cup (2,\infty)$

$=(-\infty, \infty)$

Range of $f(x)$, $R_f=\{-4\} \cup \{-1\} \cup \{3\}=\{-4,-1, 3\}$

Ex. 8 Define the function $f : A\to B$ where $A, B\subseteq \mathbb{R}$, the set of real numbers by $f(x)=\dfrac{x-3}{2x+1}$

Find the domain and range of $f(x)$. Show that f is one-one and onto. Find a formula for f^{-1} ($f : A\to B$) $A,B\subseteq \mathbb{R}$ (বাস্তব সংখ্যার জন্য একটি ফাংশান ইহার চারণ স্থল ও ব্যপ্তি নির্ণয় কর। দেখাও যে, ফাংশানটি এক-এক এবং on to f^{-1} এর জন্য একটি সূত্র লিখ।]

(D. U. 1988)

Let $y=\dfrac{x-3}{2x+1}$ for $2x+1=0$ or, $x=-\frac{1}{2}$, there is no value of y, so y is undefined for $x=-\frac{1}{2}$.

Hence domain of $f(x)$ is $D_f=(-\infty,\infty)-\{-\frac{1}{2}\}=\mathbb{R}-\{-\frac{1}{2}\}$

Again, $y=\dfrac{x-3}{2x+1}$ or, $x=\dfrac{3+y}{1-2y}$ or, $f^{-1} : x=\dfrac{3+y}{1-2y}$

x is undefined for $1-2y=0$ or $y=\frac{1}{2}$. So x is true for all real values of y, x will be found.

Range of $f(x)$ or y is $R_f=(-\infty,\infty)-\{\frac{1}{2}\}=\mathbb{R}\neq\{\frac{1}{2}\}$.

Now for each value of the set A has a corresponding one and only one value of the set B. No member is unrepresented present in sets. So, the function $f(x)$ is one-one and onto. (এক-এক এবং সাবিক) The inverse formula, f^{-1} : $x=\dfrac{3+y}{1-2y}$

Ex 9. (a) Find the domain and range of $f(x)= |x| + |x+1|$ and draw the graph. [$f(x)$ এর ফাংশানের চারণগুলি ও ব্যপ্তি বা বিস্তার নির্ণয় কর। চিত্রটি অংকন কর।]

(b) Show that $f(x)= |x| + |x+1|$ may be expressed as

$$f(x)=\begin{cases}-2x-1 ; & x<0\\ 1 ; & -1<x\leq 0\\ 2x+1 ; & x\geq 0\end{cases}$$

Sol. (a) For $x<0$, $|x| =-x$, $|x+1| =-x-1$ Then

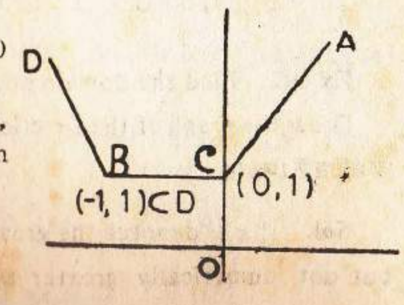


Fig 38

$y= |x| + |x+1| =-x-x-1=-2x-1$

x	...	-2	-1
y	...	3	1

Domain is the subset of D_f

(i) ∴$[-\infty,-1)\subset D_f$ and $(1, \infty)\subset R_f$, range is the subset of R_f. The graph is CD. C (0, 1),

For $-1<x\leq 0$; $x>-1$; $x+1>0$ and $|x| =-x$

Then $y= |x| + |x+1| =-x+x+1=1$.

(2) ∴Domain $[-1,0)\subset D_f$, $\{1\}\subset R_f$

x	-1	$\frac{1}{3}$	$\frac{1}{2}$	0
y	1	1	1	1

The graph $y=1$ is a straight line CB. ; C (0, 1) B $(-1, 1)$

For $x\geq 0$, $|x+1| \geq 1$ ∴ $|x+1| =x+1$, $|x| =x$

Then $y= |x| +|x+1| =x+x+1=2x+1$

(3) Domain $[0, \infty)\subset D_f$,
Range $[1,\infty)\subset R_f$

x	0	1	2	... ...
y	1	3	5	... ...

The graph is the straight line

$y=2x+1$ represents a part B A. ; B $(-1, 1)$

From (1), (2) and (3)

$D_f=(-\infty,-1)\cup[-1,0)\cup[0,\infty]=(-\infty,\infty)$

$R_f=[1,\infty)\cup\{1\}\cup[1,\infty)=[1,\infty,)$

Ex. 10. Find the domain and range of $f(x)=x-[\ |x|\ ]$

Draw the graph of the function [$f(x)$ এর চারণস্থল ব্যপ্তি নির্ণয় কর লেখচিত্রটি দেখাও।]

Sol. $|x|$ denotes the greatest integer positive or negative but not numerically greater than x.

If $0<x<1, |x|=x, [\ |x|\ ]=0.$

$\therefore\ y=x-0=x.$

when $x=1$, then $y=0$.

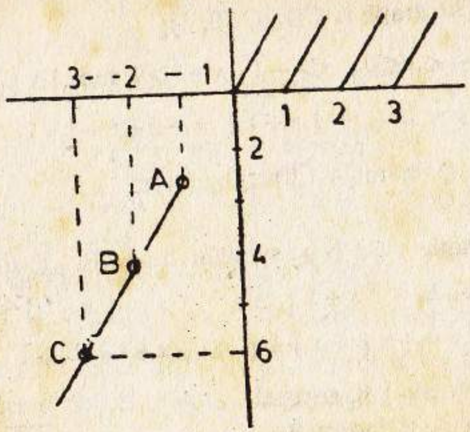


Fig 39

x	0	$0<x<1$	1	$1<x<2$	2	$2<x<3$	3	...
y	0	x	0	$x-1$	0	$x-2$	0	...

x	$-1<x<0$	$-2<x<-1$	$-3<x<-2$	...
y	$x-1$	$x-2$	$x-3$	...

The graph is shown where A$(-1,-2)$, B$(-2,-4)$, C$(-3,-6)$ etc are excluded.

where $[\ |x|\ ]$ is the largest integer in $|x|$ less than or equal to $|x|$.

Ans. Clearly, $f(x)$ is defined for all real values of x. Hence

$D_f=\mathbb{R}=(-\infty,\infty)$.

$\cdots\cup(-5,-4]\cup(-3,-2]\cup(-1,0]\subset R_f$

Alternative Method

Consider the values of $f(x)$ for $x\geq 0$.

We have $f(0)=0$.

When $0<x<1$, $[\ |x|\ ]=0$ and

$f(x)=x-0=x$

$\Rightarrow 0<f(x)<1$ when $0<x<1$.

For $1\leqslant x<2$ $[\ |x|\ ]=1$ with

$f(x)=x-1$

$\Rightarrow 0\leqslant f(x)<1$,

Since $f(1)=1-1=0$ and $f(2^-)=2^--1=1^-$.

In general, when N is zero a positive integer

and $N\leqslant x<N+1$.

$f(x)=x-N$; $f(N)=N-N=0$,

$\lim\limits_{x\to(N+1)^-} f(x)=N+1-N=1$;

So $0\leqslant f(x)<1$ for $N\leqslant x<N+1$.

Hence considering $x \geq 0$, we find that

$$[0, 1] \subset R_f$$

Now consider that $x<0$.

when $-1<x<0$, $[\,|x|\,]=0$.

$$f(x)=x-0=x \Rightarrow -1<f(x)<0\,;$$

When $-2<x\leqslant -1$, $[\,|x|\,]=1$,

$$f(x)=x-1 \Rightarrow -3<f(x)\leqslant -2\,;$$

In general, if $-(N+1)<x\leqslant -N$, where N is zero or a positive integer, $[\,|x|\,]=N$,

$$f(x)=x-N \Rightarrow -(2N+1)<f(x)\leqslant -2N$$

Hence considering $x\leqslant 0$, we see that

$$\dots \dots \cup (-5,-4] \cup (-3,-2] \cup (-1,0] \subset R_f$$

Combining the results for $x\geq 0$ and $x\leqslant 0$, we have,

$R_f=\{y : -(2N+1)<y\leqslant -N\} \cup [0, 1)$

Where N is zero or a positive integer.

Ex 11. Sketch the graph of the piecewise defined function

$$f(x)=\begin{cases} -1, & x<0 \\ 0, & x=0 \\ x+2, & x>0 \end{cases}$$ (D. H. 1987)

Let $y=f(x)=-1$, when $x<0$

x	$\dots -2$	-1	0
y	$\dots -1$	-1	$-$

For $x<0$, i. e. domain of x is $(-\infty, 0)$; $y=-1$, is a infinite straight line AB (part is shown).

For $x=0$, $y=f(x)=0$. The origin of the axis is O (0, 0) a point.

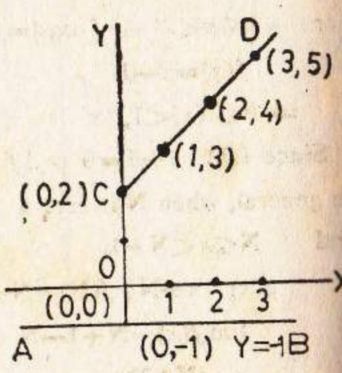


Fig 40

For $x>0$, $y=f(x)=x+2$

The domain of x is $(0, \infty)$ and the range is $(2, \infty)$

x	0	1	2	3... ...
y	2	3	4	5... ...

Therefore we get a straight line (infinite, part is shown) CD, where C(0, 2) is excluded.

Ex. 12. If R be the set of real numbers and the function $f: R\to R$ given by $f(x)=x^2-x-2$, find $f([1,2])$, $f^{-1}([-2,0])$ and $f^{-1}(\{0\})$ D. U. 1983

Sol. $f(x)=x^2-x-2$ or, $y=x^2-x-2=(x-\frac{1}{2})^2-9/4$

or, $y+9/4=(x-\frac{1}{2})^2$

It is a parabola whose vertex is at $(\frac{1}{2},-9/4)$ [$y+9/4=0$ or, $b=-9/4$, and $x-\frac{1}{2}=0$ or, $x=\frac{1}{2}$]

The graph is shown below

$f([1,2])=\{f(x) \mid 1\leqslant x\leqslant 2$, $f(x)=x^2-x-2\}$

Here domain of f is R.

for each $x\in$ R.

$f(1)=1^2-1-2=-2$

$f(2)=2^2-2-2=0$

$f([1,2])=\{f(x) \mid 1\leqslant x\leqslant 2\}$

$=[-2, 0]$

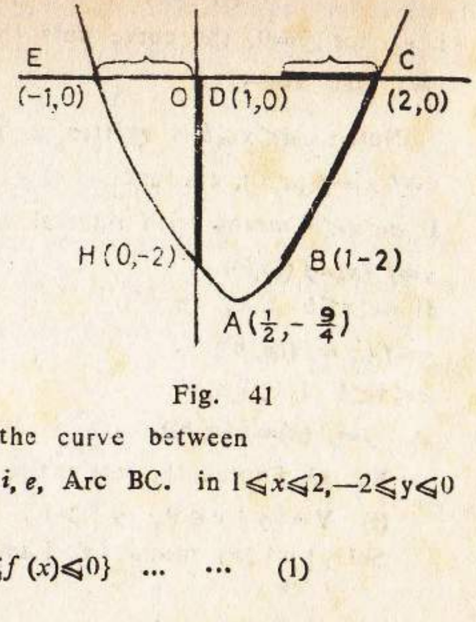


Fig. 41

i. e: the portion of the curve between B (1,−2) to C (2, 0), i, e, Arc BC. in $1\leqslant x\leqslant 2, -2\leqslant y\leqslant 0$

To find $f^{-1}([-2, 0])$;

$f^{-1}([-2, 0])=\{x \mid -2\leqslant f(x)\leqslant 0\}$ ... ... (1)

Here $f(x)=-2=x^2-x-2$ or, $x(x-1)=0$ or, $x=1,0$

Again $f(0)=0=x^2-x-2$ or, $(x-2)(x+1)=0$ or, $x=2,-1$

[The points on the graph are $(0,-2)$, $(1,-2)$, $(2,0)$, $(-1,0)$ we get the arc between E $(-1,0)$ and H $(0,-2)$ arc between B $(1,-2)$ to C $(2,0)$ *i. e.* arc EH and arc BC This can be symbolically expressed below]

From the graph we see that

$f^{-1}([-2,0])=\{x \mid -2\leqslant f(x)\leqslant 0\}$

$=\{x \mid -1\leqslant x\leqslant 0$ and $1\leqslant x\leqslant 2\}$

$=[-1,0]\cup[1,2]$

Sol : $f^{-1}(\{0\})=\{x\mid f(x)=0\}$

$=\{x\mid x^2-x-2=0\}=\{x\mid (x-2)(x+1)=0\}$

$=\{x\mid x=1,-2$ only two points $\}$

$=\{1,-2\}$

i. e. for $y=0$, the curve cuts the x—axis at two points $x=1$, and $x=-2$

Note : $a\leqslant x\leqslant b$ is written as $[a,b]$ called interval: $y=f(x)=f([a,b])$, y is function of x in the interval $x=a$ and $x=$

If $a<x<b$ means open interval of x i. e. (a, b)

$y=f(x)=f((a,b))$.

If $a<x\leqq b$ i. e, $(a, b]$

$y=f(x)=f((a, b])$

$a\leqslant x<b$ i, e, $[a, b)$

$\therefore$ $y=f(x)=f([a, b))$.

Ex. 13. Express the sets as the union of interval

(i) $Y=\{y \mid y\in R, |y|>1\}$.

Sol. ; $|y|>1$ means, $y>1$ and $-y>1$ or, $y<-1$

Graph of $y>1$ is drawn. It is the Y-axis A to infinity.

This line is in the open interval 1 and ∞ i. e., $(1, \infty)$

Again $-y>1$ or, $y<-1$ i. e., the negative y—axis from -1 to $-\infty$ i. e., in $(-\infty,-1)$

Therefore, $Y=(-\infty,-1)\cup(1,\infty)$

= The two black lines AY and BY′.

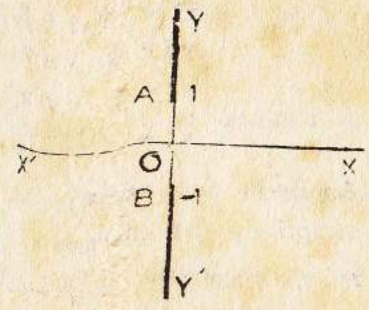


Fig. 42

$y>1$ means all valus of y bigger than 1 *i. e.*, upto $+\infty$ *i e.* all values of y between 1 and $+\infty$. It is an open interval and is $(1, \infty)$ ... ... (1)

Again—$y>1$ or, $y<-1$ means all values of y, less than -1. These values are between $-\infty$ to -1. It is an open interval and is $(-\infty, -1)$...... (2) The graph of (1) and (2) is the two portions of y—axis expressed by two intervals $(-\infty, -1)$, $(1, \infty)$. These are symbolically expressed as the union of two intervals. From the graph also the result may be obtained.

Ex. 14. Express the set $A=\{x \mid x\in R, |x|\leqslant 1\}$ as an interval and the set $B=\{y \mid y\in R, |y-2|>1\}$ as the union of two intervals. Indicate the set $A\times B$ in the cartesian plane.
D. U. 1984.

Sol. : $A=\{x \mid x\in R, |x|\leqslant 1\}$

$=\{x \mid x\in R, -1\leqslant x\leqslant 1\}$

$=[-1, 1]$

$B=\{y \mid y\in R \mid y-2 \mid >1\}$

$= \{y \mid y \in R,\ y-2>1 \text{ and} -(y-2)>1\}$

$= \{y \mid y \in R,\ y>3 \text{ and } y<1\}$

$= \{y \mid y \in R,\ 3<y<\infty \text{ and } -\infty<y<1\}$

$= (-\infty, 1) \cup (3, \infty)$

$A \times B$ = Product set comprising the area of the infinite planes by $x=-1$ and $x=1$, excluding the area LMNP, but including sides LM, PN; excluding sides LP, MN.

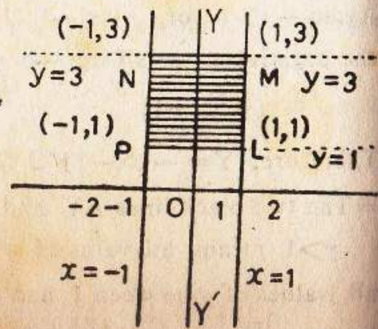


Fig. 43

EXERCISE—1

1. If $f(x)=3x^2-\dfrac{1}{x}+4x-7$

find $f(1), f(-2), f(h)$.

Ans. $-1, -5/2, 3h^2-1/h+4h-7$

2. Find the limit of the sequence (ধারার সীমা) $\{x_n\}$ For x_n as defined in each case.

(i) $2^{-1/\sqrt{n}}$ (ii) $\dfrac{1}{n}\sin\dfrac{n\pi}{2}$

(iii) $\left(100+\dfrac{1}{n}\right)^2\left(1+\dfrac{n-1}{n^2}\right)^{100}$ (iv) $\dfrac{1+(-1)^n}{n}$

Ans. (i) 1 (ii) 0 (iii) 10^4 (iv) 0.

3. Determine the domains (চারণস্থল) of definition of the following functions.

1. (i) $f(x)=\dfrac{|x|}{x}$ Ans. all values of x, except $x=0$

(ii) $f(x)=\dfrac{\sqrt{x}}{x+2}$ Ans. all +ve real values of x including zero

(iii) $f(x)=\dfrac{x^2+x}{x^2-x}=\dfrac{x+1}{x-1}$ Ans. all real values of x, except $x=0$ and 1.

(iv) $y^2=(x-x)^3$ Ans. $-\infty<x\leqq-1$ and $0\leqq x\leqq 1$

(v) $y=\log\dfrac{2+x}{2-x}$ Ans. $-2<x<2$

(vi) $y=\cos^{-1}\dfrac{2x}{1+x}$ Ans. $-\dfrac{1}{3}\leqq x\leqq 1.$

(vii) $y^2(x^2-1)=1$ Ans. $-\infty<x<-1,\ 1<x<\infty$

(viii) $y^2=\sin 2x$ Ans. $k\pi\leqq x<k\pi+\frac{1}{2}\pi (k=0,\pm1,\pm2\ldots)$

4. Find the odd and even functions of the following functions

(i) $f(x)=\frac{1}{2}(2^x+2^{-x})$ Ans. even

(ii) $f(x)=\sqrt{(1+5x+7x^2)}-\sqrt{(1-5x+7x^2)}$ Ans. odd

(iii) $f(x)=\log\dfrac{2+x}{2-x}$ Ans. odd

(iv) $f(x)=\log\{x+\sqrt{(1+x^2)}\}$ Ans. odd

5. Show that $\dfrac{\sin x}{1+\cos x}$ is not defined for $x=\pi$

6. Show that $x^{5/2}$ is not defined for all negative values of x.

7. Show that the function.

$y=\sqrt{\{(x-1)(x-2)\}}$ is non existent for every value of x lying between 1 and 2. ($1\leqq x\leqq 2$-এর জন্য y বিশ্লেষণযোগ্য নয়)।

8. Show that $\sqrt{(x^3-3x^2+2x)}$ is defined for any value of x lying between 0 and 1 but undefined for any value of x, in $1<x<2$

9. Is $\cos^{-1} x$ defined for $3\leqq x\leqq 4$ Ans. No.

10. Prove that $\frac{x^2+4x-1}{2x^2-3x-9}$ is not defined for $x=3$.

EXERCISE—1 (A)

(1) Express the sets as an interval. R is the set of real numbers. (D. U. 1984)

(i) $A=\{x\in R, |x|\leqq 1\}$

(ii) $A=\{x: x\in R, |x-3|\leqq 1\}$

(11) Express as the union of two intervals

$x=\{x|2\in R, |x-3|>1\}$

(12) In each of the following cases, decide whether the given relation F and the inverse relation F^{-1} are functions. In case F or F^{-1} is a function, decide also whether it is one to one. (D. U. 1984)

(i) $F=\{x, y)\in R^2 |x+y=1\}$

(ii) $F=\{(x, y)\in R^2 |y^2=x\}$

(iii) $F=\{x,y)\in R^2 | y=x^2\}$

(iv) $F=\{x, y)\in R^2 | x^2+y^2=1\}$

Exercise 1 (b)—Domain and Range

1. Find the domain and Range of the following

(i) $y=\sin x$ (ii) $y=x+2$. (iii) $y=\sin {}^1/x$.

Ans. (i) $D_f=(-\pi/2, \pi/2)$, $R_f=(-1, 1)$

(ii) $D_1=R_f=(-\infty, \infty)$

2. If x is any real number and $f(x)=|x|-x$, find the domain and the range of $f(x)$. Draw the graph (যদি x একটি বাস্তব সংখ্যা হয়, তাহা হইলে দেখাও যে $f(x)=|x|-x$ এর চারণক্ষেত্র এবং ব্যাপ্তি নির্ণয় কর। এবং লেখচিত্র অঙ্কন কর)। (D. U. 1987)

Ans. $D_f=[0, \infty)$, $R_f=\{-\infty, 0)$

2 a) Show that $f(x)=|x|-x$ may also be expressed as (দেখাও যে $f(x)=|x|-x$ কে লিখা যায়)

$$f(x)=\begin{cases} 0, & x\geq 0 \\ -2x, & x<0 \end{cases}$$

3. If the function $f(x)$ is defined by

$$f(x)=\begin{cases} 2x-3 ; & x<1 \\ -x^2 ; & x\geq 1 \end{cases}$$

Find the domain and range of the function $f(x)$. Draw the graph ($f(x)$ যদি উল্লিখিত ভাবে বর্ণিত হয়, তাহা হইলে $f(x)$ এর চারণক্ষেত্র ও ব্যাপ্তি নির্ণয় কর।)

Ans. $D_f=(-\infty, \infty)$, $R_f=(-\infty, -1)$

4. Find the domain and function defined by

$$f(x)=\begin{cases} x-1 , & x<2 \\ 2x+1 , & x\geq 2. \end{cases}$$ Ans. $D_f=[2, \infty)$, $R_f=[5, \infty)$

5. Find the domain and range of $f(x)= | x | + | x-1 |$ and show that $f(x)$ may be expressed as C. H. 1992

Ans. $D_f=(-\infty,\infty)$, $R f=[1, \infty)$

$$f(x)=\begin{cases} -2x+1, & x<0 \\ 1 , & 0\leqslant x< \\ 2x-1, & x\geq 1 \end{cases}$$

Draw the graph of the function. [$f(x)=|x|+|x-1|$ এর চারণক্ষেত্র এবং ব্যাপ্তি নির্ণয় কর। দেখাও যে $f(x)$ কে উল্লিখিত আকারে প্রকাশ করা যায়। $f(x)$ এর লেখচিত্র অঙ্কন কর।]

6. Find the domain and range of $f(x)=x-|x|$ in $-3\leq x\leq 3$; $[x]$=greatest integer that is $\leq x$

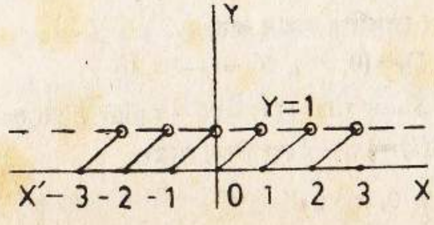


Fig 44

7. Draw the graph of the following function

$$f(x)=\begin{cases} x^2+1, & x<0 \\ x, & 0\leq x\leq 1 \\ \dfrac{1}{x}, & x>1 \end{cases}$$ D. U. 1986

Find the domain and range of $f(x)$. [$f(x)$ ফাংশানের লেখচিত্র অঙ্কন কর এবং $f(x)$ চারণক্ষেত্র ও ব্যাপ্তি নির্ণয় কর।

Ans. $D_f=(-\infty, 0)\cup[0, 1]\cup(1, \infty)$, $R_f=[1,\infty)\cup[0,1]\cup(1, 0)=[0, \infty]$

8. Fnd the domain and range of the functions

(i) $f(x)=\sqrt{\dfrac{x+1}{x-1}}$ (ii) $f(x)=x,\ 0\leq x\leq \frac{1}{2}$ $=3-x,\ \frac{1}{2}<x<3$ (C. H. 1988)

Ans. $D_f=(-\infty,-1)\cup(1,\infty)$. $R_f=(-\infty,\infty)-(-1, 1)=\mathbb{R}-\{-1, 1\}$

(ii) $D_f=(0, 3)$, $R_f=[0, \frac{1}{2}]\cup[\frac{1}{2}, 3)=[0, 3]$

9. Find the domain and range of f when

$$y=f(x)=\frac{4x+3}{x^2+1}\quad x\in\mathbb{R}\quad y\in\mathbb{R}$$

10. (i) Define the domain of a function. Find the domain of the function defined by $y=x^2$ (R. U. 1970)

(ii) Define the range of a function. Find the range of the function defined by the equation $y^2=(x-2)(x-5)$

(iii) Give an example of a function which has an inverse.

11. Find the domain and range of f when

$$y=f(x)=\frac{4x+3}{x^2+1},\ x\in\mathbb{R},\ y\in\mathbb{R}$$

Ans. $D_f=(-\infty, \infty)$ $R_f=[1, 4]$

Ex. (12) In each of the following cases, decide whether the given relation F is a function. In each case F is a function, determine its domain and range and decide whether it is one-one. (D. U. 1984)

(i) $F=\{(x,y)\in R^2 \mid y=x\}$

Ans. function, $D_f=R$, $R_f=R$. 1–1 function

(ii) $F\{(x, y)\in R^2 \mid y=x^2$ *Ans.* function, $D_f=R$, $R_f=R^+$

(iii) $F=\{(x, y)\in R^2 \mid y^2=x\}$ *Ans.* No $D_f\neq R$

(iv) $F=\{(x, y)\in R^2 \mid y=\sqrt{x}\}$ Ans. No $D_f\neq R$

13. Find the domain and range of the following function

(i) $y=\sqrt{(x^2-2x+2)}$ (ii) $y=\dfrac{1}{x^2-1}$

Ags. (i) $0\leqslant y\leqslant\infty$, $1\leqslant x\leqslant 2$ (ii) $D_f=\mathbb{R}-\{-1, 1\}$,

$R_f=[\infty, 0)\cup[-\infty,-1$

14. Determine the domain and range of the following functions.

(a) $y=[x]^2$ (b) $y=[x]+\frac{1}{2}$ in $[-4, 4]$

(c) $y=[\frac{1}{2}x]+1$ in $(-6, 6)$ (d) $y=[x+\frac{1}{3}]$ in $[-4, 4]$

Ans. $y=0,\ 0\leq x<1$; $y=1,\ 1\leqslant x<2$, and so on, $R_f=\{0, 1, 4, 9, \}$

(b) $y=\frac{1}{2}$, $0 \leqslant x < I$, $y=\frac{3}{2}$, $I \leqslant y < 2$ and so on

$R_f=\{-\frac{9}{2}, -\frac{[illegible]}{2}, -\frac{3}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}\}$

(c) $y=I$, $0 \leq x < 2$, $2=2$, $2 \leq x < 4$ and so on,

$R_f=\{-2, -I, 0, 3, 2, I\}$

(d) $y=0$, $0 \leq x+\frac{1}{3} < 1$, $y=\frac{7}{3}$, $1 \leq x < 2$, and so on,

$R_f=\{-4, -\frac{7}{3}, -1, 0, 1, \frac{8}{3}, 3\}$

EXERCISE—1 (C)

If $A=\{1, 2, 3, 4)$ and $B=\{1, 2, 3, 4)$

1. Which of the following collections of ordered pairs of numbers are functions. Find the domain and range of each it.

(i) (1, 1), (2, 1), (3, 1), (4, 1)

(ii) (1, 1), (2, 2), (3, 3), (4, 4)

(iii) (1, 1), (1, 2), (2, 1), (3, 2)

(iv) (0, 3), (1, −1), (2, 4), (3, 3), (4, 5)

2. Find the set of ordered pairs which is $\{f+g\}(x)$ and which is $\{f-g\}(x)$.

(i) $f(x)$ is ; (0, 2), (1, 1) and (2, −4) *Ans.* (0, 3), (1, 2) (2, −2)

$g(x)$ is : (0, 1), (1, 1) and (2, 2) *Ans.* (0, 1), (1, 0), (2, −6)

(ii) $f(x)$: (−3, 1), (−2, 5), (1, 4) and (3, 6) *Ans.* (−2, 2)

$g(x)$ is : (−2, −3), (−1, 7) (0, 5) and (2, 1) *Ans* (−2, 8)

(iii) $f(x)$ is : (−10, 3), (−4, 8), (0, 1) and (15, 3)

$g(x)$ is (−5, 1), (1, 3), (2, 2) and (10, 10)

3. Find the ordered pairs of Ex—2 above.

4. Which of the following relations in I are functions ? Give the domain and range of each function.

$F=\{(x, y) \mid (x, y) \in I \times I,\ y^4=x\}$ Ans. not function.

$F=\{(x, y) \mid (x, y) \in I^2, y=x^3\}$ *Ans.* $D=I$; R=set of integers expressive as the cube of integer.

$F=\{(x, y) \mid (x, y) \in I^2,\ x<y\}$ *Ans.* not function.

$F=\{(x, y) \mid (x, y) \in I^2,\ x^2-y=16\}$ *Ans.* $D=I$. R= set of all in tegers expressible as 16 less than the square.

$F=\{(x, y) \mid (x, y) \in A \times B,\ y=2x^2+3\}$

$A=\{x \mid x \in I, 1 \leqslant x \leqslant 5$, $B=\{x \mid x \in I\}, 1 \leqslant \in \leqslant 100\}$

Ans. yes, D=A, Range={5, 11, 21, 35, 53}

F=(1, 5) (2, 11), (3, 21), (4, 35), (5, 53)

$F=\{x, y) \mid (x, y \in R \times R,\ y=x^2\}$, $R=\{x \mid x \in I, |x| \leqslant 10\}$

Ans. yes, F=(0,0), (1, 1), (2, 4), (3, 2, (−1, 1), (−2, 4). (−3, 9)

$F=\{(x, y) \mid (x, y) \in R^2,\ x=y^2\}$, $R=\{x \mid x \in I, |x| \leqslant 10\}$

Ans No, F=(0, 0), (1, 1), (1−1), (4,2) (4,−2), (9,3), (9,−3)

The pairs (1, 1) and (1,−1) have same first component.

5. (i) If I is the set of all integers and $x \in I$, which of the following mapping of $I \times I$ are mapping of I onto I ? What are 1—1 of I onto I.

(a) $x \to x+3$ i, e $F=\{x, y) \in I^2 \mid y=x+3\}$

(b) $x \to x^2+x$, i, e, $F=\{(x, y) \in I^2 \mid y=x^2+x\}$

(c) $F=\{(x, y) \in I^2 \mid x=x^3\}$

(d) $F=\{x, y) \in I^2 \mid x=2x-1\}$

(e) $F=\{x, y) \in I^2 \mid x=x-4\}$

6. (i) Let $A=\{x \mid x \in R$ and $-5 \leqslant x \leqslant 5\}$

and $B=\{y \mid y \in R$ and $-2 \leqslant y \leqslant 2\}$

Find $A \times B$ and sketch $A \times B$ in certain plane

(ii) Let $A=\{x \mid x \in R$ and $|x-1| < 2\}$

and $B=\{y \mid y \in R$ and $|y+2| > 1\}$

Find A × B and sketch A × B in caertesian plane.

(iii) Let $A=\{x \mid x \in R, -6 \leqslant x \leqslant 6\}$

and define the relation R in A. Find A × A and sketch A × B in certesian plane.

7. Describe which of the relation below are functions, which into and which are 1—1 : also 1—1 correpondence.

(i) $f : \{(x, y) \in R \times R \mid y=2x+3\}$ *Ans* 1—1 Correspondence

(2) $f : \{(x, y) \in R \times R \mid y=\sqrt{x}$, R^+ be the set of all non-negative real numbers) $D_f \neq R$, it is not function.

(3) $f : \{(x, y) \in R \times R \mid y=\sin x\}$ Ans. into function

(4) $f : \{(x, y) \in R \times R \mid y=x^3\}$ *Ans.* one—one function.

(5) $f : \{(x, y) \in R \times R \mid f(x)=x^2+1\}$ *Ans.* function onto.

(6) $f : \{(x, y) \in R^2 \mid f(x)=e^x\}$ *Ans* f is one—one function.

(7) $f : \{(x, y) \in R \times R \mid y=\tan x\}$ *Ans.* f is onto,

(8) $f : \{(x, y) \in R^2 \mid f(x)=\log x$, where x is real$\}$

8. Let A and B set of real numbers $f=\{(x, y) \in A \times B \mid f(x)=6x-x^2\}$ Find $f[0, 1]$, $f(1, 4)$ $f^{-1}([0, 5])$, $f^{-1}(5, 9])$

Ans. [0, 1], [5, 9]] [0, 5]U[5,6] [1,5]

9. If $f(x)=\{(x, y) \in R^2 \mid f(x)=x^2-4x+4$ Find $f([0, 1)$, $f([3, 4])$, $f^{-1}([1, 2])$

Ans. [1, 4], [1, 4] [$2-\sqrt{2}$, 1] U[3, $2+\sqrt{2}$]

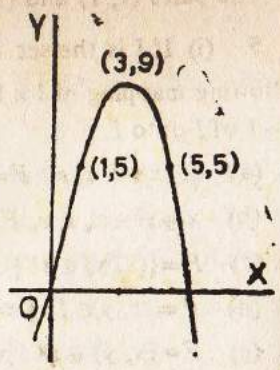


Fig 45

10. If $f : X \to Y$ is a function and $A \subset X$, then what is meant by $f^{-1}(A)$?

Ans. See Higher Algebra, set Theory Art. 9.6

Show that $f^{-1}(A \cup B)=f^{-1}(A) \cup f^{-1}(B)$

and $f^{-1}(A \cap B)=f^{-1}(A) \cap f^{-1}(B)$ (D. U. 1982)

See Higher Algebra, set Theory Art. 9. 7.

11. Show that the function

$y=\sqrt{\{(x-a)(x-b)(x-c)\}}$: $a<b<c$ is defined for all real values of x lying between a and b and is not defined for all real values of x lying between b and c.

12 Draw the graphs of the following function. Find also the domains of x and y.

(i) $f(x)=x$, $0 \leq x \leq 2$
$=3-x$ $\frac{1}{2}<x<3$

(ii) $f(x)=x^2$, $x \leq 0$
$=\sqrt{x}$, $x>0$

(iii) $f(x)=x$, $0 \leq x<\frac{1}{2}$
$=1$ $x=\frac{1}{2}$
$=1-x$, $\frac{1}{2}<x<1$

(iv) $f(x)=x$ $-2 \leq x \leq 2$
$=4-x$ $2<x \geqq 4$

(v) $f(x)=\dfrac{2x}{x-3}$, $-\infty<x<\infty$ for $x \neq 3$
$=2$ $x=3$

(vi) $f(x)=\dfrac{x^2-16}{x-4}$ for $x \neq 4$
$=2$ for $x=4$

13. Find the inverse (বিপরীত) functions of y and the regions (এলাকা) where they are defined.

(i) $y=1-2^{-x}$ Ans. $x=-\log(1-y)/\log 2, -\infty<y<1$

(ii) $y=x^2-5$ Ans. $x=\sqrt{(y+5)}, -\sqrt{(y+5)}, (-5<y<\infty)$

(iii) $y=\log_{10}x$, Ans. $x=10^y, -\infty<y<\infty$

(iv) $y=x^2$, $x>0$ Ans. $x=\sqrt{y}$, $0<y<\infty$
$=x$, $x \leqq 0$ $y=x, -\infty<y \leqq 0$.

14. Find the Domain and draw the graph.

(i) $f(x) = 1 + x; -1 \le x < 0$ Ans. (i) [-1,1]

$= 1-x; 0 \le x < 1$ (0, 1) U (1,&)

$= 0 ; 1 < x$ N.U. 1994

(ii) $f(x) = |x+1| + |x-2|$ C.H. 1992

Ans. EDOBC .... (-&,-1) U [-1, 0] U [0, -20] U [2, &]

14. চারণ ক্ষেত্র ও লেখচিত্র অঙ্কন কর

(i) $f(x) = 1+x; -1 \le x < 0$

$= 1-x; 0 \le x < 1$

$= 0 ; 1 < x$ N.U.1994

C. H. 1992

14(ii) $=x$ যখন ও ≤ 0

$f(x) = |x+1| + |x-2|$

উত্তরমালা—১

1. $-1, 5/2\ 3h^2 - \frac{1}{h} + 4h - 7$

2. (i) 1 (ii) 0 (iii) 10^4 (iv) 0.

3. (a) (i) $x = 0$ ব্যতিত x-এর সকল মান।

(ii) শূন্য সহ x-এর সকল ধনাত্মক বাস্তব মান।

(iii) $x = 0$ ব্যতিত x-এর মান সকল বাস্তব মান।

(b) $-\infty < x \le -1$ এবং $0 \le x \le 1$

(c) $-2 < x < 2$ (d) $-\frac{1}{3} \le x \le 1$

(e) $-\infty < x < -1,\ 1 < x < \infty$

(f) $k\pi \le x < k\pi + \pi/2$ ($k = 0 \pm 1 \pm 2 \dots \dots \dots$)

4. (i) জোর (ii) বিজোড় (iii) বিজোড় (iv) বিজোড়।

9. না,

13. (i) $x = -\log(1-y)/\log 2;\ -\infty < y < 1$

(ii) $x = \sqrt{(y+5)},\ -\sqrt{y+5},\ (-5 < y < \infty)$

(iii) $x = 10^y,\ -\infty < y < \infty$

(iv) $x = \sqrt{y},\ 0 < y < \infty$

$y = x,\ -\infty < y \le 0$ 14(i) $[-1,1)$; $[0,1) \cup (1,\infty)$

14(ii) EDOBC ... .$[-\infty, -1] \cup [-1, 0] \cup [0, -20] \cup [2, \infty)$

চারণক্ষেত্র $(\infty, 3) \cup (3) \cup [3, \infty)$ ব্যবধি

EXERCISE—1 (A)

(1) Express the sets as an interval. R is the set of real numb

(i) $A=\{x \in R, |x| \leqslant 1\}$ (D.U. 19

(ii) $A=(x \mid x \in P, |x-3| \leqslant 1\}$

(11) Express as the union of two intervals (দুইটি ব্যবধি যোগফলে প্রকাশ কর)

$x=\{x \mid x \in R, |x-3|>1\}$

(12) In each of the following cases, decide whether the gi relation F and the inverse relation F^{-1} are functions. In cas or F^{-1} is a function. decide also whether it is one to one. (F এবং F^{-1} ফাংশান কি ? যদি ফাংশান হয় তাহারা এক-এক কি ?)

(D. U. 198

(i) $F=\{(x, y) \in R^2 \mid x+y=1\}$

(ii) $F=\{(x, y) \in R^2 \mid y^2=x\}$

(iii) $F=\{(x, y) \in R^2 \mid y=x^2\}$

(iv) $F=\{(x, y) \in R^2 \mid x^2+y^2=1\}$

Exercise 1 (b)—Domain and Range (চারণক্ষেত্র ও ব্যাপ্তি বা বিস্তার)

1. Find the domain and Range of the following

(i) $y=\sin x$ (ii) $y=x+2$. (iii) $y=\sin {}^1/x$.

Ans. (i) $D_f=(-\pi/_2, \pi/_2)$, $R_f=(-1, 1)$

(ii) $D_f=R_f=(-\infty, \infty)$

2. If x is any real number and $f(x)=|x|-x$, find the do and the range of $f(x)$ Draw the graph (যদি x একটি বাস্ত হয়, তাহা হইলে দেখাও যে $f(x)=|x|-x$ এর চারণক্ষেত্র এবং ব্যাপ্তি নি কর। এবং লেখচিত্র অঙ্কন কর)। (D. U. 19

Ans. $D_f=[0, \infty)$, $R_f=\{-\infty, 0)$

2. (a) Show that $f(x)=|x|-x$ may also be expressed as (দেখাও যে $f(x)=|x|-x$ কে লিখা যায়)

$$f(x)=\begin{cases} 0, & x \geqslant 0 \\ -2x, & x<0 \end{cases}$$

3. If the function $f(x)$ is defined by

$$f(x)=\begin{cases} 2x-3\ ; & x<1 \\ -x^2\ ; & x \geqslant 1 \end{cases}$$

Find the domain and range of the function $f(x)$ Draw the graph ($f(x)$ যদি উল্লিখিতভাবে বর্ণিত হয়, তাহা হইলে $f(x)$ এর চারণক্ষেত্র ও ব্যাপ্তি নির্ণয় কর।)

Ans. $D_f=(-\infty, \infty)$, $R_f=(-\infty, -1]$

4. Find the domain and function defined by

$$f(x)=\begin{cases} x-1\ , & x<2 \\ 2x+1\ , & x \geqslant 2 \end{cases}$$ Ans. $D_f=[2, \infty)$, $R_f=[5, \infty)$

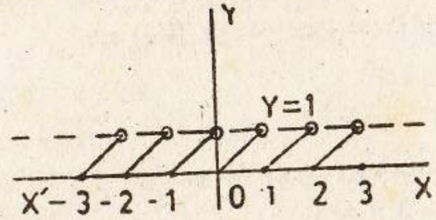
5. Find the domain and range of $f(x)=|x|+|x-1|$ and show that $f(x)$ may be expressed as

$$f(x)=\begin{cases} -2x+1, & x<0 \\ 1\ , & 0 \leqslant x<1 \\ 2x-1\ , & x \geqslant 1 \end{cases}$$

Ans. $D_f=(-\infty, \infty)$, $R_f=[1, \infty)$

Draw the graph of the function [$f(x)=|x|+|x-1|$ এর চারণ এবং ব্যাপ্তি নির্ণয় কর। দেখাও যে $f(x)$ কে উল্লিখিত আকারে প্রকাশ হয়। $f(x)$ এর লেখচিত্র অঙ্কন কর।]

6. Find the domain and range of $f(x)=x-[x]$ in $-3 \leqslant x \leqslant 3$; greatest integer that is $\leqslant x$



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7. Draw the graph of the following function

$$f(x)=\begin{cases} x^2+1, & x<0 \\ x, & 0\leq x<1 \\ \dfrac{1}{x}, & x>1 \end{cases}$$ **D. U. 19**

Find the domain and range of $f(x)$. [$f(x)$ ফাংশানের লেখচি অঙ্কন কর এবং $f(x)$ চারণক্ষেত্র ও ব্যাপ্তি নির্ণয় কর।

Ans. $D_f=(-\infty, 0) \cup [0, 1] \cup (1, \infty), R_f=[1,\infty) \cup [0, 1]$
$\cup(1, 0)=[0,$

8. Find the domain and range of the functions

(i) $f(x)=\sqrt{\dfrac{x+1}{x-1}}$

Ans. $D_f=(-\infty,-1) \cup (1, \infty)$, $R_f=(-\infty, \infty)-(-1, 1)$
$=IR-\{-1,$

9. Find the domain and range of f when

$$y=f(x)=\frac{4x+3}{x^2+1} \quad \begin{matrix} x \in IR \\ y \in IR \end{matrix}$$

10. (i) Define the domain of a function. Find the dom
of the function defined by $y=x^2$ (*R. U.* 19

(ii) Define the range of a function. Find the range of
function defined by the equation $y^2=(x-2)(x-5)$

(iii) Give an example of a function which has an inverse.

11. Find the domain and range of f when

$y = f(x) = \dfrac{5x+3}{x^2+1}$, $x \in IR; y \varepsilon IR$

Ans. $D_f = (-\infty, \infty)$; $R_f = [1, 4]$

Ex. (12) In each of the following cases, decide whether the given relation F is a function. In each case F is unction. Determine its domain and range and decide whether it is one-one. *(D. U. 198)*

(i) $F = (x, y) \varepsilon R^2 \mid y = x)$

Ans. function, $D_f = R$, $R_f = R$, *1 − 1 function*

(ii) $F \{(x, y) \varepsilon R^2 \mid y = x^2$ **Ans.** *function*, $D_f = R, R_f = R^+$

(iii) $F = \{(x, y) \varepsilon R^2 \mid y^2 = x\}$ **Ans.** *No.* $D_f \neq R$

(iv) $F = \{(x, y) \varepsilon R^2 \mid y = \sqrt{x}\}$ **Ans.** *No.* $D_f \neq R$

13. Find the domain and range of the following function

(i) $y = \sqrt{(x^2 - 2x + 2)}$ (ii) $y = \dfrac{1}{x^2 - 1}$

(iii) $f(x) = \sqrt{(x^2 - 4x + 1)}$ *N. U. 1993*

Ans. *(i)* $0 \leq y \leq \infty$, $1 \leq x \leq 2$ *(ii)* $D_f = IR - \{-1, 1\}$,
$R_f = [\infty, 0) \cup [-\infty, -1]$

(iii) $2 - \sqrt{3} \leq x \leq 2 + \sqrt{3}$, $\alpha(-\infty - \sqrt{3}] \cup$
$\sqrt{3}, -\infty]$ $R_f = [\infty, 0) \cup [-\infty, -1]$

14. Determine the domain and range of the follwoing functions with diagram (নিম্নলিখিত ফাংশন গুলির চারণক্ষেত্র ও বিস্তার নির্ণয় কর। চিত্রগুলি অঙ্কন কর।)

(a) $y = [x]^2$ *in* $[2, 4]$ Ans. $D_f = 0 \leq x < 1$, $1 \leq x < 2$, $2 \leq x < 3$
$3 \leq x < 4$ ……… $R_f = \{0, 1, 4, 9, 16 ………\}$

(b) $y = [x] + \dfrac{1}{2}$ *in* $[-4, 4]$

Ans. D_f প্রশ্ন *(a)* এর D_f

$R_f = \{-9/2, -7/2, -3/2, \frac{1}{2}, 3/2, 5/2, 7/2\}$

(c) $y=[\frac{1}{2}x]+1$ in $[-6, 6]$

Ans. D_f *Ans.* (a) এর D_f

$R_f=\{-4,-3,-2, 0,-1,-2\}$

(d) $y=[x+\frac{1}{3}]$ in $[-4, 4]$

Ans. D_f *Ans* (a) এর D_f

$R_f=\{-4,-7/3,-1, 0, 1, 8/3, 3\}$

(e) (i) $f(x)=\sqrt{\left(\frac{x+1}{x-1}\right)}$ (ii) $f(x)=x,\ 0\leqslant x\leqslant\frac{1}{2}$

C. H. 1988 $=3-x,\ \frac{1}{2}<x>3$ *C. H. 1988*

Ans. $D_f=[-\infty,\infty]-(-1, 1)$ *Ans.* $D_f=[0, 3],\ R_f=[0, 5/2)$

$R_f=(-1, \infty]-(-1, 1)$ লেখচিত্রে *OA* এবং *BC* ; *B*(3, 0) এবং $C(\frac{1}{2}, 5/2)$ বিন্দু দুইটি বহিভূর্ত।

EXERCISE—1 (C)

If $A=\{1, 2, 3, 4)$ and $B=\{1, 2, 3, 4)$

1. Which of the following collections of ordered pairs of numbers are functions. Find the domain and range of each it. [সজ্জিত জোড়ের সমাবেশ গুলির মধ্যে কোনটি ফাংশান? ফাংশানের চারণক্ষেত্র ও বিস্তার নির্ণয় কর]

(i) (1, 1), (2, 1), (3, 1), (4, 1)

(ii) (1, 1), (2, 2), (3, 3), (4, 4)

(iii) (1, 1), (1, 2), (2, 1), (3, 2)

(iv) (0, 3), (1,−1), (2, 4), (3, 3), (4, 5)

2. Find the set of ordered pairs which is $\{f+g\}(x)$ and which is $\{f-g\}(x)$. [সজ্জিত জোড়ের সেট নির্ণয় কর]

(i) $f(x)$ is : (0, 2), (1, 1) and (2,−4) *Ans.* (0, 3), (1, 2) (2,−3)

$g(x)$ is : (0, 1)), (1, 1) and (2, 2) *Ans.* (0, 1), (1, 0), (2,−6)

(ii) $f(x)$: (−3, 1), (−2, 5), (1, 4) and (3, 6) *Ans.* (−2, 2)

$g(x)$ is : (−2,−3), (−1, 7), (0, 5) and (2, 1) *Ans.* (−2, 8)

(iii) $f(x)$ is : (−10, 3), (−4, 8), (0, 1) and (15, 3)

$g(x)$ is (−5, 1), (1, 3), (2, 2) and (10, 10)

3. Find the ordered pairs of Ex−2 above.

4. Which of the following relations in I are functions ? Give the domain and range of each function. [সম্পর্ক *I* এর মধ্যে কোনটি ফাংশান। ইহাদের ডোমেন ও রেঞ্জ নির্ণয় কর।]

$F=\{(x, y) \mid (x, y) \in I\times I,\ y^4=x\}$ Ans. not function.

$F=\{(x, y) \mid (x, y) \in I^2, y=x^3\}$ *Ans.* $D=I$; R=set of integers expressive as the cube of integer.

$F=\{(x, y) \mid (x, y) \in I^2,\ x<y\}$ *Ans.* not function.

$F=\{(x, y) \mid (x, y) \in I^2, x^2-y=16\}$ *Ans.* $D=I$. R=set of all integers expressible as 16 less than the square.

$F=\{(x, y) \mid (x, y) \in A\times B, y=2x^2+3\}$

$A=\{x \mid x\in I,\ 1\leqslant x\leqslant 5,\quad B=\{x \mid x\in I;\ 1\leqslant x\leqslant 100\}$

Ans. yes, $D=A$, Range={5, 11, 21, 35, 53)

F=(1, 5) (2, 11), (3, 21), (4, 35), (5, 53)

$F=\{x, y) \mid (x, y) \in R\times R,\ y=x^2\},\ R=\{x \mid x\in I, \mid x\mid\leqslant 10\}$

Ans. yes, F=(0, 0), (1, 1), (2, 4), (3, 2), (−1, 1), (−2, 4), (−3, 9)

$F=\{(x, y) \mid (x, y) \in R^2,\ x=y^2\}\ R=\{x \mid x\in I, \mid x\mid\leqslant 10\}$

Ans. No. F=(0, 0), (1, 1), (1,−1), (4, 2) (4,−2), (9, 3), (9,−3)

The pairs (1, 1) and (1,−1) have same first component.

5. (i) if I is the set of all integers and $x \in I$, which of the following mapping of $I \times I$ are mapping of I onto I ? What are 1–1 of I onto I. [I একটি পূর্ণ সংখ্যার সেট $x \in I$. $I \times I$ চিত্রের কোনটি I এর উপর I চিত্রণ ? I এর উপর I হইলে কোনটি এক-এক চিত্রণ ?]

(a) $x \to x+3$ i, e $F=\{x, y) \in I^2 \mid y=x+3\}$

(b) $x \to x^2+x$, i, e, $F=\{(x, y) \in I^2 \mid y=x^2+x\}$

(c) $F=\{(x, y) \in I^2 \mid y=x^3\}$

(d) $F=\{X, y) \in I^2 \mid y=2x-1\}$

(e) $F=\{x, y) \in I^2 \mid y=x-4\}$

6. (i) Let $A=\{x \mid x \in R \text{ and} -5 \leqslant x \leqslant 5\}$

and $B=\{y \mid y \in R \text{ and} -2 \leqslant y \leqslant 2\}$

Find $A \times B$ and sketch $A \times 3$ in certain plane

(ii) Let $A=\{x \mid x \in R$ and $\mid x-1 \mid <2\}$

and $B=\{y \mid y \in R$ and $\mid y+2 \mid >1\}$

Find $A \times B$ and sketch $A \times B$ in cartesian plane.

(iii) Let $A=\{x \mid x \in R, -6 \leq x \leqslant 6\}$

and define the relation R in A. Find $A \times A$ and sketch $A \times B$ in certesian plane.

7. Describe which of the relation below are functions, which are into and which are 1–1 : also 1–1 correpondence,

(i) $f: \{(x, y) \in R \times R \mid y=2x+3\}$ *Ans.* 1–1 Correspondence

(2) $f: \{(x, y) \in R \times R \mid y=\sqrt{x}$, R^+ be the set of all non-negative real numbers) $D_f \neq R$, it is not function.

(3) $f: \{(x, y) \in R \times R \mid y=\sin x\}$ *Ans.* into function.

(4) $f: \{(x, y) \in R \times R \mid y=x^3\}$ *Ans.* one–one function

(5) $f: \{(x,y) \in R \times R \mid f(x)=x^2+1\}$ *Ans.* function onto.

(6) $f: \{(x, y) \in R^2 \mid f(x)=e^x\}$ *Ans.* f is one–one function.

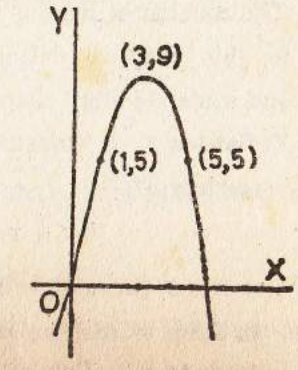
(7) $f: \{(x, y) \varepsilon R \times R \mid y = \tan x\}$ Ans. f is onto.

(8) $f: \{(x, y) \varepsilon R^2 \mid f(x) = \log x$. where x is real$\}$

8. Let A and B set of real numbers

$f = \{(x, y) \varepsilon A \times B \mid$
$f(x) = 6x - x^2\}$
find f $[0, 1], f(1, 4), f^{-1}$ $([0, 5)]$,
$f^{-1}([5, 9])$

Ans. $[0, 5], [5, 9]$ $[0, 8]$ $U [5, 6], [1, 5]$



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9. If $f(x) = \{(x, y \varepsilon R^2 \mid f(x) = x^2 - 4x + 4$ find $f([0, 1]), f([3; 4])$,
$f^{-1}([1, 2])$

Ans. $[1, 9], [1, 4], [2 - \sqrt{2}, 1]$
$U [3, 2 + \sqrt{2}]$

10. If $f: x \to y$ is a function and A X, then what is meant by $f^{-1}(A)$?

Ans. See Higher Algebra, set Theory Art. 9.6

Show that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

and $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

see Higher Algebra, set theory Art. 9.7

11. find the domain of the following (চারণ ক্ষেত্র নির্ণয় কর।

D. U. 1991

$f(x) = x^2 - 1$, $g(x) = 3x + 1$

(i) $\frac{f}{g}(x)$ (ii) $\frac{g}{f}(x)$ (iii) $f(g(x))$ (v) $g(f(x))$

Ans. $(-\infty, \infty) - \{^1/_3\}$, $(-\infty, \infty) - \{-1, 1\}$, $(-\infty, \infty)$, $(-\infty)$, $(-\infty, \infty)$

CHAPTER II

LIMITS

2.1. The concept of limit is the most basic concept of Calculus. In this chapter, definition of a limit, properties of limits and some standard theorems on limits will be discussed.

2.2. **Definition :** A constant a is said to be a limit of the variable x, if

$$0 < | x - a | < \delta$$

Where δ is a pre-assigned positive quantity as small as we please. In other words, we say that "**x approaches the constant a**, or '**x tends to a**'", Symbollically, it is denoted by

$$x \to a \quad \text{or} \quad \lim x = a,$$

Note that $x \to a$ never implies that $x = a$

When x approaches a but always remains less than a, we say that x approaches a from the left on the real axis and we write. $x \to a^-$

Similarly, when x tends to a with values always greater than a' x approaches a from the right on the real axis and this is expressed as $x \to a^+$

For example, when x takes successive numerical values, 3·9, 3.99, 3·999, 3·999...... x tends to 4 from the left or $x \to 4^-$. Smilarly, if x assumes the successive values

4·1, 4·01, 4·001, 4.0001, ... x approaches 4 from the right or $x \to 4^+$.

Considering the two sets of values

{3·9, 3·99, 3·999, 3·999.........}

and {4·1, 4·01, 4·001, 4·001...........}

assumed by x, we say that $x \to 4$

$$\text{or } 0 < | x - 4 | < \delta \text{ where } \delta = \frac{1}{10^n}, \; n = 1, 2, 3, 4, \ldots\ldots$$

Art 2·3 Limits of a function :

Definition : $\lim_{x \to a} f(x) = l$,

In common language '$\lim_{x \to a} f(x) = l$, means that $f(x)$ is very close to the fixed number l ***whenever x is very close to a.***

In terms of mathematical analysis, we give the meaning of $\lim_{x \to a} f(x) = l$, as follows :

Let S be a set of numbers and let $f(x)$ be defined for all numbers in S (that is, S is a subset of D_f, the domain of f). We assume that S is arbitrarily close to a ; i. e. given $\delta > 0$, there exists an element x of S such that $0 < | x - a | < \delta$, we shall say that $f(x)$ approaches the limit l as x tends to a if the following condition is satisfied :

Given a number $\epsilon > 0$, however small, there exists a positive number δ such that for all x in S satisfying

$$0 < | x - a | < \delta$$

we have

$$| f(x) - l | < \epsilon$$

If this is the case, we write

$$\lim_{x \to a} f(x) = l.$$

Note that *a may or may not belong to* D_f.

We can also define

$$\lim_{x \to a} f(x) = l$$

In the following way :

We write

$$\lim_{h \to 0} f(a+h) = l$$

and say that the limit of $f(a+h)$ is l if $h \to 0$, provided given $\epsilon > 0$, however small, there exists $\delta > 0$ such that whenever $0 < |h| < \delta$ and $(a+h) \in S$, then

$$|f(a+h) - l| < \epsilon$$

Cor : If the set S is such that any number $x \in S$ is less than a and for any $\epsilon > 0$, however small, there exists a positive number δ and fixed number l_1 satisfying.

$$0 < |x - a| < \delta \text{ with } |f(x) - l_1| < \epsilon,$$

Left hand Limit : We say that l_1 is the **left hand limit of** $f(x)$ as $x \to a'$ and we write

$$\lim_{x \to a^-} f(x) = l_1$$

Similarly, if each number x of S is greater than a and there is a fixed number l_2 such that whenever

$$0 < |x - a| < \delta$$

We have $|f(x) - l_2| < \epsilon$,

Righ hand Limit : Where ϵ and δ have the same meaning as before, we say that l_2 is **the right hand limit** of $f(x)$. This is expressed as $\lim_{x \to a^+} f(x) = l_2$

Note : When the set S consists of numbers less than as well as greater than a, we say that $\lim_{x \to a} f(x)$ exists,

if $l_1 = l_2 = l$ and it is equal to l.

Ex 1. Prove that C. H. 199?

$\lim_{x \to 2} (3x+4) = 10$ by (δ, ϵ) definition of a function

Let us consider an arbitrary positive number $\delta > 0$ howeve small such that

$$|3x + 4 - 10| < \epsilon \quad \text{or} \quad |3x - 6| < \epsilon$$

$$\text{i. e.,} \quad |x - 2| < \tfrac{1}{3}\epsilon,$$

$$|x - 2| < \delta \quad \text{when } \delta = \tfrac{1}{3}\epsilon > 0$$

This means that, $\lim_{x \to 2} (3x+4) = 10$

Ex. 2. Prove that $\lim_{x \to a} \dfrac{x^2 - a^2}{x - a} = 2a$

Let us consider an arbitrary positive number $\epsilon > 0$, however small. Then

$$\left|\frac{x^2 - a^2}{x - a} - 2a\right| < \epsilon. \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$\text{or} \left|\frac{(x+a)(x-a)}{x-a} - 2a\right| = |x + a - 2a| < \epsilon \quad (\because x \neq a)$$

$$\text{or,} \quad |x - a| < \epsilon$$

$$\text{or,} \quad |x - a| < \delta \quad \text{where } \delta = \epsilon \ \ldots \ldots \ (2)$$

Hence $\lim\limits_{x\to a} \dfrac{x^2-a^2}{x-a}=2a$ [From (1) and (2)]

Ex. 3. From the definition of (δ, ϵ) show that

$$\lim_{x\to 3} (2x^3-3x^2-18x+29)=2$$

Let for a given $\epsilon>0$, however small,

$$0<|x-3|<\lambda<1\ldots\ldots (1)$$

$\therefore \quad |2x^3-3x^2-18x+29-2|=|2x^3-3x^2-18+27|$

$=|2(x-3)^3+15(x-3)^2+18(x-3)|$

$\leqq 2|x-3|^3+15|x-3|^2+18|x-3|\leqslant 2\lambda^3+15\lambda^2$

$+18\lambda<35\lambda$

$(\therefore \lambda<1, \therefore \lambda^3<\lambda^2<\lambda$

Now $|(2x^3-3x^2-18x+29)-2|<\epsilon$, where $\epsilon=35\lambda$ (2)

or, $\lambda=\epsilon/35$ therefore, we can determine a small positive number δ depending on ϵ such that the limit is established. Here $\delta=\epsilon/35$

Hence $\lim\limits_{x\to 3} (2x^3-3x^2-18x+29)=2$ [From (1) and (2)]

Ex. 4. For $f(x)=x^2-3x+5$, find $\delta>0$ such that whenever $0<|x-2|<\delta$ then $|f(x)-3|<\epsilon$, when (a) $\epsilon=\frac{1}{3}$, (b) $\epsilon=0{\cdot}07$

Let ϵ be given, we are to find $\delta>0$ such that

$0<|x-2|<\lambda<1$ implies that

$|f(x)-3|=|(x^2-3x+5)-3|$

$=|(x-2)^2+(x-2)|\leqslant|x-2|^2+|x-2|<\lambda^2+\lambda<2\lambda$

$[\therefore \lambda^2<\lambda<1]$

So $|f(x)-3|<\epsilon$, if $\lambda=\epsilon/2$. Therefore we can determine a positive number δ smaller than 1 and equal to $\epsilon/2$ so that the limit exists.

Hence $\delta=\epsilon/2$

(a) If $\epsilon=\frac{1}{3}$ then $\epsilon=\dfrac{\epsilon}{2}=\dfrac{1}{3\cdot 2}=\dfrac{1}{6}$

(b) If $\epsilon=0{\cdot}07$, then $\delta=\dfrac{\epsilon}{2}=\dfrac{0.07}{2}=0{\cdot}035$

2 : 4 Distinction Between $\lim\limits_{x\to a} f(x)$ and f (a)

$f(a)$ means that the value of $f(x)$ when $x=a$.

or, the value of $f(x)$ at $x=a$ is $f(a)$; clearly $a\in D_f$.

$\lim\limits_{x\to a} f(x)$ is a statement about the values of $f(x)$ when x assumes all values of x in the neighbourhood of a except $x=a$.

When $\lim\limits_{x\to a} f(x)=f(a)$

The function $f(x)$ is said to be **continuous** at $x=a$.

Ex. 5. A function $f(x)$ is defined in the following way

$f(x)=1+2x$ for $-\frac{1}{2}\leq x<0$

$=1-2x$ for $0\leqslant x<\frac{1}{2}$

$=-1+2x$ for $x>\frac{1}{2}$

Investigate the function at $x=0$ and $x=\frac{1}{2}$

For $x=0$

$\lim\limits_{h\to 0^-} f(0+h)=\lim\limits_{h\to 0^-}\{1+2(0+h)\}=1$; take $f(x)=1+2x$ as $x<0$

$\lim\limits_{h\to 0^+} f(0+h)=\lim\limits_{h\to 0^+}\{1-2(0+h)\}=1$; take $f(x)=1-2x$. as $x>0$

and $f(0)=1-2.0=1$. take $f(x)=1-2x$; as $x=0$

Thus we get, $\lim\limits_{x\to 0} f(x)=1=f(0)$.

i. e., the value of the function $f(x)$ at $x=0$ is equal to the limit of the function $f(x)$ when $x\to 0$. Hence $f(x)$ is continuous at $x=0$

For $x=\frac{1}{2}$

$$\lim_{h\to 0^+} f(\tfrac{1}{2}+h)=\lim_{h\to 0^-}[-1+2(\tfrac{1}{2}+h)]=-1+1=0, \text{ take } f(x)=-1+2x$$

$$\lim_{h\to 0^-} f(\tfrac{1}{2}+h)=\lim_{h\to 0^+}\{1-2(\tfrac{1}{2}-h)\}=-1+1=0, \text{ take } f(x)=-1+2x$$

$$\lim_{h\to 0^-} f(\tfrac{1}{2}+h)=\lim_{h\to 0^+} f(\tfrac{1}{2}+h)=0$$

Hence the limit of the function $f(x)$ exists and is equal to zero

2. 5. Meaning of the symbols $+\infty$ and $-\infty$ (সূচকদ্বয়ের অর্থ)

If a variable x assumes all positive values and increases without limit such that it is greater than any positive number, however big which we may imagine, x is said to tend to infinity and symbolically it is written as $x\to\infty$.

Similarly if the variable possessing negative values only decreases without any limit and less than any negative number which we may imagine, x said to tend to minus infinity ($-\infty$) and it is denoted by $x\to-\infty$.

These symbols $+\infty$ and $-\infty$, by themselves do not possese any meaning and the phrases in which they occur do not take any meaning from the symbols.

The meaning of the phrase $x\to\infty$ is not found from the statement $x\to a$ by substituting $a=\infty$.

2.6 Meaning of

(i) $\lim_{x\to a} f(x)=\infty$ (ii) $\lim_{x\to a} f(x)=-\infty$

(i) A function $f(x)$ is said to tend to $+\infty$ when x approaches a, if for any pressigned positive number N, however large, we can determine another positive number δ such that $f(x)>N$ for all values of x satisfying the inequality :

$$0<|x-a|\leqq\delta \text{ with } x>a, \text{ or } a<x\leqq a+\delta$$

(ii) A function $f(x)$ is said to tend to $-\infty$ when x approaches a, if for any given positive number N, however large, we can determine a positive number δ such that

$$-f(x)>N, \text{ or, } f(x)<-N$$

for all values of x satisfying the inequality $0<x-a\leqslant\delta$

Ex. 6. Evaluate $\lim_{x\to 2}\frac{3}{(x-2)^2}$

Let $f(x)=\frac{3}{(x-2)^2}$

The limit of $f(x)$ is $+\infty$ at $x\to 2$ if both $f(a+h)$ and $f(a-h)$ become greater and greater when h approaches zero with $h>0$

$$\lim_{h\to 0^+} f(2+h)=\lim_{h\to 0} f(2+h)=\lim_{h\to 0}\frac{3}{(2+h-2)^2}=\lim_{h\to 0}\frac{3}{h^2}=\infty$$

$$\lim_{h\to 0^+} f(2-h)=\lim_{h\to 0^+}\frac{3}{(2-h-2)^2}=\lim_{h\to 0^+}\frac{3}{h^2}=\infty$$

$$\lim_{h\to 0^+} f(2+h)=\lim_{h\to 0^+} f(a-h)=\infty$$

Therefore limit exists and $\lim_{x\to 2}\frac{3}{(x-2)^2}=\infty$

Ex 7. Evaluate $\lim_{x\to 2}\frac{-1}{(x-2)^2}$

Let $f(x)=-\frac{1}{(x-2)^2}$

We have

$$\lim_{h\to 0^+} -f(2+h) = \lim_{h\to 0^+} \frac{-1}{(2+h-2)^2} = -\infty$$

$\therefore \lim_{h\to 0} f(2+h) = -\infty$

Ex. 8. **Show that**

$$\lim_{x\to 0+} \frac{1}{x} = \infty, \quad \text{and} \quad \lim_{x\to 0^-} \frac{1}{x} = -\infty$$

Let $f(x) = 1/x$

Let $x>0$, so that $f(x)$ is positive. If x diminishes gradually then $f(x)$ or $1/x$ increases gradually. If x tends to zero then $f(x)$ or $\frac{1}{x}$ tends to be infinite.

If we take N any given positive number, however large then $1/x > N$ when $x < 1/N$

Hence $\lim_{x\to 0^+} (1/x) = \infty$

Let $x<0$, so that $f(x)$ is negative.

Let N be any given positive number then for $x<0$,

$-f(x) > N$ or, $-(1/x) > N$. if $x < -1/N$,

As N increases, $\frac{1}{N}$ decreases

Hence $\lim_{x\to 0^-} (1/x) = -\infty$

Art 2·6 Meaning of $\lim_{x\to\infty} f(x) = l$.

Let a be any positive numbers, that $f(x)$ is defined for all numbers $x \geq a$. We say that $f(x)$ *approaches* l *as* x *tends to infinity*, and we write $\lim_{x\to\infty} f(x) = l$

if the following condition is satisfied; Given any $\epsilon > 0$, there exists a positive number A, such that whenever $x > A$., we have

$$|f(x) - l| < \epsilon.$$

A similar meaning is given to $\lim_{x\to -\infty} f(x) = l$.

That is, for a given $\epsilon > 0$, there exists a positive number A, such that whenever

$$x < -A, \text{ we get } |f(x) - l| < \epsilon$$

[where $(-\infty, -A)$ is a subset of D_f]

(i) when $x > 1$. Let $x = 1+h$, $0<h<1$

By Binomial theorem,

$$x^n = (1+h)^n = 1 + nh + \frac{n(n-1)h^2}{\lfloor 2} + \ldots$$

$\therefore \; x^n > 1 + nh > nh$

Let N be any positive number, such that $nh > N$ or, $n > N/h$.

Therefore $x^n > N$ where N is any large number for all $n > N/h$.

Hence $\lim_{n\to\infty} x^n = \infty$ when $x > 1$

(ii) When $-1 < x < 1$ or $|x| < 1$,

let $|x| = \frac{1}{1+h}$, where $0<h<1$.

$$\therefore |x|^n = \frac{1}{(1+h)^n} = \frac{1}{1 + nh + \frac{n(n-1)h^2}{2} + \ldots} < \frac{1}{1+nh} < \frac{1}{nh}$$

If $n\to\infty$, $nh\to\infty$ and so $\frac{1}{nh}\to 0$.

Hence $\lim\limits_{n\to\infty} |x|^n=0$

$\Rightarrow \lim\limits_{n\to\infty} x^n=0$ when $-1<x<1$.

(iii) When $x=-1$ or $+1$ according as n is an odd or even integer. Therefore when $n\to\infty$, x^n oscillates between -1 and $+1$ when $x=-1$.

Hence $\lim\limits_{n\to\infty}(-1)^n$ does not exist.

(iv) When $x<-1$,

or $-x>1$,

we have from case (ii),

$\lim\limits_{n\to\infty}(-x)^n=\infty$

or $\lim\limits_{n\to\infty} x^n=(-1)^n\infty=-\infty$ or $+\infty$ according as n is odd or even.

Hence $\lim\limits_{n\to\infty} x^n$ does not exist for $x<-1$.

(v) The trivial case:

When $x=0$, $x^n=0$ for all positive integral values of n

$\therefore$ $\lim\limits_{n\to\infty} x^n=0$ when $x=0$.

2. 7. Find $\lim\limits_{n\to\infty}\frac{x^n}{\lfloor n}$,

Where n is a positive integer and x is any real number.

Let x be any positive number which lies betwen two consecutive integers m and $m+1$, that is, $m<x<m+1$, where m is a positive integer,

We have,

$$\frac{x^n}{\lfloor n}=\frac{x}{1}\cdot\frac{x}{2}\cdot\frac{x}{3}\cdots\frac{x}{m}\cdot\frac{x}{m+1}\cdots\frac{x}{n}$$

$$=\frac{x^m}{\lfloor m}\cdot\frac{x}{m+1}\cdot\frac{x}{m+2}\cdots\frac{x}{n}$$

Now $\frac{x}{m+2}<\frac{x}{m+1}$, $\frac{x}{m+3}<\frac{x}{m+1},\ldots,\frac{x}{n}<\frac{x}{m+1}$.

$$\therefore\quad \frac{x^n}{\lfloor n}<\frac{x^m}{\lfloor m}\left[\frac{x}{m+1}\cdot\frac{x}{m+1}\cdot\frac{x}{m+1}\cdots\frac{x}{m+1}\right]$$

or $\frac{x^n}{\lfloor n}<\frac{x^m}{\lfloor m}\cdot\left(\frac{x}{m+1}\right)^{n-m}$ (since in the square bracket, there are $(n-m)$ factors').

or $\frac{x^n}{\lfloor n}<\frac{(m+1)^m}{\lfloor m}\left(\frac{x}{m+1}\right)^n$

Hence $0<\frac{x^n}{\lfloor n}<L.\left(\frac{x}{m+1}\right)^n$

Where $L=\frac{(m+1)^m}{\lfloor m}$ is a constant free from n.

As $\frac{x}{m+1}$ is positive and less than 1, that is, $0<\frac{x}{m+1}<1$,

therefore $\lim\limits_{n\to\infty}\left(\frac{x}{m+1}\right)^n=0$.

Hence $\lim\limits_{n\to\infty}\frac{x^n}{\lfloor n}=0$

Let x be any negative number say $x=-a$. so that a is a positive number.

$$\left|\frac{x^n}{Ln}\right|=\left|\frac{(-1)^n a^n}{Ln}\right|=\frac{a^n}{Ln}$$

By the previous result, $\lim\limits_{n\to\infty}\frac{a^n}{Ln}=0$

Hence $\lim\limits_{n\to\infty}\frac{x^n}{Ln}=0$ whatever be the value of x

2.8. Fundamental Theorems on limits (সীমার মৌল উপপাদ্য)

Theorem 1. The limit of a sum of any definite number of function is equal to the algebraic sum of the limits of these functions.

Let us first consider two functions of x

If $\lim\limits_{x\to a} f(x)=l$, $\lim\limits_{x\to a}\phi(x)=m$

then $\lim\limits_{x\to a}\{f(x)\pm\phi(x)\}=l\pm m$

If ϵ be a given arbitrary small positive number, we can choose another two positive smaller numbers δ_1 and δ_2 depending on ϵ such that

$|f(x)-l|<\epsilon/_2$ for $0<|x-a|<\delta_1$......(1)

$|\varphi(x)-m|<\epsilon/_2$ for $0<|x-a|<\delta_2$......(2)

Let us consider another positive number δ which is less than both δ_1 and δ_2. Then

$|f(x)-l|<\epsilon/_2$ |, $|\varphi(x)-m|<\epsilon/_2$ for $0<|x-a|<\delta$

Thus

$|f(x)+\varphi(x)-(l+m)|=|(f(x)-l)+|\varphi(x)-m)|$

$<|f(x)-l|+|\varphi(x)-m|$

$<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$ when $0<|x-a|<\delta$.

Hence $\lim\limits_{x\to a}\{f(x)+\varphi(x)\}=l+m$......(3)

Similarly we can prove that

$\lim\limits_{x\to a}\{f(x)-\varphi(x)\}=l-m$,......(4)

Now combining (3) and (4), we have

$\lim\limits_{x\to a}\{f(x)\pm\phi(x)\}=l\pm m$

Similarly we can extend it to any number of functions

$\lim\limits_{x\to a}\{f(x)\pm\phi(x)\pm\psi(x)\pm\ldots\}$

$=\lim\limits_{x\to a}f(x)\pm\lim\limits_{x\to a}\phi(x)\pm\lim\limits_{x\to a}\psi(x)\pm\ldots\ldots$

$=l_1\pm l_2\pm l_3\pm\ldots\ldots$

Theorem 2 The limit of the product of any finite number of functions is equal to the product of the limits of these functions :

If $\lim\limits_{x\to a}f_1(x)=l_1$, $\lim\limits_{x\to a}f_2(x)=l_2$,.........

then $\lim\limits_{x\to a}[f_1(x)f_2(x)\ldots\ldots]=l_1 l_2\ldots\ldots$

Proof : We prove that result for two functions first.

Let $\lim\limits_{x\to a}f(x)=l$ and $\lim\limits_{x\to a}g(x)=m$.

Then for a given number $\epsilon>0$, however small, there exists a positive number δ such that

$$\left.\begin{array}{ll}|f(x)-l|<\dfrac{\epsilon}{2|m|}, & \text{if } m\neq 0\\ |f(x)-l|<\dfrac{\epsilon}{2}, & \text{if } m=0\end{array}\right\}$$

$|g(x)-m|<\dfrac{\epsilon}{2}$ $\quad \dfrac{1}{|l|+1}\leqslant\dfrac{\epsilon}{2}$, (the equality holds when $l=0$),

for $0<|x-a|<\delta$.

From $|f(x)-l|<\dfrac{\epsilon}{2|m|}$ or $|f(x)-l|<\dfrac{\epsilon}{2}$

We have, $|f(x)|<|l|+1$

Hence, for $0<|x-a|<\delta$.

$$|f(x)\,g(x)-lm| = |\{f(x)g(x)-f(x)m\}+\{f(x)m-lm\}|$$
$$= |f(x)(g(x)-m)+m(f(x)-l)|$$
$$\leqslant |f(x)|\;|g(x)-m| + |m|\;|f(x)-l|$$
$$<(|l|+1)\frac{\epsilon}{2}\frac{1}{|l|+1}+|m|\cdot\frac{\epsilon}{2|m|}$$
$$=\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$$

Therefore $\lim_{x\to a} f(x)\,g(x)=lm=\lim_{x\to a} f(x).\lim_{x\to a} g(x)$.

Cor : It c be a constant, them

$\lim_{x\to a} cf(x)=cl$ where $\lim_{x\to a} f(x)=l$.

The theorem can be extented to any number of functions.

Theorem 3. If $\lim_{x\to a} f(x)=l$, then $\lim_{x\to a}\dfrac{1}{f(x)}=\dfrac{1}{l}$.

providid $l\neq 0$,

Proof : Given $\epsilon>0$ let ϵ_1 be the smaller of $\dfrac{|l|}{2}$ and ϵ Then there exists $\delta>0$ such that whenever

$0<|x-a|<\delta$, we have

$$|f(x)-l|<\epsilon_1 \quad \text{(i)}$$

and also

$$|f(x)-l|<\frac{\epsilon|l|^2}{2} \quad \text{(ii)}$$

From (i).

$$|f(x)|>|l|-\epsilon_1\geq|l|-\frac{|l|}{2}=\frac{|l|}{2} \quad \text{... (iii)}$$

when $0<|x-a|<\delta$. For such x,

$$\left|\frac{1}{f(x)}-\frac{1}{l}\right|=\frac{|l-f(x)|}{|f(x)||l|}<\frac{2}{|l|}\cdot\frac{1}{|l|}\cdot\frac{\epsilon|l|^2}{2}$$

[by (ii) and (iii)]

or $\left|\dfrac{1}{f(x)}-\dfrac{1}{l}\right|<\epsilon$

Hence $\lim_{x\to a}\dfrac{1}{f(x)}=\dfrac{1}{l}$.

Cor : If $\lim_{x\to a} f(x)=l$ and $\lim_{x\to a} g(x)=m$

then $\lim_{x\to a}\dfrac{f(x)}{g(x)}=\lim_{x\to a} f(x).\lim_{x\to a}\dfrac{1}{g(x)}=\dfrac{l}{m}$,

provided $m\neq 0$

The theorem can be easily extended for any definite number of functions

$\mathrm{Lt}_{x\to a} f(x)\varphi(x)\psi(x)\;\dots\;\dots=lmn\;\dots\;\dots$

Theorem 4 : Limit of a function of a function

(ফ্যাংশানের ফ্যাংশন সীমা)

If $\lim\limits_{x\to a} f(x)=t$ and $\lim\limits_{u\to t} \varphi(u)=\varphi(t)$

is in other words, if $\varphi(u)$ is continuous at $u=t$, then

$\lim\limits_{x\to a} \varphi\{f(x)\} = \varphi\{\lim\limits_{x\to a}(x)\}$

As $\lim\limits_{u\to t} \varphi(u)=\varphi(t)$, we can select a number δ, depending upon ϕ to a given small positive number $\in$, such that

$| \varphi(f(x)-t) | <\in$, for $| f(x)-t | \leqslant \delta_1$, ... (1)

Since $\lim\limits_{x\to a} f(x)=t$, a number δ_2 can be selected depending on δ_1 such that

$| f(x)-t | <\delta_1$ for $| x-a | <\delta_2$ .... (2)

Now from [1] and [2] we have

$| \varphi\{f(x)-t\} | <\in$ for $0< | x-a | <\delta_2$

i e $\lim\limits_{x\to a} \varphi\{f(x)\} = \varphi(t)=\phi\{\lim\limits_{x\to a} f(x)\}$

Ex 9. If $\lim\limits_{x\to a} f(x)=l$

[i] $\lim\limits_{x\to a} \sin f(x) = \sin\left(\lim\limits_{x\to a} f(x)\right)=\sin l$

[ii] $\lim\limits_{x\to a} e^{f(x)} = e^{\lim\limits_{x\to a} f(x)} = e^{l}$

[iii] $\lim\limits_{x\to a} (f(x))^n = \left\{\lim\limits_{x\to a} f(x)\right\}^n = l^n$

Summary on the fundamental Theorems on limits, (সীমার মৌলিক উপপাদ্য)

If $\lim\limits_{x\to a} f(x)=l$. and $\lim\limits_{x\to a} \phi(x)=m$

where l and m are finite quantities then

(i) $\lim\limits_{x\to a} \{f(x)\pm f(x)\}=l\pm m$

(ii) $\lim\limits_{x\to a} \{f(x)\phi(x)\}=lm$

(iii) $\lim\limits_{x\to a} \left\{\dfrac{f(x)}{\phi(x)}\right\}=\dfrac{l}{m}$ if $\lim\limits_{x\to a} \phi(x)\neq 0$

(iv) $\lim\limits_{x\to a} \phi\{f(x)\}=\phi\left(\lim\limits_{x\to a} f(x)\right)$

Limits of a sequence :—(See fuuction) (ধারার সীমা).

A number l is the limit of a sequence $a_1\ a_2 \cdots \ldots a_n \ldots\ldots$

or, $\lim\limits_{n\to\infty} a_n=l$

If for every positive value of $\in$ ($\in>0$) however small, there is a number N such that

$| a_n-l| <\in$ when $n\geqq N$.

we can also express limit of a sequence as

$\{a_n\}\to l$ when $n\to\infty$,

Ex, 10. Show that

$\lim\limits_{n\to\infty} \dfrac{3n+2}{n+3}=3.$

Here

$\left|\dfrac{3n+2}{n+3}-3\right|=\left|\dfrac{-7}{n+3}\right|=\dfrac{7}{n+3} <\in$ ...... (1)

if $n+3>7/\in$ or, if $n>7/\in-3=N$

Thus for every positive value of ϵ, there will be a number $N = 7/\epsilon - 3$ such that for $n > N$. ; (i) holds

$$\therefore \lim_{n \to \infty} \frac{3n+2}{n+3} = 3.$$

2. 13. Some Important Limits (কতিপয় গুরুত্বপূর্ণ সীমা)

(a) Prove that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1. \quad \text{when } x \text{ is measured in radian.}$$

Proof : This function is not continuous at $x=0$. Let us consider a circle of unit radius, Let A, B, C be three points on the circle such that

$\angle AOB = \angle AOC = x$

radians, where

$0 > x < \pi/2$

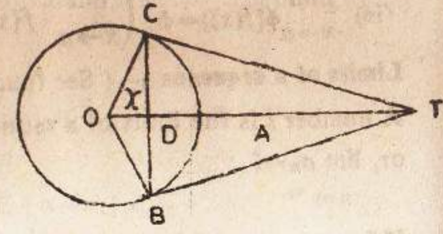

Fig. 33

Let the tangents at B and C meet at T on OT. Since BC is a chord of the circle, we have.

Chord $BC <$ arc $BAC < TC + TB$

or, $2\,CD < 2$ arc $AC < 2CT$,

or, $\frac{CD}{OC} < \frac{\text{arc } AC}{OC} < \frac{CT}{OC}$ or, $\sin x < x < \tan x$.

or, $1 < \frac{x}{\sin x} < \frac{1}{\cos x}$ or, $1 > \frac{\sin x}{x} > \cos x$ ... ... (I)

This inequality is assumes that $x > 0$, It is also true if $x > 0$

Now $\lim_{x \to 0} \cos x = 1$

Therefore from (1) we notice that in the limits both sides of $\frac{\sin x}{x}$ are the same and equal to unity,

Hence $\lim_{x \to 0} \frac{\sin x}{x} = 1$

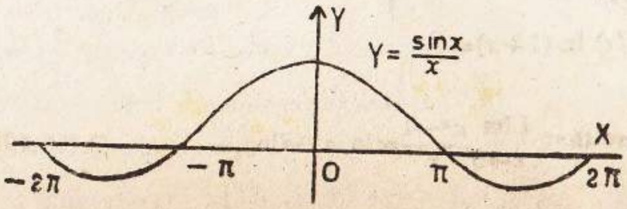

fig. 47

(b) Prove that $\lim_{x \to 0} (1+x)^{1/x} = e$

Proof :—Expanding binomially.

$$(1+x)^{\frac{1}{x}} = \left\{ 1 + \frac{1}{x}.x + \frac{(1/x)(1/x-1)}{\underline{|2}} x^2 + \dots \right\}$$

$$= \left\{ 1 + 1 + \frac{(1-x)}{\underline{|2}} + \frac{(1-x)(1-2x)}{\underline{|3}} + \dots\dots \right\}$$

Now $\lim_{x \to 0} (1-x) = 1$, $\lim_{x \to 0} (1-2x) = 1$, etc,

Hence

$$\lim_{x \to 0} (1+x)^{1/x} = 1 + \frac{1}{\underline{|1}} + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \dots\dots = e$$

(c) **Proof** $\lim_{x \to 0} (1 + 1/x)^x = e$,

The result follows on expanding $(1 + 1/x)^x$ binomially and then taking the limit.

Thus for every positive value of ϵ, there will be a number $N = 7/\epsilon - 3$ such that for $n > N$. ; (i) holds

$$\therefore \lim_{n \to \infty} \frac{3n+2}{n+3} = 3.$$

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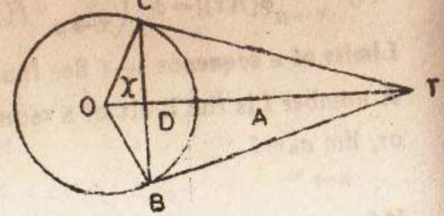

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Therefore from (1) we notice that in the limits both sides of $\frac{\sin x}{x}$ are the same and equal to unity,

Hence $\lim_{x \to 0} \frac{\sin x}{x} = 1$

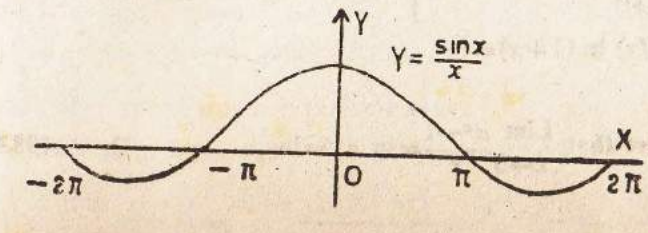


fig. 47

(b) Prove that $\lim_{x \to 0} (1+x)^{1/x} = e$

Proof :—Expanding binomially.

$$(1+x)^{\frac{1}{x}} = \left\{ 1 + \frac{1}{x}.x + \frac{(1/x)(1/x-1)}{\lfloor 2}x^2 + \dots \right\}$$

$$= \left\{ 1 + 1 + \frac{(1-x)}{\lfloor 2} + \frac{(1-x)(1-2x)}{\lfloor 3} + \dots\dots \right\}$$

Now $\lim_{x \to 0} (1-x) = 1$, $\lim_{x \to 0} (1-2x) = 1$, etc,

Hence

$$\lim_{x \to 0} (1+x)^{1/x} = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \dots \dots = e$$

(c) **Proof** $\lim_{x \to 0} (1 + 1/x)^x = e$,

The result follows on expanding $(1+1/x)^x$ binomially and then taking the limit.

(d) prove that $\lim_{x\to0}\left\{\frac{1}{x}\log(1+x)\right\}=1$

$\lim_{x\to0}(1/x)\ln(1+x)=\lim_{x\to0}\ln(1+x)^{1/x}$

$=\ln\left\{\lim_{x\to0}(1+x)^{1/x}\right\}=\ln e=\log_e e$

$\Rightarrow\lim_{x\to0}(1/x)\ln(1+x)=1.$

(e) prove that $\lim_{x\to0}\frac{a^x-1}{x}=\ln a=\log_e a$ D. U. 1983

Proof :

Put $a^x-1=y$ or, $a^x=1+y$

∴ $x\ln a=\ln(1+y)$ or, $x=\frac{\ln(1+y)}{\ln a}$,

Now $\frac{a^x-1}{x}=\frac{y\ln a}{\ln(1+y)}=\frac{\ln a}{(1/y)\ln(1+y)}$

∴ $\lim_{x\to0}\frac{a^x-1}{x}=\lim_{y\to0}\frac{\ln a}{(1/y)\ln(1+y)}$

$=\lim_{y\to0}\frac{\ln a}{\ln(1+y)^{1/y}}=\frac{\log a}{\log e}=\ln a=\log a$ ... ... by (c)

$\lim_{x\to0}\frac{a^x-1}{x}=\log a$

(f) show that $\lim_{x\to0}\frac{e^x-1}{x}=1$

Proof :—Put $e^x-1=y$ or, $e^x=1+y$.

∴ $x=\log_e(1+y)=\ln(1+y)$

Since $\ln 1=0$ therefore $y\to0$ as $x\to0$.

Now $\lim_{x\to0}\frac{e^x-1}{x}=\lim_{x\to0}\frac{y}{\ln(1+y)}=\lim_{y\to0}\frac{1}{(1/y)\ln(1+y)}$

$=\frac{1}{\ln e}=1$ [∴ $\log_e e=\ln_e=1$]

(g) **Prove that** $\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$

for all rational values of n provided a is positive.

Case. I. When n is a positive integer.

$\lim_{x\to a}\frac{x^n-a^n}{x-a}=\lim_{x\to a}\{x^{n-1}+x^{n-2}a+\ldots\ldots+a^{n-1}\}$

$=a^{n-1}+a.\,a^{n-2}+\cdots+xa^{n-1}=na^{n-1}$

Case II. When n is a negative integer.

Let $n=-m$, m being a positive integer and $a\neq0$

∴ $\lim_{x\to a}\frac{x^n-a^n}{x-a}=\lim_{x\to a}\frac{x^{-m}-a^{-m}}{x-a}=\lim_{x\to a}\left(\frac{1}{x^m}-\frac{1}{a^m}\right)$

$=\lim_{x\to1}\frac{-1}{a^m x^m}\left(\frac{x^m-a^m}{x-a}\right)$

$=-\frac{1}{a^m}\lim_{x\to a}\frac{1}{x^m}\lim_{x\to a}\left(\frac{x^m-a^m}{x-a}\right)=-\frac{1}{a^m}\cdot\frac{1}{a^m}ma^{m-1}$

[by case I]

$=(-m)\,a^{(-m)-1}$

$=na^{n-1}$ [∵ $-m=n$].

(h) Prove that $\lim_{x\to a}\frac{(1+x)^n-1}{x}=n$ D. U. 1983.

Proof :—

$\lim_{x\to0}\frac{(1+x)^n-1}{x}=\lim_{x\to0}\left[(1+nx+\frac{n(n-1)x^2}{\lfloor 2}+\ldots\ldots)-1\right]/x$

$=\lim_{x\to0}\left\{nx+\frac{n(n-1)x^2}{\lfloor 2}+\frac{n(n-1)(n-2)^2}{\lfloor 2}+\ \ldots\ \ldots\right\}/x$

$=\lim_{x\to0}\left\{n+\frac{n(n-1)x}{\lfloor 2}+\frac{n(n-1)(n-2)x^2}{\lfloor 3}+\ \ldots\ \ldots\right\}$

$= n$, since each of the remaining terms $\to 0$ as $x \to 0$

$$\therefore \lim_{x\to 0} \frac{(1+x)^n - 1}{x} = n$$

The following limits are very useful in solving problems.

(কয়েকটি প্রয়োজনীয় সীমা)

(a) $\lim_{x\to 0} \frac{\sin x}{x} = 1$ (b) $\lim_{x\to 0} (1+x)^{1/x} = e$

(c) $\lim_{x\to \infty} (1+1/x)^x = e$ (d) $\lim_{x\to 0} \frac{1}{x} \log_e (1+x) = 1$

(e) $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a$ (f) $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

(g) $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$ (h) $\lim_{x\to 0} \frac{(1+x)^n - 1}{x} = n$

Proofs of these are given in Art. 2.13

Ex. 10 (*a*) Prove that, $\lim_{x\to\infty} \frac{2x+3}{2x} = 1$

We can write the above function also in the form

$$\lim_{x\to\infty} \left(1 + \frac{3}{2x}\right) = 1$$

Now for an arbitrary ε, the inequality is satisfied

$$|(1+3/2x) - 1| < \varepsilon,\ \varepsilon \to 0$$

provided $x > N$. where N is determined by choice of ε

or, $3/2x < \varepsilon$ or, $\frac{1}{x} < \frac{2\varepsilon}{3}$ or. $x/ > 3/2\varepsilon = N$

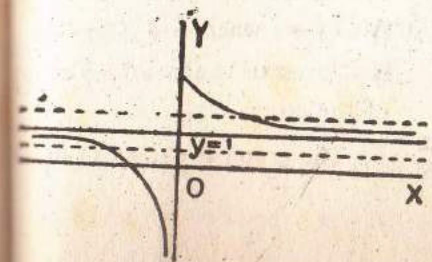
which means that

$$\lim_{x\to\infty} \left| 1 + \frac{3}{2x} \right| = \lim_{x\to\infty} \frac{2x+3}{2x} = 1$$

$$\therefore \lim_{x\to\infty} \frac{2x+3}{2x} = 1$$

It can be shown similarly that $\lim_{x\to\infty} \frac{2x+3}{2x} = 1$

This graph is not defined for $x=0$ *i.e*, y-axis is the asymptote of the curve. Again $y \to 1$ if $x \to \infty$ *i. e.*, the straight line $y=1$ is another asymptote of the curve.



The graph $y = 1 + \frac{3}{2x}$ is shown

Fig 48

Ex. 11. Prove that $\lim_{x\to 2} \frac{1}{(2-x)^2} = +\infty$

For any positive number $N > 0$, however large, we have

$\frac{1}{(2-x)^2} < N$ *i.e.*, $(2-x)^2 < \frac{1}{N}$

or, $|2-x| < \frac{1}{\sqrt{(N)}} = \delta$(say)

which depends upon N such that $|\delta| \geq 0$

The function $\frac{1}{(2-x)^2}$ assumes only positive values.

and it is greater than any large positive number

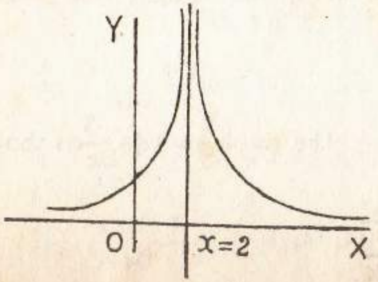
whenever $0 < |x-2| < \frac{1}{\sqrt{N}}$

Hence $\lim_{x\to 2} \frac{1}{(2-x)^2} = +\infty$

Let $y = \frac{1}{(2-x)^2}$

For all valu.s of x, y will be positive and for $x=2$, y is undifined. The line $x=2$ is an Asymptote of the graph ;

the graph is symmetrical about the line $x=2$.



The graph passes through $(0,2.5);(1,1)(-1,0.1)$ etc.

Also $y\to 0$ when $x\to\infty$ or $-\infty$

Hence x-axis is also an asymptote of the curve.,

The graph is shown in fig 49

Fig. 49

x	-2	-1	0	1	3/2	$2\pm$	2·5	3	4	5	etc
y	0·0625	0·11	.25	1	4	∞	4	1	·25	0·11	etc

Ex. 12 Show that $\lim\limits_{x\to\infty} \dfrac{\sin x}{x}=0$......

Since $|\sin x| \leqslant 1$ for all x,

therefore $\left|\dfrac{\sin x}{x}\right| \leqslant \dfrac{1}{|x|} = \dfrac{1}{x}$

which tends to 0 as $x\to\infty$

$$\therefore \lim_{x\to\infty}\left(\frac{\sin x}{x}\right)=0$$

Ex. 13. Examine the limit of $f(x)=\dfrac{1}{x}\sin\dfrac{1}{x}$

as x tends to zero.

Since $\left|\sin\dfrac{1}{x}\right| \leqslant 1$ for all real values of x except $x=0$,

therefore $\sin\dfrac{1}{x}$ oscillates between -1 and $+1$ as $x\to 0$ or

$\dfrac{1}{|x|}\to\infty$. Hence $\dfrac{1}{x}\sin\dfrac{1}{x}$ oscillates between $-\infty$ and $+\infty$ as $x\to 0$ So $\lim\limits_{x\to 0}\left(\dfrac{1}{x}\sin\dfrac{1}{x}\right)$ does not exist.

Ex. 14. Show that $\lim\limits_{x\to 0} x\sin(1/x)=0$

For non zero values of x

$|x\sin(1/x)| = |x| \cdot |\sin(1/x)| \leqslant |x|$ as $\sin(\sin(1/x)) \leqslant 1$

Thus $|x\sin(1/x)-0| = |x\sin(1/x)| \leqslant |x| < \epsilon$.

such that for all values x in,

$0<|x-0|<\epsilon$ i. e. $|x|<\epsilon$ or, $-\epsilon<x<\epsilon$ with $x\neq 0$.

Thus we notice that if ϵ is a positive number however small there is an interval $(-\epsilon, \epsilon)$ around 0 such that for all values of x, $x\neq 0$, the difference between $\sin(1/x)$ and 0 is less than a positive number ϵ, $\epsilon>0$

Hence $\lim\limits_{x\to 0} x\sin(1/x)=0$

Let $y=\sin(1/x)$; $|\sin(1/x)| \leq 1$

Thus $x\sin(1/x)$ oscillates between $y=x$ and $y=-x$, as x tends to zero.

Ex. 15. Find the limits of $\lim\limits_{x\to 0}\dfrac{1+2^{1/x}}{3+2^{1/x}}$

If $x=0+h$, where $h>0$

$\therefore \dfrac{1}{x}=\dfrac{1}{h}\to\infty$ as $h\to 0$

Now $\lim\limits_{h\to 0} f(0+h) = \lim\limits_{h\to 0} \dfrac{1+2^{1/h}}{3+2^{1/h}} = \lim\limits_{h\to 0} \dfrac{2^{-1/h}+1}{3.2^{-1/h}+1} = \dfrac{0+1}{0+1} = 1$

Again. $\lim\limits_{h\to 0} f(0-h) = \lim\limits_{h\to 0} \dfrac{1+2^{-1/h}}{3+2^{-1/h}} = \dfrac{1+0}{3+0} = \dfrac{1}{3}$

$$[\therefore\ 2^{-1/h} \to 0 \text{ as } \frac{1}{h} \to \infty\ ]$$

$\lim\limits_{h\to 0} f(0+h) \neq \lim\limits_{h\to 0} f(0-h)$

Hence the limit does not exist.,

Note : The notation $\lim\limits_{x\to a} f(x)$' will mean the same statement as, $\lim\limits_{x\to a} f(x)$

16. A real function f is defined by $f(x) = \dfrac{2x}{1-x}$

(i) Determine the domain and the range of f

(ii) Find $\lim\limits_{x\to 1+} f(x)$ and $\lim\limits_{x\to 1-} f(x)$

(iii) Find $\lim\limits_{x\to +\infty} f(x)$ and $\lim\limits_{x\to -\infty} f(x)$ R. H 1988

(iv) Find $f'(x)$ and the interval where f is increasing.

(v) Draw a sketch of the graph of f

Sol : (i) $f(x) = y = \dfrac{2x}{1-x}$ ... ... (1)

or, $y(1-x) = 2x$ or, $x = \dfrac{y}{y+2}$ ... ... (2)

If x is real then from (1), $\dfrac{2x}{1-x}$ is also real,

$\dfrac{2x}{1-x}$ is undefined at $x=1$.

So the domain of $f = (-\infty, \infty) - \{1\}$

$= (-\infty, 0]\ \text{U}\ [0, 1)\ \text{U}\ (1, \infty)$ from (2)

For range of f, $x = \dfrac{y}{y+2}$ form (2)

For any real y, $\dfrac{y}{y+2}$ is also real but f^{-1} is undefined for $y = -2$

Hence range of $f = (-\infty, \infty) - \{-2\} = (\infty, -2)\text{U}(-2, 0]$ U $[0, \infty]$

(ii) $\lim\limits_{x\to 1+} f(x) = \lim\limits_{h\to 0} \dfrac{2(1+h)}{1-(1+h)} = \lim\limits_{h\to 0} \dfrac{2-2h}{-h} = -\infty$

$\lim\limits_{x\to 1-} f(x) = \lim\limits_{h\to 0} \dfrac{2(1-h)}{1-(1-h)} = \lim\limits_{h\to 0} \dfrac{2-2h}{h} = \infty$

(iii) $f(x) = \dfrac{2x}{1-x} = \dfrac{2x}{x(1/x-1)} = \dfrac{2}{1/x-1}$

$\lim\limits_{x\to\infty} f(x) = \dfrac{2}{0-1} = -2$

$\lim\limits_{x\to -\infty} f(x) = \dfrac{2}{-0-1} = -2$

(iv) $f(x) = \dfrac{2x}{1-x}$: $f'(x) = \dfrac{(1-x).2-(-1)2x}{(1-x)^2} = \dfrac{2-2x+2x}{(1-x)^2}$

$$= \frac{2}{(1-x)^2}$$

$\therefore\ f'(x) > 0$ for all x but $x \neq 1$

When $x=0$, $f(0) = y = 0$. From (ii), (iii), (iv), $f(x) = 0$ to ∞ i. e, increasing from $x=0$ to 1 i. e. in $[0, 1)$ Also f is increasing from $-\infty$ to -2 in $(-\infty, -2)$ and $[-\infty, 0]$

(v) From the above points, the graph is $f(x)$, and passes $(0, 0)$ through increases to ∞ at $x=1$, then $f(x)$ decreases to $y=-2$ when $x=0$ to $-\infty$.

Again $f(x)$ increases from $-\infty$ to -2 when $x=1+$ to ∞.

The graph is shown

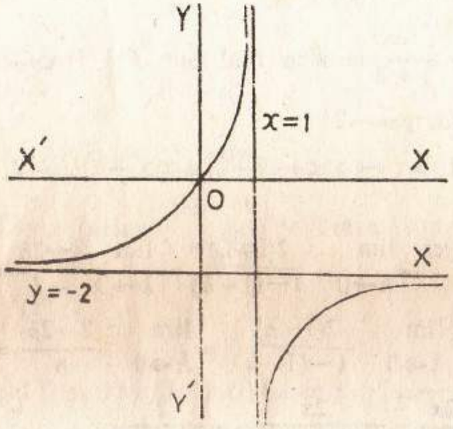


Fig 50

Exercise 11

Show that,

1. $\lim_{x\to 0} \frac{\sqrt{(1+x)}-1}{\sqrt{x}}=0$ (i) $\lim_{n\to\infty} \frac{n^2+n-1}{3n^2+1}=\frac{1}{3}$

2. $\lim_{x\to\infty} \{\sqrt{(x+1)}-\sqrt{x}\}=0$.   3. $\lim_{n\to\infty} (2^n+3^n)^{1/n}=3$

4. $\lim_{x\to 1} \frac{x-\sqrt{(2-x^2)}}{2x-\sqrt{(2+2x^2)}}=2$

5. $\lim_{x\to 1} \frac{\sqrt{x}-1}{\sqrt[3]{x}-1}=3/2$ 6. $\lim_{x\to a} \frac{x^4-a^4}{x-a}=4a^3$

7. $\lim_{x\to 0} \frac{1-\cos x}{x}=\frac{1}{2}$ 8. $\lim_{x\to a} \frac{\tan x-\sin x}{x^3}=\frac{1}{2}$

9. $\lim_{x\to 0} \frac{\sin 5x}{\sin 6x}=5/6$ 10. $\lim_{x\to\infty} \frac{5x+2}{3x+7}=5/3$

11. $\lim_{x\to\infty} \frac{x^2-2x+3}{5x^2+7x-5}=7/5$ 12. $\lim_{x\to 0} \frac{\tan x}{x}=1$

13. $\lim_{x\to 0} \frac{\sqrt{(x+1)}-1}{x}=\frac{1}{2}$ 14. $\lim_{x\to 0} \frac{1-\cos x}{x}=0$

15. $\lim_{x\to 3} \frac{x^2-4x+3}{x^2-2x-3}=\frac{1}{2}$ 15(a) $\lim_{x\to\pi/2} \frac{e^{\tan x}-1}{e^{\tan x}+1}=1$ C.H. 1993

16. Find the following limits

(i) $\lim_{x\to\infty} \frac{\sin x}{x}$ *Ans.* 0

(ii) $\lim_{x\to 0} \frac{1}{x} e^{1/x}$ *Ans.* not exist.

(iii) $\lim_{x\to a} \frac{1}{1-e^{1/x-a}}$ *Ans.* does not exist

(iv) $\lim_{x\to 0} \cos \frac{1}{x}$ *Ans.* ,,

(v) $\lim_{x\to a} \cos \frac{1}{x-a}$ *Ans.* ,,

(vi) $\lim_{x\to 0} \frac{1+5x^2}{x}$ *Ans.* ,,

(vii) $\lim_{x\to 0} e^{-1/x}$ *Ans.* ,,

(viii) $\lim_{x\to 0} \frac{1}{1-e^{1/x}}$ *Ans.* ,,

(ix) $\lim_{x\to\infty} \frac{3^x-3^{-x}}{3^x+3^{-x}}$ *Ans.* 1

17\. Show that $\lim_{x\to 0} \frac{1}{x}$ does not exist

18\. Show that

$\lim_{x\to\infty} 2^x = \infty$ and draw the graph of the function.

19\. Show that $\lim_{x\to-\infty} 2^x = 0$

20\. By (δ, ω) definition prove that

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = 2$$

21\. Use (δ, $\in$) definition to show that

$$\lim_{x\to 2} x^3-3x+7=9$$

22\. For $f(x)=x^2-2x-8$, find a $\delta>0$.

such that whenever $0<|x-3|<\delta$

then $|f(x)+5|<\in$ when (a) $\in=1/5$ (b) $\in=0.001$.

Ans. 1/25. 0.0006

23\. Find the limit of the sequence $\{a_n\}$ for a_n as defined in such case.

(i) $\frac{n^2+n-1}{4n^2-5}$ Ans. $\frac{1}{4}$

(ii) $a_n=\frac{a}{n}$ Ans. 0

(iii) $a_n=1+1/2^n$ Ans. 1

(iv) $a_n=\frac{2n^3-n}{n^4+n}$ Ans. 0

(v) If $0<r<1$, then $\lim_{n\to\infty} r^n=0$

(vi) $a^n=\frac{1}{\sqrt{n}}\cos\frac{n\pi}{2}$ Ans 0

24\. Test whether a limit exists of the followings equences.

(a) 1, 0, 1, 0, 1 .... $\frac{1-(-1)^n}{2}$ Ans. does not exist.

(b) 0.9, 0.0, 0.998, ...$1-\frac{2}{10^n}$ Ans. 1

(c) 5, 4, $\frac{11}{3}$, $\frac{7}{2}$, $\frac{17}{5}$ ... ... $3+2/n$ Ans, 34

(d) 2, 5/2, 8/3, 1/8, 14/5 ... ... Ans 3

(e) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$, 1/16, 1/32 ... ... Ans. 0

25 Consider the real function defined by

$$f(x)=\frac{x}{1-x^2}$$

a. Find $\lim_{x\to+1} f(x)$ and $\lim_{x\to-1} f(x)$

b. Find $\lim_{x\to-1+} f(x)$ and $\lim_{x\to-1-} f(x)$ D. U. 1984

c. Find $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$

d. Find $f(x)$ and the interval, where $f(x)$ increasing.

e. sketch the graph of f

26\. $\lim_{x\to a} f(x)=l \to \lim |f(x)|=|l|$

The converse will not be true. Establish it with an example.

27\. If $f(x)=x$, x is rational

$=-x$ x is irrational

Show that $\lim_{x\to a} f(x)$ exists only when $a=0$

Continuity

CHAPTER III

3. 1 : Continuity : In ordinary sense continuity means something is done without any break. For example if a boy runs for half an hour without stopping anywhere, we say that the boy has run *continuously* for half an hour. If, however, in course of half an hour running, the boy meets a ditch (say), then he has neither *to stop* or *to jump* the ditch to continue his running ; In both the cases his running is *discontinuous over this interval of half on hour.*

Mathematical definitions of continuity and discontinuity are as follows :

Continuity : A function $f(x)$ is said to be **continuous** at $x=a$, if the following conditions are satisfied

(i) $a \in D_f$ (**that is,** f **is defined at** $x=a$)

and (ii) $\lim\limits_{x \to a} f(x) = f(a)$.

If any of the above two conditions is not satisfied, then $f(x)$ is said to have a **discontinuity** at $x=a$.

3. 2. Cauchy's Definition of continuity :

A function $f(x)$ is continuous at $x=a$ if $f(a)$ is defined and for a small positive mumber $\in$, a mumber $\delta > 0$ can always be determined such that

$$| f(x) - f(a) | < \in \text{ whenever } | x - a | \leqq \delta \text{ or } a - \delta \leqq x \leqq a + \delta$$

3. 3. Discontinuous Functions. (বিচ্ছিন্ন ফাংশন)

A function $f(x)$ is said to be discotinuous for $x=a$ if $f(x)$ does not satisfy any one of the conditions of continuity in Art. 3.2. That is if $f(x)$ is not finite at $x=a$ or, $\lim\limits_{x \to 0} f(x)$ does not exist

or, $\lim\limits_{x \to 0} f(x) \neq f(a)$ then the function $f(x)$ is said to be discontinuous at $x=a$

Different classes of discontinuities have been discussed in the next Article.

3.4. Classification of discontinuites (বিচ্ছিন্নতার প্রকার)

The discontinuities are (1) Ordinary discontinuities (2) Mixed discontinuities, (3) Removal discontinuities. (4) Infinite and (5) Oscillatory discontinuities.

(A) Ordinary discontinuity or, discontinuity of the first kind (সাধারণ ছেদযুক্ততা)

If the function $f(x)$ has finite limit but

$$\lim_{h \to o} f(a+h) \neq \lim_{h \to 0} f(a-h) \neq f(a)$$

the function is said to have ordinary discontinuity or discontinuity of the first kind at $x=a$

(B) Discontinuity of the second kind.

If the limits of $f(x)$.

$$\lim_{h \to 0} f(a-h) \text{ and } \lim_{h \to 0} f(a-h)$$

do not exist for $x=a$ then $f(x)$ has a discontinuity of the second kind at $x=a$.

(C) Mixed discontinuity : (মিশ্র ছেদযুক্ততা)

If one of the limits of $f(x)$ exists then the discontinuities of the function $f(x)$ at $x=a$ is called a mixed discontinuity.

That is if $\lim\limits_{h \to 0} f(a+h) = f_L(a)$ but $\lim\limits_{h \to 0} f(a-h) \neq f(a)$, then

$f(x)$ is continuous on the right but it has a ordinary discontinuity on the left for $x=a$. Similarly if $\lim\limits_{h \to 0} f(a-h) = f(a)$ but $\lim\limits_{h \to 0} f(a+h) \neq f(a)$, then $f(x)$ is continuous on the right at $x = a$

Such types of discontinuities are called **mixed discontinuities**

(D) Removal discontinuity (অপসারণীয় ছেদযুক্ততা)

If $\lim\limits_{h \to 0} f(a+h) = \lim\limits_{h \to 0} f(a-h) \neq f(a)$

then the function $f(x)$ is said to have a **removal discontinuity** for $x = a$.

(E) Infinite discontinuity (অসীম ছেদযুক্ততা)

If both the limits of $f(x)$ are infinite, the function $f(a)$ has an infinite discontinuity. That is

if $\lim\limits_{h \to 0} f(a+h)$ and $\lim\limits_{h \to 0} f(a-h)$ tend to $+\infty$ or $-\infty$ then

$f(x)$ is said to have an infinite discontinuity at $x = a$.

Ex. 1. Prove that the function $f(x) = \sin x$ is continuous for every value of x.

$\sin x$ is defined for all values of x.

Therefore $D_f = \mathbb{R} = (-\infty, \infty)$.

For any $x \in \mathbb{R}$,

$$\lim_{h \to 0} f(x+h) = \lim_{h \to 0} \sin(x+h) = \lim_{h \to 0} (\sin x \cos^h + \cos x \sin^h)$$

$$= \sin x . \lim_{h \to 0} (\cos^h) + \cos x . \lim_{h \to 0} (\sin^h)$$

$$= \sin x . 1 + \cos x \times 0 = \sin x = f(x).$$

Hence $f(x) = \sin x$ is continuous for all real valus of x.

Ex. 2. Discuss the continuity of the function

$f(x) = \dfrac{1}{3 - e^{1/x}}$ at $x = 0$.

Since $\dfrac{1}{0}$ is undefined, therefore $f(0)$ is undefined.

Hence $f(x)$ is discontinous at $x = 0$.

Ex. 3. Show that $f(x) = \cos \dfrac{\pi}{x}$ has a discontinuity at $x = 0$.

$0 \notin D_f$, since $\dfrac{\pi}{0}$ is undefined.

Hence $f(x)$ has a discontinuity at $x = 0$.

Ex. 4. Examine the continuity of the function $f(x)$ at $x = \frac{3}{2}$ where

$f(x) = 3 - 2x$ for $0 \leqslant x < \frac{3}{2}$,

$f(x) = -3 - 2x$ for $x \geqslant \frac{3}{2}$.

Clearly $\frac{3}{2} \in Df$ and $f(\frac{3}{2}) = -3 - 2(\frac{3}{2}) = -6$.

Again $\lim\limits_{\substack{h \to 0 \\ h > 0}} f(\frac{3}{2} - h) = \lim\limits_{h \to 0} [3 - 2(\frac{3}{2} - h)] = 0$,

$\lim\limits_{\substack{h \to 0 \\ h > 0}} f(\frac{3}{2} + h) = \lim\limits_{h \to 0} [-3 - 2(\frac{3}{2} + h)] = -6$.

$\therefore \lim\limits_{h \to 0} f(\frac{3}{2} + h) = f(\frac{3}{2}) = -6 \neq \lim\limits_{h \to 0} f(\frac{3}{2} - h)$, where $h > 0$.

Although $\frac{3}{2} \in D_f$, the limit $\lim\limits_{x \to \frac{3}{2}} f(x)$ does not exist. Hence $f(x)$ is not continuous at $x = \frac{3}{2}$.

Ex. 5. Show that the function

$f(x) = \dfrac{x^2 - a^2}{x - a}$ for $x \neq a$,

$f(x) = 0$ for $x = a$,

has a removable discontinuity at $x = a$.

By **removable discontinuity** of $f(x)$ at $x = a$

We means that $f(x)$ is discontinuous at $x = a$ and continuous at any other values of $x \in D_f$ with $\lim\limits_{h \to 0^+} [f(a+h) - f(a-h)]$ finite.

The number a may or may not belong to D_f.

We have, $f(a)=0 \Rightarrow a \in D_f$.

Again $\lim_{x\to a} f(x)=\lim_{x\to a}\frac{x^2-a^2}{x-a}=\lim_{x\to a}(x+a)=2a.$

Hence $\lim_{x\to a} f(x)$ exists.

But $\lim_{x\to a} f(x)\neq f(a)$.

$\therefore$ $f(x)$ is discontinuous at $x=a$.

Again $\lim_{\substack{h\to 0\\(h>0)}}[f(a+h)-f(a-h)]=\lim_{h\to 0}[(2a+h)-(2a-h)]=0$

which is finite. Hence the discontinuity at $x=a$ is removable.

Ex. 6. Show that the function

$$f(x)=\frac{x-2}{x-1}$$

has an infinite discontinuity at $x=1$.

If $f(x)$ is discontinous at $x=a$.

and $\lim_{h\to 0}[f(a+h)-f(a-h)]=\infty$,

Lim is $|l_2-l_1|=\infty$ where l_2 is the right hand limit and l_1 is the left hand limit of $f(x)$ as $x\to a$, then $f(x)$ is said to have an **infinite discontinuity** at $x=a$.

Since $f(1)=\frac{-1}{0}$ is undifined,

$1\notin D_f$ (1 is not included in the domain)

$\therefore$ $f(x)$ is discontinuous at $x=1$.

Now $\lim_{\substack{h\to 0\\(h>0)}}|f(1+h)-f(1-h)|$

$$=\lim_{h\to 0}\left|\frac{(-1+h)}{h}-\frac{(-1-h)}{-h}\right|=\lim_{h\to 0}\left|-\frac{2}{h}\right|=\infty.$$

Hence the discontiniuty at $x=1$ is infinite.

Ex 7. show that the functim

$$f(x)=e^{-1/x}$$

has an infinite discontinuity at $x=0$.

since $f(o)=e^{-\frac{1}{0}}$ is undefined, so $f(x)$ is discontinuous at $x=0$.

Now $\lim_{\substack{h\to 0\\(h>0)}}\left|f(0+h)-f(0-h)\right|=\lim_{h\to 0}\left|e^{-1/h}-e^{1/h}\right|$

$$=\left|e^{-\infty}-e^{\infty}\right|=|0-\infty|=\infty$$

Hence $f(x)$ has an infinite discontinuity at $x=0$.

(F) Oscillatory discontinuities (দোদুল্যমান ছেদযুক্ততা)

A function $f(x)$ having a discontinuity at a point $x=a$ may oscillate finitely or does not tend to a finite limit or to ∞ or $-\infty$ as x tends to infinity. In such a case, $f(x)$ has an oscillatory discontinuity at $x=a$.

Ex. 8. $f(x)=\sin\frac{1}{x}$ oscillates between -1 and 1 and more rapidly as x approaches zero from either sides. $f(x)$ oscillates finitely at $x=0$

Ex. 9. $f(x)=\frac{1}{x-a}\sin\frac{1}{x-a}$ the function $f(x)$ oscillates infinitely as $x\to a$.

3.5. Properties of continuous Functions (অবিচ্ছিন্ন অপেক্ষকের গুণাবলী।)

(*i*, *The sum of difference of two continuous functions is a continuous function over the intersection at their domains.*

Let $f(x)$ and $\varphi(x)$ be two functions of x each being continuous at $x=a$. Then

$\lim_{x\to a} f(x)=f(a)$, $\lim_{x\to a} \varphi(x)=\phi(a)$

Thus $\lim_{x\to a} \{f(x)\pm\varphi(x)\}=\lim_{x\to a} f(x)\pm\lim_{x\to a} \varphi(x)=f(a)\pm\varphi(a)$

Hence $f(x)\pm\varphi(x)$ is continuous at $x=a$

We can extend the theorem for any finite number of functions.

(*ii*) *The product of two continuous functions is a continuous function over the intersection of* their domains.

$\lim_{x\to a} \{f(x)\,\varphi(x)\}=\lim_{x\to a} f(x)\lim_{x\to a} \varphi(x)=f(a)\,\varphi(a)$

Where $f(x)$ and $\varphi(x)$ are two continuous functions at $x=a$ and $a\in D_f\cap D\varphi$.

(*iii*) *The quotient of two continuous functions in some common domain is also a continuous function in the same domain if the denominator is not zero anywhere in it.*

Let $f(x)$ and $\varphi(x)$ be two continuous functions at $x=a$ where $a\in D_f\cap D\varphi$.

$$\text{Now} \lim_{x\to a} \frac{f(x)}{\varphi(x)}=\frac{\left(\lim_{x\to a} f(x)\right)}{\left(\lim_{x\to a} \varphi(x)\right)}=\frac{f(a)}{\varphi(a)}, \quad \text{if } \varphi(a)\neq 0.$$

Hence $\frac{f(x)}{\varphi(x)}$ is continuous at $x=a$.

(*iv*) *If a function is continuous in a closed interval, it is bounded in that interval.*

(*v*) *A function which is continuous in a closed interval attains at least once its least upper and greatest lower bounds.*

(*vi*) *A continuous function which has opposite sign at two points meets its domain vanishes at least once between these points.*

(*vii*) *A continuous function $f(x)$, in the interval (a,b), assumes at least once every value between $f(a)$ and $f(b)$, it being supposed that $f(a)\neq f(b)$.*

(*vii*) *The converse of this theorem is not true i, e, a function $f(x)$ which takes all values between $f(a)$ and $f(b)$ is not necessarily continuous in the interval (a,b).*

3.6. Continuity of some elementary functions.

(i) The function $f(x)=x^n$ is continuous for all values of x when n is any rational number, except at $x=0$ when n is negative.

Let us investigate the continuity of the function at $x=a$.

$\lim_{\substack{h\to 0\\(h>0)}} f(a+h)=\lim_{h\to 0}(a+h)^n=\lim_{h\to 0} a^n(1+h/a)^n$

$=\lim_{h\to 0} a^n\left\{1+nh/a+\frac{n(n-1)h^2/a^2}{\lfloor 1}+\dots\right\}$

$\lim_{\substack{h\to 0\\(h>0)}} f(a-h)=\lim_{h\to 0}(a-h)^n=\lim_{h\to 0} a^n(1-h/a)^n$

$=\lim_{h\to 0} a^n(1-nh/a+\dots)$

Also $f(a)=a^n$.

Hence $\lim_{x\to a} f(x)=a^n=f(a)$ for all values of n and a except $a=0$ when n is negative, When n is negative say $n=-m$, where m is positive. Then $x^n=x^{-m}=(1/x^m)$ which is undefined when $x=0$

Hence $f(x)=x^n$ is continuous for all values of x, except at $x=0$ when n is negative.

(ii) Polynomials are Continuous (অবিচ্ছিন্ন বহুপদী)

Let $f(x)=a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n$ be a polynomial in x

The polynomial $f(x)$ is the sum of a finite number of terms containing only positive integral powers of x. In the article 3.6. (i) we see that $\lim\limits_{x\to 0} x^n$ is continuous. Thus by repeated application of Art. 3.6. (i) for the terms of $f(x)$ here we see that each term is continuous.

Hence the polynoimal is itself continuous for all values of x.

(iii) Rational (আনুপাতিক) Algebraic functions are Continuous.

Let $f(x)$ and $\phi(x)$ be two polynomials which have no common factor and $\phi(x)\neq 0$

The rational algebraic function

$$R(x)=\frac{f(x)}{\phi(x)};\ \phi(x)\neq 0$$

If $f(x)$ and $\phi(x)$ are continuous for all values of x then $R(x)$ is also continuous for all values of x except for those values of x which make $\phi(x)=0$ *i. e.*, for those values for which the denominator becomes zero.

(iv) Exponential (সূচকীয়) functions $f(x)=e^x$

Let us investigate its continuity for any value of x say a. Then

$$\lim_{h\to 0} f(a+h)=\lim_{h\to 0} e^{a+h}=e^a\lim_{h\to 0} e^h$$

$$=e^a\lim_{h\to 0}\left(1+h+\frac{h^2}{\lfloor 2}+\ \ldots\ \ldots\right)=e^a$$

Also $f(a)=e^a$

Hence $\lim\limits_{h\to 0} f(a+h)=f(a)=e^a$

Therefore e^x is continuous for all values of x.

(v) Logarithmic Function

Let us consider the function $f(x)=\log x=\ln x,\ x>0$

Now we are to investigate its continuity for any vlaue of $x\ (x>0)$

Let $\log x=\ln x=y$, then $\ln(x+h)=y+k$.

or, $x=e^y$ and $x+h=e^{y+k}$ $\therefore\ h=(x+h)-x=e^{y+k}-e^y$

If $h\to 0$ then $k\to 0$. But e^y is continuous as in Art. 3.6 (v). So by definition of continuity 3.2 (6) we have:

$$\lim_{h\to 0}\left|\ln(x+h)-\ln x\right|=\lim_{k\to 0}\left|y+k-y\right|=\lim_{k\to 0}|k|=0$$

Hence $\log x=\ln x$ is continuous for all positive values of x. When $x\leq 0$. the function $\log x=\ln x$ is not defined.

3.7. Infinitely small quantities or Infinitesimals. (অণু)

Definition :—An infinitesimal is a variable quantity whose limit is zero.

Comparison of Infinitesimals.

Let α aud β be the two infinitesimals

(a) Infinitesimal of the same order.

If $\lim\limits_{\alpha\to 0}\left(\frac{\beta}{\alpha}\right)=\text{constant}=k\neq 0$,

then α and β are called the infinitesimals of the same order.

Ex. 8. If $\alpha=x$, $\beta=\sin x$, then

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

(b) Infinitesimals of different orders

If $\lim_{\alpha\to 0}\left(\frac{\beta}{\alpha}\right)=0$; that infinitesimal β is called an infinitesimal of higher order than α

If $\lim_{\alpha\to 0}\left(\frac{\beta}{\alpha}\right)=\pm\infty$, β is an infinitesimal of lower order than α.

(c) Infinitesimal of the nth order.

An infinitesimal β is said to be an infinitesimal of order n with respect to the infinitesimal α.

If $\lim_{\alpha\to 0}\frac{\beta}{\alpha^n}=k\neq 0$ i. e., β and α^n are of the same order.

(d) Equivalent Infinitesimals.

If $\lim_{\alpha\to 0}\left(\frac{\beta}{\alpha}\right)=1$,

then α and β are called equivalent infinitesimals.

(e) If α and β are equivalent infinitesimals, their difference $(\alpha-\beta)$ is an infinitesimal of higher order than that of α or β,

Proof: $\lim_{\alpha\to 0}\left(\frac{\alpha-\beta}{\alpha}\right)=\lim_{\alpha\to 0}\left(1-\frac{\beta}{\alpha}\right)=1-1=0$

The result follows from result (b)

(f) A quantity which is the product of a finite quantity and an infinitesimal of any order is an infinitesimal of that order,

$\lim_{\alpha\to 0}\left(\frac{A\alpha}{\alpha}\right)=A$, a finite quantity.

Therefore $A\alpha$ is of the same order as α [by (a)]

(g) Principal part of the Infinitesimal.

An infinitesimals β of any order may be split up into two parts one of which is of the same order as the given infintesimal α and the other is of higher order. The part which is of the same order as α is called the principal part of the infinitesimal β.

If $\lim_{\alpha\to 0}\left(\frac{\beta}{\alpha}\right)=\lim_{\alpha\to 0}\frac{A(\alpha+k)}{\alpha}=A+\lim_{\alpha\to 0}\left(\frac{Ak}{\alpha}\right)$

$=A$ as k vanishes with α.

The Principal part of infinitesimal β is A. Thus the prineipal parts of β is obtained by multiplying the infinitesimal by the finite limit of the ratio it bears to the infinitesimal α.

Ex. 9 Show that $1+\sin^2\alpha-\cos\alpha$ is of the 2nd order and its principal part $(3/2)\alpha^2$.

$$\lim_{\alpha\to 0}\frac{1+\sin^2\alpha-\cos\alpha}{\alpha^2}=\lim_{\alpha\to 0}\frac{(1-\cos\alpha)+\sin^2\alpha}{\alpha^2}$$

$$=\lim_{\alpha\to 0}\left\{\frac{2\sin^2\alpha/_2}{\alpha^2}+\frac{\sin\alpha^2}{\alpha^2}\right\}$$

$$=\tfrac{1}{2}\lim_{\alpha\to 0}\left\{\frac{\sin\alpha/_2}{\alpha/_2}\right\}^2+\lim_{\alpha\to 0}\left(\frac{\sin\alpha}{\alpha}\right)^2=\tfrac{1}{2}+1=2/3$$

which is finite and non-zero.

Therefore $1+\sin^2\alpha-\cos\alpha$ is of 2nd order w. r, α and the principal part is $\frac{3}{2}\alpha^2$,

3.8. Differentiablity of a function :

A function $f(x)$ is said to be Differetiable at $x=a$ if $a+h$ and a both belong to the domain of f as $h\to 0$

and $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ exists.

we write.

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

provided the limit exists.

If $\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0^-} \frac{f(a-h)-f(a)}{h}$

or, $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ The fun ction $f(x)$ is differentiable at $x=a$

3.9. Every finitely derivable function is Continuous.

Let $f(x)$ be differentiable at $x=a$ i. e,

$$f'(a)=\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \quad \text{exists.}$$

$\Rightarrow \lim_{h\to 0} [f(a+h)-f(a)] = \lim_{h\to 0} h\ f'(a) = 0$

or $\lim_{h\to 0} f(a+h)=f(a)$.

Hence $f(x)$ is continuous at $x=a$,

Note : $\lim_{h\to 0+} f(a+h) = \lim_{h\to 0-} f(a+h) = f(a)$

or, $\lim_{h\to 0} f(a+h)=f(a)$; $f(x)$ at $x=a$ is Continuous.

The converse of these theorems is not necessarily true i, e, a function may be continuous for a value of the variable in an interval but derivative at this point may not exist. This will be shown with an example. (উপরিউক্ত উপপাদ্যের বিপরীত কিন্তু সত্য নাও হতে পারে। একটি ফাংশান কোন চারণস্থলের কোন বিন্দুতে অবিচ্ছিন্ন হইলেও সেই বিন্দুতে ইহার ডিফারেন্সিয়েল সহগ নাও থাকিতে পারে।)

Ex. 10. Consider the function, D. U. 1983

N. U. 1993

$$f(x)=x \quad ; \quad 0\leq x<\tfrac{1}{2}$$
$$=1-x \; ; \quad \tfrac{1}{2}\leq x<1$$

Is the function continuous at $x=\frac{1}{2}$ Is it defferentiable at $x=\frac{1}{2}$.

From the conditions.

$\lim_{\substack{h\to 0\\(h>0)}} f(\tfrac{1}{2}-h) = \lim_{h\to 0} f(\tfrac{1}{2}-h) = \lim_{h\to 0} (\tfrac{1}{2}-h) = \tfrac{1}{2}$

$\lim_{\substack{h\to 0\\(h>0)}} f(\tfrac{1}{2}+h) = \lim_{h\to 0} [1-(\tfrac{1}{2}+h)] = \lim_{h\to 0} (\tfrac{1}{2}-h) = \tfrac{1}{2}$.

Also $f(\frac{1}{2})=1-\frac{1}{2}=\frac{1}{2}$.

Thus $\lim_{h\to 0+} f(\tfrac{1}{2}-h) = \lim_{h\to 0+} (\tfrac{1}{2}+h) = f(\tfrac{1}{2}) = \tfrac{1}{2}$

Hence the function $f(x)$ is continuous at $x=\frac{1}{2}$.

Now $\frac{1}{2}$ and $\frac{1}{2}+h$ as $h\to 0+$ belong to D_f.

Again. $\lim_{\substack{h\to 0\\(h>0)}} \frac{f(\frac{1}{2}+h)-f(\frac{1}{2})}{h} = \lim_{h\to 0} \frac{1-(\frac{1}{2}+h)-\frac{1}{2}}{h} = -1$ ...(a)

and $\lim_{\substack{h\to 0\\(h>0)}} \frac{f(\frac{1}{2}-h)-(\frac{1}{2})}{h} = \lim_{h\to 0} \frac{\frac{1}{2}-h-\frac{1}{2}}{-h} = 1$ (b)

Thus right hand limit (a) and the left hand limit (b) are not equal. Hence $f'(\frac{1}{2})$ does not exist i. e. $f(x)$ is not differentiable at $x=\frac{1}{2}$

Note : If $f(x)=x^n$, $(0<n<1)$ $f(x)$ is continuous at $x=0$, but $f'(x)$ does not exist at $x=0$.

Ex. 11. Test the continuity and differentiability $f(x)$ at $x=0$ when.

$$f(x)=+\sqrt{|x|}, x\geq 0$$
$$=-\sqrt{|x|} \; ; x<0$$

We have $\lim_{\substack{h\to 0\\(h>0)}} f(0+h) = \lim_{h\to 0} \sqrt{|0+h|} = 0$

$\lim_{\substack{h\to 0\\(h>0)}}$

$f(0-h) = \lim_{h\to 0} -\sqrt{|-h|} = \lim_{h\to 0} \sqrt{|h|} = 0$

Now $Rf'(0) = \lim_{\substack{h\to 0\\(h>0)}} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{\sqrt{|h|}-0}{h}$

$= \lim_{h\to 0} \frac{\sqrt{h}}{h} = \lim_{h\to 0} \frac{1}{\sqrt{h}} = \infty$

Again

$Lf'(0) = \lim_{\substack{h\to 0\\(h>0)}} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{-\sqrt{|-h|}-0}{-h}$

$= \lim_{h\to 0} \frac{\sqrt{h}}{h} = \infty$

$Rf'(0) = Lf'(0) = \infty$ Hence $f'(0)$ exis's .....(V)

Also $f(0)=0$ from the given conditions

Thus $f(0+h) = f(0-h) = f(0) = 0$ ...... (2)

This example shows that a function having an infinite derivative at a point may be continuous at that point.

Ex. 12 A function $f(x)$ is defined in the following way.

$f(x) = 0$ for $0 \leq x \leq 3$

$= 4$ for $x = 3$

$= 5$ for $3 < x \leq 4$

Investigate the continuity and differentiability at $x=3$.

we have $\lim_{\substack{h\to 0\\(h>0)}} f(3+h) = \lim_{h\to 0} 5 = 5$;

$\lim_{\substack{h\to 0\\(h>0)}} f(3-h) = \lim_{h\to 0} 0 = 0$;

$f(3) = 4$

Thus $f(3+h) \neq f(3-h) \neq f(3)$.

Hence $f(x)$ is discontinuous at $x=3$.

Now $\lim_{\substack{h\to 0\\(h>0)}} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0} \frac{5-4}{h} = \alpha$

$\lim_{\substack{h\to 0\\(h>0)}} \frac{f(3-h)-f(3)}{-h} = \lim_{h\to 0} \frac{0-4}{-h} = \lim_{h\to 0} \frac{4}{h} = \infty$ (ii)

$Rf'(3) = Lf'(3) = \infty$ *i. e.* $f'(3)$ exists ......(3)

This is a falacy. '∞' is not a finite number. Had it been a a finite number then we would have, from (i) and (ii).

$Rf'(3) = \alpha$ and $Lf'(3) = 4\infty$

$\Rightarrow Rf'(3) \neq Lf'(3)$,

that is, $f'(3)$ does not exist.

Hence $f(x)$ is not differentiable at $x=3$

Ex. 13. If $f(x) = 1$; $x < 0$

$= 1 + \sin x$, $0 < x < \frac{1}{2}\pi$

$= 2 + (x - \frac{1}{2}\pi)^2$; $\frac{1}{2}x \geq \frac{1}{2}\pi$

Discuss the continuity and differentiability of the function at $x = \pi/2$.

Now $Rf(\frac{1}{2}\pi) = \lim_{\substack{h\to 0\\(h>0)}} f(\frac{1}{2}\pi + h) = \lim_{h\to 0} [2 + \{(\frac{1}{2}\pi + h) - \frac{1}{2}\pi\}^2] = 2$

$Lf(\frac{1}{2}\pi) = \lim_{\substack{h\to 0\\(h>0)}} f(\frac{1}{2}\pi - h) = \lim_{h\to 0} [1 + \sin(\frac{1}{2}\pi - h)]$

$= \lim_{h\to 0} (1 + \cos h) = 2$

Also $f(\frac{1}{2}\pi) = 2 + (\frac{1}{2}\pi - \frac{1}{2})^2 = 2$

$\therefore$ $Rf(\frac{1}{2}\pi) = Lf(\frac{1}{2}\pi) = f(\frac{1}{2}\pi) = 2$.

The function $f(x)$ is continuous at $= \pi/2$.

For differentiablity.

$Rf'(\frac{1}{2}\pi) =$

$\lim_{\substack{h\to 0\\(h>0)}} \frac{(\frac{1}{2}\pi + h) - f(\frac{1}{2}\pi)}{h} = \lim_{h\to 0} \frac{\{2 + (\frac{1}{2}\pi + h - \frac{1}{2}\pi)^2\} - \{2 + (\frac{1}{2}\pi - \frac{1}{2}\pi)\}^2}{h}$

$$=\lim_{h\to 0}\frac{h^2}{h}=\lim_{h\to 0} h=0$$

$$Lf'(\tfrac{1}{2}\pi)=\lim_{\substack{h\to 0\\(h>0)}}\frac{f(\frac{1}{2}\pi-h)-f(\frac{1}{2}\pi)}{-h}=\lim_{h\to 0}\frac{1+\sin(\frac{1}{2}\pi-h)-(1+\sin\frac{1}{2}\pi)}{-h}$$

$$=\lim_{h\to 0}\frac{1+\cos h-2}{-h}=\lim_{h\to 0}\frac{(1-\cos h)}{h}=\lim_{h\to 0}\frac{2\sin^2 h/2}{h}=0$$

Thus $Rf'(\frac{1}{2}\pi)=Lf'(\frac{1}{2}\pi)=0$

Hence $f'(x)$ exists *i. e.*, $f(x)$ is differentiable at $x=\frac{1}{2}\pi$.

Ex. 14. Show that

$f(x)=x^2 \sin 1/x$ when $x\neq 0$

$=0,$ when $x=0$

is derivable at $x=0$ Also find $f'(0)$

Ex. 15, Draw the graph of function

$y=\sqrt{\{(x-1)(x-2)(x-3)\}}$

Sign of $(x-1)(x-2)(x-3)$:

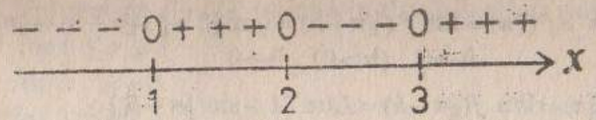


Fig. 50

From the above sign graph we see that

$y\geq 0$ for $1\leq x\leq 2$ and for $x\geq 3$.

y is imaginary for $x<1$ and for $2<x<3$.

Therefore the graph of

$y=\sqrt{(x-1)(x-2)(x-3)}$

has two branches—one betwen 1 and 2 and the other for $x\geq 3$.

we may form a table out of the values of x and y

x	1	1·1	1.2	1.5	1.7	1.9	2	3	4	5
y	0	+4	+5	.6	+0.5+	3	0	0	+2.4	+4.5

The graph may now be drawn

Ex. 15. (a) Draw the graph of the function

$f(x)=\dfrac{x}{|x|}$ N.U.1995

For $x<0$, $\dfrac{x}{|x|}=\dfrac{x}{(-x)}=-1,$

for $x>0$, $\dfrac{x}{|x|}=\dfrac{x}{x}=1$;

for $x=0, f(o)=\frac{0}{0}$ so the function is not defined at $x=0$.

Hence $f(x)$ is discontinuous at $x=0$

The graph of the fnnction is shown in fig. 51

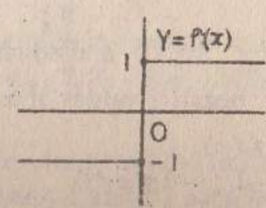


Fig. 51

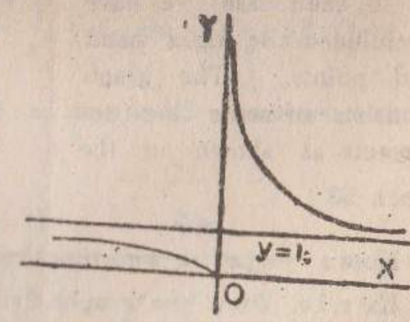
Ev. 16. Show that the $f(x)=3^{1/x}$ is discontinuous at $x=0$

$$Rf(0)=\lim_{\substack{h\to 0\\(h>0)}} f(0+h)$$

$$\lim_{h\to 0}\frac{1}{3^{0+h}}=\lim_{h\to 0} 3^{1/h}=3^{\infty}=\infty$$

$$Lf(0)=\lim_{h\to 0} f(0-h)$$

$$=\lim_{h\to 0} 3^{-1/h}=3^{-\infty}=0$$



Eig. 52

$Rf(0)\neq Lf(0)$

Now $f(0)=3^{\frac{1}{0}}$ which is undefined since $\frac{1}{0}$ is undefined.

So the function is decontinuous at $x=0$

Also note that $f(x)\to 1$ as $x\to\infty$ or $-\infty$.

The graph of function is shown in fig. 52

Ex 17. Draw the graph of $y=[x]$ where $[x]$ denotes the greatest integer positive or negative but not numerically greater than x.

The function $y=[x]$ is replaced by following statements

$$\begin{aligned} y=f(x) &= 0\ ; & 0 \leq x > 1 \\ &= 1\ ; & 1 \leq x > 2 \\ &= 2\ ; & 2 \leq x < 3 \\ &= 3\ ; & 3 \leq x < 4 \end{aligned}$$

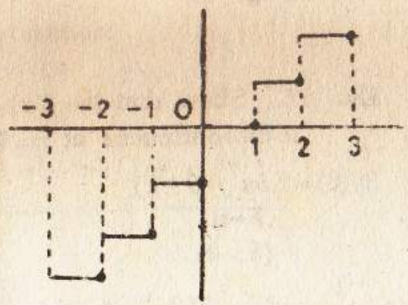
and so on.

For negative values of x.

$$\begin{aligned} y &= f(x) \\ &= -1\ ;\ -1 \leq x < 0 \\ &= -2\ ;\ -2 \leq x < 1 \\ &= -3\ ;\ -3 \leq x < -2 \end{aligned}$$

and so on.

In each case we have excluluded the right hand end points. The graph consists of some line segments as shown in the figur. 53



Eig. 53.

Note : $y=[x]$ is sometimes called a step function.

Ex : 18. Draw the graph of the function $y=f(x)=\sin 1/x$ and show that limit does not exist when x tends to zero.

$f(o)=\sin (1/o)$ is not defined.

For all other values of x, $\sin (1/x)$ exists. Let us investigate the nature of the graph.

For $x=\frac{2}{\pi},\ \frac{3}{\pi},\ \frac{4}{\pi},\ \frac{6}{\pi}$, ... (1) } ....(1)

$y=1,\ \sqrt{3}/2,\ 1/\sqrt{2},\ 1/2$

i, e, $\sin (1/x)$ decreases continuously from 1 to 0 with the increasing values of x form $2/\pi$ to ∞.

For $x=\frac{2}{\pi},\ \frac{2}{2\pi},\ \frac{2}{3\pi},\ \frac{2}{4\pi},\ \frac{2}{5\pi},\ \frac{2}{6\pi},\ \frac{2}{7\pi},\ \frac{2}{8\pi}$

$y=1,\ 0,\ -1,\ 0,\ -1\ 0,\ -1,\ 1$..........

Thus $\sin \frac{1}{x}$ Oscilletes between

$(1,-1)$ and $(-1,1)$ for the values of x in the intervals $(2/\pi, 2/3\pi)$, $(2/3\pi, 2/5\pi)$, $(2/5\pi, 2/7\pi)$...$\{2/(2n+1)\pi,\ 2/(2n-1)\pi\}$ where n is positive integer.

If $n \to \infty$ then $x \to 0$, but y does not tend to any finite limit but oscillates more frequently between $+1$ and -1 as x is nearer to zero.

Now combining (1) and (2) we get the nature of the graph of. positive values of x only.

Again $f(-x)=\sin\left(-\frac{1}{x}\right)=-\sin \frac{1}{x}=-f(x)$ so

$\sin\left(\frac{1}{x}\right)$ is an odd function and its graph is symmetrical about the origin.

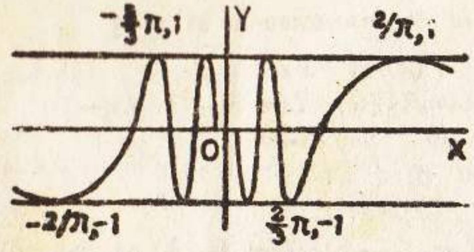


Fig. 54

Ex. 18. (a) Draw the graph of the function

$$y=\cos\frac{\pi}{x}$$

The graph has a discontinuity at $x=0$

x	∞	2	1	2/3	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	......
y	1	0	-1	0	1	-1	1	......

For $x=\dfrac{2}{(2n+1)}$; $y=0$ when $n=0, 1, 2,$

For the values of x in the interval $(0, 2/3)$ y oscillates frequently between 1 and -1 but never becomes zero.

For the values of x in the intervel $(\frac{2}{3}, 2)$, the values of y-gradually decreases to $y=-1$ and then increases to 0 when $x=2$.

The graph is continually increasing for the values of x in $(2, \infty)$ while y increases from 0 to 1. $\cos\frac{\pi}{x}$ is an even function. So similar graph will be obtained for the negative values of x. The graph has a discontinuity at $x=0$.

Ex. 19. A function $f(x)$ is defined as follows.

$f(x)=x,\ 0\leqq x<\frac{1}{2}$

$=1,\ x=\frac{1}{2}$

$=1-x,\ \frac{1}{2}<x<1$

Show that $f(x)$ is discontinuous at $x=\frac{1}{2}$ R, U. 1964

For $\frac{1}{2}<x<1, f(x)=1-x$;

$$\therefore\ Rf(\tfrac{1}{2})=\lim_{\substack{h\to 0\\(h>0)}} f(\tfrac{1}{2}+h)=\lim_{h\to 0}\ [1-(\tfrac{1}{2}+h)]=\tfrac{1}{2};$$

For $x<\frac{1}{2}$, $f(x)=x$.

$$\therefore\ Lf(\tfrac{1}{2})=\lim_{\substack{h\to 0\\(h>0)}} f(\tfrac{1}{2}-h)=\lim_{h\to 0}\ (\tfrac{1}{2}-h)=\tfrac{1}{2};$$

also $f(\frac{1}{2})=1$ (given);

$\therefore\ Rf(\frac{1}{2})=Lim\ (\frac{1}{2})\neq f(\frac{1}{2})$

Hence $f(x)$ is discontinuous at $x=\frac{1}{2}$.

Ex. 20. If each of $f(x)$ and $g(x)$ is continuous at $x=a$, show that $2f(x)+3g(x)$ is continuous at $x=a$.

As $f(x)$ and $g(x)$ are continuous at $x=0$, then

$|f(x)-f(a)|\ <\epsilon_1 \cdots\ \ldots\ (1)$

and $|g(x)-g(a)|\leqq\ \epsilon_2 \ldots\ldots\ 2$

for $0\leqq|x-a|\leqq\delta$

Now, $|\{2f(x)+3g(x)\}-\{2f(a)+3g(a)\}|$

$\leqq 2|f(x)-f(a)|-3|g(a)-g(a)|$

$<2\epsilon_1-3\epsilon_2=\epsilon$ for $0\leqq|x-a|\leqq\delta$

Hence

$$\lim_{x\to a}\ \{2f(x)+3g(x)\}=2f(a)+3g(a)$$

Therefore the function is continuous at $x=a$.

Ex. 21. A functions $f(x)$ is defined in an interval $[0, 2]$ by formulae $f(x)=x$ when $0\leqq x\leqq 1$; $f(x)=2x-1$ when $1<x\leqq 2$

Show that $f(x)$ is continuous at $x=1$ but $f'(1)$ does not exist. Draw a graph of the function.

For $x=1$

$$Rf(1)=\lim_{h\to 0^+} f(1+h)\ \lim_{h\to 0^+}[2(1+h)-1]=1$$

$$Lf(1)=\lim_{h\to 0^+} f(1-h)=\lim_{h\to 0^+}(1-h)=1$$

Again $f(1)=1$

$\lim_{h\to 0} f(0+h) = \lim_{h\to 0} |0+h| = 0$

$\lim_{h\to 0} f(0-h) = \lim_{h\to 0} |0-h| = 0$

Thus $Lf(x) = Rf(x) = f(0) = 0$

Hence $f(x)$ is continuous at $x= 0$

For differentiability,

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|h| -1}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

And $$\lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{|-h| - 0}{-h} = \lim_{h\to 0} \frac{h}{-h} = -1$$

Thus $Lf(x) \neq Rf(x)$ at $x= 0$

Hence $F(x)$ is not differentiable at $x= 0$

24. Find the differential coefficient of

$f(x) = |x|+1$ at $x=-1$

What is the domain of f' ? C.H. 1995

25. Check the continuity of the function

$f(x) = 0, \quad -\& < x \leq 0$

$= \frac{1}{x}, \quad 0 < x < 1$ C.H. 1995

$= x \quad 1 \leq x < \&$

26. Show that $f(x) = |x| + |x-1|$ is continuous but not differentiable at $x = 0, 1$ (দেখাও যে, $f(x) = |x| + |x-1|$, $x = 0, 1$ বিন্দুতে অবিচ্ছিন্ন কিন্তু অন্তরীকরণযোগ্য নহে)

27. Show that $f(x) = |x| + |x-1| + |x-2|$ is continuous at $x = 0, 1, 2$ but not differentiable at $x = 0, 1, 2$

(দেখাও যে, $f(x) = |x| + |x-1| + |x-2|$, অবিচ্ছিন্ন $x = 0, 1, 2$ বিন্দুতে কিন্তু অন্তরীকরণযোগ্য নহে)

28. Show that the function

$f(x) = \frac{x}{|x|}, \; x \neq 0$

$= 0, \; x = 0$ is discontinaus at $x=0$

See APPENDIX for Uniform Continuity

Exercise 1 (C)

1. What are the types of discontinuities, give examples of each.

2. Show that $f(x) = |x|$ is every where continuous.

3. Show that x^2+1 is continuous at $x=2$

4. A function is defined as follows

$f(x) = \cos x$ for $x \geq 0$

$= -\cos x$ for $x < 0$

Is $f(x)$ continuous at $x=0$

5. Examine the continuity of $f(x) = \frac{1}{x-3}$ at $x=3$

(i) Show that function $|x-a|$ is continuous but not differentiable at a.

6. Show that $f(x) = \frac{1}{1-e^{1/x}}$ has an ordinary discontinuity at $x=0$.

7. Show that $f(x) = \frac{2}{\pi} \lim_{n\to\infty} \tan^{-1} nx$ is discontinuous at $x=0$

8. $f(x) = e^{1/x}$ for $x \neq 0$ and $=1$ for $x=0$

Show that $f(x)$ is discontinuous at $x=0$.

9. $f(x) = (1+2)^{1/x}$ when $x \neq 0$

$= e^2$ when $x=0$

Is $f(x)$ continuous at $x=0$

10. Test the continuity of the function at $x=0$

$f(x) = \frac{e^{1/x^2}}{e^{1/x^2}-1}$ when $x \neq 0$

$= 1$ when $x=0$

11. Show that $f(x) = x^2, \; x \neq 1$

$= 2, \; x=1$ is discontinuous for $x=1$.

See APPENDIX — Uniform Continuity

12. Prove that $f(x)=\frac{\sin^2(ax)}{x^2}$ for $x\neq 0$

$=1$ for $x=0$

is discontinuous at $x=0$ unless $a=1$

13. Examin the continuity of R. U, 1965

$f(x)=\sin(1/x)$ at $x=0$.

14. Discuss whether the function defined as

$f(x)=1+x,\quad x\geqq 0$

$=0,\quad x<0$ is continuous at $x=0$. Ans. no

15. $f(x)$ is defined as follows N.U. 1995

$f(x)=0,\quad x=0$

$=x,\quad x>0$

$=-x\quad x<0$

Is $f(x)$ continuous at $=0$. Does $f'(x)$ exist at $x=0$ i. e. is $f(x)$ differentiable at $x=0$ Ans. no

16. Test the continuity of

$f(x)=\sin(\pi/x)$ for $x\neq 0$

$=1,$ for $x=0$

at the point $x=0$

17. Is the following functions continuous at $x=4$

$f(x)=4x+3$ for $x>4$ and for $x<4$

$=3x+7$ for $x=4$

18. Examine $f(x)$ posesses first derivative $[f(x)]$ at $x=0$ when $f(x)=0, x=0$

$=\frac{1}{1+e^{1/x}},\ x\neq 0$

19. Examine whether or not $e^{-1/x}$ is continuous at $x=0$

20. A function $f(x)$ defined as follows

$f(x)=x\sin(1/x)\ ;\quad x\neq 0$

$=0\quad ;\quad x=0$

Examine the continuity and differentiablity of $f(x)$ at $x=0$. Show that $f'(x)$ does not exist at $x=0$

21. Prove that $f(x)=x\sin\frac{1}{x},\ x\neq 0$

$=5\quad x=0$

is not continuous at $x=0$, Can you redefine $f(0)$ so that $f(x)$ is continuous at $x=0$.

22. Let f be defined by $f(x)=\begin{cases}\frac{|x-3|}{x-3} & \text{if } x\neq 3\\ 0 & \text{if } x=3\end{cases}$

Discuss the continuity at $x=3$. Find the domain and range of $f(x)$. Draw the graph. C. H. 1989

$\lim_{x\to 3+} f(x)=\lim_{x\to 3+}\frac{x-3}{x-3}=1\ ;\ [\because (x-3)\geq 0]$

$\lim_{x\to 3-} f(x)=\lim_{x\to 3-}-\frac{(x-3)}{x-3}=-1\ ;\ [\because (x-3)<0]$

For $x=3$

$f(x)=0$

Thus, $\lim_{x\to 3+} f(x)\neq\lim_{x\to 3-} f(x)$

The function $f(x)$ is not continuous at $x=3$

For $|x-3|>0,\ |x-3|=x-3$

$y=f(x)=\frac{|x-3|}{x-3}=\frac{x-3}{x-3}=1$

$(3,\infty)\subset D_f\ ;\ \{1\}\subset R_f$

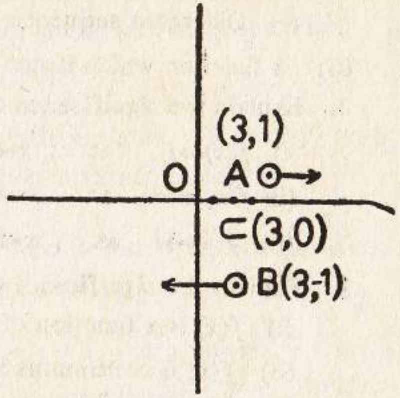


Fig 55

For $| x-3 | < 0$; $| x-3 | = -(x-3)$

$$y = \frac{| x-3 |}{x-3} = -1$$

$(-\infty, 3,) \subset D_f$, $\{-1\} \subset R_f$

For $x=3$, $f(x)=0$.

Thus domain $D_f=(-\infty, 3) \cup \{3\} \cup (3, \infty) = \mathbb{R}$

$R_f=\{-1, 0, 1\}$

The graph does not contains A (3, 1), B (3,—1). It contains two line $y=1$, $y=-1$ and an iscolated point (3, 0)

Miscellaneons Examples

(on functions, limits and continuties)

1. Give examples of a function which is not continuous at the origin.

(a) Let x be the name of a boy appearing in an examination and y be the roll number of boy x. Is the relation between y and x functional ?

(b) Give examples

(i) Divergent sequence of real numbers.

(ii) A function which is not defferentiable at the origin.

2. Explain the significance of the statements :—

(i) $f(x) \to l$ as $x \to a$

(ii($f(x) \to \infty$ as $x \to a$

(iii) $f(x) \to l$ as $x \to \infty$

3. Explain the significance of the statements :—

(i) $f(x)$ is a function of x in the interval (a, b)

(ii) $f(x)$ is continuous at $x=a$.

(iii) $f(x)$ has a differential Co-efficient at $x=a$

4. If $\lim_{x \to a} f(x)=l$, $\lim_{x \to a} g(x)=m$

Show that $\lim_{x \to a} \{f(x)+g(x)=l+m$

5. $\lim_{x \to a} f(x)=l$, $\lim_{x \to a} g(x)=m$

Show that $\lim_{x \to a} f(x)\, g(x)=lm$

6. Show that if $f(x)$ has a derivative at $x=a$, then $f(x)$ is continuous at $x=a$, but the converse is not always true.

R. H. 1964

7. Prove that continuity is a necessary condition for differentiability but not a sufficient one. Illustrate your answer with an example.

R. H. 1988

8. State any one of the fundamental properties on a continuous function and show that there is one and only one positive root of the equation $x^5=2$ by that property.

9. A function $f(x)$ is differentiable for every point of definition. What will you infer from this statement ?

10. State any one of the fundamental properties of a coutinuous function other than used in proving Rolle's Theorem.

13. Let $f(x)=\sin(1/x)$ when $x \neq 0$ and $g(x)=x \sin(1/x)$ when $x \neq 0$, prove that $\lim_{x \to 0} f(x)$ does not exist and $\lim_{x \to 0} g(x)=0$

12. Show that the limits of the following

(i) $\lim\limits_{x\to 0} \dfrac{x-\sin x \cos x}{3x}=0$

(ii) $\lim\limits_{x\to 0} \dfrac{xe^{x}-\log(1+x)}{x^2}=3/2$

(iii $\lim\limits_{x\to \frac{1}{4}\pi} \dfrac{1-\tan x}{1-\cot x}=-1$, (iv) $\lim\limits_{\theta\to 0} \dfrac{\sin\theta-\theta}{\theta^3}=-\dfrac{1}{6}$

R. U. 1966

(iii) $\lim\limits_{x\to \infty} \dfrac{1}{x}\sin x=0$ (vi) $\lim\limits_{x\to 1}\left(\dfrac{1}{1-x}-\dfrac{3}{1-x^3}\right)=-1$

(vii) $\lim\limits_{x\to 4} \dfrac{\sqrt{(2x+1)}-3}{\sqrt{(x-2)}-\sqrt{2}}=\dfrac{2\sqrt{2}}{3}$

13. If $f(x)=\dfrac{x^2+1}{x-1},\ x<3$

$=\dfrac{\sin x}{x-1}, x>3$

For what value of x the function is defined ?

14. Cite an example of a bounded function in $(1,-1)$ which is discontinuous at the single point $x=a$.

15. Show that $f(x)=(x-a)\sin\dfrac{1}{x-a}$ for $x\neq a$

$=0$ for $x=a$

is continuous and differentiable for $x=a$

16. For the function $f(x)=x\cos(1/x)$; $x\neq 0$

$=0$ $x=0$

Show that (i) the limit is zero when $x\to 0$

(ii) $f(x)$ is continuous at $x=0$

(iii) $f'(x)$ does not exist at $x=0$

17. Examine the continuity and differentiability of the function difined in the interval $(-\infty, \infty)$ C.U. 1992

such that

$f(x)=1$: $-\infty<x<0$ C. H. 1989

$=1+\sin x$ $0\leq x<\frac{1}{2}\pi$

$=2+(x-\frac{1}{2}\pi)^2$, $\frac{1}{2}\pi\leq x<\infty$

and show that $f'(x)$ exists for $x=\frac{1}{2}\pi$ and does not exist for $x=0$.

18. A function is defined as follows.

$f(x)=1+x^2$, $0<x\leq 4$

$=4$, $-1\leq x\leq 0$

$=1+x$. $-4\leq x<-1$

show that $f(x)$ is continuous at $x=0$ but discontinuous at $x=-1$

19. A function is defined as fowllows D.U. 1990

$f(x)=x^2$, $x\leq 0$

$=1$, $0<x<1$ R. U, 1980

$=1/x$, $x>1$

Show that the function $f(x)$ is not differentiable at $x=0$. What is about $f'(x)$ at $x=1$?

20. Find where the function is discontinuous

$f(x)=x^2+1$, $0\leq x<\frac{1}{2}$

$=0$ $x=\frac{1}{2}$

$=x+3$, $\frac{1}{2}<x\leq 1$ R. U. 1960

(1) Find $\dfrac{dy}{dx}$ at $x=0$ for the following function

$y=x^2+1$; $x\geq 0$

$=\cos x$; $x\geq 0$ R. U. 1986

21. If $y=x^2$ when $x\leq 1$ N.U(C-2) 1994

$=x,\quad 1<x\leq 2$ C. U. 1987 '89

$=\frac{1}{4}x^3,\quad x>2$

Show that y is continuous at $x=1$, $x=2$.

22. If $y=-x,\quad x\leq 0$

$=x,\quad 0<x\leqslant 1$ D. U. 1989

$=2-x,\quad x>1$

Show that y is continuous a $x=0$ $x=1$

23. If $f(x)=3+2x,\quad -3/2\leq x<0$ C. U. 1993

$=3-2x,\quad 0\leqslant x<3/2$

$=3+2x,\quad x\geqq 3/2$

Show that $f(x)$ is continuous at $x=0$ and discontinuous $x=3/2$

24. If $f(x)=\frac{1}{2}(b^2-a^2),\ 0\leq x\leq a$

$=\frac{1}{2}b^2-\frac{1}{6}x^2-\frac{1}{3}(a^3/x),\quad a<x\leqq b$ R. U. 1988

$=\frac{1}{3}(b^3-a^3)/x,\quad x>b$

Show that $f(x)$ and $f'(x)$ are continuous for every positive value of x.

25. A given function $y=f(x)$ is defined as follows

$f(x)=0,\quad x^2>1$

$=1\quad x^2<1$

$=\frac{1}{2},\quad x^2=1$

Show that $f(x)$ is discontinuous at $x=\pm 1$, Explain the discontinuous of the function al hough it has a value for every value of x.

26. Prove that

$\lim_{n\to\infty} y^n=0$ if y is a proper function

and n is a positive integer

27. $\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\frac{3}{n^2}\cdots+\frac{n}{n^2}\right)=\frac{1}{2}$

28. $\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^n}\right)=2$

29. $\lim_{n\to\infty}\frac{1^2+2^2+3^2\ldots+\ldots n^2}{n^3}=\frac{1}{3}$

30. Define limit of a sequence, Has the sequence defined by $S_n=(-1)^n\frac{n}{n+1}$ a limit ? R. U. 1966

31. Define a function. Find the domain of the following real functions

(i) $f(x)=\frac{1}{x}$ Ans. $0<x<\infty, -\infty<x<0$.

(ii) $f(x)=\sqrt{(x+1)}$ Ans. $x\geqq -1$.

(iii) $f(x)=\frac{x^2+11}{x^2-5x+6}$ Ans. all real values of x except $x=2, 3$

C. U. 1986

(iv) $f(x)=\frac{x^3-1}{x-1}$ Ans. all real values of x. except $x=1$.

(v) $f(x)=\frac{x}{|x|}$ Ans. all real values of x except $x=0$

(vi) $f(x)=\frac{\sqrt{(x-1)}}{(x^2-5x-1)}$. Ans. except the values of x which make $x^2-5x+1=0$ and $x<1$.

(vii) $f(x)=\frac{x^2-4}{x-2}$ (viii) $f(x)=\sqrt{(x^2-4x+3)}$

(ix) $f(x)=\frac{\sqrt{(x-2)}}{5x^2-27x+10}$ (x) $f(x)=\frac{x}{\sin(1/x)}$

R. U. 1967, C. U. 1969.

32. Determine the biggest domain and range for the following functions.

(i) $f(x)=-1$ when $x<0$ **Ans.** $|x|\geq 0$ Range $(-1, 1)$

$=0$ when $x=0$

$=1$ when $x>0$

(ii) $f(x)=2$ when $-5<x<-1$

$=\sin x$ when $0<x<2$

(iii) $f(x)=x+5; -\infty<x<\infty$ **Ans.** $-\infty<y<\infty$

(iv) $f(x)=x^2+x+1$ **Ans** $x\geq 0, y\geq 0$

(v) $f(x)=x \sin(1/x)$ **Ans.** all values of x, except $x=0$, Range. $(-1, 1)$

33. Investigate whether the following function tend to limit or not as $n\to\infty$.

(i) $f(n)=\dfrac{(-1)^n}{n}$ (ii) $f(n)=\dfrac{1}{n-(-1)^n}$

(iii) $f(n)=1+1/n$. (iv) $f(n)=n[1+(-1)^n]$ R. U. 1964.

34. Show that if α is an infinitesimal, $\sin\alpha$ and $\tan\alpha$ are infinitesimals of the same order as α and $1-\cos\alpha$ is an infinitesimal of the second order with respect to α.

we know $\lim\limits_{\alpha\to 0}\dfrac{\sin\alpha}{\alpha}=1$, $\lim\limits_{\alpha\to 0}\dfrac{\tan\alpha}{\alpha}=1$ i. e., $\sin\alpha$, $\tan\alpha$, are of same order.

Also $\lim\limits_{\alpha\to 0}\dfrac{1-\cos\alpha}{\alpha^2}=\frac{1}{2}$ which is finite other than zero.

Hence $1-\cos\alpha$ and α^2 are of same order, i. e., 2nd order,

35. Show that $3\alpha+2\alpha^2$ is an infinitesimal of the same order.

36. Prove that $\sqrt{(\sin\alpha)}$ is of lower order than α.

37. Show that $\sqrt{\alpha}$ is an infinitesimal of lower than α.

38. Show that $\sin\alpha-\tan\alpha$ is of 3rd order and its principal part is $\frac{1}{2}\alpha^2$.

39. Show that $\sin\alpha(1-\cos\alpha)$ is an infinitesimal of the 3rd order and its principal part is $\frac{1}{2}\alpha^3$,

40. Demonastrate with an example that at a given point a function may be discontinuous but its limit may exist

R. U. 1986

Answers I (C)

4. Not-continuous 5. not continuous.
9. Continuous. 10. Continuous.
13. discontinuous. 14. not cont.
15. yes, not. 16. not cont. 17. cont
18. not. 19. not. 20. cont.

Miscellaneous Exercise

1. $y=1/x$, no.
9. Function is continuous in the interval.
13. at $x=1$ 14. $f(x)=3-x$, $-1\geq x<0$

$=0$, $x=0$

$=x-3$, $0<x\geq 1$

10. undefined at $x=1$ 20. $x=0, \frac{1}{2}$ 28. 2. 30. no.

31. (vii) $-2<x<2$ (iii) $3\leqslant x\leqslant 1$

(ix) $2\leqslant x<5$, $5<x<\infty$ (x) $0<x\leqslant\dfrac{1}{n\pi}$

33. (i) 0, (ii) 0, (iii) 1, (vi) 0, n is odd, no. n is even.

Ex. 1. Locate the discontinuities of the function

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 3 \\ x+1 & \text{if } 3 < x \le 6 \\ 7 & \text{if } 6 < x < 9 \end{cases}$$

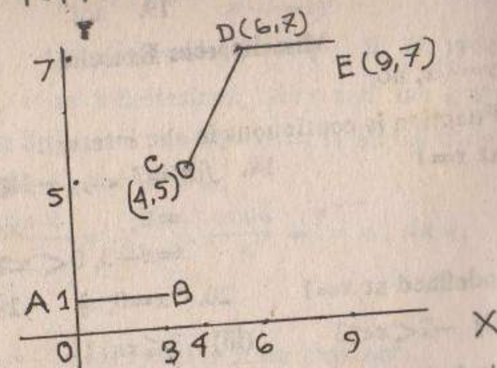
Find the domain and range of f(x)

The graph of $f(x)$ consists of the line segments such as y=1 for $0 \le x \le 3$, $y=x+1$ for $3 < x \le 6$, $y=7$ if $6 < x \le 9$ and draw the grap.

Solution : Put the points as below:

x	0	1	2	3	4	5	6	7	8	9
y	1	1	1	1	5	6	7	7	7	7

The graph is



From the figure we see that only discontinuity exists at $x=3$ for the values of x greater than 3 get closer and closer to 3, the corresponding function values $f(x)$ get closer and closer to 4. This is $\underset{x \to 3+}{Lt} f(x) = 4$.

The plus sign indicates that values of x under consideration are slightly greater than 3 i.e., $3+h$, $h \to o$

Again when the values of x are less than 3, then the values $f(x)$ gradualy approaches to 1. i.e $f(x) = 1$ for all values of x, between 0. and 3. This is expressed as $(x \to 3-h,\ h \to o)$ i. e.

Thus only discontinuity exits at x=3 i.e. the line AB fails to connect C i.e., with the line segments CDE.

The domain of $f(x)$ is $[0, 3] \cup [3, 6] \cup [6, 9]$

Range of $f(x)$ is $\{1\} \cup [4, 7] \cup \{6, 7\}$

Ex.2. The $f(x)$ is defined in the following way.

$$f(x) = \begin{cases} x^2 & \text{if } 0 \le x < 3 \\ 10 & \text{if } 3 \le x < 6 \\ x-1 & \text{if } 6 \le x < 10 \end{cases}$$

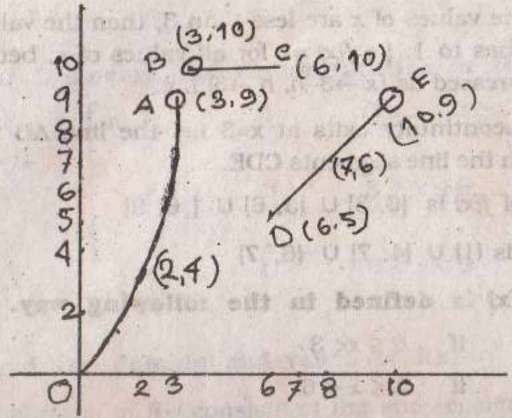
Locate the discontinuties. Also find the domain and range of $f(x)$.

Ans : Let us draw the curves for $y=x^2$ in $0 \le x < 4$, $y = 5$ in $4 \le x < 6$ and $y = x - 1$ in $6 \le x < 8$

x	0	1	2	3	for fig OA	x	4	5	6
y	0	1	4	9		y	10	10	10

$y = x^2$

x	6	7	8	9	10	fig 3
y	5	6	7	8	9	



The graphs are OA, BC and DE

From the figure we see that the curve is discontinuous at A (3,9) and E (10,9). By the formula

$\underset{x \to 3+0}{Lt} f(x) = 10$, $\underset{x \to 3-0}{Lt} f(x) = \underset{x \to 3+0}{Lt} x^2 = 9$. Limits are not equal.

$f(3+0) = f(3) = 10 \neq f(3-0)$

Hence $f(x)$ is discontinuous at $x=3$

Again

$\underset{x \to 6-0}{Lt} f(x) = 10$ $\underset{x \to 6+0}{Lt} f(x) = \underset{x \to 6+0}{Lt} x-1 = 5$ $Lt\ f(x) = 5$ when $x = 6$

$\therefore f(6-0) \neq f(6) = f(6) = 5$

Limits are not equal. Hence $f(x)$ is discontinuous at $x=6$.

Hence discontinuties of $f(x)$ are at $x = 3$ and 6

Domain of $f(x)$ is $[0,3) \cup [3,6) \cup [6,10) = [0,10)$

Range of $f(x)$ is $[0,9) \cup \{10\} \cup [5,9)- = 9]-\{3\}-\{6\}$

For Exercise

Ex.1 Show that for the function

$f(x) = x^2$ if $0 \leq x < 2$

$= 5$ if $2 \leq x < 4$

$= x-1$ if $4 \leq x < 6$

The discontinuities are at $x=2$ and $x=4$

Also find the domain and Range of $f(x)$

Ans : Domain = $[0,6)$, Range = $[0,5) - \{2\}-\{4\}$

Ex.2 Find the discontinuities of

$f(x) = 1$ if $0 \leq x \leq 2$

$= x+1$ if $2 < x \leq 5$

$= 6$ if $5 < x \leq 7$

Also find the domain and range of $f(x)$

Ans : Disconduities are at $x = 2$

Domain, $[0, 2] \cup [2, 5] \cup [5,.7] = [0, 7]$

Range = $\{1\} \cup (3, 6] \cup \{6\}$

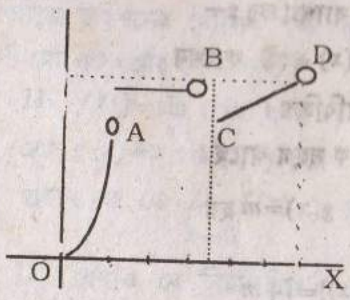


Fig. for Ex. 1

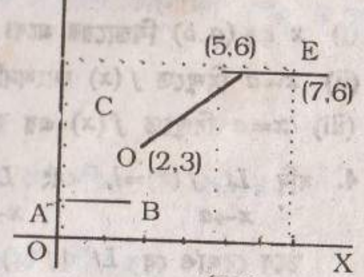


Fig 2.

বিবিধ প্রশ্নমালা

(Miscellaneous Examples)

ফাংশন, সীমা এবং অবিচ্ছিন্নতা

(on functions, limits and continuties)

1. এরূপ একটি ফাংশনের উদাহরণ দাও যাহা মূল বিন্দুতে অবিচ্ছিন্ন নয়।

(a) মনে কর কোন পরীক্ষার্থী বালকের নাম x এবং তার পরীক্ষার ক্রমিক নং y. এখন x ও y এর মধ্যে সম্পর্কটা কি ফাংশন পর্যায়ে পড়ে?

(b) নিম্নলিখিতগুলির উদাহরণ দাও।

(i) বাস্তব রাশিগুলির অপসারী অনুক্রম,
(Divergent sequence of real numbers.)

(ii) একটি ফাংশন যাহা মূলবিন্দুতে অন্তরীকরণ যোগ্য নয়।

2. নিম্নলিখিত বিবৃতিগুলির তাৎপর্য্য ব্যাখ্যা কর :—
Explain the significance of the statements :—

(i) $f(x)\to l$ যখন $x\to a$

(ii) $f(x)\to\infty$ যখন $x\to a$

(iii) $f(x)\to l$ যখন $x\to\infty$

3. নিম্নলিখিত বিবৃতিগুলির তাৎপর্য্য ব্যাখ্যা কর :—

(i) x এর (a, b) বিস্তারের মধ্যে $f(x)$ একটি ফাংশন

(ii) $x=a$ বিন্দুতে $f(x)$ ফাংশনটি অবিচ্ছিন্ন।

(iii) $x=a$ বিন্দুতে $f(x)$ এর অন্তরক সহগ আছে।

4. যদি $\underset{x\to a}{Lt}\ f(x)=l$, এবং $\underset{x\to a}{Lt}\ g(x)=m$ হয়

তবে দেখাও যে $\underset{x\to a}{Lt}\ [f(x)+g(x)]=l+m$

5. যদি $\underset{x\to a}{Lt}\ f(x)=l$, এবং $\underset{x\to a}{Lt}\ g(x)=m$ হয়

তবে দেখাও যে $\underset{x\to a}{Lt}\ f(x)\,g(x)=lm$

6. দেখাও যে $x=a$ বিন্দুতে $f(x)$ ফাংশনের বৃদ্ধিহারের (derivative) অস্তিত্ব থাকলে $x=a$ বিন্দুতে $f(x)$ অবিচ্ছিন্ন হবে; কিন্তু এর বিপরীত বিবৃতিটি সবসময় সত্য নাও হতে পারে। R. H. 1964

7. প্রমাণ কর যে অন্তরীকরণ যোগ্যতার জন্য কোন ফাংশনের অবিচ্ছিন্নতা একটি প্রয়োজনীয় শর্ত কিন্তু শর্তটি যথেষ্ট নয়। তোমার উত্তরের সাথে একটি উদাহরণ উল্লেখ কর।

8. অবিচ্ছিন্ন ফাংশনের যে কোন একটি মৌলিক ধর্মের উল্লেখ কর এবং ঐ ধর্মের সাহায্যে দেখাও যে $x^5=2$ সমীকরণের একটি এবং কেবলমাত্র একটি ধনাত্মক বীজ আছে।

9. প্রত্যেক সংজ্ঞায়িত বিন্দুতে একটি ফাংশান $f(x)$ অন্তরীকরণ যোগ্য। এ বক্তব্য থেকে কি বুঝা যায়।

(A function $f(x)$ is differentiable for every point of definition. What will you infer from this statement ?)

10. রোলের–উপপাদ্য (Rolle's Theorem.) প্রমাণের জন্য ব্যবহৃত অবিচ্ছিন্ন ফাংশনের মৌলিক ধর্মটি ব্যতিত অবিচ্ছিন্ন ফাংশনের অপর যে কোন একটি মৌলিক ধর্মের বর্ণনা দাও।

11. $f(x)=\sin(1/x)$ যখন $x\neq 0$
এবং $g(x)=x\sin(1/x)$ যখন $x\neq 0$

প্রমাণ কর যে $\underset{x\to 0}{Lt}\ f(x)$ এর অস্তিত্ব নেই কিন্তু $\underset{x\to 0}{Lt}\ g(x)=0$

12. দেখাও যে নিম্নলিখিত ফাংশনগুলির সীমা হবে

(i) $\underset{x\to 0}{Lt}\ \dfrac{x-\sin x\cos x}{3x}=0$

(ii) $\underset{x\to0}{Lt}\ \dfrac{xe^x-\log(1+x)}{x^2}=3/2$

(iii) $\underset{x\to\frac{1}{4}\pi}{Lt}\ \dfrac{1-\tan x}{1-\cot x}=-1$ (iv) $\underset{\theta\to0}{Lt}\ \dfrac{\sin\theta-\theta}{\theta^3}=-\frac{1}{6}$

R. U. 1966

(v) $\underset{x\to\infty}{Lt}\ \dfrac{1}{x}\sin x=0$ (vi) $\underset{x\to1}{Lt}\ \left[\dfrac{1}{1-x}-\dfrac{3}{1-x^3}\right]=-1$

(vii) $\underset{x\to4}{Lt}\ \dfrac{\sqrt{(2x+1)}-3}{\sqrt{(x-2)}-\sqrt{2}}=\dfrac{2\sqrt{2}}{3}$

13. যদি $f(x)=\dfrac{x^2+1}{x-1}$ যখন $x<3$

$=\dfrac{\sin x}{x-1}$ যখন $x>3$ হয়, তবে

x এর কোন মানের জন্য ফাংশনটিকে সংজ্ঞায়িত করা যাবে ?

14. $(1,-1)$ ব্যবধিতে একটি সীমাবদ্ধ ফাংশন নির্ণয় কর যাহা একটি একক বিন্দু $x=0$ তে বিচ্ছিন্ন হয়। [Cite an example of a bounded function in $(1,-1)$ Which is discontinuous at the single point $x=0$.

15. $f(x)=(x-a)\sin\dfrac{1}{x-a}$ যখন $x\neq a$

$=0$ যখন $x=a$

দেখাও যে $x=a$ বিন্দুতে ফাংশনটি অবিচ্ছিন্ন এবং অন্তরীকরণ যোগ্য।

16. ফাংশন $f(x)=x\cos(1/x)$ যখন $x\neq0$

$=0$ যখন $x=0$ হইলে

দেখাও যে (i) $x\to0$ হয় সীমা (limit) শূন্য হয়।

(ii) $x=0$ বিন্দুতে $f(x)$ অবিচ্ছিন্ন।

(iii) $x=0$ বিন্দুতে $f'(x)$ এর অস্তিত্ব থাকে না।

17. $(-\infty,\infty)$ বিস্তারের মধ্যে সংজ্ঞায়িত নিম্নলিখিত ফাংশনটির অবিচ্ছিন্নতা এবং অন্তরীকরণ যোগ্যতার পরীক্ষা কর :— C. H. 1989, C. U. 1993

$f(x)=1,$ যখন $-\infty<x<0$

$=1+\sin x$ যখন $0\leqslant x<\pi/2$

$=2+(x-\pi/2)^2$ যখন $\pi/2\leqslant x<\infty$

আরো দেখাও যে $f'(x)$ অস্তিত্ব $x=\pi/2$ বিন্দুতে থাকে কিন্তু $x=0$ বিন্দুতে থাকেনা।

18. নিম্নলিখিত উপায়ে একটি ফাংশনকে সংজ্ঞায়িত করা হলো :—

$f(x)=4+x^2$ যখন $0<x\leqslant4$

$=4,$ যখন $-1\leqslant x\leqslant0$

$=1+x,$ যখন $-4\leqq x<-1$

দেখাও যে $f(x)$ ফাংশনটি $x=0$ বিন্দুতে অবিচ্ছিন্ন কিন্তু $x=-1$ বিন্দুতে বিচ্ছিন্ন।

19 নিম্নে একটি ফাংশন সংজ্ঞায়িত করা হলো D.U. 19[illegible]

$f(x)=x^2$; যখন $x\leqslant0$ R.U. 1980

$=1$; যখন $0<x<1$

$=1/x$; যখন $x>1$

দেখাও যে $x=0$ বিন্দুতে $f(x)$ ফাংশনটি অন্তরীকরণের অযোগ্য। $x=1$ বিন্দুতে $f'(x)$-এর অবস্থা কি হবে ?

20. কোথায় ফাংশনটি অবিচ্ছিন্ন (discontinuous) তা নির্ণয় কর :—

$f(x)=x^2+1,$ যখন $0\leqslant x<\frac{1}{2}$

$=0$ যখন $x=\frac{1}{2}$ R.U. 1960

$=x+3,$ যখন $\frac{1}{2}<x\leqq1$

21. যদি $y=x^2$ যখন $x\leqslant1$ C.U. 1987, '89

$=x,$ যখন $1<x\leqslant2$

$=\frac{1}{4}x^3,$ যখন $x>2$ হয় তবে

দেখাও যে $x=1$, এবং $x=2$ বিন্দুদ্বয়ে y অবিচ্ছিন্ন।

(i) $x=0$ বিন্দুতে $\dfrac{dy}{dx}$ এর মান নির্ণয় কর

$y=x^2+1$; $x\geqq0$

$=\cos x,$ $x\leqq0$ R.U. 1986

22. যদি $y=-x$, যখন $x\leq 0$

$= x$, যখন $0<x\leq 1$

$=2-x$, যখন $x>1$ হয়, তবে   D.U. 1989

দেখাও যে $x=0$ এবং $x=1$ বিন্দুদ্বয়ে y অবিচ্ছিন্ন।

23. যদি $f(x)=3+2x$, যখন $-3/2\leq x<0$

$=3-2x$, যখন $0\leq x<3/2$ C.U. 1993

$=-3+2x$, যখন $x\geq 3/2$ হয়, তবে

দেখাও যে $x=0$ বিন্দুতে $f(x)$ অবিচ্ছিন্ন এবং $x=3/2$ বিন্দুতে $f(x)$ বিচ্ছিন্ন।

24. যদি $f(x)=\frac{1}{2}(b^2-a^2)$, যখন $0\leq x\leq a$ R.U. 1988

$=\frac{1}{2}b^2-\frac{1}{6}x^2-\frac{1}{3}(a^3/x)$, যখন $a<x\leq b$

$=\frac{1}{3}(b^3-a^3)/x$, যখন $x>b$

দেখাও যে x-এর সকল ধনাত্মক মানের জন্য $f(x)$ এবং $f'(x)$ অবিচ্ছিন্ন।

25. প্রদত্ত ফাংশন $y=f(x)$ কে নিয়ে সংজ্ঞায়িত করা হলো

$f(x)=0$, যখন $x^2>1$

$=1$, যখন $x^2<1$

$=\frac{1}{2}$, যখন $x^2=1$

দেখাও যে $x=\pm 1$, বিন্দুতে $f(x)$ বিচ্ছিন্ন। x-এর সকল মানের জন্য ফাংশনটির একটি মান থাকা সত্ত্বেও ইহা বিচ্ছিন্ন কেন তাহা ব্যাখ্যা কর।

26. প্রমাণ কর যে $\lim\limits_{x\to\infty} y^n=0$ হবে যদি y একটি প্রকৃত ভগ্নাংশ এবং n একটি ধনাত্মক পূর্ণ সংখ্যা হয়।

27. দেখাও যে

$$\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\frac{3}{n^2}+\dots+\frac{n}{n^2}\right)=\frac{1}{2}$$

28. দেখাও যে

$$\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{2^2}+\dots+\frac{1}{2^n}\right)=2$$

29. দেখাও যে $\lim\limits_{n\to\infty}\frac{1^2+2^2+3^2+\dots\dots+n^2}{n^3}=\frac{1}{3}$

30. কোন ধারার সীমার সংজ্ঞা লিখ। $S_n=(-1)^n\frac{n}{n+1}$

এভাবে সংজ্ঞায়িত একটি ধারার কি কোন সীমা আছে?

(Define limit of a sequence. Has the sequence defined by

$S_n=(-1)^n\frac{n}{n+1}$ a limit?   R.H. 1966

31. ফাংশনের সংজ্ঞা দাও। নিম্নলিখিত বাস্তব ফাংশনগুলির ধারণ ক্ষেত্র (domain) নির্ণয় কর।

(i) $f(x)=\frac{1}{x}$ উত্তর: $0<x<\infty, -\infty<x<0$

(ii) $f(x)=\sqrt{(x+1)}$ উত্তর: $x\geq -1$

(iii) $f(x)=\frac{x^2+1}{(x^2-5x+6)}$ C.U. 1986   উত্তর; $x=2, 3$ ব্যতিত x-এর সকল বাস্তব মান।

(iv) $f(x)=\frac{x^3-1}{x-1}$ উত্তর: $x=1$, ব্যতিত x-এর সকল বাস্তব মান।

(v) $f(x)=\frac{x}{|x|}$ উত্তর: $x=0$ ব্যতিত x-এর সকল বাস্তব মান।

(vi) $f(x)=\frac{+\sqrt{(x-1)}}{(x^2-5x+1)}$, উঃ x-এর যে সব মান $x^2-5x+1=0$ সমীকরণ কে সিদ্ধ করে এবং $x<1$ এসব মান ব্যতিত x-এর সকল মানের জন্য

(vii) $f(x)=\frac{x^2-4}{x-2}$ উঃ $-2<x<2$

(viii) $f(x)=+\sqrt{(x^2-4x+3)}$ উঃ $3\leqslant x\leqslant 1$

(ix) $f(x)=\dfrac{\sqrt{(x-2)}}{5x^2-27x+10}$ উঃ $2\leqslant x<5,\ 5<x<\infty$

(x) $f(x)=\dfrac{x}{\sin(1/x)}$ উঃ $0<x\leqslant\dfrac{1}{n\pi}$

(xi) $f(x)=\dfrac{x^2}{(x-1)(x-2)}$ উঃ $(-\infty,\infty)-\{1,2\}$ C.U.1992

R. U. 1967 ; C. U. 1969.

32. নিম্নলিখিত ফাংশনগুলির বৃহত্তম চারণক্ষেত্র এবং বৃহত্তম ব্যাপ্তি নির্ণয় কর :—

(i) $f(x)=-1$ যখন $x<0$ উঃ চারণ ক্ষেত্র $|x|\geqq 0$ ব্যাপ্তি $(-1, 1)$
$\quad = 0$ যখন $x=0$
$\quad = 1$ যখন $x>0$

(ii) $f(x)=2$ যখন $-5<x<-1$
$\quad =\sin x$ যখন $0<x<1$

(iii) $f(x)=x+5$ যখন $-\infty<x<\infty$ উঃ $-\infty<f(x)<\infty$

(iv) $f(x)=x^2+x+1$ উঃ $x\geqq 0,\ y\geqq 0$

(v) $f(x)=x\sin(1/x)$ উঃ $x=0$, ব্যতিত x-এর সকল মান চারণক্ষেত্র। ব্যাপ্তি $(-1, 1)$

33. $n\to\infty$ অগ্রসর হলে নিম্নলিখিত ফাংশনগুলি সীমার দিকে অগ্রসর হয় কি না হয় তাহা পরীক্ষা কর :—

(i) $f(n)=\dfrac{(-1)^n}{n}$, (ii) $f(n)=\dfrac{1}{n-(-1)^n}$

(iii) $f(n)=1+1/n$, (iv) $f(n)=n[1+(-1)^n]$ R. U. 1964.

34 দেখাও যে যদি α একটি ক্ষুদ্রাতিক্ষুদ্র রাশি হয়, তবে x এর তুলনায় $\sin\alpha$ এবং $\tan\alpha$ ও একই মাত্রার ক্ষুদ্রাতিক্ষুদ্র রাশি হবে কিন্তু x এর তুলনায় $(1-\cos\alpha)$ হবে দ্বিতীয় মাত্রার (second order) ক্ষুদ্রাতিক্ষুদ্র রাশি।

[আমরা জানি $\underset{\alpha\to 0}{Lt}\dfrac{\sin\alpha}{\alpha}=1$ এবং $\underset{\alpha\to 0}{Lt}\dfrac{\tan\alpha}{\alpha}=1$ অর্থাৎ $\sin\alpha$, $\tan\alpha$ এবং α এর মাত্রা একই।

আরো $\underset{\alpha\to 0}{Lt}\dfrac{1-\cos\alpha}{\alpha^2}=\dfrac{2\sin^2\alpha/2}{(\alpha/2)^2\times 4}=\frac{1}{2}$ যাহা শূন্য নহে এবং সসীম।

সুতরাং $1-\cos\alpha$ এবং α^2 এর মাত্রা একই এবং ইহারা উভয়ে ২য় মাত্রার।]

35. দেখাও যে $3x+2x^2$ একটি একই মাত্রার ক্ষুদ্রাতিক্ষুদ্র রাশি।

36. দেখাও যে $\sqrt{(\sin\alpha)}$ এর মাত্রা α এর মাত্রার নীচে।

37. দেখাও যে $\sqrt{\alpha}$ একটি ক্ষুদ্রাতিক্ষুদ্র রাশি যার মাত্রা α এর মাত্রার নীচে।

38. দেখাও যে $\sin x-\tan x$ এর মাত্রা তৃতীয় এবং ইহার প্রধান অংশ $\frac{1}{2}x^2$.

39. দেখাও যে $\sin\alpha\ (1-\cos\alpha)$ একটি তৃতীয় মাত্রার ক্ষুদ্রাতিক্ষুদ্র রাশি এবং ইহার প্রধান অংশ $\frac{1}{2}\alpha^3$,

40. উদাহরণের সাহায্যে দেখাও যে কোন বিন্দুতে একটি ফাংশান বিচ্ছিন্ন হইলে সীমা বিদ্যমান।

উত্তরমালা I (C)

4. অবিচ্ছিন্ন নয়।	5. অবিচ্ছিন্ন নয়।	9. অবিচ্ছিন্ন।
10. অবিচ্ছিন্ন।	13. বিচ্ছিন্ন।	14. অবিচ্ছিন্ন নয়।
15. হাঁ, না।	16. অবিচ্ছিন্ন নয়।	17. অবিচ্ছিন্ন।
18. না।	19 না।	20. অবিচ্ছিন্ন।

বিবিধ প্রশ্নের উত্তরমালা

1. $y=1/x$, না, 9. ঐ বিস্তারে ফাংশনটি অবিচ্ছিন্ন। 13. $x=1$ এ,

14. $f(x)=3-x$, যখন $-1\leqslant x<0$ 19. $x=1$ বিন্দুতে সংজ্ঞায়িত করা যায় না।
$\quad =0$, যখন $x=0$
$\quad =x-3$, যখন $0<x\leqslant 1$

20. $x=0,\ \frac{1}{2}$ 28. 2. 30. না,

33. (i) 0, (ii) 0, (iii) 1, (iv) 0, n বিজোড় না, n জোড় সংখ্যা।

CHAPTER IV

Differential Co-efficient

4.1. Differential Co-efficient :—The differential Co-efficient $f'(a)$ of a function $f(x)$ at $x=a$ is defined as

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \quad \ldots\ldots\ldots(1)$$

provided the limit exists and finite. The differential co-efficient $f'(a)$ is also known as the first derivative or simply the derivative of $f(x)$ at $x=a$.

Note that both a and $a+h$ belong to domain of f.

For the function $f: x\to y$, we can interpret the differential Co-efficient at a point as the rate of change of y with respect to x at the point. Let x change small quantity δx or h and the corresponding change in y is δy or k. That is, $f: (x+\delta x)\to y+\delta y$.

Now $k=\delta y=(y+\delta y)-y=f(x+\delta x)-f(x)$

$$\frac{\delta y}{\delta x}=\frac{f(x+\delta x)-f(x)}{\delta x}$$

Hence $f'(x)=\lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}, \ldots\ldots\ldots (2)$

provided the limit exists.

We write $\frac{dy}{dx}=\lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right)$, when the limit exists.

Thus for $y=f(x)$, $f'(x)=\frac{dy}{dx}$ is the rate of change of y with respect to x at the point x.

The process of finding the differential co-efficient or derivative of a functions is called differentiation.

It is often said that "differentiate $f(x)$ *w. r.* to x" means that differentiation is made *w.r.* to the independent variable x"

In (2) we do not consider the derivative of $f(x)$ for any particular value of x but it is considered at any $x\in D_f$

4. 2. Prove that every finitely derivable function is continuous

If $f(x)$ is differentiable at $x=a$, then

it is continuous at $x=a$.

For proof see Art-3·9

Cor. The converse of the theorem is not necessarily true *i.e*," a function may be continuous at a point $x=a$ but it is not true that finite derivative should exist for that value. An example is given below.

Ex. If $f(x)=x-a$ for $x>a$

$=a-x$ for $x<a$

$=0$ for $x=a$

Show that $f(x)$ is continuous at $x=a$ but has no differential co-efficient at $x=a$

$\lim_{x\to a^+} f(x)=\lim_{h\to 0,\ (h>0)} f(a+h)=\lim_{h\to 0}(a+h-a)=0$

$\lim_{x\to 0^-} (f(x)=\lim_{h\to 0,\ (h>0)} f(a-h)=\lim_{h\to 0}\{(a-(a-h)\}=0$

Also $f(a)=0$

$\therefore \lim_{x\to a^+} f(x)=\lim_{x\to a^-} f(x)=f(a)$

Hence $f(x)$ is continuous at $x=a$,

Now

$$Rf'(a)=\lim_{h\to 0^+}\frac{(a+h)-f(a)}{h}=\lim_{h\to 0^+}\frac{(a+h-a)-0}{h}$$

$$=\lim_{h\to 0^+}\left(\frac{h}{h}\right)=1$$

$$Lf'(a)=\lim_{h\to 0^-}\frac{f(a-h)-f(a)}{-h}=\lim_{h\to 0-}\frac{\{a-(a-h)\}-0}{-h}=-1$$

Thus $Rf'(a)\neq Lf'(a)$ *i. e.*, limit does not exist. Hence derivative of $f(x)$ at $x=a$ does not exist.

Some General Theorems on Differentiation

4.3. The differential co-efficient of any constant is zero.

Let $y=f(x)=c$ where c is a constant.

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{(x+h)-x}$$

$$=\lim_{h\to 0}\frac{c-c}{h}=\lim_{h\to 0}\frac{0}{h}=0$$

Hence $\frac{dy}{dx}(c)=0$. where c is a constant

4.4. Product of a constant and a Function

The differential co-efficient of a product of a constant and a function is equal to the product of the constant and differential co-efficient of the function

Let $y=f(x)=c\phi(x)$ then $f(x+h)=c\phi(x+h)$

$$\therefore \frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{c\phi(x+h)-c\phi(x)}{h}=c\phi'(x)$$

i. e. $\frac{dy}{dx}\{c\phi(x)\}=c\frac{dy}{dx}\phi(x)$, where c is a constant

4.5. Differential Co-efficient of a sum or difference

The differential co-efficient of the sum or difference of a set of functions is the sum or difference of the differential co-efficients of that set of function.

Let $y=f(x)=\phi(x)\pm\psi(x)$

then $f(x+h)=\phi(x+h)\pm\psi(x+h)$

$$\therefore \frac{dy}{dx}=\lim_{h\to 0}\frac{\{\phi(x+h)\pm\psi(x+h)\}-\{\phi(x)+\psi(x)\}}{h}$$

$$=\lim_{h\to 0}\frac{\phi(x+h)-\phi(x)}{h}\pm\lim_{h\to 0}\frac{\psi(x+h)-\psi(x)}{h}=\phi'(x)\pm\psi'(x)$$

Hence $\frac{d}{dx}\{\phi(x)\pm\psi(x)\}=\phi'(x)\pm\psi'(x)$

Generalisation :—By repeated applications of the above result obtained it can be proved that if

$y=u_1\pm u_2\pm u_3\pm v_4\pm \quad \ldots \quad \ldots$

and $y\pm\delta y=u_1\pm\delta u_1\pm u_2\pm\delta u_2\pm(u_3+\delta u_3)\pm(u_4\pm\delta u_4)+\ldots$

then $\delta y=\delta u_1\pm\delta u_2\pm\delta u_3\pm\delta u_4\pm \quad \ldots \quad \ldots$

$$\text{or,}\quad \frac{\delta y}{\delta x}=\frac{\delta u_1}{\delta x}\pm\frac{\delta u_2}{\delta x}\pm\frac{\delta u_3}{\delta x}\pm\frac{\delta u_4}{\delta x}+\quad\ldots$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\left(\frac{\delta u_1}{\delta x}\pm\frac{\delta u_2}{\delta x}\pm\frac{\delta u_3}{\delta x}\pm\frac{\delta u_4}{\delta x}+\ldots\right)$$

$$\text{or,}\quad \frac{dy}{dx}=\frac{du_1}{dx}\pm\frac{du_2}{dx}\pm\frac{du_3}{dx}\pm\frac{du_4}{dx}+\ldots$$

4. 7. Differential Co-efficient of a product

Let $y=uv$

where u and v are two derivable functions of x.

Let u change to $u+\delta u$; v change to $v+\delta v$ when x changes to $x+\delta x$.

Let u change to $u+\delta u$ and v changes to $v+\delta v$ when x changes to $x+\delta x$.

$$\delta y = u\delta v + v\delta u + \delta u\ \delta v$$

$$\text{Or, } \frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \delta u\ \frac{\delta v}{\delta x}$$

Let $\delta x \to 0$ then δu, δv also $\to 0$ for u and v which are derivable function of x are continuous.

$$\frac{dy}{dx} = \underset{\delta x\to 0}{\text{Lim}}\left(\frac{\delta y}{\delta x}\right) = \underset{\delta x\to 0}{\text{Lim}}\left(u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \delta u\ \frac{\delta v}{\delta x}\right)$$

$$= u\frac{dv}{dx} + v\frac{du}{dx} + 0\ \frac{dv}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\text{Hence } \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

The differential co-efficient of the product of the functions is equal to

First function × derivative of the second + second function × derivative of the first.

Cor. If $y = u.\ v$ then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \qquad \text{is divided by } u.v$$

$$\frac{1}{uv}\cdot\frac{dy}{dx} = \frac{1}{v}\frac{dv}{dx} + \frac{1}{u}\frac{du}{dx}$$

$$\text{or. } \frac{dy/dx}{y} = \frac{du/dx}{u} + \frac{dv/dx}{v}$$

Now if we consider y as the function of several variables *i.e.*, $y = u.\ v.\ w.\ t \dots \dots \dots$ then

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx} + \frac{1}{t}\frac{dt}{dx} + \dots$$

4, 8. Differential co-efficient of a quotient of the functions

Let $y = \frac{u}{v}$ where u and v are two derivable functions of x and $v \neq 0$

Let δy, δu, δv be the increments of y, u, v, respectively when x changes by δx. Then.

$$y + \delta y = \frac{u+\delta u}{v+\delta v}$$

$$\delta y = \frac{u+\delta u}{v+\delta v} - \frac{u}{v} = \frac{v\delta u - u\delta v}{v(v+\delta v)}\ .\ \text{or, } \frac{\delta y}{\delta x} = \frac{v\frac{\delta u}{\delta x} - u\frac{\delta v}{\delta x}}{v(v+\delta v)}$$

If $\delta x \to 0$ then δv also $\to 0$

$$\therefore\ \frac{dy}{dx} = \underset{\delta x\to 0}{\text{Lim}}\left(\frac{\delta y}{\delta x}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\text{Hence } \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{u'v - uv'}{v^2}$$

The differential co-efficient of the quotient of two functions is equal to

$$\frac{\text{(Differential Co-efficient of Numerator)} \times \text{(Denominator)} - \text{Differential (Co-efficient of Denominator)} \times \text{Numerator.}}{\text{(Denominator)}^2}$$

4.9. Differential Co-efficient of x^n (সংখ্যার সাহায্যে ডিফা-রেন্সিয়েল সহগ নির্ণয়)

Let $y = f(x) = x^n$ where n is a positive integer.

Then $f(x+h) = (x+h)^n$

$$\therefore \quad f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}$$

$$=\lim_{h\to 0} x^n\frac{(1+h/x)^n-1}{h}=\lim_{h\to 0} x^{n-1}\frac{(1+h/x)^n-1}{h/x}$$

$$=\lim_{h\to 0} x^{n-1}\frac{(1+h/x)^n-1}{(1+h/x)-1}\quad if\ x\neq 0$$

Put $z=1+h/x$. If $h\to 0$, then $z\to 1$.

$$\therefore \quad f'(x)=\lim_{z\to 1} x^{n-1}\frac{z^n-1}{z-1}$$

$$=x^{n-1}\lim_{z\to 1}(z^{n-1}+z^{n-2}+\dots+1)\ \ (n \text{ terms})=x^{n-1}n=nx^{n-1}$$

Hence $\frac{d}{dx}x^n=nx^{n-1}$ when n is a positive integer.

If n is not a positive integer,

$$f'(x)=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}=\lim_{h\to 0} x^n\frac{\left(1+\frac{h}{x}\right)^n-1}{h}$$

$$=x^n\lim_{h\to 0}\frac{1}{h}\left[\left\{1+\frac{n}{1}\left(\frac{h}{x}\right)+\frac{n(n-1)}{1.2}\left(\frac{h}{x}\right)^2+\dots\right\}-1\right]$$

$$=x^n\left[\frac{n}{1}\left(\frac{1}{x}\right)\right]\left[\text{using Binomial expansion with } \left|\frac{h}{x}\right|<1\right]$$

or. $f'(x)=nx^{n-1}$

$$\therefore \quad \frac{d}{dx}(x^n)=nx^{n-1}\quad \text{for any real values of } n.$$

4.10 Differential co-efficient of a^x

Let $y=f(x)=a^x$, then $f(x+h)=a^{x+h}$

$$\therefore \quad f'(x)=\frac{dy}{dx}=\lim_{h\to 0}\frac{a^{x+h}-a^x}{h}=\lim_{h\to 0} a^x\left(\frac{a^h-1}{h}\right)$$

$$=a^x\lim_{h\to 0}\left(\frac{e^{h\log a}-1}{h}\right)$$

$$=a^x\lim_{h\to 0}\frac{1}{h}\left[\left\{1+\frac{h\log a}{\lfloor 1}+\frac{(h\log a)^2}{\lfloor 2}+\dots\right\}-1\right]$$

i e. $f'(x)=a^x\log a$

Hence $\frac{d}{dx}a^x=a^x\log_e a$

4.11. If $y=e^x$, $\frac{dy}{dx}=e^x\log_e e=e^x\ \therefore\ \frac{d}{dx}e^x=e^x$

4.12 Differential Co-efficient of $\log_a x$. *w. r.* to x

Let $y=f(x)=\log_a x$. then $f(x+h)=\log_a(x+h)$

$$\therefore \quad \frac{dy}{dx}=\lim_{h\to 0}\frac{\log_a(x+h)-\log_a x}{h}=\lim_{h\to 0}\frac{1}{h}\log_a\left(1+\frac{h}{x}\right)$$

Put $\frac{x}{h}=z$. If $h\to 0$, theh $z\to\infty$

$$\therefore \quad \frac{dy}{dx}=\lim_{z\to\infty}\frac{z}{x}\log_a\left(1+\frac{1}{z}\right)=\frac{1}{x}\lim_{z\to\infty}\log_a\left(1+\frac{1}{z}\right)^z$$

or, $\frac{dy}{dx}=\frac{1}{x}\log_a e\left[\therefore \lim_{z\to\infty}\left(1+\frac{1}{z}\right)^z=e\right]$

Hence $\frac{dy}{dx}(\log_a x)=\frac{1}{x}\log_a e$

4.13 If $y=\log_e x$ then $\frac{dy}{dx}=\frac{1}{x}\log_e e=\frac{1}{x}.1=\frac{1}{x}$

$$\therefore \quad \frac{d}{dx}\log_e x=\frac{1}{x}$$

4.14 Differential Co-efficient of $\sin x$ *w. r.* ro x

Let $y=f(x)=\sin x$ then $f(x+h)=\sin(x+h)$

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$=\lim_{h\to 0}\frac{2\cos(x+\frac{1}{2}h)\sin\frac{1}{2}h}{h}=\lim_{h\to 0}\cos(x+\tfrac{1}{2}h)\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}$$

$=\cos x.\ 1$ by Art. 2. 13.

Hence $\frac{d}{dx}(\sin x) = \cos x$.

4.15 Similarly the differential co-efficient of cosx is —sin x

i. e. $\frac{d}{dx} \cos x = -\sin x$

4.16 Differential Co-efficient of tan x. *w. r* to x

Let $y = f(x) = \tan x$; then $f(x+h) = \tan(x+h)$

$$\therefore \frac{dy}{dx} = \lim_{h\to 0} \frac{\tan(x+h) - \tan x}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{h\cos(x+h)\cos x}$$

$$= \lim_{h\to 0} \frac{\sin(x+h-x)}{h\cos(x+h)\cos x} = \lim_{h\to 0} \frac{\sin h}{h} \frac{1}{\cos(x+h)\cos x}$$

$$= 1.\frac{1}{\cos x \cos x} = \frac{1}{\cos^2 x} = \sec^2 x \quad \text{by Art. 2. 13.}$$

Hence $\frac{d}{dx}(\tan x) = \sec^2 x$

4.17 Similarly the differential co-efficient of cotx is —cosec² x

i. e. $\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x$

4. 18. Differential Co-efficient of sec x. *w. r.* **to x**

Let $y = f(x) = \sec x$, then $f(x+h) = \sec(x+h)$

$$\therefore \frac{dy}{dx} = \lim_{h\to 0} \frac{\sec(x+h) - \sec x}{h} = \lim_{h\to 0} \frac{\cos x - \cos(x+h)}{h\cos x\cos(x+h)}$$

$$= \lim_{h\to 0} \frac{\sin\frac{1}{2}h}{\frac{1}{2}h} \frac{\sin(x+\frac{1}{2}h)}{\cos x\cos(x+\frac{1}{2}h)} = \frac{\sin x}{\cos^2 x} = \sec x \tan x$$

Hence $\frac{d}{dx}(\sec x) = \sec x \tan x$

4. 19. Similarly the differential co-efficient of cosecx is —cosecx cot x *i. e.,*

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x.$$

4. 20. Differential Co-efficient of $\sin^{-1} x$. *w.* **r to x**

Let $y = f(x) = \sin^{-1} x$; then $f(x+h) = \sin^{-1}(x+h) = y+k$

Then $x = \sin y$ and $x+y = \sin(y+k)$

Therefore, $h = (x+h) - x = \sin(y+k) - \sin y$

and $k = \sin^{-1}(x+h) - \sin^{-1} x$

$$\therefore \frac{dy}{dx} = \lim_{h\to 0} \frac{\sin^{-1}(x+h) - \sin^{-1} x}{h}$$

$$= \lim_{h\to 0} \frac{k}{h} = \lim_{k\to 0} \frac{k}{\sin(y+k) - \sin y}$$

$$= \lim_{k\to 0} \frac{k}{2\cos(y+\frac{1}{2}k)\sin\frac{1}{2}k} = \lim_{k\to 0}\left(\frac{\frac{1}{2}k}{\sin\frac{1}{2}k}\right)\frac{1}{\cos(y+\frac{1}{2}k)}$$

$$= 1.\frac{1}{\cos y} = \frac{1}{\sqrt{(1-\sin^2 y)}} = \frac{1}{\sqrt{(1-x^2)}}$$

[as $x = \sin y$ with $\frac{-\pi}{2} < y \leq \frac{\pi}{2}$ and so $\cos y \geqslant 0$].

Hence $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{(1-x^2)}}$

4.21. Similarly the differential Co-efficient of

$\cos^{-1} x$ is $\frac{-1}{\sqrt{(1-x^2)}}$ *i. e.,* $\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{(1-x^2)}}$

4 22. Differential Co efficient of $\tan^{-1} x$, *w. r.* **to x**

Let $y = f(x) = \tan^{-1} x$, then $f(x+h) = \tan^{-1}(x+h) = y+k$

[Note that $y \in (\frac{-\pi}{2}, \frac{\pi}{2})$]

therefore, $x = \tan y$, $x+h = \tan(y+k)$

and $h = \tan(y+k) - \tan y$; $k = \tan^{-1}(x+h) - \tan^{-1} x$

$$\therefore \frac{dy}{dx} = \lim_{h\to 0} \frac{\tan^{-1}(x+h) - \tan^{-1}x}{h} = \lim_{k\to 0} \frac{1}{\tan(y-k) - \tan y}$$

$$= \lim_{k\to 0} \frac{k\cos(y+k)\cos y}{\sin(y+k)\cos y - \cos(y+k)\sin y}$$

$$= \lim_{k\to 0} \frac{k\cos(y+k)\cos y}{\sin k} = \lim_{k\to 0} \frac{k}{\sin k}\cos(y+k)\cos y = \cos^2 y$$

$$= \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

Hence $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

4.23. Similarly the Differential co-efficient of

$\cot^{-1} x$ is $-\frac{1}{1+x^2}$ w, r, to x

i. e., $\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$

4.24. Differential co-efficient of $\sec^{-1}x$. w, r. to x.

Let $y = f(x) = \sec^{-1}x$, then $f(x+h) = \sec^{-1}(x+h) = y+k$

therefore, $x = \sec y$, $x+h = \sec(y+k)$

and $h = \sec(y+k) - \sec y$; $k = \sec^{-1}(x+h) - \sec^{-1}x$

$$\therefore \frac{dy}{dx} = \lim_{h\to 0} \frac{\sec^{-1}(x+h) - \sec^{-1}x}{h} = \lim_{h\to 0} \frac{k}{h}$$

$$= \lim_{k\to 0} \frac{k}{\sec(y+k) - \sec y} = \lim_{k\to 0} \frac{k\cos(y+k)\cos y}{\cos y - \cos(y+k)}$$

$$= \lim_{k\to 0} \frac{\frac{1}{2}k}{\sin\frac{1}{2}k} \frac{\cos(y+k)\cos y}{\sin(y+\frac{1}{2}k)} = 1\,\frac{\cos^2 y}{\sin y}$$

$$= \cot y \cos y = \frac{1}{\sec y \tan y} = \frac{1}{x\sqrt{(\sec^2 y - 1)}} = \frac{1}{x\sqrt{(x^2-1)}}$$

Hence $\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{(x^2-1)}}$

4.25. Similarly the Differential Co-efficient of $\operatorname{cosec}^{-1}x$ is

$-\frac{1}{x\sqrt{(x^2-1)}}$ i. e $\frac{d}{dx}\operatorname{cosec}^{-1}x = -\frac{1}{x\sqrt{(x^2-1)}}$

Examples

Ex. 1. Find from first principle the differential co-efficient of $\sqrt{(b-2ax)}$ w. r. to x

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{\{b-2a(x+h)\}} - \sqrt{(b-2ax)}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{\{b-2a(x+h)\} - (b-2ax)}{\sqrt{\{b-2a(x+h)\}} + (b-2ax)}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{-2ah}{\sqrt{\{b-2a(x+h)\}} + \sqrt{(b-2ax)}}$$

$$= \lim_{h\to 0} \frac{-2a}{\sqrt{\{b-2a(x+h)} + \sqrt{(b-2ax)}}$$

$$= \frac{-2a}{2\sqrt{(b-2ax)}} = -\frac{1}{\sqrt{(b-2ax)}}$$

$$\therefore \frac{d}{dx}\sqrt{(b-2ax)} = -\frac{a}{\sqrt{(b-2ax)}}$$

Ex. 2. Find from the definition the differential co-efficient of $\frac{2x-3}{2x+5}$ w. r. to x. [R. U. 1986]

Let $f(x) = \frac{2x-3}{3x+5}$

Let $f'(x = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{2(x+h)-3}{3(x+h)+5} - \frac{2x-3}{3x+5}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\cdot\frac{\{2(x+h)-3\}(3x+5)-\{3(x+h)+5\}(2x-3)\}}{\{3(x+h)+5\}(3x+5)}$$

$$=\lim_{h\to 0}\frac{1}{h}\frac{19h}{\{3(x+h)+5\}(3x+5)}=\lim_{h\to 0}\frac{19}{\{3x+h)+5\}(3x+5)}$$

$$=\frac{19}{(3x+5)}$$

$$\therefore\ \frac{d}{dx}\left(\frac{2x-3}{3x+5}\right)=\frac{19}{(3x+5)^2}$$

Ex. 3. Find from the first principle the derivative of $\cos^2 x$. *w. r.* to x. [D. U. 1965]

Let $f(x)=\cos x^2$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\cos(x+h)^2-\cos x^2}{h}$$

$$=\lim_{h\to 0}\frac{2\sin\frac{1}{2}(x+h)^2+x^2\}\sin\frac{1}{2}\{x^2-(x+h)^2\}}{h}$$

$$=\lim_{h\to 0}2\sin\tfrac{1}{2}\{(x+h)^2+x^2\}\frac{\sin\frac{1}{2}\{-2xh-h^2\}}{\frac{1}{2}\{-2xh-h^2\}}\frac{(-2xh-h^2)}{h}$$

$$=2\sin\tfrac{1}{2}(x^2+x^2).\ 1.\ \tfrac{1}{2}(-2x)=-2x\sin x^2$$

$$\therefore\ \frac{d}{dx}(\cos x^2)=-2x\sin x^2$$

Ex. 4. Differentiate $\tan^{-1}\frac{x}{a}$ from the first principle *w.* r to x

[R. U. 1952]

Let $z=\tan^{-1}\frac{x}{a}$, and $z+k=\tan^{-1}\frac{x+h}{x}$

If $h\to 0$, then $k\to 0$. Now we have

$$\frac{x+h}{a}=\tan(z+k),\ \frac{x}{a}=\tan z\text{ and}$$

$$\frac{h}{a}=\frac{x+h}{a}-\frac{x}{a}=\tan(z+k)-\tan z$$

$$\therefore\ f'(x)=\lim_{h\to 0}\frac{\tan^{-1}(x+h)/a-\tan^{-1}(x/a)}{h}$$

$$=\lim_{k\to 0}\frac{k}{a\{\tan(z+k)-\tan z\}}$$

$$=\lim_{k\to 0}\frac{1}{a}\cdot\frac{k}{\sin k}\cos(z+k)\cos z=\frac{1}{a}\ 1\cos z.\cos z$$

$$=\frac{\cos^2 z}{a}=\frac{1}{a}\frac{1}{\sec^2 z}=\frac{1}{a}\frac{1}{1+\tan^2 z}=\frac{1}{a}\frac{1}{1+(x/a)^2}$$

$$=\frac{1}{a}\frac{a^2}{x^2+a^2}=\frac{a}{a^2+x^2}$$

$$\frac{d}{dx}\left(\tan^{-1}\frac{x}{a}\right)=\frac{a}{a^2+x^2}$$

Ex. 5. Differentiate log (sin x) from the definition.

Let $z=\sin x$, $z+k=\sin(x+h)$

$k=(z+k)-z=\sin(x+h)-\sin x$

If $h\to 0$ then k also$\to 0$

$$f'(x)=\lim_{h\to 0}\frac{\log_e\sin(x+h)-\log_e\sin x}{h}$$

$$=\lim_{h\to 0,\,k\to 0}\frac{\log_e(z+k)-\log_e z}{k}\frac{k}{h}=\lim_{h\to 0,\,k\to 0}\frac{\log_e\{(z+k)/z\}}{k}\frac{k}{h}$$

$$=\lim_{k\to 0}\frac{\log(1+k/z)}{k/z}\frac{1}{z}\cdot\ \mathit{Lim}_{h\to 0}\frac{k}{h}$$

$$=\lim_{k\to 0}\frac{\log_e(1+k/z)}{k/z}\times\mathit{Lim}_{h\to 0}\frac{1}{z}\frac{\sin(x+h)-\sin x}{h}$$

$$=\lim_{h\to 0}\frac{1}{z}2\cos(x+\tfrac{1}{2}h)\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}\tfrac{1}{2}$$

$$=1.\ {}^1/z.\ 2\cos x.\ \tfrac{1}{2}=\cos x/\sin x=\cot x.$$

$$\therefore\ d(\log_e\sin x)/dx=\cot x$$

Ex. 6. Differentiate $x \sin x$ from the first principle $w.r.$ to x [C. U. 1986]

Let $f(x)=x^{\sin x}=e^{\sin x \log x}$

$$f'(x)=\lim_{h\to0}\frac{e^{\sin(x+h)\log(x+h)}-e^{\sin x\log x}}{h}$$

$$=\lim_{h\to0}\frac{e^{\sin x\log x}\left[e^{\sin(x+h)\log(x+h)-\sin x\log x}-1\right]}{h}$$

$$=e^{\sin x\log x}\lim_{h\to0}\frac{e^z-1}{h}=e^{\sin x\log x}\lim_{h\to0}\frac{e^z-1}{z}\cdot\frac{z}{h}$$

where $z=\sin(x+h)\log(x+h)-\sin x\log x$

If $h\to0$, then z also$\to0$.

$$\therefore\ f'(x)=e^{\sin x\log x}\lim_{z\to0}\left(\frac{e^z-1}{z}\right)\lim_{h\to0}\left(\frac{z}{h}\right)$$

$$=x^{\sin x}\ 1.\ \lim_{h\to0}\left(\frac{z}{h}\right).$$

$$=x^{\sin x}\lim_{h\to0}\frac{\sin(x+h)\log(x+h)-\sin x\log x}{h}=x^{\sin x}\times$$

$$\lim_{h\to0}\left[\frac{\sin x\{\cos h\log(x+h)-\log x\}+\cos x\sin h\log(x+h)}{h}\right]$$

$$=\lim_{h\to0}x^{\sin x}\left[\sin x\left\{\frac{\log(1+h/x)}{h/x}\cdot\frac{1}{x}\right\}+\lim_{h\to0}\frac{\sin h}{h}\cos x.\right.$$

$$\left.\log(x+h)\right]\qquad\left[\because\lim_{h\to0}\cosh=1\right]$$

$$=x^{\sin x}\left[\sin x\frac{1}{x}+1.\log x\cos x\right]=x^{\sin x}\left[\frac{\sin x}{x}+\log x\cos x\right]$$

$$\left[\because\ \lim_{h\to0}\frac{\log(1+h/x)}{h/x}=1\right]$$

Ex. 7. $f(x)=e^x\sin x$, find $f'(0)$ from definition.

$f(0)=e^0\sin 0=0.$

$$\therefore\ f'(0)=\lim_{h\to0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to0}\frac{e^{0+h}\sin(0+h)-e^0\sin 0}{h}$$

$$=\lim_{h\to0}\frac{e^h\sin h}{h}=\lim_{h\to0}e^h\lim_{h\to0}\frac{\sin h}{h}=1.1$$

$\therefore\ f'(0)=1$

Ex. 8. Differentiate cot x from the first principle at $x=\frac{1}{4}\pi$. $w.r.$ to x.

Let $f(x)=\cot x$

$$f'\left(\frac{\pi}{4}\right)=\lim_{h\to0}\frac{f(\frac{1}{4}\pi+h)-f(\frac{1}{4}\pi)}{h}=\lim_{h\to0}\frac{\cot(\frac{1}{4}\pi+h)-\cot\frac{1}{4}\pi}{h}$$

$$=\lim_{h\to0}\frac{\sin\frac{1}{4}\pi\cos(\frac{1}{4}\pi+h)-\cos\frac{1}{4}\pi\sin(\frac{1}{4}\pi+h)}{h\sin(\frac{1}{4}\pi+h)\sin\frac{1}{4}\pi}$$

$$=\lim_{h\to0}\frac{-\sin(\frac{1}{4}\pi+h-\frac{1}{4}\pi)}{h\sin(\frac{1}{4}\pi+h)\sin\frac{1}{4}\pi}$$

$$=\lim_{h\to0}\frac{-\sin h}{h}\cdot\frac{1}{\sin(\frac{1}{4}\pi+h)\sin\frac{1}{4}\pi}=-1.\frac{1}{\sin^2\frac{1}{4}\pi}=-2.$$

$\therefore\ f'(\frac{1}{4}\pi)=-2.$

Ex. 9. Find the differential co-efficient of $x^n\sin ax$ from the definition. [C. U. 1987]

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Let $y=f(x)=x^n\sin ax$

$$\therefore\ \frac{dy}{dx}=f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to0}\frac{(x+h)^n\sin\{a(x+h)\}-x^n\sin ax}{h}$$

$$= \lim_{h\to 0}\left[\left\{\frac{(x+h)^n - x^n}{h}\right\}\sin a(x+h) + x^n\left\{\frac{\sin a(x+h) - \sin ax}{h}\right\}\right]$$

$$= \lim_{h\to 0}[nx^{n-1} + \text{terms containing higher power of h}] \times$$

$$\sin(ax+h) + x^n \lim_{h\to 0}\frac{2\cos(ax+ah/2)\sin ah/2}{h}$$

$$= nx^{n-1}\sin ax + x^n \lim_{h\to 0}\cos(ax+ah/2)\frac{\sin ah/2}{ah/2}\,a.$$

$$= nx^{n-1}\sin ax + x^n\cos ax.\,1a = nx^{n-1}\sin ax + ax^n\cos ax$$

$$\therefore \quad \frac{d}{dx}(x^n\sin ax) = nx^{n-1}\sin ax + ax^n\cos ax.$$

Ex. 10. A function $f(x)$ is defined as follows :—

$$f(x) = x^2 \sin\frac{1}{x} \text{ when } x \neq 0.$$

$$= 0, \qquad \text{when } x = 0$$

Show that $f(x)$ is continuous and differentiable at $x=0$.

Since $\left|x^2 \sin\frac{1}{x}\right| \leqslant \left|x^2\right| 1 = x^2$

and $\lim_{h\to 0} x^2 = 0.$

$$\therefore \quad \lim_{x\to 0} x^2 \sin\frac{1}{x} = 0.$$

Also $f(0) = 0$

Hence $f(x)$ is continuous at $x=0$.

Differentiability at $x=0$

we have $\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{(0+h)^2\sin\left(\frac{1}{0+h}\right) - 0}{h}$

$$= \lim_{h\to 0}\frac{h^2\sin(1/h)}{h} = \lim_{h\to 0} h\sin(1/h) = 0$$

Hence $f(x)$ is differentiable at $x=0$

and $f'(0) = 0$

Ex. 11. Determine whether $f(x)$ is continuous and has a derivative at the origin where

$$f(x) = a + x \text{ if } x > 0$$

$$f(x) = a - x \text{ if } x < 0$$

Let us consider

$\therefore f(0) = a + 0 = a \quad (\therefore \; f(x) = a + x \text{ for } x \geqslant 0)$

Again

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+}(a+x) = a$$

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-}(a-x) = a$$

$$\therefore \quad \lim_{x\to 0+} f(x) = \lim_{x\to 0-} f(x) = f(a) = a$$

Hence the function $f(x)$ is continuous at $x=0$

Differentiability at $x=0$

$$Rf'(0) = \lim_{h\to 0}\frac{f(0+h) - f(0)}{h} = \lim_{h\to 0}\frac{a+h-a}{h} = 1 \quad (h>0)$$

$$Lf'(0) = \lim_{h\to 0}\frac{f(0-h) - f(0)}{h} = \lim_{h\to 0}\frac{a+h-a}{-h} = -1 \quad (h>0)$$

$$\therefore \quad Rf'(0) \neq Lf'(0)$$

Hence the $f(x)$ is not differentiable at $x=0$

Fx. 12. A function $f(x)$ is defined in the following way.

$$f(x)=0,\ 0\leq x<\tfrac{1}{2}$$
$$=1,\ x=\tfrac{1}{2}$$
$$=2,\ \tfrac{1}{2}<x\leq 1$$

Show that $f(x)$ is discontinuous at $x=\frac{1}{2}$;does $f'(\frac{1}{2})$ exists ?

When $x=\frac{1}{2}$, then $f(x)=1$ i. e., $f(\frac{1}{2})=1$

$$f(\tfrac{1}{2}+0)=\lim_{h\to 0} f(\tfrac{1}{2}+h)=2, \quad \text{when } x>\tfrac{1}{2}$$

$$f(\tfrac{1}{2}-0)=\lim_{h\to 0} f(\tfrac{1}{2}-h)=0, \quad \text{when } x<\tfrac{1}{2}$$

Thus $f(\frac{1}{2}+0)\neq f(\frac{1}{2}-0)\neq f(\frac{1}{2})$

Hence $f(x)$ is discontinuous at $x=\frac{1}{2}$

Differentiability at $x=\frac{1}{2}$

$$Rf'(\tfrac{1}{2})=\lim_{h\to 0}\frac{f(\frac{1}{2}+h)-f(\frac{1}{2})}{h}=\lim_{h\to 0}\frac{2-1}{h}=\infty$$

$$Lf'(\tfrac{1}{2})=\lim_{h\to 0}\frac{f(\frac{1}{2}-h)-f(\frac{1}{2})}{-h}=\lim_{h\to 0}\frac{0-1}{-h}=\infty$$

$Rf'(\frac{1}{2})=Lf'(\frac{1}{2})=\infty$

The limits are not finite.

Hence $f(\frac{1}{2})$ does not exist.

Ex.13. Discuss the Continuity of function $f(x)=[x]$ at $x={}^1/_n$ fractional, $h\neq 0$; where $[x]$ denotes the integral past of x i.e. $[x]$ denotes the greatest integer $\leq x$. Draw the graph.

Does $f(x)$ at $x={}^1/3$ or: $f({}^1/3)$ exists?

($x=1/n$ অথবা ভগ্নাংশের $f(x)=[x]$ এর জন্য ফাংশানটি অবিচ্ছিন্ন কিনা পর্যালোচনা কর। $[x]$, x এর বৃহত্তম পূর্ণ সংখ্যা নির্দেশ করে, $[x]\leq x$.)

Sol. At $x=1/n$, $n\neq 0$. an integer $f[({}^1/3]=[({}^1/3]=0$ ...(1)

Also we have

$$\lim_{x\to {}^1/_3+0} f(x)=\lim_{h\to 0} f({}^1/3+h)=\lim_{h\to 0}[{}^1/3+h]=[{}^1/3]=0 \ldots (2)$$

and $$\lim_{x\to {}^1/_3-0} f(x)=\lim_{h\to 0} f({}^1/3-h)=\lim_{h\to 0}[{}^1/3-h]=[{}^1/3]=0 \ldots (3)$$

From (1), (2), (3) we see that

$$\lim_{x\to {}^1/3-0} f(x)=\lim_{h\to \frac{1}{3}+0} f(x)=f({}^1/3)=0$$

The function in continuous at $x=1/3$ i. e; it will be continuous for fractional values of n.

For. Differentiabilty at $x=1/3$

$$Rf\left(\frac{1}{3}\right)=\text{Lt}\frac{f(1/3+h)-f(1/3)}{h}=\text{Lt}\frac{0-0}{h}=0/0$$

$$Lf\left(\frac{1}{3}\right)=\lim_{h\to 0}\frac{f(1/3-h)-f(1/3)}{-h}=\lim_{h\to 0}\frac{0-0}{-h}=-0/0$$

Limits are not finite

Hence $f(1/3)$ does not exist.

For graph see Ex17 chapter III(c)

Ex. 14. Discuss the continuity of the rational infegsal values of x for the function $f(x)=[x]$ or; integral part of x. Also draw the graph.

Sol. We know that $[x]$ denotes the greatest integer $\leq x$.

If n is an integer we conclude that

$f(x)=n-1$, for $n-1\leq x<n$...(1)

$=h$, for $n\leq x<n+1$ ...(2)

$=n+1$, for $n+1\leq x<n+2$ ....(3)

and so on.

Let us consider now $x=n$ ...(4)

Then $f(n)=n$ from (2)

$$\lim_{x\to n-h}=\lim_{h\to 0} f(n+h)=\lim_{h\to 0}[n-h]=[n-0]=n-1 \ldots (5)$$

$$\lim_{x\to n+h}=\lim_{h\to 0} f(n+h)=\lim_{h\to 0}[n+h]=[n+0]=n+1 \ldots (5)$$

From (4), (5) and (6) we see thet

$f(n-0)\neq f(n)\neq f(n+0)$

i.e.; the function in discontinuous for x=n

i.e; when x is an integer. the function is continuous when x is fractianal

For graph see Ex17 Chapter III(c)

Ex. 15. Determine the continuities and discontinuties of the following functions f(x). composite functions $f[f(x)]$ and $f[f[f(x)]]$ it $f(x) = 1/(1-2x)$.

[f(x), f[f(x)], এবং f[f[f(x)]] ফাংশান এবং সংযুক্ত ফানশানগুলির অবিচ্ছিন্নতা ও বিচ্ছিন্নতা বিন্দুগুলি নির্ণয় কর, যদি $f(x)=1/(1-2x)$]

Sol. $f(x)=1/(1-2x)$ if $1-2x=0$ or, $x=1/2$

Then $x={}^1/2$. f (x) becomes infinite So f(x) is discontinuous at $x={}^1/2$.

If $x\neq 1/2$, let $F(x)=f[f(x)]=f\left(\frac{1}{1-2x}\right)$

$$=\frac{1}{1-\{1/(1-2x)\}}=\frac{1-2x}{1-2x-1}=\frac{1-2x}{-2x}=\frac{2x-1}{2x}$$

Hwnce F(x) is disconontinuous at $x=0$

If $x\neq 0$, $x\neq 1/2$, then

$$v(x) = f[f[f(x)]] = f[f(1/(1-2x))] = f\left(\frac{2x-1}{2x}\right)$$

$$=\frac{1}{1-\frac{2x-1}{2x}}=\frac{2x}{2x-2x+1}=2x$$

Hence $v(x) = 2x$ is a continuous straight line passing through the origin. $Lf(x) = Rf(x) = f(0)=0$. and limits exist for all values of x.

Thus the points of discontinuities are $x=1/2$ for $f(x)$, $x=0$, $1/2$ for $f[f(x)]$ and $x=0$ for f[f(x)]

Exercise (IVA)

নিম্নলিখিত ফাংশনের x এর ভিত্তিতে সংজ্ঞার দ্বারা ডিফারেন্সিয়েল সহগ নির্ণয় কর।

Differentiation the following from the first principle w.r. to x.

1. x^2+3x+5

2. $7x^2+5/x$ R. U. 1960

3. $1/x$ D. U. 1957

4. $\frac{1}{\sqrt{(2+x)}}$

(4.1) $x+\sqrt{x^2+1}$ R. U. 1988

5. $\sqrt{x}$ D.U, 1965

6. $e^{\tan -x}$

7. $e^{\sqrt{x}}$

8. e^{5x+a} R.U. 1966

9. $\tan x/a$

10. $\tan x^2$ D.U. 1966

11. $\cos(ax+b)$ R. U. 1958

12. $\sin(ax+b)$ D.U. 1955

13. $a \sin x/a$

14. $\log_{10} x$

15. $\sqrt[n]{x}$

16. $\sqrt{(\sin x)}$

17. $\log \cos x$

18. $\tan^2 2x$ D.U. 1954

19. $\sin x^2$ D. U. 1964

20. $\log \sin x/a$ R. H 1988

21. x^x D. H. 1987

22. $x \sin x$

23. $\log \sin^{-1} x$

24. $\cos \log x$

25. $\frac{\cos x}{\log x}$

26. $\frac{e^{2x}}{\log x}$ (a) $\frac{x^2+3x+1}{x}$

27. $x^3 \sin x$

28. $e^{\tan x}$

(i) $\sin^{-1} x$ at $x=0$

(i) $e^{\sin x}$ at $x=a$ C. U. 1984

29. If $f(x)=\sin x$ find $f'(\pi/2)$ from the definition

30. If $f(x)=\tan x$, find $f'(\pi/4)$ from the first principle

31. Show that the function $|x|$ is continuous at $x=0$ but is not differentiable at that point.

32. A function is defined as follows :

$$f(x)=-x,\quad x\leq 0$$
$$= x,\quad x\geq 0 \qquad i.e., f(x)=|x|$$

Show that $f(x)$ is continuous but not differentiable at $x=0$

33 If $f(x)=x\tan^{-1}\frac{1}{x}$ when $x\neq 0$

$=0$ when $x=0$

Show that $f(x)$ is continuous but not differentiable at $x=0$

34. Examine whether $f(x)$ possesses $f'(x)$ at $x=0$.

where $f(x)=0$, when $x=0$

$$f(x) = \frac{1}{1+e^{1/x}} \text{ when } x \neq 0$$

35. A function $f(x)$ is defined as follows :

$$f(x) = 0, \quad x = 0$$
$$= x, \quad x > 0$$
$$= -x, \quad x < 0$$

Does $f'(0)$ exist ?

36. A function is defined as follows :--

$$f(x) = 1, \quad -\infty < x < 0$$
$$= 1 + \sin x, \; 0 < x < \tfrac{1}{2}\pi$$
$$= 2 + (x - \tfrac{1}{2}\pi)^2, \; \tfrac{1}{2}\pi \leq x < \infty$$

Show that $f(x)$ is continuous at $x = 0$ and $x = \frac{1}{2}\pi$ but $f'(x)$ exists for $x = \frac{1}{2}\pi$ and does not exist for $x = 0$

37. Discuss the continuity and differentiability of the following functions defined as

(i) $f(x) = x \sin \frac{1}{x}, \quad x \neq 0$ D. H. 1984

$$= 0, \quad x = 0$$

Investigate at $x = 0$

(i) $f(x) = x \cos 1/x, \; x \neq 0$: and $f(0) = 0$

Investigate at $x = 0$

(ii) $f(x) = (x - a) \sin \frac{1}{x - a}$ for $x \neq a$

$$= 0 \quad \text{for} \quad x = 0$$

Investigate at $x = 0$

38. Show that the function

$$f(x) = x\{1 + \tfrac{1}{3} \sin(\log x^2)\}, \; x \neq 0$$
$$= 0 \text{ when } x = 0$$

is continuous but has no derivative at $x = 0$

39. A function $f(x)$ is defined as follows

C.H. 1992

$$f(x) = 1 + x \; ; \; x \leq 0$$
$$= x \quad ; \; 0 < x < 1$$
$$= 2 - x \; ; \; 1 \leq x \leq 2$$
$$= 3x - x^2, \; x > 2$$

Show that $f'(x)$ at $x = 1$ and 2 does not exist, though $f(x)$ is continuous at these points.

39. (i) Sketch roughly th. graph of the function (ফাংশানটির চিত্র অংকন কর)।

$$f(x) = \begin{cases} x^2 + 1, & x < 0 \\ x, & 0 \leq x \leq 1 \\ 1/x, & x > 1 \end{cases}$$

D. J. 1986, 88

Discuss the continuity of the function at $x=0$ and $x = 1$ ($x = 0$, $x = 1$ বিন্দুতে ফাংশানটির অবিচ্ছিন্নতা পর্যালোচনা কর)

39. (ii) যদি

$$f(x) = \begin{cases} x^2 \cos \frac{1}{x}; & x \neq 0 \\ 0, & x = 0 \end{cases}$$

D, U. 1986

$f'(0)$ নির্ণয় কর।

40. Prove that the product of two continuous functions is continuous. Prove that if $f'(x) < 0$ in $a < x < b$, then $f(x)$ is steadily decreasing function in this interval.

Reduce that $2x/\pi < \sin x < x$. If $0 < x \leq \pi/2$.

41. Distinguish between derivability and differentiability of a function at a given point and show that a necessary and sufficient condition for the differentiability of $f(x)$ at a given point is that it possesses a finite derivative at that point.

41 (a) The function f is defined thus:-

$f(x) = e^x$ when $x < 0$

$= x^2 + 1$ when $0 \le x \le 1$ N.U.1994

$= \frac{x}{x-1}$ when $x > 1$

Find

(a) $\lim_{x \to \alpha} f(x)$, $\lim_{x \to 0-} f(x)$. $\lim_{x \to 0+} f(x)$

(a) $\lim_{x \to \alpha} f(x)$, $\lim_{x \to 0+} f(x)$, $\lim_{x \to 0+\alpha} f(x)$

(b) Determine where f is not continuous and where it is not differentiable. [f কোথায় অবিচ্ছিন্ন নয় এবং কোথায় অন্তরীকরণযোগ্য নয় তাহা নির্ধারণ কর।]

(c) Draw a rough Sketch of the graph of f and the range of f

[মোটামোটিভাবে f–এর লেখচিত্র অঙ্কন কর এবং f এর রেঞ্জ উল্লেখ কর।]

42. Let $f(x) = \sqrt{x}$ in the interval $0 \leqq x \leqq 4$. If $\in$ is a positive number, choose small at pleasure and find a $\delta > 0$ depending on $\in$, such that $| f(x_1) + f(x_2) | < \in$ whenever $| x_1 - x_2 | \leqq \delta$.

43. Show that $f(x) = | x |$ is continuous at $x=0$ but not differentiable at $x=0$

for derivability for $x=0$,

$$\frac{f(0+h)-f(0)}{h} = \frac{f(h)}{h} = \frac{|h|}{h} = \begin{cases} 1, \text{ if } h>0 \\ -1, \text{ if } h<0 \end{cases}$$

$\therefore Rf'(0) = 1 \neq Lf'(0) = -1$

Hence $f'(0)$ does not exist

For continuity at $x=0$

$| f(x) - f(0) | = | x | < \in$, when $| x - 0 | \leq \delta$

δ is any positive number less than $\in$.

Hence $f(x)$ is continuous at $x=0$

44. Examine the continuity and differentiability of the function f defined by $f(x) = | x-2 |$ at $=2$

Draw the graph of the function in $-4 \le x \le 4$. D. U. 1986

Ans. conti. at $x=2$ but not differentiable.

The graph consists of two straight lines inclined at 45^0 at $x=2$ with the x-axis, points (2, 0), (3, 1), (4. 2) ; (1, −3), (2, −4), (3, −5), (4, −6).

45. Show that $| x+1 | + | x | + | x-1 |$ is continuous but not differentiable at $x = -1, 0, 1$.

Sol : At $x = -1$,

$Rf'(-1) =$

$$\lim_{x \to -1+h} \frac{(-1+h+1-1+h-1+h-1)-0+1+1-h+1}{h}$$

$$= \lim_{h \to 0} \frac{3h}{h} = 3$$

$$Lf'(-1) = \lim_{x \to -1-h} \frac{-1-h+(-1-h-1-h-1)-(-1-2)}{h}$$

$$= \lim_{h \to 0} \frac{-3h}{h} = -3$$

$\therefore Rf'(-1) \neq Lf'(-1)$. So it is not differentiable

46. (a) Sketch roughly the graph of the function $f(x)$

where $f(x) = \begin{cases} e^{-|x|/2}, & -1 < x < 0 \\ x^2, & 0 \leqq x < 2 \end{cases}$ D.U. 1987

Find the domain and range of the function.

(b) Discuss the existence of $\lim_{x \to 0} f(x)$. where $f(x)$ is given in (a)

(c) Discuss the continuity of the above function.

$f(x)$ also at $x=0$

47. Let the function f be given by

$$f(x)=\begin{cases} x^3, & x\leq 1 \\ 3x-2, & x<1 \end{cases}$$

Does the function f have a continuous derivative at $x=0, 1$? Explain. D. U. 1987

48. With diagram discuss the continuity and differentiability of the following functions at the point indicated.

(নির্দিষ্ট বিন্দুতে $f(x)$ এর অবিচ্ছিন্নতা ও অন্তরীকরণযোগ্যতা চিত্র সহকারে বর্ণনা কর।)

(i) $f(x)=(\sin x),\ 0<x\leq 4\pi$, at $x=3\pi$

(ii) $f(x)=\begin{cases} 2x^{-1}, & 0<x\leq 1 \\ x^2-x+1, & x>1 \end{cases}$ at $x=1$ C. H. 1985

49. A function $f:\ \mathbb{R}\rightarrow\mathbb{R}^{+}\cup\{0\}$ is defined by $f(x)=(x-1)$ Is it an onto function ? Sketch the graph of f.

Does the $\lim\limits_{h\to 0}\dfrac{f(1-h)-f(1)}{h}$ exists D. U. 1989

50. Let $f(x)=5x-4$ for $0<x\leq 1$

$=4x^2-3x$ for $1<x<2$

Discuss continuity of $f(x)$ at $x=1$ and existence of $f'(x)$ for this value. R. U. 1987

51 Examine whether $f'(x)$ exists at $x=0$ and $x=2$ where

$f(x)=x,\ 0\leq x<2$

$=x-1,\ 2\leq x$ C.U. 1991

52. Draw the graph of the function defined by $y=[x]$ where $[x]$ denotes the integral part of x (largest integer not exceeding x) Discuss continuity and differentiabiltiy of the function at $x=1$ ($y=$

$[x]$ দ্বারা বর্ণিত ফাংশনটির লেখচিত্র অঙ্কন কর, যেখানে $[x]$ দ্বারা x-এর পূর্ণাংশ বুঝায় (x-এর অনূর্দ্ধ বৃহত্তম পূর্ণ সংখ্যা) $x=1$ বিন্দুতে ফাংশনটির অবিচ্ছিন্নতা ও অন্তরীকরণযোগ্যতা আলোচনা কর। D.U. 1991

53. Find the points of continuity and discontinuties of the function $f(x)=\frac{1}{1-x}$ of $f(x)$ and the composite functions $f[f(x)]$ and $f[f[f(x)]]$ Ans. At $x=1$, $f(x)$ is discontinuons.

$x=0$and 1 are discontinuties of $f[f[f(x)]]$; $x=1$ is the discontinuity of $f[f(x)$

[$f(x)={}^1/(1-x)$ ফাংশানের জন্য f(x) এর অবিচ্ছিন্নতা ও বিচ্ছিন্নতা বিন্দুগুলি নির্ণয় কর। সংযুক্ত ফাংশানের $f[(x)]$, $f[f[(x)]]$ জন্য ও বিন্দুগুলি নির্ণয় কর]

54. $f(x+y)=f(x)f(y)$ for all x and y and $f(x)=1+xg(x)$ where $\lim\limits_{x\to 0} g(x)=1$.

Show that the derivative $f'(x)$ exists and $f'(x)=f(x)$ for all x.

[যদি $f(x+y)=f(x)f(y)$ সকল x এবং y এর জন্যে এবং $f(x)=1+xg(x)$ যেখানে $\lim\limits_{x\to 0} g(x)=1$, তখন দেখাও যে অন্তরণ সহজ $f'(x)$ বিদ্যমান এবং $f'(x)=f(x)$ সকল x এর জন্য।]

54. If f is continuous function of x satisfying the functional equation $f(x+y)=f(x)+f(y)$

Show that $f(x)=ax$, a is a constant.

55.(i) Show that f is defined by $f(x)=|x|+|x-1|$

is continuous but not derivable for $x=0$, $x=1$.

(ii) Show that f defined by $f(x)=|x|+|x-1|+|x-2|$ is continous and not differentiable at $x=0$, $x=1$.: $x=2$.

56. If $\phi''(x)\geq 0$ for every value of x, then

$\phi[\frac{1}{2}(x_1+x_2)]\leq\frac{1}{2}[\phi(x_1)+\phi(x_2)]$

$\phi[\frac{1}{n}(x_1+x_2+\ldots x_n)]\leq\frac{1}{n}[\phi(x_1)+\phi(x_2)+\ldots\phi(x_n)]$

ANSWER

1. $2x+3$ 2. $14x-(5/x^2)$ 3. $-(1/x^2)$

4. $-\frac{1}{8}(2+x)^{-3/2}$ 5. $1/2\sqrt{x}$ 6. $e^{\tan x}\sec^2 x$

7. $e^{\sqrt{x}}\frac{1}{2\sqrt{x}}$ 8. $5e^{5x+a}$ 9. $2x\sec^3 x^2$

10. $\frac{1}{a}\sec^2\frac{x}{a}$ 11. $-a\sin(ax+b)$ 12. $a\cos(ax+b)$

13. $\cos\frac{x}{a}$ 14. $\frac{1}{x}\log_{10}^{e}$ 15. $\frac{1}{x}x^{(1/n)-1}$

16. $\frac{1}{2}\frac{\cos x}{\sqrt{\sin x}}$ 17. $-\tan x$ 18. $4\tan 2x\sec^2 2x$

19. $2x\cos x^2$ 20. $-\frac{1}{a}\cot\frac{x}{a}$ 21. $x^x(1+\log x)$

22. $x\cos x+\sin x$ 23. $\frac{1}{\sin^{-1}x\sqrt{(1-x^2)}}$

24. $-\frac{1}{x}\sin\log x$ 25. $\left(-\sin x\log x-\frac{1}{x}\cos x\right)(\log x)^2$

26. $\left(2e^{2x}\log x-\frac{1}{x}e^{2x}\right)(\log x)^2$ 27. $3x^2\sin x+x^3\cos x$

28. $e^{\tan x}\sec^2 x$ (i) $e^{\sin a}\cos a$.

29. 0. 30. 2. 31. $f'(x)$ exists

37. (i) Continuous, $f(o)$ does not exist.

(ii) Continuous, (o) does not exist.

(iii) Continuous, $f(o)$ does not exist.

4.27. Derivative of a function of a function.

Let $y=f(v)$, where $v=\phi(x)$. so is the function of x, then

$$\frac{dy}{dx}=\frac{dy}{dv}\cdot\frac{dv}{dx}$$

where $f(v)$ and $\phi(x)$ are continuous.

Therefore y is also continuous function of x.

Let $v+\delta v=\phi(x+h)$ and $y+\delta y=f(v+\delta v)$

Let $h=\delta x\to 0$. then δv also tends to zero.

$$\therefore \frac{dy}{dx}=\lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right)=\lim_{\delta x\to 0}\left(\frac{\delta y}{\delta v}\cdot\frac{\delta v}{\delta x}\right)$$

$$=\lim_{\delta v\to o}\left(\frac{\delta y}{\delta v}\right).\lim_{\delta x\to o}\left(\frac{\delta v}{\delta x}\right)=\frac{dy}{dv}\cdot\frac{dv}{dx},$$

Hence $\frac{dy}{dx}=\frac{dy}{dv}\cdot\frac{dv}{dx}$ provided the limits exist.

Ex. Differentiate log(sin x) *w. r, to x*

Let $y=\log(\sin x)$; put $z=\sin x$

Then $y=\log z$, $\frac{dy}{dz}=\frac{1}{z}$, $\frac{dz}{\delta x}=\cos x$.

$$\therefore \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{1}{z}\cdot\cos x=\frac{1}{\sin x}\cos x=\cot x$$

Cor. we can generalize the above differentiation

If $y=f(v)$, $v=f(t)$, $t=f(x)$,

be the three continuous functions, y is also continuous function of x. Then

$$\frac{dy}{dx}=\frac{dy}{dv}\cdot\frac{dv}{dt}\cdot\frac{dt}{dx}$$

This is known as the **chain rule of differentiation.**

In this way we can establish the theorem for any number of functions.

Ex. Differentiate $\log \sec (ax+b)^3$ *w. r.* to x

$$\frac{dy}{dx}=\frac{1}{\sec(ax+b)^3}\sec(ax+b)^3\tan(ax+b)^3.\ 3(ax+b)^2.\ a$$

4.28 Logarithmic Differentiation.

• Differentiate $x^{\cos x}$ *w. r.* to x.

Let $y=x^{\cos x}$

Take logarithm of both sides. Then

$\log y=\log x^{\cos x}=\cos x\log x$.

Differentiating both sides *w. r.* to x,

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{d}{dx}(\cos x).\log x+\cos x.\frac{d}{dx}(\log x)$$

$$=-\sin x\log x+\cos x\frac{1}{x}$$

or, $$\frac{dy}{dx}=y\left(-\sin x\log x+\frac{1}{x}\cos x\right)=x^{\cos x}\left(\sin x\log x+\frac{1}{x}\cos x\right)$$

In order to differentiate a function of the form u^v. where u and v-are variable quantities. we are to take logarithm of the expression and then differentiate. The process which is shown above is called the **logarithmic differentiation.**

In another way we can differentiate the function like $y=x^{\cos x}$ The process is shown below.

$$y=x^{\cos x}=e^{\log x^{\cos x}}=e^{\cos x\log x}—e^{v}.\quad\text{where } v=\cos x\log x$$

$$\therefore\ \frac{dy}{dv}=e^v=y=x^{\cos x};\ \frac{dv}{dx}=-\sin x\log x+\frac{1}{x}\cos x$$

$$\therefore\ \frac{dy}{dx}=\frac{dy}{dv}\frac{dv}{dx}=x^{\cos x}\{-\sin x\log x+(^1/x)\cos x\}$$

This process is not always helpful if the expression is the product of many functions. Such as $y=\sin e^x\log x\ x^x$.

In this case logarithmic differentiation is very helpful.

4.29 Derivatives of Hyperbolic function.

(পরাবৃত্তিক ফাংশনের ডিফারেন্সিয়েল গ্রহণ)

Let $y=\sin h\,x=\frac{1}{2}(e^x-e^{-x})\ \therefore(dy/dx)=\frac{1}{2}(e^x+e^{-x})=\cos h\,x$

(i) **Hence** $\frac{d}{dx}(\sin h\,x)=\cos h\,x$

Let $y=\cos hx=\frac{1}{2}(e^x+e^{-x})\ \therefore\ (dy/dx)=\frac{1}{2}(e^x-e^{-x})=\sin h\,x$

(ii) **Hence** $\frac{d}{dx}(\cos h\,x)=\sinh x$

Let $y=\tanh x=\dfrac{\sinh x}{\cos h\,x}$

$$\therefore\ \frac{dy}{dx}=\frac{(\sin h\,x)'\cosh x-(\sinh x)(\cos h\,x)'}{(\cosh x)^2}$$

$$\Rightarrow\frac{dy}{dx}=\frac{\cos h^2x-\sin h^2x}{\cos h^2x}=\frac{1}{\cosh^2x}=\operatorname{sech}^2x$$

(iii) **Hence** $\frac{d}{dx}(\tan h\,x)=\operatorname{sech}^2x$

(iv) **Similarly.** $\frac{d}{dx}(\coth x)=-\operatorname{cosec} h^2x$

Let $y=\operatorname{sech}x=\dfrac{1}{\cosh x}$

$$\therefore\ \frac{dy}{dx}=\frac{(1)^1(\cosh x-(1)(\cosh x)^1}{\cosh^2x}=\frac{-\sinh x}{\cosh^2x}$$

$$\Rightarrow\frac{dy}{dx}=-\operatorname{sech}x\tan h\,x$$

Hence $\frac{d}{dx}(\operatorname{sech}x)=-\operatorname{sech}x\tanh x$

(v) **Similarly** $\frac{d}{dx}(\operatorname{cosech}x)=-\operatorname{cosech}x\coth x$

4.30 Derivative of $\sinh^{-1} x$ w r to x

(বিপরীত পরাবৃত্তিক ফাংশনের ডিফারেন্সিয়াল সহগ)

Let $y = \sinh^{-1} x \therefore x = \sinh y \ (x \geqslant 0)$

$\therefore (dx/dy) = (\cosh y) = \sqrt{(\cosh^2 y)} = \sqrt{(1+\sinh^2 y)} = \sqrt{(1+x^2)}$

or, $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{(1+x^2)}}$ Hence $\dfrac{d}{dx}(\sinh^{-1} x) = \dfrac{1}{\sqrt{(1+x^2)}}$

Similarly,

(i) $\dfrac{d}{dx}\left(\cosh^{-1} x\right) = \dfrac{1}{\sqrt{(x^2-1)}}; \ (x>1)$

(ii) $\dfrac{d}{dx}(\tanh^{-1} x) = \dfrac{1}{1-x^2}; \ (-1<x<1)$

(iii) $\dfrac{d}{dx}(\coth^{-1} x) = -\dfrac{1}{x^2-1}, \ (|x|>1)$

(iv) $\dfrac{d}{dx}(\operatorname{cosech}^{-1} x) = -\dfrac{1}{x\sqrt{(x^2+1)}} \ (x \neq 0)$

(v) $\dfrac{d}{dx}(\operatorname{sech}^{-1} x) = \dfrac{1}{x\sqrt{(1-x^2)}}, \ (x \neq 0 \text{ and } |x|<1)$

Cor : *Let* $y = \log\{x + \sqrt{(x^2+1)}\}$

$$\therefore \frac{dy}{dx} = \frac{1}{x+\sqrt{(x^2+1)}}\left\{1+\frac{2x}{2x(x^2+1)}\right\} = \frac{1}{\sqrt{(x^2+1)}}$$

Also $\dfrac{d}{dx}(\sin h^{-1} x) = \dfrac{1}{\sqrt{x^2+1}}$

$$\therefore \sin h^{-1} x = \log(x+\sqrt{x^2+1}) + c \left[\begin{matrix}\text{when } c \text{ is a} & \frac{dc}{dx}=0 \\ \text{constant and} & \end{matrix}\right]$$

when $x = 0$, we have,

$\sin h^{-1} 0 = \log 1 + c \Rightarrow c = 0.$

Hence (i) $\sinh^{-1} x = \log(x+\sqrt{x^2+1}) \qquad (x \geqslant 0)$

Similarly, we can show that

(ii) $\cos h^{-1} x = \log(x+\sqrt{x^2-1}) \qquad x \geq 1$

(iii) $\tan h^{-1} x = \frac{1}{2}\log\left(\dfrac{1+x}{1-x}\right) \qquad -1<x<1$

4.31. Differentiation of Parametric Equations

(প্যারামেট্রিক ফাংশনের ডিফারেন্সিয়াল সহগ)

Sometimes x and y are expressed in terms of a third variable usually called a **Parameter**. In such cases we can find $\frac{dy}{dx}$ without eliminating the parameter. The process of differentiation in such cases is shown below.

Let $x = f_1(t)$ and $y = f_2(t)$ then

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}, \text{ where } \frac{dx}{dt} \neq 0$$

Ex. If $x = a\cos t$ and $y = b\sin t$, then

$\dfrac{dx}{dt} = -a \sin t, \quad \dfrac{dy}{dt} = b\cos t$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = b\cos t/(-a\sin t) = -(b/a)\cot t$$

4.32. Differentiation of Implicit Functions (অব্যক্ত ফাংশন)

If in an equation involving x and y, the variable y is not given in terms of x or it is not suitable to express **y** in terms of x by solving the equation, then y is said to be an **implicit function of** x. In this case, we may get $\frac{dy}{dx}$ by differentiating every term in the equation with respect to x as shown below :

Ex. Differentiate $ax^2+2hxy+by^2+d=0$ *w. r.* to x.

Differentiating every term *w. r.* to x,

$$a\ 2x+\left(2hy+2hx\frac{dy}{dx}\right)+2by\frac{dy}{dx}+0=0$$

or $(2hx+2by)\dfrac{dy}{dx}=-2ax-2hy$

Note : Let $f(x, y)=ax^2+2hxy+by^2+d=0$, then

$$\frac{dy}{dx}=-\frac{\delta f/\delta x}{\delta f/\delta y}=-\frac{f_x}{f_y},\ (f_y\neq 0)$$

f_x = derivative of $f(x, y)$ *w. r.* to x treating y as constant,

f_y = derivative of $f(x, y)$ *w. r.* to y treating x as constant. For detail, consult Chapter IX.

4.33. Explicit (বহুজ) Functions :— If we can express a variable y exclusively in terms of another variabale x, then y is called an **explicit function** of x. For example,

$y=\sin\log\cos x^2$, $y=f(x)$.

$$y=\frac{3x^2+5x+7}{7x^3+4x^2+3}\ \text{ete.}$$

4.34. Examples :—

Ex. 1. Find $\dfrac{dy}{dx}$ if $y=e^{\sin x}\sin a^x$ D. U. 1965

$y=e^{\sin x}\sin a^x$

$\therefore\ \log y=\log(e^{\sin x}\sin a^x)=\sin x+\log(\sin a^x)$

$$\therefore\ \frac{1}{y}\cdot\frac{dy}{dx}=\cos x+\frac{\cos a^x}{\sin a^x}a^x\log a$$

or, $\dfrac{dy}{dx}=y\,(\cos x+a^x\log a\cot a^x)=e^{\sin x}\sin a^x\ (\cos x+a^x\log a\cot a^x)$

Ex. 2. Find the differential co-efficient of

$\tan^{-1}\dfrac{a+x}{1-ax}$ *w. r.* to x. D. U. 1969 ; R U. 1959

Let $y=\tan^{-1}\left(\dfrac{a+x}{1-ax}\right)=\tan^{-1}a+\tan^{-1}x$

$$\therefore\ \frac{dy}{dx}=0+\frac{1}{1+x^2}=\frac{1}{1+x^2}$$

Ex. 3. Find $\dfrac{dy}{dx}$ if $y=\dfrac{e^{x^2}\tan^{-1}x}{\sqrt{(1+x^2)}}$ D. U. 1962 ; R. U. 1958 D. H. 1983

we have,

$$y=\frac{e^{x^2}\tan^{-1}x}{\sqrt{(1+x^2)}}$$

$\therefore\ \log y=\log e^{x^2}+\log\tan^{-1}x-\frac{1}{2}\log(1+x^2)$

$=x^2+\log\tan^{-1}x-\frac{1}{2}\log(1+x^2)$

$$\therefore\ \frac{1}{y}\frac{dy}{dx}=2x+\frac{1}{\tan^{-1}x(1+x^2)}-\frac{1}{2}\frac{2x}{1+x^2}$$

or, $\dfrac{dy}{dx}=y\left(2x+\dfrac{1}{\tan^{-1}x(1+x^2)}-\dfrac{x}{1+x^2}\right)$

Now put the value of y, the result will follow.

Ex. 4. Find (dy/dx) if $y=(\tan x)^{\cot x}+(\cot x)^{\tan x}$ D. U. 1983

Let $y=(\tan x)^{\cot x}+(\cot x)^{\tan x}=e^{\cot x\log\tan x}+e^{\tan x\log\cot x}$

$$\frac{dy}{dx}=e^{\cot x\log\tan x}\left(-\operatorname{cosec}^2 x\log\tan x+\cot x\frac{\sec^2 x}{\tan x}\right)$$

$$+e^{\tan x\log\cot x}\left(\sec^2 x\log\cot x+\tan x\frac{-\operatorname{cosec}^2 x}{\cot x}\right)$$

$=(\tan x)^{\cot x}(1-\log\tan x)\operatorname{cosec}^2 x+(\cot x)^{\tan x}(\log\cot x-1)\sec^2 x$

Ex. 5. Differentiate $\sqrt{(y/x)}+\sqrt{(x/y)}=0$ *w. r.* to x.

D. U. 1962

Now $\sqrt{(y/x)}+\sqrt{(x/y)}=0$

or, $y+x=3\sqrt{(xy)}$ or, $y+x=3\sqrt{x}\sqrt{y}$ [on simplification]

Differentiating both sides *w. r.* to x,

$$\frac{dy}{dx}+1=3\left(\frac{1}{2\sqrt{x}}\sqrt{y}+x\frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx}\right)$$

$$\text{or, } \frac{dy}{dx}\left(1-\frac{3}{2}\sqrt{\frac{x}{y}}\right)=\frac{3}{2}\sqrt{\frac{y}{x}}-1$$

$$\Rightarrow \frac{dy}{dx}=\frac{\sqrt{y}(3\sqrt{y}-2\sqrt{x})}{\sqrt{x}(2\sqrt{y}-3\sqrt{x})}=\frac{3y-2\sqrt{(xy)}}{2\sqrt{(xy)}-3x}$$

Ex. 6. Find $\frac{dy}{dx}$, if $\log(xy)=x^2+y^2$ D. U. 1981, '86.

$\log(xy)=x^2+y^2$ or, $\log x+\log y=x^2+y^2$

$$\therefore \frac{1}{x}+\frac{1}{y}\frac{dy}{dx}=2x+2y\frac{dy}{dx}$$

$$\text{or, } \frac{dy}{dx}\left(\frac{1}{y}-2y\right)=2x-\frac{1}{x} \text{ or; } \frac{dy}{dx}=\frac{(2x^2-1)y}{x(1-2y^2)}$$

Ex 7. Find $\frac{dy}{dx}$ if $x=a\cos^3 t$, $y=a\sin^3 t$ R. U. 1965

$x=a\cos^3 t$, $y=a\sin^3 t$

$$\frac{dx}{dt}=-3a\cos^2 t\sin t.\ \frac{dy}{dt}=3a\sin^2 t\cos^2 t.$$

$$\therefore \frac{dy}{dx}=\frac{dy}{dt}\Big|\frac{dx}{dt}=\frac{-3a\sin^2 t\cos t}{3a\cos^2 t\sin t}=-\tan t$$

Ex 8. Find $\frac{dy}{dx}$ if $y=\tan^{-1}\sqrt{\left\{\frac{1-\cos x}{1+\cos x}\right\}}$ D. U. 1952,'59

Let $y=\tan^{-1}\sqrt{\left(\frac{1-\cos x}{1+\cos x}\right)}=\tan^{-1}\sqrt{\left(\frac{2\sin^2 x/2}{2\cos^2 x/2}\right)}$

$=\tan^{-1}(\tan x/2)=\frac{1}{2}x$

$$\therefore \frac{dy}{dx}=\frac{1}{2}$$

Ex. 9. Differentiate $\frac{\tan}{x}\log\frac{e^x}{x^2}$ *w. r.* to x

Let $y=\frac{\tan x}{x}\log\frac{e^x}{x^x}=\frac{\tan x}{x}(x-x\log x)$

$=\tan x-\tan x\log x$

$\therefore (dy/dx)=\sec^2 x-\sec^2 x\log x-(1/x)\tan x$

$=\sec^2 x(1+\log x)-(1/x)\tan x$

Ex. 10. Differentiate $\tan^{-1}\frac{\sqrt{(1+x^2)}-1}{x}$ *w. r.* to $\tan^{-1}x$

R. U. 1981, 1987

Let $y=\tan^{-1}\frac{\sqrt{(1+x^2)}-1}{x}$ and $z=\tan^{-1}x$

we are to find dy/dz

In $y=\tan^{-1}\frac{\sqrt{(1+x^2)}-1}{x}$, put $x=\tan\theta$

$=\tan^{-1}\frac{\sqrt{(1+\tan^2\theta)}-1}{\tan\theta}$

$=\tan^{-1}\frac{\sec\theta-1}{\tan\theta}=\tan^{-1}\frac{1-\cos\theta}{\sin\theta}$

$=\tan^{-1}\left(\frac{2\sin^2\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}\right)=\tan^{-1}(\tan^{-1}\frac{1}{2}\theta)$

$=\frac{1}{2}\theta=\frac{1}{2}\tan^{-1}x=\frac{1}{2}z$ $\therefore dy/dz=\frac{1}{2}$

Ex. 11. Differentiate $x^{\sin x}$ *w. r.* $(\sin x)^x$

Let $y=x^{\sin x}$ and $z=(\sin x)^x$

we are to find $\frac{dy}{dz}$

Now $y=x^{\sin x}$ or, $\log y=\sin x\log x$

$$\frac{1}{y}\frac{dy}{dx}=\cos x\log x+\frac{1}{x}\sin x$$

or, $\frac{dy}{dx} = x^{\sin x} \cos x \log x + \frac{1}{x} \sin x \, x^{\sin x}$

Again $z = (\sin x)^x = e^{x \log \sin x}$

$\frac{dz}{dx} = (\sin x)^x (\log \sin x + x \cot x)$

$$\therefore \frac{dy}{dz} = \frac{dy}{dx}\frac{dx}{dz} = \frac{x^{\sin x} \cos x \log x + (1/x) \sin x \, x^{\sin x}}{(\sin x)^x (\log \sin x + x \cot x)}$$

Ex. 12. If $\sin y = x \sin(a+y)$ prove that

$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$ C. U. 1987

Now $\sin y = x \sin(a+y)$ ... ... (1)

or $\log \sin y = \log x + \log \sin(a+y)$

$\therefore \frac{dy}{dx}\frac{\cos y}{\sin y} = 1/x + \frac{\cos(a+y)}{\sin(a+y)}\frac{dy}{dx}$

or, $\frac{dy}{dx}\left(\frac{\cos y}{\sin y} - \frac{\cos(a+y)}{\sin(a+y)}\right) = 1/x$

or, $\frac{dy}{dx}\frac{\sin(a+y-y)}{\sin y \sin(a+y)} = 1/x$

or, $\frac{dy}{dx} = \frac{\sin y \sin(a+y)}{x \sin a} = \frac{\sin^2(a+y)}{\sin a}$ [by (1)]

Proved

Ex. 13. Find $\frac{dy}{dx}$ when $(\cos)^y + (\sin y)^x = 0$ C. H. 1985, 89 D. H. 1987

$(\cos x)^y + (\sin y)^x = 0$

$\therefore \frac{d}{dx}[(\cos x)^y + (\sin y)^x] = 0$

or, $\frac{d}{dx}\left[e^{y \log \cos x} + e^{x \log \sin y}\right] = 0$

or, $e^{y \log \cos x}\left\{\frac{dy}{dx}\log \cos x + y \frac{-\sin x}{\cos x}\right\}$

$+ e^{x \log \sin y}\left\{\log \sin y + x \frac{\cos y}{\sin y}\frac{dy}{dx}\right\} = 0$

or, $(\cos x)^y [(dy/dx) \log \cos x - y \tan x]$

$+ (\sin y)^x \left\{\log \sin y + x \cot y \frac{dy}{dx}\right\} = 0$

or, $\frac{dy}{dx}[(\cos x)^y \log \cos x + (\sin y)^x x \cot y]$

$= (\cos x)^y y \tan x - (\sin y)^x \log \sin y$

or, $\frac{dy}{dx} = \frac{(\cos x)^y y \tan x - (\sin y)^x \log \sin y}{(\cos x)^y \log \cos x + (\sin y)^x x \cot y}$

Ex. 14. $y = \sqrt{\{a^2 - b^2 \cos^2(\log x)\}}$ C. U. 1986

$$\frac{dy}{dx} = \frac{d\sqrt{\{a^2 - b^2\cos^2(\log x)\}}}{d\{a^2 - b^2\cos^2(\log x)\}} \times \frac{d\{a^2 - b^2 \cos^2(\log x)\}}{d\{\cos(\log x)\}} \times \frac{d\{\cos \log x\}}{d \log x} \times \frac{d(\log x)}{dx}$$

$$= \tfrac{1}{2}\{a^2 - b^2\cos^2(\log x)\}^{-1/2} \times \{-2b^2 \cos(\log x)\} \times \{-\sin \log x\} \times 1/x$$

$$= \frac{b^2 \sin 2(\log x)}{3x\sqrt{\{a^2 - b^2\cos^2(\log x)\}}}$$

Ex. 15. Find if $\frac{dy}{dx}$ if $x^3 + y^3 = 3axy$

$$\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(3axy)$$

or, $3x^2 + 3y^2\frac{dy}{dx} = 3ay + 3ax\frac{dy}{dx}$

or, $\frac{dy}{dx}(y^2 - ax) = ay - x^2$

$\Rightarrow \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$

Ex. 16. Differentiate $\log \dfrac{a+b\tan\frac{1}{2}x}{a-b\tan\frac{1}{2}x}$ w. r to

$$\frac{1}{a^2\cos^2\frac{1}{2}x-b^2\sin^2\frac{1}{2}x}$$ C. U. 1986

Let $y=\log\dfrac{a+b\tan\frac{1}{2}x}{a-b\tan\frac{1}{2}x}=\log(a+b\tan\frac{1}{2}x)-\log(a-b\tan\frac{1}{2}x)$

$$\frac{dy}{dx}=\frac{\frac{1}{2}b\sec^2\frac{1}{2}x}{a+b\tan\frac{1}{2}x}-\frac{-\frac{1}{2}b\sec^2\frac{1}{2}x}{a-b\tan\frac{1}{2}x}=\frac{\frac{1}{2}b\sec^2\frac{1}{2}x\,(2a)}{a^2-b^2\tan^2\frac{1}{2}x}$$

$$=\frac{ab\sec^2\frac{1}{2}x\cos^2\frac{1}{2}x}{a^2\cos^2\frac{1}{2}x-b^2\sin^2\frac{1}{2}x}=\frac{ab}{a^2\cos^2\frac{1}{2}x-b^2\sin^2\frac{1}{2}x}$$

Let $z=\dfrac{1}{a^2\cos^2\frac{1}{2}x-b^2\sin^2\frac{1}{2}x}$

$$\frac{dz}{dx}=\frac{-\frac{1}{2}a^2\sin x-\frac{1}{2}b^2\sin x}{(a^2\cos^2\frac{1}{2}x-b^2\sin^2\frac{1}{2}x)^2}=\frac{(a^2+b^2)\sin\frac{1}{2}x\cos\frac{1}{2}x}{(a^2\cos^2\frac{1}{2}x-b^2\sin^2\frac{1}{2}x)^2}$$

$$\therefore\frac{dy}{dz}=\frac{dy}{dx}\Big/\frac{dz}{dx}=\frac{ab(a^2\cos^2\frac{1}{2}x-b^2\sin^2\frac{1}{2}x)^2}{(a^2\cos^2\frac{1}{2}x-b^2\sin^2\frac{1}{2}x)(a^2+b^2)\sin\frac{1}{2}\cos\frac{1}{2}x}$$

$$=\frac{ab}{a^2+b^2}\left\{\frac{a^2\cos^2\frac{1}{2}x-b^2\sin^2\frac{1}{2}x}{\sin\frac{1}{2}x\cos\frac{1}{2}x}\right\}$$

$$=\frac{ab}{a^2+b^2}(a^2\cot\tfrac{1}{2}x-b^2\tan\tfrac{1}{2}x)$$

Ex. 17. Prove that

$$\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}+\cdots\infty=\frac{1}{1-x}$$ if $1<x$

We know that

$(1-x)(1+x)=1-x^2,\ (1-x)(1+x)(1+x^2)=1-x^4,$

$(1-x)(1+x)(1+x^2)(1+x^4)=1-x^8,$

$\therefore (1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)\ldots\infty=1-x^{\infty}=1\ldots(1)$

when $|x|<1$

and $\lim\limits_{x\to\infty} x^n=0$

Taking logarithm of both sides of (i), we have,

$\log(1-x)+\log(1+x)+\log(1+x^2)+(1+x^4)\ldots\infty=\log 1=0$

Differentating both sides w. r. to x,

$$\frac{-1}{1-x}+\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\cdots\cdots\infty=0$$

or $$\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}+\cdots\cdots\infty=\frac{1}{1-x}$$

Proved.

Exercise IV (B)

Find the differential co-efficients of the following with respect to x.

(x এর ভিত্তিতে ডিফারেন্সিয়েল সহগ নির্ণয় কর)

1. $2x+\dfrac{1}{x}+\dfrac{2}{x^3}$
2. $\log(1+\sin x)+2\log\{\sec(\frac{1}{4}\pi-x)\}$
3. $\log\dfrac{\sqrt{(4x+3)}}{2x+1}$
4. $\log\{\sqrt{(x-a)}+\sqrt{(x-b)}\}$
5. $\log\sqrt{\left(\dfrac{2+3x}{2-3x}\right)}$
6. $\log\dfrac{\sqrt{(x^2+1)}-x}{\sqrt{(x^2+1)}+x}$
7. $(\log\sin x)^2$
8. $\log\sec(ax+b)^3$
9. $\log\{x+\sqrt{(x^2+2)}\}+\sec^{-1}(x^2)+\sqrt{5}$
10. $\log(\sec x+\tan x)$
11. $\log\dfrac{a+b\tan x}{a-b\tan x}$
12. $\log_a x+\log_x a$
13. $\dfrac{x^n}{\log_a x}$
14. $\dfrac{e^x+\tan x}{\cot x-x^n}$
15. $\sec x^0$
16. $\dfrac{\sin x+\cos x}{\sqrt{(1+\sin 2x)}}$
17. $(a^{2/3}-x^{2/3})^{3/2}$

(ii) $\dfrac{(x^2+1)^3}{(x^3-1)^2},\ x\neq1$ D.U. 1988

18. $\sqrt{\left(\dfrac{a^2+ax+x^2}{a^2-ax+x^2}\right)}$

18 (i) $\tan^{-1}\dfrac{a+bx}{b-ax}$ D. U. 1984

See APPENDIX=Extra sums-142-155

19. $x(a^2+x^2)\sqrt{(a^2-x^2)}$ 20. $\frac{1-x}{\sqrt{(1+x^2)}}$ (i) $\sqrt[3]{x^2+\sqrt{3}}$ C.U. 1993

21. $\sqrt{(1+\log x \log \sin x)}$ 22. $\log\{(2x-1)+2\sqrt{(x^2-x-1)}\}$

23. $\frac{\sqrt{x}}{1+x}$ (i) $\tan^{-1}\frac{x}{a}$ R. U. 1982

24. $e^{\sin x^2}-(\sin x^2)^2$ 25. $\frac{x^3\sqrt{(x^2+4)}}{\sqrt{(x^2+3)}}$

26. $\left\{\frac{1}{1+\sqrt{(1-x^2)}}\right\}^n$ 27 $\frac{1-x}{\sqrt{(1-x^3)}}$

28. $\log\frac{\sqrt{(x+1)}-1}{\sqrt{(x+1)}+1}$ 29. $\log(\sqrt{1+\log x})-\sin x\}$

30. $\log\frac{1}{\sqrt{x}}$ 31. $\frac{1-x}{\sqrt{(1-x^2)}}$

32. $\sqrt{(a^2\cos^2 x+b^2\sin^2 x)}$ (a) $(\sin x)^{\log x}+\cot\{e^x(a+bx)\}+a^b$

33. $\sqrt{\left(\frac{3-x^2}{x^2+2}\right)}$ 34. $\frac{1}{\sqrt[3]{(3x^2-x+1)}}$

35. $\frac{x^{3/2}(1+x^2)^{3/2}}{(1-x^2)^{3/2}}$ 36. $\log\frac{x^2\sqrt{(1-x^2)}}{\sqrt{(1+x^2)}}$

37. $(\cos hx)^x$ 38. $x^{2/3}\sqrt{\frac{x-1}{x+1}}$

39. x^{x^2} 40. $x^{\cot^{-1}x}$

40. x^{x^x} 42. x^x (i) $x^x+x^{1/x}$ D.U. 1991

43. $x^{\cos^{-1}}$ 44. $(1+x)^x$ 45. $x^{2\sin x}$

46. 3.2^{x^2} 47. $(\sin x)^x$ 48. x^{e^x}

49. $\sqrt{(x+1)}^{\log(x+1)}$ 50. $(\sin x)^{\log x}$

51. $(\sin^{-1}x)^{\log x}$ 52. $(\tan^{-1}x)^{(\cos x+\sin x)}$

53. $x^x+x^{1/x}$ 54. $10^{\log \sin x}$

55. $\left(1+\frac{1}{x}\right)^x+x^{1/x}$ (i) $\sin\left(x^{\log x}\right)+\sin^2(\cos^{-1}x)$

R. U. 1982, D. U. 1981

56. $\sqrt{(\cot^{-1}e^x)}$ 57. $(\sin x)^{\cos x}+(\cos x)^{\sin x}$ C.U. 1993

58. $e^{\sin^{-1}x}$ 59. $\sin^{-1}\frac{a+b\cos x}{b+a\cos x}$ R.H. 1987 C.H. 1988 D. U. 1962, R. U. 1990

59. (i) $\sin^{-1}\frac{1-x^2}{1+x^2}$ D. U. 1986

60. $\tan^{-1}\frac{\sqrt{(1-\cos x)}}{\sqrt{(1+\cos x)}}$ (i) $\sin^{-1}\frac{x+\sqrt{(1-x^2)}}{\sqrt{2}}$ R.U. 1982 D.U. 1992

61. $\tan^{-1}\frac{1+\tan x}{1-\tan x}$ N.U. 1993 62. $x^n e^{\cos x}$

63. $\tan^{-1}\frac{a^{1/3}+x^{1/3}}{1-a^{1/3}x^{1/3}}$ (i) $\tan^{-1}\left(\frac{a-b}{a+b}\tan\frac{x}{2}\right)$ D. U. 1981

64. $\tan^{-1}(m\tan x)$ 65. $\tan(\sin^{-1}x)$

66. $\tan^{-1}\frac{x}{\sqrt{(1-x^2)}}$ 67. $\tan^{-1}\frac{\cos x}{1+\sin x}$ R. U. 1983

68. $\cos^{-1}(1-2x^2)$ (i) $x^{\cos^{-1}x}$ C. U. 1984

69. $\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)$ 70. $\tan^{-1}\frac{2x}{1+x^2}$

71. $\sin\left\{2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right\}$ D.H. 1987 72. $\tan^{-1}\frac{1}{\sqrt{(x^2-1)}}$

73. $\sin^{-1}\sqrt{(1-x^2)}$ D.H. 1987 74. $\tan(\log x^2)$

75. $\frac{3+5\cos x}{5+3\cos x}$ 76. $\sqrt{\left(\frac{\sec x+\tan x}{\sec x-\tan x}\right)}$

77. $\frac{\tan x}{x+e^x}$ 78. $\tan x+\frac{1}{3}\tan^3 x$

79. $x\cos e^x$ 80. $e^{ax}\sin^m rx$

81. $\sec\{\frac{1}{2}\log(x^2+a^2)\}$ 82. $\sin x^2\cos hx$

83. $e^{\sqrt{(\cot x)}}$ 84. $\sec^{-1}\frac{x^2+1}{x^2-1}$

85. $\cos^{-1}\{2x\sqrt{(1-x^2)}\}$ 86. $e^{(\sin^{-1}x)^2}$ D. U. 1986

87. $2\tan^{-1}\sqrt{\left\{\frac{x-a}{b-x}\right\}}$ 88 $3^{\sqrt{(1+x+x^2)}}$

89. $\sin^2(\log x^2)$

90. $x^x + (\sin x)^{\log x}$

91. $\tan^{-1} \frac{\sqrt{(1+x^2)}-1}{x}$

92. $\left\{\sin^{-1} x\right\}^{x/a}$

93. $\left\{\frac{x}{a}\right\}^{\sin^{-1} x/a}$

94. $\frac{x^4}{\sin x}$

95. $\tan x + \sec x$

96. $\sec(\tan^{-1} x)$

97. $e^{x^2\sqrt{(1-x^2)}}$

98. $(2x-3)^{2x-3}$

99. (i) $(1+x^2) \tan x + (2-\sin x) \log x$ D.H. 1983

(ii) $(x^2+1)^{\sin^{-1}} + e^{\sqrt{(1+x^2)}}$ D.H. 1984

(iii) $(1+x) \tan^{-1} \sqrt{x}-\sqrt{x}$, $x > 0$ D.H. 1984

(iv) $(x^2+1)\sqrt{(1-x^2)}+(\sin^{-1}x)^2$, $|x|<1$

(v) x^{y^x} 100 $y^2-4ax=0$

101. $x\sqrt{(1+\cos y)}$ .102. $\sqrt{(y/x)}+\sqrt{(x/y)} = 6$

(i) $y=x \log \frac{y}{a+bx}$ C.U 1986 (i) $y = \left(\frac{n}{x}\right)^{nx}\left(1+\log \frac{x}{n}\right)$. C.H 1992

103. $x^6+x^4 y - y^3 = 0$ 104) $x^y=y^x$ 105. $x^y y^x=1$

106. $3x^4-x^2y+2y^3= 0$ 107. $x^n+ u^n= a^n$

108. $y = x^y$ D.U. 1986 109. $x^4+x^2y^2+ y^4=0$

108 (i) $y\ x^{\log x} + x^{\cos^{-1}/x}$ 109 (i) $(\sin x)^{\cos y} + (\cos x)^{\sin y}=a$

(ii) $x\tan x + (\sin x)^{\cos x}$ C.H. 1992; D.U. 1990

110. $x^y+y^x=a^b$ R.U 83 N.U 93 111. $y=x\log y$

112. $x+y=\sin^{-1} (ay/x)$ 113(a) $\{\sin \sqrt{(1+x^2)}\}^{\log \cos y}$

112. (i) $\sin y=x(2+y)$ D.U. 1988 D.H.1984

113. $y=\sin^{-1} (ax/y)$ 114. $\sin (x+y) =a$

(i) $\log (x+y) = xy$ D.U. 1991

115. $y=\sin^n (ax/y)$ N.H.94 116. $x=a(\theta+\sin\theta)$, $y=(1+ \cos\theta)$ C.U.81

117. $y = \frac{3at^2}{1+t^2}$; $x = \frac{3at}{1+t^3}$ (1) $(\cos x)^y=(\sin x)^x$ C. U. 1984

118. $\tan y = \frac{2t}{1-t^2}$, $\sin x = \frac{2t}{1+t^2}$ D.U.1989

119. $x = \log t + \sin t$, $y = e^t+\cos t$

120. $x = \sin^2\theta, y = \tan\theta$

$= 0$, $(x,y)= (0,0)$ at the

Origin $f_{xy} \neq f_{yx}$

121. Differentiate $\sin x^2$. w. r. to x^2.

121. x^2- এর সাপেক্ষে $\sin x^2$-এর অন্তরক সহগ নির্ণয় কর।

(i) $\cos 3x$ এর সাপেক্ষে $e^{\sin^{-1} x}$এর অন্তরক সহগ নির্ণয় করা। N.U. 1994

Differentiate $e^{x\sin^{-1} x}$ w.r.to $\cos 3x$. C.U.1991

122. Differentiate $\tan^{-1} \frac{2x}{1-x^2}$ w.r. to $\sin^{-1} \frac{2x}{1+x^2}$

$\sin^{-1} \frac{2x}{1-x}$ এর সাপেক্ষে $\tan^{-1} \frac{2x}{1+x}$ কে অন্তরীকরণ কর।

C.H. 1989; D.H. 1984, 87

123. Differentiate $\log_{10} x$ w. r. to x^3

x^3-এর সাপেক্ষে $\log_{10} x$-এর অন্তরীকরণ কর।

124. Differentiate $x^{\sin^{-1} x}$ w.r. to $\sin^{-1} x$

$\sin^{-1} x$ -এর সাপেক্ষে $x^{\sin^{-1}}$ x-এর অন্তরীকরণ করা।

124. (i) $\log (2+x)$-এর তুলনায় $2x^2$-এর অন্তরক সহগ নির্ণয় কর।

(i) Differentiate $\cos 2x^2$ w.r.to $\log (2+x)$ C.U.1984

125. Differentiate e^t w.r. to $\sqrt{t}$.

$\sqrt{t}$ এর সাপেক্ষে e^t এর অন্তরক সহগ নির্ণয় কর।

125. (i) Find $\frac{dy}{dx}$ in trems of x and z where

$y = e^{-x^2} \sec^{-1} (x\sqrt{z})$ and $x^4+x^2z=x^5$ C.H. 1985

125. (ii) Differentiate $x^n \log \tan^{-1} x$ with respect to $\frac{\sin\sqrt{x}}{x^{3/2}}$

C.H. 1985

125. (iii) Differentiate $\tan^{-1} \frac{x}{\sqrt{(1-x^2)}}$ w.r to $\sec^{-1} \frac{1}{2x^2-1}$

C.H. 1988

125. (iv) Differentiate $\frac{\sqrt{(1+x^2)}+\sqrt{(1-x^2)}}{\sqrt{(1+x^2)}-\sqrt{(1-x^2)}}$ w. r. to $\sqrt{(1-x^2)}$

125. (v) Differentiate

$(x^2+ax+a^2)^n \log \cot \frac{1}{2} x \tan^{-1}$ (a cos bx) w. r. to (acosbx)

(vi) Differentiate $\log \frac{1+\sqrt{x}}{1-\sqrt{x}}$ w. r. to $\sqrt{x^3}$ D.H. 1989

126. $y = u^{\mathrm{u}}$-এর অন্তরীকরণ কর যেখানে u এবং v উভয়েই x-এর ফাংশন।
Differentiate $y = u^v$ where u, v are the functions of x)

127. $(x+y)^{m+n} = x^m y^n)$

128. if $y = \sin^{-1} \frac{1}{\sqrt{(1+x^2)}}$ show that $\frac{dy}{dx} = \frac{1}{1+x^2}$

129. If $y = x\sqrt{x^2+a^2}) + a^2 \log \{x + \sqrt{(a^2+x^2)}\}$. prove that $(dy/dx) = 2\sqrt{(a^2+x^2)}$

130. If $y = \cot(\cos^{-1} x)$, from that $\frac{dy}{dx} = \frac{1}{(1-x^2)^{3/2}}$

131. If $P(x) = ax^2+bx+c$, $y = \sqrt{\{P(x)\}}$
show that $4y^3 \frac{d^2y}{dx^2} = 4ac - b^2$

132. If $f = \sqrt{\left\{\frac{x^2(3-x)}{x+1}\right\}}$ Find the domain of f and f' D. H. 1983

133. If $y = \frac{\sin x}{1+} \frac{\cos x}{1+} \frac{s\ nx}{1+} . 1 \ldots \ldots \infty$

Prove that $\frac{dy}{dx} = \frac{(1+y)\cos x + y\sin x}{1+2y + \cos x - \sin x}$

134. If $c = 1 + r\cos\theta + \frac{r^2 \cos 2\theta}{\lfloor 2} + \frac{r^3 \cos 3\theta}{\lfloor 3} + \ldots \ldots$

Show that $s = r\sin\theta + \frac{r^2 \sin 2\theta}{\lfloor 2} + \frac{r^3 \sin 3\theta}{\lfloor 3} + \ldots \ldots$

$c\frac{dc}{dr} + s\frac{ds}{dr} = (c^2+s^2)\cos\theta$ and

$c\frac{ds}{dr} - s\frac{ds}{dr} = (c^2+s^2)\sin\theta$

134. (a) If $f(x) = \left(\frac{a+x}{b+x}\right)^{a+b+2x}$ show that

$f'(0) = 2\left(\log\frac{a}{b} + \frac{b^2-a^2}{ab}\right)\left(\frac{a}{b}\right)^{a+b}$ R. U. 1987

135. If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \sqrt{\text{etc to } \infty}}}}$

Prove that $\frac{dy}{dx} = \frac{\cos x}{2y-1}$

(i) $y = \sqrt{4+\sqrt{4-x}}$, $x \leq 4$

136. If $y = \sec 4x$, prove that
$\frac{dy}{dt} = \frac{16t(1-t^4)}{(1-6t^2+t^4)^2}$, $t = \tan x$.

137. If S_n = the sum of a G. P. to n terms of which r is the common ratio,
Prove that
$(r-1)\frac{dS_n}{dr} = (n-1)S_n - nS_n - 1$

138. If $x^2y + y^2x + \sqrt{(xy)} = 1$, then show that
$\frac{dy}{dx} = -\frac{y(3x^2y+xy^2+1)}{x(x^2y+3xy^2+1)}$ D. U. 1987

139. Prove that
$\frac{1-2x}{1-x+x^2} + \frac{2x-4x^3}{1+x^2+x^4} + \frac{4x^3-8x^7}{1-x^4+x^8} + \ldots\ldots\infty = \frac{1+2x}{1+x+x^2}$
if $x<1$.

140. If $\cos\frac{1}{2}x \cos\frac{x}{2^2} \cos\frac{x}{2^3} \ldots \cos\frac{x}{2^n} = \frac{\sin x}{2^n \sin(x^n/2^n)}$ R. U. 1988

then show that
$\frac{1}{2}\tan\frac{x}{2} + \frac{1}{2^2}\tan\frac{x}{2^2} + \cdots + \frac{1}{2^n}\tan\frac{x}{2^n} = \frac{1}{2^n}\cot\frac{1}{2^n} - \cot x$

and $\frac{1}{2^2}\sec^2\frac{x}{2} + \frac{1}{2^4}\sec^2\frac{x}{2^2} + \cdots + \frac{1}{2^{2n}}\sec^2\frac{x}{2^n}$
$= -\frac{1}{2^{2n}}\operatorname{cosec}^2\frac{x}{2^n} + \operatorname{cosec}^2 x$.

What happens if $n \to \infty$

141. If $(1+x)^n = c_0 + c_1x + c_2x^2 \ldots\ldots c_nx^n$
show that $c_1 + 2c_2 + 3c_3 + \cdots nc_n = n2^{n-1}$ R. U. 1987

See APPENDIX = Extra Sums = 142 → 155

31. $-\dfrac{1}{(1+x)\sqrt{(1-x^2)}}$ 32. $\dfrac{(b^2-a^2)\sin 2x}{2y}$

33. $\dfrac{-5x}{\sqrt{(x^2+2)^2}\sqrt{(3-x^2)}}$ 34. $\dfrac{-(6x-1)}{3\sqrt[3]{(4x^2-x+1)^4}}$

35. $3y\left[\frac{3}{4}x+\dfrac{2}{1-x^4}\right]$ 36. $y\,\dfrac{2(1-x^2-x^4)}{x-x^5}$

37. $y(x\tanh x+\log\cosh x)$ 38. $y\,\dfrac{2x^2+3x-2}{3x(x^2-1)}$

39. $x^{x^2}x(1+2\log x)$ 40. $y\left(\dfrac{\cot^{-1}x}{x}-\dfrac{\log x}{1+x^2}\right)$

41. $x^{x^x}\log x\left[1+\log x+\dfrac{1}{x\log x}\right]$

42. $x^x(1+\log x)$ 43. $x^{\cos^{-1}x}\left(\dfrac{\cos^{-1}x}{x}-\dfrac{\log x}{\sqrt{(1-x^2)}}\right)$

44. $(1+x)^x\left[\log(1+x)+\dfrac{x}{1+x}\right]$

45. $2x^{2\sin x}\left[\cos x\log x+\dfrac{\sin x}{x}\right]$ 46. $3.2x^2\times 2x\log 2$

47. $(\sin x)^x[x\cot x+\log\sin x]$ 48. $x^{e^x}e^x\,[1/x+\log x]$

49. $y\,\dfrac{\log(x+1)}{(x+1)}$ 50. $(\sin x)^{\log x}\left[\log x\cot x+\dfrac{1}{x}\log\sin x\right]$

51. $(\sin^{-1}x)^{\log x}\left[\dfrac{\log x}{\sin^{-1}x\sqrt{(1-x^2)}}+\dfrac{\log\sin^{-1}x}{x}\right]$

52. $y\left\{\dfrac{\cos x+\sin x}{(1+x^2)\tan^{-1}x}+(\cos x-\sin x)\log\tan^{-1}x\right\}$

54. $x^x(1+\log x)+x^{1/x}\dfrac{1-\log x}{x^2}$ 54. $10^{\log\sin x}\log 10\cot x$

ANSWERS

1. $2-\frac{1}{2}x^2-6/x^4$ 2. $\tan(\frac{1}{4}\pi-\frac{1}{2}x)-2\tan(\frac{1}{2}\pi-x)$

3. $\dfrac{-4(x+1)}{8x^2+10x+3}$ 4. $\dfrac{1}{2\sqrt{\{(x-a)(x-b)\}}}$

5. $6/(4-9x^2)$ 6. $-2/\sqrt{(x^2+1)}$ 7. $2\cot x\log\sin x$

8. $?a(ax+b)^2\tan(ax+b)^3$ 9. $1/\sqrt{(x^2+2)}+\dfrac{2}{x\sqrt{(x^4-1)}}$

10. $\sec x$ 11. $\dfrac{2ab\sec^2 x}{a^2-b^2\tan^2 x}$ 12. $\dfrac{1}{x}\log_a e-\dfrac{\log a}{x(\log x)^2}$

13. $(nx^{n-1}\log_a x-x^{n-1}\log^a e)/(\log_a x)^2$

14. $(e^x+\sec^2 x)(\cot x-x^n)+(\operatorname{cosec}^2 x+nx^{n-1})(e^x+\tan x)/(\cot x-x^n)^2$

15. $\dfrac{\pi}{180°}\,\dfrac{\sin x°}{\cot x°}$ 16. 0 17. $-\dfrac{\sqrt{(a^{2/3}-x^{2/3})}}{x^{1/3}}$

18. $\dfrac{a(a^2-x^2)}{(a^2-ax+x^2)^{3/2}\sqrt{(a^2+ax+x^2)}}$ 19. $\dfrac{a^4+a^2x^2-5x^2}{\sqrt{(a^2-x^2)}}$

(i) $1/(1+x^2)$

20. $-\dfrac{1+x}{(1+x^2)^{3/2}}$ 21. $\dfrac{\log\sin x+x\log x\cot x}{2x\sqrt{(1+\log x\log\sin x)}}$

22. $\dfrac{1}{(x^2-x-1)^{1/2}}$ 23. $\dfrac{1-x}{2\sqrt{x}(1+x)^2}$ (i) $\dfrac{a}{x^2+a^2}$

24. $2x\cos x^2(e^{\sin x^2}+2\sin x^2)$

25. $\dfrac{x^2\sqrt{(x^2+4)}}{\sqrt{(x+3)}}\left(3-\dfrac{x^2}{(x^2+3)(x^2+4)}\right)$ 26. $\dfrac{ny}{x\sqrt{(1-x^2)}}$

27. $\dfrac{(1-x^2)(x^2-2x-2)}{2(1-x^3)^{3/2}}$ 28. $\dfrac{1}{x\sqrt{(x+1)}}$

29. $\dfrac{1-2x\cos x\sqrt{(1+\log x)}}{\{\sqrt{(1+\log x)}(-\sin x)\}2x\sqrt{(1+\log x)}}$ 30. $-1/2x$

55. $(1+1/x)^x \left\{-\frac{1}{x+1}+\log(1+1/x)\right\}+x^{1+1/x}\left(\frac{x+1-\log x}{x^2}\right)$

56. $-\frac{e^x}{2\sqrt{(\cot^{-1}e^x)(1+e^{2x})}}$

57. $(\sin x)^{\cos x}\{\cot x \cos x-\sin x \log \sin x\}$

$+(\cos x)^{\sin x}(-\tan x \sin x+\cos x \log \cos x)$

58. $\frac{e^{\sin^{-1}x}}{\sqrt{(1-x^2)}}$ 59. $\frac{-\sqrt{(b^2-a^2)}}{b+a\cos x}$ 60. $\frac{1}{3}$ 61. 1.

62 $\cos x\, x^n (n/x-\sin x)$ 63. $\frac{1}{3x^{2/3}(1+x^{2/3})}$

64. $\frac{m}{\cos^2 x+m^2\sin^2 x}$ 65. $\frac{1}{\sqrt{(1-x^2)^3}}$ 66. $\frac{1}{\sqrt{(1-x^2)}}$

67. $-\frac{1}{2}$ 68. $\frac{2}{\sqrt{(1-x^2)}}$ 68. (i) $x\cos^{-1}x\left[\frac{\log x}{\sqrt{(1-x^2)}}+\frac{\cos^{-1}x}{x}\right\}$

69. -1 70. $\frac{2}{1+x^2}$

71. $-\frac{x}{\sqrt{(1-x^2)}}$ 72. $\frac{-1}{x\,.\,(x^2-1)}$ 73. $\frac{1}{\sqrt{(1-x^2)}}$

74. $\frac{2}{x}\sec^2\log x^2$ 75. $\frac{-16\sin x}{(5+3\cos x)^2}$ 76. $-\frac{1}{2}\sec^2(\pi/4-x/2)$

77. $\frac{\sec^2 x}{x+e^x}+\frac{\tan x(1+e^x)}{(x+e^x)^2}$ 78. $\sec^4 x$

79. $-xe^x\sin e^x+\cos e^x$. 80. $e^{ax}\sin^m rx(a+rm\cot rx)$

81, $\frac{x}{x^2+a^2}\sec\{\frac{1}{2}\log(x^2+a^2)\}\tan\{\frac{1}{2}\log(x^2+a^2)\}$

82. $\sin x^2\sin hx+2x\cos x^2\cos hx$. 83. $\frac{e^{\sqrt{\cot x}}}{\sin^2 x\, 2\sqrt{\cot x}}$

84. $-\frac{2}{1+x^2}$ 85. $\frac{-1}{\sqrt{(1-x^2)}}$

86. $\frac{2\sin^{-1}x\, e^{(\sin^{-1}x)^2}}{\sqrt{(1-x^2)}}$ 87. $\frac{8}{\sqrt{\{(x-a)(b-x)\}}}$

88. $\frac{(2x+1)\log 3}{2\sqrt{(1+x+x^2)}}3^{\sqrt{(1+x+x^2)}}$ 89. $\frac{2}{x}\sin(2\log x^2)$

90. $x^x(1+\log x)+(\sin x)^{\log x}\left(\cot x\log x+\frac{\log\sin x}{x}\right)$

91. $\frac{1}{2(1+x^2)}$ 92. $\frac{1}{a}\left(\sin^{-1}\frac{x}{a}\right)^{x/a}\left(\frac{x}{\sin^{-1}x/a\sqrt{(a_1-x^2)}}+\log\sin^{-1}\frac{x}{a}\right)$

93. $\left(\frac{x}{a}\right)^{\sin^{-1}x/a}\left[\frac{\sin^{-1}\frac{x}{a}}{x}+\frac{\log\frac{x}{a}}{\sqrt{(a^2-x^2)}}\right]$

94. $x^3\operatorname{cosec} x(3-x\cot x)$

95. $\sec^2 x+\sec x\tan x$. 95. $\frac{x}{\sqrt{(1+x^2)}}$

97. $\frac{x^2e^{x\sqrt{/(1-x^2)}}}{\sqrt{(1-x^2)}}(2-3x^2)$ 98. $2(2x-3)^{2x-3}\{1+\log(2x-3)\}$

99. $\frac{y^5(\log x\log y+1/x)}{1/y-y^{x-1}x\log x}$ 100. $\sqrt{(a/x)}$

101. $-2x\operatorname{cosec} y$ 102. $\frac{x-17y}{17x-y}$ 103. $\frac{5x^4+4x^5y}{x^4+3y^2}$

104. $\frac{y}{x}\frac{x\log y-y}{y\log x-x}$ 105. $\frac{y^2}{x^2}\frac{1-\log x}{1-\log y}$

106. $\frac{2x(y-6x^2)}{6y^2-x^2}$ 107. $-(x/y)^{x-1}$

108. $\frac{y^2}{x(1-y\log x)}$ 109. $-\frac{2x^3+y^2x}{2y^3+x^2y}$

110. $-\frac{y^x\log y+yx^{y-1}}{x^y\log x+xy^{x-1}}$ 111. $\frac{y\log y}{y-x}$

112. $\frac{ay+x^2\cos(x+y)}{ax-x^2\cos(x+y)}$ 113. $\frac{ay}{xa+y\sqrt{(y^2-a^2x^2)}}$

114.—1 115. $\frac{2\cos(x+y)^2(x+y)}{1-2(x+y)\cos(x+y)}$

116. $\tan\frac{1}{2}\theta$ 117. $\frac{t(2-t^3)}{1-2t^3}$ 118. 1

119 $\frac{t(e^t-\sin t)}{1+t\cos t}$ 120. $\cos^2\theta\operatorname{cosec}2\theta$

121. $\frac{\pi\cos x}{180^\circ\times 2x}$ 122. 1.

123. $\frac{1}{3}x^{-3}\log_{10}e$

124. $x^{\sin^{-1}x}\left(\sin^{-1}x\ \frac{\sqrt{(1-x^2)}}{x}+\log x\right)$(i)—$(4x^2+8x)\sin 2x^2$

125. $2\sqrt{t}\ e^t$ (i) $\frac{e^{-x^2}}{4z^3+x^2}\left[\frac{8z^4+5x^3}{2xz\sqrt{(x^2z-1)}}-(5z^4+5x^5)\sec^{-1}x\sqrt{z}\right]$

(ii) $2\ \frac{n(1+x^2)\tan^{-1}x\log\tan^{-1}x+x}{(1+x^2)\tan^{-1}x(\sqrt{x}\cos\sqrt{x}-3\sin\sqrt{x})}x^{\frac{2n+3}{2}}$

(iii) $-\frac{1}{2}$ (iv) $\frac{1}{x^4}\ \frac{\sqrt{(1+x^2)}+\sqrt{(1-x^2)}}{\sqrt{(1+x^2)}-\sqrt{(1-x^2)}}$

(v) $\frac{(1+a^2\cos^2 bx)(x^2xax+a^2)^{n-1}[n(2x+a)\log\cot\frac{1}{2}x}{-ab\sin bx}$

$\frac{-\operatorname{cosec}x(x^2+ax+a^2)]}{}$

126. $u^v\left[\frac{v}{u}\frac{du}{dx}+\log v\frac{dv}{dx}\right]$ 127. y/x 127. $\frac{1}{1+x^2}$

CHAPTER—V

APPLICATIONS OF DERIVATIVES

The concept of derivatives comes from the intutive ideas of (1) finding the velocity of a particle at an instant and (2) constructing tangent to a curve at a point.

Let us first consider the case of velocity. Let algebraic distances of a particle moving on a straight line be S and S_1 at times t and t_1 respectively where the distances one measured from a fixed point on the line. Then

$$v=\frac{S_1-S}{t_1-t}$$

is defined as the average velocity of the particle over the time interval $[t, t_1]$. Writing $t_1=t+\Delta t$, $S_1=S+\Delta S$, we have,

$$v=\frac{(S+\Delta S)-S}{(t+\Delta t)-t}=\frac{\Delta S}{\Delta t}$$

When Δt becomes infinitesimally small the length of interval $[t, t+\Delta t]$ becomes almost zero and in these case, v can be termed as the velocity of the particle of time t. Thus

$$v=\text{velocity at time } t=\lim_{\Delta t\to 0}\left(\frac{\Delta S}{\Delta t}\right)=\frac{ds}{dt}$$

For example, if $S=f(t)=t^2$

$$v=\frac{ds}{dt}=f'(t)=2t$$

If t is in seconds and S is in feet, then the velocity at t=1, 2, 3, sec., are respectively 2×1 or 2 ft/sec, 2×2 or 4ft/sec, 2×3 or 6 ft/sec.