CHAPTER IX

PARTIAL DIFFERENTIATION

9.1. Functions of Several Variables

In previous chapters, we have discussed the functions of [one] independent variable. In this chapter, we now turn to func[tions] of more than one independent variable.

Let us give two examples which are familiar to us.

The volume, v of a right circular cylinder of radius x [and] height y is given by $v=\pi x^2 y$ ... ... (1)

Thus we see that v depends upon two variables x and y. [i.e.] $v=(x, y)$; v is the function of x and y.

The volume of a Rectangular parallelopiped whose len[gth] breadth and height are respectively x, y and z is $v=xyz$......(2)

Thus in this case v is the functions of three variables x, y [and] z, and can be represented as $v=f(x, y, z)$.

Similarly if u is the functions of $x_1 x_2, x_3$ ....x_n variable[s] then we represent this function u in the form

$$u=f(x_1, x_2, x_3, \cdots x_n)$$

Art 9.1 (a) Domain

Let $u = f(x, y)$ ..... (1)

x, y are independent variables and u depends upon them [u is] the function of x and y. The ordered pair is called a point. [The] aggregate of the pair of numbers (x, y) is said to be domain [or] region of a detinition of the function. If the domain is closed [by] curve C, then f is said to be closed when it is defined for the p[oints] within and on the curve. It the function u or f is defined for p[oints] within C not on the points on the boundary, then the domain [is] called open.

Art 9.1 (b) Neighbourhood (nbd.) of a point (a, b)

A non empty subset N of R is called a nbd. of a point $x_0 \in R$ if [there] exists an open interval I such that $x_0 \in I \subseteq N$. Again we express [it as] $x_0 \in (x_0-\epsilon, x_0+\epsilon) \subseteq N$, $\epsilon>0$. If x_0 be real number (a, b) such that [...], then $(x_0-\epsilon, x_0+\epsilon)$ is a nbd. of $x_0 \in$ (a, b). This nbd. is called [ne]ighbourhood (nbd.) of x_0 and is denoted by $N(x_0, \epsilon)$.

[...] Nbd. of $x_0 \in$ (a, b) is expressed as $\{x \in R : |x_1 - x_2| < \epsilon\}$

For the case $u = f(x, y)$ at (a, b), we have the set of values x_1, y_1 [other] than a, b that satisfy $|x_1 - a| < \epsilon$, $|y - b| < \epsilon$ where $d > 0$ but [...] is said to form a nbd. of the point (a, b). Thus x may take any [value] from $(a - \delta, a + \delta; b - \epsilon, b + \epsilon)$ such that a takes values from [...] $a + \epsilon$ except a and is takes values from $b - \epsilon$ to $b + \epsilon$.

In metric space the set $N(p, r) = \{x \in X; d(p, x) < r$ is called a [nbd.] of a point p. The number r is called the sadius of N(p. r)

[...] $r(\theta^2-1)=a^r\theta^2$

[Ex.] $N(p,r)=\{x \in R : |x - p| < r\}$, R is the set of rational numbers

$\{x \in R : r-p<x<r+p\} = (p-r, p+r)$, open interval.

Thus nbd. of p is an open interval with p as centre of the circle.

[All] points inside the circle $x^2+y^2 = \epsilon^2$ may be taken as a nbd of the [point] (0,0)

A nonempty subset N of R is called a nbd. of a point x if there [exists] $\epsilon>0$ such that

$x \in (x - \epsilon, x + \epsilon) \subseteq N \subset R$

$N(x, \epsilon) = \{x \in R : |x - x| < \epsilon\}$

Art 9.1 (c) Limit Point

A real number x is called a limit point of the set $A \subset R$ if every [nbd.] of x_0 contains infinitely many points of A

A point (x_0, y_0) is called a limit point or cluster point or [condens]ation or accumulation point of A if every nbd of (x_0, y_0) [contain]s an infinite number of points of A.

The limit point itself may or may not be a point of the set.

Ex. The point (x_0, y_0) is a limit point of the set $\{\frac{1}{m}, \frac{1}{n} : m$ ... $N\}$. Limit point is not in the set.

Ex. Prove that by using $(\delta, \in)$

$\text{Lt}\ (2x^2-3y) = -2$

$(x, y) \to (x, 3)$

Sol. Let us consider a small poisitive number $\in > 0$ depending upon $\delta>0$ such that

$|2x^2-3y+1| < \in$ When $|x-(-2)| < \delta$, $|y-3|<\delta$

Now $|x-(-1)| < \delta|$, $|y-3|<\delta$

i. e. $-2-\delta < x < -2+\delta$, $3-\delta < y < 3+\delta$

Where $x \neq -2$, $y \neq 3$

Again $(-2-\delta)^2 < x^2 < (-2+\delta)^2$, $9-3\delta < 3y < 9+3\delta$

or, $4+4\delta+\delta < x^2 < 4-4\delta+\delta^2$; $9-3\delta<3y<9+3\delta$

or, $8+8\delta+2\delta^2+>2x^2 < 8-8\delta+2\delta^2$, $9-3\delta < 3y < 9+3\delta$

$\therefore$ $8+8\delta+2\delta^2-9+3\delta < x^2-3y < 8-8\delta+2\delta^2-9-3\delta$

Or, $-1+11\delta+2\delta^2 < x^2-3y < -1-11d+2\delta^2$

Or, $11\delta < x^2-3y+1 < -11\delta$

$\therefore$ $|x^2-3y+1| < 11\delta$, it $\delta = \in/11$, then

$x^2-3y+1 < 11 \cdot \in/11 = \in$

Hence $x^2-3y = -1$.

9. 2. Contiauity of a function of two variables.

The function $z=f(x, y)$ is said to be continuous for $x=a$, $y=b$ when

$$\lim_{(x, y)\to(a, b)} f(x, y) = f(a, b)$$

no matter how x and y approach their limits a and b respectively in the domain. We may explain the continuity of function in the following way also.

The function $f(x, y)$ is said to be continuous at (a, b) if and only if for each $\in>0$, there exists a number $\delta>0$ such that $|f(x, y)-f(a, b)| < \in$

for all points $(x, y) \neq (a,b)$

for which $|x-a| < \delta$ and $|y-b| < \delta$

i. e., $a-\delta \leqq x \leqq a+\delta$, and $b-\delta \leqq y \leqq b+\delta$

For the continuity of $f(x, y)$ at (a,b) there is a in a region R in the xy-plane bounded by the lines $x=a-\delta$, $x=a+\delta$, $y=b-\delta$, $y=b+\delta$ such that for any point (x, y) in R $f(x, y)$ lies between $f(a, b)-\in$ and $f(a, b)+\in$ where $\in$ is any positive number, however small.

Limit of a function of two variables

A function $z=f(x,y)$ is said to tend to the limit l as (x,y) tends to (a, b) if and only if for each positive number $\in>0$ however small, there exists a number $\delta>0$ such that.

$$|f(x, y)-l| < \in$$

for all $(x, y) \neq (a, b)$ lying in the region R given by

$$|x-a| < \delta \text{ and } |y-b| < \delta$$

i. e, $a-\delta \leqq x \leqq a+\delta$ and $b-\delta \leqq y \leqq b+\delta$.

We express this in as

$$\lim_{(x, y)\to(a, b)} f(x, y)=1$$

no matter how x and y approach their limits a and b in the domain.

9. 3. Geometrical Representation of functions of two variables.

Let us consider function of two variables x and y

$$z=f(x,y). \qquad \text{(i)}$$

Let us consider three perpendicular axis such as x—axis y— axis and z—axis meeting at point. O.

For each pair of values of x and y, there corresponds a point P on the xy plane and we can draw a perpendicular PQ to the xy—plane and (x,y) such that

$$PQ=z=f(x, y),$$

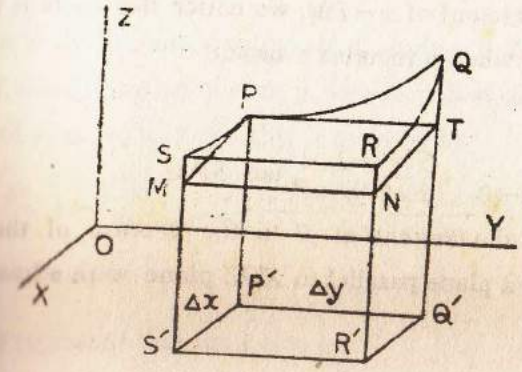
$(z>0,=a>0)$according as Q is above on or below the xy plane

Similary, lengths of the perpendiculars drawn at $P_2(x_2, y_2)$ $P_3(x_3, y_3)$......$P_n(x_n, y_n)$ are respectively

$$P_2Q_2=z_2=f(x_2,y_2)$$
$$P_3Q_3=z_3=f(x_3, y_3)$$
$$\dots \quad \dots \quad \dots$$
$$P_nQ_n=z_n=f(x_n,y_n)$$

All points $Q_1, Q_2, Q_3 \cdots\cdots Q_n$ are in space and they all lie in a surface. Thus $z=f(x, y)$ will represent a surface,

9. 4. Geometrical Representation of Partial Derivatives.



Let $z=f(x, y)$ represents a surface $PQRS$, a portion of the surface is cut off by the planes parallel to XOZ and YOZ.

Let the co-ordinates of P be (x, y, z) and these of

Q and $\{x, y+\Delta y\ f(x, y+\Delta y)\}$

R and $\{x+\Delta x, y+\Delta y, f(x+\Delta x, y+\Delta y)\}$

S and $\{x+\Delta x, y\ ;\ f(x+\Delta x, y)\}$

From the figure we notice that TQ is the increment of z and PT is the increment of y, while x remains constant. Thus from the definition, we have

$$\frac{\partial z}{\partial y}=\lim_{\Delta y\to 0} \frac{TQ}{PT}=\lim_{Q\to P} \tan \angle TPQ \quad (TQ=\delta z, PT=\delta y)$$

= Slope of the tangent at P to section of the surface $z=f(x, y)$ by a plane throug P parallel to the plane YOZ with a line parallel to y—axis.

For the increment of $x=PM$, we notice that there is an increment SM of z, when y remains constant.

Then

$$\frac{\delta z}{\delta x}=\lim_{\Delta x\to 0}\frac{SM}{PM}=\lim_{S\to P}\tan SPM$$

= Slope of the tangent at P to the section of the surface $z=f(x, y)$ by a plane parallel to XOZ plane with a line parallel to x—axis.

9. 5. Partial Derivatives

Let $z=f(x, y)$ be a function of two indepent variables defined in a domain. If y remains constant, the point (x, y) will move along a line parallel to x–axis. In this case any variation in z depends on the variable of the single variable x only and the derivate of z, w, r, to x is called the derivative af z. with r. to x provid if such derivate exists.

Symbollically we represent it by

$$\left(\frac{\delta z}{\delta x}\right) \text{ or, simply by } \frac{\delta z}{\delta x}$$

Thus

$$\frac{\delta z}{\delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x, y)-f(x,y)}{\Delta x} \text{ if limit exists.}$$

The partial derivative is represented by the following symblos also

$$\frac{\delta z}{\delta x} \text{ or, } f_x \text{ or, } \frac{\delta f(x, y)}{\delta x} \text{ or, } D_x z$$

Similarly if x remains constant during dfferentiation and y varies, the derivative of z, w, r, to y is called the partial derivative of z, w, r, to y and this is denoted by

$$\left(\frac{\delta x}{\delta y}\right)=\lim_{\Delta y\to 0}\frac{f(x, y+\Delta y)-f(x, y)}{\Delta y} \text{ ; if limit exists}$$

x = constant

We represent this limit also by

$$\frac{\delta z}{\delta y} \text{ or, } f_y \text{ or, } \frac{\delta f(x, y)}{\delta y} \text{ or, } D_y z$$

In case of a function of several variables the partial derivative of the function w, r. to a particular variable is obtained by treating all other variables as constants and differentiating the function with respect to the variable under consideration.

ART 9.5 (A) MEAN VALUE THROREM

If f_x (a,b) exists throughout a neighbourhood of a point $P(a,b)$ and $f_y(a, b)$ exists for any point $Q(a+h, b+k)$ of this neighbourbood (nbd.),

$f(a+h, b+k)—f(a, b) = hf_x(a + \theta h, b+k) + k\,[f_y(a, b) + \eta]$

Where $0 < \theta < 1$, x is the function of k if $k\to 0$, there $\pi\to 0$

Proof. We have

$f(a+h, b+k)–f(a,b)=f(a+h, b+k)–f(a, b+k)+f(a,b+k)–f(a, b)$

Since f_x exists in a nbd. of (a, b), therefore by...(1)

Lagrange's Mean Value Throrem, We have

$f(a+h,\ b+k)-f(a,\ b+k)$

$= (a + h - a)\ f_x(a+\theta_1 h.\ b+k) = hf_x(a+\theta_1 h,\ b+k),\ 0 < \theta_1 < 1$ ... (2)

Also if $f_y\ (a,b)$ exists in the same nbd. of $(a.b)$, then

$\underset{k \to 0}{\text{Lt}}\ \dfrac{f(a,b+k)-f(a,b)}{k} = f_y(a,b)$; (implies that)

$\Rightarrow f(a,\ b+k)-f(a,b) = k\ f_y\ (a,b) + \eta k$, where x is the function of k; if $k \to 0$, then $\eta \to 0$ .... (3)

From (1), (2) and (3), we have

$f(a+h,\ b+k)-f(a,b) = hf_x(a+\theta h,\ b+k) + k[f_y(a,\ b)+\eta k]$ ... (4)

Where $0<\theta<1$, $h \to 0$, $k \to 0$, also $\eta \to 0$, η is the function of k.

Sccessive Partial Derivatives of some symbols.

$\dfrac{\delta^2 u}{\delta x^2} = f_{xx}(x,y)$ may be written as $u_{xx} = f_{xx}$

$\dfrac{\delta^2 u}{\delta y^2} = f_{yy}'x,\ y)$ is replaced by $u_{yy} = f_{yy}$

$\dfrac{\delta^2 u}{\delta y \delta x} = f_{yx}$ or, $u_{yx} = f_{yx}$ and $\dfrac{\delta^2 u}{\delta x \delta y} = f_{xy}$ or, u_{xy}

How to use the above symbol u_{yx} or, f_{yx}

Differentiate the function $u = f(x,\ y)$ *w. r.* to only x first then differentiate the differentiated function again *w. r.* to y; u_{yx} or f_{yx}

Differentiate the function $u = f(x,\ y)$ *w. r.* to only y first and differentiate the differentiated function *w. r.* to only x.

In Partial Derivative of higher order say u_{yzx}, differentiation is made *w. r.* to the outer most variable first, (here, *w. r.* to y) then differentiation is made *w. r.* to a variable just before it (i, e. z) and so on.

Ex. 1. If $u = xyz + x^3 - z^3 + 2z^2 y$

then

$$\frac{\delta u}{\delta x} = yz + 3x^2,\quad \frac{\delta u}{\delta y} = xz + 3z^2,\quad \frac{\delta u}{\delta z} = xy - 3z^2 + 4yz$$

Ex. 2. If $u = ax^2 y + by^2 + czx$. find $u_{xx}, u_{yx}, u_{xy}, u_{yy}$

$$u = ax^2 y + by^2 + czx$$

$\therefore\ u_x = 2axy + cz$; $\qquad u_{xx} = 2ay$

$u_y = ax^2 + 2by$; $\qquad u_{xy} = 2ax$

$u_x = 2axy + cz$; $\qquad u_{yx} = ax$

$u_y = ax^2 + 2by$; $\qquad u_{yy} = 2b$

9.6. If the derivatives $\dfrac{\delta f}{\delta x} = f_x$ and $\dfrac{\delta f}{\delta y} = f_y$ exist in the neigbourhood of the point $(x,\ y)$ and are differentiable at that point then at $(x,\ y)$

i. e., if $f_x, f_y\ f_{xy},\ f_{yx}$ all exist at a point $(x,\ y)$ and f_{xy} or f_{yx} is continuous at the point, then $f_{xy} = f_{yx}$

We know $\dfrac{\delta f}{\delta x} = f_x(x,\ y) = \lim\limits_{h \to 0} \dfrac{f(x+h,\ y) - f(x,\ y)}{h}$

Again $\dfrac{\delta^2 f}{\delta y \delta x} = f_{yx}(x,\ y) = \lim\limits_{k \to 0} \dfrac{f_x(x,\ y+k) - f_x(x,y)}{k}$

$= \lim\limits_{k \to 0} \lim\limits_{h \to 0} \dfrac{[f(x+h,\ y+k) - f(x,\ y+k) + f(x+h,\ y) - f(x,y)]}{hk}$

$= \lim\limits_{k \to 0} \lim\limits_{h \to 0} \dfrac{\varphi(y+k) - \varphi(y)}{hk}$ — — ...(1)

where $\phi(y) = f\ (x+h,\ y) - f(x,\ y)$

and $\phi(y+k)=f(x+h, y+k)-f(x, y+k)$

$\therefore \quad \phi(y+k)-\phi(y)=(y+k-y)\phi_y(\xi)$; $0<\theta_1<1$ and $\xi=y+\theta_1 k$

$=k\phi_y(y+\theta_1 k)$ [*by Mean value theorem.*]

$=k\{f_y(x+h, y+\theta_1 k)-f_y(x, y+\theta_1 k\}$

Put $F(x)=f_y(x,y+\theta_1 k)$

then $F(x+h)=f_y(x+h, y+\theta_1 k)$

$\therefore \quad \phi(y+k)-\phi(y)=k\{F(x+h)-F(x)\}$

$=kh\ F_x(\eta)$; $\eta=x+\theta_2 h$ and $0<\theta_2<1$ [by Mean value theorem]

$=kh\ F_x(x+\theta_2 h)=kh\ f_{xy}(x+\theta_2 h, y+\theta_1 h)$

Now from (1) we have,

$$f_{yx}(x, y)=\lim_{k\to 0}\ \lim_{h\to 0}\ \frac{khf_{xy}\ (x+\theta_2 h, y+\theta_1 k)}{kh}$$

$$=f_{xy}(x,y) \text{ or, } \frac{\delta^2 f}{\delta y\delta x}=\frac{\delta^2 f}{\delta x\delta y} \text{ (proved.)}$$

Ex. 3. Find the values of f_{xy} and f_{yx} when

$$f(x, y)=e^x \cos y$$

$f_x=e^x \cos y$

$\therefore \quad f_{yx}=\frac{\delta}{\delta y}(f_x)=-e^x \sin y$

Again $f_y=-e^x \sin y$

So $f_{xy}=\frac{\delta}{\delta x}(f_y)=-e^x \sin y$

$\therefore \quad f_{xy}=f_{yx}$

Note :—It is not always true that commutativity of f_{xy} and f_{yx} will always hold good. The commutativity depends upon the commutativity of the two limits which are not always true *e. g.*

$$\lim_{x\to 0}\ \lim_{y\to 0}\ \frac{x+y}{x-y}=\lim_{x\to 0}\frac{x}{x}=1$$

$$\lim_{y\to 0}\ \lim_{x\to 0}\ \frac{x+y}{x-y}=\lim_{y\to 0}\frac{y}{-y}=-1$$

$\therefore$ Hence $\lim_{x\to 0}\ \lim_{y\to 0}\frac{x+y}{x-y}\neq\lim_{y\to 0}\ \lim_{x\to 0}\ \frac{x+y}{x-y}$

9.7. The Total Differential

Let us consider the function

$$u=f(x, y) \quad \dots \quad \dots \quad \dots \quad (1)$$

Let u change to $u+\triangle u$, when x changes to $x+\triangle x$ and y changes to $y+\triangle y$. That is

$$u+\triangle u=f(x+\triangle x, y+\triangle y) \quad \dots \quad \dots \quad \dots \quad (2)$$

or, $\triangle u=f(x+\triangle x, y+\triangle y)-f(x, y)$

$=f(x+\triangle x, y+\triangle y)-f(x,y+\triangle y)+f(x,y+\triangle y)-f(x,y)$

$=\triangle x f_x(x+\theta_1\triangle x,y+\triangle y)+\triangle y f_y(x,y+\theta_2\triangle y)$

where $0<\theta_1<1$, $0<\theta_2<1$. [by Mean value Theorem]

If $\triangle x$ and $\triangle y$ are very small, then

$$f_x(x+\theta_1\triangle x,y+\triangle y)=f_x(x,y)+\epsilon_1 \dots \quad \dots \quad \dots \quad (3)$$

where $\lim\limits_{(\triangle x,\triangle y)\to(0,0)}\epsilon_1=0$. also $\lim\limits_{(\triangle x, \triangle y)\to(0,0)} f(x+\theta_1\triangle x, y+\triangle y)=f_x(x, y)$

Similarly, $f_y(x, y+\theta_2\triangle y)=f_y(x, y)+\epsilon_2 \quad \dots \quad \dots \quad (4)$

where $\epsilon_2\to 0$ as $(\triangle x, \triangle y)\to(0, 0)$

$\therefore \quad \triangle u=f_x(x, y)\ \triangle x+\epsilon_1\triangle x+f_y(x, y)\triangle y+\epsilon_2\triangle y$

$$=\frac{\delta f}{\delta x}\triangle x+\frac{\delta f}{\delta y}\triangle y+\epsilon_1\ \triangle x+\epsilon_2\triangle y \quad \dots \quad \dots \quad (5)$$

Where ϵ_1 and ϵ_2 are infinitesimals which will be zero if $\triangle x$ and $\triangle y$ both tend to zero.

The increment of u consists of two parts, first part contains the partial derivatives and the 2nd part contains the infinitesimals with increments of x and y.

The princpial part of $\triangle u$ that is,

$$\frac{\delta f}{\delta x}\Delta x+\frac{\delta f}{\delta y}\Delta y.$$

Is called the total differential of u and it is denoted by.

$$du=\frac{\delta f}{\delta x}\Delta x+\frac{\delta f}{\delta y}\Delta y=\frac{\delta u}{\delta x}\Delta x+\frac{\delta u}{\delta y}\Delta dy \ldots\ldots (6)$$

This definition is true for all functions of u.

When $u=x$, (6) gives

$$dx=1.\ \Delta x+0=\Delta x \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

Similarly, then we take $u=y$ in (6), we get

$$dy=0+1.\ \Delta y=\Delta y \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

Hence by (7) and (8), (6) becomes

$$du=\frac{\delta f}{\delta x}dx+\frac{\delta f}{\delta y}dy=\frac{\delta u}{\delta x}dx+\frac{\delta u}{\delta y}dy \quad \ldots\Big\} \quad (9)$$

or, $du=f_x dx+f_y dy=u_x dx+u_y dy$

which is the total differential of u,

Cor. If $u=f(x, y, z)$ then by definition of total differential we have

$$du=\frac{\delta u}{\delta x}\Delta x+\frac{\delta u}{\delta y}\Delta y+\frac{\delta u}{\delta z}\Delta z$$

which, after putting

$u=x$, $u=y$ and $u=z$ successively, will give

$$du=\frac{\delta u}{\delta x}dx+\frac{\delta u}{\delta y}dy+\frac{\delta u}{\delta z}dz$$

$$\text{or, } du=f_x dx+f_y dy+f_z dz \quad \ldots \quad \ldots \quad (10)$$

If $u=f(x_1, x_2, \quad \ldots \quad \ldots \quad x_n)$ then

$$du=\frac{\delta f}{\delta x_1}dx_1+\frac{\delta f}{\delta x_2}dx_2++\cdots+\frac{\delta f}{\delta x_n}dx_n \cdots (11)$$

Total Differential Co-efficient

9.8. If $u=f(x, y)$ and $x=\phi(t)$ and $y=\psi(t)$ and $\phi'(t)$ and $\psi'(t)$ exist, then

$$\frac{du}{dt}=\frac{\delta f}{\delta x}\cdot\frac{dx}{dt}+\frac{\delta f}{\delta y}\cdot\frac{dy}{dt} \quad \ldots \quad \ldots \quad (13)$$

Proof : u is the function of x and y : x and y are functions of t.

Hence u is the function of t only

Now from (5), Art. 9. 7. we have

$$\Delta u=\frac{\delta f}{\delta x}\Delta x+\frac{\delta f}{\delta y}\Delta y+(\epsilon_1\Delta x+\epsilon_2\Delta y)\ ;\ \epsilon_1, \epsilon_2>0$$

Divide both sides by Δt,

$$\text{or, } \frac{\Delta u}{\Delta t}=\frac{\delta f}{\delta x}\frac{\Delta x}{\Delta t}+\frac{\delta f}{\delta y}\frac{\Delta y}{\Delta t}+\left(\epsilon_1\frac{\Delta x}{\Delta t}+\epsilon_2\frac{\Delta y}{\Delta t}\right)$$

$$\therefore \lim_{\Delta t\to 0}\frac{\Delta u}{\Delta t}=\frac{\delta f}{\delta x}\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t}+\frac{\delta f}{\delta y}\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t}$$

$$+\lim_{\Delta t\to 0}\left(\epsilon_1\frac{\Delta x}{\Delta t}+\epsilon_2\frac{\Delta y}{\Delta t}\right)$$

$$\text{or, } \frac{du}{dt}=\frac{\delta f}{\delta x}\frac{dx}{\delta t}+\frac{\delta f}{\delta y}\frac{dy}{dt} \ :\ \epsilon_1, \epsilon_2\to 0 \text{ as } \Delta t\to 0\ ]$$

$$\therefore \frac{du}{dt}=\frac{\delta f}{\delta x}\frac{dx}{dt}+\frac{\delta f}{\delta y}\frac{dy}{dt} \quad \ldots \quad \ldots \quad (14)$$

Which is called the Total differential co-efficient of u.

Cor. If $u\ f(x,y)$ and $y=\varphi(x)$, then

$$\frac{du}{dx}=\frac{\delta u}{\delta x}\cdot\frac{dx}{dx}+\frac{\delta u}{\delta y}\frac{dy}{dx}$$

$$\text{or, } \frac{du}{dx}=\frac{\delta u}{\delta x}+\frac{\delta u}{\delta y}\frac{dy}{dx}=\frac{\delta u}{\delta x}+\frac{\delta u}{\delta y}\ \ldots\varphi'(x).$$

Cor. If $u=f(x_1, x_2, x_3\cdots x_n)$ and

$x_1\, x_2 \ldots\ldots x_n$ are all functions of t only, and if all the derivatives are continuous, then

$$\frac{\delta u}{\delta t}=\frac{\delta u}{\delta x_1}\frac{dx_1}{dt}+\frac{\delta u}{\delta x_2}\frac{dx_2}{dt}\cdots+\frac{\delta u}{\delta x_n}\frac{dx_n}{dt}$$

9.9. If u=f(x.y) and x=ϕ(r, s, t) and y=ψ(r, s, t) then prove that

$$\frac{\delta u}{\delta r}=\frac{\delta u}{\delta x}\cdot\frac{\delta x}{\delta r}+\frac{\delta u}{\delta y}\cdot\frac{\delta y}{\delta r}$$

$$\frac{\delta u}{\delta r}=\frac{\delta x}{\delta x_1}\cdot\frac{\delta x}{\delta s}+\frac{\delta x}{\delta y}\cdot\frac{\delta y}{\delta s}$$

$$\frac{\delta u}{\delta t}=\frac{\delta u}{\delta x}\cdot\frac{\delta x}{\delta t}+\frac{\delta u}{\delta y}\cdot\frac{\delta y}{\delta t}$$

Proof : $x=\phi(r, s, t)$ and $y=\psi(r, s, t)$,

$u=f(x, y)=f\{(\phi(r, s, t), \psi(r, s, t)\}$

or, $u=F(r, s, t)$, (suppose)

The total differential of u is

$$du=\frac{\delta u}{\delta r}dr+\frac{\delta u}{\delta s}ds+\frac{\delta u}{\delta t}dt \quad \cdots \cdots \quad (1)$$

Again $x=\phi(r, s, t)$, and $y=\psi(r, s, t)$

Then $$dx=\frac{\delta x}{\delta r}dr+\frac{\delta x}{\delta s}ds+\frac{\delta x}{\delta t}dt \quad \cdots \quad (2)$$

$$dy=\frac{\delta y}{\delta r}dr+\frac{\delta y}{\delta s}ds+\frac{\delta y}{\delta t}dt \quad \cdots \quad (3)$$

But if we consider u as the function of x and y i. e.

$u=f(x, y)$, then

$$du=\frac{\delta u}{\delta x}dx+\frac{\delta u}{\delta y}dy$$

$$=\frac{\delta u}{\delta x}\left(\frac{\delta x}{\delta r}dr+\frac{\delta x}{\delta s}ds+\frac{\delta x}{\delta t}dt\right)+\frac{\delta u}{\delta y}\left(\frac{\delta y}{\delta r}dr+\frac{\delta y}{\delta s}ds+\frac{\delta y}{\delta t}dt\right)$$

by (2) and (3)

$$=\left(\frac{\delta u}{\delta x}\cdot\frac{\delta x}{\delta r}+\frac{\delta u}{\delta y}\cdot\frac{\delta y}{\delta r}\right)dr+\left(\frac{\delta u}{\delta x}\cdot\frac{\delta x}{\delta s}+\frac{\delta u}{\delta y}\cdot\frac{\delta y}{\delta s}\right)ds$$
$$+\left(\frac{\delta u}{\delta x}\cdot\frac{\delta x}{\delta t}+\frac{\delta u}{\delta y}\cdot\frac{\delta y}{\delta t}\right)dt \ldots\ldots (4)$$

Now compains the co-efficient of dr, ds, dt from (1) and (4),

$$\frac{\delta u}{\delta r}=\frac{\delta u}{\delta x}\cdot\frac{\delta x}{\delta r}+\frac{\delta u}{\delta x}\cdot\frac{\delta y}{\delta r}$$

$$\frac{\delta u}{\delta s}=\frac{\delta u}{\delta x}\cdot\frac{\delta x}{\delta s}+\frac{\delta u}{\delta y}\cdot\frac{\delta y}{\delta s}$$

$$\frac{\delta u}{\delta t}=\frac{\delta u}{\delta x}\cdot\frac{\delta u}{\delta t}+\frac{\delta u}{\delta y}\cdot\frac{\delta y}{\delta t}$$

Cor. If $u=f(x_1, x_2, x_3\cdots\cdots x_n)$ and $x_1=(t_1, t_2, t_3\cdots\cdots t_n)$

$x_2=\psi(t_1, t_2, t_3\cdots\cdots t_n)$ etc. then

$$\frac{\delta u}{\delta t_1}=\frac{\delta u}{\delta x_1}\cdot\frac{\delta x_1}{\delta t_1}+\frac{\delta u}{\delta x_2}\cdot\frac{\delta x_2}{\delta t_1}+\cdots\cdots+\frac{\delta u}{\delta x_n}\cdot\frac{\delta x_n}{\delta t_1}$$

$$\frac{\delta u}{\delta t_2}=\frac{\delta u}{\delta x_1}\cdot\frac{\delta x_1}{\delta t_2}+\frac{\delta u}{\delta x_2}\cdot\frac{\delta x_2}{\delta t_2}+\cdots\cdots+\frac{\delta u}{\delta x_n}\cdot\frac{\delta x_n}{\delta t_2}$$

... ...

... ...

... ...

$$\frac{\delta u}{\delta t_n}=\frac{\delta u}{\delta x_1}\cdot\frac{\delta x_1}{\delta t_n}+\frac{\delta u}{\delta x_2}\cdot\frac{\delta x_2}{\delta t_n}+\cdots\cdots\frac{\delta u}{\delta x_n}\cdot\frac{\delta x_n}{\delta t_n}$$

9.10. Partial Differentiation of Implicit functions

Let y defind implicitly as a function of $u=f(x, y)=c$, when c is a constant. Then $du=0\Longrightarrow$

$$0=\frac{\delta f}{\delta x}dx+\frac{\delta f}{\delta y}dy \text{ or, } \frac{\delta f}{\delta x}+\frac{\delta f}{\delta x}\cdot\frac{\delta y}{\delta x}=0$$

or. $$\frac{dy}{dx}=-\frac{(\delta f/\delta x)}{(\delta f/\delta y)}=-\frac{f_x}{f_y} \text{ : if } f_y\neq 0$$

$$\therefore \quad \frac{dy}{dx}=-\frac{f_x}{f_y} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Ex. 5. If $f(x, y)=12x^3-2xy+3y^2,=0$ find $\frac{dy}{dx}$.

We have, $f_x=36x^2-2y$; $f_y=-2x+6y$

$$\therefore \quad \frac{dy}{dx}=-f_x/f_y=-\frac{(36x^2-2y)}{-2x+6y}=\frac{18x^2-y}{3y-x}$$

9.11. Perfect or Exact Differential

Find the condition that the expression

$$P\,dx+Q\,dy \quad \ldots \quad \ldots \quad (1)$$

will be a perfect differential, when P and Q are functions of x and y.

Let $u=f(x, y)$. If f_x and f_y are continuous, the total differential of u is

$$du=\frac{\delta f}{\delta x}dx+\frac{\delta f}{\delta y}dy \ldots \quad \ldots \quad (2)$$

If $\frac{\delta f}{\delta x}dx+\frac{\delta f}{\delta y}dy$ is equal to $Pdx+Qdy$ for all values of dx and dy, then the expression $Pdx+Qdy$ will always be a perfect differential.

Now comparing (1) and (2)

$$P=\frac{\delta f}{\delta y} \text{ and } Q=\frac{\delta f}{\delta y}$$

$$\therefore \quad \frac{\delta P}{\delta y}=\frac{\delta^2 f}{\delta y\delta x} \text{ and } \frac{\delta Q}{\delta x}=\frac{\delta^2 f}{\delta x\delta y}$$

As $\frac{\delta^2 f}{\delta y\delta x}=\frac{\delta^2 f}{\delta x\delta y}$ is always true for ordinary cases, so the expression $Pdx+Qdy$ will be an exact differential if

$$\frac{\delta P}{\delta y}=\frac{\delta Q}{\delta x}$$

Cor. The necessary and sufficient condition for the expression $Pdx+Qdy+Rdz$ to be an exact is

$$\frac{\delta P}{\delta y}=\frac{\delta Q}{\delta x},\quad \frac{\delta Q}{\delta z}=\frac{\delta R}{\delta y},\quad \frac{\delta R}{\delta x}=\frac{\delta P}{\delta z}$$

where the partial derivatives are continuous.

9.12. Homogeneous Functions

A function $f(x_1, x_2, \ldots \ \ldots, x_k)$ is a homogeneous function of degree n in $x_1, x_2 \ldots, x_k$

If $f(tx_1, tx_2, \ldots, tx_k)=t^n f(x_1, x_2, \ldots, x_k)$.

For example.

$$f(x, y)=ax^2+2hxy+by^2$$

is a homogeneous function of degree two in x, y, since

$$f(tx, ty)=a(tx)^2+2h(tx)(ty)+b(ty)^2$$
$$=t^2(ax^2+2hxy+by^2)=t^2 f(x, y).$$

The function $g(x, y)=\frac{2x-3y}{x+y}$ is homogeneous function of degree **Zero** in x, y since

$$g(tx, ty)=\frac{2(tx)-3(ty)}{tx+ty}=\frac{t(2x-3y)}{t(x+y)}=\frac{2x-3y}{x+y}=t^0 g(x, y).$$

The functions

$$x^2\tan^{-1}(y/x)+2yz+z^2,\ \tfrac{1}{2}\log(x^2+y^2)-\log x,$$

$\frac{x}{\sqrt{x^2+y^2}}$ are homogeneous ; theire degrees are respectively 2, 0, $\frac{1}{2}$.

The homogenelity of $f(x_1, x_2, \ldots, x_k)$ can also be tested by writing,

$$v_1=\frac{x_2}{x_1},\quad v_2=\frac{x_3}{x_1},\ \ldots\ \ldots,\ v_{k-1}=\frac{x_k}{x_1}.$$

If $f(x_1, x_2, \ldots, x_k)$ is homogeneous and degree n, then we should have,

$$f(x_1, x_2, \ldots, x_k)=x_1{}^n\phi\,(v_1, v_2, \ldots, v_{k-1}).$$

9.13. Euler's Theorem on Homogeneous Function.

If u be a homogeneous function of degree n in x any y, then

$$x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}=uu$$

Proof: Since u is a homogeneous function of degree n in x y, we can write

$$u=x^nf(y/x)=x^n f(v),\ \text{when } v=y/x.$$

$$\frac{\delta u}{\delta x}=\frac{\delta(x^n)}{\delta x}f(v)+x^nf'(v)\frac{\delta v}{\delta x}=nx^{n-1}(v)+x^nf(v)\left(-\frac{y}{x^2}\right)$$

$$\text{or,}\quad x\frac{\delta u}{\delta x}=nx^n f(v)-yx^{n-1}f'(v) \qquad (1)$$

$$\frac{\delta u}{\delta y}=x^n f'(v)\frac{\delta v}{\delta y}=x^n f'(v)\frac{1}{x}$$

$$\text{or,}\quad y\frac{\delta u}{\delta y}=x^{n-1}\,yf'(v)\quad \ldots\quad \ldots\quad \ldots\quad (2)$$

Now adding (1) and (2), we have

$$x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}=nx^n f(v)-yx^{n-1}f'(v)+x^{n-1}yf'(v)$$

$$=nx^n f(v)=nu$$

$$\therefore\quad x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}=nu \quad \text{Proved.}$$

Note: u has continuous first partial derivatives.

i. e. $\frac{\delta u}{\delta x}$ and $\frac{\delta u}{\delta y}$ exist

9. 14. Generalisation of Euler's Theorem on Homogeneous functions.

If $f(x_1, x_2, \cdots \cdots x_n)$ be a homogeneous function of $x_1\, x_2 \cdots x_n$ of degree m and if all their first partial derivatives are continuous,

than

$$\left(x_1\frac{\delta f}{\delta x_1}+x_2\frac{\delta f}{\delta x_2}+\cdots\ \cdots\ \cdots+x_n\frac{\delta f}{\delta x_n}\right)^n=m\,(m-1)\,(m-2)\cdots\cdots\cdots(m-n+1)\,f\,(x_1, x_2\cdots\cdots x_n)$$

As $f(x_1, x_2, x_3,\ldots x_n)$ is a homogeneous function of degree m, then

$$f(tx_1, tx_2\cdots\cdots tx_n)=t^m f(x_1, x_2,\cdots\cdots x_n)\quad \cdots\quad \cdots\quad (1)$$

$$\text{If } u_1=tx_1, u_2=tx_2\ \ldots, u_n=tx_n\quad \cdots\quad \cdots\quad (2)$$

$$\text{then } f\,(u_1, u_2\ldots u_n)=t^m f\,(x_1, x_2\cdots\cdots x_n)\quad \cdots\quad (3)$$

$$\text{Also } du_1=x_1dt,\ du_2=x_2\,dt,\cdots du_n=x_ndt\quad \cdots\quad (4)$$

as $u_1, u_2\, u_3\cdots, u_n$ etc are linear functions of t

Differentiate (3) w. r. to to t then

$$\left(x_1\frac{\delta f}{\delta u_1}+x_2\frac{\delta f}{\delta u_2}+\cdots\cdots+x_n\frac{\delta f}{\delta u_n}\right)=mt^{m-1}f(x_1, x_2\cdot x_n)$$

Differentiate it again w. r. to t then

$$x_1{}^2\frac{\delta^2 f}{\delta u_1{}^2}+x_1\,x_2\frac{\delta^2 f}{\delta u_1\,\delta u_2}+x_1\,x_2\frac{\delta^2 f}{\delta u_1\,\delta u_2}+x_2{}^2\frac{\delta^2 f}{\delta u_2{}^2}+\cdots\cdots\cdots$$

$$\cdots\cdots+x^2{}_n\frac{\delta^2 f}{\delta u^2{}_n}=m(m-1)t^{m-2}f(x_1, x_2\cdot x_n)$$

$$\text{or}\left(x_1 \frac{\delta f}{\delta u_1} + x_2 \frac{\delta f}{\delta u_2} + \cdots + x_n \frac{\delta f}{\delta u_n}\right)^2 = m(m-1)t^{m-2} f(x_1, x_2 \cdots x_n)$$

Differentiate it upto n times $w. r.$ to t then

$$\left(x_1 \frac{\delta f}{\delta u_1} + x_2 \frac{\delta f}{\delta u_2} + \cdots \cdots + x_n \frac{\delta f}{\delta u_n}\right)^n$$
$$= m(m-1) \ldots (m-n+1)\, t^{m-n} f(x_1, x_2 \cdots x_n)$$

If $t=1$, then

$$\left(x_1 \frac{\delta f}{\delta x_1} + x_2 \frac{\delta f}{\delta x_2} + \cdots \cdots + x_n \frac{\delta f}{\delta x_n}\right)^n = m(m-1)(m-2) \cdots$$
$$(m-n+1) f(x_1, x_2 \cdots x_n) \text{ Proved.}$$

Cor 1. $u=1$. then

$$x_1 \frac{\delta f}{\delta x_1} + x_2 \frac{\delta f}{\delta x_2} + \cdots \cdots \cdots + x_n \frac{\delta f}{\delta x_n} = m f(x_1\ x_2 \cdots x_n) = mf$$

Cor. 2. If $u=f(x, y, z)$ is a homogeneous function of degree n in x, y and z then

$$x \frac{\delta u}{\delta x} + y \frac{\delta u}{\delta y} + z \frac{\delta u}{\delta z} = nf(x, y, z) = nu. \qquad \text{D. H. '86}$$

CONVERSE OF EULER'S THEOREM

If u be *a* differential function of x, y, z and for all x, y, z let

$$u \frac{\delta u}{\delta x} + y \frac{\delta u}{\delta y} + z \frac{\delta u}{\delta z} = nu$$

Then show that u is a homogeneous function of degree n in x, y, z,

Proof : Let $r=\lambda x$, $s=\lambda y$, $t=\lambda z$. If we consider x, y, are constant, then

$$\frac{d}{d\lambda} f(r, s\ t) = x \frac{\delta f}{\delta r} + y \frac{\delta f}{\delta s} + z \frac{\delta f}{\delta t}$$

$$= \frac{1}{\lambda}\left(r \frac{\delta f}{\delta r} + s \frac{\delta f}{\delta s} + t \frac{\delta f}{\delta t} \right) \cdots \cdots \quad (1)$$

$$\text{Since } x \frac{\delta u}{\delta x} + y \frac{\delta u}{\delta y} + z \frac{\delta u}{\delta z} = nu \cdots \cdots \quad (2)$$

is true for all x, y, z, then

$$\frac{d}{d\lambda} f(r, s, t) = \frac{n}{\lambda} f(r, s, t) \cdots \cdots \quad (3)$$

Let $v=f(r, s, t)$, then for all valuss of λ ... ... (4)

$$\frac{dv}{\lambda} = \frac{n}{\lambda} v \text{ from (3) or, } \frac{1}{v} \cdot \frac{dv}{\lambda} = \frac{n}{\lambda}$$

Integrating

$\log v = n \log \lambda + c$,

or, $v=\lambda^n e^c = A\lambda^n$, $A=e^c$ is independent of λ.

or, $f(r, s, t) = A\lambda^n$ from (4)

or, $f(\lambda x, \lambda y, \lambda z) = A\lambda^n$

When $\lambda=1$, than $f(x, y, z) = A$

Henee $f(\lambda x, \lambda y, \lambda z) = \lambda^n f(x, y, z)$

for all values of x, y, z and λ and $f(x, y, z)$ is a homogeneous function of degree n in x, y, z.

Note : The function $\frac{\delta}{\delta\lambda} f(r, s, t)$ is some function of r, s, t ;

let it be $f(r, s, t)$. Then

$\frac{\delta}{\delta\lambda} f(x, y, z)$ is $F(x, y, z)$

Art. 9.15 Generalised Taylor's Theorem.

Let $f(x, y, z)$ and all its partial derivatives upto n are continuous in the domain

$a \le x \le a+h,\ b \le y \le b+k,\ c \le z \le c+l$; then $f(a+h, b+k, c+l)$

$$= f(a, b, c) + \left(h\frac{\delta}{\delta a} + k\frac{\delta}{\delta b} + l\frac{\delta}{\delta c}\right)^{n-1} f(a, b, c)$$

$$+ \frac{1}{(n-1)!}\left\{\left(h\frac{\delta}{\delta x} + k\frac{\delta}{\delta y} + l\frac{\delta}{\delta z}\right)^n f(x, y, z)\right\} ; \begin{matrix} x = a+\theta h \\ y = b+\theta k \\ z = c+\theta l \end{matrix}$$

Where $0 < \theta < 1$

Let us suppose that $x = a + ht,\ y = b + k^t,\ z = c + lt$

So that $f(x, y, z) = f(a + ht, b+kt, c+lt) = F(t)$ say.

$$F'(t) = \frac{\delta f}{\delta x} \cdot \frac{\delta x}{\delta t} + \frac{\delta f}{\delta y} \cdot \frac{\delta y}{\delta t} + \frac{\delta f}{\delta z} \cdot \frac{\delta t}{\delta t}$$

$$= h\frac{\delta f}{\delta x} + k\frac{\delta f}{\delta y} + l\frac{\delta f}{\delta z} = \left(h\frac{\delta}{\delta x} + k\frac{\delta}{\delta y} + l\frac{\delta}{\delta z}\right) f(x, y, z) \ \ldots\ldots (1)$$

Again differentiating since Right land side is a function of x, y and z, we have

$$F''(t) = \left(h\frac{\delta}{\delta x} + k\frac{\delta}{\delta y} + l\frac{\delta}{\delta z}\right)^2 f(x, y, z) \ \ldots\ldots\ldots\ldots (2)$$

....

$$F^{n-1}(t) = \left(h\frac{\delta}{\delta x} + k\frac{\delta}{\delta z} + l\frac{\delta}{\delta z}\right)^{n-1} f(x, y, z) \ \ldots\ldots (n-1)$$

$$F^n(t) = \left(h\frac{\delta}{\delta x} + k\frac{\delta}{\delta y} + l\frac{\delta}{\delta z}\right)^n f(x, y, z) \ \ldots\ldots. (n)$$

We see from the above relations, it is clear that $f(t)$ and the first

n derivatives are continuous fuinctions in the interval (o,1) and by Maclaurin's theorem, we have,

$$F(t) = f(0) + tF'(0) + \frac{t^2}{\underline{|2}} F''(0) + \ldots\ldots$$

$$+ \frac{t^{n-1}}{(n-1)!} F^{n-1}(0) + \frac{t^n}{n!} F^n(\theta t), \ \ldots\ldots (3)$$

$0 < \theta < 1$

When $t = 0$, x, y, z become, a, b, c respectively and hence

$$\left(h\frac{\delta}{\delta x} + k\frac{\delta}{\delta y} + l\frac{\delta}{\delta z}\right)^m f(x, y, z) = \left(h\frac{\delta}{\delta a} + k\frac{\delta}{\delta b} + l\frac{\delta}{\delta c}\right)^m f(a, b, c)$$

Now applying the above relations (1), (2), ... (n) we get

$$F'(0) = \left(h\frac{\delta}{\delta x} + k\frac{\delta}{\delta b} + l\frac{\delta}{\delta c}\right) f(a,b,c)$$

$$F''(0) = \left(h\frac{\delta}{\delta a} + k\frac{\delta}{\delta b} + l\frac{\delta}{\delta c}\right)^2 f(a,b,c)$$

....

$$F^{n-1}(0) = \left(h\frac{\delta}{\delta a} + k\frac{\delta}{\delta b} + l\frac{\delta}{\delta c}\right)^{n-1} f(a,b,c)$$

Also, $F^n(\theta t) = \left(h\frac{\delta}{\delta x} + k\frac{\delta}{\delta y} + l\frac{\delta}{\delta z}\right)^n f(x,y,z)$

Where $x = a + h\theta t,\ y = b + k\theta t,\ z = c + l\theta t$

$\therefore o < \theta < 1$

Now putting these relatons in (3), we have

$F(t) = f(a + ht, b + kt, c+lt)$: if $t=1$, then

$f(a+h, b+k, c+l)$

$$= f(a,b,c) + \left(h\frac{\delta}{\delta a} + k\frac{\delta}{\delta b} + l\frac{\delta}{\delta c}\right) f(a,b,c)$$

$$+ \frac{1}{(n-1)!}\left(h\frac{\delta}{\delta a} + k\frac{\delta}{\delta b} + l\frac{\delta}{\delta c}\right)^{n-1} f(a,b,c)$$

$$+ \frac{1}{n!}\left(h\frac{\delta}{\delta x} + k\frac{\delta}{\delta y} + l\frac{\delta}{\delta z}\right)^n f(x,y,z) \ \ldots\ldots (4)$$

Where $x = a+\theta h$, $y=b+\theta k$, $z=c+\theta l$,. t=1.

The Theorem is established.

The theorem is symbolically expressd as

$$f(a+h,\ b+k,\ c+l) = \left[e^{\left(h\frac{\delta}{\delta a}+k\frac{\delta}{\delta b}+l\frac{\delta}{\delta c}\right)} \right] f(a,b,c)$$

$$\text{up to n terms} + \frac{1}{n!}\left(h\frac{\delta}{\delta x}+k\frac{\delta}{\delta y}+l\frac{\delta}{\delta z}\right) f(x,y,z) \ \ldots\ldots (5)$$

$$\text{or; } f(x+h,\ y+k,\ z+l) = \left[e^{h\frac{\delta}{\delta x}+k\frac{\delta}{\delta y}+l\frac{\delta}{\delta z}} \right] f(x,y,z) \ldots\ldots (6)$$

Art 9.15 (d) Taylor's Theorem can be stated in another form.

$$f(x,y) = f(a,b) + \left[(x-a)\frac{\delta}{\delta x} + (y-b)\frac{\delta}{\delta y} \right] f(a,b)$$

$$+ \frac{1}{2!}\left[(x-a)\frac{\delta}{\delta x} + (x-b)\frac{\delta}{\delta y} \right]^2 f(a,b) + \ldots\ldots$$

$$+ \frac{1}{(n-1)!}\left[(x-a)\frac{\delta}{\delta x} + (x-b)\frac{\delta}{\delta y} \right]^{n-1} f(a,b) + R_n$$

$$\text{Where } R_n = \frac{1}{n!}\left[(x-a)\frac{\delta}{\delta x} + (y-b)\frac{\delta}{\delta y} \right]^n f\{a + (x-a)\theta\}.$$

$$\text{Where } R_n = \frac{1}{n!}\left[(x-a)\frac{\delta}{\delta x} + (y-b)\frac{\delta}{\delta y} \right]^n f\{a + (x-a)\theta,\ b+(y-b)\theta\}$$

$0<\theta<1$ called the Tayloy's expansion of $f(x, y)$ about the point (a, b) in powers of $x-a$ and $y-b$

Alternative Form

$f(x+h,\ y+k)$

$$= f(x,y) + \left(h\frac{\delta f}{\delta x} + k\frac{\delta f}{\delta y}\right) + \frac{1}{2!}\left(h^2\frac{\delta^2 f}{\delta x^2} + 2hk\frac{\delta^2 f}{\delta x\delta y} + k^2\frac{\delta^2 f}{\delta y^2}\right)$$

Art. 9.15 (*b*) Maclaurin's Theorem for three variables.

Putting $a = 0$, $b = 0$, $c = 0$ and $h = x$, $k = y$, $l = z$ respectively in (4), we have,

$$f(x,\ y,\ z) = f(0,\ 0,\ 0) + \left(x\frac{\delta}{\delta u} + y\frac{\delta}{\delta v} + z\frac{\delta}{\delta w}\right) f(u,v,w)$$

$$\ldots\ldots + \frac{1}{(n-1)!}\left(x\frac{\delta}{\delta u} + y\frac{\delta}{\delta v} + z\frac{\delta}{\delta w}\right)^{n-1} f(u,v,w)$$

$$+ \frac{1}{n!}\left(x\frac{\delta}{\delta u} + y\frac{\delta}{\delta v} + z\frac{\delta}{\delta w}\right)^{n} f(u,v,w)$$

u,v,w are to be replaced by 0,0,0 in all the terms except the last term which is to be replced by $u = \theta x$, $v = \theta y$, $w=\theta z$, $0<\theta<1$

NOTE. Maclaurin's Theorem for two variables.

$$f(x,y) = f(0,0) + \left(x\frac{\delta}{\delta u} + y\frac{\delta}{\delta v}\right) f(u,\ v)$$

$$+ \frac{1}{2!}\left(x\frac{\delta}{\delta u} + y\frac{\delta}{\delta v}\right)^2 f(u,\ v) + \ldots\ldots$$

$$+ \frac{1}{(n-1)!}\left(x\frac{\delta}{\delta u} + y\frac{\delta}{\delta v}\right)^{n-1} f(u,v) + \frac{1}{n!}\left(x\frac{\delta}{\delta u} + y\frac{\delta}{\delta v}\right)^{n} f(u,\ v)$$

$$= \left(e^{x\frac{\delta}{\delta u}+y\frac{\delta}{\delta v}}\right) f(u,v)$$

Putting $u = 0$, $v = 0$, upto n terns and for the remainder or (n+1)th term $u = \theta x$, $v = \theta y$, the above theorem will be obtained.

NOTE. If there are n variables, then

$$f(x+h,\ y+k,\ z+l,\ \ldots..) = \left[e^{h\frac{\delta}{\delta x}+k\frac{\delta}{\delta y}+l\frac{\delta}{\delta z}} \right] + \ldots\ldots\ ]\ f(x,y,z,\ldots\ldots.)$$

NOTE. For infinite Series,

$$f(x,\ y) = \left[e^{x\frac{\delta}{\delta u}+y\frac{\delta}{\delta v}} \right] . f(u,\ v), \text{ after expansion}$$

put $u = 0$, $v = o$.

$$f(x,\ y,\ z) = \left[e^{x\frac{\delta}{\delta u}+y\frac{\delta}{\delta v}+z\frac{\delta}{\delta w}} \right] f(u,\ v,\ w).$$

after expansion, put $u = 0$, $v = 0$, $w = 0$

Art. 9.15 (*e*) DIFFERENTIABILITY.

Show that a function differentiable at a point is necessarily continuous and possesses partial derivatives thereat.

Sol. Let us consider two neighbouring points (x, y) and $(x+\delta x, y+\delta y)$ in the domain of definition of a function $f(x,y)$. The change df is given by $\delta f = f(x+\delta x,\ y+\delta y) - f(x,\ y)$

The function $f(x,y)$ is differentiable at (x,y) if the change df can be expressed as

$\delta f = A\delta x + B\delta y + \delta x\phi(\delta x, \delta y) + \delta y\psi(\delta x, \delta y)$

Where A and b are free from δx and δy and also constants. The functions ϕ and ψ are the finctions of δx and δy and $\phi(\delta x, \delta y)\to 0$. ψ $(\delta x, \delta y)\to 0$ simultaneously.

We may call $A\delta x + B\delta y$ is a differential of $f(x, y)$ at (x, y) and is denoted by $df(x, y)$ Or, df. Hence $df = A\delta x + B\delta y$

From (1) We see that $(\delta x, \delta y)\to 0$, then

$f(x+\delta x, y+\delta y) - f(x,y)\to 0$

Or: $f(x+\delta x, y+\delta y)\to f(x,y)$

We conclude that the function $f(x,y)$ is continuous at (x, y)

Thus every differentiable function is continuous.

If we put dy = 0 for y as constant in

(1), then $\delta\phi = A\,\delta x + B.o + \delta x\,\phi(\delta x, 0)$

Or; $\frac{\delta f}{\delta x} = A$ if $\lim \delta x\to 0$

Similarly $\frac{\delta f}{\delta y} = B$.

Hence the constants A and b are respectively the partial derivatives of f with respect of x and y.

Thus **a function which is differentiable at a point possesses the first order partial derivatives at that point.**

Converse of this theorem is not always true.

Functions which are continuous an may even possess partial derivatives at a point but are not differentiable at that point.

From (1),$\delta f = A\,\delta x + B\delta y = \frac{\delta f}{\delta x}\delta x + \frac{\delta f}{\delta y}\delta y$ .... (2)

If we consider $f = y$, then $dy = \delta y$

Similarly if $f = x$, then $dx = \delta x$

From (2)

$df = \frac{\delta f}{\delta x}dx + \frac{\delta f}{\delta y}dy = f_x dx + f_y dy$ .... (3)

is the differential of f at (x, y)

NOTE. 1 If the function f and its partial derivatives f_x, f_y are continuous at a point (x, y) in the domain of f, then

$\delta f = f(x+\delta x, y+\delta y) - f(x,y)$

$= f(x+\delta x, y+\delta y) - f(x, y+\delta y) + f(x, y+\delta y) - f(x, y)$

By Lagrange's Mean Value Theorem of one variable, we have

$\delta f = \delta x f_x(x+\theta\,\delta x, y+\delta y) + \delta y f_y(x, y+\theta_2\delta y)$

Where $0 < \theta. < 1$, $0 < \theta_2 < 1$

Since f_x and f_y are continuous at (x,y) therefore when $(\delta x, \delta y)\to(0,0)$, we have $\delta\psi = (f_x + \phi)\delta x + (f_y + \psi)\delta y$

Where ϕ and ψ tend to zero as $(\delta x, \delta y)\to(0,0)$

$\therefore \delta f = f_x\delta x + f_y\delta y + \delta x\,\phi(\delta x, \delta y) + \delta y\psi(\delta x, \delta y)$

$= f_x\delta x + f_y dy + \phi\delta x + \psi\delta y$ ... (4)

NOTE-2. If $\delta x = h$, $\delta y = k$ in (1) then

$df = f(a+h, b+k) - f(a, b)$

$= Ah + Bk + h\phi(h,k) + k\psi(h, k)$ .... (5)

Where $A = f_x$, $B = f_y$, and ϕ, ψ are the functions of h, k and $\phi(h,k)\to 0$, $\psi(h,k)\to 0$ if $h\to 0$, $k\to 0$

NOTE-3 Theorem of 9 15(c), which is not true always is shown below with an example.

Example. Show that the given function has partial derivatives and is contunuous at (0, 0) still the functuon is not differential at the point.

$$f(x, y) = x\frac{x^3-y^3}{x^2+y^2}, \quad (x, y) \neq (0, 0)$$

$$= 0. \qquad (x, y) = (0, 0)$$

Sol. Let $x = r\cos\theta$, $y = r\sin\theta$ in

$$\left|\frac{x^3-y^3}{x^2+y^2}\right| = \left|\frac{r^3\cos^3\theta - r^3\sin^3\theta}{r^2(\cos^2\theta+\sin^2\theta)}\right| = |r(\cos^3\theta - \sin^3\theta|$$

$\leq 2|r| = 2\sqrt{(x^2+y^2)} < \epsilon$

if $x^2 < \epsilon 2/8$, $y^2 < \epsilon^2/8$

Or: if $x < \epsilon/2\sqrt{2}$, $y < \epsilon/2\sqrt{2}$

$$\therefore \left|\frac{x^3-y^3}{x^2+y^2}-0\right| < 2\sqrt{(x^2+y^2)} < 2.\ \epsilon/2 = \epsilon$$

i.e.: $\lim\limits_{(x,y)\to 0} \frac{x^3-y^3}{x^2+y^2} = 0$

i.e.: $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

Hence the finction is continuous at (0,0).

For partial derivatives at (0,0).

$$f_x(0,0) = \lim_{h\to0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{h-0}{h} = 1.$$

$$f_y(0,0) = \lim_{k\to0}\frac{f(0,k)-f(0,0)}{k} = \lim_{k\to0}\frac{-k}{k} = -1$$

Thus the function has partial derivatives at (0,0). i.e: f_x and f_y exist at (0,0)

If the function f is differentiable at (0,0).

Then

$df = f(h,k)-f(0,0) = Ah+Bk+h\phi+k\psi$ .... (1)

Where $a = f_x(0,0) = 1$. $b = f_y(0,0) = -1$. i. 2; A and b are constants and $\phi(h,k)$, $\psi(h,k)\to0$ as $(h,k)\to(0,0)$

Now if we put $h = r_1\cos\theta$. $k = r_1\sin\theta$ in (1).

$f(h,k)-f(0,0) = 1.h+(-1)k+h\phi(h,k) + k\psi(h,k)$

Or: $\frac{r_1^3(\cos^3\theta-\sin^3\theta)}{\rho_1^2(\cos^2\theta+\sin^2\theta)} = r_1\cos_1 - r_1\sin\theta + r_1\phi\cos\theta + r_1\psi\sin\theta$

Or: $\cos^3\theta-\sin^3\theta = \cos\theta-\sin\theta+\phi\cos\theta+\psi\sin\theta$ ...... (2)

For. $\theta=\tan^{-1}(h/k)$. $r_1\to0$ implies that $(h,k)\to(0,0)$. Therefore in limit, we have $\cos^3\theta-\sin^3\theta = \cos\theta-\sin\theta$

Or; $(\cos\theta-\sin\theta)(\cos^2\theta+\sin^2\theta+\cos\theta\sin\theta-1) = 0$

Or: $\cos\theta\sin\theta(\cos\theta-\sin\theta) = 0$

Which is impossible for arbitrary θ.

Hence the function f is not differentiable at the origin.

Taylor's Theorem for two variables.

9.15(e) To expand $\phi(x+h, y+k)$ in powers of h and k, and show that

$$\phi(x+h,y+k) = e^{h\frac{\delta}{\delta x}+k\frac{\delta}{\delta y}}\phi(x,y)$$

By Taylor's Theorem, we have,

$$\phi(x+h,y+k) = \phi(x,y+k)+h\frac{\delta}{\delta x}\phi(x,y+k)+\frac{h^2}{2!}\frac{\delta^2}{\delta x^2}\phi(x,y+k)+\frac{h^3}{3!}\frac{\delta^3}{\delta x^3}\phi(x,y+k)+\cdots$$

$$=\phi(x,y)+k\frac{\delta}{\delta y}\phi(x,y)+\frac{k^2}{2!}\frac{\delta^2}{\delta y^2}\phi(x,y)+\frac{k^3}{3!}\frac{\delta^3}{\delta y^3}\phi(x,y)+$$

$$+h\frac{\delta}{\delta x}\left(\phi(x,y)+k\frac{\delta}{\delta y}\phi(x,y)+\frac{k^2}{2!}\frac{\delta^2}{\delta y^2}\phi(x,y)+\cdots\right)$$

$$+\frac{h^2}{2!}\frac{\delta^2}{\delta x^2}\left(\phi(x,y)+k\frac{\delta}{\delta y}\phi(x,y)+\cdots\right)+\frac{h^3}{3!}\frac{\delta^3}{\delta x^3}\left(\phi(x,y)+\cdots\right.$$

$$=\phi(x,y)+\left(h\frac{\delta}{\delta x}+k\frac{\delta}{\delta y}\right)\phi(x,y)+\left(\frac{h^2}{2!}\frac{\delta^2}{\delta x^2}\right.$$

$$\left.+hk\frac{\delta^2}{\delta x\delta y}+\frac{k^2}{2!}\frac{\delta^2}{\delta y^2}\right)\phi(x,y)$$

$$+\left(\frac{h^3}{3!}\frac{\delta^3}{\delta x^3}+\frac{h^2k}{2!}\frac{\delta^3}{\delta x^2\delta y}+\frac{hk^2}{2!}\frac{\delta^3}{\delta x\delta y^2}+\frac{k^3}{3!}\frac{\delta^3}{\delta y^3}\right)\phi(x,y)+\cdots$$

or, $\phi(x+h,y+k)=\phi(x,y)+\left(h\frac{\delta}{\delta x}+k\frac{\delta}{\delta y}\right)\phi(x,y)+$

$$\frac{1}{2!}\left(h^2\frac{\delta^2}{\delta x^2}+2hk\frac{\delta^2}{\delta x\delta y}+k^2\frac{\delta^2}{\delta y^2}\right)\phi(x,y)$$

$$+\frac{1}{3!}\left(h^3\frac{\delta^3}{\delta x^3}+3h^2k\frac{\delta^3}{\delta x^2\delta y}+3hk^2\frac{\delta^3}{\delta x\delta y^2}+k^3\frac{\delta^3}{\delta y^3}\right)\phi(x,y)+$$

$$=\left[1+\left(h\frac{\delta}{\delta x}+k\frac{\delta}{\delta y}\right)+\frac{1}{2!}\left(h\frac{\delta}{\delta x}+k\frac{\delta}{\delta y}\right)^2+\frac{1}{3!}\left(h\frac{\delta}{\delta x}+k\frac{\delta}{\delta y}\right)^3+\ldots\right]\phi(x,y)$$

$$\phi(x+h,y+k)=e^{h\frac{\delta}{\delta x}+k\frac{\delta}{\delta y}}\phi(x,y)$$

If there are n variables, then

$$\phi(x+h,y+k,z+l\ldots\ldots)=e^{h\frac{\delta}{\delta x}+k\frac{\delta}{\delta y}+l\frac{\delta}{\delta z}+\ldots\ldots}\phi(x,y,z,\ldots)\ldots$$

Art. 9.15(f) SUFFICIENT CONDTION FOR DIFFENTIABILITY

If $f(a,b)$ be a point in the domain of a function f such that

(i) f_x is continuous at (a, b)

(ii) f_y exists at (a, b)

then f is differentiable at (a, b)

Sol. f_x exists in a certain neighbourhood of $(a-\delta, a+\delta; b-\delta, b+\delta)$ of (a, b) since f_x is continuous by (i)

Let $(a+h, b+k)$ be a point of this neighbourhood. So

$df = f(a+h, b+k)-f(a,b)$

$= f(a+h,b+k)-f(a,b+k) + f(a, b+k)-f(a,b)$....(1)

As f_x exists in $(a-\delta, a+\delta, b-\delta, b+\delta)$. Then by Lagrange's Mean Value Theorem,

We have

$f(a+h, b+k)-f(a, b+k)=h f_x(a+\theta h, b+k)$ ..... (2)

Where $0 < \theta < 1$ and depends on h and k.

$\therefore \lim\limits_{h(h,k)\to(0,0)} f_x(a+\theta h, b+k) = f_x(a,b)$, so that we have now

$f_x(a+\theta h,b+k) = f_x(a,b) + \phi(h,k)$ ...... (3)

Where $\phi(h, k)\to 0$ as $(h,k)\to(0,0)$

By the second condition, $f_y(a,b)$ exists,

$\lim\limits_{k\to 0}\dfrac{f(a,b+k)-f(a,b)}{k} = f_y(a,b)$. So that

We can write

$\dfrac{f(a, b+k)-f(a,b)}{k} = f_y(a, b) + \psi(k)$ .... (4)

Where $\psi(k) \to 0$ as $k \to 0$

From (1), (2), (3) and (4), We have

$df = hf_x(a,b) + kf_y(a,b) + h\phi(h,k) + h\psi(h,k)$ implies that f is differentiable at (a,b).

NOTE. 4. In the same way we can show that f is differentiable at (a, b), if f_x exists and f_y is continuous at (a, b)

Art. 9.15(g) SUFFICIFNT CONDITION FOR CONTINUIUTY

A sufficient condition for a function $f(x,y)$ be continuous at (a, b) is that one of the partial derivatives exists and is bounded in a nbd of (a, b) and that the other exists

Proof. Let f_x exists and be bounded in a nbd of (a,b) and let f_y (a,b) exist, then for any point (a+h, b+k) of the nbd. of (a,b), we have

$f(a+h,b+k)-f(a, b+k) = (a+h-a) f_x(a+\theta h, b+k)$, $0<\theta<1$

Since $f_y(a, b)$ exists, so we have

$\underset{k\to 0}{\text{Lt}}\dfrac{f(a, b+k)-f(a,b)}{k} = f_y(a, b)$

Or, $f(a, b+k)-f(a,b) = k\,[f_y(a,b) + \eta]$

Where η is a functon of k and if $k\to 0$, then $\eta\to 0$

$\therefore f(a+h, b+k)-f(a,b) = hf_x(a+\theta h, b+k) + k\,[f_y(a,b)+\eta]$

If $(h, k)\to 0$. Since $f_x(a+\theta h, b+k)$ is bcunded, then

$\underset{(h,k)\to(0,0)}{\text{Lt}}(a+h,b+k) = f(a,b)$

i. e; Left hand limit = Right Land limit = functional value.

Hence $f(x, y)$ is continuous.

Conclusion : A sufficient condition that a function will be continuous in a closed region is that both the partial derivatives exist and are bounded throughout the region.

Art 9.15 (h) SCHWARZ'S THEOREM.

It f_y, exists in a certain nbd. of a point (a,b) of the domain at definition of a function f and f_y is continuous at (a, b), then

f_{xy} (a,b) exists and is equal to f_{yx} (a, b) i.e; $f_{xy} = f_{yx}$.

Proof. Under the given conditions. f_x, f_y, f_{yx} all exist in a certain nbd. of (a, b). Let (a+h, b+k) be a point of this nbd. Let

$\phi(h, k) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)$ ..... (1)

$G(x) = f(x, b+k) - f(x, b)$; $G(a+h) = f(a+h, b+k) - f(a+h, b)$

∴ [G(x) means inside the bracket the function remains the same in operation]

$\phi(h,k) = G(a+h) - G(a)$ .... (2)

$G(a) = f(a, b+k) - f(a, b)$

Since fx exists in a nbd of (a,b), the function G(x) is derivable in the open interval (a, a+h) and therefore, by Lagrange's Mean Value Theoren from (2)

$\phi(h,k) = (a+h-a)\, G'(a+\theta h)$, $0 < \theta < 1$

$= h\{f_x(a+\theta h, b+k) - f_x(a+\theta h, b)\}$ ...... (3); ∴ Change for x

Again since f_{yx} exists in a nbd. of (a, b), the function f_x is derivable with resp ect to y in (b b+k), open interval and then by Lagrange's M.V. Theorem, we get from (3)

$\phi(h,k) = h\,(b+k-b)\, f_{yx}(a+\theta h, b+\theta_1 k)$ $0 < \theta_1 < 1$

Or. $\frac{\phi(h,k)}{hk} = f_{yx}(a+\theta h, b+\theta_1 k)$

$$\therefore \lim_{h\to 0,\, k\to 0} \frac{\phi(h,k)}{hk} = \lim_{h\to 0,\, k\to 0} f_{yx}(a+\theta h, b+\theta_1 k) = f_{yx}(a,b)$$

Again

$\phi(h,k) = G(b+k) - (b) = G(b+k-b)\, G_y(b+\theta_1 k)$, $0 < \theta_1 < 1$

(By M.V. Theorem)

$= k\{f_y(a+h, b+\theta_1 k) - f_y(a, b+\theta_1 k)$

$= k(a+h-a)\{f_{xy}(a+\theta h, b+\theta_1 k)$, $0 < \theta_1 < 1$

$$\therefore \lim_{h\to 0,\, k\to 0} \frac{\phi(h,k)}{hk} = \lim_{h\to 0,\, k\to 0} f_{xy}(a+\theta h, b+\theta_1 k) = f_{xy}(a, b)$$

∴ Hence $f_{xy} = f_{yx}$ Proved.

★ 9. 16 JACOBIANS

If If $u_1, u_2, \ldots u_n$ are functions of variables $x_1, x_2, \cdots x_n$ then the determinant,

$$\begin{vmatrix} \frac{\delta u_1}{\delta x_1} & \frac{\delta u_1}{\delta x_2} & \cdots & \frac{\delta u_1}{\delta x_n} \\ \frac{\delta u_2}{\delta x_1} & \frac{\delta u_2}{\delta x_2} & \cdots & \frac{\delta u_2}{\delta x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\delta u_n}{\delta x_1} & \frac{\delta u_n}{\delta x_2} & \cdots & \frac{\delta u_n}{\delta x_n} \end{vmatrix}$$

is called the **Jacobian** of $u_1, u_2, \cdots u_n$ with r. to $x_1, x_2 \ldots\ldots x_n$

It is denoted by $\frac{\delta(u_1, u_2 \ldots u_n)}{\delta(x_1, x_2 \cdots x_n)}$ or, $J(u_1, u_2, \ldots u_n)$.

Ex. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

find $\frac{\delta(x, y, z)}{\delta(r, \theta, \phi)}$

$$\frac{\delta(x, y, z)}{\delta(r, \theta, \phi)} = \begin{vmatrix} \frac{\delta x}{\delta r} & \frac{\delta x}{\delta \theta} & \frac{\delta x}{\delta \phi} \\ \frac{\delta y}{\delta r} & \frac{\delta y}{\delta \theta} & \frac{\delta y}{\delta \phi} \\ \frac{\delta z}{\delta r} & \frac{\delta z}{\delta \theta} & \frac{\delta z}{\delta \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta \cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \cos\theta\,(r^2\sin\theta\cos\theta\cos^2\phi + r^2\sin\theta\cos\theta\sin^2\phi) +$$

$$r\sin\theta\,(r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi)$$

$$= r^2\sin\theta\cos^2\theta + r^2\sin^2\theta = r^2\sin\theta.$$

★**9.17.** Let $u_1, u_2 \cdots\cdots n_n$ be functions of independent variables $x_1, x_2, \ldots x_n$. If there exists a relation between these n functions,

$$F(u_1, u_2, \ldots u_n) = 0$$

It is necessary and sufficient condition that the Jacobian should vanish.

$$\text{i e., } \frac{\delta(u_1, u_2, \cdots u_n)}{\delta(x_1, x_2, \ldots x_n)} = 0$$

Properties of Jacobians

Some properties of Jacobians are given below without proof. For proofs Edward's Calculus may be consulted.

9.18. Jacobian of function of function:—If $u_1, u_2 \ldots\ldots u_n$ are functions of $y_1, y_2, \ldots y_n$ and $y_1, y_2 \ldots y_n$ are functions of $x_1, x_2, \ldots x_n$.

$$\frac{\delta(u_1, u_2, \ldots\ldots\ldots u_n)}{\delta(x_1, x_2, \ldots\ldots\ldots x_n)} = \frac{\delta(u_1, u_2, \ldots\ldots\ldots u_n)}{\delta(y_1, y_2, \ldots\ldots\ldots y_n)} \times \frac{\delta(y_1, y_2, \ldots\ldots\ldots y_n)}{\delta(x_1, x_2, \ldots\ldots\ldots x_n)}$$

9.19. Jacobian of Implicit Functions :—If $u_1, u_2, \ldots u_n$ be connected implicity with the independent varirables $x_1, x_2, \ldots x_n$ by the relations.

$$f_1(u_1, u_2, \ldots u_n, \quad x_1, x_2 \ldots x_n) = 0$$

$$f_2(u_1, u_2 \ldots u_n, \quad x_1, x_2 \ldots x_n) = 0$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$f_n(u_1, u_2 \ldots u_n, \quad x_1, x_2 \ldots x_n) = 0$$

$$\text{then} \frac{\delta(f_1, f_2, \cdots\cdots f_n)}{\delta(u_1, u_2 \cdots\cdots u_n)} \times \frac{\delta(u_1, u_2 \cdots\cdots u_n)}{\delta(x_1, x_2, \cdots\cdots x_n)} = (-1)^n \frac{\delta(f_1, f_2 \cdots\cdots f_n)}{\delta(x_1, x_2, \cdots\cdots x_n)}$$

Note : The above result in 9.19 be regarded as generalisation of

$$\frac{dy}{dx} = -\frac{\delta f/\delta x}{\delta f/\delta y}$$

where x and y are connected by $f(x, y) = 0$

9. 20 If J be the Jacobians of the system $u_1, u_2 \cdots u_n$ with regards to $x_1, x_2, \ldots x_n$ and J' the Jacobian of $x_1, x_2 \ldots x_n$ with regard to $u_1, u_2 \cdots u_n$ then JJ' = 1

$$\frac{\delta(u_1, u_2, \ldots\ldots u_n)}{\delta(x_1, x_2 \cdots\cdots x_n)} \times \frac{\delta(x_1, x_2 \ldots\ldots x_n)}{\delta(u_1, u_2, \cdots \ldots u_n)} = 1$$

9.21. If any set of homogeneous equations be satisfied by a common system of variables, the equation $J = 0$ is also satisfied by the same system, and if the degrees are the same, the equations.

$$\frac{\delta J}{\delta x} = 0, \; \frac{\delta J}{\delta y} = 0, \; \frac{\delta J}{\delta z} = 0 \;\; \ldots\ldots\ldots \text{etc.}$$

will also be satisfied by the same system

Cor. If $u=0$, $v=0$, $w=0$ have a common point, the curve $J=0$ will go through that point, and further, if the curves be of like degree, we shall have

$$\frac{\delta J}{\delta x}=0,\ \frac{\delta J}{\delta y}=0,\ \frac{\delta J}{\delta z}=0.$$

so that $J=0$ will have a double point there,

9.22. Covariant and Invariant.

Let u be any quantic i. e, a homogeneous function of any number of variables and of any degree. Another function ϕ is derived from u in any manner such that ϕ contains the constants and variables of u. Let u and ϕ be changed into U and ϕ respectively by any linear transformation. Then ϕ is said to be covarient of u provided the function derive from U by the same process by which the function ϕ was derived from u is merely ϕ multiplied by some powers of modulus of the transformation.

If ϕ does not contain any variables u or, ϕ contains cnly teh co-efficient of u, ϕ is called an invarient.

9. 32. The Jacobian of a system of function u, v, w, is a covariant of the system.

Let the transformation system be shown below, so that

$x=l_1x_1+m_1y_1+n_1z_1$

$y=l_2x_1+m_2y_1+n_2z_1$

$z=l_3x_1+m_3y_1+n_3z_1$

	x_1	y_1	z_1
x	l_1	m_1	n_1
y	l_2	m_2	n_2
z	l_3	m_3	n_3

Now $\frac{\delta u}{\delta x_1}=\frac{\delta u}{\delta x}\cdot\frac{\delta x}{\delta x_1}+\frac{\delta u}{\delta y}\cdot\frac{\delta v}{\delta x_1}+\frac{\delta u}{\delta z}\cdot\frac{\delta z}{\delta x_1}$

$$=\frac{\delta u}{\delta x}l_1+\frac{\delta u}{\delta y}l_2+\frac{\delta u}{\delta z}l_3$$

or, $u_{x1}=u_xl_1+u_yl_2+u_zl_3$

Similarly, u_{y1}, u_{z1}, v_{x1}, v_{y1}, v_{z1}, w_{x1}, w_{y1}, and w_{z1}

Now we have

$$\frac{\delta(u,v,w)}{\delta(x_1,y_1,z_1)}=\frac{\delta(u,v,w)}{\delta(x,y,z)}\cdot\frac{\delta(x,y,z)}{\delta(x_1,y_1,z_1)}$$

$$J_1=\begin{vmatrix} u_{x1} & u_{y1} & u_{z1}\\ v_{x1} & v_{y1} & v_{z1}\\ w_{x1} & w_{y1} & w_{z1}\end{vmatrix}=\begin{vmatrix} u_x & u_y & u_z\\ v_x & v_y & v_z\\ w_x & w_y & w_z\end{vmatrix}\times\begin{vmatrix} l_1 & l_2 & l_3\\ m_1 & m_2 & m_3\\ n_1 & n_2 & n_3\end{vmatrix}$$

or, $J_1=J\times\mu$, where $J=\frac{\delta(u,v,w)}{\delta(x,y,z)}$ is the jacobian of the original system and $\mu=\frac{\delta'(x,y,z)}{\delta(x_1,y_1,z_1)}$ is the co-efficient of transformation.

9.24. The Hessian

The Jacobian of the first differential co-efficients u_x, u_y, u_z of any function u is

$$\begin{vmatrix} u_{xx} & u_{xy} & u_{xz}\\ u_{xy} & u_{yy} & u_{yz}\\ u_{xz} & u_{yz} & u_{zz}\end{vmatrix}$$

and is called the Hessian.

Ex. 6. (a) Expand f(x, y) = log (1 + xy) by Taylors se·ies if 1 + xy > 0 at (a, b).

Ans. $f(x,y)=\log(1+xy)$, $f(a,b)=\log(1+ab)$

$$f_x(x,y)=\frac{y}{1+xy}\Rightarrow f_x(a,b)=\frac{b}{1+ab}$$

$$f_y(x,y)=\frac{x}{1+xy}\Rightarrow f_y(a,b)=\frac{a}{1+ab}$$

$$f_{xx}(x,y)=-y^2/(1+xy)^2\Rightarrow f_{xx}(a,b)=-b^2/(1+ab)^2$$

$$f_{xy}(x,y)=\{(1+xy)-xy\}/(1+xy)^2\Rightarrow f_{xy}(a,b)=1/(1+ab)^2$$

$$f_{yy}(x,y)=-x^2/(1+xy)^2\Rightarrow f_{yy}(a,b)=-a^2/(1+ab)^2$$

$$f(x,y)=f(a,b)+(x-a)f_x(a,b)+(y-b)f_y(a,b)+\frac{1}{\lfloor 2}(x-a)^2$$

$$f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2 f_{yy}(a,b)]+\ldots\ldots$$

$$\therefore\ \log(1+xy)=\log(1+ab)+(x-a)\frac{b}{1+ab}+(y-b)\frac{a}{(1+ab)^2}$$

$$+\frac{1}{\lfloor 2}\left[-(x-a)^2\frac{b^2}{(1+ab)^2}+2(x-a)(y-b)\frac{1}{(1+ab)^2}+(y-b)^2\frac{(-a^2)}{(1+ab)^2}\right]+\cdots\ldots$$

$$\text{or, } \log(1+xy)=\log(1+ab)+\frac{b}{1+ab}(x-a)+\frac{a(y-b)}{1+ab}$$

$$+\frac{1}{\lfloor 2}\left[\frac{-b^2(x-a)^2}{(1+ab)^2}+\frac{2(x-a)(y-b)}{(1+ab)^2}-\frac{a^2}{(1+ab)^2}\right]+\ldots\ldots\ldots$$

Ex. 6. (b) Show that the functions

$u=x+y-z,\ v=x-y+z,\ w=x^2+y^2+z^2-2yz$ are not independent of one another. Show that $u^2+v^2=2w$.

$$\frac{\delta(u,v,w)}{\delta(x,y,z)}=\begin{vmatrix}\frac{\delta u}{\delta x} & \frac{\delta u}{\delta y} & \frac{\delta u}{\delta z}\\ \frac{\delta v}{\delta x} & \frac{\delta v}{\delta y} & \frac{\delta v}{\delta z}\\ \frac{\delta w}{\delta x} & \frac{\delta w}{\delta y} & \frac{\delta w}{\delta z}\end{vmatrix}=\begin{vmatrix}1 & 1 & -1\\ 1 & -1 & 1\\ 2x & 2y-2z & 2z-2y\end{vmatrix}$$

$$=\begin{vmatrix}1 & 1 & 0\\ 1 & -1 & 0\\ 2x & 2(y-z) & 0\end{vmatrix}=0$$

As the Jacobian is zero, so the functions are not independent.

Again $u+v=2x,\quad u-v=2(y-z)$

Now $w=x^2+y^2+z^2-2yz=\frac{1}{4}(2x)^2+\frac{1}{4}\{2(y-z)\}^2$

$$=\tfrac{1}{4}(u+v)^2+\tfrac{1}{4}(u-v)^2$$

or, $4w=2u^2+2v^2$ or, $u^2+v^2=2w$

Ex. 7. If $z^2=x^2+y^2+1$, Prove that

$$\frac{\delta^2 z}{\delta x\delta y}=\frac{\delta^2 z}{\delta y\delta x}\quad\ldots\quad\ldots\quad [\text{D. U. } 1966]$$

Now $z^2=x^2+y^2+1$ or, $z=\sqrt{(x^2+y^2+1)}=(x^2+y^2+1)^{1/2}$

$$\therefore\quad \frac{\delta z}{\delta y}=\frac{\delta}{\delta y}(x^2+y^2+1)^{1/2}=\tfrac{1}{2}(x^2+y^2+1)^{1/2-1}+\frac{\delta(y^2)}{\delta y}$$

$$=\frac{2y}{2\sqrt{x^2+y^2+1}}=\frac{y}{\sqrt{(x^2+y^2+1)}}$$

$$\frac{\delta^2 z}{\delta x\,\delta y}=\frac{\delta}{\delta x}\left\{\frac{y}{\sqrt{(x^2+y^2+1)}}\right\}=-\tfrac{1}{2}\cdot\frac{2xy}{\sqrt{(x^2+y^2+1)^3}}$$

Again $\frac{\delta z}{\delta y}=\frac{2x}{2\sqrt{(x^2+y^2+1)}}=\frac{x}{\sqrt{(x^2+y^2+1)}}$

$\therefore \quad \frac{\delta^2 z}{\delta y \delta x}=\frac{-2xy}{2\sqrt{(x^2+y^2+1)^3}} \quad \therefore \quad \frac{\delta^2 z}{\delta x \delta y}=\frac{\delta^2 z}{\delta y \delta x}$ Proved.

Ex. 8. $v=\frac{1}{\sqrt{(x^2+y^2+z^2)}}$ Show that

$\frac{\delta^2 v}{\delta x^2}+\frac{\delta^2 v}{\delta y^2}+\frac{\delta^2 v}{\delta z^2}=0$ [R. U. '54, D. U. '60, '83]

$$v=\frac{1}{\sqrt{(x^2+y^2+z^2)}}$$

$\therefore \quad \frac{\delta v}{\delta x}=\left[-\frac{2x}{2(x^2+y^2+z^2)^{3/2}}\right]=-\frac{x}{(x^2+y^2+z^2)^{3/2}}$

Similarly $\frac{\delta v}{\delta y}=\frac{y}{\sqrt{(\Sigma x^2)^3}}, \quad \frac{\delta v}{\delta z}=-\frac{z}{\sqrt{(x^2+y^2+z^2)^3}}$

Again $\frac{\delta^2 v}{\delta x^2}=\frac{\delta}{\delta x}\left(\frac{\delta v}{\delta x}\right)=\frac{\delta}{\delta x}\left\{\frac{-x}{\sqrt{(x^2+y^2+z^2)^3}}\right\}$

$$=-\left\{\frac{\sqrt{(x^2+y^2+z^2)^3}.1-3/2x\sqrt{(x^2+y^2+z^2)}.2x}{(x^2+y^2+z^2)^3}\right\}$$

$$=-\frac{\sqrt{(x^2+y^2+z^2)}}{(x^2+y^2+z^2)^3}\{x^2+y^2+z^2-3x^2\}=-\frac{y^2+z^2-2x^2}{\sqrt{(x^2+y^2+z^2)^5}}$$

Similarly

$$\frac{\delta^2 v}{\delta y^2}=-\frac{z^2+x^2-2y^2}{\sqrt{(x^2+y^2+z^2)^5}}, \quad \frac{\delta^2 v}{\delta z^2}=-\frac{x^2+y^2-2z^2}{\sqrt{(x^2+y^2+z^2)^5}}$$

$$\frac{\delta^2 v}{\delta x^2}+\frac{\delta^2 v}{\delta y^2}+\frac{\delta^2 v}{\delta z^2}=-\frac{y^2+z^2-2x^2+z^2+x^2-2y^2+x^2+y^2-2z^2}{\sqrt{(x^2+y^2+z^2)^5}}=0$$

Hence $\frac{\delta^2 v}{\delta x^2}+\frac{\delta^2 v}{\delta y^2}+\frac{\delta^2 v}{\delta z^2}=0$ Proved.

Ex. 9. Show that $x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}=0.$

if $u=\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}$ [D. U. 1961, '84]

$$\frac{\delta u}{\delta x}=\frac{1}{\sqrt{(1-x^2/y^2)}}\cdot\frac{1}{y}+\frac{1}{1+y^2/x^2}\cdot\frac{-y}{x^2}$$

$\therefore \quad x\frac{\delta u}{\delta y}=\frac{x}{\sqrt{(y^2-x^2)}}-\frac{xy}{x^2+y^2}$ ... ... (1)

Again $\frac{\delta u}{\delta x}=\frac{1}{\sqrt{(1-x^2/y^2)}}\cdot\frac{-x}{y^2}+\frac{1}{1+y^2/x^2}\cdot\frac{1}{x}$

$\therefore \quad y\frac{\delta u}{\delta x}=\frac{-x}{\sqrt{(y^2-x^2)}}+\frac{xy}{x^2+y^2}$ ... ... (2)

Therefore adding (1) and (2) we have

$$x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}=0$$

Alternative Method

since $\frac{y}{x}$ and $\frac{x}{y}$ are each homogeneous functions of degree zero, therefore u is a homogeneous function of degree zero. Hence by Euler's theorem

$$x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}=0\times u=0.$$

Ex. 10. What is the order of u, if

$$u=\frac{x+y}{x^2+y^2}$$

Verify Euler's therem for u.

We have,

$$u(tx, ty)=\frac{tx+ty}{(tx)^2+(ty)^2}=\frac{t(x+y)}{t^2(x^2+y^2)}=t^{-1}u(x,y)$$

Hence u is a homogeneous function of degree -1.

Now $\frac{\delta u}{\delta x}=\frac{1.(x^2+y^2)-2x(x+y)}{(x^2+y^2)^2}=\frac{y^2-2xy-x^2}{(x^2+x^2)^2}$

Similarly, $\frac{\delta u}{\delta y}=\frac{x^2-2xy-y^2}{(x^2+y^2)^2}$

$\therefore\ x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}=\frac{x(y^2-2xy-x^2)+y(x^2-2xy-y^2)}{(x^2+y^2)^2}$

$=\frac{-xy^2-xy^2-x^3-y^3}{(x^2+y^2)^2}=-\frac{(x+y)(x^2+y^2)}{(x^2+y^2)^2}$

$\Rightarrow x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}=(-1)\frac{x+y}{x^2+y^2}=(-1)u$

which verifies Euler's theorem.

Ex. 11. If $z=\tan^{-1}\frac{x^3+y^3}{x-y}$ then

Prove that $x\frac{\delta z}{\delta x}+y\frac{\delta z}{\delta y}=\sin 2z$ [R. U. 1966. D.U. 1964]

Now $z=\tan^{-1}\frac{x^3+y^3}{x-y}$ i. e., $\tan z=\frac{x^3+y^3}{x-y}$

Let $v=\tan z$ then

$v(tx,\ ty)=\frac{t^3x^3+t^3y^3}{tx-ty}=t^2\frac{(x^3+y^3)}{(x-y)}=t^2v$

Therefore v is a homogeneous function of degree 2.

By Euler's theorem,

$$x\frac{\delta v}{\delta x}+y\frac{\delta v}{\delta y}=2v$$

or $x\left(\sec^2 z\frac{\delta z}{\delta x}\right)+y\left(\sec^2 z\frac{\delta z}{\delta y}\right)=2\tan z\quad\left[\therefore v=\tan z\right]$

or $x\frac{\delta z}{\delta x}+y\frac{\delta z}{\delta y}=2\tan z.\cos^2 z=2\sin z\cos z$

$\Rightarrow x\frac{\delta z}{\delta x}+y\frac{\delta z}{\delta y}=\sin 2z$ (proved)

Ex. 12. If $\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1$

prove that $u_x^2+u_y^2+u_z^2=2(xu_x+yu_y+zu_z)$

[R. U. 1964]

Ans. Differentiating both sides of

$$\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1$$

Partially with respect to x, we get

$$\frac{2x}{a^2+u}-\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]u_x=0$$

$u_x=\frac{2x}{(x^2+u)F}$

where $F=\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}$

Similarly partial differentiation w, r to y and z give

$$u_y=\frac{2y}{(b^2+u)F},\ u_z=\frac{2z}{(c^2+u)F}$$

$\therefore\ u_x^2+u_y^2+u_z^2=\frac{4}{F^2}\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]$

$=\frac{4}{F^2}.F=\frac{4}{F}$ ... ... (i)

Also $xu_x+yu_y+zu_z=\frac{2}{F}\left[\frac{x^2}{(a^2+u)}+\frac{y^2}{(b^2+u)}+\frac{z^2}{(c^2+u)}\right]$

$=\frac{2}{F}\ 1=\frac{2}{F}$ ... ... (2)

From (1) and (2), it follows that

$u_x^2+u_y^2+u_z^2=2(xu_x+yu_y+zu_z)$. (proved)

Ex. 13. If v be a function of x and y, prove that

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}$$

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where $x=r\cos\theta$ and $y=r\sin\theta$

Ans. We have,

$$\left.\begin{aligned}&x^2+y^2=r^2(\cos^2\theta+\sin^2\theta)=r^2\\ &\text{and } \tan\theta=\frac{y}{x} \text{ or } \theta=\tan^{-1}\left(\frac{y}{x}\right)\end{aligned}\right\}\quad(1)$$

hence

$$\left.\begin{aligned}&\frac{\partial r}{\partial x}=\frac{x}{r}=\cos\theta,\quad \frac{\partial r}{\partial y}=\frac{y}{r}=\sin\theta\,;\\ &\frac{\partial \theta}{\partial x}=\frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)=-\frac{y}{x^2+y^2}=-\frac{r\sin\theta}{r^2}=-\frac{\sin\theta}{r}\\ &\frac{\partial \theta}{\partial y}=\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)=\frac{x}{x^2+y^2}=\frac{r\cos\theta}{r^2}=\frac{\cos\theta}{r}\end{aligned}\right\}\ 2$$

$$\frac{\partial v}{\partial x}=\frac{\partial v}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x}=\frac{\partial v}{\partial r}\cos\theta+\frac{\partial v}{\partial \theta}\left(-\frac{\sin\theta}{r}\right)$$

$$=\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)v \qquad \text{[by (2)]}$$

$$\therefore\ \frac{\partial^2 v}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)=\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial v}{\partial r}-\frac{\sin\theta}{r}\frac{\partial v}{\partial \theta}\right)$$

$$=\cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial v}{\partial r}-\frac{\sin\theta}{r}\frac{\partial v}{\partial \theta}\right)-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial v}{\partial r}-\frac{\sin\theta}{r}\frac{\partial v}{\partial \theta}\right)$$

$$\text{or, } \frac{\partial^2 v}{\partial x^2}=\cos^2\theta\frac{\partial^2 v}{\partial r^2}-\cos\theta\sin\theta\left(-\frac{1}{r^2}\frac{\partial v}{\partial \theta}+\frac{1}{r}\frac{\partial^2 v}{\partial r\partial \theta}\right)$$

$$-\frac{\sin\theta}{r}\left(-\sin\theta\frac{\partial v}{\partial r}+\cos\theta\frac{\partial^2 v}{\partial \theta\partial r}\right)$$

$$+\frac{\sin\theta}{r^2}\left(\cos\theta\frac{\partial v}{\partial \theta}+\sin\theta\frac{\partial^2 v}{\partial \theta^2}\right)$$

$$\text{or, } \frac{\partial^2 v}{\partial x^2}=\cos^2\theta\frac{\partial^2 v}{\partial r^2}+\frac{2\cos\theta\sin\theta}{r^2}\frac{\partial v}{\partial \theta}-\frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 v}{\partial r\partial \theta}$$

$$+\frac{\sin^2\theta}{r}\frac{\partial v}{\partial r}+\frac{\sin^2\theta}{r^2}\frac{\partial^2 v}{\partial \theta^2}\ \ldots\text{(I)}$$

$$\frac{\partial v}{\partial y}=\frac{\partial v}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y}=\frac{\partial v}{\partial r}\sin\theta+\frac{\partial v}{\partial \theta}\frac{\cos\theta}{r}$$

$$=\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)v$$

$$\therefore\ \frac{\partial^2 v}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right)=\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\sin\theta\frac{\partial v}{\partial r}+\frac{\cos\theta}{r}\frac{\partial v}{\partial \theta}\right)$$

$$=\sin\theta\left[\sin\theta\frac{\partial^2 v}{\partial r^2}+\cos\theta\left(-\frac{1}{r^2}\frac{\partial v}{\partial \theta}+\frac{1}{r}\frac{\partial^2 v}{\partial r\partial \theta}\right)\right]$$

$$+\frac{\cos\theta}{r}\left[\cos\theta\frac{\partial v}{\partial r}+\sin\theta\frac{\partial^2 v}{\partial r\partial \theta}+\frac{1}{r}\left(-\sin\theta\frac{\partial v}{\partial \theta}+\cos\theta\frac{\partial^2 v}{\partial \theta^2}\right)\right]$$

$$\text{or, } \frac{\partial^2 v}{\partial y^2}=\sin^2\theta\frac{\partial^2 v}{\partial r^2}-\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial v}{\partial \theta}+\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 v}{\partial r\partial \theta}$$

$$+\frac{\cos^2\theta}{r}\frac{\partial v}{\partial r}+\frac{\cos^2\theta}{r^2}\frac{\partial^2 v}{\partial \theta^2}\ \text{(II)}$$

Adding (I) and (II) and using $\cos^2\theta+\sin^2\theta=1$,

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}\quad \text{[proved]}$$

Ex. 14. If $z=f(u,v)$, $u=x^2-2xy-y^2$ and $v=y$, show that

$(x+y)\dfrac{\partial z}{\partial x}+(x-y)\dfrac{\partial z}{\partial y}=0$ is equivalent to $\dfrac{\partial x}{\partial v}=0$.

we can write $\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x}$

But $u=x^2-2xy-y^2$; $v=y$

$\therefore \quad \frac{\delta u}{\delta x}=2x-2y\ ; \quad \frac{\delta v}{\delta x}=0\ ; \quad \frac{\delta u}{\delta y}=-2x-2y\ ; \quad \frac{\delta v}{\delta y}=1$

$\therefore \quad \frac{\delta z}{\delta x}=\frac{\delta z}{\delta u}\ 2(x-y)+\frac{\delta z}{\delta y}\ 0.=2\ (x-y)\frac{\delta z}{\delta u} \quad \ldots \ldots \quad (1)$

and $\frac{\delta z}{\delta y}=\frac{\delta z}{\delta u}\cdot\frac{\delta u}{\delta y}+\frac{\delta z}{\delta v}\cdot\frac{\delta v}{\delta y}=-2\ (x+y)\ \frac{\delta z}{\delta u}+\frac{\delta z}{\delta v} \quad (2)$

Multiply (1) by $(x+y)$ and (2) by $(x-y)$ and add. The result is

$(x+y)\ \frac{\delta z}{\delta x}+(x-y)\ \frac{\delta z}{\delta y}=2\ (x+y)(x-y)\frac{\delta z}{\delta u}$

$-2(x-y)(x+y)\ \frac{\delta z}{\delta u}+(x-y)\ \frac{\delta z}{\delta v}=(x-y)\ \frac{\delta z}{\delta v}$

Given that $(x+y)\ \frac{\delta z}{\delta x}+(x-y)\ \frac{\delta z}{\delta y}=0$

$\therefore \quad 0=(x-y)\frac{\delta z}{\delta v} \Rightarrow \frac{\delta z}{\delta v}=0 \quad (x\neq y \text{ always})$

অর্থাৎ $(x+y)\ \frac{\delta z}{\delta x}+(x-y)\ \frac{\delta z}{\delta y}=0$ এবং $\frac{\delta z}{\delta v}=0$ সমতুল্য প্রমাণিত।

Ex. 15. If $v=f\ (x-y,\ y-z,\ z-x)$, then prove that

$$\frac{\delta v}{\delta x}+\frac{\delta v}{\delta y}+\frac{\delta v}{\delta z}=0$$

Let $X=y-z,\quad Y=z-x,\quad Z=x-y$

$\frac{\delta X}{\delta x}=0,\ \frac{\delta X}{\delta y}=1,\ \frac{\delta X}{\delta z}=-1$

$\frac{\delta Y}{\delta x}=-1,\ \frac{\delta Y}{\delta y}=0,\ \frac{\delta Y}{\delta z}=1$

$\frac{\delta Z}{\delta x}=1,\ \frac{\delta Z}{\delta y}=-1,\ \frac{\delta Z}{\delta z}=0$

Now $v=f\ (X,\ Y,\ Z)$; X, Y, Z, are functions of x, y, z

$\therefore \quad \frac{\delta v}{\delta x}=\frac{\delta v}{\delta X}\cdot\frac{\delta X}{\delta x}+\frac{\delta v}{\delta Y}\cdot\frac{\delta Y}{\delta x}+\frac{\delta v}{\delta Z}\cdot\frac{\delta Z}{\delta x}$

$=\frac{\delta v}{\delta X}\times 0+\frac{\delta v}{\delta Y}\ (-1)+\frac{\delta v}{\delta Z}\cdot 1=-\frac{\delta v}{\delta Y}+\frac{\delta v}{\delta Z}$

$\therefore \quad \frac{\delta v}{\delta x}=-\frac{\delta v}{\delta Y}+\frac{\delta u}{\delta Z} \ldots \ldots \ldots \ (1)$

Similarly $\frac{\delta v}{\delta y}=\frac{\delta v}{\delta X}\cdot\frac{\delta X}{\delta y}+\frac{\delta v}{\delta Y}\cdot\frac{\delta Y}{\delta y}+\frac{\delta v}{\delta Z}\cdot\frac{\delta Z}{\delta y}$

$\Rightarrow \frac{\delta v}{\delta y}=\frac{\delta v}{\delta X}-\frac{\delta v}{\delta Z} \ldots \ldots \ (2)$

and $\frac{\delta v}{\delta z}=-\frac{\delta v}{\delta X}+\frac{\delta v}{\delta Y} \ldots \ldots (3)$

Adding, (1), (2) and (3) we have

$\frac{\delta v}{\delta x}+\frac{\delta v}{\delta y}+\frac{\delta v}{\delta z}=0$

Ex. 16. Find

(ii) $\lim\limits_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$

(ii) Show that $\lim\limits_{y\to 0,x\to 0} f(x,\ y)\neq \lim\limits_{x\to 0,\ y\to 0} f(x,y)$

where $f(x,y)=\frac{x^2-y^2}{x^2+y^2}$ D. U. 1986

Ans. let $(x,\ y)\to(0,0)$ along a line $y=mx$ in the xy plane ; then along the line

$\lim\limits_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}=\lim\limits_{x\to 0} \frac{m^2x^3}{x^2+m^4x^4}=\lim\limits_{x\to 0} \frac{m^2x}{1+m^4x^2}=0.$

which is independent of m.

(ii) $$\lim_{y\to 0}\left\{\lim_{x\to 0}\frac{x^2-y^2}{x+y^2}\right\}=\lim_{y\to 0}-\frac{y^2}{y^2}=-1$$

$$\lim_{x\to 0}\left\{\lim_{y\to 0}\frac{x^2-y^2}{x^2+y^2}\right\}=\lim_{x\to 0}\frac{x^2}{x^2}=1$$

$$\therefore \lim_{y\to 0,\ x\to 0} f(x,y)\neq \lim_{x\to 0,\ y\to 0} f(x,y)$$

Ex. 17. Examine whether $f(x,y)=\begin{cases} xy & \text{if } |x|\geq |y| \\ -xy & \text{if } |x|<|y|\end{cases}$ is continuous at the origin.

Is $\dfrac{\delta^2 f}{\delta x\delta y}=\dfrac{\delta^2 f}{\delta y\delta x}$ for this function continuous at the origin.

Give reason for your answer.

Sol. $|xy|=|x|\,|y|\leq |x^2|$ for $|x|\geq |y|$

$\to 0$ as $(x,y)\to(0,0)$

Also $|-xy|=|x|\,|y|<|y|^2$ for $|x|<|y|$

$\to 0$ as $(x,y)\to(0,0)$

Also $f(0,0)=0$

Hence $f(x,y)$ is continuous at the origin.

Again $$f_x(0,y)=\lim_{h\to 0}\frac{f(h,y)-f(0,y)}{h}=\lim_{h\to 0}\frac{-hy}{h}=-y$$

when $|h|<|y|$.

$$\therefore f_{xy}(0,0)=\lim_{k\to 0}\frac{f_x(0,k)-f_x(0,0)}{k}=\lim_{k\to 0}\frac{-k}{k}=-1.$$

Also $$f_y(x,0)=\lim_{k\to 0}\frac{f(x,k)-f(x,0)}{k}=\lim_{k\to 0}\frac{kx}{k}=x$$

$\therefore\ |x|>|k|\ ;\ f(x,0)=0$

$$\therefore f_{yx}(0,0)=\lim_{h\to 0}\frac{f_y(h,0)-f_y(0,0)}{h}=\lim_{h\to 0}\frac{h}{h}=1$$

Hence $f_{xy}(0,0)\neq f_{yx}(0,0)$

It is not continuous at the origin.

Ex. 18. If $u=\begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1\end{vmatrix}$

prove that $\dfrac{\delta u}{\delta x}+\dfrac{\delta u}{\delta y}+\dfrac{\delta u}{\delta z}=0$

Ans. $u=\begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1\end{vmatrix}=x^2(y-z)-y^2(x-z)+z^2(x-y)$

$$\therefore \frac{\delta u}{\delta x}=2x(y-z)-y^2+z^2$$

$$\frac{\delta u}{\delta y}=x^2-2y(x-z)-z^2$$

$$\frac{\delta u}{\delta z}=-x^2+y^2+2z(x-y)$$

Adding these, $\dfrac{\delta u}{\delta x}+\dfrac{\delta u}{\delta y}+\dfrac{\delta u}{\delta z}=0$

Ex. 19. Directional Derivatives.

Let $F(x,y,z)$ be defined at a point $P(x,y,z)$ on a space curve C. For a point near to P is Q where the value of the function at Q is given by $F(x+\Delta x,\ y+\Delta y,\ z+\Delta z)$. Let $PQ=\text{arc}\,\Delta S$:

Then $$\lim_{\Delta S\to 0}\frac{\Delta F}{\Delta S}=\lim_{\Delta S\to 0}\frac{F(x+\Delta x,\ y+\Delta y,\ z+\Delta z)-F(x,y,z)}{\Delta S}$$

is called the **directional derivative** of F at the point $P(x, y, z)$ along he curve C and is denoted by

$$\frac{dF}{dS}=\frac{\delta F}{\delta x}\frac{dx}{dS}+\frac{\delta F}{\delta y}\frac{dy}{dS}+\frac{\delta F}{\delta F}\frac{dz}{dS}$$

$$=\left(\frac{\delta F}{\delta x}i+\frac{\delta F}{\delta y}j+\frac{\delta F}{\delta z}k\right)\left(\frac{dx}{dS}i+\frac{dy}{dS}j+\frac{dz}{dS}k\right)$$

$=\nabla F.\frac{dR}{dS}=\nabla F.T$ where T is unit tangent. Thus the direction derivative is the component of ∇F along the unit tangent to P on C.

The maximum value of ∇F is given by $|\nabla F|$.

Ex. 19. (a) Find the directional derivative of $f(x, y)=\tan^{-1}y/x$ at $(1,-1)$ towards $(3, 0)$ D. H. 1986

[$f(x, x)=\tan^{-1}y/x$ ফাংশনটির (1,—1) বিন্দুতে (3, 0) বিন্দুর প্রতি ডাইরেকশনাল ডেরিভেটিভ নির্ণয় কর।]

Let $f(x, y)=\tan^{-1}y/x$ at $P(1,-1)$

$$\nabla f=\frac{\delta f}{\delta x}i+\frac{\delta f}{\delta y}j=\frac{x^2}{x^2+y^2}\left(-\frac{y}{x^2}\right)i+\frac{x^2}{x^2+y^2}\left(\frac{1}{x}\right)j$$

$$=\tfrac{1}{2}(1)\,i+\tfrac{1}{2}\,j \text{ at } P(1,-1)$$

The vector along $P(1,-1)$ and $Q(3, 0)$

$$PQ=r=(3-1)\,i+(0+1)\,j=2i+j$$

$\frac{dr}{dt}$ unit Tangent in the direction, $T=\frac{2i+j}{\sqrt{(4+1)}}=\frac{2i+j}{\sqrt{5}}$

Directional Derivative of $f=\nabla f.T=(\tfrac{1}{2}i+\tfrac{1}{2}j).\frac{(2i+j)}{\sqrt{5}}$

Since ∇f is positive, f is increasing $=(1+\tfrac{1}{2})/\sqrt{5}=\frac{3}{2\sqrt{5}}$

Note : [For details, please see Art 4.6 to 4.9. vector Analyses of A Text Book on Co-ordinate Geometry and Vector Analysis. By Rahman and Bhattacharjee.]

Ex. 19 (b) Find the directional derivative of $F=2xy-z^2$ at $(2,-1, 1)$ in a direction towards $(3, 1,-1)$.. Find the maximum directional derivative, also its magnitude [$F(x, y)=2xy-z^2$]
[ফাংশনটির (2,—1, 1) বিন্দুতে (3, 1,—1) বিন্দুর প্রতি ডাইরেকশনাল ডেরিভেটিভ নির্ণয় কর ইহার সর্বোচ্চ দিক কোনদিকে এবং মান কি ?]

Ans, $10/3$. $-2i+4j-2k$, $2\sqrt{6}$

Ex. 20 Find the Jacobian of the transformation

$u = r \sin s \cos t,\ v = r \sin s \sin t,\ w = r \cos s$

Sol. The Jacobian is

$$|J| = \begin{vmatrix} \frac{\delta u}{\delta r} & \frac{\delta u}{\delta s} & \frac{\delta u}{\delta t} \\ \frac{\delta u}{\delta r} & \frac{\delta u}{\delta s} & \frac{\delta u}{\delta t} \\ \frac{\delta w}{\delta r} & \frac{\delta w}{\delta s} & \frac{\delta w}{\delta t} \end{vmatrix}$$

$$= \begin{vmatrix} \sin s \cos t & r \cos t \cos t & -r \sin s \sin t \\ \sin s \sin t & r \cos s \sin t & r \sin s \cos t \\ \cos s & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} \cos s & 0 & 0 \\ \sin s \cos t & r \cos s \cos t & -r \sin s \sin t \\ \sin s \sin t & r \cos s \sin t & r \sin s \cos t \end{vmatrix}$$

$= \cos s\,(r^2 \cos s \sin s \cos^2 t + r^2 \cos s \sin s \sin^2 t)$

$= r^2 \cos^2 s \sin s\,(\cos^2 t + \sin^2 t)$

$= r^2 \cos^2 s \sin s$

EXERCISE—IX

See. APPENDIX: Ex. 97; Ex. 98., Ex. 99, Ex. 100.

1. Find $\delta f/\delta x$ when (i) $f=xy$ (ii) $f=x^y$ (iii) $f=\log(x^2+y^2)$ (iv) $f=\sin^{-1}(y/x)$ (v) $f=\tan^{-1}(x+y)$

2. Verify that $\dfrac{\delta^2 u}{\delta x\delta y}=\dfrac{\delta^2 u}{\delta y\delta x}$ where

(i) $u=\log(y\sin x+x\sin y)$ (ii $u=\dfrac{xy}{x^2+y^2}$

(ii) $u=ax^2+2hxy+by^2$ (iv) $u=\log\tan(y/x)$

2 (a) Prove that $f(x, y)=\dfrac{xy(x^2-y^2)}{x^2+y^2}$, $(x, y)\neq(0, 0)$

$f(0, 0)=0$ then $f_{xy}=f_{yx}$

(b) $\underset{x\to 0}{\text{Lt}}\cdot\underset{y\to 0)}{}\dfrac{x^2-y2}{x^3+y^3}\neq\underset{y\to 0.\ x\to 0}{\text{Lt}}\dfrac{x^2-y^2}{x^3+y^3}$

3. If $u=\log\dfrac{(1+x^2)(1+y^2)}{xy}$, find $\dfrac{\delta^2 u}{\delta x\delta y}$

4 If $f(x, y)=xy\ (x^2-y^2)/(x^2+y^2)$; when $(x, y)\neq(0, 0)$ and $f(0, 0)=0$, show that $f_{xy}(0, 0)\neq f_{yx}(0, 0)$.

5. If $(x, y)=e^{xy}\cos x\sin x$, find $f_x, f_y, f_{xx}, f_{xy}, f_{yx}$ R. U 1965; C. U. 1969

6. If $u=\log(x^2+y^2)$, show that C. U. 1979, 80

$\dfrac{\delta^2 u}{\delta x^2}+\dfrac{\delta^2 u}{\delta y^2}=0$ D. U. 1986.

7. If $u=(x, y)=e^y\cos x$, show that

$\dfrac{\delta^2 u}{\delta x^2}+\dfrac{\delta^2 u}{\delta y^2}=0$

8. If u and v are functions of x and y defined by $x=u+e^{-v}\sin u$ and $y=v+e^{-v}\cos u$, prove that

$\dfrac{\delta u}{\delta y}=\dfrac{\delta v}{\delta x}$ D. U. H. 1962

9. If $u=\sqrt{(x^2+y^2+z^2)}$, prove that D. U. 1984,

$u_{xx}+u_{yy}+u_{zz}=2/u$ D. U. H. 1960

10. If $u=\log(x^2+y^2+z^2)$, Prove that

$x\dfrac{\delta^2 u}{\delta y\delta z}=y\dfrac{\delta^2 u}{\delta z\delta u}=z\dfrac{\delta^2 u}{\delta x\delta y}$

11. If $u=2(ax+by)^2-(x^2+y^2)$ and $a^2+b^2=1$.

Prove that $\dfrac{\delta^2 u}{\delta x^2}+\dfrac{\delta^2 u}{\delta y^2}=0$

12. If $u=\sin^{-1}\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$ show that R. U. 1982

$\dfrac{\delta u}{\delta x}=-\dfrac{y}{x}\dfrac{\delta u}{\delta y}$ or, $x\dfrac{\delta u}{\delta x}+y\dfrac{\delta u}{\delta y}=0$

13. If $u=f(r)$ and $r^2=x^2+y^2$ prove that

$\dfrac{\delta^2 u}{\delta x^2}+\dfrac{\delta^2 u}{\delta y^2}=\dfrac{d^2u}{dr^2}+\dfrac{1}{r}\dfrac{du}{dr}$

14. If $u=e^{xyz}$ show that

$\dfrac{\delta^3 u}{\delta x\delta y\delta z}=(1+3xyz+x^2y^2z^2)\,e^{xyz}$ R. U. 1988

15. Find $\dfrac{\delta z}{\delta v}$ and $\dfrac{\delta z}{\delta v}$ if $z=x^2+y^2+xy$ R. U. 1967

and $x=u\cos v$, $y=u\sin v$,

15. a) Verify Euler's Theorem for the following curves

(i) $u=\frac{1}{x^2+xy+y^2}$ (ii) $u=x^n \tan(y/x)$ (iii) $u=x^2 \log y/x$

(iv) $u=e^{x/y}+e^{y/x}$ (v) if $u=\sin(\sqrt{x}+\sqrt{y})$, prove that

$$x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}=\tfrac{1}{2}(\sqrt{x}+\sqrt{y})\cos(\sqrt{x}+\sqrt{y}).$$

16. If $z=xyf(y/x)$. Show that $x\frac{\delta u}{\delta x}+y\frac{\delta v}{\delta y}=nu\frac{\delta v}{\delta u}=2z$

16 (a) If $z=\tan^{-1}\frac{x^2+y^2}{x-y}$ the $x\frac{\delta u}{\delta x}+y\frac{\delta v}{\delta y}=\frac{1}{2}=\sin 2z$ D.U. 1989

17. If $u=\sin^{-1}\frac{x^2+y^2}{x+y}$, show that N. U. 1994

$$x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}=\tan u,$$ R. U. 1978, '83 1986

18. If $u=\sin^{-1}\frac{x-y}{\sqrt{x}+\sqrt{y}}$ Show that

$$x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}=\tfrac{1}{2}\tan u$$ C. H. 1977; D. U. 1988

19. If $v=f(u)$, being homogeneous function of degree n in x, y, then $x\frac{\delta v}{\delta x}+y\frac{\delta v}{\delta y}=nu\frac{dv}{du}$

20. Prove Euler's Theorem when $u=\frac{x-y}{x+y}$

21. If $u=x^3+y^3+z^3$, then prove that

$$x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}+z\frac{\delta u}{\delta z}=3u$$

22. If $u=\log(x^3+y^3+z^3-3xyz)$ then show that C H. 1989

$$\left(\frac{\delta}{\delta x}+\frac{\delta}{\delta y}+\frac{\delta}{\delta z}\right)^2 u=-\frac{9}{(x+y+z)^2}$$ N H 1993

23. If $u=x^2-y^2-2yx+y+z$. Show that

$$(x+y)\frac{\delta u}{\delta x}+(x-y)\frac{\delta u}{\delta y}+(y-x)\frac{\delta u}{\delta z}=0$$

24. Prove that $f_{xx}+f_{yy}=f_{zz}+f_{tt}$
where $u=f(x, y)$ and $x=z\cos\alpha-t\sin\alpha$, $y=z\sin\alpha+t\cos\alpha$

25. If $u=y^2+\tan(ye^{1/x})$, Show that

$$x^2\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}=2y^2$$ D. U. 1956

26. If $z=\frac{x^2y^2}{x+y}$ then prove that

$$x\frac{\delta^2 z}{\delta x^2}+y\frac{\delta^2 z}{\delta x^2}=2\frac{\delta z}{\delta x}$$ D U. 1967

27. If $u=3(lx+my+nz)^2-(x^2+y^2+z^2)$ and $l^2+m^2+n^2=1$

Show that $\frac{\delta^2 u}{\delta x^2}+\frac{\delta^2 u}{\delta y^2}+\frac{\delta^2 u}{\delta z^2}=0$ R. U. 1987

28. If $F(x, y, z)=0$, show that

$$\frac{\delta x}{\delta y}\cdot\frac{\delta y}{\delta z}\cdot\frac{\delta z}{\delta x}=-1,$$

where each partial derivative is computed by holding the remaining variables constant. R. U. H. 1962

28. (a) If $u=\operatorname{cosec}^{-1}\sqrt{\frac{\{\sqrt{x}+\sqrt{y}\}}{\sqrt[3]{x}+\sqrt[3]{y}}}$ Show that

$$x^2\frac{\delta^2 u}{\delta x^2}+2xy\frac{\delta^2 u}{\delta x\delta y}+y^2\frac{\delta^2 u}{\delta y^2}=\frac{\tan u}{12}\left(\frac{10}{12}+\frac{\tan^2 u}{12}\right)$$

29. Prove that $xy(f_{xx}-f_{yy})-(x^2-y^2)f_{xy}=-r\frac{\delta^2 f}{\delta r\delta\theta}+\frac{\delta f}{\delta\theta}$
if $x=r\cos\theta$, $y=r\sin\theta$.

30. If $\phi(u, v)=f(x, y)$, $u=y^2-x^2$, $v=x^2+y^2$

Show that $\frac{1}{4xy}\cdot\frac{\delta^2 f}{\delta x\delta y}=\frac{\delta^2\phi}{\delta v^2}-\frac{\delta^2\phi}{\delta u^2}$ D. U. 1965 (s)

31.(i) If v is a differentiable function, the

$$x\frac{\delta v}{\delta x}+y\frac{\delta v}{\delta y}=nu\frac{dv}{du}$$

u is a differentiable homogeneous function of degree n.

(ii) If $u(x, y) = \log_e \frac{x^3 + y^3}{x + y}$, then $x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta u} = 2.$ N.U. 1994

32. If $u = \frac{x^2y^2}{x^2+y^2}$ Show that

$$x^2 \frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta^2 u}{\delta y \delta x} + y^2 \frac{\delta^2 u}{\delta y^2} = 2u$$

33. If $u=f(x^2+2yz, y^2+2zx)$, show that

$$(y^2-zx)\frac{\delta u}{\delta x} + (x^2-yz)\frac{\delta u}{\delta y} + (z^2-xy)\frac{\delta u}{\delta z} = 0$$

34. If $u=x^n F(y/x, z/x)$, Show that

$$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} + z\frac{\delta u}{\delta z} = nu.$$ [R. U. 1964, C. U. 1985]

35. If $x \cos u + y \sin u = 1$, $y = x \sin u - y \cos u$

prove that $v^2\frac{\delta^2 u}{\delta x \delta y} + v\frac{\delta u}{\delta x}. \frac{\delta u}{\delta y} = \cos 2u$ [D. U. H. 1960]

36. If $F(v^2-x^2, v^2-y^2, y^2-z^2)=0$, where $v=f(x, y, z)$

Show that $\frac{1}{x}\frac{\delta v}{\delta x} + \frac{1}{y}\frac{\delta v}{\delta y} + \frac{1}{z}\frac{\delta v}{\delta z} = \frac{1}{v}$

37. (i) If $u=x\,\phi(y/x)+\psi(y/x)$, prove that

Show that $x\left(\frac{\delta z}{\delta y}\right) - y\left(\frac{\delta z}{\delta x}\right) = x - y$ R. H. 1992

C. H. 1993

$$x^2\frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta^2 u}{\delta x \delta y} + y^2\frac{\delta^2 u}{\delta y^2} = 0$$ N. H. 1993

(ii) If $u=x\,\phi(x+y)+y\psi(x+y)$, show that

$$\frac{\delta^2 u}{\delta x^2} - 2\frac{\delta^2 u}{\delta x \delta y} + \frac{\delta^2 u}{\delta y^2} = 0$$ [D. U. 1962]

38. If $u=f(x, y)$ and $x=r\cos\theta$, $y=r\sin\theta$. Prove that

(i) $\left(\frac{\delta r}{\delta x}\right)^2 + \left(\frac{\delta r}{\delta y}\right)^2 = 1$ (ii) $x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = r\frac{\delta u}{\delta r}$

(iii) $\frac{\delta^2\theta}{\delta x^2} + \frac{\delta^2\theta}{\delta y^2} = 0.$

*39. $x=(e^u+e^{-v})$ and $y=e^v+e^{-u}$, $z=f(x, y)$;

Prove that $\frac{\delta^2 z}{\delta u^2} - 2\frac{\delta^2 z}{\delta u \delta v} + \frac{\delta^2 z}{\delta v^2}$

$$= x^2\frac{\delta^2 z}{\delta x^2} - 2xy\frac{\delta^2 z}{\delta x \delta y} + y^2\frac{\delta^2 z}{\delta y^2} + x\frac{\delta z}{\delta x} + y\frac{\delta z}{\delta y}$$

40. Denoting $\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2}$ by ∇^2, prove that $\nabla^2 t = 1/r$ and $\nabla^2 \tan^{-1} y/x = 0$ if $r^2 = x^2 + y^2$ [D. U. H. 1960]

41. If v be function of r alone, where $r^2=x^2+y^2+z^2$

Show that $\frac{\delta^2 v}{\delta x^2} + \frac{\delta^2 v}{\delta y^2} + \frac{\delta^2 v}{\delta r^2} = \frac{d^2 v}{dz^2} + \frac{2dv}{rdr}$

*42. Prove that

$$\nabla^2 u = \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} + \frac{\delta^2 u}{\delta z^2}$$

$$= \frac{\delta^2 u}{\delta r^2} + \frac{2}{r}\frac{\delta u}{\delta r} + \frac{1}{r^2}\frac{\delta^2 u}{\delta \theta^2} + \frac{\cot\theta}{r^2}.\frac{\delta u}{\delta\theta} + \frac{1}{r^2\sin^2\theta}.\frac{\delta^2 u}{\delta \phi^2}$$

where r, θ, ϕ denotes the spherical polar co-ordinates of a point.

*43. If $u = \frac{(x^2+y^2)^m}{2m(m-1)} + x\phi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)$

Prove that

$$x^2\frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta^2 u}{\delta x \delta y} + y^2\frac{\delta^2 u}{\delta y^2} = (x^2+y^2)^m$$

44. If $Pdx+Qdy+Rdz$ can be made a perfect differential of some function of x, y, z by multiplying each term by some factors, then

$$P\left(\frac{\delta Q}{\delta z}-\frac{\delta R}{\delta y}\right)+Q\left(\frac{\delta R}{\delta x}-\frac{\delta P}{\delta z}\right)+R\left(\frac{\delta P}{\delta y}-\frac{\delta Q}{\delta z}\right)=0.$$

45 If $u_1t_1=t_2t_3$, $u_2t_2=t_1t_3$, $u_3t_3=t_1t_2$ prove that

$$J(u_1, u_2, u_3)=\frac{\delta(u_1, u_2, u_3)}{\delta(t_1, t_2, t_3)}=4$$

46. Show that the function

$$u=x+3y+2z,\quad v=3x+4y-2z,\quad w=11x+18y-2z$$

are not independent and find a relation between them.

Ans. $J=0$; $w=2u+3v$ C. H. 1988

47. Show that $ax^2+by^2+cz^2+2fyz+2gzx+2hxy$ will resolve into linear factors if

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$$

Hints $u=v\,w$, $v=l_1x+m_1y+n_1z$, $w=l_2x+m_2y+n_2z$,

In $\frac{\delta(u, v, w)}{\delta(x, y, z)}=0$, equate the co-efficients of x, y, z separately to zero

48. If $u=x+y+z$, $v=x-2y+3z$, $w=2xy-xz+4yz-2z^2$

Show that $J=\frac{\delta(u, v, w)}{\delta(x, y, z)}=0$ and find a relation between u, v and ω Ans. $u^2-v^2=4\omega$

48 (i) If $u=x/\sqrt{(1-r^2)}$, $v=y/\sqrt{(1-r^2)}$, $w=z/\sqrt{(1-r^2)}$ where $r^2=x^2+y^2+z^2$, prove that $J\left(\frac{u, v, w}{x, y, z}\right)=\frac{1}{\sqrt{(1-r^2)^5}}$

R. H. 1987

*49. Show that the function $u=3x+2y-z$, $v=x-2y+z$ and $\omega=x(x+2y-z)$ are not independent, and find the relation between them. Ans. $u^2-v^2=8\omega$

*50. If $u=x+y+z$, $v=xy+yz+zx$, $w=x^3+y^3+z^3-3xyz$ show that u, v, w are connected by a functional relation $w=u^3-3uv$

*51. If $\omega=f(x, y)$ and $x=u\cosh v$, $y=u\sinh v$, then show that

$$\left(\frac{\delta w}{\delta x}\right)^2-\left(\frac{\delta w}{\delta y}\right)^2=\left(\frac{\delta w}{\delta u}\right)^2-\frac{1}{u^2}\left(\frac{\delta w}{\delta v}\right)^2$$ [D. U. H. 1970]

*52. If $x=f(u, v)$, $y=g(u, v)$, $z=h(u, v)$ and $F(u, v, w)=0$.

prove that $\frac{\delta(y, z)}{\delta(u, v)}dx+\frac{\delta(z, u)}{\delta(u, v)}dy+\frac{\delta(x, y)}{\delta(u, v)}dz=0$

*53. If $x=f(u, v, \omega)$, $y=g(u, v, \omega)$, $z=h(u, v, \omega)$ prove that

$$\frac{\delta(x, y, z)}{\delta(u, v, w)}\cdot\frac{\delta(u, v, w)}{\delta(x, y, z)}=1 \quad \text{provided } \frac{\delta(x, y, z)}{\delta(u, v, w)}\neq 0$$

54. If $u=f(v)$ where u, v are functions of $f(x, y, z)$ prove tha

$$\frac{\delta(u, v)}{\delta(y, z)}\cdot\frac{\delta z}{\delta x}+\frac{\delta(u, v)}{\delta(z, x)}\cdot\frac{\delta z}{\delta y}=\frac{\delta(u, v)}{\delta(x, y)}$$

54. (i) If $U=f(x, y)$ and $x=e^u\cos v$, $y=e^u\sin v$. then prove that

$$\frac{\delta^2 U}{\delta x^2}+\frac{\delta^2 U}{\delta y^2}=e^{-2u}\left(\frac{\delta^2 U}{\delta u^2}+\frac{\delta^2 U}{\delta v^2}\right)$$

55. By the transformation $\xi = a + \alpha x + \beta y$, $\eta = b - \beta x + \alpha y$ in which a, b, α, β are constants and $\alpha^2 + \beta^2 = 1$. The function $u(x, y)$ is transformed into a function $U(\xi, \eta)$ of ξ and η. Prove that

$$U_{\xi\xi} \; U_{\eta\eta} - U^2_{\xi\eta} = u_{xx} \; u_{yy} - u^2_{xy}$$

56. If $v = 7x^2 + 8xy + 9y^2$, then show that

$$v^2_x \, v_{yy} - 2v_x \, v_y \, v_{xy} + v^2_y \, v_{xx} = 376 \, v.$$ [D. U. 1982]

57. If $u = \phi(y + ax) + \psi(y - ax)$

Prove that $\dfrac{\delta^2 u}{\delta x^2} = a^2 \dfrac{\delta^2 u}{\delta y^2}$.

58. Given $xyz = a$, find all the differential coefficients of first and second order taking x and y are independent variables

59. Find the value of the expression

$$\frac{\delta^2 z}{\delta x^2} + \frac{\delta^2 z}{\delta y^2}, \quad \text{where } a^2x^2 + b^2y^2 - c^2z^2 = 0$$

60. If $\left(\dfrac{x}{a}\right)^n + \left(\dfrac{y}{b}\right)^n + \left(\dfrac{z}{c}\right)^n = 1$, find $\dfrac{\delta z}{\delta x}$ and $\dfrac{\delta^2 x}{\delta y \delta z}$

Also, find $\dfrac{dy}{dx}$ when the variables are connected by the two variables

(i) $\left(\dfrac{z}{c}\right)^n = \left(\dfrac{x}{a}\right)^n - \left(\dfrac{y}{b}\right)^n$ (ii) $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$

61. If $v = (1 - 2xy + y^2)^{-1/2}$ prove that

$$x \frac{\delta v}{\delta x} - y \frac{\delta v}{\delta y} = y^2 v^3$$

Also show that

$$\frac{\delta}{\delta x}\left\{(1 - x^2)\frac{\delta v}{dx}\right\} + \frac{\delta}{\delta y}(y^2)\frac{\delta v}{\delta y}\Big\} = 0$$

62. If $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, $lx + my + nz = 0$

prove that

$$\frac{dx}{ny/b^2 - mz/c^2} = \frac{dy}{lz/c^2 - nx/a^2} = \frac{dz}{mx/a^2 - ly/b^2}$$

Hints :—Diff. the conditions w. r. to x and apply cross multiplication.

63. If $z = (x \, \delta/\delta x - 1) \, \{f(y + x) - \phi(y - x)\}$

prove that

$$x\left(\frac{\delta^2 z}{\delta x^2} - \frac{\delta^2 z}{\delta y^2}\right) = 2 \frac{\delta z}{\delta x}$$

Hints $z = x(f' + \phi') - (f + \phi)$

$\delta z/\delta x = x(f'' + \phi'')$, $\delta z/\delta y = x(f'' + \phi'') - (f' + \phi')$

$\delta^2 z/\delta x^2 = x(f''' + \phi''') + (f'' + \phi'')$, $\delta^2 z/\delta y^2 = x(f''' + \phi''') - (f'' + \phi'')$

Multiply, the two equations by x and subtract the result will follow.

64. If $f(x, y, z, t) = \dfrac{f(t + r)}{r} + \dfrac{g(t - r)}{r}$, where

$r^2 = x^2 + y^2 + z^2$, prove that u satisfies the relation

$$u_{xx} + u_{yy} + u_{zz} = u_{tt}.$$

Ans. $\delta u/\delta x = f'(t + r) \, x/r^2 - f(t + r) x/r^3 + g'(t - r)(-x/r^2)$
$- g(t - r) x/r^3$

$u_{xx} = (x^2/r^3) f'' + (l/r^2 - 2x/r^4 - x^2/r^4) f' + (3x^2/r^5 - l/r^3) f$
$+ (x^2/r^3) g'' + (2x^2/r^4 + x^2/r^4 - t/r^2) g' + (3x^2/r^5 - 1/r^3) g$

$\therefore \; u_{xx} + u_{yy} + u_{zz} = (1/r)(f'' + g'') = u_{tt}$

65. If $x^2 + y^2 + z^2 - 2xyz = 1$, Show that

$$\frac{dx}{\sqrt{(1 - x^2)}} + \frac{dy}{\sqrt{(1 - y^2)}} + \frac{dz}{\sqrt{(1 - z^2)}} = 0$$

66. If $u = F(x^2 + y^2 + z^2) f(xy + yz + zx)$, prove that
$(y-z) u_x + (z-x) u_y + (x-y) u_z = 0$

66 (a) If $u=x^2-y^2-2xy+y+z$, then $(x+y)u_x+(x-y)u_y+(y-x)u_z=0$

66 (b) Show that $xu_x+yu_y+zu_z=0$ if $u=y/z + z/x + x/y$ N.U. 1995

67. If $\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1$, u is the function of x, y, z, then prove that $(u_x)^2+(u_y)^2+(u_z)^2=2(xu_x+yu_y+zu_z)$

68. Find the directional derivative of $f(x, y)=\tan^{-1}y/x$ at $(1,-1)$ towards $(3, 0)$ Ans. $\frac{3}{2\sqrt{5}}$ D. H. 1986

69. Find the total differential co-efficients of $u=(x+y+z)e^x$ D. H. 1986

70. For $f(x, y)=\log(1+xy)$, find the Taylor's series expansion about any point (x, y) upto orders 2, where $1+xy>0$.

71. State and prove Euler's Theorem on homogeneous functions of three variables. R. H, 1917

72. If $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1$, there prove that

$$\frac{x(b^2-c^2)}{dx}+\frac{y(c^2-a^2)}{dy}+\frac{z(a^2-b^2)}{dz}=0.$$ R. H. 1987

73. Find the directional derivatives of $F=2x^3y-3y^2z$ at $P(1, 2,-1)$ in a direction towards $Q(3,-1, 5)$, find also the maximum directional derivative from P and its magnitude.

Ans. 90/7, $12i+14j-12k$ at P, 22.

74. Test the continuity of the function

$$f(x, y)=\frac{2x^2y}{x^3+y^3}, \quad x+y\neq0$$
$$=0, \quad x+y=0$$ D. H. 1986

75 Consider $f(x, y) = \frac{xy}{x^2+y^2}$, $(x,y)\neq(0, 0)$ and
$=0$ $f(0, 0)=0$

Prove that $f_x(0, 0)$ and $f_y(0, 0)$ both exist but $f(x, y)$ is discontinuous at $(0, 0)$.

76. When is a function $f(x, y, z)$ said to be continuous at a pt. (a, b, c)? Examine the continuity of the function

$$f(x, y)=\frac{x^3+y^3}{x^2+y^2}, \quad x^2+y^2\neq0.$$

77. If the partial derivatives f_x, f_y of a function $f(x, y)$ exist and have the value zero at every point of R, prove that $f(x, y)$ is a constant.

78. Prove that the function $f(x, y)$ defined by

$$f(x, y)=-\frac{2xy}{x^2+y^2}, \quad x^2+y^2\neq0$$ C.U. 1992
$$f(0, 0)=0$$

has partial derivatives everywhere. Is it continuous everywhere.

79 Examine whether $f(x, y)=\sqrt{|xy|}$ is totally differentiable at the origin. Ans. no

80. If $f(x, y)=\frac{x^2}{x^2+y^2-x}$, $x^2+y^2\neq0$

$f(0, 0)=0$

81. Examine whether $f(x, y)$ is continuous at $(0, 0)$.

The partial derivatives f_x, f_y of a function $f(x, y)$ are continuous in R. prove that $f(x+h, y+k)-f(x, y)=hf_x(\xi,\eta)+kf_y f(\xi, \eta)$, where (x, y) and $(x+h, y+k)$ and the line segment joining them lie in R and (ξ, η) is an intermediate point on the line segment.

82. Find a function $f(x, y)$ which is a function of x^2+y^2 and is also a product of the form $\psi(x).\psi(y)$

Ans. $\psi(x)=a^{x^2}$, $\psi=a^{y^2}$.

83. If $f(x, y, z)$ be continuous together with derivatives of the first two orders, in the neighbourhood of the point (x_0, y_0, z_0) and if $f(x_0, y_0, z_0)=0$, and $f_\alpha(x_0, y_0, z_0)$ and

$\begin{vmatrix} f_x & f_y \\ f_{x\alpha} & f_{y\alpha} \end{vmatrix} \neq 0$, $f_{\alpha\alpha}\neq 0$ at this point, prove that the equations $f(x, y, \alpha)=0$, $f_\alpha(x, y, \alpha)=0$ define a curve which is tangent to each of the family.

84. If $f(x, y, z, t)$ has an extreme value at the point (p, q, r, s) subject to the subsidiary conditions $\phi(x, y, z, t)=0$, $\psi(x, y, z, t)=0$ and if at the point

$$\frac{\delta(\phi, \psi)}{\delta(z, t)} \neq 0$$ then two numbers λ, μ exist

such that at the point (p, q, r, s) the equations.

$$f_x+\lambda\phi_x+\mu\psi_x=0,\quad f_y+\lambda\phi_y+\mu\psi_y=0$$
$$f_z+\lambda\phi_z+\mu\psi_z=0,\quad f_t+\lambda\phi_t+\mu\psi_t=0$$

and also the subsidiary conditions are satisfied.

85. যদি $x^3+y^3+z^3-3xyz=e^u$ হয়, তাহা হইলে দেখাও যে $x=1$, $y=1$, $z=1$ বিন্দুতে $u_{xx}+u_{yy}+u_{zz}=-\frac{1}{3}$ *C. U. 1991*

86. যদি $u=f(r)$ এবং $r^2=x^2+y^2$, প্রমাণ কর যে

$$\frac{\delta^2 u}{\delta x^2}+\frac{\delta^2 u}{\delta y^2}=f''(r)+\frac{1}{r}f'(r)$$ *D. U. 1990*

87. If z and lc are the functions of x and y defined by

$$\{z-\phi(u)\}^2=x^2(y^2-u^2),\ \{z-\phi(u)\}\phi'(u)=ux^2$$

prove that $\frac{\delta z}{\delta x}, \frac{\delta z}{\delta y}=xy$ *R. U. 1991, C. H. 1986*

87 (a) If $u=f(ax^2+2hxy+by^2)$ R.H.1992

$v=\phi(ax^2+2hxy+by^2)$

Then show that $\frac{\delta}{\delta y}\left(u\frac{\delta v}{\delta x}\right)=\frac{\delta}{\delta x}\left(u\frac{\delta v}{\delta y}\right)$

88. Discuss the existence of $\underset{(x,y)\to(0,0)}{Lt} f(x,y)$

for $f(x,y)=\frac{x^3-y^3}{x^2+y^2}$, $(x, y)\neq(0,0)$ C.U. 1993

$=0$, $(x,y)=(0,0)$ nuous

89. If $u=\frac{x^2y^2}{x+y}$, then show that

(i) $x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}=3u$ C.U.1993

(ii) $x^2\frac{\delta^2 u}{\delta x^2}+2xy\frac{\delta^2 u}{\delta x\delta y}+y^2\frac{\delta^2 u}{\delta y^2}=2u$

90. If $Z=x\sin(x/y)+ye^{y/x}$, prove that $x\frac{\delta Z}{\delta x}+y\frac{\delta Z}{\delta y}=Z$ C.U. 1993

91. Show that the function f, where

$f(x,y)=\left\{\frac{xy}{\sqrt{(x^2+y^2)}}\right\}$; if $x^2+y^2\neq 0$,

$=0$; if $x=y=0$

is continuous, possesses partial derivatives but is not differentiable at the origin.

92. Prove that the function

$f(x,y)=\sqrt{|xy|}$

is not differentiable at the point (0,0), but that f_x and f_y both exist at the origin and have the value 0, Hence deduce that these two partial derivatves are continous except at the origin

93. Show that for function

$f(x,y)=\frac{xy(x^2-y^2)}{x^2+y^2}$, $(x,y)\neq(0,0)$

94. Given that

$f(x, y) = \frac{xy^2}{x^2+y^4}$, $(x, y) \neq (0, 0)$

$= 0, x = y = 0.$

Then $f(x, y)$ is continuous at the origin. Give reason.

R.U. 1985; D.U. 1980

95. Given that

$f(n, y) = (x+y) \sin\frac{1}{x} \sin\frac{1}{y}$, $x \neq 0, y \neq 0$

$= 0, \quad x = y = 0$ C.H. 1991

Test Whether $\underset{x\to 0,\ y\to 0}{\text{Lt}} f(x, y)$ exists or not Ans discontinuous.

96. For $f(x,y) = (x^2+y^2)\log(x^2+y^2)$, $x^2+y^2 \neq 0$

$= 0, \quad x = y = 0$

Prove that $f_{xy}(0, 0) = f_{yx}(0, 0)$

Show that at (0, 0) neither of the derivatives is continuous C.H. 1992

See APPENDIX
Ex. 97, 98, 99, 100

ANSWERS

1. (i) $f_x = y, f_y - x$ (ii) $f_x = yx^{y-1}, f_y = x^y \log x$

(iii) $f_x = \frac{2x}{x^2+y^2}, f_y = \frac{2y}{x^2+y^2},$

(iv) $f_x = \frac{-y}{x\sqrt{x^2-y^2}}, f_y = \frac{1}{\sqrt{x^2-y^2}}$

(v) $f_x = \frac{1}{1+(x+y)^2}, f_y = \frac{1}{1+(x+y)^2}$

3. 0, 5, $f_x = e^{xy}(\frac{1}{2}y\sin 2x + \cos 2x)$,

$f_y = e^{xy}(\frac{1}{2}x\sin 2x)$, $f_{xx} = e^{xy}(\frac{1}{2}y^2\sin 2x + 2y\cos x$

$-2\sin 2y)$, $f_{xy} = e^{xy}(xy+1)\sin 2x + e^{yx}x\cos 2x$,

$f_{yx} = e^{xy}\left(\frac{1}{2}xy + \frac{1}{2}\right)\sin 2x + e^{xy}x\cos 2x,$

$f_{yy} = \frac{1}{2}e^{xy}x^2\sin 2x.$

15. $2u + u\sin 2v, u^2\cos 2v.$

CHAPTER X (A)

TANGENTS AND NORMALS IN CARTESIAN CO-ORDINATES

Art 10. 1. Definition :—Let P be a given point on a curve and Q be any other point on it and let the point Q move along the curve nearer and nearer to the point P then the limiting position of the secant PQ, provided limit exists, when Q moves up to and ultimately coincide with P, is called the tangent to the curve at the point P.

The line through the point P perpendicular to the tangent is called the normal to the curve at the point P.

Art. 10. 2. Equation of Tangent.

To find the equation of the tangent at (x, y) of the curve $y=f(x)$.

(i) *Explicit Cartesian Equation.*

Let (x, y) be the co-ordinates of P on the curve $y=f(x)$ Let us take another point Q in the neighbourhood of P and let the co-ordinates of Q be $(x+\Delta x, y+\Delta y)$. Let (X, Y) be the current co-ordinates of a point.

The equation of the chord PQ is

$$Y-y = \frac{y+\Delta y-y}{x+\Delta x-x}(X-x)$$

or. $$Y-y = \frac{\Delta y}{\Delta x}(X-x) \quad \cdots \quad \cdots \quad (1)$$

Now, as $Q \to P$, Δx and Δy both tend to zero.

$\therefore$ from (i) the equation of the tangent at $P(x, y)$ is

$$Y-y=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}(X-x)=\frac{dy}{dx}(X-x)$$

if the limit exists.

Thus the tangent at P (x, y) on the curve $y=f(x)$ is

$$Y-y=\frac{dy}{dx}(X-x) \quad \cdots \quad \cdots \quad (1)$$

The equation of any line through (x, y) is

$$Y-y=m(X-x) \quad \cdots \quad \cdots \quad \cdots \quad (2)$$

The line (2) is perpendicular to the tangent at (x, y) if

$$m\frac{dy}{dx}=-1 \quad \text{or,} \quad m=-1\Big/\frac{dy}{dx}$$

Therefore, the straight line (2) becomes the normal at (x, y) if

$$m=-\left(\frac{dx}{dy}\right)$$

Thus the equation of the normal at (x, y) is

$$Y-y=-\frac{dx}{dy}(X-x)$$

or, $$(Y-y)\frac{dy}{dx}+(X-x)=0 \quad \cdots \quad \cdots \quad (3)$$

(ii) Implicit cartesian equation.

Find the equation of the tangent at (x, y) to the curve $f(x, y)=0$.

In this curve we have

$$\frac{dy}{dx}=-\frac{\delta f/\delta x}{\delta f/\delta y}\,;\ \frac{\delta f}{\delta y}\neq 0$$

or, $$\frac{dy}{dx}=-\frac{f_x}{f_y},\ f_y\neq 0$$

Thus the equation of the tangent at (x, y) is

$$(Y-y)=\frac{dy}{dx}(X-x) \quad \text{or,} \quad (Y-y)=\frac{-f_x}{f_y}(X-x)$$

or, $$(X-x)f_x+(Y-y)f_y=0 \quad (4)$$

The equation of the normal at (x, y) on the curve $f(x, y)=0$ is

$$(Y-y)\frac{dy}{dx}+(X-x)=0 \quad \text{or,} \quad (Y-y)\frac{-f_x}{f_y}+(X-x)=0$$

or, $$\frac{X-x}{f_x}=\frac{Y-y}{f_y} \quad (5)$$

(iii) A symmetrical form of the equation of the tangent to a rational algebraic curve. Let $f(x, y)=0$ be a function of x and y of degree n. We can make the equation homogeneous by introducing another variable z. Then the equation $f(x, y)=0$ becomes a homogeneous equation in x, y, z and the new equation of the curve can be given as $f(x, y, z)=0$. Since $f(x, y, z)=0$ is a homogeneous equation in (x, y, z) of degree n, therefore, by Euler's Theorem, we have

$$x\frac{\delta f}{\delta x}+y\frac{\delta f}{\delta y}+z\frac{\delta f}{\delta z}=nf(x, y, z)=n.\,0=0$$

or, $$x\frac{\delta f}{\delta x}+y\frac{\delta f}{\delta y}+\frac{\delta f}{\delta z}z=0. \quad (6)$$

Now the equation of the tangent (4)

$$(X-x)\frac{\delta f}{\delta x}+(Y-y)\frac{\delta f}{\delta y}=0 \text{ becomes}$$

$$X\frac{\delta f}{\delta x}+Y\frac{\delta f}{\delta y}=x\frac{\delta f}{\delta x}+y\frac{\delta f}{\delta y}=-z\frac{\delta f}{\delta z} \quad [by\ (6)]$$

or, $$X\frac{\delta f}{\delta x}+Y\frac{\delta f}{\delta y}+z\frac{\delta f}{\delta z}=0 \quad \cdots \quad \cdots \quad \cdots \quad (7)$$

or, $X\frac{\delta f}{\delta x} + Y\frac{\delta f}{\delta y} + Z\frac{\delta f}{\delta z} = 0$ ... ... (8)

or, $Xf_x + Yf_y + Zf_z = 0$ ... ... (9)

where the co-efficient of $\frac{\delta f}{\delta z}$ i. e. z is replaced by Z for the sake of symmetry.

After differentiation we are to put $Z = z = 1$ in the equation.

This method is shown below with an example.

Ex Find the tangent to the curve.

$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ at the point (x, y).

This equation can be written in a homogeneous form in x, y, z where $z = 1$.

Thus $f(x, y, z) = ax^2 + 2hxy + by^2 + 2gxz + 2fyz + cz^2$,

Now $f_x = \delta f/\delta x = 2ax + 2hy + 2gz$

$f_y = \delta f/\delta y = 2hx + 2by + 2fz$

$f_z = \delta f/\delta z = 2gz + 2fy + 2cz$

The equation of the tangent is

$Xf_x + Yf_y + Zf_z = 0$

or, $X(2ax + 2hy + 2gz) + Y(2hx + 2by + 2fz) + Z(2gx + 2fy + 2cz) = 0$

or, Putting $Z = z = 1$,

$X(ax + hy + g) + Y(hx + by + f) + (gx + fy + c) = 0.$

Thus the tangent at (x, y) to the curve $f(x, y) = 0$

is $X(ax + hy + g) + Y(hx + by + f) + gx + fy + c = 0$

Note : The use of formula (8) or (9) in finding the equation of a tangent to a rational algebraic curve is advised rather than the use of formula (4) or (5)

(iv) Parametric Equation

Find the equation of the tangent at 't' the curve represented by $x = \phi(t)$ and $y = \psi(t)$

In this case we have

$$\frac{dy}{dx} = \frac{dy}{dt}\,\frac{dt}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{\psi'(t)}{\phi'(t)} \;;\; \phi'(t) \neq 0$$

The equation of the tangent at (x, y) is

$$(Y - y) = \frac{dy}{dx}(X - x) \text{ or, } \{Y - \psi(t)\} = \frac{\psi'(t)}{\phi'(t)}\{X - \phi(t)\}$$

or, $\{X - \phi(t)\}\psi'(t) = \{X - \psi(t)\}\,\phi'(t)$ (10)

The equation of the normal to the curve $x = \phi(t)$ and $y = \psi(t)$ at (x, y) is

$$(Y - y)\frac{dy}{dx} + (X - x) = 0 \qquad \text{from (3)}$$

or, $\{Y - \psi(t)\}\,\frac{\psi'(t)}{\phi'(t)} + \{X - \phi(t)\} = 0$

or, $\{X - \phi(t)\}\,\phi'(t) + \{Y - \psi(t)\}\,\psi'(t) = 0$ (11)

Ex. . Find the equation of the tangent at 't' to the curve

$$x = t^2 - a,\; y = t^3 - b$$

Here $x = \phi(t) = t^2 - a$ $\quad \phi'(t) = 2t$

$y = \psi(t) = t^3 - b, \quad \therefore \psi'(t) = 3t^2$

Hence the equation of the tangent at 't' is

$\{X - \phi(t)\}\,\psi'(t) = \{Y - \psi(t)\}\,\phi'(t)$

or, $\{X - (t^2 - a)\}3t^2 - \{Y - (t^3 - b)\}2t = 0$

or, $3t\,X - 2\,Y + t^3 + 3at - 2b = 0.$

10. 3. Tangent at the origin,

The equation of the tangent at (x, y) to any curve is

$Y - y = (dy/dx)(X - x)$

The equation of tangent at (0, 0)

$$Y=\left(\frac{dy}{dx}\right)_0 (X-0) \quad \text{or,} \quad Y=\left(\frac{dy}{dx}\right)_0 X \quad \ldots\ldots(1)$$

where $\left(\frac{dy}{dx}\right)_0$ the value of $\frac{dy}{dx}$ at $x=0,\ y=0$

Ex. Find the tangent at the origin to the curve

$$x^2+y^2+ax+by=0$$

$$\frac{dy}{dx}=-\frac{f_x}{f_y}=-\frac{2x+a}{2y+b}=-\frac{a}{b} \text{ when } x=0,\ y=0$$

Thus the equation of the tangent at (0, 0) is

$$Y-0=\left(\frac{dy}{dx}\right)_0 (X-0) \quad \text{or,} \qquad \text{or,} \quad Y=-\frac{a}{b}X$$

or, $bY+aX=0$

Take x and y as current co-ordinates, then the equation of the tangent at the origin is $\quad ax+by=0$

which is the lowest degree terms of the given equation.

Working Rule : If a rational algebraic equation passes through the origin the equation of the tangent or tangents can be obtained by equating to zero the lowest degree terms of the given equation. By inspection such tangents can be determined.

10. 4. Angle of Intersection of two curves.

The angle between the intersection of two curves is determined by the angle between the tangents drawn to each curve at the point of their intersection.

The angle between the two straight lines $y=m_1x+c_1$ and $y=m_2x+c_2$ is $\tan\theta=\frac{m_1\sim m_2}{1+m_1m_2}$

Let us consider two curve $f(x, y)=0$ and $\phi(x, y)=0$,

The equation of tangents at (x, y) are

$$(X-x)f_x+(Y-y)f_y=0 \quad \text{or,} \quad Xf_x+Yf_y-xf_x-yf_y=0$$

and $(X-x)\phi_x+(Y-y)\phi_y=0$ or, $X\phi_x+Y\phi_y-x\phi_x-y\phi_y=0$

The slopes are

$m_1=-f_x/f_y$ and $m_2=-\phi_x/\phi_y$

Let α be angle between the two tangents drawn at (x, y)

$$\therefore \quad \tan\alpha=\frac{-f_x/f_y+\phi_x/\phi_y}{1+f_x/f_y\,\phi_x/\phi_y}=\frac{f_x\phi_y-\phi_xf_y}{f_x\phi_x+f_y\phi_y}$$

Hence the angle α of intersection of two curves is given by

$$\tan\alpha=\frac{f_x\,\phi_y-\phi_x\,f_y}{f_x\,\phi_x+f_y\,\phi_y} \quad \ldots \ldots \ldots \ldots \ldots (12)$$

Cor. 1. If the two curves touch at (x, y) then the angle between the curves is zero i. e. $\alpha=0$, is

$$f_x\phi_y\sim\phi_xf_y=0$$

$$\text{or,} \quad \frac{f_x}{\phi_x}=\frac{f_y}{\phi_y} \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

Cor. 2. If the angle between the two tangents is 90° the curves are said to intersect orthogonally.

If $\alpha=\frac{1}{2}\pi$ then

$$\tan\tfrac{1}{2}\pi=\frac{f_x\phi_y\sim\phi_xf_y}{f_x\phi_y+f_y\phi_y}=\infty$$

hence $f_x\phi_x+f_y\phi_y=0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$

Cartesian Subtangent, Subnormal, Length of tangent and normal.

Let $y=f(x)$ be the curve and $P(x,\ y)$ be any point on the curve. Let the tangent PT and normal PN meet the x-axis respectively at T and N.

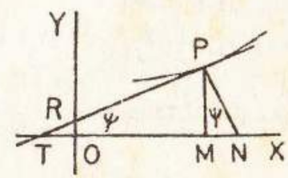


Fig 10 (a)

Draw PM perpendicular on x—axis.

The projection TM of the tangent PT on the x—axis is called **the subtangent.** While the projection, MN of the normal PN on the x—axis is called the **subnormal.** Let the tangent PT make an angle ψ with the x—axis ; then

$$\angle PTM = \psi = \angle NPM$$

$$\therefore \quad \tan \psi = \frac{dy}{dx} = y_1 \text{ and } MP = y.$$

(i) **Length of the Subtangent TM**

From the $\triangle TMP$, we have

$$TM = PM \cot\psi = y/\tan\psi = y/\frac{dy}{dx} = \frac{y}{y_1}$$

$\therefore$ The length of Subtangent $= y/y_1$ ... ... (16)

(ii) **Length of the Subnormal MN**

From the $\triangle MPN$ we have

$$MN = PM \tan \psi = y\frac{dy}{dx} = yy_1$$

$\therefore$ The length of the **Subnormal** $= y\frac{dy}{dx} = yy_1$ ... ...(17)

(iii) **Length of the tangent PT.**

From the $\triangle TMP$,

$$TP^2 = TM^2 + MP^2$$

$$= (y/y_1)^2 + y^2 = (y/y_1)^2(1+y_1^2) \quad \text{or,} \quad PT = (y/y_1)\sqrt{(1+y_1^2)}$$

$\therefore$ **Length of tangent** $= \frac{y}{y_1}\sqrt{(1+y_1^2)}$ ... ... (18)

(iv) **Length of the normal PN.**

From the $\triangle MPN$.

$$PN^2 = PM^2 + MN^2 = y^2 + (yy_1)^2 \quad \text{or.} \quad PN = y\sqrt{\{y(1+y_1^2)\}}$$

$\therefore$ **Length of the normal** $= y\sqrt{(1+y_1^2)}$ ... ... (19)

(v) **Points of Intersection —made by the tangent on axes.**

The equation to the tangent is as (x, y)

$$Y - y = (dy/dx)(X - x)$$

If the tangent meets the x—axis, then $Y=0.$ ⇒

$$-y = (dy/dx)(X-x) = y_1(X-x) \quad \text{or.} \quad X = x - y/y_1$$

(a) **The point of intersection with the x—axis is $(x - y/y_1, 0)$**

... ... (20)

If the tangent meets the Y—axis, then $X=0$, *i. e.*,

$$Y - y = \frac{dy}{dx}(0-x) = -y_1 x \quad \text{or,} \quad Y = y - y_1 x$$

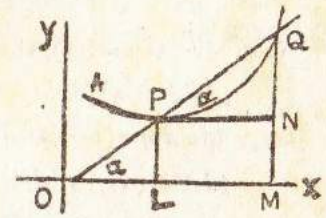
(b) $\therefore$ The point of intersection on Y—axis is $(0, y - y_1 x)$ ... ... (21)

10. 6. Arc Derivatives

To prove that for the curve $y = f(x)$:

$$\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}$$

Let P (x, y) be any point on the curve and Q$(x+\Delta x, y+\Delta y)$ be a a neighbouring point on it.



Let A be a fixed point on the curve from which the arc length AP and AQ are measured.

Let arc AP $=$ S and arc AQ $=$ S $+\Delta$S

So that arc PQ $= \Delta$S.

Draw PL, QM perpendiculars on QY and PN perpendicular to QM.

$ON = QM - NM = QM - PL = y + \Delta y - y = \Delta y$

$PN = LM = OM - OL = x + \Delta x - x = \Delta x$

From the right angled trangle QPN

$(\text{Chord } PQ)^2 = PN^2 + NQ^2 = (\Delta x)^2 + (\Delta y)^2$

or, $\left(\frac{\text{chord } PQ}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$

or, $\left(\frac{\text{chord } PQ}{\Delta x}\right)^2 \left(\frac{\text{arc } PQ}{\text{arc } PQ}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$

or $\left(\frac{\text{chord } PQ}{\text{arc } PQ}\right)^2 \left(\frac{\text{arc } PQ}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$

or, $\left(\frac{\text{chord } PQ}{\text{arc } PQ}\right)^2 \left(\frac{\Delta s}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$

If $Q \to P \frac{\text{chord } PQ}{\text{arc } PQ} \to 1$

$\therefore$ * $\lim_{Q \to P}\left(\frac{\text{chord } PQ}{\text{arc } PQ}\right)^2 \lim_{\Delta x \to 0}\left(\frac{\Delta s}{\Delta x}\right) = \lim_{\Delta x \to 0}\left\{1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right\}$

or, 1. $(ds/dx)^2 = 1 + (dy/dx)^2$

or, $\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}$ (22)

Cor. 1. If $x = f(y)$, then

$(ds/dy)^2 = 1(dx/dy)^2$ ... ... (23)

or, $(ds/dy) = \{1 + (dx/dy)^2\}$ ... ... (23i)

$(ds)^2 = (dx)^2 + (dy)^2$ ... ... (24)

Cor. 2. From the ΔPQN

$\cos QPN = \frac{PN}{PQ} = \frac{\Delta x}{PQ} = \frac{\Delta x}{\Delta s} \cdot \frac{\Delta s}{PQ}$

or, $\cos \alpha = \frac{\Delta x \text{ arc } PQ}{\Delta s \text{ chord } PQ}$

Let ψ be the angle which the tangent to the curve at P makes with the positive direction of axis of x.

if $Q \to P$, $\alpha \to \psi$ and $\Delta s \to 0$. Also $\lim_{Q \to P} \frac{\text{arc } PQ}{\text{chord } PQ} = 1$

Thus

$\cos \psi = \lim_{\Delta s \to 0} \frac{\Delta x}{\Delta s} = \frac{dx}{ds} \quad \therefore \quad \cos \psi = \frac{dx}{ds}$ (25)

Cor. 3. Similarly, $\sin PQN = \frac{QN}{PQ} = \frac{\Delta y}{\Delta s} \cdot \frac{\Delta s}{PQ}$

or, $\sin \alpha = \frac{\Delta y}{\Delta s} \cdot \frac{\text{arc } PQ}{\text{chord } PQ}$

If $P \to Q$, $\alpha \to \psi$, $\frac{\text{arc } PQ}{\text{chord } PQ} \to 1$ and hence

$\sin \psi = \lim_{\Delta s \to 0}\left(\frac{\Delta y}{\Delta s}\right) 1 = \frac{dy}{ds} \quad \therefore \quad \sin \psi = \frac{dy}{ds}$ (26)

Cor. 4. $\tan \psi = (dy/dx)$; $\cos \psi = (dx/dy)$

we have from (1)

$(ds/dx)^2 = 1 + (dy/dx)^2 = 1 + \tan^2\psi = \sec^2\psi$ or, $(ds/dx) = \sec \psi$

Similarly $\frac{ds}{dy} = \text{cosec } \psi$.

Also $(dx/ds)^2 + (dy/ds)^2 = \cos^2\psi + \sin^2\psi = 1$

$\therefore \quad (dx/ds)^2 + (dy/ds)^2 = 1$ (27)

Cor. 5. Prove that for the parametric curve $x = \phi(t)$ and $y = \phi(t)$, the arc derivative is

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

Now $\frac{ds}{dt} = \frac{dx}{ds} \cdot \frac{ds}{dt}$; $\frac{dy}{dt} = \frac{dy}{ds} \cdot \frac{ds}{dt}$

Therefore,

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\left(\frac{dx}{ds}\right)^2\left(\frac{ds}{dt}\right)^2+\left(\frac{dy}{ds}\right)^2\left(\frac{ds}{dt}\right)^2$$

$$=\left\{\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2\right\}\left(\frac{ds}{dt}\right)^2$$

$$=1\left(\frac{ds}{dt}\right)^2 \text{ [by (27)]}$$

$$\therefore \quad \left(\frac{ds}{dt}\right)^2=\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2 \quad \ldots(28)$$

$$\text{or,} \quad \frac{ds}{dt}=\sqrt{\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}} \quad \ldots \quad \ldots \quad (29)$$

provided S increases with t.

Examples

10. 7.

Ex. 1. Find the equation to the tangent to the curve

$$(x/a)^m+(y/b)^m=1 \quad \text{at} \quad (x, y)$$

The equation can be written as $x^m/a^m+y^m/b^m-1=0 \ldots\ldots(1)$

Now $f_x=\frac{m}{a^m}x^{m-1}=\frac{m}{a}\left(\frac{x}{a}\right)^{m-1}$ and $f_y=\frac{m}{b^m}y^{m-1}$

$$=\frac{m}{b}\left(\frac{y}{b}\right)^{m-1}$$

The equation of the tangent at (x, y) is

$$(X-x)f_x+(Y-y)f_y=0,$$

$$\text{or,} \quad (X-x)\frac{m}{a}\left(\frac{x}{a}\right)^{m-1}+(Y-y)\frac{m}{b}\left(\frac{y}{b}\right)^{m-1}=0$$

$$\text{or,} \quad \frac{X}{a}\left(\frac{x}{a}\right)^{m-1}+\frac{Y}{b}\left(\frac{y}{b}\right)^{m-1}=\left(\frac{x}{a}\right)^m+\left(\frac{y}{b}\right)^m=1 \text{ by (1)}$$

$$(X/a)(x/a)^{m-1}+(Y/b)(y/b)^{m-1}=1.$$

Ex. 2. Find the angle of intersection of the curves

$$x^3+2xy^2-10a^2x+12a^2y+3a^3=0$$

and $y^3+2xy^2-5a^2x-a^3=0$ at $(3a -2a)$

Let $f(x, y)=x^3+2xy^2-10a^2x+12a^2y+3a^3=0 \quad \ldots \quad \ldots (1)$

$\phi(x, y)=y^3-2xy^2-5a^2x-a^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots (2)$

Then $f_x=3x^2+2y^2-10a^2, f_y=4xy+12a^2$

$$\phi_x=2y^2-5a^2, \quad \phi_y=3y^2+4xy$$

At $(3a, -2a)$

$f_x=27a^2+8a^2-10a^2=25a^2, f_y=-24a^2+12a^2=-12a^2$

$\phi_x=8a^2-5a^2=3a^2, \phi_y=12a^2-24a^2=-12a^2$

Let α be the angle of their intersection. Then,

$$\tan\alpha=\left[\frac{f_x\phi_y-f_y\phi_x}{f_x\phi_x+f_y\phi_y}\right] \text{ When } x=3a, \ y=-2a$$

$$=\frac{25a^2(-12a^2)-(-12a^2 \cdot 3a^2)}{25a^2.3a^2+(-12a^2)(-12a^2)}=\frac{264a^4}{219a^4}=\frac{88}{73}$$

$$\therefore \quad \alpha=\tan^{-1}(88/73)$$

Ex. 3. Show that in curve $by^2=(x+a)^3$ the square of the subtangent varies as the subnormal. [R. U. 1961, '83]

We have, $by^2=(x+a)^3$ or, $y^2=\frac{(x+a)^3}{b} \quad \ldots \quad \ldots \quad (1)$

$$\therefore \quad 2y\frac{dy}{dx}=\frac{3(x+a)^2}{b} \text{ or, } y_1=\frac{dy}{dx}=\frac{3(x+a)^2}{2yb}$$

Let m=length of subtangent$=y/y_1$, n=length of the subnormal$=yy_1$

$$\frac{m^2}{n}=\frac{(y/y_1)^2}{(yy_1)}=\frac{y}{y_1^3}=\frac{yy^3b^3}{27(x+a)^6}$$

$$=\frac{8b}{27}\frac{y^4}{\{(x+a)^2/b\}^2}=\frac{8b\,y^4}{27\,y^4}=\frac{8b}{27}=\text{constant}$$

$$\therefore \quad m^2 \propto n]$$

Hence square of the subtangent varies as subnormal.

Ex. 4. Show that the subnormal at any point of the curve $y^2x^2=a^2(x^2-y^2)$ varies inversely as the cube of the abscissa

[R. U, 1966 D. U. 1961]

$$y^2x^2=a^2(x^2-a^2) \quad \text{or,} \quad y^2=a^2-a^4/x^2$$

$$\therefore \quad 2y\frac{dy}{dx}=\frac{2a^4}{x^3} \text{ or } y_1=\frac{a^4}{yx^3}$$

$$n=\text{subnormal}=y\frac{dy}{dx}=yy_1=y\frac{a^4}{yx^3}=\frac{a^4}{x^3}$$

$$n \propto 1/x^3$$

Hence subnormal varies inversely as the cube of the abscissa.

Ex. 5. Prove that the segment (between the co-ordinates axes) of a tangent to astroid $x^{2/3}+y^{2/3}=a^{2/3}$ is of constant length.

C.U. 1992 [D. U 1962, 1969, 83, R. H, 1966]

From the equation $f(x,y)=x^{2/3}+y^{2/3}-a^{2/3}=0$ ... ... (1)

$$f_x=\tfrac{2}{3}(1/x)^{-1/3},\ f_y=\tfrac{2}{3}(1/y)^{-1/3}$$

The equation of the tangent at (x, y) is

$$(X-x)f_x+(Y-y)f_y=0$$

or, $(X-x)\frac{2}{3}x^{-\frac{1}{3}}+(Y-y)\frac{2}{3}y^{-\frac{1}{3}}=0$

or $\frac{X}{x^{1/3}}+\frac{Y}{y^{1/3}}=(x)^{2/3}+(y)^{2/3}=a^{2/3}$ by (1)

Let the tangent line meet the axes at A and B. Then the co-ordinates of A and B are respectively,

$A(a^{2/3}.\ x^{1/3}, 0)$ and $B(0,\ a^{2/3}\ y^{1/3})$

Now $OA=a^{2/3}x^{1/3}$, $OB=a^{2/3}\ y^{1/3}$

Now $AB^2=OA^2+OB^2=a^{4/3}x^{2/3}+a^{4/3}y^{2/3}=a^{4/3}(x^{2/3}+y^{2/3})$

$=a^{4/3}a^{2/3}=a^3=$ constant.

Ex. 6. Show that in the curve $y=a\log(x^2-a^2)$, sum of the lengths of the tangent and the subtangent varies as the product of the co-ordinates of the Point of contact.

Let $m=$length of the tangent$=\frac{y\sqrt{(1+y_1^2)}}{y_1}$

$\triangle^2$ $n=$Length of the subtangent$=y/y_1$

Here $y=a\log(x^2-a^2)\ \therefore\ y_1=\frac{a.2x}{x^2-a^2}$

Thus $m+n=\left\{\frac{y\sqrt{(1+y_1^2)}}{y_1}+\frac{y}{y_1}\right\}=\frac{y}{y_1}\left\{\sqrt{(1+y_1^2)}+1\right\}$

But $\sqrt{(1+y_1^2)}=\sqrt{\left\{1+\frac{a^2x^2}{(x^2-a^2)^2}\right\}}=\frac{x^2+a^2}{x^2-a^2}$

$\therefore\ m+n=\frac{y}{2ax}(x^2-a^2)\left\{\frac{x^2+a^2}{x^2-a^2}+1\right\}=\frac{y}{2ax}2x^2=\frac{xy}{a}$

$\therefore\ (m+n)\propto xy$, as a is constant.

Ex. 7. Find the equations of the tangent at the origin to the curve $y^2(a+x)=x^2(3a-x)$.

Rewrite the equation $y^2(a+x)=x^2(3a-x)$

or, $x^3+xy^2-3ax^2+ay^2=0$ or, $-x^3-x^2y+a(3x^2-y^2)=0$

Equate to zero the lowest degree term in the above equation to get

$$3x^2-y^2=0 \quad \text{or,} \quad y=\pm\sqrt{3}x$$

Hence the tangents at the origin are $y=\sqrt{3}x$ and $y=-\sqrt{3}x$.

Ex. 8. Find the angle of intersection of the curves

$2y^2=x^3$ and $y^2=32x$.

Let us find the points of intersection of the curves

$2y^2=x^3$ ... ... (1) and $y^2=32x$ ... ... (2)

From (1) & (2) $2 \cdot 32x = x^3$

or, $x^3 - 64x = 0$ or, $x(x^2-64)=0$ or. $x=0,\ 8,-8$

But $x=-8$ is inadmissable because that will give imaginary value of y,

From (2),

when $x=0$, $y=0$, and

when $x=8$, $y^2=32.8=(\pm 16)^2$ or, $y=\pm 16$

$\therefore$ The points of the intersection of the curve are

$O(0,0)$, $P(8,16)$, $Q(8,-16)$

Let $f(x,y)=2y^2-x^3$ and $\phi(x,y)=y^2-32x$

$\therefore$ $f_x=-3x^2, f_y=4y$; $\phi_x=-32$ and $\phi_y=2y$

Let θ be the angle of intersection, then

$$\text{At } O(0,0);\ \tan\theta = \frac{f_x\phi_y - f_y\phi_x}{f_x\phi_x + f_y\phi_y} = \frac{(-3x^2)(2y)+4y(-32)}{(-3x^2)(-32)+4y\ 2y}$$

$$= \frac{(-3\times 0)(2,0)-4,0(-32)}{(-3)(-32)+4.\ 0,\ 2.\ 0.} = \frac{0}{0}$$

which is the indeterminate form $\therefore$ no tangent is possible.

$$\text{At } P(8,16),\ \tan\theta = \frac{-6x^2y+128y}{96x^2+8y^2} = \frac{-6\times 8^2\times 16+128.16}{96.\ 8^2+8.\ 16^2}$$

$$= \frac{8^2\times 16(-6+2)}{8^2\times 16\ (6+2)} = \frac{-4}{8} = -\tfrac{1}{2}$$

$$\Rightarrow \theta = \tan^{-1}(-\tfrac{1}{2}) \quad \text{at } P(8,16)$$

$$\text{At } Q(8,-16),\ \tan\theta = \frac{-6.8^2(-16)+12.8(-16)}{96,\ 8^2+8\ (-16)^2}$$

$$= \frac{8^2\times 16(6-2)}{8^2\times 16(6+2)} = \frac{4}{8} = \tfrac{1}{2}$$

$$\Rightarrow \theta = \tan^{-1}(\tfrac{1}{2}) \text{ at } Q\ [(8,-16)]$$

Exercise X (A)

1. Find the equation to the tangent at the point (x_1, y_1) of the following curves.

(i) $x^2/a^2+y^2/b^2=1$ at (x_1, y_1) (ii) $y=a \log \sin x$ at (x_1, y_1)

(iii) $(x^2+y^2)^2=a^2(x^2-y^2)$ at (x_1, y_1)

1. (a) Prove that tangent at (a, b) to the curve $(x/a)^3+(y/b)^3=2$ is $(x/a)+(y/b)=2$ [R. H. 1988. D. H. 1986]

2. Find the equations of the normals at the point (x_1, y_1) of the following curves.

(i) $y^2=4ax$ (ii) $x^3-3axy+y^3=1$

(iii) $x^2(x-y)+a^2(x+y)=0$ at $(0,0)$

3. Find the tangent and normol at the point determined by θ on.

(i) $x=a\cos\theta$, $y=b\sin\theta$

(ii) $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$

(iii) $x=a\cos^3\theta$, $y=b\sin^3\theta$

3 (a) If $f(x)=x^2+x-6$ find the equation to the tangent and the normal to the curve of $f(x)$ at the point $x=1$. Draw a rough sketch. D. U. 1987

(যদি $f(x)=x^2+x-6$ একটি বক্ররেখা হয় তবে $x=1$ বিন্দুতে উক্ত বক্ররেখার স্পর্শক ও অভিলম্বের সমীকরণ নির্ণয় কর। মোটামুটি লেখচিত্র অঙ্কন কর।) Ans. $3x-y-7=0$, $x+3y+11=0$

At $x=1$, $y=x^2+x-6=-4$

we are to find tangent and normal at $(1,-4)$

$$\frac{dy}{dx}=2x+1=2.1+1=3 \text{ at } (1,-4)$$

The eq of the tangent at $(1,-4)$ is

$y+4=(dy/dx)(x-1)$ or, $y+4=3(x-1)$

or, $3x-y-7=0$

Normal at $(1,-4)$ is

$(y+4)(dy/dx)+(x-1)=0$ or, $(y+4)3+(x-1)=0$

or, $3y+x+11=0$

3(b) Find the eq of the tangent and normal at $(2,-2)$ of the curve $y=x^3-3x+2$ (বক্ররেখাটির $(2,-2)$ বিন্দুতে স্পর্শক ও অভিলম্ব নির্ণয় কর। C. U. 1981

Ans. $9x-y-20=0$, $x+9x+16=0$

4. Find the angle of intersection of the curves

(i) $x^2-y^2=a^2$ and $x^2+y^2=a^2\sqrt{2}$

(ii) $x^2=4ay$ and $2y^2=ax$

(iii) $y=4-x^2$ and $y=x^2$

(iv) $x^2-y^2=2a^2$, $x^2+y^2=4a^2$ D.U. 1993

5. For the curve $y=c\cos h\dfrac{x}{c}$ find the subtangent and the subnormal at any point. [R. U. 1962]

6. Show that the portion of the tangent at any point on the curve $x=27\cos^3\theta$, $y=27\sin^3\theta$ intercepted between the axis is of constant length.

7. Prove that the subtangent is of constant length at any pt. of the curve $\log y=x\log a$ and that the subnormal is constant at any point on the parabola $y^2=4ax$.

8. Show that the abscissa of the point on the curve. $\sqrt{(xy)}=a+x$, at which the normal makes equal intercepts from the co-ordinate axes is $a/\sqrt{2}$.

9. Find the condition that the conics shall cut orthogonally.

$ax^2+by^2=1$

$a_1x^2+b_1y^2=1$ [R. U 1964]

9. (i) Show that $xy=4$, $x^2-y^2=15$ cut each other orthogonally. [D. U. 1986]

(ii) prove that the curves $x^2/a+y^2/b=1$

$x^2/a_1+y^2/b_1=1$ cut orthogonally if $a-b=a_1-b_1$ R. U. 1987

(iii) Show that the curves $x^3-3xy^2+2=0$

$3x^2y-y^3=2$ cut orthgonally. D. U. 1989

10. Find the equation of the tangent and normal to the curve $x=e^{-t}\cos t$, $y=e^t\sin t$ at the point $t=\pi$.

10 (i) $x^2+2y^2=3$ at $(1,-1)$ D. U. 1980, '88

11. Show that the curve represented by

$$(x/a)^n+(y/b)^n=2$$

for different values of n have a common tangent at the point (a,b)

Hence show that the equation of the tangent is $\dfrac{x}{a}+\dfrac{y}{b}=2$

12. Show that all the points of the curve

$y^2=4a\left\{x+a\sin\dfrac{x}{a}\right\}$ [R. U. 1982]

at which the tangent is parallel to the axis of x lie on a parabola $y^2=4ax$.

13. Prove that $\dfrac{x}{a}+\dfrac{y}{b}=1$. touches the curve $y=be^{-x/a}$ at the point where the curve crosses the axis of y.

14. If $p=x\cos\alpha+y\sin\alpha$ touches the curves

$(x/a)^{n/n-1}+(y/b)^{n/n-1}=1$.

prove that $p^n=(a\cos\alpha)^n+(b\sin\alpha)^n$.

15. If $ax+by=1$ is a normal to the parabola $y^2=4cx$ then prove that $ca^3+2acb^2=b^2$.

16. Show that the condition that the curves $x^2/\alpha+y^2/\beta=c^2\beta$ and $x^2/a^2+y^2/b^2=1$ may touch is $c=a+b$.

17. For the catenary $y=c\cos h\,x/c$, prove that the length of the portion of the normal at (x, y) intercepted between the curve and the axis of x is y^2/c. [R. U. 1967, '78 ; C. H. 1969, '83]

18. If $p=x\cos\alpha+y\sin\alpha$ touch the curve,

$$\frac{x^m}{a^m}+\frac{y^m}{b^m}=1.\text{ show that}$$

$$p^{\frac{m}{m-1}}=(a\cos\alpha)^{\frac{m}{m-1}}+(b\sin\alpha)^{\frac{m}{m-1}}$$ [D. U. 1952]

19. Show that in the curve $a^2y^3=k(bx+e)^4$ the cube of the subtangent varies as the fifth power of the subnormal. D. U. 1962

20. Tangents are drawn from the origin to the curve $y=\sin x$. Prove that points of contract lie on $x^2y^2=x^2-y^2$,

21. Show that the length of the portion of the normal to the curve. $x=a(4\cos^3\theta-3\cos\theta)$; $y=a(4\sin^3\theta-9\sin\theta)$. intercepted between the co-ordinate axes is constant.

22. Find the condition that the line $p=x\cos\alpha+y\sin\alpha$ may be a tangent to the curve

$$x^my^n=a^{m+n}$$

23. Show that the curves cut orthogonally.

(i) $x^3-3xy^2+2=0$ and $3x^2y-y^3=2$.

(ii) $y=x^2$ and $x^3+6y=7$

24. If ϕ be the angle between the tangent to a curve and the radius vectors drawn from the origin of co-ordinates to the point of contact, prove that

$$\tan\phi=\left(x\frac{dy}{dx}-y\right)\Big/\left(x+y\frac{dy}{dx}\right)$$

25. Show that for curve $by^2=(x+a)^2$, the square of the sub-tangent varies as the subnormal. D. H. 1986

26. Show that the value of $n=-2$, so that the subnormal of the curve $xy^n=a^{n+1}$ may be of constant length.

27. Prove that mth power of the subtangent varies as nth power of the sub-normal of the curve. C. H. 1988

$$x^{m+n}=a^{m+n}y^{2n}$$

28. Show that subtangent at any point of the curve.

$$x^my^n=a^{m+n}$$

varies as the abscissa of the point

29. Show that for the curve
$x=a+b\log[b^2+\sqrt{(b^2-y^2)}]-\sqrt{(b^2-y^2)}$
sum of subnormal and subtangent is constant.

30. Find $\frac{ds}{dx}$ for the following curve.

(i) $y^2=4ax$ (ii) $x^2/^3+y^2/^3=a^2/^3$

(iii) $x=t^2$, $y=t-1$ (iv) $x=2\sin t$, $y=\cos 2t$.

31. Prove that in the curve $x=\frac{y^2-a^2}{4a}+\frac{a}{2}\log\frac{a}{y}$ the difference between the lengths of the tangent and subtangent is constant.

32. If the normal at any point to the curve $x^{2/3}+y^{2/3}=a^{2/3}$ makes an angle ϕ with the x-axis. show that its equation is $y\cos\phi-x\sin\phi=a\cos 2\phi$.

33. If x_1, y_1 be the portions of the axes of x and y intercepted by the tangent at any point (x,y) on the curve $(x/a)^{2/3}+(y/b)^{2/3}=1$.

show that $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$

34. Show that tangents at the origin of the following curve

(i) $x^3+y^3=3axy$ are $x=0$ and $y=0$

(ii) $a^2y^2=a^2x^2-x^4$ are $y=\pm x$

(iii) $y^3(x+3a)=x(x-a)(x-2a)$ is $x=0$

(iv) $x(y-x)^2=ay^2$ is $y=0$

(v) $x^2(x^2-4a^2)=y^2(x^2-a^2)$ are $y=\pm 2x$

(vi) $x^4-2x^2y^2+x^2+x-3y=0$ is $x=3y$

(vii) $x^5+y^5+x^3y+x^2y^2-6xy^2=0$

are $x=0$, $y=0$, $x=2y$, $x=-3y$

35. Find the area of the triangle formed by the axes and the tangent to the curve

$$x^{2/3}+y^{2/3}=a^{2/3}$$ [R. U, 1966]

36. Show that in the curve $ay^4=b^2(cx+f)^3$ the square of the subnormal varies as the subtangent. C. H. 1989

37. Find the tangent and normal to the curve

(a) $y(x-1)(2x-3)-x+4=0$ at the points where it cuts the x-axis N.H. 1994

(b) $y(x-2)(x-3)-x+7=0$ where it cuts the x-axis N.H. 1994

(c) $y(x^2+a^2)=ax^2$ at $y=a/4$ N.U. 1995

প্রশ্নমালা X (A)

1. নিম্নলিখিত বক্ররেখা সমূহের (x_1, y_1) বিন্দুতে স্পর্শকের সমীকরণ নির্ণয় কর।

(i) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ এর (x_1, y_1) বিন্দুতে। (ii) $y=a\log\sin x$-এর (x_1, y_1) বিন্দুতে। (iii) $(x^2+y^2)^2=a^2(x^2-a^2)$ এর (x_1, y_1) বিন্দুতে।

2. নিম্নলিখিত বক্ররেখা সমূহের (x_1, y_1) বিন্দুতে অভিলম্বের সমীকরণ নির্ণয় কর।

(i) $y^2=4ax$ (ii) $x^3-3axy+y^3=1$

(iii) $x^2(x-y)+a^2(x+y)=0$ এর $(0, 0)$ বিন্দুতে।

3. নিম্নলিখিত পরামিতিক সমীকরণগুলির 'θ' বিন্দুতে স্পর্শক এবং অভিলম্ব নির্ণয় কর।

(i) $x=a\cos\theta$, $y=b\sin\theta$.

(ii) $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$.

(iii) $x=a\cos^3\theta$, $y=b\sin^3\theta$.

3 (a) If $f(x)=x^2+x-6$ find the equation to the tangent and the normal to the curve of $f(x)$ at the point $x=1$. Draw a rough sketch. D. U. 1987

(যদি $f(x)=x^2+x-6$ একটি বক্ররেখা হয় তবে $x=1$ বিন্দুতে উক্ত বক্ররেখার স্পর্শক b অভিলম্বের সমীকরণ নির্ণয় কর। মোটামুটি লেখচিত্র অঙ্কন কর।) Ans. $3x-y-7=0$, $x+3y+11=0$

At $x=1$, $y=x^2+x-6=-4$

we are to find tangent and normal at $(1,-4)$

$$\frac{dy}{dx}=2x+1=2.1+1=3 \text{ at } (1,-4)$$

The eq of the tangent at $(1,-4)$ is

$y+4=(dy/dx)(x-1)$ or, $y+4=3(x-1)$

or, $3x - y - 7 = 0$

Normal at $(1, -4)$ is

$(y + 4)(dy/dx) + (x - 1) = 0$ or, $(y + 4)3 + (x - 1) = 0$

or, $3y + x + 11 = 0$

3 (b) Find the eq of the tangent and normal at $(2, -2)$ of the curve $y = x^3 - 3x + 2$ [বক্ররেখাটির (2,-2) বিন্দুতে স্পর্শক ও অভিলম্ব নির্ণয় কর।] C. U. 1981

Ans. $9x - y + 20 = 0,\ x + 9x + 16 = 0$

4. নিম্নলিখিত বক্ররেখাগুলির ছেদক কোণ (Argle of intersection) নির্ণয় কর।

(i) $x^2 - y^2 = a^2$ এবং $x^2 + y^2 = a^2 \sqrt{2}$. (ii) $x^2 = 4ay$ এবং $2y^2 = ax$

(iii) $y = 4 - x^2$ এবং $y = x^2$. (iv) $x^2 - y^2 = 2a^2,\ x^2 + y^2 = 4a^2$

D. U. 1991

5. সমীকরণ $y = c \cos h \dfrac{x}{c}$ এর জন্য যে কোন বিন্দুতে উপস্পর্শক ও উপলম্ব নির্ণয় কর। R. U. 1962

6. দেখাও যে $x = 27 \cos^3 \theta,\ y = 27 \sin^3\theta$ বক্ররেখার যে কোন বিন্দুতে অঙ্কিত স্পর্শক দুই অক্ষরেখাদ্বারা সীমাবদ্ধ স্পর্শকের দৈর্ঘ্য সর্বদা ধ্রুব।

7. দেখাও যে $\log y = x \log a$ বক্ররেখার যে কোন বিন্দুতে অঙ্কিত উপস্পর্শকের দৈর্ঘ্য ধ্রুব। এবং $y^2 = 4ax$ অধিবৃত্তে যে কোন বিন্দুতে অঙ্কিত উপলম্বের দৈর্ঘ্য ও ধ্রুব।

8. দেখাও যে $\sqrt{(xy)} = a + x$ বক্ররেখার যে বিন্দুতে অঙ্কিত অভিলম্ব অক্ষদ্বয় হইতে সমান সমান অংশ ছেদ করে ঐ বিন্দুর ভুজ হইবে $a/\sqrt{2}$.

9. $ax^2 + by^2 = 1$ এবং $a_1x^2 + b_1y^2 = 1$ কণিকদ্বয় পরস্পর লম্বভাবে ছেদ করিবার শর্ত নির্ণয় কর। R. U. 1964

9. (i) Show that $xy = 4,\ x^2 - y^2 = 15$ cut each other ortnogonally. [R. U. 1964]

(ii) prove that the curves $x^2/a + y^2/b = 1$

$x^2/a_1 + y^2/b_1 = 1$ cut orthogonally if $a - b = a_1 - b_1$ [R. U. 1987]

(iii) Show that the curves $x^3 - 3xy^2 + 2 = 0$

$3x^2y - y^3 = 2$ cut orthogonally. [D. U. 1989]

10. বক্ররেখা $x = e^{-t} \cos t,\ y = e^t \sin t$ এর $t = \pi$ বিন্দুতে স্পর্শক এবং অভিলম্বের সমীকরণ নির্ণয় কর।

10. (i) $x^2 + 2y^2 = 3$ at $(1, -1)$ [D. U. 1980, '88]

11. দেখাও যে n-এর বিভিন্ন মানের জন্য (a, b) বিন্দুতে

$$\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2.$$

বক্ররেখার উপর একটি সাধারণ স্পর্শক আঁকা যায়। ইহা হইতে দেখাও যে স্পর্শকের সমীকরণ $\dfrac{x}{a} + \dfrac{y}{b} = 2$.

12. দেখাও যে $y^2 = 4a\{x + a \sin x/a\}$ বক্ররেখার যে সকল বিন্দুতে স্পর্শকগুলি x- অক্ষের সমান্তরাল তাহার অধিবৃত্ত (Parabola) $y^2 = 4ax$ এর উপর থাকিবে।

13. বক্ররেখা $y = be^{-x/a}$ যে বিন্দুতে y-অক্ষকে অতিক্রম করে, দেখাও যে, সেই বিন্দুতে $x/a + y/b = 1$ সরলরেখাটি ঐ বক্ররেখাকে স্পর্শ করে।

14. যদি $P = x \cos \alpha + y \sin \alpha$ রেখাটি যদি বক্ররেখা

$\left(\dfrac{x}{a}\right)^{n/n-1} + \left(\dfrac{y}{b}\right)^{n/n-1} = 1$ কে স্পর্শ করে তবে দেখাও যে

$p = (a \cos \alpha)^n + (b \sin \alpha)^n$.

15. যদি $ax + by = 1$ রেখাটি অধিবৃত্ত (Parabola) $y^2 = 4cx$ এর উপর অভিলম্ব হয় তবে প্রমাণ কর যে $ca^3 + 2acb^2 = b^3$.

16. দেখাও যে $x^{2/3} + y^{2/3} = c^{2/3}$ এবং $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

বক্ররেখাদ্বয় পরস্পরকে স্পর্শ করার শর্ত হইল $c = a + b$.

17. কেটেনারী (Catenary) $y=c\cosh x/c$ এর যে কোন বিন্দু (x, y) এ যে অভিলম্ব অংকিত করা যায়, তাহার x-অক্ষ এবং ঐ বক্ররেখা দ্বারা কর্তিত অংশের দৈর্ঘ্য হইবে y^2/c. ইহা প্রমাণ কর।

[R. U. 1967, 78 C. U. H. 1969, 83]

18. যদি $P=x\cos\alpha+y\sin\alpha$ সরলরেখা

$\left(\frac{x}{a}\right)^m\left(\frac{y}{b}\right)^m=1$ বক্ররেখাকে স্পর্শ করে তবে দেখাও যে

$$P^{\frac{m}{m-1}}=(a\cos\alpha)^{\frac{m}{m-1}}+(b\sin\alpha)^{\frac{m}{m-1}}$$ [D. U. 1962]

19. দেখাও যে $a^2y^5=k(bx+c)^4$ বক্ররেখার যে কোন বিন্দুতে উপস্পর্শকের ঘন ঐ বিন্দুতে অঙ্কিত উপলম্বের পঞ্চম ঘাতের সমানুপাতিক।

[D. U. 1962]

20. $y=\sin x$ রেখার উপর মূল বিন্দু হইতে স্পর্শক অঙ্কিত করা হইলে, দেখাও যে স্পর্শ বিন্দুগুলি $x^2y^2=x^2-y^2$-এর উপর থাকিবে।

21. দেখাও যে বক্ররেখা $x=a(4\cos^3\theta-3\cos\theta)$; $y=a(4\sin^3\theta-9\sin\theta)$ এর যে কোন বিন্দুতে অঙ্কিত অভিলম্ব হইতে অক্ষ দ্বয় দ্বারা কর্তিত অংশের দৈর্ঘ্য ধ্রুব থাকে।

22. $P=x\cos\alpha+y\sin\alpha$ রেখাটি $x^my^n=a^{m+m}$ বক্ররেখাকে স্পর্শ করার শর্ত নির্ণয় কর।

23. দেখাও যে নিম্নলিখিত বক্ররেখাদ্বয় পরস্পরকে লম্বভাবে ছেদ করে। (cut orthogonally).

(i) $x^3-3xy^2+2=0$ এবং $3x^2y-y^3=2$.

(ii) $y=x^2$ এবং $x^3+6y=7$.

24. কোন বক্ররেখার কোন বিন্দুতে অঙ্কিত স্পর্শক এবং মূলবিন্দু এবং স্পর্শ বিন্দুর মধ্যে যোজিত ব্যাসার্ধ (Radius vector) ভেক্টরের মধ্যে কোণ ϕ হইলে প্রমাণ কর যে

$$\tan\phi=\left(x\frac{dy}{dx}-y\right)\Big/\left(x+y\frac{dy}{dx}\right).$$

25. $by^2=(x+a)^3$ বক্ররেখার জন্য দেখাও যে উপস্পর্শকের বর্গ উপলম্বের সমানুপাতিক।

26. দেখাও যে $n=-2$ হইলে বক্ররেখা $xy^n=a^{n+1}$ এর যে কোন বিন্দুতে উপলম্বের দৈর্ঘ্য ধ্রুব হইবে।

27. প্রমাণ কর যে $x^{m+n}=a^{m-n}y^{2n}$ বক্ররেখার যে কোন বিন্দুতে অঙ্কিত উপ-স্পর্শকের m-তম মাত্রা ঐ বিন্দুতে অংকিত উপলম্বের n-তম মাত্রার সমানুপাতিক। [C. H. 1988]

28. দেখাও যে $x^my^n=a^{m+n}$ বক্ররেখার যে কোন বিন্দুতে অঙ্কিত উপ-স্পর্শক ঐ বিন্দুর ভুজের সমানুপাতিক।

29. দেখাও যে $x=a+b\log[b^2+\sqrt{(b^2-y^2)}]-\sqrt{(b^2-y^2)}$ বক্ররেখার যে কোন বিন্দুতে অঙ্কিত উপস্পর্শক এবং উপলম্বের দৈর্ঘ্যের যোগফল ধ্রুব হইবে।

30. নিম্নলিখিত বক্ররেখাগুলির জন্য $\frac{ds}{dx}$ নির্ণয় কর।

(i) $y^2=4ax$ (ii) $x^{2/3}+y^{2/3}=a^{2/3}$

(iii) $x=t^2$, $y=t-1$ (iv) $x=2\sin t$, $y=\cos 2t$.

31. দেখাও যে, $x=\frac{y^2-a^2}{4a}+\frac{a}{2}\log\frac{a}{y}$ বক্ররেখার যে কোন বিন্দুতে অঙ্কিত স্পর্শকের এবং উপস্পর্শকে দৈর্ঘ্যের অন্তরফল ধ্রুব হইবে। D. U. H. 1960

32. যদি $x^{2/3}+y^{2/3}=a^{2/3}$ বক্ররেখার কোন বিন্দুতে অভিলম্ব অঙ্কিত করিলে ইহা x-অক্ষের সহিত ϕ কোণ উৎপন্ন করে তবে দেখাও যে উহার সমীকরণ হইবে $y\cos\phi-x\sin\phi=a\cos 2\phi$.

33. $\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}=1$ বক্ররেখার যে কোন বিন্দু (x, y) এ অংকিত স্পর্শক যদি x ও y অক্ষ হইতে যথাক্রমে x_1 ও y_1 অংশ কর্তন করে তবে প্রমাণ কর যে $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$.

34. দেখাও যে নিম্নলিখিত বক্ররেখাগুলির উপর মূলবিন্দুতে অংকিত স্পর্শক সমূহ হইবে

(i) $x^3 + y^3 = 3axy$ এর জন্য $x = 0$ এবং $y = 0$
(ii) $a^2y^2 = a^2x^2 - x^4$ এর জন্য $y = \pm x$
(iii) $y^2 (x + 3a) = x (x - a) (x - 2a)$ এর জন্য $x = 0$
(iv) $x (y - x)^2 = ay^2$ এর জন্য $y = 0$.
(v) $x^2 (x^2 - 4a^2) = y^2 (x^2 - a^2)$ এর জন্য $y = \pm 2x$.
(vi) $x^4 - 2x^2 y^2 + x^2 + x - 3y = 0$ এর জন্য $x = 3y$.
(vii) $x^5 + y^5 + x^3y + x^2 y^3 - 6xy^2 = 0$ এর জন্য $x = 0, y = 0, x = 2y, x = -3y$.

35. বক্ররেখা $x^{2/3} + y^{2/3} = a^{2/3}$ এর যে কোন বিন্দুতে অংকিত স্পর্শক এবং অক্ষদ্বয়ের দ্বারা সৃষ্ট ত্রিভুজের ক্ষেত্রফল নির্ণয় কর। [R. U. H. 1966]

36. Show that in the curve $ay^4 = b^2 (cx + f)^3$ the square of the subnormal varies as the subtangent. C. H. 1989

37(a) $y (x - 1) (2x - 3) - x + 4 = 0$ এই রেখাটিতে স্পর্শক ও লম্ব বাহির কর যেখানে x-অক্ষ ছেদ করে (Find the tangent and normal to the curve $y (x - 1) (2x - 3) - x + 4 = 0$ at the point where it cuts the x-axis.) (b) $y(x-2)(x-3)-x+7=0$ N.U. 1994 C. U. 1991

উত্তরমালা

1. (i) $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$. (ii) $y - y_1 = a \cot x (x - x_1)$
(iii) $(2y_1 (x_1^2 + y_1^2) + a^2y_1)y + (2x_1 (x_1^2 + y_1^2) + a^2x_1) x = a^2 (x_1^2 - y_1^2 - y_1^2)$
2. (i) $(x - x_1) y_1 + 2a (y - y_1) = 0$
(ii) $x (ax_1 - y_1^2) + y (x_1^2 - ay_1) = (x_1 - y_1) (ax_1 + ay_1 + x_1y_1)$
(iii) $x - y = 0$.
3. (i) $(x/a) \cos \theta + (y/b) \sin \theta = 1$
by $\cos \theta - ax \sin \theta + (a^2 - b^2) \sin \theta \cos \theta = 0$.
(ii) $y = (x - a\theta) \tan \theta/2; x + y \tan \theta/2 = a\theta + 2a \tan \theta/2$.

(ii) $y = (x - a\theta) \tan \theta/2$; $x + y \tan \theta/2 = a\theta + 2a \tan \theta/2$

(iii) $\frac{x}{a \cos \theta} + \frac{y}{b \sin \theta} = 1$, $yb \sin\theta - xa \cos\theta = b^2 \sin^4\theta - a^2\cos^4\theta$

4. (i) $\pi/4$, (ii) $\pi/2$, $\tan^{-1} 3/5$
(iii) $\tan \alpha = -4\sqrt{2/7}$ (iv) $\tan \alpha = -\frac{1}{2}$

5. $cy/\sqrt{(y^2 - c^2)}$; $(c/y)\sqrt{(y^2 - c^2)} = 9$, $aa_1x^2 + bb_1y^2 = 0$

10. $ye^{\frac{3\pi}{\pi}} = e(x+1)$; $y + xe^{\frac{2\pi}{\pi}} + e = 0$

22. $p^{m+n}m^mn^n = a^{m+n}(m+n)^{m+n} \cos^m\alpha \sin^n\alpha$

30. (i) $\sqrt{\frac{x+a}{x}}$ (iii) $(a/x)^{1/3}$ (iii) $\sqrt{(1 + 1/4x^2)}$ (iv) $y(1 + x^2)$

35. $\Delta = \frac{1}{2}a^{4/3}(xy)^{1/3}$

POLAR CO-ORDINATE X (B)

10. 8. Angle Between radius vector and Tangent.

Let P be any given point on the curve $r = f(\theta)$. Take any other Q on the curve very near to P.

Let the co-ordinates of P and Q be represented by (r, θ) and $(r + \Delta r, \theta + \Delta\theta)$ respectively.

Join PQ and produce. Then PQ is a secant of the curve $r = f(\theta)$ through P and Q. Draw PN perpendicular to OQ.

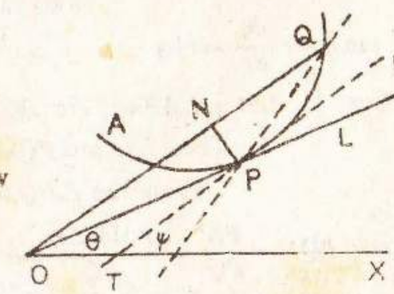


Fig No 13

If $Q \to P$, then $\angle OQP \to \angle OPT = \phi$, where ϕ is the angle between the radius vector OP and the tangent, PT at P.

We are to find ψ. We have,

$OP=r,\ OQ=r+\Delta r$

$\angle XOQ=\theta+\Delta\theta,\quad \angle XOP=\theta\ ;\ \angle POQ=\Delta\theta$

From $\triangle$OPN,

$PN=OP\sin\Delta\theta=r\sin\Delta\theta\ ;\quad ON=r\cos\Delta\theta$

Now from the $\triangle PQN$,

$$\tan PQN=\frac{PN}{QN}=\frac{PN}{OQ-ON}=\frac{r\sin\Delta\theta}{r+\Delta r-r\cos\Delta\theta}$$

$$=\frac{r\sin\Delta\theta}{\Delta r+(1-\cos\Delta\theta)r}=\frac{r\sin\Delta\theta}{\Delta r+2r\sin^2\frac{1}{2}\Delta\theta}$$

or, $$\tan PQN=\frac{r(\sin\Delta\theta/\Delta\theta)}{\frac{1}{2}r\Delta\theta\left(\sin\frac{\Delta\theta}{2}\Big/\frac{\Delta\theta}{2}\right)^2+\Delta r/\Delta\theta}$$

If $Q\to P$, then $\Delta\theta\to 0$ and $\angle PQN\to\angle OPT=\phi$

$$\tan\phi=\lim_{\Delta\theta\to 0}\left[\frac{r(\sin\Delta\theta/\Delta\theta)}{\frac{1}{2}r\Delta\theta(\sin\frac{1}{2}\Delta\theta/\frac{1}{2}\Delta\theta)^2+\Delta r/\Delta\theta}\right]$$

$$=\lim_{\Delta\theta\to 0}\frac{r}{\frac{1}{2}\Delta\theta\ 1+r/\Delta\theta}=\frac{r}{dr/d\theta}=r/r_1$$

$$\tan\phi=r\frac{d\theta}{dr}=r/r_1 \quad \dots\ \dots\ \dots \quad (31)$$

Cor. 1. Let arc $AP=s$, arc $AQ=s+\Delta s$.

Therefore arc $PQ=\Delta s$

From the $\triangle PQN$,

$$\sin PQN=\frac{PN}{PQ}=\frac{r\sin\Delta\theta}{PQ}=r\ \frac{\sin\Delta\theta}{\Delta\theta}\cdot\frac{\Delta\theta}{\Delta s}\cdot\frac{\Delta s}{PQ}$$

If $Q\to P$, then $\Delta\theta\to 0$, $\Delta s\to 0$, $\angle PQN\to\phi$ and

$$\lim_{Q\to P}\frac{\Delta s}{PQ}=\lim_{Q\to P}\frac{\text{arc } PQ}{\text{chord } PQ}=1$$

Thus $$\sin\phi=\lim_{\Delta\theta\to 0} r\left(\frac{\sin\Delta\theta}{\Delta\theta}\right)\lim_{\Delta s\to 0}\frac{\Delta\theta}{\Delta s}$$

$$\Rightarrow\ \sin\phi=r\frac{d\theta}{ds} \quad (32)$$

Cor. 2 Similarly from the $\triangle PQN$

$$\cos PQN=\frac{QN}{PQ}=\frac{OQ-ON}{PQ}$$

$$=\frac{r+\Delta r-r\cos\Delta\theta}{PQ}=\frac{\Delta r+r\,2\sin^2\frac{1}{2}\Delta\theta}{PQ}$$

$$=\left[\frac{\Delta r}{\Delta s}+2r.(\sin\tfrac{1}{2}\Delta\theta/\tfrac{1}{2}\Delta\theta)^2\frac{(\frac{1}{2}\Delta\theta)^2}{\Delta s}\right]\frac{\Delta s}{PQ}$$

In the limit, when $Q\to P$, then $\Delta s\to 0$, $\Delta\theta\to 0$ and so

$\cos\phi=[\,dr/ds+2r.1.0\,]=dr/ds$

$\cos\phi=(dr/ds)\quad \dots\quad \dots\quad \dots\quad (33)$

We may get the value of $\cos\phi$ with the help of $\sin\phi$ and $\cot\phi$

$$\cos\phi=\frac{\cos\phi}{\sin\phi}\sin\phi=\cot\phi.\sin\phi$$

$$=\frac{dr}{rd\theta}r\ \frac{d\theta}{ds}\ \text{[by (31) and (32)}$$

$$=\frac{dr}{ds}$$

Cor. 3. We have

$$\cos\phi=\frac{dr}{ds},\ \sin\phi=r\ \frac{d\theta}{ds}$$

$$\therefore\ \left(\frac{dr}{ds}\right)^2+\left(r\ \frac{d\theta}{ds}\right)^2=\cos^2\phi+\sin^2\phi$$

or, $$\left(\frac{dr}{ds}\right)^2+\left(r\frac{d\theta}{ds}\right)^2=1$$

or, $$(dr)^2+(rd\theta)^2=(ds)^2 \quad (34)$$

or, $(ds/d\theta)=\sqrt{\{r^2+(dr/d\theta)^2\}}$

provided s increases with θ.

Cor. 4. From the figure 13 we see that $\angle PTX=\psi$ is the angle made by the tangent PT with the positive direction of x axis. $\angle XOP=\theta$ is the angle by the radius vector OP with the (x-axis) initial line, $\angle TPO=\phi$ is the angle between radius vector and the tangent. We have from the figure, $\angle XTP=\angle TPO+\angle TOP$.

or, $\psi=\phi+\theta$ (35)

$\therefore$ $\tan\psi=\tan(\phi+\theta)$

or, $\tan\psi=\dfrac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$ (36)

Alternative Method

10. 8. (a) Angle between radius vector and Tangent.

Let $P(r,\theta)$ be any point on the curve $r=f(\theta)$. Another point $Q(r+\Delta r,\theta+\Delta\theta)$ is taken on the curve very near to P.

Let PT be tangent at P which meets the initial line at T. Join PQ. Produce OP to L.

Let $\angle LPQ=\alpha$ and $\angle LPM=\phi$, the angle between the radius vector and tangent at P.

If $Q\to P$, then $\angle LPQ\to\angle LPM$ i. e, $\alpha\to\phi$.

Therefore $\angle OPQ=\pi-\alpha$

and $\angle OQP=\angle LPQ-\angle POQ=\alpha-\Delta\theta$

Now by sin Rule, from $\triangle OPQ$

$$\frac{OQ}{OP}=\frac{\sin OPQ}{\sin OQP}=\frac{\sin(\pi-\alpha)}{\sin(\alpha-\Delta\theta)}$$

or, $\dfrac{r+\Delta r}{r}=\dfrac{\sin\alpha}{\sin(\alpha-\Delta\theta)}$ or, $\dfrac{\Delta r}{r}=\dfrac{\sin\alpha}{\sin(\alpha\Delta\theta)}-1$

or, $\dfrac{r\Delta}{r}=\dfrac{\sin\alpha-\sin(\alpha-\Delta\theta)}{\sin(\alpha-\Delta\theta)}=\dfrac{2\cos(\alpha+\frac{1}{2}\Delta\theta)\sin\frac{1}{2}\Delta\theta}{\sin(\alpha-\Delta\theta)}$

or, $\dfrac{1}{r}\dfrac{\Delta r}{\Delta\theta}=\dfrac{\cos(\alpha+\frac{1}{2}\Delta\theta)}{\sin(\alpha-\Delta\theta)}\cdot\dfrac{\sin\frac{1}{2}\Delta\theta}{\frac{1}{2}(\Delta\theta)}$

If $Q\to P$, then $\alpha\to\phi$, $\Delta\theta\to 0$, thus

$$\frac{1}{r}\frac{dr}{d\theta}=\frac{\cos\phi}{\sin\phi}\cdot 1=\cot\phi.$$

or. $\tan\phi=r\dfrac{d\theta}{dr}$

10. 9. Angle of intersection of two curves

Let the two polar curves whose equations are $r=f(\theta)$, $r=\phi(\theta)$ intersect at P. Let α be the angle of their intersection. Find ϕ_1 the angle between the radius vector and the tangent of the first curve $r=f(\theta)$. Similarly ϕ_2 for the 2nd curve $r=\psi(\theta)$.

Thus the angle of intersection of two curves is given by

$\alpha=\phi_1\sim\phi_2$ (37)

If $\tan\phi_1=n_1$ and $\tan\phi_2=n_2$ then

$$\tan\alpha=\tan(\phi_1\sim\phi_2)=\frac{\tan\phi_1\sim\tan\phi_2}{1+\tan\phi_1\tan\phi_2}$$

$\therefore$ $\tan\alpha=\dfrac{n_1\sim n_2}{1+n_1n_2}$ (38)

Cor. 1. If the two curves intersect at right angles, then $\alpha=\pi/2$.

$\therefore$ $\phi_1\sim\phi_2=\frac{1}{2}\pi$ (39)

or, $n_1 n_2=-1$ (40)

10.10. Polar subtangent and Polar subnormal.

Let $r=f(\theta)$ be any Polar curve. O is the Pole and OX the initial line. Draw the tangent PT at P and the normal PN at P. Thus the tangent PT and normal PN meet the straight line TON through the pole at right angles to the radius vector OP.

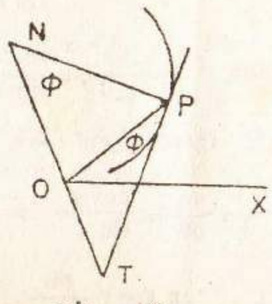


Fig. 15

Then

(i) **OT is called the polar subtangent and ON is called the polar subnormal.**

$\angle OPT=\phi$. the angle between the radius vector and the tangent at P, $OP=r$. From the $\triangle OPT$.

$$OT=OP\tan\phi=r\,\frac{r\,d\theta}{dr}\text{ [by (31)] or, } OT=r^2\frac{d\theta}{dr}$$

$\therefore$ **Polar subtangent** $=r^2\dfrac{d\theta}{dr}$ (41)

(ii) Again from the $\triangle OPN$, we have

$$ON=OP\cot\phi=r\,\frac{dr}{r\,d\theta}=\frac{dr}{d\theta}$$

$\therefore$ **Polar subnormal** $=\dfrac{dr}{d\theta}$ (42)

(iii) Length of the tangent.

$$PT^2=OT^2+OP^2=\left(r^2\frac{d\theta}{dr}\right)^2+r^2$$

or, $PT^2=(r^4/r_1^2+r^2)=\dfrac{r^2}{r_1^2}(r^2+r_1^2)$

$\therefore$ $PT=\dfrac{r}{r_1}\sqrt{(r^2+r_1^2)}$ (43)

(iv) Length of the normal

$$PN^2=ON^2+OP^2=\left(\frac{dr}{d\theta}\right)^2+r^2=r_1^2+r^2$$

$\therefore$ $PN=\sqrt{(r_1^2+r^2)}$ (44)

Examples

Ex. 1. Express ϕ in terms of θ, form the curve $r^2=a^2\cos 2\theta$

Take logarithm of both sides of

$r^2=a^2\cos 2\theta$

Then, $\log r^2=\log(a^2\cos 2\theta)=\log a^2+\log\cos 2\theta$

or, $2\log r=2\log a+\log\cos 2\theta$

$\therefore$ $\dfrac{2}{r}\dfrac{dr}{d\theta}=-\dfrac{2\sin 2\theta}{\cos 2\theta}=-2\tan 2\theta$

or, $\cot\phi=-\tan 2\theta=\cot(\pi)/2+2\theta)$. $\left[\because r\dfrac{d\theta}{dr}=\tan\phi\right]$

$\therefore$ $\phi=\pi/2+2\theta$

Ex. 2. Show that the curves $r^2\sin 2\theta=a^2$ and $r^2\cos 2\theta=b^2$ intersect orthogonally.

Consider $r^2\sin 2\theta=a^2$

Taking logarithm of both sides,

$2\log r+\log\sin 2\theta=2\log a$

$\therefore$ $\dfrac{2}{r}\dfrac{dr}{d\theta}+\dfrac{2\cos 2\theta}{\sin 2\theta}=0$

or. $\cot\phi=-\cot 2\theta=\cot(\pi-2\theta)$

$\therefore$ $\phi=\pi-2\theta$ (1)

Similarly for the 2nd curve, $r^2\cos 2\theta=b^2$

we have $\dfrac{2}{r}\dfrac{dr}{d\theta}-\dfrac{2\sin 2\theta}{\cos 2\theta}=0$. Differentiate w. r. to θ

or, $\cot \phi_1 = \tan 2\theta$. $\therefore$ $\phi_1 = \frac{1}{2}\pi - 2\theta$.

Hence the angle of intersection of the curves is

$\angle = \phi - \phi_1 = \pi - 2\theta - \frac{1}{2}\pi + 2\theta = \frac{1}{2}\pi$.

Thus the curves intersect orthogonally (at right angles).

Ex. 3. Prove that locus of the extremity of the polar subtangent for the curve.

$$\frac{1}{r} + f(\theta) = 0 \text{ is } \frac{1}{r} = f\left(\frac{\pi}{2} + \theta\right)$$

Let the entrimity of the polar subtangent be (r_1, θ_1). then $r_1 = OT$ = polar subtangent (see fig. 15)

$$= r^2 \frac{d\theta}{dr} \qquad (1)$$

From the figure

$$-\theta_1 = \angle XOT = \angle POT - \angle POX = \tfrac{1}{2}\pi - \theta$$

or, $\theta = \frac{1}{2}\pi + \theta_1$ (2)

From the given equation $\frac{1}{r} = -f(\theta)$

$$\therefore \quad -\frac{1}{r^2} \cdot \frac{dr}{d\theta} = -f'(\theta) \quad \text{or,} \quad r^2 \frac{d\theta}{dr} = \frac{1}{f'(\theta)}$$

or, $r_1 = \dfrac{1}{f'(\frac{1}{2}\pi + \theta_1)}$ [by (1) and (2)]

or, $\dfrac{1}{r_1} = f'(\frac{1}{2}\pi + \theta_1)$

Drop the suffixes from r and θ getting equation is

$$\frac{1}{r} = f'(\tfrac{1}{2}\pi + \theta)$$

which is the reauired locus.

Ex. 4. Show that the tangents to the cardioide $r = a(1 + \cos\theta)$ at the points whose vectorial angles are $\pi/3$ and $2\pi/3$ are respectively parallel and perpendicular to the initial line.

Let the angle between the radius vector and the tangent at (r, θ) be ϕ, then

$$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{(dr/d\theta)}$$

$$= a(1 + \cos\theta) \times \frac{1}{-a\sin\theta}$$

$$= -\cot\tfrac{1}{2}\theta = \tan(\tfrac{1}{2}\pi + \tfrac{1}{2}\theta)$$

$\therefore$ $\phi = \frac{1}{2}\pi + \frac{1}{2}\theta$.

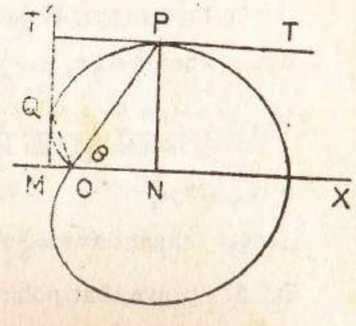


Fig. 16

If the tangent QT is parallel to the initial line OX then

$\phi = \pi - \theta \Rightarrow \frac{1}{2}\pi + \frac{1}{2}\theta = \pi - \theta$

or, $3\theta/2 = \frac{1}{2}\pi$ or, $\theta = \frac{1}{3}\pi$

Hence the tangent is parallel to the initial line at $\theta = \pi/3$

Let the tangent be perp. to the initial line at θ, when

$\phi = \angle OQT' = \angle QMO + \angle QOM = \frac{1}{2}\pi + \pi - \theta = 3\pi/2 - \theta$

$\Rightarrow \frac{1}{2}\pi + \frac{1}{2}\theta = 3\pi/2 - \theta$ or, $3\theta/2 = \pi$ or, $\theta = \frac{2}{3}\pi$

Hence the tangent at $\theta = \frac{2}{3}\pi$ is perp to the initial line

Alternative method

Let ψ be the angle made by the tangent with the x-axis (here initial line). Then $\psi = \theta + \phi$; $r = a(1 + \cos\theta)$.

$$\therefore \quad \frac{1}{r}\frac{dr}{d\theta} = \frac{-a\sin\theta}{a(1+\cos\theta)} = -\tan\tfrac{1}{2}\theta$$

or, $\cot\phi = \cot(\frac{1}{2}\pi + \frac{1}{2}\theta)$ $\therefore$ $\phi = \frac{1}{2}\pi + \frac{1}{2}\theta$

Now $\tan\psi = \tan(\theta + \phi) = \dfrac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$

At $\theta=\frac{1}{3}\pi$. then $\phi=\frac{1}{2}\pi+\frac{1}{2}.\frac{1}{3}\pi=\frac{2}{3}\pi$

$$\tan\psi=\frac{\tan\frac{1}{3}\pi+\tan\frac{2}{3}\pi}{1-\tan\frac{1}{3}\pi\tan\frac{2}{3}\pi}=\frac{\sqrt{3}-\sqrt{3}}{1+3}=0 \quad \therefore \quad \psi=0$$

Hence the tangent is parallel to the initial line at $\theta=\pi/3$.

Again when $\theta=\frac{2}{3}\pi$, $\phi=\frac{1}{2}\pi+\frac{1}{2}\ \frac{2}{3}\pi=5\pi/6$

$$\tan\psi=\frac{\tan\frac{2}{3}\pi+\tan 5\pi/6}{3-\tan\frac{2}{3}\pi\tan 5\pi/6}=\frac{-\sqrt{3}-1/\sqrt{3}}{1-\sqrt{3}(1/\sqrt{3})}=\infty$$

$\therefore\ \psi=90^\circ$,

Hence tangent at $\theta=\frac{2}{3}\pi$ is perp. to the initial line.

Ex. 5. prove that polar subtangent and the polar subnormal of the equiangular spiral $r=e^{a\theta}$ varies as the radius vector r.

Here $r=e^{a\theta}$ $\quad\therefore\quad \frac{dr}{d\theta}=ae^{a\theta}=ar$

Polar subtangent $r^2\frac{d\theta}{dr}=r^2\frac{1}{ar}=\frac{r}{a}$

$\therefore$ Polar subtangent varies as r as a is constant.

Polar subnormal$=\frac{dr}{d\theta}=ar$

$\therefore$ Polar subnormal varies as r as a is constant.

Exercise X (B)

1. Express ϕ in terms of θ for the following curves

(i) $r=a\theta$ (ii) $2a=r(1-\cos\theta)$ (iii) $r=a(1-\cos\theta)$

(iv) $r^2=\sin 2\theta$ (v) $r=ae^{\theta\cot\alpha}$ (vi) $r^n=a^n\cos n\theta$ C. U. 1983

2. Find the angle of intersection of the following curves

(i) $r=a(1+\cos\theta)$, $r=a(1-\cos\theta)$ (ii) $r=2\cos\theta$, $r\cos\theta=-\frac{1}{2}$

(iii) $r=4(1+\sin\theta)$, $r=3(1-\sin\theta)$ (iv) $r=1+\sin\theta$, $r=1-\sin\theta$

(v) $r^n=a^n\cos n\theta$, $r^n=a^n\sin n\theta$

3. Show that the following curves intersect orthogonally (at right angles).

(i) $r=a\sec^2\theta/2$, $r=b\operatorname{cosec}^2\theta/2$

(ii) $r^m=a^m\cos m\theta$, $r^m=b^m\sin m\theta$

(iii) $r=a(1+\cos\theta)$, $r=b(1-\cos\theta)$

(iv) $r^2=a^2\cos 2\theta$; $r^2=a^2\sin 2\theta$.

4. Show that polar subtangent is of constant length for the curve. $r\theta=a$

5. Show that in the equiangular spiral $r=ae^{\theta\cot a}$ the tangent is inclined at a constant angle to the radius vector.

6. Show that for the curve $r=a\theta$ the polar subnormal is constant and for the curve $r\theta=a$ the polar subtangent is constant a being a constant.

7. Prove that locus of the extremity of the polar subnormal of the curve $r=f(\theta)$ is $r=f(\theta-\frac{1}{2}\pi)$

8. Show that the curves

$$r^n=a^n \sec(n\theta+\alpha) \text{ and } r^n=b^n \sec(n\theta+\beta)$$

intersect at an angle $(\alpha-\beta)$

9. Show that the locus of the extremity of the polar subnormal of the equiangular spiral $r=ae^{m\theta}$ is another equiangular spiral.

10. Show that Polar subtangent for the curve,

$r=a\ (1+\cos\theta)$ is $-\dfrac{2a\cos^3\theta/2}{\sin\theta/2}$

11. Prove that

$$\frac{ds}{d\theta}=a(\sec n\theta)^{(n-1)/n}$$

for the curve $r^n=a^n\cos n\theta$

12. For the curve $r^n=a^n\cos n\theta$, prove that

$$\alpha^{2n}\cdot\frac{d^2r}{ds^2}+nr^{2n-1}=0$$

13. Show that tangent drawn at the extremities of any chord of the cardioide $r=a(1+\cos\theta)$ which passes through the pole are perpendicular to each other.

14. Show that in the curve $r^5=a(1-\cos\theta)$ the radius vector of the pt. whose vectorial angle is $2\cos^{-1}\frac{5}{\sqrt{26}}$ makes an angle of 45° with the tangent at the point. [D. U. 1954]

15. Find the polar subtangents, polar subnormals, length of the tangent, length of normal of the following curves.

(i) $r=-6\sin\theta$ at $\pi/3$

(ii) $r=2\sec\theta$ at $\pi/4$

(iii) $r=5+2\sin\theta$ at $\pi/6$

(iv) $r^2=4\cos\theta$ at $\pi/6$

(v) $r=\dfrac{4}{1+\sin\theta}$ at $\pi/4$.

16. Show that the locus in polar co-ordinates of the intersection of two perpendicular tangents to the curve

$x=a\cos^3\phi,\ y=a\sin^3\phi$ is

$r^2=\frac{1}{2}a^2\cos^2 2\theta$.

প্রশ্নমালা X (B)

1. নিম্নলিখিত বক্ররেখাগুলির জন্য ϕ কে θ-এর মাধ্যমে প্রকাশ কর।

(i) $r=a\theta$ (ii) $2a=r(1-\cos\theta)$ (iii) $r=a(1-\cos\theta)$

(iv) $r^2=\sin 2\theta$ (v) $r=ae^{\theta\cot x}$ (vi) $r^n=a^n\cos n\theta$

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2. নিম্নলিখিত বক্ররেখাগুলির ছেদক কোণ নির্ণয় কর।

(i) $r=a(1+\cos\theta)$, $r=a(1-\cos\theta)$ (ii) $r=2\cos\theta$, $r\cos\theta=-\frac{1}{2}$. (iii) $r=4(1+\sin\theta)$, $r=3(1-\sin\theta)$

(iv) $r=1+\sin\theta$, $r=1-\sin\theta$

(v) $r^n=a^n\cos n\theta$, $r^n=a^n\sin n\theta$

3. দেখাও যে নিম্নলিখিত বক্ররেখাগুলি পরস্পরকে লম্বভাবে ছেদ করে।

(i) $r=a\sec^2\theta/2,\quad r=b\operatorname{cosec}^2\theta/2$

(ii) $r^m=c^m\cos m\theta,\quad r^m=b^m\sin m\theta$

(iii) $r=a(1+\cos\theta),\quad r=b(1-\cos\theta)$

(iv) $r^2=a^2\cos 2\theta\ ;\quad r^2=a^2\sin 2\theta$.

4. দেখাও যে বক্ররেখার $r\theta=a$ এরজন্য পোলার উপস্পর্শকের দৈর্ঘ্য ধ্রুব থাকে।

5. দেখাও যে সমান-কৌণিক কুণ্ডলী $r=ae^{\theta\cot\alpha}$ (eqiangular spiral) এর যে কোন বিন্দুতে স্পর্শক ঘূর্ণায়মান রেখার সহিত সর্বদা সমান কোণ উৎপন্ন করে।

6. যদি 'a' একটি ধ্রুব সংখ্যা হয় তবে দেখাও যে $r=a\theta$ বক্ররেখার পোলার উপলম্ব এবং $r\theta=a$ রেখায় পোলার উপস্পর্শকের দৈর্ঘ্য ধ্রুব থাকে।

7. প্রমাণ কর যে $r=f(\theta)$ বক্ররেখার যে কোন বিন্দুতে পোলার উপলম্বের প্রান্তবিন্দুর সঞ্চার পথ $r=f'(\theta+\pi/2)$.

8. দেখাও যে $r^n=a^n\sec(n\theta+\alpha)$ এবং $r^n=b^n\sec(n\theta+\beta)$ বক্ররেখাদ্বয়ের মধ্যে কোন হইল $(\alpha-\beta)$.

9. দেখাও যে সমানকৌণিক কুণ্ডলী $r=ae^{m\theta}$ এর যে কোন বিন্দুতে পোলার উপলম্বের প্রান্তবিন্দুর সঞ্চার পথ অপর একটি সমান-কৌণিক কুণ্ডলী (Equiangular spiral).

10. দেখাও যে $r=a(1+\cos\theta)$ বক্ররেখার পোলার উপস্পর্শক হইবে

$$-\frac{2a\cos^3(\theta/2)}{\sin(\theta/2)}$$

11. প্রমাণ কর যে $r^n=a^n\cos n\theta$ এর জন্য

$$\frac{ds}{d\theta}=a(\sec n\theta)^{\frac{n-1}{n}}$$

12. বক্ররেখা $r^n=a^n\cos n\theta$, এর জন্য প্রমাণ কর যে

$$a^{2n}.\frac{d^2r}{ds^2}+nr^{2n-1}=0$$

13. দেখাও যে মেরুবিন্দু দিয়া অতিক্রান্ত কার্ডিওইড $r=a(1+\cos\theta)$ এর কোন জ্যা-এর প্রান্তবিন্দুদ্বয়ে অংকিত স্পর্শক পরস্পর লম্ব হইবে।

14. দেখাও যে বক্ররেখা $r^2=a(1-\cos\theta)$ এর যে বিন্দুর ভেক্টর কোন $\cos^{-1}\frac{5}{\sqrt{26}}$, সে বিন্দুতে ঘূর্ণায়মান রেখা এবং ঐ বিন্দুতে স্পর্শক-এর মধ্যে কোণ হইবে 45°. D. U. 1954

15. নিম্নলিখিত বক্ররেখাগুলির পোলার উপস্পর্শক, পোলার উপলম্ব, স্পর্শকের দৈর্ঘ্য, অভিলম্বের দৈর্ঘ্য নির্ণয় কর।

(i) $r=-6\sin\theta$ এর $\pi/3$ বিন্দুতে।

(ii) $r=2\sec\theta$ এর $\pi/4$ বিন্দুতে।

(iii) $r=5+2\sin\theta$ এর $\pi/6$ বিন্দুতে।

(iv) $r^2=4\cos\theta$ এর $\pi/6$ বিন্দুতে।

(v) $r=\dfrac{4}{1+\sin\theta}$ এর $\pi/4$ বিন্দুতে।

16. দেখাও যে বক্ররেখা $x=a\cos^3\phi,\ y=a\sin^3\phi$ এর উপর অংকিত দুইটি স্পর্শক পরস্পরকে যদি লম্বভাবে ছেদ করে তবে ছেদবিন্দুর পোলার স্থানাংকে সঞ্চারপথ হইবে $r^2=\frac{1}{2}a^2\cos^2 2\theta$.

ANSWERS

1. (i) $\tan\phi=\theta$, (ii) $-\theta/2,\ \pi-\theta/2$, (iii) $\theta/2$, (iv) 2θ, (v) α.

2. (i) $\pi/2$ (ii) 2θ (iii) $\pi/2$ (iv) $\pi/2$ (v) $\pi/2$.

15. (i) $-9, -3, \sqrt{108}, 6$, (ii) $2\sqrt{2}, 2\sqrt{2}, 4, 4$.

(iii) $12\sqrt{3},\ \sqrt{3},\ 6\sqrt{13},\ \sqrt{39}$

(iv) $\sqrt[4]{3},\ \sqrt[4]{3},\ 1/\sqrt{(2/\sqrt[4]{3})}\ 26\sqrt{3},\ \left(\frac{13}{\sqrt{3}}\right)^{1/2}$

(v) $-4\sqrt{2},\quad \dfrac{-4\sqrt{2}}{3+2\sqrt{2}}$, 6.15, 3 2 প্রায়।

PEDAL EQUATIONS X (C)

10. 11. Pedal equation :—The relation between p and r for a given curve is called the pedal equation of the curve, p is the length of the perpendicular from the pole or origin to the tangent and r is the distance of any point on the curve from the pole. It is denoted by $f(p, r)=0$.

10. 12. The Pedal equation of a curve given by r=f(θ).

Let O be the pole and OA the initial line. Let PT be the tangent at P(r, θ) to the curve $r=f(\theta)$. The radius vector OP=r.

Draw OT perpendicular to the tangent PT

Let OT=p, ∠OPT=ϕ

From the right angled △OPT,

OT=OP sin OPT=r sinϕ

or, $P=r \sin\phi$...... (45)

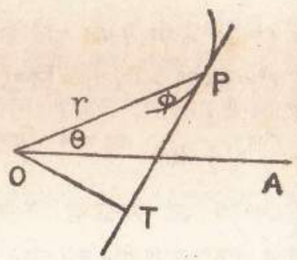


Fig. 17

$$\text{or, } \frac{1}{p}=\frac{1}{r}\operatorname{cosec}\phi \text{ or, } \frac{1}{p^2}=\frac{1}{r^2}\operatorname{cosec}^2\phi=\frac{1}{r^2}(1+\cot^2\phi)$$

$$=\frac{1}{r^2}\left\{1+\left(\frac{dr}{rd\theta}\right)^2\right\}\left[\because \tan\phi=\frac{rd\theta}{dr}\right]$$

$$\text{or, } \frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \qquad (46)$$

If we put $u=\frac{1}{r}$, then $\frac{du}{d\theta}=-\frac{1}{r^2}\frac{dr}{d\theta}$

and hence

$$\frac{1}{p^2}=u^2+\left(\frac{du}{d\theta}\right)^2 \qquad (47)$$

By eliminating θ from (46) or (47) with the help of $r=f(\theta)$ the required pedal equation is obtained.

10. 13. To determine the pedal equation of a curve whose cartesian equation is given by f(x, y)=0.

The equation of the tangent at (x, y) on the curve $f(x, y)=0$ is

$$Y-y=\frac{dy}{dx}(X-x), \text{ or, } Y-y-(X-x)\frac{dy}{dx}=0$$

The perpendicular distance from the origin (0, 0) to the tangent is

$$p=\frac{-y+x(dy/dx)}{\sqrt{\{1+(dy/dx)^2\}}}=\frac{xy_1-y}{\sqrt{(1+y_1^2)}}$$

Also $r^2=x^2+y^2$ ... ... (ii)

and $f(x, y)=0$ ... ... (iii)

Now eliminating x, y from (i), (ii) and (iii) we get a relation between p and r. This relation is the required pedal equation. See example 3,

Pedal Curves

10. 14. First Positive Pedal. If a perpendicular is drawn from a fixed point on a movable tangent to a given, the locus of the foot of the perpendicular is called the First Positive pedal of the orginal curve $w. r.$ to the given point.

The pedal of the first positive pedal is called the 2nd positive pedal and so on.

P is a point on the curve APQ. A tangent PT is drawn at P on the curve APQ and OT is perpendicular from O to the tangent PT.

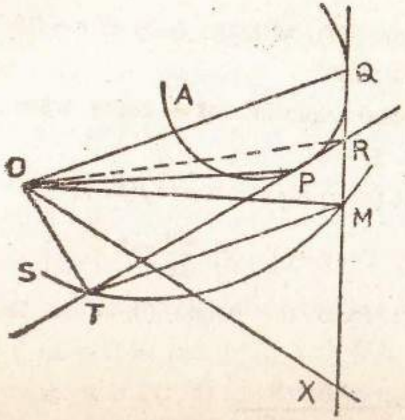


Fig. 18

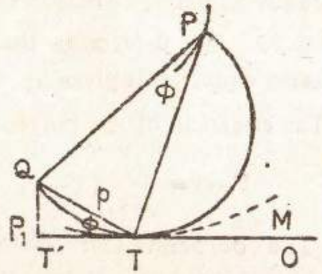


Fig. 19

T is the foot of the perpendicular from the origin to tangent at P of curve APQ. Similarly we get feet of the perpendiculars from the origin to the tangents drawn at the points on the curves APQ. All feet of the perpendiculars will lie on a curve STM

The curve STM is called the pedal curve of the curve APQ.

Consider two adjacent points P and Q on the curve APQ. Draw tangents PT and QM at P and Q respectively of the curve. Draw OT and OM perpendiculars from O to the tangents PT and QM respectively. The two tangents PT and QM intersect at R. Join OR and TM is a chord of the circle described on OR as diameter. P, Q, R coincide, when the chord TM becomes a tangent to the circle described on OP as diameter at a point where the circle cuts the tangent PT, i e, TM is tangent to the circle as well as the pedal curve at T.

Let OT_1 be perpendicular drawn from O the tangent T_1TM i, e, on first Pedal.

$\angle OPT=\phi$, then $\angle OTT_1=\phi$

Let $OP=r$, $OT_1=p_1$, $OT=p$

From $\triangle OTT_1$, $OT_1=OT\sin\phi$ or, $p_1=p\sin\phi$

From $\triangle OTP$, $OT=OP\sin\phi$ or $p=r\sin\phi$

Therefore, from the above two equations. we get a relation between p and p_1.

$\therefore\ p^2=p_1r$ or, $r=p^2/p_1$

(If $f(p,r)=0$ be the equation of the original curve, we will get a relation between p and p_1 by putting the value fo r in $f(p,r)=0$ i, e, $f(p, p^2/p_1)=0$.

$f(p,p^2/p_1)=0$ is the pedal equation of the pedal curve

put $p=r$, $p_1=r^2p$ in the pedal equation,

then $f(r,r^2/p)=0$

is the pedal curve of the pedal equation $f(p,p^2/p_1)=0$

Working Rule for determining pedal curve.

Find the pedal equation of the given curve say $f(p,r)=0$ Now put $p=r$, and $r=r^2/p$ in the pedal equation $f(p,r)=0$, to get $f(r,r^2/p)=0$ which is the pedal curve of the original pedal equation.

$f(r,r^2/p)=0$ is called positive pedal of the original equation. Similarly by applying the above rule we may get 1st positive pedal, 2nd positive pedal, 3rd positive pedal and so on,

Ex. Given the pedal equation of the circle $ap=r^2$. Determine the 4th positive pedal.

The pedal equation of the circle is

$$f(p, r)=ap-r^2=0 \quad \cdots \quad \cdots \quad \cdots \quad (1)$$

For the 1st positive pedal

put $p=r$, and $r=r^2/p$ in (1)

Then $f(r, r^2/p)=ar-(r^2/p)^2=0$ or, $p^2a=r^3$ ... (2)

The equation of the Ist positive pedal of (1)

$r^2a=(r^2/p)^3$ or, $p^3a=r^4$ ... ... (3)

Similarly, 3rd positive pedal is $p^4a=r^5$

Therefore the 4th positive pedal is $p^5a=r^6$

10. 15. To determine the positive pedal w, r, to the origin of any curve where cartesian equation is given.

Let the equation of the curve be $f(x, y)=0$ ... ... (1)

Let $P=X\cos a+Y\sin a$ (2)

be only straight line which touches the curve.

But the equation of the tangent to the curve $f(x, y)=0$ at (X, Y)

is $\frac{\delta f}{\delta x}+Y\frac{\delta f}{\delta y}+Z\frac{\delta f}{\delta z}=0$ (3) sec Art. 10.2 (iii)

If (2) and (3) are identical then their co-efficient will be proportional,

Hence $\frac{\delta f}{\delta x}\Big|\cos\alpha=\frac{\delta f}{\delta y}\Big|\sin\alpha=\frac{\delta f}{\delta z}\Big|(-p)=\lambda$ (say) ... (4)

∴ From the above four equation

we can eliminate x, y and λ get the result in terms of p and α.

p and α are the polar co-ordinates of the foot of the perpendicular.

If we replace p by r and α by θ then we get the polar equation

which is the required locus of the foot of the perpendicular. The locus is called the first positive pedal of the given curve w, r. to the origin.

Ex. Show that the first positive pedal of the parabola $y^2=4ax$ w. r to the vertex is $x(x^2+y^2)+ay^2=0$

$$y^2=4ax \quad \cdots \quad \cdots \quad \cdots \quad (1)$$

The equation to the tangent at (X, Y) of the curve (1) is

$Yy=2a(X+x)$ or, $2aX-Yy+2ax=0$ ... ... (2)

Let $p=X\cos\alpha+Y\sin\alpha$ ... ... (3)

be any straight line which touches the curve (1)

Thus two straight lines are identical if their co-efficients are proportional. Compare the co-efficients of these equations. Then

$\frac{\cos\alpha}{2a}=\frac{\sin\alpha}{-y}=\frac{-p}{2ax}$ or, $y=-2a\tan a$, $x=-p\sec\alpha$

and $4a^2\tan\alpha=-4ap\sec\alpha$

Now from (1) we have

$4a^2\tan^2\alpha=-4ap\sec\alpha$ or, $a\sin^2\alpha+p\cos\alpha=0$ ... ... (4)

Now replace α by θ and p by r then the locus is

$a\sin^2\theta+r\cos\theta=0$ or, $a.\ y^2/r^2+x=0$ ∴ $r^2=x^2+y^2$

or, $ay^2+xr^2=0$ or, $ay^2+x(x^2+y^2)=0$ ∴ $x=r\cos\theta$, $y=r\sin\theta$

Thus the first positive pedal w. r. to the vertex is

$$x(x^2+y^2)+ay^2=0.$$

***10. 16. Determine the first positive pedal w, r, to the pole of any curve whose polar equation is given.**

Let the equation to the curve be $f(r, \theta)=0$ ... ... (1)

Let PT be the tangent to the curve at $P(r, \theta)$

OQ is drawn perpendicular to the tangent. Let $Q(r_1, \theta_1)$ be the foot of the perpendicular. Let OX be the initial line.

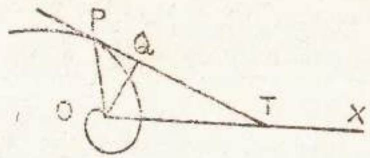


Fig. 20

$\triangle XOP=\theta$. $\triangle OPT=\phi$, $PTX=\psi$

$\therefore$ $\theta=\angle XOP=\angle XOQ+\angle POQ$

$\theta=\theta_1+\frac{1}{2}\pi-\angle OPQ=\theta_1+\frac{1}{2}\pi-\phi$ or, $\theta=\theta_1+\frac{1}{2}\pi-\phi$ ... ... (2)

আবার আমরা জানি, $\tan\phi=r\dfrac{d\theta}{dr}$ ... ... (3)

$\triangle OPQ$ হইতে $OQ=OP\sin\angle OPQ=r\sin\phi$

or, $r_1=r\sin\phi$ ... ... (4)

or, $\dfrac{1}{r_1^2}=\dfrac{1}{r^2}+\dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$ ... ... (5)

From the equations (1), (2), (3), (4) or, (5) we can eliminate r, θ, ϕ and the result will be in terms of r_1 and θ_1.

The dashes may be dropped and the required locus will be obtained in r and θ only.

The required locus is called the first positive pedal (pedal curve) w. r. to the pole.

Ex. Show that the equation of first positive pedal of

$r^n=a^n\cos n\theta$ w. r. to the pole is

$$r^{\frac{n}{n+1}}=a^{\frac{n}{n+1}}\cos\frac{n}{n+1}\theta.$$

$r^n=a^n\cos n\theta$ ... ... (1)

or, $n\log r=n\log a+\log\cos n\theta$

$n.\ \dfrac{1}{r}\dfrac{dr}{d\theta}=0+\dfrac{1}{\cos n\theta}.(-\sin n\theta).\ n=-n\tan n\theta$

or, $\cot\phi=-\tan n\theta=\cot(\pi/2+n\theta)$

$\therefore$ $\phi=\pi/2+n\theta$ ... ... (2)

But $\theta=\theta_1+\pi/2-\phi=\theta_1+\pi/2-\pi/2-n\theta=\theta_1-n\theta$

$\therefore$ $(n+1)\theta=\theta_1$ or, $\theta=\dfrac{\theta_1}{n+1}$ ... ... (3)

we know $r_1=r\sin\phi=\sin(\pi/2+n\theta)$

$=r\cos n\theta=a(\cos n\theta)^{1/n}.\cos n\theta$ by (1)

$\therefore$ $r_1=a(\cos n\theta)^{\frac{n+1}{n}}$

But $(r_1)^{\frac{n}{n+1}}=a^{\frac{n}{n+1}}\cos\dfrac{n\theta_1}{n+1}$

or, $(r_1)^{\frac{n}{n+1}}=a^{\frac{n}{n+1}}\cos\left(\dfrac{n\theta_1}{n+1}\right)$ by (3)

Now make r_1 and θ_1 as current, then the required first positive pedal is

$$r^{\frac{n}{n+1}}=a^{\frac{n}{n+1}}\cos\frac{n\theta}{n+1}\ \text{Proved}$$

*10.17. Negative pedals

Let C, C_1, C_2, C_3, C_4, C_5 ... ... C_n be a series of curves. C_1 is called the first positive pedal of C, C_2 called the 2nd positive pedal of C. Similarly C_5 is called the 5th positive pedal of C.

Now if we consider C_4 as the original curve, then C_5 is called the first positive pedal, C_6 the 2nd positive pedal of C_4.

The curve C_3 is called the first negative pedal of C_4, C_2 the 2nd negative pedal of C_4, C_1, the 3rd negative pedal of C_4, C the 4th negative pedal of C_4. In this way we get negative pedal from a series of curves. see Ex. 4 and Ex. 5.

Working Rule for determining negative pedals

Put $r=p$ and $p=p^2/r$ in the given pedal equation $f(p, r)=0$ i. e. $f(r, p^2/r)=0$ is the first negative pedal. Repeat this process for the 2nd negative pedal, 3rd negative pedal and so on.

*Ex. Determine the pth negative pedal of the curve $p^4a=r^5$

Put $r=p$ and $p=p^2/r$ in $p^4a=r^5$

The first negative pedal is $(p^2/r)^4\, a=p^5$ or, $p^3a=r^4$

2nd negative pedal is $(p^2/r)^3 a=p^4$ or, $p^2a=r^3$

3rd negative pedal is $(p^2/r)^2 a=p^3$ or, $pa=r^2$

The nth negative pedal is $p^{n-4}=ar^{n-5}$

***10. 18. Inverse curve**

Let P be a point on a curve and O be the origin. Another point Q is taken on OP such that OP. OQ$=a$ constant, say k^2.

The locus of Q is called the inverse of the curve along which P moves, with respect to a circle of radius k and centre O.

***(a) To find the inverse of a given curve whose cartesian equation is given.**

Let P (x, y) be the co-ordinates of any point on the curve APC $f(x\ y)=0$. Let Q (x', y') be another point on OP such that OP. OQ$=k^2$ ... ... (1)

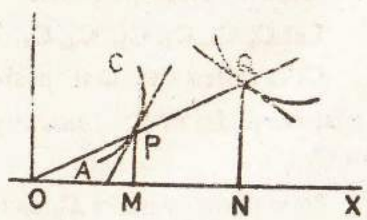


Fig. 21

PM and QN perpendicular are drawn from P and Q on OX respectively.

Ex. 1. Find the polar reciprocal of the hyperbola

Now from $\triangle$OPM and $\triangle$OQN

$$\frac{x}{x'}=\frac{OM}{ON}=\frac{OP}{OQ}=\frac{OP.OQ}{OQ^2}=\frac{k^2}{OQ^2} \text{ [by (1)]}$$

$$\text{or, } x^2=\frac{k^2x'}{OQ^2}=\frac{k^2x'}{x'^2+y'^2} \text{ as [} \therefore \ OQ^2=ON^2+NQ^2 \text{]}$$

$$\text{Similarly } y=\frac{k^2y'}{x'^2+y'^2}$$

The equation of the given curve is $f(x. y)=0$.

Replace x and y by their new values.

$$\text{or, } f\left(\frac{k^2x'}{x'^2+y'^2}, \frac{k^2y'}{x'^2+y'^2}\right)=0 \quad \dots \quad \dots \quad \dots \quad (2)$$

The locus of Q is another curve which is obtained by removing dashes from eq. (2)

$$i.\ e.\ f\left(\frac{k^2x}{x^2+y^2}, \frac{k^2y}{x^2+y^2}\right)=0.$$

Working Rule : The inverse of a curve is obtained by replacing x by $\frac{k^2x}{x^2+y^2}$ and y by $\frac{k^2y}{x^2+y^2}$ in the cartesian equation of the curve.

(b) Polar equation :—The inverse of a curve is obtained by replacing r by k^2/r in the polar equation $f(r, \theta)=0$ of the curve i. e.

$$f\left(\frac{k^2}{r}, \theta\right)=0$$

(c) Inverse of a curve when its **pedal equation is given**

Let $p=f(r)$ be the pedal equation of a curve

The pedal equation of the inverse of a curve whose equation is $y=f(r)$ is given by

$$p=f\left(\frac{k^2}{r}\right)$$

(d) The polar reciprocal of a curve is the inverse of its pedal.

*** 10. 19. Polar Reciprocal**

(A) Polar reciprocal of a curve w. r. to a given circle.

Let OP be the perpendicular from the pole, O to a tangent to the curve, a point Q is taken on OP or OP produced such that

$OP.OQ$=constant$=k^2$ (say)

The locus of Q is called the polar reciprocal of the given curve *w. r.* to a circle of radius k and centre at O.

The polar reciprocal of a curve is the inverse of its First pedal. The equation of the polar reciprocal of a given curve is obtained by finding of its pedals.

Working Rule :—Find the condition that $p=x\cos\alpha+y\sin\alpha$ touches the given curve. Then replace p by k^2/r and α by θ from the condition.

The result will be in terms of r and θ is the required polar reciprocal *w. r.* to the circle of radius k and centre at the origin.

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with regard to circle of radius k and the centre the origin of the curve.

Let $p=x\cos\alpha+y\sin\alpha$ ... ... (1)

touch the conic $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ... ... (2)

The equation of tangent at (x_1, y_1) to (2) is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1 \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

If (1) and (3) are identical, then their co-efficients will be proportional. Then compare their co-efficients. Thus

$$\frac{x_1}{a^2\cos\alpha}=-\frac{y_1}{b^2\sin\alpha}=\frac{1}{p}$$

$$\therefore \quad x_1=\frac{a^2}{p}\cos\alpha,\; y_1=-\frac{b^2\sin\alpha}{p}$$

The required condintion of tangency of line (1) is

$$p=\frac{a^2\cos^2\alpha}{p}-\frac{b^2\sin^2\alpha}{p} \text{ or, } p^2=a^2\cos^2\alpha-b^2\sin^2\alpha$$

Now remove p by $\frac{k^2}{r}$ and α by θ

$$\therefore \quad \frac{k^4}{r^2}=a^2\cos^2\theta-b^2\sin^2\theta$$

or, $k^4=a^2r^2\cos^2\theta-b^2r^2\sin^2\theta$ ... ... (4)

which is the required polar reciprocal of the curve, (2)

If $x=r\cos\theta$, $y=r\sin\theta$, then (4)

becomes $k^4=a^4x^2-b^4y^2$ or, $a^2x^2-b^2y^2=k^4$.

which is also the polar reciprocal of the hyperbola *w. r.* to the circle of radius k and centre at the origin.

***10.20. (B) Polar Reciprocal with respect to a given conic.**

Let $S=0$ be any curve and F=0 be the given conic. The locus of the poles *w. r.* to the conic F=0 of tangents to $S=0$ is called the polar reciprocal of the curve $S=0$ *w. r.* to the conic F=0.

Let $p=X\cos\alpha+Y\sin\alpha$ ... ... (1)

touch the curve $S=0$ and the condition of tangency is in terms of p and α i. e. $p=f(\alpha)$ ... ... (2)

Let (x, y) be the pole of the tangent (1) *w. r.* to the conic F=0. The polar of the conic F=0 *w. r.* to the pole (x, y) is

$XF_x+YF_y+ZF_z=0$ (3) $\therefore$ (Z=1)

This polar must coincide with the tangent (1) and their co-efficients are proportional. Now compare their co-efficients,

$$\frac{\cos\alpha}{F_x}=\frac{\sin\alpha}{F_y}=-\frac{p}{F_z} \text{ or, } \frac{\cos\alpha}{p}=-\frac{F_x}{F_z},\ \frac{\sin\alpha}{p}=-\frac{F_y}{F_z}$$

$$\therefore\ \frac{\cos^2\alpha}{p^2}+\frac{\sin^2\alpha}{p^2}=\frac{F_x^2+F_y^2}{F_z^2} \text{ or, } \frac{1}{p^2}=\frac{F_x^2+F_y^2}{F_z^2}$$

Also $\tan\alpha=\frac{F_y}{F_x}$

Put them in (2), then

$$\sqrt{\left\{\frac{F_x^2+F_y^2}{F_z^2}\right\}^{-1}}=f\left\{\tan^{-1}\frac{F_y}{F_x}\right\}^2$$

or, $F_z^2+F_y^2=F_z^2\{f\tan^{-1}F_y/F_x\}^2$

which is required polar reciprocal $w.\ r.$ to the conic $F=0$

see Ex. 7

Examples

Ex. 1. Show that the pedal equation of the ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is C. H. 1992

$\frac{b^2}{p^2}=\frac{2a}{r}-1$ R. U. 1960, D. U. H. 1963, C. H. 1965.

we know the polar equation of the ellipse by taking one of the focus as pole is $l/r=1+e\cos\theta$ ... ... (1)

From (1), we have $\log l-\log r=\log(1+e\cos\theta)$

$$\therefore\ 0-\frac{1}{r}\cdot\frac{dr}{d\theta}=-\frac{e\sin\theta}{1+e\cos\theta} \text{ or, } \cot\phi=\frac{e\sin\theta}{1+e\cos\theta}\cdots(3)$$

$$\left[\therefore\ \frac{1}{r}\frac{dr}{d\theta}=\cot\phi\right]$$

But we know, $p=r\sin\phi$

$$\text{or, } \frac{1}{p^2}=\frac{1}{r^2\sin^2\phi}=\frac{\operatorname{cosec}^2\phi}{r^2}=\frac{1}{r^2}(1+\cot^2\phi)$$

$$=\frac{1}{r^2}\left[1+\frac{e^2\sin^2\theta}{(1+e\cos\theta)^2}\right]=\frac{1+2e\cos\theta+e^2(\sin^2\theta+\cos^2\theta)}{r^2(1+e\cos\theta)^2}$$

$$\text{or, } \frac{1}{p^2}=\frac{2+2e\cos\theta+e^2-1}{r^2(2+e\cos\theta)^2}=\frac{1}{r^2}\left[\frac{2(1+e\cos\theta)}{(1+e\cos\theta)^2}+\frac{e^2-1}{(1+e\cos\theta)^2}\right]$$

$$=\frac{1}{r^2}\left[\frac{2}{1+e\cos\theta}+\frac{e^2-1}{(1+e\cos\theta)^2}\right]=\frac{1}{r^2}\left[\frac{2}{l/r}+\frac{e^2-1}{l^2/r^2}\right]$$

$$=\frac{1}{r^2}\left[\frac{2r}{l}+\frac{r^2}{l^2}(e^2-1)\right]=\frac{1}{r^2}\left[\frac{2ra}{b^2}+\frac{r^2a^2}{b^4}(e^2-1)\right]$$

$$=\frac{2a}{b^2r}+\frac{a^2}{b^4}\left[-\frac{b^2}{a^2}-1\right]=\frac{2a}{b^2r}-\frac{b^2}{a^2}\times\frac{a^2}{b^4}$$

$$\therefore\ \frac{1}{p^2}=\frac{2a}{b^2r}-\frac{1}{b^2} \text{ or, } \frac{b^2}{p^2}=\frac{2a}{r}-1 \text{ Proved.}$$

Ex. 2. Obtain the pedal equation of the curve.

$r^m=a^m\sin m\theta$ R. U. 1967

Take logarithm of both sides $m\log r=m\log a+\log\sin m\theta$

$$\therefore\ m\cdot\frac{1}{r}\frac{dr}{d\theta}=0+\frac{1}{\cos m\theta}(-\sin m\theta)m$$

or, $\cot\phi=\cot m\theta\ \therefore\ \phi=m\theta,$ ... ... (1)

But $p=r\sin\phi=r\sin m\theta$ ... ... (2)

From the given equation $r^m=a^m\sin m\theta$

We have $\sin m\theta=r^m/a^m$ ... ... (3)

$\therefore\ p=r.\ r^m/a^m$ ... ... by (4)

or, $pa^m=r^{m+1}$

which is the required pedal equation.

Ex. 3 Find the pedal equation of the curve $x^2+y^2-2ax=0$

The equation of the tangent to the curve

$x^2+y^2-2ax=0$ ... ... (1)

at (x_1, y_1) is $xx_1+yy_1-a(x+x_1)=0$

or, $x(x_1-a)+yy_1-ax_1=0$ ... ... (2)

The length of the perp. from the contre (0, 0) to the tangent is

$$p=\frac{-ax_1}{\sqrt{\{(x_1-a)^2+y_1^2\}}}=\frac{-ax_1}{\sqrt{(x_1^2+y_1^2-2ax_1+a^2)}}=\frac{-ax_1}{-a}=x_1$$

Again from (1), $x_1^2+y_1^2-2ax_1=0$, or, $x_1=\frac{x_1^2+y_1^2}{2a}=\frac{r^2}{2a}$.

$\therefore$ $2ap=r^2$.

* **Ex. 4.** Find the kth positive pedal of the cardioide $r=a(1+\cos\theta)$.

The equation can be written as $r=2a\cos^2\theta/2$ ... ... (1)

or, $\log r=\log 2a+2\log\cos\frac{1}{2}\theta$

$$\therefore \frac{1}{r}\frac{dr}{d\theta}=2\cdot\frac{1}{2}\cdot\frac{-\sin\theta/2}{\cos\theta/2}=-\tan\tfrac{1}{2}\theta=\cot(\tfrac{1}{2}\pi+\tfrac{1}{2}\theta)$$

or, $\cot\phi_1=\cot(\frac{1}{2}\pi+\frac{1}{2}\theta)$

$\therefore$ $\phi_1=\frac{1}{2}\pi+\frac{1}{2}\theta$ ... ... (2)

But from the Art 10.17 ... we have

$\theta=\theta'+\frac{1}{2}\pi-\phi_1=\theta'+\frac{1}{2}\pi-\pi/2-\frac{1}{2}\theta$ or, $\theta=\frac{2}{3}\theta'$ ... (3)

Again we know

$r_1=r\sin\phi_1=r\sin(\frac{1}{2}\pi+\frac{1}{2}\theta)=r\cos\frac{1}{2}\theta$

$=2a\cos^2\theta\cdot 2\cos\frac{1}{2}\theta=2a\cos^3\theta/2=2a\cos^3\frac{1}{2}\cdot\frac{2}{3}\theta_1$

or, $r_1=2a\cos^3\frac{1}{3}\theta_1$ ... — (4)

which is the first positive pedal Simplify as before, then

$$\frac{1}{r_1}\cdot\frac{dr_1}{d\theta_1}=-\tan\theta_1/3=\cot(\pi/2+\theta_1/3)$$

or, $\cot\phi_2=\cot(\pi/2+\theta_1/3)$

$\therefore$ $\phi_2=\theta/2+\theta_1/3$, $\theta_1=\theta_2+\pi/2-\phi_2$ or, $\theta_1=3\theta_2/4$

Again $r_2=r_1\sin\phi_2=r_1\cos\theta_1/3=2a\cos^3\theta_1/3\cos\theta_1/3$

$=2a\cos^4\theta_1/3=2a\cos^4(\theta_2/4)$

$\therefore$ $r_2=2a\cos^4\theta_2/4$ ... ... (5)

which is the 2nd positive pedal of (1). In the same way we get the k th positive pedal of (1), and this is

$$r_k=2a\cos^{k+2}\left(\frac{\theta_k}{k+2}\right)$$

Alternative method

$r=2a\cos^2\theta/2$ ... ... (1)

Let us consider it as

$r=2a\cos^m(\theta/m)$ ... ... (2)

or, $\log r=\log 2a+m\log\cos\theta/m$.

$$\therefore \frac{1}{r}\cdot\frac{dr}{d\theta}=m\frac{-\sin(\theta/m)}{\cos(\theta/m)}\cdot\frac{1}{m}=-\tan\theta/m$$

or, $\cot\phi=\cot(\pi/2+\theta/m)$ $\therefore \phi=\pi/2+\theta/m$

and $\theta=\theta_1+\pi/2-\phi$ or, $\theta=\theta_1+\pi/2-\pi/2-\theta/m$

or, $\theta+\frac{\theta}{m}=\theta_1$ $\therefore$ $\theta\frac{(1+m)}{m}=\theta_1$ $\therefore$ $\theta=\frac{m\,\theta_1}{m+1}$

But we know $r_1=r\sin\phi=r\sin(\pi/2+\theta/m)=r\cos\theta/m$

$=2a\cos^m(\theta/m)\cos(\theta/m)=2a\cos^{m+1}(\theta/m)$

or, $r_1=2a\cos^{m+1}\left(\frac{\theta}{m+1}\right)$ ... ... (3)

Therefore first positive pedal equation of (2)

$$r=2a \cos^{m+1}\left(\frac{\theta}{m+1}\right) \quad \cdots \quad \cdots \quad (4)$$

If we put $m_1=m+1$, then

$$r=2a \cos^{m_1} \frac{\theta}{m_1}, \; m_1=m+1$$

which is the first positive pedal of the curve (2)

Similarly the 2nd posittve pedal of (2) is

$$r=2a \cos^{m_2} \frac{\theta}{m_2}, \quad m_2=m_1+1=m+2$$

and the kth positive pedal is

$$r_k=2a \cos^{m_k} \frac{\theta}{m_k}, \; m_k=m+k$$

Put $m=2$, then $m_k=k+2$.

Thus $r=2a \cos^{k+2} \dfrac{\theta}{k+2}$ is the pedal equation of

$$r=2a \cos^2 \theta/2=a\,(1+\cos\theta)$$

Ex. 5. Find the pth negative pedal of the curve $r=a(1+\cos\theta)$

The equation can be written as

$r=2a \cos^2 \theta/2=2a \cos^m \theta/m$, $m=2$.

The kth positive pedal of $r=2a \cos^n (\theta/n)$ is

$r=2a \cos^m (\theta/m)$

where $m=n+k$ or, $n=m-k$

Hence kth negative pedal

of $r=2a \cos^m \theta/m$ is $r=2a \cos^n\theta/m$

where $n=m-k$

Similarly the pth negative pedal of $r=2a \cos^m\theta/m$ is

$$r=2a \cos^{m-p} \frac{\theta}{m-1}$$

Hence the pth negative pedal for $r=2a \cos^2\theta/2$

is $r=2a \cos^{2-p} \dfrac{\theta}{2-p}$ or, $r=2a \cos^n \dfrac{\theta}{n}$, $n=2-p$

Ex. 6. Show that the positive pedal of the curve

$r^{2/9} \cos 2\theta/9=a^{2/9}$ is $r^2 \cos 2\theta=a^2$

Let $r^{2/9} \cos (2/9)\theta=a^{2/9}$ become $r^m \cos m\theta=a^m$, $m=2/9$

or, $m \log r+\log \cos m\theta=m \log a$

$$\therefore \quad \frac{m}{r}\frac{dr}{d\theta}\,\theta-m\frac{\sin (2/9)\theta}{\cos (2/9)\theta}=0$$

or, $\cot\phi=\tan m\theta=\cot(\frac{1}{2}\pi-m\theta)$, $\phi=\frac{1}{2}\pi-m\theta$.

But $\theta=\theta_1+\frac{1}{2}\pi-\phi=\theta_1+\frac{1}{2}\pi-\frac{1}{2}\pi+m\theta$ or, $\theta=\theta_1+m\theta$

or, $\theta(1-m)=\theta_1$ or, $\theta=\dfrac{\theta_1}{1-m}$

Again $r_1=r \sin\phi=r \sin(\frac{1}{2}\pi-m\theta)=r \cos m\theta$

or, $r_1^m=r^m \cos^m m\theta=a^m \cos^{m-1} m\theta$

or, $r_1^{m/(m-1)}=a^{m/(m-1)} \cos \dfrac{m}{1-m}\theta_1$

or, $\imath^{m/(1-m)} \cos \dfrac{m}{1-m}\,\theta=a^{m/(1-m)}$

Put $m_1=\dfrac{m}{1-m}$

The first positive pedal of (1) is $r_1^{m_1} \cos m_1 \theta = a^{m_1}$

Similarly the 2nd positive pedal is

$$a^{m_2} = r^{m_2} \cos m_2\theta \;;\; m_2 = \frac{m}{1-m_1} = \frac{m}{1-2m}$$

Similarly 4th positive pedal is

$$a^{m_4} = r^{m_4} \cos m_4 \theta, \quad m_4 = \frac{m}{1-4m}$$

But $m = \frac{2}{9}$, then $m_4 = \frac{2/9}{1-8/9} = 2$

Therefore the 4th positive pedal of $r^{2/9} \cos \frac{2}{9}\theta = a^{2/9}$ is $a^2 = r^2 \cos 2\theta$.

Ex. 7. Show that inverse of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with regard to the origin is $(x^2+y^2)^2 = k^4\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)$

Put $\frac{k^2X}{X^2+Y^2}$ for x, $\frac{k^2Y}{X^2+Y^2}$ for y in the equation of ellipse.

Then $$\frac{k^4 X^2}{a^2(X^2+Y^2)^2} + \frac{k^4 Y^2}{b^2(X^2+Y^2)^2} = 1$$

or, $$k^4\left(\frac{X^2}{a^2} + \frac{Y^2}{b^2}\right) = (X^2+Y^2)^2 \quad \ldots \ldots (1)$$

for the equation of inverse of the ellipse.

Now replace X by x, Y by y in (1), then

$$k^4\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = (x^2+y^2)^2.$$

Ex. 8. Show that the inverse of the equiangular spiral $r = ae^{\theta \cot\alpha}$ is $r = \frac{k^2}{a} e^{-\theta \cot\alpha}$ another equiangular spiral.

Replace r by $\frac{k^2}{r_1}$ in $r = ae^{\theta \cot\alpha}$ then $\frac{k^2}{r_1} = ae^{\theta \cot\alpha}$ for the equation of the inverse of $r = ae^{\theta \cot\alpha}$

Now replace r_1 by r. then $\frac{k^2}{r} = ae^{\theta \cot\alpha}$ or, $r = \frac{k^2}{a} e^{-\theta \cot\alpha}$

which is the inverse of the given conic.

Ex. 9. Find the polar reciprocal of the parabola $y^2 = 4ax$ with regard to its focus.

The eq. of the tangent to curve $y^2 = 4ax$ at (x_1, y_1) is

$$yy_1 = 2a(x+x_1) \text{ or, } 2ax - yy_1 - 2ax_1 = 0 \qquad (1)$$

If $p = x \cos\alpha + y \sin\alpha$ touches the curve $y^2 = 4ax$ then it will be identical to (1) if the co-efficients are proportional.

Thus $2ax - yy_1 + 2ax_1 = 0$; $x \cos\alpha + y \sin\alpha - p = 0$

Comparing co-efficients, we have

$$\frac{2a}{\cos\alpha} = -\frac{y_1}{\sin\alpha} = -\frac{2ax_1}{p}$$

or, $$x_1 = -\frac{2ap}{2a\cos\alpha}, y_1 = -\frac{2a \sin\alpha}{\cos\alpha}$$

Thus the condition of tangency of the line $p = x_1 \cos\alpha + y_1 \sin\alpha$

is $$p = \frac{-p\cos\alpha}{\cos\alpha} - \frac{2a\sin^2 a}{\cos\alpha}$$

or, $2p = -2a \sin\alpha \tan\alpha$ or, $p = -a \sin\alpha \tan\alpha \quad \ldots \ldots (1)$

The polar equation of the pedal with respect to the vertex is

$r = -a \sin\theta \tan\theta \quad \ldots \quad \ldots \quad \ldots \quad (2)$

The inverse of this curve is

$$k^2/r = -a \sin\theta \tan\theta \qquad (rr_1 = k^2)$$

which is the polar reciprocal of $y^2 = 4ax$.

Put $x = r\cos\theta$, $y = r\sin\theta$, $k^2 = -r\sin\theta\tan\theta$

or, $k^2 = -y\dfrac{y}{x}$ or, $y^2 + k^2x = 0$

Thus $r\sin\theta \tan\theta = -k^2$ or, $y^2 + k^2x = 0$

is the polar reciprocal of the parabola w. r. to a circle with centre at the vertex and radius k.

* Ex. 10. Show that the polar reciprocal of the curve $r^m = a^m \cos m\theta$ with regard to the hyperbola $r^2\cos 2\theta = a^2$ is

$$r^{\frac{m}{m+1}} \cos\frac{m\theta}{m+1} = a^{\frac{m}{m+1}}$$

Let the pedal equation of $r^m = a^m \cos m\theta$ ... (1)

be $p = r\sin\phi$

Take logarithmic differentiation of (1)

then $r\dfrac{dr}{d\theta} = -\tan m\theta$

or, $\cot\phi = \cot(\frac{1}{2}\pi + m\theta)$ $\therefore$ $\phi = (\frac{1}{2}\pi + m\theta)$

Hence $p = r\sin(\frac{1}{2}\pi + m\theta) = r\cos m\theta$

$$p = a\cos^{(m+1).m} \frac{m\theta_1}{m+1}$$

$\therefore$ $\theta = \theta'_1 + \frac{1}{2}\pi - \phi = \theta'_1 + \frac{1}{2}\pi - \frac{1}{2}\pi - m\theta$ or, $\theta = \dfrac{m}{m+1}\theta_1$

$$\therefore \quad p^{m/(m+1)} = a^{m/(m+1)} \cos\frac{m\theta_1}{m+1} \qquad (2)$$

Let $p = X\cos\alpha + Y\sin\alpha$ ... ... (3)

touch the curve (1)

From the 2nd equation we have

$r^2\cos 2\theta = a^2$ or, $r^2\cos^2\theta - r^2\sin^2\theta = a^2$

or, $x^2 - y^2 = a^2$ ... ... (4)

If (x, y) be the pole of tangent (3) w. r. to $x^2 - y^2 = a^2$ then the tangent must coincide with the polar of (x, y).

Therefore $Xx - Yy = a^2$. ... ... (5)

Compare (3) and (5)

$$\frac{\cos\alpha}{x} = \frac{-\sin\alpha}{y} = \frac{p}{a^2} \quad \text{or,} \quad \frac{\cos\alpha}{p} = \frac{\alpha}{a^2}, \quad \frac{\sin\alpha}{p} = -\frac{y}{a^2}$$

$$\therefore \frac{1}{p^2}(\cos^2\alpha + \sin^2\alpha) = \frac{x^2+y^2}{a^4} = \frac{r^2}{a^4}$$

or, $p = \dfrac{a^2}{r}$ ... ... (6)

From (2) and (6), we have

$$\left(\frac{a^2}{r}\right)^{m/(m+1)} = a^{m/(m+1)} \cos\left(\frac{m\theta}{m+1}\right)$$

or, $a^{m/(m+1)} = r^{m/(m+1)} \cos\left(\dfrac{m\theta}{m+1}\right)$

$$r^{m/(m+1)} \cos\frac{m\theta}{m+1} = a^{m\theta/(m+1)} \quad \text{Proved.}$$

Exercise X (C)

1. Find the pedal equation of a parabola either from its polar equation or from cartesian with the focus as origin of co-ordinates. D. U. 1952, '56

2. Find the pedal equation of the curve

$r^4 = a^4 \cos 4\theta$ D. U. 1965.

4. Find the pedal equation of an ellipse w. r to one extremity of the major axis. D U. 1953

5. Find the pedal equation of the parabola

$y^2=4a(x+a)$ D. U. 1961

6. Show that the pedal equation of the parabola $y^2=4ax$ with its vertex is $a^2(r^2-p^2)=p^2(r^2+4a^2)(p^2+4a^2)$ R.U.H.1964

7. Show that the pedal equation of the astroid

$x^{2/3}+y^{2/3}=a^{2/3}$ is $r^2+3p^2=a^2$ R. H. 67, C.U. 1982

8. Show that the pedal equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with regard to the origin is $a^2b^2/p^2=a+b^2-r^2$. N. U. 1995

*9. Show that the first positive pedal with regard to the vertx of the parabola $y^2+4bx=0$ can be written as R. U. 1959

$y^2(2a-x)=x^2$ where $b=2a$

$$\left(\frac{\cos \alpha}{p}\right)^{\frac{m}{m-1}} a^{\frac{1}{1-m}}+\left(\frac{\sin \alpha}{p}\right)^{\frac{m}{m-1}} b^{\frac{1}{m-1}}=1$$

*11. Show that the first positive pedal of the curve

$x^3+y^3=a^3$ is $(x^2+y^2)^{3/2}=(x^{3/2}+a^{3/2})$

*12. Prove that the kth positive pedal of $r^m=a^m\cos m\theta$ is $r^{m_k}=a^{m_k}\cos m_k\theta$. where $m_k=\frac{m}{1+km}$ D. U. 1965

*13. Show that the 5th negative pedal of the cardioide

$r=a(1-\cos\theta)$ is $8a/r=\cos\theta+3\cos\frac{1}{3}\theta$

*14. Show that nth positive pedal of the spiral $r=ae^{\theta\cot\alpha}$ is $r=a\sin^n\alpha\, e^{n(\frac{1}{2}\pi-\alpha)\cot\alpha}\, e^{\theta\cot\alpha}$

*15. Show that kth negative pedal of the curve

$r^m=a^m\cos m\theta$ is $r^n=a^n\cos n\theta$ when $n=\frac{m}{1-km}$

*16. Show that pedal equation of the curve

$x=a(3\cos\theta-\cos^3\theta),\quad y=a(3\sin\theta-\sin^3\theta)$

is $3p^2(7a^2-r^2)=(10a^2-r^2)^2$

*17. Show that the pedal equation of the curve

$c^2(x^2+y^2)=x^2y^2$ is $\frac{1}{p^2}+\frac{3}{r^2}=\frac{1}{c^2}$

*18. Find the pedal equations of the following curves

(ii $r=a+b\cos\theta$ (ii) $r^m=a^m\sin m\theta+b^m\cos m\theta$

(iii) $l/r=a(1+\cos\theta)$ (iv) $r^2\cos 2\theta=a^2$

(v) $r^2=a^2\cos 2\theta$, (iv) $r=a(1-\cos\theta)$ N.U. 1995

*19. Show that the pedal equation of the curve

$y^2(3a-x)=(x-a)^3$ is $p^2=9a^2(r^2-a^2)/r^2+(5a^2)$

*20. Show that 5th negative pedal of $r^2=a^2\cos 2\theta$

is $r^{2/9}\cos 2/9\theta=a^{2/9}$

21. Write down the Ist, 2nd and nth positive and negative pedal of the curves

(i) $p+r=a$. (ii) $a^2r=p^3$ (iii) $r/p=a$

22. Show that the pedal equation of the hyperbolic spiral

$r\theta=a$ is $\frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{a^2}$

23. Show that pedal equation of the spiral of Achimedes $r=a\theta$ is $r^4=p^2(a^2+r^2)$

24. Show that the inverse of a straight line is a circle and that of circle is another circle.

25. Show that the inverse of the conic $ax^2+2hxy+by^2=5y$ is the cubic $k^2(ax^2+2hxy+by^2)=5y(x^2+y^2)$

26. Show that the inverse of the parabola $\frac{l}{r}=1+\cos\theta$ is cardoide $r=a(1+\cos\theta)$ where $a=k^2/l$

27. Show that polar reciprocal $r=a\cos\theta$ with regard to a circle of radius k and centre at the origin of the curve is

$$\frac{l}{r}=1+\cos\theta$$

28. Show that polar reciprocal of $x^m y^m=a^{m+n}$ with regard to the circle of radius k and centre at the origin of the curve is

$$x^m y^m \left\{\frac{a(m+n)}{k^2}\right\}^{m+n}=m^m n^n$$

*29. Show that the polar reciprocal of ellipse with regard its centre is $a^2x^2+b^2y^2=k^4$

where k is the radius of the circle.

30. Show that first positive pedal of the curve

$p=\frac{r^{m+1}}{a^m}$ is $p^{m+1}a^m=r^{m+1}$

and that its polar reciprocal with regard to a circle of radius a whose centre is at the origin $p^{m+1}=a^m r$

*31. Show that the polar reciprocal of the curve $x^n+y^n=a^n$ with regard to the circle of radius k and centre at the origin of the given curve is $x^{\frac{n}{n-1}}+y^{\frac{n}{n-1}}=\left(\frac{k^2}{a}\right)^{\frac{n}{n-1}}$

প্রশ্নমালা X (C)

1. মূল বিন্দুকে উপকেন্দ্র (focus) ধরিয়া পোলার বা কার্টেসীয় স্থানাংকে অধিবৃত্তের সমীকরণ হইতে পেডেল সমীকরণ নির্ণয় কর। D. U. 1952, '56

2. কার্ডিয়য়েড (Cardioide) $r=a(1-\cos\theta)$ এর পেডেল (Pedal) সমীকরণ নির্ণয় কর। C. U. 1983, D. U. 1958.

3. বক্ররেখা $r^4=a^4\cos 4\theta$ এর (p, r) সমীকরণ নির্ণয় কর। D. U. 196

4. কোন উপবৃত্তের বৃহদক্ষের একপ্রান্তের সাপেক্ষে উহার পেডেল সমীকরণ নির্ণয় কর। D. U. 1953

5. পরাবৃত্ত (Parabola) $y^2=4a(x+a)$ এর পেডেল সমীকরণ নির্ণয় কর। D. U. 1961

6. দেখাও যে পরাবৃত্ত $y^2=4ax$ এর শীর্ষবিন্দুর সাপেক্ষে ইহার পেডেল সমীকরণ হইবে

$a^2(r^2-p^2)^2=p^2(r^2+4a^2)(p^2+4a^2)$ R. U. H. 1964

7. দেখাও যে এষ্ট্রয়েড (Astroid) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ এর পেডেল সমীকরণ $r^2+3p^2=a^2$ R. U. H. 67, C. U. 1982

8. দেখাও যে মূলবিন্দুর সাপেক্ষে উপবৃত্ত

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ এর পেডেল সমীকরণ $a^2b^2/p^2=a^2+b^2-r^2$.

*9. দেখাও যে শীর্ষবিন্দুর সাপেক্ষে পরাবৃত্ত $y^2+4bx=0$ এর প্রথম ধনাত্মক পেডেল সমীকরণকে লিখা যায় $y^2(2a-x)=x^3$ যেখানে $b=2a$ R. U. '59

*10. দেখাও যে বক্ররেখা $ax^m+by^m=1$ এর প্রথম ধনাত্মক পেডেল সমীকরণ হইবে $\left(\frac{\cos\alpha}{p}\right)^{m/(m-1)} a^{1/(1-m)}+\left(\frac{\sin\alpha}{p}\right)^{m/(m-1)} b^{1/(1-m)}=1.$

*11. দেখাও যে বক্ররেখা $x^3+y^3=a^3$ এর প্রথম ধনাত্মক পেডেল সমীকরণ হইবে $(x^2+y^2)^{3/2}=a^{3/4}(x^{3/2}+y^{3/2})$

*12. দেখাও যে $r^m=a^m\cos m\theta$ সমীকরণের k-তম

ধনাত্মক পেডেল সমীকরণ হইবে $r^{m_k}=a^{m_k}\cos m_k\theta$

যেখানে $m_k=\dfrac{m}{1+km}$

D. U. 1965

*13. দেখাও যে কার্ডিয়য়েড (cardioide) $r=a\,(1-\cos\theta)$ এর পঞ্চম ঋণাত্মক পেডেল সমীকরণ হইবে

$$\frac{8a}{r}=\cos\theta_1+3\cos\tfrac{1}{3}\theta_1,\ \theta_1=\tfrac{1}{2}\pi-\tfrac{1}{2}\theta$$

*14. কুণ্ডলী $r=ae^{\theta\cot\alpha}$ এর জন্য দেখাও যে
n-তম ধনাত্মক পেডেলের সমীকরণ হইবে

$$r=a\sin^n\alpha\, e^{n(\pi/2-\alpha)\cot\alpha}\ e^{\theta\cot\alpha}$$

*15. দেখাও যে বক্ররেখা $r^m=a^m\cos m\theta$-এর k তম ঋণাত্মক পেডেল সমীকরণ হইবে $r^n=a^n\cos n\theta$ যেখানে $n=\dfrac{m}{1-km}$

*16. দেখাও যে বক্ররেখা $x=a(3\cos\theta-\cos^3\theta)$, $y=a\,(3\sin\theta-\sin^3\theta)$ এর পেডেল সমীকরণ হইবে $3p^2(7a^2-r^2)=(10\,a^2-r^2)^2$

*17. দেখাও যে বক্ররেখা $c^2(x^2+y^2)=x^2y^2$ এর পেডেল সমীকরণ হইবে

$$1/p^2+3/r^2=1/c^2$$

*18. নিম্নলিখিত বক্ররেখাগুলির পেডেল সমীকরণ নির্ণয় কর :-

(i) $r=a+b\cos\theta$ (ii) $r^m=a^m\sin m\theta+b^m\cos m\theta$

(ii) $l/r=a(1+\cos\theta)$ (iv) $r^2\cos 2\theta=a^2$ (v) $r^2=a^2\cos 2\theta$

*19. দেখাও যে বক্ররেখা $y^2(3a-x)=(x-a)^3$ এর পেডেল সমীকরণ হইবে

$$p^2=9a^2(r^2-a^2)/(r^2+15a^2)$$

*20. দেখাও যে বক্ররেখা $r^2=a^2\cos 2\theta$ এর পঞ্চম ঋনাত্মক পেডেল সমীকরণ হইবে $r^{2/p}\cos 2/9\,\theta=a^{2/p}n$

*21. নিম্নলিখিত বক্ররেখাগুলির ১ম, ২য় এবং n-তম ধনাত্মক এবং ঋণাত্মক পেডেলস নির্ণয় কর।

(i) $p+r=a$ (ii) $a^2r=p^3$ (iii) $r/p=a$

22. দেখাও যে পরাবৃত্তিয় কুণ্ডলী $r\theta=a$-এর পেডেল সমীকরণ

$$1/p^2=1/r^2+1/a^2.$$

23. দেখাও যে আর্কিমিডিসের কুণ্ডলী $r=a\theta$ এর পেডেল সমীকরণ

$$r^4=p^2(a^2+r^2).$$

24. দেখাও যে একটি সরলরেখার বিপরীত সমীকরণ একটি বৃত্ত এবং একটি বৃত্তের বিপরীত সমীকরণ অপর একটি বৃত্ত।

25. দেখাও যে কণিক $ax^2+2hxy+by^2=5y$ এর বিপরীত সমীকরণ হইল ত্রিঘাত সমীকরণ

$$k^2(ax^2+2hxy+by^2)=5y\,(x^2+y^2)$$

26. দেখাও যে অধিবৃত্ত $l/r=1+\cos\theta$ এর বিপরীত সমীকরণ হইল কার্ডিয়য়েড $r=a(1+\cos\theta)$, এখানে $a=k^2/l$.

27. দেখাও যে বক্ররেখার মূলবিন্দুকে কেন্দ্র এবং k ব্যাসার্ধ বিশিষ্ট বৃত্তের সাপেক্ষে বক্ররেখা $r=a\cos\theta$ এর পোলার উল্টা সমীকরণ হইবে $l/r=1+\cos\theta$.

28. দেখাও যে বক্ররেখার মূল বিন্দুকে কেন্দ্র এবং k ব্যাসার্ধ বিশিষ্ট বৃত্তের সাপেক্ষে বক্ররেখা

$x^m y^m=a^{m+n}$ এর পোলার উল্টা (Reciprocal) সমীকরণ হইবে

$$x^m y^m\left\{\frac{a\,(m+n)}{k^2}\right\}^{m+n}=m^m n^n.$$

*29. দেখাও যে উপবৃত্তের কেন্দ্রের সাপেক্ষে (with regard to its centre) উপবৃত্তের পোলার উল্টো সমীকরণ হইবে $a^2x^2+b^2y^2=k^4$, এখানে k হইল বৃত্তের ব্যাসার্ধ।

*30. দেখাও যে বক্ররেখা $p=\dfrac{r^{m+1}}{a^m}$ এর প্রথম ধনাত্মক পেডেল সমীকরণ হইবে $p^{m+1}a^m=r^{2m+1}$

এবং মূল বিন্দুতে কেন্দ্র ও 'a' ব্যাসার্ধ বিশিষ্ট বৃত্তের সাপেক্ষে ইহার পোলারের উল্টো সমীকরণ হইবে $p^{m+1}=a^m r$.

*31. দেখাও যে বক্ররেখা $x^n+y^n=a^n$ এর মূলবিন্দুতে কেন্দ্র এবং k ব্যাসার্ধ বিশিষ্ট বৃত্তের সাপেক্ষে বক্ররেখাটির পোলারের উল্টা সমীকরণকে লিখা যায়

$$x^{\frac{n}{n-1}}+y^{\frac{n}{n-1}}=\left(\frac{k^2}{a}\right)^{\frac{n}{n-1}}$$

Answers

1. $p^2=ar$ 2. $r^3=2ap^2$ 3. $r^5=a^4p$ 5. $p^2=ar$

18. (i) $r^4=(b^2-a^2+2ar)p^3$ (ii) $r^{m+1}=p\sqrt{(a^{2m}+b^{2m})}$

(iii) $rl=2ap^2$ (iv) $pr=a^2$ (v) $r^3=a^2p$.

21. nth + ve pedal negative pedal

(i) nth $\frac{r^n}{p^n}(p+r)=a$ $\frac{p^n}{r^n}(p+r)=a$

(ii) $p^{2n+3}\ a^2=r^{2n-1}$, $r^{2n+1}\ a^2=p^{2n-3}$

(iii) $\frac{r}{p}=a.$ $r/p=a$

Chapter xi

MAXIMA AND MINIMA

11. 1. Definition : A function $f(x)$ is said to have a maximum at $x=a$ if $f(a)\geqslant f(x)$ for every x in the neighbourhood of a.

Thus if h is any small positive number, then

$f(a)\geqslant f(x)$ for $a-h\leqslant x\leqslant a+h$.

This implies that

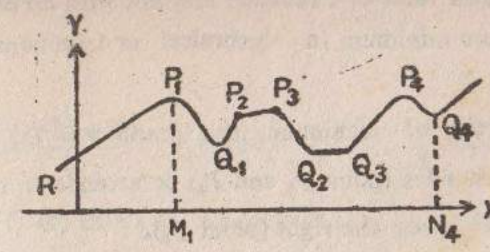
$f(a)\geqslant f(a-h)$ and $f(a)\geqslant f(a+h)$.

Similary, function $f(x)$ is minimum at $x=a$, if

$f(a)\leqslant f(x)$ for $a-h\leqslant x\leqslant a+h$

when $h>0$ and $h\to 0$. In this case

$f(a)\leqslant f(a-h)$ and $f(a)\leqslant f(a+h)$.



Feg.—1

Let Fig 1. represent the graph of some function $y=f(x)$. The function is maximum at P_1 P_2, P_3, P_4 and minimum at Q_1, Q_2 Q_3, Q_4.

(i) We note that a function may have several maxima and minima in an interval where the function is defined.

(ii) It is not necessary that a maximum value of a function is always greater than minimum value of the function. Maximum value of a function may be less than minimum value of a functions. For example the minimum value $Q_4 N_4$ at Q_4 is greater than the maximum value $P_1 M_1$ at P_1.

(iii) In between two maxima, there should be at least one minimum value of the function. Similary at least one maximum value of the function must lie between two minimum values of the function. There is a minimum value of the function between two consecutive maximum values and vice versa.

Thus we observe that maxmimum and minimum values of a function (continuous) occur alternately.

(iv) In Calculus we are concerned with a relative maximum or a relative minimum value of a function and not with an absolute maximum or absolute minimum in algebraical or trigonometrical examples.

(v) From a point of maximum, the graph $y=f(x)$ either descends on both sides (point P_1 and P_4) or ascends on the left if (point P_2) or descends on the right (point P_3).

From a point of minimum, the curve $y=f(x)$ either ascends on both sides (as at Q_1 and Q_4) or descends on the left (as at Q_2) or ascends on the right (as at Q_3)

A point of maximum or minimum of a function is called a ***turning point*** or a ***stationary point*** when the function is differentiable at the point.

11.2. Necessary Condition for Maxima or minima

If a function $f(x)$ is maximum or minimum at $x=a$ and if $f'(a)$ exist, then $f'(a)=0$.

Let $f(x)$ be a finite and continuous function of x in the neighbourghood of $x=a$

From the definition of maxima or minima, $f(x)$ is maximum or minimum at $x=a$ according as $f(a+h)-f(a)$ and $f(a-h)-f(a)$ are $+ve$ or both negative, h being indefinitely small $(h \to 0)$ and positive.

Proof : For the maximum value of $f(x)$ at $x=a$

$f(a+h)-f(a) \leqslant 0$, $f(a-h)-f(a) \leqslant 0$

$$\therefore \quad \frac{f(a+h)-f(a)}{h} \leqslant 0 \qquad \frac{f(a-h)-f(a)}{-h} \geqslant 0 \quad (\because \ h>0)$$

Hence

$$\lim_{h \to 0} \frac{f(a+h)-f(a)}{h} \leqslant 0 \text{ and } \lim_{h \to 0} \frac{f(a-h)-f(a)}{-h} \geqslant 0$$

Now if $f'(a)$ exists at $x=a$, then the above two limits must be equal. Hence $f'(a)=0$

Similary we can establish that $f'(x)=0$ is the necessary condition for a minimum value of $f(x)$ at $x=a$

Hence the necessary condition for maximum or minimum of $f(x)$ at $x=a$ is $f'(a)=0$

11.3.(A) Determination of Maxima and minima

If $f(x)$ *is finite and continuous function of* x *in the vicinity of* $x=a$ *and if* $f'(a)=0$ *and* $f''(a)\neq 0$, *then*

(i) $f(a)$ is maximum if $f''(a)$ is negative

(ii) $f(a)$ is maximum if $f''(a)$ is positive.

For $f(x)$ has maximum or minimum value at $x=a$, we have $f(a+h)-f(a)$ and $f(a-h)-f(a)$ are both negative or both positive h being indefinitely small ($h\to 0$).

By Taylor's Theorem

$$\begin{cases} f(a+h)=f(a)+hf'(a)+\dfrac{h^2}{\underline{|2}}f''(a)+\dfrac{h^3}{\underline{|3}}f'''(a+\theta h), 0<\theta<1 \\ f(a-h)=f(a)-hf'(a)+\dfrac{h^2}{\underline{|2}}f''(a)-\dfrac{h^3}{\underline{|3}}f'''(a-\theta h), 0<\theta<1 \end{cases}$$

$$\begin{cases} f(a+h)-f(a)=\dfrac{h^2}{\underline{|2}}f''(a)+\dfrac{h^3}{\underline{|3}}f'''(a+\theta h) \\ f(a-h)-f(a)=\dfrac{h^2}{\underline{|2}}f''(a)-\dfrac{h^3}{\underline{|3}}f'''(a-\theta h) \end{cases} \quad [\because f'(a)=0]$$

Since h is very small, we can neglect the 2nd term. Then

$$f(a+h)-f(a)=\frac{h^2}{\underline{|2}}f''(a), \; f(a-h)-f(a)=\frac{h^2}{\underline{|2}}f''(a)$$

approximately

As h^2 is always positive, so the sign of $f(a\pm h)-f(a)$ depends upon $f''(a)$

Since $f(a\pm h)-f(a)<0$ for a maximum of $f(x)$ at $x=a$, $f''(a)$ must be negative and similarly for a minimum value of $f(x)$ at $x=a$, $f''(a)$ must be positive.

(B) When $f''(a)=0$ at $x=a$ with $f'(a)=0$

we have from Taylor's Theorem ;

$$\begin{cases} f(a+h)-f(a)=hf'(a)+\dfrac{h^2}{\underline{|2}}f''(a)+\dfrac{h^3}{\underline{|3}}f'''(a)+\dfrac{h^4}{\underline{|4}}f^{iv}(a+\theta h) \\ f(a-h)-f(a)=-hf'(a)+\dfrac{h^2}{\underline{|2}}f''(a)-\dfrac{h^3}{\underline{|3}}f'''(a)+\dfrac{h^4}{\underline{|4}}f^{iv}(a+\theta h) \end{cases}$$

where $0<\theta<h$

$$\begin{cases} f(a+h)-f(a)=\dfrac{h^3}{\underline{|3}}f'''(a)+\dfrac{h^4}{\underline{|4}}f^{iv}(a+\theta h) \\ f(a-h)-f(a)=-\dfrac{h^3}{\underline{|3}}f'''(a)+\dfrac{h^4}{\underline{|4}}f^{iv}(a-\theta h) \end{cases} \quad (\because f''(a)=0)$$

In general,

(C) If $f'(a)=f''(a)=\ldots=f^{(n-1)}(a)=0$ and $f^{n}(a)\neq 0$, then

(I) $f(x)$ is maximum or minimum if n is even ;

$f(x)$ is maximum if $(f)^{(n)}(a)$ is negative and

$f(x)$ is minimum if $f^{(n)}(a)$ is positive.

(II) $f(x)$ is neither a maximum nor a minimum if n is odd.

Proof :— By Taylor's Theorem, we have

$$f(a+h)-f(a)=hf'(a)+\frac{h^2}{\underline{|2}}f''(a)+\ldots+\frac{h^{n-1}}{\underline{|(n-1)}}f^{n-1}(a)+\frac{h^n}{\underline{|n}}f^n(a+\theta h).\ 0<\theta<h$$

$$f(a-h)-f(a)=-hf'(a)+\frac{h^2}{\underline{|2}}f''(a)\ldots(-1)^{n-1}\frac{h^{n-1}}{\underline{|(n-1)}}f^{(n-1)}(a)+(-1)^n\frac{h^n}{\underline{|n}}f^{(n)}(a-\theta h),\ 0<\theta<h$$

or,

$$\left.\begin{array}{l} f(a+h)-f(a)=\dfrac{h^n}{\underline{|n}}f^{(n)}(a+\theta h) \\ f(a-h)-f(a)=(-1)^n\dfrac{h^n}{\underline{|n}}f^{(n)}(a-\theta h) \end{array}\right\}\text{by condition } (c)$$

Now we can write $f^n(a+\theta h)=f^n(a)+\epsilon$, where $\in$ is a very small quantity ($\in \to 0$). So the sign of $f^n(a+\theta h)$ is the same as the sign of $f^n(a)$

Similarly for $f^n(a-\theta h)$ has the same sign of $f^n(a)$

$$\therefore \quad f(a+h)-f(a) = \frac{h^n}{\lfloor n} f^n(a)$$

$$f(a-h)-f(a)=(-1)^n \frac{h^n}{\lfloor n} f^n(a) \text{ approximately}$$

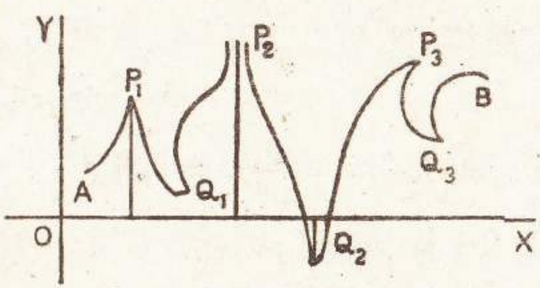
If n is even, then $f(a+h)-f(a)$ and $f(a-h)-f(a)$ have the same sign. So $f(a+h)-f(a)$ and $f(a-h)-f(a)$ are both negative if $f^{(n)}(a)$ is negative.

Similarly $f(x)$ has a minimum value at $x=a$ if $f^{(n)}(a)$ is positive.

If n is odd then $f(a+h)-f(a)$ and $f(a-h)-f(a)$ have different signs and $f(a)$ is neither a maximum nor a minimum if n is odd.

11.5. Maximum and minimum values when dy/dx or, $f'(x)$ is discontinuous.

Let $y=f(x)$ be a function of x The function is not continuous at points shown in the figure and $f'(x)$ is not continuous at some points, P_1, Q_1 and Q_2. (dy/dx) is undefined but y is finite



At P_2, both dy/dx and y are infinite At P_3 and Q_3, dy/dx is discontinuous.

How can we determine the maxima or minima values of $f(x)$ at these points?

Immediately before P_1, the curve rises and so $\frac{dy}{dx} > 0$; immediately after P_1, $\frac{dy}{dx} < 0$ since the curve falls. At P_1 the tangent to the curve is vertical and so $\frac{dy}{dx}$ is undefined. Thus at P_1, $y=f(x)$ is *maximum* and $\frac{dy}{dx}$ *changes sign from* +ve to −ve.

Immediately before Q_1, the curve $y=f(x)$ falls and immediately after Q the curve rises. Hence Q is a point of *minimum* of $y=f(x)$ where $\frac{dy}{dx}$ *changes sign from* −ve *to* +ve, although $\frac{dy}{dx}$ does not exist at Q.

Similar conclusions hold for at maxima and minima at points where $\frac{dy}{dx}$ is either dicontinuous or undefined.

11.6. *In a rational integral algebraical function of the nth degree the greatest number of critical values is $n-1$ and these are alternately maxima and minima.*

Let $y=f(x)=a_0x^n+ax^{n-1}+\ldots+a_{n-1}x+a_n$ ... (1)

be an integral algebraical function (integral means there is no x in the denominator)

$\therefore \quad dy/dx=f'(x)=a_0nx^{n-1}+a_1(n-1)x^{n-2}+\ldots+a_{n-1}$ ... (2)

The necessary condition for maxima and minima is $f'(x)=0$

or, $a_0nx^{n-1}+a_1(n-1)x^{n-2}+\ldots+a^{n-1}=0$ ... (3)

The eq (3) is of $(n-1)$th degree, so it has $(n-1)$ roots, real or complex.

So, $f(x)$ has $(n-1)$ critical values if all the roots of $f(x)$ are real. From (2)

$$f'(x) = dy/dx = a_0 n(x-\alpha_1)^2(x-\alpha_3)\cdots(x-\alpha_{n-1}) \cdots (4)$$

where $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ are the roots of $f'(x)=0$

Let the roots be different and

$\alpha_1 < \alpha_2 < \alpha_3 < \ldots\ldots < \alpha_{n-1}$

Sign of $\left(\frac{dy}{dx}\right)$ ---0 ++++0 ----0---0 ---0++++ → x at α_1, α_2, α_3, α_n, α_{n-1}

Supposing $\frac{dy}{dx} < 0$ for $x \to \alpha_1 -$ (which is true n is even), we see $\frac{dy}{dx}$ changes sign from $-ve$ to $+ve$ at α_1, from $+$ve to $-$ve at α_2, from $-ve$ to $+ve$ or α_3 and so on. Hence $y=f(x)$ is minimum at α_1, maximum at α_2, minimum at α_3 and so on. Hence maxima and minima occur alternately.

11.7. Inflexions

The necessary condition for maxima and minima if a function $y=f(x)$ is $f'(x)=0$

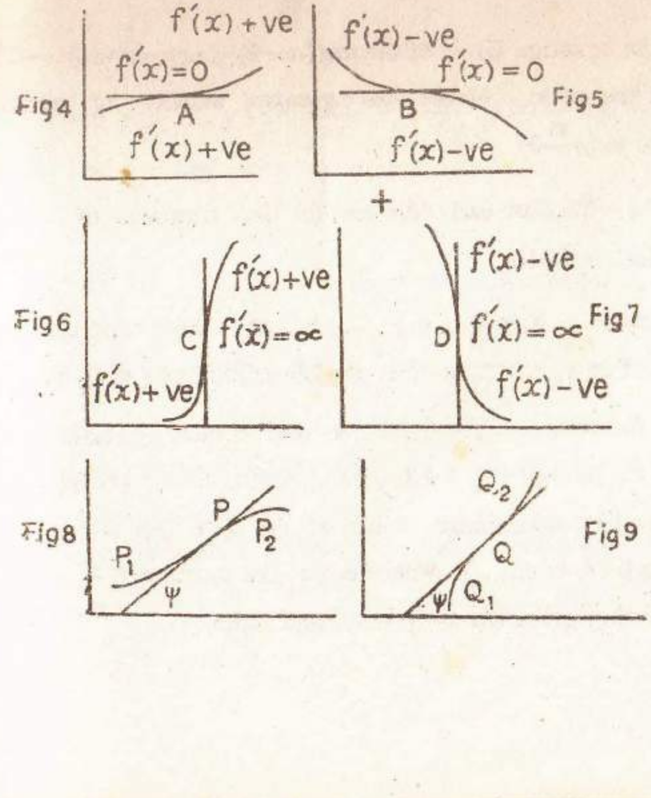
but this condition is not sufficient.

At a point P, $f'(x)$ may be zero, but it is *either* positive *or* negative on both sides of it, that is $f'(x)$ *does not change its sign* although it becomes zero at P. Such a point is called a **point of inflexion.**

In general, a point of inflexion is one at which a curve changes the sign of curvature or the direction of bending. Let $x=\alpha$ be point of inflexion. Then, if $f''(\alpha-h)>0$, we must have $f''(\alpha+h)<0$ or vice versa, where h is a small number, If the curve is smooth at $x=\alpha$, then $f''(\alpha)=0$. Hence $f''(\alpha)=0$ *is the necessary condition for the point* $x=\alpha$ *to be a point of inflexion.*

If d^2y/dx^2 is first $+ve$ for a given value of x say $x=\alpha$ at P then d^2y/dx^2 passes through zero and then becomes $-ve$, dy/dx first increases, then becomes stationary and then decreases, then the point P where $d^2y/dx^2=0$ is called a point of inflexion (Fig 8.)

Similarly if d^2y/dx^2 is first $-ve$, then passes through zero at Q and becomes $+ve$, then dy/dx first decreases then becomes stationary at Q and then increases. The necessary condition that dy/dx is maximum or minimum is $d^2y/dx^2=0$



Cor. The greatest number of points of inflexion is $(n-2)$ in an intergral algebraical function.

Let $y = a_0x^n + a_1x^{n-1} + \dots\dots\dots + a_{n-1}\,x + a_n$

Then

$$\frac{dy}{dx} = a_0nx^{n-1} + a_1(n-1)x^{n-2} + \dots\dots + a_{n-1}2x + a_{n-1}$$

$$\frac{d^2y}{dx^2} = a_0n(n-1)x^{n-2} + \dots\dots + 2a_{n-1} \dots\dots\dots\dots (1)$$

The condition for the existence of a point of inflexion is

$$\frac{d^2y}{dx^2} = 0$$

As the equation (i) is of degree $(n-2)$ if may have $(n-2)$ real roots at the most. **Hence the greatest number of points of inflexions is** $(n-2)$

11. 18. Maxima and Minima for the function of several independent variables.

Definition : Let $\phi(x, y, z\dots\dots)$ be any finite and continuous function of $x, y, z\dots\dots$ at the neighbourhood of $(a, b, c, \dots\dots\dots)$.

The function $\phi(x, y, z, \dots\dots\dots)$ is said to have a maximum value at $(a, b. c\dots)$ if $\phi(a+h, b+k, c+l\dots) < \phi(a, b, c\dots)$ and $\phi(x, y, z\dots)$ is said to have a minimum value at $(a, b, c\dots)$ if $\phi(a+h, b+k, c+l\dots) > \phi(a, b, c\dots\dots)$, whatever be the increments $h, k, l, \dots$etc. provided they are sufficiently small and finite.

11. 19. Necessary conditions for the existence of Maxima and Minima.

A function $\phi(x, y, z\dots)$ has maximum or minimum value at $(a, b, c, \dots\dots)$ if the partial dervatives exist and

$$\frac{\delta\phi}{\delta x} = \frac{\delta\phi}{\delta y} = \frac{\delta\phi}{\delta z\phi} = \dots\dots\dots = 0$$

As the function $\phi(x, y, z, \dots\dots)$ is continuous at the neighbourhocd of $(a, b, c\dots\dots)$ then we have by extended Taylors's Theorem

$$\phi(x+h, y+k, z+l\dots) = e^{\left(h\frac{\delta}{\delta x} + k\frac{\delta}{\delta y} + l\frac{\delta}{\delta z} + \dots\right)} \phi(x, y, z\dots)$$

$$= [1 + (h\frac{\delta}{\delta x} + k\frac{\delta}{\delta y} + l\frac{\delta}{\delta z} + \dots) + \frac{1}{\underline{|2}}(h\frac{\delta}{\delta x} + k\frac{\delta}{\delta y} + l\frac{\delta}{\delta z} + \dots)^2 + \dots\dots)\phi(x, y, z\dots\dots)$$

$$= \phi(x, y, z, \dots\dots) + (h\frac{\delta\phi}{\delta x} + k\frac{\delta\phi}{\delta y} + l\frac{\delta\phi}{\delta z} + \dots\dots) + \text{terms}$$

of the 2nd and higher orders.

or, $(x+h, y+k, z+l\dots) - \phi(x, y, z\dots) = h\frac{\delta\phi}{\delta x} + k\frac{\delta\phi}{\delta y}$

$+ l\frac{\delta\phi}{\delta z} + \dots +$ terms of the 2nd higher order $h, k, l \dots\dots(1)$

As $h, k, l\dots$are very small, then $h\frac{\delta\phi}{\delta z} + k\frac{\delta\phi}{\delta y} + l\frac{\delta\phi}{\delta y} + \dots\dots$ dominates the sign of Right handside of (1),

If we change the sign of $h, k, l\dots\dots$, the sign of the Right side of (1) will change.

Hence the necessary condition for a maximum or minimum value is $h\frac{\delta\phi}{\delta x} + k\frac{\delta\phi}{\delta y} + l\frac{\delta\phi}{\delta z} + \dots\dots + = 0 \dots\dots(2)$

Since (2) is always true whatever be the values of h, k, l .. (......as as h, k, l......etc, are all independent of one other,)

we have

$$\frac{\delta \phi}{\delta x}=0, \frac{\delta \phi}{\delta y}=0, \frac{\delta \phi}{\delta z}=0 \quad \ldots \quad \ldots \qquad (3)$$

As there are n independent variables, so we will get n equations. Solving these equations we get the value of $a, b, c,$...... ..etc. Now put the values of x, y, z......in $\phi(x, y, z, \ldots)$ which is either a maximum or a minimum.

The conditions

$$\frac{\delta \phi}{\delta x}=0, \frac{\delta \phi}{\delta y}=0, \frac{\delta \phi}{\delta z}=0, \text{ etc.}$$

are necessary but not sufficient for the existence of maxima or minima.

11.19. Determination of the sign of quadratic expressions.

(i) $ax^2+2hxy+by^2$

(ii) $ax^2+2hxy+by^2+cz^2=2fyz+2gzx$

Let $I_2=ax^2+2hxy+by^2$

$=(1/a)(a^2x^2+2haxy+aby^2)=(1/a)\{(ax+hy)^2+(ab-h^2)y^2\}$

$(ax+hy)^2+(ab-h^2)y^2$ is positive if $ab-h^2>0$. Thus the sign of I_2 depends on the sign of a the co efficient of x^2

If $ab-h^2$ is negative, we cannot say anything about the sign of the expression. Let

$I_3=ax^2+by^2+cz^2+2hxy+2fyz+2gzx$

$=(1/a)\{a^2x^2+aby^2+acz^2+2ahxy+2afyz+2agzx\}$

$=(1/a)\{a^2x^2+2ax(gz+hy)+aby^2+acz^2+2fazy\}$

$=(1/a)\{(ax+hy+gz)^2+(ab-h^2)y^2+2(af-gh)yz+(ac-g^2)z^2\}$

I_3 will have the same sign as a for real values of x, y, z if $(ab-h^2)y^2+2(af-gh)yz+(ac-g^2)z^2$ is positive

i. e., $(ab-h^2)$ and $\{(ab-h^2)(ab-g^2)-(af-gh)^2\}$ are positive

i e., $(ab-h^2)$ and $a(abc+2fgh-af^2-bg^2-ch^2)$ are all +ve

Thus I_3 will be positive

if a, $\begin{vmatrix} a & h \\ h & b \end{vmatrix}$, $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ are all positive

I_3 will be negative if these determinants are alternately negative and positive.

These findings will be used in determining maximum or minimum value of a function of two and three variables.

11.21. Two Independent variables. Lagrange's Conditions.

Let $\phi(x, y)$ be function of two variables in x and y

Let $r=\frac{\delta^2\phi}{\delta x^2}$, $s=\frac{\delta^2\phi}{\delta x\delta y}$, $t=\frac{\delta^2\phi}{\delta y^2}$, For $x=a$ and $y=b$,

By Taylors theorem, we have

$$\phi(a+h, b+k)=\phi(a, b)+h\left(\frac{\delta\phi}{\delta x}+\frac{\delta\phi}{\delta y}\right)+\frac{1}{\underline{|2}}\left(h^2\frac{\delta^2\phi}{\delta x^2}+k^2\frac{\delta^2\phi}{\delta y^2}+2hk\frac{\delta^2\phi}{\delta x dy}\right)+ \ldots \ldots \ldots$$

or, $\phi(a+h, b+k)-\phi(a, b)=\frac{1}{2}(h^2r+2hks+tk^2)+R_3 \quad \ldots (1)$

where $\frac{\delta\phi}{\delta x}=\frac{\delta\phi}{\delta y}=0$

and R_3 contains terms with higher pawers of h and k and h k are very small. So the sign of $\phi(a+h, b+k)-\phi(a, b)$ depends upon the sign of

$I_2=(h^2r+2hks+tk^2)$

If the expression I_2 is positive, then $\phi(a, b)$ is minimum If I_2 is negative then $\phi(a, b)$ is maximum. Now,

I_2 will be positive if $h^2r+2hks+tk^2$ is positive *i. e,*

$rt-s^2$ is positive and r is positive. (See Art. 11 20)
Not that $rt-s^2>0 \Rightarrow rt>s^2>0$.

Thus if $rt-s^2$ is positive, $\phi(a, b)$ is maximum or minimum according as r and t are both negative or are both positive.

These are called Lagrange's condittons as these conditions were pointed out by him first.

If $rt-s^2$ is negative, $\phi(a,b)$ is neither a maximum nor a minimum.

If $rt=s^2>0$ then $I_2=rh^2+2shk+tk^2=(1/r)(hr+ks)^2$

∴ Thus the sign of I_2 is the same as the sign of r or t. So $\phi(a b)$ will be maximum or minimum if r is positive or negative.

If $rt=s^2$ and $(h/k)=-(s/r)=\beta$ (say) the 2nd degree terms in (1) will vanish. We are to consider the cubic terms of (1) which will again vanish as $h/k=\beta$ and then consider the 4th degree terms of (1) where we will get the sign of the entire 4th degree term depending on the signs of r and t when $(h/k)=\beta$.

11.22. Maximum and Minimum values of a function of two variables x and y, when x and y are connected by a relation.

Let $u=\phi(x, y)$ (1)

be a finite and continuous function of x and y. Let variables x and y be connected by a relation $f(x, y)=0$ ... ... (2)

Now eliminate y from (1) and (2), then a relations between u and x will be obtained. Now u is the function of x only.

From (2), we have $\dfrac{\delta f}{\delta x}dx+\dfrac{\delta f}{\delta y}dy=0$ or, $f_x dx+f_y dy=0$

or, $\dfrac{dy}{dx}=-\dfrac{\delta f}{\delta x}\Big/\dfrac{\delta f}{\delta y}=-f_x/f_y$... ... (3)

But from (1), the total differential is $du=\dfrac{\delta\phi}{\delta x}dx+\dfrac{\delta\phi}{\delta y}dy$

$$\frac{du}{dx}=\frac{\delta\phi}{\delta x}+\frac{\delta\phi}{\delta y}\frac{dy}{\delta x}=\frac{\delta\phi}{\delta x}+\frac{\delta\phi}{\delta y}(-f_x/f_y) \qquad [by\ (3)\ (4)]$$

As u is the function of x, so u will be maximum or minimum if

$$\frac{du}{dx}=0 \text{ i. e., } \frac{\delta\phi}{\delta x}+\frac{\delta\phi}{\delta y}(-f_x/f_y)=0$$

$$\text{or, } \frac{\delta\phi}{\delta x}f_y+\frac{\delta\phi}{\delta y}f_x=0 \quad \ldots \quad \ldots \quad (5)$$

Now solve (2) and (5) for x and y. These values are required for determining the maximum and minimum of u.

Now find d^2u/dx^2 put the value of x and y in d^2u/dx^2.

If d^2u/dx^2 is positive, then u is minimum.

If d^2u/dx^2 is negative, then u is maximum.

11.23. To determine the maximum or minimum values of a function of three independent variables at a point.

Let $\phi(x, y, z)$ be a function of x, y, z. We are to investigate at (a, b, c) whether $\phi(x, y, z)$ is maximum or minimum.

Let $A=\dfrac{\delta^2\phi}{\delta x^2}$, $B=\dfrac{\delta^2\phi}{\delta y^2}$, $C=\dfrac{\delta^2\phi}{\delta z^2}$. $F=\dfrac{\delta^2\phi}{\delta y\delta z}$, $G=\dfrac{\delta^2\phi}{\delta z\delta x}$, $H=\dfrac{\delta^2\phi}{\delta x\delta y}$

For the existence of a maximum of a minimum at $(a\ b,\ c)$

$$\frac{\delta\phi}{\delta x}=\frac{\delta\phi}{\delta y}=\frac{\delta\phi}{\delta z}=0$$

By Taylor's Theorem

$$\phi(a+h, b+k, c+l)-\phi(a, b, c)=\tfrac{1}{2}(Ah^2+Bk^2+Cl^2+2Fkl+2Glh+2Hk)+R_3.$$

R_3 consists of terms of 3rd and higher powers of h, k, l which are very small. So the sign of (1) depends on the sign of

$I_3=Ah^2+Bk^2+Cl^2+2Fkl+2Glh+2Hhk$

Therefore ϕ (a b, b) is minimum

if $A, \begin{vmatrix} A & H \\ H & B \end{vmatrix}, \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$ are all positive.

and ϕ (a, b, c) is maximum if

$A, \begin{vmatrix} A & H \\ H & B \end{vmatrix}, \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$

are alternately negative and positive.

If these conditions are not satisfied, then ϕ (a, b, c) is neither a maximum nor a minimum.

11.24 Lagrange's method of Finding Undetermined Multipliers.
For Several Independent variables

Let $u = \phi(x_1, x_2, \ldots\ldots x_n)$ be a function of n variables $x_1, x_2, \ldots x_n$

Let these variables be connected by m equations

$f_1(x_1, x_2 \ldots\ldots x_n) = 0,\ f_2(x_1, x_2 \ldots\ldots\ldots x_n) = 0 \ldots\ldots\ldots$
$f_m(x_1, x_2, \ldots\ldots\ldots x_n) = 0$

So, only $(n-m)$ of the variables are independent. We are to find equations for $(n-m)$ remaining variables which will determine a point where u is minimum or maximum.

When u is a maximum or minimum.

$$\left.\begin{array}{l} du = \dfrac{\delta u}{\delta x_1} dx_1 + \dfrac{\delta u}{\delta x_2} dx_2 + \ldots\ldots\ldots + \dfrac{\delta u}{\delta x_n} dx_n = 0 \\ df_1 = \dfrac{\delta f_1}{\delta x_1} dx_1 + \dfrac{\delta f_1}{\delta x_2} \delta x_2 + \ldots + \dfrac{\delta f_1}{\delta x_n} dx_n = 0 \\ df_2 = \dfrac{\delta f_2}{\delta x_1} dx_1 + \dfrac{\delta f_2}{\delta x_2} dx_2 + \ldots\ldots + \dfrac{\delta f_2}{\delta x_n} dx_n = 0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ df_m = \dfrac{\delta f_m}{\delta x_1} dx_1 + \dfrac{\delta f_m}{\delta x_2} dx_2 + \ldots + \dfrac{\delta f_m}{\delta x_n} dx_n = 0 \end{array}\right\} \ldots(1)$$

In order to avoid total elimination we may make use of undetermined multipliers. Now multiply equations of (1) by $1, \lambda_1, \lambda_2 \ldots\ldots, \lambda_m$ respectively and add then $P_1 dx_1 + P_2 dx_2 + \ldots P_n dx_n = 0 \ldots (2)$

Where $P_r = \dfrac{\delta u}{\delta x_r} + \lambda_1 \dfrac{\delta f_1}{\delta x_r} + \dfrac{\delta f_2}{\delta x_r}\lambda_2 \ldots\ldots + \lambda_m \dfrac{\delta f_m}{\delta x_r}$ (3)

Let us select $\lambda_1, \lambda_2 \ldots \ldots \ldots \lambda_m$ in such a way that they satisfy m linear equations.

If $P_1 = P_2 = P_3 = \ldots \ldots \ldots = P_m = 0 \ldots \ldots (4)$

then the equation ( ) reduces to

$P_{m+1}\, dx_{m+1} + P_{m+2} dx_{m+2} + \ldots \ldots + P_n\, dx_n = 0 \quad \ldots \quad \ldots \quad (5)$

Let $n-m$ quantities $x_{m+1}, x_{m+2} \ldots \ldots x_n$ be independent variables, As $n-m$ quantities.

$dx_{m+1}, dx_{m+2}, \ldots \ldots \ldots dx_n$ are all independent, so their co-efficients must be separately zero. Thus from (5)

$P_{m+1} = P_{m+2} = \ldots \ldots = P_n = 0 \quad \ldots \quad \ldots \quad (6)$

Therefore we get

$$\left.\begin{array}{l} P_1 = P_2 = P_3 \quad \ldots \quad \ldots \quad = P_m = 0 \\ P_{m+1} = P_{m+2} = \quad \ldots \quad \ldots \quad = P_n = 0 \\ \text{along with } f_1 = f_2 = f_3 \quad \ldots \quad = f_n = 0 \end{array}\right\}$$

These constitute a set of $m+n$ equations.

These will determime m multipliers $\lambda_1, \lambda_2 \ldots \ldots \lambda_m$, the value of u and a point where u is maximum or minimum.

Art 24(a) The significance of the Lagranges multiplier

Let m be the maximum (or minimum) value of $f(x,y)$ subject to the contraint $g(x,y) = k$ As the value of k, of the constraint function values, so does the corresponding optimal value m of the function f. The Lagrange multiplier λ is the rate of change of m with respect to k. That is $\lambda = dm/dk$

Art.11.25 If $F(x, y, z)$ be subject to constraint $G(x, y, z)=0$, prove that a necessary condition that $F(x, y, z)$ have an extreme value is $F_x G_y - F_y G_x = 0$

As $G(x, y, z)=0$, so we may consider z as function of x and y, $z=f(x, y)$;

Now $F(x, y, z)$ is equivalent to $F[x, y, f(x, y)]$

A necessary condition that $F[x, y, f(x, y)]$ have an extreme value is that partial derivatives with respect to x and y be zero. Thus

$F_x + F_z Z_x = 0$ ... (1), $F_y + F_z Z_y = 0$ ... (2)

Again $G(x, y, z)=0$, so

$G_x + G_z Z_x = 0$ ... (3) ; $G_y + G_z Z_y = 0$ ... (4)

Now eliminate Z_x from (1) and (3),

$F_x G_z - F_z G_x = 0$ ... ... (5)

Eliminate Z_y from (2) and (4), $F_y G_z - F_z G_y = 0$ ... (6)

Eliminate G_z and F_z from (5) then

$\begin{vmatrix} F_x & -G_x \\ F_y & -G_y \end{vmatrix} = 0$ or, $F_x G_y - F_y G_x = 0$ D. H. 1987

Alternative Method

Let $\phi = F + \lambda G$ λ is a constant

The necessary conditions for extreme value of ϕ are

$\phi_x = 0,\ \phi_y = 0$ i. e.

$F_x + \lambda G_x = 0$, $F_y + \lambda G_y = 0$. Eliminate λ, then the condition is

$F_x G_y - F_y G_x = 0$

Ex. 1. Find the maximum and minimum values of

$2x^3 - 9x^2 + 12x - 3$ C. U. 1989

Let $f(x) = 2x^3 - 9x^2 + 12x - 3$ ... ... (1)

$\therefore$ $f'(x) = 6x^2 - 18x + 12$ ... ... (2)

As $f(x)$ is maximum or minimum $f'(x)=0$ i. e.,

$6x^2 - 18x + 12 = 0$

or, $x^2 - 3x + 2 = 0$ or, $(x-1)(x-2) = 0 \Rightarrow x = 1, 2$

Again $f''(x) = 12x - 18$

when $x=1$, $f''(x) = 12 - 18 = -ve$

when $x=2$, $f''(x) = 24 - 18 = 6 = +ve$.

Hence $f(x)$ will be maximum when $x=1$ and the maximum value is $f(1) = 2.1 - 9.1 + 12.1 - 3 = 2$

$f(x)$ will be minimum when $x=2$ and the minimum value is $f(2) = 2.8 - 9.4 + 12.2 - 3 = 1$

Ex. 2. Find the maximum and minimum values of

$x^3 - 3x^2 + 3x + 1$ R. U. 1966

Let $f(x) = x^3 - 3x^2 + 3x + 1$ ... ... (1)

$\therefore$ $f'(x) = 3x^2 - 6x + 3$ ... ... (2)

The necessary condition for the existence of maxima or minima is $f'(x)=0$ or, $3x^2 - 6x + 3 = 0$ or, $x^2 - 2x + 1 = 0$

or, $x=1$ Again $f''(x) = 6x - 6$

when $x=1$, $f''(x) = 0$ Again $f'''(x) = 6$

Thus $f(x)$ has no maximum or minimum.

Alternative method ;—

Let $y=x^3-3x^2+2x+1$

$\therefore \quad \frac{dy}{dx}=3x^2-6x+3=3(x^2-2x+1)=3(x-1)^2$

which is always positive can never change sign.

Hence y has neither a maximum nor a minimum.

Ex. 3. Discuss the maximum and minimum values of the function $f(x)=x^5-5x^4+5x-1$ R. U. 1987, D.U. 1983

$f(x)=x^5-5x^4+5x^3-1$ ... ... (1)

$\therefore \quad f'(x)=5x^4-20x^3+15x^2$ ... ... (2)

For the existence of maximum or minimum ; $f'(x)=0$

or, $5x^4-20x^3+15x^2=0$

or, $x^2(x^2-4x+3)=0$ or, $x^2(x-1)(x-3)=0 \therefore x=0, 1, 3$

Therefore $f(x)$ may have exterme values for $x=0, 1, 3$ only.

Now we are to investigate for the maximum or minimum at $x=0$.

$f'(x)=5x^4-20x^3+15x^2=5x^2(x-1)(x-3)$ ... ... (3)

sign of $f'(x)$

$$\begin{array}{c} + + + + + + 0 + + + + 0 - - - - - 0 + + + \to 0 \\ \hline 0 \qquad\qquad 1 \qquad\qquad 3 \end{array}$$

Now for $x<0, f'(x)=(+)(-)(-)=+ve$ i. e. $x<0, f'(x)>0$.

for $0<x<1$ i. e : $f'(x)=(+)(-)(-)=+ve$

Thus $f'(x)$ does not change sign when passes through $x=0$.

Therefore $f(x)$ has no maximum or minimum at $x=0$

At $x=1$, from (3), if $x<1, f'(x)=(+)(-)(-)=+ve$

if $1<x<3$ i. e. for $x>1$ but $x<3$, $f'(x)=(+)(+)(-1)=-ve$

Thus $f'(x)$ changes sign from $+ve$ to $-ve$ when passes through $x=1$

Hence $f(x)$ is maximum at $x=1$ and the maximum value of $f(x)$ is $f(1)=1-5+5-1=0$ from (1) At $x=3, f'(x)=0$;

if $1<x<3$, $f'(x)=(+)(+)(-1)=-ve$,

if $x>3$, $f'(x)=(+)(+1)(+)=+ve$.

Thus $f'(x)$ changes sign from $-ve$ to $+ve$ when passes through $x=3$

Hence $f'(x)$ is minimum for $x=3$ and the minimum value is $f(3)=-28$ [from (1)]

Ex. 4. Find the maximum or the minimum value of y when $y^7=(x-3)^4$ ... ... (1)

Let $f(x)=y=(x-3)^{4/7}$ $\quad\therefore\quad f'(x)=4/7 \; \frac{1}{(x-3)^{3/7}}$ ...(2)

Here $f'(x)=0$ means $4/7=0$ which is absurd

If $dy/dx=f'(x)=\infty$, then $x-3=0$ or, $x=3$

We are to investigate maximum or minimum for $x=3$

If $x<3$ i. e., $x=3-h$ where $h\to 0$, then from (2)

$$f'(x)=\frac{4}{7}\frac{1}{(3-h-3)^{3/7}}=\frac{4}{7}\left(-\frac{1}{h}\right)^{3/7}=-ve$$

If $x>3$ i. e $x=3+h$ $h\to 0$

$$f'(x)=\frac{4}{7}\frac{1}{(3+h-3)^{3/7}}=\left(\frac{1}{h}\right)^{3/7}=+ve$$

Thus dy/dx or, $f'(x)$ changes sign from $-ve$ to $+ve$ when passes through $x=3$. Hence $f(x)$ is minimum for $x=3$.

Ex.5. Find the maximum or the minimum value of $y=f(x)$, if

$$f'(x)=\frac{2^{1/x}+1}{1-2^{1/x}}$$

If $x<0$ i, e , $x=0-h, h\to 0$, then

$$f'(x)=\lim_{h\to 0}\frac{2^{-1/h}+1}{1-2^{-1/h}}=\frac{2^{-\infty}+1}{1-2^{-\infty}}=\frac{0+1}{1-0}=1 \text{ i. e., } f'(x)$$

is positive $x=0+h, h\to 0^+$ then

$$f'(x)=\lim_{h\to 0}\frac{2^{1/h}+1}{1-2^{1/h}}=\lim_{h\to 0}\frac{2^{1/h}(1+2^{-1/h})}{2^{1/h}(2^{-1/h}-1)}=\frac{1+0}{0-1}=-1$$

i. e., $f'(x)$ is negative.

L. H. Limit $\neq$ *R. H.* Limit. Here $f'(x)$ or, dy/dx is discontinuous at $x=0$ but dy/dx changes from $+ve$ to $-ve$ when passes through $x=0$. Hence $f'(x)$ or, y is maximum at $x=0$.

Ex. 6. Show that $e^x+e^{-x}-x^2$ has a minimum value for $x=0$.

Let $f(x)=e^x+e^{-x}-x^2$ ... ... (1)

$\therefore\ f'(x)=e^x-e^{-x}-2x$ ... ... (2)

Putting $f'(x)=0$ we have $e^x-e^{-x}-2x=0$ ... (3)

or. $\left(1+x+\dfrac{x^2}{\lfloor 2}+\dots\dots\right)-\left(1-x+\dfrac{x^2}{\lfloor 2}-\dots\right)-2x=0$

Thus we see that $x=0$ is a solution of (3)

Again $f''(x)=e^x+e^{-x}-2,\ f'''(x)=e^x-e^{-x},\ f^{iv}(x)=e^x+e^{-x}$

When $x=0$ $f''(x)=1+1-2=0,\ f'''(x)=0,\ f^{iv}(x)=2=$ positive. Thus we se that $f^{iv}(x)$ is the first derivative which does not vanish and $f^{iv}(x)$ is positive *i. e.*, $f(x)$ is minimum at $x=0$

Ex. 7. Determine the maximum and minimum value of the function $\sin x+\cos 2x$; $0<x<2\pi$

Let $f(x)=\sin x+\cos 2x$ ... ... (1)

$\therefore\ f'(x)=\cos x-2\sin 2x=\cos x(1-4\sin x)$

Put $f'(x)=0$. then $\cos x(1-4\sin x)+0\cos x=0$ or, $\sin x=\frac{1}{4}$

Now $\cos x=0=\cos \pi/2=\cos 3\pi/2$ $\quad x=\pi/2,\ 3\pi/2$

When $\sin x=\frac{1}{4}=\sin\alpha$ (say) then $x=\alpha,\ \pi-\alpha,\ \alpha=\sin^{-1}\frac{1}{4}$

Again $f''(x)=-\sin x-4\cos 2x$

when $x=\pi/2$, $f''(x)=-\sin\pi/2-4\cos 2\,\pi/2=3$

When $x=\pi/2$! $f''(x)=-\sin 3\pi/2-4\cos 2.\,3\pi/2=5$

Thus $f(x)$ is minimum for $x=\pi/2$ and $x=3\pi/2$.

From (1), values are $f(\frac{1}{2}\pi)=0,\ f(3\pi/2)=-2$

Again for $x=\alpha$, or, $\sin\alpha=\frac{1}{4}$.

$f''(x)=-\sin x-4\cos x2x=-\frac{1}{4}-4(1+2.\ 1/16)=-15/4$

for $x=\pi-\alpha=\pi-\sin^{-1}\frac{1}{4}$

$f''(x)=-\sin(\pi-\sin^{-1}\frac{1}{4})-4\cos(2\pi-2\sin^{-1}\frac{1}{4})]$

$=-\sin^{-1}\frac{1}{4}-4\cos(2\sin^{-1}\frac{1}{4})=-\frac{1}{4}-4[1-2\sin^2(\sin^{-1}\frac{1}{4})]$

$=-\frac{1}{4}-4(1-2.\ 1/16)=-\frac{1}{4}-7/2=-15/4$

Thus $f'(x)$ is maximum for $x=\sin^{-1}\frac{1}{4}$ and $x=\pi-\sin^{-1}\frac{1}{4}$

And from (1) the values are

$f(\sin^{-1}\frac{1}{4})=\sin(\sin^{-1}\frac{1}{4})+1-2\sin^2(\sin^{-1}\frac{1}{4})=9/8.$

$f(\pi-\sin^{-1}\frac{1}{4})=\sin(\pi-\sin^{-1}\frac{1}{4})+1-2\sin^2(\pi-\sin^{-1}\frac{1}{4})=\frac{1}{4}+1-\frac{1}{8}$

$=9/8.$

Ex. 8. Show that the maximum value of $(1/x)^x$ is $e^{1/e}$

Let $y=(1/x)^x$ ... ... (1)

or, $\log y=x\log(1/x)=-x\log x$

$\therefore\ \dfrac{1}{y}\dfrac{dy}{dx}=-x\times 1/x-\log x=-1-\log x$

$\therefore\ \dfrac{dy}{dx}=y(-1-\log x)$ ... — (2)

For the maximum or minimum value of y, $dy/dx=0$

or, $y(-1-\log x)=0$. or, $\log x=-1$ as $y\neq 0$, $x=e^{-1}$

Again $\dfrac{d^2y}{dx^2}=\dfrac{dy}{dx}(-1-\log x)-(y/x)$

$=(1/x^x)(-1-\log x)^2-(1/x)^x\,1/x$ by (1) and (2) when $x=e^{-1}$, then $d^2y/dx^2=e^{1/e}\{(-1-\log e^{-1})^2-e\}=-e^{1/e}e$,

which is negative.

Hence y is maximum for $x=e^{-1}$, the maximum value is

$y=\left(\dfrac{1}{e^{-1}}\right)^{e^{-1}}=e^{e^{-1}}=e^{1/e}$ Proved

Ex. 9. Find the maximum and minimum values of

$x^2+2y^2-4x+4y-3$ R. U, 1965

Let $\phi(x,y)=x^2+2y^2-4x+4y-3$

$\phi_x=2x-4$; $\phi_y=4y+4$; $r=\phi_{xx}=2$

$t=\phi_{yy}=4$; $s=\phi_{xy}=0$

$\phi(x,y)$ may have critical values if $\phi_x=0$ and $\phi_y=0$

or, $2x-4=0$ and $4y+4=0$ $\therefore$ $x=2$ and $y=-1$

Thus $\phi(x,y)$ may have a critical value at $(2,-1)$

Now $rt-s^2=2.4-0=8=+ve$ and r and t are both positive

Hence $\phi(x,y)$ has a minimum value at $(2,-1)$

Ex. 10. Show that the maximum values of $xy(a-x-y)$ is $1/27a^3$

Let $\phi(x,y)=xy(a-x-y)=axy-x^2y-xy^2$ ... ... (1)

$\therefore$ $\phi_x=ay-3xy-y^2$, $\phi_y=ax-x^2-2xy$

$\phi(x,y)$ is maximum. or, minimum if

$\phi_x=0$ and $\phi_y=0$ i.e. ; $ay-2xy-y^2=0$, $ax-x^2-2xy=0$

Solve for x and y and the points are $(0,0)$, $(0,a)$, $(a,0)$, $(\frac{1}{3}a, \frac{1}{3}a)$

Now we are to investigate the nature of the critical points at these points

Again $r=\phi_{xx}=-2y$, $s=\phi_{xy}=a-2x-2y$, $t=-2x$

For $(0, 0)$

$rt-s^2=(-2y)(-2x)-(a-2x-2y)^2=0-a^2=-a^2=-ve$

There is no maximum or, minimum value of $\phi(x,y)$ at $(0,0)$

For $(0,a)$ $rt-s^2=0-(a-2a)^2=-ve$

No critical value of $\phi(x,y)$ at $(a,0)$

For $(\frac{1}{3}a, \frac{1}{3}a)$ $rt-s^2=(-\frac{2}{3}a)(-\frac{2}{3}a)$ $(a-\frac{2}{3}a-\frac{2}{3}a)^2=\frac{1}{3}a^2$

Thus $rt-s^2$ is positive and r and t both negative

Hence $\phi(x,y)$ is maximum at $(\frac{1}{3}a, \frac{1}{3}a)$ and the maximum value is $\phi(\frac{1}{3}a, \frac{1}{3}a)=1/27a^3$ from (1)

Ex. 11. Show that the function ϕ, where

$\phi(x,y,z)=x^2+y^2+z^2+x-2z-xy$ has a minimum value at $(-2/3, -1/3, 1)$

Let $\phi=x^2+y^2+z^2+x-2z-xy$

$\therefore$ $\phi x=2x+1-y$, $\phi y=2y-x$, $\phi z=2z-2$

$\phi(x,y,z)$ will be maximum or minimum if $\phi_x=0$, $\phi_y=0$, $\phi_z=0$

or $2x-y+1=0$, $-x+2y=0$, $2z-2=0$

Solve for x, y and z, then $x=-2/3, y=-\frac{1}{3}$, $z=1$

We are to investigate the nature of critical point at $(-2/3, -1/3, 1)$ if there is any.

Again $A=\phi_{xx}=2$, $B=\phi_{yy}=2$, $C=\phi_{zz}=2$,

$F=\phi_{yz}=0$, $G=\phi_{zx}=0$, $H=\phi_{xy}=-1$

Hence from Art. 11.23

$$A=2. \begin{vmatrix} A & H \\ H & B \end{vmatrix}=\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}=3 \text{ and}$$

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}=\begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix}=6$$

As all them are positive, $\phi(x,y,z)$ is a minimum at $(-2/3,-1/3,1)$

The minimum value is $\phi(-\frac{2}{3},-\frac{1}{3}, 1)=-4/3$

Ex. 12. Show that the minimum value of C. H, 1977

$x^2+y^2+z^2$ when $ax+by+cz=p$ is $p^2/(a^2+b^2+c^2)$

Let $\phi=x^2+y^2+z^2$ (1)

and $ax+by+cz=p$ or, $z=(p-ax-by)/c$ (2)

Now $\phi=x^2+y^2+(p-ax-by)^2/c^2$ [by (2)]

$\therefore$ $\phi_x=2x-(2a/c^2)(p-ax-by)$, $\phi_y=2y-(2b/c^2)(p-ax-by)$

ϕ may be maximum or minimum if $\phi_x=0$ and $\phi_y=0$

or, $2x-(2a/c^2)(p-ax-by)=0$ and $2y-(2b/c^2)(p-ax-by)=0$

or, $x(c^2+a^2)-ap+aby=0$ ... ... (3)

and $y(c^2+b^2)-bp+abx=0$ ... ... (4)

solving these

$x=ap/(a^2+b^2+c^2),\quad y=bp/(c^2+b^2+a^2)$ (5)

Again $r=\phi_{xx}=2+2a^3/c^2$, $s=\phi_{xy}=2ab/c^2$, $t=\phi_{yy}=2+2b^2/c^2$

and $rt-s^2=4(1+a^2/c^2+b^2/c^2)=+ve$

Also r and t are both positive. Hence ϕ is minimum.

For values of x and y given by (5).

$z=\{p-a^2p/(a^2+b^2+c^2)-b^2p/(a^2+b^2+c^2)\}/c=cp/(a^2+b^2+c^2)$

Now from (1),

$$\phi=\frac{a^2p^2+b^2p^2+c^2p^2}{(a^2+b^2+c^2)^2}=\frac{p^2(a^2+b^2+c^2)}{(a^2+b^2+c^2)^2}=p^2/(a^2+b^2+c^2)$$

The minimum value of ϕ is $p^2/(a^2+b^2+c^2)$.

Ex. 13. Find the minimum value of $x^2+y^2+z^2$ with conditions $ax+by+cz=1$, $a_1x+b_1y+c_1z=1$ and interpret the result geometrically. R, H, 1966, D, H. 1961.

Let $u=x^2+y^2+z^2$ ... ... (1)

$ax+by+cz=1$ ... ... (2)

$a_1x+b_1y+c_1z=1$ ... ... (3)

$\therefore\ du=2xdx+2ydy+2zdz=0$ ... (4)

$adx+bdy+cdz=0$ ... ... (5)

$a_1dx+b_1dy+c_1dz=0$ ... ... (6)

Taking $(4)\times\frac{1}{2}+(5)\times\lambda+(6)\times\lambda_1$.

$(x+a\lambda+a_1\lambda_1)\,dx+(b+b\lambda+b_1\lambda_1)dy+(z+c\lambda+c_1\lambda_1)dz=0$

or, $\left.\begin{aligned}x+a\lambda+a_1\lambda_1=0\\ y+b\lambda+b_1\lambda_1=0\\ z+c\lambda+c_1\lambda_1=0\end{aligned}\right\}$ ... ... (7)

Now multiply equations of (7) by x, y, z respectively and then add

$x^2+y^2+z^2+\lambda(ax+by+cz)+\lambda_1(a_1x+b_1+c_1z)=0$

or, $u+\lambda+\lambda_1=0$ ... ... (8)

Again multiply the equations of (7) by a, b, c, respectively and then add

$(ax+by+cz)+(a^2+b^2+c^2)\lambda+(aa_1+bb_1+cc_1)\lambda_1=0$

or, $1+\lambda\Sigma a^2+\lambda\ \Sigma aa_1=0$ ... ... (9)

Again multiply the equations of (7) by $a, b_1\ c_1$ respectively and add :

$a_1x+b_1y+c_1z+(aa_1+bb_1+cc_1)\lambda+(a^2+b^2-c^2)\lambda_1=0$

or, $1+\lambda\Sigma aa_1+\lambda_1\Sigma a^2=0$ ... ... (10)

Thus $u+\lambda+\lambda_1=0$, $1+\lambda\Sigma a^2+\lambda_1\Sigma aa_1=0$, $1+\lambda\Sigma aa_1+\lambda_1\Sigma a^2=0$

Eliminate λ and λ_1 from the above equations.

$$\begin{vmatrix} u & 1 & 1\\ 1 & \Sigma a^2 & \Sigma aa_1\\ 1 & \Sigma aa_1 & \Sigma a^2\end{vmatrix}=0$$

which gives the maximum or minimum value of u.

λ, λ_1, can be obtained by solving (9) and (10) and then x, y, z can be found from (7)

Ex. 14, Divide a number a into three parts such that their product shall be maximum.

Let a be the number. Let $x=$first part, $y=$2nd part and $z=a-x-y=$third part. Their product is

$f(x,y)=xy(a-x-y)=axy-x^2y-xy^2$

$\delta f/\delta x=ay-2xy-y^2$ and $\delta f/\delta y=ax-x^2-2xy$

$f(x, y)$ will be maximum or minimum if $\delta f/\delta x=0$ and $\delta f/\delta y=0$

or, $ay-2xy-y^2=0$ xnd $ax-x^2-2yx=0$

or, $a-2x-y=0$ and $a-x-2y=0$

Solve for x and y. They are $x=\frac{1}{3}a$ ard $y=\frac{1}{3}a$

Again $\delta^2 f/\delta x^2=r=2y$, $\delta^2 f/\delta y^2=t=-2x$,

and $\delta^2 f/\delta x\delta y=s=a-2x-2y$ at $x=\frac{1}{3}a$, $y=\frac{1}{3}a$

$r=-2a/3$, $t=-2a/3$, $s=-a/3$

$\therefore$ $rt-s^2=(-2a/3)(-2a/3)-a^2/9=a^2/3=+ve$ and r and t are negative. Hence maximum if $x=y=z=\frac{1}{3}a$.

This is the same problem as Ex10

Ex. 15. Find the shape of a quart-can open at the top which requires for its construction the least amount of tin.

Let r = radius of the base h = depth of the cone.

Area of the base and curved surface $=\pi r^2+2\pi rh$ ... ... (1)

Let v be the volume of the cone

$\therefore$ $v=\pi r^2h$, v is constant or, $h=v/\pi r^2$ ... ... (2)

$\therefore$ $A=\pi r^2+2\pi rv/\pi r^2=\pi r^2+2v/r$... ... ... ($x$)

$\delta A/\delta r=2\pi r-2v/r^2$, area is to be minimum.

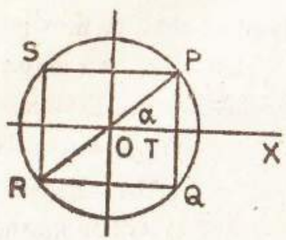
$\delta A/\delta r=0$ i, e., $2\pi r=(2v/r^2)=0$ or. $r=(v/\pi)^1/^3$

$\therefore$ $\delta^2A/\delta r^2=2\pi+4v/r^2$

$\therefore$ $\delta^2A/\delta r^2=2\pi+4\pi=+ve$ for $r(v/\pi)^1/^3$

Hence Area is minimum when $r=(v/\pi)^1/^3$ and $h=(v/\pi)^1/^3$,

Ex. 16. Prove that of all rec angles that can be inscribed in a given circle the square has the greatest area. Also show that the square will have the maximum perimeter as well. D. U. 1967, 83



Let $P(a\cos\theta, a\sin\theta)$ be any point on the circle $x^2+y^2=a^2$ ... (1)

It is evidened that

$PQ=2PT=2OP\sin\alpha=2a\sin\alpha$

$QR=2a\cos\alpha$

Now area $PQRS=2a\sin\alpha\ 2a\cos\alpha$

$=2a^2\sin 2\alpha$

Area will be maximum if $\sin 2\alpha$ is maximum *i, e.,*

$\sin 2\alpha=1=\sin\frac{1}{2}\pi$ or. $\alpha=\frac{1}{4}\pi$

In this case

$PQ=2a\sin\alpha=2a\sin\frac{1}{4}\pi=2a/\sqrt{2}=\sqrt{2}a$

also $QR=2a\cos\alpha=2a/\sqrt{2}=\sqrt{2}a$. Hence $PQ=QR$,

Thus the rectangle PQRS is a square

again perimeter$=C=2(PQ+QR)=2\{2a\sin\alpha+2a\cos\alpha\}$

$=4a(\sin\alpha+\cos\alpha)=4a\{\sin\alpha+\sin(\pi/2-\alpha)\}$

$=4a\ .\ 2\sin\left\{\frac{\pi/2-\alpha+\alpha}{2}\right\}\cos\left\{\frac{\pi/2-\alpha-\alpha}{2}\right\}$

$\therefore$ $C=8a\sin\pi/4\cos(\pi/4-\alpha)$ $\therefore$ Value of C (Perimeter) would be greatest if $\cos(\pi/4-\alpha)$ be greatest. i, e $\cos(\pi/4-\alpha)=1=\cos 0$

$\therefore$ $\pi/4-\alpha=0$ or ; $\alpha=\frac{1}{4}\pi$

$\therefore$ $PQ=2a\sin\alpha=2a\sin\pi/4=2a\ .\ 1/\sqrt{2}=\sqrt{2}a$

$PR=2a\cos\alpha=2a\cos\pi/4=2a\ .\ 1/\sqrt{2}=\sqrt{2}a$ $\therefore$ PQ=PR i.e. when a rectangle becomes a square, it will have the maximum perimeter.

Ex. 17. If a triangle has a given base and if the sum of the other two sides be given, prove that the area is greatest when these two sides ar. equal. R. U. 1954

Let ABC be the triangle, BC=base of the triangle$=a$ (given)

Let $AB=x$, $CA=y$ where $x+y$=constant$=k$ (say). We know

$2S$=Perimeter of the triangle$=AB+BC+CA$

$=a+x+y=a+k$= constant, *i. e.* S constant.

Area of the ABC.

$\Delta = \sqrt{\{s(s-a)(s-x)(s-y)\}} = \sqrt{\{m(s-x)(s-y)\}}$

where $m = s(s-a) =$ constant.

Thus Δ will be greatest if the product $(s-x)(s-y)$ is greatest and $(s-x)(s-y)$ will be greatest if factors are equal

i. e ; $s-x = s-y$ or, $x = y$. i. e ; $AB = CA$.

Hence area of Δ ABC will be greatest if $AB = CA$.

Ex. 18. Show that the volume of greatest cylinder which can be inscribed in a cone of height h and semivertical angle α is $4/27 \pi h^3 \tan^2 \alpha$.

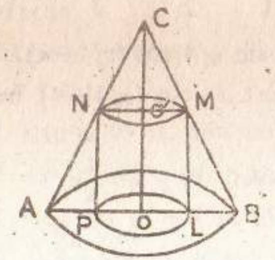
Let $LMNP$ be the inscribed cylinder of the cone ABC

Let

height of the cone $= h$

radius of the cone $= a$

radius of the cylinder $= x$



height of the cylinder $= ML = LB \cot \alpha = (OB - OL) \cot \alpha$

$= (a-x) \cot \alpha$ where a and α are constants.

Let $V =$ volume of the cylinder.

$V = \pi . OL^2 . LM = \pi^2 x(a-x) \cot \alpha = (\pi \cot \alpha)(ax^2 - x^3)$.

$dV/dx = (\pi \cot \alpha)(2ax - 3x^2)$. If V is to be maximum or minimum

$dV/dx = 0$ i. e. $2ax - 3x^2 = 0$ or, $x = 0$, $2a/3$

Again $(d^2V/dx^2) = (x \cot \alpha)(2a - 6x)$: When $x = 2a/3$

(rejecting $x = 0$,) $d^2V/dx^2 = (\pi \cot \alpha)(2a - 6.2\ a.3) = -ve$

Hence V is maximum when $x = 2a/3$

Then cylinder of greatest volume.

$V = \pi / x^2(a-x) \cot \alpha = \pi(4a^2/9)(a - 2a/3) \cot \alpha = 4/27\ \pi a^2 \cot \alpha$

is inscrived in the cone. But $a = h \tan \alpha$.

$$\therefore\ V = \frac{4}{27} \cdot \pi (h \tan \alpha)^3 \frac{1}{\tan \alpha} = \frac{4}{27} \pi h^3 \tan^2 \alpha \text{(Proved)}$$

Ex. 19. Prove that of all rectangalar parallelopipeds of the same volume the cube has the least surface.

Let x, y, z be the length, breadth and height of the cuboid.

Let V be the volume and S, the surface of the cuboid.

Then

$V = xyz$ ... ... (1) ; $S = 2xy + 2yz + 2zx$ ... ... (2)

$\therefore$ $dS = 2(y+z)dx + 2(z+x)dy + 2(x+y)dz$ ... ... (3)

$dV = 0 = yzdx + zxdy + xydz$ ... ... (4)

as the volume V is constant

For the surface S is to be maximum or minimum if $dS = 0$,

or, $(y+z)dx + (z+x)dy + (x+y)dz = 0$ ... (5)

Also $yzdx + zxdy + xydz = 0$ ... ... (6) from (4)

Now multiply (5) $\times$ 1. and (6) $\times \lambda$ and add.

$(y + z + \lambda yz)dx + (z + x + \lambda xz)dy + (x + y + \lambda xy)dz = 0$

Equate the co-efficients of dx, dy, dz to zero.

$\therefore$ $y + z + \lambda yx = 0$ ... (7), $z + x + \lambda zx = 0$ ... ... (8)

$x + y + \lambda xy = 0$ ... (2)

or, $1/z + 1/y = 1/x + 1/z = 1/y + 1/x = -\lambda$

From two equations we have $\therefore$ $1/y = 1/x$ or. $x = y$

Similarly $y = z$;

Now from (1) $x=y=z=V^{1/3}$ ... ... (10)

Let us assume that x and y are independent of each other.

Then $S=2xy+2yz+2zx$

$\therefore$ $(\delta S/\delta x)=2y+2y(\delta z/\delta x)+2z+2x(\delta z/\delta x)\cdots$ ... (11)

Now from (1), differentiating w. r. to x

$yz+xy(\delta z/\delta x)=0$ or. $(\delta z/\delta x)=-z/x$

$\therefore$ $(\delta S/\delta x)=2y-2y(z/x)+2z-2zx/x=2y-2yz/x$ ... (12)

Again $r=(\delta^2 s/\delta x^2)=2yz/x^2-2y/x\ (\delta z/\delta x)=2yz/x^2-2y/x(-z/x)$

$=(2yz/x^2)+(2yz/x^2)=4yz/x^2=4$ at $x=y=z$

Similarly $t=\delta^2 S/\delta x^2=4$ at $x=y=z$

$s=(\delta^2 S/\delta y\delta x)=2-2z/x-(2y/x)(\delta z/\delta x)=2-2z/x-2y/x(-z/x)$

$=2-2+2=2$ at $x=y=z$

Hence S is least when $x=y=z$.

Ex. 20. The cost of fuel for running a train is proportional to the square of the speed generated in kms. per hour and costs Tk. 48 per/hr. at 16 kms/hr. What is the most economical speed if the fixed charges are Tk. 300/hr.

[একটি রেলগাড়ীতে তেলের খরচ পড়ে ঘণ্টায় গতির বর্গের সমানুপাতিক ; ঘণ্টায় গতি ১৬ কিমি হইলে খরচ পড়ে ৪৮ টাকা ঘণ্টায়। কত কম বেগে গাড়িটি চলিলে ঘণ্টায় খরচ ৩০০ টাকা পড়িবে]

Sol : Let the speed of the train be v km/hr. and the distance travelled d kms. The time$=d/v$ hrs. The given cost of fuel according to the law$=kv^2$, where k is a constant. When cost$=48$; $v=16$, then

$48=k(16)^2$ or. $k=3/16$.

Hence the cost of fuel$=\frac{3}{16}v$ rupees per hour

For d kms, the cost $=\frac{3}{16}v^2 .d=\frac{3}{16}v^2 .\ \frac{d}{v}=\frac{3}{16}vd.$

Also the cost of fixed charges$=300/$ per hour$=300\times\frac{d}{v}$

If p be the total cost of running per km in Taka, then

$p=(3/16)dv+300\ (d/v)$

$\therefore\ \frac{dp}{dv}=\frac{3}{16}d-\frac{300d}{v^2},\quad \frac{d^2p}{dv^2}=\frac{600d}{v^3}$

For extremum value, $dp/dv=0$

or, $\frac{3}{16}d-\frac{300d}{v^2}=0$ or, $v^2=16\times100$ or, $v=40$ km/hr

$\frac{d^2p}{dv^2}=\frac{600}{40}d=+ve.$

Hence p is minimum for $v=40$km/hr. for most economical speed.

Ex. 21. Show that $f(x,y)=4+x^2-y^2$ has a saddle point at (0, 0,4)

Let $z=f(x, y)=4+x^2-y^2$

$f_x=2x.\ f_{xx}=2,\ f_{xy}=0,\ f_y=-2y,\ f_{yy}=-2$

Critical points are $f_x=0\ f_y=0$ $\therefore$ $x=0\ y=0$

$D(x,y)=f_{xx}\ f_{yy}-f_{xy}^2=rt-s^2=2(-2)-0=-4<0$

So, no critical point at (0,0) we know if

$D(x\ y)<0$ i. e., $rt<s^2$, then the point $\{a, b, f(a,b)\}$ is a saddle point.

$z=f(x,0)=4+x^2$, $z=f(0,y)=4-y^2$, $z=(0, 0)=4$

The saddle point is at $\{0,0,\ f(0,0)\}=(0,0,4)$

Ex. 22 Find the maximum and minimum values of the function $f(x,y)=xy$ subject to the condition $x^2+y^2=8$

Ans. [$f(x,y)=xy$ এর গুরুমান ও লঘুমান নির্ণয় কর যদি $x^2+y^2=8$ হয়]

শর্ত $g(x,y)=8=x^2+y^2,\ f(x,y)=xy$

$\therefore$ $f_x=y,\ f_y=x,\ g_x=2x,\ g_y=2y$

Lagrang's এর সমীকরণগুলি

$\frac{\delta f}{\delta x} = y = 2\lambda x, \quad \frac{\delta f}{\delta y} = x = 2\lambda y$ এবং $x^2 + y^2 = 8$

$\therefore \quad 2\lambda = y/x, \ x \neq 0 \ ; \ 2\lambda = x/y, \ y \neq 0$

$\therefore \quad y/x = x/y \qquad x^2 = y^2$

Lagrange এর সমীকরণগুলির অস্তিত্ব থাকতে হলে, $x \neq 0$, অতএব $2x^2 = 8$ or, $x^2 = 4$ or. $x = \pm 2$

যদি $x = 2$, তখন $y = 2, -2$

যদি $x = -2, y = 2, -2$

$\therefore \quad f(2, 2) = 4, f(2, -2) = 4, f(2, -2) = -4, f(-2, 2) = -4$

$f(x,y)$ এর মান সর্বোচ্চ হবে এবং ইহা 4 ; (2, 2) এবং (−2, −2) বিন্দুতে এবং −4 হবে (2, −2), (−2, 2) বিন্দুতে. লঘুমান হইবে।

Ex. 23. A publisher has been alloted Tk. 60,000/- to spend on the improvement and development of a new Calculus Text. He estimates that if he spends x thousand taka on improvement and y thousand taka for development,, approximately $20\,x^{3/2}y$ copies of the books will be sold. How much money should the Publisher allocate to improvement and how much to development in order to maximize sales ?

If the allotment is Tk. 61000/-, what is the effect of the sale of the maximum number of books for the addition of Tk. 1000 /-

(একজন প্রকাশককে একটি নূতন Calculus বই এর মানোন্নয়ন এবং উন্নতির জন্য Tk. 60,000/- দেওয়া হল। তিনি মানোন্নয়নের জন্য x হাজার টাকা এবং উন্নয়নের জন্য y হাজার টাকা বরাদ্দ করিলে, $20\,x^{3/2}y$ কপি বই বিক্রয় হয়। মানোন্নয়ন ও উন্নতির জন্য বরাদ্দের টাকার পরিমাণ কত ?

যদি বরাদ্দ Tk. 61,000/- হয়, এই বর্দ্ধিত Tk. 1000/- জন্য সর্বোচ্চ কত কপি বিক্রয়কে প্রভাবান্বিত করিবে)

Sol : $g(x,y) = 60, \ g(x,y) = x + y, \ M = 20x^{3/2}y$. The Lagranges equations are

$\frac{\delta M}{\delta x} = 30\sqrt{x}\,y = \lambda \ ; \quad \frac{\delta M}{\delta y} = 20\,x^{3/2} = \lambda$

and $x + y = 60$

Solving, $30\sqrt{x}\,y = 20\,x^{3/2}$ or, $x = \frac{3}{2}y$.

Put it in $x + y = 60$ or, $3/2y + y = 60$ or, $y = 24$ and the $x = 36$

Hence to maximize the sales, the publisher should spend Tk. 36,000/- for inprovement and Tk. 24000/- for development.

$M = f(36, 24) = 20x^{3/2}y = 20\,(36)^{3/2}\ 24 = 103680.$

The value of λ is from

$\lambda = \frac{\delta M}{\delta y} = 20\,x^{3/2} = 20\,(36)^{3/2}$, when $x = 36$

$= 20 \times 216 =$ Tk. 4320 copies

$\lambda = dM/dk$, it follows that the unit increase in k from $k = x + y = 60$ to $k = 61$ will increase the maximal sales M of the book by approximately 4320 copies.

Exercise XI

1. Find the maximum and minimum values of the following experessions.

(i) $x^2 - 3x^3 + 22x^3 - 24x + 1$ — C. U. 1980

(ii) $2x^3 - 21x^2 + 36x - 20$ — (a) $5x^6 - 18x^5 + 15x^4 - 10$. N. U. 1994

(iii) $x^6 - 5x^4 + 5x^2 - 1$ — R. U. 1987

(iv) $x^3 - 3x$ — (a) $x^4 - 4x^3 + 10$ — D. U. 1991

(v) $x^3 - 3x^2 - 93$ — C. U. 1983

(vi) $2x^3 - 6x^2 - 18x + 7$ — C. U. 1984

(vii) $x^6 - 12x^6 - 36x^4 + 4 = 4$ — C. U. 1986

2. Show that $y = x + 1/x$ has a maximum and a minimum values and that the latter is greater than the former. D. U. 1961

3. Show that the curve $y = xe^x$ has a minimum ordinate where $x = -1$ R. U. 1962 '82

3. (a) If $f(x) = |x|$ Show that $f(o)$ is minimum although $f'(o)$ does not exist.

Sol : $f(x) = x, \quad x>0$
$\quad = 0 \quad x = 0$
$\quad = -x \quad x<0$

$\lim_{x\to 0+} f(x) = \lim_{h\to 0} h = 0, \ \lim_{x\to 0-} f(x) = \lim_{h\to 0} -h = 0, \ f(o) = 0$

Then $f(a+h) = f(a-h) = f(0) = 0$

Hence $f(0)$ is continuous.

$$f'(x) = \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^+} \frac{(0+h)-0}{h} = 1$$

$$f'(x) = \lim_{h\to 0^-} \frac{f(x-h)-f(x)}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$$

L.H Lim $\neq$ R.H Lim

Hince $f'(0)$ does not exist.

4. (a) Find the maximum and minimum value of $\sin x \cos^2 x$

(b) $\dfrac{x^4}{(x-1)(x-3)^3}$ R. U.. 1978

5. Find the maximum and minimum value of the function.

$t^3 = 3m^2t + 2n$ when m and n are real constants. D.U. 1966

6. Find the maximum and minimum values of $\cos^4 x - \sin^4 x$.

7. Find the minimum value of $\dfrac{a^2}{\sin^2 x} + \dfrac{b^2}{\cos^2 x}$

8. Find the minimum value of $y = \frac{1}{2}a(e^{x/a} + e^{-x/a})$

R. U. 1958

9. Determine the values of x which will make $\dfrac{a^2}{x} + \dfrac{b^2}{a-x}$ a maximum and minimum

10. Show that $4\cos x + \cos 2x$ is maximum or minimum whenever $\cos x$ is maximum or minimum.

11. Prove that $x^{1/x}$ is a maximum when $x = e$

D. U. 1984 , C. U. 1988

12. Show that $\dfrac{x^2 - 7x + 6}{x-1}$ has a maximum value when $x = 4$ and a minimum value $x = 16$.

13. Prove that $\sin x(1+\cos x)$ has a maximum for $x = \frac{1}{3}\pi$

R. U. 1983.

(i) In what intervals is the function $f(x) = 17 - 15x + 9x^2 - x^3$ increasing and in what intervals decreasing ?

Also find the relative maximum and maximum values of the function. Sketch the graph of f.

(উল্লেখিত ফাংশানের কোন ব্যবধিতে হ্রাস এবং কোন ব্যবধিতে বৃদ্ধি পায় নির্ণয় কর। ফাংশানটির আপেক্ষিক সর্বোচ্চ মান ও আপেক্ষিক সর্বনিম্ন মানও নির্ণয় কর। লেখচিত্র অংকন কর) D. U. 1987

Ans. $(-\infty, 1) \cup (5, \infty)$; $x = 1$, the greatest Value is 42, least is 10 at $x = 5$

13 (ii) Find in what intervals the function $f(x) = x^2 - 2x + 1$, $-1 \leqslant x \leqslant 4$ is decreasing and in what intervals the function is increasing. Find the maximum and minimum values $f(x)$

13 (iii) Find the intervals where the function $f(x) = x^4 + 2x^3 - 4x + 4$ is increasing and decreasing. Also find the relative maximum and minimum values of the function and hence sktch its graph

$(fx) = x^4 + 2x^3 - 3x^2 - 4x + 4$ ফাংশনটির কোন্ ব্যবধিতে বর্ধমান এবং কোন্ ব্যবধিতে হ্রাসমান নির্ণয় কর। ইহার আপেক্ষিক গরিষ্ঠ ও লঘিষ্ঠমান নির্ণয় কর এবং চিত্র অঙ্কন কর)

C. U. 1992

14. Show that $(\log x/x)$ is a maximum for $x=e$ and the value is $1/e$.

15. Show that x^x is a minimum for $x=1/e$ and the value is $(1/e)^{1/e}$ D. U. 1984

16. Find the maximum and minimum values of

(i) $\sin 2x - x$ (ii) $\sin^n x \sin nx$, n being a +ve integer.

17. Show that x^3-3x^2+3x+3 has neither a maximum nor a minimum value.

(i) Examine maxima and minima $f(x)=\frac{1}{5}x^5-\frac{1}{4}x^4$ R. U. 1988

18. Show that $(3-x)e^{2x}-4xe^x-x$ has no maximum or minimum value for $x=0$.

19. Find the extreme values of

(i) $a^{x+1}-a^x-x$, when $a>1$. (ii) $4x-8x \log 2$.

20. Show that the following function has neither a maximum nor a minimum value $x-\sin x$.

21. Test the following function for maxima and minima.

(i) $x^2-3xy+y^2+13x-12y+13$ (ii) $x^2-y^2+2x-4y-2$

(iii) $x^3+3xy^2-3x^2-3y^2+4$ (iv) $xy+(4/x)+(2/y)$

(v) $x^3y^2(1-x-y)$ (vi) $\sin x+\sin y+\sin(x+y)$

(vii) x^2+xy+y^2-6x+2 (viii) $9y^2+6xy+4x^2-24y-8x+4=0$

(ix) $x^2y^2-5x^2-8xy-5y^2=u$ C. H. 1985

(x) $f(x, y)=xy+8/x+8/y$. Also determine there nature.

D. H. 1986 Ans. $f(2,2)=12$

(xi) Show that $f(x,y)=4+x^2-y^2$ has a saddle point at $(0,0,4)$

22. Find the maximum and minimum values of the following functions.

(i) $\sin 2x-\sin 2x \cos 2x$ (ii) $4\cos x+4\sin^2 x-2$

(iii) $(3-x)\{\sqrt{(1+x^2)}-x\}$

23. Show that y is a minimum at $x=2$ of the function

$$y^3=(x-2)^2$$

24. Find the maximum or minimum value of y, when $y^3=(x-3)^4$

*26. Find the maximum or minimum value of y if

$$\frac{dy}{dx}=\frac{3^{1/x}-2}{1+3^{1/x}}$$

*26. Show that y is max at $x=0$, point of inflexion for $x=1$ and minimum at $x=2$ of the function $dy/dx=x(x-1)^2(x-2)^3$

27. Show that the point $x=a$ is a maximum point of the $f(x)=b-\sqrt[3]{(x-a)^2}$ D. H. 1960

28. Show that points of inflexion of the curve $y^2=(x-a)^2(x-b)$ lie on the line $3x+a=4b$

29. A cubic function of x has a maximum value equal to 15 when $x=-3$, and a minimum value -17 when $x=1$ Find the function. D. U. 1963

(a) Show that the function $y=x+1/x$ has precisely one local maximum and a local minimum and that the latter is greater than the former. Give a rough graph of the function.

*30. Show that the curve $y=2+(1+\sin x)\cos x$ has a maximum, a minimum and four points of inflexions.

*31. Show that $e^x + e^{-x} + 2\cos x$ has a minimum value at $x=0$

*32. Show that $\tan^m x \tan^n(a-x)$ is a maximum when

$$\tan(a-2x)=\frac{n-m}{n+m}\tan \alpha$$

*33 Show that $4x^2+\cos 2x-\frac{1}{2}(e^{2x}+e^{-2x})$ has a maximum value for $x=3$

34 Find the maximum, minimum points and the points of inflexion of $y=2x^3-6x^2+18x+7$ and show that point of inflexion lies between maximum and minimum points.

* 5 Show that the function $f(x)=(2x+3)(x+4)(x-2)(x-1)^3$ change signs from $+ve$ to $-ve$ as x passes through -4 and 1. and from $-ve$ to $+ve$ as x passes through $-3/2$ and 2,

36. Find the critical points of the following curves

(i) $f(x)=x-x^2-x^3$

(ii) $y^3=(x-a)^2$ (iii) $f'(x)=2e^{\frac{-1}{x-a}}-1$.

37. Show that if $f(x)=(x^2+3x+2)^{2/5}+x^{2/5}$, $f'(x)=\infty$ gives a minimum for $x=-2$ $x=-1$ and $x=0$; and $f(x)=0$ gives two intermediate maxima

*38. Examine the function $z=3axy-x^3-y^3$ for maxima and minima

*39 Show that the maximum value of $u=\sin A \sin B \sin C$ is when $A=B=C=\pi/3$

*40. Find the maximum and minimum values of $ax+by$ when $xy=c^2$.

*41. If $z=a^2\{x+b^2/y\}$ where $x+y$ show that z has a minimum value when $x=a^2/(a+b)$ and a maximum value when $x=a^2/(a-b)$.

*42 Find the maximum and minimum values of u of the following curves,

(i) $u=x^3+2y^2+3x^2-2xy-2yz-2=0$

(ii) $u=2a^2xy-3ax^2y-ay^3+x^3y+xy^3$.

(iii) $u=\dfrac{xyz}{(a+x)(x+y)(y+z)(z+b)}$

(iv) $u=2xyz-4zx-2yz+x^2+y^2+z^2-2x-4y+4z$

(v) $u=x^2y^2-5x^2-8xy-5y^2$

(vi) $u=axy^2z^3-z^3x^2y^2-xy^3z^3-xy^3z^4$

(vii) Show that $u=2xyz+x^3+y^2+z^2$ min. at (0,0,0)

(viii) $u = 2x^2 + 3y^2 + 4z^2-3xy + 8z$ mini. at (0, 0, -1)

(ix) $u = x^2-3xy + y^2+13x-12y+13$. D.U. 1989

42. (a) Prove that a necessary condition that $F(x, y, z)$ have an extreme value is that $F_xG_y-F_yG_x=0$ subject to the constraint condition $G(x, y, z)=0$ D. H. 1987

42(b) Use the method of Lagrange multiplyer to find the maximum and minimum values of the function.

(i) $f(x, y)=x^2+2y^2+2x+3$

subject to the condition $x^2+y^2=4$

Ans. Max. $f(1, \sqrt{3})=f(1, -\sqrt{3})=12$ min. $f(-2,0)=3$

(ii) $f(x, y)=8x^2-24xy+y^2$

subject to the condition $x^2+y^2=1$

Ans, max. $f(4/5, -3/5)=f(-4/5, 3/5)=17$, min. $f(3/5, 4/5)=f(-3/5, -4/5)=8$

(iii) $f(x, y)=x^2+2y^2-xy$ subject to the condition $2x+y=2$

Ans. min. $f(9,4)=77$

(iv) $f(x,y)=xy$ subject to the condition $x+y=1$

Ans. max. $f(\frac{1}{2}, \frac{1}{2})=\frac{1}{4}$

*43. If $x^2/a^2+y^2/b^2+z^2/c^2=1$ Show that maximum value of xyz is abc. Interpret the result geometrically. R. H. 1967

43. (a) Show that $(x+y+z)^3-3(x+y+z)-24xyz+a^3$ is min. at (1, 1, 1) and max. at (−1, −1, −1)

*44. Show that maximum and minimum values of

$$u=4x+y+y^2 \text{ where } x^2+y^2+2x+y-1=0$$

are at $(\frac{1}{2},-\frac{1}{2})$ and at $(-5/2,-1/2)$ respectively Is it always possible to express u as a function of x to have the result ?

*45. Show that the minimum value of $u=x+y+z$ when $a/x+b/y+c/z=1$ is $x/\sqrt{a}=y/\sqrt{b}=z/\sqrt{c}=\sqrt{a}+\sqrt{b}+\sqrt{c}$

*46. Show that maximum and minimum values of $u=x^2+y^2+z^2$, when $ax^2+by^2+cz^2=1$. are given y the roots of $(1/a-u)(1/b-u)(1/c-u)=0$.

47. Show that the minimum value of $u=x^4+y^4+z^4$, is $xyz=c^3$ when $x=y=z=c$.

47. (a) Show that $F(x,y,z)=xy^2z^3$ is max and its value is 108.

*48. Find the maximum and minimum value of $x^2+y^2+z^2$ subject to the following conditions $ax^2+by^2+cz^2=1$ and $lx+my+nz=0$ C. H. 1972. D. U. H. 1958.

*49. Show that the maximum and minimum values of $x^2+y^2+z^2$ with the condition C. H. 1988

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1$$

are given by the roots of

$$\begin{vmatrix} a-1/u & h & g \\ h & b-1/u & f \\ g & f & c-1/u \end{vmatrix}$$

*50. Find the minimum value of $u=x^2+y^2+z^2$ when $xy+yz+zx=3a^2$.

*51. Find the maximum and minimum values of xy when $x^2+xy+y^2=a^2$. (ii) $\frac{x}{3}+\frac{y}{3}=1$ N. H. 1995

*52. Show that the maximum and minimum value of x^2+y^2 where $ax^2+2hxy+by^2=1$ are given by the roots of the quadratic $(a-1/r^2)(b-1/r^2)=h^2$

*53. Show that critical values of $u=x^py^qz^r$ when $a/x+b/y+c/z=1$ is maximum and the value is $(p+q+r)^{p+q+r}(a/p)^p(b/q)^q(c/r)^r$

*54. Find the maximum value of $x^2+y^2+z^2$ subject to $6x^2+3y^2+2z=12$, $3x+2y+z=0$

*55. Find the critical value of $u=x^2+y^2+z^2$ when $x+y+z=3a$ 55(a) See Bengali version

*56. Show that the critical values of u^2, when $u^2=a^2x^2+b^2y^2+c^2z^2$, $x^2+y^2+z^2=1$ and $lx+my+nz=0$. are the roots of

$$\frac{l^2}{u^2-a^2}+\frac{m^2}{u^2-b^2}+\frac{n^2}{u^2-b^2}=0. \quad \text{C.U. 1993}$$

57. Show that maximum and minimum values of $u=x^2+y^2+z^2$. if $px+qy+rz=0$ and $x^2/a^2+y^2/b^2+z^2/c^2=1$ are given by the roots of

$$\frac{a^2p^2}{u-a^2}+\frac{b^2q^2}{u-b^2}+\frac{c^2r^2}{u-c^2}=0$$

For 57(a) See Bengali version

*58. If (x_1, y_1, z_1) : (x_2, y_2, z_2) are two points on the curve of intersection of $lx+my+nz=0$ and $ax^2+by^2+cz^2=1$ the distance r between these points is stationary when

$$\frac{l^2}{1-ar^2}+\frac{m^2}{1-br^2}+\frac{n^2}{1-cr^2}=0$$

*59. Show that the maximum and minimum values of $u=x^2/a^4+y^2/b^4+z^2/b^4$, when $lx+my+nz=0$ and $x^2/a^2+y^2/b^2+z^2/c^2=1$ are given by the roots of

$$\frac{l^2a^4}{1-a\ u}+\frac{m^2b^4}{1-b^2u}+\frac{n^2c^4}{1-c^2u}=0$$

59 (a) Find the minimum value of $yz+zx+xy$ if $xyz=a^2(x+y+z)$ Ans. $9a^2$ D. H. '87

(b) Find the points where the value of $F=x^2y^2z^2$ will be maximum $x^2+y^2+z^2=c^2$ and find the greatest value of F.

60. Show that the semivertical angle of the right cone of a given curved surface and maximum volume is $\sin^{-1} 1/\sqrt{3}$.

61. A farmer can afford for buying 8800 ft. of wire fencing. He wishes to enclose a rectangular field of largest possible area. What should be the dimensions of the field ?

*62. Find the fraction which exceeds its second power by the greatest number possible.

63. Show that of all rectangles of a given area, the square has the smallest perimeter.

64. Show that semivertical angle of a cone of maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

65. Given the total surface area $2\pi a^2$ of a right circular cylinder, show that the cylinder of maximum volume is $2\pi a^3/3\sqrt{3}$

66. Find the surface area of the right circular cylinder of great st surface which can be inserted into a sphere of radius r.

67. A ladder is to be taken in a horizontal position round a right angled corner from a passage of width a to a passage of width b Show that the greatest possible length of it is $(a^{2/3}+b^{2/3})^{3/2}$. D. U. 1265

68. Prove that the least perimeter of isosceles triangle in which a circle of radius r can be inscribed is $6r\sqrt{3}$.

69. Show that the maximum right cone inserted into a given sphere is $(32/81)\pi a^3$, a being the radius of the sphere.

70. Find the maximum cylinder that can be inserted into a sphere of radius a. D. U. 1986

71. Show that the radius of the right circular cylinder of greatest curved surface which can be inserted within a given cone is half that of the cone.

72. Determine the cone of minimum volume described about of a given sphere of radius r.

*73. Divide a number into two parts such that square of one part multiplied by the cube of the other should give the greatest possible product.

74. What is the height of a right cone of greatest volume which can be kept within a sphere of radius 'a' ?

75. The sum of the surfaces of a cube and sphere is given, show that when the sum of their volume is least the diameter of the sphere is equal to the edge of the cube.

75 A thin closed rectangular box has one edge n times the length of another edge and the volume of the box is V. Prove that the least surface S is given by $ns^2 = 54(n+1)^2V^2$.

77. P is a point on the ellipse whose centre is C and N the foot of the perpendicular from C upon the tangent to the ellipse at P, find the maximum value of PN.

*78. A variable sphere is described with its centre on the surface of a fixed sphere. Find for what value of the radius the area of its surface intercepted by the fixed sphere is greatest.

*79. Through a given point $P(h, k)$, a straight line is drawn meeting the co-ordinate axes OX and OY in A and B respectively, determine the position of the line in order that $OA+OB$ may be a minimum.

80. If r_1, r_2 be the local distances of a point on an ellipse whose major axis is $2a$, find the maximum and minimum values of $r_1 r_2 (r_1 - r_2)$; $(r_1 > r_2)$ distinguishing between the cases where the eccentricity is greater than or less than $1/\sqrt{3}$ D. H. 1962

*81. What is the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid,

*82. Find the dimensions of the rectangular box open at the top, which has a maximum volume if its surface is 12.

*83. Find a point such that the sum of the squares of the perpendiculars drawn on the sides of a given triangle shall be a minimum.

*84. Show that the points such that the sum of the squares of its distance from n given points shall be minimum is the centre of mean position of the given points.

*85. P is a point in the $\triangle ABC$, PA; PB, PC are joined. If $\angle APB = \angle BPC = \angle CPA = 120^0$, show that the angular distance is maximum

*86. Prove that, of all polygons of a given number of sides circumscribed to a circle the regular polygon is of minimum area, and of all polygones inscribed in a circle, the regular polygon has a maximum area.

87. During the course of on epidemic, the number of people inficted at the time t is given by

$$N = f(t) = kt^{5/2}e^{-t}$$

where t is measured in weeks from start of the epidemic, and k is a constant. When is the maximum number of people infected ? Find also the maximum value of N. R. H. 1988

Ans. $t = 5/2$ weeks, $N = k(5/2)^{5/2}e^{-5/2}$

88. Show that the surface area of all ellipsoid of constant volume is minimum when it a sphere. R. H. 1987

89. A firm sells all the product it makes at Tk. 12.00 per unit. The cost of making x units $c = 20 + 0.6x + 0.01x^2$. Find the maximum profit. D H. 1989

(একটি ফার্ম উৎপাদিত দ্রব্যের প্রত্যেকটির দাম ১২.০০ টাকা দরে বিক্রয় করে। x একক উৎপাদিত ব্যয় $c = 20 + 0.6x + 0.01x^2$ হইলে সর্বোচ্চ মুনাফা নির্ণয় কর।) Ans. Tk, 3229/-

90. A farmer with a field adjacent to a straight river wishes to fence a rectangular area for grazing. If no fence is needed along the river, and has 1600m of fencing, what should be the dimensions of the field in order that it have a maximum area ?

95. The percel post regulation restrict parcels to be such that the length plus the girth ($=2\pi r$, r is the radius of the cylinder) must not exceed 180 cms. Determine the percel of greatest volume that can be sent by post at the form of the percel be a right circular cylinder,

[পোষ্ট অফিসে পার্শেল করিবার নিয়ম পার্শেলের দৈর্ঘ্য এবং পার্শেলের তলের পরিসীমার যোগফল ১৮০ সেমি বেশী হইবে না এবং দৈর্ঘ্য ১০০ সেমি এর বেশী হইবে না। সুষম চোঙা আকৃতি একটি পার্শেল পোষ্টে পাঠাতে ইহার বৃহত্তম ঘন কত হইবে।]

উঃ চোঙের দৈর্ঘ্য 60 cms, $2\pi r = 120$ cm,

96. A farmer can afford for 800 metre of wire fencing. He wishes to enclose a rectangular field to largest possible area. What should the dimensions of the field be? (একজন কৃষক ৮০০ মিটার দীর্ঘ তারের বেড়া দেওয়ার সামর্থ রাখে, সে সম্ভাব্য বৃহত্তম আয়তাকার মাঠকে ঘেরার ইচ্ছা রাখে, মাঠের দৈর্ঘ্য-প্রস্থ কিরূপ হবে) C.H. 1992

97. Show that th ratio of the height of a right circular cone of greatest curved surface which can be inscribed in a given sphere to the radius of the sphere is 4 : 3 (দেখাও যে, একটি গোলকের মধ্যে সুষম বৃত্তাকার কোণকের বৃহত্তম বক্রতলের উচ্চতা এবং গোলকের ব্যাসার্ধের অনুপাত 4 : 3 হইবে)

C.U. 1993

98. Find all the local maximum or minimum values (whichever exists) of the function $f(x) = x^4 + 2x^3 + 3$. Determine the inflexion points and the concavity of the graph of this fuinction and draw a rough sktch of the graph. N.U. 1994

($f(x) = x^4 + 2x^3 + 3$ ফাংশনের স্থানীয় আপেক্ষিক গুরু ও লঘুমান নির্ণয় কর, যাহা বিদ্যমান। ইনফ্লেকশান বা বাক বিন্দুগুলি নির্ণয় কর এবং ফাংশনটির চিত্রের অবতল নির্ণয় করিয়া লেখচিত্রটি অঙ্কন কর।)

অনুশীলনী– XI

1. নিম্নলিখিত রাশিমালাগুলির বৃহত্তম ও ক্ষুদ্রতম মান নির্ণয় কর।

(i) $x^4 - 8x^3 + 22x^2 - 24x + 1$ C. U. 1980

(ii) $2x^3 - 21x^2 + 36x - 20$ C. U. 1979

(iii) $x^5 - 5x^4 + 5x^2 - 1$ ... ... R. U. 1987

(iv) $x^3 - 3x$ (a) $x^4 - 4x^3 + 10$ D. U. 1991

(v) $x^3 - 3x^2 - 93$ C. U. 1983

(vi) $2x^3 - 6x^2 - 18x + 7$ C. U. 1984

(vii) $x^6 - 12x^5 - 36x^4 + 4 = 0$ C. U. 1986

(viii) $5x^6 - 18x^5 + 15x^4 - 10$ C. U. 1991

2. দেখাও যে $y = x + 1/x$ -এর বৃহত্তম চরমমান ও ক্ষুদ্রতম চরমমান আছে। আরো দেখাও যে ক্ষুদ্রতম মান বৃহত্তম মানের চেয়ে বৃহত্তম। D. U. 1961

3. দেখাও যে বক্ররেখা $y = xe^x$ -এর একটি সর্বনিম্ন মানের কোটি (ordinate) আছে যেখানে $x = -1$. R. U. 1961

4. (a) $4 \sin x \cos^2 x$ (b) $\dfrac{x^4}{(x-1)(x-3)^3}$ এর বৃহত্তম ও ক্ষুদ্রতম চরমমান নির্ণয় কর। R. U. 1987

5. ফাংশন $t^3 = 3m^2 t + 2n$, এখানে m ও n বাস্তব এবং ধ্রুব। এই ফাংশনের বৃহত্তম এবং ক্ষুদ্রতম চরমমান নির্ণয় কর। D. U. 1961

6. $\cos^4 x - \sin^4 x$ এর বৃহত্তম ও ক্ষুদ্রতম চরমমান নির্ণয় কর।

7. $y = \frac{1}{2} a(e^{x/a} - e^{-x/a})$ এর ক্ষুদ্রতম চরমমান নির্ণয় কর। R. U. 1953

8. $\dfrac{a^2}{\sin^2 x} + \dfrac{b^2}{\cos^2 x}$ এর ক্ষুদ্রতম চরমমান নির্ণয় কর।

9. x-এর কোন মানের জন্য $\dfrac{a^2}{x} + \dfrac{b^2}{a-x}$ এর বৃহত্তম ও ক্ষুদ্রতম মান পাওয়া যাইবে।

10. দেখাও যে $4\cos x + \cos 2x$ এর মান বৃহত্তম অথবা ক্ষুদ্রতম হয় যখন $\cos x$ এর মান বৃহত্তম বা ক্ষুদ্রতম হইবে।

11. প্রমাণ কর যে যখন $x = e$ হয় তখন $x^{1/x}$ মান বৃহত্তম।

D. U. 1984

12. দেখাও যে $\frac{x^2-7x+6}{x-10}$ এর মান বৃহত্তম হইবে যখন $x=4$ এবং ক্ষুদ্রতম হইবে যখন $x=16$ হয়।

13. প্রমাণ কর যে $x=\pi/3$ এর জন্য $\sin x(1+\cos x)$ এর মান বৃহত্তম হইবে।

R. U. 1983

(i) In what intervals is the function $f(x)=17-15x+9x^2-x^3$ increasing and in what intervals decreasing ?

Also find the relative maximum and maximum values of the function. Sketch the graph of f.

(উল্লেখিত ফাংশানের কোন ব্যবধিতে হ্রাস এবং কোন ব্যবধিতে বৃদ্ধি পায় নির্ণয় কর। ফাংশানটির আপেক্ষিক সর্বোচ্চ মান ও আপেক্ষিক সর্বনিম্ন মানও নির্ণয় কর। লেখচিত্র অংকন কর) D. U. 1987

Ans. $(-\infty,1)\cup(5,\infty)$; $x=1$, the greatest value is 42, least is 10 at $x=5$

13. (iii) Determine the maximum and minimum values of $f(x) = x^3 - 3x + 2$ on the intervals $[-3, 3/2]$ Hence sketch the graph. ($f(x)$ এর জন্য ব্যবধি $[-3, 2/2]$ এর মধ্যে লঘিষ্ঠ ও গরিষ্ঠ মান নির্ণয় কর এবং চিত্রটি অংকন কর। D. U. 1990

(ii) Find in what intervals the function $f(x) = x^2 = x^2 - 2x + 1$, $-1 \le x \le 4$ is decreasing and in what intervals the function is increasing. Find the maximum and minimum values $f(x)$

(ফাংশনটি কোন ব্যবধিতে হ্রাস পায়, কোন ব্যবধিতে বৃদ্ধি পায় নির্ণয় কর। ফাংশনটির গরিষ্ঠমান এবং লঘিষ্ঠমান নির্ণয় কর) D. U. 1986

Ans. $[-1,1)$, $x = 1$, greatest value 4 and 9 and at $x = 4$, least value is zero

14. দেখাও যে $x=e$ এর জন্য $\left(\frac{\log x}{x}\right)$ এর মান ক্ষুদ্রতম এবং ক্ষুদ্রতম মান হইবে $1/e$.

15. দেখাও যে, $x=1/e$ এর জন্য x^x এর মান ক্ষুদ্রতম এবং এই ক্ষুদ্রতম মান হইল $(1/e)^{1/e}$. D. U. 1984

16. (i) $\sin 2x - x$ (ii) $\sin^n x \sin nx$ (এখানে n একটি $+ve$ পূর্ণ সংখ্যা) এর বৃহত্তম এবং ক্ষুদ্রতম মান নির্ণয় কর।

17. দেখাও যে $x^3 - 3x^2 + 3x + 3$ এর বৃহত্তম বা ক্ষুদ্রতম কোন মানই নাই।

(i) Examine maxima and minima $f(x) = \frac{1}{5}x^5 - \frac{1}{4}x^4$

R. U. 1988

18. দেখাও যে, $x = 0$ এর জন্য $(3-x)e^{2x} - 4xe^x - x$ এর কোন বৃহত্তম বা ক্ষুদ্রতম মান নাই।

19. (i) $a^{x+1} - a^x - x$ যখন $a > 1$ (ii) $4^x - 8x\log 2$. ইহাদের চরম মান সমূহ নির্ণয় কর।

20. দেখাও যে; $x - \sin x$ এর বৃহত্তম বা ক্ষুদ্রতম মান নাই।

21. বৃহত্তম এবং ক্ষুদ্রতম মানের জন্য নিম্নলিখিত ফাংশনগুলি পরীক্ষা কর।

(i) $x^2 - 3xy + y^2 + 13x - 12y + 13$ (ii) $x^2 - y^2 + 2x - 4y - 2$

(iii) $x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$ (iv) $xy + (4/x) + (2/y)$

(v) $x^3y^2(1-x-y)$ (vi) $\sin x + \sin y + \sin(x+y)$

(vii) $x^2 + xy + y^2 - 6x + 2$ (viii) $9y^2 + 6xy + 4x^2 - 24y - 8x + 4 = 0$

(ix) $x^2y^2 - 5x^2 - 8xy - 5y^2 = u$ C. U. 1985

(x) $f(x,y)=xy+8/x+8/y$, Also determine theie nature.

D. H. 1986 Ans. $f(2,2)=12$

(xi) Show that $f(x,y)=4+x^2-y^2$ has *a* saddle point at (0, 0, 4)

22. নিম্নলিখিত ফাংশনগুলির বৃহত্তম ও ক্ষুদ্রতম চরমমান নির্ণয় কর।

(i) $\sin 2x - \sin 2x \cos 2x$ (ii) $4\cos x + 4\sin^2 x - 2$

(iii) $(3-x)\{\sqrt{(1+x^2)}-x\}$

23. দেখাও যে $y^3=(x-2)^2$ ফাংশানটির $x=2$ বিন্দুতে y এর মান ক্ষুদ্রতম।

24. y এর বৃহত্তম বা ক্ষুদ্রতম মান বাহির কর যখন $y^3=(x-3)^2$

*25. y-এর বৃহত্তম বা ক্ষুদ্রতম মান নির্ণয় কর যখন

$$\frac{dy}{dx}=\frac{3^{1/x}-2}{1+3^{1/x}}$$

*26. দেখাও যে $dy/dx=x(x-1)^2(x-2)^3$ ফাংশনের $x=0$ বিন্দুতে বৃহত্তম, $x=1$ বিন্দুতে বক্রতার বিন্দু (point of inflexion) এবং $x=2$ বিন্দুতে ক্ষুদ্রতম মান আছে।

*27. দেখাও যে $f(x)=b-\sqrt[3]{(x-a)^2}$ ফাংশানে $x=a$ বিন্দুটি একটি বৃহত্তম চরমমান বিন্দুতে অবস্থিত। D. H. 196

28. দেখাও যে বক্ররেখা $y^2=(x-a)^2(x-b)$ এর বক্রতার বিন্দুগুলি (Point of inflexion) $3x+a=4b$ এই সরলরেখাটির উপর অবস্থিত।

29. x এর একটি ত্রিঘাত সমীকরণের বৃহত্তম মান 15 যখন $x=-3$, এবং ক্ষুদ্রতম মান -17 যখন $x=1$. ফাংশনটি নির্ণয় কর। D. U. 1963

29. (a) $u=x\phi\left(\frac{y}{x}\right)+\phi\left(\frac{y}{x}\right)$ হয়, তবে দেখাও যে

$$x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}=x\phi\left(\frac{y}{x}\right)$$

(b) দেখাও যে ফাংশন $y=x+1/x$ এর নির্দিষ্ট একটি স্থানীয় বৃহত্তম ও স্থানীয় ক্ষুদ্রতম মান আছে। আরো দেখাও যে ক্ষুদ্রতম মানটি বৃহত্তম মান অপেক্ষা বৃহত্তর। ফাংশনটির মোটামুটি একটি লেখচিত্র অংকন কর।

[Show that the function $y=x+1/x$ has precisely one local maximum and a local minimum and that the letter is greater than the former. Give a rough sketch of the graph of the function]

*30. দেখাও যে বক্ররেখা $y=2+(1+\sin x)\cos x$-এর একটি বৃহত্তম ও ক্ষুদ্রতম চরমমান এবং চারটি বক্রতার বিন্দু (four points of inflexions) আছে।

*31. দেখাও যে $x=0$ বিন্দুতে $e^x+e^{-x}+2\cos x$-এর একটি ক্ষুদ্রতম মান আছে।

*32. দেখাও যে যখন $\tan(a-2x)=\frac{n-m}{n+m}\tan\alpha$ হয় তখন

$\tan^m x\tan^n(a-x)$ এর মান বৃহত্তম হইবে।

*33. দেখাও যে $x=3$ হইলে রাশিমালা $4x^2+\cos 2x-\frac{1}{2}(e^{2x}+e^{-2x})$ এর মান বৃহত্তম হইবে।

34. $y=2x^3-6x^2-18x+7$ এর বৃহত্তম, বা ক্ষুদ্রতম এবং বক্রতার বিন্দুসমূহ (Points of inflexion) নির্ণয় কর। দেখাও যে বক্রতার বিন্দু সমূহ বৃহত্তম ও ক্ষুদ্রতম বিন্দুর মাঝখানে অবস্থিত।

*35. দেখাও যে x, -4 এবং 1 বিন্দু দিয়া অতিক্রম করার সময় ফাংশন $f(x)=(2x+3)(x+4)(x-2)(x-1)^3$ এর চিহ্ন $+ve$ হইতে $-ve$ এর পরিবর্তিত হয়। এবং x-এর মান $-3/2$ হইতে 2-এ পরিবর্তিত হওয়ার সময় চিহ্ন $-ve$ হইতে $+ve$ এ পরিবর্তিত হয়।

36. নিম্নলিখিত বক্ররেখাগুলির জন্য সন্ধি বিন্দু (Critical points) নির্ণয় কর।

(i) $f(x)=x-x^2-x^3$ (ii) $y^3=(x-a)^2$, (iii) $f'(x)=2e^{-\frac{1}{x-a}}-1$.

37. দেখাও যে যদি $f(x)=(x^2+3x+2)^{2/5}+x^{2/5}$ এবং $f'(x)=\infty$ হয়, তবে $x=-2$, $x=-1$ এবং $x=0$ মানের জন্য $f(x)$ ক্ষুদ্রতম হইবে। আবার $f(x)=0$ ক্ষুদ্রতম মানগুলির মধ্যবর্তী বৃহত্তম মানগুলি প্রদান করিবে।

*38. বৃহত্তম ও ক্ষুদ্রতম চরমমানের জন্য ফাংশন $z=3axy-x^3-y^3$ পরীক্ষা কর।

39. দেখাও যে ফাংশন $u=\sin A\sin B\sin C$-এর মান বৃহত্তম চরম হইবে যখন $A=B=C=\pi/3$ হইবে।

40. $ax+by$-এর বৃহত্তম এবং ক্ষুদ্রতম চরমমান নির্ণয় কর যখন $xy=c^2$

41. যদি $z=\frac{a^2}{x}+b^2/y$ যেখানে $x+y=a$ হয় তবে দেখাও যে,

$x=a^2/(a+b)$ এর জন্য z এর মান ক্ষুদ্রতম হইবে এবং $x'=a^2/(a-b)$. এর জন্য z বৃহত্তম হইবে।

42. নিম্নলিখিত বক্ররেখাগুলির জন্য u-এর বৃহত্তম এবং ক্ষুদ্রতম চরমমান নির্ণয়কর।

(i) $u=x^3+2y^2+3z^2-2xy-2yz-2=0$

(ii) $u=2a^2xy-3ax^2y-ay^3+x^3y+xy^3$.

(iii) $u=\dfrac{xyz}{(a+x)(x+y)(y+z)(z+b)}$

(iv) $u = 2xyz - 4zx - 2yz + x^2 + y^2 + z^2 - 2x - 4y + 4z.$

(v) $u = x^2y^2 - 5x^2 - 8xy - 5y^2$

(vi) $u = axy^2z^3 - z^3x^2y^2 - xy^3z^3 - xy^3z^4$

(vii) দেখাও যে $(0, 0, 0)$ বিন্দুতে $u = 2xyz + x^2 + y^2 + z^2$ ক্ষুদ্রতম।

(viii) দেখাও যে $(0, 0, -1)$ বিন্দুতে $u = 2x^2 + 3y^2 + 4z^2 - 3xy + 8z$ এর মান ক্ষুদ্রতম

(ix) $x^2 - 3xy + y^2 + 13x - 12y + 13$ D. U. 1989

42. (a) প্রমাণ কর যে $F(x, y, z)$ ফাংশনের সর্বোচ্চমান থাকিতে হইলে $F_xG_y - F_y G_x = 0$ একটি প্রয়োজনীয় শর্ত হইবে এবং বাধ্যতামূলক (constraint) শর্ত হইবে $G(x, y, z) = 0$.

43. যদি $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. হয় তবে দেখাও যে xyz এর বৃহত্তম মান হইবে abc. এই ফলের জ্যামিতিক রূপ দাও। R. U. 1967

43. (a) দেখাও যে $(x + y + z)^3 - 3(x + y + z) - 24xyz + a^3$ এর মান $(1, 1, 1)$ বিন্দুতে ক্ষুদ্রতম এবং $(-1, -1, -1)$ বিন্দুতে বৃহত্তম।

*44 দেখাও যে $u = 4x + y + y^2$ সর্বোচ্চ ও সর্বনিম্ন মান যথাক্রমে $(\frac{1}{2}, -\frac{1}{2})$ এবং $(-5/2, -\frac{1}{2})$ বিন্দুতে যখন $x^2 + y^2 + 2x + y - 1 = 0$. এই ফল পাইতে হইলে u কে x-এর ফাংশনরূপে সর্বদা প্রকাশ করা সম্ভব কি?

*45. দেখাও যে $u = x + y + z$ এর ক্ষুদ্রতম মান $\frac{x}{\sqrt{a}} = \frac{y}{\sqrt{b}} = \frac{z}{\sqrt{c}}$

$= \sqrt{a} + \sqrt{b} + \sqrt{c}$ যখন $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$ হয়।

*46. দেখাও যে $u = x^2 + y^2 + z^2$ এর বৃহত্তম ও ক্ষুদ্রতম মান

$$\left(\frac{1}{a} - u\right)\left(\frac{1}{b} - u\right)\left(\frac{1}{c} - u\right) = 0$$

সমীকরণের মূল হইতে পাওয়া যাইবে যখন $ax^2 + by^2 + cz^2 = 1$ হইবে।

47. দেখাও যে যখন $x = y = z = c$ যখন তখন $u = x^4 + y^4 + z^4$ এর ক্ষুদ্রতম মান হইবে $xyz = c^3$.

47. (a) দেখাও যে $F(x, y, z) = xy^2z^3$ এর কেবল বৃহত্তম মান আছে এবং এই মান হইবে 108.

*48. $ax^2 + by^2 + cz^2 = 1$ এবং $lx + my + nz = 0$

এই শর্তে $x^2 + y^2 + z^2$ এর বৃহত্তম ও ক্ষুদ্রতম মান নির্ণয় কর।

C. H. 1972, D. U. H. 1958.

49. $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1$।

এই শর্তে $\begin{vmatrix} a - \frac{1}{u} & h & g \\ h & b - \frac{1}{u} & f \\ g & f & c - \frac{1}{u} \end{vmatrix} = 0$ এর মূলগুলি হইতে C. H. 1988

$x^2 + y^2 + z^2$ এর বৃহত্তম ও ক্ষুদ্রতম মান পাওয়া যাইবে, ইহা দেখাও।

*50. $u = x^2 + y^2 + z^2$ এর ক্ষুদ্রতম মান নির্ণয় কর

যখন $xy + yz + zx = 3a^2$ হয়।

51. xy এর বৃহত্তম ও ক্ষুদ্রতম মান নির্ণয় কর যখন $x^2 + xy + y^2 = a^2$ হয়।

52. দেখাও যে $(a - 1/r^2)(b - 1/r^2) = h^2$ এর মূলগুলিদ্বারা $x^2 + y^2$ এর বৃহত্তম ও ক্ষুদ্রতম মান সমূহ দেওয়া যায়।

যখন $ax^2 + 2hxy + by^2 = 1$ হয়।

53. দেখাও যে যখন $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$ হয় তখন $u = x^py^qz^r$ এর সন্ধি মানই বৃহত্তম মান হইবে এবং এই বৃহত্তম মান

$$(p + q = r)^{p+q+r} \left(\frac{a}{p}\right)^p \left(\frac{b}{q}\right)^q \left(\frac{c}{r}\right)^r$$

54. $6x^2 + 3y^2 + 2z^2 = 12, 3x + 2y + z = 0$
শর্ত সাপেক্ষে $x^2 + y^2 + z^2$ এর বৃহত্তম মান নির্ণয় কর।

55. যখন $x + y + z = 3a$ হয় তখন $u = x^2 + y^2 + z^2$ এর সন্ধিমান নির্ণয় কর। (critical values)

(a) $2x^2 + 2y^2 + 9a^3 - 6ax - 6ay + 2xy$

56. দেখাও যে যখন $u = a^2x^2 + b^2y^2 + c^2z^2$, $x^2 + y^2 + z^2 = 1$ এবং $lx + my + nz = 0$ হয়, তখন

$$\frac{l^2}{u^2-a^2}+\frac{m^2}{u^2-b^2}+\frac{n^2}{u^2-c^2}=0.$$

এর মূলগুলিই হইবে u^2 এর সন্ধিমান সমূহ। (Critical values)

57. দেখাও যে যদি $px+qy+rz=0$ এবং

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ হয় তবে

$\frac{a^2p^2}{u-a^2}+\frac{b^2q^2}{u-b^2}+\frac{c^2r^2}{u-c^2}=0$ এর মূলগুলি হইবে $u=x^2+y^2+z^2$ বৃহত্তম ও ক্ষুদ্রতম মান সমূহ।

57. (a). find the maximum and minimum values of $x^2+y^2+z^2$ Subject to the conditions $x^2/4+y^2/5+z^2/z=1$ and $z=x+y$

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58. সরলরেখা $lx+my+nz=0$ এবং বক্ররেখা $ax^2+by^2+cz^2=1$ এর মধ্যে ছেদ বিন্দু (x_1, y_1, z_1) x (x_2, y_2, z_2).

এই দুই বিন্দুর মধ্যে দূরত্ব স্থীর হইবে যদি

$$\frac{l^2}{1-ar^2}+\frac{m^2}{1-br^2}+\frac{n^2}{1-cr^2}=0.$$

*59. দেখাও যে যখন $lx+my+nz=0$ এবং $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ হয় তখন $u=\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}$, এর বৃহত্তম ও ক্ষুদ্রতম মান পাওয়া যাইবে

$$\frac{l^2a^4}{1-a^2u}+\frac{m^2b^4}{1-b^2u}+\frac{n^2c^4}{1-c^2u}=0$$ এর মূলগুলি হইতে।

(*a*) Find the minimum value of

$yz+zx+xy$ if $xyz=a^2(x+y+z)$ Ans. $9a^2$ D. H. 1987

(*b*) Find the points where the value of $F=x^2y^2z^2$ will be maximum $x^2+y^2+z^2=c^2$ and find the greatest value of F.

60. দেখাও যে নির্দিষ্ট ক্ষেত্রফল বিশিষ্ট বক্রতল ও বৃহত্তম মানের আয়তন বিশিষ্ট কোন খাড়া বৃত্তাকার কোণকের (right cone) অর্ধ-শির কোণ $=\sin^{-1}(1/\sqrt{3})$ হইবে।

61. একজন কৃষক ৮০০ ফুট তারের বেড়া কিনিতে পারে। সে ঐ বেড়া দিয়া সর্বাধিক ক্ষেত্রফল যুক্ত একখণ্ড জমি ঘিড়িতে মনস্থ করিল। ঐ জমির পরিমান কি হইবে নির্ণয় কর।

62. এমন একটি ভগ্নাংশ নির্ণয় কর যাহা হইতে ইহার বর্গ বিয়োগ করিলে বিয়োগফল বৃহত্তম হইবে।

$$\left[\text{সংকেত } f(x,y)=\frac{y}{x}-\left(\frac{y}{x}\right)^2\right]$$

63. দেখাও যে প্রদত্ত ক্ষেত্রফলের সকল আয়তক্ষেত্রের মধ্যে বর্গক্ষেত্রের পরিসীমাই ক্ষুদ্রতম।

64. দেখাও যে প্রদত্ত হেলানো উন্নতি এবং বৃহত্তম আয়তন বিশিষ্ট কোন কোণের (cone) অর্ধ-শির-কোণ $\tan^{-1}\sqrt{2}$ হইবে।

(Show that semivertical angle of a cone of maximum volume and given slant heigh is $\tan^{-1}\sqrt{2}$.)

65. একটি বেলন বা সমবৃত্তভূমিক সিলিণ্ডারের মোট ক্ষেত্রফল দেওয়া আছে $2\pi a^2$. ইহার বৃহত্তম আয়তন হইবে $2\pi a^3/3\sqrt{3}$. ইহা প্রমাণ কর।

[Given the total surface area $2\pi a^2$ of a right circular cylinder, show that the maximum volume of such a cylinder is $2\pi a^3/3\sqrt{3}$.]

66. 'r' ব্যাসার্ধ বিশিষ্ট একটি গোলকের মধ্যে রক্ষিত সর্বাপেক্ষা বৃহৎ ক্ষেত্রফল বিশিষ্ট সমবৃত্তভূমিক সিলিণ্ডারের l পৃষ্ঠের মোট ক্ষেত্রফলের পরিমাণ নির্ণয় কর।

67. 'a' প্রস্থ বিশিষ্ট একটি রাস্তা, 'b' প্রস্থ বিশিষ্ট অপর একটি রাস্তার সহিত সমকোণে সংযুক্ত। একটি মইকে আনুভূমিকভাবে ঐ সংযোগস্থল পার করাইতে হইবে। দেখাও যে ঐ মইের সর্ব্বাবৃহৎ দৈর্ঘ্য $(a^{2/3}+b^{2/3})^{3/2}$ এর বেশী হইবে না।

68. r ব্যাসার্ধ বিশিষ্ট একটি বৃত্ত একটি সমদ্বিবাহু ত্রিভুজের মধ্যে অঙ্কিত হইলে, দেখাও যে ত্রিভুজের ন্যূনতম পরিসীমা হইবে $6r/\sqrt{3}$।

69. দেখাও যে a ব্যাসার্ধ বিশিষ্ট একটি গোলকের মধ্যে অস্থিত বৃহত্তম সমবৃত্তভূমিক কোণের আয়তন হইবে $\frac{32}{81}\pi a^3$.

70. a ব্যাসার্ধ বিশিষ্ট একটি গোলকের মধ্যে রক্ষিত সর্ববৃহৎ সিলিণ্ডারের আয়তন কত ?

71. দেখাও যে প্রদত্ত কোণের (cone) মধ্যে রক্ষিত সর্বাপেক্ষা বৃহত্তম বক্রতল বিশিষ্ট সম বৃত্তভূমিক সিলিণ্ডারের ব্যাসার্ধ প্রদত্ত কোণের ব্যাসার্ধের অর্দ্ধেক হইবে। D. U. 1986

72. একটি কোণ (cone) দ্বারা, 'r' ব্যাসার্ধ বিশিষ্ট একটি গোলককে (sphere) পরিবেষ্টিত করিতে ন্যূনতম কোণের আয়তন নির্ণয় কর।

*73. একটি প্রদত্ত সংখ্যাকে এমন দুই অংশে বিভক্ত কর যেন প্রথম অংশের বর্গের সহিত ২য় অংশের ঘণফল গুণ করিলে গুণফল সর্ব্ববৃহৎ হয়।

73. (a) 100 কে এমন দুই অংশে বিভক্ত কর যেন ইহাদের গুণফল বৃহত্তম হয়।

74. 'a' ব্যাসার্ধের একটি গোলকের মধ্যে রক্ষিত বৃহত্তম সমবৃত্ত ভূমির কোণের উচ্চতা কত হইবে ?

75. একটি ঘনক এবং একটি গোলকের পৃষ্ঠদ্বয়ের ক্ষেত্রফলের যোগফল দেওয়া আছে। দেখাও যে যখন ইহাদের আয়তনের যোগফল ন্যূনতম হয় তখন গোলকের ব্যাসার্ধ ঘনকের ধারের সমান হইবে।

76. একটি পাতলা আয়তাকার বন্ধ বাক্সের একটি ধার অপর ধারের দৈর্ঘ্যের n গুণ এবং আয়তন V, দেখাও যে ঐ বাক্সের পৃষ্ঠের ন্যূনতম ক্ষেত্রফল s পাওয়া যাইবে $ns^3=54(n+1)^2V^2$ সমীকরণ হইতে।

77. C কেন্দ্র বিশিষ্ট কোন উপবৃত্তের উপর P একটি বিন্দু এবং P বিন্দুতে অংকিত স্পর্শকের উপর C হইতে অংকিত লম্বের পাদ বিন্দু N হইলে PN এর ক্ষুদ্রতম মান নির্ণয় কর।

*78. স্থির একটি গোলকের পৃষ্ঠের যে কোন বিন্দুকে কেন্দ্র করিয়া একটি সম্প্রসারণশীল গোলক নেওয়া হইল। ব্যাসার্ধ কত হইলে স্থির গোলক পৃষ্ঠদ্বারা কর্তিত পৃষ্ঠের ক্ষেত্রফল বৃহত্তম হইবে ?

*79. একটি নির্দিষ্ট বিন্দু $P(h, k)$ এর মধ্য দিয়া একটি সরল রেখা এমন-

ভাবে আঁকা হইল যেন উহা অক্ষদ্বয় OX ও OY কে যথাক্রমে A ও B বিন্দুতে ছেদ করে। সরলরেখাটির এরূপ একটি অবস্থান নির্ণয় কর যেখানে $OA+OB$ এর মান ক্ষুদ্রতম হইবে।

*80. একটি উপবৃত্তের (Ellipse) বৃহদাক্ষ $2a$, ইহার উপর কোন বিন্দুর ফোকাস দূরত্ব r_1 ও r_2 ; রাশিমালা $r_1 r_2(r_1-r_2)$ $[r_1>r_2]$ এর বৃহত্তম ও ক্ষুদ্রতম চরমমান নির্ণয় কর। এই মান উৎকেন্দ্রতার (eccentricity) মান $1/\sqrt{3}$ অপেক্ষা বৃহত্তর বা ক্ষুদ্রতর হওয়ার সাথে সাথে কি রকম পরিবর্তিত হয় তাহা দেখাও। D. H. 1962

*81. একটি উপবৃত্তকের (Ellipsoid) মধ্যে অবস্থিত সর্ববৃহৎ আয়তাকার ঘনবস্তুর (rectangular parallelopiped) আয়তন নির্ণয় কর।

*82. একটি আয়তাকার বাক্সের উপরিতল খোলা। যদি ইহার তলসমূহের ক্ষেত্রফলের যোগফল 12 হয় এবং আয়তন বৃহত্তম হয় তবে ইহার দৈর্ঘ্য, প্রস্থ ও উচ্চতা নির্ণয় কর।

*83. এমন একটি বিন্দুর অবস্থান নির্ণয় কর যাহা হইতে একটি নির্দিষ্ট ত্রিভুজের বাহুগুলির উপর অংকিত লম্বগুলির বর্গের যোগফল ন্যূনতম হয়।

*84. দেখাও যে, যে বিন্দু হইতে nটি বিন্দুর দূরত্বের বর্গের যোগফল ক্ষুদ্রতম হয়, সে বিন্দুটি প্রদত্ত n বিন্দুগুলির গড় অবস্থানের কেন্দ্র হইবে।

*85. $\triangle ABC$ ত্রিভুজের মধ্যে P যে কোন বিন্দু। PA, PB, PC যুক্ত করা হইল। দেখাও যে কৌণিক দূরত্ব সর্বাধিক হইবে যদি
$\angle APB=\angle BPC=\angle CPA=120°$ হয়।

*86. প্রমাণ কর যে কোন বৃত্তে পরিলিখিত কোন নির্দিষ্ট সংখ্যক বাহুদ্বারা গঠিত বহুভূজগুলির মধ্যে সুষম বহুভূজের ক্ষেত্রফল ক্ষুদ্রতম এবং কোন বৃত্তে অন্তর্লিখিত কোন নির্দিষ্ট সংখ্যক বাহুদ্বারা গঠিত বহুভূজগুলির মধ্যে সুষম বহুভূজের ক্ষেত্রফল বৃহত্তম হইবে।

Exercise XI

Answers

1. (i) max $x=2$ min $x=1$, and 3

(i) max $x=1$, min $x=6$

(iii) max $x=-1$ min $x=1$

(iv) max $x=1$ min $x=3$, none. $x=0$

5. mini. for $=\pi/2$, max for $\tan x=1/\sqrt{2}$ in the Ist quadrant

(i) no. max or mini.

5. max. value $2m^3+2n$ when $t=-m$

min. value $-2m^2+2n$ when $t=m$

6. max. 1, min. 1

7. minimum value is a when $x=0$ 8. $(a+b)^2$

9. max. $\dfrac{a^2}{a+b}=x$, min $x=\dfrac{a^2}{a-b}$

16. (i) max $x=n\pi+\pi/6$, mini $x=n\pi-\pi/6$,

(ii) max. and mini. alternately for $x=\dfrac{k\pi}{n+1}$

starting from $k=1$, omiting even values of n for which $k=0$ or multiples $(n+1)$

19. (i) min value $=\{\log(ae-e)/\log a\}\log a$

(ii) min. at $x=1$, value $=4-8\log 2$.

21. (i) nothing

(ii) no maximum or minimum value at $(-1, 2)$

(iii) max $(0, 0)$, min. $(2, 0)$

(iv) min. at $(2, 1)$ (v) max at $(\frac{1}{2}, \frac{1}{3})$

(iv) $x=y=\pi/3$ max, mini; $x=y=5\pi/3$

(vii) min at $(4, -2)$ (viii) mini. at $(0, 4/3)$.

22. (ii) max $x=\pi/3$

(ii) max at $x=\pi/3$, $5\pi/3$, min at $x=0, \pi, 2\pi$

(iii) max. value $=3$, $x=4/3$ 24. min at $x=3$

26. min at $x=0$ 29. $x^3+3x^2-9x-12$

34. max at $(-1, 17)$, min at $(3, -47)$ inflexion at $(1, -15)$

36. (i) max at $(\frac{1}{3}, 5/27)$, mini at $(-1, -1)$ (i) min, at $x=a$

(iii) $f'(a)$ is discontinuous but $f(a)$ is max at $x=a$

38. mini. at (a, a) nothing at $(0, 0)$

40. mini at $x=\sqrt{(b/a)}$ max. at $x=-c\sqrt{(b/a)}$

42. (i) min at $(2/5, 6/25, 2/25)$, $u=-254/125$,

(ii) min. at $(a/2, a/2)$ max. at $(3a/2, -a/2)$

max at $(a/2, a/2)$. min at $(a/2, -a/2)$

(iii) min value is $(a^{1/4}+b^{1/4})^{-4}$ at (ar, ar^2, ar^3), $r=(b/a)^{1/4}$

(iv) min. at $(1, 2, 0)$ (v) max at $(0, 0)$

(vi) max value $=108a^7/7^7$.

44. This example illustrates how it is not sufficient to express u as a function of x say $2x-x^2+1$

48. $\dfrac{l^2}{au-1}+\dfrac{m^2}{bu-1}+\dfrac{n^2}{cu-1}=0$

50. $3a^2$, at (a, a, a) and at $(-a, -a, -a)$

51. $\frac{1}{3}a^2$, max. at $(a^2/\sqrt{3}, a/\sqrt{3})$, $(-a/\sqrt{3}, -a/\sqrt{3})$.

$-a^2$, min, at $(a, -a)$; $(-a, a)$.

54. $28/5$, 3 55. (i) min at $(a\ a\ a)$ $3a^2$

61. 200, 200. 62. $\frac{1}{2}$. 66. max surface $=\pi r^2(5+\sqrt{5})/\sqrt{5}$

67. $h=\frac{2}{3}a\sqrt{3}$ 71. $h=2r$.

72. v is mini. for $x=3r$, r is the radius of the sphere.

73. $x=2a/5$, a being a number y maxi, 74. $h=4a/3$

79. $a-b$. 78. $r=4a/3$

79. $\dfrac{x}{b+\sqrt{(hk)}}+\dfrac{y}{k+\sqrt{(hk)}}=1$

81. $8abc/3\sqrt{3}$

82. V max at $(2, 2)$ 83. $\dfrac{4\Delta^2}{a^2+b^2+c^2}$

CHAPTER XII

ASYMPTOTES

12. **Definition.** *If a straight line meets a curve into two coincident points at infinity, and yet is not itself wholly at infinity, it is called an asymptote to the curve.*

An asymptote can also be defined as "a straight line whose distance d from the moving point P on the curve to the straight line approaches zero as the point recedes to infinity.

When we say that (Px, y) moves along the curve to infinity, we mean that at least one of the Co-ordinates x and y tends to $+\infty$ or $-\infty$ and this is denoted by $P \to \infty$

12.1 Determination of asymptotes (*not parallel to the axes*)

Let $y = mx + c$ (1)

where m and c are finite, be an asymptote of a curve.

Then $m = \lim\limits_{x \to \infty} \dfrac{y}{x}$ and $c = \lim\limits_{x \to \infty} (y - mx)$

where x, y are the co-ordinates of a moving point P on the curve.

Let d be perpendicular distance between the point $P(x, y)$ of the curve and the straight line $y - mx - c = 0$. Then

$$PM = d = \frac{y - mx - c}{\sqrt{(1 + m^2)}} \qquad (2)$$

$\therefore \lim\limits_{x \to \infty} (y - mx - c) = \lim\limits_{d \to 0} d\sqrt{(1 + m^2)} = 0$; Since m is finite.

Again putting $d\sqrt{(1 + m^2)} = z$. from (2) ; we have

$y - mx - c = z$ or, $y - mx = z + c$ or ; $y/x - m = (z + c)/x$

$\therefore \lim\limits_{x \to \infty} (y/x - m) = \lim\limits_{x \to \infty} (z + c)/x = 0$

Since $z \to 0$ as $x \to \infty$ and c is finite with

$$\text{or } m = \lim_{x \to \infty} \frac{y}{x} \qquad (4)$$

From (3) and (4) we notice that we are to find out $m = \lim\limits_{x \to \infty} (y/x)$ first from the equation ; m may have more than one value. For each value of m, we get a corresponding value of $c = \lim\limits_{x \to \infty} (y - mx)$

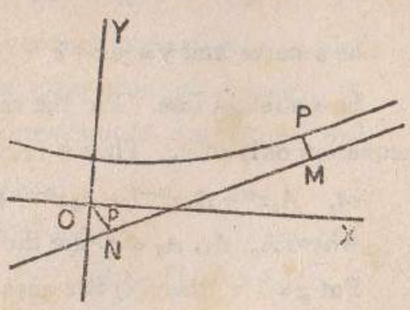
If c is finite, then $y = mx + c$ will be an asymptote of the curve.

This method will determine all asymptotes of the curve not parallel to the co-ordinate axes. Seperate methods will be discussed in Art. 124 for parallel asymptotes

Alternative Definition

An Asymptote is defined as a tangent whose points of contact are at infinity and again it is not itself wholly at infinity.

Let the equation of the tangent at (x, y) of the curve $y = f(x)$ be



Fig—12

$$Y - y = \frac{dy}{dx}(X - x) \text{ or. } Y = \frac{dy}{dx} X + \left(y - x\frac{dy}{dx} \right) \quad (1)$$

Exclude the asymptote parallel to the co-ordinate axes (that is $\frac{dy}{dx}\neq 0, \infty$). The tangent (1) will be asymptote if

$\lim_{x\to\infty} \frac{dy}{dx}=m$ and $\lim_{x\to\infty}\left(y-x\frac{dy}{dx}\right)=c$ are finite. If these conditions are satisfied, then the asymptote is given by **y=mx+c.**

NOTE : The converse is not always true.

12.2. To determine the Asymptotes of an algebraic curve.

In. Art. 12.1. We have discussed definition of asymptotes and also discussed how to determine asymptotes.

In this Article we are giving another method for determining asymptotes.

$$\text{Let } \phi(x, y)=(a_0y^n+a_1y^{n-1}x+\dots\dots+a_nx^n)+(b_0y^{n-1}$$
$$b_1y^{n-2}x+\dots+b_n{}^{-1})+(c_0y^{n-2}+c_0y^{n-3}x+\dots+c_{n-2}x^{n-2})+$$
$$\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad =0 \quad\dots\quad\dots\quad (1)$$

be a curve and $y=mx+c$ (2)

be a straight line. Put the value of y in (1), then we get an equation only in x. Thus $\varphi(x, mx+c)=0$ ... ... (3)

or, $A_0x^n+A_1x^{n-1}+A_2x^{n-2}+ \cdots +A_n=0$ ... (4)

where A_0, A_1, A_2 etc. are the functions of m and c.

Put $x=1/z$, then (4) becomes

$$A_0+A_1z+A_2z^2+\dots+A_{n-1}z^{n-1}+A_nz^n=0$$

If two roots of this equation are zero, then

$$A_0=0 \text{ and } A_1=0$$

But $x=1/z$, when $z\to 0$, x becomes infinite.

Thus we see that we are to select the values of m and c in such a way that two of the roots of x in (4) become infinite. In this case the straight line $y=mx+c$ cuts the curve at two coincident points on the curve whose distances from the origin are infinite.

From $A_0=0$ and $A_1=0$ we will get values of m and c. Putting these values in $y=mx+c$ we will get the equations of asymptote of the curve (1)

12.3. The asymptotes of the general algebraic curve.

(A) Let the equation to the curve be

$$(a_0y^n+a_1y^{n-1}x+a_2y^{n-2}x^2+\dots+a_{n-1}yx^{n-1}+a_nx^n)$$
$$+(b_1y^{n-1}+b_2y^{n-2}x+\dots+b^{n-1}yx^{n-2}+b_nx^{n-1})$$
$$+(c_2y^{n-2}+\dots+c_nx^{n-2})+=0 \quad\dots\quad (1)$$

or ; $P_n+P_{n-1}+P_{n-2}+ \cdots \cdots +P_0=0$ ... (2)

Now we see that P_n is homogeneous expression in x. y of degree n ; P_{n-1}, is a homogeneous expression of degree $n-1$, P_{n-2} is also an homogeneous expression of degree $n-2$ and so on.

$$P_n=a_0y^n+a_1y^{n-1}x+\dots+a_{n-1}yx^{n-1}+a_nx^n$$
$$=x^n\{a_0(y/x)^n+a_1(y/x)^{n-1}+\dots\dots\dots a_{n-1}(y/x)+a_n\}$$
$$=x^n\phi_n(y/x) \quad\dots\quad\dots\quad\dots\quad (3)$$
$$=x^n\phi(m), \text{ where } (y/x)=m \quad\dots\quad (4)$$

$\phi_n(m)$ is a homogenous expression of degree n, and so $\phi_n(m)=0$ has n roots. Let $m_1, m_2 \dots m_n$ be the roots of $\phi_n(m)=0$

Thus $\phi_n(m)=(m-m_1)(m-m_2)\dots(m-m_n)$

$=(y/x-m_1)(y/x-m_2)\dots(y/x-m_n)$

$\therefore\ P_n=x^n\phi_n(m)=(y-m_1x)(y-m_2x)(y-m_3x)\ \dots(y-m_nx)$

The possible asymptotes of the curve are parallel to $y-m_1x=0$

$y-m_2x=0$ ... ... $y-m_nx=0$.

Case 1. If P_n has no repeated linear factor, then the equation (2) can be written as

$P_n+(P_{n-1}+P_{n-2}\ \dots\ \dots\ +P_o)=0$ or, $P_n+F_{n-1}=0$ ... ... (6)

Let $y-m_1x$ be a non-repeated factor of P_n

Then $P_n=(y-m_1x)\ Q_{n-1}$,

where $Q_{n-1}=(y-m_2x)(y-m_3x)\dots(y-m_nx)$ and

F_{n-1} contains all the terms of $(n-1)$ the degree and lower degree.

$\therefore\ (y-m_1x)\ Q_{n-1}+F_{n-1}=0$ ... ... (7)

It there is an asymptote parallel to $y-m_1x=0$ let it be

$$y-m_1x=c_1$$

From the definition of asymptotes

$$c_1=\lim_{x\to\infty}(y-m_1x)=\lim_{x\to\infty}\frac{-F_{n-1}}{Q_{n-1}}$$

Therefore, the asymptote parallel to $y-m_1x=0$ is

$$y-m_1x+\lim_{x\to\infty}\frac{F_{n-1}}{Q_{n-1}}=0 \quad \dots \quad \dots \quad (8)$$

In this way other asymptotes parallel to $y-m_2x=0$.

$y-m_3x=0$ etc can be determined.

Cese II. Let P_n consist of repeated factors, say $(y-m_1x)^2$

Then Eq (7) becomes

$(y-m_1x)^2\ Q_{n-2}+P_{n-1}+(P_{n-2}+\dots+P_n)=0$

or, $(y-m_1x)^2\ Q_{n-2}+P_{n-1}+F_{n-2}=0$ ... ... (9)

$Q_{n-2}=(y-m_3x)(y-m_4x)\dots(y-m_nx)=$ terms of $(n-2)$th degree,

$P_{n-1}=$ terms of $(n-1)$th degree

$F_{n-2}=$ terms of $(n-2)$th degree and lower degree terms.

Asymptotes parallel to $y-m_1x=0$ is given by

$y-m_1x=c_2$, Where c_2 is obtained from (9)

$$c_2=\lim_{x\to\infty}(y-m_1x)^2=-\lim_{x\to\infty}\frac{P_{n-1}+F_{n-2}}{Q_{n-2}}$$

If P_{n-1} does not contain $y-m_1x$ as a factor, then c_2 does not tend to a finite limit, so there will be no asymptote parallel to $y-m_1x=0$,

If P_{n-1} has a factor $y-m_1x$. then eq. (9) becomes

$(y-m_1x)^2\ Q_{n-2}+(y-m_1x)\ R_{n-1}+F_{n-2}=0$

If there are asymptotes parallel to $y-m_1x=0$, they are given by

$$(y-m_1x)^2+(y-m_1x)\lim_{x\to\infty}\frac{R_{n-1}}{Q_{n-2}}+\lim_{x\to\infty}\frac{F_{n-2}}{Q_{n-2}}=0 \ \dots \quad (10)$$

In the same way we can find out asymptotes parallel to the factors of $P_n=0$.

All the asymptotes of curve (1) can be determined by the method shown above.

Cor. If P_{n-1} in eq (2) is absent, then all the asymptotes parallel to the factors of P_n are given by $P_n=0$ if there are repeated factors in P_n;

i, e ; if the eq, (2) is written in the form.

$P_n+(P_{n-2}+P_{n-3}+\dots P_0)=0$

or. $P_n+R_{n-2}=0$ ... ... (10a)

then all the asympototes are given by $P_n=0$ ... ... (1)

Ex. Find the asympotes of $x^4-xy^3+x^2+y^2-a^2=0$

The eq can be written as $(x^4-xy^3)+(x^2+y^2)-a^2=0$

or ; $P_4+(P_2+P_0)=0$ or ; $P_4+R_2=0$

Here P_2 is absent. So all the asymptotes are given by

$P_n=0$ or, $(x^4-xy^3)=0$ or, $x(x-y)(x^2+xy+y^2)=0$ are the asymptotes of given curve.

Working Rule :—

(a) Group the highest homogeneous degree terms in bracket It is P_n. Equate $P_n=0$. The asymptotes of the curve will be parallel to the factors of $P_n=0$.

Let $y=m_1x$ be of the factor of P_n. Then the asynptotes parallel to $y-m_1x=0$ is from (7)

$$y=m_1x+\lim_{\substack{x\to\infty\\ y=m_1x}}\frac{F_{n-1}}{Q_{n-1}}=0$$

Similarly for other asymptotes parallel to the factors of P_n.

(b) If P_n has repeated factors say $(y-m_1x)^2$ then the asymptotes parallel to $y-m_1x=0$ are given by

$$(y-m_1x)^2+(y-m_1x)\lim_{\substack{x\to\infty\\ y=m_1x}}\frac{R_{n-2}}{Q_{n-2}}+\lim_{\substack{x\to\infty\\ y=m_1x}}\frac{F_{n-2}}{Q_{n-2}}=0$$

12.3. (B) Let th eq. (1) ...... (A) be written as

$$P_n+P_{n-1}+P_{n-2}+\dots+P_n=0$$

or, $x^n\phi_n\left(\frac{y}{x}\right)+x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right)+$ ... ... $=0$ ...(12)

where $\phi_n(y/x)$ represent an expession of nth degree in y/x

Let $y=mx+c$ ... ... (13)

be one of the asymptotes of the curve (12)

$\therefore$ $y/x=m+c/x$ ... ... (14)

Put the value of y/x in (12) then

$$x^n\phi_n(m+c/x)+x^{n-1}\phi_{n-1}(m+c/x)+\dots=0 \quad \dots (15)$$

Expand it by Taylors Theorem.

$$x^n\{\phi_n(m)+\frac{c}{x}\phi'_n(m)+\frac{c^2}{\lfloor 2x^2}\phi''(m)+\dots\}$$

$$+x^{n-1}\{\phi_{n-2}(m)+\frac{c}{x}\phi'_{n-1}(m)+\frac{c^2}{\lfloor 2x^2}\phi''_{n-1}(m)+\dots\}$$

$$+x^{n-2}\{\phi_{n-2}(m)+\frac{c}{x}\phi'_{n-2}(m)+\frac{c^2}{\lfloor 2x^2}\phi''_{n-2}(m)+\dots\}+\dots=0$$

or, $x^n\phi_n(m)+x^{n-1}\{c\phi'_n(m)+\phi_{n-1}(m)\}+$

$$x^{n-2}\left\{\frac{c^2}{\lfloor 2}\phi''_n(m)+c\phi'_{n-1}(m)+\phi_{n-2}(m)\right\}+\dots=0 \quad \dots(16)$$

Since $y=mx+c$ is an asymptote of the curve (12) therefore two roots of the equations become infinite for which we are to equate to zero the co-efficients of first two highest degree terms.

Now from (16) we have

$\phi_n(m)=0$ ... ... (17)

$c\phi'_n(m)+\phi_{n-1}(m)=0$ ... (18)

Equation $\phi_n(m)=0$ is of nth degree in m, so it gives us roots say, $m_1, m_2 \dots m_n$. From (18) we get the values c corresponding to the values of m say m_1, m_2 etc.

Thus the asymptotes are $y=m_1x+c_1$, $y=m_2x+c_2$ etc.

Cor, *If* $\phi_n(m)=0$ *has two equal roots.* From (16) we have

$$c_1=-\phi_{n-1}(m_1)/\phi'_n(m_1)$$

If $m_1=m_2$ in $\phi(m)=0$, then $\phi_n'(m)=0$

Hence $c_1=-\phi_{n-1}(m_1)/\phi_n'(m_1)$ is infinite if $\phi_{n-1}(m_1)\neq 0$,

The straight line $y=m_1x+c_1$ will intersect the y—axis at infinity, so is not an asymptote, through m_1 is a roots of $\phi_n'(m)=0$

In this case if $\phi_n(m)=0$ has two equal roots, then $\phi_n'(m_1)=0$ and $\phi_{n-1}(m_1)=0$. The eq. (18) vanish independently.

In such cases c will be obtained by equation to zero the co-efficient of x^{n-2} in (16).

Thus

$$\frac{c^2}{\lfloor 2}\phi''_n(m)+c\phi'_{n-1}(m)+\phi_{n-2}(m)=0 \quad \cdots \quad \cdots \quad (19)$$

The eq. (17) is a quadratic in c so it will give two values of c for the value of m_1.

Therefore, the asymptotes are $y=m_1x+c_1$, $y=m_1x+c_2$,

Cor. 2. If the equation $\phi_n(m)=0$ has three equal roots then we are to equate to zero the co-efficient of x^{n-2}, of eq. (16)

Thus $\phi_n(m)=0$

and $(c^3/\lfloor 3\phi'''_n(m)+c^2/\lfloor 2\phi''_{n-1}(m)+c\phi'_{n-2}(m)+\phi_{n-3}(m)=0$

There will be three values of c say c_1, c_2, c_3 for m_1. Then the asymptotes are $y=m_1x+c_1$, $y=m_1x+c_2$, $y=m_1x+c_3$.

Cor. 3. For imaginary roots, there will be no asymptotes of the curve.

Working Rule :—

(a) First group the highest homogeneous degree terms in a bracket : which is $\phi_n(y/x)$.

Put $x=1$, $y=m$ Then $\phi_n(m)$ is obtained.

Similarly find $\phi_{n-1}(m)$. Differentiate $\phi_n(m)$ w. r. to m.

Then $\phi'_n(m)$ is obtained

(b) Put $\phi_n(m)=0$. Find the roots of m. Consider only the real roots of m. Put one by one in $c\phi'_n(m)+\phi_{n-1}(m)=0$,

then for each value of m, there will be a value of c.

Put the values of m and c in $y=mx+c$. which will be an asymptote. In this way all the asymptotes can be determined for real values of m.

(c) If $\phi(m)=0$ has two eqal roots, then equate to zero the co-efficient of x^{n-1} i. e. ;

$$\frac{c^2}{\lfloor 2}\phi_n''(m)+c\phi'_{n-1}(m)+\phi_{n-2}(m)=0.$$

Two values of c will be obtained and thus two parallel asymptotes will be determined.

(d) If $\phi_n(m)=0$ has three equal roots, then

$$\frac{c^3}{\lfloor 3}\phi_n'''(m)+\frac{c^2}{\lfloor 2}\phi''_{n-1}(m)+c\phi'_{n-2}(m)+\phi_{n-3}(m)=0$$

Thus there will be three parallel asymptotes.

———

Asymptotes parallel to the co-ordinates axes

12.4. (A) **Asymptotes parallel to the x—axis.**

Let $\phi(x, y) = 0$

be the equation as shown in Art. 12.2

The equation can be arranged in the descending powers of x *i.e.*

$$a_0x^n + (a_1y + b_1)x^{n-1} + (a_2y^2 + b_2y + c_1)x^{n-2} + \quad \ldots \quad = 0 \quad \ldots (1)$$

If $a_0 = 0$ and y be selected in such a way that $a_1y + b_1 = 0$ ...(2)

Then there are two infinite roots of x.

Hence $a_1y + b_1 = 0$ is an asymptote of the curve (1)

$a_1y + b_1 = 0$ or. $y = -b_1/a$; $a_1 \neq 0$, or, $y = k$, (say)

which is a straight line parallel to x-axis.

If $a_0 = 0$, $a_1 = 0$, $b_1 = 0$. then there are three infinite roots of x so $a_2y^2 + b_2y + c_1 = 0$ giving two asymptotes parallel to x—axis if the roots of y are not imaginary.

Hence to determine the asymptotes parallel to the x-axis prcceed as follows ;

Rule : In an algebraic equation of a degree if the highest power term of x (say x^n) is absent, all the asymptotes parallel to the x-axis are obtained by equating to zero the co-efficients to the next available highest power of x (say x^{n-1}) in the equation.

If the co-efficient is constant, then equate to zero the next available higher power.

If the co-efficients give imaginary factors or constant then there will be no asymptotes parallel to x—axis.

Ex. Determine the asymptote of the curve.

$$y^3 - yx^2 + y^2x + x^2 - 4 = 0$$

In this equation highest power of x. *i. e.* x^3 is absent. So there may be some asymptotes parallel to x—axis.

Arrange the equation in descending powers of x. Then

$$x^2(1 - y) + xy^2 + y^3 - 4 = 0 \quad \ldots \quad \ldots \quad (1)$$

Equate to zero the Co-efficiente of available highest power of x (i. e., x^2). Then the asymptot parallel to x—given by

$$1 - y = 0 \quad \text{or,} \quad y = 1$$

12.4. (B) **Asmptotes parallel to y—axis.**

In an equation of nth degree in x and y, if the highest power of y (i. e., y^n) is absent, then equate the co-efficients of next available highest powers of y (i. e, y^{n-1}) to zero ; all asymptotes parallel to y—axis will be obtained.

Ex. Determine the asymptotes of the curve

$$x^4 - x^2y^2 + x^2 + y^2 - a^2 = 0$$

In this equation the highest powers of y (*i. e.*, y^4) is absent. There may be some asymptotes parallel to y—axis.

Arrange the equation in descending powers of y. Then

$$y^2(1 - x^2) + x^4 + x^2 - a^2 = 0 \quad \ldots \quad \ldots \quad (1)$$

Equate to zero the available highest power of y in the equation (1), Thus $1 - x^2 = 0$ or, $x = \pm 1$

Thus asymptotes parallel to y axis are given by

$$x - 1 = 0 \text{ and } x + 1 = 0$$

12.5. Intersection of a curve with its asymptotes

Let us consider an equation of nth degree. This equation has n asymptotes. If there are no two parallel asymptotes, we can write the equations of this curve and the asymptotes as in Cor, Art. 12.3

$$F_n=0 \quad \ldots \quad \cdots \quad (1)$$

$$F_n+F_{n-2}=0 \quad \cdots \quad (2)$$

If $S_1=0$ and $S_2=0$ be two curves then any other curve is represented by $S_1-\lambda S_2=0$.

If $\lambda=1$, then the curve is $S_1-S_2=0$

If $S_1=F_n+F_{n-2}$ and $S_2=F_n$

then $(F_n+F_{n-2})-F_n=0$ or, $F_{n-2}=0$

represents curve of intersection of the two curves $F_n+F_{n-2}=0$ and $F_{n-2}=0$. Thus we see that all points of intersection of the asymptotes and curve be on a curve $F_{n-2}=0$.

We know a straight line cuts generally a curve of nth degree in n points (real, imaginary). If this line is an asymptote it cuts a curve in two points at infinity. Hence it cuts the curve of nth degree in $(n-2)$ points. As there are n asymptotes for curve of nth degree so all asymptotes cut the curve in $n\ (n-2)$ point and all the points will lie on a curve. $F_{n-2}=0$

The equations of all asymptotes are given by $F_n=0$.

Thus the given curve is obtained by combining the equations $F_n+F_{n-2}=0$.

For example

(1) If the curve is a cubic equation the points of intersection of the asymptotes and the curve will lie on a curve.

$F_{n-2}=0$ or, $F_{3-2}=0$ or, $F_1=0$

which is a first degree equation in x and y. The asymptotes meet the curve at $n(n-2)$ i. e ; $3(3-2)=3$ points.

Hence three points will lie in a straight line.

(ii) If the curve is a 4th degree equation then $n=4$.

The points of intersection of the asymptotes and the curve will lie on a curve.

$F_{n-2}=0$ or, $F_{4-2}=0$ or $F_2=0$ which is an equation degree 2. This will represent a conic section.

Moreover the points of intersection of the asymptotes and the curve are

$n(n-2)=4(4-2)=8$ *i. e.*, 8 points will lie on a curve $F_2=0$ Similarly for curves of higher degree.

12.6. Asymptotes in Polar Co-ordinates.

If α be a root of the eqation $f(\theta)=0$, then

$r \sin(\theta-\alpha)=1/f'(\alpha)$ (1)

is a asymptote of the curve $1/r=f(\theta)$ ·· (2)

If $r\to\infty$, then from the eq. 2) we have $f(\theta)=0\ldots$ (4)

Let the roots of the equation $f(\theta)=0$ be α, β, γ ··

Let $P(r, \theta)$ be any point on a curve near the asymptote CM.

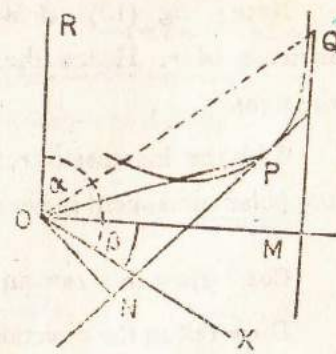


Fig. 13

Draw

$OP=$ the radius vector

$PN=$ tangent at P

NO is perpendicular to $P.N.$

$ON=$ the polar subtangent $=r^2\dfrac{d\theta}{dr}$

If P moves along the curve up to infinity, the tangent PN becomes an asymptote QM and consquently ON becomes perpendicular to QM, ON equals OM, and becomes parallel to QM,

If $p = \lim_{P \to Q} ON = \lim_{r \to \infty} r^2 \frac{d\theta}{dr}$ and $\beta = \alpha - \frac{1}{2}\pi$ where $\lim_{r \to \infty} \theta = \alpha$,

The polar equation of the asymptote is $p = r \sin(\theta - \beta)$

or, $1/f'(\alpha) = r \sin(\theta - \alpha + \frac{1}{2}\pi) = r \sin(\theta - \alpha)$

or, $r \sin(\theta - \alpha) = 1/f'(\alpha)$

Working Rule: Arrange the given equation in the form $1/r = f(\theta)$

Equate $f(\theta) = 0$. Find the roots of θ, say α, β, γ etc. Differentiate $f(\theta)$, put $\theta = \alpha, \beta, \gamma$ etc in $f'(\theta)$ so that $f'(\alpha)$, $f'(\beta)$ etc, are obtained.

Put the values of α and $f'(\alpha)$ to get $r \sin(\theta - \alpha) = 1/f'(\alpha)$ which will give us the polar equation of an asymptote.

Note: fig (13). $dr/d\theta$ is positive as θ increase with the increase of r. Hence the polar subtangent is positive and is to the right.

With the increase of r, θ decreases, so $dr/d\theta$ is negative, so the polar subtangent is negative and is drawn to the left.

Cor. How to draw an asymptote in polar Co-ordinates.

Draw OR in the direction indicated by α

Draw OM parpendicular to OR where $OM = \lim_{r \to \infty} r^2 \frac{d\theta}{dr} = p$

If p positive, draw OM to the right side of OR, and p is negative draw to left side of OR.

Now draw MQ parallel to OR which dedermines the asymptote of the curve.

12.7. Circular Asymptotes.

In many polar equation when θ increases indefinitely and r remains finite, the equation involves only r. Such type of curve posseses circular asymptotes since r is ultimately constant.

In this case the curve represents one or more concentric circles.

Ex. 1. Find the asymptotes of

$$x^3 + 2x^2y - xy^2 - 2y^3 + xy - y^2 - 1 = 0 \quad \cdots \quad \cdots \quad (1)$$

Let $y = mx + c \quad \cdots \quad \cdots \quad (2)$

be one of the asymptotes.

Put the value of y in (1), then

$$x^2 + 2x^2(mx+c) - x(mx+c)^2 - 2(mx+c)^3 + x(mx+c) - (mx+c)^2 - 1 = 0$$

or, $x^3(1 + 2m - m^2 - 2m^3) + x^2(2c - 2mc - 6m^2c + m - m^2) + x(c - c^2 - 6mc^2 + 2mc) + (-2c^3 - c^2 - 1) = 0$,

or, $A_0x^3 + A_1x^2 + A_2x + A_3 = 0$.

Now equate to zero the co-efficients of x^3 and x^2 seperately.

$\therefore$ $A_0 = 0$ or, $1 + 2m - m^2 - 2m^3 = 0$

and $A_1 = 0$ or, $2c - 2mc - 6m^2c + m - m^2 = 0$

The first equation gives

$1 + 2m - m^2 - 2m^3 = 0$ or, $(1 + 2m)(1 + m)(1 - m) = 0$

$\therefore$ $m = 1, -1, -\frac{1}{2}$.

From the 2nd equation $2c-2mc-5m^2c+m-m^2=0$

or, $c=\dfrac{m^2-m}{2-2m-6m^2}$

For, $m=1,\ -1,\ -\frac{1}{2},\ c=0,\ -1,\ \frac{1}{2}$ respectively.

Hence the asymptotes are obtained from (2) by putting the values of m and c.

There are $y=x,\ y=-x-1,$ and $y=-\frac{1}{2}x+\frac{1}{2}$

or: $y=x,\ x+y+1=0$ and $x+2y-1=0$

Ex. 2, Find the asymptotes of D. H. 1987

$$4x^3-x^2y-4xy^2+y^3+3x^2+2xy-y^2-7x+5=0$$

The equation can be written as

$$(4x^3-x^2y-4xy^2+y^3)+(3x^2+2xy-y^2)-7x+5=0$$

or: $x^3\phi_3(y/x)+x^2\phi_2(y/x)-7x+5=0$ ... ... (1)

Let $y=mx+c$ ... ... (2)

be one of the asymptotes of (1)

Then Put $x=1$ and $y=m$ in $\phi_3(y/x)$ and $\phi_2(y/x)$ in (1) we have

$\phi_3(m)=4-m-4m^2+m^3$; $\phi_2(m)=3+2m-m^2$

$\phi'_3(m)=-1-8m+3m^2$

Now $\phi_3(m)=0$ or, $4-m-4m^2+m^3=0$

or, $(4-m)(1+m)(1-m)=0$ or, $m=1,-1,4$

Also $c=-\dfrac{\phi_{n-1}(m)}{\phi'_n(m)}=-\dfrac{\phi_2(m)}{\phi'_3(m)}=-\dfrac{3+2m-m^2}{-1-8m+3m^2}$

For $m=1,-1\ 4,$ the values of $c=\frac{2}{3},\ 0,\frac{1}{3}$ respectively

Hence required asymptotes are, from (2).

$y=x+\frac{2}{3},\ y=-x+0,\ y=4x+\frac{1}{3}$

or, $3y=3x+2,\ y+x=0,\ 3y=12x+1$

Ex. 3. Find the asymptotes of

$x^3-x^2y-xy^2+y^3+2x^2-4y^2+2xy+x+y+1=0$ R. U. 1954

Write the equation as

$(x^3-x^2y-xy^2+y^3)+(2x^2-4y^2+2xy)+(x+y)+1=0$ ... (1)

or, $P_3+P_2+P_1+P_0=0$

Now $P_3=x^3-x^2y-xy+y^3=(y-x)^2(y+x)$

[Put $x=1,\ y=m,\ 1-m-m^2+m^3=(1-m)^2(1+m)$.

Then put $m=y/x$. The factors of P_3 will be obtained].

The asymptotes of the curve (1) are parallel to $y-x=0$.

$y+x=0$

Now the asymptote parallel to $y+x=0$

$$y+x+\lim_{\substack{x\to\infty\\ y=-x}}\frac{2x^2-4y^2+2xy}{(y-x)^2}+\lim_{\substack{x\to\infty\\ y=-x}}\frac{x+y+1}{(y-x)^2}=0$$

or, $y+x+\lim_{x\to\infty}\dfrac{2x^2-4x^2-2x^2}{(-x-x)^2}+\lim_{x\to\infty}\dfrac{x-x+1}{(-x-x)^2}=0$

or, $y+x-1+0=0$

or, $y+x=1$

Again the asymptote parallel to $y-x=0$ is

$$(y-x)^2-(y-x)\lim_{\substack{x\to\infty\\ y=x}}\frac{2(x+2y)}{y+x}+\lim_{\substack{x\to\infty\\ y=x}}\frac{x+y}{x+y}+\lim_{\substack{x\to\infty\\ y=x}}\frac{1}{x+y}=0$$

or, $(y-x)^2-(y-x).\ 3+1+0=0$

or, $(y-x)^2-3(y-x)+1=0$

Hence asymptotes are $x+y-1=0$ $y-x=\frac{1}{2}(3\pm\sqrt{5})$

Ex. 4. Find the asymptotes of the curve N.U. 1994

$(y-x)^2x-3y(y-x)+2x=0$ R. U. 1951, D.H. 1965.

The equation is of 3rd degree but y^3 is absent

So, there are asymptotes parallel to y—axis. The highest power of y available here is y^2

$$y^2x-2x^2y+x^3-3y^2+3xy+2x=0$$

or, $y^2(x-3)-2x^2y+x^2+3xy+2x=0$

Now equate the Co-efficients of y^2 to zero. Thus

$x-3=0$ is an asymptote.

For other asymptotes, from the original equation we notice that

$$P_3=(y-x)^2x$$

Hence the asymptotes are parallel to $y-x=0$, $x=0$.

The asymptotes parallel to $y-x=0$ are

$$(y-x)^2+(y-x)\lim_{\substack{x\to\infty\\y=x}}\frac{-3y}{x}+\lim_{\substack{x\to\infty\\y=x}}\frac{2x}{x}=0$$

or, $(y-x)^2+(y-x)(-3)+2=0$ or ; $(y-x-2)(y-x-1)=0$

or, $y-x-2=0$, $y-x-1=0$

Hence the asymptotes are $x=3$, $y-x-2=0$, $y-x-1=0$

Ex. 5. Find the asymptotes of the curve $y\sqrt{(x^2-1)}=x^2$

C. U. '88

The equation is written as $y^2(x^2-1)=x^4$ ... ... (1)

The equation (1) is of 4th degree, so it may have four asymptotes. As the highest power of y, *i. e.* :

y^4 is absent, so there are some asymptotes parallel to y—axis.

Now equate the co-efficient of y^2 (as y^2 is available higher power) to zero, then $x^2-1=0$ or, $x=\pm1$.

For two more asymptotes. the equation (1) is written as

$$x^4-y^2x^2-y^2=0$$

or, $x^2(x^2-y^2)-y^2=0$ ... (2)

or, $x^2(x+y)(x-y)-y^2=0$ ... (3)

The other two asymptotes are parllel to $x+y=0$, $x-y=0$

Now asymptote parallel to $x+y=0$ is

$$x+y+\underset{\substack{x\to\infty\\y=-x}}{\text{Lt}}\frac{-y^2}{x^2(y-x)}=0$$

or, $x+y+0=0$ or, $x+y=0$.

Similarly other asymptote is $x-y=0$.

Hence asymptotes are $x+1=0$, $x-1=0$, $x+y=0$, $x-y=0$

Ex. 6. Find the asymptotes of the curve

$$xy^2(x-y)-5x^2y-y^3+6x^2-5=0$$

The equation is of 4th degree, so there are four asymptotes. From the equation, the asymptotes are parallel to $x=0, y=0$, $x-y=0$

The asymptotes parallel to $x-y=0$ is

$$x-y+\lim_{\substack{x\to\infty\\y=x}}\frac{-y(5x^2+y^2)}{xy^2}+\lim_{\substack{x\to\infty\\y=x}}\frac{6x^2-5}{xy^2}=0$$

or, $x-y-6=0$ or, $x-y=6$

The equation is written as

$$x^2y^2-xy^3-5x^2y-y^3+6x^2-5=0 \quad \cdots \quad \cdots \quad (1)$$

or, $x^2(y^2-5y+6)-xy^3-y^3-5=0$ ... ... (2)

or, $y^3(-x-1)+y^2x^2-5yx^2+6x^2-5=0$ ... (3)

The equation is of 4th degree. The highest powers of x and y. *i. e*, x^4 and y^4 are absent. So there are some asymptotes parallel to co-ordinate axes.

Now equate the co-efficient of x^2 (available highest power of x) to zero from (2) then the asymptotes are

$y^2-5y+6=0$ or, $(y-3)(y-2)=0$ ∴ $y-2=0$ and $y-3=0$.

Again equate the co-efficient of y^2 (available highest power of y) to zero from (3). then the asymptote parallel to y axis is

$$-x-1=0 \quad \text{or,} \quad x+1=0$$

Thus the asymptotes are $y=2$, $y=3$, $x+1=0$, $x-y=6$.

Ex. 7. Find the asymptotes of the curve $y=\dfrac{\log x}{x}$

Let us define x in the interval $0<x<+\infty$

The function y is not discontinuous in the defined domain but when x tends to zero, then

$$\lim_{x\to 0} y = \lim_{x\to 0} \frac{\log x}{x} = -\infty$$

Hence the straight line $x=0$ *i. e.*; y—axis is the asymptotes of the given curve.

For an asymptote of the from $y=mx+c$, where m and c finite we have,

$$m=\lim_{x\to+\infty} (y/x)=\lim_{x\to\infty} (\log x)/x^2=0$$

$$\text{and } c=\lim_{x\to+\infty} (y-mx)=\lim_{x\to\infty}\left(\frac{\log x}{x}-mx\right)$$

$$=\lim_{x\to\infty}\left(\frac{\log x}{x^2}x-0\right)=0$$

$\therefore$ There is another asymptote which is $y=0$, *i. e.* the x—axis is also an asymptote of the curve.

The asymptotes are $x=0$, $y=0$.

Note : Note the asymptotes of the following curves-

(i) $y=a^x$; x—axis is the asymptote.

(ii) $y=\log_a x$; y—axis is the asymptote See Art. 1. 15.

Ex. 8. Find the asymptotes of the curve $y=e^{-x}\sin x+x$

There are no asymptotes parallel to the co-ordinate axes.

Let us try for obeique asymptote of the form $y=mx+c$

$$m=\lim_{x\to\infty}\frac{y}{x}=\lim_{x\to\infty}\frac{e^{-x}\sin x+x}{x}=0+1=1$$

$$c=\lim_{x\to\infty}(y-mx)=\lim_{x\to\infty}(e^{-x}\sin x+x-x)$$

$$=\lim_{x\to\infty} e^{-x}\sin x=0$$

Hence $y=x$ is the asymptote of the curve as $x\to\infty$.

When $x\to-\infty$.

$$m=\lim_{x\to-\infty}\frac{y}{x}=\lim_{x\to\infty}\left(\frac{e^{-x}\sin x}{x}+1\right)=\text{does not exist as}$$

the first term increases to infinity as $x\to-\infty$

So, there is no asymptote when $x\to-\infty$

Ex. 9. Find the asymptote of the curve $r(3\theta-\pi)=a\sin\theta$

The equation is written as

$$\frac{1}{r}=\frac{3\theta-\pi}{a\sin\theta}=f(\theta) \quad \ldots \quad (1)$$

For the asymptotes $r\to\infty$ and so $f(\theta)\to 0$

$$\Rightarrow \frac{3\theta-\pi}{a\sin\theta}=0 \quad \text{or,} \quad \theta=\tfrac{1}{3}\pi$$

We are to investigate the possibility of an asymptote at $\theta=\pi/3$

Differentiate (1) $w.r$ to θ, $f'(\theta)=\dfrac{3\sin\theta-(3\theta-\pi)\cos\theta}{a\sin^2\theta}$

When $\theta = \pi/3$,

$$f'(\pi/3) = \frac{3 \sin \pi/3 - (3\pi/3 - \pi) \cos \pi/3}{a \sin^2 \pi/3} = \frac{3\sqrt{3/4}}{2a.3} = \frac{2\sqrt{3}}{a}$$

Therefore the asymptote is

$$r \sin(\theta - \tfrac{1}{3}\pi) = 1/f'(\pi/3) = \frac{a}{2\sqrt{3}}$$

or ; $2\sqrt{3}\, r \sin(\theta - \pi/3) = a$

Ex. 10. Find the asymptotes of $r = \sec\theta + b \tan\theta$.

The equation is written as $r = a/\cos\theta + (b \sin\theta)/\cos\theta$

$$= (a + b \sin\theta)/\cos\theta$$

or ; $1/r = \cos\theta/(a + b \sin\theta) = f(\theta)$

If $f(\theta) = 0$, then $\cos\theta = 0$

$\therefore$ $\theta = (2n+1)\pi/2$, n is any integer.

$\therefore$ $\theta = \frac{1}{2}\pi, 3\pi/2, 5\pi/2, 7\pi/2$ and so on.

$$\text{Now } f'(\theta) = \frac{-\sin\theta\,(a + b \sin\theta) - \cos\theta\, b \cos\theta}{(a + b \sin\theta)^2} = \frac{-a \sin\theta - b}{(a + b \sin\theta)^2}$$

$f'(\pi/2) = -1/(a+b)$, $f'(3\pi/2) = (a-b)/(a-b)^2 = 1/(a-b)$,

$f'(5\pi/2) = -1/(a+b)$, $f'(7\pi/2) = 1/(a-b)$

Thus the equation of the asymptote at $\theta = \pi/2$ is

$r \sin(\theta - \pi/2) = 1/f'(\pi/2) = -(a+b)$ or ; $r \cos\theta = a + b$

the asymptote at $\theta = 3\pi/2$ is

$$r \sin(\theta - 3\pi/2) = a - b \quad \text{or ;} \quad r \cos\theta = a - b$$

Proceding this way we see that the asymptotes at $\theta = \pi/2, 5\pi/2, 9\pi/2$ each has equation

$$r \cos\theta = a + b$$

and asymptotes at $\theta = 3\pi/2$, $7\pi/2$, ete. each has equation

$$r \cos\theta = a - b.$$

Therefore there are only two different asymptotes whose equations are

$$r \cos\theta = a + b \text{ and } r \cos\theta = a - b$$

Ex. 11. Find the circular asymptotes of $(r-2)\,\theta = \sin\theta$

The circular asymptote is

$$\therefore \quad r - 2 = \lim_{\theta \to \infty} (3 \sin\theta)/\theta = 0 \quad \text{as } |\sin\theta| \leq 1$$

or ; $r = 2$

Ex. 12. Show that the asymptotes of the curve

$$(x + a)\, y^2 - (y + b) x^2 = 0$$

cut the curve in three points which lie on the straight line

$$b^2 (x + a) = a^2 (y + b)$$

The given equation is of 3rd degree and is written as

$$(x + a) y^2 - (y + b)\, x^2 = 0 \quad \dots \quad \dots \quad (1)$$

$$\text{or ;} \quad xy^2 - yx^2 + ay^2 - bx^2 = 0 \quad \dots \quad \dots \quad (2)$$

The asymptotes parallel to x axis is given by equating the co-efficient of x^2 to zero i. e $y + b = 0$ ... ... (3)

Similarly asymptote parallel to y axis is given by

$x + a = 0$ ... ... (4)

From eq. (2), $xy(y - x) + ay^2 - bx^2 = 0$ ... (5)

Now asymptote parallel to $y - x = 0$ is

$$y - x + \lim_{\substack{x \to \infty \\ y = x}} \frac{ay^2 - bx^2}{xy} = 0$$

or ; $y - x + a - b = 0$ ... ... (6)

Hence the asymptotes are

$x + a = 0$, $y + b = 0$, $y - x + a - b = 0$.

The joint equation of the asymptotes is

$$(x+a)(y+b)(y-x+a-b)=0$$

or ; $xy^2-x^2y-bx^2+ay^2-b^2x+a^2b-ab^2+a^2y=0$

or ; $(x+a)y^2-(y+b)x^2-b^2(x+a)+a^2(y+b)=0$

The equation of the given curve is written as

$(x+a)y^2-(y+b)x^2+\{a^2(y+b)-b^2(x+a)\}=0$

which is of the from $F_n+F_{n-2}=0$ i. e. $F_3+F_1=0$

Hence $3(3-2)$ i. e. ; 3 points of their intersection lie on the line $F_{n-2}=0$ or, $F_{n-2}=0$ or, $F_1=0$ i. e.

$$a^2(y+b)-b^2(x+a)=0$$

$\therefore$ $a^2(y+b)-b^2(x+a)=0$ Proved.

Ex. 13. Show that the asymptotes of the curve

$$x^2y^2-a^2(x^2+y^2)-a^3(x+y)+a^4=0$$

form a square, two of whose angular points lie on the curve.

The equation is of 4th degree and highest powers of x and y are absent. The equation is written as

$$x^2(y^2-a^2)-a^2y^2-a(x+y)+a^4=0 \quad \cdots \quad \cdots \quad (1)$$

or, $y^2(x^2-a^2)-a^2x^2-a(x+y)+a^4=0 \quad \cdots \quad \cdots \quad (2)$

The asymptotes parallel to x axis from (1). are $y=\pm a$

and asymptotes parallel to y-axis, from (2), are $x=\pm a$

The four asymptotes are $x=a$, $x=-a$, $y=a$, $y=-a$.

which form a square of sides $2a$.

Thus $A(a, a)$, $B(-a, a)$, $C(-a, -a)$, $D(a, -a)$ are vertices of the square $ABCD$

Put the co-ordinates of $B(-a, a)$ and $D(a, -a)$ satisfy the equation of the curve.

Hence $(a, -a)$ and $(-a, a)$ are on the curve.

Ex. 14. Show that the asymptotes of the curve

$$4(x^4+y^4)-17x^2y^2-4x(4y^2-x^2)+2(x^2-2)=0$$

pass through the point of intersection of the ellipse

$x^2+4y^2=4$ D. U. H. 1958, C. H. 77.

The equation is written as

$$(4x^4+4y^4-17x^2y^2)-4x(4y^2-x^2)+2(x^2-2)=0 \quad \cdots \quad (1)$$

Then put $x=1$, $y=m$ in the first and 2nd terms of (1)

$\phi_4(m)=4+4m^4-17m^2$; $\phi_2(m)=4(4m^2-1)$

$\phi_4(m)=(2-m)(2+m)(2m+1)(1-2m)$

$\therefore$ $\phi_4(m)=0 \Rightarrow$

or, $m=2, -2, -\frac{1}{2}, \frac{1}{2}$

$\phi'_4(m)=16m^3-34m$

Now $c=-\dfrac{\phi_3(m)}{\phi'_4(m)}=-\dfrac{+16m^3-4}{16m^2-34m}$

When $m=2, -2, -\frac{1}{2}, \frac{1}{2}$, then values of $c=1, +1, 0, 0$

Thus the asymptotes are $(y=mx+c)$

$y=2x-1$, $y=-2x+1$, $y=-\frac{1}{2}x$, $y=\frac{1}{2}x$

or ; $y-2x+1=0$. $y+2x-1=0$, $2y+x=0$, $2y-x=0$

The combined equation of all asymptotes is

$(y-2x+1)(y+2x-1)(2y+x)(2y-x)=0$

or ; $(y-2x)(y+2x)(2y+x)(2y-x)+4x(4y^2-x^2)-(4y^2-x^2)=0$

or , $4(x^4+y^4)-17x^2y^2+4x(4y^2-x^2)-(4y^2-x^2)=0 \quad \cdots \quad (2)$

The equation of the given curve is written as

$4(x^4+y^4)-17x^2y^2+4x(4y^2-x^2)-4y^2+x^2+(4y^2+x^2-4)=0$

or ; $F_4+F_2=0$

The asymptotes meet the curve at $n(n-2)=4(4-2)=8$ points and these points lie on the curve $F_{n-2}=0$ or, $F_{4-2}=F_2=0$

or, $4y^2+x^2-4=0$ or, $4y^2+x^2=4$.

Ex. 15. Show that the equation of the cubic which has the same asymptotes as the curve $x^2y-xy^2+xy+y^2+x-y=0$ and which passes through the points (0,0), (0,1), (1,0) is

$$yx^2-xy^2+xy+y^2+3y=0$$

The given equation is of 3rd degree and x^3 and y^3 both the terms are absent. There are some asymptotes parallel to co-ordinate axes.

The equation is written as

$$yx^2-xy^2+xy+y^2+3y=0 \quad \cdots \quad \cdots \quad (1)$$

$$\text{or ; } xy(x-y)+xy+y^2+3y=0 \quad \cdots \quad \cdots \quad (2)$$

$$y^2(-x+1)+yx^2+xy+3y=0 \quad \cdots \quad \cdots \quad (3)$$

From (2), the asymptotes are parallel to $x=0$, $y=0$, $x-y=0$.

Now asymptotes parallel to x-axis from (1) is $y=0$

Aaymptotes parallel to $y=x$ from (2) is

$$x-y+\lim_{\substack{x\to\infty \\ y=x}}\frac{xy+y^2}{xy}+\lim_{\substack{x\to\infty \\ y=x}}\frac{3y}{yx}=0 \quad \text{or,} \quad x-y+2=0$$

And asymptotes parallel to y-axis from (3) is $x=1$

$\therefore$ The asymptotes are $y=0$, $x=1$, $x-y+2=0$

and the joint equation of asymptotes is

$$y(x-1)(x-y+2)=0 \quad \cdots \quad \cdots \quad (4)$$

Let the equation of the new curve be

$$y(x-1)(x-y+2)+ax+by+c=0 \quad \cdots \quad (5)$$

Since it passes through (0,0), (1,0) (0,1)

then, $c=0$, $a=0$, $b=1$

Hence the new curve which has the same asymptotes as (1) is

$$y(x-1)(x-y+2)+y=0.$$

Exercise X11

Find the asymptotes of following curves.

1. $2y^3-2x^2y-4xy^2+4x^3-14xy+6y^2+4x^2+6y+1=0$
2. $3x^3+2x^2y-7xy^2+2y^3-14xy+7y^2+4x+5y=0$
3. $y^3-6xy^2+11x^2y-6x^3+x+y=0$
4. $x^3+x^2y-xy^2-y^3+x^2-y^2-2=0$
5. $x^3-2y^3+xy(2x-y)+y(x-y)+1=0$

6. $y^3-xy^2-x^2y+x^3+x^2-y^2-1=0$

(i) $x^3+2x^2y-xy^2-2y^3+4y^2+2xy-5y+6=0$ D.U. 1991

7. $x^3-5x^2y+8xy^2-4y^3+x^2-3xy+2y^2-7=0$

(i) $x^3-y^3-x^2+2y^2=0$ N.U. 1994

8. $xy^2-x^2y=a^2(x+y)+b^2$ (i) $y^2(x^2-y^2)-2ay^3+2a^3x=0$ N.U. 1995

9. $x^3+2x^2y+xy^2-x^2-xy+2=0$ D.U. 1959

10. $x^2y^2=a^2y^2-b^2x^2$ (ii) $y(x^2-y^2)=y(x-y)+2$ N.H. 1994

(i) $x^3-2x^2y+xy^2+x^2-xy+2=0$ D.U.1977

11. $xy^2+x^2y+xy+y^2+3x=0$
12. $x^3-xy^2+6y^2=0$
13. $y^2(x^2-a^2)=x^2(x^2-4a^2)$
14. $y^2(x^2-a^2)=x$ R. U. 1979
15. $y^3+x^2y+2xy^2-y+1=0$ C. U. 1969

(i) $x^4-x^2y^2+x^2+y^2-a^2=0$

16. $x^2y-xy^2+xy+y^2+x-y=0$
17. $x^2y^2=x^2+y^2$ 18. $x^2y=x^3+x+y$
19. $y^2=x^3+ax^2$ 20. $x^2y^2-xy^2+x+y+1=0$
21. $x^3+y^3=3axy$ 22. $x^3-y^3=a^2xy$
23. $\{y/(x+a)\}^3=(x-a)/(x-2a)$
24. $(x-y)^2(x-2y)(x-3y)-2a(x^3-y^3)-2a^2(x-2y)(x+y)=0$ D. H, 1961
25. $y^4-2xy^3+2x^3y-x^4-3x^3+3x^2y+3xy^3-3y^3-2x^2+2y^2-1=0$

26. $x^2(y^2-4)-4y^2+8xy+2y-3=0$

27. $x^2y^2-9(x^2+y^2)-5(x+y)+27=0$

28. $x^2(y+3)-y^2(x+2)=0.$

29. $x^2(x^2+y^2-2xy)-2x^3-2y^2=0$ parallel to $y=x$.

30. $x^2(x-y)^2-a^2(x^2+y^2)=0$

31. $(x+3y)^2(2x+y)+2x-3y+7=0$ D. U. 1984

32. $x(y-3)^3=4y(x-1)^3$ 33. $2x(y-3)^3-3y(x-1)^2=0$

34. $xy(x^2-y^2)(x^2-4y^2)+3xy(x^2-y^2)+x^2+y^2-7=0$

35. $x^3+3x^2y-xy^2-3y^3+x^2-2xy+3y^2+4x+7=0$

36. $y=e^{1/x}$ 37. $y=x+\log x$

38. $y=e^{-x^2}$ 39. $y=e^{ax}$

40. $y=e^{1/x}-1$ 41. $y=\log x$

42. $y=e^{-2x}\sin x$ 43. $y=xe^{1/x^2}$

44. $y=e^{-x}\sin 2x+x.$

Find the rectilinear asymptotes of the following curves.

45 $r\theta=a$ 46. $r\cos\theta=a\theta$

47. $r\sin\theta/2=a$ 48. $r\cos 3(\theta+\alpha=a\sin(\theta+\alpha)$

49 $r\sin n\theta=a$ 50. $r=4(1-\sec 2\theta)$

51. $r=a(\sec\theta+\cos\theta)$ 52. $r=a\dfrac{\theta-\alpha}{\theta+\alpha}$

53. $r=a\dfrac{\sin^2\theta}{\cos\theta}$ 54. $r\tan 3\theta=\alpha$

55. $r^n\sin n\theta=a^n$ 56. $r\cos\theta=a\sin\theta$

57. $r(\frac{1}{2}-\cos\theta)=a$

Find the circular asymptotes of the following curves.

58. $r=\dfrac{\theta}{\theta+1}$ 59. $r=\dfrac{\theta^2-1}{\theta^3+1}$

60. $r=\dfrac{5\theta^2+3\theta+3}{3\theta^2+7\theta+5}$ 61. $(r-1)(\theta-1)=1$

62. $r(e^\theta-1)=a(e^\theta+1).$

63. (i) Show that the asymptotes of

$(x^2-y^2)^2=2(x^2+y^2)$ form a square

63. (ii) Show that the asymptotes of the curve

$(x^2-y^2)y-2ay^2+5x-7=0$

form a triangle of area equal to a^2 R.U.H. 1969. C. H. 1969

(iii) Show that the asymptotes of the curve $x^2y^2=a^2(x^2+y^2)$ form a square of the side $2a$ R. H. 1986 ; R. U. 1972 '87

(iv) Show that the asymptotes of $x^2y^2=9(x^2+y^2)$ form a square of area 36 square units. R. H. 1988

64. Show that the asymptotes of the curve

$x^3-2y^3+xy(2x-y)+y(x-y)+1=0$

cut the curve in three points which lie on the straight line, $x-y+1=0.$

65. Show that the asymptotes of the curve

$x^2y-xy^2+xy+y^2+x-y=0$

cut the curve in three points which lie on the straight lins $x+y=0$

66. Show that points of intersection of the curve

$2y^3-2x^2y-4xy^2+4x^3-14xy+6y^2+4x^2+6y+1=0$

and its asymptotes lie on the straight line, $8x+2y+1=0$

67. Show that the asymptotes of the curve

$x^4-5x^2y^2+4y^4+x^2-y^2+x+y+1=0$

cut the curve again in eight point lying upon a rectangular hyperbola. $x^2-y^2+x+y=0.$

68. Show that the asymptotes of the curve

$(x^2-4y^2)(x^2-9y^2)+5x^2y-5xy^2-30y^3+xy+7y^2-1=0$

cut the curve in the eight points lying on a circle, $x^2+y^2=1.$

69. Show that the points of intersection of the curve

$$4x^4 - 13x^2y^2 + 9y^4 + 32x^2y - 42y^3 - 20x^2 + 74y^2 - 56y + 4x + 16 = 0$$

and its asymptotes lie on the curve $y^2 + 4x = 0$.

70. Show that the equation of the curve which has the same asymptote as the curve

$$x^3 - 6x^2y + 11xy^2 - 6y^3 + 4x + 5y + 7 = 0$$

and which passes through the points (0, 0), (2, 0), (0, 2) is

$$x^3 - 6x^2y + 11xy^2 - 6y^3 - 4x + 24y = 0$$

71. Show that there is an infinite series of parallel asymptotes to the curve $r = \frac{a}{\theta \sin \theta} + b$, and show that their distances from the pole are in Harmonic progression.

72. Show that all the asymptote of the curve $r \tan n\theta = a$ touch a circle $r = a/n$.

73. Find the equation of the curve which has $x = 0$, $y = 0$, $y = x$, $y + x = 0$ for asymptotes, which passes through the point (a, b) and which cuts its asymptotes again in lying upon the circle $x^2 + y^2 = a^2$.

Find the asymptote of the following curves.

74. (i) $xy(x^2 - y^2) = x^2 + y^2$ (a) $x^2y^2 - 9(x^2+y^2) - 5(x+y) + 27 = 0$ C.H. 1992

(ii) $y^2x^2 - 3yx^2 - 5xy^2 + 2x^2 + 6y^2 - x - 3y + 2 = 0$ R. U. 1982

(iii) $x^2y^2 - x^2y - xy^2 + 2x - 3y + 4 = 0$ R. U. 1983

(iv) $xy + (x^2 - 4y^2) = 2x^2 + 7y^2$ R. U. 1984

(v) $x^3 = 8y^3 + 3x^2 + y^2 - 7x + 2 = 0$ C. U. 1984, 92

(vi) $y = a \log \sec x/a$ D. H. 1983

(vii) $2x(y-5)^2 = 3(y-2)(x-1)^2$ C. U. 1987

(viii) $x^3 - x^2y - xy^2 + y^3 + 2x^2 - 4y^2 + 2xy + x + y + 1 = 0$ C.H. 1993

প্রশ্নমালা XII

নিম্নলিখিত বক্ররেখাগুলির তটরেখা সমূহ নির্ণয় কর।

1. $2y^3 - 2x^2y - 4xy^2 + 4x^3 - 14xy + 6y^2 + 4x^2 + 6y + 1 = 0$
2. $3x^3 + 2x^2y - 7xy^2 + 2y^3 - 14xy + 7y^2 + 4x + 5y = 0$
3. $y^3 - 6xy^2 + 11x^2y - 6x^3 + x + y = 0$
4. $x^3 + x^2y - xy^2 - y^3 + x^2 - y^2 - 2 = 0$
5. $x^3 - 2y^3 + xy(2x - y) + y(x - y) + 1 = 0$
6. $y^3 - xy^2 - x^2y + x^3 + x^2 - y^2 - 1 = 0$

6(i) $x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy - 5y + 6 = 0$ D.U 1991

7. $x^3 - 5x^2y + 8xy^2 - 4y^3 + x^2 - 3xy + 2y^2 - 7 = 0$ R. U. 1962
8. $xy^2 - x^2y = a^2(x + y) + b^2$ D. U. 1959
9. $x^3 + 2x^2y + xy^2 - x^2 - xy + 2 = 0$
10. (i) $x^2y^2 = a^2y^2 - b^2x^2$ D. U. 1977

 (ii) $x^3 - 2x^2y + xy^2 + x^2 - xy + 2 = 0$

11. $xy^2 + x^2y + xy + y^2 + 3x = 0$
12. $x^3 - xy^2 + 6y^2 = 0$
13. $y^2(x^2 - a^2) = x^2(x^2 - 4a^2)$ (i) $y^2(x^2 - y^2) - 2ay^2 + 2a^3x = 0$ N.U. 1995
14. $y^2(x^2 - a^2) = x$ (i) $y(x^2 - y^2) = y(x - y) + 2$ N.U. 1994. R. U. 1979
15. (i) $y^3 + x^2y + 2xy^2 - y + 1 = 0$ R. U. 1960

 (ii) $x^4 - x^2y^2 + x^2 + y^2 - a^2 = 0$

16. $x^2y - xy^2 + xy + y^2 + x - y = 0$
17. $x^2y^2 = x^2 + y^2$
18. $x^2y = x^3 + x + y$
19. $y^3 = x^3 + ax^2$
20. $x^2y^2 - xy^2 + x + y + 1 - x^2y = 0$
21. $x^3 + y^3 = 3axy$
22. $x^5 - y^5 = a^2xy$
23. $\left\{\frac{y}{x+a}\right\}^3 = \frac{x-a}{x+2a}$
24. $(x - y)^2(x - 2y)(x - 3y) - 2a(x^3 - y^3) - 2a^2(x - 2y)(x + y) = 0$ D. H. 1961
25. $y^4 - 2xy^3 + 2x^3y - x^4 - 3x^3 + 3x^2y + 3xy^2 - 3y^3 - 2x^2 + 2y^2 - 1 = 0$
26. $x^2(y^2 - 4) - 4y^2 + 8xy + 2y - 3 = 0$
27. $x^2y^2 - 9(x^2 + y^2) - 5(x + y) + 27 = 0$
28. $x^2(y + 3) - y^2(x + 2) = 0.$

29. $x^2(x^2+y^2-2xy)-2x^3-2y^2=0$, $y=x$ এর সমান্তরাল।

30. $x^2(x-y)^2-a^2(x^2+y^2)=0$

31. $(x+3y)^2(2x+y)+2x-3y+7=0$ D. H. 1954

32. $x(y-3)^3=4y(x-1)^3$, 33. $2x(y-3)^2-3y(x-1)^2=0$

34. $xy(x^2-y^2)(x^2-4y^2)+3xy(x^2-y^2)+x^2+y^2-7=0$

35. $x^3+3x^2y-xy^2-3y^3+x^2-2xy+3y^2+4x+7=0$

36. $y=e^{1/x}$ 37. $y=x+\log x$

38. $y=e^{-x^2}$ 39. $y=e^{ax}$

40. $y=e^{1/x}-1$ 41. $y=\log x$

42. $y=e^{-2x}\sin x$ 43. $y=xe^{1/x^2}$

44. $y=e^{-x}\sin 2x+x$.

নিম্নলিখিত বক্ররেখাগুলির সরলরৈখিক তটরেখাসমূহ নির্ণয় কর।

45. $r\theta=a$ 46 $r\cos\theta=a\theta$

47. $r\sin\theta/2=a$ 48. $r\cos 3(\theta+\alpha)=a\sin(\theta+\alpha)$

49. $r\sin n\theta=a$ 50. $r=4(1-\sec 2\theta)$

51. $r=a(\sec\theta+\cos\theta)$ 52. $r=a\dfrac{\theta-\alpha}{\theta+\alpha}$

53. $r=a\dfrac{\sin^3\theta}{\cos\theta}$ 54. $r\tan 3\theta=\alpha$

55. $r^n\sin n\theta=a^n$ 56. $r\cos\theta=a\sin\theta$

57. $r(\frac{1}{2}-\cos\theta)=a$

নিম্নলিখিত বক্ররেখাগুলির বৃত্তীয় তটরেখা সমূহ নির্ণয় কর।

58. $r=\dfrac{\theta}{\theta+1}$ 59. $r=\dfrac{\theta^3-1}{\theta^3+1}$

60. $r=\dfrac{5\theta^2+3\theta+3}{3\theta^2+7\theta+5}$ 61. $(r-1)(\theta-1)=1$

62. $r(e^\theta-1)=a(e^\theta+1)$.

63. (i) দেখাও যে বক্ররেখা $(x^2-y^2)^2=2(x^2+y^2)$ এর তটরেখা সমূহ একটি বর্গক্ষেত্র তৈরী করে। C.U. 1993

(ii) দেখাও যে বক্ররেখা $(x^2-y^2)y-2ay^2+6x-7=0$ এর তটরেখা সমূহ একটি ত্রিভুজ তৈরী করে যাহার ক্ষেত্রফল সমান a^2, N.H. 1993.

R. U. H. 1969, 92, C. H. '69

(iii) দেখাও যে, বক্ররেখা $x^2y^2=a^2(x^2+y^2)$ এর তটরেখাসমূহ একটি বর্গক্ষেত্র তৈরী করে যাহার বাহু $2a$ R. U. 1972

64 দেখাও যে বক্ররেখা $x^3-2y^3+xy(2x-y)+y(x-y)+1=0$ এর তটরেখা সমূহ বক্ররেখাটিকে এমন তিনটি বিন্দুতে ছেদ করে যাহারা সরলরেখা $x-y+1=0$ এর উপর থাকে।

65 দেখাও যে বক্ররেখা $x^2y-xy^2+xy+y^2+x-y=0$ এর তটরেখা সমূহ বক্ররেখাটিকে এমন তিনটি বিন্দুতে ছেদ করে যাহারা সরলরেখা $x+y=0$ এর উপর অবস্থিত।

66. দেখাও যে বক্ররেখা $2y^3-2x^2y-4xy^2+4x^3-14xy+6y^2+4x^2+6y+1=0$ এবং এর তটরেখা সমূহের ছেদবিন্দুগুলি $8x+2y+1=0$ সরলরেখার উপর অবস্থিত।

67. দেখাও যে বক্ররেখা $x^4-5x^2y^2+4y^4+x^2-y^2+x+y+1=0$ এর তটরেখা সমূহ বক্ররেখাটিকে আটটি বিন্দুতে ছেদ করে যাহারা $x^2-y^2+x+y+1=0$ এই আয়ত অধিবৃত্তটির (Rectangular hyperbola) উপর অবস্থিত।

68. দেখাও যে বক্ররেখা $(x^2-4y^2)(x^2-9y^2)+5x^2y-5xy^2-30y^3+xy+7y^2-1=0$ এবং এর তটরেখা সমূহের আটটি ছেদবিন্দু আছে যাহার $x^2+y^2=1$ এই বৃত্তটির উপর অবস্থিত।

69. দেখাও যে বক্ররেখা $4x^4-13x^2y^2+9y^4+32x^2y-42y^3-20x^2+74y^2-56y+4x+16=0$ এবং ইহার তটরেখা সমূহের ছেদবিন্দুগুলি বক্ররেখা $y^2+4x=0$ এর উপর অবস্থিত।

70. একটি বক্ররেখার তটরেখাগুলি এবং $x^3-6x^2y+11xy^2-6y^3+4x+5y+7=0$ বক্ররেখার তটরেখাগুলি একই। ঐ বক্ররেখা (0, 0), (2, 0), (0, 2) বিন্দুগুলি দিয়া অতিক্রম করিলে দেখাও যে ইহার সমীকরণ হইবে

$$x^3-6x^2y+11xy^2-6y^3-4x+24y=0.$$

71. দেখাও যে বক্ররেখা $r=\dfrac{a}{\theta\sin\theta}+b$ এর অসীম সংখ্যক সমান্তরাল তটরেখা আছে এবং আরো দেখাও যে মেরু বিন্দু বা মূল বিন্দু হইতে (Pole) ইহাদের দূরত্ব একটি ধারাবাহিক শ্রেণী গঠন করে।

72. দেখাও যে বক্ররেখা $r\tan n\theta = a$ এর সমস্ত তটরেখাগুলি বৃত্ত $r=a/n$ কে স্পর্শ করে।

73. একটি বক্ররেখার সমীকরণ নির্ণয় কর যাহার তটরেখাগুলি হইল $x=0, y=0, y=x, y+x=0$, যাহা (a, b) বিন্দুদিয়া যাইবে এবং ঐ বক্ররেখা ও তটরেখাগুলির ছেদ বিন্দু বৃত্ত $x^2+y^2=a^2$ এর উপর অবস্থিত হইবে।

74. নিম্নলিখিত বক্ররেখাগুলির তটরেখাগুলি নির্ণয় কর :—

(i) $xy(x^2-y^2)=x^3+y^2$

(ii) $y^2x^2-3yx^2-5xy^2+2x^2+6y^2-x-3y+2=0$ R. U. 1982

(iii) $x^2y^2-x^2y-xy^2+2x-3y+4=0$ R. U. 1983

(iv) $xy+(x^2-4y^2)=2x^2+7y^2$ R. U. 1984

(v) $x^3-8y^3+3x^2+y^2-7x+2=0$ C. U. 1984

(vi) $y=a\log\sec x/a$. D. U. 1983

(vii) $2x(y-5)^2=3(y-5)(x-1)^2$ C. U. 1987

(viii) $x^3-x^2y-xy^2+y^3+2x^2-4y^2+2xy+x+y+1=0$ C. H-1993

উত্তরমালা XII

1. $y=2x, y=-x-2, y=x-1$
2. $6y=6x-7, 2y=6x+15, 2y+x+1=0$
3. $y=x, y=2x, y=3x$, 4. $y=x, y+x=0, y+x+1=0$
5. $x-y=0, x+y+1=0, x+2y-1=0$
6. $y+x=0, y=x, y-x=1$
7. $y-x=0, 2y=x, 2y=x+1$
8. $x=0, y=0, y=x$.
9. $x=0, x+y=0, x+y=1$

10(i) $x=\pm a$. (ii) $x=0, x-y=0, x-y+1=0$

11. $y=0, x+1=0, x+y=0$
12. $x=6$, $y+x=3, y=x+3$
13. $x=\pm a, y=\pm x$. (i) $y=0, y=0, x+y+a=0, x-y-a=0$

14. $y=0, y=0, x=a, x=-a$
15. (i) $x+y=1, x+y+1=0$
15. (ii) $x=\pm 1, x\pm y=0$
16. $y=0, x-1=0, y=x+2$
17. $y=\pm 1, x=\pm 1$.
18. $x=\pm 1, y=x$ 19. $y=x+a/3$
20. $x=0, x=1, y=0, y=1$ 21. $x+y=a$
22. $y=x$ 23. $x+2a=0, y=x$
24. $x-y=a, x-y=2a, x-2y=13a, x-2y+14a=0$
25. $y=\pm x, y=x+1, y=x+2$ 26. $y=\pm 2, x=\pm 2$
27. $x=\pm 3, y=\pm 3$, 28. $x+2=0, y=x+1, y+3=0$
29. $x-y=\pm 2$ 30. $x=\pm a; x-y=\pm\sqrt{2}a$
31. $2x+y=0$ 32. $x=0, y=0. 2y=4x+3, 2y+4x=15$
33. $y=0, x=0, 2y=3x+9$ 34. $y=\pm x, 2y=\pm x, x=0, y=0$
35. $4x-4y+1=0, 2x+2y-3=0, 4x+12y+9=0$
36. $x=0$ 37. $x=0$ 38. $y=0$ 39. $y=0$
40. $x=0, y=0$ 41. $x=0$ 42. $y=0$ 43. $y=x, x=0$
44. $y=x$ 45. $r\sin\theta=a$ 46. $r\cos\theta=-a\pi/2, 3a\pi/2$
47. $r\sin\theta=2a$ 48. $a=6r\sin(\frac{1}{6}\pi-\alpha-\theta), -a=3r\sin(\frac{1}{2}\pi-\alpha-\theta)$; $a=6r\sin\left(\frac{5\pi}{6}-\alpha-\theta\right)$ 49. $rn\sin\left(\theta-\frac{m}{n}\pi\right)=a\sec m\pi$
50. $r\sin(\theta-\pi/4)=\pm 2$, $r\cos(\theta-\frac{1}{2}\pi)=\pm 2$. 51. $r\cos\theta=a$
52. $2ax=-r\sin(\theta+a)$ 53. $a=r\cos\theta$
54. $\theta=0, \pi/3, 2\pi/3$ ইত্যাদি।

এখানে ছয়টি তটরেখা আছে যারা একটি সুষম ষড়ভুজ তৈরী করে যাহার কেন্দ্র মূলবিন্দু O

55. তটরেখাগুলির সমীকরণ $\theta=\frac{k\pi}{n}$ যদি $n\geq 1$ হয়। কিন্তু যদি $n<1$ হয় তবে কোন তটরেখা হইবেনা।

56. $r\cos\theta=\pm a$. 57. $\pm 4a=\sqrt{r}(3\sin\theta\pm 3\cos\theta)$

58. $r=1$ 59. $r=1$ 60. $r=5/3$. 61. $r=1$ 62. $r=\mp a$.

73. $bxy(y^2-x^2)+a(a^2-b^2)(x^2+y^2-a^2)=0$

74. (i) $x=0,\ y=0,\ x+y-2=0,\ x+y+2=0$

(ii) $x-2=0,\ x-3=0,\ y-1=0,\ y-2=0$

(iii) $x=0,\ y=0,\ x=1,\ y=1.$

(iv) $x=0,\ y=0,\ x\pm 2y=0$

(v) $24x-48y+13=0$

(vi) $x=\pm\pi a/2$. (vii) $x=0,\ y=2,\ 2y-3x-14=0$

— 0 —

CHAPTER XIII

CURVATURE

13.1. Definitions :—

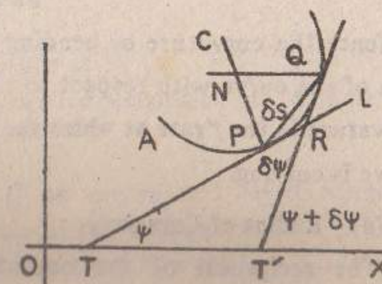
(*a*) **Angle of Contignance.**

Let P and Q be two neighbouring points on the curve APQ.

Let arc $AP=s$,

arc $AQ=s+\delta s$

then arc $PQ=\delta s$.



Let the tangents PT and QT' at P and Q of the curve make angles ψ and $\psi+\delta\psi$ respectively with the positive direction of X-axis

$$\angle QRL=\angle TRT'=\angle RT'X-\angle RTT'=\psi+\delta\psi-\psi=\delta\psi$$

Thus $\delta\psi=(\psi+\delta\psi-\psi)$ is the change in inclination of the tangent line as the point of contact of the tangent describes the arc δs. The angle $\delta\psi$ is called the **angle of contignance** of PQ provided the bending of the curve between P and Q is continuous in one direction only.

(b) Average Curvature or Average bending

The average curvature of arc PQ is the ratio of the corresponding angle of contignence $\delta\psi$ to the length of the arc δs, that is average curvature or average bending of the arc $PQ=\frac{\delta\psi}{\delta s}$

For one and the same curve the average curvature of its difference parts may be different.