108. $\frac{y^2}{x(1-y \log x)}$

109. $-\frac{2x^3+y^2x}{2y^3+x^2y}$

110. $-\frac{y^x \log y+yx^{y-1}}{x^y \log x+xy^{x-1}}$

111. $\frac{y \log y}{y-x}$

112. $\frac{ay+x^2 \cos (x+y)}{ax-x^2 \cos (x+y)}$

113. $\frac{ay}{xa+y\sqrt{(y^2-a^2x^2)}}$

114. -1

115. $\frac{2 \cos (x+y)^2(x+y)}{1-2(x+y)\cos(x+y)}$

116. $\tan \frac{1}{2} \theta$

117. $\frac{t(2-t^3)}{1-2t^3}$

118. 1

119 $\frac{t(e^t-\sin t)}{1+t \cos t}$

120. $\cos^2\theta \operatorname{cosec} 2\theta$

121. $\frac{\pi \cos x}{180^\circ \times 2x}$

122. 1.

123. $\frac{1}{3}x^{-3} \log_{10} e$

124. $x^{\sin^{-1}x}\left(\sin^{-1}x \frac{\sqrt{(1-x^2)}}{x}+\log x\right)$ (i) $-(4x^2+8x) \sin 2x^2$

125. $2\sqrt{t}\, e^t$ (i) $\frac{e^{-x^2}}{4z^3+x^2}\left[\frac{8z^4+5x^3}{2xz\sqrt{(x^2z-1)}} -(5z^4+5x^5) \sec^{-1} x\sqrt{z}\right]$

(ii) $2\frac{n(1+x^2) \tan^{-1}x \log \tan^{-1}x+x}{(1+x^2) \tan^{-1}x(\sqrt{x} \cos \sqrt{x}-3 \sin\sqrt{x})} x^{\frac{2n+3}{2}}$

(iii) $-\frac{1}{2}$ (iv) $\frac{1}{x^4} \frac{\sqrt{(1+x^2)}+\sqrt{(1-x^2)}}{\sqrt{(1+x^2)}-\sqrt{(1-x^2)}}$

(v) $\frac{(1+a^2 \cos^2 bx)(x^2xax+a^2)^{n-1} [n(2x+a) \log \cot \frac{1}{2}x}{-ab \sin bx}$

$-\operatorname{cosec} x(x^2+ax+a^2)]$

126. $u^v\left[\frac{v}{u}\frac{du}{dx}+\log v\frac{dv}{dx}\right]$

127. y/x

127. $\frac{1}{1+x^2}$

CHAPTER—V

APPLICATIONS OF DERIVATIVES

The concept of derivatives comes from the intutive ideas of (1) finding the velocity of a particle at an instant and (2) constructing tangent to a curve at a point.

Let us first consider the case of velocity. Let algebraic distances of a particle moving on a straight line be S and S_1 at times t and t_1 respectively where the distances one measured from a fixed point on the line. Then

$$v=\frac{S_1-S}{t_1-t}$$

is defined as the average velocity of the particle over the time interval $[t, t_1]$. Writing $t_1=t+\Delta t$, $S_1=S+\Delta S$, we have,

$$v=\frac{(S+\Delta S)-S}{(t+\Delta t)-t}=\frac{\Delta S}{\Delta t}$$

When Δt becomes infinitesimally small the length of interval $[t, t+\Delta t]$ becomes almost zero and in these case, v can be termed as the velocity of the particle of time t. Thus

$$v=\text{velocity at time } t=\lim_{\Delta t\to 0}\left(\frac{\Delta S}{\Delta t}\right)=\frac{ds}{dt}$$

For example, if $S=f(t)=t^2$

$$v=\frac{ds}{dt}=f'(t)=2t$$

If t is in seconds and S is in feet, then the velocity at $t=1$, 2, 3, sec., are respectively 2×1 or 2 ft/sec, 2×2 or 4ft/sec, 2×3 or 6 ft/sec.

Similarly the acceleration of a particle is the derivative of velocity ; that is,

f=acceleration of time $t=\frac{dv}{dt}$

5. 2. Geometrical Interpretation of the derivative $\frac{dy}{dx}$

Let $y=f(x)$ represent the curve APQ. Let $P(x, y)$ be any point on the curve, and $Q(x+\Delta x, y+\Delta y)$.

be any point in the neighbourhood of P. Thus

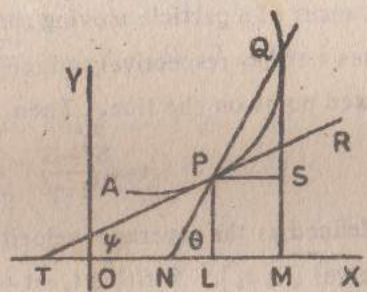
$OL=x$, $PL=SM=y$

$OM=x+\Delta x$, $QM=y+\Delta y$

Let $\angle QPS=\angle QNM=\theta$

$\angle XTR=\psi$

Now from $\triangle PQS$,



Fig

$$\tan QPS=\frac{QS}{PS}=\frac{QM-SM}{LM}=\frac{QM-PL}{OM-OL}$$

$$\text{or, } \tan\theta=\frac{y+\Delta y-y}{x+\Delta x-x}=\frac{\Delta y}{\Delta x}$$

If Q approaches P along the curve, Δx tends to zero. The straight line QPN becomes the straight line TPR in the limit as $Q\to P$ or $\Delta x\to 0$. In this case, $\theta\to\psi$, which is the inclination of the tangent at P with the positive direction of x—axis.

Therefore

$$\lim_{\theta\to\psi}\tan\theta=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\Rightarrow \tan\psi=\frac{dy}{dx} \text{ if the limit exist.}$$

5.9 Differentials

If $f'(x)$ is the derivative or differential co-efficient of $f(x)$ and Δx is an increment of x, the product $f'(x)\,\Delta x$ is denoted by $df(x)$ and is called the differential of $f(x)$. Symbolically it is written as

$df(x)=f'(x)\,\Delta x$......(1)

If x is an independent variable, $f(x)=x$, then $f'(x)=1$

$\therefore\ dx=\Delta x$ ... ... (2)

If $y=f(x)$ then (1) reduces to

$dy=f'(x)\,dx$ ... ... (3)

which is the differential of the function $y=f(x)$

Proof. From the limit of a function, we have

$$\left|\frac{f(x+\Delta x)-f(x)}{\Delta x}-f'(x)\right|<\epsilon,\ 0<|\Delta x|<0$$

where ϵ_1 is arbitrarily small and Δ depends on ϵ and x, From this we have.

$\Delta f(x)=f(x+\Delta x)-f(x)=\{f'(x)+\epsilon_1\}\,\Delta x$

when $\epsilon_1\to 0$ as $\Delta x\to 0$ ... ...(2)

$\Delta f(x)$ is almost equal to $f(x)\,\Delta x$ since $\epsilon_1\to 0$. We represent the product $f'(x)\Delta x$ by $df(x)$ and the differential is $df(x)=f'(x)\,\Delta x$ when $f(x)=x$, then $f'(x)=1$.

5.5. Geometrical representation of the Differential of a function. (ডিফারেন্সিয়ালের জ্যামিতিক ব্যাখ্যা)

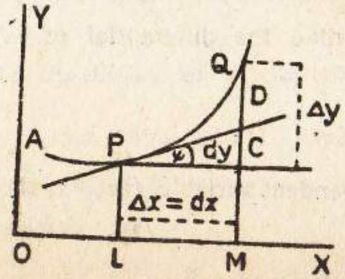


Fig 54

Let $y=f(x)$ represent the curve APQ and the derivatives $f'(x)$ exists at every point of the curve. Let the co-ordinates of P and θ be (x, y) and $(x+\Delta x, \ y+\Delta y)$ respectively, PD is the tangent at P.

Let $\angle CPD=\psi, PC=\Delta x=dx, QC=\Delta y, \ DC=dy$

Thus $\tan \psi=\frac{CD}{PC}=\frac{CD}{\Delta x}=f'(x)$ ... ... [4]

Since the value of the derivative at a point on a curve equal to the slope of the tangent at that point.

From (4) we have $CD=f'(x)\,\Delta(x)=df(x)=dy$

or, $dy=f'(x)dx$

The symbols dx and dy are also called differentials (differential x and differential y)

If $dx\neq 0$; then $\frac{dy}{dx}=f'(x)$ or $\frac{\text{differential } y}{\text{differential } x}$ = derivatives of $f(x)$ at x.

Examples

Ex. 1. $d(u+v+w)=du+dv+dw$

Ex. 2. $u=xy$

$du=d(xy)=ydx+xdy$

Ex. 3 $d(uv)=udv+vdu$

Ex. 4. $A=\pi r^2$

$\therefore \ dA=2\pi r dr$

Ex. 5. $x=\cos\theta, \ y=\sin\theta$

$\therefore \ dx=-\sin\theta\, d\theta, \ dy=\cos\theta\, d\theta$

Note : In the above theorem we notice that the increment Δx of x is equal to the differential dx but is not generally the case with the dependent variable i. e. the increment Δy of y is not equal to dy i. e., $\Delta y\neq dy$.

If $y=f(x)=x^4-3x^2$, then

$\Delta y=f(x+\Delta x)-f(x)=(x+\Delta x)^4-3(x+\Delta x)^2-(x^4-3x^2)$

$=x^4+4x^3\Delta x+6x^2(\Delta x)^2+4x(\Delta x)^3+(\Delta x^4-3x^2-6x\Delta x+3(\Delta x)^2-x^4+3x^2=(4x^3-6x)2x+(16x^2-3)(\Delta x)^2+4x(\Delta x)^3 + \Delta(x)^3$

dy = Principal part of $\Delta y=(4x^3-6x)\Delta x=(4x^3-6x)dx$, since $\Delta x=dx$.

$\therefore \ f'(x)=\frac{dy}{dx}=4x^3-6x$ and $dy=(4x^3-6x)dx$ i. e ,

$dy/dx=4x^3-6x$ which shows that dy and dx are not necessarily small.

5. 6. Increasing and decreasing functions (ঊর্ধ এবং অধঃক্রমিক ফাংশন)

Theorem : If y is a function of x, then y increases as x

increases if (dy/dx) is positive, and y decreases as x increases if $\frac{dy}{dx}$ is negative.

For an small increment of x, say Δx, let Δy be the corresponding small increment in y.

As x change by Δx and y by Δy and if both of them have the same sign (i. e , if x and y both increase or decrease together) $\frac{\Delta y}{\Delta x}$ is positive. On the other hand if Δx and Δy have opposite signs (if one increases and other decreases) $\frac{\Delta y}{\Delta x}$ is negative.

$$\text{Now } \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

If we ignore the limit then

$\frac{\Delta y}{\Delta x} = \frac{dy}{dx} + \alpha$ where α is an infinitesimal which tends to zero as $\Delta x \to 0$.

Thus the sign of (dy/dx) is the same as the sign of $(\Delta y/\Delta x)$. Hence y increases with increase of x if (dy/dx) is positive and y decreases with the increase of x if (dy/dx) is negative.

Therefore $y=f(x)$ is called an ***increasing function*** of x for an interval of x if y increases for increasing values of x in that interval. The function $y=f(x)$ is called a ***decreasing function*** of x in an interval if the value of v decreases with the increasing values of x in the interval.

Increasing and Decreasing Functions

Art. 5.7 Prove that it $f'(a) < 0$ in $a < x < b$, then $f(x)$ is steadilly decreasing in this interval.

Ans. $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$. $f'(a) = \lim_{-h \to 0} \frac{f(a-h)-f(a)}{-h}$

But given $f'(a) < 0$ $\therefore$ $f(a+h)-f(a) < 0$ or $f(a+h)$ ...

Also $-f(a-h) + f(a) < 0$ Or. $f(a-h) > f(a)$ ... (2)

From (1) and (2). $f(a-h) > f(a) > f(a+h)$

The function is steadily decreasing in the nbd of ...

Alternative Method :—

Let a be a point in (c, d) such that $c < a < d$.

Consider two points x_1. x_2 in the interval such ... d. By the Mean value Theorem, we have.

$f(x_2)-f(x_1) = (x_2-x_1) f'(a)$ where $x_1 < a < x_2$ i.e. ...

But $f'(a) < 0$ in (c, d) i.e.: $f(x_2)-f(x_1) < 0$ or. $f(x_2)$ ... steadily decreases in (c, d)

Art. 5.8 State the meaning of derivative at ... signify the increasing or decreasing of a function?

Ans. Let $f(x)$ be differentiable in (a, b). Let c ... point of (a, b). Then

$$f'(c) = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$$

$$\therefore \left| \frac{f(c+h)-f(c)}{h} - f'(c) \right| < \epsilon > 0 \text{ Where } 0 < \ldots$$

$$\text{Or. } f'(c) - \epsilon < \frac{f(c+h)-f(c)}{h} < f'(c) + \epsilon$$

Let $f'(c) > 0$. Choose $\epsilon < f'(c)$. Then $f'(c) - \epsilon$ α ... positive. Thus $\frac{f(c+g)-f(c)}{h} > 0$ When $0 < |h| \le \delta$

The function $f(x)$ will then be called an increa... the interval $(c-\delta, c+\delta)$

Next, let $f'(c) < 0$. chose $\epsilon < |f'(c)|$. Then $f'(c)$ — ... are both negative

Now $\frac{f(c+h)-f'(c)}{h} < 0$ so that $f(c+h)-f(c$ ... apposite signs.

If $c + h = x$, then $c-\delta \le x \le c+\delta$ and hence we ...

$f(c)$, it $x > c$ and $f(x) > f(c)$ it $x < c$.

Thus, it $f'(c) < 0$, we can find a nbd. of c such that $f(x_2) < f(x'_1)$ when $c-\delta < x_1 < x_2 < c + \delta_2$

The function $f(x)$ will, be called a decreasing function in $(c-\delta, c + \delta)$.

For the end points a and b, we only restrict the intervals about them.

Or, $f(c+h) > f(c)$ it $0 < h \leq \delta$

and $f(c+h) < f(c)$ it $-\delta \leq h < 0$

If $c+h = x \therefore c-\delta \leq x \leq c + \delta$ it $x < c$

Hence we see that $f(x) > f(c)$ it $x > c$ and $f(x) < f(c)$ it $x < c$

From the above discussions, we have come to the conclusion that $f'(x)$ is positive or negative, $f(x)$ is an increasing or, decreasing function in a suitably restlicted ndb. of c.

Ex. 1 Show that $y=x^3-6x^2+15x+3$ is an increasing function of x.

Let $y=x^3-6x^2+15x+3, \dfrac{dy}{dx}=3x^2-12x+15$

$$=3(x^3-4x+5)=3\{x-2)^2+1\}$$

$\Rightarrow(dy/dx)$ is positive for all real values of x. Hence y is an increasing function of x.

Ex. 2 If $y=2x^3-9x^2+12x-6$, find the range of values of x for which y is increasing and that in which y is decreasing.

$$\frac{dy}{dx}=6x^2-18x+12$$

$$=6(x^2-3x+2)=6(x-1)(x-2)$$

$\therefore \dfrac{dy}{dx}$ is positive for $x<1$ or $x>2$.

So, y is an increasing function of x as x increases from $-\infty$ to $+1$ or from $+2$ to ∞.

Again dy/dx is negative for $1<x<2$, so y is a decreasing function of x as x increases from $+1$ to $+2$.

Hence we have a tangent at a point, not parallel to y—axis, if and only if the derivative (dy/dx) or, f′(x) exists, in that case, the derivative is the slope or, gardient of the tangent line).

If ψ be the inclination of the tangent to the curve $y=f(x)$ then $\tan\psi=\dfrac{dy}{dx}$. So ψ is acute if $\dfrac{dy}{dx}>0$ or if y increases with x or the curve is *rising* with x and ψ is obtuse, if $\dfrac{dy}{dx}<0$ or y decreases as x increase implying that the curve is *falling* with increasing values of x.

If $\dfrac{dy}{dx}=0$. then $\psi=0$ and the tangent is parallel to the x-axis.

If $\tan\psi=\dfrac{dy}{dx}=\lim_{\Delta x\to 0}\left(\dfrac{\Delta y}{\Delta x}\right)=\pm\infty$, then $\cos\psi=0$ or $\psi=90$ and so the tangent prependicular to the x-axis.

Ex. 3. If $0\leqslant x<\frac{1}{2}\pi$, prove that $\dfrac{2}{\pi}\leqslant\dfrac{\text{sni } x}{x}<1$

Let $f(x)=\dfrac{\sin x}{x}$, then

$$f'(x)=\frac{x\cos x-\sin x}{x^2}=\frac{(x-\tan x)\cos x}{x^2}$$

When x is a positive acute angle, we know that $x<\tan x$. Also in the given range $\cos x$ and x^2 are both positive.

Hence $f'(x)\leqslant 0$ when $0\leqslant x\leqslant\frac{1}{2}\pi$. Thus $f(x)$ is a decreasing function of x throughout the interval $(0, \frac{1}{2}\pi)$ Hence its greatest value occurs at $x=0^+$ and least at $x=\frac{1}{2}\pi$. When

$$x\to 0,\ f(x)=\frac{\sin x}{x}\to 1 \text{ and } f(\tfrac{1}{2}\pi)=2/\pi$$

Thus we have $\dfrac{2}{\pi}\leqslant\dfrac{\sin x}{x}\leqslant 1$ for $0\leqslant x\leqslant\frac{1}{2}/\pi$

Ex. 4 Use differential calculus, calculate the value *of* $\sqrt{48}$

Let $y = \sqrt{x}$, $\delta y = \dfrac{1}{2\sqrt{x}}\,\delta x$

See APPENDIX
Worked out Examples
NO5 NO.6; NO.7

If $x = 49$. and $\delta x = -1$. Then

$$y = \sqrt{49} = 7,\ \delta y = -\frac{1}{2.7} = -.071$$

$$\therefore \sqrt{49} = 7,\ \sqrt{(x + \delta x)} = y + \delta y = 7 - .071 = 6.929$$

প্রশ্নমালা V

1. দেখাও যে, $y = x^2 - 10x + 3$ হলে y কমবে যদি $x < 5$ হয় এবং y বাড়বে যদি $x > 5$ হয়।

2. দেখাও যে $y = (x-2)e^x + x + 2$ এর মান x এর সকল ধনাত্মক মানের জন্য সর্বদা ধনাত্মক হবে।

3. দেখাও যে $-x^3 + 9x^2 - 30$ এর মান x এর সকল ধনাত্মক মানের জন্য সর্বদা ঋনাত্মক হবে।

4. দেওয়া আছে $y = 2x - \tan^{-1} x - \log\{x + \sqrt{(1+x^2)}\}$; x-এর কম বর্ধমান মানের জন্য y সর্বদা ধনাত্মক হবে x-এর সেই বিস্তারটি নির্ণয় কর।

C. H. 1972 উঃ *(0,* থেকে ∞)

5. If $f(x) = (x-1)e^x + 1$, show that $f(x)$ is positive for all positive values x.

6. Prove that $x - \frac{1}{2}x^2 < \log(1+x) < x - \dfrac{x^2}{2(1+x)}$ for $x>0$

7. If x is not equal to zero, prove that

(i) $\dfrac{x}{1+x} < 1 - e^{-x} < x$ for $x > -1$ (ii) $x < e^x - 1 < \dfrac{x}{1-x}$ for $x < 1$

8. Use differentials to calculate approximately $\sqrt{99}$, $\sqrt[3]{533}$

Ans. 9.95, 8,005

9. In what interval is the function $f(x) = 3x^5 - 25x^3 + 60x$ increasing and decreasing? Sketch ($f(x)$ ফাংশনটি কোন্ ব্যবধিতে বৃদ্ধিমান আর কোন্ ব্যবধিতে হ্রাসমান তাহা নির্ণয় কর। চিত্রটি অঙ্কন কর

10. Determine the gradient of the tangent to the curve $\sqrt{(1-x^2)} - \sin y = 0$ at the point whose coordinate is $1/\sqrt{2}$ N.U. 1991

CHAPTER—VI

SUCCESSIVE DIFFERENTIATION

(ক্রমিক ডিফারেন্সিয়েশন)

6.1. Definition :—If $y = f(x)$, its differential co-efficient $f'(x) = dy/dx$ will be in general a function of x, $f'(x)$ is called the **first derivative** of $f(x)$. The differential co-efficient of $f'(x)$ is called the **2nd derivative** of $f(x)$ i, e.

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} \text{ provided the limits exist.}$$

Similarly the derivative of $f''(x)$ is called the 3rd derivative of $f(x)$ and denoted by $f'''(x)$ and so on. If $f(x)$ is differentiated n times with respect to x then we get what is called the nth derivative of $f(x)$ and this is denoted by $f^n(x)$.

The successive derivative of $y=f(x)$ are denoted by

$f'(x), f''(x), f'''(x), \ldots\ldots f^n(x), \ldots\ldots$

6. 2. Notation :—The first derivative of $y=f(x)$ is denoted by $f'(x) = \dfrac{dy}{dx} = y_1$

The first derivative of $f'(x) = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d^2y}{dx^2} = \left(\dfrac{d}{dx}\right)^2 y$

$$f''(x) = \frac{d^2}{dx^2}(y) = \frac{d^2y}{dx^2} = y_2$$

It is convenient to denote the operator

$\dfrac{d}{dx}$ by D i. e. $D = \dfrac{d}{dx}$

Therefore $\dfrac{d^2}{dx^2} = \dfrac{d}{dx}\left(\dfrac{d}{dx}\right) = D.\ D = D^2$

If the operator $\frac{d}{dx}$ is applied n times on $y=f(x)$ then n th derivative of $y=f(x)$ is denoted by

$$y_n=\left(\frac{d}{dx}\right)^n y=D^n y=y_n.$$

Thus for $y=$ the successive derivatives are denoted by

$$f'(x), f''(x), f'''(x), \cdots \cdots \cdots f^{(n)}(x),$$

$$y', y'', y''', \cdots \cdots \cdots y^{(n)}$$

or, $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3} \cdots \cdots \cdots \frac{d^ny}{dx^n}$

or, $Dy, D^2y, D^3y, \cdots \cdots \cdots D^ny$

or, $y_1, y_2, y_3, \cdots \cdots \cdots y_n$

Ex. Find the 3rd derivative of $y=x^7$.

Let $y=x^7$

$\therefore \frac{dy}{dx}=y_1=Dy=7x^6 \therefore \frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}=y_2=D^2y=7.6x^5$

$\therefore \frac{d^3y}{dx^3}=D^3y=7.6.5x^4$

6. 3. Standard Results (প্রামাণ্য ফল)

(1) Find the nth derivative of $y=(ax+b)^m$.

$y=(ax+b)^m$

$\therefore y_1=m.a(ax+b)^{m-1}, y_2=m(m-1)a^2(ax+b)^{m-2}$

$y_3=m(m-1)(m-2)a^3(ax+b)^{m-3}$ and so on.

Hence,

$D^n(ax+b)^m=m(m-1)(m-2)\cdots\cdots(m-n+1)a^n(ax+b)^{m-n}$

$\therefore y_n=D^n(ax+b)^m=\frac{\lfloor m}{\lfloor(m-n)}a^n(ax+b)^{m-n}$

⇒ **Cor : (A)** Let m be a positive integer.

(i) If $n<m$, then

$y_n=m(m-1)(m-2)\cdots\cdots(m-n+1)a^n(ax+b)^{m-n}$

or $y_n=\frac{\lfloor m}{\lfloor(m-n)}a^n(ax+b)^{m-n}$

(ii) If $n=m$, then

$y_n=D^n(ax+b)^n=n(n-1)(n-2)\cdots(n-n+1)a^n(ax+b)^{n-m}$

$=\lfloor n\, a^n.$

(iii) If $n>m$ or $n=m+r$, then

$y_n=D^{m+R}(ax+b)^m=D^R\{D^m(ax+b)^m\}$

$=D^R(\lfloor m.a^m)=0.$

(iv) If m is a negative integer,

Let $m=-r$, where R is a positive integer, Then

$y_n=D^n(ax+b)^m=D^n\left(\frac{1}{ax+b)^r}\right)$

$=(-r)(-r-1)(-r-2)\cdots(r-n+1)a^n(ax+b)^{-r-n}$

$=(-1)^n r(r+1)(r+2)\cdots(r+n-1)\frac{a^n}{(ax+b)^{r+n}}$

$\Rightarrow D^n\left(\frac{1}{(ax+b)^r}\right)=(-1)^n\frac{\lfloor(r+n-1)}{\lfloor(r-1)}\frac{a^n}{(ax+b)^{r+n}}$

When $r=1$,

$D^n\left(\frac{1}{ax+b}\right)=(-1)^n\lfloor n\frac{a^n}{(ax+b)^{n+1}}$

(v) Find the nth differential of $y=e^{ax}$

$\therefore y_1=ae^{ax}, y_2=a^2e^{ax}, y_3=a^3e^{ax}$ so on.

Hence $y_n=D^n(e^{ax})=a^ne^{ax}$

Cor :— If $a=1$, then

$y_n=D^n(e^x)=e^x$

Cor :—If $y=a^x=e^{x\log a}$

∴ $y_1=(\log a)\,e^{x\log a}$, $y_2=(\log a)^2e^{x\log a}$ and so on

Hence $y_n=D^n(a^x)=(\log a)^n\,e^{x\log a}=(\log a)^n\,a^x$

∴ $y_n=D^n(a^x)=(\log a)^n a^x$

(vi) Find the nth derivative of $y=\log(ax+b)$

∴ $y_1=\frac{a}{ax+b}=a(ax+b)^{-1}$, $y_2=a^2(-1)(ax+b)^{-2}$

$y_3=(-1)(-2)(a^3)(ax+b)^{-3}=(-1)^2\underline{|2}\;a^3(ax+b)^{-3}$

$=(-1)^{3-1}\underline{|(3-1)}\;a^3(ax+b)^{-3}$

$y_4=(-1)^{4-1}\underline{|(4-1)}\,a^4(ax+b)^{-4}=\frac{(-1)^{4-1}\underline{|(4-1)}\,a^4}{(ax+b)^4}$

and so on,

$y_n=D^n\{\log(ax+b)\}=\frac{(-1)^{n-1}\underline{|(n-1)}\,a^n}{(ax+b)^n}$

Cor : Put $b=0$ and $a=1$ in $\log(ax+b)$ then $y=\log_e x$.

The nth derivative is

$y_n=D^n(\log x)=\frac{(-1)^{n-1}\underline{|(n-1)}}{x^n}$

(vii) Find the nth derivative of $y=\sin(ax+b)$ C. U. 1983

∴ $y_1=a\cos(ax+b)=a\sin\{\frac{1}{2}\pi+(ax+b)\}$

$y_2=a^2\cos\{\frac{1}{2}\pi+(ax+b)\}=a^2\sin\{\frac{1}{2}\pi+\frac{1}{2}\pi+(ax+b)\}$

$=a^2\sin\{2.\frac{1}{2}\pi+(ax+b)\}$

$y_3=a^3\cos\{2.\frac{1}{2}\pi+(ax+b)\}=a^3\sin\left\{\frac{\pi}{2}+2.\frac{\pi}{2}+(ax+b)\right\}$

$=a^3\sin\{3.\frac{\pi}{2}+(ax+b)\}$, and so on,

Hence $y_n=D^n\{\sin(ax+b)\}=a^n\sin\{\frac{1}{2}n\pi+(ax+b)\}$

(viii) Find the derivative of $y=\cos(ax+b)$

Proceeding as in (iv), we get

$y_n=D^n\{\cos(ax+b)\}=a^n\cos\{\frac{1}{2}n\pi+(ax+b)\}$

Cor : If $a=1$, $b=0$, then

$D^n(\sin x)=\sin(\frac{1}{2}n\pi+x)$,

$D^n(\cos x)=\cos(\frac{1}{2}n\pi+x)$.

6.4. (1) Find the nth differential co-efficient of

$y=e^{ax}\sin(bx+c)$

∴ $y_1=ae^{ax}\sin(bx+c)+e^{ax}b\cos(bx+c)$

Put $a=r\cos\phi$ and $b=r\sin\phi$ then

$r^2=a^2+b^2$ and $\tan\phi=b/a$ or $\phi=\tan^{-1}\left(\frac{b}{a}\right)$

Now y_1 becomes

$y_1=re^{ax}\sin(bx+c)\cos\phi+e^{ax}r\cos(bx+c)\sin\phi$

$=re^{ax}\sin(bx+c+\phi)$

Similarly

$y_2=r^2e^{ax}\sin(bx+c+2\phi)$ and so on.

$y_n=r^ne^{ax}\sin(bx+c+n\phi)=(a^2+b^2)^{n/2}\sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$

If $y=e^{ax}\sin bx$, then

$y_n=D^n(e^{ax}\sin bx)=(a^2+b^2)^{n/2}\sin(bx+n\tan^{-1}b/a)$

Similarly.

$y_n=D^n(e^{ax}\cos bx)=(a^2+b^2)^{n/2}\cos(bx+n\tan^{-1}b/a)$

6.5 Partial Fractions. Expression of the form

$\frac{\phi(x)}{f(x)}$, where $\phi(x)$ and $f(x)$ are both rational integral algebraic functions, can be differentiated easily if the expression is first resolved into partial fractions. This is explained with examples below.

Ex. Differentiate x times the expression

$$y=\frac{x^2-6x+1}{(x-1)(3x-2)(2x+3)}=\frac{A}{x-1}+\frac{B}{3x-2}+\frac{C}{2x+3}$$

$$\Rightarrow\ y=\frac{-4/5}{x-1}-\frac{23/13}{3x-2}+\frac{49/65}{2x+3}$$

Therefore the nth derivative of y is

$$\therefore\ y_n=\frac{4}{5}\frac{(-1)^n|\underline{n}}{(x-1)^{n+1}}-\frac{23(-1)^n3^n|\underline{n}}{13(3x-2)^{n+1}}+\frac{49(-1)^n|\underline{n}\,2^{n+1}}{65(2x+3)^{n+1}}$$

[See Ex. 6.3(1)]

Use of De Moivre's Theorem

6.6 Find the nth derivative of $y=\frac{1}{x^2+a^2}$ C. U. 1986

$$\therefore\ y=\frac{1}{x^2+a^2}=\frac{1}{(x+ai)(x-ai)}=\frac{1}{2ia}\left(\frac{1}{x-ia}-\frac{1}{x+ia}\right)$$

$$\text{or, } y=\frac{1}{2ia}(x-ia)^{-1}-\frac{1}{2ai}(x+ia)^{-1}$$

$$y_1=\frac{1}{2ia}(-1)(x-ia)^{-2}-\frac{1}{2ia}(-1)(x+ia)^{-2}$$

$$y_2=\frac{1}{2ia}(-1)(-2)(x-ia)^{-3}-\frac{1}{2ia}(-1)(-2)(x+ia)^{-3}$$

and so on

$$y_n=\frac{1}{2ia}(-1)^n n!\,(x-ia)^{-n-1}-\frac{1}{2ia}(-1)^n n!\,(x+ia)^{-n-1}$$

Foot Note : See Higher Algebra and Trigonometry by Shahidulla & Bhattacharjee

$$=\frac{(-1)^n|\underline{n}}{2ia}\left\{\frac{1}{(x-ia)^{n+1}}-\frac{1}{(x+ia)^{n+1}}\right\}$$

Put $x=r\cos\theta$, $a=r\sin\theta$, $r^2=x^2+a^2$, $\theta=\tan^{-1}\left(\frac{a}{x}\right)$

$$y_n=\frac{(-1)^n n)}{2ia}\frac{1}{r^{n+1}}\left\{\frac{1}{(\cos\theta-i\sin\theta)^{n+1}}-\frac{1}{(\cos\theta+\sin\theta)^{n+1}}\right\}$$

$$=\frac{(-1)^n|\underline{n}}{2iar^{n+1}}\{(\cos\theta-i\sin\theta)^{-n-1}-(\cos\theta+\sin\theta)^{-n-1}$$

$$=\frac{(-1)^n|\underline{n}}{2iar^{n+1}}\,i2\sin(n+1)\theta=\frac{(-1)^n n!}{2_i a^{n+1}a}\sin^{n+1}\theta,\ \sin(n+1)\theta$$

$$\therefore\ D^n\left(\frac{1}{x^2+a^2}\right)=\frac{(-1)^n n!}{a^{n+2}}\sin(n+1)\theta\,\sin^{n+1}\theta,\ \theta=\tan^{-1}a/x$$

6.7 Trigonometrical Transformation.

In finding nth derivative of expressions like $\sin^p\theta\ \cos^q\theta$, $\sin^q\theta$ $\sin^n\theta$, $\cos^n\theta$ etc, we are to transform the expression into a sum by Trigonometry. In such cases the substitutions are

$z=\cos\theta+i\sin\theta$, then $z^{-1}=\cos\theta-i\sin\theta$

$\therefore\ 2\cos\theta=z+1/z$, $2i\sin\theta=z-1/z$

The method is explained below with an example.

Ex. Find the nth derivative of $\cos^3x\sin^2x$

Let $z=\cos x+i\sin x$, then $z^{-1}=\cos x-i\sin x$

$\therefore\ 2\cos x=z+z^{-1}$, $2i\sin x=z-z^{-1}$……(1)

Also $z^n=(\cos x+i\sin x)^n=\cos nx+i\sin nx$

$z^{-n}=(\cos x+i\sin x)^{-n}=\cos nx-i\sin nx$

$\therefore\ 2\cos nx=z^n+z^{-n}$, $2i\sin nx=z^n-z^{-n}$… … [2]

Thus $2^2 \cos^2 x\ 2^3 i^3 \sin^3 x=(z+z^{-1})^2(z-z^{-1})^3$

or; $2^5 i^3 \cos^2 x \sin^3 x=\left(z^5-\frac{1}{z^5}\right)-\left(z^3-\frac{1}{z^3}\right)-2\left(z-\frac{1}{z}\right)$

$$=2i \sin 5x-2i \sin 3x-2.2i \sin x$$

or, $-2^5 \cos^2 x \sin^3 x=-2 (\sin 5x-\sin 3x-2 \sin x)$

$\therefore\ D^n (\cos^2 x \sin^3 x)=-2^{-5}. 2\{5^n \sin (\frac{1}{2}n\pi+5x)$

$$-3^n \sin (\tfrac{1}{2}n\pi+3x)-2 \sin (\tfrac{1}{2}\pi+x)\}$$

$=(1/16)[2 \sin (\frac{1}{2}n\pi+x)+3^n \sin (\frac{1}{2}n\pi+3x)-5^n \sin (\frac{1}{2}n\pi+5x)]$

6. 8. Leibnitz's Theorem (লিবনীটজের উপপাদ্য)

With the help of this theorem we can find the *n*th differential co-efficient of a product of two functions.

The theorem states that if u and v are two function of x and each has derivatives upto order n, then the nth derivatives of the product of the functions is given by

$$D^n(uv)=(D^n u)v+{}^nC_1 D^{n-1} uDv+{}^nC_2 D^{n-2}u D^2 v+$$
$$\dots\dots\dots+{}^nC_r D^{n-r}u D^r v+\dots+u. D^n v$$

or, $(uv)_n=u_n v+{}^nC_1 u_{n-1} v_1+{}^nC_2 u_{n-2} v_2+\dots\dots$

$$+{}^nC_r u_{n-r} v_r+\dots\dots+uv_n$$

Where the suffixes of u and v indicate the order of derivatives with respect to x.

Let $y=uv$. Differentiate it *w. r.* to x

$$\frac{dy}{dx}=\frac{d}{dx}(uv)=v\frac{du}{dx}+u\frac{dv}{dx}=vu_1+uv_1 \text{ or. } y_1=u_1 v+uv_1$$

$$\therefore\ y_2=\frac{dy_1}{dx}=\frac{d}{dx}(u_1 v)+\frac{d}{dx}(uv_1)$$

or, $y_2=(u_2 v+u_1 v_1)+(u_1 v_1+uv_2)=u_2 v+2u_1 v_1+uv_2$

$$=u_2 v+{}^2C_1 u_{2-1} v_1+uv_2$$

Similarly,

$y_3=u_3 v+3u_2 v_1+3u_1 v_2+uv_3=u_3 v+{}^3C_1 u_{3-1} v_1+{}^3C_2 u_{3-2} v_2+v_3$

Because ${}^3C_1={}^3C_2=3$.

Thus we see that the theorem holds good for $n=1$, 2 and 3.

Let us assume that the theorem is true for $n=m$ (a positive integer). Then

$$y_m=u_m v+{}^mC_1 u_{m-1} v_1+{}^mC_2 u_{m-2} v_2+\dots\dots+{}^mC_r u_{m-r} v_r+\dots$$
$$\dots\dots+uv_m\dots\dots\dots \quad (1)$$

Differentiate (1) with respect to x :

$$y_{m+1}=u_{m+1}v+(u_m v_1+{}^mc_1 u_m v_1)+({}^mc_1 u_{m-1} v_2+{}^mc_2 u_{m-2} v_2)$$
$$+\dots\dots+{}^mc_{r-1} u_{m-r+1} v_r+{}^mc_r u_{m-r+1} v_r+\dots\dots$$
$$=u_{m+1}v+({}^mc_1+1) u_m v_1+({}^mc_2+{}^mc_1) u_{m+1} v_2+\dots\dots$$
$$+({}^mc_r+{}^mc_{r-1})u_{m-r+1} v_r+\dots\dots+uv_{m+1}$$

But ${}^mc_r+{}^mc_{r-1}={}^{m+1}c_r$ and ${}^mc_r+1={}^{m+1}c_1$

Therefore

$$y_{m+1}=u_{m+1} v+{}^{m+1}c_1 u_m v_1+{}^{m+1}c_2 u_{m-1} v_2+\dots\dots$$
$$+{}^{m+1}c_r u_{m-r+1} v_r+\dots\dots uv_{m+1}\dots\dots(2)$$

If the theorem holds good for $n=m$, then, it, will also hold good for $n=m+1$. But it was proved that the theorem is true for $n=1$,2 and 3......When the theorem is true for $n=3$, it is also true for $n=4$. When it is true for $n=4$, it is true for $n=5$ and so on. Hence the theorem must be true for every positive integral value of n.

Ex. Find the nth derivative of $x^2 e^{2x}$

Let $y=uv=x^2 e^{2x}$

Where $u=e^{2x}$, $v=x^2$.

$(uv)_n$ ... $+uv_n$

Then $u_1=2e^{2x},\ u_2=2^2e^{2x},\ \ldots\ldots,\ u_r=2^re^{2x}\ldots\ldots$

$v_1=2x,\ v_2=2,\ v_3=v_4=\ldots\ldots 0$

$$\therefore\quad y_n=u_nv+{}^nC_1u_{n-1}v_1+{}^nC_2u_{n-2}v_2+{}^nC_3u_{n-3}v_3+\ldots\ldots$$
$$+{}^nC_ru_{n-r}v_r+\ldots\ldots+uv_n$$

$$=2^ne^{2x}x^2+\frac{n}{1}2^{n-1}.\ e^{2x}\ 2x+\frac{n(n-1)}{1.2}2^{n-2}e^{2x}.2+0$$

$$=2^ne^{2x}\left[x^2+nx+\frac{n(n-1)}{4}\right]$$

6.9. Some Important Symbolic Operators.

$F(D)=A_nD^n+A_{n-1}D^{n-1}+\ldots\ldots\ldots+A_1D+A.$

$=\Sigma A_rD^r$, where A_r is independent of D

$F(D)$ is any rational integral algebraic function of D or $\frac{d}{dx}$

The following result may be obtained from the above function

(i) $F(D)e^{ax}=F(a)e^{ax}$

(ii) $F(D)e^{ax}V=e^{ax}F(D+a)V$, where V is the function of a

(iii) $F(D^2)\sin(ax+b)=F(-a^2)\sin(ax+b)$

(iv) $F(D^2)\cos(ax+b)=F(-a^2)\cos(ax+b)$

Proof : (1) $F(D)e^{ax}=(A_nD^n+A_{n-1}D^{n-1}+\cdots+A_1D+A_0)e^{ax}$

We know that $D^re^{ax}=a^re^{ax}$

$\therefore F(D)e^{ax}=e^{ax}(a^nA_n+\ldots+a^{n-1}A_{n-1}\ldots+aA_1+A_0)=F(a)e^{ax}$

Proof. (ii) By Leibnitz's Theorem we have

$$D^n(e^{ax}V)=VD^ne^{ax}+{}^nC_1D^{n-1}e^{ax}DV+{}^nC_2D^{n-2}e^{ax}D^2V+\ldots\ldots+e^{ax}D^nV$$

$$=e^{ax}(a^nV+{}^nC_1a^{n-1}DV+{}^nC_2a^{n-2}D^2V+\ldots\ldots+D^nV)$$

By $D^re^{ax}=a^re^{ax}$

$$=e^{ax}(a+D)^nV$$

V কে ডানদিকে রাখা হইয়াছে কারণ ইহা দ্বারা D, D^2, D^3 ইত্যাদির সঙ্গে এর অপারেশন হইবে

$$D^r(e^{ax}V)=e^{ax}(a+D)^rV$$

এখন $F(D)\{e^{ax}V\}=\Sigma A_rD(e^{ax}V)=e^{ax}\Sigma A(a+D)^rV$

$$=e^{ax}F(D+a)V$$

Proof. (iii) $F(D^2)$ ফাংশানটির চলক D^2

মনে করি $F(D^2)=\sum_{r=0}^{n}A_rD^{2r}$

এখন $D\sin(ax+b)=a\cos(ax+b)$

$D^2\sin(ax+b)=(-a^2)\sin(ax+b)=(-a^2)\sin(ax+b)$

$D^3\sin(ax+b)=-a^3\cos(ax+b)$

$D^4\sin(ax+b)=(-a^2)^2\sin(ax+b)$

... ...

$D^{2r}\sin(ax+b)=(-a^2)^r\sin(ax+b)$

$\therefore\ F(D)^2\sin(ax+b)=\Sigma A_r(-a^2)^r\sin(ax+b)$

$$=F(-a^2)\sin(ax+b).$$

অনুরূপ ভাবে প্রমাণ করা যায় যে

(iv) $F(D^2)\cos(ax+b)=F(-a^2)\cos(ax+b)$

Examples

Ex. Find nth differential co-efficient of $\sin^2x$. D. U. 1954

Let $y=\sin^3x=\frac{1}{4}(3\sin x-\sin 3x)$

$\therefore\ y_n=\frac{3}{4}\sin(\frac{1}{2}n\pi+x)-\frac{1}{4}.3^n\sin(\frac{1}{2}n\pi+3x)$

Ex. 2. Find the nth derivative of $\frac{1}{(x-1)^2(x-2)}$

Let $y=\frac{1}{(x-1)^2(x-2)}$

Put $x-1=z$ or, $x=1+z$; $x-2=1+z-2=z-1$

$$y=\frac{1}{z^2(z-1)}=\frac{A}{z}+\frac{B}{z^2}+\frac{C}{z-1}$$

or, $1=Az(z-1)+B(z-1)+Cz^2$

Equate co-efficients of z^2, z and constant terms from both sides. Then $A=-1$, $B=-1$, $C=1$,

$$\text{Now } y=\frac{-1}{z}-\frac{1}{z^2}+\frac{1}{z-1}=-\frac{1}{x-1}-\frac{1}{(x-1)^2}+\frac{1}{x-2}$$

Differentiating n times we have

$$y_n=-\frac{(-1)^n n!}{(x-1)^{n+1}}-\frac{(-1)^n n!}{(x-1)^{n+3}}+\frac{(-1)^n n!}{(x-2)^{n+1}}$$

$$=(-1)^{n+1} n!\left[\frac{1}{(x-1)^{n+1}}+\frac{1}{(x-1)^{n+3}}-\frac{1}{(x-2)^{n+1}}\right]$$

Ex. 3. Find the nth differential co-efficient of $y=\sin^4 x \cos^3 x$

Let $\cos x+i\sin x=z$, then $\cos x-i\sin x=z^{-1}$

$\cos rx+i\sin rx=z^r$, $\cos rx-i\sin rx=z^{-r}$

$\therefore$ $2\cos x=z+z^{-1}$ $\quad 2i\sin x=z-z^{-1}$

$2\cos rx=z^r+z^{-r}$ $\quad 2i\sin rx=z^r-z^{-r}$

Now $(2i\sin x)^4(2\cos x)^3=(z-z^{-1})^4(z+z^{-1})^3$

or, $2^7 i^4 \sin^4 x\cos^3 x=(z-4z^2+6-4/z^2+1/z^4)(z^3+3z+3/z+1/z^3)$

or, $2^7\sin^4 x\cos^3 x=(2^7+1/z^7)+(-4+3)(z^5+1/z^5)$

$+(6+3-12)(z^3+1/z^3)+(18-12-4+1)(z+1/z)$

or, $2^7\sin^4 x\cos^3 x=2\cos 7x-2\cos 5x-3.2\cos 3x+3.2\cos x$

or, $\sin^4 x\cos^3 x=(\frac{1}{2})^6(\cos 7x-\cos 5x-3\cos 3x+3\cos x)$

Hence nth derivative of y is

$y^n=(\frac{1}{2})^6(7^n\cos(\frac{1}{2}n\pi+7x)-5^n\cos(\frac{1}{2}n\pi+5x)$

$-3.3^n\cos(\frac{1}{2}n\pi+3x)+3\cos(\frac{1}{2}n\pi+x)\}]$

Ex. 4. Find nth derivative of $\sin^7 x$.

Let $z=\cos x+i\sin x$, $1/z=\cos x-i\sin x$

$\therefore$ $2\cos x=z+z^{-1}$, $2i\sin x=z-z^{-1}$

$\therefore$ $2^7 i^7\sin^7 x=(z-z^{-1})^7=(z^7-1/z^7)-7(z^5-1/z^5)$

$+21(z^3-1/x^3)-35(z-1/z)$

$=2i\sin 7x-7.2i\sin 5x+2.21i\sin 3x-35.2i\sin x$

or, $\sin^7 x=(\frac{1}{2})^6(\sin 7x-7\sin 5x+21\sin 3x-35\sin x)$

$\therefore$ $y_n=(\frac{1}{2})^6(7^n\sin\{\frac{1}{2}n\pi+7x)-7.5^n\sin\frac{1}{2}(n\pi+5)$

$+21.3^n\sin(\frac{1}{2}n\pi+3x)-35\sin(\frac{1}{2}n\pi+x)\}$

Ex. 5. If $y=\tan^{-1}x$, show that $y_3=2\cos(\pi+3y)\cos^3 y$

we have $x=\tan y$ ... ... (1)

Differentiate it w. r. to x :

$1=\sec^2 y\frac{dy}{dx}\Rightarrow y_1=\cos^2 y$ ... ... (2)

$=-2\cos y\sin y\frac{dy}{dx}=-2\cos^3 y\sin y=y_2$ [by (2)]

$\therefore$ $y_3=-\left(2\cos^4 y\frac{dy}{dx}-6\cos^2 y\sin^2 y\frac{dy}{dx}\right)$

$=-2\cos^2 y(\cos^2 y-3\sin^2 y)\cos^2 y$ [by (2)]

$=-2\cos^3 y\{\cos^3 y-3\cos y(1-\cos^2 y)\}$

$=-2\cos^3 y(4\cos^3 y-3\cos y)=-2\cos^3 y\cos 3y$

$=2\cos(\pi+3y)\cos^3 y$ (Proved.)

Ex. 6. If $y=e^{ax}\cos bx$, prove that $y_2-2ay_1+(a^2+b^2)y=0$

Let $y=e^{ax}\cos bx$ (1)

$y_1 = ae^{ax}\cos bx - be^{ax}\sin bx = ay - be^{ax}\sin bx$ by (1)

$\therefore\quad y_2 = ay_1 - abe^{ax}\sin bx - b^2 e^{ax}\cos bx$ ... (2)

$= ay_1 - a(be^{ax}\sin bx) - b^2y$ [by (1)]

$= ay_1 - a(ay - y_1) - b^2y$ ... [by (2)]

or, $y_2 - 2ay_1 + (a^2+b^2)y = 0$ (Proved)

Ex. 7. Find the nth derivative of $x^3 \sin x$

Let $y = x^3 \sin x$, Then

$y_n = D^n(\sin x.\ x^3)$

$= D^n(\sin x).\ x^3 + {}^nC_1\, D^{n-1}(\sin x).(3x^2)$

$+ {}^nC_2\, D^{n-2}(\sin x).\ 6x + {}^nC_3\, D^{n-3}(\sin x)6$

$= \sin(\frac{1}{2}n\pi + x)x^3 + {}^nC_1 \sin\{\frac{1}{2}n-1)\pi + x\}3x^2$

$+ {}^nC_2 \sin\{\frac{1}{2}(n-2)\pi + x\}6x + {}^nC_3 \sin\{\frac{1}{2}(n-3)\pi + x\}6$

or, $y_n = x^3 \sin\{\frac{1}{2}n\pi + x) + 3nx^2 \sin\{\frac{1}{2}(n-1)\pi + x\} +$

$3xn(n-1)\sin\{\frac{1}{2}(n-2)\pi + x) + n(n-1)(n-2)\sin\{\frac{1}{2}(n-3)\pi + x\}$

Ex. 8; If $y = x^2e^x$ show that

$y_n = \frac{1}{2}n(n-1)y_2 - n(n-2)y_1 + \frac{1}{2}(n-1)(n-2)y$

Differentiate $y = x^2e^x$ *w. r.* to x

$\therefore\quad y_1 = e^x x^2 + e^x 2x = e^x(x^2+2x)$

$y_2 = e^x(x^2+2x) + e^x(2x+2) = e^x(x^2+4x+2)$

$\therefore\quad y_n = e^x x + ne^x 2x + \frac{n(n-1)}{2}e^x 2 = e^x\{x^2 + 2nx + n(n-1)\}$ ...(4)

Now $e^x\{x^2 + 2nx + n)(n-1)\} = e^x(x^2+4x+2)\frac{1}{2}(n^2-n) -$

$e^x(x^2+2x)(n^2-2n) + e^x\,\frac{n^2-3n+2}{2}n^2$

$= \frac{1}{2}n(n-1)y_2 - n(n-2)y_1 + \frac{1}{2}(n-1)(n-2)y$ by [1], [2], [3]

$y_n = \frac{1}{2}n(n-1)y_2 - n(n-2)y_1 + \frac{1}{2}(n-1)(n-2)y$. **Proved.**

Ex. 9. Differentiate n times the equation

$(1-x^2)y_2 - xy_1 + a^2y = 0$ D. U. 1982

$D^n\{(1-x^2)y_2\} = (1-x^2)y_{n+2} - ny_{n+1}2x - n(n-1)y_n l;$

$D^n xy_1 = y_{n+1}x + ny_n \qquad \therefore D_n a^2y = a^2y_n$

Therefore $D^n(1-x^2)y_2 - D_1xy_1 + D^na^2y = (1-x^2)y_{n+2}$

$-(2n+1)xy_{n+1} - (n^2-a^2)y_n = 0$

or, $(1-x^2)y_{n+2} - \{2n+1)xy_{n+1} - (n^2-a^2)y_n = 0$

Ex. 10. If $y = \sin(m \sin^{-1}x)$, show that

(i) $(1-x^2)y_2 = xy_1 - m^2y$ D. U. 1983

(ii) $(1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2-m^2)y_n$ D. H. '62, R. U.67

$y = \sin(m\sin^{-1}x)$ ... ... (1)

$y_1 = \cos(m\sin^{-1}x)\ \frac{m}{\sqrt{(1-x^2)}}$ ... ... (2)

or, $y_1\sqrt{(1-x^2)} = m\cos(m\sin^{-1}x)$

or, Squaring,

$y_1^2(1-x^2) = m^2\cos^2(m\sin^{-1}x) = m^2[1-\sin^2(m\sin^{-1}x)]$

or, $y_1^2(1-x^2) = m^2(1-y^2)$

Differentiating both sides *w. r.* to x

$2y_1y_2(1-x^2) + y_1^2(-2x) = m^2(-2yy_1)$

or, dividing throughout by $2y_1$,

$y_2(1-x^2) - xy_1 = -m^2y$

$\Rightarrow (1-x^2)y_2 = xy_1 - m^2y.$ (proved)

Differentiate it n times :

$(1-x^2)y_{n+2} - n(2x)y_{n+1} - \frac{n(n-1)}{2}(2)y_n = y_{n+1}x + ny_n(1) - m^2y_n$

or, $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$ (proved).

Ex. 11. If $y = e^{a\sin^{-1}x}$ prove that **C. U, 1984**

C. H. 1992

(i) $(1-x^2)y_2 - xy_1 - a^2y = 0$

(ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (x^2+a^2)y_n = 0$

and hence find the value of y_n where $x = 0$

R. U. 1964, '66 ; D. U. 1952

Differentiate $y=e^{a\sin^{-1}x}$ ... ... (1)

∴ $y_1=e^{a\sin^{-1}x}\dfrac{a}{\sqrt{(1-x^2)}}=\dfrac{ay}{\sqrt{(1-x^2)}}$ ... ... (2)

or, $y_1^2(1-x^2)=a^2y^2$

∴ $2y_1y_2(1-x^2)-2xy_1^2=2y_1a^2$ [Differentiating both sides w, r, to x]

$(1-x^2)y_2-xy_1-ya^2=0$ ... (proved)

Differentiate it n times :

$$\therefore\ (1-x^2)y_{n+2}-n\cdot 2x\,y_{n+1}-\frac{n(n-1)}{2}\cdot 2y_n-xy_{n+1}-ny_n-a^2y_n=0$$

or, $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+a^2)y_n=0$ (4) proved

Put $x=0$ in (1), (2), (3), (4), then

$(y)_0=1$, $(y_1)_0=a$ and $(y_{n+2})_0=(n^2+a^2)(y_n)_0$ ... ... (5)

Putting $n=n-2$ in (5),

$(y_n)_0=\{(n-2)^2+a^2\}(y_{n-2})_0$ ... ... (5)

$=\{(n-2)^2+a^2\}\{(n-4)^2+a^2\}(y_{n-4})_0$ etc.

But $(y)_0=1$, $(y_2)_0=a^2$, $(y_4)_0=(2^2+a^2)a^2$

when n is even.

∴ $(y_n)_0=\{(n-2)^2+a^2\}\{(n-4)^2+a^2\}\ldots(4^2+a^2)(2^2+a^2)a^2$

(when n is even)

When n is odd, $(y_1)_0=a$, $(y_3)_0=(1^2+a^2)a$

∴ $(y_n)_0=\{(n-2)^2+a^2\}\{(n-4)^2+a^2\}\{\ldots(3^2+a^2)(1^2+a^2)(y_1)_0$

$=\{(n-2)^2+a^2\}\{(n-4)^2)+a^2\}\cdots\cdots(3^2+a^2)(1^2+a^2)a$

Ex. 12. If u, v, w be functions of t, and if suffixes denote differentiations with regard to t, prove that.

$$\frac{d}{dt}\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}=\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_4 & v_4 & w_4 \end{vmatrix}$$

[For the differentiation of Determinant see authors's Higher Algebra.

When a determinant is differentiated, there will be as many determinants as the order of the determinant and each determinant contains only one row or one column differentiated once only.]

$$\frac{d}{dt}\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}=\begin{vmatrix} \frac{d}{dt}u_1 & \frac{d}{dt}v_1 & \frac{d}{dt}w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}$$

$$+\begin{vmatrix} u_1 & v_1 & w_1 \\ \frac{d}{dt}u_2 & \frac{d}{dt}v_2 & \frac{d}{dt}w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}+\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ \frac{d}{dt}u_3 & \frac{d}{dt}v_3 & \frac{d}{dt}w_3 \end{vmatrix}$$

$$=\begin{vmatrix} u_2 & v_2 & w_2 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}+\begin{vmatrix} u_1 & v_1 & w_1 \\ u_3 & v_3 & w_3 \\ u_3 & v_3 & w_3 \end{vmatrix}+\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_4 & v_4 & w_4 \end{vmatrix}$$

$$=\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_4 & v_4 & w_4 \end{vmatrix}\ \text{proved.}$$

Ex. 13. If $y=px$ and $z=qx$, all the variables are the funcions of t, then prove that

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}=x^3\begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}$$

: p_1, p_2 etc. indicate the successive differentiation.

Now $y=px$ ∴ $y_1=p_1x+x_1p$, $y_2=px_2+2p_1x_1+p_2x$

$$z=qx \quad \therefore \quad z_1=qx_1+q_1x,\ z_2=qx_2+2q_1x_1+q_2x$$

Here $\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} x & px & qx \\ x_1 & px_1+p_1x & qx_1+q_1x \\ x_2 & px_2+2p_1x_1+p_2x_1 & qx_2+2q_1x_1+q_2x \end{vmatrix}$

$c_2-pc_1,\ c_3-qc_1$

$$=\begin{vmatrix} x & 0 & 0 \\ x_1 & p_1x & q_1x \\ x_2 & 2p_1x_1+p_2x & 2q_1x_1+q_2x \end{vmatrix} = x\begin{vmatrix} p_1x & q_1x \\ 2p_1x_1+p_2x & 2q_1x_1+p_2x \end{vmatrix}$$

$$=x^2\begin{vmatrix} p_1 & q_1 \\ 2p_1x_1+p_2x & 2q_1x_1+q_2x \end{vmatrix} = x^2\begin{vmatrix} p_1 & q_1 \\ p_2x & q_2x \end{vmatrix};\quad R_1-2x_1R_1$$

$$=x^3\begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}$$

Exercise

Find y_n in the following cases.

1. $y=(2x+3)^m$ 2. $y=\dfrac{x^3}{(x-1)(x-2)}$

3. $y=\dfrac{c}{(3-5x)}$ 4. $y=\dfrac{x^2}{(x-1)^3(x-2)}$

5. $y=\dfrac{x^2}{(x-1)^2(x+2)}$ 6. $y=\dfrac{1}{1-5x+6x^2}$

7. $y=\sin^5 x \cos^8 x$ 8. $y=\sin 2x \sin 3x$

9. $y=e^{ax+b}$ 10. $y=\sin^2 x \sin 2x$

11. $y=\sin^6 x$ 12. $y=\cos^5 x$

13. $y=e^{2x}\cos^2 x$ 14. (i) $y=e^{3x}\sin 4x$

13(i) If $y=\dfrac{ax^2+bx+c}{1-x}$ (ii) $y=e^{ax}\sin(p-qx)$

Then prove that $(1-x)y_3=3y_2$ **R. U. 1988**

15. $y=\dfrac{x}{x^2+a^2}$ 16. $y=\tan^{-1}\dfrac{x}{a}$ R.H. 1988

17. (i) $y=\tan^{-1}\dfrac{1+x}{1-x}$ (ii) $y=\dfrac{1-x}{1+x}$ 18. $y=\tan^{-1}\dfrac{2x}{1-x^2}$

19. $y=\sin^{-1}\dfrac{2x}{1+x^2}$ 20. $y=e^x \log x$

21. $y=e^x(ax+b)^3$ 22. (i) $y=x^2\cos x$

(ii) $y=x^n \log bx$ **D. U. 1954**

23. If $y=\sin x$, prove that $4\dfrac{d^3}{dy^3}\cos^7 x=105\sin 4x$

24. If $y=\sin nx+\cos nx$, show that **R.U. 19988**

$y_r=n^r\sqrt{\{1+(-1)^r\sin 2nx\}}$

25. **If $y=ax^{n+1}+bx^{-n}$, prove that $x^2y_2=n(n+1)y$**

26. **If $y=\tan^{-1}x$, show that $(y_{10})=0$**

27 **If $y=x^4\log x$, prove that $y=24\log x+50$**

(i) If $y=x^{n-1}\log x$. prove that $y_n=\underline{|(n-1)}/x$ **R. U. 1987**

28. Find $(y_n)_0$ if $y=e^{m\cos^{-1}x}$

29. **Find the derivative at $x=0$, if**

$y=\{x+\sqrt{(1+x^2)}\}^m$

30. Find $(y_n)_n$ when $y=\sin(a\sin^{-1}x)$

31. If $y=x^{2n}$ where n is a positive integer, show that

$y_n=2^n\{1.3\,5\ldots\ldots(2n-1)\}\,x^n$ **R. U. 1987**

31. (i) Show that $y=e^x\sin x$ is the solution of $y_4+4y=0$

(ii) Show that $y=\sin h\,(m\sin h^{-1}x)$ satisfies the equation $(1+x)^2y_2+xn_1=m^2y$

32. If $y=\sin^{-1}x$, show that $(y_n)_0=0$. or,

$(n-2)^2(n-4)^2\ldots5^2.3^2.1^2$ when n is even or odd

33. If $y=A\tan\frac{1}{2}\theta+B(2+\theta\tan\frac{1}{2}\theta)$, when A and B are any

constants, prove that

$(1+\cos\theta)y_2=y$ R. U. (H) 1962

34. Prove that $\frac{1}{n!}\left(\frac{d}{dx}\right)^n \frac{1}{x(1-x)}=\frac{(-1)^n}{x^{n+1}}+\frac{1}{(1+x)^{n+1}}$

R. U. (H) 1962

35. Show that $\frac{d^{n+1}}{dx^{n+1}}(x^n \log x)=\frac{n!}{x}$ D. U. 1957, D. U, 1969

36. If $y=\sin(\sin y)$, prove that

$\frac{d^2y}{dx^2}+\tan x\frac{dy}{dx}+y\cos^2x=0$ D. U. 1954

37. If $y=m\cos^{-1}x$, show that

$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+m^2)y_n=0$

38. If $y=ax\sin x$, prove that $x^2y_2-2xy_1+(x^2+2)y=0$

39. If $y=\sin\{a\log(x+b)\}$ prove that

(i) $(x+b)^2y_2+(x+b)y_1+a^2y=0$ N.U. 1995, 94

(ii) $(x+b)^2y_{n+2}+(2n+1)(x+b)y_{n+1}+(n^2+a^2)y_n=0$

C. U 1980, D. U. 1954

40. If $y=A\cos\{m\sin^{-1}(ax+b)\}$ prove that

$\{1-(ax+b)^2\}y_{n+2}-(2n+1)a(ax+b)y_{n+1}+(m^2-n^2)a^2y_n=0$

D. U. 1964

41. If $y=\tan^{-1}x$, prove that

$(1+x^2)y_{n+2}+2(n+1)xy_{n+1}+n(n+1)y_n=0$

42. If $y=a\cos(\log x)+b\sin(\log x)$, show that C. U. 1993

$x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2+1)y_n=0$ D. H. 1961 D. U. 1991

43. If $y=e^{x\cos a}\sin(x\sin a)$ prove that

$y_{n+2}-2y_{n+1}\cos a+y_n=0$ D. H, 1955

44 If $y=(\sinh^{-1}x)^2$, prove that

$(1+x^2)y_{n+2}+(2n+1)xy_{n+2}+x^2y_n=0$ D. H. 1959

45. If $y\sqrt{(1-x^2)}=\sin^{-1}x$, prove that

$(1-x^2)y_{n+1}-(2n+3)xy_n-n^2y_{n-1}=0$ D. U. 1958

46. If $y=\cos\{\log(1+x)^2\}$ prove that N.U. 1994

$(1+x)^2y_{n+2}+(2n+1)(1+x)y_{n+1}+(n^2+1)y_n=0$

D. U. 1951

47. Show that the nth derivative of the differential equation.

$x^2y_2+xy_1+(a^2-m^2)y=0$ is

$x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2-a^2-m^2)y_n=0$

(i) If $\log y=\tan^{-1}x$, show that

$(1+x^2)y_{n+2}+(2nx+2x-1)y_{n+1}+n(n+1)y=0$ C.U. 1980, '84

48. If $y=x^2\sin x$, prove that N.U. 1994

$y_n=(x^2-n^2+n)\sin(x+n\pi/2)-2nx\cos(x+n\pi/2)$

49. Differentiate n times the equation.

[i] $x^2(d^2y/dx^2)+x(dy/dx)+y=0$ [ii] $(1+x^2)y_2+(2n-1)y_1=0$

50. If $x=\sin\left(\frac{1}{m}\log y\right)$ or, $y=e^{m\sin^{-1}x}$

Show that

$(1-x^2)y_{n+2}-(2n+1)y_{n+1}-(n^2+m^2)y_n=0$ D. U. 1986

Show that the value of y_n at $x=0$ i. e., D. H. 87

$(y_n)_0=m(m^2+2^2)(m^2+3^2)\ldots\{m^2+(n-2)^2\}$ when n is odd

or, $m^2(m^2+2^2)(m^2+4^2)\ldots(m^2+\{n-2\}^2\}$ when n is even

50. If $y=e^{2\sin^{-1}x}$ prove that C. H. 1987

(i) $(1-x^2)y_2-xy_1-4y=0$

(ii) $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+4)y=0$

51. If $y=(x^2-1)^n$ prove that

$$(y^2-1)y_{n+2}+2xy_{n+1}-n(n+1)y_n=0$$

Hence if $u=\frac{d^n}{dx^n}(y^2-1)^n$

Show that $\frac{d}{dx}\left\{(1-x^2)\frac{du}{dx}\right\}+n(n+1)u=0$

52. If $y^{1/m}+y^{-1/m}=2x$, prove that D. H. 1989, D.U.1990

$$(x^2-1)y_{n+2}+(2n+1)yx_{n+1}+(n^2-m^2)y_n=0$$ R.U. 1960

53. If $x=(A+Bt)e^{-nt}$, prove that D.U. 1984

$$\frac{d^2x}{dt^2}+2n\frac{dx}{dt}+n^2x=0$$ D.U. 1962

54. If $y=\sin^{-1}x$, then prove that

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n=0$$

(i) If $y=(a\sin^{-1}bx)^2$, obtain an equation connecting y, y_1, and y_2

Apply Leibnitz Theorem on this equation and a relation connecting y_n, y_{n+1}, y_{n+2}.

55. If $n=\tan^{-1}x$, prove that

$$(1+x^2)\frac{d^2u}{dx^2}+2x\frac{du}{dx}=0$$ C. U. 1992

Hence determine the nth derivatives of u, with respect to x when $x=0$ R.U. 1958

56. If $y=x^m \log x$, show that $xy_1=my+x^m$ where $y_1=dy/dx$

Differentiate this equation n times where $n>m$

57. If $y=\{\log\{x+\sqrt{(1+x^2)}\}^2$ Prove that

[i] $(1+x^2)y_2+xy_1=2$

[ii] $(1+x^2)y_{n+2}+(2n+1)xy_{n+1}+n^2y_n=0$

58. If $y=x^2+\frac{1}{x^2}$ show that $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-4y=0$ R.U.1962

59. Show that $\frac{d^n}{dx^n}\left(e^{\frac{1}{2}x^2}\right)=u_n(x)e^{\frac{1}{2}x^2}$ where $u_n(x)$ is a polynomial of degree n. Establish the recurrence relation. D.H. 1962

$u_{n+1}=xu_n+nu_{n-1}$ and hence obtain the differential equation $u''_n+xu'_n-nu_n=0$ satisfied by $u_n(x)$

(i) If $x=\tan(\log y)$, prove that

$(1+x^2)y_{n+1}+(2nx-1)y_n+n(n-1)y_{n-1}=0$ C.H. 1977

60. If $y=x/(x^2+a^2)$ and $x=a\cot\theta$. show that

$$y_n=(-1)^n\frac{\lfloor n}{a^{n+1}}(\sin\theta)^{n+1}\cos(n+1)\theta$$ R.H. 1967

61. Show that

$$\frac{d^n}{dx^n}\left(\frac{\sin x}{x}\right)=\frac{P\sin(x+\frac{1}{2}n\pi)+Q\cos(x+\frac{1}{2}n\pi)}{x^{n+1}}$$

$$\frac{d}{dy^n}\left(\frac{\cos x}{y}\right)=\frac{P\cos(x+\frac{1}{2}n\pi)-Q\sin(y+\frac{1}{2}n\pi)}{y^{n+1}}$$

where $P=y^n-n(n-1)y^{n-2}+n(n-1)(n-2)(n-3)y^{n-4}\ \dots$

$Q=ny^{n-1}-n(n-1)(n-2)y^{n-3}+\cdots$

62. If $y=e^{\tan^{-1}x}=a_0+a_1x+a_2x^2+\ \cdots\ \dots$

Show that

(i) $(1+x^2)y_2+(2x-1)y_1=0$

(ii) $(1+x^2)y_{n+2}+\{2(n+1)x-1\}y_{n+1}+n(n+1)y_n=0$

(ii) $(n+2)a_{n+2}+na_n=a_{n+1}$

Sol :—See worked out example 14 of chapter VII General Theorems.

63. If $y=\sin(m \sin^{-1}x)=a_0+a_1x+a_2x^2+ \cdots \cdots$

Show that D. H. 1986

N. H. 1994

(i) $(1-x^2)y_2=xy_1-m^2y$

(ii) $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2-m^2)y_n=0$

(iii) $(n+1)(n+2)a_{n+2}=(n^2-m^2)a_n$

Sol :—see worked out Ex. 10 for (i) and (ii) and for (iii) differentiate y, then put the values of y_1 and y_2 in (i), equate the co-efficients of x^n from both sides. The result will follow.

64. If $y=e^{a \sin^{-1}x}=a_0+a_1x+a_2x^2+ \cdots \cdots$

Show that

$(n+1)(n+2)\,a_{n+2}=(n^2+a^2)a_n$

Hints : see worked out Ex. 11, then equate co-efficients of x^n from both sides after putting the values of y_1 and y_2 in (1)

65. If $y=(\sin^{-1}x)^2=a_0+a_1x+a_2x^2+ \cdots \cdots$

Show that $(n+1)(n+2)\,a_{n+2}=n^2a_n$

66. Prove that

$$\left(\frac{d}{dx}\right)^r e^{ax} x^n = a^{r-n} x^{n-r}\left(\frac{d}{dx}\right)^n e^{ax} x^r$$

67. Prove that if $x+y=1$.

$$\frac{d^n}{dx^n}(x^ny^n)=n!\,(y^n-{}^nc_1{}^2\,y^{n-1}x+{}^nc_2{}^2y^{n-2}x^2+\cdots \;\ldots)$$

98. Prove that if $y=\dfrac{\log x}{y}$, then

$$y_n=\frac{(-1)^n\,n!}{x^{n+1}}(\log x-1-\tfrac{1}{2}-\tfrac{1}{3}\cdots \cdots \cdots 1/n)$$

69. By forming in two different ways the nth derivative of x^{2n}, prove that

$$1+\frac{n^2}{1^2}+\frac{n^2(n-1)^2}{1^2.2^2}+\frac{n^2(n-1)^2(n-2)^2}{1^2.2^2 3^2}+\cdots\cdots=\frac{(2n)!}{(n!)^2}$$

Hint : If $y=x^{2n}$, $y_n=\dfrac{(2n)!}{n!}x^n$

Again $y=x^n.\,x^n$

$D^n(x^{2n})=D^n(x^n.\,x^n)=x^n\,n!\,[\text{ given series }]=\left(\dfrac{(2n)!}{n!}\right)^2$

70. If $\cot y = x$ then show that

$$\frac{d^n}{dx^n}\left\{\frac{x^n}{1+x^2}\right\}=n!\sin y\,[\sin y-{}^nc_1\cos y\sin 2y+{}^nc_2\cos^2 y\sin^3 y\cdots]$$

প্রশ্নমালা VI

নিম্নলিখিত ফাংশনগুলির y_n নির্ণয় কর :

1. $y=(2x+3)^m$
2. $y=\dfrac{x^3}{(x-1)(x-2)}$
3. $y=\dfrac{c}{(4-5x)}$
4. $y=\dfrac{x^2}{(x-1)^3(x-2)}$
5. $y=\dfrac{x^2}{(x-1)^2(x+2)}$
6. $y=\dfrac{1}{1-5x+6x^2}$
7. $y=\sin^5 x\cos^3 x$
8. $y=\sin 2x\sin 3x$

9. $y=e^{ax+b}$ 10. $y=\sin^2 x \sin 2x$

11. $y=\sin^6 x$ 12. $y=\cos^5 x$

13. $y=e^{2x}\cos^2 x$ 14. (i) $=e^{3x}\sin 4x$

13. (i) যদি $y=\dfrac{ax^2-bx+c}{1-x}$ (ii) $y=e^{ax}\sin(p-qx)$

প্রমাণ কর যে $(1-x)y_3-3y_2$ *R. U. 1933* *D. U. 1954*

15. $y=\dfrac{x}{x^2+a^2}$ 16. $y=\tan^{-1}\dfrac{x}{a}$ *R. H. 1988*

17. (i) $y=\tan^{-1}\dfrac{1+x}{1-x}$ (ii) $y=\dfrac{1-x}{1+x}$ 18. $y=\tan^{-1}\dfrac{2x}{1-x^2}$

19. $y=\sin^{-1}\dfrac{2x}{1+x^2}$ 20. $y=e^x \log x$

21. $y=e^x(ax+b)^3$ 22. (i) $y=x^2\cos x$

(ii) $y=x^5\log bx$ *D. U. 1954*

23. যদি $y=\sin x$, হয় তবে প্রমাণ কর যে $4\dfrac{d^3}{dy^3}\cos^7 x=105\sin 4x$

24. যদি $y=\sin nx+\cos nx$ হয় তবে দেখাও যে

$y_r=n^r\sqrt{\{1+(-1)^r\sin 2nx\}}$ C.H.1993 *R. U. 1988*

25. যদি $y=ax^{n+1}+bx^{-n}$ হয় তবে প্রমাণ কর যে $x^2y_2=n(n+1)y$

26. যদি $y=\tan^{-1}x$ হয় তবে দেখাও যে $(y_{10})=0$

27. যদি $y=x^4\log x$ হয় তবে প্রমাণ কর যে $y=24\log x+50$

28. যদি $y=e^{m\cos^{-1}x}$ হয় তবে $(y_n)_0$ এর মান নির্ণয় কর।

29. যদি $y=\{x+\sqrt{(1+x^2)}\}^m$ হয়, তবে $x=0$ বিন্দুতে n তম অন্তরক সহগ নির্ণয় কর।

30. $y=\sin(a\sin^{-1}x)$ হইলে $(y_n)_0$ নির্ণয় কর।

31. (i) দেখাও যে $y_4+4y=0$ এর সমাধান হল $y=e^x\sin x$

(ii) দেখাও যে $(1+x^2)y_2+xy_1=m^2y$ সমীকরণকে $y=\sin h(m\sin h^{-1}x)$ সিদ্ধ করে। *R. U. 19*

32. যদি $y=\sin^{-1}x$ হয়, তবে দেখাও যে $(y_n)_0=0$ বা, $(n-2)^2(n-4)^2\ldots5^2.3^2.1^2$ যখন n জোর অথবা বিজোড় হয়।

33. যদি $y=A\tan\frac{1}{2}\theta+B(2+\theta\tan\frac{1}{2}\theta)$ হয়, যেখানে A এবং B দুইটি ধ্রুবক, তবে প্রমাণ কর যে $(1+\cos\theta)y_2=y$ R.U. (H) 1962

34. প্রমাণ কর যে $\dfrac{1}{n!}\left(\dfrac{d}{dx}\right)^n\dfrac{1}{x(1-x)}=\dfrac{(-1)^n}{x^{n+1}}+\dfrac{1}{(1+x)^{n+2}}$

R. U. (H) 1962

35. দেখাও যে $\dfrac{d^{n+1}}{dx^{n+1}}(x^n\log x)=\dfrac{n!}{x}$ D. U. 1957 D. U. 1969

36. যদি $y=\sin(\sin y)$ হয় তবে প্রমাণ কর যে

$\dfrac{d^2y}{dx^2}+\tan x\dfrac{dy}{dx}+y\cos^2x=0$ D. U. 1954

37. যদি $y=e^{c\sin^{-1}x}$, হয় তবে দেখাও যে

$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+m^2)y_n=0$

38. যদি $y=ax\sin x$, হয় তবে প্রমাণ কর যে $x^2y_2-2xy_1+(x^2+2)y=0$

39. যদি $y=\sin\{a\log(x+b)\}$ হয় তবে প্রমাণ কর যে

(i) $(x+b)^2y_2+(x+b)y_1+a^2y=0$

(ii) $(x+b)^2y_{n+2}+(2n+1)(x+b)y_{n+1}+(n^2+a^2)y_n=0$

N. U. 1994. C. U. 1980. D. U. 1953

40. যদি $y=A\cos\{n\sin^{-1}(ax+b)\}$ হয় তবে প্রমাণ কর যে

$\{1-(ax+b)^2\}y_{n+2}-(2n+1)a(ax+b)y_{n+1}+(m^2-n^2)a^2y_n=0$

D. U. 1964

41. যদি $y=\tan^{-1}x$ হয় তবে প্রমাণ কর যে

$(1+x^2)y_{n+2}+2(n+1)xy_{n+1}+n(n+1)y_n=0$

42. যদি $y=a\cos(\log x)+b\sin(\log x)$, হয় তবে দেখাও যে

$x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2+1)y_n=0$ D. H. 1961

43. যদি $y=e^{x\cos a}\sin(x\sin a)$ হয়, তবে প্রমাণ কর যে D. H. 1950

$y_{n+2}-2y_{n+1}\cos a+y_n=0$

44. যদি $y=(\sin h^{-1}x)^2$ হয় তবে প্রমাণ কর যে D. H. 1959

$(1+x^2)y_{n+2}+(2n+1)xy_{n+1}+x^2y_n=0$

45. যদি $y\sqrt{(1-x^2)}=\sin^{-1}x$ হয় তবে প্রমাণ কর যে D. U. 195

$(1-x^2)y_{n+1}-(2n+1)xy_n-n^2y_{n-1}=0$

46. যদি $y=\cos\{\log(1+x)\}$ হলে প্রমাণ কর যে

$(1+x)^2y_{n+2}+(2n+1)(1+x)y_{n+1}+(n^2+1)y_n=0$ D. U. 195

N.U. 1994.

47. দেখাও যে অন্তরক সমীকরণ $x^2y_2+xy_1+(a^2-m^2)y=0$ এর n

বৃদ্ধিহার হবে,

$x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2-a^2-m^2)y_n=0$

(i) যদি $\log y=\tan^{-1}x$ হয়, তবে দেখাও যে C. U. 1980,

$(1+x^2)y_{n+2}+(2nx+2x-1)y_{n+1}+n(n+1)y_n=0$

43. যদি $y=x^2\sin x$ হয়, তবে প্রমাণ কর যে

$y_n=(x^2-n^2+n)\sin(x+n\pi/2)-2nx\cos(x+n\pi/2)$

49. নিম্নলিখিত সমীকরণগুলিকে n বার অন্তরীকরণ কর,

[i] $x^2(d^2y/dx^2)+x(dy/dx)+y=0$ [ii] $(1+x^2)y_2+(2n-1)y_1=0$

50. $x=\sin\left(\frac{1}{m}\log y\right)$ হলে দেখাও যে D. U. 19

$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+m^2)y_n=0$ D. H. 19

আরো দেখাও যে $x=0$ বিন্দুতে y_n এর মান হবে

$(y_n)_0=m(m^2+1^2)(m^2+3^2)\ldots\{m^2+(n-2)^2\}$ যখন n বিজো

বা, $m^2(m^2+2^2)(m^2+4^2)\ldots(m^2+\{n-2)^2\}$ যখন n জোড়

50. (i) If $y=e^{2\sin^{-1}x}$ prove that C. H. 1987

(i) $(1-x^2)y_2-xy_1-4y=0$

(ii) $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+4)y=0$

51. যদি $y=(x^2-1)^n$ হয়, তবে প্রমাণ কর যে

$(x^2-1)y_{n+2}+2xy_{n+1}-n(n+1)y_n=0$

তাহাতে যদি $u=\frac{d^n}{dx^n}(x^2-1)^n$ হয়, তবে দেখাও যে

$\frac{d}{dx}\left\{(1-x^2)\frac{du}{dx}\right\}+n(n+1)u=0$

52. $y^{1/m}+y^{-1/m}=2x$, হলে প্রমাণ কর যে D.U. 1990, D.H. 1989.

$(x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0$ R. U. 1960

53. যদি $x=(A+B)e^{-nt}$ হয়, তবে প্রমাণ কর যে D. U. 1984

$\frac{d^2x}{dt^2}+2n\frac{dx}{dt}+n^2x=0$ D. U. 1962

54. যদি $y=\sin^{-1}x$ হয় তবে প্রমাণ কর যে

$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n=0$

(i) If $y=(a\sin^{-1}bx)^2$, obtain an equation connecting y, y_1, and y_2

Apply Leibnitz Theorem on this equation and find a relation connecting y_n, y_{n+1}, y_{n+2},

55. যদি $u=\tan^{-1}x$ হয়, তবে প্রমাণ কর যে

$(1+x^2)\frac{d^2u}{dx^2}+2x\frac{du}{dx}=0$ C.U. 1992.

এবং $x=0$ হলে n তম বৃদ্ধিহার নির্ণয় কর। R. U. 1958

56. যদি $y=x^m\log x$, হয় তবে দেখাও যে

$xy_1=my+x^m$ যেখানে $y_1=dy/dx$

এ সমীকরণের n-তম বৃদ্ধিহার নির্ণয় কর যখন $n>m$ হয়।

57. যদি $y=\{\log\{x+\sqrt{(1+x^2)}\}^2$ হয়, তবে প্রমাণ কর যে

[i] $(1+x^2)y_2+xy_1=2$

[ii] $(1+x^2)y_{n+2}+(2n+1)xy_{n+1}+n^2y_n=0$

58. যদি $y=x^2+\frac{1}{x^2}$ হয় তবে দেখাও যে

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-4y=0$$ R. U. 1962

59. দেখাও যে $\frac{d^n}{dx^n}\left(e^{\frac{1}{2}x^2}\right)=u_n(x)e^{\frac{1}{2}x^2}$ এখান $u_n(x)$ হল n মাত্রার একটি বহুপদী রাশি। পৌনঃপুনিক সম্পর্কটি প্রতিষ্ঠা কর।

D. H. 19

$u_{n+1}=xu_n+nu_{n-1}$ এবং ইহা হতে $u_n(x)$ দ্বারা সিদ্ধ অন্তরক সমীক

$u''_n+xu'_n-nu_n=0$ প্রতিষ্ঠা কর।

(i) যদি $x=\tan(\log y)$ হয় তবে প্রমাণ কর যে

$(1+x^2)y_{n+1}+(2nx-1)y_n+n(n-1)y_n=0$ C. H. 19

60. যদি $y=x/(x^2+a^2)$ এবং $x=a\cot\theta$. হয় তবে দেখাও যে

$$y_n=(-1)^n\frac{\lfloor n}{a^{n+1}}(\sin\theta)^{n+1}\cos(n+1)\theta$$ R. H. 19

61. দেখাও যে

$$\frac{d^n}{dx^n}\left(\frac{\sin x}{x}\right)=\frac{P\sin(x+\frac{1}{2}n\pi)+Q\cos(x+\frac{1}{2}n\pi)}{x^{n+1}}$$

$$\frac{d}{dy^n}\left(\frac{\cos x}{y}\right)=\frac{P\cos(x+\frac{1}{2}n\pi)-Q\sin(y+\frac{1}{2}n\pi)}{x^{n+1}}$$

যেখানে $P=y^n-n(n-1)y^{n-2}+n(n-1)(n-2)(n-3)y^{n-4}\ \ldots$

$Q=ny^{n-1}-n(n-1)(n-2)y^{n-3}+\ldots$

62. If $y=e^{\tan^{-1}x}=a_0+a_1x+a_2x^2+\ \ldots\ \ldots$

Show that

(i) $(1+x^2)y_2+(2x-1)y_1=0$

(ii) $(1+x^2)y_{n+2}+\{2(n+1)x-1\}y_{n+1}+n(n+1)y_n=0$

(iii) $(n+2)a_{n+2}+na_n=y_{n+1}$

Sol: see worked out example 14 of chapter VII. General Theorems.

63. If $y=\sin(m\sin^{-1}x)=a_0+a_1x+a_2x^2+\ \ldots\ \ldots$

N.H. 1994. *D. H. 1986*

Show that

(i) $(1-x^2)y_2=xy_1-m^2y$

(ii) $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2-m^2)y_n=0$

(iii) $(n+1)(n+2)a_{n+2}=(n^2-m^2)a_n$

Sol:—see workedout Ex. 10 for (i) and (ii) and for (iii) differentiate y, then put the values of y_1 and y_2 in (i), equate the coefficients of x^n from both sides. The result will follow.

64. If $y=e^{a\sin^{-1}x}=a_0+a_1x+a_1x^2+\ \ldots\ \ldots$

Show that

$(n+1)(n+2)a_{n+2}=(n^2+a^2)a_n$

Hints: see worked out Ex. 11, then aquate co-efficients of x^n from both sides after putting the values of y_1 and y_2 in (1).

65. If $y=(\sin^{-1}x)^2=a_0+a_1x+a_2x^2+\ldots\ \ldots$

Show that $(n+1)(n+2)a_{n+2}=n^2a_n$

66. Prove that

$$\left(\frac{d}{dx}\right)^r e^{ax}x^n=a^{r-n}x^{n-r}\left(\frac{d}{dx}\right)^n e^{ax}x^r$$

67. Prove that if $x+y=1$,

$$\frac{d^n}{dx^n}(x^ny^n)=n!\,(y^n-{}^nc_1^2y^{n-1}x+{}^nc_2^2y^{n-2}x^2-\ \ldots\ \ldots)$$

68. Prove that *if* $y = \dfrac{\log x}{y}$, then

$$y_n = \frac{(-1)^n n!}{x^{n+1}} \left(\log x - 1 - \frac{1}{2} - \frac{1}{3} \ldots\ldots - 1/n\right)$$

69. By forming in two different ways the nth derivative of x^{2n}, prove that

$$1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{1^2.2^2} + \frac{n^2(n-1)^2(n-2)^2}{1^2.2^2.3^2} + \ldots\ldots\ldots = \frac{(2n)!}{(n!)^2}$$

Hint : If $y = x^{2n}$, $y_n = \dfrac{(2n)}{n!} x^n$

Again $y = x^n . x^n$

$D^n(x^{2n}) = D^n(x^n . x^n) = x^n n!$ [Given series]= $\dfrac{(2n\,!)^2}{n!}$

70. If $x = \cot y$, then show that

$$\frac{d^n}{dx^n}\left(\frac{x^n}{1+x^2}\right) = n! \sin y \,[\, \sin y - {}^nc_1 \cos y \sin 2y + {}^nc_2 \cos^2 y \sin 3y \ldots\ldots\ldots]$$

উত্তরমালা VI

1. $2^n m!/(m-n)!\,(2x+3)^{m-n}$

2. $(-1)^{n-1} n! \{(x-1)^{-n-1} - (8x-x)^{-n-1}\}$

3. $c \lfloor n\, 5^n (4-5x)^{-n-1}$

4. $(-1)^{n-1} n! \left[\dfrac{(n+2)(n+1)}{2(x-1)^{n+3}} + \dfrac{3(+1)}{(x-1)^{n+1}} + \dfrac{4}{(x-1)^{n-1}} + \dfrac{4}{(x-2)^{n+1}}\right]$

5. $\dfrac{(n+1)!\,(-1)^n}{3(x-1)^{n+1}} + \dfrac{5n!\,(-1)^n}{9(x-1)^{n+1}} + \dfrac{4n!\,(-1)^n}{9(x+2)^{n+1}}$

6. $n! \left[\dfrac{3^{n+1}}{(1-3x)^{n+1}} + \dfrac{3n+1}{(1-2x)^{n+1}}\right]$

7. $2^{-7}\{8^n \sin(8x+\frac{1}{2}n\pi) - 2.6^n \sin(6x+\frac{1}{2}n\pi)$
$-2.4^n \sin(\frac{1}{2}x+\frac{1}{2}n\pi) + 6.2^n \sin(2x+\frac{1}{2}n\pi)\}$

8. $y\{\cos(x+\frac{1}{2}n\pi) - 5^n \cos(5x+\frac{1}{2}n\pi)\}$

9. $a^n e^{ax+b}$

10. $2^{n-1}\sin(2x+\frac{1}{2}n\pi) - 4^{n-1}\sin(4x+\frac{1}{2}n\pi)$

11. $-(\frac{1}{2})^5\{6^n\cos(6x+\frac{1}{2}n\pi) - 6.4^n\cos(4x+\frac{1}{2}n\pi)$
$+15.2^n\cos(\frac{1}{2}n\pi+2x)\}$

12. $(\frac{1}{2})^4\{5^n\cos(5x+\frac{1}{2}n\pi) + 5.3^n\cos(3x+\frac{1}{2}n\pi)$
$+10\cos(x+\frac{1}{2}n\pi)\}$

13. $2^{n-1}e^{2x} + (\frac{1}{3})^{n/2}\cos(\frac{1}{2}n\pi+2x)$

14. (i) $5^n e^{3x}\sin(4x+n\tan^{-1}4/3)$

(ii) $(a^2+q^2)^{n/2} e^{ax}\sin(p-qx-n\tan^{-1}q/a)$

15. $\{(-1)^n n!\cos(n+1)\theta\sin^{n+1}\theta\}/a^{n+1}$, $x = a\cos\theta$

16. $\dfrac{(-1)^{n-1}(n-1)!}{a^n}\sin n\theta\sin^n\theta$, যেখানে $x = a\cot\theta$

17. (i) $(-1)^{n-1}\lfloor(n-1)!\,\sin^n\theta\sin n\theta$, এখানে $x = \cot\theta$

(ii) $(-1)^n \dfrac{2\lfloor n}{(1+x)^{n+1}}$

18. 17 নং এর দুইগুণ

19. 18 নং এর মত।

20. $e^x\{\log x + {}^nc_1 x^{-1} + (-1)\,{}^nc_2 x^{-2} + \ldots (-1)^{n-1}\lfloor(n-1)\,x^{-n}\}$

21. $y_n = e^x\{(ax+b)^3 + 3\,{}^nc_1\, a(ax+b)^2 + 6a^2\,{}^nc_2(ax+b) + 6a^3\}$

22. (i) $(x^2+n-n^2)\cos(\frac{1}{2}n\pi+x) + 2ax\sin(\frac{1}{2}n\pi+x)$

(ii) $y_n = \dfrac{(-1)^{n-1}}{x^{n-1}}\{-(n-1)! + 5\,{}^nc_1(n-2)!\ldots + {}^nc_3 !\, 5(n-6)!$

28. $\{(n-2)^2+m^2\}\{(n-4)^2+m^2\}\ldots\ \ldots$
$\ldots(4^2+m^2)(2^2+m^2)m^2 e^{m\frac{1}{2}\pi}$ যখন n জোড় সংখ্যা হয়।

বা, $\{(n-2)^2+m\}^2\{(n-4)^2+m^2\} \ldots \ldots$ 249

$\ldots(3^2+m^2)(1^2+m^2)me^{m\frac{1}{2}\pi}$ যখন n বিজোর সংখ্যা হয়।

29. $\{m^2-(n-2)^2\{m^2-(n-4)^2\}\ldots(m^2-1)\ 1^2$ যখন n জোড় হয়।

বা, $\{m^2(n-2)^2\}\{m^2-(n-4)^2\}\ldots(m^2-1^2)\ m$ যখন n বিজোড় সংখ্যা হয়।

49. (i) $x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2+1)y_n=0$

(ii) $(1+x)y_{n+2}+\{2x(n+1)-1\}y_{n+1}+n(n+1)y_n=0$

54. (i) $y_2(1-b^2x^2)-y_1 b^2x=2a^2b^2$

$(1-b^2x^2)y_{n+2}-(2n+1)b^2x\, y_{n+1}-b^2(n^2-1)y_n=0$

55. $(u_n)_0=0$ or, $\frac{n-1}{(-1)^2}(n-1)\,!$ যখন n জোড় বা বিজোড়

56. $xy_{n+1}+(n-m)y_n=0$

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CHHPTER VII

GENERAL THEOREMS AND EXPANSIONS

Art. 7.1. Introduction : – That a function $f(x)$ is continuous in the closed interval $[a,b]$ means that the function $f(x)$ is continuous at every point in the interval $[a,b]$ including the end point a and b and the interval is given by $a \leqslant x \leqslant b$.

The function $f'(x)$ is continuous in the open interval (a, b) means that $f(x)$ is derivable at every point in the open interval (a, b) satisfying the condition $a<x<b$.

7.2. Rolle's Theorem

If a function $f(x)$ is contiouous in the closed interval $a \leqslant x \leqslant b$, $f'(x)$ exists in the open interval $a<x<b$, and $f(b)=f(a)$, then there exist at least one point, say $x=c$, $a<c<b$ at which the derivative $f'(x)$ vanishes i. e., $f'(c)=0$

Proof : As the function $f(x)$ is continuous is the interval $a \leqslant x \leqslant b$, and $f(a)=f(b)$, $f(x)$ has at least a maximum or a minimum or both in the interval. Either $f(x)$ has minimum value M at $x=c$ i. e., $f(c)=M$, or a minimum value m at $x=c$ i. e., $f(c)=m$, c lying in the interval (a, b)

There are two cases

[i] $M=m$ [ii] $M \neq m$.

[i] When $M=m$

let $f(x)$ be constant in the interval $[a, b]$ with

$f(x)=m$, The derivative of $f(x)$ is zero for every point in the interval. So $f'(x)$ in also zero for $x=c$, $a<c<b$,

Hence $f'(c)=0$

(ii) When $M \neq m$, then at least one value of $f(x)$ is different from $f(b)$ and $f(a)$ in the interval. Let $f(c)=M$ be different from them, where c lies within the interval.

Since $f(c)$ is the maximum value of the function $f(x)$ then $f(c+h)-f(c) \leqslant 0$

whether h is positive or negative

Thus we have

$$\frac{f(c+h)-f(c)}{h} \leqslant 0 \text{ when } h>0,$$

and $$\frac{f(c+h)-f(c)}{h} \geq 0, \text{ when } h<0$$

But from the statement of the theorem we see that $f'(x)$ exist at $x=c$

Now $\lim\limits_{h\to 0} \frac{f(c+h)-f(c)}{h} \leqslant 0$ and $\lim\limits_{h\to 0^-} \frac{f(c+h)-f(c)}{h} \geq 0$

$\Rightarrow f'(c) \geq 0$ ... ... (1)

and $f'(c) \leqslant 0$... ... ... (2)

The relation (1) and (2) are true if and only if $f'(c)=0$

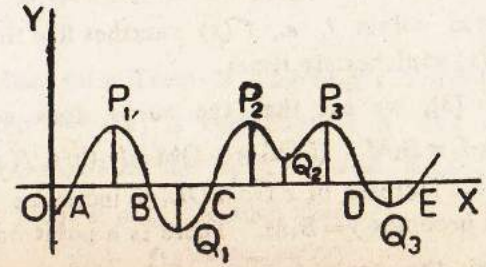
Consequently there is a point c inside the interval $a<c<b$, at which the derivative $f'(x)$ is equal to zero *i. e.* $f'(c)=0$.

We can reach the same conclusion if $f(x)$ has a minimum value which differs from $f(a)$ and $f(b)$.

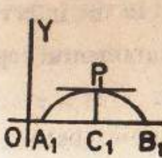
Hence the theorem is established.

73 Geometric Interpretation of Rolle's Theorem

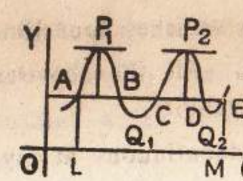
Statement :— *If $f(x)$ be a continuous curve which has tangent at every point in an interval (a, b) and the ordinates of the extremeties of it are equal, i. e. $f(a)=f(b)$, then there exists at least one point c, $a<c<b$. at which the tangent to the curve is parallel to the x-axis.*



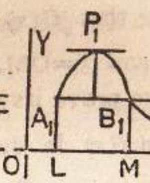
Fig—2



Fig—1



Fig—4



Fig—3

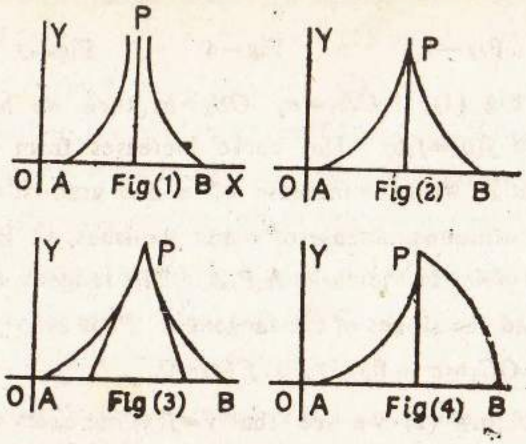
In the Fig (1), if $OA_1=a$, $OB_1=b$, then we have $f(a)=0$, $f(b)=0$, So $f(a)=f(b)$. The curve increases from A_1 becomes maximum at P_1 with the increase of x and gradually diminishes with the continuous increase of x and vanishes at B_1. Thus P_1 is a highest point in the curve $A_1P_1B_1$. The tangent P_1 is parallel to x-axis and the slopes of the tangent at P_1 is zero. *i. e.* $f'(x)=0$ for $x=OC_1=c$ in fig (1) *i. e*, $f'(c)=0$.

In the figure (2) we see that $y=f(x)$ increases with the increase of x, attains its maximum. then diminishes and becomes zero at B. For further increase, of x from OB to OC, y diminishes first and then increase, and at some point Q_1, $\bar{y}$ is minimum and $f'(x)$ vanishes at Q_1 i. e., $f'(x)=0$.

If we consider the interval OD, we get three maxima, and two minima at these points tangents are parallel to x-axis i. e. $f'(x)=0$ at these points *i, e.,* $f'(x)$ vanishes five times. In the interval OE, $f(x)$ vanishes six times.

Is the figure [3], we see that the curve does not meet the x—axis, but $A_1L=B_1M$. If $OL=a$, $OM=b$ then $f(a)=f(b)$. In this case with the increase of x from OL, y increases first, then diminishes and becomes $y=B_1M$. There is a point on the curve. $A_1\,P_1\,B_1$ where the tangent is parallel to x—axis *i, e.* $f'(x)$ vanishes. Hence $f'(x)=0$ at a point x where $OL<x<OM$. In fig. (4), we see that $f'(x)$ vanishes four times in the interval between $x=OL$ and $x=OM$. All the above statements regarding Roll's theorem prove its validity.

If function $f(x)$ is discontinuous at any point between $x=a$



and $x=b$ or, $f'(x)$ does not exist at any point within the interval, then the theorem does not hold good.

In the above figures we see that $f'(x)$ is discontinuous at P in fig. (1) while $f'(x)$ does not exist at P in fig, (2), (3) and (3), and (4) ; $f'(x)$ does not vanish at the point P in the above figures. There is no point in the interval (a, b), where the tangent is parallel to x-axis.

7. 5. Mean value Theorem (*Lagrange's Theorem*).

If a function $f(x)$ is continuous in the closed interval $a \leqslant x \leqslant b$, $f'(x)$ exists in the open interval $a<x<b$, there exists at least one point c, $a<c<b$ such that

$$f(b)-f(a)=(b-a)f'(c)$$

To prove this theorem let us consider another function $\phi(x)$ involving $f(x)$ such that $\phi(x)$ satisfies all the conditions of Rolle's Theorem. Let $\phi(x)=f(x)+Ax$ ... ... (1)

Where A is a constant which is to be determined by the condition $\phi(b)=\phi(a)$.

From (1),

$$\phi(b)=f(b)+Ab \text{ and } \phi(a)=f(a)+Aa$$

Now $\quad \phi(b)=\phi(a)$

$$\therefore \quad f(b)+Ab=f(a)+Aa \Rightarrow A=-\frac{f(b)-f(a)}{b-a} \quad \ldots \quad \ldots \quad (2)$$

Since $f(x)$ and x are continuous in the closed interval and derivable in the open interval, then

$\phi(x)$ is also continuous in the interval $a \leqslant x \leqslant b$, and derivable in $a<x<b$ and also $\phi(a)=\phi(b)$,

Thus $\phi(x)$ satisfies all the condition of Rolle's Theorem,

Then there is at least one point c in the interval $a<c<b$ such that $\quad \phi'(c)=0 \quad \ldots \quad \ldots \quad (3)$

Now differentiate (1) w. r. to x :

$\phi'(x)=f'(x)+A$

$\therefore$ $\phi'(c)=f'(c)+A$. or, $0=f'(c)+A$ by (5)

or, $0=f'(c)-\dfrac{f(b)-f(a)}{b-a}$ by (2)

or. $f(b)-f(a)=(b-a)\ f'(c)$ [Proved]

or, $f'(c)=\dfrac{f(b)-f(a)}{b-a}$ ... ... (4)

This is called First mean value Theorem.

Cor. 1. If we put $b-a=h$ or, $b=a+h$, then the relation [4] becomes,

$f(a+h)-f(a)=hf(a+\theta h)$, $c=a+\theta h$

where θ is a number such that $0<\theta<1$, so that $a<c<a+h$.

So the statement of mean value Theorem also runs as follows.

If a function $f(x)$ is continuons in the closed interval $a\leq x\leq a+h$, $f'(x)$ exists in the open interval $a<x<a+h$, then there exists at least one number θ lying between 0 and 1 such that

$f(a+h)-f(a)=hf'(a+\theta h)$, $0<\theta<1$,

or, $f(a+h)=f(a)+hf'(a+\theta h)$

Note : The Theorem proved in the Article 7·5 is called First mean value theorem The order of the Theorem depends on the highest order of the derivative of $f(x)$ involved.

Cor. 2. If $f'(x)=0$ for all points in (a, b), then $f(x)$ is constant throughout (a, b),

For any interval (a, x) lying in (a, b), we have by Mean value theorem,

$f(x)=f(a)+(x-a)f'\{a+\theta(x-a)\}$, $0<\theta<1$,

According to the given condition

$f(x)$ is zero throughout the interval (a, b) so $f\{a+\theta(x-a)\}=0$.

Hence. $f(x)=f(a)=$ constant

If $f(x)$ and $g(x)$ are two functions such that $f'(x)=g'(x)$ for all points in (a, b), then $f(x)$ and $g(x)$ differ by a constant in (a, b).

For $f(x)-g(x)=$ constant in (a, b)

Differentiate it w. r. to x,

$f'(x)-g'(x)=0$ or, $f'(x)=g'(x)$ in (a, b)

Cor 3 If $f'(x)=0$ in the closed interval $a\leq x\leq b$, show that $f(x)$ is steadily increasing in $a\leq x\leq b$,

Let us consider an internal $[c,d]$ of $[a,b]$ such that $a\leq c<d\leq b$

Then $f(x)$ exists in the interval $c\leq x\leq d$ and is greater than zero i. e., $f'(x)$ exists and remains greater than zero in $a\leq x\leq b$.

As $f'(x)$ exists in $c\leq x\leq d$, so $f(x)$ is continuous in $c\leq x\leq d$ then by Mean value Theorem, we have.

$f(d)-f(c)=(d-c)\,f'(\xi)$, $c<\xi<d$

But $f'(x)>0$ and $d>c$. Therefore $f(d)-f(c)$ is positive

i. e, $f(d)-f(c)>0$ for $d>c$

or $f(d)>f(c)$ for any interval $[c.d]$ with $a\leq c<\xi\leq d$.

Thus $f(x)$ is steadily (monotonically) increasing in the given interval, $a\leq x\leq b$

Similarly $f(x)$ is steadily (or monotonically) decreasing if $f'(x)$ is negative in the interval.

Cor. 4. If the function $f(x)$ and $g(x)$ have the same derivative at all points in an interval $a\leq x\leq b$, show that $f(x)-g(x)$ is constant in $a\leq x\leq b$.

Let $\phi(x)=f(x)-g(x)$ ... ... (i)

$\therefore$ $\phi'(x)=f'(x)-g'(x)$ ... ... (ii)

According to the given condition $f'(x)-g'(x)$ for all value of x in $a\leq x\leq b$.

Hence from [ii], $\phi'(x)=0$

or, $\phi(x)=$ constant *i. e.*, $f(x)-g(x)=$ constant.

7.6. Second Mean Value Theorem (দ্বিতীয় গড়মান উপপাদ্য)

If $f(x)$ and $f'(x)$ are continuous in the closed interval $a\leq x\leq a+h$, $f''(a)$ exits in the open interval $a<x<a+h$, then

$f(a+h)=f(a)+hf'(a)+\frac{1}{2}h^2 f''(a+\theta h)$.

where θ lies between 0 and 1 *i. e.*, $0<\theta<1$.

Let us consider the function.

$\phi(x)=f(x)-f(a)-(x-a)f'(a)+A(x-a)^2$ ... ... [1]

where A is a constant the constant A is chosen in such a way that

$\phi(a+x)=\phi(a)$ ... ... (2)

or, $f(a+h)-f(a)-hf'(a)+Ah^2=\phi(a)=0$ from (1)

or, $A=\frac{\{f(a+h)-f(a)-hf'(a)\}}{h^2}$ ... ... (3)

Now $\phi(x)$ satisfies the conditions of Rolle's Theorem in the interval $(a, a+h)$ Then x is a point in the interval such that $a<x_1<a+h$,

where $\phi'(x_1)=0$ ... ... (4)

Now differentiate (1) w. r. to x,

$\phi'(x)=f'(x)-f'(a)+2A(x-a)$ ... (5)

$\phi''(x)=f''(x)+2A$ ... (6)

when $x=a$ in (5), we get

$\phi'(a)=f'(a)-f'(a)+2A.\ 0=0$

or, $\phi'(a)=0$ ... ... (7)

Therefore, we get another interval (a,x_1) such that $\phi'(x)$ is continuous, in $a\leq x_1\leq x$, $\phi'(x)$ exists in $a<x_1<x$

and moreover $\phi'(a)=\phi'(x_1)=0$

Thus $\phi'(x)$ satisfied all the conditions of Rolle's theorem so, there is at least one point x_1 lying between a and x_2 such that

$\phi''(x_1)=0,\ a<x_2<x_1$

or, $\phi''(x_2)=f''(x_2)+2A=0$ from (7)

or. $f''(x_2)=-2A$

or, $f''(x_2)=\frac{f(a+h)-f(a)-hf'(a)}{h^2}.\ 2$ from (3)

where $a<x_2<a+h$

or, $\frac{h^2}{2}f''(a+\theta h)=f(a+h)-f(a)-hf'(a)$

where $0<\theta<1$, $x_2=a+\theta h$

or, $f(a+h)=f(a)+hf'(a)+\frac{h^2}{2}f''(a+\theta h)$

Hence the theorem is established

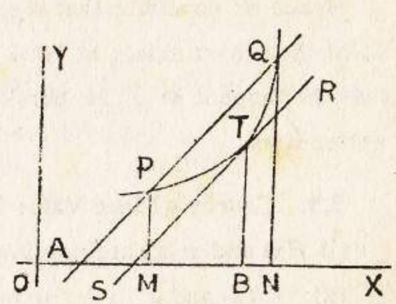
7.7. Geometrical Interpretation of Mean value Theorem

Let $y=f(x)$ represent the curve PTQ in the interval (a, b) Then

$OM=a$, $PM=f(a)$

$ON=b$ $QN=f(b)$

Thus the equation of chord PQ passing through $P\{a, f(a)\}$ and $Q\{b, f(b)\}$ is



Fig—9

$$\frac{y-f(b)}{f(b)-f(a)}=\frac{x-b}{b-a} \text{ or, } y=\frac{f(b)-f(a)}{b-a}(x-b)$$

Let PQ make angle ψ with the x-axis, then

$$\tan\psi=\frac{f(b)-f(a)}{b-a} \quad \dots \quad (1)$$

But the curve $y=f(x)$ is continuous in the closed interval $a\leqslant x\leqslant b$ and $f(x)$ is differentiable in the open interval $a<x<b$ i.e., $f'(x)$ exists in the interval $a<x<b$. It means that the curve has a tangent at every point between P and Q. Then there is a point T in the curve where the tangent STR is parallel to the chord PQ. Let the point T corresponds to $x=c$. The tangent STR at $T\{c, f(c)\}$ makes angle ψ with the x-axis where

$$\tan\psi=\frac{dy}{dx}=f'(x) \text{ at } x=c$$

that is, $\tan\psi=f'(c)$ (1)

From [1] and [2] we have

$$f'(c)=\frac{f(b)-f(a)}{b-a} \text{ or, } f(b)-f(a)=(b-a)f'(c),\ a<c<b$$

Hence we conclude that if a curve has tangent at each of its point then there exists at least one point T on the curve such that the tangent at T is parallel to the chord PQ joining its extremities.

7.8. Cauchy's Mean Value Theorem

(i) $f(x)$ and $g(x)$ are continuous in the closed interval $a\leqslant x\leqslant b$

(ii) $f'(x)$ and $g'(x)$ exist in the open interval $a<x<b$,

(iii) $g(b)\neq g(a)$

and (iv) $f'(x)$ and $g'(x)$ do not vanish for the same value of x then there exists at least one point c in the interval (a, b) such that

$$\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}$$

Let us consider a function $\phi(x)$ defined in the following way

$$\phi(x)=f(x)+Ag(x) \quad \dots \quad \dots \quad (1)$$

where A is a constant. We select A in such a way that

$$\phi(a)=\phi(b)$$

Then from (1).

$$f(a)+Ag(a)=f(b)+Ag(b)$$

$$\Rightarrow A=-\frac{f(b)-f(a)}{g(b)-g(a)} \quad \dots \quad \dots \quad (2)$$

Again

$$\phi'(x)=f'(x)+Ag'(x) \quad \dots \quad \dots \quad (3)$$

Now we see that $\phi(x)$ is continuous in the closed interval $a\leqslant x\leqslant b$, $\phi'(x)$ exists in the open interval $a<x<b$ as $f'(x)$ and $g'(x)$ exist also $\phi(a)=\phi(b)$. Hence by Rolle's theorem, there exists at least one point c in the interval (a, b) such that

$$\phi'(c)=0$$

or, $\phi'(c)=f'(c)+Ag'(c)=0$ from [3]

or, $f'(c)+Ag'(c)=0$

But $g'(c)\neq 0$, otherwise : $f'(c)$ would also be zero which is contrary to the assumption (iv) of the hypothesis.

$$\text{Thus } A=-\frac{f'(c)}{g'(c)} \quad \dots \quad \dots (4)$$

From (2) and (4), we have

$$\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)} \quad ; (a<c<b)$$

Hence the theorem is established.

7.9. Taylor's Theorem with Remainder.

If a function $f(x)$ and all of its derivatives upto $(n-1)$th order [i. e., $f^{n-1}(x)$] are continuous in the closed interval $a \leq x \leq a+h$ and the nth derivative $f^n(x)$ exists in the open interval $a<x<a+h$, then

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{\lfloor 2}f''(a)+\dots+\frac{h^{n-1}}{\lfloor n-1}f^{n-1}(x)+R_n$$

where R_n is the remainder after n terms.

Let us consider a function $\varphi(x)$ in the interval $[a, a+h]$ such that $\varphi(x)=f(x)+(a+h-x)f'(x)-\frac{(a+h-x)^2}{\lfloor 2}f''(x)+$

$$\dots \dots +\frac{(a+h-x)^{n-1}}{(n-1)!}f^{n-1}(x)+(a+h-x)^m A$$

where A is a constant and $m>0$ ... (1)

We are to select A in such a way that

$$\varphi(a)=\varphi(a+h) \quad \dots \dots \quad (2)$$

Now

$$\varphi(a)=f(a)+hf'(a)+\frac{h^2}{\lfloor 2}f''(a)+\frac{h^3}{\lfloor 3}f'''(a)+ \dots \dots \dots +\frac{h^{n-1}}{\lfloor (n-1)}f^{n-1}(a)+h^m A \dots \dots (3)$$

$$\varphi(a+h)=f(a+h) \quad \dots \quad \dots \quad \dots \quad (4)$$

Putting (3) and (4) in (2),

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{\lfloor 2}f''(a)+\dots+\frac{h^{n-1}(a)}{\lfloor (n-1)}f^{n-1}(a) + h^m A \quad \dots \dots \quad (5)$$

Again

$$\varphi'(x)=f'(x)-f'(x)+(a+h-x)f''(x)-\frac{2}{\lfloor 2}(a+h-x)f''(x) + \frac{(a+h-x)^2}{\lfloor 2}f'''(x) \dots + \frac{(a+h-x)^{n-1}}{\lfloor (n-1)}f^n(x)-m(a+h-x)^{m-1}A$$

$$\text{or, } \varphi'(x)=\frac{(a+h-x)^{n-1}}{\lfloor (n-1)}f^n(x)-m(a+h-x)^{m-1}A \quad \dots\dots (6)$$

Since $\varphi(x)$ is the sum of $(n+1)$ continuous terms in the interval $[a, b]$. Therefore $\varphi(x)$ is continuous in $[a, b]$; as $f^n(x)$ and $(a+h-x)^{m-1}$ are defined for $a<x<a+h$, so $\varphi'(x)$ exists in the open interval (a, b). Also $\varphi(a)=\varphi(b$ Hence all the conditions of Rolle's theorem are satisfied and so there exists a point c such that

$$\varphi'(c)=0 \quad \text{where} \quad a<c<a+h$$

$$\text{or, } \varphi'(a+\theta h)=0, \quad c=a+\theta h \text{ and, } 0<\theta<1$$

Now from (6)

$$\varphi'(a+\theta h) = \frac{(a+h-a-\theta h)^{n-1}}{\lfloor (n-1)}f^n(a+\theta h) - m(a+h-a-\theta h)^{m-1} A=0$$

$$\text{or, } \frac{h^{n-1}(1-\theta)^{n-1}}{\lfloor (n-1)}f^n(a+\theta h)=mh^{m-1}(1-\theta h)^{m-1}A$$

$$\text{or, } A=\frac{(1-\theta)^{n-m}h^{n-m}}{\lfloor m(n-1)}f^n(a+\theta h)$$

Put the value of A in (5) then

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\dots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^m h^{n-m}(1-\theta)^{n-m}}{m(n-1)!}f^n(a+\theta h)$$

$$=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\quad\dots\quad\dots$$

$$+\frac{h^{n-1}f^{n-1}(a)}{(n-1)!}+\frac{h^n(1-\theta)^{n-m}}{m(n-1)!}f^n(a+\theta h)\dots\dots(7)$$

or, $f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\dots\dots$

$$+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+R_n\dots\dots(8)$$

where R_n is called the Remainder after n terms in the expansion of $f(a+h)$.

$$R_n=\frac{h^n(1-\theta)^{m-n}}{m(n-1)!}f^n(a+\theta h)\dots\dots\dots(9)$$

which is called **Schomilche and Roche's Remainder.**

When $n=m$ in [9]

$$R_n=\frac{h^n}{n(n-1)!}f^n(a+\theta h)=\frac{h^n}{n!}f^n(a+\theta h)\dots\dots(10)$$

which is called **Lagrange's Remainder.**

When $m=1$ in [9].

$$R_n=\frac{h^n(1-\theta)^{n-1}}{(n-1)!}f^n(a+\theta h)\dots\dots(11)$$

which is called **Cauchy's form of Remainder.**

Changing a to x we have, from (9), (10), (11) respectively.

(i) **Taylor's series with Schlomiche and Roche's remainder :**

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\dots\dots\frac{h^{n-1}}{(n-1)!}f^{n-1}(x)+$$

$$\frac{h^n(1-\theta)^{n-m}}{m(n-1)!}f^n(x+\theta h)\dots(12)$$

(ii) **Taylors Series with Lagranges Remainder**

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\dots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(x)+$$

$$\frac{h^n}{n!}f^n(x+\theta h)\cdots\dots(13)$$

(iii) **Taylors Series with Cauchy's Form of Remainder**

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\cdots\cdots$$

$$+\frac{h^{n-1}}{(n-1)!}f^{n-1}(x)+\frac{h^n(1-\theta)^{n-1}}{(n-1)!}f^n(x+\theta h)\dots\cdots(14)$$

Cor. In the Taylors Expansion with Lagranges **Remainder**, when $n=1$,

$$f(x+h)=f(x)+hf'(x+\theta h)\quad\dots\quad\dots\quad\dots\quad(15)$$

which is called **Lagrange's Mean value Theorem** of the First Mean Value Theorem, See Art. 7·5

If in the expansion, we have $n=2$, then

$$f(x+h)=f(x)+f'(x)+\frac{h^2}{2!}f''(x+\theta h)\quad\dots\quad\dots(16)$$

which is called the Second Mean Value Theorem. See Art. 7.7

7.10. Taylor Series (infinite form)

If $f(x)$ and $f^n(x)$ be finite and continuous in the interval $[a,a+h]$ for every positive integral value of n and if the remainder R_n tends to zero when n tends to infinity, *i. e.*, $\lim_{n\to 0} R_n=0$, then

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)\dots+\frac{h^n}{n!}f^n(a)+\dots\dots$$

If $a=x$, then the Taylor series becomes.

$$(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)\dots+\frac{h^nf^n(x)}{n!}+\cdots$$

The necessary and sufficient condition that $f(x+h)$ can be expanded in an infinite series is $\lim_{n \to \infty} R_n = 0$

R_n may be any one of the form (9), (10), (11) in Art. 7.9.

2.11. Failure of Taylor's theorem

In the following cases, Taylor's theorem fails.

(i) If $f(x)$ or one of its derivatives becomes undefined in the given interval.

(ii) If $f(x)$ or one of its derivatives becomes discontinuous in the same interval

(iii) If the remainder R_n cannot be made to vanish in the limit when n is taken sufficiently large so that R_n does not tend to any finite limit for, a given n.

7.12. Maclaurin's series with Remainder after n terms ;

If we put $x=0$ and $h=x$ in Taylor's seris (12)

$$f(x)=f(o)+xf'(o)+\frac{x^2}{\lfloor 2}f''(o)+\ldots+x^{n-1}\frac{f^{n-1}(o)}{\lfloor (n-1)}$$

$$+\frac{x^n(1-\theta)^{n-m}}{\lfloor m(n-1)}f^n(\theta h) \text{ where } 0<\theta<1 \ldots\ldots(17)$$

$$\text{or, } f(x)=f(o)+xf'(o)+\frac{x^2}{\lfloor 2}f''(o)\ldots\ldots+\frac{x^{n-1}f^{n-1}(o)}{\lfloor (n-1)}+R_n$$

$$\ldots \quad \ldots \quad (18)$$

where R_n is the remainder after n terms in the Maclaurins expansion. Now

$$\text{(i)} \quad R_n=\frac{x^n(1-\theta)^{n-m}}{\lfloor m(n-1)}f^n(\theta x)\ \ldots\ \ldots \quad (19)$$

is called Schlomilch and Roche form of Remainder.

(ii) when $n=m$ in (19) then

$$R_n=\frac{x^n}{\lfloor n(n-1)}f^n(\theta x)=\frac{x^n}{\lfloor n}f^n(\theta x)$$

$$\text{or, } R_n=\frac{x^n}{\lfloor n}f^n(\theta x) \quad \ldots \quad \ldots \ldots (20)$$

which is called Lagranges Remainder.

(iii) When $m=1$ in (19), then

$$R_n=\frac{x^n}{\lfloor n(n-1)}(1-\theta)^{n-1}f^n(\theta x)\ \ldots \quad \ldots \quad (21)$$

which is called Cauchys form of Remainder.

Cor. when $n=1$, the Maclaurin series with Lagranges Remainder is

$$f(x)=f(o)+xf'(\theta x) \quad \ldots \quad \ldots \quad \ldots (22)$$

when $n=2$ in the above expansion then

$$f(x)=f(o)+xf'(o)+\frac{x^2}{\lfloor 2}f''(\theta x)\ \ldots \quad \ldots (23)$$

7.13. Maclaurins Series (*infinite form*)

If $f(x)$ and $f^n(x)$ are continuous and limit at $x=0$ for every positive integral value of n and the remainder R_n tends to zero as n tends to infinity then,

$$f(x)=f(o)+xf'(o)+\frac{x^2}{\lfloor 2}f''(o)\ldots+\frac{x^n}{\lfloor n}f^n(o)+\ldots\ldots$$

Where R_n may be any one of the forms (17), (19), (20) of Art. 7.12

7.14. Failure of Maclaurin's theorem.

In the following cases Maclaurin's theorem fails.

(i) If any of the expressions $f'(o), f''(o)\ldots f^n(o)$ becomes undefined

(ii) If $f(x)$ or any one of its derivative is discontinuous at $x=0$

(iii) If $\lim_{n\to\infty} \frac{x^n}{\lfloor n} f^n(\theta x)\neq 0$ *i. e.,*

If R_n does not tend to a limit when n tends to infinity

7.15. Expansions

Students are already familiar with the expansions of given explicit functions in ascending powers of a variable. For example the expansions of $(a+x)^n$, $\log(1+x)$, e^{ax}, $\tan^{-1}x$ etc. are already known to them in Algebra and Trigonometry. These expansions are generally made with the following Principal Methods.

(i) By purely Algebaric and Trigonometric Methods See. Ex. 7

(ii) By Taylor's and Maclaurin's Theorems See Ex. 8, Ex. 2, Ex. 10.

(iii) By differentiation or, Integration of known series See Ex. 11. Ex. 12, Ex. 13.

(iv) By the use of differential equations. See Ex. 14.

The above methods will be explained with examples.

7. 16. Determination of the co-efficients in the expansion of $f(x)$ and $f(x+h)$.

Taylor's Theorem (infinite form)

(A) If $f(x+h)$ can be expanded in a convergent series of positive integral powers of h in an interval, prove that

$$(x+h)=f(x)+hf'(x)+\frac{h^2}{\lfloor 2} f''(x)+\text{to infinity}$$

$$\text{Let } f(x+h)=a_0+a_1h+a_2h^2+a_3h^3+ \quad \ldots \quad \ldots (1)$$

Where a_0, c_1, a_2, a_3 etc are all free from h, but are functions of x only.

Now differentiate (1) successively $w.\ r.$ to h, treating x as constants and hence a_0, a_1, a_2, etc are all constants.... ...

Then

$$f'(x+h)=a_1+2a_2h+3.a_3h^2+4.a_4h^3+ \quad \ldots \ldots (2)$$

$$f''(x+h)=2a_2+2.3a_3h+3.4a_4h^2+ \quad \ldots \ldots (3)$$

$$f'''(x+h)=3.2a_3+2.34a_4h+ \quad \ldots \ldots (4)$$

and so on

Putting $h=0$ in (1), (2), (3), (4), we have,

$f(x)=a_0, f'(x)=a_1, f''(x)=2.1\, a_2, f'''(x)=\lfloor 3 a_3$

or, $a_0=f(x), a_1=f'(x), a_2=f''(x)/2\,!$

$a_3=f'''(x)/3\,!$

Now from (1)

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{\lfloor 2} f''(x)+\frac{h^3}{\lfloor 3} f'''(x)+ + \frac{h^r}{\lfloor r} f^r(x) \quad \ldots \ldots (5)$$

$$\text{or, } f(x+h)=f(x)+h\frac{d}{dx}f(x)+\frac{h^2}{\lfloor 2}\frac{d^2}{dx^2}f(x) + \frac{h^3}{\lfloor 3}\frac{d^3}{dx^3}f(x)+$$

$$=\left\{1+h\frac{d}{dx}+\frac{h^2}{\lfloor 2}\left(\frac{d}{dx}\right)^2+\frac{h^3}{\lfloor 3}\left(\frac{d}{dc}\right)^3+\ldots\right\}f(x)$$

$$\Rightarrow f(x+h)=e^{hd/dx} f(x)$$

Note :

Put $x=a$, then we get a series of $f(a)$.

If $h=x-a$ in (5), then

We have,

$F(a)=0=F(b)$.

Then Rolle's theorem, there is a value of $x=\xi$, $a<\xi<b$ such that $F'(\xi)=0$.

$$\text{Now } F'(x)=\begin{vmatrix} f(a) & f'(x) \\ \varphi(a) & \varphi'(x)\end{vmatrix}-\frac{1}{(b-a)}\begin{vmatrix} f(a) & f(b) \\ \varphi(a) & \varphi(b)\end{vmatrix}$$

Hence

$$F'(\xi)=0=\begin{vmatrix} f'(a) & f'(\xi) \\ \varphi'(a) & \varphi'(a)\end{vmatrix}-\frac{1}{b-a}\begin{vmatrix} f(a) & f(b) \\ \varphi(a) & \varphi(b)\end{vmatrix}$$

$$\begin{vmatrix} f'(a) & f(b) \\ \varphi(a) & \varphi(b)\end{vmatrix}=(b-a)\begin{vmatrix} f(a) & f'(\xi) \\ \varphi(a) & \varphi'(\xi)\end{vmatrix}$$

Art 17.19. If $f(x)$, $\varphi(x)$, $\psi(x)$ have derivatives when $a\leqslant x\leqslant b$, show shat there is a value ξ of x lying between a and b such that

$$\begin{vmatrix} f(a) & \varphi(a) & \psi(a) \\ f(b) & \varphi(b) & \psi(b) \\ f'(\xi) & \varphi'(\xi) & \psi'(\xi)\end{vmatrix}=0$$

and deduce $\dfrac{f(b)-f(a)}{\varphi(b)-\varphi(a)}=\dfrac{f'(\xi)}{\varphi'(\xi)}$

Art. 17. 20. If α, β lie between the least and the greatest of the numbers a, b and c; show that

$$\begin{vmatrix} f(a) & f(b) & f(c) \\ \varphi(a) & \varphi(b) & \varphi(c) \\ \psi(a) & \psi(b) & \psi(c)\end{vmatrix}=k\begin{vmatrix} f(a) & f'(\alpha) & f''(\beta) \\ \phi(a) & \varphi'(\alpha) & \varphi''(a) \\ \psi(a) & \psi'(\alpha) & \psi''(\beta)\end{vmatrix}$$

Where $k=\frac{1}{2}(b-c)(c-a)(a-b)$

Sol Let $f(x)$, $\varphi(x)$, $\psi(x)$ be continuous in the closed interval (a, c) and twice differentiable in the open interval (a, c) $a<b<c$.

Let us consider the function $F(x)=$

$$\begin{vmatrix} f(a) & f(b) & f(x) \\ \varphi(a) & \varphi(b) & \varphi(x) \\ \psi(a) & \psi(b) & \psi(x)\end{vmatrix}-\frac{(x-a)(x-b)}{(c-a)(c-b)}\begin{vmatrix} f(a) & f(b) & f(c) \\ \varphi(a) & \varphi(b) & \varphi(c) \\ \psi(a) & \psi(b) & \psi(c)\end{vmatrix}$$

Here $F(a)=F(b)=F(c)=0$. By Rolle's theorem

$$F'(x_1)=F'(x_2)=0,\ a<x_1<b,\ b<x_2<c$$

Using Rolle's Theorem once again on $F'(x)$,

$F''(\beta)=0$ where $x_1<\beta<x_2$

$$\therefore \begin{vmatrix} f(a) & f(b) & f''(\beta) \\ \varphi(a) & \varphi(b) & \varphi''(\beta) \\ \psi(a) & \psi(b) & \psi''(\beta)\end{vmatrix}=\frac{2}{(c-a)(c-b)}\begin{vmatrix} f(a) & f(b) & f(c) \\ \varphi(b) & \varphi(b) & \varphi(c) \\ \psi(a) & \psi(b) & \psi(c)\end{vmatrix} \quad \ldots (1)$$

Again consider the function

$$F_1(x)=\begin{vmatrix} f(a) & f(x) & f''(\beta) \\ \varphi(a) & \varphi(x) & \varphi''(\beta) \\ \psi(a) & \psi(x) & \psi''(\beta)\end{vmatrix}-\frac{2(x-a)}{(c-a)(c-b)(b-a)}\begin{vmatrix} f(a) & f(b) & f(c) \\ \varphi(a) & \varphi(b) & \varphi(c) \\ \psi(a) & \psi(b) & \psi(c)\end{vmatrix}$$

$F_1(a)=0$ from (2) $F_1(b)=0$ [by (1)] ......(2)

From Rolle's Theorem, $F_1'(\alpha)=0$, $a<\alpha<b$

$$\text{or, } \begin{vmatrix} f(a) & f'(\alpha) & f''(\beta) \\ \varphi(a) & \varphi'(\alpha) & \varphi''(\beta) \\ \psi(a) & \psi'(\alpha) & \psi''(\beta)\end{vmatrix}=\frac{2}{(c-a)(c-b)(b-a)}\begin{vmatrix} f(a) & f(b) & f(c) \\ \varphi(a) & \varphi(b) & \varphi(c) \\ \psi(a) & \psi(b) & \psi(c)\end{vmatrix}$$

Art 7. 21. If $\varphi(x)=f(x)+f(1-x)$ and $f''(x)>0$ in $0\leqslant x\leqslant 1$, Show that $\varphi(x)$ increases in

$0\leqslant x\leqslant\frac{1}{2}$ and decreases in $\frac{1}{2}\leqslant x\leqslant 1$

and then prove that

$$\pi<\frac{\sin \pi x}{x(1-x)}<4,\ 0<x<1.$$

Given
$$\left.\begin{aligned} \varphi(x)&=f(x)+f(1-x) \\ \varphi'(x)&=f'(x)-f'(1-x) \\ \varphi''(x)&=f''(x)+f''(1-x)\end{aligned}\right\} \quad \ldots\ \ldots (1)$$

If x varies from 0 to 1, then $1-x$ varies from 1 to 0, so $f''(1-x)$ is also negative as $f''(x)$ is negative so $\varphi''(x)$ is negative throughout the interval (0, 1) by hypothesis.

$\therefore$ $\varphi''(x)$ is monotonically decreasing in (0, 1)

Again $\varphi'(0)=f'(0)-f'(1)$, $\varphi'(\frac{1}{2})=f'(\frac{1}{2})-f'(\frac{1}{2})=0$

$$\varphi'(1)=f'(1)-f'(0) \quad \ldots \quad \ldots \quad (2)$$

Since $f''(x)<0$ throughout the interval $0\leqslant x\leqslant 1$ and on that account $f'(x)$ is monotone decreasing in (0, 1), then $f'(0)>f'(1)$

So $\varphi'(x)$ is positive in $(0, \frac{1}{2})$ and negative in $(\frac{1}{2}, 1)$ by (2)

$\therefore$ $\varphi(x)$ is monotone increasing in $(0, \frac{1}{2})$ and monotone decreasing in $(\frac{1}{2}, 1)$

Let $\varphi(x)=\dfrac{\sin \pi x}{x(1-x)}=\dfrac{2\sin \pi x/2 \sin \pi(1-x)/2}{x(1-x)}$ ... ...(3)

$\therefore$ $\log\frac{1}{2}\varphi(x)=\log\dfrac{\sin \pi x/2}{x}+\log\dfrac{\sin \pi(1-x)/2}{1-x}$

$$=f(x)+f(1-x)$$

where $f(x)=\log\dfrac{\sin \pi x/2}{x}$

But $f'(x)=\dfrac{\cos \pi x/2}{\sin \pi x/2}\dfrac{\pi}{2}-\dfrac{1}{x}$

$$f''(x)=-\frac{\pi^2}{4}\operatorname{cosec}^2 \pi x/2+\frac{1}{x^2}$$

$$=\frac{(-\pi^2x^2)/4+\sin^2 \pi x/2}{x^2 \sin^2(\pi x/2)}<0,\ 0<x<1$$

$\therefore$ $\log\dfrac{\phi(x)}{2}$ or, $\phi(x)$ increases in $(0, \frac{1}{2})$

and decreases in $(\frac{1}{2}, 1)$

As $\phi(x)=\dfrac{\sin \pi x}{x(1-x)}$ ... ... (4)

As $x\to+0+$, $\phi(x)\to\pi$

As $x\to 1=0$, $\phi(x)\to\pi$ as $\sin \pi(1-h)\phi=\sin \pi h$

Also $x=\frac{1}{2}$, $\phi(x)=4$, from (4)

$\therefore$ $\pi<\phi(x)<4$ in $(0, \frac{1}{2})$

and $4<\phi(x)<\pi$ in $(\frac{1}{2}, 1)$

Hence $\pi<\dfrac{\sin \pi x}{x(1-x)}\leqslant 4$

Art. 7. 22. A Function is twice differentiable and satisfies the inequalities.

$|f(x)|<A$, $|f''(x)|<B$, in the range $x>a$;

Where A and B are constants Prove that $|f(x)|<2\sqrt{(AB)}$.

Ans. For positive number h, and $x>a$;

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{\lfloor 2}f''(x+\theta h),\ 0<\theta<1$$

$$\therefore\ |hf'(x)|=\left|f(x+h)-f(x)-\frac{h^2}{\lfloor 2}f''(x+\theta h)\right|$$

$$\leqslant |f(x+h)|+|f(x)|+\left|\frac{h^2}{\lfloor 2}f''(x+\theta h)\right|$$

$$\leqslant A+A+Bh^2/2$$

or, $|f'(x)|<\dfrac{2A}{h}+B\,h/2$; h is $+$ve

Thus $|f'(x)|$ must be less than the least value of $(2A/h+Bh/2)$

Thus $(2A/h+Bh/2)=\{\sqrt{(2A/h)}-\sqrt{(Bh/2)}\}^2+2\sqrt{(AB)}\geqslant 2\sqrt{AB}$

Square of a quantity is positive.

$(\because\ [\sqrt{(2A/h)}-\sqrt{2Bh/2)}]^2>0)$

Hence $|f(x)|<(2A/h+Bh/2)$ then

$\Rightarrow |f'(x)|<2\sqrt{(AB)}$

Examples

Ex. 1. Verify the truth of Rolle's theorem for the function $f(x)=x^2-3x+2$ in the interval (1, 2).

when $x=1$, then $f(1)=1-3+2=0$

when $x=2$, then $f(2)=4-6+2=0$

$\therefore\ f(1)=f(2)$.

Now $f'(x)=2x-3$

If $f'(x)=0$ then $2x-3=0$ or, $x=3/2=1.5$.

Thus we see that $f(x)$ is continuous in $1\leqslant x\leqslant 2$.

$f'(x)$ exists in $1<x<2$

and $f(1)=f(2)$ There exists a point $x=1.5$.

within the interval (1, 2) such that, $1<1.5<2$

where $f'(x)=0$ i e., $f(1.5)=0$

Hence Rolle's theorem is verified.

Ex. 2. Verify Rolle's theorem for the function.

$f(x)=x^3-x^2-4x+4$

$f(x)=x^3-x^2-4x+4$

If $f(x)=0$, then $x^3-x^2+4x+4=0$ or, $(x-1)(x+2)(x-2)=0$

$\therefore\ x=1, 2, -2$

So $f(x)=0$ for $x=1$, $x=2$, and $x=-2$

i. e. $f(1)=0$ $f(2)=0$, $f(-2)=0$

Now $Rf'(x)=\lim\limits_{h\to 0+}\dfrac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{(x+h)^3-(x+h)^2-4(x+h)+4-x^3+x^2+4x-4}{h}$$

$=3x^2-2x-4$

$Lf'(x)=\lim\limits_{h\to 0}\dfrac{f(x-h)-f(x)}{h}=3x^2-2x-4,\quad (h>0)$

Thus $f'(x)$ exists as a finite differential co-efficient in the interval $-2<x<2$.

$\therefore\ f'(x)=3x^2-2x-4$

$f(x)$ is continuous in the interval $-2\leqslant x\leqslant 2$.

Now $f(x)$ is also continuous in $-2\leqslant x\leqslant 1$ and $1\leqslant x\leqslant 2$

$f(x)$ is differentiable in $2<x<1$ and $1<x<2$

Also $f(-2)=f(1)=f(2)=0$

Hence all conditions of Rolle's theorem are satisfied.

Therefore $f'(x)=0$ at a point in (—2, 1) and also at a point in (1, 2)

Now $f'(x)=0\Rightarrow 3x^2-2x-4=0$ giving

$$x=\frac{2\pm\sqrt{(4+4.4.3)}}{6}=\frac{2\pm 2\sqrt{(13)}}{6}=\frac{1\pm 3.6}{2}=0.87,\ 1.55$$

Thus $f'(x)=0$ for $x=-0.87$ which lies between -2 and 1 and $f'(x)=0$ again for $x=1.55$ which lies between 1 and 2,

Hence Rolle's theorem is verified.

Ex 3. Verify the truth of Rolle's theorem for the function.

$f(x)=1-\sqrt[5]{x^2}$

The function $f(x)$ is continuous in the interval $-1\leqslant x\leqslant 1$.

$f(1)=0$ and $f(-1)=0$

Again $f'(x)=-\dfrac{2}{5}x^{-3/5}=-\dfrac{2}{5}\,\dfrac{1}{\sqrt[5]{x^3}}$

But $f'(x)$ does not vanish in the interval $-1<x<1$, besides $f'(x)$ does not exist at $x=0$

Hence Rolle's Theorem does not hold good.

Ex. 4. Verify Mean Value Theorem for the function.

$f(x)=2x-x^2$ in the interval (0, 1).

The function $f(x)=2x-x^2$ ... ... [1]

is continuous in th: intervl [0,1] and differentiable for all values of x in the interval (a, b),

$f'(x) = 2-2x$ ... ... (2)

Clearly $f'(x)$ is continuous for $0<x<1$

By the Mean value theorem, we have

$f(1)-f(0)=(1-0)f'(c)$ where c is a point such that $0<c<1$

or, $(2-1)-0=f'(c)$ or, $f'(c)=1$

or, $2-2c=1$ [from (2)] or. $c=\frac{1}{2}$

Since $0<\frac{1}{2}<1$,

Hence the Mean value Theorem verified.

Ex. 5. At what point is the tangent by the curve $y=x^3$ parallel to the chord joining the points (1,1) to (2, 8,)

we have $f(x)=x^3$ ... ... (1)

The function $f(x)$ is continuous in [1,2] and differentiable in the interval (1, 2).

Now $f'(x)=3x^2$ ... ... (2)

exists at every point in the interval (1, 2)

By Mean value theorem, we have

$f(2)-f(1)=(2-1)f'(c)$ we have

c is a point in $1<c<2$,

or, $8-1=3c^2$ or, $c=1.15$

Now $f(1.15)=(1.15)^3=2.68$

The tangent at (1.15 ; 2.68) is parallel to the chord passing through the points (1,1) and (2,8)

Ex. 5. (a) show that $(x-\sin x)$ is a steadily increasing function in $0\leqslant x\leqslant\frac{1}{2}\pi$.

Let $y=x-\sin x$ ... ... (1)

$\therefore$ $dy/dx=f'(x)=1-\cos x$ ... (2)

Now for values of x in $0\leqslant x\leqslant\frac{1}{2}\pi$, the values of $\cos x$ change from 1 to 0 i.e,. $0\leqslant\cos x\leqslant 1$.

Therefore $1-\cos x$ is always positive

Hence $f'(x)$ is always positive i. e., $f'(x)>0$ for $0\leqslant x\leqslant\pi/2$

Then by cor Art. 7.5 we see that $f(x)$ is a steadily (monotonically) increasing function of x in the given interval.

Ex. 6. If $f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x+\theta h)$

where $f(x)=(x-a)^{5/2}$ then

show that $\theta = \frac{64}{225}$ for $x=a$

$f(x+h)=f(x)+hf'(x)+\frac{h}{2!}f''(x+\theta h)$ ... ... (1)

and $f(x)=(x-a)^{5/2}$ ... ... (2)

$\therefore$ $f'(x)=5(x-a)^{3/2}/2$

$f''(x)=(5/2)(3/2)\sqrt{(x-a)}=15\sqrt{(x-a)}/4$

when $x=a$, then $f(a)=0$, $f(a+h)=h^{5/2}$

$f''(a+\theta h)=15\sqrt{(\theta h)}/4$; $f'(a)=0$

In (1), put $x=a$; then

$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a+\theta h)$

$\therefore$ $(h)^{5/2}=\frac{h^2}{2!}\cdot\frac{15}{4}\theta^{1/2}h^{1/2}$ or, $\sqrt{\theta}=8/15$ or, $\theta=64/225$

Ex. 7. Expand $x\operatorname{cosec}x$ in a series of ascending powers up to the fourth power of x inclusive. D. U. 1966

$\frac{x}{\sin x}$ is not defiened at $x=0$, Hence

$\left(\frac{x}{\sin x}\right)$ does not possess a Taylor siries expansion.

However, $\left(\dfrac{x}{\sin x}\right)\to 1$ as $x\to 0$, Hence a sine expansion of $\left(\dfrac{x}{\sin x}\right)$ can be obtained by Binomial theorem :

$$\frac{x}{\sin x}=\frac{x}{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}\cdots\cdots}$$

$$=\left\{1-\left(\frac{x^2}{3!}-\frac{x^4}{5!}+\frac{x^6}{7!}\cdots\cdots\right)\right\}^{-1}$$

$$=1+\left(\frac{x^2}{3!}-\frac{x^4}{5!}+\cdots\cdots\right)+\left(\frac{x^2}{3!}-\frac{x^4}{5!}+\cdots\cdots\right)$$

$$+\ \cdots\ \cdots\ \cdots$$

$$=1+\frac{x^2}{6}-\frac{x^4}{120}+\frac{x^4}{30}+\text{terms}$$

containing x^5 and higher powers of x

or, $x\operatorname{cosec}x=1+\dfrac{x^2}{6}+\dfrac{7x^4}{360}+\cdots\cdots\cdots$

Ex. 8. Expand $e^x\cos x$ in a finite series in powers of x with Lagrange's form of Remainder. R.U. 1950

Let $f(x)=e^x\cos x=f(o)+xf'(o)+\dfrac{x^2}{2!}f''(o)+\cdots\cdots$

$$+\frac{x^n}{n!}f^n(\theta x)\quad\cdots\cdots\cdots\quad(1)$$

$f(x)=e^x\cos x\quad\cdots\cdots\cdots\quad(2)$

$f^n(x)=2^{n/2}\,e^x\cos\left(x+\tfrac{1}{4}n\pi\right)\cdots\quad\cdots\quad(3)$

$f^n(\theta x)=2^{n/2}\,e^{\theta x}\cos\left(\theta x+\tfrac{1}{4}n\pi\right)$

From [1] and [2] when $x=0$

$f(o)=1,\ f'(o)=\sqrt{2}\cos\pi/4=1$

$f''(o)=2\cos 2\pi/4=0.\quad f'''(o)=-2^{3/2}\dfrac{1}{\sqrt{2}}=-2$

$f^{iv}(o)=2^{4/2}\cos\pi=-4$ and so on.

Now from (1), we have

$e^x\cos x=1+x-\dfrac{x^3}{3}-\dfrac{x^6}{6}\cdots-\dfrac{x^n}{n!}f^n(\theta x)$ where $0<\theta<1$

Lagrange's Remainder is $\dfrac{x^n}{n!}f^n(\theta x)=\dfrac{x^n}{n!}2^{n/2}e^{\theta x}\cos\left(\tfrac{1}{2}n\pi+\theta x\right)$

Ex.9. Expand $\log(1+x)$ in powers of x by Maclaurin's Theorem R. U. 1962

Let $f(x)=\log(1+x)=f(o)+xf'(o)+\dfrac{x^2}{2!}f''(o)\cdots\cdots$

$$+\frac{x^n}{n!}f^n(\theta x),\ 0<\theta<1$$

Here $f(x)=\log(1+x)$, $f'(x)=\dfrac{1}{1+x}=(1+x)^{-1}$,

$f^n(x)=\dfrac{(-1)^{n-1}(n-1)!}{(1+x)^n}$ which exists for all values of n for $x\neq-1$

$f^n(o)=(-1)^{n-1}(n-1)!$ when $x=0$

Therefore $f(o)=0,\ f'(o)=1,\ f''(o)=-1$

$f'''(o)=2,\ f^{iv}(o)=-3$, and so on

Now $R_n=\dfrac{x^n}{n!}f^n(\theta x)=\dfrac{x^n}{n!}(-1)^{n-1}(n-1)!\ \dfrac{1}{(1+\theta x)^n}$

$$=(-1)^{n-1}\frac{1}{n}\left(\frac{x}{1+\theta x}\right)^n$$

$\therefore\ \log(1+x)=x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\cdots+(-1)^{n-1}\dfrac{1}{n}\left(\dfrac{x}{1+\theta x}\right)^n$

(i) If $0 < x < 1$ then

$$\lim_{n \to \infty} \left(\frac{x}{1+\theta x}\right)^n = 0 \text{ as } \left|\frac{x}{1+\theta x}\right| < 1$$

Hence $R_n = 0$ if $n \to \infty$;

(ii) If $-1 < x < 0$. in this case $\frac{x}{1+\theta x}$ may not be numerically less than 1

So $\lim_{n \to \infty} \left(\frac{x}{1+\theta x}\right)^n \neq 0$

Thus we fail to get the definite value of R_n from Lagrange's form of Remainder.

From Cauchy's form of Remainder.

$$R_n = \frac{x^n}{\lfloor(n-1)}(1-\theta)^{n-1} f^n(\theta x)$$

$$= \frac{x^n}{\lfloor(n-1)}(1-\theta)^{n-1} \lfloor(n-1) \frac{1}{(1+\theta x)^n}$$

$$= (-1)^{n-1} \frac{x^n}{1+\theta x}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}$$

As $\frac{1-\theta}{1+\theta x} < 1$ then $\lim_{n \to \infty} \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} = 0$

So $R_n = 0$ if $n \to \infty$ for $-1 < x < 1$

Hence the expansion of

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \dots (-1)^{n-1} \frac{x^n}{n} + \dots$$

Ex. 10. Use Maclaurin's theorem to expand $\tan^{-1}x$ into an infinite series of ascending powers of x. [R U. 58, 65]

Let $f(x) = \tan^{-1}x = f(o) + xf'(o) + \frac{x^2}{2!}f''(o) + \frac{x^3 f'''(o)}{3!} + \dots$ (1)

Here $f(x) = \tan^{-1}x$, $f'(x) = \frac{1}{1+x^2}$

$f^n(x) = (-1)^{n-1}(n-1)!\, \sin^n\phi \sin n\phi$ see Art. 6.6

where $\tan\phi = \frac{1}{x}$. $\sin\phi = \frac{1}{\sqrt{(1+x^2)}}$

$$\therefore f^n(o) = (-1)^{n-1}(n-1)!\left(\frac{1}{1+x^2}\right)^{n/2} \sin\left(n \sin^{-1}\frac{1}{\sqrt{(1+x^2)}}\right)$$

$$f^n(o) = (-1)^{n-1}(n-1)!\, \sin(n\pi/2)$$

$\therefore f(o) = 0, f'(o) = 1, f''(o) = 0, f'''(o) = -(2!) = -2!$

$f^{iv}(o) = 0, f^{v}(o) = 4!$ and so on

Now from (1)

$\tan^{-1}x = x - \frac{1}{3}x^3 + x^5/5$

We may get the expansion of $\tan^{-1} x$ in an ascending powers of x by integrating a known series. (See below)

Ex. 11. Expand $\tan^{-1}x$ in an ascending powers of x by integrating a series.

Let $y = \tan^{-1}x$.

$$\therefore \frac{dy}{dx} = \frac{1}{1+x^2} = (1+x^2)^{-1}$$

or, $\frac{dy}{dx} = 1 - x^2 + x^4 - x^6 + x^8 - \dots + (-1)^n x^{2n} + \dots [\dots \text{if } |x^2| < 1)$

or, $dy = \{1 - x^2 + x^4 - x^6 + x^8 - \dots + (-1)^n x^{2n} + \dots\dots\} dx$

Now integrate both sides.

$$y = x - \frac{1}{3}x^3 + x^5/5 - x^7/7 + \dots + \frac{(-1)^n x^{2n+1}}{2n+1} + \dots\dots$$

(The constant of integration is zero since $\tan^{-1}0 = 0$)

or, $\tan^{-1} x = x - \frac{1}{3}x^3 + x^5/5 - \dots + \frac{(-1)^n x^{2n+1}}{2n+1} + \dots \dots$

Ex. 12. If the expansion of

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots \dots$$

Show that the expansion of

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \dots$$

Since

$\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + x^8/8! - \dots \dots$

Integrating both sides w. r to x.

$$\sin x = x - \frac{x^3}{2!\,3} + \frac{x^5}{5\,4!} - \frac{x^7}{7.6!} + \dots \dots$$

or, $\sin x = x - x^3/3! + x^5/5! - x^7/7! + \dots \dots$

[for all values of x, the series is convergent]

Ex. 13. Find the expansion of $\tan x$ if

$$\log\sec x = \frac{x^2}{2!} + \frac{x^4}{4!} + 16\frac{x^6}{6!} + \dots \dots$$

$$\text{Since } \log\sec x = \frac{x^2}{2!} + \frac{2x^4}{4!} + 16\frac{x^6}{6!} + \dots \dots$$

Differentiate both sides w. r. to x then.

$$\frac{\sec x\tan x}{\sec x} = \frac{2x}{2!} + \frac{2\,4x^3}{4!} + 16\,6\frac{x^5}{6!} + \dots \dots$$

or, $\tan x = x + \frac{1}{3}x^3 + (2/15)x^5 + \dots\dots$

Ex. 14. If $y = e^{\tan^{-1}x} = a_0 + a_1x + a_2x^2 + \dots\dots + a_nx^n + \dots\dots$

Show that (i) $(n+2)a_{n+2} + na_n = a_{n+1}$

(ii) $e^{\tan^{-1}x} = 1 + x + \frac{1}{2}x^2 - \frac{1}{6}x^3 \dots\dots$

Since

$$y_1 = e^{\tan^{-1}x} = a_0 + a_1x + a_2x^2 + \dots\dots + a_nx^n + a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \dots \dots \quad (1)$$

$$\therefore\ y_1 = e^{\tan^{-1}x}\frac{1}{1+x^2} = a_1 + 2a_2x + 3a_3x^2 + \dots\dots + na_nx^{n-1} + (n+1)a_{n+1}x^n + (n+2)a_{n+2}x^{n+1} + \dots\dots (2)$$

$$\therefore\ y_1' = 2a_2 + 6a_3x + \dots + n(n-1)a_nx^{n-2} + (n+1)na_{n+1}x^{n-1} + (n+1)(n+2)a_{n+2}x^n + \dots\dots (3)$$

From (2).

$$y_1(1+x^2) = e^{\tan^{-1}x} = y$$

Differentiating

$y_2(1+x^2) + 2xy_1 = y_1$ or, $y_2(1+x^2) + (2x-1)y_1 = 0 \dots\dots (4)$

Put the values of y_1 and y from (2) and (3) in (4) then

$(1+x)^2\{2a_2 + \dots\dots n(n-1)a_nx^{n-1} + n(n+1)a_{n+1}x^{n+1} +$

$(n+1)(n+2)(a_{n+2}x^n) + \dots\dots\} + (2x-1)\{a_1 + \dots\dots$

$+ na_nx^{n-1} + (n+1)a_{n+1}x^n + (n+2)a_{n+2}x^{n+1} + \dots\dots\} = 0$

Equating the co-efficients of x^n from both sides,

$(n+1)(n+2)a_{n+2} + n(n-1)a_n + 2na_n - (n+1)a_{n+1} = 0$

or, $(n+1)(n+2)a_{n+2} + n(n+1)a_n = (n+1)a_{n+1}$

or, $(n+2)a_{n+2} + na_n = a_{n+1}$ ... ... (5)

when $x = 0$ in (1), and (2),

$1 = a_0, 1 = a_1$

Putting $n = 0, 1, 2, \dots$ ...successively in (5),

$2a_2 = a_1$ or, $2a_2 = 1$ or, $a_2 = \frac{1}{2}$

$3a_3 + 1a_1 = a_2$ or, $3a_3 + 1 = \frac{1}{2}$ or, $a_3 = -\frac{1}{6}$

Continuting this way we may get as many terms as we like

Thus

$e^{\tan^{-1}x} = 1 + x + \frac{1}{2}x^2 - \frac{1}{6}x^3 \dots \dots$

Ex. 15. If $\varphi(x)$ is continuous for $a \leq x \leq b$ and $\varphi''(x)$ exists and positive in $a < x < b$ then $\dfrac{\varphi(x)-\varphi(a)}{x-a}$ increases steadily and strictly for $a < x < b$.

Hints. $f(x)=\dfrac{\varphi(x)-\varphi(a)}{x-a}$

Since $\varphi''(x)$ exists obviously as a finite differential co efficient $\varphi'(x)$ is continuous and therefore finite in $a<x<b$. Also $\varphi(x)$ is given as a continuous function. Hence by the second mean value theorem.

$$f(x)=\frac{\varphi(x)-\varphi(a)}{x-a}=\varphi'(a)+\frac{x-a}{\lfloor 2}\varphi''(\xi)$$

where $a<x<b$ and $a<\xi<b$

Hence $f'(x)=\dfrac{\varphi''(\xi)}{\lfloor 2}$ = a positive quantity by hypothesis throughout the interval $a<x<b$.

Now $f(x)=\dfrac{\varphi'(x)-\varphi(a)}{x-a}$ increases steadily and strictly throughout the same interval $a<x<b$.

Ex. 16 If $\varphi(x)=\begin{vmatrix} f(a) & f(b) & f(x) \\ g(a) & g(b) & g(x) \\ h(a) & h(b) & h(x) \end{vmatrix}$

Where $a<c<b$ and $f(x)$, $g(x)$, $h(x)$ are continuous in the closed interval $(a\ b)$ and twice differentiable in the open interval (a,b) Prove that there exists a number ξ such that $a<\xi<b$ and $\varphi(c)=\frac{1}{2}(c-a)(c-b)\varphi''(\xi)$

Let $\psi(x)=\begin{vmatrix} f(a) & f(b) & f(x) \\ g(a) & g(b) & g(x) \\ h(a) & h(b) & h(x) \end{vmatrix}-\dfrac{(x-a)(x-b)}{(c-a)(c-b)}\begin{vmatrix} f(a) & f(b) & f(c) \\ g(a) & g(b) & g(c) \\ h(a) & h(b) & h(c) \end{vmatrix}$

Here $\psi(a)=\psi(c)=\psi(b)=0$

By Rolle's Theorem

$\psi'(\xi_1)=0,\ a<\xi_1<c$

$\psi'(\xi_2)=0,\ c<\xi_2<b$

Now for (1)

$$\psi(x)=\varphi(x)-\frac{(x-a)(x-b)}{(c-a)(c-b)}\varphi(c)$$

$$\because\quad \psi'(x)=\varphi'(x)-\left\{\frac{(x-a)}{(c-a)(c-b)}+\frac{x-b}{(c-a)(c-b)}\right\}\varphi(c)$$

and $\psi''(x)=\varphi''(x)-\dfrac{2}{(c-a)(c-b)}\varphi(c)$ ... ... (2)

As $\psi'(\xi_1)=\psi'(\xi_2)=0$, then by Rolle's theorem, $\psi''(\xi)=0,\ \xi_1<\xi<\xi_2$

or, $\varphi''(\xi)-\dfrac{2}{(c-a)(c-b)}\varphi(c)=0$

or, $\varphi(c)=\frac{1}{2}(c-a)(c-b)\varphi''(\xi)$

Ex. 17. Show that

$f(b)-f(a)=\xi f'(\xi)\log(b/a)$, where $f(x)$ is continuous and differentiable in (a,b) and $a<\xi<b$.

From Cauchy's Mean Value Theorem

$$\frac{f'(\xi)}{g'(\xi)}=\frac{f(b)-f(a)}{g(b)-g(a)}\quad ...\ ...\ (1)$$

If $g(x)=\log x$, $g'(x)=1/x$ i.e. $g'(\xi)=1/\xi$.

putting $g'(\xi)=1/\xi$ in (1)

$$\frac{f'(\xi)}{1/\xi}=\frac{f(b)-f(a)}{\log(b)-\log(a)}=\frac{f(b)-f(a)}{\log(b/a)}$$

or, $f(b)-f(a)=\xi f'(\xi)\log(b/a)$

Ex. 18. If $y=\sqrt{(1-x^2)}\sin^{-1}x$, then prove that

$(1-x^2)y_n=(2n-3)xy_{n-1}+(n-1)(n-3)y_{n-2}$

and find the co-efficient x^7 in the expansion. C. H, 1986

Sol. $y=\sqrt{(1-x^2)}\sin^{-1}x$ ... ... (1)

$$y_1=\sqrt{(1-x^2)}\times\frac{1}{\sqrt{(1-x^2)}}+\frac{\sin^{-1}x(-2x)}{2\sqrt{(1-x^2)}}$$

$$=1-\frac{x\sqrt{(1-x^2)}\ \sin^{-1}x}{(1-x^2)}$$

or $y_1(1-x^2)=(1-x^2)-xy$ [by (1)] ... (2)

Differentiating this,

$y_2(1-x^2)-2xy_1=-2x-xy_1-y$

or, $y_2(1-x^2)=-2x+xy_1-y$ ... ... (3)

$\therefore\ y_3(1-x^2)-2xy_2=-2+xy_2+y_1-y_1$

or $y_3(1-x^2)=-2+3xy_2$ ... ... (4)

Differentiating both sides of (4) $(n-3)$ times $w.\ r.$ to x,

$$y_n(1-x^2))+\frac{(n-3)}{1}y_{n-1}(-2x)+\frac{(n-3)(n-4)}{1.2}y_{n-2}(-2)$$

$$=3xy_{n-1}+3\ \frac{(n-3)}{1}y_{n-2}\quad (1)$$

$\Rightarrow(1-x^2)y_n=(2n-3)\ xy_{n-1}+(n-1)(n-3)y_{n-2}$...(5) (proved)

where n -3< or $n\geqslant4$

From (1)—(4)

$y(o)=0,\ y_1(o)=1,\ y_2(o)=-y(o)=0,\ y_3(o)=-2.$ ...(6)

Now from (5), putting $n=4,5,6,7$ successively,

$y_4(o)=(3)(1)y_2(o)=0$

$y_5(o)=(4)\ (2)\ y_3(o)=8(-2)=-16$

$y_6(o)=(5)(\ \ )\ y_4(o)=0$

$y_7(o)=(6)(4)y_5(o)=(24)(-16)=-384,$

Hence the coefficient of x^7 in the expansion of $\sqrt{1-x^2}\ \sin^{-1}x$ is

$$\frac{y_7(o)}{7\,!}=\frac{-384}{7\,!}=-\frac{8}{105}\quad \text{[by Maclaurin's theorem]}$$

Ex. 19. Expand $2x^3+7x^2+x-1$ in powers of $(x-2)$

We know that (ii) log x in powers of (x-2) C.U. 1992

$$f(x)=f(a)+(x-a)\ f'(a)+\frac{(x-a)^2}{2\,!}f''(a)+\frac{(x-a)^3}{3\,!}f'''(a)+\ldots$$

Here $f(x)=2x^3+7x^2+x+1$ and $a=2$

$\therefore\ f(a)=f(2)=2\ 2^3+7.2^2+2-1=45$

$f'(x)=6x^2+14x+1$, $f'(a)=f'(2)=6.2^2+14.2+1=53$

$f''(x)=12x+14x$; $f''(a)=f''(2)=12.2+14=38$

$f'''(x)=12$; $f'''(a)=f'''(2)=12$

Hence $2x^3+7x^2+x-1$

$$=45+53(x-2)+\frac{38}{2\,!}(x-2)^2+\frac{12}{3\,!}(x-2)^3$$

$$=45+53(x-2)+19(x-2)^2+2(x-2)^3$$

Ex. 20. Expand $\sin x$ in powers of $x-\frac{1}{2}\pi$.

We have

$f(x)=f(\frac{1}{2}\pi+x-\frac{1}{2}\pi)$ and $a=\frac{1}{2}\pi$

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{\underline{|2}}f''(a)+\ldots$$

$$\Rightarrow f(x)=\sin x=1-\frac{1}{2\,!}(x-\tfrac{1}{2}\pi)^2+\frac{1}{4\,!}(x-\tfrac{1}{2}\pi)^4+\ldots$$

Ex. 21. $a=-1$ $b>1$ and $f(x)=\dfrac{2}{|x|}$, show that the conditions of Lagrange's mean value theorem are not satisfied in the interval (a,b), but the conclusion of the theorem is true, if and only if $b>1+\sqrt{2}$.

Sol : For h +ve,

$$f(0+h)=\frac{1}{|0+h|}=\frac{1}{h}\ ;\ f(0-h)=\frac{1}{|0-h|}=\frac{1}{h}\ldots\ldots(1)$$

$f(0)$ is not defined. Let $f(0)=A$, according to the definition of the function A is definite finite number.

Hence, $\lim_{h\to 0}\dfrac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\dfrac{1/h-A}{h}=\lim_{h\to 0}\dfrac{1}{h}$

$$\times \lim_{h\to 0}\left(\frac{1}{h}-A\right)=\infty\times\infty=\infty \quad \ldots \quad \ldots \quad (2)$$

Aso $\lim \dfrac{(0-h)-f(0)}{-h}=\lim \dfrac{1/h-A}{-h}=\lim_{h\to 0}\dfrac{1}{-h}\times\lim_{h\to 0}\left(\dfrac{1}{h}-A\right)$

$$=-\infty\times\infty=\infty \quad \ldots \quad \ldots \quad (3)$$

From (2) and (3), it is noticed that the function is not differentiable at $x=0$

The conditions of *M.V.* Theorem are not satisfied (a, b) which includes 0 also. Hence the conclusion is

$$\frac{f(b)-f(a)}{b-a}=f'(c),\ a<c<b \quad \ldots \quad \ldots \quad (4)$$

It is true for $x=c$, $a<c<b$,

$$\frac{\dfrac{1}{|b|}-\dfrac{1}{|a|}}{b-a}=\frac{d}{dc}\left(\frac{1}{|c|}\right)=-\frac{1}{|c|^2}=-\frac{1}{c^2}$$

or, $\dfrac{1/b-1/1}{b-(-1)}=\dfrac{1}{c^2}$ or, $\dfrac{1-b}{b(b+1)}=-\dfrac{1}{c^2}$ or, $c^2=\dfrac{b^2+b}{b-1}$

or, $\dfrac{b^2+b}{b-1}<b^2$ or, $b^2<c^2$

or, $\dfrac{b+1}{b-1}<b$ or, b is +ve, which can be divided by b,

or, $b^2-2b>1$ or, $b^2-2b+1>2$

or, $(b-1)^2>2$ or, $b=1+\sqrt{2}$

Under this condition the conlusion of *M. V.T.* is true although the conditions for the validity of the theorem are not fulfilled.

See APPENDIX— Art. 17.23, Art. 17.24, Art. 17.25

Exercise VII

1. Verify Rolle's Theorem for the following forms

[i] $f(x)=x^2+6x-6$ in the interval $[-6, 1]$

[ii] $f(x)=x^2$ in $[-2, 2]$

[iii] $f(x)=\sin x/e^x$ in $[0\ \pi]$

[iv] $f(x)=(x-2)(x-3)(x-4)$ in $[2\ 3]$

2. Verify Rolle's theorem for the function

$f(x)=x(x+5)\,e^{-x/2}$

3. Verify Rolle's theorem for the function

$f(x)=2x^3+x^2-4x-2$

4. Verify Rolle's theorem for the function

$f(x)=3x^3+7x^2-11x-15$

5. Verify the truth of Rolle's theorem for the function

(i) $f(x)=x^3-7x^2+36$ (ii) $f(x)=\log\dfrac{x^2+ab}{a+b}$

6. Verify Rolle's theorem for the function in $(-\pi/4, \pi/4)$

(i) $f(x)=\cos^2 x$ (ii) $f(x)=e^x(\sin x-\cos x)$

7. Verify Rolle's Theorem for the following functions.

(i) $f(x)=1-x^{4/5}$ in $[0, 2]$

(ii) $f(x)=\sqrt[3]{(x^2-5x+6)}$ in $[2, 3]$

(iii) $f(x)=2+(x-1)^{2/3}$

8. Verify the Mean Value Theorem for the function

(i) $f(x)=x-x^3$ in the interval $[-2, 1]$

(ii) $f(x)=3+2x-x^2$ in $[0,1]$ D. U. 1988

9. Using Rolle's Theorem, show that $f(x)$ cannot have equal values for two distinct values of x for the function

$f(x)=2x^3+x^2+6x$.

10. Verify the Mean Value Theorem for the function

$f(x)=lx^2+mx+n$ in the interval $[a, b]$

11. At what point is the tangent to the curve $y=\log x$ parall[illegible] to the chord passing through the points $(1,0)$ and $(c,0)$.

Ans. $c=$[illegible]

12. Does the Mean Value Theorem hold good for the functi[illegible] $f(x)=1/x, (x\neq 0), f(o)=0$ in the interval $[-1,1]$?

13. AB is a chord of the parabola $y=x^2$ passing through t[illegible] points $(1, 1)$ and $(3, 9)$; Show that tangent at $(2, 4)$ to the cur[illegible] is parallel to the chord AB.

14. Find the value of c in the Mean Value Theorem

$f(b)-f(a)=(b-a) f'(c)$

(i) if $f(x)=x^{4/3}$ in $(-1, 1)$ (ii) If $f(x)=e^x$ in $(0, 1)$

(iii) if $f(x)=x^3-2x^2+3x-2$ in $[0, 2]$

(iv) if $f(x)=x^4-2x^3+x^2-2x$ in $[-1, 2]$

(v) if $f(x)=x^2$ in $(1,2)$ (vi) if $f(x)=\sin \pi x/2$ in $[0,1]$

(vii) if $f(x)=\sqrt{(x^2-4)}$ in $[5, 3]$

(viii) if $f(x)=\dfrac{-2x+3}{3x-2}$ in $[1,4]$ (ix) If $f(x)=(x-1)($[illegible] in $[0,4]$

15. Verify Cauchy's Mean Value Theorem for the functi[illegible]

$f(x)=x^2+2$ and $g(x)=x^3-1$ in the interval $[1, 2]$

16. Verify that Cauchy's Mean Value Theorem fails for [illegible] functions, $f(x)=x^2$ and $g(x)=x^3$ in the interval $[-1, 1]$

16. (i) State and prove Cauchy's Mean Value Theorem [illegible] deduce that

$f(b)-f(a)=cf'(c) \log(b/a)$ where $f(x)$ is continuous [illegible] differentiable in (a,b) and $a<x<b$

Hints : Put $g(x)=\log x$.

17. If $f'(x)=0$ in $1\leqslant x\leqslant 3$ and if $f(2)=2$, show that f[illegible] throughout the interval $1\leqslant x\leqslant 3$

[illegible] If $y=2x-\tan^{-1}x-\log\{x+\sqrt{(1+x^2)}\}$, show that y [illegible] with x in $0\leqslant x\leqslant\infty$

[illegible] In the Mean value Theorem.

$f(a+h)=f(a)+hf'(a+\theta h)$, $0<\theta<1$,

if $f(x)=\frac{1}{3}x^3-\frac{3}{2}x^2$ find the value of θ in the interval $[0,3]$

[illegible] (i) [illegible] Applying mean value Theorem show that

$$\frac{a-b}{1+a^2}<\tan^{-1}a-\tan^{-1}b<\frac{a-b}{1+b^2}$$ D. H. 1986

[illegible] Show that

$$(x+h)^{3/2}=x^{3/2}+\tfrac{3}{2}x^{1/2}h+\frac{3.1\,h^2}{2.2.2!}\frac{1}{\sqrt{(x+\theta h)}},\ 0<\theta<1,$$

and find the value of θ when $x=0$.

[illegible] (i) If $f''(x)$ exists then show that

$$f''(x)=\lim_{h\to 0}\frac{f(x+h)+f(x-h)-f(x)}{h^2}$$ D. H. 1984

[illegible] Let $f(x)=\dfrac{2x+3}{3x+2}$ in $[1, 0]$

[illegible] that there is no number c between -1 and 0.

[illegible] (a) Expand the following functions in powers of h with [illegible]'s (Schlomiltch, Lagrange and Cauchy's form)

[illegible] $\sin(x+h)$ (ii) e^{x+h} (iii) $\log(x+h)$ (vi) $(x+h)^m$ (v) $e^{m\tan^{-1}(x+h)}$

[illegible] (b) Expand the following functions in powers of x with [illegible]'s (Schlomiltch Lagrange and Cauchy's form)

[illegible] [illegible], (ii) $\cos^{-1}x$ (iii) e^x (iv) $e^{2x}\cos bx$

[illegible] Show that expansion of $x\cot x$ as far as the term [illegible] is $1-x^2/3-x^4/45+\ldots\ \ldots\ \ldots$

[illegible] Prove that $e^x \log(1+x)=x+x^2/2!+2x^3/3!+9x^5/5!+\ldots$

[illegible] Prove that the expansion of $e^{\sin x}$ in a series up to the [illegible] is $1+x+\frac{1}{2}x^2-\frac{1}{8}x^4-\ldots\ \ldots$

[illegible] Prove that $\log\sec x=\frac{1}{2}x^2+\frac{1}{12}x^4+\frac{1}{45}x^6+\ldots$ [illegible] log cos x N.U.1994.

26. Show that $\cos^2 x = 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \ldots\ldots$

27. Prove that

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{x^2}{6.2\,!} - \frac{1}{30}\frac{x^4}{4\,!} + \ldots\ldots + \qquad \text{D. U. 195[illegible]}$$

28. Show that

$\sin^{-1}x = x + \frac{1}{2}.\frac{1}{3}x^3 + \frac{1}{2}.\frac{3}{4}\frac{1}{5}x^5 + \ldots\ldots$

29. Prove that

$e^x \cos x = 1 + x + x^2/2 - x^3/3 - 11x^4/24 - \ \ldots \ \ldots$

30. Prove that $(1+x)^x = 1 + x^2 - \frac{1}{2}x^3 + \frac{5}{6}x^4 - \frac{3}{4}x^5 \ldots\ldots$

31. Prove that $\log(1+\sin x) = x - x^2/2 + x^3/6 - x^4/12 + \ldots$ N. U. (C.A.) [illegible]

32. Find the expansion of $\sin(e^x - 1)$ upto and including the term x^4.

Ans. $x + \frac{1}{2}x^2 - (\frac{5}{24})x^4 + \ldots\ldots$

32. (i) If $\tan^{-1}x$ in powers of $x - \frac{1}{4}\pi$ expanded, then

$$\tan^{-1}x = \tan^{-1}\frac{\pi}{4} + \left(x - \frac{\pi}{4}\right)\frac{1}{(1+\pi^2/16)} - \frac{(x_1 - \frac{1}{2}\pi)^2\pi}{4(1+\pi^2/16)^2} + \ldots$$

(ii) Prove by Taylor's Theorem

(a) $\log(x+h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \ldots$

(b) $\frac{1}{x+h} = \frac{1}{x} - \frac{h}{x^2} - \frac{h^2}{x^3} - \frac{h^3}{x^4} \ \ldots \ \ldots$

(c) $f\left(\frac{x}{1+x}\right) = f(x) - \frac{x}{1+x}f'(x) + \frac{x^2}{(1+x^2)}\frac{f''(x)}{2\,!} + \ldots$

(d) $\log \sin x = \log \sin 2 + (x-2)\cot 2 - \frac{1}{2}(x-2)^2 \operatorname{cosec}^2 2 + \frac{1}{3}(x-2)^3 \operatorname{cosec}^2 2 \cot 2 + \ \ldots \ \ldots \ \ldots$

33. Show that Maclaurin's theorem fails to expand sin [illegible]

34. Show that θ which occurs in the Lagrange's form of remainder, $R_n = \frac{h^n}{n\,!} f^n(a+\theta h)$ tends to the limits $\frac{1}{n+1}$ as $h \to 0$ Provided that $f^{n+1}(x)$ is continuous at a and $f^{n+1}(a) \neq 0$

[illegible] If $f''(x)$ exists for all points in (a, b) and $\frac{f(c)-f(a)}{c-a} = \frac{f(b)-f(c)}{b-c}$ [illegible] $a<c<b$ then there is a number such that $a<\xi<b$ and $f''(\xi) = 0$

[illegible] If $y = \sin \log(x^2+2x+1)$, prove that

$(x+1)^2 y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0$

[illegible] expand y in ascending powers of x as far as x^4

[illegible] If $y = \tan(m \tan^{-1}x)$ show that first three terms in the [illegible]'s series for y are

$mx + m(m^2-1)x^3/3 + m(m^2-1)(2m^2-3)x^5/15$

[illegible] If $y = \sin(m \sin^{-1}x)$ show that

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$ and hence find [illegible] of (i) $\sin(m \sin^{-1}x)$ (ii) $\sin^{-1}x$.

[illegible] If $a_n x^n + a_{n+1}x^{n+1} + a_{n+2}x^{n+2}$ be three consecutive terms [illegible] expansion of $\sqrt{(1-x^2)}\sin^{-1}x$ in powers of x,

[illegible] that $a_{n+2} = \frac{n-1}{n+2}a_n$

[illegible] show that all even terms vanish, and that the expansion is

$x - \frac{1}{3}x^3 - \frac{2}{3.5}x^5 - \frac{2.4}{3.5.7}x^7 - \ldots\ldots$ R. H. 1955, C. H. 1986

[illegible] If $y = e^{a \sin^{-1}x} = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n + \ldots$ [illegible] that

(i) $(1-x^2)y_2 = xy_1 + a^2 y$ (ii) $(n+1)(n+2)a_{n+2} = (n^2+a^2)a_n$

[illegible] State any one of the fundamental properties of a continuous [illegible] and show that there is one and only one positive root of [illegible] $x^8 = 2$ by the property.

[illegible] State any one of the fundamental properties of a continu- [illegible] other than one used in proving Rolle's Theorem,

43. The function u, v (of x) and their derivatives u', v' are continuous in $a \leqslant x \leqslant b$ and $uv' - u'v \neq 0$ in $a \leqslant x \leq b$. Show that between any two roots of $u=0$ lies one of v.

44. If $f'(x) > 0$ in $a \leqslant x \leqslant b$ then show that $f(x)$ is strictly increasing in $a \leqslant x \leqslant b$. Deduce $a^x > x^a$ if $x > a \geqslant e$.

45. If $\log_e y = \tan^{-1} x$, prove that

$$(1+x^2)y_n = \{1-2(n-1)x\}y_{n-1} - (n-1)(n-2)y_{n-2}$$ and

hence find the co-efficients of x^5 in the expansion of y by Maclaurin's Theorem.

46. If $\dfrac{(\tan^{-1}x)^2}{\lfloor 2} = \dfrac{a_2x^2}{2} - \dfrac{a_4x^4}{4} + \dfrac{a_6x^6}{6} \cdots \cdots$

Prove that $a_{2n} - a_{2n-2} = \dfrac{1}{2n-1}$

47. If $\sin^{-1}x = \sum_{n=1}^{\infty} \dfrac{b_n x^n}{n!}$ and $\dfrac{(\sin^{-1}x)^3}{3!} = \sum_{n=1}^{\infty} \dfrac{a_n x^n}{n!}$

Show that $a_{n+1} = n^2 a_n + b_n$

48. If $x^2 + 2x = 2\log\left(c \dfrac{dx}{dy}\right)$

and $y = a_0 + a_1 x + \dfrac{a^2}{\lfloor 2} x^2 + \cdots \cdots$

Show that $a_{n+2} + a_{n+1} + n^2 a_n = 0$

49. If $f^n(x)$ is continuous in the given equation

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(x + \theta h)$$

then prove that $\lim_{h \to 0} \theta = \dfrac{1}{n}$

50. Deduce from Cauchy's mean value theorem that

$f(b) - f(a) = g\, f'(g) \log\left(\dfrac{b}{a}\right)$ where $f(x)$ is continuous and differentiable in (a, b) and $a < g < b$.

51. If $\varphi''(x) > 0$ for all values of x, prove that

$$\varphi\left(\frac{x_1+x_2}{2}\right) < \frac{1}{2}\varphi(x_1) + \frac{1}{2}\varphi(x_2)$$

SOL. If we consider $\phi''(x)$ to be finite so that $\phi'(x)$ and $\phi(x)$ are both continuous and finite for all values of x. Now

$\phi(x_1) = \phi\{(x_1+x_2)/2 + (x_1-x_2)/2$ of the form

$\phi(x_1+h) = \phi\{(x_1+x_2)/2\} + \{(x_1-x_2)/2\}\,\phi'\{(x_1+x_2)/2\} +$

$\frac{1}{2!}\{(x_1-x_2)/2\}^2\,\phi''\{(x_1+x_2)/2 + \theta_1(x_1-x_2)/2\} + \ldots (1);\ 0 < \theta_1 < 1$

Similarly

$\phi(x_2) = \phi\{(x_1+x_2)/2\} + \{(x_2-x_1)/2\}\,\phi'\{(x_1+x_2)/2 +$

$\frac{1}{2!}\left(\frac{x_2-x_1}{2}\right)^2 \phi''\left\{\frac{x_1+x_2}{2} + \theta_2 \frac{x_2-x_1}{2}\right\} + \ldots\ldots (2),\ 0 < \theta_2 < 1$

Adding (1) and (2)

$\phi(x_1) + \phi(x_2) = 2\phi\left\{\frac{1}{2}(x_1+x_2)\right\} + \frac{1}{2!}(x_1-x_2)^2 [\phi''\left\{\frac{1}{2}(x_1+x_2) + \theta_1 \frac{1}{2}(x_1-x_2)\right\} +$

$\phi''\left[\frac{1}{2}(x_1+x_2) + \theta_2 \frac{1}{2}(x_2-x_1)\right]$

But $\frac{1}{2}(x_1+x_2)$ is always positive. Also $\phi''(x)$ is positive for all values of x as a given condition.

$\therefore \phi(x_1) + \phi(x_2) = 2\phi\left\{\frac{1}{2}(x_1+x_2)\right\}$ + a positive quantity

i.e. $2\phi\left\{\frac{1}{2}(x_1+x_2)\right\} < \phi(x_1) + \phi(x_2)$ or; $\phi\frac{(x_1+x_2)}{2} < \frac{1}{2}\{\phi(x_1)+\phi(x_2)\}$

See APPENDIX ~ Extra Sums

NO.52, NO.53, NO.54. NO.55. NO.56, NO.57

Art. 17.22; Art. 17.24. Art. 17.25.

প্রশ্নমালা VII

1. নিম্নলিখিত ফাংশনগুলির উপর রোলের উপপাদ্য পরীক্ষা কর।

(i) $[-6, 1]$ বিস্তারে ফাংশন $f(x)=x^2+6x-6$

(ii) $[-2, 2]$ বিস্তারে ফাংশন $f(x)=x^2$

(iii) $[0, \pi]$ বিস্তারে ফাংশন $f(x)=\sin x/e^x$

(iv) $[2, 3]$ বিস্তারে ফাংশন $f(x)=(x-2)(x-3)(x-4)$

2. $f(x)=x(x+5)\, e^{-x/2}$ এর জন্য রোলের উপপাদ্য প্রতিপাদন কর।

3. $f(x)=2x^3+x^2-4x-2$-এর জন্য রোলের উপপাদ্য প্রতিপাদন কর

4. $f(x)=3x^3+7x^2-11x-15$ ফাংশনটির জন্য রোলের উপপাদ্য প্রতিপাদন কর।

(Verify Rolle's theorem for the function)

$f(x)=3x^3+7x^2-11x-15$.)

5. (i) $f(x)=x^3-7x^2+36$ (ii) $f(x)=\log \dfrac{x^2+ab}{a+b}$

ফাংশনগুলির জন্য রোলের উপপাদ্যটির সত্যতা প্রতিপাদন কর।

6. $(-\pi/4, \pi/4)$ বিস্তারে ফাংশনদ্বয়

(i) $f(x)=\cos^2 x$ এবং (ii) $f(x)=e^x (\sin x-\cos x)$-এর

রোলের উপপাদ্যের সত্যতা প্রতিপাদন কর।

7. নিম্নলিখিত ফাংশনগুলির জন্য রোলের উপপাদ্যের সত্যতা যাচাই কর।

(i) $[0, 2]$ বিস্তারে $f(x)=1-x^{4/5}$

(ii) $[2, 3]$ বিস্তারে ফাংশন $f(x)=\sqrt[3]{(x^2-5x+6)}$

(iii) $f(x)=2+(x-1)^{2/3}$

8. $[-2, 1]$ বিস্তারে ফাংশন $f(x)=x-x^3$ এর জন্য গড় মান উপপাদ্যের সত্যতা প্রতিপাদন কর।

(i) $[0, 1]$ বিস্তারে $f(x)=3+2x-x^2$ এর জন্য গড়মান উপপাদ্যের সত্যতা প্রমাণ কর। D. U.

9. রোলের উপপাদ্যটির ব্যবহার করে দেখাও যে, x-এর দুটি নির্দিষ্ট মানের জন্য $f(x)=2x^3+x^2+6x$ ফাংশনের দুটি সমান মান থাকতে পারেনা।

[Using Rolle's Theorem, show that $f(x)$ cannot have equal values for two distinct values of x for the function $f(x)=2x^3+x^2+6x$]

10. (a, b) বিস্তারে $f(x)=lx^2+mx+n$ ফাংশনের জন্য গড়মান উপপাদ্যের সত্যতা প্রতিপাদন কর। [Verify the Mean Value Theorem for the function $f(x)=lx^2+mx+n$ in the interval (a, b).]

11. $y=\log x$ এই বক্ররেখার কোন বিন্দুতে স্পর্শক $(1, o)$ এবং (c, o) গামী জ্যা-এর সহিত সমান্তরাল হবে?

12. $(-1, 1)$ বিস্তারে $f(x)=1/x$, $(x\neq 0)$ এবং $f(o)=0$-এর জন্য গড়মান উপপাদ্যটি সত্য হবে কি?

13. $(1, 1)$ এবং $(3, 9)$ বিন্দুগামী AB একটি $y=x^2$ পরাবৃত্তের জ্যা। দেখাও যে $(2, 4)$ বিন্দুগামী স্পর্শক AB জ্যা এর সহিত সমান্তরাল।

14. গড়মান উপপাদ্যে $f(b)-f(a)=(b-a)f'(c)$ হতে c-এর মান নির্ণয় কর যখন (a) $[0,4]$ বিস্তারে $f(x)=(x-1)(x-2)(x-3)$ N.U. 1994.

(i) $(-1, 1)$ বিস্তারে $f(x)=x^{4/3}$ হয়।

(ii) $(0, 1)$ বিস্তারে $f(x)=e^x$ হয়।

(iii) $(0, 2)$ বিস্তারে $f(x)=x^3-2x^2+3x-2$ হয়।

(iv) $(-1, 2)$ বিস্তারে $f(x)=x^4-2x^3+x^2-2x$ হয়।

(v) $(1, 2)$ বিস্তারে $f(x)=x^2$ হয়।

(vi) $(0, 1)$ বিস্তারে $f(x)=\sin \pi x/2$ হয়।

(vii) $(2, 3)$ বিস্তারে $f(x)=\sqrt{(x^2-4)}$ হয়।

(viii) $(1, 4)$ বিস্তারে $f(x)=\dfrac{-2x+3}{5x-2}$ হয়।

15. (1, 2) বিস্তারে $f(x)=x^2+2$ এবং $g(x)=x^3-1$ ফাংশনদ্বয়ের জন্য কাওচির (cauchy) গড়মান উপপাদ্যটির সত্যতা প্রতিপাদন কর।

16. (−1, 1) বিস্তারে $f(x)=x^2$ এবং $g(x)=x^3$ ফাংশনদ্বয়ের জন্য কাওচির গড়মান উপপাদ্য সত্য নহে—ইহা প্রমাণ কর।

16. (i) কাওচির গড়মান উপপাদ্যটি বর্ণনা কর এবং প্রমাণ কর।

(a, b) এবং $a<x<b$ বিস্তারে $f(x)$ অবিছিন্ন ও অন্তরীকরণ যোগ্য হলে প্রমাণ কর যে

$$f(b)-f(a)=cf'(c)\log(b/a)$$

[ইঙ্গিত $g(x)=\log x$ বসাতে হবে।]

17. $1\leqslant x\leqslant 3$ বদ্ধবিস্তারে যদি $f'(x)=0$ হয় এবং $f(2)=2$, তবে দেখাও যে $1\leqslant x\leqq 3$ বিস্তারের সর্বত্রই $f(x)=1$ হবে।

18. যদি $y=2x-\tan^{-1}x-\log\{x+\sqrt{(1+x^2)}\}$ হয় তবে দেখাও যে $0\leqslant x\leqslant\infty$ বিস্তারে x বৃদ্ধির সাথে সাথে y ও বৃদ্ধি পাবে।

19. গড়মান উপপাদ্যে $f(a+h)=f(a)+hf'(a+\theta h)$, $0<\theta<1$, যদি $f(x)=\frac{1}{3}x^3-\frac{3}{2}x^2$ হয় তবে (0, 3) বিস্তারে, θ-এর মান নির্ণয় কর।

19. (i) গড়মান উপপাদ্য প্রয়োগ করে দেখাও যে

$$\frac{a-b}{1+a^2}<\tan^{-1}a-\tan^{-1}b<\frac{a-b}{1+b^2}$$

D. H. 1986

20. দেখাও যে

$$(x+h)^{3/2}=x^{3/2}+3/2\,x^{1/2}\,h+\frac{3.1.h^2}{2.2.2\,!}\,\frac{1}{\sqrt{(x+\theta h)}},\quad 0<\theta<1,$$

এবং $x=0$ যখন তখন θ-এর মান নির্ণয় কর।

20. (i) If $f''(x)$ exists then show that

$$f''(x)=\underset{h\to 0}{Lt}\ \frac{f(x+h)+f(x-h)-f(x)}{h^2}$$

D. H. 1984

21. (−1, 0) বিস্তারে মনে কর $f(x)=\dfrac{2x+3}{3x+2}$ দেখাও যে −1 এবং 0 এর মধ্যে কোন সংখ্যা c নেই।

21. (a) ম্যোমিচির, ল্যাগ্রেঞ্জের এবং কাওচির অবশিষ্ট-সহ নিম্নলিখিত ফাংশনগুলিকে h-এর শক্তিতে সম্প্রসারণ কর?

(i) $\sin(x+h)$ (ii) e^{x+h} (iii) $\log(x+h)$, (iv) $(x+k)^m$ (v) $e^{m\tan^{-1}(x+h)}$

21. (b) ম্যোমিচির, ল্যাগ্রেঞ্জের এবং কাওচির অবশিষ্টসহ নিম্নলিখিত ফাংশনগুলিকে x-এর শক্তিতে সম্প্রসারণ কর:—

(i) $\cos x$, (ii) $\cos^{-1}x$ (iii) e^x (iv) $e^{ax}\cos bx$

22. দেখাও যে $x\cot x$ কে x^4 সম্বলিত পদ পর্যান্ত বিস্তৃত করলে পাওয়া যাবে $1-x^2/3-x^4/45+\ \dots\ \dots\ \dots$

23. প্রমাণ কর যে

$$e^x\log(1+x)=x+x^2/2\,!+2x^3/3\,!+9x^5/5\,!+\dots\ \dots$$

24. প্রমাণ কর যে $e^{\sin x}$ কে x^4 সম্বলিত পদ পর্যান্ত বিস্তৃত করলে পাওয়া যাবে $1+x+\frac{1}{2}x^2-\frac{1}{8}x^4-\ \dots\ \dots\ \dots$

25. প্রমাণ কর যে $\log\sec x=\frac{1}{2}x^2+1/12x^4+1/45x^6+\dots$ *D. U. 1964*

26. দেখাও যে $\cos^2 x=1-x^2+\frac{1}{3}x^4-2/45x^6+\dots\ \dots\ \dots$

27. প্রমাণ কর যে

$$\frac{x}{e^x-1}=1-\frac{x}{2}+\frac{x^2}{6.2\,!}-\frac{1}{30}\cdot\frac{x^4}{4\,!}+\ \dots\ \dots\ +$$

D. U. 1955

28. দেখাও যে $\sin^{-1}x=x+\frac{1}{2}\cdot\frac{1}{3}x^3+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{5}x^5+\ \dots\ \dots$

29. প্রমাণ কর যে

$$e^x\cos x=1+x+x^2/2-x^3/3-11x^4/24-x^5/5\ \dots\ \dots$$

30. প্রমাণ কর যে $(1+x)^x=1+x^2-\frac{1}{2}x^3+5/6x^4-\frac{3}{4}x^5\ \dots\ \dots$

31. প্রমাণ কর যে $\log(1+\sin x)=x-x^2/2+x^3/6-x^4/12+\ \dots$

32. $\sin(e^x-1)$ কে x^4 সম্বলিত পদ পর্যান্ত বিস্তৃত কর।

32. (i) If $\tan^{-1}x$ in powers of $x-\frac{1}{4}\pi$ expanded, then

$$\tan^{-1}x=\tan^{-1}\frac{\pi}{4}+\left(x-\frac{\pi}{4}\right)\frac{1}{(1+\pi^2/16)}-\frac{(x-\frac{1}{4}\pi)^2\pi}{4(1+\pi^2/16)^2}+\ \dots\ \dots$$

(ii) Prove by Taylor's Theorem

(a) $\log(x+h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \ldots$

(b) $\frac{1}{x+h} = \frac{1}{x} - \frac{h}{x^2} + \frac{h^2}{x^3} - \frac{h^3}{x^4} \ldots \ldots$

(c) $f\left(\frac{x^2}{1+x}\right) = f(x) - \frac{x}{1+x} f'(x) + \frac{x^2}{(1+x^2)} \frac{f''(x)}{\lfloor 2} + \ldots \ldots$

(d) $\log \sin x = \log \sin 2 + (x-2) \cot 2 - \frac{1}{2}(x-2)^2 \operatorname{cosec}^2 2 + \frac{1}{3}(x-2)^3 \operatorname{cosec}^2 2 \cot 2 + \ldots\ldots$

33. দেখাও যে ম্যাকলরিনের উপপাদ্য $\sin(x\sqrt{\pi})$ কে বিস্তৃত করণে অপারগ।

34. দেখাও যে ল্যাগ্রেঞ্জের আকারের অবশিষ্ট $R_n = \frac{h^n}{\lfloor n} f^n(a+\theta h)$-এ যে θ আছে তাহা সীমা $\frac{1}{n+1}$-এর দিকে অগ্রসর হবে যখন $h \to 0$, a বিন্দুতে $f^{n+1}(x)$ অবিচ্ছিন্ন এবং $f^{n+1}(a) \neq 0$

35. (a, b) বিস্তারে যদি $f'(x)$-এর অস্তিত্ব থাকে এবং $\frac{f(c)-f(a)}{c-a} = \frac{f(b)-f(c)}{b-c}$ যেখানে $a<c<b$ তবে এমন একটি সংখ্যা পাওয়া যাবে যেন $a<\xi<b$ এবং $f'(\xi)=0$ হয়।

36. যদি $y = \sin \log(x^2+2x+1)$ হয় তবে প্রমাণ কর যে

$(x+1)^2 y_{n+2} + (2n+1)(x+1) y_{n+1} + (n^2+4) y_n = 0$

এ হতে y-কে x-এর চতুর্থ শক্তি (x^4) পর্যন্ত বিস্তৃত কর।

37. যদি $y = \tan(m \tan^{-1} x)$ হয়, তবে ম্যাকলরিনের ধারায় ইহার তিনটি পদ হবে

$mx + m(m^2-1)x^3/3 + m(m^2-1)(2m^2-3)x^5/15$

38. যদি $y = \sin(m \sin^{-1} x)$ হয় তবে দেখাও যে

$(1-x^2) y_{n+2} - (2n+1) x y_{n+1} + (m^2-n^2) y_n = 0$ এবং ইহা হতে (i) $\sin(m \sin^{-1} x)$ ও (ii) $\sin^{-1} x$-এর বিস্তৃতি নির্ণয় কর।

39. x-এর শক্তিতে এবং $\sqrt{(1-x^2)} \sin^{-1} x$-এর বিস্তৃতিতে পরপর তিনটি পদ যদি $a_n x^n + a_{n+1} x^{n+1} + a_{n+2} x^{n+2}$ হয়

তবে প্রমাণ কর যে $a_{n+2} = \frac{n-1}{n+2} a_n$

আরো দেখাও যে সকল জোড় পদগুলি শূন্য হয় এবং ধারাটি হয়

$x - \frac{1}{3}x^3 - \frac{2}{3.5}x^5 - \frac{2.4}{3.5.7}x^7 - \ldots \ldots$ [R. U. 1964] C. H. 1986

40. যদি $y = e^{a \sin^{-1} x} = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n + \ldots$ হয় তবে প্রমাণ কর যে

$(1-x^2) y_2 = x y_1 + a^2 y$ (ii) $(n+1)(n+2) a_{n+2} = (n^2+a^2) a_n$

41. অবিচ্ছিন্ন ফাংশনের একটি মৌলিক ধর্মের উল্লেখ কর এবং দেখাও যে এই ধর্মের জন্য $x^5 = 2$ এই সমীকরণের একটি এবং কেবলমাত্র একটি ধনাত্মক বীজ পাওয়া যাবে।

42. রোলের উপপাদ্য প্রমাণের জন্য যে ধর্ম ব্যবহৃত হয় তাহা ব্যতিত অবিচ্ছিন্ন ফাংশনের অপর একটি মৌলিক ধর্মের উল্লেখ কর।

43. $u(x)$, $v(x)$, $u'(x)$, $v'(x)$ এবং $uv' - u'v \neq 0$ আবদ্ধ ব্যবধি (a, b) তে অবিচ্ছিন্ন। (a, b) ব্যবধিতে মনে করি p_1, p_2 দুইটি $u(x)=0$ এর মূল যাহার (a, b) ব্যবধির মধ্যে অবস্থান। $u(p_1)=0$ এবং $u(p_2)=0$

দেখাতে হইবে যে (p_1, p_2) ব্যবধিতে $v(x)=0$ এর একটি মূল বিদ্যমান, মনে করি $v(x)=0$-এর কোন মূল (p_1, p_2) ব্যবধিতে নাই।

অর্থাৎ (p_1, p_2)-এর মধ্যে কোন মানের জন্য $v(x) \neq 0$ ... ... (1)

প্রদত্ত সম্পর্ক হইতে $u(x)v'(x) - v(x)u'(x) \neq 0$ ... ... (2)

মনে করি $F(x) = u(x)/v(x)$

(i) হইতে (p_1, p_2) ব্যবধিতে $F(x)$ অবিচ্ছিন্ন।

$F'(x) = \frac{u'(x)v(x) - v'(x)u(x)}{[v(x)]^2}$ বিদ্যমান (1) এবং (2) হইতে

আরও $F(p_1)=F(p_2)=0$.

$\therefore$ রোলের উপপাদ্য হইতে, x-এর একটি মান ξ, $p_1<\xi<p_2$ এমন যে $F'(\xi)=0$

or, $d/dx\,[F(x)]=0$, $x=\xi$ বিন্দুতে

or, $\dfrac{u'(\xi)v(\xi)-u'(\xi)v'(\xi)}{[v(\xi)]^2}=0$, ইহা কল্পনা বিরুদ্ধে (*a* contradiction)

সুতরাং আমাদের সিদ্ধান্ত ভুল। $v(x)=0$-এর একটি মূল (p_1, p_2) ব্যবধিতে থাকিবে।

44. If $f'(x)>0$, in $a\leqslant x\leqq b$, then show that $f(x)$ is strictly increasing in $a\leqq x\leqslant b$, Deduce $a^x>x^a$ in $x>a>e$.

Mean Value Theorem হইতে

$f(d)-f(c)=(d-c)f'(\xi)$, $c>\xi<d$.

কিন্তু $d-c>0$ এবং $f'(\xi)=0$ $[\therefore\ a<\xi<b]$

$\therefore$ $f(d)-f(c)>0$ or, $f(d)>f(c)$ or

$\therefore$ $f(x)$ ফাংশনটি $[a, b]$ ব্যবধিতে একটি অভিসত্য বর্দ্ধমান (Strictly increasing) ফাংশন.

$a^x=x^a$ বা, $x\log a-a\log x=0$

মনে করি $f(x)=x\log a-a\log x$

$\therefore$ $f'(x)=\log a-a/x=(x\log a-a)/x>0$

$\therefore$ $f(x)>0$ বা, $x\log a>a\log x$ বা, $a^x>x^a$

45. If $\log_e y=\tan^{-1}x$, prove that

$(1+x^2)y_n=\{1-2(n-1)x\}y_{n-1}-(n-1)(n-2)\,y_{n-2}$ and hence find the co-efficients of x^5 in the expansion of y by Machlaurin's Theorem.

46. If $\dfrac{(\tan^{-1}x)^2}{\lfloor 2}=\dfrac{a_2x^2}{2}-\dfrac{a_4x^4}{4}+\dfrac{a_6x^6}{6}$.........

Prove that $a_{2n}-a_{2n-2}=\dfrac{1}{2n-1}$

ANSWERS VII

1. (i) yes (ii) yes (iii) yes (iv) yes.

2. yes 3. yes 4. yes for $(-1,-3)$

5. (i) true for $-2<0<3$; $3<14/3<6$

(ii) yes, $-\sqrt{(a+b-ab)}<0<\sqrt{(a+b-ab)}$

6. (i) yes, $-\pi/4\leq\theta\leqslant\pi/4$ (ii) no

7. (i) no, (ii) yes $2<5/2<3$ (iii) no.

8. $-2<-1<1$. 9. fails, 10 $a<\dfrac{a+b}{2}<b$

11. $e-1$, $\log(e-1)$, 12. no 13. (i) 0, (ii) $\log(e-1)$

(iii) 4/3, (iv) 0, $\frac{1}{2}$, 1, (v) 3/2 (vi) $\dfrac{2}{\pi}\cos^{-1}\dfrac{2}{\pi}$

(vii) $\sqrt{5}$ (viii) $(2+\sqrt{10})/3$. 14. $1<4/9<2$

18. failed as $g'(0)=f'(0)=0$ 19. $\frac{1}{6}(3\pm\sqrt{3})$ 20. 9/64

17. (a) (i) $\sin x+h\sin(\pi/2+x)+\dfrac{h^{n-1}}{\lfloor(n-1)}\sin\{(n-1)\pi/2+x\}+R_n$

$R_n=\dfrac{h^n(1-\theta)^{n-m}}{m\lfloor(n-1)}\sin(n\pi/2+x+\theta h)$ then put $m=n$

then put $m=1$ giving successively Lagrange's and Cauchy's Reminders.

(ii) $e^x+he^x+h^2e^x/2$ ... ...$+\dfrac{h^{n-1}}{(n-1)!}e^x+R_n$

$R_n=\dfrac{h^n(1-\theta)^{n-m}}{m(n-1)!}e^{x+\theta h}$

Put $m=n$, and $m=1$. Lagrange's and Cauchy's Reminders

(iii) $\log x+h/x+(-1)\left(\dfrac{h^2}{\lfloor 2}(1/x^2)+(-1)^2\dfrac{h^3}{\lfloor 3}1/x^3\right)+\ldots$

$\ldots+(-1)^{n-2}\dfrac{h^{n-1}}{\lfloor(n-1)}\dfrac{1}{x^{n-1}}+R_n$

$$R_n = \frac{h^m(1-\theta)^{n-m}}{m\lfloor(n-1)}(-1)^{n-1}\frac{1}{x^n} \therefore \quad 0<\theta<1$$

Lagranges and Cauchy's Remainders by putting $m=n$, $n=1$ respectively.

(iv) . $x^n + mhx^{m-1} \ldots + \quad + \frac{h^{n-1}}{\lfloor(n-1)}\{m(m-1)\ldots\ldots(m-n+2)\}$
$x^{n-n+1} + R_n$

$$R_n = \frac{h^n(1-\theta)^{n-p}}{p\lfloor(n-1)} m(m-1)\ldots\ldots(m-n+1)(x+\theta h)^{m-n}$$

$p=n$, Lagrange's Remainder.

$p=1$, Cauchy's Remainder.

21. (b) (1) $\cos x = 1 - x^2/\lfloor 2 + x^4/\lfloor 4 \ldots\ldots + \frac{x^{n-1}}{\lfloor(n-1)}$ $\cos\{(n-1)\pi/2\} +$

Schlemiltch Remainder R_n,

$$R_n = \frac{x^n(1-\theta)^{n-m}}{m\lfloor(n-1)} ; f^n(\theta x) = \frac{x^n(1-\theta)^{n-m}}{m\lfloor(n-1)} \cos\{\tfrac{1}{2}n\pi + \theta\}$$

$R_n = \frac{x^n}{n!}\cos(\frac{1}{2}n\pi + \theta x)$ Lagranges's $m=n$

$R_n = \frac{x^2(1-\theta)^{n-1}}{\lfloor(n-1)} \cos\{\frac{1}{2}n\pi + \theta x\}$ $\quad m=1$, Cauchy's

$\cos^{-1} x = 1 - x - x^3$ and so on.

(iii) $e^x = 1 + x + \frac{x^2}{\lfloor 2} + \ldots\ldots + \frac{x^{n-1}}{\lfloor(n-1)} + R_n$

$$R_n = \frac{x^n(1-\theta)^{n-m}}{m\lfloor(n-1)} e^{\theta x}$$

$n=m$, Lagrange's $m=1$, Cauchy's Remainder.

(iv) $e^{ax}\cos bx = 1 + x(a^2+b^2)^{1/2}\cos(\tan^{-1} b/a) + \quad \ldots \quad \ldots$

$+ \frac{x^n - 1}{\lfloor(n-1)} \sqrt{(a^2+b^2)^{n-1}} \cos(n-1)\tan^{-1}b/a + R_n$

36. $2x - x^2 - \frac{2}{3}x^3 + \frac{2}{3}x^4 \quad \ldots \quad \ldots$

38. $\sin(m\sin^{-1}x) = x - m(m^2-1)\frac{x^3}{3!} + m(m^2-1^2)(m^2-2^2)$
$(m^2-3^2)x^5/\lfloor 5 + \quad \ldots \quad \ldots$
$\sin^{-1}x = x + \frac{1}{2}.\frac{1}{2}.x^3 + \frac{1}{2}.\frac{3}{4}\, x^5/5 + \ldots \quad \ldots$

CHAPTER VIII

INDETERMINATE FORMS

8. 1. In Chapter II we have already discussed limits in some cases of functions. It is shown that limit of a certain function (continuous) can be determined by direct substitution of that value of the independent variable. But there are cases in which direct substitutions reduce the function forms like $\frac{0}{0}$, $\frac{\infty}{\infty}$ etc. which are meaningless. In this chapter attempts have been made to evaluate limits of such meaningless forms. Examples of meaningless form are shown below.

$\lim_{x\to 0} \frac{\sin x}{x}$ the expression becomes $\frac{0}{0}$ for $x=0$

$\lim_{x\to 3} \frac{x-3}{x^2-9}$. When $x=3$, the expression becomes $\frac{0}{0}$

$\lim_{x\to\infty} \frac{e^x}{x^n}, n>0$; When $x=\infty$, the expression becomes $\frac{\infty}{\infty}$

If $F(x) = \frac{f(x)}{\phi(x)}$, when $f(a)=0$, $g(a)=0$

Then $F(a) = \frac{0}{0}$' which is meaningless. The form $\frac{0}{0}$, $\frac{\infty}{\infty}$ etc. are called **Indeterminate forms.**

In some books these forms are also called **Singular form, Undetermined form, or Illusory forms.**

8. 2. List of Indeterminate forms.

There are many indeterminate forms of limit ; which after proper substitution reduce to any one of the forms given below,

$$\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 0^0, \infty^0, 1^\infty$$

8. 3. Method of determining the indeterminate forms.

There are two methods for determing indeterminate forms of limits.

(1) Algebraical Method

(2) Application of Differential Calculus.

The algebraical method has been displayed with an example and the application of Differential Calculus will need the establishment of some Theorems. The second method will be applied after the proof of these Theorems in Art. 8.4.

Ex. 1. Evaluate $\lim_{x\to 0} \frac{\log(1+kx^2)}{1-\cos x}$

Let $y=\lim_{x\to 0}\frac{\log(1+kx^2)}{1-\cos x}$ ($\therefore$ when $x=0$, y is of the form 0/0)

$$=\lim_{x\to 0}\frac{kx^2-\frac{1}{2}k^2x^4+\ \ldots}{1-(1-x^2/2\,!+x^4/4\,!\ldots)} \text{ by expansion}$$

$$=\lim_{x\to 0}\frac{kx^2-\frac{1}{2}k^2x^4+\ldots}{x^2/2\,!-x^4/4\,!+\ldots}$$

$$=\lim_{x\to 0}\frac{k-\frac{1}{2}k^2x^2+\ldots}{1/2\,!-x^2/4\,!+\ldots\ldots}=\frac{k}{\frac{1}{2}}=k$$

Hence the limit is $2k$

This limit can also be evaluated by the 2nd method.

8.4. Form $\frac{0}{0}$ (L. Hopsitals Theorem) R. H. 198[illegible]

Let $\phi(x)$, $\psi(x)$ and their derivatives $\phi(x)$ are all continuous at $x=a$, and also $\phi(a)=\psi(a)=0$, and $\psi'(a)\neq 0$, then

$$\lim_{x\to a}\frac{\phi(x)}{\psi(x)}=\lim_{x\to a}\frac{\phi'(x)}{\psi'(x)}=\frac{\phi'(a)}{\psi'(a)}$$

Sine $\varphi(a)=0$, $\psi(a)=0$, we have

$$\lim_{x\to a}\frac{\varphi(x)}{\psi(x)}=\lim_{x\to a}\frac{\varphi(x)-\varphi(a)}{\psi(x)-\psi(a)}$$

$$=\lim_{h\to 0}\frac{\varphi(a+h)-\varphi(a)}{\psi(a+h)-\psi(a)}\text{ — (taking } x=a+h)$$

$$=\lim_{h\to 0}\left(\frac{\frac{\varphi(a+h)-\varphi(a)}{h}}{\frac{\psi(a+h)-\psi(a)}{h}}\right)=\frac{\varphi'(a)}{\psi'(a)}$$

$$\lim_{x\to a}\frac{\varphi(x)}{\psi(x)}=\frac{\varphi'(a)}{\psi'(a)}\text{(proved)}$$

where $\psi'(a)\neq 0$

If $\varphi'(a)=0$, $\psi'(a)=0$, but $\psi''(a)\neq 0$ then proceeding as before with φ' and ψ' in place of φ and ψ we get

$$\lim_{x\to a}\frac{\varphi(x)}{\psi(x)}=\lim_{x\to a}\frac{\varphi'(x)}{\psi'(x)}=\frac{\varphi''(a)}{\psi''(a)}\text{ when } \psi''(a)\neq 0$$

Proceed in this way untill the given expression is free from the 0/0 form.

Mathematically we explain the above statement in this form.

If $\phi'(a)=\phi''(a)\ldots\ldots=\phi^{n-1}(a)=0$ and

$$\psi'(a)=\psi''(a)\ldots=\phi^{n-1}(a)=0$$

but $\psi^n(a)\neq 0$, then

$$\lim_{x\to 0}\frac{\varphi(x)}{\psi(x)}=\lim_{x\to 0}\frac{\phi^n(x)}{\psi^n(x)}=\frac{\varphi^n(a)}{\psi^n(a)}$$

Note : The proposition of Art. 8.4. is even true when $x\to\infty$ instead of $x\to a$. In this case we are to put $x=1/t$.

Then

$$\lim_{x\to\infty} \phi(x)=\lim_{t\to 0} \phi(1/t) \text{ and } \lim_{x\to\infty} \psi(x)=\lim_{t\to 0} \psi(1/t)$$

$$\therefore \lim_{x\to\infty}\frac{\phi(x)}{\psi(x)}=\lim_{t\to 0}\frac{\phi(1/t)}{\psi(1/t)}=\lim_{t\to 0}\frac{\phi'(1/t)(-1/t^2)}{\psi'(1/t)(-1/t^2)}$$

$$=\lim_{t\to 0}\frac{\phi'(1/t)}{\psi'(1/t)}=\lim_{x\to\infty}\frac{\phi'(x)}{\psi'(x)}$$

Note : Students are advised to note the fact that the differentiation of the numerator and denominator made separately. They should not be confused by seeing the form which shows generally a fraction, the differentiation of which is made by the rule of differentiating the quotient of two functions.

Alternative proof : Let us consider a curve which passes through the origin and be defined by the equations.

$$\left.\begin{aligned} x&=\psi(t)\\ y&=\phi(t)\end{aligned}\right\}$$

Let, $P(x, y)$ be point very near to the origin O. Suppose when $t=a$, $x=0$ and $y=0$; *i. e.* $\phi(a)=0$, $\psi(a)=0$

$$\lim_{x\to 0}\left(\frac{y}{x}\right)=\lim_{\theta\to 0}\left(\tan\theta\right)=\lim_{x\to 0}\left(\frac{dy}{dx}\right)=\lim_{t\to a}\frac{\phi'(t)}{\psi'(t)}$$

$$\therefore \lim_{t\to a}\frac{\phi(t)}{\psi(x)}=\lim_{t\to a}\frac{\phi'(t)}{\phi'(t)}=\frac{\phi'(a)}{\psi'(a)}, \quad \text{if } \psi'(a)\neq 0$$

Note : In the above theorem the functions and their derivative are continuous. If the functions are not continuous at the point concerned still the theorem holds good.

In fact, if

$\lim_{x\to a}\phi(x)=\lim_{x\to a}\psi(x)=0$ and $\lim_{x\to a}\frac{\phi'(x)}{\psi'(x)}$

exists and equals L, then

$$\lim_{x\to a}\frac{\phi(x)}{\psi(x)}=L$$

Ex. 2. Evaluate $\lim_{x\to 0}\frac{x-\tan x}{x^3}$

$$\lim_{x\to 0}\frac{x-\tan x}{x^3} \quad [\text{ form } \frac{0}{0}]$$

$$=\lim_{x\to 0}\frac{1-\sec^2 x}{3x^2}\left[\text{from }\frac{0}{0}\right.,$$ differentiating both numerator and denominator separately in each case.]

$$=\lim_{x\to 0}\frac{-2\sec^2 x\tan x}{6x}=\lim_{x\to 0}\frac{-1}{3}\sec^2 x\left(\frac{\tan x}{x}\right)$$

$$=\lim_{x\to 0}\left(-\frac{1}{3}\sec^2 x\right)\lim_{x\to 0}\left(\frac{\tan x}{x}\right)=-\tfrac{1}{3}(1)=-\tfrac{1}{3}$$

§ 5 Form $\frac{\infty}{\infty}$.

Let $\lim_{x\to a}\phi(x)=\infty$ and $\lim_{x\to a}\psi(x)=\infty$. Then

$\lim_{x\to a}\frac{\phi(x)}{\psi(x)}$ takes the form $\frac{\infty}{\infty}$. If limit exists, then

$$\lim_{x\to a}\left(\frac{\phi(x)}{\psi(x)}\right)=\lim_{x\to a}\frac{\phi'(x)}{\psi'(x)}=\frac{\phi'(a)}{\psi'(a)}$$

Proof : We can write the expression

$$=\lim_{x\to a}\frac{\phi(x)}{\psi(x)}=\lim_{x\to a}\frac{\left(\frac{1}{\psi(x)}\right)}{\left(\frac{1}{\phi(x)}\right)} \quad [\text{ form } \frac{0}{0}]$$

$$= \lim_{x\to a} \frac{-\frac{1}{\{\psi(x)\}^2}\psi'(x)}{-\frac{1}{\{\phi(x)\}^2}\phi'(x)} \quad \text{[by Art. 8.4.]}$$

$$= \lim_{x\to a} \frac{\psi'(x)}{\phi'(x)}\left[\frac{\phi(x)}{\psi(x)}\right]^2$$

$$= \lim_{x\to a} \frac{\phi(x)}{\phi(x)} = \lim_{x\to a} \frac{\psi'(x)}{\phi'(x)} \lim_{x\to a} \left[\frac{\phi(x)}{\psi(x)}\right]^2$$

If $\lim_{x\to a} \frac{\phi(x)}{\psi(x)} = l$ then

$$l = \lim_{x\to a} \frac{\psi'(x)}{\phi'(x)} l^2 \Rightarrow l = \lim_{x\to a} \frac{\phi'(x)}{\psi'(x)} \quad \ldots \ldots \ldots (1)$$

Case 1. If $l \neq 0$ $l \neq \infty$, then

$$= l \lim_{x\to a} \frac{\phi'(x)}{\psi'(x)} \text{ i.e., } \lim_{x\to a} \frac{\phi(x)}{\psi(x)} = \lim_{x\to a} \frac{\phi'(x)}{\psi'(x)} \quad \ldots \ldots (2)$$

Case II. If $l = 0$, then adding 1 to each side of equation (2) we have.

$$l = \lim_{x\to a} \frac{\phi(x)}{\psi(x)} \text{ or, } l+1 = \lim_{x\to a} \frac{\phi(x)}{\psi(x)} + 1 = \lim_{x\to a} \frac{\phi(x)+\psi(x)}{\phi(x)}$$

$$\left(\text{Form } \frac{\infty}{\infty}\right)$$

$$= \lim_{x\to a} \frac{\phi'(x)+\psi'(x)}{\psi'(x)} = \lim_{x\to a} \frac{\phi'(x)}{\psi'(x)} + 1 \quad [\text{ by (1) as } l+1 \neq 0]$$

$$l = \lim_{x\to a} \frac{\phi(x)}{\psi x} = \operatorname{Lim}_{x\to a} \frac{\phi'(x)}{\psi'(x)}$$

Case III. If $l = \infty$. then

$$\therefore \frac{1}{l} = \lim_{x\to a} \frac{1}{\frac{\phi(x)}{\psi(x)}} = \lim_{x\to a} \frac{\psi'x}{\phi(x)} = \lim_{x\to a} \frac{\psi'(x)}{\phi'(x)} \quad \left(\text{by case II} \atop 1/l = 0\right)$$

or, $\lim_{x\to a} \frac{\phi(x)}{\psi(x)} = \lim_{x\to a} \frac{\phi'(x)}{\phi'(x)}$

From cases I, II & III, it is seen that the theorem is ture of all cases. Thus, if $\lim_{x\to a} \phi(x) = \infty$ ' $\lim_{x\to a} \psi'(x) = \infty$

$$\lim_{x\to a} \frac{\phi(x)}{\psi(x)} = \lim_{x\to a} \frac{\phi'(x)}{\psi'(x)}$$

Hence the theorem is proved for all cases.

Ex. 3. Evaluate

$$\lim_{x\to 1} \frac{\log(1-x)}{\cot \pi x}$$

$$y = \lim_{x\to 1} \frac{\log(1-x)}{\cot \pi x} \quad \left[\text{Form } \frac{\infty}{\infty}\right]$$

$$= \lim_{x\to 1} \frac{-\frac{1}{1-x}}{-\pi \operatorname{cosec}^2 \pi x} = \lim_{x\to 1} \frac{\sin^2 \pi x}{\pi(1-x)} \text{[form 0/0]}$$

$$= \lim_{x\to 1} \frac{2 \sin \pi x \cos \pi x}{-\pi} \pi = 0$$

8. 6. Form $0 \times \infty$

This form can be converted into the form $\frac{0}{0}$ or, $\frac{\infty}{\infty}$

Ex. 4. Evaluate $\lim_{x\to 0} \log(1+x) \frac{1}{\sin x}$.

$$y = \lim_{x\to 0} \log(1+x) \cdot \frac{1}{\sin x} \quad (\text{form } 0 \times \infty)$$

$$= \lim_{x\to 0} \frac{\log(1+x)}{\sin x} \quad \left(\text{form } \frac{0}{0}\right)$$

$$= \operatorname{Lim}_{x\to 0} \frac{1/(1+x)}{\cos x} = \lim_{x\to 0} \frac{1}{(1+x)\cos x} = \frac{1}{(1)(1)} = 1$$

8. 7. Form $0^0, \infty^0, 1^\infty$

These forms can be converted to the form 0/0 or ∞/∞ with the help of logarithm.

The process is better to be demonstrated with some examples given below.

Ex. 5. Evaluate $\lim_{x\to 0} (\sin x)^x$ if $x>0$

Let $y=\lim_{x\to 0} (\sin x)^x$ [form 0^0]

$\because \log y=\lim_{x\to 0} \log (\sin x)^x=\lim_{x\to 0} x \log (\sin x)$ (form $0\times\infty$)

$=\lim_{x\to 0}\dfrac{\log \sin x}{1/x}=\lim_{x\to 0} -\dfrac{\cot x}{1/x^2}$ $\left[\text{form } \dfrac{\infty}{\infty}\right]$

$=\lim_{x\to 0}\dfrac{-x^2}{\tan x}=\lim_{x\to 0}\left(\dfrac{x}{\tan x}\right)\lim_{x\to 0}(-x)=1.\ 0=0$

or, $\log y=0=\log 1$ or, $y=1$ that is,

$\lim_{x\to 0} (\sin x)^x=1$

Ex 6. Evaluate $\lim_{x\to \frac{1}{2}\pi} (\sin x)^{\tan x}$

Let $y=\lim_{x\to\frac{1}{2}\pi} (\sin x)^{\tan x}$ $\therefore \left(\text{form } 1^\infty\right)$

$\therefore \log y=\lim_{x\to\frac{1}{2}\pi} \log (\sin x)^{\tan x}$

$=\lim_{x\to\frac{1}{2}\pi} \tan x \log \sin x$ (form $\infty\times 0$)

$=\lim_{x\to\frac{1}{2}\pi}\dfrac{\log \sin x}{\cot x}$ $\left(\text{form } \dfrac{0}{0}\right)$

$\log y=\lim_{x\to\frac{1}{2}\pi}\dfrac{\cot x}{-\operatorname{cosec}^2 x}=\lim_{x\to\frac{1}{2}\pi} -\cos x \sin x=0$

$\Rightarrow y=e^0=1$

or $\lim_{x\to\frac{1}{2}\pi} (\sin x)^{\tan x}=1$

Ex. 7. Evaluate $\lim_{x\to 1} x^{1/(1-x)}$

Let $y=\lim_{x\to 1} x^{1/(1-x)}$ (form 1^∞)

Then. $\log y=\lim_{x\to 1} \log x^{\frac{1}{1-x}}$

$=\lim_{x\to 1}\dfrac{1}{(1-x)}\log x$ (form $\therefore \infty\times 0$)

$=\lim_{x\to 1}\dfrac{\log x}{1-x}\left(\text{form } \dfrac{0}{0}\right)=\lim_{x\to 1}\left(\dfrac{1}{-x}\right)=-1$

$\Rightarrow y=e^{-1}=\dfrac{1}{e}$

$\therefore \lim_{x\to 1} x^{\frac{1}{1-x}}=\dfrac{1}{e}$

Ex. 8. Find the limit of the expression

$\lim_{x\to 0}\dfrac{x-\sin x}{x^3}$

Let $y=\lim_{x\to 0}\dfrac{x-\sin x}{x^3}\left(\text{form } \dfrac{0}{0}\right)$

$y=\lim_{x\to 0}\dfrac{1-\cos x}{3x^2}\left(\text{form}\dfrac{0}{0}\right)=\lim_{x\to 0}\dfrac{2\sin^2 x/2}{3x^2}$

$$= \lim_{x \to 0} \left(\frac{\sin x/2}{x/2} \right)^2 = \frac{1}{6}.$$

Ex. 9. Fvaluate $\lim_{x \to 0} \frac{\sin x - \log (e^x \cos x)}{x \sin x}$ C. U. 1984

Let $y = \lim_{x \to 0} \frac{\sin x \log (e^x \cos x)}{x \sin x}$

$= \lim_{x \to 0} \frac{\sin x - x - \log \cos x}{x \sin x}$ $\therefore \left(\text{form} \frac{0}{0} \right)$

$[\therefore \log e^x = x \log_e = x]$

$= \lim_{x \to 0} \frac{\cos x - 1 + \tan x}{\sin x + \cos x}$ $\left(\text{form} \frac{0}{0} \right)$

$= \lim_{x \to 0} \frac{-\sin x + \sec^2 x}{\cos x + \cos x - x \sin x} = \frac{1}{2}$

Ex. 10. Evaluate $\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\log x} \right)$ D. U. 1966

$\lim_{x \to 1} \left\{ \frac{x}{x-1} - \frac{1}{\log x} \right\}$ $(\tan \infty - \infty)$

$= \lim_{x \to 1} \frac{x \log x - x + 1}{(x-1) \log x} \left(\text{form} \frac{0}{0} \right) = \lim_{x \to 1} \frac{1 + \log x - 1}{\log x + (x-1)/x} \left(\text{form} \frac{0}{0} \right)$

$= \lim_{x \to 1} \frac{\log x}{\log x + 1 - 1/x} = \lim_{x \to 1} \frac{1/x}{1/x + 1/x^2} = \frac{1}{2}$

Ex. 11 Find the values of a, b and c, if

$$\lim_{x \to 0} \frac{ae^x - b \cos x + ce^{-x}}{x \sin x} = 2$$

Ans. Here $\varphi(x) = a e^x - b \cos x + c e^{-x}$

$\psi(x) = x \sin x$

Since $\psi(0) = 0$,

therefore $\varphi(0) = 0 \Rightarrow a - b + c = 0$ (i)

The given limit $= \lim_{x \to 0} \frac{\varphi(x)}{\psi(x)} = \lim_{x \to 0} \frac{\varphi'(x)}{\psi'(x)}$

Now $\psi'(x) = x \cos x + \sin x$

$\Rightarrow \psi'(0) = 0.$

$\therefore \varphi'(0) = 0$

But $\varphi'(x) = ae^x + b \sin x - ce^{-x}$

$\varphi'(0) = 0 \Rightarrow a - c = 0$ (2)

By L' Hospitals Rule, the given limit is then

$$\lim_{x \to 0} \frac{\varphi''(x)}{\psi''(x)}$$

We have, $\psi''(x) = -x \sin x + \cos x + \cos x$

$\psi''(0) = 0 + 1 + 1 = 2 \neq 0$

Hence $\lim_{x \to 0} \frac{\varphi''(x)}{\psi''(x)} = 2$ (given)

$\Rightarrow \frac{\varphi''(0)}{\psi''(0)} = 2$

or $\varphi''(0) = 2 \times \psi''(0) = 2 \times 2 = 4$

Now $\varphi''(x) = ae^x + b \cos x + ce^x$

$\therefore a + b + c = 4$ (3)

Solving (1), (2), and (3), we get

$a = 1$, $b = 2$, $c = 1$.

Second Method :

Let $y = \lim_{x \to 0} \frac{ae^x - b \cos x + ce^{-x}}{x \sin x} = 2.$

or $\lim\limits_{x\to 0} \dfrac{a(1+x+\frac{x^2}{2!}\cdots)-b(1+\frac{x^2}{2}+\ldots)+c(1-x.+\frac{x^2}{2!}+\ldots)}{x(x-\frac{x^3}{3!}+\ldots)}$

or, $\lim\limits_{x\to 0} \dfrac{(a-b+c)+x(a-c)+\frac{1}{2}x^2(a+b+c)+\ldots\ldots}{x^2(1-\frac{1}{6}x^2+\ldots\ldots)}=2$

As the limit is 2, so

$a-b+c=0,\ a-c=0$ and $\frac{1}{2}(a+b+c)=2$

Now solve for a, b, c.

Hence $a=1, b=2, c=1$.

Ex. 12. Evaluate $\lim\limits_{x\to 0} \dfrac{\sin x-\tan^{-1}x}{x^2 \log(1+x)}$.

The form is $\dfrac{0}{0}$.

We get the result by using L' Hospital's Rule several times. So, expansion by taylor's Theorem is helpful.

$\lim\limits_{x\to 0} \dfrac{\sin x-\tan^{-1}x}{x^2 \log(1+x)}$

$=\lim\limits_{x\to 0} \dfrac{x-x^3/\lfloor 3+x^5/\lfloor 5- \quad -(x-\frac{1}{3}x^3+\frac{1}{15}x^5\ \ldots)}{x^2(x-x^2/2 \quad x^3/3- \quad)}$

$\lim\limits_{x\to 0} \dfrac{(\frac{1}{3}-1/\lfloor 3)\,x^3+x^5\,(1/\lfloor 5-1/15+\ \ldots}{x^3(1-x/2+x^2/3-\ldots)}$

$=\lim\limits_{x\to 0} \dfrac{1/6+x^2(1/\lfloor 5-1/15)+}{1+x/2+x^2/3-\ldots} \quad =\frac{1}{6}$

Ex. 13. (i) Find $\lim\limits_{x\to 0} x^{x^k}$ when $k>0$.

let $u=\lim\limits_{x\to 0} x^{x^k}$.

Now $\log u=\lim\limits_{x\to 0} x^k \log x$ (form $0\times\infty$; $x>0$)

$=\lim\limits_{x\to 0} \dfrac{\log x}{x^{-k}}$ ∴ form ∞/∞

$=\lim\limits_{x\to 0} \dfrac{1/x}{-kx^{-k-1}} = \lim\limits_{x\to 0} \dfrac{x^k}{-k}=0 \Rightarrow u=e^0=1$ (Ans)

13. (ii) show that $\lim\limits_{x\to\infty} (x^{1/x^k})=1$; $k>0$

Let $u=\lim\,(x^{1/x^k})$

Then $\log u=\lim\limits_{x\to\infty} \dfrac{1}{x^k} \log x$

$=\lim\limits_{x\to\infty} \dfrac{\log x}{x^k} \left(\text{form} \dfrac{\infty}{\infty}\right)$

$=\lim\limits_{x\to\infty} \dfrac{1/x}{kx^{k-1}} = \lim\limits_{x\to\infty} \dfrac{1}{kx^k}=0$

$\Rightarrow u=e^0=1$ (Ans)

Exercise VIII

Evaluate each of the following limits.

1. $\lim_{x \to a} \dfrac{x-a}{x^5-a^5}$ 2. $\lim_{x \to 1} \dfrac{\sin \pi x}{\tan 4\pi x}$

1. (i) $\lim_{x \to 0} \dfrac{\tan^{-1} x}{\sin^{-1} x}$ D. U. 1960. 2. (i) $\lim_{x \to 0} \dfrac{x - \sin x}{x}$

3. $\lim_{x \to 0} \dfrac{3 \tan x - 3x - x^3}{x^5}$ 4. $\lim_{x \to 0} \dfrac{\sqrt{x} \tan x}{(e^x - 1)^{3/2}}$

5. $\lim_{x \to 0} \dfrac{e^x - e^{-x} + 2 \sin x - 4x}{x^3}$ 6. $\lim_{x \to 1} \dfrac{1 + \cos \pi x}{\tan^2 \pi x}$ R. H. '66

7. $\lim_{x \to 0} \dfrac{\tan x - x}{x^2 \tan x}$ 8. $\lim_{x \to 0} \dfrac{e^x - e^{\sin x}}{x - \sin x}$

9. $\lim_{x \to 0} \dfrac{e^x - e^{-x} - 2 \log (1+x)}{x \sin x}$ 10. $\lim_{x \to \frac{1}{2}} \dfrac{\cos^2 \pi x}{e^{2x} - 2e^x}$

11. $\lim_{x \to \infty} \dfrac{x + \cos x}{1 + x}$ R.H. '67 12. $\lim_{x \to 0} \dfrac{\log \sin 3x}{\log \sin x}$ D. U. 1967

13. $\lim_{x \to \infty} x^4/e^x$ D. U. 1964 (a) $\mathrm{Lt}_{x \to \pi/2} \dfrac{e^{\tan x} - 1}{e^{\tan x} + 1} = 1$ C.H. 1993

(i) $\lim_{x \to 0} \dfrac{\tan x - \sin x}{\tan x - x}$ D. U. '60 (ii) $\lim_{x \to 0} \dfrac{\sin x - \tan^{-1} x}{x^2 \log (1+x)}$

(14) $\lim_{x \to 0} \dfrac{\log \tan x}{\log x}$ (15) $\lim_{x \to \pi/2} \dfrac{\log (x - \pi/2)}{\tan x}$ D.U.'61

16. $\lim_{x \to 0} \dfrac{\log \tan 2x}{\log \tan x}$ 17. $\dfrac{\log (x-1) + \tan \frac{1}{2} \pi x}{\cot \pi x}$

18. $\lim_{x \to 0} x^m (\log x)^n$ $m, n > 0$ 19. $\lim_{x \to 0} \dfrac{\log (1-x^2)}{\log \cos x}$ D.U. '61

(i) $\lim_{x \to 0} + x^2 \log x$ (ii) $\lim_{x \to \frac{1}{2}\pi} (1 - \sin x) \tan x$ C. U. '87

20 $\mathrm{Lt}_{x \to \alpha} \dfrac{x^n}{e^x}$, $x > 0$ (i) $\mathrm{Lt}_{x \to 0} \dfrac{\log x^2}{\log \cot^2 x}$ (ii) $\mathrm{Lt}_{x \to 1} x(2^{1/x} - 1)$ N.U. 1994

21. $\lim_{x \to 0} \dfrac{e^x - 1 - \log (1+x)}{x^2}$ 22. $\lim_{x \to \pi} \left(\dfrac{\pi}{x - \pi} - \dfrac{x - \sin x}{x - \pi} \right)$

23 $\mathrm{Lt}_{x \to 0} \left(\dfrac{1}{x^2} - \dfrac{1}{\sin^2 x} \right)$ (i) $\mathrm{Lt}_{x \to 1} \dfrac{\sqrt{x} - 1}{\sqrt{x - 1}} = \dfrac{3}{2}$ (ii) $\mathrm{Lt}_{x \to 0} \left(\dfrac{1}{x^2} - \dfrac{1}{\tan^2 x} \right)$ C.U. 1993

24. $\lim_{x \to 0} x \log \sin x$ 25. $\lim_{x \to \infty} 2^x \sin (a/2^x)$

26. $\lim_{x \to 1} \left(\dfrac{1}{\log x} - \dfrac{x}{\log x} \right)$ 27. $\lim_{x \to 0} \left(\dfrac{1}{x^2} - \cot^2 x \right)$ D.U. 67

28. $\lim_{x \to \frac{1}{2}\pi} \sec x \, (x \sin x - \frac{1}{2}\pi)$ 28. (a) $\lim_{x \to 0} \left\{ \dfrac{1}{x} - \dfrac{1}{\tan^{-1} x} \right.$ D. U. 1968

28(a) $\lim_{x \to 4} \dfrac{\sqrt{x} - 2}{\sqrt{7 + \sqrt{x}} - 3} = 6$ C.U. 1993

29. $\lim_{x \to 0} \cot x \log \dfrac{1-x}{1+x}$ D.H.'62, C.H.'77'89 30. $\lim_{x \to 0} \tan x \log x$

31. $\lim_{x \to 0} \dfrac{2 \tan^{-1} x - x}{2x - \sin^{-1} x}$ 31. (a) $\lim_{x \to 0+} x^{1/x}$ D. U. '83

32. $\lim_{x \to a} (a - x) \tan (x\pi/2a)$ 33. $\lim_{x \to 0} \left(\dfrac{1}{x} \right)^{\sin x}$

34. $\lim_{x \to 0} (\sec x)^{\tan x}$ 35. $\lim_{x \to \pi} \left(\dfrac{2\pi}{x} - 1 \right)^{\tan x/2}$

36. $\lim_{x \to e} (\log x)^{1/(x-e)}$ 37. $\lim_{x \to 0} \dfrac{\sqrt[n]{(1+x)} - 1}{x}$ R. H. '66

38. $\lim_{x \to 0} \log (\cos x)^{1/x^2}$ R.H. '62, 39. $\lim_{x \to 0} \left(\dfrac{\tan x}{x} \right)^{1/x^2}$ D.H. '62

40. $\lim_{x \to 0} (\cos x)^{\operatorname{cosec}^2 x}$ 40. (i) $\lim_{x \to 1} \dfrac{x - \sqrt{(2 - x^2)}}{2x - \sqrt{(2 + 2x^2)}} = 2$

41 $\mathrm{Lt}_{x \to 0} \dfrac{\log (1+x^2)}{\sin x^2}$ C.U. 1993 42 $\mathrm{Lt}_{x \to 0} x^x$ (i) $\mathrm{Lt}_{x \to 0} (\cos x)^{1/x}$

43. $\underset{x\to 0}{\text{Lt}} \left(\frac{\tan x}{x}\right)^{1/x}$ D.U. '68 43 (a) $\underset{x\to 1}{\text{Lt}} (1+x)^{1/x}$ D.U.1991; C.U.1992

44. $\lim_{x\to 0} (\cos x)^{1/x^2}$ 45 $\lim_{x\to 0} (1+\sin x)^{\cot x}$

46. $\lim_{x\to 1} (2-x)^{\tan\frac{1}{2}\pi x}$

47. $\lim_{x\to \pi/2} (\tan x)^{\tan 2x}$ 48. $\lim_{x\to 0} (\cos x)^{\cot^2 x}$ C. U. 1980

49. $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}$ D. U. 1962 D. U. 1987 49 (a) $\lim_{x\to 0}\frac{\cosh x-\cos x}{x\sin x}$

50. $\lim_{x\to e} (\log x)^{\frac{1}{1-\log x}}$ 51. $\lim_{x\to 0}\frac{\sin x\sin^{-1}x}{x^2}$ D. U '69

52. $\lim_{x\to 0}\frac{\log(0+x^3)}{\sin^2 x}$ 53 $\lim_{x\to 0}(\coth x)^{\sinh x}$

54. $\lim_{x\to 0}(\cot^2 x)^{\sin x}$ 55. $\lim_{x\to 0}\frac{x-\sin^{-1}x}{\sin^3 x}$

56. $\lim_{x\to 0}\frac{\tan x\tan^{-1}x-x^2}{x^6}$ 56 (a) $\lim_{x\to 0}\frac{4^x-2^x}{x}$ R. U. 1961

57. $\lim_{x\to 0}\frac{e^x\sin x-x-x^2}{x^2+x\log(1-x)}$ 58. $\lim_{x\to 0}\frac{e^x-e^{x\cos x}}{x-\sin x}$

59. $\lim_{x\to \pi/4}\frac{\sec^2 x-2\tan x}{1+\cos 4x}$ 60. $\lim_{x\to \pi/2}(\cos x)^{\cos x}$

61. $\lim_{x\to 0}\frac{\sin 2x+2\sin\frac{1}{2}x-2\sin x}{\cos x-\cos^2 x}$ 62. $\lim_{x\to 0}\frac{x\cos x-\log(1+x)}{x^2}$

63. $\lim_{x\to\infty}\left[x\left(1+\frac{1}{x}\right)^x-ex^2\log\left(1+\frac{1}{x}\right)\right]$

64. Find the values of a, b, c; $\underset{x\to 0}{\text{Lt}}\frac{x(a+b\cos x)+c\sin x}{x^5}=1$

(a) $\underset{n\to\alpha}{\text{Lt}}\frac{1}{n}\sum_{i=1}^{n} e^{i/n}$ C.H. 1992 (b) $\underset{x\to 0}{\text{Lt}}\frac{e^{-1/x^2}}{x^n}$ C.H. 1992

$\lim_{x\to 2}\frac{x^2-4}{\sqrt{(x+2)}-\sqrt{(3x+2)}}=1$

Find the values of a and b so that

$\lim_{x\to 0}\frac{x+ax\cos x-b\sin x}{x^3}=1$

Find the value of m if $\lim_{x\to 0}\frac{\sin x(2\cos x+m)}{x^2}$ is finite

Does limit exist for 68. $\lim_{x\to 0+} x^{1/x}=0$

$\lim_{x\to 1}\frac{\log(x-1)-\tan\frac{1}{2}\pi x}{\cot\pi x}$

Prove that

$\lim_{n\to\infty}\frac{1^m+2^m+3^m\ldots+4n^m}{n^{m+1}}=\frac{1}{m+1},\ m>0$

Does limit exist for 71. $\lim_{x\to 0} +x^m(\log x)^n=0$ D. U. 1984

$\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^3}$ R. H. 1988

If $u=\frac{\cos xy}{\cos y}$ and $x=\sin y$

Prove that

$\lim_{x\to 0}\frac{\frac{d^{n+1}u}{dx^{n+1}}}{\frac{d^{n-1}u}{dx^{n-1}}}=n^2-m^2$

71 (a) $\underset{x\to 0}{\text{Lt}}\frac{x^2\sin(x/4)}{\sin x}$ N.U. (C-2) 1994

(b) $\underset{x\to 0}{\text{Lt}}\frac{e^x-e^{-x}-2x}{x-\sin x}$ D.U. 1991

If $y=(\sin^{-1}x)^2$, Prove that

$\lim_{x\to 0}\frac{\frac{d^{n+2}y}{dx^{n+2}}}{\frac{d^ny}{dx^n}}=x^2$

72. If $u = \dfrac{\text{Cos}xy}{\cos y}$ and $x = \sin y$

Prove that $\underset{x\to 0}{\text{Lt}} \dfrac{d^{n+1}u}{dx^{n+1}} \Big/ \dfrac{d^{n-1}u}{dx^{x-1}} = n^2 - m^2$

(a) If $y = (\sin^{-1}x)^2$, Prove that

$\underset{x\to 0}{\text{Lt}} \dfrac{d^{x+2}y}{dx^{n+2}} \Big/ \dfrac{d^n y}{dx^n} = x^2$

73. Examine whether the function $f(x)$ is continuous at $x=0$
Where $f(x) = x^{2x}$, $x \neq 0$, $f(0) = 1$

Sol. প্রদত্ত রাশি, $f(x) = x^{2x}$

$\therefore \log f(x) = 2x \log x = 2 \dfrac{\log x}{1/x}$

$\therefore \lim\limits_{x\to 0} \log f(x) = 2 \lim\limits_{x\to 0} \dfrac{\log x}{1/x}$ $\therefore$ form $\frac{\infty}{\infty}$

$= -2 \lim\limits_{x\to 0} \dfrac{1/x}{1/x^2} = 0$

এখন $\lim\limits_{x\to 0} \log f(x) = \log \lim\limits_{x\to 0} f(x)$

অতএব $\log \lim\limits_{x\to 0} \log f(x) = 0$, ফলে

$\lim\limits_{x\to 0} f(x) = e^0 = 1$

দেওয়া আছে $f(0) = 1$. ফলে $\lim\limits_{x\to 0} f(x) = f(0)$

অতএব $x = 0$ বিন্দুতে $f(x)$ অবিচ্ছিন্ন।

73. Evaluate $\lim\limits_{a\to b} \dfrac{a^x \sin bx - b^x \sin ax}{\tan bx - \tan ax}$ R. U. 19..

ANSWERS

1. $1/5a^4$ (i) 1	2. $-\frac{1}{4}$ (i) 0	3. 2/5
4. 1.	5. 0	6. $\frac{1}{2}$
7. $\frac{1}{3}$	8. 1	9. 1
10. $\pi^2/2e$	11. 1	12. 1
13. 0, (i) 3/2, (ii) 1/6	14. 1	15. 0
16. 1.	17. -2.	18. 0 (i) 0
19. 2. (i) 0	20. 0. (i) -1	21. 1
22. -2.	23. $-\frac{1}{3}$ (i) 0	24. 0
25. a	26. -1	27. 2/3
28. -1, (a) 0	29. -2	30. 1
31. 1	32. $2a/\pi$	33. 1
34. 1	35. 1	36. c
37. $1/n$	38. $-\frac{1}{2}$.	39. $e^{1/3}$
40. $1/\sqrt{e}$	41. $1/c$	42. 1. (i) $1/\sqrt{e}$
43. 1.	44. $e^{-1/2}$	45. e
46. $\frac{2/\pi}{e}$	47. 1	48. $\frac{-1/2}{e}$
49. $e^{-1/6}$ 49. (a) 1	50. $1/e$	51. 1
52. 1	53. 1	54. 1
55. $-\frac{1}{6}$	56. 2/9 (a) log2	57. $-2/3$
58. 3	59. $-\frac{1}{2}$	60. 1
61. 4	62. $\frac{1}{2}$	63. 0

64. $a=120$, $b=60$, $c=180$. 65—8

66. $a=-5/2$, $b=-3/2$

67. $m=-2$, limit$=-1$

68. 2

69. Limit does not exist. 70. $b^x(b \cos bx - \sin bx) \cos^2 bx$